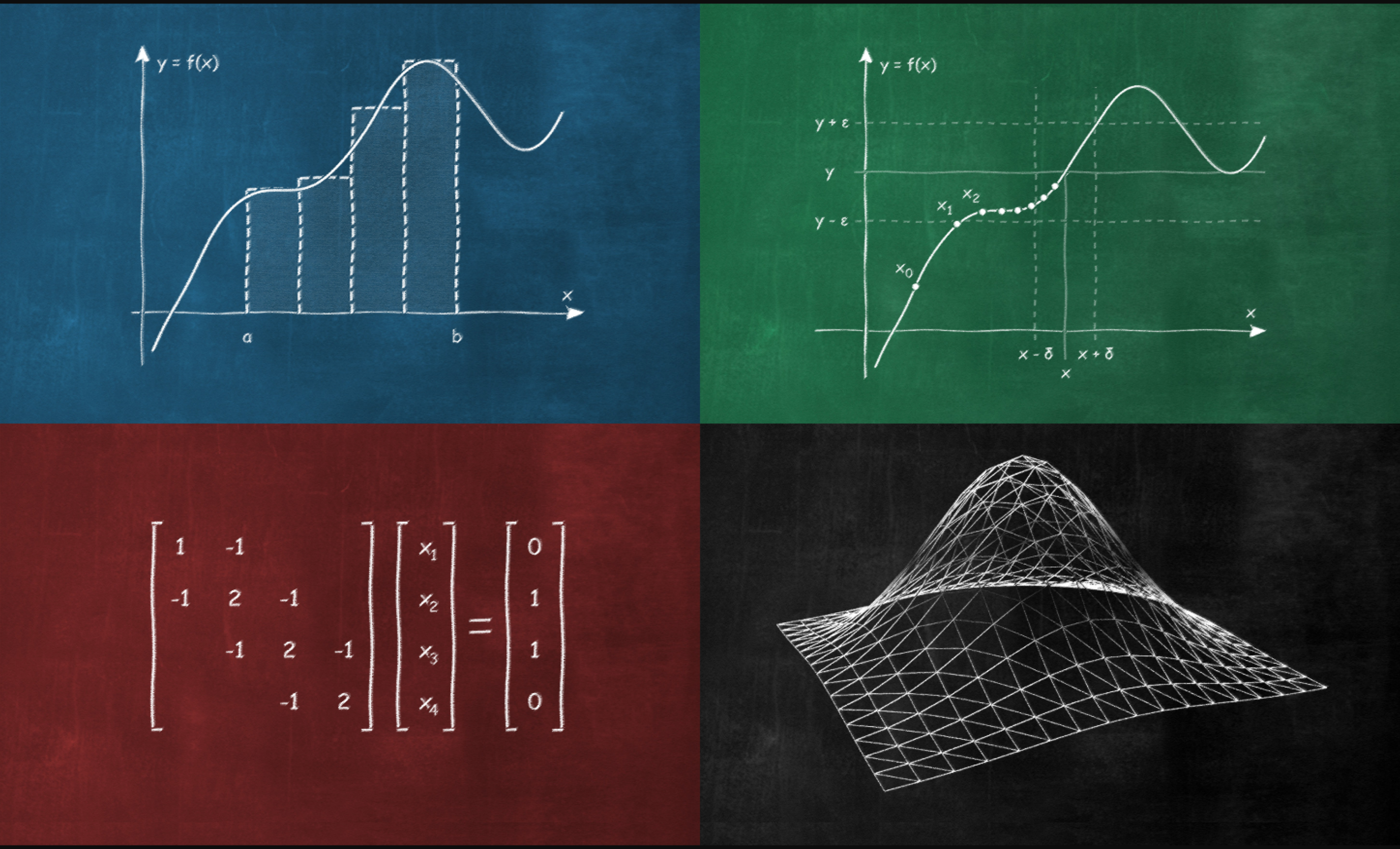
Analys & linjär algebra



Lösningsmanual

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Innehåll

Kapitel 1

Övningar

Ö1.1 (a) 1 (b) $\sqrt{14}$ (c) $\sqrt{n}$ (d) $\sqrt{\frac{(2n+1)n(n+1)}{6}}$

Ö1.2 (a) 1 (b) 3 (c) 1 (d) n

Ö1.3 —

Ö1.4 —

Ö1.5 (a) $\mathcal{D}(f) = \{x \in \mathbb{R}^2 \mid x_1 \neq x_2\}$, $\mathcal{R}(f) = \mathbb{R}$
(b) $\mathcal{D}f = \mathbb{R}^2$, $\mathcal{R}f = \mathbb{R}^+$
(c) $\mathcal{D}f = \mathbb{R}^2$, $\mathcal{R}f = (0, 1]$
(d) $\mathcal{D}(f) = \{x \in \mathbb{R}^2 \mid (x_1, x_2) \neq (0, 0)\}$, $\mathcal{R}f = \mathbb{R}^+$

Ö1.6 (a) En cirkel i origo med radie 1
(b) En cirkel i $(2, 0)$ med radie 2
(c) En kvadrat med hörn i $(0, -1)$, $(1, 0)$, $(0, 1)$ och $(-1, 0)$
(d) Punkten $(0, 0)$

Ö1.7 Inre punkt, yttre punkt, randpunkt, öppen, sluten, begränsad:
(a) $(1, 1)$, $(0, 0)$, $(1, 0)$, ja, nej, nej
(b) $(1, 0)$, $(0, 1)$, $(0, 0)$, nej, ja, ja
(c) $(3, 0)$, $(3/2, 0)$, (1.0), nej, nej, nej
(d) $(1, 0)$, saknas, $(0, 0)$, ja, nej, nej

Ö1.8 (a) 0 (b) saknas (titta på $f(x_1, 0)$ och $f(0, x_2)$)
(c) 0 (d) saknas (titta på $x_2 = -x_1 + kx_1^2$)

Lösningar:

$$\text{(c) } |f(x) - 0| = \left|\frac{x_1 x_2^2}{x_1^2 + x_2^2} - 0\right| = |x_1| \underbrace{\left|\frac{x_2^2}{x_1^2 + x_2^2}\right|}_{\leq 1} \leq |x_1| \to 0$$

Ö1.9 (a) existerar (b) existerar ej (c) existerar ej (d) existerar ej

Lösningar: Vi använder det kända gränsvärdet $\lim\limits_{t\to 0}\dfrac{\sin(t)}{t}=1$.

(a) $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin(x^2+y^2)}{x^2+y^2}$
$=\left\{t=x^2+y^2\right\}=\lim\limits_{t\to 0}\dfrac{\sin(t)}{t}=1$

(b) $\lim\limits_{x\to 0} f(x,kx)=\lim\limits_{x\to 0}\dfrac{\sin(x^2+k^2x^2)}{x^2}$
$=\lim\limits_{x\to 0}(1+k^2)\dfrac{\sin((1+k^2)x^2)}{(1+k^2)x^2}$
$=\left\{t=(1+k^2)x^2\right\}=\lim\limits_{t\to 0}(1+k^2)\dfrac{\sin(t)}{t}=1+k^2,$
olika värden för olika k

(c) $\lim\limits_{x\to 0} f(x,x)=\lim\limits_{x\to 0}\dfrac{x^3}{x^4+x^2}=\lim\limits_{x\to 0}\dfrac{x}{x^2+1}$
$=\lim\limits_{x\to 0} x\lim\limits_{x\to 0}\dfrac{1}{x^2+1}=0\cdot 1=0$
$\lim\limits_{x\to 0} f(x,x^2)=\lim\limits_{x\to 0}\dfrac{x^4}{x^4+x^4}=\lim\limits_{x\to 0}\dfrac{1}{2}=\dfrac{1}{2}$
olika värden då $(x,y)\to(0,0)$ längs $y=x$ och $y=x^2$

(d) $f(0,y)=0\to 0$ då $y\to 0$, $f(x,0)=\dfrac{|x|}{\sqrt{x^2}}=1\to 1$ då $x\to 0$

Ö1.10 (a) ja (b) nej (c) ja (d) nej

Ö1.11 (a) $3/5$ (b) $-4/5$ (c) $16/125$ (d) $9/125$

Ö1.12 (a) $27/8$ (b) $-3/20$ (c) 18
(d) $u'_x=6x\pi^{-1}\cos(3\pi x^2)\cos(12\pi y^2)\sin(7\pi z^2)$
$u''_{zx}=84xz\cos(3\pi x^2)\cos(12\pi y^2)\cos(7\pi z^2)$
$u''_{zx}(1,1,1)=84\cdot(-1)\cdot 1\cdot(-1)=84$

Ö1.13 (a)

$$f'(x)=\begin{bmatrix}-\mathrm{e}^{-x_1}\sin(x_2), & \mathrm{e}^{-x_1}\cos(x_2)\end{bmatrix}$$
$$L[f,\bar{x}](x)=f(\bar{x})+f'(\bar{x})(x-\bar{x})=0+\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=x_2$$

(b)

$$f'(x) = \begin{bmatrix} 2x_1 & 2x_3 & 2x_3 \end{bmatrix}$$

$$L[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = 3 + \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{bmatrix}$$

$$= -3 + 2x_1 + 2x_2 + 2x_3$$

(c)

$$f'(x) = \begin{bmatrix} 2 + x_2 & x_1 \\ 3x_2 & 1 + 3x_1 \end{bmatrix}$$

$$L[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$$

(d)

$$f'(x) = \begin{bmatrix} \cos(x_1) & -\sin(x_2) \\ -\sin(x_1) & \cos(x_2) \end{bmatrix}$$

$$L[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ö1.14 (a)

$$f'(x) = \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}$$

$$L_{\bar{x}}[f](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} (x - \pi/2)$$

(b)

$$f'(x) = \begin{bmatrix} 1 \\ 2x \end{bmatrix}$$

$$L_{\bar{x}}[f](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

(c)

$$f'(x) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \mathrm{e}^{x_2} & x_1\mathrm{e}^{x_2} \end{bmatrix}$$

$$L[f,\bar{x}](x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) = \begin{bmatrix} 1 \\ 2 \\ 1 + \mathrm{e} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \mathrm{e} & \mathrm{e} \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

(d) $\begin{bmatrix} 2x_2 - 1 \\ 2x_1 - 1 \end{bmatrix}$

Ö1.15 (a) $z = 1$ (b) $z = x$ (c) $z = 3+2(x-1)+4(y-1)$ (d) $z = \frac{1}{2}-\frac{1}{2}(x-1)-\frac{1}{2}(y-1)$

Lösningar:

$$\begin{aligned}
&\text{(c) } f(x,y) = x^2 + 2y^2,\ f(1,1) = 3\\
&\quad f'_x(x,y) = 2x,\ f'_x(1,1) = 2,\ f'_y(x,y) = 4y,\ f'_y(1,1) = 4\\
&\quad z = 3 + \begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}\\
&\text{(d) } f(x,y) = (x^2+y^2)^{-1},\ f(1,1) = \frac{1}{2}\\
&\quad f'_x(x,y) = -(x^2+y^2)^{-2}\,2x,\ f'_x(1,1) = -\frac{1}{2}\\
&\quad f'_y(x,y) = -(x^2+y^2)^{-2}\,2y,\ f'_y(1,1) = -\frac{1}{2}\\
&\quad z = \frac{1}{2} + \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}
\end{aligned}$$

Ö1.16 (a) $6xz^2$ (b) 0 (c) $6x^2$ (d) 0

Ö1.17 (a) $f''_{13}(x,y,z) = 4xy^2z\mathrm{e}^{x^2y^2}\cos(z^2)$
(b) $f''_{32}(x,y,z) = 4x^2yz\mathrm{e}^{x^2y^2}\cos(z^2)$
(c) $f'''_{113}(x,y,z) = 4y^2z(1+2x^2y^2)\mathrm{e}^{x^2y^2}\cos(z^2)$
(d) $f'''_{131}(x,y,z) = 4y^2z(1+2x^2y^2)\mathrm{e}^{x^2y^2}\cos(z^2)$

Ö1.18 (a) ja (b) ja (c) ja (d) nej

Ö1.19 (a) $g(\pi) = (0,-1)$
$f'(x,y) = (y,x)$ ger $f'(0,-1) = (-1,0)$
$g'(t) = (\cos(t), -\sin(t))$ ger $g'(\pi) = (-1,0)$
Kedjeregeln ger $(f \circ g)'(\pi) = (-1,0)\cdot(-1,0) = 1$
(b) 0 (c) 7 (d) 16

Ö1.20 (a) 0 (b) $2t(1-t^2)\mathrm{e}^{-t^2}$ (c) $3t^2\cos(t^3)$ (d) e^t

Ö1.21 (a) $\partial f/\partial u = (u+v)(u-v)^2(5u+v)$, $\partial f/\partial v = -(u+v)(u-v)^2(u+5v)$
(b) $\partial f/\partial u = 28(v-u)$, $\partial f/\partial v = 14(2u-v)$
(c) $\partial f/\partial u = 2\cos(2u-v)$, $\partial f/\partial v = -\cos(2u-v)$
(d) $\partial f/\partial u = \mathrm{e}^v$, $\partial f/\partial v = u\mathrm{e}^v$

Ö1.22 (a) $x_1^2 + x_2^2 - x_1x_2$ (b) $\frac{1}{3} - x_1 - 4x_1x_2 + x_1^2 + x_2^2 + x_3^2$

(c) $P_2[f,\bar{x}](x_1,x_2) = 3 + 2x_1 - 2x_1^2 - 4x_1(x_2-1) - (x_2-1)^2$
(d) $4 - 9x_1 + 9x_1^2 - x_2^2$

Ö1.23 (a) $f(\bar{x}) = 1 + 1 + 1 = 3$

$f'(x) = (2x_1, 2x_2, 2x_3)$ ger $f'(\bar{x}) = (2, 2, 2)$

$$f''(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$h = x - \bar{x} = (2, 2, 2) - (1, 1, 1) = (1, 1, 1)$
ger $P_2[f, \bar{x}](x) = 3 + (2, 2, 2) \cdot (1, 1, 1) + \frac{1}{2} h^\top f'' h = 3 + 6 + \frac{1}{2} 6 = 9 + 3 = 12$
(b) $f(\bar{x}) = 1 + 8 + 27 = 36$

$f'(x) = (3x_1^2, 3x_2^2, 3x_3^2)$ ger $f'(\bar{x}) = (3, 12, 27)$

$$f''(x) = \begin{bmatrix} 6x_1 & 0 & 0 \\ 0 & 6x_2 & 0 \\ 0 & 0 & 6x_3 \end{bmatrix} \text{ ger } f''(\bar{x}) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$h = x - \bar{x} = (0, 0, 0) - (1, 2, 3) = (-1, -2, -3)$
ger $P_2[f, \bar{x}](x) = 36 + (3, 12, 27) \cdot (-1, -2, -3) + \frac{1}{2} h^\top f'' h = 36 - 108 + \frac{1}{2} 216 = 36$
(c) $f(\bar{x}) = 0$

$f'(x) = ((1 - x_1)x_2 \exp(-x_1), x_1 \exp(-x_1))$ ger $f'(\bar{x}) = (0, 0)$

$$f''(x) = \begin{bmatrix} (x_1 - 2)x_2 \exp(-x_1) & (1 - x_1)\exp(-x_1) \\ (1 - x_1)\exp(-x_1) & 0 \end{bmatrix} \text{ ger } f''(\bar{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$h = x - \bar{x} = (6, 7) - (0, 0) = (6, 7)$
ger $P_2[f, \bar{x}](x) = 0 + (0, 0) \cdot (6, 7) + \frac{1}{2} h^\top f'' h = 0 + \frac{1}{2}(6, 7) \cdot (7, 6) = 42$
(d) $f(\bar{x}) = 1 + 6 + 27 + 5 = 39$

$f'(x) = (3x_1^2, 3, 3x_3^2)$ ger $f'(\bar{x}) = (3, 3, 27)$

$$f''(x) = \begin{bmatrix} 6x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6x_3 \end{bmatrix} \text{ ger } f''(\bar{x}) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$h = x - \bar{x} = (0, 0, 0) - (1, 2, 3) = (-1, -2, -3)$
ger $P_2[f, \bar{x}](x) = 39 + (3, 3, 27) \cdot (-1, -2, -3) + \frac{1}{2} h^\top f'' h = 39 - 90 + \frac{1}{2} 168 = 33$

Ö1.24 (a) 4 (b) 0
(c) Jacobi-matrisen ges av $f'(x, y) = \begin{bmatrix} 2xy^3 & 3x^2y^2 \end{bmatrix}$ och Hesse-matrisen

$$f''(x, y) = \begin{bmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y \end{bmatrix}, \quad f''(3, 4) = \begin{bmatrix} 128 & 288 \\ 288 & 216 \end{bmatrix}$$

Detta ger determinanten $128 \cdot 216 - 288^2 = -55\,296$.
(d) -1

Ö1.25 (a) $\begin{bmatrix} 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2x+y & 2y+x \end{bmatrix}$ (c) $\begin{bmatrix} 3x^2+h^2 & 0 \end{bmatrix}$
(d) $\begin{bmatrix} 0 & \frac{\sin(y+h)-\sin(y-h)}{2h} \end{bmatrix}$

Problem

P1.4 $f'(x) = A$, $L[f,\bar{x}](x) = Ax$, $E[f,\bar{x}](x) = 0$ för alla $\bar{x}, x \in \mathbb{R}^n$

P1.6 Med hjälp av kedjeregeln för funktioner av en variabel får vi

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\big(xF(y/x)\big) = F(y/x) + xF'(y/x)\frac{-y}{x^2} \\ \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\big(xF(y/x)\big) = xF'(y/x)\frac{1}{x} \end{aligned}$$

så att

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xF(y/x) - yF'(y/x) + yF'(y/x) = u$$

P1.7 Kedjeregeln ger (med $\frac{\partial x}{\partial r} = \cos(\varphi)$, $\frac{\partial y}{\partial r} = \sin(\varphi)$)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos(\varphi)\frac{\partial u}{\partial x} + \sin(\varphi)\frac{\partial u}{\partial y}$$

och

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\Big(\cos(\varphi)\frac{\partial u}{\partial x} + \sin(\varphi)\frac{\partial u}{\partial y}\Big) \\ &= \cos(\varphi)\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \sin(\varphi)\frac{\partial}{\partial r}\frac{\partial u}{\partial y} \\ &= \cos(\varphi)\Big(\cos(\varphi)\frac{\partial^2 u}{\partial x^2} + \sin(\varphi)\frac{\partial^2 u}{\partial y\,\partial x}\Big) + \sin(\varphi)\Big(\cos(\varphi)\frac{\partial^2 u}{\partial x\,\partial y} + \sin(\varphi)\frac{\partial^2 u}{\partial y^2}\Big) \\ &= \cos^2(\varphi)\frac{\partial^2 u}{\partial x^2} + 2\cos(\varphi)\sin(\varphi)\frac{\partial^2 u}{\partial x\,\partial y} + \sin^2(\varphi)\frac{\partial^2 u}{\partial y^2} \end{aligned}$$

Kedjeregeln ger (med $\frac{\partial x}{\partial \varphi} = -r\sin(\varphi)$, $\frac{\partial y}{\partial \varphi} = r\cos(\varphi)$)

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \varphi} = -r\sin(\varphi)\frac{\partial u}{\partial x} + r\cos(\varphi)\frac{\partial u}{\partial y}$$

och (med $\cos(\varphi)\frac{\partial u}{\partial x} + \sin(\varphi)\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$)

$$
\begin{aligned}
\frac{\partial^2 u}{\partial \varphi^2} &= -r\frac{\partial}{\partial \varphi}\Big(\sin(\varphi)\frac{\partial u}{\partial x}\Big) + r\frac{\partial}{\partial \varphi}\Big(\cos(\varphi)\frac{\partial u}{\partial y}\Big) \\
&= -r\cos(\varphi)\frac{\partial u}{\partial x} - r\sin(\varphi)\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial x} \\
&\quad - r\sin(\varphi)\frac{\partial u}{\partial y} + r\cos(\varphi)\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial y} \\
&= -r\frac{\partial u}{\partial r} - r\sin(\varphi)\Big(-r\sin(\varphi)\frac{\partial^2 u}{\partial x^2} + r\cos(\varphi)\frac{\partial^2 u}{\partial y\,\partial x}\Big) \\
&\quad + r\cos(\varphi)\Big(-r\sin(\varphi)\frac{\partial^2 u}{\partial x\,\partial y} + r\cos(\varphi)\frac{\partial^2 u}{\partial y^2}\Big) \\
&= -r\frac{\partial u}{\partial r} + r^2\Big(\sin^2(\varphi)\frac{\partial^2 u}{\partial x^2} - 2\cos(\varphi)\sin(\varphi)\frac{\partial^2 u}{\partial x\,\partial y} + \cos^2(\varphi)\frac{\partial^2 u}{\partial y^2}\Big)
\end{aligned}
$$

Alltså:

$$
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \Big(\cos^2(\varphi) + \sin^2(\varphi)\Big)\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}
$$

P1.8

$$
Df(x)^\top = \begin{bmatrix} f_1'(x) \\ \vdots \\ f_n'(x) \end{bmatrix}, \quad D(Df^\top)(x) = \begin{bmatrix} f_{11}''(x) & \cdots & f_{1n}''(x) \\ \vdots & \ddots & \vdots \\ f_{n1}''(x) & \cdots & f_{nn}''(x) \end{bmatrix} = D^2 f(x)
$$

P1.9 Imitera härledningen av Taylors formel av ordning 2.

Datorövningar

D1.1

Python (3.x)

```
from pylab import *

t = linspace(0, 12 * pi, 10000)
x = sin(t) * (exp(cos(t)) - 2 * cos(4 * t) - (sin(t / 12)) ** 5)
y = cos(t) * (exp(cos(t)) - 2 * cos(4 * t) - (sin(t / 12)) ** 5)

plot(x, y)
xlabel("$x$")
ylabel("$y$")
axis("equal")
grid(True)
show()
```

MATLAB

```
t = linspace(0, 12 * pi, 10000);
x = sin(t) .* (exp(cos(t)) - 2 * cos(4 * t) - (sin(t / 12)) .^ 5);
y = cos(t) .* (exp(cos(t)) - 2 * cos(4 * t) - (sin(t / 12)) .^ 5);

plot(x, y)
xlabel("x")
ylabel("y")
axis equal
grid on
```

D1.2

Python (3.x)

```
from pylab import *

t = linspace(0, 2 * pi, 100)
x = sin(t) + 2 * sin(2 * t)
y = cos(t) - 2 * cos(2 * t)
z = -sin(3 * t)

ax = axes(projection="3d")
ax.plot(x, y, z)
ax.set_xlabel("$x$")
ax.set_ylabel("$y$")
ax.set_zlabel("$z$")
show()
```

MATLAB

```
t = linspace(0, 2 * pi, 100);
x = sin(t) + 2 * sin(2 * t);
y = cos(t) - 2 * cos(2 * t);
z = -sin(3 * t);

plot3(x, y, z)
xlabel("x")
ylabel("y")
zlabel("z")
```

D1.3

Python (3.x)

```
from pylab import *

x = linspace(-2, 2, 200)
y = linspace(-1, 3, 200)
X, Y = meshgrid(x, y)
a = 1
b = 100
Z = (a - X) ** 2 + b * (Y - X**2) ** 2

c = contourf(X, Y, Z)
```

```
colorbar(c)
contour(X, Y, Z, levels=1000)
xlabel("$x$")
ylabel("$y$")
show()
```

MATLAB

```
x = linspace(-2, 2, 200);
y = linspace(-1, 3, 200);
[X, Y] = meshgrid(x, y);
a = 1;
b = 100;
Z = (a - X).^2 + b * (Y - X.^2).^2;

%contourf(X, Y, Z); hold on
contour(X, Y, Z, 1000)
colorbar
xlabel("x")
ylabel("y")
```

D1.4

Python (3.x)

```
from pylab import *

x = linspace(-2, 2, 200)
y = linspace(-1, 3, 200)
X, Y = meshgrid(x, y)
a = 1
b = 100
Z = (a - X) ** 2 + b * (Y - X**2) ** 2

ax = axes(projection="3d")
ax.plot_surface(X, Y, Z)
ax.set_xlabel("$x$")
ax.set_ylabel("$y$")
ax.set_zlabel("$z$")
show()
```

MATLAB

```
x = linspace(-2, 2, 200);
y = linspace(-1, 3, 200);
[X, Y] = meshgrid(x, y);
a = 1;
b = 100;
Z = (a - X).^2 + b * (Y - X.^2).^2;

surf(X, Y, Z)
xlabel("x")
ylabel("y")
zlabel("z")
```

D1.5

Python (3.x)

```
from pylab import *

x = linspace(-5, 5, 100)
y = linspace(-5, 5, 100)
X, Y = meshgrid(x, y)
Z = sin(X**2 / 2 - Y**2 / 4 + 3) * cos(2 * X + 1 - exp(Y))

figure()
c = contourf(X, Y, Z)
colorbar(c)
contour(X, Y, Z)
xlabel("$x$")
ylabel("$y$")

figure()
ax = axes(projection="3d")
ax.plot_surface(X, Y, Z)
ax.set_xlabel("$x$")
ax.set_ylabel("$y$")
ax.set_zlabel("$z$")

show()
```

MATLAB

```
x = linspace(-5, 5, 100);
y = linspace(-5, 5, 100);
[X, Y] = meshgrid(x, y);
Z = sin(X.^2 / 2 - Y.^2 / 4 + 3) * cos(2 * X + 1 - exp(Y));

contourf(X, Y, Z)
colorbar
contour(X, Y, Z, 10)
xlabel("x")
ylabel("y")

figure()
surf(X, Y, Z)
xlabel("x")
ylabel("y")
zlabel("z")
```

D1.6

Python (3.x)

```
from pylab import *

x = linspace(-5, 5, 20)
y = linspace(-5, 5, 20)
X, Y = meshgrid(x, y)
U = X**2 - Y**2 - 4
```

```
V = 2 * X * Y
M = sqrt(U**2 + V**2)

quiver(X, Y, U, V, M)
xlabel("$x$")
ylabel("$y$")
show()
```

MATLAB

```
x = linspace(-5, 5, 20);
y = linspace(-5, 5, 20);
[X, Y] = meshgrid(x, y);
U = X.^2 - Y.^2 - 4;
V = 2 * X .* Y;

quiver(X, Y, U, V)
xlabel("x")
ylabel("y")
```

D1.7

Python (3.x)

```
from pylab import *

x = linspace(-5, 5, 20)
y = linspace(-5, 5, 20)
X, Y = meshgrid(x, y)
U = X**2 - Y**2 - 4
V = 2 * X * Y
M = sqrt(U**2 + V**2)

streamplot(X, Y, U, V, color=M)
xlabel("$x$")
ylabel("$y$")
show()
```

MATLAB

```
x = linspace(-5, 5, 20);
y = linspace(-5, 5, 20);
[X, Y] = meshgrid(x, y);
U = X.^2 - Y.^2 - 4;
V = 2 * X .* Y;

streamslice(X, Y, U, V)
xlabel("x")
ylabel("y")
```

D1.8

Python (3.x)

```
from numpy import isscalar, zeros
```

```
def jacobi(f, x):

    h = 1e-6
    y = f(x)
    m = 1 if isscalar(y) else len(y)
    n = 1 if isscalar(x) else len(x)
    A = zeros((m, n))

    for j in range(n):
        a = x.astype(float)  # Copy x and convert to float
        b = x.astype(float)  # Copy x and convert to float
        a[j] = a[j] - h
        b[j] = b[j] + h
        A[:, j] = (f(b) - f(a)) / (2 * h)

    if m == 1 or n == 1:
        A = A.flatten()

    return A
```

Python (3.x)

```
from numpy import array
from jacobi import *


def f(x):
    A = array(((1, 2, 3, 4, 5, 6), (7, 8, 9, 10, 11, 12), (13, 14, 15,
       16, 17, 18)))
    return A.dot(x)


A = jacobi(f, array((3.14, 1.41, 2.72, 1.62, 1.73, 1.0)))
print(A)
```

MATLAB

```
function A = jacobi(f, x)

    h = 1e-5;
    m = length(f(x));
    n = length(x);
    A = zeros(m, n);

    for j = 1:n
        a = x;
        b = x;
        a(j) = a(j) - h;
        b(j) = b(j) + h;
        A(:, j) = (f(b) - f(a)) / (2 * h);
    end

end
```

MATLAB

```
f = @(x) [1 2 3 4 5 6; 7 8 9 10 11 12; 13 14 15 16 17 18] * x

A = jacobi(f, [3.14, 1.41, 2.72, 1.62, 1.73, 1.0]')
```

D1.9

Python (3.x)

```
from numpy import array
from jacobi import *

A = array(((1, 2, 3, 4), (5, 6, 7, 8), (9, 10, 11, 12), (13, 14, 15, 16)))


def f(x):
    return x.dot(A.dot(x))


def Df(x):
    return jacobi(f, x).T


x = array((1, 2, 3, 4))
H = jacobi(Df, x)

print(H)
print(A + A.T)
```

MATLAB

```
A = [1, 2, 3, 4; 5, 6, 7, 8; 9, 10, 11, 12; 13, 14, 15, 16]
f = @(x) x'*A*x;
Df = @(x) jacobi(f, x)';

x = [1, 2, 3, 4]';
H = jacobi(Df, x)
A + A'
```

D1.10

Python (3.x)

```
from numpy import array
from jacobi import *

a = 1
b = 100


def f(x):
    return (a - x[0]) ** 2 + b * (x[1] - x[0] ** 2) ** 2


def Df(x):
```

```
    return jacobi(f, x).T


x = array((1, 1))
H = jacobi(Df, x)

print(H)
print([[2 + 12 * b * x[0] ** 2 - 4 * b * x[1], -4 * b * x[0]], [-4 * b *
    x[0], 2 * b]])
```

MATLAB

```
a = 1;
b = 100;

f = @(x) (a - x(1))^2 + b * (x(2) - x(1)^2)^2;
Df = @(x) jacobi(f, x)';

x = [1, 1]';
H = jacobi(Df, x)
[2 + 12 * b * x(1)^2 - 4 * b * x(2), -4 * b * x(1); -4 * b * x(1), 2 * b]
```

Kapitel 2

Övningar

Ö2.1 (a) $x - \cos(x) = 0$ (b) $x - \mathrm{e}^{-x} = 0$

(c) $\begin{bmatrix} x_1 - x_1^2 x_1 \\ x_2 - x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} x_1 - \cos(x_2) \\ x_2 - \cos(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Ö2.2 (a) 1 (b) $\frac{6}{5}$ (c) $\frac{2}{1+\mathrm{e}}$ (d) $\frac{2\cos(1)}{\sin(1)+\cos(1)}$

Ö2.3 (a)

$$f(u) = \begin{bmatrix} u_2(1 - u_1^2) \\ 2 - u_1 u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) Vi finner två lösningar: $\bar{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ och $\bar{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

(c) Jacobi-matrisen är

$$\mathrm{D}f(u) = \begin{bmatrix} -2u_1u_2 & 1 - u_1^2 \\ -u_2 & -u_1 \end{bmatrix}$$

(d) Första steget av Newtons metod:

Beräkna $A = \mathrm{D}f(1,1) = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}$ och $b = -f(1,1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Lös $Ah = b$, $\begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{cases} -2h_1 = 0, \\ -h_1 - h_2 = -1, \end{cases} \quad h = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Uppdatera $u^{(1)} = u^{(0)} + h = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \bar{u}$

Bingo! Vi hittade en av lösningarna.

Ö2.4 (a)

$$f(u) = \begin{bmatrix} u_1(1-u_2) \\ u_2(1-u_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) Vi finner två lösningar: $\bar{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ och $\bar{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Jacobi-matrisen är:

$$\mathrm{D}f(u) = \begin{bmatrix} 1-u_2 & -u_1 \\ -u_2 & 1-u_1 \end{bmatrix}$$

(d) Första steget av Newtons metod:

Beräkna $A = \mathrm{D}f(2,2) = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$ och $b = -f(2,2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Lös $Ah = b$, $\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\begin{cases} -h_1 - 2h_2 = 2, \\ -2h_1 - h_2 = 2, \end{cases} \quad h = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix}$

Uppdatera $u^{(1)} = u^{(0)} + h = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix}$

Vi närmar oss en av lösningarna $\bar{u}$.

Ö2.5 (a) $\begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ (b) $\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ (c) $\begin{bmatrix} \frac{5}{4} & \frac{1}{4} \end{bmatrix}^\top$ (d) $\begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix}^\top$

Lösning: (d) Vi har $f(x,y) = (x\cos(6y) + 1, x\sin(y) - \frac{1}{2})$ vilket ger

$$f'(x,y) = \begin{bmatrix} \cos(6y) & -6x\sin(y) \\ \sin(y) & x\cos(y) \end{bmatrix}$$

I startpunkten $(1,0)$ har vi därför $f(x,y) = (1+1, 0-\frac{1}{2}) = (2, -\frac{1}{2})$ och $f'(x,y) = I$ (enhetsmatrisen), vilket ger

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$$

Ö2.6 (a) Ja, $y = \sqrt{4-x^2}$ (b) Ja, $y = \sqrt{4-x^2}$ (c) Nej (d) Ja, $y = -\sqrt{4-x^2}$

Ö2.7 (a) Ja (b) Nej (c) Nej (d) Ja

Ö2.8 (a) $\begin{bmatrix} \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

Ö2.9 Kom ihåg att normera riktningsvektorn, $\hat{\boldsymbol{v}} = \boldsymbol{v}/|\boldsymbol{v}|$.
(a) $\nabla f(1,1) = (\cos(1), -\cos(1))$, $\mathrm{D}_{\hat{\boldsymbol{v}}}(1,1) = \frac{\cos(1)}{\sqrt{5}}$
(b) $\nabla f(-1,0,2) = (1,-1,1)$, $\mathrm{D}_{\hat{\boldsymbol{v}}}(-1,0,2) = 1$
(c) $\nabla f(-1,0,2) = (1,-1,1)$, $\mathrm{D}_{\hat{\boldsymbol{v}}}(-1,0,2) = 0$
(d) $\nabla f(7,2) = (14,-24)$, $\mathrm{D}_{\hat{\boldsymbol{v}}}(7,2) = -5\sqrt{2}$

Ö2.10 (a) 7 och -7 (b) 2 och -2 (c) $\sqrt{3}$ och $-\sqrt{3}$ (d) $2\sqrt{3}$ och $-2\sqrt{3}$

Ö2.11 (a) $x - y = 2$ (b) $x + 3y + 2z = 0$ (c) $2x - z = -1$ (d) $6x + 3y + 2z = 18$

Ö2.12 (a) $\begin{bmatrix} 10 & 5 & 0 & 0 & 2 \end{bmatrix}^\top$ (b) $\begin{bmatrix} 0 & 2 & 3 & 1 \end{bmatrix}^\top$ (c) $\begin{bmatrix} 0 & 1 & -1 & 3 \end{bmatrix}^\top$ (d) $\begin{bmatrix} 2 & 0 & 2 \end{bmatrix}^\top$

Ö2.13 (a) $\frac{r}{|r|^2}$ (b) $-\frac{r}{|r|^3}$ (c) $\frac{r}{|r|}$ (d) $-\frac{r}{|r|}\mathrm{e}^{-|r|}$

Ö2.15 (a) Maximum 1 och minimum 0.
(b) Maximum 1 och minimum $-\frac{1}{8}$.
(c) Maximum 0 i punkten $(0,0)$, minimum -2 i punkterna $(0,2)$ och $(2,0)$.

Lösning: Stationära punkter ges av

$$f'_x(x,y) = 2xy - 1 = 0$$
$$f'_y(x,y) = x^2 - 1 = 0$$

Vi får $x = \pm 1$, $y = \pm\frac{1}{2}$. Två stationära punkter: $(-1, -\frac{1}{2})$ är yttre punkt och därför ointressant och $(1, \frac{1}{2})$ är inre punkt och av intresse med $f(1, \frac{1}{2}) = -1$.
Rand 1: $x = 0$, $0 \leqslant y \leqslant 2$, $g(y) = f(0,y) = -y$, $g'(y) = -1 \neq 0$, $g(0) = 0$, $g(2) = -2$, max 0 i $y = 0$, min -2 i $y = 2$.
Rand 2: $y = 0$, $0 \leqslant x \leqslant 2$, $g(x) = f(x,0) = -x$, $g'(x) = -1 \neq 0$, $g(0) = 0$, $g(2) = -2$, max 0 i $x = 0$, min -2 i $x = 2$.
Rand 3: $y = 2 - x$, $0 \leqslant x \leqslant 2$, $g(x) = f(x, 2-x) = \cdots = 2x^2 - x^3 - 2$, $g'(x) = 4x - 3x^2 = x(4 - 3x) = 0$ ger $x = 0$ och $x = \frac{4}{3}$, $g(0) = 0$, $g(\frac{4}{3}) = -\frac{22}{27}$, $g(2) = -2$, max 0, min -2.
Jämförelse av värdena i de intressanta punkterna $(1, \frac{1}{2})$, $(0,0)$, $(0,2)$, $(2,0)$ och $(\frac{4}{3}, \frac{2}{3})$

ger att minimum -2 inträffar i punkterna $(0,2)$ och $(2,0)$, maximum 0 i punkten $(0,0)$.
(d) Maximum 3 inträffar i punkterna $(0,-1)$ och $(2,0)$, minimum $\frac{3}{28}$ i punkten $(\frac{5}{7}, \frac{9}{14})$.

Lösning: Stationära punkter ges av

$$\begin{aligned} f'_x(x,y) &= 2x - y - 1 = 0 \\ f'_y(x,y) &= -x + 2y - 1 = 0 \end{aligned}$$

Den enda stationära punkten $(1,1)$ är yttre punkt och därför ointressant.
Rand 1: $x = 0$, $-1 \leqslant y \leqslant 1$, $g(y) = f(0,y) = y^2 - y + 1$, $g'(y) = 2y - 1 = 0$ ger $y = \frac{1}{2}$, $g(-1) = 3$, $g(\frac{1}{2}) = \frac{3}{4}$, $g(1) = 1$, max 3 i $y = -1$, min $\frac{3}{4}$ i $y = \frac{1}{2}$.
Rand 2: $y = 1 - \frac{1}{2}x$, $0 \leqslant x \leqslant 2$, $g(x) = f(x, 1 - \frac{1}{2}x) = \cdots = \frac{7}{4}x^2 - \frac{5}{2}x + 1$, $g'(x) = \frac{1}{2}(7x - 5) = 0$ ger $x = \frac{5}{7}$, $g(0) = 1$, $g(\frac{5}{7}) = \frac{3}{28}$, $g(2) = 3$, max 3 i $x = 2$, min $\frac{3}{28}$ i $x = \frac{5}{7}$.
Rand 3: $y = -1 + \frac{1}{2}x$, $0 \leqslant x \leqslant 2$, $g(x) = f(x, -1 + \frac{1}{2}x) = \cdots = \frac{3}{4}x^2 - \frac{3}{2}x + 3$, $g'(x) = \frac{3}{2}(x - 1) = 0$ ger $x = 1$, $g(0) = 3$, $g(1) = \frac{9}{4}$, $g(2) = 3$, max 3, min $\frac{9}{4}$.
Jämförelse av värdena i dessa punkter ger att maximum 3 inträffar i punkterna $(0,-1)$ och $(2,0)$, minimum $\frac{3}{28}$ i punkten $(\frac{5}{7}, \frac{9}{14})$.

Ö2.16 (a) $(0,0)$ min (b) $(-\frac{1}{2}, 0)$ sadel (c) $(0,0)$ max (d) $(0,0)$ max

Ö2.17 (a) 1 (b) $\frac{1}{4}$ (c) 7 (d) $\frac{5}{4}$

Ö2.18 (a) $(1,1)$ min
(b) $f(x) = x_1^3 + x_2^2 - x_1 x_2$. Stationära punkter ges av

$$f'(x)^\top = \begin{bmatrix} f'_{x_1}(x) \\ f'_{x_2}(x) \end{bmatrix} = \begin{bmatrix} 3x_1^2 - x_2 \\ 2x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Den andra ekvationen ger $x_1 = 2x_2$ vilket insättes i den första ekvationen: $12x_2^2 - x_2 = 0$, vilken har två lösningar $x_2 = 0$ och $x_2 = 1/12$. Vi har alltså två stationära punkter nämligen $(0,0)$ och $(1/6, 1/12)$. Hesse-matrisen är

$$f''(x) = \begin{bmatrix} f''_{x_1 x_1} & f''_{x_1 x_2} \\ f''_{x_1 x_2} & f''_{x_2 x_2} \end{bmatrix} = \begin{bmatrix} 6x_1 & -1 \\ -1 & 2 \end{bmatrix}$$

I de två punkterna får vi $f''(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ med egenvärdena $1 \pm \sqrt{2}$, så att matrisen är indefinit, och $f''(1/6, 1/12) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ med egenvärdena $(3 \pm \sqrt{5})/2$, så att matrisen är positivt definit. Vi drar slutsatsen att $(0,0)$ är en sadelpunkt och att $(1/6, 1/12)$ är en minimipunkt.
(c)

$$f(x_1, x_2, x_3) = \tfrac{1}{3}x_1^3 + x_2^2 + x_3^2 - 4x_1 x_2$$

Stationära punkter ges av

$$f'(x)^\top = \begin{bmatrix} x_1^2 - 4x_2 \\ 2x_2 - 4x_1 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

De stationära punkterna är $x = (0,0,0)$ och $x = (8,16,0)$. Hesse-matrisen är

$$f''(x) = \begin{bmatrix} 2x_1 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

I den ena stationära punkten:

$$f''(0,0,0) = \begin{bmatrix} 0 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

med egenvärdena $1 \pm \sqrt{17}, 2$, olika tecken, sadelpunkt.

I den andra stationära punkten:

$$f''(8,16,0) = \begin{bmatrix} 16 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

med egenvärdena $9 \pm \sqrt{65}, 2$, alla positiva, lokalt minimum.
(d) Inga singulära punkter, inga randpunkter. Stationära punkter ges av

$$f'(x,y)^\top = \begin{bmatrix} 3x^2 + 3y^2 - 15 \\ 6xy - 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

De stationära punkterna är $(1,2)$, $(2,1)$, $(-1,-2)$ och $(-2,-1)$. Hesse-matrisen är

$$f''(x,y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

Vi beräknar egenvärdena till Hesse-matrisen i de fyra stationära punkterna. Vi finner

$$f''(1,2) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix}, \ \lambda = -6,\ 18, \quad \text{sadelpunkt}$$

$$f''(2,1) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix}, \ \lambda = 6,\ 18, \quad \text{lokalt minimum}$$

$$f''(-1,-2) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix}, \ \lambda = 6,\ -18, \quad \text{sadelpunkt}$$

$$f''(-2,-1) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix}, \ \lambda = -6,\ -18, \quad \text{lokalt maximum}$$

Ö2.19 a) $(0,0,1)$ (b) $(1,1,1)$ (c) $(0,0,1)$ (d) $(2,2,1)$ och $(-2,-2,1)$

Ö2.20 (a) 2 (b) $\sqrt{2}$ (c) $\sqrt{10}+1$ (d) $\sqrt{2}$

Ö2.21 (a) $4\sqrt{2}/9$ (b) $\sqrt{n}$ (c) $\frac{4\sqrt{5}}{125}$ (d) 84375
(c) Lösning: Minimum är 0 och fås då minst en av x, y, z är 0, dvs på linjerna $x+y=1$, $x+z=1$ eller $y+z=1$. Maximum är $\frac{4\sqrt{5}}{125}$ och fås i punkten $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$.

Lösning: $f(x,y,z) = xy\sqrt{z}$ skall optimeras för $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ och $x+y+z=1$, dvs över den del D av planet $x+y+z=1$ som ligger i första oktanten. Detta är en snedställd triangel med hörn i $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Mängden D är kompakt (sluten och begränsad) så funktionen har både maximum och minimum i D enligt sats 2.6. Det är klart att $f(x,y,z) = xy\sqrt{z} \geqslant 0$ och att $f(x,y,z) = 0$ på randen av D där minst en av x, y, z är lika med 0. Alltså: $\min_D f = 0$ inträffar på ∂D.

Vi söker maximum med Lagranges metod. Målfunktion $f(x,y,z) = xy\sqrt{z}$, bivillkor $g(x,y,z) = x+y+z-1 = 0$. Lagrange-funktionen är

$$L(x,y,z,\lambda) = xy\sqrt{z} + \lambda(x+y+z-1)$$

Stationära punkter ges av $L'(x,y,z,\lambda)^\top = 0$:

$$\begin{aligned} L'_x(x,y,z,\lambda) &= y\sqrt{z} + \lambda = 0 \\ L'_y(x,y,z,\lambda) &= x\sqrt{z} + \lambda = 0 \\ L'_z(x,y,z,\lambda) &= \frac{xy}{2\sqrt{z}} + \lambda = 0 \\ L'_\lambda(x,y,z,\lambda) &= x+y+z-1 = 0 \end{aligned}$$

Vi eliminerar λ:

$$-\lambda = y\sqrt{z} = x\sqrt{z} = \frac{xy}{2\sqrt{z}}$$

Vi kan antaga att $x > 0$, $y > 0$, $z > 0$, annars är $f(x,y,z) = 0$. Ekvationen $y\sqrt{z} = x\sqrt{z}$ ger $y = x$. Insättning i ekvationen $x\sqrt{z} = \frac{xy}{2\sqrt{z}}$ ger $z = \frac{1}{2}x$. Insättning i bivillkoret $x+y+z-1 = 0$ ger sedan $x + x + \frac{1}{2}x = 1$, dvs $x = \frac{2}{5}$. Till sist: $y = \frac{2}{5}$, $z = \frac{1}{5}$, $\lambda = -\frac{2}{5\sqrt{5}}$. Alltså: vi har en stationär punkt $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, -\frac{2}{5\sqrt{5}})$. Maximum är $f(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}) = \frac{4}{25}\frac{1}{\sqrt{5}} = \frac{4\sqrt{5}}{125}$.

Ö2.22 (a) $\begin{bmatrix} -\sin(x)y & \cos(x) \\ \cos(x) & 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & 2y \\ 2y & 2x \end{bmatrix} + \lambda \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -\cos(x)y & -\sin(x) \\ -\sin(x) & 0 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Ö2.23 a) $\frac{1}{12\sqrt{3}}S^2$ (b) $\frac{1}{16}S^2$ (c) $\left(\frac{S}{6}\right)^{3/2}$ (d) $\frac{1}{6}\left(\frac{2S}{3+\sqrt{3}}\right)^{3/2}$

Ö2.24 (a) $(0, 0)$ (b) $(0, \frac{1}{2})$ (c) $(1, 0)$ (d) $(0, \frac{1}{2}, 0)$

Ö2.25 (a) $\frac{5}{18}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

Problem

P2.1

$$\begin{aligned}
&x_{j+1} - \bar{x} = g(x_j) - g(\bar{x}) = g'(\bar{x})(x_j - \bar{x}) + E_1[g, \bar{x}](x_j) = E_1[g, \bar{x}](x_j) \\
&\|x_{j+1} - \bar{x}\| = \|E_1[g, \bar{x}](x_j)\| \leqslant K_1 \|x_j - \bar{x}\|^2
\end{aligned}$$

P2.2 Vi delar upp felet i två delar:

$$x_{j+1} - \bar{x} = (x_{j+1} - x_j) + (x_j - \bar{x})$$

Taylors formel av ordning 1 runt x_j ger

$$\begin{aligned}
0 = f(\bar{x}) &= f(x_j) + f'(x_j)(\bar{x} - x_j) + E_1[f, x_j](\bar{x}) \\
x_j - \bar{x} &= f'(x_j)^{-1} f(x_j) + f'(x_j)^{-1} E_1[f, x_j](\bar{x})
\end{aligned}$$

Vi har även, enligt Newtons metod,

$$x_{j+1} - x_j = -f'(x_j)^{-1} f(x_j)$$

så att

$$\begin{aligned}
x_{j+1} - \bar{x} &= (x_{j+1} - x_j) + (x_j - \bar{x}) \\
&= -f'(x_j)^{-1} f(x_j) + f'(x_j)^{-1} f(x_j) + f'(x_j)^{-1} E_1[f, x_j](\bar{x}) \\
&= f'(x_j)^{-1} E_1[f, x_j](\bar{x}) \\
\|x_{j+1} - \bar{x}\| &= \|f'(x_j)^{-1} E_1[f, x_j](\bar{x})\| \\
&\leqslant M \|E_1[f, x_j](\bar{x})\| \\
&\leqslant M K_1 \|x_j - \bar{x}\|^2
\end{aligned}$$

P2.5 Vi beräknar vinkeln mellan normalvektorerna i skärningspunkten. Svar: $\pi/3$

P2.6 $a = 1, b = -1/6$. Tips: skriv $f(a, b) = \displaystyle\int_0^1 (x^2 - ax - b)^2 \, \mathrm{d}x$ och utveckla kvadraten.

P2.7 (a) T ex $x = 0, y = 18, z = 2$.
(b) $x = y = 5, z = 10$.
Tips: $f(x, y) = xyz^2 = (20 - y - z)yz^2$ med $y \geqslant 0, z \geqslant 0, y + z \leqslant 20$.

P2.8 Rosenbrocks funktion: $f(x) = (1-x_1)^2 + a(x_2 - x_1^2)^2$. Stationära punkter ges av

$$\nabla f(x) = \begin{bmatrix} -2(1-x_1) - 4a(x_2 - x_1^2)x_1 \\ 2a(x_2 - x_1^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Den andra ekvationen ger $(x_2 - x_1^2) = 0$, vilket insatt i den första ekvationen leder till $2(1-x_1) = 0$. Det ger $x_1 = 1$ och sedan $x_2 = x_1^2 = 1$. Vi har en unik stationär punkt: $(1,1)$.

Hesse-matrisen är

$$f''(x) = \begin{bmatrix} 2 - 4ax_2 + 12ax_1^2 & -4ax_1 \\ -4ax_1 & 2a \end{bmatrix}$$
$$H = f''(1,1) = \begin{bmatrix} 2+8a & -4a \\ -4a & 2a \end{bmatrix}$$

Karakteristiska ekvationen är

$$\begin{aligned} \det(H - \lambda I) &= \begin{vmatrix} 2+8a-\lambda & -4a \\ -4a & 2a-\lambda \end{vmatrix} \\ &= (2+8a-\lambda)(2a-\lambda) - 16a^2 \\ &= \lambda^2 - (2+10a)\lambda + (2+8a)2a - 16a^2 \\ &= (\lambda - (1+5a))^2 - (1+5a)^2 + 4a = 0 \\ \lambda &= 1 + 5a \pm \sqrt{(1+5a)^2 - 4a} \end{aligned}$$

Båda egenvärdena är positiva, eftersom $\sqrt{(1+5a)^2 - 4a} < 1 + 5a$ då $a > 0$. Vi har lokalt minimum i punkten $(1,1)$. Eftersom inga andra extrempunkter finns, så är det globalt minimum.

P2.9 —

P2.10 (a) Skalär multiplikation av $\dot{x}(t) = -\nabla f(x(t))$ med $\dot{x}(t)$ och användning av kedjeregeln ger

$$\begin{aligned} \dot{x}(t) \cdot \dot{x}(t) &= -\dot{x}(t) \cdot \nabla f(x(t)) \\ \|\dot{x}(t)\|^2 &= -\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) \end{aligned}$$

dvs $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t)) = -\|\dot{x}(t)\|^2 \leqslant 0$.

(b) Sats 2.5 säger att $\nabla f(x)$ är en normalvektor till nivåmängden $f(x) = C$. Det betyder att kurvtangenten $\dot{x}(t) = -\nabla f(x(t))$ är vinkelrät mot nivåmängden i punkten $x(t)$ och pekar dit funktionen minskar.

(c) Explicit Euler:

$$x_{j+1} = x_j - h_j \nabla f(x_j), \quad j = 0, 1, 2, \ldots$$

Detta är detsamma som gradientmetoden för att minimera funktionen f.

Datorövningar

D2.1

Python (3.x)

```python
from numpy import *
from jacobi import *

def newton(f, x0, tol):

    x =  x0
    b = -f(x)

    while linalg.norm(b)>tol:
        A = jacobi(f,x)
        b = - f(x)
        h = linalg.solve(A,b)
        x = x + h

    return x
```

Python (3.x)

```python
from numpy import *
from jacobi import *
from newton import *

# övning IV.2.4
# f(x)=[x_1(1-x_2); x_2(1-x_1)]
# två lösningar (0,0) och (1,1)

def f(x):
    A = array(( x[0]*(1-x[1]), x[1]*(1-x[0]) ))
    return A

x0 = array((2, 2))
print(x0)

b = - f(x0)
print(b)
print(linalg.norm(b))

A = jacobi(f, x0)
print(A)

tol = 1e-3
x = newton(f,x0,tol)
print(x)
```

MATLAB

```matlab
function x = newton(f,x0,tol)

x = x0;
```

```
b = -f(x);

while norm(b)>tol
   A = jacobi(f,x);       % evaluate the Jacobian A=Df(x)
   b = -f(x);             % evaluate the residual b=-f(x)
   h = A\b;               % solve the linearized equation
   x = x + h;             % update
end
```

MATLAB

```
% övning IV 2.4
% två lösningar (0,0) och (1,1)

f=@(x) [x(1)*(1-x(2)); x(2)*(1-x(2))];
x0 = [2; 2];
tol = 1e-3;
x = newton(f,x0,tol)
```

D2.2

MATLAB

```
%%
% Visualisera funktionen.
clear all; clf(1); clf(2);
f = @(x,y) (x.^2+x.*y+5*y.^2+x-y).*exp(-(x.^2+y.^2));
[X,Y]=meshgrid(linspace(-2,2));
Z=f(X,Y);
figure(1); surfc(X,Y,Z)
figure(2); contour(X,Y,Z,10); grid on
% Verkar ha ungefär följande stationära punkter:
% (0,-1) max
% (-0.5,0.2) min
% (0.4,1) max
% (1,0) sadel
% Dessa blir startpunkter i nästa sektion.

%%
% Undersök stationära punkter.
clear all
f = @(x) (x(1)^2+x(1)*x(2)+5*x(2)^2+x(1)-x(2))*exp(-(x(1)^2+x(2)^2));
gradf = @(x) jacobi(f,x)';
x0 = [0;-1];
% x0 = [-0.5;0.2];
% x0 = [0.4;1];
% x0 = [1;0];

x = newton(gradf,x0,1e-6)
y = f(x)
D2f = @(x) jacobi(gradf,x);
H = D2f(x);
lambda = eig(H)
```

D2.3

MATLAB

```
%%
% Visualisera funktionen
clear all; clf(1); clf(2);
f=@(x,y) ( sin(0.5*x.^2-0.25*y.^2+3).*cos(2*x+1-exp(y)) );
b=5; c=5;
[X,Y]=meshgrid(linspace(-b,c));
Z=f(X,Y);
figure(1)
surfc(X,Y,Z)
figure(2)
contour(X,Y,Z,10); grid on

% Verkar bland annat ha minimum nära punkten (1,-3).
% Denna blir startpunkt i nästa sektion.

%%
% Undersök stationära punkter.
clear all
f=@(x) ( sin(0.5*x(1)^2-0.25*x(2)^2+3)*cos(2*x(1)+1-exp(x(2))) );
gradf = @(x) jacobi(f,x)';
x0 = [1;-3];

x = newton(gradf,x0,1e-10)
y = f(x)
D2f = @(x) jacobi(gradf,x);
H = D2f(x);
lambda = eig(H)
```

D2.4

MATLAB

```
%%
% Undersök stationära punkter, Rosenbrocks funktion.
clear all
a = 100;
f = @(x) ( (1-x(1))^2+a*(x(2)-x(1)^2)^2 );
gradf = @(x) jacobi(f,x)';
x0 = [0;0];

x = newton(gradf,x0,1e-10)
y = f(x)
D2f = @(x) jacobi(gradf,x);
H = D2f(x);
lambda = eig(H)
```

D2.5

MATLAB

```
function x = gradientmetod(f,x0,h,tol)

x = x0;
y = f(x);                  % funktionsvärdet
```

```
y1 = y+2*tol;             % påhittat värde

while abs(y1-y) > tol
    y1 = y;               % gammalt funktionsvärde
    gradf = jacobi(f,x)'; % gradienten
    x = x - h*gradf;      % nytt x
    y = f(x);             % nytt funktionsvärde
end
```

D2.6

MATLAB

```
clear all; clf
f = @(x,y) (x.^2+x.*y+5*y.^2+x-y).*exp(-(x.^2+y.^2));
[X,Y]=meshgrid(linspace(-2,2));
Z=f(X,Y);
figure(1);
contour(X,Y,Z,10)
hold on; grid on
% Verkar ha minimum nära (-0.5,0.2).

%%
% Gradientmetoden
clear all
f = @(x) (x(1)^2+x(1)*x(2)+5*x(2)^2+x(1)-x(2))*exp(-(x(1)^2+x(2)^2));

x0 = [0;0];
% x0 = [0;1];
% x0 = [0;-1];

h = 0.2;
% h = 0.1;
tol = 1e-9;

x = gradientmetod(f,x0,h,tol)
plot(x(:,1),x(:,2),'-.r')
```

D2.7

MATLAB

```
clear all; clf
f = @(x,y) (x.^2+x.*y+5*y.^2+x-y).*exp(-(x.^2+y.^2));
[X,Y]=meshgrid(linspace(-2,2));
Z=f(X,Y);
figure(1);
contour(X,Y,Z,10)
hold on; grid on
% Verkar ha minimum nära (-0.5,0.2).

%%
% Gradientflöde.
clear all
f = @(x) (x(1)^2+x(1)*x(2)+5*x(2)^2+x(1)-x(2))*exp(-(x(1)^2+x(2)^2));
```

```
gradf = @(t,x) -jacobi(f,x)';

x0 = [0;0];
x0 = [0;1];
% x0 = [0;-1];

[t,x] = ode45(gradf,[0,10],x0);
plot(x(:,1),x(:,2),'-.b')
```

D2.8

MATLAB

```
%%
% Visualisera Rosenbrocks funktion
clear all; clf
a=100;
f=@(x,y) ( (1-x).^2+a*(y-x.^2).^2 );
x=linspace(-2,2);
y=linspace(-1,4);
[X,Y]=meshgrid(x,y);
Z=f(X,Y);
figure(1)
contour(X,Y,Z,[1,5,10])
hold on; grid on

%%
% Gradientmetoden
clear all
a = 100;
f = @(x) ( (1-x(1))^2+a*(x(2)-x(1)^2)^2 );

% x0 = [0;1];
x0 = [-1;.5];
% x0 = [0;-1];

h = 1e-3;
% h = 0.1; % fungerar ej
tol = 1e-8;

x = gradientmetod(f,x0,h,tol)
plot(x(:,1),x(:,2),'-.r')

%%
% Newtons metod

gradf = @(x) jacobi(f,x)';
x0 = [0;1];
% x0 = [-1;.5];
% x0 = [0;-1];

% Modifierad newton-funktion som ger ut alla iterationer i xx.
xx = newtonxx(gradf,x0,tol);
plot(xx(:,1),xx(:,2),'-.b')
```

```
% Newton går en annan väg och är mycket snabbare än gradientmetoden.
```

D2.9

MATLAB

```
%% Optimera funktion över ett plan.
clear all
p = 5
f = @(x) ( x(1)^2+3*x(2)^2+2*x(3)^2 );   % funktionen
g = @(x) ( x(1)+x(2)+x(3) - p );         % bivillkoret
L = @(x) ( f(x(1:3)) + x(4)*g(x(1:3)) ); % Lagrange-funktionen

gradL = @(x) ( jacobi(L,x)' );   % gradienten

x0 = [1;1;1;1];                  % startgissning for newton
gradL(x0)                        % test av gradienten
%% Sök stationära punkter.

x0 = [1;1;1;1];
% x0 = [-1;-1;-1;1];
% x0 = [1;-1;1;-1];
x = newton(gradL,x0,1e-6)     % stationär punkt
y = f(x(1:3))                 % funktionsvärdet

%  Funktionen är obegränsad på planet: inget maximum.
%  Vi hittar bara en stationär punkt: detta måste vara minimipunkten.


%% Undersök Hesse-matrisen.  Intressant men behövs ej.

D2L = @(x) ( jacobi(gradL,x) ) ;   % Hessematrisen
H = D2L(x);
lambda = eig(H)
% sadelpunkt i 4 dimensioner, typiskt for Lagrange-problem
```

D2.10

MATLAB

```
%% Optimera funktion över en sfär.
clear all
C = 12;
f = @(x) ( x(1)*x(2)*x(3));                  % funktionen
g = @(x) ( x(1)^2+x(2)^2+x(3)^2  - C );      % bivillkoret
L = @(x) ( f(x(1:3)) + x(4)*g(x(1:3)) );     % Lagrange-funktionen

gradL = @(x) ( jacobi(L,x)' );   % gradienten

x0 = [1;1;1;1];               % startgissning for newton
gradL(x0)                        % test av gradienten
%% Sök stationära punkter.

x0 = [1;1;1;1];
x0 = [-1;-1;-1;1];
```

```
x0 = [1;-1;1;-1];
x = newton(gradL,x0,1e-6)     % stationär punkt
y = f(x(1:3))              % funktionsvärdet

%  Vi hittar två funktionsvärden: +8 och -8.
%  Detta måste vara maximum respektive minimum.


%% Undersök Hesse-matrisen.  Intressant men behövs ej.

D2L = @(x) ( jacobi(gradL,x) ) ;    % Hessematrisen
H = D2L(x);
lambda = eig(H)
% sadelpunkt i 4 dimensioner, typiskt for Lagrange-problem
```

Kapitel 3

Övningar

Ö3.1 (a) 1 (b) 0 (c) 1
(d) Vi kan ta $\{x_i\}_{i=0}^4 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, $\{y_j\}_{j=0}^4 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, så att $\Delta x_i = \Delta y_j = \frac{1}{4}$. Vi evaluerar i rektanglarnas mittpunkter $(\tilde{x}_i, \tilde{y}_j) = (\frac{2i-1}{8}, \frac{2j-1}{8})$, $i, j = 1, 2, 3, 4$. Vi får

$$\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{2i-1}{8}\frac{2j-1}{8}\frac{1}{4}\frac{1}{4} = \frac{16 \cdot 16}{64}\frac{1}{16} = \frac{1}{4}$$

vilket råkar vara integralens exakta värde.

Ö3.2 (a) $\frac{1}{4}$ (b) $\frac{8}{15}(2\sqrt{2}-1)$ (c) $\frac{2}{3}$ (d) $\ln(\frac{9}{8})$

Ö3.3 (a) $\frac{4}{3}$ (b) $\frac{1}{12}$ (c) π (d) $\frac{1}{16}(1+3\mathrm{e}^4)$
Lösningar:

$$\text{(a)} \int_{-1}^{1}\Big(\int_{-y-2}^{y} y^2\,\mathrm{d}x\Big)\,\mathrm{d}y = \int_{-1}^{1} y^2\Big(\int_{-y-2}^{y}\mathrm{d}x\Big)\,\mathrm{d}y = \int_{-1}^{1} y^2(2y+2)\,\mathrm{d}y = \frac{4}{3}$$

Ö3.4 (a) $\frac{1}{24}$ (b) $\frac{31}{8}$ (c) 0 (d) $\frac{1}{8}$

Ö3.5 (a) $\displaystyle\int_0^1\Big(\int_x^1 f(x,y)\,\mathrm{d}y\Big)\,\mathrm{d}x$ (b) $\displaystyle\int_0^1\Big(\int_0^{\arccos(y)} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y$
(c) $\displaystyle\int_0^1\Big(\int_0^y f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y$ (d) $\displaystyle\int_0^1\Big(\int_0^{\sqrt{1-y^2}} f(x,y)\,\mathrm{d}x\Big)\,\mathrm{d}y$

Ö3.6 (a) 0 (b) 0 (c) $\frac{1}{6}(\mathrm{e}^9-1)$ (d) $\frac{1}{3}\ln(9)$

Ö3.7 (a) 0 (b)
Polära koordinater ger

$$\iint_\Omega \frac{x^2+y^2}{\pi}\,\mathrm{d}x\,\mathrm{d}y = \iint_R \frac{r^2}{\pi}\,r\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{1}{\pi}\int_0^1 \Big(\int_{\pi/2}^{\pi} r^3\,\mathrm{d}\varphi\Big)\,\mathrm{d}r = \frac{\pi}{2}\frac{1}{\pi}\int_0^1 r^3\,\mathrm{d}r = \frac{1}{8}$$

(c) Området utgör första kvadranten av cirkelskivan med radie $\sqrt{\ln(31)}$. Symmetri ger att integralen är en $\frac{1}{4}$ av integralen över hela cirkelskivan. Polära koordinater ger

$$\begin{aligned}\frac{1}{4}\iint_\Omega \frac{1}{\pi}\exp(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{4}\iint_R \frac{1}{\pi}\exp(r^2)\,r\,\mathrm{d}r\,\mathrm{d}\varphi \\ &= \frac{1}{4\pi}\int_0^{2\pi}\Big(\int_0^{\sqrt{\ln(31)}} \exp(r^2)\,r\ \mathrm{d}r\Big)\,\mathrm{d}\varphi \\ &= \frac{1}{4\pi}\cdot 2\pi\cdot\left[\frac{1}{2}\exp(r^2)\right]_0^{\sqrt{\ln(31)}} \\ &= \frac{1}{4}\cdot(\exp(\ln(31))-1) = \frac{15}{2}\end{aligned}$$

(d) Området utgörs av den övre delen av cirkelskivan med radie 7. Polära koordinater ger

$$\begin{aligned}\int_0^7\int_0^\pi \frac{r\cos(\varphi)+r\sin(\varphi)}{r}\,r\,\mathrm{d}\varphi\,\mathrm{d}r &= \int_0^7 r\,\mathrm{d}r\cdot\int_0^\pi(\cos(\varphi)+\sin(\varphi))\,\mathrm{d}\varphi \\ &= \frac{49}{2}\cdot(0+2) = 49\end{aligned}$$

Ö3.8 (a) Vi erinrar oss den trigonometriska identiteten $\sin^2(\varphi) = \frac{1}{2}(1-\cos(2\varphi))$.

$$\begin{aligned}\iint_R \frac{r^2\sin^2(\varphi)}{r^2}\,r\,\mathrm{d}r\,\mathrm{d}\varphi &= \int_0^{2\pi}\Big(\int_a^b \sin^2(\varphi)\,r\,\mathrm{d}r\Big)\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\tfrac{1}{2}(1-\cos(2\varphi))\,\mathrm{d}\varphi\int_a^b r\,\mathrm{d}r = \tfrac{1}{2}\pi(b^2-a^2) = \tfrac{3}{2}\pi\end{aligned}$$

(b) Polära koordinater ger

$$\begin{aligned}
\iint_\Omega \frac{1}{\pi}(x+y)^2\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{\pi}\iint_\Omega (x^2+y^2+2xy)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{\pi}\iint_R (r^2+2r^2\cos(\varphi)\sin(\varphi))\,r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= \frac{1}{\pi}\int_3^5 \Big(\int_0^{2\pi} r^3(1+2\cos(\varphi)\sin(\varphi))\,\mathrm{d}\varphi\Big)\,\mathrm{d}r \\
&= \frac{1}{\pi}\int_3^5\int_0^{2\pi} r^3\,\mathrm{d}\varphi\,\mathrm{d}r + \frac{1}{\pi}\int_3^5\int_0^{2\pi} r^3\sin(2\varphi)\,\mathrm{d}\varphi\,\mathrm{d}r \\
&= \frac{2}{4}(5^4-3^4) + \frac{1}{\pi}\int_3^5 r^3\,\mathrm{d}r\cdot\underbrace{\int_0^{2\pi}\sin(2\varphi)\,\mathrm{d}\varphi}_{=0} \\
&= 272
\end{aligned}$$

(c)

$$\begin{aligned}
\iint_\Omega x^2\,\mathrm{d}x\,\mathrm{d}y &= \iint_R r^2\cos^2(\varphi)\,r\,\mathrm{d}r\,\mathrm{d}\varphi \\
&= \int_1^2\Big(\int_0^{2\pi} r^3\cos^2(\varphi)\,\mathrm{d}\varphi\Big)\,\mathrm{d}r \\
&= \int_1^2 r^3\,\mathrm{d}r\int_0^{2\pi}\frac{1+\cos(2\varphi)}{2}\,\mathrm{d}\varphi \\
&= \frac{1}{4}(2^4-1^4)\pi = \frac{15\pi}{4}
\end{aligned}$$

(d)

$$\begin{aligned}
\iint_\Omega \exp(-x^2-y^2)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^R\int_0^{2\pi}\exp(-r^2)\,r\,\mathrm{d}\varphi\,\mathrm{d}r \\
&= 2\pi\int_0^R r\exp(-r^2)\,\mathrm{d}r \\
&= \pi(1-\exp(-R^2))
\end{aligned}$$

Ö3.9 (a) $\frac{2\pi}{3}$
(b) D ges av $x^2+(y-1)^2 \leqslant 1$ dvs $x^2+y^2 \leqslant 2y$. I polära koordinater blir detta $r^2 \leqslant 2r\sin(\varphi)$, dvs $r \leqslant 2\sin(\varphi)$. $E = \{(r,\varphi) \mid 0 \leqslant r \leqslant 2\sin(\varphi),\ 0 \leqslant \varphi \leqslant \pi\}$, rita figur!

$$\begin{aligned}
\iint_\Omega f\,\mathrm{d}A = \iint_E \sqrt{r^2}\,r\,\mathrm{d}r\,\mathrm{d}\varphi &= \int_0^\pi\Big(\int_0^{2\sin(\varphi)} r^2\,\mathrm{d}r\Big)\,\mathrm{d}\varphi \\
= \frac{8}{3}\int_0^\pi \sin^3(\varphi)\,\mathrm{d}\varphi &= \frac{16}{3}\int_0^{\frac{1}{2}\pi}\sin^3(\varphi)\,\mathrm{d}\varphi = \frac{32}{9}
\end{aligned}$$

där $\int_0^{\frac{1}{2}\pi} \sin^3(\varphi)\,\mathrm{d}\varphi = \int_0^{\frac{1}{2}\pi} \sin(\varphi)(1-\cos^2(\varphi))\,\mathrm{d}\varphi = \frac{2}{3}$.
(c) D ges av $(x-1)^2+(y-1)^2 \leqslant 1$ dvs $x^2+y^2 \leqslant 2x$. I polära koordinater blir detta $r^2 \leqslant 2r\cos(\varphi)$, dvs $r \leqslant 2\cos(\varphi)$. $E = \{(r,\varphi) \mid 0 \leqslant r \leqslant 2\sin(\varphi),\ -\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}\}$, rita figur!

$$\begin{aligned}\iint_D f\,\mathrm{d}A &= \iint_E \sqrt{r^2}\,r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{-\pi/2}^{\pi/2} \Big(\int_0^{2\cos(\varphi)} r^2\,\mathrm{d}r\Big)\,\mathrm{d}\varphi \\ &= \frac{8}{3}\int_{-\pi/2}^{\pi/} \cos^3(\varphi)\,\mathrm{d}\varphi = \frac{16}{3}\int_0^{\frac{1}{2}\pi} \sin^3(\varphi)\,\mathrm{d}\varphi = \frac{32}{9}\end{aligned}$$

(d) D ges av $x^2+(y-2)^2 \leqslant 4$ dvs $x^2+y^2 \leqslant 4y$. I polära koordinater blir detta $r^2 \leqslant 4r\sin(\varphi)$, dvs $r \leqslant 4\sin(\varphi)$. $E = \{(r,\varphi) \mid 0 \leqslant r \leqslant 4\sin(\varphi),\ 0 \leqslant \varphi \leqslant \pi\}$, rita figur!

$$\begin{aligned}\iint_D f\,\mathrm{d}A &= \iint_E \sqrt{r^2}\,r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\pi} \Big(\int_0^{4\sin(\varphi)} r^2\,\mathrm{d}r\Big)\,\mathrm{d}\varphi \\ &= \frac{64}{3}\int_0^{\pi} \sin^3(\varphi)\,\mathrm{d}\varphi = \frac{128}{3}\int_0^{\frac{1}{2}\pi} \sin^3(\varphi)\,\mathrm{d}\varphi = \frac{256}{9}\end{aligned}$$

Ö3.10 (a) 0 (b) $\frac{1}{3}$
(c) $u = x+y$, $v = x-2y$, $R = [0,3]\times[1,4]$, $x = \frac{2}{3}u+\frac{1}{3}v$, $y = \frac{1}{3}u-\frac{1}{3}v$
$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$, $\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}\,\mathrm{d}u\,\mathrm{d}v$

$$\begin{aligned}\iint_D \frac{y}{x-2y}\,\mathrm{d}x\,\mathrm{d}y &= \iint_R \frac{\frac{1}{3}(u-v)}{v}\frac{1}{3}\,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{9}\int_0^3 \Big(\int_1^4 (\frac{u}{v}-1)\,\mathrm{d}v\Big)\,\mathrm{d}u = \ln(2)-1\end{aligned}$$

(d) $\Omega = \{(x,y) \mid 1 \leqslant xy \leqslant 2,\ 2 \leqslant x^2y \leqslant 4\}$, $u = xy$, $v = x^2y$
$x = v/u$, $y = u^2/v$, $\frac{\partial(x,y)}{\partial(u,v)} = -1/v$, $\mathrm{d}x\,\mathrm{d}y = \frac{1}{v}\,\mathrm{d}u\,\mathrm{d}v$, $R = [1,2]\times[2,4]$

$$\operatorname{area}(\Omega) = \iint_\Omega \mathrm{d}x\,\mathrm{d}y = \iint_R \frac{1}{v}\,\mathrm{d}u\,\mathrm{d}v = \ln(2)$$

Ö3.11 (a) 27 (b) $\frac{1}{24}$ (c) $\frac{9}{8}\pi$ (d) $\frac{1}{24}$

Ö3.12 (a) $\frac{1}{6}$ (b) $\frac{1}{60}$ (c) $\frac{1}{120}$ (d) $\frac{1}{20}$

Ö3.13 (a) $\frac{1}{8}$ (b) $\frac{\pi}{2}$ (c) 19 (d) $\frac{128}{15}\pi$.
Tips. Enkelt i y:

$$\iint_{B_4} \Big(\int_{x^2+z^2}^4 \sqrt{x^2+z^2}\,\mathrm{d}y\Big)\,\mathrm{d}x\,\mathrm{d}z$$

eller skivmetoden:

$$\int_0^4 \left(\iint_{B_{\sqrt{y}}} \sqrt{x^2+z^2}\,\mathrm{d}x\,\mathrm{d}z \right) \mathrm{d}y$$

där B_R cirkelskiva med radie R.

Ö3.14 (a) $\frac{4}{3}\pi R^3$ Tips: sfäriska koordinater. (b) $\frac{148\pi}{3}$ (c) $\frac{4}{5}\pi$ (d) 0

Ö3.15 (a) $\frac{4}{3}\pi$ (b) 8π (c) $\frac{4\pi}{3}(4-\frac{5}{\sqrt(2)})$ (d) $\frac{8\pi}{3}(2-\sqrt{2})$

Ö3.16 (a) Cylindriska koordinater:

$$\iiint_\Omega \sin(\pi z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 \int_0^2 \int_0^{2\pi} \sin(\pi z)\,\rho\,\mathrm{d}\varphi\,\mathrm{d}\rho\,\mathrm{d}z = 4\pi \int_0^1 \sin(\pi z)\,\mathrm{d}z = 8$$

(b) Sfäriska koordinater: $\Omega = [0,1]\times[0,\pi/4]\times[0,2\pi]$

$$\iiint_\Omega r\,r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \int_0^1 r^3\,\mathrm{d}r \int_0^{\pi/4} \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi = \tfrac{1}{4}(2-\sqrt{2})\pi$$

(c) $\frac{12}{5}\pi$ Tips: cylindriska koordinater.
(d) $\frac{\pi}{8}$

Ö3.17 (a) Cylindriska koordinater ger

$$\int_0^1 \int_0^{2\pi} \int_0^z \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = 2\pi \int_0^1 \int_0^z \rho\,\mathrm{d}\rho\,\mathrm{d}z = \pi \int_0^1 z^2\,\mathrm{d}z = \frac{\pi}{3}$$

(b) Cylindriska koordinater ger

$$\int_0^4 \int_0^{2\pi} \int_0^z \frac{z}{\pi}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = 2\int_0^4 z\cdot\frac{z^2}{2}\,\mathrm{d}z = 4^4/4 = 64$$

(c) Cylindriska koordinater ger

$$\int_0^4 \int_0^{2\pi} \int_0^z \frac{5z^2}{\pi}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = 10\int_0^4 z^2\cdot\frac{z^2}{2}\,\mathrm{d}z = 4^5 = 1024$$

(d) Cylindriska koordinater ger

$$\begin{aligned}\int_0^1 \int_0^{2\pi} \int_0^z \rho^2\cos^2(\varphi) z\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z &= \int_0^1 \int_0^{2\pi} \cos^2(\varphi)\frac{z^5}{4}\,\mathrm{d}\varphi\,\mathrm{d}z \\ &= \frac{1}{24}\int_0^{2\pi} \frac{1+\cos(2\varphi)}{2}\,\mathrm{d}\varphi = \frac{\pi}{24}\end{aligned}$$

Ö3.18 Använd substitutionen $x = ar\sin(\theta)\cos(\varphi)$, $y = br\sin(\theta)\sin(\varphi)$, $z = cr\cos(\theta)$ med $\mathrm{d}V = abcr^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$.
(a) 8π (b) 0 (c) $\frac{24\pi}{5}$ (d) $\frac{24\pi}{15}$

Ö3.19 (a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

Ö3.20 Enheter $M = ka^3$ [kg], a [m], k [kg/m^3]. (a) $\bar{\boldsymbol{r}} = (\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12})$,
(b) $\bar{\boldsymbol{r}} = (\frac{5a}{9}, \frac{a}{2}, \frac{a}{2})$ (c) $\bar{\boldsymbol{r}} = (\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$ (d) $\bar{\boldsymbol{r}} = (\frac{2a}{3}, \frac{2a}{3}, \frac{a}{2})$

Ö3.21 (a) $\frac{1}{2}\pi\varrho h R^4$ (b) $\varrho h R\pi(\frac{R^3}{4} + \frac{Rh^2}{3})$ (c) $\varrho h R\pi(\frac{R^3}{4} + \frac{Rh^2}{3})$ (d) $\frac{3}{2}\pi\varrho h R^4$

Ö3.22 (a) $\frac{2}{3}\varrho L^5$ (b) $\frac{2}{3}\varrho L^5$ (c) $\frac{2}{3}\varrho L^5$ (d) $\frac{1}{6}\varrho L^5$

Ö3.23 (a) divergent $+\infty$
(b) Positiv integrand, upprepad integration tillåten. Använd polära koordinater. Divergent mot $+\infty$.
(c) Insättning av integrationsgränserna ger

$$\begin{aligned}\int_0^\infty \int_{x-1}^{x+1} \exp(-x-y)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^\infty [-\exp(-x-y)]_{x-1}^{x+1}\,\mathrm{d}x \\ &= \int_0^\infty \exp(-2x)\cdot(\mathrm{e}-\mathrm{e}^{-1})\,\mathrm{d}x \\ &= \frac{\mathrm{e}-\mathrm{e}^{-1}}{2}\end{aligned}$$

Konvergent med värdet $\frac{\mathrm{e}-\mathrm{e}^{-1}}{2}$.
(d) Positiv integrand, polära koordinater ger

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{2\pi} r^{-1}\,r\,\mathrm{d}\varphi\,\mathrm{d}r = 2\pi$$

Konvergent med värdet 2π.

Ö3.24 (a) $p < 1$ (b) $p > 1$ (c) alla p (d) inga p

Ö3.25 (a) 2π (b) 4π (c) π^2 (d) 2π

Problem

P3.9 Positiv integrand, upprepad integration tillåten. Sfäriska koordinater:

$$\begin{aligned}&\int_\Omega |\boldsymbol{r}|^\alpha\,\mathrm{d}V = \int_0^1 \int_0^\pi \int_0^{2\pi} r^\alpha\,r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 r^{2+\alpha}\,\mathrm{d}r = 4\pi\Big[\frac{r^{3+\alpha}}{3+\alpha}\Big]_0^1 = 4\pi\frac{1}{2+\alpha}\end{aligned}$$

är konvergent om och endast om $2+\alpha > -1$, dvs $\alpha > -3$.

P3.10 På samma vis:

$$\int_\Omega |\boldsymbol{r}|^\alpha \,\mathrm{d}V = \int_1^\infty \int_0^\pi \int_0^{2\pi} r^\alpha\, r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$$
$$= \int_0^\pi \sin(\theta)\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \int_1^\infty r^{2+\alpha}\,\mathrm{d}r = 4\pi\Big[-\frac{r^{3+\alpha}}{3+\alpha}\Big]_1^\infty = -4\pi\frac{1}{2+\alpha}$$

är konvergent om och endast om $2+\alpha < -1$, dvs $\alpha < -3$.

Kapitel 4

Övningar

Ö4.1 (a) $(t, \sqrt{4-t^2})$ (b) $(\sqrt{4-t^2}, t)$ (c) $(2\cos(t), 2\sin(t))$ (d) $(\sqrt{t}, \sqrt{4-t})$

Ö4.2 (a) $(\cos(t), \sin(t), 0), 0 \leqslant t \leqslant 2\pi$
(b) $(\cos(t), \sin(t), -\cos(t)-\sin(t)), 0 \leqslant t \leqslant 2\pi$
(c) $(\cos(t), \sin(t), 1-\cos(t)-\sin(t)), 0 \leqslant t \leqslant 2\pi$
(d) $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, t)$ och $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, t), t \in \mathbb{R}$

Ö4.3 (a) $x=-2t, y=1, z=\frac{1}{2}\pi+t; \quad t \in \mathbb{R}$
(b) $x=1, y=t, z=2t; \quad t \in \mathbb{R}$
(c) $x=1+t, y=3+2t; \quad t \in \mathbb{R}$
(d) $x=-1, y=4t, z=\pi^2+2\pi t; \quad t \in \mathbb{R}$

Ö4.4 (a) $x=t, y=\frac{1}{2}t+\frac{1}{2}$
(b) $x=t, y=1-t, z=2t-1$
(c) $x=t, y=1, z=0$
(d) $x=t, y=3t+2$

Ö4.5 (a) $2\pi\sqrt{a^2+b^2}$ (b) $\mathrm{e}-\mathrm{e}^{-1}$ (c) $\frac{1}{27}(13^{3/2}-8)$ (d) $\frac{14}{3}$

Ö4.6 (a) $2\pi+\frac{2}{3}$ (b) $\frac{5}{2}\sqrt{42}$ (c) $\frac{5\sqrt{5}-1}{6}+2$
(d) $\boldsymbol{r}(t) = (\cos(t), \sin(t), t^3)$
$\implies \boldsymbol{r}'(t) = (-\sin(t), \cos(t), 3t^2)$
$\implies \|\boldsymbol{r}'(t)\| = \sqrt{\sin^2(t)+\cos^2(t)+9t^4} = \sqrt{1+9t^4}$

Kurvintegralen ges därmed av

$$\begin{aligned}\int_C f\,\mathrm{d}s &= \int_a^b f(x(t),y(t),z(t))\|\boldsymbol{r}'(t)\|\,\mathrm{d}t\\ &= 3\int_0^{\pi/3}\sin^3(t)(1+9t^4)^{-1/2}(1+9t^4)^{1/2}\,\mathrm{d}t\\ &= 3\int_0^{\pi/3}\sin^3(t)\,\mathrm{d}t\end{aligned}$$

Variabelsubstitutionen $u=\cos(t)$ ger $\mathrm{d}u=-\sin(t)\,\mathrm{d}t$ och därmed

$$\begin{aligned}3\int_0^{\pi/3}\sin^3(t)\,\mathrm{d}t &= -3\int_1^{1/2}(1-u^2)\,\mathrm{d}u\\ &= 3\int_{1/2}^{1}(1-u^2)\,\mathrm{d}u\\ &= \frac{3}{2}-(1-1/8)\\ &= \frac{12}{8}-\frac{8}{8}+\frac{1}{8}=\frac{5}{8}=0.625\end{aligned}$$

Ö4.7 (a) $-\frac{2}{3}$ (b) $\frac{27}{28}$ (c) $-mg$ (d) 2π

Ö4.8 (a) 12 (b) $\frac{3}{2}$ (c) $\frac{3}{2}$ (d) 0

Ö4.9 (a) $\nabla\times\boldsymbol{F}=0 \implies \boldsymbol{F}=\nabla\phi$
$\frac{\partial\phi}{\partial x}=y+z \implies \phi(x,y,z)=(y+z)x+f(y,z)$
$\frac{\partial\phi}{\partial y}=x+z \implies \phi(x,y,z)=(x+z)y+g(x,z)$
$\frac{\partial\phi}{\partial z}=x+y \implies \phi(x,y,z)=(x+y)z+h(x,y)$
$\implies \phi(x,y,z)=xy+xz+yz+C$
$\implies \int_C \boldsymbol{F}\cdot\hat{T}\,\mathrm{d}s=\phi(4,5,6)-\phi(1,2,3)=74-11=63$
(b) -5
(c) $\nabla\times\boldsymbol{F}=0 \implies \boldsymbol{F}=\nabla\phi$
$\frac{\partial\phi}{\partial x}=x+y \implies \phi(x,y,z)=x^2/2+xy+f(y,z)$
$\frac{\partial\phi}{\partial y}=x-z \implies \phi(x,y,z)=xy-yz+g(x,z)$
$\frac{\partial\phi}{\partial z}=z-y \implies \phi(x,y,z)=z^2/2-yz+h(x,y)$
$\implies \phi(x,y,z)=x^2/2+z^2/2+xy-yz+C$
$\implies \int_C \boldsymbol{F}\cdot\hat{T}\,\mathrm{d}s=\phi(3,0,3)-\phi(2,-3,-4)=9-(-8)=17$
(d) 24

Ö4.10 (a) $x+y+z=1$ (b) $2x-3z=2$ (c) $x-y=1$ (d) $z=0$

Ö4.11 (a) $S\colon \boldsymbol{r}=(\rho\cos(\theta),\rho\sin(\theta),0), 0\leqslant\rho\leqslant 1, 0\leqslant\theta\leqslant 2\pi$
(b) $S\colon \boldsymbol{r}=(\rho\cos(\theta),\rho\sin(\theta),\rho\cos(\theta)), 0\leqslant\rho\leqslant 1, 0\leqslant\theta\leqslant 2\pi$
(c) $S\colon \boldsymbol{r}=(\rho\cos(\theta),\rho\sin(\theta),1-\rho\cos(\theta)-\rho\sin(\theta)), 0\leqslant\rho\leqslant 1, 0\leqslant\theta\leqslant 2\pi$
(d) $S\colon \boldsymbol{r}=(u,0,v), -1\leqslant u\leqslant 1, -\infty<v<\infty$

Ö4.12 (a) $x = 1$ (b) $x + y + z = \sqrt{3}$ (c) $z = 1$ (d) $x - y + \sqrt{2}z = 2$

Ö4.13 (a) $x + 2y - 2z = -3$ (b) $-x + 2z = 1$ (c) $-4y + z + 3 = 0$ (d) $x - z = 0$

Ö4.14 (a) 8π (b) $3\sqrt{14}$ (c) $\sqrt{14}\pi$ (d) $\frac{2}{3}(2\sqrt{2} - 1)\pi$

Lösningar:

(a) se exempel **??**

$$\begin{cases} x = u, \\ y = v, & (u,v) \in D = \{(u,v) \mid u^2 + v^2 \leqslant 9\} \\ z = u^2 + v^2, \end{cases}$$

$$\boldsymbol{r}'_u \times \boldsymbol{r}'_v = (-2u, -2v, 1),\ \mathrm{d}S = \sqrt{1 + 4u^2 + 4v^2}\,\mathrm{d}u\,\mathrm{d}v$$
$$E = \{(r,\varphi) \mid 0 \leqslant r \leqslant 3,\ 0 \leqslant \varphi \leqslant 2\pi\}$$
$$\iint_S \mathrm{d}S = \iint_D \sqrt{1 + 4u^2 + 4v^2}\,\mathrm{d}u\,\mathrm{d}v = \iint_E \sqrt{1 + 4r^2}\,r\,\mathrm{d}r\,\mathrm{d}\varphi$$
$$= \int_0^3 (1 + 4r^2)^{1/2}\,r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi$$
$$= 2\pi\Big[\frac{1}{12}(1 + 4r^2)^{3/2}\Big]_0^3 = \frac{1}{6}(37^{3/2} - 1)\pi$$

Ö4.15 (a) 4π (b) $\frac{8}{3}\pi$ (c) $\frac{4}{3}\pi$ (d) $\frac{4}{3}\pi$

Ö4.16 (a) 2π (b) 0 (c) $\frac{1}{2}\pi$
(d) Sidoytan S_1: $\iint_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{2}\pi$. Toppytan S_2: $\iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0$.
$\iint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \iint_{S_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{2}\pi$.

Ö4.17 (a) 0 (b) $\frac{1}{2}\pi$ (c) 0
(d) Parametrisering av den buktiga delen S_1 av ytan (i cylindriska koordinater har vi $z = x^2 + y^2 = \rho^2$, dvs $\rho = \sqrt{z}$):

$$S_1\colon \begin{cases} x = \sqrt{z}\cos(\varphi), \\ y = \sqrt{z}\sin(\varphi), & (\varphi, z) \in R = [0, 2\pi] \times [0, 1] \\ z = z, \end{cases}$$

Tangenter:

$$\frac{\partial \boldsymbol{r}}{\partial \varphi} = -\sqrt{z}\sin(\varphi)\boldsymbol{e}_x + \sqrt{z}\cos(\varphi)\boldsymbol{e}_y + 0\boldsymbol{e}_z$$
$$\frac{\partial \boldsymbol{r}}{\partial z} = \frac{1}{2\sqrt{z}}\cos(\varphi)\boldsymbol{e}_x + \frac{1}{2\sqrt{z}}\sin(\varphi)\boldsymbol{e}_y + \boldsymbol{e}_z$$

En normalvektor:

$$\boldsymbol{N} = \frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial z} = \sqrt{z}\cos(\varphi)\boldsymbol{e}_x + \sqrt{z}\sin(\varphi)\boldsymbol{e}_y - \tfrac{1}{2}\boldsymbol{e}_z = x\boldsymbol{e}_x + y\boldsymbol{e}_y - \tfrac{1}{2}\boldsymbol{e}_z$$

Vi ser att $\boldsymbol{N}$ pekar utåt (ty $N_z = -\frac{1}{2} < 0$) och, se (4.201),

$$\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{N}}\,\mathrm{d}S = \Big(\frac{\partial \boldsymbol{r}}{\partial \varphi} \times \frac{\partial \boldsymbol{r}}{\partial z}\Big)\,\mathrm{d}\varphi\,\mathrm{d}z = \boldsymbol{N}\,\mathrm{d}\varphi\,\mathrm{d}z$$

så att

$$\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S = \boldsymbol{F} \cdot \boldsymbol{N}\,\mathrm{d}\varphi\,\mathrm{d}z$$

där

$$\begin{aligned}\boldsymbol{F} \cdot \boldsymbol{N} &= (x\boldsymbol{e}_x + y\boldsymbol{e}_y + z^2\boldsymbol{e}_z) \cdot (x\boldsymbol{e}_x + y\boldsymbol{e}_y - \tfrac{1}{2}\boldsymbol{e}_z) \\ &= x^2 + y^2 - \tfrac{1}{2}z^2 = z - \tfrac{1}{2}z^2 \quad \text{på } S_1\end{aligned}$$

Flödet ut genom S_1 blir, se (4.202),

$$\begin{aligned}\iint_{S_1} \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S &= \iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{N}\,\mathrm{d}\varphi\,\mathrm{d}z = \iint_R (z - \tfrac{1}{2}z^2)\,\mathrm{d}\varphi\,\mathrm{d}z = \{\text{Fubini}\} \\ &= \int_0^1 (z - \tfrac{1}{2}z^2)\,\mathrm{d}z \int_0^{2\pi} \mathrm{d}\varphi = 2\pi(\tfrac{1}{2} - \tfrac{1}{6}) = \frac{2}{3}\pi\end{aligned}$$

(Inom parentes noterar vi att $\|\boldsymbol{N}\| = \sqrt{x^2 + y^2 + \frac{1}{4}} = \sqrt{z + \frac{1}{4}}$ så att flödestätheten blir $\boldsymbol{F} \cdot \hat{\boldsymbol{N}} = \boldsymbol{F} \cdot \boldsymbol{N}/\|\boldsymbol{N}\| = (z - \frac{1}{2}z^2)/\sqrt{z + \frac{1}{4}}$. Detta behövs dock ej för att beräkna flödesintegralen.)

På toppytan S_2 (med polära koordinater):

$$S_2\colon \begin{cases} x = r\cos(\varphi), \\ y = r\sin(\varphi), \qquad (r, \varphi) \in E = [0, 1] \times [0, 2\pi] \\ z = 1, \end{cases}$$

Parametriseringen behövs inte, vi ser helt enkelt

$$\hat{\boldsymbol{N}} = \boldsymbol{e}_z,\ \boldsymbol{F} \cdot \hat{\boldsymbol{N}} = (x\boldsymbol{e}_x + y\boldsymbol{e}_y + z^2\boldsymbol{e}_z) \cdot \boldsymbol{e}_z = z^2 = 1$$

så att

$$\iint_{S_2} \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S = \iint_{S_2} \mathrm{d}S = \mathrm{area}(S_2) = \pi$$

Alltså blir totala utflödet

$$\iint_S \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S = \iint_{S_1} \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S + \iint_{S_2} \boldsymbol{F} \cdot \hat{\boldsymbol{N}}\,\mathrm{d}S = \frac{2}{3}\pi + \pi = \frac{5}{3}\pi$$

Alternativ parametrisering (bättre?):

$$S_1\colon \begin{cases} x = \rho\cos(\varphi), \\ y = \rho\sin(\varphi), \qquad (\rho, \varphi) \in R = [0, 1] \times [0, 2\pi] \\ z = \rho^2, \end{cases}$$

Genomför räkningarna!

Ö4.18 (a) Integranden är 0 då $x = 0$, $y = 0$ eller $z = 0$ vilket gör att vi får ett bidrag endast från tre av kubens sidor.

$$x = 1\colon \int_0^1 \int_0^1 yz \,\mathrm{d}y\,\mathrm{d}z = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$
$$y = 1\colon \int_0^1 \int_0^1 xz \,\mathrm{d}x\,\mathrm{d}z = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$
$$z = 1\colon \int_0^1 \int_0^1 xy \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Summan av bidragen ges av $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$.
(b) Enhetskuben har sex sidor och på tre av dessa ($x = 0, y = 0, z = 0$) är integranden noll. Vi skalar (tillfälligt) bort faktorn 480 och integrerar över de övriga tre sidorna:

$$\iint_S x^3y^5z^7 \mathrm{d}S = \iint_{x=1} y^5z^7 \,\mathrm{d}y\,\mathrm{d}z + \iint_{y=1} x^3z^7 \,\mathrm{d}x\,\mathrm{d}z + \iint_{z=1} x^3y^5 \,\mathrm{d}x\,\mathrm{d}y$$

Första integralen är separabel (Fubini) och ges av

$$\begin{aligned}\iint_{x=1} y^5z^7 \,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \int_0^1 y^5z^7 \,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^1 y^5 \,\mathrm{d}y \cdot \int_0^1 z^7 \,\mathrm{d}z = \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{48}\end{aligned}$$

På samma sätt ges den andra integralen av $\frac{1}{4\cdot 8} = \frac{1}{32}$ och den tredje av $\frac{1}{4\cdot 6} = \frac{1}{24}$.
Ytintegralen ges därmed av summan

$$\frac{480}{48} + \frac{480}{32} + \frac{480}{24} = 10 + 15 + 20 = 45$$

(c) Integranden är 0 då $x = 0$, $y = 0$ eller $z = 0$ vilket gör att vi får bidrag endast från tre av rätblockets sidor.

$$x = 1\colon 1 \cdot \int_0^2 \int_0^3 yz \,\mathrm{d}y\,\mathrm{d}z = 1 \cdot \frac{2^2}{2} \cdot \frac{3^2}{2} = 9$$
$$y = 2\colon 2 \cdot \int_0^1 \int_0^3 xz \,\mathrm{d}x\,\mathrm{d}z = 2 \cdot \frac{1^2}{2} \cdot \frac{3^2}{2} = \frac{9}{2}$$
$$z = 3\colon 3 \cdot \int_0^1 \int_0^2 xy \,\mathrm{d}x\,\mathrm{d}y = 3 \cdot \frac{1^2}{2} \cdot \frac{2^2}{2} = 3$$

Summan av bidragen ges av $9 + \frac{9}{2} + 3 = 16.5$.
(d) 33.

Ö4.19 (a) (yz, xz, xy) (b) $3\|\boldsymbol{r}\|\boldsymbol{r}$. Ledning: $f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$
(c) $(\sin(y), x\cos(y), 0)$ (d) $(y/z, x/z, -xy/z^2)$

Ö4.20 (a) $\nabla \cdot \boldsymbol{F} = 0$, $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, $\phi = xyz$

(b) $\nabla \cdot \boldsymbol{F} = 0$, $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, $\phi = xy + yz + zx$

(c) $\nabla \cdot \boldsymbol{F} = 0$, $\nabla \times \boldsymbol{F} = -2x\boldsymbol{e}_z$, potential finns ej.

(d) $\nabla \cdot \boldsymbol{F} = 2yz + 2$, $\nabla \times \boldsymbol{F} = \boldsymbol{0}$, $\phi = x^2yz + z^2$

Ö4.21 (a) $\Delta f = 4$ (b) $\Delta f = 0$ (c) $\Delta f = \frac{1}{\sqrt{x^2+y^2}}$ (d) $\Delta f = 2(y^2z^2 + x^2z^2 + x^2y^2)$

Lösningar:

$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
(a) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 2 \implies \Delta f = 4$ (b) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial z^2} = 0 \implies \Delta f = 0$
(c) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{1}{2\sqrt{x^2+y^2}} \implies \Delta f = \frac{1}{\sqrt{x^2+y^2}}$
(d) $\frac{\partial^2 f}{\partial x^2} = 2y^2z^2$, $\frac{\partial^2 f}{\partial y^2} = 2x^2z^2$, $\frac{\partial^2 f}{\partial z^2} = 2x^2y^2 \implies \Delta f = 2(y^2z^2 + x^2z^2 + x^2y^2)$

Ö4.22 (a) 0 (b) 22 (c) 11

Lösning:
$\boldsymbol{v} = (3,2,2) - (1,2,-1) = (2,0,3)$
$\boldsymbol{r} = (1,2,-1) + t(2,0,3)$, $t \in [0,1]$, $\mathrm{d}\boldsymbol{r} = (2,0,3)\,\mathrm{d}t$
(b) Enligt Ö4.20: $\nabla \times \boldsymbol{F} = -2x\boldsymbol{e}_z$, potential finns ej.

$$\begin{aligned} W &= \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \big(2x(t)y(t), -y(t)^2, 2\big) \cdot (2,0,3)\,\mathrm{d}t \\ &= \int_0^1 \big(4x(t)y(t) + 6\big)\,\mathrm{d}t \\ &= \int_0^1 \big(4(1+2t)2 + 6\big)\,\mathrm{d}t = \int_0^1 (14 + 16t)\,\mathrm{d}t = 22 \end{aligned}$$

(c) $\nabla \times \boldsymbol{F} = (z - x, 0, z + 1 - x)$, potential finns ej.

$$\begin{aligned} W &= \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \big(x(t)y(t), x(t)z(t) + x(t), y(t)z(t)\big) \cdot (2,0,3)\,\mathrm{d}t \\ &= \int_0^1 \big(2x(t)y(t) + 3y(t)z(t)\big)\,\mathrm{d}t = \int_0^1 \big(2(1+2t)2 + 6(-1+3t)\big)\,\mathrm{d}t \\ &= \int_0^1 (-2 + 26t)\,\mathrm{d}t = 11 \end{aligned}$$

(d) Konservativt fält med potential $\phi = \frac{x^2}{2} + \frac{z^2}{2} + xy - yz + C$. Vi får $\phi(3,2,2) - \phi(1,2,-1) = \frac{7}{2}$

Ö4.23 (a) π (b) 0 (c) 2π (d) 0

Ö4.24 (a) 0 (b) $\frac{1}{2}\pi$ (c) 0
(d)

$$\nabla \cdot \boldsymbol{F} = 1 + 1 + 2z = 2(1+z)$$

Cylindriska koordinater:

$$E = \{(r, \varphi, z) \mid 0 \leqslant r \leqslant \sqrt{z},\ 0 \leqslant \varphi \leqslant 2\pi\}, \quad D = \{(r, z) \mid 0 \leqslant r \leqslant \sqrt{z}\}$$

Divergenssatsen:

$$\begin{aligned}
\iint_S \boldsymbol{F} \cdot \hat{\boldsymbol{N}} \, \mathrm{d}S &= \iiint_D \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \iiint_D 2(1+z) \, \mathrm{d}V = \iiint_E 2(1+z) \, r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z \\
&= \{\text{skivning}\} = \int_0^{2\pi} \Big(\iint_D 2(1+z) r \, \mathrm{d}r \, \mathrm{d}z \Big) \, \mathrm{d}\varphi \\
&= \{D \text{ enkel i } r\} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^1 \Big(2(1+z) \int_0^{\sqrt{z}} r \, \mathrm{d}r \Big) \, \mathrm{d}z \\
&= 2\pi \int_0^1 2(1+z) \frac{z}{2} \, \mathrm{d}z = 2\pi \int_0^1 (z + z^2) \, \mathrm{d}z = 2\pi(\tfrac{1}{2} + \tfrac{1}{3}) = \tfrac{5}{3}\pi
\end{aligned}$$

Ö4.25 (a) Gauss divergenssats ger att utflödet är

$$\iint_S \boldsymbol{F} \cdot \hat{\boldsymbol{N}} \, \mathrm{d}S = \iiint_B \nabla \cdot \boldsymbol{F} \, \mathrm{d}V$$

Med sfäriska koordinater

$$\begin{cases} x = r \sin(\theta) \cos(\varphi) \\ y = r \sin(\theta) \sin(\varphi) \\ z = r \cos(\theta) \end{cases} \quad 0 \leqslant r \leqslant 1,\ 0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi < 2\pi,$$

Volymselementet: $\mathrm{d}V = r^2 \sin(\theta) \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi$. Integranden: $\nabla \cdot \boldsymbol{F} = 3x^2 + 3y^2 + 1 = 3r^2 \sin^2(\theta) + 1$. Integralen blir

$$\begin{aligned}
\iiint_B \nabla \cdot \boldsymbol{F} \, \mathrm{d}V &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{1} (3r^2 \sin^2(\theta) + 1) r^2 \sin(\theta) \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi \\
&= 3 \int_0^1 r^4 \, \mathrm{d}r \int_0^\pi \sin^3(\theta) \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi + \int_0^1 r^2 \, \mathrm{d}r \int_0^\pi \sin(\theta) \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \\
&= 3 \, \frac{1}{5} \, \frac{4}{3} \, 2\pi + \frac{1}{3} \cdot 2 \cdot 2\pi = \frac{8}{5}\pi + \frac{4}{3}\pi = \frac{44}{15}\pi
\end{aligned}$$

där vi använt

$$\int_0^\pi \sin^3(\theta) \, \mathrm{d}\theta = \int_0^\pi (1 - \cos^2(\theta)) \sin(\theta) \, \mathrm{d}\theta = \Big\{ s = -\cos(\theta) \Big\} = \int_{-1}^1 (1 - s^2) \, \mathrm{d}s = \frac{4}{3}$$

(b) 0
(c) Gauss sats ger

$$Q = \iint_{\partial\Omega} \boldsymbol{F} \cdot \hat{\boldsymbol{N}} \, \mathrm{d}S = \iiint_{\Omega} \nabla \cdot \boldsymbol{F} \, \mathrm{d}V = \iiint_{\Omega} 3x^2 + 3y^2 + 1 \, \mathrm{d}V$$

Integralen av den tredje termen blir $4\pi/3$ (volymen av enhetssfären).
Sfäriska koordinater för de första två termerna ger

$$\begin{aligned}
Q &= \iiint_{\Omega} 3x^2 + 3y^2 \, \mathrm{d}V \\
&= 3 \int_0^1 \int_0^{\pi} \int_0^{2\pi} r^2(\cos^2(\varphi) + \sin^2(\varphi)) \sin^2(\varphi) r^2 \sin(\varphi) \, \mathrm{d}r \, \mathrm{d}theta \, \mathrm{d}\theta \\
&= 3 \int_0^1 r^4 \, \mathrm{d}r \int_0^{\pi} \sin^3(\varphi) \, \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta = \{\text{Låt } u = \cos(\varphi)\} \\
&= 3 \cdot \frac{1}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{8\pi}{5}
\end{aligned}$$

Flödet är således $\frac{4\pi}{3} + \frac{8\pi}{5} = \frac{44\pi}{15}$.
(d) $\nabla \cdot \boldsymbol{F} = \frac{3}{4\pi}$ så Gauss sats ger att $Q = 1$.

Problem

P4.1 Tips: derivera båda sidorna i likheten $\|\boldsymbol{r}(t)\|^2 = \boldsymbol{r}(t) \cdot \boldsymbol{r}(t)$.

P4.2

$$\begin{aligned}
M &= \int_C \delta \, \mathrm{d}s = \int_C k(1 - y/R) \, \mathrm{d}s = kR \int_0^{\pi} (1 - \sin(t)) \, \mathrm{d}t = kR(\pi - 2) \quad [\text{kg}] \\
\bar{y} &= \frac{1}{M} \int_C y\delta \, \mathrm{d}s = \frac{1}{M} \int_C yk(1 - y/R) \, \mathrm{d}s \\
&= \frac{1}{kR(\pi - 2)} kR^2 \int_0^{\pi} (\sin(t) - \sin^2(t)) \, \mathrm{d}t = R\frac{\pi - 4}{\pi - 2} \quad [\text{m}] \\
\bar{x} &= 0 \quad (\text{pga symmetri})
\end{aligned}$$

P4.3 Tips: sätt in rörelseekvationen $\boldsymbol{F}(\boldsymbol{r}(t)) = m\boldsymbol{r}''(t)$ i integralen och använd $\boldsymbol{r}''(t) \cdot \boldsymbol{r}'(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{r}'(t)\|^2$.

P4.4 Bevisa (4.40). Med $t = g(s)$ ger (3.65) att

$$\int_a^b f(\boldsymbol{r}(t)) \, \|\boldsymbol{r}'(t)\| \, \mathrm{d}t = \int_A^B f(\boldsymbol{r}(g(s))) \, \|\boldsymbol{r}'(g(s))\| \, |g'(s)| \, \mathrm{d}s$$

Men kedjeregeln ger

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{r}(g(s)) = \boldsymbol{r}'(g(s))g'(s)$$

så att

$$\left\| \frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{r}(g(s)) \right\| = \|\boldsymbol{r}'(g(s))g'(s)\| = \|\boldsymbol{r}'(g(s))\| \, |g'(s)|$$

vilket bevisar (4.40).

P4.5 $\displaystyle\iint_{S_1} z \, \mathrm{d}S = \frac{3}{2}\pi, \quad \iint_{S_2} z \, \mathrm{d}S = 0, \quad \iint_{S_3} z \, \mathrm{d}S = \sqrt{2}\pi, \quad \iint_{S} z \, \mathrm{d}S = (\tfrac{3}{2} + \sqrt{2})\pi$

P4.6 Konstanten a måste ha enheten s^{-1} och b har enheten m. Vi har

$$\boldsymbol{v} = a(x, y, z), \hat{\boldsymbol{N}} = \frac{-1}{\sqrt{3}}(1, 1, 1), \text{area}(S) = \frac{1}{2}\sqrt{3}\, b^2$$

Flödet blir

$$\begin{aligned}
\iint_S \boldsymbol{v} \cdot \hat{\boldsymbol{N}} \, \mathrm{d}S &= \iint_S a(x, y, z) \cdot \frac{-1}{\sqrt{3}}(1, 1, 1) \, \mathrm{d}S = \frac{-1}{\sqrt{3}} a \iint_S (x + y + z) \, \mathrm{d}S \\
&= \frac{-1}{\sqrt{3}} a \iint_S b \, \mathrm{d}S = \frac{-1}{\sqrt{3}} ab \, \text{area}(S) \\
&= \frac{-1}{\sqrt{3}} ab \frac{1}{2}\sqrt{3}\, b^2 = -\frac{1}{2} ab^3 \quad \mathrm{m}^3/\mathrm{s}
\end{aligned}$$

P4.7 —

P4.8 $\nabla \cdot \boldsymbol{v} = 0, \nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{u}}$, det är hastighetsfältet för stelkroppsrotation med vinkelhastigheten ω [1/s] (radianer per sekund) kring en axel genom origo med riktningsvektorn $\hat{\boldsymbol{u}}$, jämför med exempel 4.33.

P4.9 (a) 2π (b) $(\pi, -\pi, 0)$ (c) 0

P4.10 $P = cr^{a+1}/(a + 1) + C$ om $a \neq -1$, $\phi = c \ln(r) + C$ om $a = -1$
Lösning:
$\boldsymbol{F} = -cr^a \hat{\boldsymbol{r}} = -cr^a \frac{\boldsymbol{r}}{r} = -cr^{a-1}\boldsymbol{r}$, där c är en proportionalitetskonstant. Enligt (4.229):

$$\begin{aligned}
\nabla \times \boldsymbol{F} &= -c\nabla \times (r^{a-1}\boldsymbol{r}) = -c\Big((\nabla r^{a-1}) \times \boldsymbol{r} + r^{a-1} (\nabla \times \boldsymbol{r}) \Big) \\
&= -c\Big((a - 1)r^{a-2}(\nabla r \times \boldsymbol{r}) + r^{a-1} (\nabla \times \boldsymbol{r}) \Big) = \boldsymbol{0}
\end{aligned}$$

eftersom $\nabla r = \hat{\boldsymbol{r}}$, $\hat{\boldsymbol{r}} \times \boldsymbol{r} = \boldsymbol{0}$ och $\nabla \times \boldsymbol{r} = \boldsymbol{0}$. Alltså: kraften är konservativ.

En potential $\phi(\boldsymbol{r})$ fås genom att integrera längs en godtycklig kurva fram till punkten $\boldsymbol{r}$, se (4.86):

$$\phi(\boldsymbol{r}) = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

Vi har en fysikalisk kraft $\boldsymbol{F} = -\nabla P = \nabla\phi$ (med minustecken), så att genom att integrera längs den räta från origo till $\boldsymbol{r}$ får vi

$$P(r) = -\phi(r) = c\int_C r^a \hat{\boldsymbol{r}}\cdot \mathrm{d}\boldsymbol{r} = c\int_0^r r^a\,\mathrm{d}r = \begin{cases} c\frac{r^{a+1}}{a+1}, & a \neq -1 \\ c\ln(r), & a = -1 \end{cases}$$

Med $a = -2$ får vi gravitationsfältet $P = -cr^{-1}$,

$$\boldsymbol{F} = -\nabla P = c\nabla r^{-1} = -cr^{-2}\nabla r = -cr^{-2}\hat{\boldsymbol{r}}$$

jämför exempel 4.35.

P4.11 (a) $\phi = \frac{1}{2}\ln(x^2+y^2)$ om $(x,y)\neq(0,0)$ (b) $\phi = -\arctan(x/y)$ om $y \neq 0$

P4.12 Låt $S = \partial\Omega$ och låt $\boldsymbol{F}$ vara ett godtyckligt konstant vektorfält och använd divergenssatsen (4.324):

$$0 = \iiint_\Omega \nabla\cdot\boldsymbol{F}\,\mathrm{d}V = \iint_{\partial\Omega}\boldsymbol{F}\cdot\hat{\boldsymbol{N}}\,\mathrm{d}S = \boldsymbol{F}\cdot\iint_{\partial\Omega}\hat{\boldsymbol{N}}\,\mathrm{d}S$$

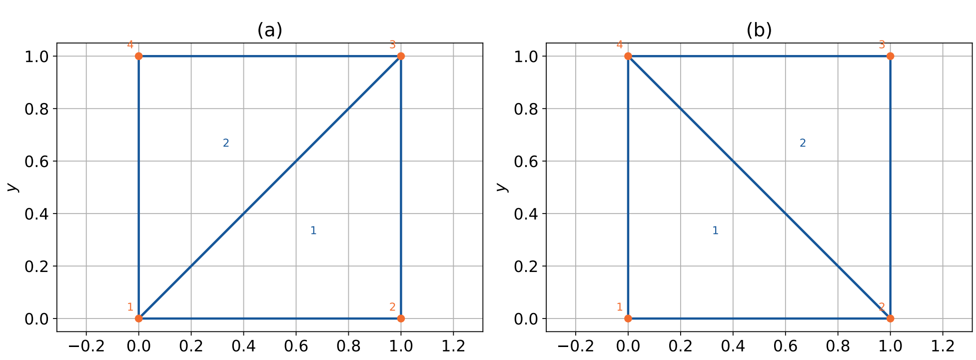
Eftersom $\boldsymbol{F}$ är godtycklig leder detta till $\iint_{\partial\Omega}\hat{\boldsymbol{N}}\,\mathrm{d}S = \boldsymbol{0}$. Alternativt: med $\boldsymbol{F} = \iint_{\partial\Omega}\hat{\boldsymbol{N}}\,\mathrm{d}S$ får vi $\|\iint_{\partial\Omega}\hat{\boldsymbol{N}}\,\mathrm{d}S\|^2 = 0$.

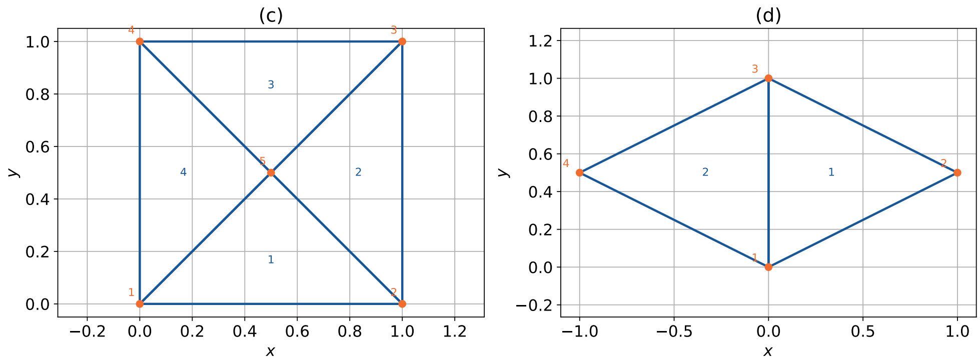
P4.13 Använd divergenssatsen och $\nabla\cdot\boldsymbol{r} = 3$.

Kapitel 5

Övningar

Ö5.1 Se figur.

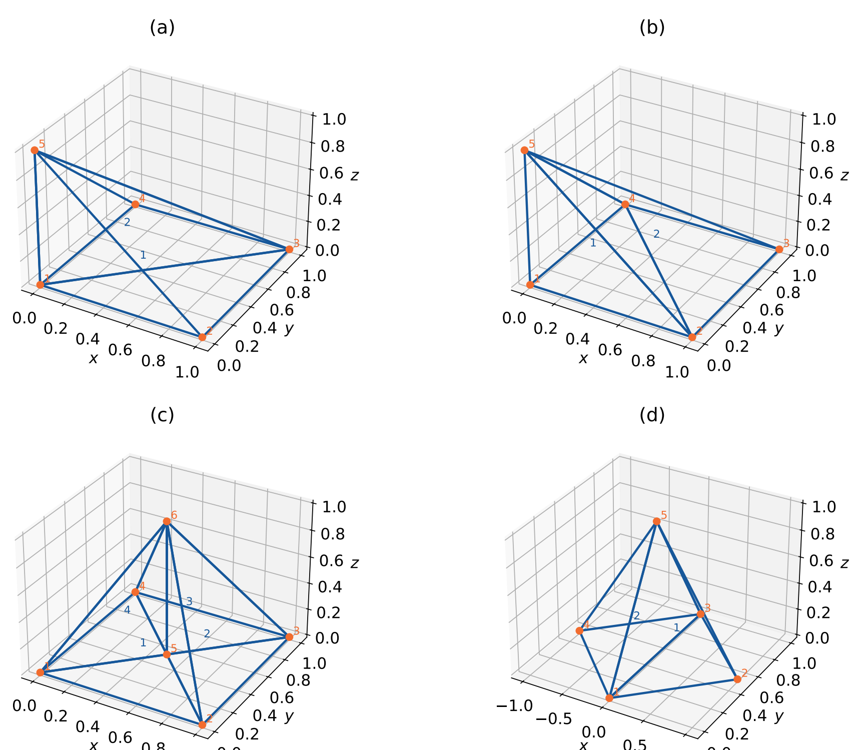




Ö5.2 (a) $\mathcal{V} = \{(1,0),(1,4),(2,1),(3,1)\}, \mathcal{K} = \{(1,4,3),(2,3,4),(1,3,2)\}$
(b) $\mathcal{V} = \{(-1,-1),(0,0),(1,1),(0,1)\}, \mathcal{K} = \{(2,3,4),(1,2,4)\}$
(c) $\mathcal{V} = \{(10,10),(11,11),(11,12),(12,10)\}, \mathcal{K} = \{(1,4,2),(2,4,3),(1,2,3)\}$
(d) $\mathcal{V} = \{(-1,0),(0,0),(1,0),(-1/2,1),(1/2,1),(0,2)\}$
$\mathcal{K} = \{(1,2,4),(2,3,5),(2,5,4),(4,5,6)\}$

Ö5.3 (a) $\mathcal{E} = \{(1,2),(2,3),(1,4),(3,4),(1,3),(2,4)\}$
(b) $\mathcal{E} = \{(1,2),(2,3),(1,4),(3,4),(2,4)\}$
(c) $\mathcal{E} = \{(1,2),(2,3),(1,4),(3,4),(1,3),(2,4)\}$
(d) $\mathcal{E} = \{(1,2),(3,5),(2,3),(4,5),(1,4),(2,5),(5,6),(2,4),(4,6)\}$

Ö5.4 Se figur.



Ö5.5 (a) $\mathcal{V} = \{(1,0,0),(1,4,0),(2,1,0),(3,1,0),(2,1,1)\}$
$\mathcal{K} = \{(1,4,3,5),(2,3,4,5),(1,3,2,5)\}$
(b) $\mathcal{V} = \{(-1,-1,0),(0,0,0),(1,1,0),(0,1,0),(0,0,1)\}$
$\mathcal{K} = \{(2,3,4,5),(1,2,4,5)\}$
(c) $\mathcal{V} = \{(10,10,0),(11,11,0),(11,12,0),(12,10,0),(11,11,1)\}$
$\mathcal{K} = \{(1,4,2,5),(2,4,3,5),(1,2,3,5)\}$
(d) $\mathcal{V} = \{(-1,0,0),(0,0,0),(1,0,0),(-\frac{1}{2},1,0),(\frac{1}{2},1,0),(0,2,0),(0,1,1)\}$
$\mathcal{K} = \{(1,2,4,7),(2,3,5,7),(2,5,4,7),(4,5,6,7)\}$

Ö5.6 (a) $\mathcal{F} = \{(1,2,5),(1,3,5),(1,4,5),(2,3,4),(1,3,4),(2,4,5),(3,4,5),\ldots$
$\ldots,(1,2,3),(2,3,5)\}$
(b) $\mathcal{F} = \{(1,2,5),(1,4,5),(1,2,4),(2,3,4),(2,4,5),(3,4,5),\ldots$
$\ldots,(2,3,5)\}$
(c) $\mathcal{F} = \{(1,2,5),(1,3,5),(1,4,5),(1,2,4),(2,3,4),(2,4,5),(3,4,5),\ldots$
$\ldots,(1,2,3),(2,3,5)\}$
(d) $\mathcal{F} = \{(4,5,7),(2,3,7),(4,6,7),(1,2,4),(2,5,7),(1,2,7),(2,4,5),\ldots$
$\ldots,(1,4,7),(4,5,6),(5,6,7),(2,4,7),(3,5,7),(2,3,5)\}$

Ö5.7 Dimensionen ges av antalet hörn, dvs $\dim V_h = |\mathcal{V}|$.
(a) 9 (b) 81 (c) 48 (d) 32

Ö5.8 Dimensionen ges av antalet hörn och kanter, dvs $\dim V_h = |\mathcal{V}| + |\mathcal{E}|$.
(a) 25 (b) 289 (c) 165 (d) 105

Ö5.9 Evaluera funktionen i varje hörn i beräkningsnätet.
(a) $F = (0, 0.5, 1, 0.5, 1, 1.5, 1, 1.5, 2)$
(b) $F = (0, 0, 0, 0, 0.25, 0.5, 0, 0.5, 1)$
(c) $F = (0, 0.5, 1, -0.5, 0, 0.5, -1, -0.5, 0)$
(d) $F = (0, 0.25, 1, 0.25, 0.5, 1.25, 1, 1.25, 2)$

Ö5.10 Evaluera funktionen i varje hörn i beräkningsnätet.
(a) $F = (0, 0.5, 1, 0.5, 1, 1.5, 1, 1.5, 2, 0.5, 1, 1.5, 1, 1.5, 2, \ldots$
$\ldots, 1.5, 2, 2.5, 1, 1.5, 2, 1.5, 2, 2.5, 2, 2.5, 3)$
(b) $F = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.125, 0.25, \ldots$
$\ldots, 0, 0.25, 0.5, 0, 0, 0, 0, 0.25, 0.5, 0, 0.5, 1)$
(c) $F = (0, 0.5, 1, -0.5, 0, 0.5, -1, -0.5, 0, 0.5, 1, 1.5, 0, 0.5, 1, \ldots$
$\ldots, -0.5, 0, 0.5, 1, 1.5, 2, 0.5, 1, 1.5, 0, 0.5, 1)$
(d) $F = (0, 0.25, 1, 0.25, 0.5, 1.25, 1, 1.25, 2, 0.25, 0.5, 1.25, 0.5, 0.75, 1.5, \ldots$
$\ldots, 1.25, 1.5, 2.25, 1, 1.25, 2, 1.25, 1.5, 2.25, 2, 2.25, 3)$

Ö5.11 Den affina avbildningen ges av $F_K(\hat{x}) = \hat{\lambda}_1(\hat{x})v_1 + \hat{\lambda}_2(\hat{x})v_2 + \hat{\lambda}_3(\hat{x})v_3$.

(a)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2)\begin{bmatrix}0\\0\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}0\\1\end{bmatrix} \tag{1.1}$$

$$= \begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} \quad \text{(identitetsavbildningen)} \tag{1.2}$$

(b)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2)\begin{bmatrix}1\\1\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}0\\1\end{bmatrix} \tag{1.3}$$

$$= \begin{bmatrix}1 - \hat{x}_1 - \hat{x}_2 + \hat{x}_1\\1 - \hat{x}_1 - \hat{x}_2 + \hat{x}_2\end{bmatrix} = \begin{bmatrix}1 - \hat{x}_2\\1 - \hat{x}_1\end{bmatrix} \tag{1.4}$$

(c)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2)\begin{bmatrix}0\\2\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}0\\1\end{bmatrix} \tag{1.5}$$

$$= \begin{bmatrix}\hat{x}_1\\2(1 - \hat{x}_1 - \hat{x}_2) + \hat{x}_2\end{bmatrix} = \begin{bmatrix}\hat{x}_1\\2 - 2\hat{x}_1 - \hat{x}_2\end{bmatrix} \tag{1.6}$$

(d)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2)\begin{bmatrix}1\\2\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}2\\1\end{bmatrix} \tag{1.7}$$

$$= \begin{bmatrix}1 - \hat{x}_1 - \hat{x}_2 + \hat{x}_1 + 2\hat{x}_2\\2(1 - \hat{x}_1 - \hat{x}_2) + \hat{x}_2\end{bmatrix} = \begin{bmatrix}1 + \hat{x}_2\\2 - 2\hat{x}_1 - \hat{x}_2\end{bmatrix} \tag{1.8}$$

Ö5.12 Den affina avbildningen ges av $F_K(\hat{x}) = \lambda_1(\hat{x})v_1 + \lambda_2(\hat{x})v_2 + \lambda_3(\hat{x})v_3 + \lambda_4(\hat{x})v$.

(a)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)\begin{bmatrix}0\\0\\0\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \hat{x}_3\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{1.9}$$

$$= \begin{bmatrix}\hat{x}_1\\\hat{x}_2\\\hat{x}_3\end{bmatrix} \quad \text{(identitetsavbildningen)} \tag{1.10}$$

(b)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3)\begin{bmatrix}1\\1\\0\end{bmatrix} + \hat{x}_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \hat{x}_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \hat{x}_3\begin{bmatrix}1\\0\\1\end{bmatrix} \tag{1.11}$$

$$= \begin{bmatrix}1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_1 + \hat{x}_3\\1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_2\\\hat{x}_3\end{bmatrix} = \begin{bmatrix}1 - \hat{x}_2\\1 - \hat{x}_1 - \hat{x}_3\\\hat{x}_3\end{bmatrix} \tag{1.12}$$

(c)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \hat{x}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \hat{x}_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \hat{x}_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.13}$$

$$= \begin{bmatrix} \hat{x}_1 + \hat{x}_3 \\ 2(1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) + \hat{x}_2 + \hat{x}_3 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 + \hat{x}_3 \\ 2 - 2\hat{x}_1 - \hat{x}_2 - \hat{x}_3 \\ \hat{x}_3 \end{bmatrix} \tag{1.14}$$

(d)

$$F_K(\hat{x}) = (1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \hat{x}_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \hat{x}_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \hat{x}_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1.15}$$

$$= \begin{bmatrix} 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 + \hat{x}_1 + 2\hat{x}_2 + \hat{x}_3 \\ 2(1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3) + \hat{x}_2 + \hat{x}_3 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 + \hat{x}_2 \\ 2 - 2\hat{x}_1 - \hat{x}_2 - \hat{x}_3 \\ \hat{x}_3 \end{bmatrix} \tag{1.16}$$

Ö5.13 (a) $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ (b) $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 0$
(c) $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1$ (d) $\lambda_1 = \frac{6}{10}, \lambda_2 = \frac{1}{10}, \lambda_3 = \frac{3}{10}$

Ö5.14 (a) $\frac{1}{2}x_1$ (b) 1 (c) $\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$

Ö5.15 (a) Ja (b) Nej (c) Ja (d) Ja

Ö5.16 (a) $\frac{1}{6}$ (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{2}$

Ö5.17 Interpolanten ges av $\pi_h f(x) = f(v_1)\hat{\lambda}_1(x) + f(v_2)\hat{\lambda}_2(x) + f(v_3)\hat{\lambda}_3(x)$.

(a) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + 1 \cdot x_2 = x_1 + x_2$

(b) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 0 \cdot x_1 + 0 \cdot x_2 = 0$

(c) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + (-1) \cdot x_2 = x_1 - x_2$

(d) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + 1 \cdot x_2 = x_1 + x_2$

Ö5.18 Interpolanten ges av $\pi_h f(x) = f(v_1)\hat{\lambda}_1(x) + f(v_2)\hat{\lambda}_2(x) + f(v_3)\hat{\lambda}_3(x) + f(v_4)\hat{\lambda}_4(x)$.

(a) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = x_1 + x_2 + x_3$

(b) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$

(c) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + (-1) \cdot x_2 + 1 \cdot x_3 = x_1 - x_2 + x_3$

(d) $\pi_h f(x) = 0 \cdot (1 - x_1 - x_2) + 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = x_1 + x_2 + x_3$

Ö5.19 (a) Eftersom $f \in V_h$ följer direkt att $P_h f = f = x$.

(b) Matrisen A ges av

$$A = \begin{bmatrix} \int_0^1 1 \cdot 1 \,\mathrm{d}x & \int_0^1 x \cdot 1 \,\mathrm{d}x \\ \int_0^1 1 \cdot x \,\mathrm{d}x & \int_0^1 x \cdot x \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_0^1 1 \,\mathrm{d}x & \int_0^1 x \,\mathrm{d}x \\ \int_0^1 x \,\mathrm{d}x & \int_0^1 x^2 \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \tag{1.17}$$

Matrisens invers ges av

$$A^{-1} = \frac{1}{1 \cdot \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2}} \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = 12 \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -6 & 12 \end{bmatrix} \tag{1.18}$$

Vektorn b ges av

$$b = \begin{bmatrix} \int_0^1 x^2 \cdot 1 \,\mathrm{d}x \\ \int_0^1 x^2 \cdot x \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_0^1 x^2 \,\mathrm{d}x \\ \int_0^1 x^3 \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \tag{1.19}$$

Koefficientvektorn U ges därför av $U = A^{-1}b = [-\frac{1}{6}, 1]^\top$, vilket ger projektionen

$$P_h f(x) = -\tfrac{1}{6} \cdot 1 + 1 \cdot x = x - \tfrac{1}{6} \tag{1.20}$$

(c) Vektorn b ges av

$$b = \begin{bmatrix} \int_0^1 x^3 \cdot 1 \,\mathrm{d}x \\ \int_0^1 x^3 \cdot x \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_0^1 x^3 \,\mathrm{d}x \\ \int_0^1 x^4 \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} \tag{1.21}$$

Matrisen och dess invers är densamma som i uppgift (b). Vi får koefficientvektorn $U = A^{-1}b = \begin{bmatrix} -\frac{1}{5} & \frac{9}{10} \end{bmatrix}^\top$, vilket ger projektionen

$$P_h f(x) = -\tfrac{1}{5} \cdot 1 + \tfrac{9}{10} \cdot x = \tfrac{9x}{10} - \tfrac{1}{5} \tag{1.22}$$

(d) Vektorn b ges av

$$b = \begin{bmatrix} \int_0^1 x^4 \cdot 1 \,\mathrm{d}x \\ \int_0^1 x^4 \cdot x \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \int_0^1 x^4 \,\mathrm{d}x \\ \int_0^1 x^5 \,\mathrm{d}x \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{6} \end{bmatrix} \tag{1.23}$$

Matrisen och dess invers är densamma som i uppgift (b). Vi får koefficientvektorn $U = A^{-1}b = \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \end{bmatrix}^\top$, vilket ger projektionen

$$P_h f(x) = -\tfrac{1}{5} \cdot 1 + \tfrac{4}{5} \cdot x = \tfrac{4x}{5} - \tfrac{1}{5} \tag{1.24}$$

Ö5.20 (a) Eftersom $f \in V_h$ följer direkt att $P_h f = f = x$.
(b) $U = \begin{bmatrix} \frac{1}{3} & 0 \end{bmatrix}^\top$, vilket ger $P_h f(x) = \frac{1}{3} \cdot 1 + 0x = \frac{1}{3}$.
(c) $U = \begin{bmatrix} 0 & \frac{3}{5} \end{bmatrix}^\top$, vilket ger $P_h f(x) = 0 \cdot 1 + \frac{3}{5}x = \frac{3x}{5}$.
(c) $U = \begin{bmatrix} \frac{1}{5} & 0 \end{bmatrix}^\top$, vilket ger $P_h f(x) = \frac{1}{5} \cdot 1 + 0x = \frac{1}{5}$.

Ö5.21 (a) Eftersom $f \in V_h$ följer direkt att $P_h f = f = x$.
(b) $U = \begin{bmatrix} -\frac{1}{6} & \frac{5}{6} \end{bmatrix}^\top$, vilket ger $P_h f(x) = -\frac{1}{6}(1-x) + \frac{5}{6}x = x - \frac{1}{6}$.
(c) $U = \begin{bmatrix} -\frac{1}{5} & \frac{7}{10} \end{bmatrix}^\top$, vilket ger $P_h f(x) = -\frac{1}{5}(1-x) + \frac{7}{10}x = \frac{9x}{10} - \frac{1}{5}$.
(d) $U = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \end{bmatrix}^\top$, vilket ger $P_h f(x) = -\frac{1}{5}(1-x) + \frac{3}{5}x = \frac{4x}{5} - \frac{1}{5}$.
Notera att svaren är desamma som i övning 5.21 eftersom basfunktionerna spänner samma rum.

Ö5.22 (a) $30\frac{2}{3}$ (b) $69\frac{1}{3}$ (c) $1\frac{1}{3}$ (d) $177\frac{1}{3}$

Ö5.23 (a) $30\frac{2}{3}$ (b) $61\frac{1}{3}$ (c) $1\frac{1}{3}$ (d) $132\frac{2}{3}$

Ö5.24 (a) 78 (b) $355\frac{1}{2}$ (c) 27 (d) 444

Ö5.25 (a) 78 (b) $372\frac{15}{16}$ (c) 27 (d) $392\frac{1}{4}$

Problem

P5.5 Basfunktionerna erhålls genom att multiplicera de linjära basfunktionerna på ett sådant sätt att produkterna antar värdet 1 i något av de tre hörnen eller mittpunkterna av de tre kanterna och värdet 0 i övriga hörn och mittpunkter. Basfunktionerna ges av

$$\varphi_1 = \lambda_1(2\lambda_1 - 1) \qquad \varphi_4 = 4\lambda_2\lambda_3 \tag{1.25}$$
$$\varphi_2 = \lambda_2(2\lambda_2 - 1) \qquad \varphi_5 = 4\lambda_1\lambda_3 \tag{1.26}$$
$$\varphi_3 = \lambda_3(2\lambda_3 - 1) \qquad \varphi_6 = 4\lambda_1\lambda_2 \tag{1.27}$$

(Numreringen av kanterna kan variera och följer här en lexikografisk ordning av de icke ingående hörnen.)

P5.6 Basfunktionerna erhålls genom att multiplicera de linjära basfunktionerna på ett sådant sätt att produkterna antar värdet 1 i något av de fyra hörnen eller mittpunkterna av de sex kanterna och värdet 0 i övriga hörn och mittpunkter. Basfunktionerna ges av

$$\varphi_1 = \lambda_1(2\lambda_1 - 1) \qquad \varphi_5 = 4\lambda_3\lambda_4 \tag{1.28}$$
$$\varphi_2 = \lambda_2(2\lambda_2 - 1) \qquad \varphi_6 = 4\lambda_2\lambda_4 \tag{1.29}$$
$$\varphi_3 = \lambda_3(2\lambda_3 - 1) \qquad \varphi_7 = 4\lambda_1\lambda_2 \tag{1.30}$$
$$\varphi_4 = \lambda_4(2\lambda_4 - 1) \qquad \varphi_8 = 4\lambda_1\lambda_2 \tag{1.31}$$
$$\varphi_9 = 4\lambda_1\lambda_3 \tag{1.32}$$
$$\varphi_{10} = 4\lambda_1\lambda_2 \tag{1.33}$$

(Numreringen av kanterna kan variera och följer här en lexikografisk ordning av de icke ingående hörnen.)

P5.7 Dimensionen ges av $(q+1)(q+2)/2$.

P5.8 Dimensionen ges av $(q+1)(q+2)(q+3)/6$.

Datorövningar

D5.1

Python (3.x)

```python
from numpy import array

def generate_mesh_2d(nx, ny):

    hx = 1 / nx
    hy = 1 / ny

    V = []
    K = []

    for iy in range(ny + 1):
        for ix in range(nx + 1):
            x = ix*hx
            y = iy*hy
            V.append((x, y))

    for iy in range(ny):
        for ix in range(nx):
            v0 = iy*(nx + 1) + ix
            v1 = v0 + 1
            v2 = v0 + (nx + 1)
            v3 = v1 + (nx + 1)
            K.append((v0, v1, v3))
            K.append((v0, v3, v2))

    return array(V), array(K)
```

MATLAB

```matlab
function [V, K] = generate_mesh_2d(nx, ny)

    hx = 1 / nx;
    hy = 1 / ny;

    V = [];
    K = [];

    for iy = 0:ny
        for ix = 0:nx
            x = ix*hx;
            y = iy*hy;
            V = [V; [x, y]];
        end
    end

    for iy = 1:ny
        for ix = 1:nx
            v1 = (iy - 1)*(nx + 1) + ix;
            v2 = v1 + 1;
```

```
            v3 = v1 + (nx + 1);
            v4 = v2 + (nx + 1);
            K = [K; [v1, v2, v4]];
            K = [K; [v1, v4, v3]];
        end
    end

end
```

D5.2

Python (3.x)

```
from numpy import array

def generate_mesh_3d(nx, ny, nz):

    hx = 1 / nx
    hy = 1 / ny
    hz = 1 / nz

    V = []
    K = []

    for iz in range(nz + 1):
        for iy in range(ny + 1):
            for ix in range(nx + 1):
                x = ix*hx
                y = iy*hy
                z = iz*hz
                V.append((x, y, z))

    for iz in range(nz):
        for iy in range(ny):
            for ix in range(nx):
                v0 = iz*(nx + 1)*(ny + 1) + iy*(nx + 1) + ix
                v1 = v0 + 1
                v2 = v0 + (nx + 1)
                v3 = v1 + (nx + 1)
                v4 = v0 + (nx + 1)*(ny + 1)
                v5 = v1 + (nx + 1)*(ny + 1)
                v6 = v2 + (nx + 1)*(ny + 1)
                v7 = v3 + (nx + 1)*(ny + 1)
                K.append((v0, v1, v3, v7))
                K.append((v0, v1, v7, v5))
                K.append((v0, v5, v7, v4))
                K.append((v0, v3, v2, v7))
                K.append((v0, v6, v4, v7))
                K.append((v0, v2, v6, v7))

    return array(V), array(K)
```

MATLAB

```
function [V, K] = generate_mesh_3d(nx, ny, nz)
```

```
    hx = 1 / nx;
    hy = 1 / ny;
    hz = 1 / nz;

    V = [];
    K = [];

    for iz = 0:nz
        for iy = 0:ny
            for ix = 0:nx
                x = ix*hx;
                y = iy*hy;
                z = iz*hz;
                V = [V; [x, y, z]];
            end
        end
    end

    for iz = 1:iz
        for iy = 1:ny
            for ix = 1:nx
                v1 = (iz - 1)*(nx + 1)*(ny + 1) + (iy - 1)*(nx + 1) + ix;
                v2 = v1 + 1;
                v3 = v1 + (nx + 1);
                v4 = v2 + (nx + 1);
                v5 = v1 + (nx + 1)*(ny + 1);
                v6 = v2 + (nx + 1)*(ny + 1);
                v7 = v3 + (nx + 1)*(ny + 1);
                v8 = v4 + (nx + 1)*(ny + 1);
                K = [K; [v1, v2, v4, v8]];
                K = [K; [v1, v2, v8, v6]];
                K = [K; [v1, v6, v8, v5]];
                K = [K; [v1, v4, v3, v8]];
                K = [K; [v1, v7, v5, v8]];
                K = [K; [v1, v3, v7, v8]];
            end
        end
    end

end
```

D5.3

Python (3.x)

```
def generate_edges_2d(K):

    E = set()

    for t in K:
        edges = [(t[0], t[1]), (t[0], t[2]), (t[1], t[2])]
        for e in edges:
            E.add(tuple(sorted(e)))
```

```
    return list(E)
```

MATLAB

```
function E = generate_edges_2d(K)

    E = [];

    for i = 1:size(K, 1);
        t = K(i, :);
        edges = [t(1), t(2); t(1), t(3); t(2), t(3)];
        for j = 1:3
            e = sort(edges(j, :));
            if isempty(E) || ~ismember(e, E, 'rows')
                E = [E; e];
            end
        end
    end

end
```

D5.4

Python (3.x)

```
def generate_faces_3d(K):

    F = set()

    for t in K:
        faces = [(t[0], t[1], t[2]),
                 (t[0], t[1], t[3]),
                 (t[0], t[2], t[3]),
                 (t[1], t[2], t[3])]
        for f in faces:
            F.add(tuple(sorted(f)))

    return list(F)
```

MATLAB

```
function F = generate_faces_3d(K)

    F = []

    for i = 1:size(K, 1)
        t = K(i, :);
        faces = [t(1), t(2), t(3);
                 t(1), t(2), t(4);
                 t(1), t(3), t(4);
                 t(2), t(3), t(4)];
        for j = 1:4
            f = sort(faces(j, :));
            if isempty(F) || ~ismember(f, F, 'rows')
                F = [F; f];
```

```
            end
        end
    end

end
```

D5.5

Python (3.x)

```python
from pylab import *

def plot_mesh_2d(V, K):

    for t in enumerate(K):
        for j in range(3):
            for k in range(j, 3):
                x = [V[t[j]][0], V[t[k]][0]]
                y = [V[t[j]][1], V[t[k]][1]]
                plot(x, y)

    for v in enumerate(V):
        plot(v[0], v[1], 'o')
```

MATLAB

```matlab
function plot_mesh_2d(V, K)

    hold on

    for i = 1:size(K, 1)
        t = K(i, :);
        for j = 1:3
            for k = j:3
                x = [V(t(j), 1), V(t(k), 1)];
                y = [V(t(j), 2), V(t(k), 2)];
                plot(x, y)
            end
        end
    end

    for i = 1:size(V, 1)
        v = V(i, :);
        plot(v(1), v(2), 'o')
    end

end
```

D5.6

Python (3.x)

```python
from pylab import *

def plot_mesh_3d(V, K):

```

```
    ax = axes(projection='3d')

    for t in K:
        for j in range(4):
            for k in range(j, 4):
                x = [V[t[j]][0], V[t[k]][0]]
                y = [V[t[j]][1], V[t[k]][1]]
                z = [V[t[j]][2], V[t[k]][2]]
                ax.plot3D(x, y, z)

    for v in V:
        ax.plot3D(v[0], v[1], v[2], 'o')
```

MATLAB

```
function plot_mesh_3d(V, K)

    hold on

    for i = 1:size(K, 1)
        t = K(i, :);
        for j = 1:4
            for k = j:4
                x = [V(t(j), 1), V(t(k), 1)];
                y = [V(t(j), 2), V(t(k), 2)];
                z = [V(t(j), 3), V(t(k), 3)];
                plot3(x, y, z)
            end
        end
    end

    for i = 1:size(V, 1)
        v = V(i, :);
        plot3(v(1), v(2), v(3), 'o')
    end

    view(3)

end
```

D5.7

Python (3.x)

```
from numpy import zeros, dot
from numpy.linalg import inv

def evaluate_basis_2d(i, x, X):

    # Compute Jacobian (linear part) of affine map
    J = zeros((2, 2))
    J[:, 0] = X[:, 1] - X[:, 0]
    J[:, 1] = X[:, 2] - X[:, 0]

    # Compute inverse of Jacobian
```

```
    Jinv = inv(J)

    # Map point to reference element
    xhat = dot(Jinv, x - X[:, 0])

    # Evaluate basis function
    if i == 0:
        phi = 1 - xhat[0] - xhat[1]
        grad_phi = dot(Jinv.T, (-1, -1))
    elif i == 1:
        phi = xhat[0]
        grad_phi = dot(Jinv.T, (1, 0))
    else:
        phi = xhat[1]
        grad_phi = dot(Jinv.T, (0, 1))

    return phi, grad_phi
```

Python (3.x)

```
from numpy import array

from evaluate_basis_2d import *

X = array([[0, 0], [1, 0], [0, 1]]).T
x = array([0.5, 0.5]).T

print(evaluate_basis_2d(0, x, X))
print(evaluate_basis_2d(1, x, X))
print(evaluate_basis_2d(2, x, X))
```

MATLAB

```
function [phi, grad_phi] = evaluate_basis_2d(i, x, X)

    % Compute Jacobian (linear part) of affine map
    J = zeros(2, 2);
    J(:, 1) = X(:, 2) - X(:, 1);
    J(:, 2) = X(:, 3) - X(:, 1);

    % Compute inverse of Jacobian
    Jinv = inv(J);

    % Map point to reference element
    xhat = Jinv*(x - X(:, 1));

    % Evaluate basis function
    if i == 1
        phi = 1 - xhat(1) - xhat(2);
        grad_phi = Jinv'*[-1; -1];
    elseif i == 2
         phi = xhat(1);
         grad_phi = Jinv'*[1; 0];
    else
```

```
        phi = xhat(2);
        grad_phi = Jinv'*[0; 1];
    end

end
```

MATLAB

```
X = [0, 0; 1, 0; 0, 1]';
x = [0.5, 0.5]';

[phi_1, grad_phi_1] = evaluate_basis_2d(1, x, X)
[phi_2, grad_phi_2] = evaluate_basis_2d(2, x, X)
[phi_2, grad_phi_3] = evaluate_basis_2d(3, x, X)
```

D5.9

Python (3.x)

```
from numpy import zeros, cross, array

def generate_quadrature_2d(X):

        # Compute midpoints of edges
        points = zeros((2, 3))
        points[:, 0] = 0.5*(X[:, 0] + X[:, 1])
        points[:, 1] = 0.5*(X[:, 0] + X[:, 2])
        points[:, 2] = 0.5*(X[:, 1] + X[:, 2])

        # Compute area of triangle
        a = X[:, 1] - X[:, 0]
        b = X[:, 2] - X[:, 0]
        area = 0.5*abs(cross(a, b))

        # Compute quadrature weights
        weights = array([area/3, area/3, area/3])

        return points, weights
```

Python (3.x)

```
from numpy import array

from generate_quadrature_2d import *

X = array([[0, 0], [1, 0], [0, 1]]).T

print(generate_quadrature_2d(X))
```

MATLAB

```
function [points, weights] = generate_quadrature_2d(X)

    % Compute midpoints of edges
    points = zeros(2, 3);
```

```
    points(:, 1) = 0.5*(X(:, 1) + X(:, 2));
    points(:, 2) = 0.5*(X(:, 1) + X(:, 3));
    points(:, 3) = 0.5*(X(:, 2) + X(:, 3));

    % Compute area of triangle
    a = [X(:, 2) - X(:, 1); 0];
    b = [X(:, 3) - X(:, 1); 0];
    c = cross(a, b);
    area = 0.5*abs(c(3));

    % Compute quadrature weights
    weights = [area/3, area/3, area/3];

end
```

MATLAB

```
X = [0, 0; 1, 0; 0, 1]';

[points, weights] = generate_quadrature_2d(X)
```

Kapitel 6

Övningar

Ö6.1 (a) 0 (b) -4 (c) $2\pi^2 \sin(\pi x)\sin(\pi y)$ (d) $2x(1-x)+2y(1-y))$

Ö6.2 (a) 0 (b) x^2+y^2-4 (c) $x^4-y^4-3x^2+3y^2-4xy$ (d) $-10-10xy+3y^2-3x^2$

Ö6.3 —

Ö6.4 —

Ö6.5 (a) x^2 (b) $-\sin(x)$ (c) $-x$ (d) x

Ö6.6 (a) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$, $L(v) = \int_\Omega v \, \mathrm{d}x$
(b) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$, $L(v) = \int_\Omega v \, \mathrm{d}x + \int_{\Gamma_R} v \, \mathrm{d}s$
(c) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\Gamma_R} uv \, \mathrm{d}s$, $L(v) = \int_\Omega v \, \mathrm{d}x + \int_{\Gamma_R} xv \, \mathrm{d}s$
(d) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_{\Gamma_R} 10uv \, \mathrm{d}s$, $L(v) = \int_\Omega v \, \mathrm{d}x + \int_{\Gamma_R} x^2 v \, \mathrm{d}s$

Ö6.7 Hitta $u_h \in V_{h,0}$ sådan att $a(u_h, v) = L(v)$ för alla testfunktioner $v \in V_{h,0}$ med a och L enligt facit till Övning 6.6 ovan.

Ö6.8 (a) $-\frac{1}{2}$ (b) $\frac{1}{6}$ (c) 1 (d) $\frac{1}{8}$

Ö6.9 a) 0 (b) $\frac{2}{3}$ (c) 1 (d) $\frac{1}{24}$

Ö6.10 (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) 1

Ö6.11 (a) 0 (b) $\frac{1}{27}$ (c) 0 (d) $\frac{1}{6}$

Ö6.12 (a) $\frac{1}{6}$ (b) $-\frac{1}{6}$ (c) $\frac{1}{2}$ (d) 0

Ö6.13 (a) $\frac{1}{60}$ (b) $\frac{1}{120}$ (c) $\frac{1}{60}$ (d) $\frac{1}{120}$

Ö6.14 Hitta $u_h \in V_{h,0}$ sådan att $a(u,v) = \int_\Omega fv \, \mathrm{d}x$ för alla $v \in V_{h,0}$ där
(a) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$ (b) $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + cuv \, \mathrm{d}x$
(c) $a(u,v) = \int_\Omega \kappa \nabla u \cdot \nabla v + \beta \cdot \nabla uv \, \mathrm{d}x$ (d) $a(u,v) = \int_\Omega \kappa \nabla u \cdot \nabla v + \nabla \cdot (\beta u) v \, \mathrm{d}x$

Ö6.15 (a) $-\nabla \cdot \kappa \nabla u = f$ i Ω, $u = 0$ på Γ (b) $-\nabla \cdot \nabla u + u = f$ i Ω, $u = 0$ på Γ
(c) $-\nabla \cdot \kappa \nabla u = f$ i Ω, $\kappa \partial_n u + \gamma u = 0$ på Γ (d) $-\nabla \cdot \nabla u = f$ i Ω, $\partial_n u + u = 1$ på Γ

Ö6.16 (a) $-\frac{1}{6}$ (b) $\frac{1}{12}$ (c) $\frac{1}{12}$ (d) $\frac{1}{12}$

Ö6.17 (a) $(I - 0.01M^{-1}A)^k U_0$ (b) $(I + 0.1M^{-1}A)^{-k} U_0$
(c) $((I+0.1M^{-1}A)^{-1}(I-0.1M^{-1}A))^k U_0$ (d) $(I-0.1M^{-1}A+0.005(M^{-1}A)^2)^k U_0$

Ö6.18 (a) $(I - 0.1M^{-1}A)U_0 + 0.1M^{-1}b(t_0)$ (b) $(I + 0.1M^{-1}A)^{-1}(U_0 + 0.1M^{-1}b(t_1))$
(c) $(I + 0.05M^{-1}A)^{-1}((I - 0.05M^{-1}A)U_0 + 0.1M^{-1}b(t_{1/2}))$
(d) $U_0 + 0.05(f(0,U_0) + f(0.1, U_0 + 0.1f(0,U_0)))$, där $f(t,U) = -M^{-1}AU + M^{-1}b(t)$

Ö6.19 (a) $\frac{1}{60}$ (b) $\frac{1}{24}$ (c) 0 (d) 24

Ö6.20 (a) $-\frac{1}{2}$ (b) $\frac{2}{3}$ (c) 0 (d) $\frac{5}{6}$

Ö6.21 (a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} \hat{x}$ (b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \hat{x}$
(c) $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \hat{x}$ (d) $\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \hat{x}$

Ö6.22 (a) $\frac{1}{3}$ (b) $\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$

Ö6.23 (a) $-\frac{1}{8}$ (b) $-\frac{1}{6}$ (c) 0 (d) $-\frac{3}{8}$

Ö6.24 (a) $\int_\Omega \nabla u \cdot \nabla v + \lambda(u^3 - u)v \, \mathrm{d}x$ (b) $\int_\Omega \kappa \nabla u \cdot \nabla v + u^2 v \, \mathrm{d}x$
(c) $\int_\Omega \kappa(u) \nabla u \cdot \nabla v - \lambda u^p v \, \mathrm{d}x$ (d) $\int_\Omega \nabla u \cdot \nabla v - \mathrm{e}^{\lambda u} v \, \mathrm{d}x$

Ö6.25 (a) $\int_\Omega \nabla \delta u \cdot \nabla v \, \mathrm{d}x + \lambda \int_\Omega (3u^2 - 1)\delta uv \, \mathrm{d}x$
(b) $\int_\Omega \kappa \nabla \delta u \cdot \nabla v \, \mathrm{d}x + \lambda \int_\Omega 2u\delta uv \, \mathrm{d}x$
(c) $\int_\Omega \kappa(u) \nabla \delta u \cdot \nabla v \, \mathrm{d}x + \int_\Omega \kappa'(u) \delta u \nabla u \cdot \nabla v \, \mathrm{d}x - \lambda p \int_\Omega u^{p-1} \delta uv \, \mathrm{d}x$
(d) $\int_\Omega \nabla \delta u \cdot \nabla v \, \mathrm{d}x - \lambda \int_\Omega \mathrm{e}^{\lambda u} \delta uv \, \mathrm{d}x$

Problem

Datorövningar

D6.1

Python (3.x)

```
from numpy import dot

def poisson_lhs(u, v, grad_u, grad_v, x, dx):
    return dot(grad_u, grad_v)*dx
```

Python (3.x)

```
from poisson_lhs import *

u = v = dx = 1
x = grad_u = grad_v = [1, 1]

print(poisson_lhs(u, v, grad_u, grad_v, x, dx))
```

MATLAB

```
function a = poisson_lhs(u, v, grad_u, grad_v, x, dx)
    a = dot(grad_u, grad_v)*dx;
end
```

MATLAB

```
u = 1;
v = 1;
dx = 1;

x = [1; 1];
grad_u = [1; 1];
grad_v = [1; 1];

a = poisson_lhs(u, v, grad_u, grad_v, x, dx)
```

D6.2

Python (3.x)

```
from numpy import dot, exp

def poisson_rhs(v, grad_v, x, dx):
    a = 0.25
    return exp(-((x[0] - 0.5)**2 + (x[1] - 0.5)**2) / (2*a**2))*v*dx
```

Python (3.x)

```
from poisson_rhs import *

v = dx = 1
x = grad_v = [1, 1]
```

```
print(poisson_rhs(v, grad_v, x, dx))
print(exp(-4))
```

MATLAB

```
function L = poisson_rhs(v, grad_v, x, dx)
    a = 0.25;
    L = exp(-((x(1) - 0.5)^2 + (x(2) - 0.5)^2) / (2*a^2))*v*dx;
end
```

MATLAB

```
v = 1;
dx = 1;

x = [1; 1];
grad_v = [1; 1];

L = poisson_rhs(v, grad_v, x, dx);

exp(-4)
```

D6.3

Python (3.x)

```
from numpy import zeros

from generate_quadrature_2d import *
from evaluate_basis_2d import *

def compute_element_matrix_2d(lhs, X):

    # Create empty element matrix
    A_K = zeros((3, 3))

    # Compute quadrature points and weights
    points, weights = generate_quadrature_2d(X)

    # Iterate over quadrature points
    for k in range(len(weights)):

        # Get quadrature point and weight
        x = points[:, k]
        dx = weights[k]

        # Iterate over rows in element matrix
        for i in range(3):

            # Evaluate basis function
            v, grad_v = evaluate_basis_2d(i, x, X)

            # Iterate over columns in element matrix
            for j in range(3):
```

```
                # Evaluate basis function
                u, grad_u = evaluate_basis_2d(j, x, X)

                # Evaluate integral at current quadrature point
                A_K[i, j] += lhs(u, v, grad_u, grad_v, x, dx)

    return A_K
```

Python (3.x)

```
from compute_element_matrix_2d import *
from poisson_lhs import *

X = array([[0, 0], [1, 0], [0, 1]]).T

print(compute_element_matrix_2d(poisson_lhs, X))
```

MATLAB

```
function A_K = compute_element_matrix_2d(lhs, X)

    % Create empty element matrix
    A_K = zeros(3, 3);

    % Compute quadrature points and weights
    [points, weights] = generate_quadrature_2d(X);

    % Iterate over quadrature points
    for k = 1:length(weights)

        % Get quadrature point and weight
        x = points(:, k);
        dx = weights(k);

        % Iterate over rows in element matrix
        for i = 1:3

            % Evaluate basis function
            [v, grad_v] = evaluate_basis_2d(i, x, X);

            % Iterate over columns in element matrix
            for j = 1:3

                % Evaluate basis function
                [u, grad_u] = evaluate_basis_2d(j, x, X);

                % Evaluate integral at current quadrature point
                A_K(i, j) = A_K(i, j) + feval(lhs, u, v, grad_u, grad_v,
                    x, dx);

            end

        end
```

```
    end
end
```

MATLAB

```
X = [0, 0; 1, 0; 0, 1]';

A_K = compute_element_matrix_2d('poisson_lhs', X)
```

D6.4

Python (3.x)

```python
from numpy import zeros

from generate_quadrature_2d import *
from evaluate_basis_2d import *

def compute_element_vector_2d(rhs, X):

      # Create empty element vector
      b_K = zeros(3)

      # Compute quadrature points and weights
      points, weights = generate_quadrature_2d(X)

      # Iterate over quadrature points
      for k in range(len(weights)):

          # Get quadrature point and weight
          x = points[:, k]
          dx = weights[k]

          # Iterate over rows in element vector
          for i in range(3):

              # Evaluate basis function
              v, grad_v = evaluate_basis_2d(i, x, X)

              # Evaluate integral at current quadrature point
              b_K[i] += rhs(v, grad_v, x, dx)

      return b_K
```

Python (3.x)

```python
from compute_element_vector_2d import *
from poisson_rhs import *

X = array([[0, 0], [1, 0], [0, 1]]).T

print(compute_element_vector_2d(poisson_rhs, X))
print(exp(-2) / 6)
```

```
print((1 + exp(-2)) / 12)
```

MATLAB

```
function b_K = compute_element_vector_2d(rhs, X)

    % Create empty element vector
    b_K = zeros(3, 1);

    % Compute quadrature points and weights
    [points, weights] = generate_quadrature_2d(X);

    % Iterate over quadrature points
    for k = 1:length(weights)

        % Get quadrature point and weight
        x = points(:, k);
        dx = weights(k);

        % Iterate over rows in element vector
        for i = 1:3

            % Evaluate basis function
            [v, grad_v] = evaluate_basis_2d(i, x, X);

            % Evaluate integral at current quadrature point
            b_K(i) = b_K(i) + feval(rhs, v, grad_v, x, dx);

        end

    end

end
```

MATLAB

```
X = [0, 0; 1, 0; 0, 1]';

b_K = compute_element_vector_2d('poisson_rhs', X)

exp(-2) / 6
(1 + exp(-2)) / 12
```

D6.5

Python (3.x)

```
from numpy import array
from scipy.sparse import lil_matrix

from compute_element_matrix_2d import *

def assemble_matrix_2d(lhs, V, K):

    # Create empty stiffness matrix
```

```
    N = len(V)
    A = lil_matrix((N, N))

    # Iterate over elements
    for K_ in K:

        # Get element vertex coordinates
        X = array([V[K_[0]], V[K_[1]], V[K_[2]]]).T

        # Compute element matrix
        A_K = compute_element_matrix_2d(lhs, X)

        # Add element matrix to stiffness matrix
        for i in range(3):
            I = K_[i];
            for j in range(3):
                J = K_[j];
                A[I, J] += A_K[i, j]

    return A
```

Python (3.x)

```
from assemble_matrix_2d import *
from generate_mesh_2d import *
from poisson_lhs import *

V, K = generate_mesh_2d(1, 1)

print(assemble_matrix_2d(poisson_lhs, V, K).todense())
```

MATLAB

```
function A = assemble_matrix_2d(lhs, V, K)

    % Create empty stiffness matrix
    N = size(V, 1);
    A = sparse(N, N);

    % Iterate over elements
    for k = 1:size(K, 1)

        % Get element vertex coordinates
        K_ = K(k, :);
        X = [V(K_(1), :); V(K_(2), :); V(K_(3), :)]';

        % Compute element matrix
        A_K = compute_element_matrix_2d(lhs, X);

        % Add element matrix to stiffness matrix
        for i = 1:3
            I = K_(i);
            for j = 1:3
                J = K_(j);
```

```
                A(I, J) = A(I, J) + A_K(i, j);
            end
        end

    end

end
```

MATLAB

```
[V, K] = generate_mesh_2d(1, 1);

A = assemble_matrix_2d('poisson_lhs', V, K);

full(A)
```

D6.6

Python (3.x)

```
from numpy import array
from scipy.sparse import lil_matrix

from compute_element_vector_2d import *

def assemble_vector_2d(rhs, V, K):

    # Create empty load vector
    N = len(V)
    b = zeros(N)

    # Iterate over elements
    for K_ in K:

        # Get element vertex coordinates
        X = array([V[K_[0]], V[K_[1]], V[K_[2]]]).T

        # Compute element vector
        b_K = compute_element_vector_2d(rhs, X)

        # Add element vector to load vector
        for i in range(3):
            I = K_[i]
            b[I] += b_K[i]

    return b
```

Python (3.x)

```
from assemble_vector_2d import *
from generate_mesh_2d import *
from poisson_rhs import *

V, K = generate_mesh_2d(1, 1)
```

```
print(assemble_vector_2d(poisson_rhs, V, K))
print((1 + exp(-2)) / 6)
print(exp(-2) / 6)
```

MATLAB

```
function b = assemble_vector_2d(rhs, V, K)

    % Create empty load vector
    N = size(V, 1);
    b = zeros(N, 1);

    % Iterate over elements
    for k = 1:size(K, 1)

        % Get element vertex coordinates
        K_ = K(k, :);
        X = [V(K_(1), :); V(K_(2), :); V(K_(3), :)]';

        % Compute element vector
        b_K = compute_element_vector_2d(rhs, X);

        % Add element vector to load vector
        for i = 1:3
            I = K_(i);
            b(I) = b(I) + b_K(i);
        end

    end

end
```

MATLAB

```
[V, K] = generate_mesh_2d(1, 1);

b = assemble_vector_2d('poisson_rhs', V, K)

(1 + exp(-2)) / 6
exp(-2) / 6
```

D6.7

Python (3.x)

```
from numpy import zeros, shape

def apply_dirichlet_bc_2d(A, b, V):

    # Create zero row
    N = shape(V)[0]
    zero = zeros(N)

    # Iterate over vertices
    for i, v in enumerate(V):
```

```
        # Check if we are on the boundary
        eps = 1e-6
        if v[0] < eps or v[0] > 1 - eps or v[1] < eps or v[1] > 1 - eps:

            # Zero row
            A[i, :] = zero

            # Insert 1 on the diagonal
            A[i, i] = 1

            # Set Dirichlet value in vector
            b[i] = 0
```

Python (3.x)

```
from numpy import ones

from apply_dirichlet_bc_2d import *
from generate_mesh_2d import *

A = ones((9, 9))
b = ones((9, 1))
V, K = generate_mesh_2d(2, 2)

apply_dirichlet_bc_2d(A, b, V)

print(A)
print(b)
```

MATLAB

```
function [A, b] = apply_dirichlet_bc_2d(A, b, V)

    % Create zero row
    N = size(V, 1);
    zero = zeros(1, N);

    % Iterate over vertices
    for i = 1:N

        % Check if we are on the boundary
        eps = 1e-6;
        v = V(i, :);
        if v(1) < eps || v(1) > 1 - eps || v(2) < eps || v(2) > 1 - eps

            % Zero row
            A(i, :) = zero;

            % Insert 1 on the diagonal
            A(i, i) = 1;

            % Set Dirichlet value in vector
            b(i) = 0;
```

```
        end

    end

end
```

MATLAB

```
A = ones(9, 9);
b = ones(9, 1);
[V, K] = generate_mesh_2d(2, 2);

[A, b] = apply_dirichlet_bc_2d(A, b, V)
```