

# Formas Diferenciables

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Sea  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . El pullback de  $f$  en un punto  $p$  es

$$f^* : \Lambda^k(\mathbb{R}_{f(p)}^m)^* \rightarrow \Lambda^k(\mathbb{R}_p^n)^*$$

tal que

$$f^*(w)(p)(v_1, \dots, v_k) = w(f(p))(df_p(v_1), \dots, df_p(v_k))$$

Sea  $w = -y \, dx + x \, dy$  y  $f : U \rightarrow \mathbb{R}^2$  tal que  
 $(r, \theta) \mapsto (r \cos(\theta), r \sin(\theta))$ , siendo  $U = \{r > 0, \theta \in (0, 2\pi)\}$ ,  
entonces

$$f^*(w) = w(f(r, \theta)) = -r \sin(\theta) \, dx + r \cos(\theta) \, dy$$

$$\begin{aligned} dx &= d(r \cos(\theta)) = \frac{\partial(r \cos(\theta))}{\partial r} dr + \frac{\partial(r \cos(\theta))}{\partial \theta} d\theta \\ &= \cos(\theta) dr - r \sin(\theta) d\theta \end{aligned}$$

$$\begin{aligned} dy &= d(r \sin(\theta)) = \frac{\partial(r \sin(\theta))}{\partial r} dr + \frac{\partial(r \sin(\theta))}{\partial \theta} d\theta \\ &= \sin(\theta) dr + r \cos(\theta) d\theta \end{aligned}$$

$$\begin{aligned} f^*(w) &= -r \sin(\theta)(\cos(\theta) dr - r \sin(\theta) d\theta) \\ &\quad + r \cos(\theta)(\sin(\theta) dr + r \cos(\theta) d\theta) \\ &= r^2 d\theta \end{aligned}$$

Sea  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  un mapeo diferenciable dado por

$$f(x_1, x_2) = (y_1, y_2),$$

y sea  $w = dy_1 \wedge dy_2$ . Muestre que

$$f^*w = \det(df_p) dx_1 \wedge dx_2.$$

Entonces  $f^*(w) = w(f(x_1, x_2))$  tenemos

$$f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (y_1, y_2)$$

$$\begin{aligned} dy_1 &= d(f_1(x_1, x_2)) \\ &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \end{aligned}$$

$$\begin{aligned} dy_2 &= d(f_2(x_1, x_2)) \\ &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \end{aligned}$$

$$\begin{aligned}f^*(w) &= w(f(x_1, x_2)) \\&= dy_1 \wedge dy_2 \\&= \left( \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \right) \wedge \left( \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \right) \\&= \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} dx_2 \wedge dx_1 \\&= \left( \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \right) dx_1 \wedge dx_2 \\&= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} dx_1 \wedge dx_2 = \det(df_p) dx_1 \wedge dx_2\end{aligned}$$

## Ejercicio 8

Sea  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  un mapeo diferenciable dado por

$$f(x_1, \dots, x_n) = (y_1, \dots, y_n),$$

y sea  $w = dy_1 \wedge \dots \wedge dy_n$ . Muestre que

$$f^*w = \det(df_p) dx_1 \wedge \dots \wedge dx_n.$$



Para cada  $i$  de 1 a  $n$ , calculamos  $dy_i$  como la diferencial de  $f_i(x_1, \dots, x_n)$ :

$$dy_i = d(f_i(x_1, x_2, \dots, x_n)) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j$$

# Expansión del Producto Exterior

El pullback  $f^*(w)$  se obtiene expandiendo el producto exterior de las diferenciales  $dy_i$ :

$$\begin{aligned} f^*(w) &= \bigwedge_{i=1}^n \left( \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j \right) \\ &= \left( \sum_{j=1}^n \frac{\partial f_1}{\partial x_j} dx_j \right) \wedge \left( \sum_{j=1}^n \frac{\partial f_2}{\partial x_j} dx_j \right) \wedge \cdots \wedge \left( \sum_{j=1}^n \frac{\partial f_n}{\partial x_j} dx_j \right) \\ &= \begin{vmatrix} \sum_{j=1}^n \frac{\partial f_1}{\partial x_j} dx_j(v_1) & \cdots & \sum_{j=1}^n \frac{\partial f_1}{\partial x_j} dx_j(v_n) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n \frac{\partial f_n}{\partial x_j} dx_j(v_1) & \cdots & \sum_{j=1}^n \frac{\partial f_n}{\partial x_j} dx_j(v_n) \end{vmatrix} \end{aligned}$$

$$f^*(w) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_j} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \begin{vmatrix} dx_1(v_1) & \cdots & dx_1(v_n) \\ \vdots & \ddots & \vdots \\ dx_n(v_1) & \cdots & dx_n(v_n) \end{vmatrix} \\ = \det(df_p) dx_1 \wedge \cdots \wedge dx_n(v_1, \dots, v_n)$$

El resultado final es que el pullback de  $w$  bajo  $f$  es el determinante del Jacobiano multiplicado por el producto exterior de las diferenciales  $dx_1 \wedge \cdots \wedge dx_n$ :

$$f^*(w) = \det(df_p) dx_1 \wedge \cdots \wedge dx_n$$

[https://sites.ualberta.ca/~vbouchar/MATH215/section\\_pullback\\_two\\_form.html](https://sites.ualberta.ca/~vbouchar/MATH215/section_pullback_two_form.html)

<https://math.stackexchange.com/questions/576638/how-to-calculate-the-pullback-of-a-k-form-explicitly>

<https://math.stackexchange.com/questions/1302562/proving-that-the-pullback-map-commutes-with-the-exterior-d>