# CONTROLLABILITY TO SYSTEMS OF QUASILINEAR WAVE EQUATIONS WITH FEWER CONTROLS

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ABSTRACT. This study addresses challenges in solving exact boundary controllability problems of quasilinear wave equations with Dirichlet type boundary conditions satisfying some kind of Kalman rank conditions. Utilizing techniques such as characteristic method and a specially designed linearization iteration scheme, we developed controls that use fewer boundary controls than the classical theory.

## 1. INTRODUCTION

Consider the controllability for system of the quasilinear wave equations below

$$\begin{cases} U_{tt} - a^2\left(U, U_t, U_x\right) U_{xx} = 0, & (t,x) \in \mathbb{R}^+ \times [0, L], \\ t = 0 : (U, U_t) = (\varphi(x), \psi(x)), & x \in [0, L], \\ x = L : U = 0, & t \in \mathbb{R}^+, \\ x = 0 : U_t - D\left(U, U_t, U_x\right) U_x = H(t), & t \in \mathbb{R}^+, \\ t = T : (U, U_t) = (\Phi(x), \Psi(x)), & x \in [0, L], \end{cases} \tag{1.1}$$

where

$$U = (u_1, \ldots, u_n)^T (t,x) \in C^2([0,T] \times [0,L]).$$

$a\left(U, U_t, U_x\right) > 0$ is the $C^1$ uniform wave speed. $D\left(U, U_t, U_x\right) = \left(d_{ij}\left(U, U_t, U_x\right)\right)_{i,j=1}^n$ is an $n \times n$ matrix with $C^1$ regularity. Part of the components of $H(t) = (h_1(t), \ldots, h_n(t))^T$ would be chosen as boundary controls. We denote $D^{\pm}\left(U, U_t, U_x\right) = aI_n \pm D$, which we request are invertible. We also request that

$$\begin{aligned} &\|(\varphi(x), \psi(x))\|_{(C^{2,1}[0,L])^n \times (C^{1,1}[0,L])^n} < \varepsilon, \\ &\|(\Phi(x), \Psi(x))\|_{(C^{2,1}[0,L])^n \times (C^{1,1}[0,L])^n} < \varepsilon, \end{aligned}$$

where $0 < \varepsilon < \varepsilon_0 \ll 1$. Without loss of generosity, we set $a(0,0,0) = 1$, or we consider the equations of $\bar{U}(t,x) = U(t, a(0,0,0)x)$, and $H(t) = (h_1(t), \ldots, h_p(t), 0, \ldots, 0)^T$. Now we try to reduce the wave equation to a hyperbolic equation system. Denote

$$\begin{aligned} &\tilde{U} = D^+(0,0,0)U, \\ &\tilde{V}^{\pm} = \partial_t \tilde{U} \mp \tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \partial_x \tilde{U}. \end{aligned}$$

where

$$\tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) = a\left(D^+(0,0,0)^{-1}U, D^+(0,0,0)^{-1}U_t, D^+(0,0,0)^{-1}U_x\right).$$

Denote

$$\tilde{D}^{\pm}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) = D^{\pm}\left(D^+(0,0,0)^{-1}U, D^+(0,0,0)^{-1}U_t, D^+(0,0,0)^{-1}U_x\right).$$

The original equations can be changed into

$$\begin{cases} \partial_t \tilde{V}^{\pm} \pm \tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \partial_x \tilde{V}^{\pm} = \mp \partial_x \tilde{U}\left(\partial_t \pm \tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \partial_x\right)\left(\tilde{a}\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right)\right), \\ x = L : \tilde{V}^- = -\tilde{V}^+, \\ x = 0 : \tilde{D}^-\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \tilde{D}^+(0,0,0)^{-1} \tilde{V}^- = -\tilde{D}^+\left(\tilde{U}, \tilde{U}_t, \tilde{U}_x\right) \tilde{D}^+(0,0,0)^{-1} \tilde{V}^+ + \tilde{H}(t), \end{cases} \tag{1.2}$$

where $\tilde{H}(t) = 2aH(t)$. Then the boundary condition can be written as

$$x = 0 : \widetilde{V}^{+} = B_0\widetilde{V}^{-} + \widetilde{H}(t) - \left(\widetilde{D}^{-}\left(\widetilde{U}, \widetilde{U}_t, \widetilde{U}_x\right) - \widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+}$$
$$- \left(\widetilde{D}^{-}\left(\widetilde{U}, \widetilde{U}_t, \widetilde{U}_x\right) - \widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-},$$

where the coupling matrix

$$B_0 = -\left(a(0,0,0)I_n - D(0,0,0)\right)\left(a(0,0,0)I_n + D(0,0,0)\right)^{-1}.$$

Define the controlling matrix as

$$C = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

and then the Kalman type condition as

$$\operatorname{Rank}\left[C, B_0C, \ldots, B_0^{k-1}C\right] = n, \tag{1.3}$$

where $k$ is a positive integer.

The following theorem can be obtained.

**Theorem 1.** *Under Assumption* (1.3)*, if $T > kL$, there exists a small $\varepsilon_0$ and a set of $C^1$ smooth control functions $h_i(t)$, $i = 1, \cdots, p$, such that for any $\varepsilon < \varepsilon_0$ and any $\Phi(x)$, $\varphi(x)$ which are $C^{2,1}$ smooth and satisfy the $C^2$ compatibility condition and $\Psi(x)$, $\psi(x)$ which are $C^{1,1}$ smooth and satisfy the $C^1$ compatibility condition, there exists a $C^2$ smooth solution $U(t,x)$ to the control problem* (1.1).

The proof of Theorem 1 requires the following proposition.

**Proposition 1.1.** *Taking $U^{(0)} \equiv 0$, consider the following iterative problem*

$$\begin{cases} \partial_t V^{\pm(m+1)} \pm a^{(m)}\partial_x V^{\pm(m+1)} = \mp\partial_x U^{(m)}\left(\partial_t \pm a^{(m)}\partial_x\right)\left(a^{(m)}\right), \\ x = L : V^{-(m+1)} = -V^{+(m+1)}, \quad t \in \mathbb{R}^+, \\ x = 0 : V^{-(m+1)} = B_0V^{+(m+1)} + \widetilde{H}^{(m+1)}(t) \\ + \left(\widetilde{D}^{+}\left(U^{(m)}, U_t^{(m)}, U_x^{(m)}\right) - \widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}V^{-(m)} \\ - \left(\widetilde{D}^{-}\left(U^{(m)}, U_t^{(m)}, U_x^{(m)}\right) - \widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}V^{+(m)}. \\ t = 0 : \left(U^{(m+1)}, U_t^{(m+1)}\right) = \left(\varphi(x), \psi(x)\right), \\ t = T : \left(U^{(m+1)}, U_t^{(m+1)}\right) = \left(\Phi(x), \Psi(x)\right), \end{cases} \tag{1.4}$$

*where*

$$a^{(m)} = a\left(U^{(m)}, U_t^{(m)}, U_x^{(m)}\right).$$

*One can get uniform bounds*

$$\left\|U^{(m)}\right\|_{C^{2,1}} \le C_P\varepsilon, \tag{1.5}$$

$$\left\|H^{(m)}\right\|_{C^{1,1}} \le C_P\varepsilon, \tag{1.6}$$

*as well as Cauchy sequence*

$$\left\|U^{(m)} - U^{(m-1)}\right\|_{C^2} \le C_P\varepsilon\alpha^{m-1}, \tag{1.7}$$

*where $\alpha$ is a constant satisfying $0 < \alpha < 1$ and $C_P$ is a large constant. The sequence $\{H^{(m)}\}_{m=1}^{\infty}$ is also a Cauchy sequence in $C^1$ space, i.e.*

$$\left\|H^{(m)} - H^{(m-1)}\right\|_{C^1} \le C_P\varepsilon\alpha^{m-1}. \tag{1.8}$$

Theorem 1 follows after obtaining Proposition 1.1. In fact, The sequence $\{U^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence in $C^2$ space, and thus converges to some $C^2$ function $U$ uniformly. Therefore, $U$ is $C^2$ smooth and satisfies the control condition, where $U^{(m+1)}$ for each step can be obtained using the Dirichlet boundary condition as

$$U^{(m+1)}(t,x) = -\int_x^L \partial_x U^{(m+1)} \mathrm{d}x,$$

where

$$\partial_x U^{(m+1)} = \frac{V^{-(m+1)} - V^{+(m+1)}}{2a\left(U^{(m)}, U_t^{(m)}, U_x^{(m)}\right)}.$$

## 2. Linear Controllability Problem

We need to introduce a lemma about the linear equation. Consider the following linear equation system

$$\begin{cases} \partial_t u_i - a(t,x)\partial_x u_i = f_i^{(1)}(t,x)(\partial_t - a(t,x)\partial_x) f_i^{(2)}(t,x), \ i = 1, \cdots, r, \\ \partial_t u_i + a(t,x)\partial_x u_i = f_i^{(1)}(t,x)(\partial_t + a(t,x)\partial_x) f_i^{(2)}(t,x), \ i = r+1, \cdots, r+s, \end{cases} \tag{2.1}$$

where $r+s=n$ and $a(t,x)$ is a $C^{1,1}$ function satisfying that $\|a(t,x)-1\|_{C^{1,1}} \le \varepsilon_0$. Denote $\tilde{a} = a-1$. $\varepsilon_0$ is a constant small enough. $f_i^{(\kappa)}(t,x)$, $i = 1, \cdots, r+s$, $\kappa = 1, 2$ are $C^{1,1}$ functions. The boundary conditions are given as follows:

$$\begin{cases} x = 0: \ u_i = \sum_{j=1}^{r} a_{ij} u_j + g_i(t) + h_i(t), \ i = r+1, \cdots, n, \\ x = L: \ u_i = \sum_{j=r+1}^{n} a_{ij} u_j + g_i(t) + h_i(t), \ i = 1, \cdots, r, \end{cases} \tag{2.2}$$

where $g_i(t)$, $i = 1, \cdots, n$, are $C^{1,1}$ function, $h_i(t)$, $i = 1, \cdots, n$, are $C^{1,1}$ controlling function. The matrix $A = (a_{ij})$, $i = 1, \cdots, n$, $j = 1, \cdots, n$, is a matrix. $u(t,x)$ satisfies

$$u(0,x) = \varphi(x). \tag{2.3}$$

The matrix $B$ is defined as follows:

$$B = \begin{pmatrix} I_{r_1} & & & \\ & O_{r_2} & & \\ & & I_{s_1} & \\ & & & O_{s_2} \end{pmatrix},$$

where $r_1 + r_2 = r$, $s_1 + s_2 = s$, $r + s = n$. The Kalman rank condition is given as follows:

$$\text{Rank}\left[B, AB, \dots, A^{k-1}B\right] = n, \tag{2.4}$$

where $k$ is a positive integer. The following lemma is the main result of this section.

**Lemma 2.1.** *Under the conditions of* (2.4)*, if $T > kL$, then there exists a set of $C^{1,1}$ smooth control functions $h_i(t)$, $i = 1, \cdots, r_1, r+1, \cdots, r+s_1$, such that for any $C^{1,1}$ smooth functions $g_i(t), f_i^{\kappa}(t,x)$, $i = 1, \cdots, n$ ,$\kappa = 1, 2$ and any $C^{1,1}$ smooth functions $\varphi(x)$, $\psi(x)$, which satisfy the $C^1$ compatibility conditions, there exists a $C^{1,1}$ smooth solution $u(t,x)$ to the control problem* (2.1)-(2.3) *satisfying*

$$u(T,x) = \psi(x). \tag{2.5}$$

*Proof.* Without loss of generality, we can assume that $k$ is odd, $T - kL < \frac{L}{4}$ and $\varphi(x) = 0$, or take any set of $\tilde{h}_i(t)$ that satisfies the compatibility conditions at $(0,0)$ and $(0,L)$ and consider the equations of $\tilde{u}(t,x)$ satisfying $\tilde{u}(0,x) = \varphi(x)$ below:

$$\begin{cases} \partial_t \tilde{u}_i - a(t,x)\partial_x \tilde{u}_i = \chi(t) f_i^{(1)}(t,x)(\partial_t - a(t,x)\partial_x) f_i^{(2)}(t,x), \ i = 1, \cdots, r, \\ \partial_t \tilde{u}_i + a(t,x)\partial_x \tilde{u}_i = \chi(t) f_i^{(1)}(t,x)(\partial_t + a(t,x)\partial_x) f_i^{(2)}(t,x), \ i = r+1, \cdots, r+s, \end{cases} \tag{2.6}$$

where $\chi(t)$ is a smooth function that satisfies $\chi(t)=1$ for $t\in[0,\frac{L}{2}]$ and $\chi(t)=0$ for $t\in[T-\frac{L}{2},T]$. The boundary conditions are given as follows:

$$\left\{\begin{array}{l} x=0:\ \tilde{u}_i=\sum_{j=1}^{r} a_{ij}\tilde{u}_j+\chi(t)g_i(t)+\tilde{h}_i\left(t\right),\ i=r+1,\cdots,r+s, \\ x=L:\ \tilde{u}_i=\sum_{j=r+1}^{n} a_{ij}\tilde{u}_j+\chi(t)g_i(t)+\tilde{h}_i\left(t\right),\ i=1,\cdots,r. \end{array}\right. \tag{2.7}$$

Then we can get that $\tilde{u}(T,x)$ satisfies the $C^1$ compatibility conditions at $(T,0)$ and $(T,L)$ and consider the following controllability problem:

$$\left\{\begin{array}{l} \partial_t u_i-a(t,x)\partial_x u_i=(1-\chi(t))f_i^{(1)}(t,x)(\partial_t-a(t,x)\partial_x)f_i^{(2)}(t,x),\ i=1,\cdots,r, \\ \partial_t u_i+a(t,x)\partial_x u_i=(1-\chi(t))f_i^{(1)}(t,x)(\partial_t+a(t,x)\partial_x)f_i^{(2)}(t,x),\ i=r+1,\cdots,r+s, \end{array}\right. \tag{2.8}$$

where the boundary conditions are given as follows:

$$\left\{\begin{array}{l} x=0:\ u_i=\sum_{j=1}^{r} a_{ij}u_j+(1-\chi(t))g_i(t)+h_i\left(t\right),\ i=r+1,\cdots,r+s, \\ x=L:\ u_i=\sum_{j=r+1}^{n} a_{ij}u_j+(1-\chi(t))g_i(t)+h_i\left(t\right),\ i=1,\cdots,r. \end{array}\right. \tag{2.9}$$

The initial and final conditions are given as follows:

$$u(0,x)=0,\quad u(T,x)=\psi(x)-\tilde{u}(T,x). \tag{2.10}$$

Finally, the two sets of control functions and solutions obtained can be added separately.

In the following we first study the characteristic lines of equations. Let

$$b(t,x)=\frac{1}{a(t,x)}.$$

The characteristic lines of the system (2.1) are given as follows:

$$\left\{\begin{array}{l} \frac{\mathrm{d}\tilde{t}_p}{\mathrm{d}x}=-b\left(\tilde{t}_p,x\right), \\ \tilde{t}_p(0;x_0)=t_{p-1}(0;x_0),\ p=2,\cdots,k+1, \\ \tilde{t}_1(x_0;x_0)=T, \end{array}\right. \tag{2.11}$$

$$\left\{\begin{array}{l} \frac{\mathrm{d}t_p}{\mathrm{d}x}=b\left(t_p,x\right), \\ t_p(L;x_0)=\tilde{t}_{p-1}(L;x_0),\ p=2,\cdots,k+1, \\ t_1(x_0;x_0)=T, \end{array}\right. \tag{2.12}$$

thereby obtaining the following equations:

$$\begin{aligned} t_1\left(x;x_0\right)&=T+\int_{x_0}^{x} b\left(t_1\left(s;x_0\right),s\right)\mathrm{d}s, \\ t_p\left(x;x_0\right)&=t_{p-1}\left(L;x_0\right)+\int_{L}^{x} b\left(t_p\left(s;x_0\right),s\right)\mathrm{d}s,\ p=2,\cdots,k+1, \\ \partial_{x_0}t_1\left(x;x_0\right)&=-b\left(T,x_0\right)+\int_{x_0}^{x}\partial_t b\left(t_1\left(s;x_0\right),s\right)\partial_{x_0}t_1\left(s;x_0\right)\mathrm{d}s, \\ \partial_{x_0}t_p\left(x;x_0\right)&=\partial_{x_0}t_{p-1}\left(L;x_0\right)+\int_{L}^{x}\partial_t b\left(t_p\left(s;x_0\right),s\right)\partial_{x_0}t_p\left(s;x_0\right)\mathrm{d}s,\ p=2,\cdots,k+1. \end{aligned} \tag{2.13}$$

Denote $z_p(x;x_0)=\partial_{x_0}t_p\left(x;x_0\right)$, which conduces that

$$\frac{\mathrm{d}}{\mathrm{d}x}z_p\left(x;x_0\right)=\partial_t b_p\left(t\left(x;x_0\right),x\right)z_p\left(x;x_0\right).$$

Then we can get the following equations:

$$\begin{aligned} z_1\left(x;x_0\right)&=-b\left(T,x_0\right)\exp\left(\int_{x_0}^{x}\partial_t b\left(t_1\left(s;x_0\right),s\right)\mathrm{d}s\right), \\ z_p\left(x;x_0\right)&=\partial_{x_0}t_{p-1}\left(L;x_0\right)\exp\left(\int_{L}^{x}\partial_t b\left(t_p\left(s;x_0\right),s\right)\mathrm{d}s\right). \end{aligned} \tag{2.14}$$

Similarly one can obtain that

$$\begin{aligned}
&\tilde{t}_1(x;x_0) = T - \int_{x_0}^{x} b\left(\tilde{t}_1(s;x_0), s\right) \mathrm{d}s,\\
&\tilde{t}_p(x;x_0) = t_{p-1}(0;x_0) - \int_{0}^{x} b\left(\tilde{t}_p(s;x_0), s\right) \mathrm{d}s,\ p = 2,\cdots,k+1,\\
&\tilde{z}_1(x;x_0) = b(T,x_0) \exp\left(-\int_{x_0}^{x} \partial_t b\left(\tilde{t}_1(s;x_0), s\right) \mathrm{d}s\right),\\
&\tilde{z}_p(x;x_0) = \partial_{x_0} t_{p-1}(0;x_0) \exp\left(\int_{0}^{x} \partial_t b\left(\tilde{t}_p(s;x_0), s\right) \mathrm{d}s\right),\ p = 2,\cdots,k+1.
\end{aligned}$$

Therefore

$$\begin{cases} t_1(x;0) = T,\\ t_{2m+1}(x;0) = t_{2m}(x;0),\\ t_{2m}(x;L) = t_{2m-1}(x;L), \end{cases} \tag{2.15}$$

$$\begin{cases} \tilde{t}_1(x;L) = T,\\ \tilde{t}_{2m+1}(x;L) = \tilde{t}_{2m}(x;L),\\ \tilde{t}_{2m}(x;0) = \tilde{t}_{2m-1}(x;0), \end{cases} \tag{2.16}$$

and

$$\begin{cases} \partial_{x_0} t_1(x;0) = -b(T,0),\\ \partial_{x_0} t_{2m+1}(x;0) = -\partial_{x_0} t_{2m}(x;0),\\ \partial_{x_0} t_{2m}(x;L) = -\partial_{x_0} t_{2m-1}(x;L), \end{cases} \tag{2.17}$$

$$\begin{cases} \partial_{x_0} \tilde{t}_1(x;L) = b(T,L),\\ \partial_{x_0} \tilde{t}_{2m+1}(x;L) = -\partial_{x_0} \tilde{t}_{2m}(x;L),\\ \partial_{x_0} \tilde{t}_{2m}(x;0) = -\partial_{x_0} \tilde{t}_{2m-1}(x;0). \end{cases} \tag{2.18}$$

The following defines a set of vector functions:

$$H_p(x) = \left(H^-\left(\tilde{t}_p(L;x)\right), H^+\left(t_p(0;x)\right)\right)^T,\ p = 1,\cdots,k+1,$$

where $H^-(t)$ is an $r$-dimensional vector function and $H^+(t)$ is an $s$-dimensional vector function defined as follows:

$$\begin{aligned}
H^-(t) &= \left(h_1(t),\cdots,h_{r_1}(t), 0,\cdots,0\right),\\
H^+(t) &= \left(h_{r+1}(t),\cdots,h_{r+s_1}(t), 0,\cdots,0\right).
\end{aligned}$$

We require that $\tilde{t}_{k+1}(x;x_0) = 0$ if there exists such $\tilde{x} \in [0,L]$ such that $\tilde{x} \le x$ and $\tilde{t}_{k+1}(\tilde{x};x_0) = 0$, and $t_{k+1}(x;x_0) = 0$ if there exists such $\hat{x} \in [0,L]$ such that $\hat{x} \ge x$ and $t_{k+1}(\hat{x};x_0) = 0$. Denote

$$u(t,x) = \left(u^-(t,x), u^+(t,x)\right)^T,$$

$$H(t) = \left(H^-(t), H^+(t)\right)^T,$$

where

$$\begin{aligned}
u^-(t,x) &= \left(u_1(t,x),\cdots,u_r(t,x)\right),\\
u^+(t,x) &= \left(u_{r+1}(t,x),\cdots,u_{r+s}(t,x)\right).
\end{aligned}$$

Define

$$G_p(x) = \left(G^-\left(\tilde{t}_p(L;x)\right), G^+\left(t_p(0;x)\right)\right)^T,$$

where

$$\begin{aligned}
G^-(t) &= \left(g_1(t),\cdots,g_r(t)\right),\\
G^+(t) &= \left(g_{r+1}(t),\cdots,g_{r+s}(t)\right).
\end{aligned}$$

Define

$$F_1(x) = \left(\int_x^L F^-\left(\tilde{t}_1(s;x), s\right) \mathrm{d}s, \int_0^x F^+\left(t_1(s;x), s\right) \mathrm{d}s\right)^T,$$

$$F_p(x) = \left(\int_0^L F^-\left(\tilde{t}_p(s;x), s\right) \mathrm{d}s, \int_0^L F^+\left(t_p(s;x), s\right) \mathrm{d}s\right)^T,\ p = 2,\cdots,k+1,$$

where

$$F^{-}(t,x)=\left(f_1^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_1^{(2)}(t,x),\cdots,f_r^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_r^{(2)}(t,x)\right),$$

$$F^{+}(t,x)=\left(f_{r+1}^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_1^{(2)}(t,x),\cdots,f_{r+s}^{(1)}(t,x)(b(t,x)\partial_t-\partial_x)f_{r+s}^{(2)}(t,x)\right).$$

If a solution $u(t,x)$ satisfying the requirement exists, for any $x\in[0,L]$, we can get the following equations with characteristic lines method:

$$\begin{aligned}\psi(x)&=u(T,x)\\&=(u^-(\tilde t_1(L;x),L),u^+(t_1(0;x),0))^T+F_1(x)\\&=A(u^-(t_1(0;x),0),u^+(\tilde t_2(L;x),L))^T+BH_1(x)+G_1(x)+F_1(x)\\&=A(u^-(t_2,L),u^+(\tilde t_2,0))^T+AF_2(x)+BH_1(x)+G_1(x)+F_1(x)\\&\cdots\\&=A^kB(H_{k+1}(x)+G_{k+1}(x)+F_{k+1}(x))+\cdots+B(H_1(x)+G_1(x)+F_1(x)),\ x\in[0,L].\end{aligned}\tag{2.19}$$

Using this equation, we first determine the value and derivative of $h_i$ at $(t_p(0;0))$, $(t_p(0;L))$, $(t_p(L;0))$ and $(t_p(L;L))$, $(i=1,\cdots,n;\ p=1,\cdots,k+1)$. Consider the vector $\tilde G(x)=\sum_{l=0}^k A^lG_{l+1}(x)$.

$$\tilde G^-(L)=G_1^-(L)+A_1G_2^+(L)+A_1A_2G_3^-(L)+\cdots+A_1A_2\cdots A_1G_{k+1}^+(L),$$

$$\tilde G^+(L)=G_1^+(L)+A_2G_2^-(L)+A_2A_1G_3^+(L)+\cdots+A_2A_1\cdots A_2G_{k+1}^-(L).$$

Recognizing (2.15)-(2.16) and $T-kL<\frac{L}{4}$, we can get

$$G_{k+1}^-(L)=0,$$

$$G_m^+(L)=G_{m+1}^+(L),\ m\ is\ odd of 1,\cdots,k,$$

$$G_m^+(L)=G_{m+1}^+(L),\ m\ is\ even of 1,\cdots,k.$$

Therefore,

$$\begin{aligned}\tilde G^+(L)&=G_1^+(L)+A_2G_2^-(L)+A_2A_1G_3^+(L)+\cdots+A_2A_1\cdots A_1G_k^+(L)\\&=G_2^+(L)+A_2G_3^-(L)+A_2A_1G_4^+(L)+\cdots+A_2A_1\cdots A_1G_{k+1}^+(L),\end{aligned}\tag{2.20}$$

and thereby we can get the following equations:

$$\tilde G^-(L)=G_1^-(L)+A_1\tilde G^+(L).$$

Similarly,

$$\tilde G^+(0)=G_1^+(0)+A_2\tilde G^-(0).$$

Namely,

$$\begin{pmatrix}\tilde G^-(L)\\ \tilde G^+(0)\end{pmatrix}=\begin{pmatrix} & A_1\\ A_2 & \end{pmatrix}\begin{pmatrix}\tilde G^-(0)\\ \tilde G^+(L)\end{pmatrix}+\begin{pmatrix}g^-(T)\\ g^+(T)\end{pmatrix}.\tag{2.21}$$

Now consider the vector $\tilde F(x)=\sum_{l=0}^k A^lF_{l+1}(x)$.

$$\tilde F^-(L)=F_1^-(L)+A_1F_2^+(L)+A_1A_2F_3^-(L)+\cdots+A_1A_2\cdots A_1F_{k+1}^+(L),$$

$$\tilde F^+(L)=F_1^+(L)+A_2F_2^-(L)+A_2A_1F_3^+(L)+\cdots+A_2A_1\cdots A_2F_{k+1}^-(L).$$

From the defination of $\chi(t)$, we can assume that $f_i^{(\kappa)}(t,x)=0,\ t<\frac{L}{2},\ \kappa=1,2,\ i=1,\cdots,n.$ Recognizing (2.15)-(2.16) and $T-kL<\frac{L}{4}$, we can get

$$F_1^-(L)=0,$$

$$F_{k+1}^-(L)=0,$$

$$F_m^+(L)=F_{m+1}^+(L),\ m\ is\ odd of 1,\cdots,k,$$

$$F_m^+(L)=F_{m+1}^+(L),\ m\ is\ even of 1,\cdots,k.$$

Therefore,

$$\begin{aligned}\tilde{F}^{+}(L) &= F_1^{+}(L) + A_2F_2^{-}(L) + A_2A_1F_3^{+}(L) + \cdots + A_2A_1\cdots A_1F_k^{+}(L)\\ &= F_2^{+}(L) + A_2F_3^{-}(L) + A_2A_1F_4^{+}(L) + \cdots + A_2A_1\cdots A_1F_{k+1}^{+}(L),\end{aligned} \tag{2.22}$$

and thereby we can get the following equations:

$$\tilde{F}^{-}(L) = A_1\tilde{F}^{+}(L).$$

Similarly,

$$\tilde{F}^{+}(0) = A_2\tilde{F}^{-}(0).$$

Namely,

$$\begin{pmatrix}\tilde{F}^{-}(L)\\ \tilde{F}^{+}(0)\end{pmatrix} = \begin{pmatrix} & A_1\\ A_2 & \end{pmatrix}\begin{pmatrix}\tilde{F}^{-}(0)\\ \tilde{F}^{+}(L)\end{pmatrix}. \tag{2.23}$$

The $C^0$ compatibility condition of $\psi(x)$ is

$$\psi_i\left(0\right) = \sum_{j=1}^{r} a_{ij}\psi_j\left(0\right) + g_i(T),\ \ i = r+s_1+1,\cdots,r+s, \tag{2.24}$$

$$\psi_i\left(L\right) = \sum_{j=r+1}^{n} a_{ij}\psi_j\left(L\right) + g_i(T),\ \ i = r_1+1,\cdots,r. \tag{2.25}$$

So $\psi - \tilde{G} - \tilde{F}$ satisfies the conditions of Lemma 4.1. Therefore, the following equations have a solution:

$$\psi\left(0\right) = A^k\left(H_{k+1}\left(0\right) + G_{k+1}\left(0\right) + F_{k+1}\left(0\right)\right) + \cdots + \left(H_1\left(0\right) + G_1\left(0\right) + F_1\left(0\right)\right), \tag{2.26}$$

$$\psi\left(L\right) = A^k\left(H_{k+1}\left(L\right) + G_{k+1}\left(L\right) + F_{k+1}\left(L\right)\right) + \cdots + \left(H_1\left(L\right) + G_1\left(L\right) + F_1\left(L\right)\right), \tag{2.27}$$

to determine the values of $h_i$ at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L)$, $(i = 1,\cdots,n; p = 1,\cdots,k+1)$. Derivation of (2.19) with respect to $x$ gives the following equations:

$$\psi'\left(x\right) = A^kB(H'_{k+1}\left(x\right) + G'_{k+1}\left(x\right) + F'_{k+1}\left(x\right)) + \cdots + B(H'_1\left(x\right) + G'_1\left(x\right) + F'_1\left(x\right)),\ \ x \in \left[0,L\right], \tag{2.28}$$

where

$$H'_p\left(x\right) = \left(\frac{\mathrm{d}H^{-}}{\mathrm{d}t}\left(\tilde{t}_p(L;x)\right)\frac{\partial\tilde{t}_p(L;x)}{\partial x}, \frac{\mathrm{d}H^{+}}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$G'_p\left(x\right) = \left(\frac{\mathrm{d}G^{-}}{\mathrm{d}t}\left(\tilde{t}_p(L;x)\right)\frac{\partial\tilde{t}_p(L;x)}{\partial x}, \frac{\mathrm{d}G^{+}}{\mathrm{d}t}\left(t_p(0;x)\right)\frac{\partial t_p(0;x)}{\partial x}\right)^T,$$

$$\begin{aligned}F'_1(x) = \Big(&-f^{(1)-}(b\partial_t - \partial_x)f^{(2)-}(T,x) + \int_x^L \frac{\partial f^{(1)-}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(2)-}\left(\tilde{t}_1(s,x),s\right)\frac{\partial\tilde{t}_1}{\partial x}(s;x)\mathrm{d}s\\ &- \int_x^L \frac{\partial f^{(2)-}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(1)-}\left(\tilde{t}_1(s,x),s\right)\frac{\partial\tilde{t}_1}{\partial x}(s;x)\mathrm{d}s + f^{(1)-}\frac{\partial f^{(2)-}}{\partial t}\left(\tilde{t}_1(s;x),s\right)\frac{\partial\tilde{t}_1}{\partial x}(s;x)\Big|_x^L,\\ &f^{(1)+}(b\partial_t - \partial_x)f^{(2)+}(T,x) + \int_0^x \frac{\partial f^{(1)+}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(2)+}\left(t_1(s,x),s\right)\frac{\partial t_1}{\partial x}(s;x)\,\mathrm{d}s\\ &- \int_0^x \frac{\partial f^{(2)+}}{\partial t}\left(b\partial_t - \partial_x\right)f^{(1)+}\left(t_1(s,x),s\right)\frac{\partial t_1}{\partial x}(s;x)\mathrm{d}s + f^{(1)+}\frac{\partial f^{(2)+}}{\partial t}\left(t_1(s;x),s\right)\frac{\partial t_1}{\partial x}(s;x)\Big|_0^x\Big)^T,\end{aligned}$$

$$F_p'(x) = \left( f^{(1)-}\frac{\partial f^{(2)-}}{\partial t}\left(\tilde{t}_p(s;x), s\right)\frac{\partial \tilde{t}_p}{\partial x}(s;x)\Big|_0^L + \int_0^L \frac{\partial f^{(1)-}}{\partial t}(b\partial_t - \partial_x) f^{(2)-}\left(\tilde{t}_p(s,x), s\right)\frac{\partial \tilde{t}_p}{\partial x}(s;x)\,\mathrm{d}s \right.$$
$$- \int_0^L \frac{\partial f^{(2)-}}{\partial t}(b\partial_t - \partial_x) f^{(1)-}\left(\tilde{t}_p(s,x), s\right)\frac{\partial \tilde{t}_p}{\partial x}(s;x)\,\mathrm{d}s,$$
$$f^{(1)+}\frac{\partial f^{(2)+}}{\partial t}\left(t_p(s;x), s\right)\frac{\partial t_p}{\partial x}(s;x)\Big|_0^L + \int_0^L \frac{\partial f^{(1)+}}{\partial t}(b\partial_t - \partial_x) f^{(2)+}\left(t_p(s,x), s\right)\frac{\partial t_p}{\partial x}(s;x)\,\mathrm{d}s$$
$$\left. - \int_0^L \frac{\partial f^{(2)+}}{\partial t}(b\partial_t - \partial_x) f^{(1)+}\left(t_p(s,x), s\right)\frac{\partial t_p}{\partial x}(s;x)\,\mathrm{d}s \right)^T.$$

Consider the vector $\tilde{G}'(x) = \sum_{l=0}^k A^l G_{l+1}'(x)$ and $\tilde{F}'(x) = \sum_{l=0}^k A^l F_{l+1}'(x)$. Similar to the previous case using (2.15)-(2.18), we can get the following equations:

$$\tilde{G}^{-\prime}(L) = b(T,L)\, g^{-\prime}(T) - A_1 \tilde{G}^{+\prime}(L),$$
$$\tilde{G}^{+\prime}(0) = -b(T,L)\, g^{+\prime}(T) - A_2 \tilde{G}^{-\prime}(0),$$
$$\tilde{F}^{-\prime}(L) + f^{(1)-\prime}(T,L)\,\partial_s f^{(2)-\prime}(T,L) = A_1\left(-\tilde{F}^{+\prime}(L) + f^{(1)+\prime}(T,L)\,\partial_s f^{(2)+\prime}(T,L)\right),$$
$$-\tilde{F}^{+\prime}(0) + f^{(1)+\prime}(T,0)\,\partial_s f^{(2)+\prime}(T,0) = A_2\left(\tilde{F}\prime_{-}(L) + f^{(1)-\prime}(T,0)\,\partial_s f^{(2)-\prime}(T,0)\right).$$

Combined with the $C^1$ compatibility condition satisfied by $\psi(x)$:

$$\begin{aligned} &a(T,L)\,\psi^{-\prime}(L) + f^{(1)-}\big(\partial_t - a(T,L)\,\partial_x\big) f^{(2)-}(T,L) \\ &= A_2\big(-a(T,L)\,\psi^{+\prime}(L) + f^{(1)+}\big(\partial_t + a(T,L)\,\partial_x\big) f^{(2)+}(T,L)\big) \\ &\quad + g^{-\prime}(T) + h^{-\prime}(T). \end{aligned}$$

$$\begin{aligned} &-a(T,0)\,\psi^{+\prime}(0) + f^{(1)+}\big(\partial_t + a(T,0)\,\partial_x\big) f^{(2)+}(T,0) \\ &= A_2\big(a(T,0)\,\psi^{-\prime}(0) + f^{(1)-}\big(\partial_t - a(T,0)\,\partial_x\big) f^{(2)-}(T,0)\big) \\ &\quad + g^{+\prime}(T) + h^{+\prime}(T). \end{aligned}$$

we know that the vector $(\psi^{-\prime} - \tilde{G}^{-\prime} - \tilde{F}^{-\prime}, -\psi^{+\prime} + \tilde{G}^{+\prime} + \tilde{F}^{+\prime})^T$ stastisfies the conditions of Lemma 4.1.Therefore, the following equations have a solution:

$$\psi'(0) = A^k B(H_{k+1}'(0) + G_{k+1}'(0) + F_{k+1}'(0)) + \cdots + B(H_1'(0) + G_1'(0) + F_1'(0)), \tag{2.29}$$

$$\psi'(L) = A^k B(H_{k+1}'(L) + G_{k+1}'(L) + F_{k+1}'(L)) + \cdots + B(H_1'(L) + G_1'(L) + F_1'(L)), \tag{2.30}$$

to determine the values of $h_i'$ at $t_p(0;0), t_p(0;L), t_p(L;0), t_p(L;L)$,$i = 1,\cdots,n;\ p = 1,\cdots,k+1$. Now we continue to determine the other values of $h_i(t)$. (2.19) can be written as

$$\psi(x) - \tilde{G}(x) - \tilde{F}(x) = A^k B H_{k+1}(x) + \left[B, AB, ..., A^{k-1}B\right]\hat{H}(x), x \in (0,L), \tag{2.31}$$

where

$$\hat{H}(x) = \left(H_1^T(x), \cdots, H_k^T(x)\right)^T.$$

From (2.4) and the implicit function existence theorem, we know that there exist $n$ components of $\hat{H}(x)$ denoted as $H[n](x)$, s.t.

$$H[n](x) = \mathcal{H}\left(\psi(x) - \tilde{G}(x) - \tilde{F}(x), H_{k+1}(x), H[\hat{n}](x)\right), \tag{2.32}$$

where $H[\hat{n}](x)$ is the other components of $\hat{H}(x)$, and $\mathcal{H}$ is a $C^1$ continuous function. For $H_{k+1}(x)$, we utilize the interpolation method to obtain the following equations:

$$H^-(t_{k+1}(L;x)) = p\left(L - x; H^-(t_{k+1}(L;0)), -H^{-\prime}(t_{k+1}(L;0))\right), \tag{2.33}$$

$$H^+(t_{k+1}(0;x)) = p\left(x; H^+(t_{k+1}(0;L)), H^{+\prime}(t_{k+1}(0;L))\right), \tag{2.34}$$

where $p(x;\alpha,\alpha'),\ x\in[0,L]$ is a polynomial determined by $\alpha,\alpha'$ satisfying $p(x;\alpha,\alpha')=0,\ x\in[0,T-\frac{t_0}{4}]$ and $p(L)=\alpha,\ p'(L)=\alpha'$ .Similarly, for the components of $H\left[\hat{n}\right](x)$, we can also use the interpolation method to obtain the following equations:

$$H^-\left(\tilde{t}_p\left(L;x\right)\right)=p\left(x;H^-\left(\tilde{t}_p\left(L;0\right)\right),H^{-\prime}\left(\tilde{t}_p\left(L;0\right)\right),H^-\left(\tilde{t}_p\left(L;L\right)\right),H^{-\prime}\left(\tilde{t}_p\left(L;L\right)\right)\right),p=1,\cdots,k, \tag{2.35}$$

$$H^+\left(t_p\left(0;x\right)\right)=p\left(x;H^+\left(t_p\left(0;0\right)\right),H^{+\prime}\left(t_p\left(0;0\right)\right),H^+\left(t_p\left(0;L\right)\right),H^{+\prime}\left(t_p\left(0;L\right)\right)\right),p=1,\cdots,k, \tag{2.36}$$

where $p(x;\alpha,\alpha',\beta,\beta'),\ x\in[0,L]$ is a polynomial determined by $\alpha,\alpha',\beta,\beta'$ satisfying $p(0)=\alpha,\ p'(0)=\alpha',\ p(L)=\beta,\ p'(L)=\beta'$ . Finally, we use (2.32) to obtain the values of $H\left[n\right](x)\ ,\ x\in(0,L)$.From the $C^1$ continuity of $\mathcal{H}$ , (2.26) and (2.27), (2.29) and (2.30) and (2.4) ,we know that $h_i(t)$ is $C^1$ continuous in $[0,T]$ . A solution $u(t,x)$ is obtained using the obtained boundary conditions and initial value conditions. Then

$$\begin{aligned}
u(T,x)&=(u^-(\tilde{t}_1(L;x),L),u^+(t_1(0;x),0))^T+F_1(x)\\
&=A(u^-(t_1(0;x),0),u^+(\tilde{t}_2(L;x),L))^T+BH_1(x)+G_1(x)+F_1(x)\\
&=A(u^-(t_2,L),u^+(\tilde{t}_2,0))^T+AF_2(x)+BH_1(x)+G_1(x)+F_1(x)\\
&\cdots\\
&=A^kB(H_{k+1}\left(x\right)+G_{k+1}\left(x\right)+F_{k+1}\left(x\right))+\cdots+B(H_1\left(x\right)+G_1\left(x\right)+F_1\left(x\right)),\ x\in\left[0,L\right].
\end{aligned} \tag{2.37}$$

Therefore, $u(T,x)=\psi(x)$ and thereby $h_i(t),i=1,\cdots,n$ are exactly the control function that satisfies the requirement. □

From the proof above, it is not difficult to obtain the $C^1$ estimate for $u(t,x)$ and $h_i(t)$, $i=1,\cdots,n$. From (2.14), we get

$$\left\|z_p\left(x;x_0\right)-1\right\|_{C^0}\leq\left\|\tilde{a}\right\|_{C^1}\left(1+C\left\|\tilde{a}\right\|_{C^0}\right)^p. \tag{2.38}$$

Using (2.26), (2.27), (2.29), (2.30) and (2.38), we know the estimate of $h_i(t),\ i=1,\cdots,n$ at $t_p(0;0),t_p(0;L),t_p(L;0),t_p(L;L)$,$(p=1,\cdots,k+1)$.

$$\|h_i(t_p(\lambda_1;\lambda_2))\|_{C^1}\leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\lambda_1,\lambda_2\in\{0,L\}. \tag{2.39}$$

From the defination $H_{k+1}(x)$ and $H\left[\hat{n}\right](x)$, we know that

$$\|H_{k+1}(x)\|_{C^1}\leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L], \tag{2.40}$$

and

$$\|H\left[\hat{n}\right](x)\|_{C^1}\leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L]. \tag{2.41}$$

Using (2.32), we can get the estimate of $H\left[n\right](x)$:

$$\|H\left[n\right](x)\|_{C^1}\leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ x\in[0,L]. \tag{2.42}$$

Namely,

$$\|H(t)\|_{C^1}\leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1}+\|\varphi\|_{C^1}+\|g\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}). \tag{2.43}$$

Then we try to get the estimate of $u(t,x)$: From (2.1), we know that

$$\frac{\mathrm{d}u^+}{\mathrm{d}x}\left(t_1(x;\xi),x\right)=f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}x}\left(t_1(x;\xi),x\right).$$

Thereby,

$$u^+\left(t_1\left(x;\xi\right),x\right)=u^+(0,\xi)+\int_\xi^x f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}s}\left(t_1\left(s;\xi\right),s\right)\mathrm{d}s.$$

Therefore,

$$\left|u^+\left(t_1(x;\xi),x\right)\right|\leq C_{\varepsilon_0}\left(\|\varphi\|_{C^0}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Similarly, we can get the estimate of $u^-\left(\tilde{t}_1(x;\xi),x\right)$:

$$\left|u^-\left(\tilde{t}_1(x;\xi),x\right)\right|\leq C_{\varepsilon_0}\left(\|\varphi\|_{C^0}+\left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Below we make inductive assumptions that for $j = 1, \cdots, p-1$, the following estimates hold:

$$\left|u^+\left(t_j(x;\xi),x\right)\right| + \left|u^-\left(\tilde{t}_j(x;\xi),x\right)\right| \leq C_{A,k,\varepsilon_0}\left(\|\varphi\|_{C^1} + \|\psi\|_{C^1} + \|g\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right). \tag{2.44}$$

Then for $j = p$, we have

$$\frac{\mathrm{d}u^+}{\mathrm{d}x}\left(t_p(x;\xi),x\right) = f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}x}\left(t_p(x;\xi),x\right),$$

and thereby

$$u^+\left(t_p\left(x;\xi\right),x\right) = u^+(t_p(0,\xi),0) + \int_0^x f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}s}\left(t_p\left(s;\xi\right),s\right)\mathrm{d}s.$$

Using (2.7) and the inductive assumption, we can get

$$u^+\left(t_p(0;\xi),0\right) = A_2u^-\left(\tilde{t}_{p-1}(0;\xi),0\right) + G^+ + H^+,$$

and then

$$\left|u^+\left(t_p(x;\xi),x\right)\right| \leq C_{A,k,\varepsilon_0}\left(\|\varphi\|_{C^1} + \|\psi\|_{C^1} + \|g\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Similarly, we can get the same estimate of $u^-\left(\tilde{t}_p(x;\xi),x\right)$. Therefore, (2.44) holds for $j = p$. Multiply both sides of (2.6) by $b(t,x)$ and then derive for $t$, denoting $w^+$ as the derivative of $u^+$ with respect to $t$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}x}w^+ + \partial_t b w^+ = \partial_t\left(f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}x}\right).$$

Then we can get

$$w^+\left(t_1(x;\xi),x\right) = w^+(0,\xi) + \int_\xi^x \partial_t b w^+\mathrm{d}s + \int_\xi^x \partial_t f^{(1)+}\frac{\mathrm{d}f^{(2)}}{\mathrm{d}s}\mathrm{d}s + f^{(1)+}\partial_t f^{(2)+}\Big|_\xi^x - \int_\xi^x \partial_t f^{(2)+}\frac{\mathrm{d}f^{(1)+}}{\mathrm{d}s}\mathrm{d}s. \tag{2.45}$$

where

$$w^+(0,\xi) = -a\varphi'(\xi) + f^{(1)+}\left(\partial_t + a\partial_x\right)f^{(2)+}. \tag{2.46}$$

Thereby

$$\left|w^+(0,\xi)\right| \leq C_{\varepsilon_0}\left(\|\varphi\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Then we can get the estimate of $w^+\left(t_1(x;\xi),x\right)$:

$$\left|w^+\left(t_1(x;\xi),x\right)\right| \leq \|\tilde{a}\|_{C^1}\int_\xi^x \left|w^+\right|\mathrm{d}s + C_{\varepsilon_0}\left(\|\varphi\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Using Gronwall's inequality, we can get

$$\left|w^+\left(t_1(x;\xi),x\right)\right| \leq C_{\varepsilon_0}\left(\|\varphi\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Similarly, we can get the estimate of $w^-\left(\tilde{t}_1(x;\xi),x\right)$:

$$\left|w^-\left(\tilde{t}_1(x;\xi),x\right)\right| \leq C_{\varepsilon_0}\left(\|\varphi\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right).$$

Below we make inductive assumptions that for $j = 1, \cdots, p-1$, the following estimates hold:

$$\left|w\left(t_j(x;\xi),x\right)\right| + \left|w^-\left(\tilde{t}_j(x;\xi),x\right)\right| \leq C_{A,k,\varepsilon_0}\left(\|\varphi\|_{C^1} + \|\psi\|_{C^1} + \|g\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right), \tag{2.47}$$

Then for $j = p$, we have

$$\begin{aligned} w^+\left(t_p(x;\xi),x\right) = w^+(t_p(0,\xi),0) &+ \int_0^x \partial_t b w^+\mathrm{d}s + \int_0^x \partial_t f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}s}\mathrm{d}s \\ &+ f^{(1)+}\partial_t f^{(2)+}\Big|_0^x - \int_0^x \partial_t f^{(2)+}\frac{\mathrm{d}f^{(1)+}}{\mathrm{d}s}\mathrm{d}s, \end{aligned} \tag{2.48}$$

where

$$w^{+}(t_p(0;\xi),0) = A_2 w^{-}\left(\tilde{t}_{p-1}(0;\xi),0\right) + G^{+\prime} + H^{+\prime}. \tag{2.49}$$

and then using (2.47) and Gronwall' inequality, we can get (2.47) holds for $j = p$. Finally, we can get the estimate of $u(t,x)$:

$$\|u(t,x)\|_{C^1} \leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1} + \|\varphi\|_{C^1} + \|g\|_{C^1} + \|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}),\ t \in [0,T],\ x \in [0,L]. \tag{2.50}$$

Below we estimate the continuity paradigm of $h_i'(t)$, $\partial_x u(t,x)$ and $\partial_t u(t,x)$ . For $H\left[\hat{n}\right]$ and $H_{k+1}$ , using (2.33)- (2.36) , we have

$$\left|h_i'\left(t_1\right) - h_i'\left(t_2\right)\right| \leq C_{k,\varepsilon_0}(\|p\|_{C^2}\left|x_1 - x_2\right| + \|p\|_{C^1}\left\|\tilde{a}\right\|_{C^{1,1}}\left|x_1 - x_2\right|),$$

where $t_1 = t_p(\lambda; x_1)$, $t_2 = t_p(\lambda; x_2)$, $\lambda \in 0, L$, $x_1, x_2 \in [0,L]$ and

$$\|p\|_{C^2} \leq C\|h\|_{C^1} \leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^1} + \|\varphi\|_{C^1} + \|g\|_{C^1} + \|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}).$$

Using (2.38), we can get

$$\left|x_1 - x_2\right| \leq C_{k,\varepsilon_0}\left|t_p\left(\lambda; x_1\right) - t_p\left(\lambda; x_2\right)\right| = C_{k,\varepsilon_0}\left|t_1 - t_2\right|.$$

For $H\left[n\right]$, using (2.32) and (2.42), we can get

$$\left|h_i'\left(t_1\right) - h_i'\left(t_2\right)\right| \leq C_A(\|\psi\|_{C^{1,1}} + \|\varphi\|_{C^{1,1}} + \|H\left[\hat{n}\right]\|_{C^{1,1}} + \|H_{k+1}\|_{C^{1,1}}).$$

Thereby, we can get the $C^{1,1}$ estimate of $h_i(t)$:

$$\|H(t)\|_{C^{1,1}} \leq C_{A,k,\varepsilon_0}(\|\psi\|_{C^{1,1}} + \|\varphi\|_{C^{1,1}} + \|g\|_{C^{1,1}} + \left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}).$$

Then we try to get the $C^{1,1}$ estimate of $u(t,x)$.We first consider $\omega\left(\eta \mid \partial_t u(\cdot,x)\right)$. From (2.48) and (2.46), we know that

$$\begin{aligned}&\left|w^{+}\left(t_1\left(x;\xi_1\right),x\right) - w^{+}\left(t_1\left(x;\xi_2\right),x\right)\right| \leq C\left(\left|w^{+}\left(0,\xi_1\right) - w^{+}\left(0,\xi_2\right)\right| + \|u\|_{C^1}\left\|\tilde{a}\right\|_{C^{1,1}}\left|\xi_1 - \xi_2\right|\right.\\&\left.+\left\|\tilde{a}\right\|_{C^1}\int_0^x\left|w^{+}\left(t_1\left(s;\xi_1\right),s\right) - w^{+}\left(t_1\left(s;\xi_2\right),s\right)\right|\mathrm{d}s + \left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}\left|\xi_1 - \xi_2\right|\right),\end{aligned}$$

where

$$\begin{aligned}\left|w^{+}\left(0,\xi_1\right) - w^{+}\left(0,\xi_2\right)\right| \leq C&\left(\|\varphi\|_{C^{1,1}} + \left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}\right.\\&\left.+\left(\|\varphi\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1}\right)\left\|\tilde{a}\right\|_{C^{1,1}}\right)\left|\xi_1 - \xi_2\right|.\end{aligned}$$

Using Gronwall' inequality ,(2.14) and (2.38), we can get

$$\left|\xi_1 - \xi_2\right| \leq C_{\varepsilon_0}\left|t_1\left(x;\xi_1\right) - t_1\left(x;\xi_2\right)\right|,$$

and then

$$\begin{aligned}\omega\left(\eta \mid w^{+}\left(t_1\left(x;\cdot\right),x\right)\right) \leq C_{A,k,\varepsilon_0}&\left(\|\varphi\|_{C^{1,1}} + \left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}\right.\\&\left.+\ (\|\psi\|_{C^1} + \|\varphi\|_{C^1} + \|g\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1})\left\|\tilde{a}\right\|_{C^{1,1}}\right)\eta\end{aligned}$$

Similarly, we can get the same estimate of $\omega\left(\eta \mid w^{-}\left(\tilde{t}_1\left(x;\cdot\right),x\right)\right)$. Below we make inductive assumptions that for $j = 1, \cdots, p-1$, the following estimates hold:

$$\begin{aligned}&\omega\left(\eta \mid w^{+}\left(t_j\left(x;\cdot\right),x\right)\right) + \omega\left(\eta \mid w^{-}\left(\tilde{t}_j\left(x;\cdot\right),x\right)\right) \leq C_{A,k,\varepsilon_0}\left(\|\varphi\|_{C^{1,1}} + \|\psi\|_{C^{1,1}}\right.\\&\left.+\left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}} + \|g\|_{C^{1,1}} + (\|\psi\|_{C^1} + \|\varphi\|_{C^1} + \|g\|_{C^1} + \left\|f^{(1)}\right\|_{C^1}\left\|f^{(2)}\right\|_{C^1})\left\|\tilde{a}\right\|_{C^{1,1}}\right)\eta\end{aligned} \tag{2.51}$$

Then for $j = p$, from (2.48), we have

$$\begin{aligned}\left|w^{+}\left(t_p\left(x;\xi_1\right),x\right) - w^{+}\left(t_p\left(x;\xi_2\right),x\right)\right| \leq C&\left(\left|w^{+}\left(t_p\left(0;\xi_1\right),0\right) - w^{+}\left(t_p\left(0;\xi_2\right),0\right)\right|\right.\\&+\left\|\tilde{a}\right\|_{C^1}\int_0^x\left|w^{+}\left(\xi_1,s\right) - w^{+}\left(\xi_2,s\right)\right|\mathrm{d}s\\&\left.+\left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}\left|\xi_1 - \xi_2\right| + \|u\|_{C^1}\left\|\tilde{a}\right\|_{C^{1,1}}\left|\xi_1 - \xi_2\right|\right)\end{aligned}$$

where from (2.49), we know that

$$\omega\left(\eta \mid w^{+}\left(t_{p}(0; \cdot), 0\right)\right) \leq C_{A}\left(\omega\left(\eta \mid w\left(t_{p-1}(x; \cdot), x\right)\right)+\left(\|g\|_{C^{1,1}}+\|h\|_{C^{1,1}}\right) \eta\right).$$

Then using Gronwall's inequality and (2.51) and similar estimate of $\omega\left(\eta \mid w^{-}\left(\tilde{t}_{p}(x; \cdot), x\right)\right)$, we can get (2.51) holds for $j=p$. Next we consider $\omega\left(\eta \mid w^{+}\left(t_{p}(\cdot, \xi), \cdot\right)\right)$. From (2.45) and (2.48), we know that

$$\omega\left(\eta \mid w\left(t_{p}(\cdot ; \xi), \cdot\right)\right) \leq\left(\|\tilde{a}\|_{C^{1}}\|u\|_{C^{1}}+\left\|f^{(1)}\right\|_{C^{1}}\left\|f^{(2)}\right\|_{C^{1}}\right) \eta.$$

Similarly, we can get the same estimate of $\omega\left(\eta \mid w^{-}\left(\tilde{t}_{p}(\cdot ; \xi), \cdot\right)\right)$. Therefore, we get

$$\begin{aligned}\left\|\partial_{t} u\right\|_{C^{0,1}} \leq & C_{A, k, \varepsilon_{0}}\Big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^{1,1}}+\left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}+\|g\|_{C^{1,1}} \\ & +\left(\|\psi\|_{C^{1}}+\|\varphi\|_{C^{1}}+\|g\|_{C^{1}}+\left\|f^{(1)}\right\|_{C^{1}}\left\|f^{(2)}\right\|_{C^{1}}\right)\|\tilde{a}\|_{C^{1,1}}\Big)\end{aligned} \tag{2.52}$$

For $\partial_{x} u(t, x)$, from (2.1) we have

$$\left\|\partial_{x} u\right\|_{C^{0,1}} \leq C\left(\left\|\partial_{t} u\right\|_{C^{0,1}}+\|u\|_{C^{1}}\|\tilde{a}\|_{C^{1}}+\left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}\right).$$

Thereby, we can get

$$\begin{aligned}\|u\|_{C^{1,1}} \leq & C_{A, k, \varepsilon_{0}}\Big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^{1,1}}+\left\|f^{(1)}\right\|_{C^{1,1}}\left\|f^{(2)}\right\|_{C^{1,1}}+\|g\|_{C^{1,1}} \\ & +\left(\|\psi\|_{C^{1}}+\|\varphi\|_{C^{1}}+\|g\|_{C^{1}}+\left\|f^{(1)}\right\|_{C^{1}}\left\|f^{(2)}\right\|_{C^{1}}\right)\|\tilde{a}\|_{C^{1,1}}\Big)\end{aligned} \tag{2.53}$$

Then we consider the continuity of control functions and solutions with regard to the coefficients of equations. Consider another controlling problem with the coefficients $\bar{a}(t, x)$, $\bar{f}^{(1)}(t, x)$, $\bar{f}^{(2)}(t, x)$, $\bar{g}(t, x)$ and its solution $v(t, x)$, denoting $y=\partial_{t} v$, where we assume that the values of these coefficients are the same as those of $a(t, x)$, $f^{(1)}(t, x)$, $f^{(2)}(t, x)$, $g(t, x)$ at $t=0$ and $t=L$ and thereby we can select the same set of functions $\widetilde{H}(t)$ to satisfy the compatibility. We first consider the initial boundary value problem (2.6) and (2.7) with the coefficients $\bar{a}(t, x)$, $\bar{f}^{(1)}(t, x)$, $\bar{f}^{(2)}(t, x)$, $\bar{g}(t)$.From (2.14), we know that

$$\left\|t_{1}\left(x ; x_{0}\right)-\bar{t}_{1}\left(x ; x_{0}\right)\right\|_{C^{0}} \leq C_{\varepsilon_{0}}\|a-\bar{a}\|_{C^{0}}.$$

Select a point $(t_{0}, L)$, where $t_{0}=t_{1}(L ; \xi)$ for some $\xi$, and consider the characteristic curve passing through this point, denoting $\xi$ as the point where this characteristic curve $t_{1}(x ; t_{0})$ intersects with $t=0$ and $\bar{\xi}$ as the point where this characteristic curve $\bar{t}_{1}(x ; t_{0})$ intersects with $t=0$, we have

$$\left|x_{1}-\bar{x}_{1}\right| \leq C_{\varepsilon_{0}}\|a-\bar{a}\|_{C^{0}},$$

and

$$\begin{aligned}\tilde{u}^{+}\left(t_{1}(L ; \xi), L\right) & =\tilde{u}^{+}(0, \xi)+\int_{\xi}^{L} f^{(1)+} \frac{\mathrm{d} f^{(2)+}}{\mathrm{d} s}\left(t_{1}(s ; \xi), s\right) \mathrm{d} s, \\ \tilde{v}^{+}\left(\bar{t}_{1}(L ; \bar{\xi}), L\right) & =\tilde{v}^{+}(0, \bar{\xi})+\int_{\bar{\xi}}^{L} \bar{f}^{(1)+} \frac{\mathrm{d} \bar{f}^{(2)+}}{\mathrm{d} s}\left(\bar{t}_{1}(s ; \bar{\xi}), s\right) \mathrm{d} s.\end{aligned}$$

Thereby,

$$\begin{aligned}\left|\tilde{u}^{+}\big(t_{1}(L ; \xi), L\big)-\tilde{v}^{+}\big(\bar{t}_{1}(L ; \bar{\xi}), L\big)\right| \leq & \left|\tilde{v}^{+}(0, \bar{\xi})-\tilde{u}^{+}(0, \xi)\right| \\ & +C\Big(\big(\|f^{(1)}\|_{C^{1}}\|f^{(2)}\|_{C^{1}}+\|\bar{f}^{(1)}\|_{C^{1}}\|\bar{f}^{(2)}\|_{C^{1}}\big)|\xi-\bar{\xi}| \\ & \quad+\|\bar{f}^{(2)}\|_{C^{1}}\|f^{(1)}-\bar{f}^{(1)}\|_{C^{0}}+\|\bar{f}^{(2)}\|_{C^{1}}\|f^{(1)}\|_{C^{1}}\|t_{1}-\bar{t}_{1}\|_{C^{0}} \\ & \quad+\|f^{(1)}\|_{C^{0}}\|f^{(2)}-\bar{f}^{(2)}\|_{C^{1}}+\|f^{(1)}\|_{C^{0}}\|\bar{f}^{(2)}\|_{C^{1,1}}\|t_{1}-\bar{t}_{1}\|_{C^{0}} \\ & \quad+\|f^{(1)}\|_{C^{0}}\|f^{(2)}\|_{C^{1}}\|a-\bar{a}\|_{C^{0}}\Big).\end{aligned}$$

where

$$\left|\tilde{v}^{+}(0, \bar{\xi})-\tilde{u}^{+}(0, \xi)\right|=\left|\varphi(\xi)-\varphi(\bar{\xi})\right| \leq\|\varphi\|_{C^{1}}\left|\xi-\bar{\xi}\right|,$$

$$\|t_1-\bar{t}_1\|_{C^0}=\sup_{0\le s\le L}\left|\bar{t}_1(s;\bar{\xi})-t_1(s;\xi)\right|\le C_{\varepsilon_0}\|a-\bar{a}\|_{C^0}.$$

If the characteristic curve $\bar{t}_2(x;\bar{\xi})$ intersects with $x=0$ at $(\bar{t}_2(0;\bar{\xi}),0)$, we have

$$\tilde{v}^+\left(\bar{t}_2\left(L;\bar{\xi}\right),L\right)=\tilde{v}^+\left(\bar{t}_2\left(0;\bar{\xi}\right),0\right)+\int_0^L\bar{f}^{(1)+}\frac{\mathrm{d}\bar{f}^{(2)+}}{\mathrm{d}s}\left(\bar{t}_2\left(s;\bar{\xi}\right),s\right)\mathrm{d}s,$$

and

$$|\xi|\le C\|a-\bar{a}\|_{C^0},$$

$$\left|\bar{t}_2(0;\bar{\xi})\right|\le C\|a-\bar{a}\|_{C^0},$$

$$\begin{aligned}\left|\tilde{u}^+(0,\xi)-\tilde{v}^+\left(\bar{t}_2(0;\bar{\xi}),0\right)\right|&\le\left|\tilde{u}^+(0,\xi)-\tilde{u}^+(0,0)\right|+\left|\tilde{u}^+(0,0)-\tilde{v}^+(0,0)\right|+\left|\tilde{v}^+(0,0)-\tilde{v}^+\left(\bar{t}_2(0;\bar{\xi}),0\right)\right|\\&\le\|\varphi\|_{C^1}|\xi|+\|\tilde{v}\|_{C^{\bar{1}}}\left|\bar{t}_2(0;\bar{\xi})\right|.\end{aligned}$$

Thereby, we can get

$$\begin{aligned}\left|\tilde{u}^+(t_0,L)-\tilde{v}^+(t_0,L)\right|\le\ &C\Big(\left(\|\varphi\|_{C^1}+\|g\|_{C^1}+\|\bar{g}\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^0}\\&+\|\bar{f}^{(2)}\|_{C^1}\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}+\|f^{(1)}\|_{C^0}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\Big).\end{aligned}$$

Similarly, we can get the same estimate of $\left|\tilde{u}^-(\tilde{t}_0,0)-\tilde{v}^-(\tilde{t}_0,0)\right|$. Then we make inductive assumptions that for $j=1,\cdots,p-1$, the following estimates hold:

$$\begin{aligned}\left|\tilde{u}^+(t_0,L)-\tilde{v}^+(t_0,L)\right|\le\ &C\Big(\left(\|\varphi\|_{C^1}+\|g\|_{C^1}+\|\bar{g}\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^0}\\&+\|g-\bar{g}\|_{C^0}+\|\bar{f}^{(2)}\|_{C^1}\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}+\|f^{(1)}\|_{C^0}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\Big).\end{aligned}$$

and

$$\begin{aligned}\left|\tilde{u}^-(\tilde{t}_0,0)-\tilde{v}^-(\tilde{t}_0,0)\right|\le\ &C\Big(\left(\|\varphi\|_{C^1}+\|g\|_{C^1}+\|\bar{g}\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^0}\\&+\|g-\bar{g}\|_{C^0}+\|\bar{f}^{(2)}\|_{C^1}\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}+\|f^{(1)}\|_{C^0}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\Big).\end{aligned}$$

where $t_0=t_j(L;\xi)$ and $\tilde{t}_0=\tilde{t}_j(0;\xi)$ for some $\xi$. Then for $j=p$, from (2.50), we have

$$\tilde{u}^+\left(t_p\left(L;\xi\right),L\right)=\tilde{u}^+\left(t_p\left(0;\xi\right),0\right)+\int_0^L f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}s}\left(t_p\left(s;\xi\right),s\right)\mathrm{d}s,$$

$$\tilde{v}^+\left(\bar{t}_p\left(L;\bar{\xi}\right),L\right)=\tilde{v}^+\left(t_p\left(0;\xi\right),0\right)+\int_0^L\bar{f}^{(1)+}\frac{\mathrm{d}\bar{f}^{(2)+}}{\mathrm{d}s}\left(\bar{t}_p\left(s;\bar{\xi}\right),s\right)\mathrm{d}s,$$

where

$$\tilde{u}^+\left(t_p\left(0;\xi\right),0\right)=A_2\tilde{u}^-\left(\tilde{t}_{p-1}\left(0;\xi\right),0\right)+G^+\left(t_p(0;\xi)\right)+H^+\left(t_p(0;\xi)\right),$$

$$\tilde{v}^+\left(\bar{t}_p\left(0;\bar{\xi}\right),0\right)=A_2\tilde{v}^-\left(\bar{\tilde{t}}_{p-1}\left(0;\bar{\xi}\right),0\right)+\bar{g}_s\left(\bar{t}_p(0;\bar{\xi})\right)+H^+\left(\bar{t}_p(0;\bar{\xi})\right).$$

Therefore, using the inductive assumptions, we can get

$$\begin{aligned}\left|\tilde{u}^+\left(t_p(0;\xi),0\right)-\tilde{v}^+\left(\bar{t}_p(0;\bar{\xi}),0\right)\right|\le\ &C\Big(\left(\|\varphi\|_{C^1}+\|g\|_{C^1}+\|\bar{g}\|_{C^1}\right)\|a-\bar{a}\|_{C^0}\\&+\|g-\bar{g}\|_{C^0}+\|\bar{f}^{(2)}\|_{C^1}\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}+\|f^{(1)}\|_{C^0}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\Big).\end{aligned}$$

Then using the same method as $j=1,\cdots,p-1$, we can get that the above estimate holds for $j=p$. And thereby we can get

$$\begin{aligned}\|\tilde{u}-\tilde{v}\|_{C^0}\le\ &C\Big(\left(\|\varphi\|_{C^1}+\|g\|_{C^1}+\|\bar{g}\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^0}\\&+\|g-\bar{g}\|_{C^0}+\|\bar{f}^{(2)}\|_{C^1}\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}+\|f^{(1)}\|_{C^0}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\Big).\end{aligned}$$

Considering the point $(t_0, L)$, where $t_0 = t_0(L;\xi)$ for some $\xi$, and the characteristic curve $t_1(x;t_0)$ passing through this point and intersecting $t=0$ at $\xi$, as well as the characteristic curve $\bar{t}_1(x;t_0)$ passing through this point and intersecting $t=0$ at $\bar{\xi}$, we have

$$\tilde{w}^+\left(t_1(L;\xi),L\right) = \tilde{w}^+(0,\xi)+\int_\xi^L \partial_t b\tilde{w}^+ \mathrm{d}s+\int_\xi^L \partial_t f^{(1)+}\frac{\mathrm{d}f^{(2)+}}{\mathrm{d}s}\mathrm{d}s+f^{(1)+}\partial_t f^{(2)+}\Big|_\xi^L-\int_\xi^L \partial_t f^{(2)+}\frac{\mathrm{d}f^{(1)+}}{\mathrm{d}s}\mathrm{d}s,$$

$$\tilde{y}^+\left(\bar{t}_1(L;\bar{\xi}),L\right) = \tilde{y}^+(0,\bar{\xi})+\int_{\bar{\xi}}^L \partial_t \bar{b}\tilde{y}^+ \mathrm{d}s+\int_{\bar{\xi}}^L \partial_t \bar{f}^{(1)+}\frac{\mathrm{d}\bar{f}^{(2)+}}{\mathrm{d}s}\mathrm{d}s+\bar{f}^{(1)+}\partial_t \bar{f}^{(2)+}\Big|_{\bar{\xi}}^L-\int_{\bar{\xi}}^L \partial_t \bar{f}^{(2)+}\frac{\mathrm{d}\bar{f}^{(1)+}}{\mathrm{d}s}\mathrm{d}s,$$

where

$$\begin{aligned}\tilde{w}^+(0,\xi) &= -a\varphi^{+\prime}(\xi)+f^{(1)+}\left(\partial_t + a\partial_x\right)f^{(2)+},\\ \tilde{y}^+(0,\bar{\xi}) &= -\bar{a}\varphi^{+\prime}(\bar{\xi})+\bar{f}^{(1)+}\left(\partial_t + \bar{a}\partial_x\right)\bar{f}^{(2)+}.\end{aligned}$$

Then we have

$$\begin{aligned}&\left|\tilde{w}^+\big(t_1(L;\xi),L\big)-\tilde{y}^+\big(\bar{t}_1(L;\bar{\xi}),L\big)\right|\\
\le{}& C_{\varepsilon_0}\Big(\left|\tilde{w}^+(0,\xi)-\tilde{y}^+(0,\bar{\xi})\right| + \|\tilde{a}\|_{C^1}\int_\xi^L\left|\tilde{w}^+\big(t_1(s;\xi),s\big)-\tilde{y}^+\big(\bar{t}_1(s;\bar{\xi}),s\big)\right|\mathrm{d}s\\
&+\left(\|\tilde{a}\|_{C^1}\|\tilde{v}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^0}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)|\xi-\bar{\xi}|\\
&+\|\tilde{a}\|_{C^{1,1}}\|\tilde{u}\|_{C^1}\|t_1-\bar{t}_1\|_{C^0}+\|\tilde{u}\|_{C^1}\|a-\bar{a}\|_{C^1}\\
&+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}\\
&+\|\bar{f}^{(1)}\|_{C^1}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}\|_{C^1}\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\\
&+\|f^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}\|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\|f^{(2)}\|_{C^1}\|t_1-\bar{t}_1\|_{C^0}\\
&+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}\|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}\|a-\bar{a}\|_{C^1}\\
&+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^{1,1}}\|t_1-\bar{t}_1\|_{C^0}\Big).\end{aligned}$$

where

$$\begin{aligned}\left|\tilde{w}^+(0,\xi)-\tilde{y}^+(0,\bar{\xi})\right| \le{}& C_{\varepsilon_0}\Big(\|a-\bar{a}\|_{C^0}\ \|\varphi\|_{C^1}+\|\bar{a}\|_{C^1}\ \|\varphi\|_{C^1}\ |\xi-\bar{\xi}|+\|\varphi\|_{C^{1,1}}\ |\xi-\bar{\xi}|\\
&+\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}\ \|\bar{f}^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^0}\ \|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\\
&+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^0}\ \|\bar{f}^{(2)}\|_{C^{1,1}}\ \|t_1-\bar{t}_1\|_{C^0}\\
&+\|\bar{f}^{(1)}\|_{C^0}\ \|\bar{f}^{(2)}\|_{C^1}\ \|a-\bar{a}\|_{C^0}\Big).\end{aligned}$$

Using Gronwall's inequality, we can get

$$\begin{aligned}\left|\tilde{w}^+\big(t_1(L;\xi),L\big)-\tilde{y}^+\big(\bar{t}_1(L;\bar{\xi}),L\big)\right| \le{}& C_{\varepsilon_0}\Big(\|a-\bar{a}\|_{C^0}\ \|\varphi\|_{C^1}+\|\bar{a}\|_{C^1}\ \|\varphi\|_{C^1}\ |\xi-\bar{\xi}|+\|\varphi\|_{C^{1,1}}\ |\xi-\bar{\xi}|\\
&+\left(\|\tilde{a}\|_{C^1}\|\tilde{v}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^0}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)|\xi-\bar{\xi}|\\
&+\|\tilde{a}\|_{C^{1,1}}\ \|\tilde{u}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\tilde{u}\|_{C^1}\ \|a-\bar{a}\|_{C^1}\\
&+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1}\\
&+\|\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}\|_{C^1}\ \|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\\
&+\|f^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}\\
&+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}\ \|a-\bar{a}\|_{C^1}\\
&+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^{1,1}}\ \|t_1-\bar{t}_1\|_{C^0}\Big).\end{aligned}$$

If the characteristic curve $\bar{t}_2(x;\bar{\xi})$ intersects with $x=0$ at $(\bar{t}_2(0;\bar{\xi}),0)$, we have

$$\tilde{y}^-\left(\bar{t}_2(L;\bar{\xi}),L\right) = \tilde{y}^-(\bar{t}_2(0,\bar{\xi}),0)+\int_0^L \partial_t \bar{b}\tilde{y}^- \mathrm{d}s+\int_0^L \partial_t \bar{f}^{(1)+}\frac{\mathrm{d}\bar{f}^{(2)}}{\mathrm{d}s}\mathrm{d}s+\bar{f}^{(1)+}\partial_t \bar{f}^{(2)+}\Big|_0^L-\int_0^L \partial_t \bar{f}^{(2)+}\frac{\mathrm{d}\bar{f}^{(1)+}}{\mathrm{d}s}\mathrm{d}s,$$

and

$$\begin{aligned}\left|\tilde{y}^-(\bar{t}_2(0,\bar{\xi}),0)-\tilde{w}^+(0,\xi)\right| &\le \left|\tilde{y}^-(\bar{t}_2(0,\bar{\xi}),0)-\tilde{y}^-(0,0)\right|+\left|\tilde{w}^+(0,0)-\tilde{w}^+(0,\xi)\right| \\ &\le C_{\varepsilon_0}\left(\|\tilde{v}\|_{C^{1,1}}+\|\tilde{u}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^1}\end{aligned}$$

Therefore, we can get

$$\begin{aligned}\left|\tilde{w}^+\big(t_1(L;\xi),L\big)-\tilde{y}^+\big(\bar{t}_1(L;\bar{\xi}),L\big)\right| \le\ & C_{\varepsilon_0}\Big(\|a-\bar{a}\|_{C^0}\ \|\varphi\|_{C^1}+\|\bar{a}\|_{C^1}\ \|\varphi\|_{C^1}\ |\xi-\bar{\xi}|+\|\varphi\|_{C^{1,1}}\ |\xi-\bar{\xi}| \\ &+\left(\|\tilde{a}\|_{C^1}\|\tilde{v}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}\|\bar{f}^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^0}\|\bar{f}^{(2)}\|_{C^{1,1}}\right)|\xi-\bar{\xi}| \\ &+\|\tilde{a}\|_{C^{1,1}}\ \|\tilde{u}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\big(\|\tilde{u}\|_{C^{1,1}}+\|\tilde{v}\|_{C^{1,1}}\big)\ \|a-\bar{a}\|_{C^1} \\ &+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1} \\ &+\|\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}\|_{C^1}\ \|f^{(2)}-\bar{f}^{(2)}\|_{C^1} \\ &+\|f^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\ \|f^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0} \\ &+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}\ \|t_1-\bar{t}_1\|_{C^0}+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^1}\ \|a-\bar{a}\|_{C^1} \\ &+\|\bar{f}^{(1)}\|_{C^1}\ \|\bar{f}^{(2)}\|_{C^{1,1}}\ \|t_1-\bar{t}_1\|_{C^0}\Big).\end{aligned}$$

Similarly, we can get the same estimate of $\left|\tilde{w}^-\big(\tilde{t}_1(0;\xi),0\big)-\tilde{y}^-\big(\bar{\tilde{t}}_1(0;\bar{\xi}),0\big)\right|$. Then we make inductive assumptions that for $j=1,\cdots,p-1$, the following estimates hold:

$$\begin{aligned}&\left|\tilde{w}^+(t_0,L)-\tilde{y}^+(t_0,L)\right| \\ &\le\ C_{\varepsilon_0}\Big(\big(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1}\big)\big(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\big) \\ &\quad+\big(\|\varphi\|_{C^{1,1}}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\big)\ \|a-\bar{a}\|_{C^1} \\ &\quad+\|g^{(1)}-\bar{g}^{(1)}\|_{C^1}\Big),\end{aligned}$$

and

$$\begin{aligned}&\left|\tilde{w}^-(\tilde{t}_0,0)-\tilde{y}^-(\tilde{t}_0,0)\right| \\ &\le\ C_{\varepsilon_0}\Big(\big(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1}\big)\big(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\big) \\ &\quad+\big(\|\varphi\|_{C^{1,1}}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\big)\ \|a-\bar{a}\|_{C^1} \\ &\quad+\|g^{(1)}-\bar{g}^{(1)}\|_{C^1}\Big).\end{aligned}$$

where $t_0=t_j(L;\xi)$ for some $\xi$. Then for $j=p$, we have

$$\tilde{w}^+\left(t_p(L;\xi),L\right)=\tilde{w}^+(t_p(0,\xi),0)+\int_0^L\partial_t b\tilde{w}^+\mathrm{d}s+\int_0^L\partial_t f^{(1)+}\frac{\mathrm{d}f^{(2)}}{\mathrm{d}s}\mathrm{d}s+f^{(1)+}\partial_t f^{(2)+}\Big|_0^L-\int_0^L\partial_t f^{(2)+}\frac{\mathrm{d}f^{(1)+}}{\mathrm{d}s}\mathrm{d}s,$$

$$\tilde{y}^-\left(\bar{t}_p(L;\bar{\xi}),L\right)=\tilde{y}^-(\bar{t}_p(0,\bar{\xi}),0)+\int_0^L\partial_t\bar{b}\tilde{y}^-\mathrm{d}s+\int_0^L\partial_t\bar{f}^{(1)+}\frac{\mathrm{d}\bar{f}^{(2)}}{\mathrm{d}s}\mathrm{d}s+\bar{f}^{(1)+}\partial_t\bar{f}^{(2)+}\Big|_0^L-\int_0^L\partial_t\bar{f}^{(2)+}\frac{\mathrm{d}\bar{f}^{(1)+}}{\mathrm{d}s}\mathrm{d}s,$$

where

$$\begin{aligned}\tilde{w}^+\left(t_p\left(0;\xi\right),0\right)&=A_2\tilde{w}^-\left(t_{p-1}\left(0;\xi\right),0\right)+G^{+\prime}\left(t_p(0;\xi)\right)+H^{+\prime}\left(t_p(0;\xi)\right),\\ \tilde{y}^-\left(\bar{t}_p\left(0;\bar{\xi}\right),0\right)&=A_2\tilde{y}^-\left(\bar{t}_{p-1}\left(0;\bar{\xi}\right),0\right)+\bar{G}^{+\prime}\left(\bar{t}_p(0;\bar{\xi})\right)+H^{+\prime}\left(\bar{t}_p(0;\bar{\xi})\right).\end{aligned}$$

Therefore, using the inductive assumptions, we can get

$$\begin{aligned}&\left|\tilde{w}^+\big(t_p(0;\xi),0\big)-\tilde{y}^+\big(\bar{t}_p(0;\bar{\xi}),0\big)\right| \le\ C_{A,k,\varepsilon_0}\Big(\left(\|\tilde{u}\|_{C^{1,1}}+\|\tilde{v}\|_{C^{1,1}}\right)\|a-\bar{a}\|_{C^0} \\ &\quad+\left(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1}\right)\left(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\right) \\ &\quad+\big(\|\varphi\|_{C^{1,1}}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\big)\ \|a-\bar{a}\|_{C^1} \\ &\quad+\|g^{(1)}-\bar{g}^{(1)}\|_{C^1}\Big).\end{aligned}$$

Then using the same method as $j=1,\cdots,p-1$, we can get that the above estimate holds for $j=p$. And thereby we can get

$$\begin{aligned}\|\tilde{u}-\tilde{v}\|_{C^1}\leq\ & C_{A,k,\varepsilon_0}\Big((\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1})\,(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1})\\ &+\big(\|\varphi\|_{C^{1,1}}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\big)\,\|a-\bar{a}\|_{C^1}\\ &+\|g-\bar{g}\|_{C^1}\Big).\end{aligned}$$

Then we consider the value and derivative of $h_i(t_p(0;0))$, $h_i(t_p(0;L))$, $h_i(t_p(L;0))$, $h_i(t_p(L;L))$. From (2.19), we know that

$$\left|h_i\left(t_p\left(\lambda_1;\lambda_2\right)\right)-\bar{h}_i\left(\bar{t}_p\left(\lambda_1;\lambda_2\right)\right)\right|\leq\|\tilde{u}-\tilde{v}\|_{C^0}+\sum_{p=1}^{k+1}\left|G_p\left(\lambda_2\right)-\bar{G}_p\left(\lambda_2\right)\right|+\sum_{p=1}^{k+1}\left|F_p\left(\lambda_2\right)-\bar{F}_p\left(\lambda_2\right)\right|,$$

where $\lambda_1,\lambda_2=0,L$. And

$$\left|G_p\left(\lambda_2\right)-\bar{G}_p\left(\lambda_2\right)\right|\leq\|g-\bar{g}\|_{C^0}+\|g\|_{C^1}\,\|a-\bar{a}\|_{C^0}\,,$$

$$\begin{aligned}\left|F_p(\lambda_2)-\bar{F}_p(\lambda_2)\right|\leq\ & C_{\varepsilon_0,k}\Big((\|f^{(1)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\|a-\bar{a}\|_{C^0}\\ &+\|f^{(1)}-\bar{f}^{(1)}\|_{C^0}\ \|f^{(2)}\|_{C^1}+\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\ \|\bar{f}^{(1)}\|_{C^0}\Big).\end{aligned}$$

For the derivative, we have

$$\left|h_i'\left(t_p\left(\lambda_1;\lambda_2\right)\right)-\bar{h}_i'\left(\bar{t}_p\left(\lambda_1;\lambda_2\right)\right)\right|\leq C_{\varepsilon_0,k}\left(\sum_{p=1}^{k+1}\left|H_p'\left(\lambda_2\right)-\bar{H}_p'\left(\lambda_2\right)\right|+\|h\|_{C^1}\left(\sum_{p=1}^{k+1}\left|z_p'\left(\lambda_1;\lambda_2\right)-\bar{z}_p'\left(\lambda_1;\lambda_2\right)\right|\right)\right)$$

where

$$\left|H_p'\left(\lambda_2\right)-\bar{H}_p'\left(\lambda_2\right)\right|\leq C_{A,\varepsilon_0,k}\left(\|\tilde{u}-\tilde{v}\|_{C^1}+\sum_{p=1}^{k+1}\left|G_p'\left(\lambda_2\right)-\bar{G}_p'\left(\lambda_2\right)\right|+\sum_{p=1}^{k+1}\left|F_p'\left(\lambda_2\right)-\bar{F}_p'\left(\lambda_2\right)\right|\right),$$

$$\left|z_p'\left(\lambda_1;\lambda_2\right)-\bar{z}_p'\left(\lambda_1;\lambda_2\right)\right|\leq C_{\varepsilon_0,k}\left(\|a-\bar{a}\|_{C^1}+\|\tilde{a}\|_{C^{1,1}}\,\|a-\bar{a}\|_{C^0}\right),$$

and

$$\left|G_p'\left(\lambda_2\right)-\bar{G}_p'\left(\lambda_2\right)\right|\leq C_{\varepsilon_0,k}\left(\|g-\bar{g}\|_{C^1}+\|g\|_{C^{1,1}}\,\|a-\bar{a}\|_{C^0}+\|g\|_{C^1}\left(\sum_{p=1}^{k+1}\left|z_p'\left(\lambda_1;\lambda_2\right)-\bar{z}_p'\left(\lambda_1;\lambda_2\right)\right|\right)\right),$$

$$\begin{aligned}\left|F_p'(\lambda_2)-\bar{F}_p'(\lambda_2)\right|\leq\ & C_{\varepsilon_0,k}\Big((\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\,\|a-\bar{a}\|_{C^0}\\ &+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\,\|f^{(2)}\|_{C^1}+\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}\,\|\bar{f}^{(1)}\|_{C^1}\\ &+\|f^{(1)}\|_{C^1}\,\|f^{(2)}\|_{C^1}\Big(\sum_{p=1}^{k+1}\big|z_p'(\lambda_1;\lambda_2)-\bar{z}_p'(\lambda_1;\lambda_2)\big|\Big)\Big).\end{aligned}$$

Therefore, we can get

$$\begin{aligned}\big\|\,h\big(t_p(\lambda_1;\lambda_2)\big)-\bar{h}\big(\bar{t}_p(\lambda_1;\lambda_2)\big)\,\big\|_{C^1}\leq\ & C_{A,k,\varepsilon_0}\Big(\big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^1}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}\big)\|a-\bar{a}\|_{C^1}\\ &+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\,\|a-\bar{a}\|_{C^1}\\ &+\big(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1}\big)\big(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1}\big)\\ &+\|g-\bar{g}\|_{C^1}\Big).\end{aligned}$$

For $H[\hat{n}]$, from (2.33)-(2.36), we know that

$$\left|H^+\left(t_p(0;x)\right)-\bar{H}^+\left(\bar{t}_p(0;x)\right)\right|\leq C\left\|h\left(t_p\left(\lambda_1;\lambda_2\right)\right)-\bar{h}\left(\bar{t}_p\left(\lambda_1;\lambda_2\right)\right)\right\|_{C^1},$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}H^+\left(t_p(0;x)\right)-\frac{\mathrm{d}}{\mathrm{d}x}\bar{H}^+\left(\bar{t}_p(0;x)\right)\right|\leq C\left\|h\left(t_p\left(\lambda_1;\lambda_2\right)\right)-\bar{h}\left(\bar{t}_p\left(\lambda_1;\lambda_2\right)\right)\right\|_{C^1}.$$

Thereby

$$\left|H^{+\prime}\left(t_p(0;x)\right)-\bar{H}^{+\prime}\left(\bar{t}_p(0;x)\right)\right| \le C_{\varepsilon_0,k}\left(\left\|h\left(t_p\left(\lambda_1;\lambda_2\right)\right)-\bar{h}\left(\bar{t}_p\left(\lambda_1;\lambda_2\right)\right)\right\|_{C^1}+\|h\|_{C^1}\|a-\bar{a}\|_{C^1}\right).$$

For $H[n]$, from (2.32), we know that

$$\begin{aligned}\left|\, H\big(t_p(0;x)\big)-\bar{H}\big(\bar{t}_p(0;x)\big)\,\right| \le\ & C_{\varepsilon_0,k,A}\Big(\|\tilde{u}-\tilde{v}\|_{C^0}+\left\|H[\hat{n}]\big(t_p(0;x)\big)-\bar{H}[\hat{n}]\big(\bar{t}_p(0;x)\big)\right\|_{C^0}\\ &+\left\|G\big(t_p(0;x)\big)-\bar{G}\big(\bar{t}_p(0;x)\big)\right\|_{C^0}+\left\|F\big(t_p(0;x)\big)-\bar{F}\big(\bar{t}_p(0;x)\big)\right\|_{C^0}\Big).\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}x}H\big(t_p(0;x)\big)-\frac{\mathrm{d}}{\mathrm{d}x}\bar{H}\big(\bar{t}_p(0;x)\big)\right| \le\ & C_{\varepsilon_0,k,A}\Big(\|\tilde{u}-\tilde{v}\|_{C^1}\\ &+\left\|H[\hat{n}]\big(t_p(0;x)\big)-\bar{H}[\hat{n}]\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\\ &+\left\|G\big(t_p(0;x)\big)-\bar{G}\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\\ &+\left\|F\big(t_p(0;x)\big)-\bar{F}\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\Big).\end{aligned}$$

Thereby

$$\begin{aligned}\left|\, H'\big(t_p(0;x)\big)-\bar{H}'\big(\bar{t}_p(0;x)\big)\,\right| \le\ & C_{\varepsilon_0,k,A}\Big(\|h\|_{C^1}\ \|a-\bar{a}\|_{C^1}+\|\tilde{u}-\tilde{v}\|_{C^1}\\ &+\left\|H[\hat{n}]\big(t_p(0;x)\big)-\bar{H}[\hat{n}]\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\\ &+\left\|G\big(t_p(0;x)\big)-\bar{G}\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\\ &+\left\|F\big(t_p(0;x)\big)-\bar{F}\big(\bar{t}_p(0;x)\big)\right\|_{C^1}\Big).\end{aligned}$$

Therefore, we can get

$$\begin{aligned}\left\|\, H\big(t_p(0;x)\big)-\bar{H}\big(\bar{t}_p(0;x)\big)\,\right\|_{C^1} \le\ & C_{\varepsilon_0,k,A}\Big(\big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^1}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}\big)\|a-\bar{a}\|_{C^1}\\ &+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\,\|a-\bar{a}\|_{C^1}\\ &+(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1})\,(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1})\\ &+\|g-\bar{g}\|_{C^1}\Big).\end{aligned}$$

For any $t\in[0,T]$, there exists some $p$ and $x\in[0,L]$ such that $t=t_p(0;x)$. Therefore, we can get

$$\left\|H-\bar{H}\right\|_{C^1} \le \left\|H\left(t_p(0;x)\right)-\bar{H}\left(\bar{t}_p(0;x)\right)\right\|_{C^1}+\|H\|_{C^{1,1}}\left|t_p(0;x)-\bar{t}_p(0;x)\right|.$$

Thereby

$$\begin{aligned}\left\|\, H-\bar{H}\right\|_{C^1} \le\ & C_{\varepsilon_0,k,A}\Big(\big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^1}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}\big)\|a-\bar{a}\|_{C^1}\\ &+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\,\|a-\bar{a}\|_{C^1}\\ &+(\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1})\,(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1})\\ &+\|g-\bar{g}\|_{C^1}\Big).\end{aligned}$$

For $\|u-v\|_{C^1}$, similar to the estimate of $\|\tilde{u}-\tilde{v}\|_{C^1}$, we can get

$$\begin{aligned}\|u-v\|_{C^1} \le\ & C_{\varepsilon_0,k,A}\Big((\|f^{(1)}\|_{C^1}+\|f^{(2)}\|_{C^1}+\|\bar{f}^{(1)}\|_{C^1}+\|\bar{f}^{(2)}\|_{C^1})\,(\|f^{(2)}-\bar{f}^{(2)}\|_{C^1}+\|f^{(1)}-\bar{f}^{(1)}\|_{C^1})\\ &+\big(\|\varphi\|_{C^{1,1}}+\|\psi\|_{C^{1,1}}+\|g\|_{C^{1,1}}+\|\bar{g}\|_{C^{1,1}}+(\|f^{(1)}\|_{C^{1,1}}+\|\bar{f}^{(1)}\|_{C^{1,1}})(\|f^{(2)}\|_{C^{1,1}}+\|\bar{f}^{(2)}\|_{C^{1,1}})\big)\,\|a-\bar{a}\|_{C^1}\\ &+\|g-\bar{g}\|_{C^1}\Big).\end{aligned}$$

## 3. LINEARIZATION ITERATION

In this section, we will show the proof of Proposition 1.1 with inductive reasoning. Denote $\tilde{a}^{(m)} = a^{(m)} - 1$, and we know that

$$\left\|\tilde{a}^{(m)}\right\|_{C^{1,1}} \leq C\left\|\tilde{U}^{(m)}\right\|_{C^{2,1}} \leq CC_P\varepsilon,$$

using the inductive assumption. Denote

$$\begin{aligned} G^{(m)} = &\left(\widetilde{D}^{+}\left(\widetilde{U}^{(m)}, \widetilde{U}_t^{(m)}, \widetilde{U}_x^{(m)}\right) - \widetilde{D}^{+}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{-(m)} \\ &- \left(\widetilde{D}^{-}\left(\widetilde{U}^{(m)}, \widetilde{U}_t^{(m)}, \widetilde{U}_x^{(m)}\right) - \widetilde{D}^{-}(0,0,0)\right)\widetilde{D}^{+}(0,0,0)^{-1}\widetilde{V}^{+(m)}. \end{aligned}$$

Then we have

$$\left\|G^{(m)}\right\|_{C^{1,1}} \leq C\left\|\tilde{U}^{(m)}\right\|_{C^{2,1}}^2 \leq CC_P^2\varepsilon^2,$$

Denote

$$\begin{aligned} f^{(m,1)} &= \partial_x\tilde{U}^{(m)}, \\ f^{(m,2)} &= a^{(m)}. \end{aligned}$$

Then we have

$$\begin{aligned} \left\|f^{(m,1)}\right\|_{C^{1,1}} &\leq \left\|\tilde{U}^{(m)}\right\|_{C^{2,1}} \leq C_P\varepsilon, \\ \left\|f^{(m,2)}\right\|_{C^{1,1}} &\leq \left\|a^{(m)}\right\|_{C^{1,1}} \leq CC_P\varepsilon. \end{aligned}$$

Using (2.43) and (2.50), we can get

$$\begin{aligned} \left\|H^{(m+1)}\right\|_{C^{1,1}} &\leq 2C\left(1 + CC_P^2\varepsilon\right)\varepsilon, \\ \left\|\tilde{V}^{(m+1)}\right\|_{C^{1,1}} &\leq 2C\left(1 + CC_P^2\varepsilon\right)\varepsilon, \end{aligned}$$

and then

$$\left\|\tilde{U}^{(m+1)}\right\|_{C^{2,1}} \leq 2C\left(1 + CC_P^2\varepsilon\right)\varepsilon.$$

Select $\varepsilon_0$ small enough and $C_P$ large enough such that

$$2C\left(1 + CC_P^2\varepsilon_0\right) \leq C_P.$$

Then we can get

$$\left\|H^{(m+1)}\right\|_{C^{1,1}} \leq C_P\varepsilon,$$

and

$$\left\|\tilde{U}^{(m+1)}\right\|_{C^{2,1}} \leq C_P\varepsilon.$$

Then we try to get the cauchy property. Provided the method of obtaining the controls $h_i(t)$, we first condider the cauchy property of $\tilde{\varphi}^{(m)}(x)$, where $\tilde{\varphi}^{(m)}(x) = \tilde{u}^{(m)}(T,x)$ and $\tilde{u}^{(m)}(t,x)$ is the solution of the equation below:

$$\begin{cases} \partial_t\widetilde{u}^{\pm(m+1)} \pm a^{(m)}\partial_x\widetilde{u}^{\pm(m+1)} = \mp\chi\left(t\right)\partial_x\widetilde{U}^{(m)}\left(\partial_t \pm a^{(m)}\partial_x\right)\left(a^{(m)}\right), \\ x = L : \widetilde{u}^{-(m+1)} = -\widetilde{u}^{+(m+1)}, \quad t \in \mathbb{R}^+, \\ x = 0 : \widetilde{u}^{-(m+1)} = B_0\widetilde{u}^{+(m+1)} + \widetilde{H}(t) + \chi\left(t\right)G^{(m)}\left(t\right), \\ t = 0 : \widetilde{u}^{(m+1)} = \varphi\left(x\right), \end{cases}$$

where $\chi(t)$ is the same as in section 2 and $\widetilde{H}(t)$ is a set of functions selected that satisfy the compatibility condition. Provided that $G^{(m)}\left(0\right)$ depends on $\varphi(x)$ and is independent of $m$, we can select the same set of functions $\widetilde{H}(t)$ for all $m$. Using the inductive assumption, we can get

$$\begin{aligned} \left\|a^{(m)} - a^{(m-1)}\right\|_{C^1} &\leq C\left(\left\|\tilde{U}^{(m)} - \tilde{U}^{(m-1)}\right\|_{C^2}\right), \\ \left\|G^{(m)} - G^{(m-1)}\right\|_{C^1} &\leq C\left(\left(\left\|\tilde{U}^{(m)}\right\|_{C^2} + \left\|\tilde{U}^{(m-1)}\right\|_{C^2}\right)\left\|\tilde{U}^{(m)} - \tilde{U}^{(m-1)}\right\|_{C^2}\right), \\ \left\|f^{(m,1)} - f^{(m-1,1)}\right\|_{C^1} &\leq C\left(\left\|\tilde{U}^{(m)} - \tilde{U}^{(m-1)}\right\|_{C^2}\right), \end{aligned}$$

$$\left\| f^{(m,2)} - f^{(m-1,2)} \right\|_{C^1} \le C \left( \left\| \tilde{U}^{(m)} - \tilde{U}^{(m-1)} \right\|_{C^2} \right).$$

Using (2.43) and (2.50), we can get

$$\left\| H^{(m+1)} - H^{(m)} \right\|_{C^1} \le 2C^2\varepsilon \left(1 + 4CC_p + C_p\varepsilon + CC_p^2\varepsilon + 2C^2C_p^2\varepsilon\right) C_P\varepsilon\alpha^{m-1},$$

and then

$$\left\| \tilde{V}^{(m+1)} - \tilde{V}^{(m)} \right\|_{C^1} \le 2C^2\varepsilon \left(1 + 4CC_p + C_p\varepsilon + CC_p^2\varepsilon + 2C^2C_p^2\varepsilon\right) C_P\varepsilon\alpha^{m-1},$$

and

$$\left\| \tilde{U}^{(m+1)} - \tilde{U}^{(m)} \right\|_{C^2} \le 2C^2\varepsilon \left(1 + 4CC_p + C_p\varepsilon + CC_p^2\varepsilon + 2C^2C_p^2\varepsilon\right) C_P\varepsilon\alpha^{m-1}.$$

Select $\varepsilon_0$ small enough such that

$$2C^2\varepsilon_0 \left(1 + 4CC_p + C_p\varepsilon_0 + CC_p^2\varepsilon_0 + 2C^2C_p^2\varepsilon_0\right) < \alpha < 1.$$

The proof of Theorem 1.1 is completed.

## 4. Appendice

**Lemma 4.1.** *Consider the equation*

$$\psi = A^k B H_{k+1} + \cdots + ABH_2 + BH_1. \tag{4.1}$$

$\psi, H_p \in \mathbb{R}^n, p = 1, \cdots, k+1,$

$$A = \begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix},$$

$$B = \begin{pmatrix} I_{r_1} & & & \\ & O_{r_2} & & \\ & & I_{s_1} & \\ & & & O_{s_2} \end{pmatrix},$$

*where* $r_1 + r_2 = r,\ s_1 + s_2 = s,\ r + s = n,\ A_1 = (a_{ij}),\ i = 1, \cdots, r,\ j = r+1, \cdots, n,$ $A_2 = (a_{ij}),\ i = r+1, \cdots, n,\ j = 1, \cdots, r.$ *If* $\psi$ *satisfies the following condition*

$$\psi_i = \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r + s_1 + 1, \cdots, r+s. \tag{4.2}$$

*and the Kalman rank condition holds as follows,*

$$\text{Rank}\left[B, A^k B, \ldots, A^{k-1}B\right] = n, \tag{4.3}$$

*then there exists a set of solutions* $H_p(x),\ p = 1, \cdots, k+1$ *of equation* (4.1) *satisfying*

$$H_{2m-1}^- = H_{2m}^-, \tag{4.4}$$

$$H_{2m+1}^+ = H_{2m}^+, \tag{4.5}$$

where $H_p = \left(H_p^-, H_p^+\right)^T$, $H_p^- \in \mathbb{R}^r$, $H_p^+ \in \mathbb{R}^s$.

*Proof.* Without loss of generality, we can assume that $k$ is odd. The condition 4.2 can be written as

$$\psi^+ = A_2\psi^- + \bar{H}. \tag{4.6}$$

$\bar{H}$ is the difference term, i.e.

$$\bar{H}_i = \psi_i - \sum_{j=1}^{r} a_{ij}\psi_j,\ i = r+1, \cdots, r+s_1. \tag{4.7}$$

By the definition of the matrix $B$, the $i$-th $(i = r_1 + 1, \cdots, r, s_1 + 1, \cdots, r+s)$ components of $H_p,\ p = 1, \cdots, k+1$ can be taken to be zero. The following is an attempt to write conditions (4.4)-(4.5) as matrix equations. Denote

$$H = \left(H_{k+1}^-, H_{k+1}^+, \cdots, H_1^-, H_1^+\right)^T.$$

Then the equation (4.1) ,(4.4)-(4.5) can be written as a matrix equation as follows:

$$\begin{pmatrix}\psi\\0\end{pmatrix}=\begin{pmatrix} & A_1A_2\cdots A_2 & A_1\cdots A_1 & & \cdots & & A_1 & I_r & \\ A_2A_1\cdots A_1 & & & A_2\cdots A_2 & \cdots & A_2 & & & I_s\\ I_r & & -I_r & & & & & & \\ & & & I_s & & & & & \\ & & & & \cdots & & & & \\ & & & & & & -I_s & & \\ & & & & & I_r & & -I_r & \end{pmatrix}H.$$

After appropriate matrix row transformations, the first row of the equation can be written as

$$\psi=\begin{pmatrix}O & A_1A_2\cdots A_2 & A_1\cdots A_1 & O & \cdots & O & A_1 & I_r & \\ & & A_2A_1\cdots A_1 & O & \cdots & O & A_2A_1 & A_2 & I_s\end{pmatrix}H.$$

Let $H^+_{k+1}=0$. Combined with (4.6), equation (4.1) can be rewritten as

$$\begin{pmatrix}\psi^-\\ \bar H\end{pmatrix}=\begin{pmatrix} O & A_1\cdots A_1 & O & \cdots & O & A_1 & I_r & \\ & & & \cdots & & & & I_s\end{pmatrix}H. \tag{4.8}$$

Notice that (4.3) yields that the rank of the equation generalization matrix is equal to the rank of the original coefficient matrix, and thus (4.8) has a solution. ☐

**Corollary.** *Under the conditions of the above lemma, if $\psi$ satisfies*

$$\psi_i=\sum_{j=r+1}^{r+s}a_{ij}\psi_j,\ i=r_1+1,\cdots,r, \tag{4.9}$$

*then there exists a set of solutions $H_p(x),\ p=1,\cdots,k+1,$ of equation (4.1) satisfying*

$$H^+_{2m-1}=H^+_{2m}, \tag{4.10}$$

$$H^-_{2m+1}=H^-_{2m}. \tag{4.11}$$

*Proof.* Define $\tilde\psi$ as $\tilde\psi^-=\psi^+$, $\tilde\psi^+=\psi^-$, $\tilde A$ as $\tilde A^1=A^2$, $\tilde A^2=A^1$, and $\tilde H_p$ as $\tilde H^-_p=H^+_p$, $\tilde H^+_p=H^-_p$, which are the same as the above lemma. ☐

**Corollary.** *In the proof of Lemma 4.1 one obtains $H^+_1=\bar H$, and in the conditions of Corollary 4 one obtains $H^-_1=\bar H$.*

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