CONTROLLABILITY TO SYSTEMS OF QUASILINEAR WAVE EQUATIONS WITH FEWER CONTROLS

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ABSTRACT. This study addresses challenges in solving exact boundary controllability problems of quasilinear wave equations with Dirichlet type boundary conditions satisfying some kind of Kalman rank conditions. Utilizing techniques such as characteristic method and a specially designed linearization iteration scheme, we developed controls that use fewer boundary controls than the classical theory.

1. Introduction

Consider the controllability for system of the quasilinear wave equations below

$$\begin{cases} U_{tt} - a^2\left(U, U_t, U_x\right) U_{xx} = 0, & (t,x) \in \mathbb{R}^+ \times [0,L]\,, \\ t = 0 : (U, U_t) = \left(\varphi\left(x\right), \psi\left(x\right)\right), & x \in [0,L]\,, \\ x = L : U = 0, & t \in \mathbb{R}^+, \\ x = 0 : U_t - D\left(U, U_t, U_x\right) U_x = H\left(t\right), & t \in \mathbb{R}^+, \\ t = T : (U, U_t) = \left(\Phi\left(x\right), \Psi\left(x\right)\right), & x \in [0,L]\,. \end{cases} \tag{1.1}$$

where

$$U = \left(u_1, \dots, u_n\right)^T (t,x) \in C^2.$$

$a\left(U, U_t, U_x\right) > 0$ is the C^1 uniform wave speed. $D\left(u_1, u_2, u_3\right) = \left(d_{ij}\left(u_1, u_2, u_3\right)\right)_{i,j=1}^n$ is an $n \times n$ matrix with C^1 regularity. Part of the components of $H(t) = \left(h_1(t), \dots, h_n(t)\right)^T$ would be chosen as boundary controls. We denote $D^{\pm}\left(U, U_t, U_x\right) = aI_n \pm D$, which we request are invertible. We also request that

$$\|\left(\varphi\left(x\right), \psi\left(x\right)\right)\|_{(C^2[0,L])^n \times (C^1[0,L])^n} < \varepsilon,$$
$$\|\left(\varphi\left(x\right), \Psi\left(x\right)\right)\|_{(C^2[0,L])^n \times (C^1[0,L])^n} < \varepsilon.$$

where $0 < \varepsilon \ll 1$. Without loss of generosity, we set $a(0,0,0) = 1$, $H(t) = \left(h_1(t), \dots, h_p(t), 0, \dots, 0\right)^T$.Now we try to reduce the wave equation to a hyperbolic equation group. Denote

$$\tilde{U} = D^+(0,0,0)U,$$
$$\tilde{V}^{\pm} = \partial_t \tilde{U} \mp \tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \partial_x \tilde{U}.$$

where

$$\tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) = a\left(D^+(0,0,0)^{-1}U, D^+(0,0,0)^{-1}U_t, D^+(0,0,0)^{-1}U_x\right),$$

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$$\tilde{D}^{\pm}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)=D^{\pm}\left(D^{+}(0,0,0)^{-1}U,D^{+}(0,0,0)^{-1}U_t,D^{+}(0,0,0)^{-1}U_x\right).$$

The original equations can be changed into

$$\begin{cases}\partial_t\tilde{V}^{\pm}\pm\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\partial_x\tilde{V}^{\pm}=\mp\partial_x\tilde{U}\left(\partial_t\pm\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\partial_x\right)\left(\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\right),\\ x=L:\tilde{V}^{-}=-\tilde{V}^{+},\\ x=0:\tilde{D}^{-}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\tilde{D}^{+}(0,0,0)^{-1}\tilde{V}^{-}=-\tilde{D}^{+}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\tilde{D}^{+}(0,0,0)^{-1}\tilde{V}^{+}+\tilde{H}(t).\end{cases}\tag{1.2}$$

where $\tilde{H}(t)=2aH(t)$. Then the boundary condition can be written as

$$\begin{aligned}x=0:\widetilde{V}^{+}=B_0\widetilde{V}^{-}+\widetilde{H}(t)&-\left(\widetilde{D}^{-}\left(\widetilde{U},\widetilde{U}_t,\widetilde{U}_x\right)-\widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+}\\ &-\left(\widetilde{D}^{-}\left(\widetilde{U},\widetilde{U}_t,\widetilde{U}_x\right)-\widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-}.\end{aligned}$$

where the coupling matrix

$$B_0=-\left(a(0,0,0)I_n-D(0,0,0)\right)\left(a(0,0,0)I_n+D(0,0,0)\right)^{-1}.$$

Define the controlling matrix as

$$C=\begin{pmatrix}I_p & 0\\ 0 & 0\end{pmatrix},$$

then the Kalman type condition as

$$\operatorname{Rank}\left[C,B_0C,\ldots,B_0^{k-1}C\right]=n,\tag{1.3}$$

where k is a positive integer.

The following theorem can be obtained.

Theorem 1. *Under Assumption* (1.3)*, if $T>kL$, there exists a small ε_0 and a set of C^1 smooth control functions $H_i(t)$, $i=1,\cdots,p$, such that for any $\varepsilon<\varepsilon_0$ and any $\Phi(x)$, $\varphi(x)$ which are C^2 smooth and satisfy the C^2 compatibility condition and $\Psi(x)$, $\psi(x)$ which are C^1 smooth and satisfy the C^1 compatibility condition, there exists a unique C^2 smooth solution $U(t,x)$ to the control problem* (1.1)*.*

The proof of Theorem 1 requires the following proposition.

Proposition 1.1. *Taking $U^{(0)}\equiv 0$, consider the following iterative problem*

$$\begin{cases}\partial_t\widetilde{V}^{\pm(m+1)}\pm a^{(m)}\partial_x\widetilde{V}^{\pm(m+1)}=\mp\partial_x\widetilde{U}^{(m)}\left(\partial_t\pm a^{(m)}\partial_x\right)\left(a^{(m)}\right),\\ x=L:\widetilde{V}^{-(m+1)}=-\widetilde{V}^{+(m+1)},\quad t\in\mathbb{R}^{+},\\ x=0:\widetilde{V}^{-(m+1)}=B_0\widetilde{V}^{+(m+1)}+\widetilde{H}^{(m+1)}(t)\\ +\left(\widetilde{D}^{+}\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-(m)}\\ -\left(\widetilde{D}^{-}\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+(m)}.\\ t=0:\left(U^{(m+1)},U_t^{(m+1)}\right)=\left(\varphi\left(x\right),\psi\left(x\right)\right),\\ t=T:\left(U^{(m+1)},U_t^{(m+1)}\right)=\left(\Phi\left(x\right),\Psi\left(x\right)\right).\end{cases}\tag{1.4}$$

where

$$a^{(m)}=a\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right).$$

One can get uniform bounds

$$\left\|U^{(m)}\right\|_{C^2}\le C_P\varepsilon,\tag{1.5}$$

$$\left\|H^{(m)}\right\|_{C^1}\le C_P\varepsilon.\tag{1.6}$$

as well as Cauchy sequence

$$\left\|U^{(m)}-U^{(m-1)}\right\|_{C^1}\le C_P\varepsilon\alpha^{m-1},\tag{1.7}$$

where α is a constant satisfying $0<\alpha<1$ and C_P is a large constant. The sequence $\{H^{(m)}\}_{m=1}^{\infty}$ is also a Cauchy sequence in C^0 space, i.e.,

$$\left\|H^{(m)}-H^{(m-1)}\right\|_{C^0}\leq C_P\varepsilon\alpha^{m-1}. \tag{1.8}$$

with additional equi-continuity by estimates on modulus of continuity

$$\omega\left(\eta\left|\frac{\partial H^{(m)}}{\partial t}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial t^2}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial x^2}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m)}}{\partial t\partial x}\right.\right)\leq\Omega\left(\eta\right). \tag{1.9}$$

Where $\Omega(\eta)$ is a monotonically increasing continuous function satisfying $\Omega(0)=0$.

Theorem 1 follows after obtaining Proposition 1.1.In fact, The sequence $\{U^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence in C^1 space, and thus converges to some C^1 function U uniformly, applying Arzelà-Ascoli theorem, we know that there exists a subsequence of $\{U^{(m)}\}_{m=1}^{\infty}$,which converges uniformly in C^2 space. Thus we know that the whole original sequence$\{U^{(m)}\}_{m=1}^{\infty}$ converges to U in C^2 space. Therefore, U is C^2 smooth and satisfies the control condition, where $U^(m+1)$ for each step can be obtained using the Dirichlet boundary condition as

$$U^{(m+1)}(t,x)=-\int_x^L\partial_x U^{(m+1)}\mathrm{d}x,$$

where

$$\partial_x U^{(m+1)}=\frac{V^{-(m+1)}-V^{+(m+1)}}{2a\left(U^{(m)},U_t^{(m)},U_x^{(m)}\right)}.$$

2. Proof of Theorem 2.1

We need to introduce a lemma about the linear equation.Consider the following linear equation system

$$\begin{cases}\partial_t u_i-a(t,x)\partial_x u_i=f_i^{(1)}(t,x)(\partial_t-a(t,x)\partial_x)f_i^{(2)}(t,x),i=1,\cdots,r,\\ \partial_t u_i+a(t,x)\partial_x u_i=f_i^{(1)}(t,x)(\partial_t+a(t,x)\partial_x)f_i^{(2)}(t,x),i=r+1,\cdots,r+s.\end{cases} \tag{2.1}$$

where $a(t,x)$ is a C^1 function satisfying that $\|a(t,x)-1\|_{C^1}\leq\varepsilon_0.\varepsilon_0$ is small enough. $f_i^{(\kappa)}(t,x),i=1,\cdots,r+s,\kappa=1,2$ are C^1 functions. The boundary conditions are given as follows:

$$\begin{cases}x=0:\ u_i=\sum_{j=1}^r a_{ij}u_j+g_i(t)+h_i\left(t\right),\ i=r+1,\cdots,n,\\ x=L:\ u_i=\sum_{j=r+1}^n a_{ij}u_j+g_i(t)+h_i\left(t\right),\ i=1,\cdots,r.\end{cases} \tag{2.2}$$

where $g_i(t)$ is a C^1 function, $h_i(t)$ is a C^1 controlling function. The matrix $A=(a_{ij})\,,\ i=1,\cdots,n,\ j=1,\cdots,n$ is a C^1 matrix. $u(t,x)$ satisfies

$$u\left(0,x\right)=\varphi\left(x\right). \tag{2.3}$$

The matrix B is defined as follows:

$$B=\begin{pmatrix}I_{r_1}&&&\\&O_{r_2}&&\\&&I_{s_1}&\\&&&O_{s_2}\end{pmatrix},$$

where $r_1+r_2=r,\ s_1+s_2=s,r+s=n$. The Kalman rank condition is given as follows:

$$\text{Rank}\left[B,AB,\ldots,A^{k-1}B\right]=n, \tag{2.4}$$

where k is a positive integer. The following lemma is the main result of this section.

Lemma 2.1. *Under the conditions of* (2.4)*, if $T > kL$, then there exists a set of C^1 smooth control functions $h_i(t)$, $i = 1, \cdots, r_1, r+1, \cdots, r+s_1$, such that for any C^1 smooth functions $g_i(t), f_i(t,x)$, $i = 1, \cdots, n$ and any C^1 smooth functions $\varphi(x), \psi(x)$, which satisfy the C^1 compatibility conditions, there exists a unique C^1 smooth solution $u(t,x)$ to the control problem* (2.1)-(2.3) *satisfying*

$$u(T,x) = \psi(x). \tag{2.5}$$

Proof. Without loss of generality, we can assume that k is odd, $T - kL < \frac{L}{4}$ and $\varphi(x) = 0$,or take any set of $\tilde{h}_i(t)$ that satisfies the compatibility conditions at $(0,0)$ and $(0,L)$ and consider the equations of $\tilde{u}(t,x)$ satisfying $\tilde{u}(0,x) = \varphi(x)$ below:

$$\begin{cases} \partial_t \tilde{u}_i - a(t,x)\partial_x \tilde{u}_i = \chi(t) f_i^{(1)}(t,x)(\partial_t - a(t,x)\partial_x) f_i^{(2)}(t,x), i = 1, \cdots, r, \\ \partial_t \tilde{u}_i + a(t,x)\partial_x \tilde{u}_i = \chi(t) f_i^{(1)}(t,x)(\partial_t + a(t,x)\partial_x) f_i^{(2)}(t,x), i = r+1, \cdots, r+s. \end{cases} \tag{2.6}$$

where $\chi(t)$ is a smooth function that satisfies $\chi(t) = 1$ for $t \in [0, \frac{L}{2}]$ and $\chi(t) = 0$ for $t \in [T - \frac{L}{2}, T]$. The boundary conditions are given as follows:

$$\begin{cases} x = 0: \ \tilde{u}_i = \sum_{j=1}^{r} a_{ij}\tilde{u}_j + \chi(t) g_i(t) + \tilde{h}_i(t), \ i = r+1, \cdots, n, \\ x = L: \ \tilde{u}_i = \sum_{j=r+1}^{n} a_{ij}\tilde{u}_j + \chi(t) g_i(t) + \tilde{h}_i(t), \ i = 1, \cdots, r. \end{cases} \tag{2.7}$$

Then we can get that $\tilde{u}(T,x)$ satisfies the C^1 compatibility conditions at $(T,0)$ and (T,L) and consider the following controllability problem:

$$\begin{cases} \partial_t u_i - a(t,x)\partial_x u_i = (1-\chi(t)) f_i^{(1)}(t,x)(\partial_t - a(t,x)\partial_x) f_i^{(2)}(t,x), i = 1, \cdots, r, \\ \partial_t u_i + a(t,x)\partial_x u_i = (1-\chi(t)) f_i^{(1)}(t,x)(\partial_t + a(t,x)\partial_x) f_i^{(2)}(t,x), i = r+1, \cdots, r+s. \end{cases} \tag{2.8}$$

where the boundary conditions are given as follows:

$$\begin{cases} x = 0: \ u_i = \sum_{j=1}^{r} a_{ij} u_j + (1-\chi(t)) g_i(t) + h_i(t), \ i = r+1, \cdots, n, \\ x = L: \ u_i = \sum_{j=r+1}^{n} a_{ij} u_j + (1-\chi(t)) g_i(t) + h_i(t), \ i = 1, \cdots, r. \end{cases} \tag{2.9}$$

The initial and final conditions are given as follows:

$$u(0,x) = 0, \quad u(T,x) = \psi(x) - \tilde{u}(T,x). \tag{2.10}$$

Finally, the two sets of control functions and solutions obtained can be added separately. We determine $h_i(t)$ in the following order: In the following we first study the equations of characteristic lines. Let

$$b(t,x) = \frac{1}{a(t,x)}.$$

The characteristic lines of the system (2.1) are given as follows:

$$\begin{cases} \frac{\mathrm{d}\tilde{t}_p}{\mathrm{d}x} = -b\left(\tilde{t}_p, x\right), \\ \tilde{t}_p(0; x_0) = t_{p-1}(0; x_0), \ p = 2, \cdots, k+1, \\ \tilde{t}_1(x_0; x_0) = T, \end{cases} \tag{2.11}$$

$$\begin{cases} \frac{\mathrm{d}t_p}{\mathrm{d}x} = b\left(t_p, x\right), \\ t_p(L; x_0) = \tilde{t}_{p-1}(L; x_0), \ p = 2, \cdots, k+1, \\ t_1(x_0; x_0) = T, \end{cases} \tag{2.12}$$

thereby obtaining the following equations:

$$\begin{aligned} t_1(x; x_0) &= T + \int_{x_0}^{x} b\left(t_1(s; x_0), s\right) \mathrm{d}s, \\ t_p(x; x_0) &= t_{p-1}(L; x_0) + \int_{L}^{x} b\left(t_p(s; x_0), s\right) \mathrm{d}s, p = 2, \cdots, k+1, \\ \partial_{x_0} t_1(x; x_0) &= -b(T, x_0) + \int_{x_0}^{x} \partial_t b\left(t_1(s; x_0), s\right) \partial_{x_0} t_1(s; x_0) \mathrm{d}s, \\ \partial_{x_0} t_p(x; x_0) &= \partial_{x_0} t_{p-1}(L; x_0) + \int_{L}^{x} \partial_t b\left(t_p(s; x_0), s\right) \partial_{x_0} t_p(s; x_0) \mathrm{d}s, p = 2, \cdots, k+1. \end{aligned} \tag{2.13}$$

Denote $z_p(x;x_0)=\partial_{x_0}t_p\left(x;x_0\right)$, which conduces that

$$\frac{\mathrm{d}}{\mathrm{d}x}z_p\left(x;x_0\right)=\partial_t b_p\left(t\left(x;x_0\right),x\right)z_p\left(x;x_0\right).$$

Then we can get the following equations:

$$\begin{aligned}
z_1\left(x;x_0\right)&=-b\left(T,x_0\right)\exp\left(\int_{x_0}^{x}\partial_t b\left(t_1\left(s;x_0\right),s\right)\mathrm{d}s\right),\\
z_p\left(x;x_0\right)&=\partial_{x_0}t_{p-1}\left(L;x_0\right)\exp\left(\int_{L}^{x}\partial_t b\left(t_p\left(s;x_0\right),s\right)\mathrm{d}s\right).
\end{aligned}\tag{2.14}$$

Similarly one can obtain that

$$\begin{aligned}
\tilde{t}_1\left(x;x_0\right)&=T-\int_{x_0}^{x}b\left(\tilde{t}_1\left(s;x_0\right),s\right)\mathrm{d}s\\
\tilde{t}_p\left(x;x_0\right)&=t_{p-1}\left(0;x_0\right)-\int_{0}^{x}b\left(\tilde{t}_p\left(s;x_0\right),s\right)\mathrm{d}s,l=2,\cdots,k+1,\\
\tilde{z}_1\left(x;x_0\right)&=b\left(T,x_0\right)\exp\left(-\int_{x_0}^{x}\partial_t b\left(\tilde{t}_1\left(s;x_0\right),s\right)\mathrm{d}s\right),\\
\tilde{z}_p\left(x;x_0\right)&=\partial_{x_0}t_{p-1}\left(0;x_0\right)\exp\left(\int_{0}^{x}\partial_t b\left(\tilde{t}_p\left(s;x_0\right),s\right)\mathrm{d}s\right).
\end{aligned}$$

Therefore

$$\begin{cases}
t_1(x;0)=T,\\
t_{2m+1}(x;0)=t_{2m}(x;0),\\
t_{2m}(x;L)=t_{2m-1}(x;L),
\end{cases}\tag{2.15}$$

$$\begin{cases}
\tilde{t}_1(x;L)=T,\\
\tilde{t}_{2m+1}(x;L)=\tilde{t}_{2m}(x;L),\\
\tilde{t}_{2m}(x;0)=\tilde{t}_{2m-1}(x;0).
\end{cases}\tag{2.16}$$

and

$$\begin{cases}
\partial_{x_0}t_1(x;0)=-b(T,0),\\
\partial_{x_0}t_{2m+1}(x;0)=-\partial_{x_0}t_{2m}(x;0),\\
\partial_{x_0}t_{2m}(x;L)=-\partial_{x_0}t_{2m-1}(x;L),
\end{cases}\tag{2.17}$$

$$\begin{cases}
\partial_{x_0}\tilde{t}_1(x;L)=b(T,L),\\
\partial_{x_0}\tilde{t}_{2m+1}(x;L)=-\partial_{x_0}\tilde{t}_{2m}(x;L),\\
\partial_{x_0}\tilde{t}_{2m}(x;0)=-\partial_{x_0}\tilde{t}_{2m-1}(x;0).
\end{cases}\tag{2.18}$$

The following defines a set of vector functions:

$$H_p\left(x\right)=\left(H^-\left(\tilde{t}_p(L;x)\right),H^+\left(t_p(0;x)\right)\right)^T,$$

where

$$\begin{aligned}
H^-\left(t\right)&=\left(h_1\left(t\right),\cdots,h_{r_1}\left(t\right),0,\cdots,0\right),\\
H^+\left(t\right)&=\left(h_{r+1}\left(t\right),\cdots,h_{r+s_1}\left(t\right),0,\cdots,0\right).
\end{aligned}$$

Let

$$h_i\left(x\right)=0,\ x<0,i=1,\ldots,n.$$

Define

$$u\left(t,x\right)=\left(u^-\left(t,x\right),u^+\left(t,x\right)\right)^T,$$

where

$$\begin{aligned}
u^-\left(t,x\right)&=\left(u_1\left(t,x\right),\cdots,u_r\left(t,x\right)\right),\\
u^+\left(t,x\right)&=\left(u_{r+1}\left(t,x\right),\cdots,u_{r+s}\left(t,x\right)\right).
\end{aligned}$$

Define

$$G_p\left(x\right)=\left(G^-\left(\tilde{t}_p(L;x)\right),G^+\left(t_p(0;x)\right)\right)^T,$$

where

$$\begin{aligned}
G^-\left(t\right)&=\left(g_1\left(t\right),\cdots,g_r\left(t\right)\right),\\
G^+\left(t\right)&=\left(g_{r+1}\left(t\right),\cdots,g_{r+s}\left(t\right)\right).
\end{aligned}$$

Define

$$F_1(x) = \left(\int_x^L F^-\left(\tilde{t}_1(s;x), s\right) \mathrm{d}s, \int_0^x F^+\left(t_1(s;x), s\right) \mathrm{d}s \right)^T,$$

$$F_p(x) = \left(\int_0^L F^-\left(\tilde{t}_p(s;x), s\right) \mathrm{d}s, \int_0^L F^+\left(t_p(s;x), s\right) \mathrm{d}s \right)^T, p = 2, \cdots, k+1,$$

where

$$F^-(t,x) = \left(f_1^{(1)}(t,x)(b(t,x)\partial_t - \partial_x) f_1^{(2)}(t,x), \cdots, f_r^{(1)}(t,x)(b(t,x)\partial_t - \partial_x) f_r^{(2)}(t,x) \right),$$

$$F^+(t,x) = \left(f_{r+1}^{(1)}(t,x)(b(t,x)\partial_t - \partial_x) f_1^{(2)}(t,x), \cdots, f_{r+s}^{(1)}(t,x)(b(t,x)\partial_t - \partial_x) f_{r+s}^{(2)}(t,x) \right),$$

If a solution $u(t,x)$ satisfying the requirement exists,for any $x \in [0,L]$, we can get the following equations with characteristic lines method:

$$\begin{aligned}
\psi(x) &= u(T,x) \\
&= (u^-(\tilde{t}_1(L;x),L), u^+(t_1(0;x),0))^T + F_1(x) \\
&= A(u^-(t_1(0;x),0), u^+(\tilde{t}_2(L;x),L))^T + BH_1(x) + G_1(x) + F_1(x) \\
&= A(u^-(t_2,L), u^+(\tilde{t}_2,0))^T + AF_2(x) + BH_1(x) + G_1(x) + F_1(x) \\
&\cdots \\
&= A^k B(H_{k+1}(x) + G_{k+1}(x) + F_{k+1}(x)) + \cdots + B(H_1(x) + G_1(x) + F_1(x)), \ x \in [0,L].
\end{aligned} \tag{2.19}$$

Using this equation, we first determine the value of the function and derivative of $h_i(t_p(0;0)), h_i(t_p(0;L)), h_i(t_p(L$ $1, \cdots, n; p = 1, \cdots, k+1)$. Consider the vector $\tilde{G}(x) = \sum_{l=0}^k A^l G_{l+1}(x)$.

$$\tilde{G}^-(L) = G_1^-(L) + A_1 G_2^+(L) + A_1 A_2 G_3^-(L) + \cdots + A_1 A_2 \cdots A_1 G_{k+1}^+(L),$$

$$\tilde{G}^+(L) = G_1^+(L) + A_2 G_2^-(L) + A_2 A_1 G_3^+(L) + \cdots + A_2 A_1 \cdots A_2 G_{k+1}^-(L).$$

Recognizing (2.15)-(2.16) and $T - kL < \frac{L}{4}$, we can get

$$G_{k+1}^-(L) = 0,$$

$$G_m^+(L) = G_{m+1}^+(L), \ misodd, \ m = 1, \cdots, k,$$

$$G_m^+(L) = G_{m+1}^+(L), \ miseven, \ m = 1, \cdots, k.$$

Therefore,

$$\begin{aligned}
\tilde{G}^+(L) &= G_1^+(L) + A_2 G_2^-(L) + A_2 A_1 G_3^+(L) + \cdots + A_2 A_1 \cdots A_1 G_k^+(L) \\
&= G_2^+(L) + A_2 G_3^-(L) + A_2 A_1 G_4^+(L) + \cdots + A_2 A_1 \cdots A_1 G_{k+1}^+(L),
\end{aligned} \tag{2.20}$$

and thereby we can get the following equations:

$$\tilde{G}^-(L) = G_1^-(L) + A_1 \tilde{G}^+(L).$$

Similarly,

$$\tilde{G}^+(0) = G_1^+(0) + A_2 \tilde{G}^-(0).$$

Namely,

$$\begin{pmatrix} \tilde{G}^-(L) \\ \tilde{G}^+(0) \end{pmatrix} = \begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix} \begin{pmatrix} \tilde{G}^-(0) \\ \tilde{G}^+(L) \end{pmatrix} + \begin{pmatrix} g^-(T) \\ g^+(T) \end{pmatrix}. \tag{2.21}$$

Now consider the vector $\tilde{F}(x) = \sum_{l=0}^k A^l F_{l+1}(x)$.

$$\tilde{F}^-(L) = F_1^-(L) + A_1 F_2^+(L) + A_1 A_2 F_3^-(L) + \cdots + A_1 A_2 \cdots A_1 F_{k+1}^+(L),$$

$$\tilde{F}^+(L) = F_1^+(L) + A_2 F_2^-(L) + A_2 A_1 F_3^+(L) + \cdots + A_2 A_1 \cdots A_2 F_{k+1}^-(L).$$

From the defination of $\chi(t)$,we can assume that $f_i^{(j)}(t,x) = 0, t < T - \frac{L}{4}, j = 1,2, i = 1, \cdots, n$.Recognizing (2.15)-(2.16) and $T - kL < \frac{L}{4}$, we can get

$$F_1^-(L) = 0,$$

$$F^-_{k+1}(L) = 0,$$
$$F^+_m(L) = F^+_{m+1}(L),\ misodd,\ m = 1, \cdots, k,$$
$$F^+_m(L) = F^+_{m+1}(L),\ miseven,\ m = 1, \cdots, k.$$

Therefore,

$$\begin{aligned} \tilde{F}^+(L) &= F^+_1(L) + A_2F^-_2(L) + A_2A_1F^+_3(L) + \cdots + A_2A_1 \cdots A_1F^+_k(L) \\ &= F^+_2(L) + A_2F^-_3(L) + A_2A_1F^+_4(L) + \cdots + A_2A_1 \cdots A_1F^+_{k+1}(L), \end{aligned} \tag{2.22}$$

and thereby we can get the following equations:

$$\tilde{F}^-(L) = A_1\tilde{F}^+(L).$$

Similarly,

$$\tilde{F}^+(0) = A_2\tilde{F}^-(0).$$

Namely,

$$\begin{pmatrix} \tilde{F}^-(L) \\ \tilde{F}^+(0) \end{pmatrix} = \begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix} \begin{pmatrix} \tilde{F}^-(0) \\ \tilde{F}^+(L) \end{pmatrix}. \tag{2.23}$$

The C^0 compatibility condition satisfied by $\psi(x)$ is

$$\psi_i\left(0\right) = \sum_{j=1}^{r} a_{ij}\psi_j\left(0\right) + g_i(T),\ i = r + s_1 + 1, \cdots, n, \tag{2.24}$$

$$\psi_i\left(L\right) = \sum_{j=r+1}^{n} a_{ij}\psi_j\left(L\right) + g_i(T),\ i = r_1 + 1, \cdots, r. \tag{2.25}$$

So $\psi - \tilde{G} - \tilde{F}$ satisfies the conditions of Lemma 4.1. Therefore, the following equations have a solution:

$$\psi\left(0\right) = A^k\left(H_{k+1}\left(0\right) + G_{k+1}\left(0\right) + F_{k+1}\left(0\right)\right) + \cdots + \left(H_1\left(0\right) + G_1\left(0\right) + F_1\left(0\right)\right), \tag{2.26}$$

$$\psi\left(L\right) = A^k\left(H_{k+1}\left(L\right) + G_{k+1}\left(L\right) + F_{k+1}\left(L\right)\right) + \cdots + \left(H_1\left(L\right) + G_1\left(L\right) + F_1\left(L\right)\right), \tag{2.27}$$

to determine the values of h_i at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L), (i = 1, \cdots, n; p = 1, \cdots, k+1)$. Derivation of (2.19) with respect to x gives the following equations:

$$\psi'\left(x\right) = A^kB(H'_{k+1}\left(x\right) + G'_{k+1}\left(x\right) + F'_{k+1}\left(x\right)) + \cdots + B(H'_1\left(x\right) + G'_1\left(x\right) + F'_1\left(x\right)),\ x \in \left[0, L\right], \tag{2.28}$$

where

$$H'_p\left(x\right) = \left(\frac{\mathrm{d}H^-}{\mathrm{d}t}\left(\tilde{t}_p(L;x)\right)\frac{\partial\tilde{t}_p(L;x)}{\partial x}, \frac{\mathrm{d}H^+}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$G'_p\left(x\right) = \left(\frac{\mathrm{d}G^-}{\mathrm{d}t}\left(\tilde{t}_p(L;x)\right)\frac{\partial\tilde{t}_p(L;x)}{\partial x}, \frac{\mathrm{d}G^+}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$F_1^{'}\left(x\right) = \Big(-f^{(1)-}\left(b\partial_t - \partial_x\right)f^{(2)-}\left(T, x\right) + \textstyle\int_x^L \frac{\partial f^{(1)-}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(2)-}\left(\tilde{t}_1\left(s, x\right), s\right)\frac{\partial\tilde{t}_1}{\partial x}\left(s; x\right)ds - \int_x^L \frac{\partial f^{(1)-}}{\partial t}\left(b\partial_t - \right.$$

$$F_p^{'}\left(x\right) = \Big(\textstyle\int_0^L \frac{\partial f^{(1)-}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(2)-}\left(\tilde{t}_p\left(s, x\right), s\right)\frac{\partial\tilde{t}_p}{\partial x}\left(s; x\right)ds - \int_0^L \frac{\partial f^{(1)-}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(2)-}\left(\tilde{t}_p\left(s, x\right), s\right)\frac{\partial\tilde{t}_p}{\partial x}\left(s; x\right)d$$

Consider the vector $\tilde{G}'(x) = \sum_{l=0}^k A^lG'_{l+1}(x)$ and $\tilde{F}'(x) = \sum_{l=0}^k A^lF'_{l+1}(x)$.Similar to the previous case using (2.15)-(2.18), we can get the following equations:

$$\tilde{G}'_-\left(L\right) = b\left(T, L\right)g'_r\left(T\right) - A_1\tilde{G}'_+\left(L\right)$$

$$\tilde{G}'_+\left(0\right) = -b\left(T, L\right)g'_s\left(T\right) - A_2\tilde{G}'_-\left(0\right)$$

$$\tilde{F}'_-\left(L\right) + f_r^{(1)}\left(T, L\right)\partial_s f_r^{(2)}\left(T, L\right) = A_1\left(-\tilde{F}'_+\left(L\right) + f_s^{(1)}\left(T, L\right)\partial_s f_s^{(2)}\left(T, L\right)\right)$$

$$-\tilde{F}'_+\left(0\right) + f_s^{(1)}\left(T, 0\right)\partial_s f_s^{(2)}\left(T, 0\right) = A_2\left(\tilde{F}'_-\left(L\right) + f_r^{(1)}\left(T, 0\right)\partial_s f_r^{(2)}\left(T, 0\right)\right)$$

Combined with the C^1 compatibility condition satisfied by $\psi(x)$:

$$a\left(T,L\right)\psi_r'\left(L\right)+f_r^{(1)}\left(\partial_t-a\left(T,L\right)\partial_x\right)f_r^{(2)}\left(T,L\right)=A_2\left(-a\left(T,L\right)\psi_s'\left(L\right)+f_s^{(1)}\left(\partial_t+a\left(T,L\right)\partial_x\right)f_s^{(2)}\left(T,L\right)\right)+g_r'$$

$$-a\left(T,0\right)\psi_s'\left(0\right)+f_s^{(1)}\left(\partial_t+a\left(T,0\right)\partial_x\right)f_s^{(2)}\left(T,0\right)=A_2\left(a\left(T,0\right)\psi_r'\left(0\right)+f_r^{(1)}\left(\partial_t-a\left(T,0\right)\partial_x\right)f_r^{(2)}\left(T,0\right)\right)+g_s'\left(T\right)$$

we know that the vector $(\psi_r'-\tilde{G}_-'-\tilde{F}_-',-\psi_s'+\tilde{G}_+'+\tilde{F}_+')^T$ stastisfies the conditions of Lemma 4.1.Therefore, the following equations have a solution:

$$\psi'\left(0\right)=A^kB(H_{k+1}'\left(0\right)+G_{k+1}'\left(0\right)+F_{k+1}'\left(0\right))+\cdots+B(H_1'\left(0\right)+G_1'\left(0\right)+F_1'\left(0\right)),\tag{2.29}$$

$$\psi'\left(L\right)=A^kB(H_{k+1}'\left(L\right)+G_{k+1}'\left(L\right)+F_{k+1}'\left(L\right))+\cdots+B(H_1'\left(L\right)+G_1'\left(L\right)+F_1'\left(L\right)),\tag{2.30}$$

to determine the values of h_i' at $t_p(0;0),t_p(0;L),t_p(L;0),t_p(L;L),(i=1,\cdots,n;p=1,\cdots,k+1)$. Now we continue to determine the other values of $h_i(t)$.(2.19) can be written as

$$\psi\left(x\right)=A^kBH_{k+1}\left(x\right)+\left[B,AB,\ldots,A^{k-1}B\right]\hat{H}(x),\ x\in\left(0,L\right),\tag{2.31}$$

where

$$\hat{H}(x)=\left(H_1^T\left(x\right),\cdots,H_k^T\left(x\right)\right)^T.$$

From (2.4) and the implicit function existence theorem, we know that there exist n components of $\hat{H}(x)$ denoted as $H[n](x)$,s.t.

$$H\left[n\right]\left(x\right)=\mathcal{H}\left(\psi\left(x\right),H_{k+1}\left(x\right),\hat{H}\left[\hat{n}\right]\left(x\right)\right),\tag{2.32}$$

where $\hat{H}\left[\hat{n}\right]\left(x\right)$ is the other components of $\hat{H}(x)$, and $\mathcal{H}$ is a C^1 continuous function. For $H_{k+1}\left(x\right)$, we utilize the interpolation method to obtain the following equations:

$$h_r\left(t_{k+1}\left(L;x\right)\right)=p\left(L-x;h_r\left(t_{k+1}\left(L;0\right)\right),-h_r'\left(t_{k+1}\left(L;0\right)\right)\right)\tag{2.33}$$

$$h_s\left(t_{k+1}\left(0;x\right)\right)=p\left(x;h_s\left(t_{k+1}\left(0;L\right)\right),h_s'\left(t_{k+1}\left(0;L\right)\right)\right)\tag{2.34}$$

where $p(x;\alpha,\alpha'),\ x\in[0,L]$ is a polynomial determined by α,α' satisfying $p(x;\alpha,\alpha')=0,\ x\in[0,\frac{7L}{8}]$ and $p(L)=\alpha,\ p'(L)=\alpha'$ .Similarly,for the components of $\hat{H}\left[\hat{n}\right]\left(x\right)$, we can also use the interpolation method to obtain the following equations:

$$h_r\left(\tilde{t}_p\left(L;x\right)\right)=p\left(x;h_r\left(\tilde{t}_p\left(L;0\right)\right),h_r'\left(\tilde{t}_p\left(L;0\right)\right),h_r\left(\tilde{t}_p\left(L;L\right)\right),h_r'\left(\tilde{t}_p\left(L;L\right)\right)\right),p=1,\cdots,k.\tag{2.35}$$

$$h_s\left(t_p\left(0;x\right)\right)=p\left(x;h_s\left(t_p\left(0;0\right)\right),h_s'\left(t_p\left(0;0\right)\right),h_s\left(t_p\left(0;L\right)\right),h_s'\left(t_p\left(0;L\right)\right)\right),p=1,\cdots,k.\tag{2.36}$$

where $p(x;\alpha,\alpha',\beta,\beta'),\ x\in[0,L]$ is a polynomial determined by $\alpha,\alpha',\beta,\beta'$ satisfying $p(0)=\alpha,\ p'(0)=\alpha',\ p(L)=\beta,\ p'(L)=\beta'$. Finally, we use (2.32) to obtain the values of $H\left[n\right]\left(x\right),\ x\in(0,L)$.From the C^1 continuity of $\mathcal{H}$, (2.26) and (2.27), (2.29) and (2.30) and (2.4) ,we know that $h_i(t)$ is C^1 continuous in $[0,T]$. A solution $u(t,x)$ is obtained using the obtained boundary conditions and initial value conditions. Then

$$\begin{aligned}u(T,x)&=(u^-(\tilde{t}_1(L;x),L),u^+(t_1(0;x),0))^T+F_1(x)\\&=A(u^-(t_1(0;x),0),u^+(\tilde{t}_2(L;x),L))^T+BH_1(x)+G_1(x)+F_1(x)\\&=A(u^-(t_2,L),u^+(\tilde{t}_2,0))^T+AF_2(x)+BH_1(x)+G_1(x)+F_1(x)\\&\cdots\\&=A^kB(H_{k+1}\left(x\right)+G_{k+1}\left(x\right)+F_{k+1}\left(x\right))+\cdots+B(H_1\left(x\right)+G_1\left(x\right)+F_1\left(x\right)),\ x\in\left[0,L\right].\end{aligned}\tag{2.37}$$

Therefore, $u(T,x)=\psi(x)$ and thereby $h_i(t),i=1,\cdots,n$ are exactly the control function that satisfies the requirement. □

From the proof above, it is not difficult to obtain the C^1 estimate for $u(t,x)$ and $h_i(t)$, $i=1,\cdots,n$. From (2.14), we get

$$\|z_p(x;x_0)-1\|_{c^0} \le \|b-1\|_{c^1}\left(1+C\|\partial_t b\|_{c^0}\right)^p \tag{2.38}$$

Using (2.26), (2.27), (2.29), (2.30) and (2.38), we know the estimate of $h_i(t)$, $i=1,\cdots,n$ at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L)$,$(p=1,\cdots,k+1)$.

$$\|h_i(t_p(\lambda_1;\lambda_2))\|_{C^1} \le C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}), \lambda_1,\lambda_2\in\{0,L\},\ p=1,\cdots,k+1. \tag{2.39}$$

From the defination $H_{k+1}(x)$ and $\hat{H}\left[\hat{n}\right](x)$, we know that

$$\|H_{k+1}(x)\|_{C^1} \le C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L]. \tag{2.40}$$

and

$$\left\|\hat{H}\left[\hat{n}\right](x)\right\|_{C^1} \le C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L]. \tag{2.41}$$

Using (2.32), we can get the estimate of $H\left[n\right](x)$:

$$\|H\left[n\right](x)\|_{C^1} \le C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L]. \tag{2.42}$$

Namely,

$$\|h_i(t)\|_{C^1} \le C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}), t\in[0,T], i=1,\cdots,n. \tag{2.43}$$

Then we try to get the estimate of $u(t,x)$: From (2.1), we know that

$$\frac{\mathrm{d}u_s}{\mathrm{d}x}\left(t_1(x;\xi),x\right) = f_s^{(1)}\frac{\mathrm{d}f_s^{(2)}}{\mathrm{d}x}\left(t_1(x;\xi),x\right).$$

Thereby,

$$u_s\left(t_1\left(x;\xi\right),x\right) = u_s(0,\xi)+\int_\xi^x f_s^{(1)}\frac{\mathrm{d}f_s^{(2)}}{\mathrm{d}s}\left(t_1\left(s;\xi\right),s\right)\mathrm{d}s$$

Therefore,

$$\left|u_s\left(t_1(x;\xi),x\right)\right| \leqslant c\left\|\varphi\right\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\Big)$$

Similarly, we can get the estimate of $u_r\left(t_1(x;\xi),x\right)$:

$$\left|u_r\left(t_1(x;\xi),x\right)\right| \leqslant c\left\|\varphi\right\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\Big).$$

Below we make inductive assumptions that for $j=1,\cdots,p-1$, the following estimates hold:

$$\left|u\left(t_j(x;\xi),x\right)\right| \leqslant C_A\left\|\varphi\right\|_{c^1}+\left\|\psi\right\|_{c^1}+\left\|g\right\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\Big) \tag{2.44}$$

Then for $j=p$, we have

$$\frac{\mathrm{d}u_s}{\mathrm{d}x}\left(t_p(x;\xi),x\right) = f_s^{(1)}\frac{\mathrm{d}f_s^{(2)}}{\mathrm{d}x}\left(t_p(x;\xi),x\right).$$

and thereby

$$u_s\left(t_p\left(x;\xi\right),x\right) = u_s(t_p(0,\xi),0)+\int_0^x f_s^{(1)}\frac{\mathrm{d}f_s^{(2)}}{\mathrm{d}s}\left(t_p\left(s;\xi\right),s\right)\mathrm{d}s$$

Using (2.7) and the inductive assumption, we can get

$$u_s\left(t_p(0;\xi),0\right) = A_2u_r\left(\tilde{t}_{p-1}(0;\xi),0\right)+g_s+h_s.$$

and then

$$\left|u_s\left(t_p(0;\xi),0\right)\right| \leqslant C_A\left\|\varphi\right\|_{c^1}+\left\|\psi\right\|_{c^1}+\left\|g\right\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\Big)$$

Therefore, (2.44) holds for $j=p$. Multiply both sides of (2.6) by $b(t,x)$ and then derive for t,denoting w_s as the derivative of u_s with respect to t, and we have

$$\frac{\mathrm{d}}{\mathrm{d}x}w_s+\partial_t bw_s = \partial_t\left(f_s^{(1)}\frac{\mathrm{d}f^{(2)}}{\mathrm{d}x}\right)$$

Then we can get

$$w_s\left(t_1(x;\xi),x\right)=w_s(0,\xi)+\int_{\xi}^{x}\partial_t b w_s \mathrm{d}s+\int_{\xi}^{x}\partial_t f_s^{(1)}\frac{\mathrm{d}f^{(2)}}{\mathrm{d}s}\mathrm{d}s+\left.f_s^{(1)}\partial_t f_s^{(2)}\right|_{\xi}^{x}-\int_{\xi}^{x}\partial_t f_s{}^{(2)}\frac{\mathrm{d}f_s{}^{(1)}}{\mathrm{d}s}\mathrm{d}s \tag{2.45}$$

where

$$w_s(0,\xi)=-a\varphi'(\xi)+f_s^{(1)}\left(\partial_t+a\partial_x\right)f_s^{(2)}. \tag{2.46}$$

Thereby

$$|w_s(0,\xi)|\leqslant C\left(\|\varphi\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\right)$$

Then we can get the estimate of $w_s\left(t_1(x;\xi),x\right)$:

$$\left|w_s\left(t_1(x;\xi),x\right)\right|\leqslant\|\partial_t b\|_{c^0}\int_{\xi}^{x}|w_s|\,\mathrm{d}s+C\left(\|\varphi\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\right)$$

Using Gronwall's inequality, we can get

$$\left|w_s\left(t_1(x;\xi),x\right)\right|\leqslant C_{\varepsilon_0}\left(\|\varphi\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\right)$$

Below we make inductive assumptions that for $j=1,\cdots,p-1$, the following estimates hold:

$$\left|w\left(t_j(x;\xi),x\right)\right|\leqslant C_{A,\varepsilon_0}\left\|\varphi\right\|_{c^1}+\|\psi\|_{c^1}+\|g\|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\Big) \tag{2.47}$$

Then for $j=p$, we have

$$w_s\left(t_p(x;\xi),x\right)=w_s(t_p(0,\xi),0)+\int_{0}^{x}\partial_t b w_s \mathrm{d}s+\int_{0}^{x}\partial_t f_s^{(1)}\frac{\mathrm{d}f^{(2)}}{\mathrm{d}s}\mathrm{d}s+\left.f_s^{(1)}\partial_t f_s^{(2)}\right|_{0}^{x}-\int_{0}^{x}\partial_t f_s{}^{(2)}\frac{\mathrm{d}f_s{}^{(1)}}{\mathrm{d}s}\mathrm{d}s \tag{2.48}$$

where

$$w_s(t_p(0;\xi),0)=A_2 w_r\left(\tilde{t}_{p-1}(0;\xi),0\right)+g_s'+h_s'. \tag{2.49}$$

and then using (2.47) and Gronwall' inequality, we can get (2.47) holds for $j=p$. Finally, we can get the estimate of $u(t,x)$:

$$\|u(t,x)\|_{C^1}\leq C_{A,k,\|a-1\|_{C^1}}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ t\in[0,T],x\in[0,L]. \tag{2.50}$$

Below we estimate the continuity paradigm of $h_i'(t)$,$\partial_x u(t,x)$and $\partial_t u(t,x)$. For $h_i(t)\in\hat{H}\left[\hat{n}\right]$ and H_{k+1} ,where $t=t_p(\lambda;\xi)$for some $p\in 1,\cdots,k+1$, $\lambda\in\{0,L\}$, using (2.33)- (2.36) and (**??**), we have

$$\left|h_i'\left(t_1\right)-h_i'\left(t_2\right)\right|\leqslant C\left\|p\right\|_{C^2}\left(\left|x_1-x_2\right|+\omega(\left|x_1-x_2\right|\left|a(T,\cdot)\right)\right)$$

where $t_1=t_p(\lambda_1;x_1)$, $t_2=t_p(\lambda_2;x_2)$, $\lambda_1,\lambda_2\in\{0,L\}$, $x_1,x_2\in[0,L]$ and C is a constant depending on A, k and ε_0 and

$$\|p\|_{c^2}\leqslant C\|h\|_{c^1}\leqslant C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1})$$

Using (2.38), we can get

$$|x_1-x_2|\leq C_{\|a-1\|_{c_1}}\left|t_p\left(\lambda;x_1\right)-t_p\left(\lambda;x_2\right)\right|=|t_1-t_2|$$

For $h_i(t)\in H\left[n\right]$, using (2.32) and (2.42), we can get

$$\left|h_i'\left(t_1\right)-h_i'\left(t_2\right)\right|\leqslant C_A(\omega\left(|t_1-t_2|\mid\psi'\right)+\omega\left(|t_1-t_2|\mid\varphi'\right)+\omega\left(|t_1-t_2|\mid h_i'(t)\right)),h_i\in\hat{H}\left[\hat{n}\right].$$

Thereby, we can get the continuity paradigm of $h_i'(t)$:

$$\omega\left(\eta\mid h_i'(t)\right)\leqslant C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1})\left(\omega\left(\eta\mid\varphi'\right)+\omega\left(\eta\mid\psi'\right)+\omega(\eta\mid a(T,\cdot))\right)$$

Then we try to get the continuity paradigm of $\partial_x u(t,x)$ and $\partial_t u(t,x)$.We consider $\omega\left(\eta\mid\partial_t u(\cdot,x)\right)$. From (**??**) and (2.46), we know that

$$\left|w_s\left(t_1\left(x;\xi_1\right),x\right)-w_s\left(t_1\left(x;\xi_2\right),x\right)\right|\leqslant C\Bigg(\left|w_s\left(0,\xi_1\right)-w_s\left(0,\xi_2\right)\right|+\|u\|_{C^1}\,\omega\left(|\xi_1-\xi_2|\mid\partial_t b\right)+\|\partial_t b\|_{C^0}\int_0^x\omega\left(|\right.$$

where

$$\begin{aligned}|w_s(0,\xi_1)-w_s(0,\xi_2)| \leqslant \quad & C\Big(\omega\left(|\xi_1-\xi_2| \mid \varphi'\right)+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}|\xi_1-\xi_2| \\ & +\left(\|\varphi\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right)\omega\left(|\xi_1-\xi_2| \mid a\right) \\ & +\left\|f^{(1)}\right\|_{C^1}\left(\omega\left(|\xi_1-\xi_2| \mid \partial_x {f_s}^{(2)}(0,\cdot)\right)+\omega\left(|\xi_1-\xi_2| \mid \partial_x f_s^{(2)}(0,\cdot)\right)\right)\Big)\end{aligned}$$

Using Gronwall' inequality ,(2.14) and (2.38), we can get

$$|\xi_1-\xi_2| \leqslant C\left|t_1(x;\xi_1)-t_1(x;\xi_2)\right|$$

and then

$$\begin{aligned}\omega\left(|\xi_1-\xi_2| \mid w_s\left(t_1(x;\cdot),x\right)\right) \leqslant \quad & C\Big(\omega\left(|\xi_1-\xi_2| \mid \varphi'\right)+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}|\xi_1-\xi_2|+\|u\|_{C^1}\,\omega\left(|\xi_1-\xi_2| \mid \partial_t b\right) \\ & +\left(\|\varphi\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right)\omega\left(|\xi_1-\xi_2| \mid a\right)+\omega\left(|\xi_1-\xi_2| \mid \partial f^{(1)}\partial f^{(2)}\right) \\ & +\left\|f^{(1)}\right\|_{C^1}\left(\omega\left(|\xi_1-\xi_2| \mid \partial_x {f_s}^{(2)}(0,\cdot)\right)+\omega\left(|\xi_1-\xi_2| \mid \partial_x f_s^{(2)}(0,\cdot)\right)\right)\Big)\end{aligned}$$

Below we make inductive assumptions that for $j=1,\cdots,p-1$, the following estimates hold:

$$\begin{aligned}\omega\left(\eta \mid \omega\left(t_j(x;\cdot),x\right)\right) \leqslant \quad & C\Big(\omega\left(\eta \mid \varphi'\right)+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\eta+\|u\|_{C^1}\,\omega\left(\eta \mid \partial_t b\right) \\ & +\left(\|\varphi\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right)\omega\left(\eta \mid a\right)+\omega\left(\eta \mid \partial f^{(1)}\partial f^{(2)}\right) \\ & +\left\|f^{(1)}\right\|_{C^1}\left(\omega\left(\eta \mid \partial_x {f_s}^{(2)}(0,\cdot)\right)+\omega\left(\eta \mid \partial_x f_s^{(2)}(0,\cdot)\right)\right)\Big) \\ +\omega\left(\eta \mid g'\right)+\omega\left(\eta \mid h'\right) &\end{aligned} \tag{2.51}$$

Then for $j=p$, from (2.48), we have

$$|w_s\left(t_p(x;\xi_1),x\right)-w_s\left(t_p(x;\xi_2),x\right)| \leqslant C\Big(|w_s\left(t_p(0;\xi_1),0\right)-w_s\left(t_p(0;\xi_2),0\right)|+\|u\|_{C^1}\,\omega\left(|\xi_1-\xi_2| \mid \partial_t b\right)+\|\ \partial_t$$

where from (2.49), we know that

$$\omega\left(\eta \mid w_s\left(t_p(0;\cdot),0\right)\right) \leqslant C\left(\omega\left(\eta \mid w\left(t_{p-1}(x;\cdot),x\right)+\omega\left(\eta \mid g'\right)+\omega\left(\eta \mid h'\right)\right)\right.$$

Then using Gronwall's inequality and (2.51), we can get (2.51) holds for $j=p$. Next we consider $\omega\left(\eta \mid \omega\left(t_p(\cdot,\xi),\cdot\right)\right)$. From (2.45) and (2.48), we know that

$$\omega\left(\eta \mid w\left(t_p(\cdot;\xi),\cdot\right)\right) \leqslant \left(\|\partial_t a\|_{c^0}\ \|\ u\ \|_{c^1}+\left\|f^{(1)}\right\|_{c^1}\left\|f^{(2)}\right\|_{c^1}\right)\eta$$

Therefore, we take

$$\begin{aligned}\varOmega(\eta)= \quad & C_0 C_{A,k,\varepsilon_0}\Big(\omega\left(\eta \mid \varphi'\right)+(\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}+\|\partial_t a\|_{C^0}\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1})\eta \\ & +\left(\|\varphi\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right)\omega\left(\eta \mid a\right)+\omega\left(\eta \mid \partial f^{(1)}\partial f^{(2)}\right)+\omega\left(\eta \mid \psi'\right) \\ & +\left\|f^{(1)}\right\|_{C^1}\left(\omega\left(\eta \mid \partial_x {f_s}^{(2)}(0,\cdot)\right)+\omega\left(\eta \mid \partial_x f_s^{(2)}(0,\cdot)\right)\right)\Big)\end{aligned} \tag{2.52}$$

where $C_0 \gg 1$ is a constant, and then we know that

$$\omega\left(\eta \mid h_i'(t)\right) \leqslant \varOmega(\eta).$$

$$\omega\left(\eta \mid \partial_t u(t,x)\right) \leqslant \varOmega(\eta).$$

For $\partial_x u(t,x)$, from (2.1) we have

$$\omega\left(\eta \mid \partial_x u\right) \leqslant C\left(\omega\left(\eta \mid \partial_t u\right)+\|\ u\ \|_{C^1}\ \omega(\eta \mid a)+\omega\left(\eta \mid \partial f^{(1)}\partial f^{(2)}\right)\right)$$

Thereby, after adjusting C_0, we can get

$$\omega\left(\eta \mid \partial_x u(t,x)\right) \leqslant \varOmega(\eta).$$

Remark:. *From the above analysis, we can see that the continuity paradigm of $h_i'(t)$, $\partial_x u(t,x)$ and $\partial_t u(t,x)$ is uniform in t and x.*

Then we consider the continuity of control functions and solutions with regard to the coefficients of equations. Consider another controlling problem with the coefficients $\tilde{a}(t,x)$, $\tilde{f}^{(1)}(t,x)$, $\tilde{f}^{(2)}(t,x)$, $\tilde{g}(t,x)$ and its solution $v(t,x)$,where we assume that the values of these coefficients are the same as those of $a(t,x)$, $f^{(1)}(t,x)$, $f^{(2)}(t,x)$, $g(t,x)$ at $t=0$ and $t=L$ and thereby we can select the same set of functions $\widetilde{H}(t)$ to satisfy the compatibility. We first consider the initial boundary value problem (2.6) and (2.7) with the coefficients $\tilde{a}(t,x)$, $\tilde{f}^{(1)}(t,x)$, $\tilde{f}^{(2)}(t,x)$, $\tilde{g}(t,x)$.We can easily prove that

$$\begin{aligned}\| \tilde{u}(T,x)-\tilde{v}(T,x) \|_{C^0} \leqslant C \Big(\big(\|\varphi\|_{C^1}+\|g\|_{C^1}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\big)\|a-\tilde{a}\|_{C^0} \\ +\|g-\tilde{g}\|_{C^0}+\left\|f^{(2)}\right\|_{C^1}\left\|f^{(1)}-\tilde{f}^{(1)}\right\|_{C^0}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}-\tilde{f}^{(2)}\right\|_{C^0}\Big)\end{aligned}$$

From (2.14), we know that

$$\left\|t\left(x;x_0\right)-\tilde{t}\left(x;x_0\right)\right\|_{c^0} \leqslant C \parallel a-\tilde{a} \parallel_{c^0}$$

From (2.33)-(2.36), we know that

3. LINEARIZATION ITERATION

In this section, we will show the proof of Theorem 1.1 with inductive reasoning. Denote $\tilde{a}^{(m)}=a^{(m)}-1$,and we know that

$$\left\|\tilde{a}^{(m)}\right\|_{C^1} \leqslant C\left\|\tilde{U}^{(m)}\right\|_{C^2} \leqslant CC_P\varepsilon$$

using the inductive assumption (**??**). Denote

$$\begin{aligned}G^{(m)}= & \left(\widetilde{D}^+\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^+(0,0,0)\right)\widetilde{D}^+(0,0,0)^{-1}\widetilde{V}^{-(m)} \\ & -\left(\widetilde{D}^-\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^-(0,0,0)\right)\widetilde{D}^+(0,0,0)^{-1}\widetilde{V}^{+(m)}.\end{aligned}$$

Then we have

$$\left\|G^{(m)}\right\|_{C^1} \leqslant C\left\|\tilde{U}^{(m)}\right\|_{C^2}^2 \leqslant CC_P^2\varepsilon^2$$

Using (2.43) and (2.50), we can get

$$\left\|H^{(m+1)}\right\|_{C^1} \leqslant 2C\left(1+CC_P^2\varepsilon\right)\varepsilon$$

$$\left\|\tilde{V}^{(m+1)}\right\|_{C^1} \leqslant 2C\left(1+CC_P^2\varepsilon\right)\varepsilon$$

and then

$$\left\|\tilde{U}^{(m+1)}\right\|_{C^2} \leqslant 2C\left(1+CC_P^2\varepsilon\right)\varepsilon$$

Select ε_0 small enough and C_P large enough such that

$$2C\left(1+CC_P^2\varepsilon_0\right) \leqslant C_P$$

Then we can get

$$\left\|H^{(m+1)}\right\|_{C^1} \leqslant C_P\varepsilon.$$

and

$$\left\|\tilde{U}^{(m+1)}\right\|_{C^2} \leqslant C_P\varepsilon.$$

Using (**??**) and inductive assumptions, we know that

$$\omega\left(\eta\left|\frac{\partial H^{(m+1)}}{\partial t}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m+1)}}{\partial t^2}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m+1)}}{\partial x^2}\right.\right)+\omega\left(\eta\left|\frac{\partial^2 U^{(m+1)}}{\partial t\partial x}\right.\right) \leq \tilde{\Omega}(\eta). \quad (3.1)$$

where

$$\widetilde{\Omega}(\eta)=C\left(\omega\left(\eta\mid\varphi'\right)+\omega\left(\eta\mid\psi'\right)+\left(1+C_P^2\varepsilon\right)\varepsilon\eta+C_P\varepsilon\Omega(\eta)\right)$$

Select ε_0 small enough and C_P large enough such that

$$\widetilde{\Omega}(\eta) \leqslant \Omega(\eta)$$

Then we can get the uniform continuity paradigm of $H^{(m+1)}$ and $\tilde{U}^{(m+1)}$. Then we try to get the cauchy property (**??**) and (**??**). Provided the method of obtaining the controls $h_i(t)$, we first condider the cauchy property of $\tilde{\varphi}^{(m)}(x)$, where $\tilde{\varphi}^{(m)}(x) = \tilde{u}^{(m)}(T,x)$ and $\tilde{u}^{(m)}(t,x)$ is the solution of the equation below:

$$\begin{cases} \partial_t \widetilde{u}^{\pm(m+1)} \pm a^{(m)} \partial_x \widetilde{u}^{\pm(m+1)} = \mp\chi\left(t\right) \partial_x \widetilde{U}^{(m)} \left(\partial_t \pm a^{(m)} \partial_x\right) \left(a^{(m)}\right), \\ x = L : \widetilde{u}^{-(m+1)} = -\widetilde{u}^{+(m+1)}, \quad t \in \mathbb{R}^+, \\ x = 0 : \widetilde{u}^{-(m+1)} = B_0 \widetilde{u}^{+(m+1)} + \widetilde{H}(t) + G^{(m)}\left(t\right), \\ t = 0 : \widetilde{u}^{(m+1)} = \varphi\left(x\right), \end{cases}$$

where $\chi(t)$ is the same as in section 2 and $\widetilde{H}(t)$ is a set of functions selected that satisfy the compatibility condition. Provided that $G^{(m)}(0)$ depends on $\varphi(x)$ and is independent of m, we can select the same set of functions $\widetilde{H}(t)$ for all m. When x is close to 0 or L, we can use the same method as (2.19) to obtain the equation of $\tilde{\varphi}^{(m)}(x)$:

4. Appendice

Lemma 4.1. *For the equation*

$$\psi = A^k B H_{k+1} + \cdots + ABH_2 + BH_1. \tag{4.1}$$

$\psi, H_i \in \mathbb{R}^n, i = 1, \cdots, k+1,$

$$A = \begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix},$$

$$B = \begin{pmatrix} I_{r_1} & & & \\ & O_{r_2} & & \\ & & I_{s_1} & \\ & & & O_{s_2} \end{pmatrix},$$

where $r_1 + r_2 = r,\ s_1 + s_2 = s, r + s = n,\ A_1 = (a_{ij}),\ i = 1, \cdots, r,\ j = r+1, \cdots, n,$ $A_2 = (a_{ij}),\ i = r+1, \cdots, n,\ j = 1, \cdots, r$*.If* ψ *satisfies the following condition*

$$\psi_i = \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r + s_1 + 1, \cdots, r+s. \tag{4.2}$$

and the Kalman rank condition holds as follows,

$$\text{Rank}\left[B, A^k B, \ldots, A^{k-1}B\right] = n, \tag{4.3}$$

where $r + s = n$*,then there exists a set of solutions* $H_p(x), (p = 1, \cdots, k+1)$ *of equation* (4.1) *satisfying*

$$H_{2m-1}^- = H_{2m}^-, \tag{4.4}$$
$$H_{2m+1}^+ = H_{2m}^+. \tag{4.5}$$

Proof. Without loss of generality, we can assume that k is odd. The condition 4.2 can be written as

$$\psi^+ = A_2\psi^- + \bar{H}. \tag{4.6}$$

$\bar{H}$ is the difference term, i.e.

$$\bar{H}_i = \psi_i - \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r+1, \cdots, r+s_1. \tag{4.7}$$

By the definition of the matrix B, the i-th $(i = r_1 + 1, \cdots, r, s_1 + 1, \cdots, r+s)$ components of $H_p(p = 1, \cdots, k+1)$ can be taken to be zero.The following is an attempt to write conditions (4.4)-(4.5) as matrix equations. Denote

$$H = \left(H_{k+1}^-, H_{k+1}^+, \cdots, H_1^-, H_1^+\right)^T.$$

Then the equation (4.1) ,(**??**)-(4.4)can be written as a matrix equation as follows:

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} & A_1A_2\cdots A_2 & A_1\cdots A_1 & & \cdots & & A_1 & I_r & \\ A_2A_1\cdots A_1 & & & A_2\cdots A_2 & \cdots & A_2 & & & I_s \\ I_r & & -I_r & & & & & & \\ & & & I_s & & & & & \\ & & & & \cdots & & & & \\ & & & & & & -I_s & & \\ & & & & & I_r & & -I_r & \end{pmatrix} H.$$

After appropriate matrix row transformations, the first row of the equation can be written as

$$\psi = \begin{pmatrix} A_1A_2\cdots A_2 & A_1\cdots A_1 & \cdots & A_1 & I_r & \\ & A_2A_1\cdots A_1 & \cdots & A_2A_1 & A_2 & I_s \end{pmatrix} H.$$

Let $H_{k+1}^+ = 0$. Combined with (4.6), equation (4.1) can be rewritten as

$$\begin{pmatrix} \psi^- \\ \bar{H} \end{pmatrix} = \begin{pmatrix} A_1\cdots A_1 & \cdots & A_1 & I_r & \\ & \cdots & & & I_s \end{pmatrix} H. \tag{4.8}$$

Notice that (**??**) yields that the rank of the equation generalization matrix is equal to the rank of the original coefficient matrix, and thus (4.8) has a solution. $\square$

Corollary. *Under the conditions of the above lemma, if ψ satisfies*

$$\psi_i = \sum_{j=r+1}^{r+s} a_{ij}\psi_j, \ i = r_1 + 1, \cdots, r, \tag{4.9}$$

then there exists a set of solutions $H_p(x), (p = 1, \cdots, k+1)$ of equation (4.1) satisfying

$$H_{2m-1}^+ = H_{2m}^+, \tag{4.10}$$

$$H_{2m+1}^- = H_{2m}^-. \tag{4.11}$$

Proof. Define $\tilde{\psi}$ as $\tilde{\psi}^- = \psi^+$,$\tilde{\psi} = \psi^-$,$\tilde{A}$ as $\tilde{A}^1 = A^2$,$\tilde{A}^2 = A^1$, and $\tilde{H}_p$ as $\tilde{H}_p^- = H_p^+$,$\tilde{H}_p^+ = H_p^-$, which are the same as the above lemma. $\square$

Corollary. *In the proof of Lemma 4.1 one obtains $H_1^+ = \bar{H}$, and in the conditions of Corollary 4 one obtains $H_1^- = \bar{H}$.*

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