LECTURE 5

Last time we constructed a homomorphism $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{t})$ fitting into the following diagram

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\phi"] \arrow[dr, "\chi_\lambda = \xi_{-,\lambda}"'] & \mathsf{Sym}(\mathfrak{t}) \arrow[d, "\mathsf{ev}_\lambda"] \\ & k_\lambda, \end{tikzcd} \tag{0.1}$$

where for any $\lambda \in \mathfrak{t}^*$, $\chi_\lambda = \xi_{-,\lambda}$ is the central character of the Verma module M_λ. We stated the following result, which will be proved today.

Theorem 1 (Harish-Chandra)**.** *The homomorphism ϕ induces an isomorphism*

$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \mathsf{Sym}(\mathfrak{t})^{W_\bullet}$$

from $Z(\mathfrak{g})$ to the invariance of $\mathsf{Sym}(\mathfrak{t})$ with respect to the dotted W-action.

1. Step 1: image is $W_\bullet$-invariant

Let us first prove the image of ϕ is indeed contained in $\mathsf{Sym}(\mathfrak{t})^{W_\bullet}$. Since W is generated by simple reflections s_α, $\alpha \in \Delta$, we only need to show $\phi(z) = s_\alpha \cdot \phi(z)$ for any $z \in Z(\mathfrak{g})$. In other words, we need to show

$$\mathsf{ev}_\lambda(\phi(z)) = \mathsf{ev}_\lambda(s_\alpha \cdot \phi(z)) \tag{1.1}$$

for any $\lambda \in \mathfrak{t}^*$. In fact, we only need to prove this for a Zariski dense subset of $\mathfrak{t}^*$. Let us first give this dense subset. The following result is obvious after drawing a picture.

Lemma 2. *Let $\alpha \in \Delta$ be a simple root. The subset $\{\lambda \in \mathfrak{t}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}\}$ is a Zariski dense subset of $\mathfrak{t}^*$.*

Example 3. For $\mathfrak{g} = \mathfrak{sl}_2$, this subset is $\mathbb{Z}^{\geq -1} \subset \mathbb{A}^1$. In fact, in [Exam. 15, Lect. 4], we have used this subset to show the image of $\phi : Z(\mathfrak{sl}_2) \to k[h]$ is invariant under $h \mapsto -h-2$. The argument below is an immediate generalization.

Lemma 4. *Let $\alpha \in \Delta$ be a simple root and $\lambda \in \mathfrak{t}^*$. Suppose $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}$. Then M_λ contains $M_{s_\alpha \cdot \lambda}$ as a submodule.*

Corollary 5. *The image of $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{t})$ is indeed contained in $\mathsf{Sym}(\mathfrak{t})^{W_\bullet}$.*

Proof. By previous discussion, we only need to prove (1.1) for α and λ satisfying the assumption of Lemma 4. By (0.1), the LHS and RHS of (1.1) are exactly the central character of M_λ and $M_{s_\alpha \cdot \lambda}$. Now Lemma 4 implies they are equal because the central character of a module is equal to the central character of any nonzero submodule of it. □

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Proof of Lemma 4. Recall
$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha = \lambda - \langle\lambda,\check{\alpha}\rangle\alpha.$$
It follows that
$$s_\alpha\cdot\lambda = \lambda - \langle\lambda+\rho,\check{\alpha}\rangle\alpha = \lambda - m\alpha$$
for $m\in\mathbb{Z}^{\geq 0}$. The lemma is obvious for $m=0$. We assume $m>0$. Then we have $\langle\lambda,\check{\alpha}\rangle = m-1$ because $\langle\rho,\check{\alpha}\rangle = 1$ (see [H1, Cor. to Lem. 10.2(B)]).

Let $f_\alpha\in\mathfrak{n}^-$ be a nonzero vector of weight $-\alpha$. Consider the vector $f_\alpha^m\cdot v_\lambda$, which is of weight $\lambda - m\alpha = s_\alpha\cdot\lambda$. We only need to show
$$\mathfrak{n}\cdot(f_\alpha^m\cdot v_\lambda) = 0.$$
Indeed, if this is true, the map $k_{s_\alpha\cdot\lambda}\to M_\lambda,\ c\mapsto c(f_\alpha^m\cdot v_\lambda)$ is $\mathfrak{b}$-linear, and thereby induces a $\mathfrak{g}$-linear map $M_{s_\alpha\cdot\lambda}\to M_\lambda$. This map is injective because as a morphism between $U(\mathfrak{n}^-)$-modules, it is given by $-\cdot f_\alpha^m : U(\mathfrak{n}^-)\to U(\mathfrak{n}^-)$, which is injective by the PBW theorem.

It remains to show $\mathfrak{n}$ annihilates $f_\alpha^m\cdot v_\lambda$. For each simple root $\beta\in\Delta$, let $e_\beta\in\mathfrak{n}$ be a nonzero vector of weight β. Note that the vectors $(e_\beta)_{\beta\in\Delta}$ generate $\mathfrak{n}$ under Lie brackets[1]. Hence we only need to show $e_\beta\cdot f_\alpha^m\cdot v_\lambda = 0$. There are two cases:

- If $\alpha\neq\beta$, then $[e_\beta,f_\alpha] = 0$[2] and
$$e_\beta\cdot f_\alpha^m\cdot v_\lambda = f_\alpha^m\cdot e_\beta\cdot v_\lambda = 0$$
because $\mathfrak{n}\cdot v_\lambda = 0$.
- If $\alpha=\beta$, $[e_\alpha,f_\alpha]\in\mathfrak{t}$ is propotionate to $\check{\alpha}$ (see [H1, Prop. 8.3(d)]). Rescale e_α, we may assume $[e_\alpha,f_\alpha]=\check{\alpha}$. Then $[\check{\alpha},f_\alpha] = \langle-\alpha,\check{\alpha}\rangle f_\alpha = -2f_\alpha$. Now the following calculation is essentially that in [Exe. 24, Lect. 2]. We have
$$e_\alpha\cdot f_\alpha^m\cdot v_\lambda = \sum_{1\leq i\leq m} f_\alpha^{m-i}\cdot[e_\alpha,f_\alpha]\cdot f_\alpha^{i-1}\cdot v_\lambda + f_\alpha^m\cdot e_\alpha\cdot v_\lambda.$$
Recall we have $e_\alpha\cdot v_\lambda = 0$. Also,
$$\check{\alpha}\cdot f_\alpha^j = \sum_{1\leq i\leq j} f_\alpha^{j-i}\cdot[\check{\alpha},f_\alpha]\cdot f_\alpha^{i-1} + f_\alpha^j\cdot\check{\alpha} = -2jf_\alpha^j + f_\alpha^j\cdot\check{\alpha}.$$
Hence
$$e_\alpha\cdot f_\alpha^m\cdot v_\lambda = \sum_{1\leq i\leq m}(-2(i-1)f_\alpha^{m-1} + f_\alpha^{m-1}\cdot\check{\alpha})v_\lambda = \big(-m(m-1) + m\langle\lambda,\check{\alpha}\rangle\big)v_\lambda = 0$$
as desired.

□[Lemma 4]

2. Step 2: Filtrations

By Step 1, we have a homomorphism
$$\phi_{\mathsf{HC}} : Z(\mathfrak{g})\to\mathsf{Sym}(\mathfrak{t})^{W_\bullet}.$$
In this step, we equip both sides with filtrations, and show ϕ_{HC} is compatible with them. The punchline is the following easy fact:

Fact 6. *Let V_1 and V_2 be two vector spaces equipped with $\mathbb{Z}^{\geq 0}$-indexed (exhausted) filtrations. Suppose $\varphi : V_1\to V_2$ is a k-linear map compatible with the filtrations. Then φ is an isomorphism iff $\mathsf{gr}^\bullet\varphi : \mathsf{gr}^\bullet V_1\to\mathsf{gr}^\bullet V_2$ is so.*

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[1] Because $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] = \mathfrak{g}_{[\alpha,\beta]}$ whenever $\alpha,\beta,\alpha+\beta\in\Phi^+$.

[2] Otherwise it is a nonzero vector with weight $\beta-\alpha$ but the latter is not a root because $\Phi = \Phi^+\sqcup\Phi^-$.

Construction 7. *The PBW filtration on $U(\mathfrak{g})$ induces a filtration on $Z(\mathfrak{g})$ with $\mathsf{F}^{\leq i}Z(\mathfrak{g}) := Z(\mathfrak{g}) \cap \mathsf{F}^{\leq i}U(\mathfrak{g})$. Note that $\mathsf{gr}^\bullet Z(\mathfrak{g})$ is a subalgebra of $\mathsf{gr}^\bullet U(\mathfrak{g}) \simeq \mathsf{Sym}(\mathfrak{g})$.*

Construction 8. *The PBW filtration on $U(\mathfrak{t}) \simeq \mathsf{Sym}(\mathfrak{t})$ is preserved by the dotted W-action. Hence it induces a filtration on $U(\mathfrak{t})^{W_\bullet}$*[3].

Warning 9. *The* dotted *W-action on $\mathsf{Sym}(\mathfrak{t})$ does not preserve the grading. But the usual (linear) W-action does.*

Lemma 10. *The homomorphism $\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \to U(\mathfrak{t})^{W_\bullet}$ is compatible with the above filtrations.*

Proof. This is obvious from the description of ϕ as

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow k \underset{U(\mathfrak{n}^-)}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} k \simeq U(\mathfrak{t}).$$

□

3. Step 3: Calculating the graded pieces

Construction 11. *Recall $\mathsf{Sym}(\mathfrak{g})$ has a natural $\mathfrak{g}$-module structure constructed as follows. For any $V \in \mathfrak{g}$, there is a natural $\mathfrak{g}$-module structure on $V^{\otimes n}$ given by*

$$\mathfrak{g} \times V^{\otimes n} \to V^{\otimes n},\ (x, \underset{i}{\otimes} v_i) \mapsto \sum_i (v_1 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \cdots \otimes v_n).$$

This action is compatible with the symmetric group Σ_n-action on $V^{\otimes n}$ and thereby induces a $\mathfrak{g}$-module structure on $\mathsf{Sym}^n(V)$. Taking direct sum, we obtain a $\mathfrak{g}$-module structure on $\mathsf{Sym}(V)$.

In the case $V = \mathfrak{g}$, to distinguish with the multiplication structure on $\mathsf{Sym}(\mathfrak{g})$, we denote this action by

$$\mathfrak{g} \times \mathsf{Sym}(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{g}),\ (x, u) \mapsto \mathsf{ad}_x(u),$$

and call it the ***adjoint action****.*

Note that by definition, for $x \in \mathfrak{g}$ and $u, v \in \mathsf{Sym}(\mathfrak{g})$, we have

$$\mathsf{ad}_x(u \cdot v) = \mathsf{ad}_x(u) \cdot v + u \cdot \mathsf{ad}_x(v).$$

In particular, the $\mathfrak{g}$-invariance

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} := \{u \in \mathsf{Sym}(\mathfrak{g}) \,|\, \mathsf{ad}_x(u) = 0 \text{ for any } x \in \mathfrak{g}\}$$

is a subalgebra of $\mathsf{Sym}(\mathfrak{g})$.

Lemma 12. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{tikzcd} \mathsf{gr}^\bullet Z(\mathfrak{g}) \arrow[r, dotted, "\simeq"] \arrow[d, "\subset"] & \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\subset"] \\ \mathsf{gr}^\bullet U(\mathfrak{g}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{g}), \end{tikzcd}$$

where the bottom isomorphism is given by the PBW theorem.

To prove this lemma, we use the following exercise:

Exercise 13. This is Homework 2, Problem 4. Prove: the adjoint $\mathfrak{g}$-action on $U(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \to U(\mathfrak{g}),\ (x, u) \mapsto \mathsf{ad}_x(u) = [x, u],$$

preserves each $\mathsf{F}^{\leq n}U(\mathfrak{g})$, and the induced $\mathfrak{g}$-action on $\mathsf{gr}^\bullet(U(\mathfrak{g})) \simeq \mathsf{Sym}(\mathfrak{g})$ is the adjoint action in Construction 11.

[3] Note that $\mathsf{gr}^\bullet U(\mathfrak{t})$ is also isomorphic to $\mathsf{Sym}(\mathfrak{t})$. To distinguish them, we always use $U(\mathfrak{t})$ to denote the filtered commutative ring while use $\mathsf{Sym}(\mathfrak{t})$ to denote the graded commutative ring.

Proof of Lemma 12. By the above exercise, we have a short exact sequence of *finite-dimensional* $\mathfrak{g}$-modules:

$$0 \to \mathsf{F}^{\leq n-1}U(\mathfrak{g}) \to \mathsf{F}^{\leq n}U(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g}) \to 0.$$

Since $\mathfrak{g}\text{–}\mathsf{mod}_{\mathsf{fd}}$ is semisimple, this short exact sequence splits. Hence taking $\mathfrak{g}$-invariance, we obtain[4]

$$0 \to \mathsf{F}^{\leq n-1}Z(\mathfrak{g}) \to \mathsf{F}^{\leq n}Z(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}} \to 0$$

This gives the desired isomorphism $\mathsf{gr}^n Z(\mathfrak{g}) \simeq \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}}$.

□[Lemma 12]

A similar proof[5] gives:

Lemma 14. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{tikzcd} \mathsf{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) \arrow[r, dotted, "\simeq"] \arrow[d, "\subset"] & \mathsf{Sym}(\mathfrak{t})^W \arrow[d, "\subset"] \\ \mathsf{gr}^\bullet U(\mathfrak{t}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{t}), \end{tikzcd}$$

where the right-top corner is the invariance for the linear W*-action on* $\mathsf{Sym}(\mathfrak{t})$*.*

Combining the above two lemmas, we obtain:

Corollary 15. *There is a unique dotted graded homomorphism making the following diagram commute*

$$\begin{tikzcd} \mathsf{gr}^\bullet Z(\mathfrak{g}) \arrow[r, "\simeq"] \arrow[d, "\mathsf{gr}^\bullet \phi_{\mathsf{HC}}"] & \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[d, dotted, "\phi_{\mathsf{cl}}"] \\ \mathsf{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{t})^W. \end{tikzcd}$$

4. Step 4: Chevalley isomorphism

It remains to show

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^W$$

is an isomorphism. Let us first give an explicit construction of this homomorphism. We need the following characterization of ϕ.

Construction 16. *Consider the composition*

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} k.$$

With respect to the adjoint $\mathfrak{t}$*-action, the source has weight* 0*. Hence the composition factors through the* 0*-weight subspace of the target, which is exactly* $U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t})$*. By definition, the obtained map*

$$Z(\mathfrak{g}) \to U(\mathfrak{t})$$

is just ϕ*.*

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[4] Warning: in general, taking invariance is only *left exact*. (Memory method: it is given by $\mathsf{Hom}_{\mathfrak{g}}(k, -)$.) Hence we need the existence of a splitting.

[5] Note that $\mathsf{Rep}(W)_{\mathsf{fd}}$ is also semisimple

Construction 17. *It follows ϕ_{cl} can be constructed as follows. Consider the composition*

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow \mathsf{Sym}(\mathfrak{g}) \twoheadrightarrow \mathsf{Sym}(\mathfrak{g}/\mathfrak{n}). \tag{4.1}$$

It factors through $\mathsf{Sym}(\mathfrak{b}/\mathfrak{n}) \simeq \mathsf{Sym}(\mathfrak{t})$. The obtained map

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})$$

can be identified with $\mathsf{gr}^\bullet\phi$. Since ϕ factors through $U(\mathfrak{t})^{W\bullet}$, the map $\mathsf{gr}^\bullet\phi$ factors through $\mathsf{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) \simeq \mathsf{Sym}(\mathfrak{t})^W$. The obtained map

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^W$$

is just ϕ_{cl}.

Remark 18. The geometric meaning of the above construction is as follows.

Note that $\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} = \mathsf{Sym}(\mathfrak{g})^G$ because G-invariance is equal to $\mathfrak{g}$-invariance[6]. Hence (4.1) corresponds to the morphisms

$$(\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^* \to \mathfrak{g}^*/\!/G.$$

Since $\mathsf{gr}^\bullet\phi$ factors through $\mathsf{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) \simeq \mathsf{Sym}(\mathfrak{t})^W$. The above composition factors through $(\mathfrak{g}/\mathfrak{n})^* \to (\mathfrak{b}/\mathfrak{n})^* \simeq \mathfrak{t}^* \to \mathfrak{t}^*/\!/W$. In other words, we have

$$\begin{tikzcd} (\mathfrak{g}/\mathfrak{n})^* \arrow[r] \arrow[d] & \mathfrak{g}^*/\!/G \\ \mathfrak{t}^* \arrow[r] & \mathfrak{t}^*/\!/W \arrow[u, dotted] \end{tikzcd}$$

such that the dotted arrow is given by $\mathsf{Spec}(\phi_{\mathsf{cl}})$.

It is convenient to get rid of the dual spaces using the Killing form. Namely, Kil induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ compatible with the G-actions, while $\mathsf{Kil}|_{\mathfrak{t}}$ induces an isomorphism $\mathfrak{t} \simeq \mathfrak{t}^*$ compatible with the W-actions[7]. Via the first isomorphism, the subspaces $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ corresponds to $(\mathfrak{g}/\mathfrak{b})^* \subset (\mathfrak{g}/\mathfrak{n})^* \subset \mathfrak{g}$. Then the above commutative diagram is identified with

$$\begin{tikzcd} \mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g}/\!/G \\ \mathfrak{t} \arrow[r] & \mathfrak{t}/\!/W. \arrow[u, dotted] \end{tikzcd} \tag{4.2}$$

We abuse notations and also view the dotted arrow as $\mathsf{Spec}(\phi_{\mathsf{cl}})$.

Remark 19. Since the projection $\mathfrak{b} \to \mathfrak{t}$ has a splitting $\mathfrak{t} \to \mathfrak{b}$. The above claim implies $\mathfrak{t} \to \mathfrak{g} \to \mathfrak{g}/\!/G$ factors through $\mathfrak{t}/\!/W$. We can also prove this using group-theoretic method. Namely, consider the normalizer $N_G(T)$ of T insider G. Recall we have $W \simeq N_G(T)/T$ such that the linear W-action on $\mathfrak{t}$ can be identified with the adjoint action of $N_G(T)/T$. Then the morphism $\mathfrak{t} \to \mathfrak{g}/\!/G$ factors through $\mathfrak{t}/\!/W$ because $N_G(T)$ is a subgroup of G.

Warning 20. *I do not know any group-theoretic proof of* (4.2)*. This is because not $\mathfrak{b} \to \mathfrak{g}/G$ (the quotient stack) does not factor through $\mathfrak{t}$: two elements in $\mathfrak{b}$ that have the same image in $\mathfrak{t}$ are not necessarily conjugate to each other. In fact, the 0-fiber of the map $\mathfrak{g} \to \mathfrak{g}/\!/G$ contains exactly the nilpotent elements in $\mathfrak{g}$.*

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[6]Because $\mathsf{Rep}(G) \to \mathfrak{g}\text{-}\mathsf{mod}$ is fully faithful when G is connected.

[7]Both claims follow from the fact that Kil is invariant with respect to the adjoint $\mathfrak{g}$-action and thereby to the adjoint G-action. Here we use $W \simeq N_G(T)/T$.

Theorem 21 (Chevalley)**.** *The homomorphism*

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^W$$

is an isomorphism. In other words, the natural morphism $\mathfrak{t}/\!/W \to \mathfrak{g}/\!/G$ *is an isomorphism.*

Proof. As in Remark 18, we can identify ϕ_{cl} with the restriction map $\mathsf{Fun}(\mathfrak{g})^G \to \mathsf{Fun}(\mathfrak{t})^W$, which is also $\mathsf{Sym}(\mathfrak{g}^*)^G \to \mathsf{Sym}(\mathfrak{t}^*)^W$.

This map is injective because if an adjoint-invariant function on $\mathfrak{g}$ vanishes on $\mathfrak{t}$, then it vanishes on each semisimple elements. But the latter are Zariski dense in $\mathfrak{g}$.

To prove the surjectivity, we first find generators of $\mathsf{Sym}(\mathfrak{t}^*)^W$ as follows. Recall the subset P^+ of **dominant integral weights**, i.e.,

$$P^+ := \{\lambda \in \mathfrak{t}^* \,|\, \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0} \text{ for all } \alpha \in \Delta\}.$$

Note that P^+ spans $\mathfrak{t}^*$. In fact, for each $\alpha \in \Delta$, we can find **fundamental dominant weights** ω_α such that

$$\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha\beta},\ \alpha, \beta \in \Delta.$$

Then $\{\omega_\alpha\}$ is a basis of $\mathfrak{t}^*$ and $P^+ = \mathbb{Z}^{\geq 0}\{\omega_\alpha\}$. A direct calculation shows that $\{\lambda^n \,|\, \lambda \in P^+\}$ span $\mathsf{Sym}^n(\mathfrak{t}^*)$. Hence the sums

$$b_{\lambda,n} := \sum_{w \in W} w(\lambda^n),\ \lambda \in P^+, n \geq 0$$

span $\mathsf{Sym}(\mathfrak{t}^*)^W$.

It remains to show each $b_{\lambda,n}$ is contained in the image of ϕ_{cl}. We need the following well-known fact.

Theorem 22 (Weyl)**.** *For any* $\lambda \in P^+$*, there is a unique finite-dimensional irreducible* $\mathfrak{g}$*-module* L_λ *with higheset weight* λ*.*

For $\lambda \in P^+, n \geq 0$, consider the function $a_{\lambda,n} \in \mathsf{Fun}(\mathfrak{g})$ defined by

$$a_{\lambda,n}(x) := \mathsf{tr}(x^n; L_\lambda),$$

i.e., its value at any $x \in \mathfrak{g}$ is the trace of the action of x^n on L_λ. It is easy to see $a_{\lambda,n}$ is $\mathfrak{g}$-invariant[8] and thereby G-invariant.

Now the following exercise implies each $b_{\lambda,n}$ is contained in the image of ϕ_{cl}. Indeed, this follows from induction on λ with respect to the partial ordering $\prec$[9].

Exercise 23. This is Homework 2, Problem 5. Let $\lambda \in P^+$ be a dominant integral weight and $n \geq 0$. Prove there exists scalars $c_{\lambda'} \in k$, $\lambda' \prec \lambda$ such that

$$\phi_{\mathsf{cl}}(a_{\lambda,n}) = a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\mathsf{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n},$$

where $\mathsf{Stab}_W(\lambda) \subset W$ is the stablizer of the W-action at λ.

□

Combining all the previous discussion, we finish the proof of Theorem 1.

[8] Recall L_λ is G_{sc}-integrable ([Thm. 47, Lect. 3]). Hence $a_{\lambda,n}$ is G_{sc}-invariant, and thereby $\mathfrak{g}$-invariant.

[9] Recall $\lambda' \preceq \lambda$ iff $\lambda - \lambda' \in \mathbb{Z}^{\geq 0}\Phi^+$. See [Defn. 22, Lect. 2].

5. More on the quotient $\mathfrak{t}/\!/W$

We state some results on $\mathfrak{t}/\!/W \simeq \mathfrak{t}/\!/W_\bullet$ without proof.

Proposition 24 ([H2, Sect. 1.10])**.** *The morphism $\varpi : \mathfrak{t} \to \mathfrak{t}/\!/W_\bullet$ is surjective, and $W_\bullet$ acts transitively on the fiber at each closed point.*

Remark 25. In other words, any character $\chi : Z(\mathfrak{g}) \to k$ is the central character χ_λ for some Verma module V_λ, and $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.

Corollary 26 (Linkage principle)**.** *Verma modules M_λ and M_μ belong to the same block iff $\mu = w \cdot \lambda$ for some $w \in W$. In particular, there are only finitely many Verma modules and irreducible objects*[10] *in each block.*

Proposition 27 ([G, Sect. 7])**.** *We have*
(1) *The schemes $\mathfrak{t}/\!/W \simeq \mathfrak{g}/\!/G$ are smooth.*
(2) *The map $\mathfrak{t} \to \mathfrak{t}/\!/W$ is flat.*
(3) *The map $\mathfrak{g} \to \mathfrak{g}/\!/G$ is flat.*

In fact, we have:

Theorem 28 (Chevalley–Shephard–Todd)**.** *The scheme $\mathfrak{t}/\!/W$ is an affine space whose dimension is equal to that of $\mathfrak{t}$.*

This is true for any finite group acting on an affine space as long as the action is given by reflections. You should convince yourself this is true in the case $\mathfrak{g} = \mathfrak{sl}_n$.

References

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/catO.pdf`.

[H1] Humphreys, James E. Introduction to Lie algebras and representation theory. Vol. 9. Springer Science & Business Media, 2012.

[H2] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[10] Recall M_λ has a unique irreducible quotient L_λ, and all irreducible objects of $\mathcal{O}$ are of this form. See [Thm. 25, Prop. 31, Lec. 2].