§1 An abstract GAGA theorem

<u>prop.</u> $X = \mathrm{LRS}$, $\mathcal{O}(1) =$ l.b. on X s.t. $\forall$ v.b. $\mathcal{E}$ on X, for large enough n:

(*) $\mathcal{E}(n)$ is globally gen,

(**) $H^i(X, \mathcal{E}(n)) = 0 \quad \forall i \geq 1$.

then $\exists$ a natural $f: X \longrightarrow X^{alg} := \mathrm{Proj} \underbrace{\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}(n))}_{P_\bullet}$ such that f^* yields an equiv of cats of v.b. and preserves cohom of v.b.s.

<u>proof.</u> what's f? $g \in P_n \rightsquigarrow$ on $D(g)$, $g: \mathcal{O}_{D(g)} \xrightarrow{\sim} \mathcal{O}_{D(g)}(n)$. $\Rightarrow \forall \frac{a}{g^k} \in P_\bullet[g^{-1}]_0$, $\frac{a}{g^k} \longmapsto g^{-k} a$ yields $P_\bullet[g^{-1}]_0 \longrightarrow \mathcal{O}_X(D(g)) \rightsquigarrow D(g) \longrightarrow D_+(g)$. glue these maps together. (*) $\Rightarrow$ the $D(g)$ cover X, so we get $f: X \longrightarrow X^{alg}$.

write $\mathcal{E}^{alg} := \mathrm{Proj} \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{E}(n))$. (**) $\Rightarrow (-)^{alg}$ is exact. why is $\mathcal{E}^{alg}$ a v.b.? (*) $\Rightarrow \mathcal{O}_X^m \twoheadrightarrow \mathcal{E}(n)$. now take $f' \in P_{n'}$.

<u>claim.</u> n' large enough $\Rightarrow \exists\, \mathcal{E}(n-n') \longrightarrow \mathcal{O}_X^m$ such that
$$\begin{tikzcd} \mathcal{E}(n-n') \arrow[r] \arrow[dr, "f'"'] & \mathcal{O}_X^m \arrow[d] \\ & \mathcal{E}(n) \end{tikzcd}$$
commutes.

$$\left(\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d, equal] & \mathcal{G} \arrow[r, dashed] \arrow[d] & \mathcal{E}(n-n') \arrow[d, "f'"] \\ \mathcal{F} \arrow[r] & \mathcal{O}_X^m \arrow[r] & \mathcal{E}(n). \end{tikzcd} \right.$$

$(\ast\ast) \Rightarrow H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F})(n'-n)) = 0$ for n' large enough.)

$\Rightarrow$ on $D_+(f')$, $\mathcal{E}^{alg}$ is a v.b. (*) $\Rightarrow \mathcal{E}^{alg}$ is a v.b. on X^{alg}.

adjunction $\Rightarrow$ get map $f^* \mathcal{E}^{alg} \xrightarrow{\sim} \mathcal{E} \Rightarrow f^*$ is essentially surj. now need to show f^* preserves cohom. ampleness of $\mathcal{O}_{X^{alg}}(1) \Rightarrow$ inj $\mathcal{F} \hookrightarrow \mathcal{O}_{X^{alg}}(n)^m$.

$\mathcal{F} =$ v.b. on X^{alg}

$$\begin{tikzcd} H^0(X^{alg}, \mathcal{F}) \arrow[r, hook] \arrow[d] & H^0(X^{alg}, \mathcal{O}_{X^{alg}}(n)^m) \arrow[d, "s"] \\ H^0(X, f^*\mathcal{F}) \arrow[r, hook] & H^0(X, \mathcal{O}(n)^m). \end{tikzcd}$$

$\Rightarrow$ for n large enough.

$\Rightarrow$ inj on H^0.

apply inj to $\mathrm{coker}(\mathcal{F} \longrightarrow \mathcal{O}_{X^{alg}}(n)^m) \Rightarrow$ surj on H^0.

$\Rightarrow f^*$ is fully faithful.

for higher cohom, follows from Čech cohom + $f^* \mathcal{E}^{alg} \xrightarrow{\sim} \mathcal{E}$. □

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§2 Ampleness on relative FF curve

$E =$ nonarch local field, $\pi =$ unif, $\mathcal{O}_E / \pi = \mathbb{F}_q$.

$S = \mathrm{spa}(R, R^+)$ aff perfectoid $/ \mathbb{F}_q$.

$\varpi =$ pseudo-unif of R

$\rightsquigarrow X_S =$ rel FF curve.

what is it? rad: $|Y_S| \longrightarrow (0, \infty)$. $\rightsquigarrow$ rational affinoids $Y_{S,[a,b]} = \mathrm{spa}(B_{R,[a,b]}, B^+_{R,[a,b]})$.

X_S is $Y_{S,[q^{-1},q]}$, where we glue $Y_{S,[q^{-1},1]} \xrightarrow{\sim} Y_{S,[1,q]}$ using φ.

<u>thm.</u> (*) and (**) hold for $X = X_S$ and $\mathcal{O}(1) = \mathcal{O}(1)$.

<u>proof.</u> v.b. on $X_S \Leftrightarrow$ loc free fin rk M on $B_{R,[q^{-1},q]}$ equipped w/ an iso $M_{[q^{-1},1]} \xrightarrow{\sim} \varphi^*(M_{[1,q]}) =: \varphi_M$. want to reduce to M free.

take a surj $B^m_{R,[q^{-1},q]} \twoheadrightarrow M \Rightarrow$ split this to get $B^m_{R,[q^{-1},q]} \cong N \oplus M$. need an iso $N_{[q^{-1},1]} \xrightarrow{\sim} \varphi^*(N_{[1,q]})$. but in $K_0(B_{R,[q^{-1},1]})$, both equal $[\underbrace{B^m_{R,[q^{-1},1]}}] - [M_{[q^{-1},1]}]$, so we get the iso after adding enough $B_{R,[q^{-1},1]}$ to both sides.

$B^m_{R,[q^{-1},q]}$
$\|$
so can assume M is free $\rightsquigarrow \varphi_M = A^{-1}\varphi$ for $A \in GL_m(B_{R,[q^{-1},1]})$. choose N and N' such that

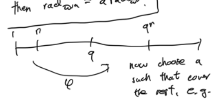
- A has entries in $\pi^N W_{\mathcal{O}_E}(R^+) \langle (\frac{[\varpi]}{\pi})^{\pm 1} \rangle$,
- A^{-1} " " $\pi^{-N'} W_{\mathcal{O}_E}(R^+) \langle (\frac{[\varpi]}{\pi})^{\pm 1} \rangle$.

note twisting by $\mathcal{O}(1)$ replaces A w/ πA. so can assume $N \geq 1$ and $qN > N'$.

<u>WTS:</u> fix any rational r w/ $1 < r \leq q$. then $\exists v_1, \ldots, v_m \in (B^m_{R,[1,1]})^{\varphi = A}$ that are a basis of $B^m_{R,[r,q]}$. (if we had this, we'd be done by applying the argument to a different pseudo-unif.)

$\boxed{\text{if you replace } \varpi \text{ w/ } \varpi_1^a \text{ where } a \in \mathbb{Z}[\tfrac{1}{p}], \text{ then } \mathrm{rad}_{\varpi_1} = a\, \mathrm{rad}_\varpi.}$

($\|\cdot\|$ is normalized such that $\|[\varpi]\| = \frac{1}{q}$)



now choose a such that cover the rest, e.g. $a \geq r$ and $ra \leq q$.

<u>claim.</u> $\varphi - A : B^m_{R,[1,q]} \longrightarrow B^m_{R,[1,1]}$ satisfies: for large enough M, if $w \in \pi^M W_{\mathcal{O}_E}(R^+) \langle (\frac{[\varpi]}{\pi})^{\pm 1} \rangle^m$, then $\exists v \in B^m_{R,[1,q]}$ such that $(\varphi - A)v = w$ and $\|v\| \leq q^{-M-1}$.

(take $w_i = (\varphi - A)([\varpi]^M e_i)$ and get v_i' s.t. $w_i = (\varphi - A)(v_i')$ and $\|v_i'\| \leq q^{-M-1}$ then $v_i = [\varpi]^M e_i - v_i' \in (B^m_{R,[1,q]})^{\varphi = A}$, and they form a basis since the change of basis matrix lies in $[\varpi]^M(\mathrm{id} + [\varpi] M_m(B^\circ_{R,[r,q]})) \in GL_m(B_{R,[r,q]})$.)

i-th standard basis vector

write $w = w_1 + w_2$, where
$$w_1 \in [\varpi]^{N-1} \pi^{M-N+1} W_{\mathcal{O}_E}(R^+) \langle \tfrac{\pi}{[\varpi]} \rangle^m$$
$$w_2 \in [\varpi]^N \pi^{M-N} W_{\mathcal{O}_E}(R^+) \langle \tfrac{[\varpi]}{\pi} \rangle^m.$$

attempt to find v

take $v = \varphi^{-1}(w_1) - A^{-1} w_2$, and $w' := w - \varphi(v) + Av = \varphi(A^{-1} w_2) + A\varphi^{-1}(w_1)$.

$$N \geq 1 \Rightarrow A\varphi^{-1}(w_1) \in [\varpi]^{\frac{N-1}{q}} \pi^{M+1} W_{\mathcal{O}_E}(R^+) \langle \tfrac{\pi}{[\varpi]} \rangle^m \subseteq \pi^{M+1} W_{\mathcal{O}_E}(R^+) \langle (\tfrac{\pi}{[\varpi]})^{\pm 1} \rangle^m$$

$$qN > N' \Rightarrow \varphi(A^{-1} w_2) \in [\varpi]^{qN} \pi^{M-N'} W_{\mathcal{O}_E}(R^+) \langle \tfrac{[\varpi]}{\pi} \rangle^m \subseteq \pi^{M+1} W_{\mathcal{O}_E}(R^+) \langle (\tfrac{[\varpi]}{\pi})^{\pm 1} \rangle^m.$$

$$\Rightarrow w' \in \pi^{M+1} W_{\mathcal{O}_E}(R^+) \langle (\tfrac{[\varpi]}{\pi})^{\pm 1} \rangle^m.$$

so suffices to show $\|v\| \leq q^{-M-1}$ (then just iterate the above process.)

$$\varphi^{-1}(w_1) \in [\varpi]^{\frac{N-1}{q}} \pi^{M-N+1} W_{\mathcal{O}_E}(R^+) \langle \tfrac{\pi}{[\varpi]} \rangle^m \quad \left(\|\pi\| = \tfrac{1}{q^r} \right)$$

$$\Rightarrow \|\varphi^{-1}(w_1)\| \leq q^{-\frac{N-1}{q} - rM + rN - r} \Rightarrow \leq q^{-M-1} \text{ for large enough } M.$$

$$A^{-1} w_2 \in [\varpi]^N \pi^{M-N'} W_{\mathcal{O}_E}(R^+) \langle \tfrac{[\varpi]}{\pi} \rangle^m$$

$$\Rightarrow \|A^{-1} w_2\| \leq q^{-N - rM + rN'} \Rightarrow \leq q^{-M-1} \text{ for large enough } M.$$

□