Talk 0: Overview

Sect 1: Zeta functions

$K/\mathbb{Q}$ number field, $\mathcal{O}_K^{/\mathbb{Z}}$ ring of integers

Dedekind zeta function

Euler product

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K \\ \text{prime ideal}}} \frac{1}{1 - |\mathcal{O}_K/\mathfrak{p}|^{-s}} \quad (\mathrm{Re}(s) > 1)$$

$\mathcal{O}_K$ is a Dedekind domain

$$= \sum_{\substack{I \subseteq \mathcal{O}_K \\ \text{ideal}}} |\mathcal{O}_K/I|^{-s}$$

Dirichlet series

prime ideal $\mathfrak{p} \subseteq \mathcal{O}_K \longleftrightarrow$ finite place of K

dense embedding into complete fields

$K \overset{\text{dense}}{\hookrightarrow} \mathbb{R}$ or $\mathbb{C} \longleftrightarrow$ infinite place of K

Λ-factors: $\pi^{-s/2}\Gamma(s/2)$ for each $\mathbb{Q} \hookrightarrow \mathbb{R}$

$2 \cdot (2\pi)^{-s}\Gamma(s)$ for each $\mathbb{Q} \hookrightarrow \mathbb{C}$

$\Gamma(s)$: Gamma function $\int_0^\infty t^{s-1}e^{-t}\,dt$

no zero; simple pole: $\mathbb{Z}_{\leq 0}$

Completed zeta function

$K = \mathbb{Q}$

$\Lambda_{\mathbb{Q}}(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx$

$$\Lambda_K(s) := \zeta_K(s) \cdot (\Lambda\text{-factors})$$

Thm (Hecke)

- $\zeta_K(s)$ extends meromorphically to $\mathbb{C}$, with a simple pole at $s=1$.
- $\Lambda_K(s) = \Lambda_K(1-s)$ functional equation

Conj (Riemann Hypothesis)

- $\zeta_K(s)$ has no $\underline{\text{nontrivial zeros}}$ outside $\mathrm{Re}(s)=\frac{1}{2}$.

  forced by functional equation

$\iff$ $\Lambda_K(s)$ has no zeros outside $\mathrm{Re}(s)=\frac{1}{2}$

E. Artin: analogy between $\underline{\text{number fields}}$ and $\underline{\text{function fields}}$

$K/\mathbb{F}_q(t)$. $\mathcal{O}_K/\mathbb{F}_q[t]$

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_K \\ \text{prime ideal}}} \frac{1}{1-|\mathcal{O}_K/\mathfrak{p}|^{-s}} \qquad (\mathrm{Re}(s)>1)$$

$$= \sum_{\substack{I\subseteq\mathcal{O}_K \\ \text{ideal}}} |\mathcal{O}_K/I|^{-s}$$

power series in $T:=q^{-s}$ b/c $\mathcal{O}_K/\mathfrak{p}$ is a field over $\mathbb{F}_q$.

Geometry: $\mathbb{A}^1_{/\mathbb{F}_q} = \operatorname{Spec} \mathbb{F}_q[t]$

$U \longrightarrow \mathbb{A}^1$ normalization in K

smooth affine curve

$$\zeta_K(s) = \prod_{\substack{x \in U \\ \text{closed point}}} \frac{1}{1 - |\kappa(x)|^{-s}}$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{q^{-ns}}{n} |U(\mathbb{F}_{q^n})| \right)$$

$$\ln \zeta_K(s) = \sum_{x \in U} \left(\kappa(x)^{-s} + \frac{1}{2} \kappa(x)^{-2s} + \cdots \right)$$

$X \longrightarrow \mathbb{P}^1$ normalization in K

smooth projective curve

closed points of $X \longleftrightarrow$ (finite) places of K.

$\Lambda_K(s)$: same formula with U replaced by X.

In general, for any smooth projective $X/\mathbb{F}_q$

Hasse-Weil zeta function

$$\zeta_X(s) = \prod_{\substack{x \in X \\ \text{closed point}}} \frac{1}{1 - |\kappa(x)|^{-s}}$$

$$= \exp\left(\sum_{n=1}^{\infty} \frac{q^{-ns}}{n} |X(\mathbb{F}_{q^n})| \right)$$

$$= Z_X(T) \qquad T = q^{-s}$$

$Z_X(T) \in \mathbb{Q}[[T]]$.

Conj (Weil)

- Rationality $Z_X(T)$ is a rational function in T.
- Functional equation If $\dim X = n$ pure dim.

$$Z_X(q^{-n}T^{-1}) = \pm q^{nE/2} T^E Z_X(T)$$

for some integer E.

- Riemann Hypothesis

$$Z_X(T) = \frac{P_1(T) \cdots\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)} \qquad P_i(T) = \prod_j (1 - \underset{\sim}{\alpha_{ij}} T)$$

algebraic integers

such that $|\alpha_{ij}| = q^{i/2}$

- Betti numbers:

Suppose number field $K/\mathbb{Q}$, $\mathfrak{p} \subseteq O_K$ with $O_K/\mathfrak{p} \simeq \mathbb{F}_q$. Y smooth projective over K w/ "good reduction" X smooth projective over $\mathbb{F}_q$.

Then:

complex manifold

$$\deg(P_i) = i\text{-th Betti number of } Y_{\mathbb{C}}$$

(for any/all $K \hookrightarrow \mathbb{C}$)

E = Euler characteristic.

$$|X(\mathbb{F}_{q^n})| = \sum_{i,j} (-1)^i \alpha_{ij}^n$$

Example of $\mathbb{P}^n$

$$\mathbb{A}^n \hookrightarrow \mathbb{P}^n \hookleftarrow \mathbb{P}^{n-1}$$

$$\Rightarrow Z_{\mathbb{P}^n}(T) = Z_{\mathbb{A}^n}(T) \cdot Z_{\mathbb{P}^{n-1}}(T)$$

$$Z_{\mathbb{A}^n}(T) = \frac{1}{1-q^n T}$$

$$\Rightarrow Z_{\mathbb{P}^n}(T) = \frac{1}{(1-T)(1-qT)\cdots(1-q^n T)}$$

compare with: $H^i(\mathbb{C}\mathbb{P}^n) = \begin{cases} \mathbb{Q} & i=0,2,\dots,2n \\ 0 & \text{otherwise} \end{cases}$

Example (Elliptic curve, Hasse)

$$A \longrightarrow A$$
$$x \longmapsto x^q$$

$E/\mathbb{F}_q$ group structure

$\mathrm{id} - \mathrm{Frob}^n$ isogeny with kernel $= E(\mathbb{F}_{q^n})$

$$|E(\mathbb{F}_{q^n})| = \deg(\mathrm{id} - \mathrm{Frob}^n)$$
$$= (\mathrm{id} - \mathrm{Frob}^n)(\mathrm{id} - \mathrm{Frob}^n)^*$$
$$= 1 - \mathrm{Frob}^n - (\mathrm{Frob}^n)^* + q^n$$

$q = \deg \mathrm{Frob} = \mathrm{Frob} \cdot \mathrm{Frob}^*$.

Weil: generalise this to abelian varieties and curves (using Jacobian's)

also did Fermat hypersurfaces

$$\sum a_i x_i^n = 0$$

$P_i(T)$ should be related to $\underset{\sim}{H^i(X)}$

(some cohomology theory)

$$\mathrm{Frob}: H^i(X) \longrightarrow H^i(X)$$

$$P_i(T) = \det(1 - \mathrm{Frob}\, T)$$

characteristic polynomial.

$$|X(\mathbb{F}_{q^n})| = |\text{fixed points of } \mathrm{Frob}^n|$$

(some version of Lefschetz fixed point formula)

as if a 2d-dim real manifold

$$= \sum_{i=0}^{2d} (-1)^i \,\mathrm{Trace}(\mathrm{Frob}^n \mid H^i(X))$$

$$\Rightarrow \zeta_X(s) = Z_X(T) = \prod P_i(T)^{(-1)^i}.$$

Meta Conjecture (Weil)

There is a contravariant functor

$$H^*: \{\text{sm. proj. over } \mathbb{F}_q\} \longrightarrow \{\text{graded } K\text{-algebras}\}$$

satisfying certain axioms

(Poincare duality, Lefschetz fixed point, Comparison with Betti ....

Weil cohomology theory

$\mathrm{End}(X) \curvearrowright H^i(X)$.

Consider supersingular elliptic curve $(\dim_K H^1(X) = 2)$

$\rightsquigarrow$ $K \neq \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{Q}_p$

Grothendieck (school):

- constructed a Weil coh. theory with $K = \mathbb{Q}_\ell$, $\ell \neq p$.

  ℓ-adic cohomology

  (relative version ℓ-adic sheaf) Dwork did rationality before

- proved Weil conjectures except for RH.

- made standard conjectures about motives

$$\text{standard conjectures} \Downarrow \text{RH} \qquad \text{motives} \Downarrow \text{motivic theory}$$

Deligne

- proved RH without standard conjecture
  Weil I

- generalized Weil's conj's to ℓ-adic sheaves
  Weil II

  $\Rightarrow$ weight theory, perverse sheaves ......

Berthelot:

rigid cohomology for $K = \overline{\mathbb{Q}_p}$