4CM20 - Hybrid Systems and Control

Take-home exam 2022-2023, Part II

- Make a report in Word or LaTeX containing your solutions, and put both your email address and student number on your report. In case you are working in pairs, please provide this information for both of you.
- All solutions should be clearly motivated. Only end results of calculations are not sufficient!
- You should hand in your report either individually or make a report in a team of two persons. Afterwards you will be invited for an *individual* oral exam.
- Hand in your report as follows:

  Electronically (as a PDF) to k.j.a.scheres@tue.nl under the file name
  `[Name1_Name2]_[IDnos.]_4CM20_THE_PART2.pdf` (e.g., Scheres_Eijnden_0123_4567_4CM20_THE_PART2.pdf)
- **DEADLINE: June 25, 2023**
- **Good luck with the take-home exam!**

Problem 1: Stability of a Switched Linear System with time-based switching.
Consider the *continuous-time* switched linear system (SLS)

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{1}$$

with active mode $\sigma(t) \in \{1, 2\}$ and continuous state variable $x(t)$ at time $t \in \mathbb{R}_{\geqslant 0}$. The matrices in (1) are given by

$$A_1 = \begin{bmatrix} 1.6 & 2 \\ -4 & -1.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.8 & 4 \\ -2 & 1.6 \end{bmatrix}.$$

The goal of this problem is to design a stabilizing time-based switching signal $\sigma(t) \in \{1, 2\}$, $t \in \mathbb{R}_{\geqslant 0}$.

(a) Show that, for any $\alpha \in [0, 1]$, $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz (you may use MATLAB for this).

(b) Given that all convex combinations of A_1 and A_2 are Hurwitz, is the system stable for any time-based switching signal $\sigma(t)$, $t \in \mathbb{R}_{\geqslant 0}$? In other words, does (a) imply that (1) is globally asymptotically stable (GAS) under arbitrary switching? Provide a (mathematical) proof, if this is the case, or a counter-example, if it is not the case.

(c) Now we consider the case where we stay in the first mode for τ_1 time-units, and in the second mode for τ_2 time-units, after which we switch back to the first mode again, where we stay again τ_1 time-units, etc. Use MATLAB to find all (τ_1, τ_2) pairs that result in the system being GAS for the values $\tau_1, \tau_2 \in [0, 1.5]$ and plot the results in a figure.

(d) Suppose that the time we stay in each mode is uncertain but can take two values, *i.e.*, we stay in mode 1 for $\tau_1 \in \{0.15, 0.2\}$ time-units, then we switch to mode 2 for $\tau_2 \in \{1.5, \theta\}$ time-units, after which we switch back to the first mode again. Here, $\theta \in [1.5, 2]$ is a (constant) parameter that is to be determined. Derive LMIs that show GAS of the switched system (for fixed θ) when the time we stay in each mode is uncertain. Solve the LMIs using MATLAB and find the largest value for $\theta \in [1.5, 2]$ for which the SLS is GAS. Provide your solutions.

(e) Now suppose that τ_1 switches periodically between 1.2 and 1.25, *i.e.*, the value of τ_1 is given by the sequence

$$\tau_1 = \tau_{1,k} := \begin{cases} 1.2, & \text{if } k \text{ is even,} \\ 1.25, & \text{if } k \text{ is odd,} \end{cases}$$

where $k \in \mathbb{N}$ denotes the k^{th} time mode 1 is active, and that τ_2 is uncertain but can take two values, *i.e.*, $\tau_2 \in \{1, \theta\}$ where $\theta \in [2, 3]$ is again a parameter which is to be determined. Derive LMIs that show GAS of the SLS for fixed θ. Try to relax your conditions as much as possible, *i.e.*, exploit as much structure as you can from the problem, and find the largest value for $\theta \in [2, 3]$ for which the SLS is GAS.

Problem 2: Observer design and output-based controller design for switched systems.
Consider the *discrete-time* switched linear system (SLS)

$$\begin{aligned} x_{k+1} &= A_{\sigma_k} x_k + B_{\sigma_k} u_k, \\ y_k &= C_{\sigma_k} x_k, \end{aligned} \tag{2}$$

with active mode $\sigma_k \in \{1,2\}$, continuous state variable $x_k \in \mathbb{R}^2$, control input $u_k \in \mathbb{R}$ and measured output $y_k \in \mathbb{R}$ for discrete time $k \in \mathbb{N}$. Moreover, the matrices in (2) are given by

$$A_1 = \begin{bmatrix} 1/2 & 1/2 \\ -1/4 & 5/4 \end{bmatrix}; \;\; A_2 = \begin{bmatrix} 5/6 & 5/3 \\ -2/3 & 1/6 \end{bmatrix}; \;\; B_1 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}; \;\; B_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}; \;\; C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}; \;\; C_2 = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}.$$

Throughout this problem, we will assume that the mode σ_k can be arbitrary (hence, arbitrary switching), but we assume that the mode σ_k is *known* at time $k \in \mathbb{N}$. The goal will be to first design a stabilizing state feedback controller for this SLS (while assuming full state feedback is available), then an observer, and finally show that this results in a stabilizing observer-based controller that only uses the output signal y_k and known mode σ_k. Stability is meant here under arbitrary switching.

(a) Suppose for the moment that we measure the state x_k of the switched linear system (2) completely. Under this assumption, we would like to design a controller $u_k = K_{\sigma_k} x_k$, $k \in \mathbb{N}$, such that it stabilizes (2), *i.e.*, it makes the system

$$x_{k+1} = (A_i + B_i K_i) x_k, \;\; i = 1, 2 \tag{3}$$

globally asymptotically stable (GAS) under arbitrary switching.

Provide the LMIs that would solve this problem using a common quadratic Lyapunov function of the form $V(x_k) = x_k^\top P x_k$, $k \in \mathbb{N}$ and give numerical values of K_i, $i = 1, 2$ and the corresponding Lyapunov function such that (3) is indeed GAS under arbitrary switching.

(b) Now we consider the system (2) including the output y_k. Design an observer of the form

$$\begin{aligned} \hat{x}_{k+1} &= A_{\sigma_k} \hat{x}_k + B_{\sigma_k} u_k + L_{\sigma_k}(y_k - \hat{y}_k), \\ \hat{y}_k &= C_{\sigma_k} \hat{x}_k, \end{aligned} \tag{4}$$

that asymptotically recovers the state of the SLS (2). Derive LMIs to design the observer gains L_1 and L_2 using a *mode-dependent quadratic Lyapunov function* (depending on both e_k and σ_k) such that $\lim_{k\to\infty} e_k = 0$ is guaranteed, where e_k is the estimation error $e_k := x_k - \hat{x}_k$, $k \in \mathbb{N}$. Provide numerical values of L_1 and L_2 and the corresponding Lyapunov function given by P (corresponding to the error dynamics) for the particular SLS (2) under study.

(c) Take now the observer that you constructed in subproblem (b) that provides an estimate $\hat{x}_k$ for the state x_k of the original switched linear system (2) ($\lim_{k\to\infty} e_k = 0$) and use the corresponding estimate $\hat{x}_k$ in the state feedback controller using the so-called *certainty equivalence controller*

$$u_k = K_{\sigma_k} \hat{x}_k = K_{\sigma_k} x_k - K_{\sigma_k} e_k \tag{5}$$

for $k \in \mathbb{N}$. Derive a model for the closed-loop system consisting of the observer of subproblem (b), the state feedback controller (5) using the estimated state and the original switched linear system (2) (assuming that we know the active mode). Take the state variable $\begin{pmatrix} x_k \\ e_k \end{pmatrix}$ for the closed-loop system, where $e_k = x_k - \hat{x}_k$ is, as before, the state estimation error at time $k \in \mathbb{N}$.

(d) Using the numerical values for K_1, K_2, L_1, and L_2 calculated in subproblems (a) and (b), show that the closed-loop system (as determined in the previous subproblem) consisting of the observer-based controller and the original switched linear system (2) is GAS under arbitrary switching (and knowing the mode) by constructing a *mode-dependent quadratic Lyapunov function* using MATLAB. Explain how you construct such a Lyapunov function and provide numerical values for this Lyapunov function.

Problem 3: Split-path nonlinear integrator.
Consider the linear system given in state-space form by

$$\mathcal{P}:\begin{cases}\dot{x}_p = A_p x_p + B_p(u+d)\\ e = -C_p x_p + D_p r,\end{cases} \tag{6}$$

where

$$A = \begin{bmatrix}-0.2 & 0\\ 1 & 0\end{bmatrix}, \qquad B_p = \begin{bmatrix}1\\0\end{bmatrix}, \qquad C_p = \begin{bmatrix}1 & 0.1\end{bmatrix}, \qquad D_p = 1, \tag{7}$$

and where $r:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ and $d:\mathbb{R}_{\geqslant 0}\to\mathbb{R}$ are a reference signal and an exogenous input disturbance signal, respectively.

In this exercise, we are going to control the plant by a specific hybrid controller known as the *split-path nonlinear* (SPAN) filter, which was originally proposed by Foster in 1966[1], but without having proper tools at that time for stability analysis. The key philosophy underlying a SPAN filter is to enable independent tuning of gain and phase characteristics of a signal by making use of a smart switching strategy, and with that potentially overcome limitations of linear control imposed by Bode's gain-phase relationship. As shown in Fig. 1, the input signal

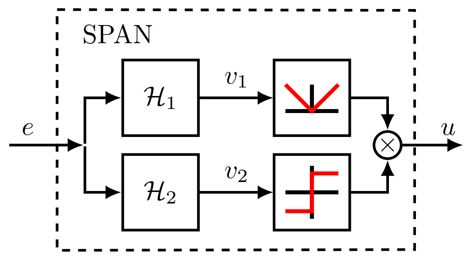

Figure 1: SPAN filter.

$e := r - y$ of a SPAN filter is split into two branches. The output of the filter is formed by multiplying the output of the two branches, *i.e.*, "$\otimes$" denotes multiplication in Fig. 1. The upper branch, referred to as the *magnitude branch*, retains all magnitude information (gain of the linear filter $\mathcal{H}_1$) and removes all sign information, *i.e.*, it outputs $|v_1|$. The lower branch on the other hand, referred to as the *sign branch*, retains only sign information (phase of the filter $\mathcal{H}_2$) and removes all magnitude information, *i.e.*, it outputs ± 1 depending on the sign of v_2, *i.e.*,

$$\operatorname{sign}(v_2)\in\begin{cases}\{1\}, & \text{if } v_2>0,\\ \{-1,1\}, & \text{if } v_2=0,\\ \{-1\}, & \text{if } v_2<0.\end{cases}$$

Suppose we design the SPAN filter with

$$\mathcal{H}_1:\begin{cases}\dot{x}_1 = a x_1 + e\\ v_1 = x_1,\end{cases} \qquad \text{and} \qquad \mathcal{H}_2 = 1, \tag{8}$$

with $a\in\mathbb{R}$. That is, we consider a (weak) *integrator* with the sign of its output equal to the sign of the input.

(a) Write the closed-loop system resulting from the feedback interconnection of the plant $\mathcal{P}$ and the SPAN filter with $\mathcal{H}_n$, $n=1,2$, given in (8) as a piecewise linear system of the form

$$\dot{x} = A_i x + B_i w \quad \text{if } (x,w)\in\mathcal{X}_i, \quad i\in\{1,2\}, \tag{9}$$

where $w = [r,d]^\top$ the vector of external input signals.

HINT: note that the output of the SPAN filter (see Fig. 1) can be written as $u(t)=v_1(t)$ if $v_1(t)v_2(t)\geq 0$ and $u(t)=-v_1(t)$ if $v_1(t)v_2(t)\leqslant 0$.

[1] W. C. Foster, D. L. Gieseking, and W. K. Waymeyer, "A Nonlinear Filter for Independent Gain and Phase (With Applications)," Journal of Basic Engineering, vol. 88, no. 2. ASME International, pp. 457–462, Jun. 01, 1966.

(b) Suppose for the time-being that $d(t) = 0$, $t \in \mathbb{R}_{\geqslant 0}$. Simulate the resulting closed-loop system with $a = 0$ when subject to a step-input $r(t) = 1$ for $t \in \mathbb{R}_{\geqslant 0}$ and $r(t) = 0$, otherwise. Compare the step-response to the step-response of a linear equivalent system, *i.e.*, the plant $\mathcal{P}$ in closed-loop with a linear time-invariant integrator. Can you (intuitively) explain the benefits of a SPAN integrator?

(c) Suppose for the time-being that $r(t) = d(t) = 0$, $t \in \mathbb{R}_{\geqslant 0}$. Provide the least conservative LMI conditions that guarantee the quadratic function $V(x) = x^\top P x$ to be a quadratic Lyapunov function for the closed-loop system. Include as many relaxations as you can to obtain the least conservative stability test. Try to solve your LMIs numerically for a range of $a \in [-3, 3]$, and indicate for which values of a the LMIs are feasible.

(d) Now suppose that $r = 0$ and $d \neq 0$. Provide conditions in the form of LMIs based on a quadratic storage function $W(x) = x^\top P x$ for computing an upper-bound on the $\mathcal{L}_2$-gain from the disturbance d to the output e, *i.e.*, provide an upper-bound on the ratio

$$\gamma := \sup_{d(t) \neq 0} \frac{\|e\|_2}{\|d\|_2}, \tag{10}$$

where $\|x\|_2 = \sqrt{\int_0^\infty x(t)^\top x(t) dt}$. Solve your LMIs numerically for the range $a \in [-3, 3]$ and provide a plot of the computed upper-bound on $\mathcal{L}_2$ as a function of a. Can you give an interpretation of the $\mathcal{L}_2$-gain as a performance measure for this particular system?

(**BONUS**) Can you come up with a procedure to refine the conditions in question (d) even further and provide a more accurate estimate of γ? Explain your procedure and provide your conditions in terms of LMIs. You do not have to solve the LMIs numerically.