PROBLEM SET 4

Due: Nov 26, noon

100 credits + 50 bonus

Problem 1 (20 credits). For a morphism $f : X \to Y$ between schemes and $\mathcal{N} \in \mathcal{O}_Y\text{-}\mathsf{mod}$, consider the following presheaf on X:

$$U \mapsto \mathcal{O}_X(U) \underset{(f^{-1}\mathcal{O}_Y)(U)}{\otimes} (f^{-1}\mathcal{N})(U). \tag{1.1}$$

(1) (10 credits) Let k be a field and I be an infinite set. Let

$$f : \bigsqcup_I \mathsf{Spec}(k) \to \mathsf{Spec}(k)$$

be the obvious morphism, and $\mathcal{N} \in \mathcal{O}_Y\text{-}\mathsf{mod} \simeq k\text{-}\mathsf{mod}$ be an infinite-dimensional object. Show that (1.1) is not a sheaf.

(2) (10 credits) Let k be a field. Let

$$f : \mathbb{A}^2_k \to \mathbb{A}^1_k$$

be the projection morphism, and $\mathcal{N} := i_* \mathcal{O}_{\mathsf{Spec}(k)}$ be the skyscraper sheaf at the point $0 \in \mathbb{A}^1_k$. Show that (1.1) is not a sheaf.

Problem 2 (10 credits). Let k be a field and $X := \mathbb{A}^1_k$. Let I be an infinite set and consider the obvious morphism

$$(\mathsf{id})_{i \in I} : \bigsqcup_{i \in I} \mathbb{A}^1_k \to \mathbb{A}^1_k.$$

Show that pushforward along this morphism does not preserve quasi-coherent modules.

Problem 3 (10 credits). Let X be a scheme and $\mathcal{M}, \mathcal{N} \in \mathsf{QCoh}(X)$. Show that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathsf{QCoh}(X)$.

Problem 4 (10 bonus credits). Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. Show that the following functors are inverse to each other:

$$\begin{array}{ccc} A\text{-}\mathsf{alg} & \rightleftarrows & \mathcal{O}_X\text{-}\mathsf{alg}_{\mathsf{qcoh}} \\ B & \mapsto & \widetilde{B} \\ \mathcal{B}(X) & \leftarrow\!\shortmid & \mathcal{B}. \end{array}$$

Problem 5 (10 credits). Let $f : S' \to S$ be a morphism between schemes. For $\mathcal{A} \in \mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}}$, consider

$$\mathcal{A}' := f^* \mathcal{A} \in \mathcal{O}_{S'}\text{-}\mathsf{alg}_{\mathsf{qcoh}}.$$

Show that

$$\mathsf{Spec}_S(\mathcal{A}) \underset{S}{\times} S' \simeq \mathsf{Spec}_{S'}(\mathcal{A}').$$

Problem 6 (10 bonus credits)**.** Let S be a scheme and $\mathcal{A} \in \mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}}$. Show that for *any* S-scheme $p : X \to S$, the canonical map

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, \mathsf{Spec}_S(\mathcal{A})) \to \mathsf{Hom}_{\mathcal{O}_S\text{-}\mathsf{alg}}(\mathcal{A}, p_*\mathcal{O}_X)$$

is a bijection. Hint: base-change along $X \to S$.

Problem 7 (10 credits)**.** Show that a closed immersion between schemes is quasi-compact.

Problem 8 (10 credits)**.** Let $f : X \to Y$ be a morphism between schemes. Show that f is a monomorphism iff the diagonal morphism $\Delta_f : X \to X \times_Y X$ is an isomorphism.

Problem 9 (10 credits)**.** Show that a morphism out of an affine scheme is separated.

Problem 10 (10 credits)**.** Let X be a separated scheme. Show that the intersection of two affine open subsets of X is affine.

Problem 11 (10+10 credits)**.** Let $f : X \to Y$ be a morphism between S-schemes.

(1) (5 credits) Show that the graph morphism $\Gamma_f : X \to X \times_S Y$ is a locally closed immersion.
(2) (5 credits) If $Y \to S$ is separated, show that Γ_f is a closed immersion.
(3) (10 bonus credits) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is quasi-separated and $g \circ f$ is quasi-compact. Show that f is quasi-compact.

Problem 12 (20 bonus credits)**.** Consider $\mathbb{A}^\infty_{\mathbb{Z}} := \mathsf{Spec}(\mathbb{Z}[t_1, t_2, \cdots])$ and its closed subscheme Z corresponding to the ideal $(t_1, t_2, \cdots)$. Let $U := \mathbb{A}^\infty_{\mathbb{Z}} \setminus Z$ be the complementary open subscheme. Define X to be the scheme glued from two pieces of $\mathbb{A}^\infty_{\mathbb{Z}}$ via the identity morphism on U.

(1) (10 bonus credits) Show that X is quasi-compact, but is not quasi-separated.
(2) (10 bonus credits) Consider the unique morphism $p : X \to \mathsf{Spec}(\mathbb{Z})$. Show that p_* does not preserve quasi-coherent modules.