Formas Diferenciables

Wilson Jerez

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Definición

Sea $f : \mathbb{R}^n \to \mathbb{R}^m$. El pullback de f en un punto p es

$$f^* : \Lambda^k(\mathbb{R}^m_{f(p)})^* \to \Lambda^k(\mathbb{R}^n_p)^*$$

tal que

$$f^*(w)(p)(v_1, \dots, v_k) = w(f(p))(df_p(v_1), \dots, df_p(v_k))$$

Ejemplo

Sea $w = -y\,dx + x\,dy$ y $f : U \to \mathbb{R}^2$ tal que $(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$, siendo $U = \{r > 0, \theta \in (0, 2\pi)\}$, entonces

$$f^*(w) = w(f(r,\theta)) = -r\sin(\theta)\,dx + r\cos(\theta)\,dy$$

$$\begin{aligned} dx = d(r\cos(\theta)) &= \frac{\partial(r\cos(\theta))}{\partial r}\,dr + \frac{\partial(r\cos(\theta))}{\partial\theta}\,d\theta \\ &= \cos(\theta)\,dr - r\sin(\theta)\,d\theta \end{aligned}$$

$$\begin{aligned} dy = d(r\sin(\theta)) &= \frac{\partial(r\sin(\theta))}{\partial r}\,dr + \frac{\partial(r\sin(\theta))}{\partial\theta}\,d\theta \\ &= \sin(\theta)\,dr + r\cos(\theta)\,d\theta \end{aligned}$$

$$\begin{aligned} f^*(w) &= -r\sin(\theta)(\cos(\theta)\,dr - r\sin(\theta)\,d\theta) \\ &\qquad + r\cos(\theta)(\sin(\theta)\,dr + r\cos(\theta)\,d\theta) \\ &= r^2\,d\theta \end{aligned}$$

Ilustración

Sea $f : \mathbb{R}^2 \to \mathbb{R}^2$ un mapeo diferenciable dado por

$$f(x_1, x_2) = (y_1, y_2),$$

y sea $w = dy_1 \wedge dy_2$. Muestre que

$$f^*w = \det(df_p)\, dx_1 \wedge dx_2.$$

Entonces $f^*(w) = w(f(x_1, x_2))$ tenemos
$f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)) = (y_1, y_2)$

Cálculo de dy_1 y dy_2

$$\begin{aligned} dy_1 &= d(f_1(x_1, x_2)) \\ &= \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \end{aligned}$$

$$\begin{aligned} dy_2 &= d(f_2(x_1, x_2)) \\ &= \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \end{aligned}$$

Cálculo del Pullback

$$
\begin{aligned}
f^*(w) &= w(f(x_1, x_2)) \\
&= dy_1 \wedge dy_2 \\
&= \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \right) \wedge \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \right) \\
&= \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} dx_2 \wedge dx_1 \\
&= \left(\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \right) dx_1 \wedge dx_2 \\
&= \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} dx_1 \wedge dx_2 = \det(df_p) dx_1 \wedge dx_2
\end{aligned}
$$

Ejercicio 8

Sea $f : \mathbb{R}^n \to \mathbb{R}^n$ un mapeo diferenciable dado por

$$f(x_1, \ldots, x_n) = (y_1, \ldots, y_n),$$

y sea $w = dy_1 \wedge \cdots \wedge dy_n$. Muestre que

$$f^* w = \det(df_p)\, dx_1 \wedge \cdots \wedge dx_n.$$

Cálculo de dy_i

Para cada i de 1 a n, calculamos dy_i como la diferencial de $f_i(x_1, \ldots, x_n)$:

$$dy_i = d(f_i(x_1, x_2, \ldots, x_n)) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j$$

Expansión del Producto Exterior

El pullback $f^*(w)$ se obtiene expandiendo el producto exterior de las diferenciales dy_i:

$$
\begin{aligned}
f^*(w) &= \bigwedge_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j \right) \\
&= \left(\sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} dx_j \right) \wedge \left(\sum_{j=1}^{n} \frac{\partial f_2}{\partial x_j} dx_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} dx_j \right) \\
&= \begin{vmatrix} \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} dx_j(v_1) & \dots & \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} dx_j(v_n) \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} dx_j(v_1) & \dots & \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} dx_j(v_n) \end{vmatrix}
\end{aligned}
$$

$$f^*(w) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_j} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \ddots & \frac{\partial f_n}{\partial x_n} \end{vmatrix} \begin{vmatrix} dx_1(v_1) & \dots & dx_1(v_n) \\ \vdots & \ddots & \vdots \\ dx_n(v_1) & \dots & dx_n(v_n) \end{vmatrix}$$

$$= det(df_p) dx_1 \wedge \cdots \wedge dx_n(v_1, \dots, v_n Do)$$

Resultado Final

El resultado final es que el pullback de w bajo f es el determinante del Jacobiano multiplicado por el producto exterior de las diferenciales $dx_1 \wedge \cdots \wedge dx_n$:

$$f^*(w) = \det(df_p)\, dx_1 \wedge \cdots \wedge dx_n$$

Links de interes

https://sites.ualberta.ca/~vbouchar/MATH215/section_pullback_two_form.html

https://math.stackexchange.com/questions/576638/how-to-calculate-the-pullback-of-a-k-form-explicitly

https://math.stackexchange.com/questions/1302562/proving-that-the-pullback-map-commutes-with-the-exterior-d