4CM20 - Hybrid Systems and Control

Take-home exam 2022-2023, Part I

- Make a report in Word or LaTeX containing your solutions, and put both your email address and student number on your report. In case you are working in pairs, please provide this information for both of you.
- All solutions should be clearly motivated. Only end results of calculations are not sufficient!
- You should hand in your report either individually or make a report in a team of two persons. Afterwards you will be invited for an *individual* oral exam.
- Hand in your report as follows:

  Electronically (as a PDF) to k.j.a.scheres@tue.nl under the file name
  `[Name1_Name2]_[IDnos.]_4CM20_THE_PART1.pdf` (e.g., Scheres_Eijnden_0123_4567_4CM20_THE_PARTI.pdf)
- **DEADLINE: June 16, 2023**
- **Good luck with the take-home exam!**

Problem 1: Stability of a Hybrid Automaton.
Consider the hybrid automata in Figure 1 and Figure 2.

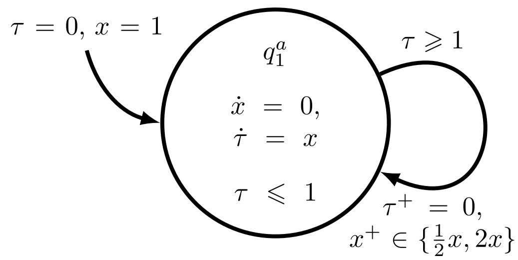

Figure 1: Hybrid Automaton $\mathscr{H}^a$.

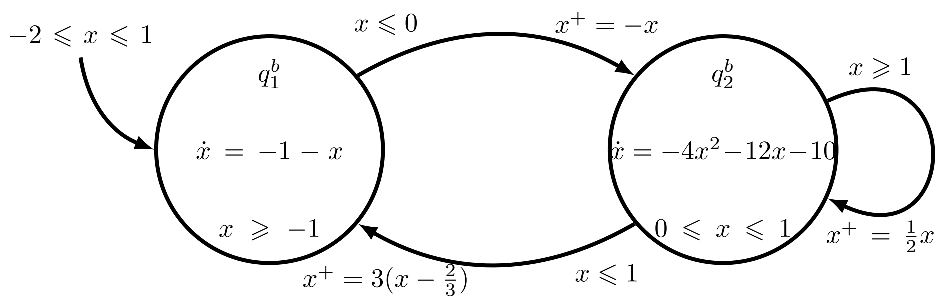

Figure 2: Hybrid Automaton $\mathscr{H}^b$.

(a) Write automaton $\mathscr{H}^a$ as $\mathscr{H}^a = (Q^a, X^a, f^a, \mathrm{Init}^a, \mathrm{Inv}^a, E^a, G^a, R^a)$, i.e., provide explicit expressions for the octuple $(Q^a, X^a, f^a, \mathrm{Init}^a, \mathrm{Inv}^a, E^a, G^a, R^a)$.

(b) Write automaton $\mathscr{H}^b$ as $\mathscr{H}^b = (Q^b, X^b, f^b, \mathrm{Init}^b, \mathrm{Inv}^b, E^b, G^b, R^b)$, i.e., provide explicit expressions for the octuple $(Q^b, X^b, f^b, \mathrm{Init}^b, \mathrm{Inv}^b, E^b, G^b, R^b)$.

Consider a third hybrid automaton $\mathscr{H}^c$ described by $\mathscr{H}^c = (Q^c, X^c, f^c, \mathrm{Init}^c, \mathrm{Inv}^c, E^c, G^c, R^c)$ with the octuple $(Q^c, X^c, f^c, \mathrm{Init}^c, \mathrm{Inv}^c, E^c, G^c, R^c)$ given by

- $Q^c = \{q_1^c, q_2^c\}$
- $X^c = \mathbb{R}$

- $f^c(q_1^c, x) = x^3$, $f^c(q_2, x) = x$
- $\text{Init}^c = \{(q_1^c, x) \mid -\alpha < x < 0\}$
- $\text{Inv}^c(q_1^c) = \text{Inv}^c(q_2^c) = \{-\alpha \leqslant x \leqslant \alpha\}$
- $E^c = \{(q_1^c, q_2^c), (q_2^c, q_1^c)\}$
- $G^c((q_1^c, q_2^c)) = \{x \in X^c \mid x \leqslant -\alpha \vee x \geqslant \alpha\}$, $G^c((q_2^c, q_1^c)) = \{x \in X^c \mid x \leqslant -\alpha\}$
- $R^c((q_1^c, q_2^c)) = \{(x^-, x^+) \in X^c \times X^c \mid x^+ = \frac{1}{2}x^-\}$, $R^c((q_2^c, q_1^c)) = \{(x^-, x^+) \in X^c \times X^c \mid x^+ = -\frac{1}{2}x^-\}$

where $\alpha \in \mathbb{R}_{>0}$ is a positive real number.

(c) Draw a picture of $\mathscr{H}^c$ (similar as done in Figure 1 and 2)

For all hybrid automata $\mathscr{H}^a$, $\mathscr{H}^b$ and $\mathscr{H}^c$ considered in (a), (b) and (c), answer the following questions:

(d) Compute the sets Reach and Out.

(e) Use the results of (d) to conclude

(e.1) Whether or not each hybrid automaton is non-blocking.

(e.2) Whether or not each hybrid automaton is deterministic.

(f) Are all maximal solutions to each hybrid automaton defined for all times $t \in \mathbb{R}_{\geqslant 0}$? Explain why or why not.

(g) Consider a general hybrid automata $\mathscr{H} = (Q, X, f, \text{Init}, \text{Inv}, E, G, R)$. Give a condition in terms of Reach and Out and the defining octuple $(Q, X, f, \text{Init}, \text{Inv}, E, G, R)$, that is *necessary and sufficient* for the property that $\mathscr{H}^M_{(q_0,x_0)} \subseteq \mathscr{H}^\infty_{(q_0,x_0)}$ holds for all $(q_0, x_0) \in \text{Init}$, where $\mathscr{H}^M_{(q_0,x_0)}$ and $\mathscr{H}^\infty_{(q_0,x_0)}$ denote the set of maximal and infinite executions for initial hybrid states $(q_0, x_0) \in \text{Init}$, respectively. Is this condition true for $\mathscr{H}^a$, $\mathscr{H}^b$ and $\mathscr{H}^c$ above?

Problem 2: Stability analysis of a piecewise linear system.
Consider a piecewise linear system of the form

$$\dot{x} = A_i x \qquad \text{when } x \in \mathcal{S}_i,$$

where the matrices A_i, $i \in \{1, 2\}$ are given by

$$A_1 = \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} -4 & -7 \\ 4 & -5 \end{bmatrix},$$

and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ denotes the state vector. The active dynamics of the system are dictated by the sets $\mathcal{S}_i$, $i \in \{1, 2\}$, which are given by

$$\mathcal{S}_1 = \{x \in \mathbb{R}^2 \mid x_2 \geqslant x_1^2 + 1\} \qquad \text{and} \qquad \mathcal{S}_2 = \{x \in \mathbb{R}^2 \mid x_2 \leqslant x_1^2 + 1\}.$$

(a) Simulate the system for various initial conditions. Plot the solutions in the phase plane (i.e., plot x_1 versus x_2, where x_1 and x_2 are the first and second component of x, respectively.) Clearly indicate the switching plane in your figures.

We now consider the dynamics on the switching surface.

(b) Compute the dynamics on the switching surface according to Filippov's convex definition. For which part(s) of the switching surface do sliding modes exist?

(c) Are the sliding mode(s) attractive, repulsive, or both (explain your reasoning)?

(d) Given an initial condition $x(0) = x_0$, are the corresponding Filippov solutions unique, and why/why not?

(e) Provide LMI conditions that guarantee the quadratic function $V(x) = x^\top P x$ to be a common quadratic Lyapunov function for the system $\dot{x} = A_i x$, $i \in \{1, 2\}$. Are the LMIs feasible? If yes, provide a solution. If not, explain why.

(f) Consider the figure that you created in (a). Derive a conic region that includes the region $\mathcal{S}_1$, i.e., find the (2-by-2) matrix E such that $\mathcal{S}_1$ is contained in the region given by $Ex \geqslant 0$.

(g) Exploit the conic region that you obtained in (f) to construct a piecewise quadratic Lyapunov function $V_i(x) = x^\top P_i x$ using S-procedure relaxations. Explain how you ensure continuity of the Lyapunov function on the switching surface and explain the LMIs that you have derived. Solve your obtained LMIs accordingly using MATLAB.

Problem 3: Dwell time computations.
Consider the switched linear system

$$\dot{x} = A_i x, \quad i \in \{1, 2\}, \tag{1}$$

with

$$A_1 = \begin{bmatrix} -7 & 4 & 6 \\ 8 & -47 & -60 \\ 0 & 36 & 45 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} -1\frac{1}{3} & 4\frac{1}{3} & 6 \\ -\frac{1}{3} & -2\frac{2}{3} & -1 \\ 0 & 0 & -2 \end{bmatrix}. \tag{2}$$

Note that all matrices A_i, $i \in \{1, 2\}$, in (2) are Hurwitz.

(a) Suppose that the switching system (1) is *periodic* in the sense that it operates in each mode for exactly τ_p time units. In other words, the system evolves according to $\dot{x} = A_1 x$ for τ_p time units, then according to $\dot{x} = A_2 x$ for τ_p time units, after which it switches back to $\dot{x} = A_1 x$ and operates in this mode for τ_p time units and so on. Compute a lower-bound $\tau_p^\star$ on τ_p such that the switched linear system is GAS for all $\tau_p \geqslant \tau_p^\star$. Clearly explain your reasoning and the procedure that you follow for the computations.

(b) Derive now for each mode a *minimal mode-dependent dwell-time* $\tau_1^\star$ and $\tau_2^\star$ for mode 1 and mode 2, respectively, that guarantees GAS for all switching signals that have a minimal dwell-time in mode 1 of $\tau_1^\star$, and in mode 2 of $\tau_2^\star$. Once again clearly explain and motivate your answer and computations.

(c) Compute an *averaged dwell time* τ_a that guarantees GAS of the switched system based on using quadratic Lyapunov functions for each mode. Try to get the least conservative estimate for τ_a. Once again clearly explain your reasoning and the procedure that you follow for the computations.

(d) Compare the answers you had in parts (a), (b), and (c). Is there a difference between the conditions that you found? Does this match your expectations? Explain your answer.

Consider now the switched linear system (1) with the matrices given by

$$A_1 = \begin{bmatrix} -7 & 4 & 6 \\ 8 & -47 & -60 \\ 0 & 36 & 45 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 7 & 16 & 8 \\ 3 & 5 & -1 \\ 1 & 6 & 11 \end{bmatrix}, \tag{3}$$

and note that in this case the matrix A_1 is Hurwitz, whereas the matrix A_2 is not.

(e) Suppose that for the switched system (1) with matrices given in (3), the second mode (i.e., mode 2) always remains active for a *fixed* time of $\tau_2^\star = 2$ time units. Derive a *minimal dwell-time* $\tau_1^\star$ for mode 1 that guarantees GAS of the overall switched system. Clearly explain your answer.

(**BONUS**) Again suppose that mode 2 remains active for $\tau_2^\star = 2$ time units. Derive an *average dwell-time* τ_a for mode 1 that guarantees GAS of the system. Once again clearly explain and motivate your answer.

(f) Now suppose that the times in which both mode 1 and mode 2 are active are not necessarily fixed anymore. Can you derive a condition (in terms of a lower-bound) on the ratio τ_1/τ_2, i.e., the ratio of the times that mode 1 and mode 2 can be active that guarantees GAS of the switched system? Clearly motivate your answer *Hint: Try to make use of a common quadratic Lyapunov function.*

Consider now the jump-flow system

$$\dot{x} = Ax, \tag{4}$$

$$x^+ = Rx, \tag{5}$$

with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \qquad R = \begin{bmatrix} 0.6 & 0.3 \\ 1 & 0.2 \end{bmatrix}. \tag{6}$$

(g) Provide a time-based specification on jumping/flowing (average or reverse average inter-jump times) such that the jump flow system (6) is GAS. Motivate your answer by providing a proof of stability for the resulting system under the given specification.