# CONTROLLABILITY TO SYSTEMS OF QUASILINEAR WAVE EQUATIONS WITH FEWER CONTROLS

LONG HU, PENG QU, AND JIAXIN TONG

Abstract. This study addresses challenges in solving exact boundary controllability problems of quasilinear wave equations with Dirichlet type boundary conditions satisfying some kind of Kalman rank conditions. Utilizing techniques such as characteristic method and a specially designed linearization iteration scheme, we developed controls that use fewer boundary controls than the classical theory.

## 1. Introduction

Consider the controllability for system of the quasilinear wave equations below

$$\begin{cases} U_{tt} - a^2\left(U, U_t, U_x\right) U_{xx} = 0, & (t,x) \in \mathbb{R}^+ \times [0,L], \\ t = 0 : (U, U_t) = \left(\varphi(x), \psi(x)\right), & x \in [0,L], \\ x = L : U = 0, & t \in \mathbb{R}^+, \\ x = 0 : U_t - D\left(U, U_t, U_x\right) U_x = H(t), & t \in \mathbb{R}^+, \\ t = T : (U, U_t) = \left(\Phi(x), \Psi(x)\right), & x \in [0,L]. \end{cases} \tag{1.1}$$

where

$$U = \left(u_1, \dots, u_n\right)^T (t,x) \in C^2.$$

$a\left(U, U_t, U_x\right) > 0$ is the $C^1$ uniform wave speed. $D\left(u_1, u_2, u_3\right) = \left(d_{ij}\left(u_1, u_2, u_3\right)\right)_{i,j=1}^n$ is an $n \times n$ matrix with $C^1$ regularity. Part of the components of $H(t) = \left(h_1(t), \dots, h_n(t)\right)^T$ would be chosen as boundary controls. We denote $D^{\pm}\left(U, U_t, U_x\right) = aI_n \pm D$ , which we request are invertible. We also request that

$$\left\|\left(\varphi(x), \psi(x)\right)\right\|_{(C^2[0,L])^n \times (C^1[0,L])^n} < \varepsilon,$$
$$\left\|\left(\varphi(x), \Psi(x)\right)\right\|_{(C^2[0,L])^n \times (C^1[0,L])^n} < \varepsilon.$$

where $0 < \varepsilon \ll 1$. Without loss of generosity, we set $a(0,0,0) = 1$, $H(t) = \left(h_1(t), \dots, h_p(t), 0, \dots, 0\right)^T$.Now we try to reduce the wave equation to a hyperbolic equation group. Denote

$$\tilde{U} = D^+(0,0,0)U,$$
$$\tilde{V}^{\pm} = \partial_t \tilde{U} \mp \tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \partial_x \tilde{U}.$$

where

$$\tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) = a\left(D^+(0,0,0)^{-1}U, D^+(0,0,0)^{-1}U_t, D^+(0,0,0)^{-1}U_x\right),$$

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$$\tilde{D}^{\pm}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)=D^{\pm}\left(D^{+}(0,0,0)^{-1}U,D^{+}(0,0,0)^{-1}U_t,D^{+}(0,0,0)^{-1}U_x\right).$$

The original equations can be changed into

$$\begin{cases}\partial_t\tilde{V}^{\pm}\pm\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\partial_x\tilde{V}^{\pm}=\mp\partial_x\tilde{U}\left(\partial_t\pm\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\partial_x\right)\left(\tilde{a}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\right),\\ x=L:\tilde{V}^{-}=-\tilde{V}^{+},\\ x=0:\tilde{D}^{-}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\tilde{D}^{+}(0,0,0)^{-1}\tilde{V}^{-}=-\tilde{D}^{+}\left(\tilde{U},\tilde{U}_t,\tilde{U}_x\right)\tilde{D}^{+}(0,0,0)^{-1}\tilde{V}^{+}+\tilde{H}(t).\end{cases}\tag{1.2}$$

where $\tilde{H}(t)=2aH(t)$. Then the boundary condition can be written as

$$\begin{aligned}x=0:\widetilde{V}^{+}=B_0\widetilde{V}^{-}+\widetilde{H}(t)&-\left(\widetilde{D}^{-}\left(\widetilde{U},\widetilde{U}_t,\widetilde{U}_x\right)-\widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+}\\ &-\left(\widetilde{D}^{-}\left(\widetilde{U},\widetilde{U}_t,\widetilde{U}_x\right)-\widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-}.\end{aligned}$$

where the coupling matrix

$$B_0=-\left(a(0,0,0)I_n-D(0,0,0)\right)\left(a(0,0,0)I_n+D(0,0,0)\right)^{-1}.$$

Define the controlling matrix as

$$C=\begin{pmatrix}I_p & 0\\ 0 & 0\end{pmatrix},$$

then the Kalman type condition as

$$\operatorname{Rank}\left[C,B_0C,\ldots,B_0^{k-1}C\right]=n,\tag{1.3}$$

where $k$ is a positive integer.

The following theorem can be obtained.

**Theorem 1.** *Under Assumption* (1.3), *if $T>kL$, there exists a small $\varepsilon_0$ and a set of $C^1$ smooth control functions $H_i(t)$, $i=1,\cdots,p$, such that for any $\varepsilon<\varepsilon_0$ and any $\Phi(x)$, $\varphi(x)$ which are $C^2$ smooth and satisfy the $C^2$ compatibility condition and $\Psi(x)$, $\psi(x)$ which are $C^1$ smooth and satisfy the $C^1$ compatibility condition, there exists a unique $C^2$ smooth solution $U(t,x)$ to the control problem* (1.1).

The proof of Theorem 1 requires the following proposition.

**Proposition 1.1.** *Taking $U^{(0)}\equiv 0$, consider the following iterative problem*

$$\begin{cases}\partial_t\widetilde{V}^{\pm(m+1)}\pm a^{(m)}\partial_x\widetilde{V}^{\pm(m+1)}=\mp\partial_x\widetilde{U}^{(m)}\left(\partial_t\pm a^{(m)}\partial_x\right)\left(a^{(m)}\right),\\ x=L:\widetilde{V}^{-(m+1)}=-\widetilde{V}^{+(m+1)},\quad t\in\mathbb{R}^{+},\\ x=0:\widetilde{V}^{-(m+1)}=B_0\widetilde{V}^{+(m+1)}+\widetilde{H}^{(m+1)}(t)\\ +\left(\widetilde{D}^{+}\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-(m)}\\ -\left(\widetilde{D}^{-}\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right)-\widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+(m)}.\\ t=0:\left(U^{(m+1)},U_t^{(m+1)}\right)=\left(\varphi\left(x\right),\psi\left(x\right)\right),\\ t=T:\left(U^{(m+1)},U_t^{(m+1)}\right)=\left(\Phi\left(x\right),\Psi\left(x\right)\right).\end{cases}\tag{1.4}$$

*where*

$$a^{(m)}=a\left(\widetilde{U}^{(m)},\widetilde{U}_t^{(m)},\widetilde{U}_x^{(m)}\right).$$

*One can get uniform bounds*

$$\left\|U^{(m)}\right\|_{C^2}\le C\varepsilon,$$
$$\left\|H^{(m)}\right\|_{C^1}\le C\varepsilon.$$

*as well as Cauchy sequence*

$$\left\|U^{(m)}-U^{(m-1)}\right\|_{C^1}\le C\varepsilon\alpha^{m-1},$$

$$\left\| H^{(m)} - H^{(m-1)} \right\|_{C^0} \leq C\varepsilon\alpha^{m-1}.$$

*with additional equi-continuity by estimates on modulus of continuity*

$$\omega\left(\eta \left| \frac{\partial H^{(m)}}{\partial t}\right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial t^2}\right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial x^2}\right.\right) + \omega\left(\eta \left| \frac{\partial^2 U^{(m)}}{\partial t \partial x}\right.\right) \leq \Omega\left(\eta\right).$$

Where $\Omega(\eta)$ is a monotonically increasing continuous function satisfying $\Omega(0) = 0$. Theorem 1 follows after obtaining Proposition 1.1.In fact, The sequence $\{U^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence in $C^1$ space, and thus converges to some $C^1$ function $U$ uniformly, applying Arzelà-Ascoli theorem, we know that there exists a subsequence of $\{U^{(m)}\}_{m=1}^{\infty}$ ,which converges uniformly in $C^2$ space. Thus we know that the whole original sequence$\{U^{(m)}\}_{m=1}^{\infty}$ converges to $U$ in $C^2$ space. Therefore, $U$ is $C^2$ smooth and satisfies the control condition, where $U^(m+1)$ for each step can be obtained using the Dirichlet boundary condition as

$$U^{(m+1)}(t,x) = -\int_x^L \partial_x U^{(m+1)} \mathrm{d}x,$$

where

$$\partial_x U^{(m+1)} = \frac{V^{-(m+1)} - V^{+(m+1)}}{2a\left(U^{(m)}, U_t^{(m)}, U_x^{(m)}\right)}.$$

## 2. Proof of Theorem 2.1

We need to introduce a lemma about the linear equation.Consider the following linear equation system

$$\begin{cases} \partial_t u_i - a(t,x)\partial_x u_i = f_i^{(1)}(t,x)(\partial_t - a(t,x)\partial_x) f_i^{(2)}(t,x), i = 1, \cdots, r, \\ \partial_t u_i + a(t,x)\partial_x u_i = f_i^{(1)}(t,x)(\partial_t + a(t,x)\partial_x) f_i^{(2)}(t,x), i = r+1, \cdots, r+s. \end{cases} \tag{2.1}$$

where $a(t,x)$ is a $C^1$ function satisfying that $\|a(t,x) - 1\|_{C^1} \leq \varepsilon_0.\varepsilon_0$ is small enough. $f_i^{(\kappa)}(t,x), i = 1, \cdots, r+s, \kappa = 1, 2$ are $C^1$ functions. The boundary conditions are given as follows:

$$\begin{cases} x = 0: \ u_i = \sum_{j=1}^{r} a_{ij} u_j + g_i(t) + h_i\left(t\right), \ i = r+1, \cdots, n, \\ x = L: \ u_i = \sum_{j=r+1}^{n} a_{ij} u_j + g_i(t) + h_i\left(t\right), \ i = 1, \cdots, r. \end{cases} \tag{2.2}$$

where $g_i(t)$ is a $C^1$ function, $h_i(t)$ is a $C^1$ controlling function. The matrix $A = (a_{ij})$, $i = 1, \cdots, n$, $j = 1, \cdots, n$ is a $C^1$ matrix. $u(t,x)$ satisfies

$$u\left(0, x\right) = \varphi\left(x\right). \tag{2.3}$$

The matrix $B$ is defined as follows:

$$B = \begin{pmatrix} I_{r_1} & & & \\ & O_{r_2} & & \\ & & I_{s_1} & \\ & & & O_{s_2} \end{pmatrix},$$

where $r_1 + r_2 = r$, $s_1 + s_2 = s, r + s = n$. The Kalman rank condition is given as follows:

$$\text{Rank}\left[B, AB, \ldots, A^{k-1}B\right] = n, \tag{2.4}$$

where $k$ is a positive integer. The following lemma is the main result of this section.

**Lemma 2.1.** *Under the conditions of* (2.4)*, if $T > kL$, then there exists a set of $C^1$ smooth control functions $h_i(t)$, $i = 1, \cdots, r_1, r+1, \cdots, r+s_1$, such that for any $C^1$ smooth functions $g_i(t), f_i(t,x)$, $i = 1, \cdots, n$ and any $C^1$ smooth functions $\varphi(x), \psi(x)$, which satisfy the $C^1$ compatibility conditions, there exists a unique $C^1$ smooth solution $u(t,x)$ to the control problem* (2.1)-(2.3) *satisfying*

$$u(T,x) = \psi(x). \tag{2.5}$$

*Proof.* Without loss of generality, we can assume that $k$ is odd, $T-kL<\frac{L}{4}$ and $\varphi(x)=0$ ,or take any set of $\tilde{h}_i(t)$ that satisfies the compatibility conditions at $(0,0)$ and $(0,L)$ and consider the equations of $\tilde{u}(t,x)$ satisfying $\tilde{u}(0,x)=\varphi(x)$ below:

$$\left\{\begin{array}{l}\partial_t\tilde{u}_i-a(t,x)\partial_x\tilde{u}_i=\chi(t)f_i^{(1)}(t,x)(\partial_t-a(t,x)\partial_x)f_i^{(2)}(t,x), i=1,\cdots,r,\\ \partial_t\tilde{u}_i+a(t,x)\partial_x\tilde{u}_i=\chi(t)f_i^{(1)}(t,x)(\partial_t+a(t,x)\partial_x)f_i^{(2)}(t,x), i=r+1,\cdots,r+s.\end{array}\right. \tag{2.6}$$

where $\chi(t)$ is a smooth function that satisfies $\chi(t)=1$ for $t\in[0,\frac{L}{2}]$ and $\chi(t)=0$ for $t\in[T-\frac{L}{2},T]$. The boundary conditions are given as follows:

$$\left\{\begin{array}{l}x=0:\ \tilde{u}_i=\sum_{j=1}^{r}a_{ij}\tilde{u}_j+\chi(t)g_i(t)+\tilde{h}_i\left(t\right),\ i=r+1,\cdots,n,\\ x=L:\ \tilde{u}_i=\sum_{j=r+1}^{n}a_{ij}\tilde{u}_j+\chi(t)g_i(t)+\tilde{h}_i\left(t\right),\ i=1,\cdots,r.\end{array}\right. \tag{2.7}$$

Then we can get that $\tilde{u}(T,x)$ satisfies the $C^1$ compatibility conditions at $(T,0)$ and $(T,L)$ and consider the following controllability problem:

$$\left\{\begin{array}{l}\partial_tu_i-a(t,x)\partial_xu_i=(1-\chi(t))f_i^{(1)}(t,x)(\partial_t-a(t,x)\partial_x)f_i^{(2)}(t,x), i=1,\cdots,r,\\ \partial_tu_i+a(t,x)\partial_xu_i=(1-\chi(t))f_i^{(1)}(t,x)(\partial_t+a(t,x)\partial_x)f_i^{(2)}(t,x), i=r+1,\cdots,r+s.\end{array}\right. \tag{2.8}$$

where the boundary conditions are given as follows:

$$\left\{\begin{array}{l}x=0:\ u_i=\sum_{j=1}^{r}a_{ij}u_j+(1-\chi(t))g_i(t)+h_i\left(t\right),\ i=r+1,\cdots,n,\\ x=L:\ u_i=\sum_{j=r+1}^{n}a_{ij}u_j+(1-\chi(t))g_i(t)+h_i\left(t\right),\ i=1,\cdots,r.\end{array}\right. \tag{2.9}$$

The initial and final conditions are given as follows:

$$u(0,x)=0,\quad u(T,x)=\psi(x)-\tilde{u}(T,x). \tag{2.10}$$

Finally, the two sets of control functions and solutions obtained can be added separately. We determine $h_i\left(t\right)$ in the following order: In the following we first study the equations of characteristic lines. Let

$$b(t,x)=\frac{1}{a(t,x)}.$$

The characteristic lines of the system (2.1) are given as follows:

$$\left\{\begin{array}{l}\frac{\mathrm{d}\tilde{t}_p}{\mathrm{d}x}=-b\left(\tilde{t}_p,x\right),\\ \tilde{t}_p(0;x_0)=t_{p-1}(0;x_0),\ p=2,\cdots,k+1,\\ \tilde{t}_1(x_0;x_0)=T,\end{array}\right. \tag{2.11}$$

$$\left\{\begin{array}{l}\frac{\mathrm{d}t_p}{\mathrm{d}x}=b\left(t_p,x\right),\\ t_p(L;x_0)=\tilde{t}_{p-1}(L;x_0),\ p=2,\cdots,k+1,\\ t_1(x_0;x_0)=T,\end{array}\right. \tag{2.12}$$

thereby obtaining the following equations:

$$\begin{aligned}t_1\left(x;x_0\right)&=T+\int_{x_0}^{x}b\left(t_1\left(s;x_0\right),s\right)\mathrm{d}s,\\ t_p\left(x;x_0\right)&=t_{p-1}\left(L;x_0\right)+\int_{L}^{x}b\left(t_p\left(s;x_0\right),s\right)\mathrm{d}s, p=2,\cdots,k+1,\\ \partial_{x_0}t_1\left(x;x_0\right)&=-b\left(T,x_0\right)+\int_{x_0}^{x}\partial_tb\left(t_1\left(s;x_0\right),s\right)\partial_{x_0}t_1\left(s;x_0\right)\mathrm{d}s,\\ \partial_{x_0}t_p\left(x;x_0\right)&=\partial_{x_0}t_{p-1}\left(L;x_0\right)+\int_{L}^{x}\partial_tb\left(t_p\left(s;x_0\right),s\right)\partial_{x_0}t_p\left(s;x_0\right)\mathrm{d}s, p=2,\cdots,k+1.\end{aligned}$$

Denote $z_p(x;x_0)=\partial_{x_0}t_p\left(x;x_0\right)$, which conduces that

$$\frac{\mathrm{d}}{\mathrm{d}x}z_p\left(x;x_0\right)=\partial_tb_p\left(t\left(x;x_0\right),x\right)z_p\left(x;x_0\right).$$

Then we can get the following equations:

$$\begin{aligned} z_1(x;x_0) &= -b(T,x_0)\exp\left(\int_{x_0}^{x} \partial_t b(t_1(s;x_0),s)\,\mathrm{d}s\right), \\ z_p(x;x_0) &= \partial_{x_0} t_{p-1}(L;x_0)\exp\left(\int_{L}^{x} \partial_t b(t_p(s;x_0),s)\,\mathrm{d}s\right). \end{aligned} \tag{2.13}$$

Similarly one can obtain that

$$\begin{aligned} \tilde{t}_1(x;x_0) &= T - \int_{x_0}^{x} b\left(\tilde{t}_1(s;x_0),s\right)\mathrm{d}s \\ \tilde{t}_p(x;x_0) &= t_{p-1}(0;x_0) - \int_0^x b\left(\tilde{t}_p(s;x_0),s\right)\mathrm{d}s, l=2,\cdots,k+1, \\ \tilde{z}_1(x;x_0) &= b(T,x_0)\exp\left(-\int_{x_0}^{x} \partial_t b\left(\tilde{t}_1(s;x_0),s\right)\mathrm{d}s\right), \\ \tilde{z}_p(x;x_0) &= \partial_{x_0} t_{p-1}(0;x_0)\exp\left(\int_0^x \partial_t b\left(\tilde{t}_p(s;x_0),s\right)\mathrm{d}s\right). \end{aligned}$$

Therefore

$$\begin{cases} t_1(x;0) = T, \\ t_{2m+1}(x;0) = t_{2m}(x;0), \\ t_{2m}(x;L) = t_{2m-1}(x;L), \end{cases} \tag{2.14}$$

$$\begin{cases} \tilde{t}_1(x;L) = T, \\ \tilde{t}_{2m+1}(x;L) = \tilde{t}_{2m}(x;L), \\ \tilde{t}_{2m}(x;0) = \tilde{t}_{2m-1}(x;0). \end{cases} \tag{2.15}$$

and

$$\begin{cases} \partial_{x_0} t_1(x;0) = -b(T,0), \\ \partial_{x_0} t_{2m+1}(x;0) = -\partial_{x_0} t_{2m}(x;0), \\ \partial_{x_0} t_{2m}(x;L) = -\partial_{x_0} t_{2m-1}(x;L), \end{cases} \tag{2.16}$$

$$\begin{cases} \partial_{x_0} \tilde{t}_1(x;L) = b(T,L), \\ \partial_{x_0} \tilde{t}_{2m+1}(x;L) = -\partial_{x_0} \tilde{t}_{2m}(x;L), \\ \partial_{x_0} \tilde{t}_{2m}(x;0) = -\partial_{x_0} \tilde{t}_{2m-1}(x;0). \end{cases} \tag{2.17}$$

The following defines a set of vector functions:

$$H_p(x) = \left(H^-\left(\tilde{t}_p(L;x)\right), H^+\left(t_p(0;x)\right)\right)^T,$$

where

$$\begin{aligned} H^-(t) &= (h_1(t),\cdots,h_{r_1}(t),0,\cdots,0), \\ H^+(t) &= (h_{r+1}(t),\cdots,h_{r+s_1}(t),0,\cdots,0). \end{aligned}$$

Let

$$h_i(x) = 0,\ x<0, i=1,\ldots,n.$$

Define

$$u(t,x) = \left(u^-(t,x), u^+(t,x)\right)^T,$$

where

$$\begin{aligned} u^-(t,x) &= (u_1(t,x),\cdots,u_r(t,x)), \\ u^+(t,x) &= (u_{r+1}(t,x),\cdots,u_{r+s}(t,x)). \end{aligned}$$

Define

$$G_p(x) = \left(G^-\left(\tilde{t}_p(L;x)\right), G^+\left(t_p(0;x)\right)\right)^T,$$

where

$$\begin{aligned} G^-(t) &= (g_1(t),\cdots,g_r(t)), \\ G^+(t) &= (g_{r+1}(t),\cdots,g_{r+s}(t)). \end{aligned}$$

Define

$$F_1(x) = \left(\int_x^L F^-\left(\tilde{t}_1(s;x),s\right)\mathrm{d}s, \int_0^x F^+\left(t_1(s;x),s\right)\mathrm{d}s\right)^T,$$

$$F_p\left(x\right)=\left(\int_0^L F^-\left(\tilde{t}_p\left(s;x\right),s\right)\mathrm{d}s,\int_0^L F^+\left(t_p\left(s;x\right),s\right)\mathrm{d}s\right)^T,p=2,\cdots,k+1,$$

where

$$F^-\left(t,x\right)=\left(f_1^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_1^{(2)}(t,x),\cdots,f_r^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_r^{(2)}(t,x)\right),$$

$$F^+\left(t,x\right)=\left(f_{r+1}^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_1^{(2)}(t,x),\cdots,f_{r+s}^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_{r+s}^{(2)}(t,x)\right),$$

If a solution $u(t,x)$ satisfying the requirement exists,for any $x\in[0,L]$, we can get the following equations with characteristic lines method:

$$\begin{aligned}\psi\left(x\right)&=u(T,x)\\&=(u^-(\tilde{t}_1(L;x),L),u^+(t_1(0;x),0))^T+F_1(x)\\&=A(u^-(t_1(0;x),0),u^+(\tilde{t}_2(L;x),L))^T+BH_1(x)+G_1(x)+F_1(x)\\&=A(u^-(t_2,L),u^+(\tilde{t}_2,0))^T+AF_2(x)+BH_1(x)+G_1(x)+F_1(x)\\&\cdots\\&=A^kB(H_{k+1}\left(x\right)+G_{k+1}\left(x\right)+F_{k+1}\left(x\right))+\cdots+B(H_1\left(x\right)+G_1\left(x\right)+F_1\left(x\right)),\ x\in\left[0,L\right].\end{aligned}\tag{2.18}$$

Using this equation, we first determine the value of the function and derivative of $h_i(t_p(0;0)),h_i(t_p(0;L)),h_i(t_p(L$ $1,\cdots,n;p=1,\cdots,k+1)$. Consider the vector $\tilde{G}(x)=\sum_{l=0}^k A^lG_{l+1}(x)$.

$$\tilde{G}^-(L)=G_1^-(L)+A_1G_2^+(L)+A_1A_2G_3^-(L)+\cdots+A_1A_2\cdots A_1G_{k+1}^+(L),$$

$$\tilde{G}^+(L)=G_1^+(L)+A_2G_2^-(L)+A_2A_1G_3^+(L)+\cdots+A_2A_1\cdots A_2G_{k+1}^-(L).$$

Recognizing (2.14)-(2.15) and $T-kL<\frac{L}{4}$, we can get

$$G_{k+1}^-(L)=0,$$

$$G_m^+(L)=G_{m+1}^+(L),\ misodd,\ m=1,\cdots,k,$$

$$G_m^+(L)=G_{m+1}^+(L),\ miseven,\ m=1,\cdots,k.$$

Therefore,

$$\begin{aligned}\tilde{G}^+(L)&=G_1^+(L)+A_2G_2^-(L)+A_2A_1G_3^+(L)+\cdots+A_2A_1\cdots A_1G_k^+(L)\\&=G_2^+(L)+A_2G_3^-(L)+A_2A_1G_4^+(L)+\cdots+A_2A_1\cdots A_1G_{k+1}^+(L),\end{aligned}\tag{2.19}$$

and thereby we can get the following equations:

$$\tilde{G}^-(L)=G_1^-(L)+A_1\tilde{G}^+(L).$$

Similarly,

$$\tilde{G}^+(0)=G_1^+(0)+A_2\tilde{G}^-(0).$$

Namely,

$$\begin{pmatrix}\tilde{G}^-(L)\\\tilde{G}^+(0)\end{pmatrix}=\begin{pmatrix}&A_1\\A_2&\end{pmatrix}\begin{pmatrix}\tilde{G}^-(0)\\\tilde{G}^+(L)\end{pmatrix}+\begin{pmatrix}g^-(T)\\g^+(T)\end{pmatrix}.\tag{2.20}$$

Now consider the vector $\tilde{F}(x)=\sum_{l=0}^k A^lF_{l+1}(x)$.

$$\tilde{F}^-(L)=F_1^-(L)+A_1F_2^+(L)+A_1A_2F_3^-(L)+\cdots+A_1A_2\cdots A_1F_{k+1}^+(L),$$

$$\tilde{F}^+(L)=F_1^+(L)+A_2F_2^-(L)+A_2A_1F_3^+(L)+\cdots+A_2A_1\cdots A_2F_{k+1}^-(L).$$

From the defination of $\chi(t)$ ,we can assume that $f_i^{(j)}(t,x)=0,t<T-\frac{L}{4},j=1,2,i=1,\cdots,n$.Recognizing (2.14)-(2.15) and $T-kL<\frac{L}{4}$, we can get

$$F_1^-(L)=0,$$

$$F_{k+1}^-(L)=0,$$

$$F_m^+(L)=F_{m+1}^+(L),\ misodd,\ m=1,\cdots,k,$$

$$F_m^+(L)=F_{m+1}^+(L),\ miseven,\ m=1,\cdots,k.$$

Therefore,

$$\begin{aligned}\tilde F^{+}(L) &= F_1^{+}(L)+A_2F_2^{-}(L)+A_2A_1F_3^{+}(L)+\cdots+A_2A_1\cdots A_1F_k^{+}(L)\\ &= F_2^{+}(L)+A_2F_3^{-}(L)+A_2A_1F_4^{+}(L)+\cdots+A_2A_1\cdots A_1F_{k+1}^{+}(L),\end{aligned} \tag{2.21}$$

and thereby we can get the following equations:

$$\tilde F^{-}(L)=A_1\tilde F^{+}(L).$$

Similarly,

$$\tilde F^{+}(0)=A_2\tilde F^{-}(0).$$

Namely,

$$\begin{pmatrix}\tilde F^{-}(L)\\ \tilde F^{+}(0)\end{pmatrix}=\begin{pmatrix} & A_1\\ A_2 & \end{pmatrix}\begin{pmatrix}\tilde F^{-}(0)\\ \tilde F^{+}(L)\end{pmatrix}. \tag{2.22}$$

The $C^0$ compatibility condition satisfied by $\psi(x)$ is

$$\psi_i(0)=\sum_{j=1}^{r}a_{ij}\psi_j(0)+g_i(T),\ \ i=r+s_1+1,\cdots,n, \tag{2.23}$$

$$\psi_i(L)=\sum_{j=r+1}^{n}a_{ij}\psi_j(L)+g_i(T),\ \ i=r_1+1,\cdots,r. \tag{2.24}$$

So $\psi-\tilde G-\tilde F$ satisfies the conditions of Lemma 3.1. Therefore, the following equations have a solution:

$$\psi(0)=A^k\left(H_{k+1}(0)+G_{k+1}(0)+F_{k+1}(0)\right)+\cdots+\left(H_1(0)+G_1(0)+F_1(0)\right), \tag{2.25}$$

$$\psi(L)=A^k\left(H_{k+1}(L)+G_{k+1}(L)+F_{k+1}(L)\right)+\cdots+\left(H_1(L)+G_1(L)+F_1(L)\right), \tag{2.26}$$

to determine the values of $h_i$ at $t_p(0;0),t_p(0;L),t_p(L;0),t_p(L;L)$,$(i=1,\cdots,n;p=1,\cdots,k+1)$. Derivation of (2.18) with respect to $x$ gives the following equations:

$$\psi'(x)=A^kB(H'_{k+1}(x)+G'_{k+1}(x)+F'_{k+1}(x))+\cdots+B(H'_1(x)+G'_1(x)+F'_1(x)),\ \ x\in[0,L], \tag{2.27}$$

where

$$H'_p(x)=\left(\frac{\mathrm{d}H^-}{\mathrm{d}t}\left(\tilde t_p(L;x)\right)\frac{\partial\tilde t_p(L;x)}{\partial x},\frac{\mathrm{d}H^+}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$G'_p(x)=\left(\frac{\mathrm{d}G^-}{\mathrm{d}t}\left(\tilde t_p(L;x)\right)\frac{\partial\tilde t_p(L;x)}{\partial x},\frac{\mathrm{d}G^+}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$F_1'(x)=\Big(-f^{(1)-}(b\partial_t-\partial_x)f^{(2)-}(T,x)+\int_x^L\frac{\partial f^{(1)-}}{\partial t}(b\partial_t-\partial_x)f^{(2)-}\left(\tilde t_1(s,x),s\right)\frac{\partial\tilde t_1}{\partial x}(s;x)\,ds-\int_x^L\frac{\partial f^{(1)-}}{\partial t}(b\partial_t-$$

$$F_p'(x)=\Big(\int_0^L\frac{\partial f^{(1)-}}{\partial t}(b\partial_t-\partial_x)f^{(2)-}\left(\tilde t_p(s,x),s\right)\frac{\partial\tilde t_p}{\partial x}(s;x)\,ds-\int_0^L\frac{\partial f^{(1)-}}{\partial t}(b\partial_t-\partial_x)f^{(2)-}\left(\tilde t_p(s,x),s\right)\frac{\partial\tilde t_p}{\partial x}(s;x)\,d$$

Consider the vector $\tilde G'(x)=\sum_{l=0}^{k}A^lG'_{l+1}(x)$ and $\tilde F'(x)=\sum_{l=0}^{k}A^lF'_{l+1}(x)$.Similar to the previous case using (2.14)-(2.17), we can get the following equations:

$$\tilde G'_-(L)=b(T,L)\,g'_r(T)-A_1\tilde G'_+(L)$$

$$\tilde G'_+(0)=-b(T,L)\,g'_s(T)-A_2\tilde G'_-(0)$$

$$\tilde F'_-(L)+f_r^{(1)}(T,L)\,\partial_sf_r^{(2)}(T,L)=A_1\left(-\tilde F'_+(L)+f_s^{(1)}(T,L)\,\partial_sf_s^{(2)}(T,L)\right)$$

$$-\tilde F'_+(0)+f_s^{(1)}(T,0)\,\partial_sf_s^{(2)}(T,0)=A_2\left(\tilde F'_-(L)+f_r^{(1)}(T,0)\,\partial_sf_r^{(2)}(T,0)\right)$$

Combined with the $C^1$ compatibility condition satisfied by $\psi(x)$:

$$a(T,L)\,\psi'_r(L)+f_r^{(1)}(\partial_t-a(T,L)\,\partial_x)\,f_r^{(2)}(T,L)=A_2\left(-a(T,L)\,\psi'_s(L)+f_s^{(1)}(\partial_t+a(T,L)\,\partial_x)\,f_s^{(2)}(T,L)\right)+g'_r$$

$$-a(T,0)\,\psi'_s(0)+f_s^{(1)}(\partial_t+a(T,0)\,\partial_x)\,f_s^{(2)}(T,0)=A_2\left(a(T,0)\,\psi'_r(0)+f_r^{(1)}(\partial_t-a(T,0)\,\partial_x)\,f_r^{(2)}(T,0)\right)+g'_s(T)$$

we know that the vector $(\psi_r' - \tilde{G}_-' - \tilde{F}_-', -\psi_s' + \tilde{G}_+' + \tilde{F}_+')^T$ stastisfies the conditions of Lemma 3.1.Therefore, the following equations have a solution:

$$\psi'(0) = A^k B(H_{k+1}'(0) + G_{k+1}'(0) + F_{k+1}'(0)) + \cdots + B(H_1'(0) + G_1'(0) + F_1'(0)), \tag{2.28}$$

$$\psi'(L) = A^k B(H_{k+1}'(L) + G_{k+1}'(L) + F_{k+1}'(L)) + \cdots + B(H_1'(L) + G_1'(L) + F_1'(L)), \tag{2.29}$$

to determine the values of $h_i'$ at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L),(i = 1, \cdots, n; p = 1, \cdots, k+1)$. Now we continue to determine the other values of $h_i(t)$.(2.18) can be written as

$$\psi(x) = A^k B H_{k+1}(x) + \left[B, AB, \ldots, A^{k-1}B\right] \hat{H}(x),\ x \in (0, L), \tag{2.30}$$

where

$$\hat{H}(x) = \left(H_1^T(x), \cdots, H_k^T(x)\right)^T.$$

From (2.4) and the implicit function existence theorem, we know that there exist $n$ components of $\hat{H}(x)$ denoted as $H[n](x)$,s.t.

$$H[n](x) = \mathcal{H}\left(\psi(x), H_{k+1}(x), \hat{H}[\hat{n}](x)\right), \tag{2.31}$$

where $\hat{H}[\hat{n}](x)$ is the other components of $\hat{H}(x)$, and $\mathcal{H}$ is a $C^1$ continuous function. For $H_{k+1}(x)$, we utilize the interpolation method to obtain the following equations:

$$h_r(t_{k+1}(L;x)) = p\left(L - x; h_r(t_{k+1}(L;0)), -h_r'(t_{k+1}(L;0))\right) \tag{2.32}$$

$$h_s(t_{k+1}(0;x)) = p\left(x; h_s(t_{k+1}(0;L)), h_s'(t_{k+1}(0;L))\right) \tag{2.33}$$

where $p(x;\alpha,\alpha'),\ x \in [0, L]$ is a polynomial determined by $\alpha, \alpha'$ satisfying $p(x;\alpha,\alpha') = 0,\ x \in [0, \frac{7L}{8}]$ and $p(L) = \alpha,\ p'(L) = \alpha'$ .Similarly,for the components of $\hat{H}[\hat{n}](x)$, we can also use the interpolation method to obtain the following equations:

$$h_r\left(\tilde{t}_p(L;x)\right) = p\left(x; h_r\left(\tilde{t}_p(L;0)\right), h_r'\left(\tilde{t}_p(L;0)\right), h_r\left(\tilde{t}_p(L;L)\right), h_r'\left(\tilde{t}_p(L;L)\right)\right), p = 1, \cdots, k. \tag{2.34}$$

$$h_s(t_p(0;x)) = p\left(x; h_s(t_p(0;0)), h_s'(t_p(0;0)), h_s(t_p(0;L)), h_s'(t_p(0;L))\right), p = 1, \cdots, k. \tag{2.35}$$

where $p(x;\alpha,\alpha',\beta,\beta'),\ x \in [0, L]$ is a polynomial determined by $\alpha, \alpha', \beta, \beta'$ satisfying $p(0) = \alpha,\ p'(0) = \alpha',\ p(L) = \beta,\ p'(L) = \beta'$ . Finally, we use (2.31) to obtain the values of $H[n](x),\ x \in (0, L)$.From the $C^1$ continuity of $\mathcal{H}$ , (2.25) and (2.26), (2.28) and (2.29) and (2.4) ,we know that $h_i(t)$ is $C^1$ continuous in $[0, T]$ . A solution $u(t, x)$ is obtained using the obtained boundary conditions and initial value conditions. Then

$$\begin{aligned}
u(T, x) &= (u^-(\tilde{t}_1(L;x), L), u^+(t_1(0;x), 0))^T + F_1(x) \\
&= A(u^-(t_1(0;x), 0), u^+(\tilde{t}_2(L;x), L))^T + BH_1(x) + G_1(x) + F_1(x) \\
&= A(u^-(t_2, L), u^+(\tilde{t}_2, 0))^T + AF_2(x) + BH_1(x) + G_1(x) + F_1(x) \\
&\cdots \\
&= A^k B(H_{k+1}(x) + G_{k+1}(x) + F_{k+1}(x)) + \cdots + B(H_1(x) + G_1(x) + F_1(x)),\ x \in [0, L].
\end{aligned} \tag{2.36}$$

Therefore, $u(T, x) = \psi(x)$ and thereby $h_i(t), i = 1, \cdots, n$ are exactly the control function that satisfies the requirement. □

From the proof above, it is not difficult to obtain the $C^1$ estimate for $u(t, x)$ and $h_i(t)$, $i = 1, \cdots, n$. From (2.13), we get

$$\|z_p(x;x_0) - 1\|_{c^0} \le \|b - 1\|_{c^1}(1 + C\|\partial_t b\|_{c^0})^p \tag{2.37}$$

Using (2.25), (2.26), (2.28), (2.29) and (2.37), we know the estimate of $h_i(t)$, $i = 1, \cdots, n$ at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L),(p = 1, \cdots, k+1)$.

$$\|h_i(t_p(\lambda_1;\lambda_2))\|_{C^1} \le C_{A,k}(\|\psi\|_{C^1} + \|\varphi\|_{C^1} + \|g\|_{C^1} + \|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}), \lambda_1, \lambda_2 \in \{0, L\},\ p = 1, \cdots, k+1. \tag{2.38}$$

## 3. Appendice

**Lemma 3.1.** *For the equation*

$$\psi = A^k BH_{k+1} + \cdots + ABH_2 + BH_1. \tag{3.1}$$

$\psi, H_i \in \mathbb{R}^n, i = 1, \cdots, k+1,$

$$A = \begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix},$$

$$B = \begin{pmatrix} I_{r_1} & & & \\ & O_{r_2} & & \\ & & I_{s_1} & \\ & & & O_{s_2} \end{pmatrix},$$

*where* $r_1 + r_2 = r,\ s_1 + s_2 = s, r + s = n,\ A_1 = (a_{ij}),\ i = 1, \cdots, r,\ j = r+1, \cdots, n,$ $A_2 = (a_{ij}),\ i = r+1, \cdots, n,\ j = 1, \cdots, r$.*If* $\psi$ *satisfies the following condition*

$$\psi_i = \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r + s_1 + 1, \cdots, r+s. \tag{3.2}$$

*and the Kalman rank condition holds as follows,*

$$\text{Rank}\left[B, A^k B, \ldots, A^{k-1}B\right] = n, \tag{3.3}$$

*where* $r + s = n$,*then there exists a set of solutions* $H_p(x), (p = 1, \cdots, k+1)$ *of equation* (3.1) *satisfying*

$$H^-_{2m-1} = H^-_{2m}, \tag{3.4}$$

$$H^+_{2m+1} = H^+_{2m}. \tag{3.5}$$

*Proof.* Without loss of generality, we can assume that $k$ is odd. The condition 3.2 can be written as

$$\psi^+ = A_2\psi^- + \bar{H}. \tag{3.6}$$

$\bar{H}$ is the difference term, i.e.

$$\bar{H}_i = \psi_i - \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r+1, \cdots, r+s_1. \tag{3.7}$$

By the definition of the matrix $B$, the $i$-th $(i = r_1 + 1, \cdots, r, s_1 + 1, \cdots, r+s)$ components of $H_p(p = 1, \cdots, k+1)$ can be taken to be zero.The following is an attempt to write conditions (3.4)-(3.5) as matrix equations. Denote

$$H = \left(H^-_{k+1}, H^+_{k+1}, \cdots, H^-_1, H^+_1\right)^T.$$

Then the equation (3.1) ,(**??**)-(3.4)can be written as a matrix equation as follows:

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = \begin{pmatrix} & A_1A_2\cdots A_2 & A_1\cdots A_1 & & \cdots & & A_1 & I_r & \\ A_2A_1\cdots A_1 & & & A_2\cdots A_2 & \cdots & A_2 & & & I_s \\ I_r & & -I_r & & & & & & \\ & & & I_s & & & & & \\ & & & & \cdots & & & & \\ & & & & & & -I_s & & \\ & & & & & I_r & & -I_r & \end{pmatrix} H.$$

After appropriate matrix row transformations, the first row of the equation can be written as

$$\psi = \begin{pmatrix} A_1A_2\cdots A_2 & A_1\cdots A_1 & \cdots & A_1 & I_r & \\ & A_2A_1\cdots A_1 & \cdots & A_2A_1 & A_2 & I_s \end{pmatrix} H.$$

Let $H^+_{k+1} = 0$. Combined with (3.6), equation (3.1) can be rewritten as

$$\begin{pmatrix} \psi^- \\ \bar{H} \end{pmatrix} = \begin{pmatrix} A_1\cdots A_1 & \cdots & A_1 & I_r & \\ & \cdots & & & I_s \end{pmatrix} H. \tag{3.8}$$

Notice that (**??**) yields that the rank of the equation generalization matrix is equal to the rank of the original coefficient matrix, and thus (3.8) has a solution. □

**Corollary.** *Under the conditions of the above lemma, if $\psi$ satisfies*

$$\psi_i = \sum_{j=r+1}^{r+s} a_{ij}\psi_j,\ i = r_1+1,\cdots,r, \tag{3.9}$$

*then there exists a set of solutions $H_p(x), (p=1,\cdots,k+1)$ of equation (3.1) satisfying*

$$H^+_{2m-1} = H^+_{2m}, \tag{3.10}$$

$$H^-_{2m+1} = H^-_{2m}. \tag{3.11}$$

*Proof.* Define $\tilde{\psi}$ as $\tilde{\psi}^- = \psi^+$,$\tilde{\psi} = \psi^-$,$\tilde{A}$ as $\tilde{A}^1 = A^2$,$\tilde{A}^2 = A^1$, and $\tilde{H}_p$ as $\tilde{H}^-_p = H^+_p$,$\tilde{H}^+_p = H^-_p$, which are the same as the above lemma. □

**Corollary.** *In the proof of Lemma 3.1 one obtains $H^+_1 = \bar{H}$, and in the conditions of Corollary 3 one obtains $H^-_1 = \bar{H}$.*

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