第七章 图的控制集理论及应用

问题： 欲在 n 个地点 $v_1, v_2, ..., v_n$ 中设置若干个应急中心，使得每个地点都与至少一个应急中心相邻（有直达通路）。同时，为了减少造价，应急中心的数目要尽可能少。问应设多少个？如何设置？

§7.1 控制集(dominating set)
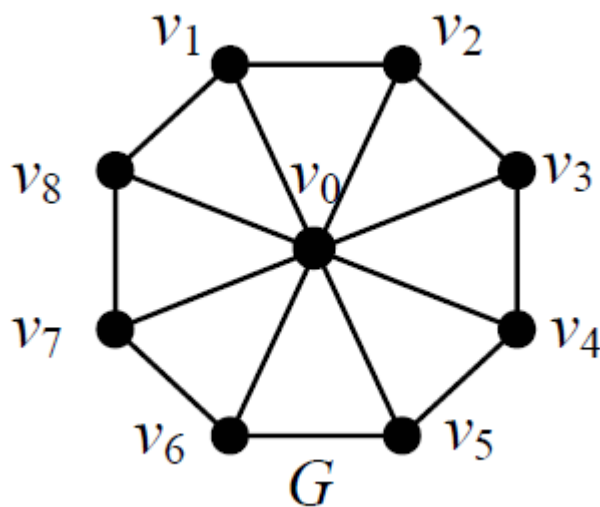
定义7.1.1 设$D \subseteq V(G)$，若对$\forall u \in V(G)$，要么$u \in D$，要么u与D中的某些顶点相邻，则称D为图G的一个控制集。

如果一个控制集的任何真子集都不是控制集，则称它为极小控制集。

定义7.1.2 图G的含顶点最少的控制集称为G的 最小控制集。最小控制集的顶点个数称为G 的 控制数(domination number)，记为γ (G)或γ 。

例如，在下图中， $D_0 = \{v_0\}$ ， $D_1 = \{v_1, v_4, v_7\}$ ， $D_2 = \{v_1, v_3, v_5, v_6\}$ 都是G的控制集，前两个是极小控制集， D_0 是最小控制集。控制数$\gamma(G) = 1$ 。

G

（1）最小控制集必是一个极小控制集，反之不然。

（2）任一控制集必含有一个极小控制集。

（3）极小控制集不唯一，最小控制集一般也不唯一。

（4）对二部图$G = (X, Y)$，X和Y都是控制集。

（5）若图G 有完美匹配M*，则从M*中每边取一个端点构成的顶点集是一个控制集。

定理 **7.1.1** 设图G中无孤立顶点，则存在控制集D，使得$D=V(G)-D$ 也是一个控制集。

证明：不妨设G是连通图。于是G有生成树T。任取$v_0 \in V(G)$。令

$D = \{v \mid v \in V(G)$且 $d_T(v_0, v) =$偶数$\}$，

则$D' = V(G) - D = \{v \mid v \in V(G)$且$d_T(v_0, v) =$奇数$\}$，且D 和$D'$ 都是控制集。

定理 **7.1.2** 设图G 无孤立顶点，D 是一个极小控制集，则 $D' = V(G) - D$ 也是一个控制集。

证明（反证法）：若不然，则存在 $v_0 \in D$，它与 D' 中所有顶点都无边相连，但它又不是孤立顶点，故必与 D 中顶点连边，因此 $D - v_0$ 仍是控制集。这与D 是极小控制集矛盾。

推论 **7.1.3** 设图G 中无孤立顶点。对G 的任一个极小控制集D_1，必存在另一个极小控制集 D_2，使得$D_1 \cap D_2 = \emptyset$ 。

证明：由定理7.1.2， $D'_1 = V(G) - D_1$ 也是一个控制集，且$D_1 \cap D'_1 = \emptyset$ 。D'_1 中必含有一个极小控制集D_2。显然$D_1 \cap D_2 = \emptyset$ 。

定理**7.1.4** 图 G 的控制集D 是一个极小控制集当且仅当D 中每个顶点v 满足下列条件之一：

（1）存在$u \in V(G) - D$ 使得$N(u) \cap D = \{v\}$；

（2）$N(v) \cap D = \emptyset$ 。

定理**7.1.5** 如果图G 无孤立顶点，则

$$γ(G)≤|V(G)|/2。$$

证明：设 D是G的一个极小控制集，则由定理7.1.2，V(G) − D也是G的控制集。因此，

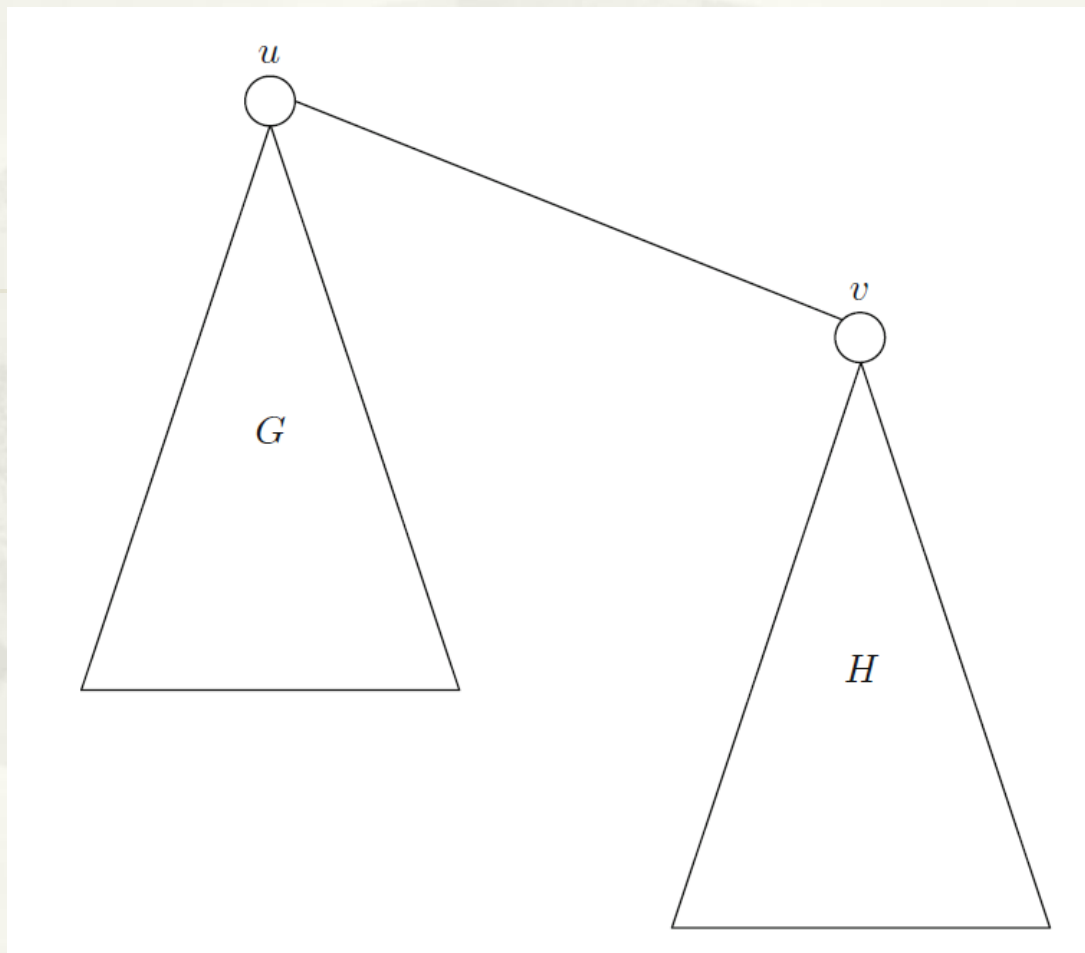$$γ(G) ≤\min\{| D |, V(G) -D |\}$$
$$≤ |V(G)|/2 。$$

定理 **7.1.6** 对任何图**G**，有

$$\left\lceil \frac{|V(G)|}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq |V(G)| - \Delta(G)。$$

证明：设D 是G的一个最小控制集，则V(G) −D ⊆ $\bigcup_{v \in D} N(v)$，因此|V(G)−D|≤|D|·Δ(G)。而|V(G)−D|= |V(G)|−|D|，|D|=γ(G)，故 |V(G)| −γ(G) ≤γ(G)Δ(G)，于是γ(G) ≥

$$\left\lceil \frac{|V(G)|}{1+\Delta(G)} \right\rceil。$$

§7.2 计算树的控制数的动态规划算法

$$\gamma^1(G, u) = \min\{|D| : D \text{ is a dominating set of } G \text{ and } u \in D\}.$$

$$\gamma^0(G, u) = \min\{|D| : D \text{ is a dominating set of } G \text{ and } u \notin D\}.$$

Lemma 1 $\gamma(G) = \min\{\gamma^1(G, u), \gamma^0(G, u)\}$ *for any graph G with a specific vertex u.*

The aim is to use $\gamma^1(G,u)$, $\gamma^0(G,u)$, $\gamma^1(H,v)$, and $\gamma^0(H,v)$ to find $\gamma^1(I,u)$ and $\gamma^0(I,u)$. Suppose D is a dominating set of I with $u \in I$. Then $D = D' \cup D''$, where D' is a dominating set of G with $v \in D'$ and D'' is a subset of $V(H)$ which dominates $V(H) - \{v\}$. There are two cases. In the case of $v \in D''$, D'' is a dominating set of H. On the other hand, if $v \notin D''$, then D'' is a dominating set of $H - v$. In order to cover the latter case, the following new problem is introduced:

$$\gamma^{00}(G,u) = \min\{|D| : D \text{ is a dominating set of } G - u\}.$$

Note that $\gamma^{00}(G,u) \leq \gamma^0(G,u)$, since a dominating set D of G with $u \notin D$ is also a dominating set of $G - u$.

Theorem 2 *Suppose G and H are graphs with specific vertices u and v, respectively. Let I be the graph with the specific vertex u, which is obtained from the disjoint union of G and H by joining a new edge uv. Then the following statements hold:*

(1) $\gamma^1(I, u) = \gamma^1(G, u) + \min\{\gamma^1(H, v), \gamma^{00}(H, v)\}$.

(2) $\gamma^0(I, u) = \min\{\gamma^0(G, u) + \gamma^0(H, v), \gamma^{00}(G, u) + \gamma^1(H, v)\}$.

(3) $\gamma^{00}(I, u) = \gamma^{00}(G, u) + \gamma(H) = \gamma^{00}(G, u) + \min\{\gamma^1(H, v), \gamma^0(H, v)\}$.

Algorithm DomTreeD. Determine the domination number of a tree.
Input. A tree T with a tree ordering $[v_1, v_2, \ldots, v_n]$.
Output. The domination number $\gamma(T)$ of T.
Method.

for $i = 1$ **to** n **do**
    $\gamma^1(v_i) \leftarrow 1;$
    $\gamma^0(v_i) \leftarrow \infty;$
    $\gamma^{00}(v) \leftarrow 0;$
    end do;
for $i = 1$ **to** $n - 1$ **do**
    let v_j be the parent of v_i;
    $\gamma^1(v_j) \leftarrow \gamma^1(v_j) + \min\{\gamma^1(v_i), \gamma^{00}(v_i)\};$
    $\gamma^0(v_j) \leftarrow \min\{\gamma^0(v_j) + \gamma^0(v_i), \gamma^{00}(v_j) + \gamma^1(v_i)\};$
    $\gamma^{00}(v_j) \leftarrow \gamma^{00}(v_j) + \min\{\gamma^1(v_i), \gamma^0(v_i)\};$
    end do;
$\gamma(T) \leftarrow \min\{\gamma^1(v_n), \gamma^0(v_n)\}.$

§7.3 电力网络控制

Electric power companies need to continually monitor their system's state as defined by a set of state variables. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. Because of the high cost of a PMU, it is desirable to minimize the number of PMUs while maintaining the ability to monitor (observe) the entire system.

A system is said to be observed if all of the state variables of the system can be determined from a set of measurements (e.g., voltages and currents).

Let G=(V,E) be a graph representing an electric power system, where a vertex represents an electrical node (a substation bus where transmission lines, loads, and generators are connected) and an edge represents a transmission line joining two electrical nodes. The problem of locating a smallest set of PMUs to monitor the entire system is a graph theory problem closely related to the well-known vertex covering and domination problems.

A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their end vertices (these vertices and edges are said to be observed.) The other observation rules are as follows:

1. Any vertex that is incident to an observed edge is observed.

2. Any edge joining two observed vertices is observed.

3. If a vertex is incident to a total of k>1 edges and if k-1 of these edges are observed, then all k of these edges are observed.

Considering the power system monitoring problem as a variation of the dominating set problem, we define a set S to be a *power dominating set* (PDS) if every vertex and every edge in G is observed by S. The *power domination number* of G, denoted by $\gamma_P(G)$, is the minimum cardinality of a power dominating set of G.

Note that all vertices and edges of G are observed if and only if all vertices of G are observed. Hence, the following definition is equivalent to power domination.

A subset S⊆V is a power dominating set of G if all vertices of V can be observed recursively by the following rules: (i) all vertices in N[S] are observed initially, and (ii) if an observed vertex u has all neighbors observed except one neighbor v, then v is observed (by u).

Observation 1. For any graph G,

$$1 \leq \gamma_P(G) \leq \gamma(G).$$

Observation 2. For the graph G, where G $\in$ $\{K_n, C_n, P_n, K_{2,n}\}$,$\gamma_P(G) = 1$.

Observation 3. If G is a graph with maximum degree at least 3, then G contains a $\gamma_P(G)$-set in which every vertex has degree at least 3.

Theorem 4. For any tree,γ_P (G)=1 if and only if T is a spider.

The spider number of a tree T, denoted by sp(T), to be the minimum number of subsets into which V(T) can be partitioned so that each subset induces a spider.
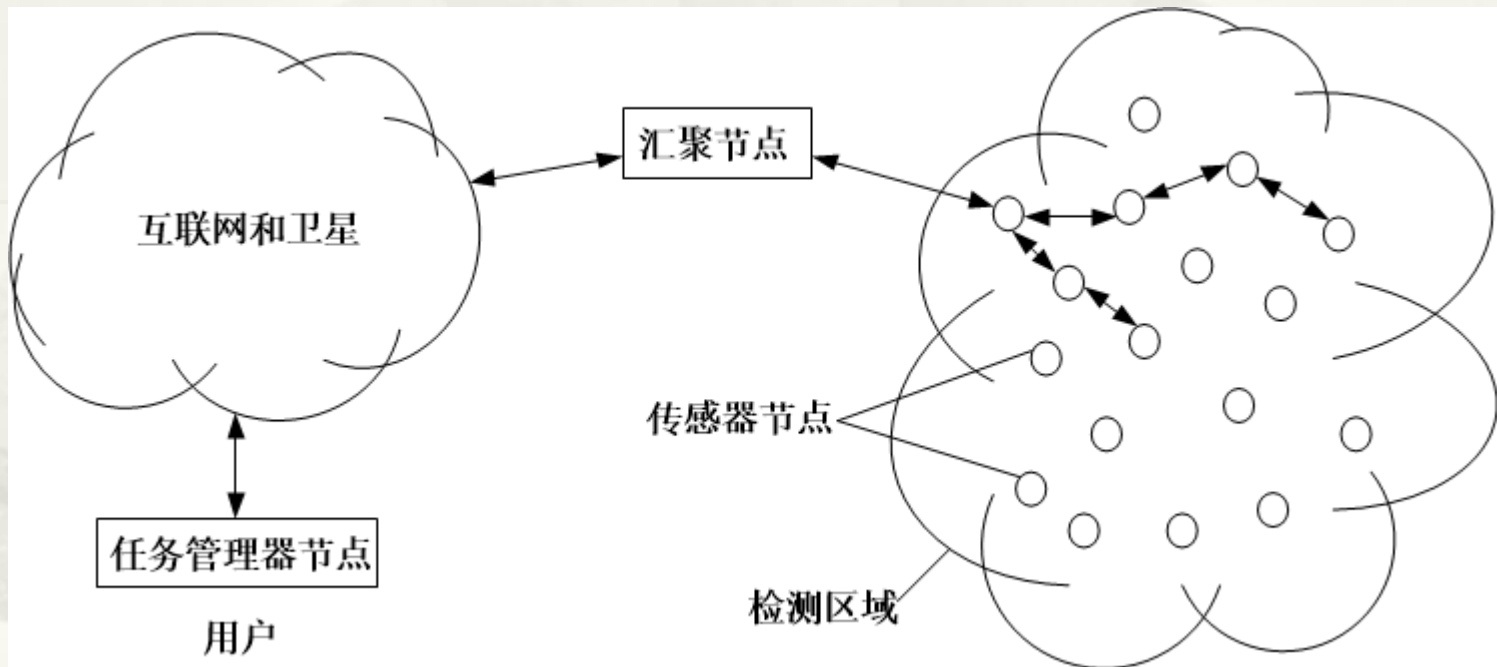
Theorem 5. For any tree T, sp(T)=γ_P (G).

Theorem 6. If G=(V,E) is a connected graph of order n⩾3, then $\gamma_P(G) \leqslant n/3$ with equality if and only if G∈F∪{$K_{3,3}$}.

Where F be the family of graphs obtained from connected graphs H by adding two new vertices v′ and v″ to each vertex v of H and new edges vv′ and vv″, while v′v″ may be added or not.

§7.4 无线传感器网络拓扑控制

In the wireless sensor network, each sensor is not only a mobile host but also a router. In other words, the sensors are able to forward the received data packages according to routing protocols. We assume that all sensors have the same power, that is, every sensor can communicate with others within a unit distance. Under this assumption, the topology of the sensor network can be formulated as a unit disk graph.

A unit disk is a disk with radius one. A *unit disk graph* is associated with a set of unit disks in the Euclidean plane. Each node is the center of a unit disk. An edge exists between two nodes u and v if and only if $|uv| \leqslant 1$, where $|uv|$ is the Euclidean distance between u and v. This means that two nodes connecting with an edge if and only if u's disk covers v and v's disk covers u.

Multicasting is to send messages to a group of receivers at the same time. For example, when one sensor wants to send out topology update information to a group of other sensors in the network, it will use multicasting. Multicasting reduces the network traffic by combining multiple unicast data stream into one. The advancement of multicasting in the network is driven by emerging applications, such as net meeting and video conference.

One of efficient ways to support multicasting is to use virtual backbone in wireless networks. A virtual backbone is a *connected dominating set* in the network, that is, it is a subset of sensors such that they form a connected sub-network and every sensor is either in the subset or adjacent to a sensor in the subset.

Since multicasting can be performed first within virtual backbone and then to others, it has been recommended to manage and updating the topology of virtual backbone instead of the topology of the whole network, which reduces both storage and message complexities.

Clearly, the smaller virtual backbone gives the better performance.

It is a popular idea to construct a connected dominating set in two steps:

In the first step a dominating set is constructed; in the second step, connect the dominating set into a connected dominating set.

Steiner Tree with Minimum Number of Steiner Nodes (ST-MSN): Given a unit disk graph G and a subset P of nodes, compute a Steiner tree for P with the minimum number of Steiner nodes.

ALGORITHM A. Input a maximal independent set and mark all its nodes in black and others in grey. In the following, we will change some grey nodes to black in certain rules. A *black component* is a connected component of the subgraph induced by black nodes.

Stage 1. **while** there exists a grey node adjacent to at least three black components **do**

change its color from grey to black;

   end-while;

Stage 2. **while** there exists a grey node adjacent to at least two black components **do**

change its color from grey to black;

   end-while;

   return all black nodes.

Theorem 1. *The two step algorithm with Steiner tree produces 6.8-approximation for the minimum connected dominating set.*