UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 01
Fall 2023

1. **Deriving Facts from the Axioms**

    a. Let $n \in \mathbb{Z}_{>0}$ and $A_1, \dots, A_n$ be any events. Prove the **union bound**: $\Pr(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n \Pr(A_i)$.

    b. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \Pr(A_n) = \Pr(\bigcup_{i=1}^\infty A_i)$.

    [This can be viewed as a **continuity** property for probability measures.]

    c. Let $A_1, A_2, \dots$ be a sequence of events. Prove that the union bound holds for countably many events: $\Pr(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \Pr(A_i)$.

2. **Superhero Basketball**

   Superman and Captain America are playing a game of basketball. At the end of the game, Captain America scored n points and Superman scored m points, where $n > m$ are positive integers. Supposing that each basket counts for exactly one point, what is the probability that after the start of the game (when they are initially tied), Captain America was always *strictly* ahead of Superman?

   (Assume that all sequences of baskets which result in the final score of n baskets for Captain America and m baskets for Superman are equally likely.)

   Hint: Think about symmetry. Is it more likely that Superman scored the first point and there was a tie at some point in the game, or Captain America scored the first point and there was a tie at some point in the game?

3. **Independence and Pairwise Independence**

A collection of events $\{A_i\}_{i \in I}$ is said to be *pairwise independent* if $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all distinct indices $i \neq j$.

You flip a fair coin 99 times, where the result of each flip is independent of all other flips. For $i = 1, \ldots, 99$, let A_i be the event that the ith flip comes up heads. Let B be the event that in total, an *odd* number of heads are seen. Show that the events $A_1, \ldots, A_{99}, B$ are pairwise independent but *not* independent.

4. **The Probabilistic Method**

We introduce a proof technique — the *probabilistic method*. If we wish to show that there exists an element with property A in a set $\mathcal{X}$, it suffices to show that there exists a probability distribution p over $\mathcal{X}$ such that under p, the probability assigned to elements with property A is greater than 0.

(Why does this work? If there is no element with property A, then there cannot possibly exist a p that assigns a positive probability to elements with property A, because we require that $p(\varnothing) = 0$.) Such a proof method is nonconstructive, meaning that it doesn't provide a method for finding such an element, yet it demonstrates the element exists.

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest colored red. Show that no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

Hint: If we sample an inscribed cube uniformly at random among all possible inscribed cubes, what is (an upper bound on) the probability that the sampled cube has at least one blue vertex?

5. **Joint Occurrence**

You know that at least one of the events A_i, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. n is an integer ≥ 2. Show that if the probability of occurrence of any single event is p, and the probability of joint occurrence of any two distinct events is q, we have $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n(n-1)}$.

6. **Suspicious Game**

   You are playing a card game with your friend in which you take turns picking a card from a deck. (Assume that you never run out of cards.) If you draw one of the special *bullet* cards, then you lose the game. Unfortunately, you do not know the contents of the deck. Your friend claims that $\frac{1}{3}$ of the deck is filled with bullet cards. However, you don't fully trust your friend: you believe he is lying with probability $\frac{1}{4}$. Assume that if your friend is lying, then the opposite is true: $\frac{2}{3}$ of the deck is filled with bullet cards!

   What is the probability that you win the game if you go first?