Talk II: Adic Spaces

§1: Why adic spaces?

Huber: adic spaces are generalizations of formal schemes & rigid analytic varieties.

Def: A Noetherian adic ring A is a topological ring equipped with I-adic topology for $I \subset A$

$$\mathrm{Spf}\, A := \mathrm{colim}\, \mathrm{Spec}\, A/I^n.$$

Ex: $\mathrm{Spf}\, \mathbb{Z}_p$ only has 1 point.

$$\mathrm{Spec}\, \mathbb{F}_p \longrightarrow \mathrm{Spf}\, \mathbb{Z}_p$$

Can't take generic fibers naively

(no map $\mathrm{Spec}\, \mathbb{Q}_p \to \mathrm{Spf}\, \mathbb{Z}_p$).

Tate's rigid geometry

Def: A non-Arch. field is K complete with respect to a nontrivial non-Arch valuation

$$|\cdot|_K : K \longrightarrow 0 \cup \mathbb{R}^+$$

K° its valuation ring

" $\{x \in K \mid |x| \leq 1\}$

ϖ : pseudo-uniformizer

$\Updownarrow$

a topologically nilpotent unit

$\Updownarrow$

$|\varpi| \in (0,1)$. "\varpi"

Ex: $\mathbb{Q}_p$ is a non-Arch field.

$\mathbb{Q}_p^\circ = \mathbb{Z}_p \quad \varpi := p$

Ex: $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ is algebraically closed.

Rmk: Try to invert rigid geometry

$K = \overline{K}$ non-Arch field.

Want to define $(\mathbb{A}^1_K)^{rigid}$

$\{\text{its points}\} \longleftrightarrow \{x \in K\}$.

- Zariski opens
- disk-like opens (Weierstrass opens)

$D_{\leq r}$

want "$\{x \in K \mid |x| \leq r\}$ to be open

because it is open in K.

Question: What should $\Gamma(D_r, \mathcal{O})$

Def: For any topological algebra R,

Tate algebra:

$$R\langle T\rangle := \{f = \sum a_i T^i \in R[[T]] \mid \lim a_i = 0 \text{ in } R\}$$

$$\begin{array}{c} R\langle T\rangle \\ \cap \\ R[[T]] \end{array}$$

$$R\langle T_1, T_2, \dots T_n\rangle = (R\langle T_1, \dots, T_m\rangle)\langle T_{m+1}, \dots T_n\rangle$$

A **topologically finitely presented** (tfp) **R-affinoid algebra** is a topological R-algebra $A = R\langle T_1, \dots, T_n\rangle / I$, I to be finitely generated.

Ex: If the topology of R is given by a norm $|-|$, then $R\langle T\rangle$ is the completion of $R[T]$ wrt the Gauss norm

$$\left|\sum a_i T^i\right|_{\mathrm{Gauss}} := \max |a_i|_R$$

Ex: $K^\circ\langle T\rangle$ = ϖ-adic completion of $K^\circ[T]$

$$\mathbb{Z}_p\langle T\rangle = \varprojlim_n \mathbb{Z}/p^n[T]$$

$$K\langle T\rangle = K^\circ\langle T\rangle[\varpi^{-1}].$$

Rmk: For $x \in K$.

$f(x)$ converges $\forall f \in K\langle T\rangle$

iff $|x|_K \leq 1$.

($\lim_{n\to\infty} (a_n)^{1/n}$ ...)

view anything inside $\langle - \rangle$ as $|\_| \leq 1$.

$\Rightarrow$ $K\langle T\rangle$ $\Gamma(D_{\leq 1}, \mathcal{O})$

Try to define $D_{\leq 1} := \mathrm{MaxSpec}\, K\langle T\rangle$

<u>Def</u>: $\mathrm{Sp}\, A = \mathrm{MaxSpec}\, A$

for any tfp K-affinoid algebra.

Want to give a topology to $\mathrm{Sp}\, A$.

- Zariski one. $\{g \neq 0 \mid g \in A\}$

A_g

- Weierstrass one: $\{|f| \leq 1 \mid f \in A\}$.

$A\langle f\rangle := A\langle T\rangle/(T-f)$

If we use them to generate a topology on $\mathrm{Sp}A$, then $D_{\le 1}$ is not connected (b.c. $\mathbb{Z}_p$ is not connected)

But we want $D_{\le 1}$ to be connected.

<u>Solution</u> : G-topology.

<u>Def</u> : For ^the^ K-affinoid algebra A,

and $f_1, \dots f_n, g \in A$ generate (1),

define <u>rational domain</u>

$$U(f_i/g) \subseteq \mathrm{Sp}A$$

$$\mathrm{Sp}\, A\langle f_i/g\rangle = \mathrm{Sp}\, A\langle T_i\rangle/(gT_i - f_i)$$

$$\{x \in \mathrm{Sp}A \mid |f_i(x)| \le |g(x)| \ne 0\}$$

"Tate topology on Sp A":

$\vdots$

quasi-compact G-topology with "opens" given by $U(f_i/g)$

(coverings are required to be finite).

Globalize it to define non-affine rigid spaces...

Rmk:

1) Take $f_i = 0$ & $g \neq 0$

Zariski open.

2) Take $g=1$ $\cap\{|f_i|\le 1\}$

Weierstrass gives.

3) $(f_1, \dots f_n) = A$

$(f_1, \dots f_n, g) = A$.

define the same topology.

(you can always $f_0 = g$)

4). $\{ x \mid |f(x)| \le |g(x)| \neq 0 \}$

The condition $|g(x)| \neq 0$

is redundant.

If $g(x)=0$

$\Rightarrow f_i(x)=0 \quad \forall i$

$\Rightarrow 1=0$ being

$(1)=(f_1,\dots f_n)$

5) If we keep $g\neq 0$

and drop $(1)=(f_1,\dots f_n)$,

then we still get

$$U(f_i/g)\subset \operatorname{Sp} A$$

its still an open subtopi

$\underline{U(f_i/g)}$ natural domain

$$=\bigcup_n U(f_1,\dots f_n, \omega^n/g)$$

[If $g(x) \neq 0$, $\exists$ n such that
$|\varpi^n_{(x)}| \leq |g(x)|$.

[This is indeed a cover bc.
for any rational domain

$U' \longrightarrow U(f_i/g)$
restriction of
$\bigcup_m U(f_i, \varpi^m/g)$ on U'
has a finite refinement.

However: $U(f_i/g)$ is
not quasi-cpt.

$\Longrightarrow$ Do need $(f_i) = A$

to make sure

$U(f_i | g)$ is quasi-cpt.

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6) Why we use non-strict inequalities ?

$\{x \mid |f_i(x)| \mid g(x)| \neq 0\}$

is still open.

but it's not quasi-cpt.

$$\left[\quad K\langle T\rangle, \qquad \begin{array}{l} f = T \\ g > 1 \end{array} \quad\right]$$

$$\left[\begin{array}{ll} D_{<1} \text{ is not quasi-cpt} & \\ \quad \shortparallel & \\ \displaystyle\bigcup_{r<1} D_{\leq r} & K = \mathbb{C}_p \end{array}\right]$$

7) Now $D_{\leq 1}$

$$\cong$$

$\mathrm{Sp}\ K\langle T\rangle$

is connected.

The cover $\underline{D_{<1}}$ and $D_{=1}$

does not $\overline{\text{ruin}}$ the connectedness

- $\underline{D_{<1}}$ is open
- $D_{=1} = D_{\geq 1} \cap D_{\leq 1}$ is also open.

but $D_{<1}$ is not quasi-cpt

it's not an object inside this site

$D_{=1} \sqcup \underline{D_{<1}} \to \underline{D_{\leq 1}}$
is not a cover of
topos.

Spa A

<u>Rmk</u>: Shortcomings of Tate's theory:

All of the following are correct but <u>cannot</u> be proved within the theory:

- flat morphisms are open (under finiteness assumptions)

- flatness is preserved by change of base field
  $k \to k'$
- fpqc descents
- $\exists$ a robust theory of étale sheaves.
  ($(\mathrm{Sp}\,A)_{\text{ét}}$ is well-defined).

- - - - - .

$\rightsquigarrow$ evolution of theories

Raynaud's formal models

$\Downarrow$

Berthelot's formal models

$\Downarrow$

Berkovich spaces

$\Downarrow$

$\boxed{\text{Huber's adic space}}$

$\Downarrow$

Clausen-Scholze's condensed mathematics

<u>Rmk</u>:

1) Raynaud's:

For any tfp K^0-algebra A. flat over K^0.

$\operatorname{Spf} A \rightsquigarrow (\operatorname{Spf} A)_\eta$ generic fiber affine rigid space.

- $A = K^0\langle T_i \rangle$

  $\mathrm{Sp}\ K\langle T_i\rangle \simeq : (\mathrm{Spf}\ A)_\eta$

- $A = K^0\langle T_i\rangle / I$

  - - - - .

This is not bijective.

Ex: you can blowup A at the special fiber, but the resulting $(\mathrm{Spf}\ A)_\eta$ does not change.

Ex: $A := \mathbb{Z}_p\langle T\rangle$ blowup (p, T)

...

Raynaud: any qcqs rigid space

X has a formal model $\mathfrak{X}$,

($\mathfrak{X}_\eta = X$) unique up

to blow up at special fiber.

2) Berthelot's theory

A: $K^\circ[[T]]$

can take generic fiber for Spf A.

[$K[[T]]$ is not

a tfp K-algebra

~~Sp $K[[T]]$~~

3) Berkovich theory

views points as semi-norms of A.

$$\left[\ \forall\, m \in \operatorname{Sp} A, \quad \begin{array}{c} A \to A/m \xrightarrow{1-1} 0 \cup \mathbb{R}^{+} \\ \uparrow \text{finite extension} \\ K \end{array} \right.$$

4) Huber: why $\mathbb{R}^+$?

why not all the valuations?

$$A \xrightarrow{1-1} 0 \cup \Gamma$$

Γ is any totally ordered abelian group.

1) Spa A is a topological space rather than a G-top. sp.

2) This theory uniformises formal schemes & rigid spaces.

- It uses topology can be taken on the nose

[Spa $\mathbb{Z}_p$ has 2 points !]

3) $$|\operatorname{Sp} A|$$ for tfp K-affinoid algebra

$$\Big\updownarrow \text{ bijection}$$

$\{$ generic points of the top. sp. Spa A $\}$.

4) "$\mathrm{Spa}\,A$" $= \lim\limits_{\mathfrak{X}_\eta = \mathrm{Spa}\,A} \mathfrak{X}$".

5) This allows $\mathrm{Spa}\,A$ for infinite type A.

Allows perfectoid geometry.

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§2: Continuous valuations.

Def: A topological ring A is Huber if it admits an open subring $A_0 \subset A$.

which is I-adic with

respect to $I \subset A_0$.

A_0 : ring of definition

I : ideal of definition

Rmk:

1) A_0, I only play

auxiliary roles.

no constructions depend on them.

Ex: $\underset{A}{\mathbb{Q}_p}$, $\underset{A_0}{\mathbb{Z}_p}$ $\underset{I}{(p\mathbb{Z}_p)}$.

2) A subring $A_0' \subset A$ is

a ring of definition

iff A_0 is open & bounded.

S is bounded means the following:

$\forall\, U \subset A$ open neighborhood of 0

$\exists\, V \subset A$ — — — .

s.t. $S \cdot V \subset U$

3). $A^0 := \{ x \in A \mid \{x, x^2, \dots\}$ is bounded $\}$

Subring of power bounded elements

($\mathbb{Q}_p^0 = \mathbb{Z}_p$ $K^0 = K^0$).

$A_0 \subset A^0$

In fact

$$A^\circ = \bigcup \text{rings of definitions.}$$

But A° is not a ring of definition in general.

$$\left[\ \mathbb{Q}_p[\varepsilon]/(\varepsilon^2)\ \right]$$

4). All constructions only depend on $\hat{A}$.

$$\left[\ \hat{A} = \widehat{A_0} \otimes_{A_0} A\ \right]$$

as plain rings.

Def: A is **Tate** if it is Huber and has a ϖ (topologically nilpotent unit)

A is **analytic** if the ideal generated by top. nil. elements $= A$

(Tate $\Rightarrow$ analytic)

$\left[\quad \text{Spa } \mathbb{Z}_p \quad \begin{array}{l} \cdot \text{ analytic generic point} \\ \cdot \text{ nonanalytic closed point} \end{array} \right]$

Warning:

Tate $\not\Rightarrow$ ϖ-adic
(topology is ϖ-adic)

eg $\mathbb{Q}_p[[T]] \supset \mathbb{Z}_p[[T]]$ with (p, T)-adic topology..

$\underline{\mathbb{Q}_p[[T]]}$, $\varpi = p$

Ex: Any tfp K-alg is Tate.

Def: For a Huber A, a <u>continuous valuation</u> is

$|\cdot| : A \longrightarrow 0 \cup \Gamma$

(Γ: tot. ordered. abelian gp.)

such that it is

multiplicative, non Arch. & continuous.

- $|0|=0$ $|1|=1_\Gamma$ $|ab|=|a||b|$
- $|a+b| \leq \max(|a|, |b|)$
- $\forall \gamma \in \Gamma$, $\{a \in A \mid |a| < \gamma\}$ is open.

If $\Gamma = \mathbb{R}^+$,

(being continuous means the usual one)

We can

(Replace Γ by the subgrp generated by its image, and we view it as the same valuation).

Def: $|-| \Leftrightarrow |-|'$

iff $\forall f, g \in A$
$|f| \le |g|$
iff $|f|' \le |g|'$.

Def: $\mathrm{Cont}(A)$

$= \{$ continuous valuations of $A \}$

equipped with topology generated by

$\{ \underline{x} \in \mathrm{Cont}(A) \mid |f(x)| \le |g(x)| \ne 0 \}$

$[\ |f(x)| = x(f)\ ]$

(But they are not quasi-cpt opens)

quasi-cpt opens:

you need rational domains:

$\forall \underbrace{(f_1, \dots f_n, g) \subseteq A}_{\text{being an open ideal}}$

$$\boxed{U(f_i/g) := \{x \in \mathrm{Cont} \mid |f_i(x)| \leq |g(x)| \neq 0\}}$$

then $\underline{U(f_i/g)}$ is a quasi-cpt. open.

<u>Thm</u>: (Huber): $\mathrm{Cont}(A)$ is a spectral space.

Y is spectral iff

1) $Y = \operatorname{Spec} B$ for some B

$\Leftarrow$ 2) Y is quasi-cpt

has a topology generated by

quasi-cpt opens stabled

under finite intersections,

and Y is sober

(any irr. component has a generic point.)

$\Rightarrow$ 3) $Y = \varprojlim$ (finite T_0-space)

Moreover: $U(f_i/g)$ is

quasi-cpt.

<u>CEx</u>: If $C(f, \cdot)$ is not open,

then $\cdots$ is not quasi-cpt

$$\begin{cases} A = \mathbb{C}_p\langle T\rangle \\ f = 0 \\ g = T \end{cases}$$

<u>Def</u>: $\mathrm{Cont}(A) \longrightarrow \mathrm{Spec}\, A.$

$$|\cdot| \longmapsto \ker |\cdot|.$$

For any $x \in \mathrm{Cont}(A)$,

$\mathbb{K}(x)$ as the completion of the residue field

$$\mathrm{Frac}(\, A/\ker(x)\,)$$

A point x is <u>analytic</u>

iff $K(x)$ is analytic.

iff $K(x)$ is Tate.

iff $K(x)$ is nondiscrete

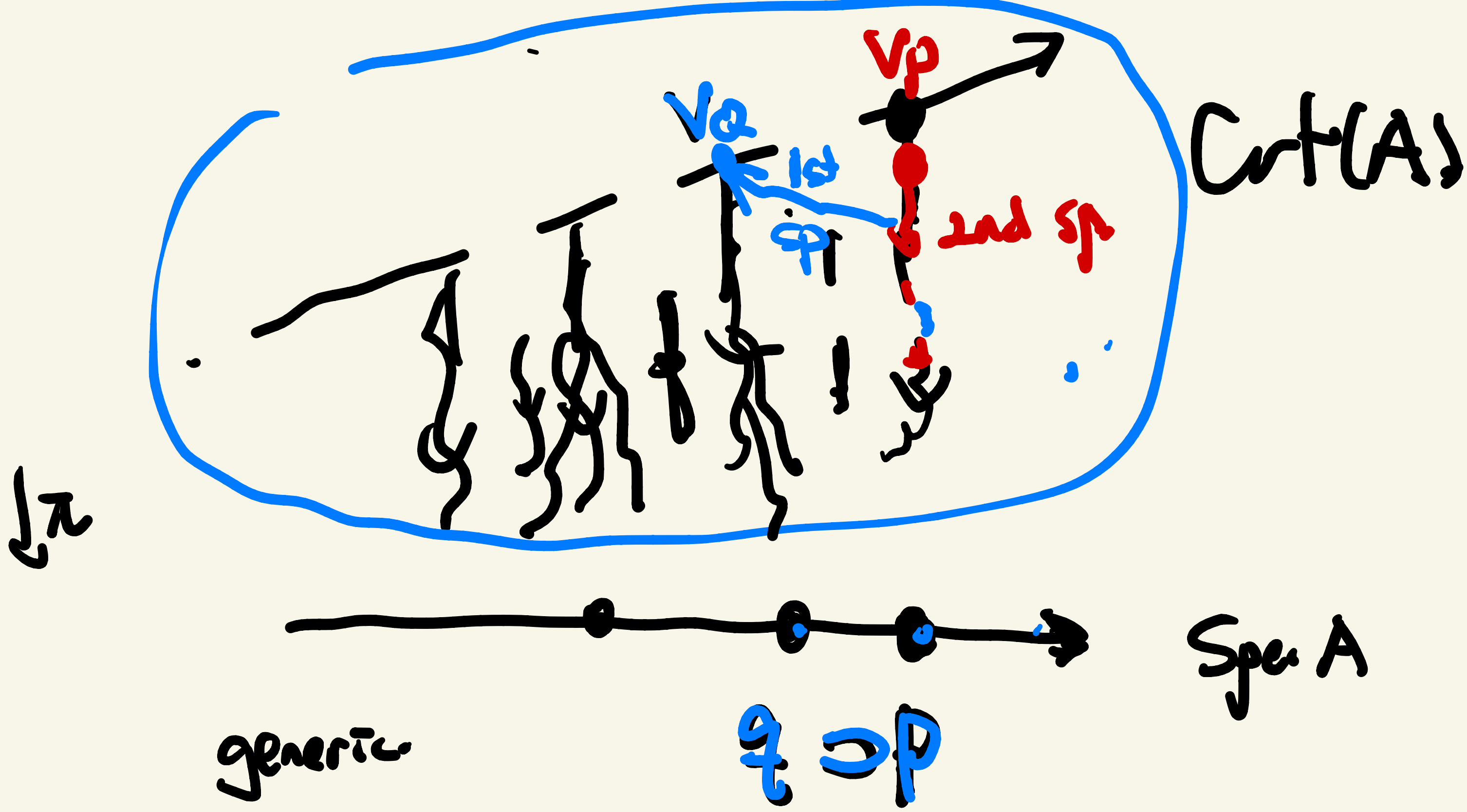
$$\begin{tikzcd} A \arrow[r] \arrow[dr, "x"', "1-1"'] & \boxed{A/\ker x} \arrow[d, "1-1"] \\ & 0 \cup \Gamma_x \end{tikzcd}$$

is valuation

$A/\ker x$ is equipped with

the valuation topology

$$\boxed{\mathrm{Cont}(A)} \longrightarrow \operatorname{Spec} A.$$



If p is open,

$v_p : A \to A/p \to 0 \cup 1$

the trivial valuation with $\ker = p$

is continuous

this is a non-analytic point

Any non-analytic point is of this form

If p is open, then

$\mathcal{V}_p$ is the generic part of

$\pi^{-1}(p)$

For $x, y \in \mathrm{Cot}(A)$,

If $x \rightsquigarrow y$ $(y = \bar{x})$

and $\pi(x) = \pi(y)$.

then Hutton calls

y a <u>secondary</u> <u>specialization</u>

of x

$y : A \longrightarrow OUP_y$

$x : A \longrightarrow OUP_x$: Tyl H

then y is 2nd sp. of x

iff $\exists\, H \subset P_y$

such that

$\boxed{x := y/H}$ { H is convex $\forall u, v \in H$ $[u,v] \subset H$ }

$(x(f) = y(f) \in O \cup P_y/H$

y is a primary specialization of x iff $\pi(y) \supset \pi(x)$ and the valuation of y is universal inside these points.

For $H \subset P_x$

$\boxed{y := x|_H}$

$$y(f) = \begin{cases} x(f) & \text{if } x(f) \in H \\ 0 & \text{else} \end{cases}$$

Thm: Any specialization in Cont(A) is a primary specialization of a secondary specialization.

Thm: Any analytic point only has 2nd generizations. All the generizations of an analytic point x form a chain. □