Technical Report for Description-Similarity Rules

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Proof for Theorem 1 Consider the event ξ defined by

$$\xi = \{j \in \{1, ..., K\}, |\frac{1}{n_k}\sum_{s=1}^{n_k} X_s - f_j| \leq \frac{1}{2}\Delta_{K+1-k}\} \tag{1}$$

By Hoeffding's Inequality and an union bound, the probability of the complementary event $\overline{\xi}$ can be bounded as follows

$$\begin{aligned} P(\overline{\xi}) &\leq \sum_{j=1}^{K}\sum_{k=1}^{K-1} P(|\frac{1}{n_k}\sum_{s=1}^{n_k} X_s - f_j| \leq \frac{1}{2}\Delta_{K+1-k}) \\ &\leq \sum_{j=1}^{K}\sum_{k=1}^{K-1} 2exp(2n_k(\Delta_{K+1-k})/2)^2) \\ &\leq 2K^2 exp(-\frac{n-K}{2\overline{log}K * H}) \end{aligned} \tag{2}$$

where the last inequality comes from the fact that

$$\begin{aligned} & n_k(\Delta_{K+1-k})^2 \\ \geq & \frac{n-K}{\overline{log}(K)(K+1-H)(\Delta_{K+1-k})^{-2}} \\ & \geq \frac{n-K}{\overline{log}(K) * H} \end{aligned} \tag{3}$$

Thus, it suffices to show that on the event ξ, the algorithm does not make any error. We prove this by induction on k. Let $k \geq 1$. Assume the algorithm makes no error in all previous $k-1$ stages, that is no bad arm $\mu_i < \theta$ has been accepted and no good arm $\mu_i \geq \theta$ has been rejected. Note that event ξ implies that at the end of stage k, all empirical means are within $\frac{1}{2}(\Delta_{K+1-k})^{-2}$ of the respective true means.

Let $A_k = \{a_1, ..., a_{K+1-k}\}$ be the set of active arms during phase k. We order the a_i's such that $\mu_{a_1} > \mu_{a_2} > ... > \mu_{a_{K+1-k}}$. Denote $m' = m(k)$ for the number of arms that are left to find in phase k. The assumption that no error occurs in the first $k-1$ stages implies that

$$a_1, a_2, ..., a_{m'} \in \{1, ..., m\} \tag{4}$$

and

$$a_{m'+1}, ..., a_{K+1-k} \in \{m+1, ..., K\} \tag{5}$$

If an error is made at stage k, it can be one of the following two types:

1. The algorithm accepts a_j at stage k for some $k \geq m' + 1$
2. The algorithm rejects a_j at stage k for some $j \leq m'$

Denote $\sigma = \sigma_k$ for the bijection (from $\{1, ..., K+1-k\}$ to A_k) such that $\overline{\mu}_{\sigma(1),n_k} \geq \overline{\mu}_{\sigma(2),n_k} \geq ... \geq \overline{\mu}_{\sigma(K+1-k),n_k}$. Suppose Type 1 error occurs. Then $a_j = \sigma(1)$ since if algorithm accepts, it must accept the empirical best arm. Furthermore we also have

$$\overline{\mu}_{a_j,n_k} - \theta \geq \theta - \overline{\mu}_{\sigma(K+1-k),n_k} \tag{6}$$

since otherwise the algorithm would rather reject arm $\sigma(K+1-k)$. The condition $a_j = \sigma(1)$ and the event ξ implies that

$$\begin{aligned} \overline{\mu}_{a_j,n_k} &\geq \overline{\mu}_{a_j,n_k}, \\ \mu_{a_j} + \frac{1}{2}(\Delta_{K+1-k}) &\geq \mu_{a_1} - \frac{1}{2}(\Delta_{K+1-k}), \\ (\Delta_{K+1-k}) &\geq \mu_{a_1} - \mu_{a_j} \geq \mu_{a_1} - \theta \end{aligned} \tag{7}$$

We then look at the condition (16). In the event of ξ, for all $i \leq m'$ we have

$$\begin{aligned} \overline{\mu}_{a_j,n_k} &\geq \mu_{a_j} - \frac{1}{2}\Delta_{(K+1-k)} \\ &\geq \mu_{a_{m'}} - \frac{1}{2}\Delta_{(K+1-k)} \\ &\geq \theta - \frac{1}{2}\Delta_{(K+1-k)} \end{aligned} \tag{8}$$

On the other hand, $\overline{\mu}_{\sigma(K+1-k),n_k} \leq \overline{\mu}_{a_{K+1-k},n_k} \leq \overline{\mu}_{a_{K+1-k},n_k} + \frac{1}{2}\Delta_{(K+1-k)}$. Therefore, using those two observations and (16) we deduce

$$\begin{aligned} (\mu_{a_j} + \frac{1}{2}\Delta_{(K+1-k)}) - \theta &\geq \theta - (\mu_{a_{K+1-k}} + \frac{1}{2}\Delta_{(K+1-k)}), \\ \Delta_{(K+1-k)} &\geq 2\theta - \mu_{a_j} - \mu_{a_{K+1-k}} > \theta - \mu_{a_{K+1-k}} \end{aligned} \tag{9}$$

Thus so far we proved that if there is a Type 1 error, then

$$\Delta_{(K+1-k)} > max(\mu_{a_1} - \theta, \theta - \mu_{a_{K+1-k}}) \tag{10}$$

But at stage k, only k-1 arms have been accepted or rejected, thus $\Delta_{(K+1-k)} \leq max(\mu_{a_1} - \theta, \theta - \mu_{a_{K+1-k}})$. By contradiction, we conclude that Type 1 error does not occur.

Suppose Type 2 error occurs. The reasoning is symmetric to Type 1. This completes the induction and consequently the proof of the theorem. □

Proof for Theorem 2 Consider the events ξ_d for each pre-trained MAB defined by

$$\begin{aligned}
\xi_{d_1} &= \{j \in \{1,...,K\}, |\frac{1}{n_k}\sum_{s=1}^{n_k} X_{s,d_1} - f_{j,d_1}| \le \frac{1}{2}\Delta_{K+1-k}\} \\
\xi_{d_2} &= \{j \in \{1,...,K\}, |\frac{1}{n_k}\sum_{s=1}^{n_k} X_{s,d_2} - f_{j,d_2}| \le \frac{1}{2}\Delta_{K+1-k}\} \\
&\qquad ... \\
\xi_{d_{|D|}} &= \{j \in \{1,...,K\}, |\frac{1}{n_k}\sum_{s=1}^{n_k} X_{s,d_{|D|}} - f_{j,d_{|D|}}| \le \frac{1}{2}\Delta_{K+1-k}\}
\end{aligned} \tag{11}$$

Also, consider the event ξ defined by

$$\xi = \{j \in \{1,...,K\}, |\frac{1}{n_k}\sum_{s=1}^{n_k} X_s - f_j| \le \frac{1}{2}\Delta_{K+1-k}\} \tag{12}$$

where f_j is defined as:

$$\begin{aligned}
f_j = \frac{\sum_{sim_j \in t, A' \in D} cos < \vec{D}, \vec{D'} > * p(t, A')}{\sum_{sim_j \in t, A' \in D} cos < \vec{D}, \vec{D'} >} \\
= \frac{\sum_d cos < \vec{D}, \vec{d} > * p(t, d)}{\sum_d cos < \vec{D}, \vec{d} >} \\
= \frac{\sum_d cos < \vec{D}, \vec{d} > * f_{j,d}}{\sum_d cos < \vec{D}, \vec{d} >}
\end{aligned} \tag{13}$$

Setting $w_d = cos < \vec{D}, \vec{d} >$, we can rewrite (26) as:

$$\begin{aligned}
\xi = \{1 \le j \le K, |\frac{1}{n_k}\sum_{s=1}^{n_k} \frac{\sum_d w_d * X_{s,d}}{\sum_{d\in D} w_d} - \frac{\sum_d w_d * f_{j,d}}{\sum_{d \in D} w_d}| \le \frac{1}{2}\Delta_{K+1-k}\} \\
= \{1 \le j \le K, |\frac{1}{n_k}\sum_{s=1}^{n_k}\sum_d w_d * X_{s,d} - \sum_d w_d * f_{j,d}| \le \frac{1}{2}\sum_{d\in D} w_d \Delta_{K+1-k}\}
\end{aligned} \tag{14}$$

Suppose (25) is true. Using the absolute value inequality, for any $1 \le j \le K$, we have:

$$\begin{aligned}
&|\frac{1}{n_k}\sum_{s=1}^{n_k}\sum_{d} w_d * X_{s,d} - \sum_{d} w_d * f_{j,d}| \\
&= |\sum_{d}(\frac{1}{n_k}\sum_{s=1}^{n_k} w_d * X_{s,d} - w_d * f_{j,d})| \\
&\leq \sum_{d} w_d * |\frac{1}{n_k}\sum_{s=1}^{n_k} X_{s,d} - f_{j,d}| \\
&\leq \frac{1}{2}\sum_{d\in D} w_d \Delta_{K+1-k}
\end{aligned} \tag{15}$$

This means when $\xi_{d_1} \cap ... \cap \xi_{d_{|D|}}$ is true, ξ has to hold true regardless of w_d. Its contrapositive implies that when $\bar{\xi}$ is true, $\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}$ has to hold true regardless of w_d. By full probability law, we have

$$\begin{aligned}
P(\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}) = P(\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}|\xi) * P(\xi)+ \\
P(\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}|\bar{\xi}) * P(\bar{\xi}) \\
\geq P(\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}|\bar{\xi}) * P(\bar{\xi}) = P(\bar{\xi})
\end{aligned} \tag{16}$$

In algorithm 2, for each $MAB_i \in \{MAB_1, ..., MAB_n\}$ we have

$$P(\overline{\xi_d}) \leq 2K^2 exp(-\frac{\frac{n}{a} - K}{2\overline{log}K * H(i)}) \tag{17}$$

By union bound, we come to the conclusion that

$$\begin{aligned}
P(\bar{\xi}) \leq P(\overline{\xi_{d_1}} \cup ... \cup \overline{\xi_{d_{|D|}}}) \leq \sum_{i=1}^{a} 2K^2 exp(-\frac{\frac{n}{a} - K}{2\overline{log}K * H(i)}) \\
\leq 2aK^2 exp(-\frac{\frac{n}{a} - K}{2\overline{log}K * H(a)})
\end{aligned} \tag{18}$$

Thus, it suffices to show that, by probability of $2aK^2 exp(-\frac{\frac{n}{a}-K}{2\overline{log}K*H(a)})$, the algorithm does not make any error. □