$\boxed{\text{F12.B}}$ • Integralsatser (forts)
• Avsnitt 4.6

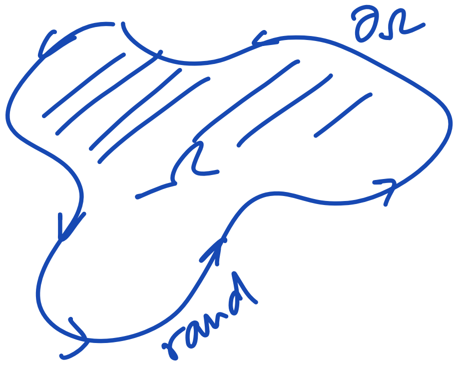
Sammanfattning/repetition av integraler (6st)

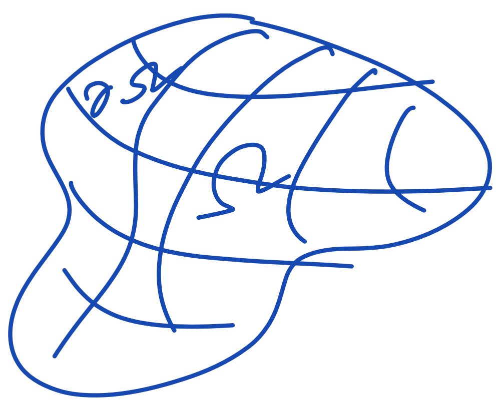
$$\iint_\Omega dA \qquad \iiint_\Omega dV \qquad = \qquad \int_\Omega dx$$

$$\int_C ds \qquad \int_C \cdot\, d\mathbf{r} \qquad = \qquad \int_{\partial\Omega} ds \qquad \int_{\partial\Omega} \cdot\, ds$$

$$\iint_S dS \qquad \iint_S \cdot\, d\mathbf{S} \qquad = \qquad \int_{\partial\Omega} ds \qquad \int_{\partial\Omega} \cdot\, \mathbf{n}\, ds$$

2D





Mål:

1) Kunna utföra enkla integraler:

$$\underline{\|\mathbf{r}'(t)\|\,dt} \qquad \underline{\cdot\, \mathbf{r}'(t)\,dt}$$

$$\underline{\|\mathbf{r}'_u \times \mathbf{r}'_v\|\,du\,dv} \qquad \cdot\,\underline{(\mathbf{r}'_u \times \mathbf{r}'_v)}\,du\,dv$$

2) Förstå "the big picture"

- Integralsatser

Skall visa 5 integralsatser:

derivata

$$\int_{\Omega} df = \int_{\partial\Omega} f$$

område   rand

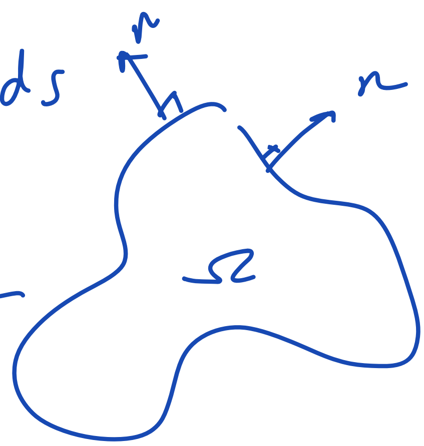
1. Fundamentalsatsen
2. Partiell integration
3. Greens sats
4. Stokes sats
5. Gauss sats

Sats: Fundamentalsatsen

$\Omega \subseteq \mathbb{R}^n$ område med rand $\partial\Omega$ och utåtriktad enhetsnormal n

$$\int_{\Omega} \frac{\partial f}{\partial x_i}\, dx = \int_{\partial\Omega} f\, n_i\, ds$$

$i = 1, 2, \ldots, n$

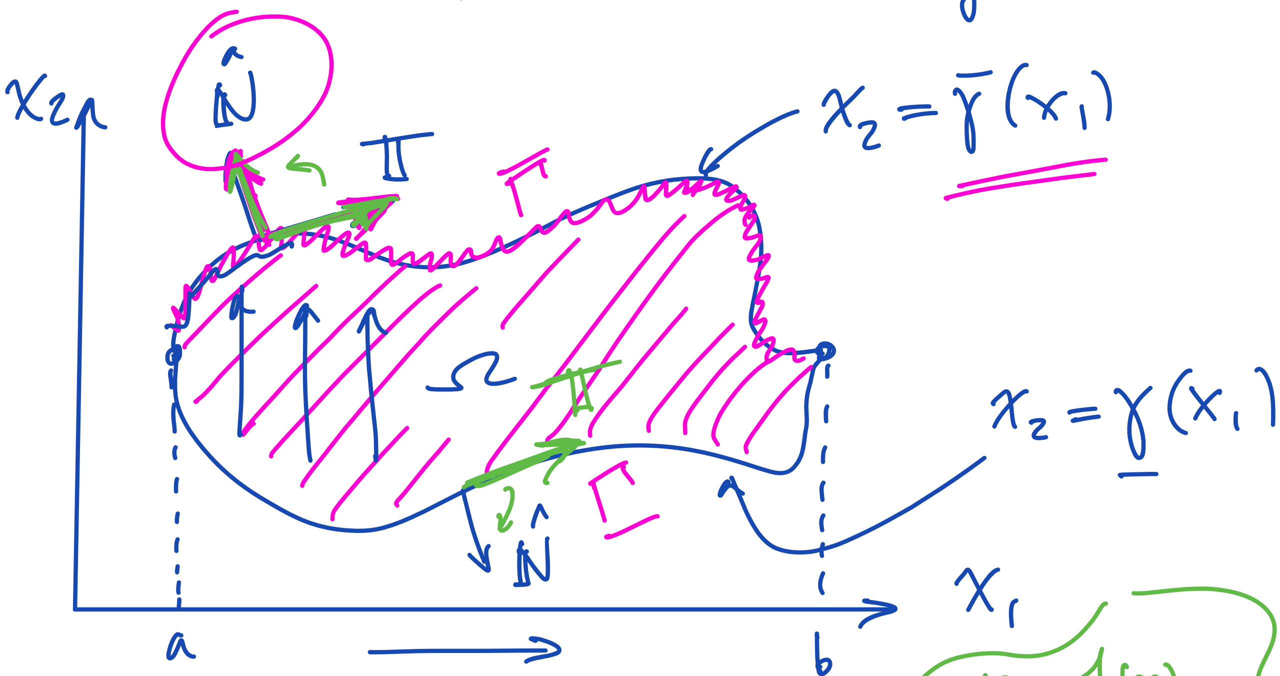


Exempel:

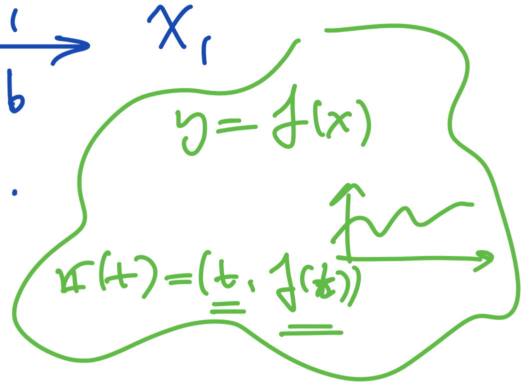
$$\iint_{\Omega} \frac{\partial f}{\partial x}\, dx\,dy = \int_{\partial\Omega} f\, n_x\, ds$$

Dubbelintegral Kurvintegral

Bevis: Antag: $n=2$ och Ω är "enkelt i y"



Vi behöver enhetsnormalen $\hat{N}$.



På övre rande $\bar{\Gamma}$ har vi:

$$\mathbf{r}(t) = (\overset{x}{t}, \overset{\gamma}{\bar{\gamma}(t)}) \quad \text{(parametrisering)}$$

$$\Rightarrow \mathbb{T} = \mathbf{r}'(t) = (1, \bar{\gamma}'(t))$$

$$\Rightarrow \mathbb{N} = (-\bar{\gamma}', 1) \qquad \text{normal}$$

$$\Rightarrow \quad \underline{\underline{\hat{N}}} = \frac{N}{\|N\|} = \frac{(-\bar{\gamma}', 1)}{\sqrt{1+(\bar{\gamma}')^2}} = n_2$$

Normal på övre randen

På undre randen:

$$\hat{N} = \frac{(\underline{\gamma}', -1)}{\sqrt{1+(\underline{\gamma}')^2}}$$

Vill visa att

$$\int_\Omega \frac{\partial f}{\partial x_2}\,dx = \int_{\partial\Omega} f\, n_2\, dx$$

(Motsvarande gäller för $i = 1$)

$$VL = \int_\Omega \frac{\partial f}{\partial x_2}\,dx = \int_a^b \left(\int_{\underline{\gamma}(x_1)}^{\bar{\gamma}(x_1)} \underline{\underline{\frac{\partial f}{\partial x_2}}}\, \underline{\underline{dx_2}} \right) dx_1$$

$$= \int_a^b \left[f \right]_{\underline{\gamma}(x_1)}^{\bar{\gamma}(x_1)} dx_1 = \int_a^b \left(f(x_1, \bar{\gamma}(x_1)) - f(x_1, \underline{\gamma}(x_1)) \right) dx_1$$

$$= \underbrace{\int_a^b f(x_1, \bar{\gamma}(x_1))\,dx_1}_{\text{Första}} - \underbrace{\int_a^b f(x_1, \underline{\gamma}(x_1))\,dx_1}_{\text{Andra}}$$

Första:

$$\int_a^b f(x_1, \bar{\gamma}(x_1))\, dx_1 = \left\{ \text{Förläng med } \sqrt{1+(\bar{\gamma}')^2} \right\}$$

$$= \int_a^b f \underbrace{\frac{1}{\sqrt{1+(\bar{\gamma}')^2}}}_{= n_2} \cdot \underbrace{\sqrt{1+(\bar{\gamma}')^2}\, dx_1}_{= ds}$$

$$= \int_a^b f n_2\, ds = \text{Kurvintegralen på övre randen}$$

P.s.s.

Andra: Kurvintegralen på undre randen

$$\therefore \underbrace{\int_\Omega \frac{\partial f}{\partial x_2}\, dx}_{\text{Dubbelintegral}} = \int_{\bar{\Gamma}} f n_2\, ds + \int_{\underline{\Gamma}} f n_2\, ds$$

$$= \underbrace{\int_{\partial\Omega} f n_2\, ds}_{\text{Kurvintegralen}}$$

$\blacksquare$

Hur vet vi att $\int_\Omega f\,dx$ är en dubbelintegral?

$\Omega \subseteq \mathbb{R}^2$

Fundamentalsatsen:

$$\int_\Omega \frac{\partial f}{\partial x_i}\,dx = \int_{\partial\Omega} f n_i\,ds$$

$i = 1, 2, \dots, n$

n n

→ Partiell integration

→ (Greens första identitet)

→ (Greens sats)

→ Stokes sats

→ Gauss sats

2D: VL = dubbelintegral
HL = kurvintegral

3D: VL = trippelintegral
HL = ytintegral

Sats: Partiell integration

$$\int_\Omega \frac{\partial f}{\partial x_i} g\,dx = \int_{\partial\Omega} f g n_i\,ds - \int_{\partial\Omega} f \frac{\partial g}{\partial x_i}\,dx$$

Jämför 1D:

$$\int_a^b f' g\,dx = \Big[fg\Big]_a^b - \int_a^b f g'\,dx$$

Bevis: Använd fundamentalsatsen på produkten fg:

$$\int_\Omega \frac{\partial}{\partial x_i}(fg)\,dx = \int_{\partial\Omega} fg\, n_i\, ds$$

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f\frac{\partial g}{\partial x_i}$$

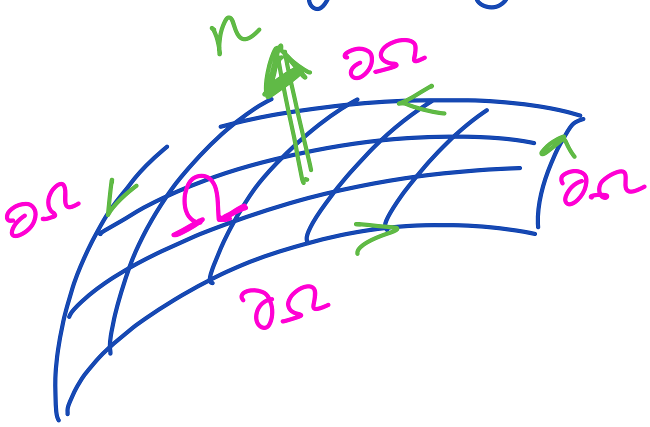
HL

$\blacksquare$

Stokes sats:

$f : \mathbb{R}^3 \to \mathbb{R}^3$, Ω yta i $\mathbb{R}^3$

$$\underbrace{\int_\Omega \nabla \times f \cdot ds}_{\text{Normalytintegral}} = \underbrace{\int_{\partial\Omega} f \cdot dr}_{\text{Tangentkurvintegral}}$$

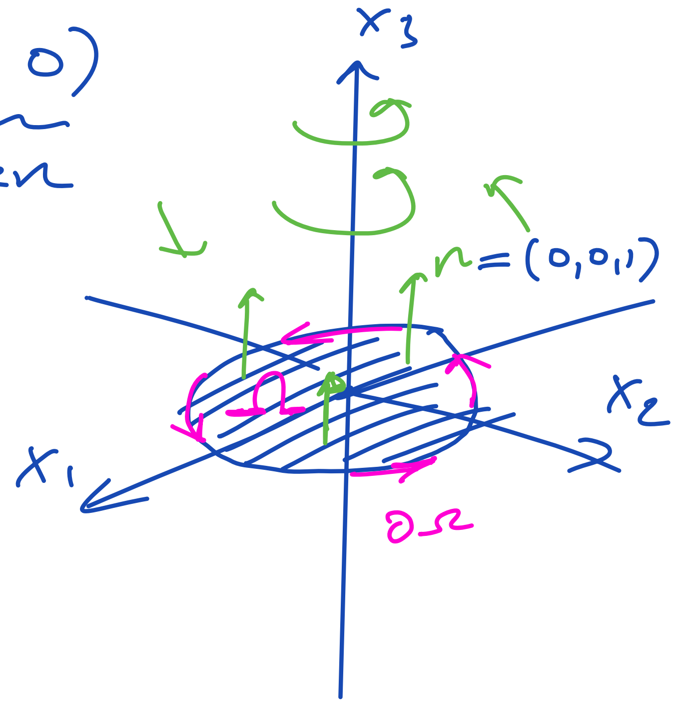


Exempel:

$$\nabla \times f = (0,0,2)$$

$$f(x) = (-x_2, x_1, 0)$$

Ω = enhetsdisken



Stokes sats:

$$\int_{\Omega} \nabla \times f \cdot dS = \int_{\partial\Omega} f \cdot dr$$

Kontroll:

VL = normalytintegral

$$\int_{\Omega} \nabla \times f \, dS = \int_{\Omega} (0,0,2) \cdot (0,0,1) \, dS$$

$$= 2 \cdot |\Omega| = \underline{\underline{2\pi}}$$

Kom ihåg:

$$\nabla \times f = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right)$$

$$0 - 0 \qquad 0 - 0 \qquad 1 - (-1)$$

$$= (0,0,2)$$

HL = tangentkurvintegral

$$= \int_0^{2\pi} \underbrace{(-\sin(t), \cos(t), 0)}_{= f = (-x_2, x_1, 0)} \cdot \underbrace{(-\sin(t), \cos(t), 0)}_{r'(t)} dt$$

$$\begin{cases} x_1 = \cos t \\ x_2 = \sin t \end{cases}$$

$$= \int_0^{2\pi} 1 \, dt = \underline{\underline{2\pi}}$$

$\therefore$ VL = HL enligt Stokes

* Hur kopplar Stokes sats till cirkulation och skalär potential?

Kom ihåg från måndagens föreläsning:

(i)

Cirkulationen alltid = 0 (Konservativt fält)

$$\oint_C f \cdot ds = 0 \quad \forall C$$

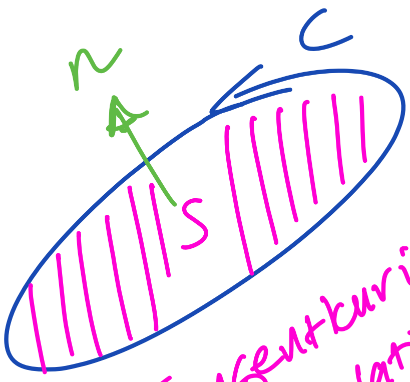
$\Longleftrightarrow$

(ii)

Finns en skalär potential ϕ

$$f = \nabla \phi$$

$\nabla \times f = 0$



Stokes sats

Tangentkurvintegral = Cirkulation = Arbete

$$\oint_C f \cdot ds = \int_S \nabla \times f \cdot ds$$

Notera:

1) $\nabla \times f = 0 \implies \oint_C f \cdot ds = 0$

2) $f = \nabla \phi \implies \nabla \times f = \nabla \times \nabla \phi = 0$

Följer att:

$$\begin{cases} \text{(i) Konservativt vektorfält (cirkulationsfritt)} \\ \Longleftrightarrow \\ \text{(ii) Finns skalär potential: } f = \nabla \phi \\ \Longleftrightarrow \\ \text{(iii) Rotationsfritt: } \nabla \times f = 0 \end{cases}$$

(i ett enkelt sammanhängande område)

Sats: $f : \mathbb{R}^n \to \mathbb{R}^n$

Volymintegral

Normalytintegral = Flöde

$$\int_{\Omega} \nabla \cdot f \, dx = \int_{\partial\Omega} f \cdot n \, ds$$

Bevis:

$$\int_{\Omega} \nabla \cdot f \, dx = \int_{\Omega} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx$$

$$= \sum_{i=1}^{n} \int_{\Omega} \frac{\partial f_i}{\partial x_i} dx$$

$$= \sum_{i=1}^{n} \int_{\partial\Omega} f_i n_i \, ds$$

$$= \int_{\partial\Omega} f \cdot n \, ds = \int_{\partial\Omega} f \cdot ds$$

$\blacksquare$

Hur hänger Gauss sats ihop med flöde och existens av vektorpotential?

Inkompressibelt

Flödet genom varje yta alltid $= 0$

$$\oint_{S \text{ sluten}} f \cdot ds = 0$$

$\Longleftrightarrow$

Finns en vektorpotential:

$$f = \nabla \times \psi$$

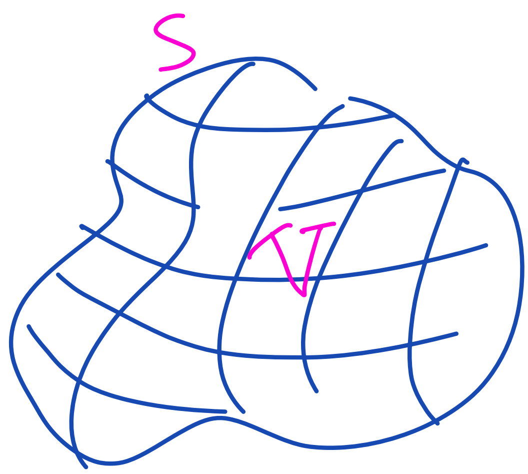
$$\nabla \cdot f = 0$$

$$\nabla \cdot \nabla \times \psi = 0$$

Gauss sats:

$$\oint_S f \cdot ds = \int_V \nabla \cdot f \, dx$$

$\nabla \cdot f = 0 \longrightarrow$ Flödet $= 0$



Följer att:

(i) Inkompressibelt (nettoflöde alltid 0)

$\Longleftrightarrow$

(ii) Finns vektorpotential: $\underline{f} = \nabla \times \underline{f}$

$\Longleftrightarrow$

(iii) Divergensfritt: $\nabla \cdot \underline{f} = 0$