$$
\begin{aligned}
\Delta\varepsilon &= -\frac{\beta v_*^{5/3}}{10}\left[5(\rho_2-\rho_1)^2-\rho_2^2+\rho_1^2\right]+[H_*-H(v_*)]\,\Delta m(v_*)\\
&= \frac{\beta v_*^{5/3}}{10}\left[-5(\rho_2-\rho_1)^2+\rho_2^2-\rho_1^2+5\rho_1(\rho_2-\rho_1)\right]\\
&= \frac{3\beta v_*^{5/3}}{5}\,(\rho_2-\rho_1)\left(\frac{3}{2}\,\rho_1-\rho_2\right).
\end{aligned}
\tag{12.101}
$$

Since $\rho_2 > \rho_1$, the stability condition $\Delta\varepsilon > 0$ reduces to $\rho_2 < (3/2)\,\rho_1$. This conclusion was obtained in [638, 893] by a more complicated method. Note that the derivation of the condition (12.101) by direct evaluation of the energy in Lagrangian coordinates failed because the nonintegral terms in (12.86) become of a lower order of smallness and thereby predominant for the result.

12.5 Dynamic Stabilization of NonSpherical Bodies Against Unlimited Collapse

The dynamic stability of spherical stars is determined by an average adiabatic power

$$
\gamma = \frac{\partial \log P}{\partial \log \rho}|_S.
$$

For a density distribution $\rho = \rho_0\varphi(m/M)$, the star in Newtonian gravity is stable against dynamical collapse when

$$
\int_0^R \left(\gamma - \frac{4}{3}\right) P\,\frac{\mathrm{d}m}{\varphi(m/M)} > 0,
$$

see (12.39). This approximate criterion becomes exact for adiabatic stars with constant γ. The collapse of a spherical star can be stopped only by a stiffening of the equation of state, such as neutron star formation at the late stages of evolution, or formation of a fully ionized stellar core with $\gamma = 5/3$ at the collapse of clouds during star formation. Without a stiffening, a spherical star in Newtonian theory would collapse into a point with an infinite density (singularity).

In the presence of a rotation, a star becomes more dynamically stable against collapse. Because of the more rapid increase of the centrifugal force during contraction, in comparison with the Newtonian gravitational force, the collapse of a rotating star will always be stopped at finite density by centrifugal forces. It is shown in [142] that deviation from spherical symmetry in a nonrotating star with zero angular momentum leads to a similar stabilization, and a nonspherical star without dissipative processes will never reach a singularity. Therefore, the collapse to a singularity is connected with a secular type of instability, even without rotation. This conclusion is based on calculations of the dynamical behaviour of a nonspherical, nonrotating star after its loss of linear stability, at the nonlinear stages. An approximate system of dynamic equations derived in [140, 141] is used, describing three degrees of freedom of a uniform self-gravitating compressible ellipsoidal body. The development of instability leads to the formation of a regularly or chaotically oscillating body, in which dynamical motion prevents the formation of the singularity. The regions of chaotic and regular pulsations are found by constructing a Poincaré diagram for different values of the initial eccentricity and initial entropy. The calculations have been done for spheroidal figures with $\gamma = 4/3$ and $\gamma = 6/5$.

12.5.1 Equations of Motion

Let us consider 3-axis ellipsoid with semi-axes $a \neq b \neq c$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{12.102}$$

and uniform density ρ. A mass m of the uniform ellipsoid is written as (V is the volume of the ellipsoid)

$$m = \rho V = \frac{4\pi}{3} \rho abc. \tag{12.103}$$

Let us assume a linear dependence of velocities on coordinates

$$\upsilon_x = \frac{\dot{a}x}{a}, \quad \upsilon_y = \frac{\dot{b}y}{b}, \quad \upsilon_z = \frac{\dot{c}z}{c}. \tag{12.104}$$

The gravitational energy of the uniform ellipsoid is defined as [615]:

$$U_{\mathrm{g}} = -\frac{3Gm^2}{10} \int\limits_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \tag{12.105}$$

The equation of state $P = K\rho^\gamma$ is considered here, with $\gamma = 4/3$. A spherical star with $\gamma = 4/3$ collapses to singularity at small enough K, and deviations from a spherical form prevent formation of any singularity. For $\gamma = 4/3$, the thermal energy of the ellipsoid is $E_{\mathrm{th}} \sim V^{-1/3} \sim (abc)^{-1/3}$, and the value

$$\varepsilon = E_{\mathrm{th}}(abc)^{1/3} = 3\left(\frac{3m}{4\pi}\right)^{1/3} K$$

remains constant in time. A Lagrange function of the ellipsoid is written as

$$L = U_{\mathrm{kin}} - U_{\mathrm{pot}}, \quad U_{\mathrm{pot}} = U_{\mathrm{g}} + E_{\mathrm{th}}, \tag{12.106}$$

$$U_{\mathrm{kin}} = \frac{1}{2}\rho \int\limits_V \left(\upsilon_x^2 + \upsilon_y^2 + \upsilon_z^2\right) \mathrm{d}v = \frac{m}{10}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2), \tag{12.107}$$

$$E_{\mathrm{th}} = \frac{\varepsilon}{(abc)^{1/3}}. \tag{12.108}$$

Equations of motion describing behavior of three semiaxes (a, b, c) is obtained from the Lagrange function (12.106) in the form

$$\ddot{a} = -\frac{3Gm}{2}\, a \int_0^\infty \frac{\mathrm{d}u}{(a^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{a}\,\frac{\varepsilon}{(abc)^{1/3}}, \tag{12.109}$$

$$\ddot{b} = -\frac{3Gm}{2}\, b \int_0^\infty \frac{\mathrm{d}u}{(b^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{b}\,\frac{\varepsilon}{(abc)^{1/3}}, \tag{12.110}$$

$$\ddot{c} = -\frac{3Gm}{2}\, c \int_0^\infty \frac{\mathrm{d}u}{(c^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{c}\,\frac{\varepsilon}{(abc)^{1/3}}, \tag{12.111}$$

$$\Delta^2 = (a^2+u)(b^2+u)(c^2+u).$$

12.5.2 Dimensionless Equations

To obtain a numerical solution of equations, they are written in nondimensional variables

$$\tilde{t} = \frac{t}{t_0},\ \tilde{a} = \frac{a}{a_0},\ \tilde{b} = \frac{b}{a_0},\ \tilde{c} = \frac{c}{a_0},$$
$$\tilde{m} = \frac{m}{m_0},\ \tilde{\rho} = \frac{\rho}{\rho_0},\ \tilde{U} = \frac{U}{U_0},\ \tilde{E}_{\rm th} = \frac{E_{\rm th}}{U_0},\ \tilde{\varepsilon} = \frac{\varepsilon}{\varepsilon_0}.$$

The scaling parameters $t_0,\ a_0,\ m_0,\ \rho_0,\ U_0,\ and\, \varepsilon_0$ are connected by the following relations:

$$t_0^2 = \frac{a_0^3}{Gm_0},\ U_0 = \frac{Gm_0^2}{a_0},\ \rho_0 = \frac{m_0}{a_0^3},\ \varepsilon_0 = U_0 a_0. \tag{12.112}$$

The system of non-dimensional equations is:

$$\ddot{a} = -\frac{3m}{2}\, a \int_0^\infty \frac{\mathrm{d}u}{(a^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{a}\,\frac{\varepsilon}{(abc)^{1/3}}, \tag{12.113}$$

$$\ddot{b} = -\frac{3m}{2}\, b \int_0^\infty \frac{\mathrm{d}u}{(b^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{b}\,\frac{\varepsilon}{(abc)^{1/3}}, \tag{12.114}$$

$$\ddot{c} = -\frac{3m}{2}\,c\int_0^\infty \frac{\mathrm{d}u}{(c^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{c}\,\frac{\varepsilon}{(abc)^{1/3}},$$

$$\Delta^2 = (a^2+u)(b^2+u)(c^2+u). \tag{12.115}$$

In equations (12.113)–(12.115) only nondimensional variables are used, and "tilde" sign is omitted for simplicity in this section. The nondimensional Hamiltonian (or nondimensional total energy) is:

$$H = U_{\text{kin}} + U_{\text{g}} + E_{\text{th}} = \frac{m}{10}\,(\dot{a}^2+\dot{b}^2+\dot{c}^2)$$
$$-\frac{3m^2}{10}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} + \frac{\varepsilon}{(abc)^{1/3}}. \tag{12.116}$$

In the case of a sphere ($a = b = c$, $\dot{a} = \dot{b} = \dot{c}$), the nondimensional Hamiltonian and nondimensional equations of motion reduce to:

$$H = \frac{3}{10}\,m\dot{a}^2 - \frac{3}{5a}\left(m^2 - \frac{5}{3}\,\varepsilon\right), \tag{12.117}$$

$$\ddot{a} = -\frac{1}{ma^2}\left(m^2 - \frac{5}{3}\,\varepsilon\right). \tag{12.118}$$

As follows from (12.117) and (12.118) for the given mass, there is only one equilibrium value of ε

$$\varepsilon_{\text{eq}} = \frac{3m^2}{5}, \tag{12.119}$$

at which the spherical star has zero total energy, and it may have an arbitrary radius. For smaller $\varepsilon < \varepsilon_{\text{eq}}$, the sphere should contract to singularity, and for $\varepsilon > \varepsilon_{\text{eq}}$ there will be a total disruption of the star with an expansion to infinity. The equations of motion have been solved numerically in [142] for a spheroid with $a = b \neq c$. Using (12.113)–(12.115), these are written for the oblate spheroid with $k = c/a < 1$ as

$$\ddot{a} = \frac{3}{2}\,\frac{m}{a^2(1-k^2)}\left[k - \frac{\arccos k}{\sqrt{1-k^2}}\right] + \frac{5}{3m}\,\frac{1}{a}\,\frac{\varepsilon}{(a^2c)^{1/3}}, \tag{12.120}$$

$$\ddot{c} = -3\frac{m}{a^2(1-k^2)}\left[1 - \frac{k\arccos k}{\sqrt{1-k^2}}\right] + \frac{5}{3m}\,\frac{1}{c}\,\frac{\varepsilon}{(a^2c)^{1/3}}; \tag{12.121}$$

and for the prolate spheroid with $k = c/a > 1$ as

$$\ddot{a} = -\frac{3}{2}\,\frac{m}{a^2(k^2-1)}\left[k - \frac{\cosh^{-1} k}{\sqrt{k^2-1}}\right] + \frac{5}{3m}\,\frac{1}{a}\,\frac{\varepsilon}{(a^2c)^{1/3}}, \tag{12.122}$$

$$\ddot{c} = 3\frac{m}{a^2(k^2-1)}\left[1 - \frac{k\cosh^{-1}k}{\sqrt{k^2-1}}\right] + \frac{5}{3m}\frac{1}{c}\frac{\varepsilon}{(a^2c)^{1/3}}. \tag{12.123}$$

It is convenient to introduce variables

$$\varepsilon_* = \frac{5}{3}\frac{\varepsilon}{m^2}, \quad t_* = t\sqrt{m}. \tag{12.124}$$

In these variables, the (12.122)–(12.123), (12.118) are written as (omitting the subscript "*")

$$\ddot{a} = \frac{3}{2a^2(1-k^2)}\left[k - \frac{\arccos k}{\sqrt{1-k^2}}\right] + \frac{1}{a}\frac{\varepsilon}{(a^2c)^{1/3}}, \tag{12.125}$$

$$\ddot{c} = -\frac{3}{a^2(1-k^2)}\left[1 - \frac{k\arccos k}{\sqrt{1-k^2}}\right] + \frac{1}{c}\frac{\varepsilon}{(a^2c)^{1/3}} \tag{12.126}$$

for the oblate spheroid with $k = c/a < 1$, and as

$$\ddot{a} = -\frac{3}{2a^2(k^2-1)}\left[k - \frac{\cosh^{-1}k}{\sqrt{k^2-1}}\right] + \frac{1}{a}\frac{\varepsilon}{(a^2c)^{1/3}}, \tag{12.127}$$

$$\ddot{c} = \frac{3}{a^2(k^2-1)}\left[1 - \frac{k\cosh^{-1}k}{\sqrt{k^2-1}}\right] + \frac{1}{c}\frac{\varepsilon}{(a^2c)^{1/3}} \tag{12.128}$$

for the prolate spheroid with $k = c/a > 1$. For the sphere, where the equilibrium corresponds to $\varepsilon_{\mathrm{eq}} = 1$, the (12.122)–(12.123) and (12.118) are reduced to

$$\ddot{a} = -\frac{1-\varepsilon}{a^2}. \tag{12.129}$$

Near the spherical shape, we should use expansions around $k = 1$, which leads to equations of motion valid for both oblate and prolate cases

$$\ddot{a} = -\frac{1-\varepsilon}{a^2} + \left(\frac{\varepsilon}{3} + \frac{3}{5}\right)\frac{1-k}{a^2},$$

$$\ddot{c} = -\frac{1-\varepsilon}{a^2} + \left(\frac{4\varepsilon}{3} - \frac{4}{5}\right)\frac{1-k}{a^2}. \tag{12.130}$$

In these variables, the total energy is written as

$$H_* = \frac{H}{m^2}, \tag{12.131}$$

and omitting "*", we have

$$
\begin{aligned}
H &= \frac{\dot a^2}{5} + \frac{\dot c^2}{10} - \frac{3}{5a}\frac{\arccos k}{\sqrt{1-k^2}} + \frac{3}{5}\frac{\varepsilon}{(a^2c)^{1/3}}, \quad \text{(oblate)}\\
H &= \frac{\dot a^2}{5} + \frac{\dot c^2}{10} - \frac{3}{5a}\frac{\cosh^{-1} k}{\sqrt{k^2-1}} + \frac{3}{5}\frac{\varepsilon}{(a^2c)^{1/3}}, \quad \text{(prolate)}\\
H &= \frac{3\dot a^2}{10} - \frac{3}{5a}(1-\varepsilon), \quad \text{(sphere)}\\
H &= \frac{\dot a^2}{5} + \frac{\dot c^2}{10} - \frac{3}{5a}\left(1+\frac{\delta}{3}+\frac{2\delta^2}{15}\right) + \frac{3\varepsilon}{5a}\left(1+\frac{\delta}{3}+\frac{2\delta^2}{9}\right),\\
&\delta = 1-k, \ \text{(around the sphere)}, \ |\delta| \ll 1.
\end{aligned}
\tag{12.132}
$$

Solution of the system of (12.122)–(12.123) was performed in [142] for initial conditions at $t = 0$: $\dot c_0 = 0$, different values of initial $a_0, \ \dot a_0, \ k_0$, and different values of the constant parameter ε. Evidently, at $k_0 = 1$, $\dot a_0 = 0$, $\varepsilon < 1$, we have the spherical collapse to singularity. No singularity was reached at $k_0 \neq 1$, for any $\varepsilon > 0$. At $\varepsilon = 0$, a weak singularity is reached during formation of a pancake with infinite volume density and finite gravitational force. At $\varepsilon > 0$, the behavior depends on the value of the total energy H: at $H > 0$, we obtain a full disruption of the body, and at $H < 0$ the oscillatory regime is established at any value of $\varepsilon < 1$. At $\varepsilon \geq 1$, the total energy of spheroid is positive, $H > 0$, determining the full disruption. The case with $H = 0$ is described separately.

At $H < 0$, the type of oscillatory regime depends on initial conditions and may be represented either by regular periodic oscillations or by chaotic behavior. Examples of two types of such oscillations are represented in Figs. 12.6 and 12.7 (regular, periodic), and in Fig. 12.8 (chaotic). For rigorous separation between these kinds of oscillations, a method, developed by Poincaré [637], was used.

12.5.3 Numerical Results for the Case $H = 0$

In the case of a sphere with zero initial radial velocity, we have an equilibrium state with $\varepsilon = \varepsilon_{eq} = 1$. Let us consider the case of $\dot a \neq 0$ and $\varepsilon < 1$. It follows from (12.132) that for non-zero initial velocity $\dot a$, entropy function ε is strictly less than unity. The fate of such sphere depends on the sigh of $\dot a$. At $\dot a < 0$, the star collapses to singularity, and at $\dot a > 0$ there total disruption happens, with zero velocity at infinity. For the initial spheroid with $a = b \neq c$, and zero initial velocities $\dot a = \dot c = 0$, the entropy function is also less than 1. In this case, the value of the entropy function

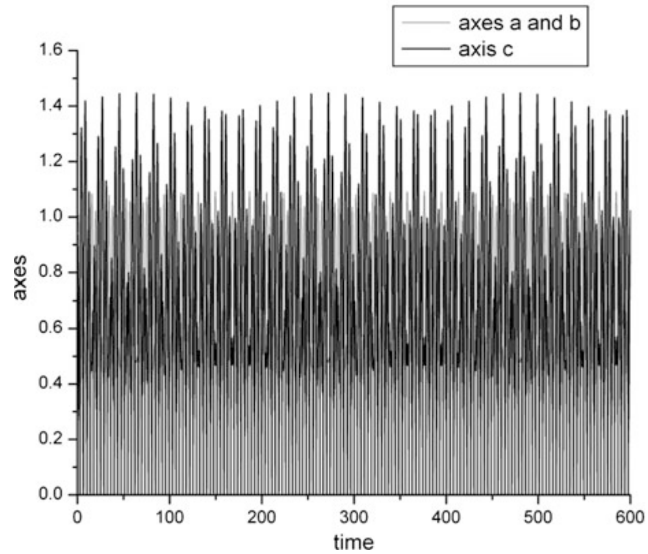


Fig. 12.6 Example of regular motion of spheroid with $\gamma = 4/3$, $H = -1/5$ and $\varepsilon = 2/3$. This motion corresponds to full line on the Poincaré map in Fig. 12.9, from [142]

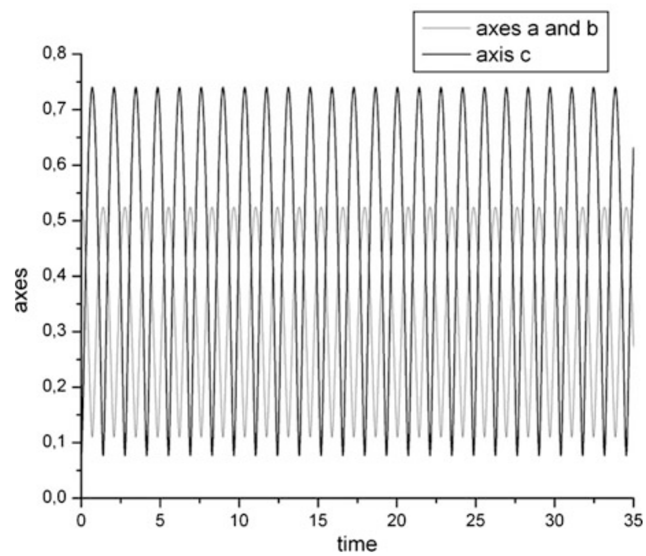


Fig. 12.7 Example of regular motion of spheroid with $\gamma = 4/3$, $H = -1/5$ and $\varepsilon = 2/3$. This motion corresponds to the point inside the regular region on the Poincaré map in Fig. 12.9, from [142]

ε is uniquely determined by the deformation $\delta = (a - c)/a$. This dependence can be found explicitly from (12.132); for small δ, we obtain

$$\varepsilon = 1 - \frac{4}{45}\,\delta^2.$$

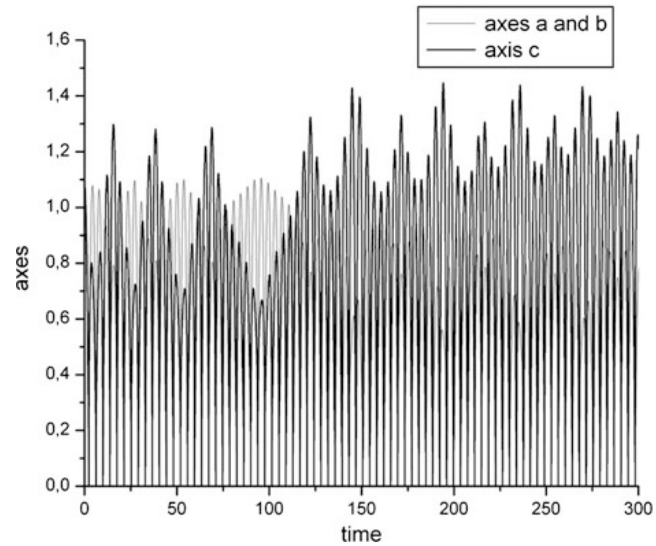


Fig. 12.8 Example of chaotic motion of spheroid with $\gamma = 4/3$, $H = -1/5$ and $\varepsilon = 2/3$. This motion corresponds to grey points on the Poincaré map in Fig. 12.9, from [142]

Thus, even in the case of zero initial velocities, ε is less than unity for a spheroid and reaches unity only for a sphere. For the same ε, the deformation δ of a zero energy body at rest has two states with

$$\delta = \pm\frac{3}{2}\sqrt{5(1-\varepsilon)},$$

corresponding to oblate and prolate spheroids. If we set non-zero initial velocities in the case of a spheroid, the entropy function will be even less. The calculation of the motion in this case leads finally to the expansion of the body, with an oscillating behaviour of δ around zero (oblateprolate oscillations). This takes place under the conditions of both initial contraction or initial expansion (Fig. 12.8).

12.5.4 Poincaré Section

To investigate regular and chaotic dynamics, the method of Poincaré section was used [637], and the Poincare map for different values of the total energy H were obtained. Let us consider a spheroid with semi-axes $a = b \neq c$. This system has two degrees of freedom. Therefore, in this case the phase space is four-dimensional: a, $\dot{a}$, c, $\dot{c}$. If we choose a value of the Hamiltonian H_0, we fix a three-dimensional energy surface $H(a, \dot{a}, c, \dot{c}) = H_0$. During the integration of (12.122) and (12.123), which preserve the constant H, we fix moments t_i, when $\dot{c} = 0$. At these moments, there are only two independent values (i.e. a and $\dot{a}$) because the value of c is determined uniquely from the relation for the Hamiltonian at constant H. At each moment t_i, we put a dot on the plane $(a, \dot{a})$.

For the same values of H and ε, we solve the equations of motion (12.122) and (12.123) at initial $\dot{c} = 0$, and different a and $\dot{a}$. For each integration, we put the points on the plane $(a, \dot{a})$ at the moments t_i. These points are the intersection points of the trajectories on the three-dimensional energy surface with a two-dimensional plane $\dot{c} = 0$, called the Poincaré section.

For each fixed combination of ε and H, we obtain the Poincaré map, represented in Figs. 12.9–12.13. The condition $\dot{c} = 0$ splits into two cases, of a minimum and of a maximum of c. The Poincaré maps are drawn separately, either for the minimum or for the maximum of c, and both maps lead to identical results. The regular oscillations are represented by closed lines on the Poincaré map, and chaotic behaviour fills regions of finite square with dots. These regions are separated from the regions of regular oscillations by a separatrix line.

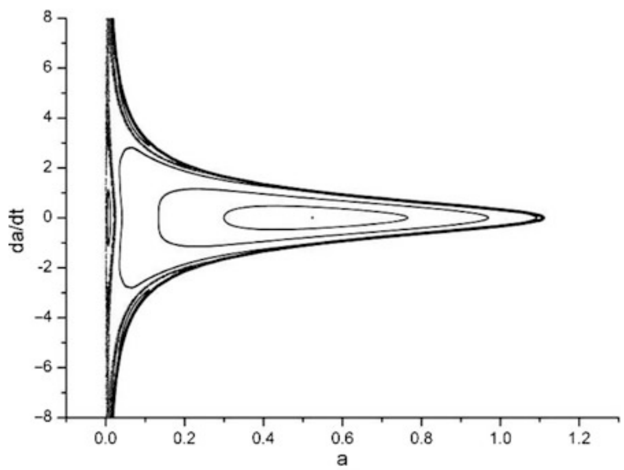


Fig. 12.9 The Poincaré map for five regular and two chaotic trajectories in case of $\gamma = 4/3$, $H = -1/5$, and $\varepsilon = 2/3$. The $(a, \dot{a})$ values are taken in the minimum of c. Full *black line* is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, represented in Fig. 12.7

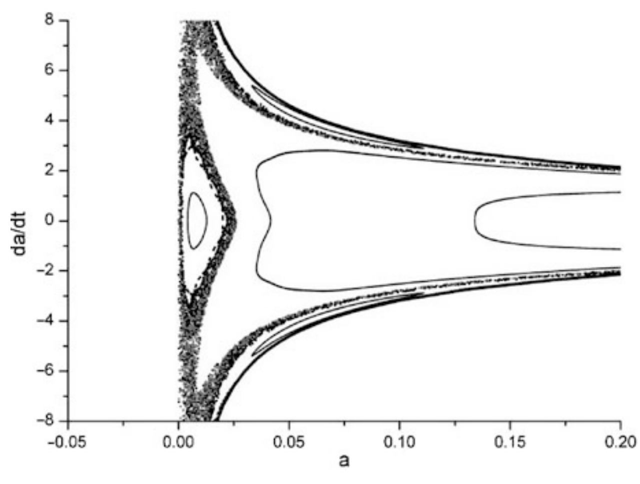
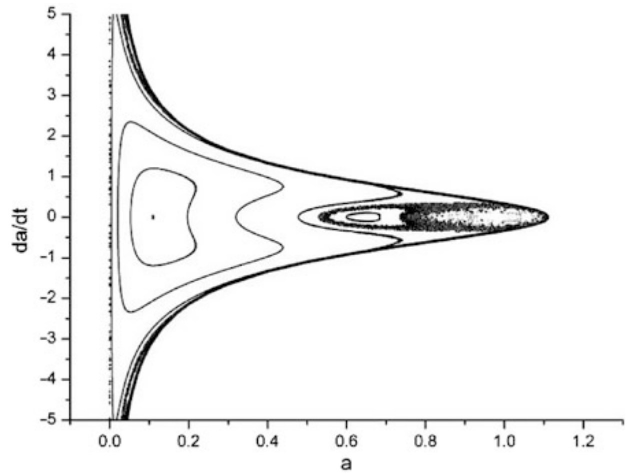


Fig. 12.10 Zoom of Fig. dfig4

Fig. 12.11 The Poincaré map for five regular and two chaotic trajectories in case of $\gamma = 4/3$, $H = -1/5$, and $\varepsilon = 2/3$. The $(a, \dot{a})$ values are taken in the maximum of c. Full black line is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, represented in Fig. 12.7



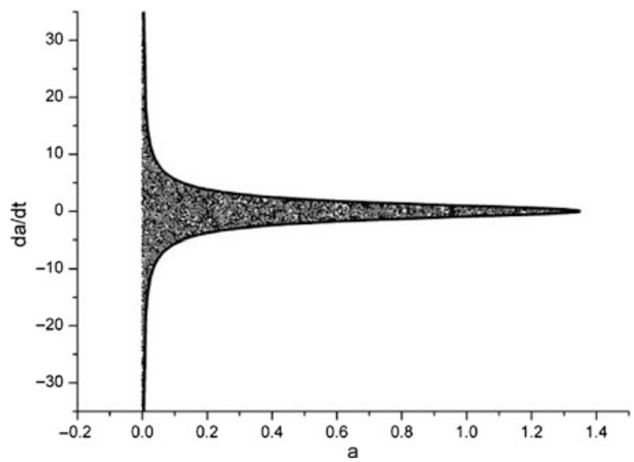


Fig. 12.12 The Poincaré map for two chaotic trajectories in case of $\gamma = 4/3$, $H = -1/2$, $\varepsilon = 1/6$. The $(a, \dot{a})$ values are taken in the minimum of c. Full *black line* is the bounding curve

12.5.4.1 The Bounding Curve

Actually, the variables a and $\dot{a}$ cannot occupy the whole plane $(a, \dot{a})$: $0 < a < \infty, -\infty < \dot{a} < +\infty$. Let us obtain a curve bounding the area of the values a and $\dot{a}$. Let a function $\Phi(a, \dot{a}, c)$ in the variables (12.124) and (12.131) be:

$$\Phi(a, \dot{a}, c) = \frac{1}{10}(\dot{a}^2 + \dot{b}^2 + \dot{c}^2) - \\ - \frac{3}{10}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} + \frac{3\varepsilon}{5(abc)^{1/3}} - H \quad (12.133)$$

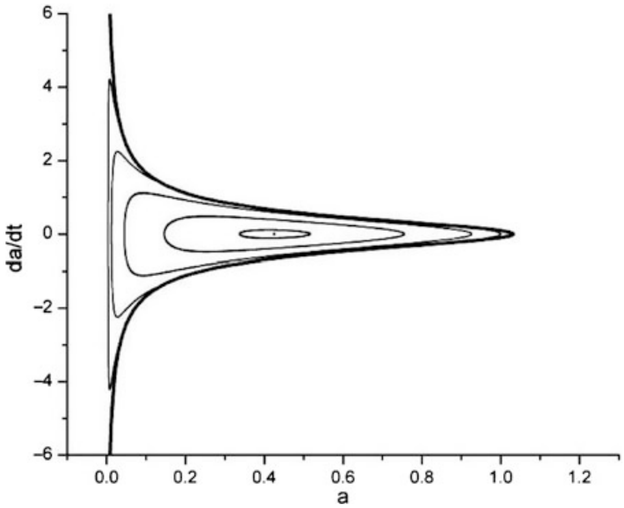


Fig. 12.13 The Poincaré map for six regular trajectories in the case of $\gamma = 4/3$, $H = -3/50$, and $\varepsilon = 9/10$. The $(a, \dot{a})$ values are taken in the minimum of c. Full *black line* is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, similar to those represented in Fig. 12.7

at $a = b$, $\dot{a} = \dot{b}$, $\dot{c} = 0$, and fixed value of ε and H. The equation for the bounding curve $f(a, \dot{a}) = 0$ at given ε and H is determined from the following system of equation:

$$\Phi(a, \dot{a}, c) = 0, \quad \frac{\partial}{\partial c}\Phi(a, \dot{a}, c) = 0. \tag{12.134}$$

For $c < a$, this system of equations has the forms

$$\frac{1}{5}\dot{a}^2 - \frac{3}{5}\frac{\arccos(c/a)}{\sqrt{a^2 - c^2}} + \frac{3\varepsilon}{5(a^2c)^{1/3}} - H = 0, \tag{12.135}$$

$$-\frac{3}{5}c\frac{\arccos(c/a)}{(a^2 - c^2)^{3/2}} + \frac{3}{5}\frac{1}{a^2 - c^2} - \frac{1}{3}\frac{1}{c}\frac{3\varepsilon}{5(a^2c)^{1/3}} = 0. \tag{12.136}$$

This system is solved numerically. The second equation does not depend on $\dot{a}$. We set a and obtain a corresponding value of c from the second equation. Then we substitute a and c into the first equation and find $\dot{a}$. Thus, we obtain the point $(a, \dot{a})$. Changing a, we obtain numerically the curve $f(a, \dot{a}) = 0$. This bounding curve is shown in Figs. 12.9–12.13 by a bold line.

12.6 General Picture

The main result following from above calculations is the indication of the degenerate nature of the formation of a singularity in unstable Newtonian self-gravitating gaseous bodies. Only pure spherical models can collapse to singularity, but any kind