LECTURE 6

Last time we proved the Harish-Chandra isomorphism

$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \mathsf{Sym}(\mathfrak{t})^{W_\bullet}.$$

We obtain isomorphisms between k-schemes:

$$\mathsf{Spec}(Z(\mathfrak{g})) \simeq \mathfrak{t}^*/\!/W_\bullet \simeq \mathfrak{t}^*/\!/W \simeq \mathfrak{t}/\!/W \simeq \mathfrak{g}/\!/G,$$

where

- The 1st isomorphism is due to the Harish-Chandra isomorphism;
- The 2nd isomorphism is given by the tranlation $\mathfrak{t}^* \to \mathfrak{t}^*$, $\lambda \mapsto \lambda + \rho$;
- The 3rd isomorphism is given by the W-equivariant isomorphism $\mathfrak{t}^* \simeq \mathfrak{t}$ provided by the Killing form;
- The 4th isomorphism is due to Chevalley's restriction theorem.

This time, we first give more algebro-geometric results about the above schemes. Then we study the structure of $\mathcal{O}$.

1. More on $\mathsf{Spec}(Z(\mathfrak{g}))$

Proposition 1 ([H, Sect. 1.10]). *The morphism $\varpi : \mathfrak{t} \to \mathfrak{t}/\!/W_\bullet \simeq \mathsf{Spec}(Z(\mathfrak{g}))$ is surjective, and $W_\bullet$ acts transitively on the fiber at each closed point.*

Remark 2. In other words, any character $\chi : Z(\mathfrak{g}) \to k$ is the central character χ_λ for some Verma module V_λ, and $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.

Remark 3. Similar claim is true for any finite group action on an affine scheme. See [SGA1, Exp. V, Prop. 1.1].

Theorem 4 (Chevalley–Shephard–Todd). *Consider the homomorphism $\mathsf{Sym}(\mathfrak{t})^W \to \mathsf{Sym}(\mathfrak{t})$. We have:*

1. *There exists homogeneous elements $c_1, \cdots, c_r \in \mathsf{Sym}(\mathfrak{t})^W$, $r = \dim(\mathfrak{t})$ such that $\mathsf{Sym}(\mathfrak{t})^W$ is isomorphic to $k[c_1, \cdots, c_r]$ as graded algebras. In other words, $\mathsf{Sym}(\mathfrak{t})^W$ is a* graded polynomial algebra *of transcendence degree* $\dim(\mathfrak{t})$.
2. *There exists homogeneous elements $a_w \in \mathsf{Sym}(\mathfrak{t})$, $w \in W$ such that $\mathsf{Sym}(\mathfrak{t})$ is isomorphic to the free $\mathsf{Sym}(\mathfrak{t})^W$-module generated by them. In other words, $\mathsf{Sym}(\mathfrak{t})$ is a* graded free $\mathsf{Sym}(\mathfrak{t})^W$*-module of rank $\#W$.*

Corollary 5. *The scheme $\mathfrak{t}/\!/W$ is an affine space whose dimension is equal to* $\dim(\mathfrak{t})$. *Also, the map $\mathfrak{t} \to \mathfrak{t}/\!/W$ is finite and flat.*

Remark 6. Similar claim is true for any linear action of a finite group H on a k-vector space V as long as:

- The order $\#H$ of the group is relatively prime to $\mathsf{char}(k)$ (which is true by our assumption);

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- The group H admits a set of generators consisting of elements w that act as pseudoreflections[1] on V.

For a proof, see [B, Sect. 5].

Remark 7. There is no canonical choice for the generators $c_1, \cdots, c_r$ of the polynomial algebra $\mathsf{Sym}(\mathfrak{t})^W$. However, the degrees $d_1, \cdots, d_r \in \mathbb{Z}^{\geq 0}$ of them are determined up to order. Also, we have $\#W = \prod_i d_i$.

Similarly, there is no canonical choice for the generators of $\mathsf{Sym}(\mathfrak{t})$ as a free $\mathsf{Sym}(\mathfrak{t})^W$-module. However, the degrees of these generators are determined up to order.

Example 8. For $\mathfrak{g} = \mathfrak{sl}_n$, recall $W \simeq \Sigma_n$ acts on $\mathfrak{t} \simeq \ker(k^{\oplus n} \xrightarrow{\Sigma} k)$ in the standard way. It follows that $\mathsf{Sym}(\mathfrak{t}^*) \simeq \mathsf{Fun}(\mathfrak{t}) \simeq k[x_1, \cdots, x_n]/(\sigma_1)$ and $\mathsf{Sym}(\mathfrak{t}^*)^W \simeq k[\sigma_1, \cdots, \sigma_n]/(\sigma_1)$. Here σ_i is the basic symmetric polynomial of degree i, i.e., $\prod_{j=1}^n (x + x_j) = x^n + \sigma_1 x^{m-1} + \cdots \sigma_n$.

Our next goal is to prove the following:

Theorem 9 (Kostant)**.** *As a $Z(\mathfrak{g})$-module, $U(\mathfrak{g})$ is free.*

Remark 10. This classical result was first proved by Kostant in the 60's ([K]) and used to be considered a hard theorem. The simplified proof below was due to Bernstein–Lunts ([BL]), after more than 30 years[2].

Construction 11. *We equip $U(\mathfrak{g})$ with a $U(\mathfrak{n}^-) \otimes Z(\mathfrak{g}) \otimes U(\mathfrak{n})^{\mathsf{op}}$-module structure by the formula*

$$(u^- \otimes z \otimes u)(x) := u^- xzu.$$

Here $u^- \otimes z \otimes u \in U(\mathfrak{n}^-) \otimes Z(\mathfrak{g}) \otimes U(\mathfrak{n})^{\mathsf{op}}$ and $x \in U(\mathfrak{g})$.

We will prove the following stronger result.

Theorem 12. *Let $\{a_w\}$ be a homogeneous free basis of $\mathsf{Sym}(\mathfrak{t})$ over $\mathsf{Sym}(\mathfrak{t})^W$. We view a_w as elements in $U(\mathfrak{g})$ via the embedding $\mathsf{Sym}(\mathfrak{t}) \hookrightarrow U(\mathfrak{g})$. Then $\{a_w\}$ is also a free basis of $U(\mathfrak{g})$ over $U(\mathfrak{n}^-) \otimes Z(\mathfrak{g}) \otimes U(\mathfrak{n})^{\mathsf{op}}$.*

Construction 13. *Consider the PBW filtrations on $U(\mathfrak{g})$ and $U(\mathfrak{n}^-) \otimes Z(\mathfrak{g}) \otimes U(\mathfrak{n})^{\mathsf{op}}$. Taking associated graded spaces, we obtain a graded $\mathsf{Sym}(\mathfrak{n}^-) \otimes \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \otimes \mathsf{Sym}(\mathfrak{n})$-module structure on $\mathsf{Sym}(\mathfrak{g})$. By definition, this module structure comes from the obvious graded homomorphism*

$$\mathsf{Sym}(\mathfrak{n}^-) \otimes \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \otimes \mathsf{Sym}(\mathfrak{n}) \xrightarrow{\mathsf{mult}} \mathsf{Sym}(\mathfrak{g}). \tag{1.1}$$

This reduces Theorem 12 to the following result:

Proposition 14. *Let $\{a_w\}$ be a homogeneous free basis of $\mathsf{Sym}(\mathfrak{t})$ over $\mathsf{Sym}(\mathfrak{t})^W$. We view a_w as homogeneous elements in $\mathsf{Sym}(\mathfrak{g})$ via the embedding $\mathsf{Sym}(\mathfrak{t}) \hookrightarrow \mathsf{Sym}(\mathfrak{g})$. Then $\{a_w\}$ is also a free basis of $\mathsf{Sym}(\mathfrak{g})$ over $\mathsf{Sym}(\mathfrak{n}^-) \otimes \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \otimes \mathsf{Sym}(\mathfrak{n})$.*

We also record the following corollary of the proposition.

Corollary 15. *The map $\mathfrak{g} \to \mathfrak{n} \times \mathfrak{g}/\!/G \times \mathfrak{n}^-$ is flat. In particular, the map $\mathfrak{g} \to \mathfrak{g}/\!/G$ is flat.*

Example 16. For $\mathfrak{g} = \mathfrak{sl}_2$ and the standard basis e, f, h. Recall $\mathsf{Sym}(\mathfrak{t})^W = k[h^2]$ and $\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} = k[\Omega]$, where $\Omega = h^2 + 4ef$ is the image of the Casimir element. Then the proposition says $k[e, f, h]$ is free over $k[e, f, h^2 + 4ef]$ and any homogeneous basis of $k[h]$ over $k[h^2]$ is also a free basis of $k[e, f, h]$ over $k[e, f, h^2 + 4ef]$.

[1] This means $\mathsf{Id} - w$ is of rank 1.

[2] Gaitsgory also sketched a proof in [G, Cor. 7.3], but I think there is a gap in the second paragaph. It is not clear what is the logic behind *"... it is enough to show that $\mathsf{Sym}(\mathfrak{g}/\mathfrak{n})$ is free as a $\mathsf{Sym}(\mathfrak{h})^W$-module."* In fact, this reduction is the main point of [BL].

Warning 17. *Proposition 14 does* not *follow obviously from Theorem 4. Namely, we have two homomorphisms*

$$\begin{aligned} \mathsf{Sym}(\mathfrak{n}^-)\otimes\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}\otimes\mathsf{Sym}(\mathfrak{n}) &\to \mathsf{Sym}(\mathfrak{g}) \\ \mathsf{Sym}(\mathfrak{n}^-)\otimes\mathsf{Sym}(\mathfrak{t})^{W}\otimes\mathsf{Sym}(\mathfrak{n}) &\to \mathsf{Sym}(\mathfrak{g}). \end{aligned}$$

Theorem 4 implies $\mathsf{Sym}(\mathfrak{g})$ *is a free module over the source of the second map, but the images of these two homomorphisms are in general not the same.*

To see this, it is better to pass to dualities. Via the isomorphisms $\mathfrak{g}\simeq\mathfrak{g}^*$, $\mathfrak{n}\simeq(\mathfrak{n}^-)^*$ *and* $\mathfrak{t}\simeq\mathfrak{t}^*$ *provided by the Killing form, the above two homomorphisms are given by*

$$\mathsf{Fun}(\mathfrak{n})\otimes\mathsf{Fun}(\mathfrak{g})^{\mathfrak{g}}\otimes\mathsf{Fun}(\mathfrak{n}^-) \to \mathsf{Fun}(\mathfrak{g}) \tag{1.2}$$

$$\mathsf{Fun}(\mathfrak{n})\otimes\mathsf{Fun}(\mathfrak{t})^{W}\otimes\mathsf{Fun}(\mathfrak{n}^-) \to \mathsf{Fun}(\mathfrak{g}), \tag{1.3}$$

which are induced by the decomposition $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{t}\oplus\mathfrak{n}$. *Now the following exercise says the two images are not the same even for* $\mathfrak{g}=\mathfrak{sl}_3$, *although they happen to be the same for* $\mathfrak{sl}_2$.

Exercise 18. This is Homework 3, Problem 1. Let $\mathfrak{g}=\mathfrak{sl}_n$ and $\sigma_i\in\mathsf{Fun}(\mathfrak{t})^W$ be as in Example 8.

(1) For each $1<i\le n$, find the unique element $\widetilde{\sigma}_i\in\mathsf{Fun}(\mathfrak{g})^{\mathfrak{g}}$ corresponding to σ_i via the Chevalley isomorphim $\mathsf{Fun}(\mathfrak{g})^{\mathfrak{g}}\xrightarrow{\simeq}\mathsf{Fun}(\mathfrak{t})^W$. In other words, find the unique extension of σ_i to an adjoint invariant polynomial function on $\mathfrak{g}$.
(2) For $\mathfrak{g}=\mathfrak{sl}_2$, prove that (1.2) and (1.3) are both injective and have the same image.
(3) For $\mathfrak{g}=\mathfrak{sl}_3$, prove that $\widetilde{\sigma}_3$ is contained in the image of (1.2) but not in the image of (1.3).

Let us return to Bernstein–Lunts's proof. We need the following general result, which is an easy exercise in linear algebra.

Lemma 19. *Let* $A=\bigcup_{n\ge0}\Phi^{\le n}A$ *be a filtered algebra and* $M=\bigcup_{n\ge0}\Phi^{\le n}M$ *be a filtered* A*-module. Let* $\{b_i\}$ *be a family of elements of* M *such that their symbols*[3] $\{\sigma_\Phi(b_i)\}$ *form a free basis of the* $\mathsf{gr}^\bullet_\Phi A$*-module* $\mathsf{gr}^\bullet_\Phi M$. *Then* $\{b_i\}$ *is a free basis of the* A*-module* M.

Proof of Proposition 14. The strategy is as follows. We will construct compatible new filtrations on the source and the target of the homomorphism (1.1):

$$\mathsf{Sym}(\mathfrak{n}^-\oplus\mathfrak{n})\otimes\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}\to\mathsf{Sym}(\mathfrak{g}).$$

Hence we can view $\mathsf{Sym}(\mathfrak{g})$ as a filtered module over $\mathsf{Sym}(\mathfrak{n}^-\oplus\mathfrak{n})\otimes\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}$ with respect to these new filtrations. Then we apply Lemma 19.

The desired filtration on $\mathsf{Sym}(\mathfrak{g})$ can be summarized in a sentence: we ignore the indeterminates contained in $\mathfrak{n}^-\oplus\mathfrak{n}$ and only count the degree for those contained in $\mathfrak{t}$. In other words, we define

$$\Phi^{\le i}\mathsf{Sym}(\mathfrak{g}) := \mathsf{Sym}(\mathfrak{n}^-\oplus\mathfrak{n})\cdot\mathsf{Sym}^{\le i}(\mathfrak{t}),$$

where $\mathsf{Sym}^{\le\bullet}(\mathfrak{t})$ is the standard filtration on $\mathsf{Sym}(\mathfrak{t})$ given by degrees. For $f\in\mathsf{Sym}(\mathfrak{g})$, let $\deg_{\mathfrak{t}}(f)$ be the minimal index i such that $f\in\Phi^{\le i}\mathsf{Sym}(\mathfrak{g})$. In other words, it is the $\mathfrak{t}$-degree of the polynomial f.

Note that this filtraiton is compatible with the multiplication. Also note that

$$\mathsf{gr}^\bullet_\Phi\mathsf{Sym}(\mathfrak{g})\simeq\mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)\otimes\mathsf{Sym}^\bullet(\mathfrak{t}) \tag{1.4}$$

[3] For $b\in M$, let $n\ge0$ be the smallest index such that $b\in\Phi^{\le n}M$. Then the **symbol** $\sigma_\Phi(b)$ is the image of b under the projection $\Phi^{\le n}M\to\mathsf{gr}^n_\Phi M$. Note that σ_Φ is *not* a linear map!

as graded commutative algebras, where the grading on the RHS is the standard grading on $\mathsf{Sym}^\bullet(\mathfrak{t})$.

Consider the injective homomorphisms

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{g}) \leftarrow \mathsf{Sym}(\mathfrak{t})^W.$$

We equip the sources with the induced Φ-filtrations. Note that the Φ-filtration on $\mathsf{Sym}(\mathfrak{t})^W$ is the standard one. By definition, the following diagram commutes[4]

$$\begin{tikzcd} \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "\subset"] \arrow[d, "\sigma_\Phi"] & \mathsf{Sym}(\mathfrak{g}) \arrow[d, "\sigma_\Phi"] & \mathsf{Sym}(\mathfrak{t})^W \arrow[l, "\supset"'] \arrow[d, "\sigma_\Phi"] \\ \mathsf{gr}^\bullet_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}) \arrow[r, "\subset"] & \mathsf{gr}^\bullet_\Phi\mathsf{Sym}(\mathfrak{g}) & \mathsf{Sym}^\bullet(\mathfrak{t})^W. \arrow[l, "\supset"'] \end{tikzcd}$$

We have the following key observation:

Lemma 20. *The injective homomorphisms*

$$\mathsf{gr}^\bullet_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}) \to \mathsf{gr}^\bullet_\Phi\mathsf{Sym}(\mathfrak{g}) \leftarrow \mathsf{Sym}^\bullet(\mathfrak{t})^W$$

have the same image.

Let us first finish the proof of the proposition assuming this lemma. We can tensor the sources with the factor $\mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)$ and consider the homomorphisms

$$\begin{aligned} \mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)\otimes\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} &\xrightarrow{m_1} \mathsf{Sym}(\mathfrak{g}), \\ \mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)\otimes\mathsf{Sym}(\mathfrak{t})^W &\xrightarrow{m_2} \mathsf{Sym}(\mathfrak{g}). \end{aligned}$$

Via the isomorphism (1.4), the graded homomorphism $\mathsf{gr}^\bullet_\Phi(m_2)$ is given by the obvious embedding

$$\mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)\otimes\mathsf{Sym}^\bullet(\mathfrak{t})^W \to \mathsf{Sym}(\mathfrak{n}\oplus\mathfrak{n}^-)\otimes\mathsf{Sym}^\bullet(\mathfrak{t}).$$

Hence the above lemma implies $\mathsf{gr}^\bullet_\Phi(m_1)$ is also an embedding with the same image. By Theorem 4, the graded homomorphism $\mathsf{gr}^\bullet_\Phi(m_2)$ exhibits its target as a graded free module over its source, and the images of $\{a_w\}$ form a basis. Hence the same is true for the graded homomorphism $\mathsf{gr}^\bullet_\Phi(m_1)$. Then we win by Lemma 19.

□[Proposition 14]

It remains to prove Lemma 20. We first prove the following elementary result.

Lemma 21. *For any $i \geq 0$, we have:*

(1) *For any nonzero $f \in \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}}$, we have $\deg_{\mathfrak{t}}(f) = i$.*

(2) *By (2), taking Φ-symbols gives a map*

$$\sigma^i_\Phi : \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{gr}^i_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}).$$

We claim the following diagram commutes:

(1.5)
$$\begin{tikzcd} \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}} \arrow[rr, "\phi_{\mathsf{cl}}", "\simeq"'] \arrow[d, "\sigma^i_\Phi"] & & \mathsf{Sym}^i(\mathfrak{t})^W \arrow[d, equal] \\ \mathsf{gr}^i_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}) \arrow[r, "\subset"] & \mathsf{gr}^i_\Phi\mathsf{Sym}(\mathfrak{g}) & \mathsf{Sym}^i(\mathfrak{t})^W. \arrow[l, "\supset"'] \end{tikzcd}$$

(3) *The map $\sigma^i_\Phi : \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{gr}^i_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}})$ is bijective.*

[4]Warning: the right vercial map is not the identity map. It abandons all non-highest degree terms in a polynomial.

Proof. Let $f \in \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}}$ be any nonzero element. By definition, we have

$$f \in \bigoplus_{0 \le j \le i} \mathsf{Sym}^{i-j}(\mathfrak{n}^- \oplus \mathfrak{n}) \cdot \mathsf{Sym}^j(\mathfrak{t}).$$

Let $f_j \in \mathsf{Sym}^{i-j}(\mathfrak{n}^- \oplus \mathfrak{n}) \cdot \mathsf{Sym}^j(\mathfrak{t})$ be the j-th entry of f with respect to the above direct sum decomposition, i.e., the part of f whose $\mathfrak{t}$-degree is j.

By the construction of ϕ_{cl}[5], we have

$$\phi_{\mathsf{cl}}(f) = f_i \in \mathsf{Sym}^i(\mathfrak{t})$$

Since ϕ_{cl} is an isomorphism, we obtain $f_i \ne 0$. In particular, $\deg_{\mathfrak{t}}(f) = i$. This proves (1).

By definition, we also have $\sigma^i_\Phi(f) = f_i$ because this is the sum of the highest $\mathfrak{t}$-degree terms. This proves (2).

The commutative diagram in (2) implies σ^i_Φ is injective. It remains to show it is surjective. Let $\overline{h} \in \mathsf{gr}^i_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}})$ be any nonzero element and $h \in \Phi^{\le i}(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}})$ be a lifting of it, i.e., $\overline{h} = \sigma_\Phi(h)$. Note that $\deg_{\mathfrak{t}}(h) = i$. Write $h = h_0 + h_1 + \cdots + h_d$ such that $h_j \in \mathsf{Sym}^j(\mathfrak{g})^{\mathfrak{g}}$ and $h_d \ne 0$. By (1) and (2), either $h_j = 0$ or $\deg_{\mathfrak{t}}(h_j) = j$. It follows that we must have $d = i$ and $\deg_t(h - h_i) < i$. This implies $\overline{h} = \sigma_\Phi(h) = \sigma^i_\Phi(h_i)$ with $h_i \in \mathsf{Sym}^i(\mathfrak{g})^{\mathfrak{g}}$. In other words, the given map is surjective as desired.

□

Warning 22. *The digram* (1.5) *would* not *commute if we dropped the superscripts i from* $\mathsf{Sym}^i(-)$. *In other words, the following diagram does not commute*

$$\begin{tikzcd}
\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[rr, "\phi_{\mathsf{cl}}", "\simeq"'] \arrow[d, "\sigma_\Phi"] & & \mathsf{Sym}(\mathfrak{t})^W \arrow[d, "\sigma_\Phi"] \\
\mathsf{gr}^\bullet_\Phi(\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}) \arrow[r, "\subset"] & \mathsf{gr}^\bullet_\Phi \mathsf{Sym}(\mathfrak{g}) & \mathsf{Sym}^\bullet(\mathfrak{t})^W . \arrow[l, "\supset"']
\end{tikzcd}$$

Proof of Lemma 20. This follows from Lemma 21 by a diagram chasing.

□[Lemma 20]

2. Category $\mathcal{O}$ is Artinian and Noetherian

The next few lectures are devoted to the algebraic study of $\mathcal{O}$. In this section, we prove any object in $\mathcal{O}$ has finite length. This was promised in the second lecture.

We first recall the following corollary of Proposition 1.

Proposition 23 (Linkage principle)**.** *Verma modules M_λ and M_μ belong to the same block iff $\mu = w \cdot \lambda$ for some $w \in W$.*

We also recall the following corollary of [Thm. 25, Prop. 31, Lect. 2]:

Proposition 24. *Each Verma module M_λ has a unique irreducible quotient L_λ, whose highest weight is equal to λ. Any irreducible object in $\mathcal{O}$ is of such form.*

Corollary 25. *If the irreducible module L_μ is isomorphic to a subquotient of the Verma module M_λ, then $\mu = w \cdot \lambda$ for some $w \in W$. In particular, M_λ has only finitely many non-isomorphic irreducible subquotients.*

Theorem 26. *Each object $M \in \mathcal{O}$ is both Artinian and Noetherian.*

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[5] Recall it kills any factor in $\mathsf{Sym}^{\ge 1}(\mathfrak{n}^- \oplus \mathfrak{n})$.

Remark 27. By Jordan–Hölder, M is both Artinian and Noetherian iff there exists a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each M_i/M_{i-1} is a (nonzero) irreducible object. Moreover, for each irreducible object $L_\lambda \in \mathcal{O}$, its multiplicity in the collection of such quotients does not depend on the choice of the filtration, and is denoted by $[M : L_\lambda]$. This implies n does not depend on the choice of the filtration, and is called the **length** of M, i.e., $\mathsf{length}(M)$. The numbers $[M : L_\lambda]$ and $\mathsf{length}(M)$ are basic objects in the study of $\mathcal{O}$ (and any representation theory).

Proof. Since M can be written as a quotient of a successive extension of Verma modules ([Prop. 31, Lect. 2]), we only need to prove the theorem for Verma modules $M = M_\lambda$. By Corollary 25, we only need to prove for *any* finite filtration of M_λ, the multiplicity of each L_μ ($\mu = w \cdot \lambda$, $w \in W$) that appears in the graded pieces is uniformly bounded. Recall L_μ is a weight module with highest weight μ. Since any module in $\mathcal{O}$ is a weight module (with respect to the $\mathfrak{t}$-action), the aforementioned multiplicity is bounded by the dimension of the μ-weight subspace of M_λ. But this is finite by ([Cor. 33, Lect. 2]).

□

Proposition 28. *For any $M, N \in \mathcal{O}$, the vector space $\mathsf{Hom}_{\mathcal{O}}(M, N)$ is finite dimensional.*

Proof. By dévissage, we only need to show $\mathsf{Hom}_{\mathcal{O}}(L_\lambda, L_\mu)$ is finite dimensional. This is a subspace of $\mathsf{Hom}_{\mathcal{O}}(M_\lambda, L_\mu)$. And the latter is a subspace of the λ-weight subspace of L_μ, which is finite-dimensional.

□

Note that the above argument actually shows

Lemma 29. *For any $\lambda \in \mathfrak{t}^*$, $\mathsf{Hom}_{\mathcal{O}}(L_\lambda, L_\lambda) = k \cdot \mathsf{Id}$ is 1-dimensional.*

Exercise 30. This is Homework 3, Problem 2. Consider $\mathfrak{g} = \mathfrak{sl}_3$ and its standard Borel $\mathfrak{b}$ and Cartan subalgebras $\mathfrak{t}$. Let α_1 and α_2 be the two simple positive roots.

(1) Prove $W \cdot 0 = \{0, -\alpha_1, -\alpha_2, -2\alpha_1 - \alpha_2, -\alpha_1 - 2\alpha_2, -2\alpha_1 - 2\alpha_2\}$.
(2) Prove: $M_{-2\alpha_1-2\alpha_2}$ is irreducible.
(3) Prove: $M_{-\alpha_1-2\alpha_2}$ contains $M_{-2\alpha_1-2\alpha_2}$ as a submodule[6] and the quotient is irreducible[7]. Deduce $\mathsf{length}(M_{-\alpha_1-2\alpha_2}) = 2$ and $[M_{-\alpha_1-2\alpha_2} : L_{-\alpha_1-2\alpha_2}] = [M_{-\alpha_1-2\alpha_2} : L_{-2\alpha_1-2\alpha_2}] = 1$.

Remark 31. By symmetry, $M_{-2\alpha_1-\alpha_2}$ has length 2, and contains $M_{-2\alpha_1-2\alpha_2}$ as a submodule.

Exercise 32. This is Homework 3, Problem 3. We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$.

(1) Prove: M_0 contains $M_{-\alpha_1}$ and $M_{-\alpha_2}$ as submodules and $[M_0 : L_{-\alpha_1}] = [M_0 : L_{-\alpha_2}] = 1$.
(2) Prove: $M_{-\alpha_2}$ contains $M_{-2\alpha_1-\alpha_2}$ as a submodule and $[M_{-\alpha_2} : L_{-2\alpha_1-\alpha_2}] = 1$.
(3) Prove: there exists a (unique) dotted arrow making the following diagram commutes[8]:

$$\begin{tikzcd} & M_{-\alpha_1} \arrow[r, "\subset"] & M_0 & \\ M_{-2\alpha_1-\alpha_2} \arrow[ur, dotted] \arrow[rrr, "\subset"] & & & M_{-\alpha_2}. \arrow[ul, "\subset"'] \end{tikzcd}$$

Exercise 33. This is Homework 3, Problem 4. We continue with the case $\mathfrak{g} = \mathfrak{sl}_3$. Prove: for $\lambda, \mu \in W \cdot 0$, $[M_\lambda : L_\mu] \neq 0$ iff $\lambda \geq \mu$.

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[6]Hint: [Lem. 4, Lect. 5].

[7]Hint: count the dimension of the $(-2\alpha_1 - 2\alpha_2)$-weight subspace of $M_{-\alpha_1-2\alpha_2}$.

[8]Hint: show $f_1^2 f_2 \cdot v_0 = u \cdot v_{-\alpha_1}$ for some u. Here v_0 is the highest weight of M_0, $v_{-\alpha_1} = f_1 \cdot v_0$ is the highest weight of $M_{-\alpha_1}$, and $f_i \in \mathfrak{n}^-$ is the root vector corresponding to α_i.

You may want to play with the above example for more time. E.g., can you find upper or lower bounds, or even the exact value, of $\mathsf{length}(M_0)$?

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