Lecture 1

Sep. 10, 2024

A slogan

$$\infty\text{-category theory} \;=\; \text{category theory} \;+\; \text{homotopy theory}.$$

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects:* $a \in \mathrm{Ob}(\mathcal{C})$.

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects:* $a \in \mathrm{Ob}(\mathcal{C})$.
- *Morphisms:* $f : a \to b$, $f \in \mathrm{Hom}(a, b)$.

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects:* $a \in \mathrm{Ob}(\mathcal{C})$.
- *Morphisms:* $f : a \to b$, $f \in \mathrm{Hom}(a, b)$.
- *Composition of morphisms:* $f, g \Rightarrow g \circ f$.

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects:* $a \in \mathrm{Ob}(\mathcal{C})$.
- *Morphisms:* $f : a \to b$, $f \in \mathrm{Hom}(a, b)$.
- *Composition of morphisms:* $f, g \Rightarrow g \circ f$.

Axioms:

- *Associativity:*
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects:* $a \in \mathrm{Ob}(\mathcal{C})$.
- *Morphisms:* $f : a \to b$, $f \in \mathrm{Hom}(a, b)$.
- *Composition of morphisms:* $f, g \Rightarrow g \circ f$.

Axioms:

- *Associativity:*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- *Identity:* $f : a \to b$,

$$\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a.$$

Classical category theory

Definition 1

A *category* $\mathcal{C}$ *consists of the following data:*

- *Objects*: $a \in \mathrm{Ob}(\mathcal{C})$.
- *Morphisms*: $f : a \to b$, $f \in \mathrm{Hom}(a, b)$.
- *Composition of morphisms*: $f, g \Rightarrow g \circ f$.

Axioms:

- *Associativity*:
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- *Identity*: $f : a \to b$,
$$\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a.$$

Classical category theory

- *In order to study a collection of objects one should also consider suitably defined morphisms between such objects.*

Classical category theory

- *In order to study a collection of objects one should also consider suitably defined morphisms between such objects.*
- Examples: Set, Grp, Ring, Top...

Three doctrines

(1) Morphisms are discrete: $f = g$ or $f \neq g$.

Three doctrines

(1) Morphisms are discrete: $f = g$ or $f \neq g$.

(2) Associativity is strict: $h \circ (g \circ f) = (h \circ g) \circ f$.

Three doctrines

(1) Morphisms are discrete: $f = g$ or $f \neq g$.

(2) Associativity is strict: $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Composition is concrete: there is no ambiguity for $g \circ f$.

Three doctrines

(1) Morphisms are discrete: $f = g$ or $f \neq g$.

(2) Associativity is strict: $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) Composition is concrete: there is no ambiguity for $g \circ f$.

We will abandon all of them.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.
- Richer structures on $\mathrm{Hom}(-,-)$.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.
- Richer structures on $\mathrm{Hom}(-,-)$.
- topological spaces $\Rightarrow$ topological categories.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.
- Richer structures on $\mathrm{Hom}(-,-)$.
- topological spaces $\Rightarrow$ topological categories.
- categories $\Rightarrow$ strict 2-categories.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.
- Richer structures on $\mathrm{Hom}(-,-)$.
- topological spaces $\Rightarrow$ topological categories.
- categories $\Rightarrow$ strict 2-categories.
- strict $(n-1)$-categories $\Rightarrow$ strict n-categories.

Enrichment

- ~~Doctrine (1): Morphisms are discrete~~.
- Richer structures on $\mathrm{Hom}(-,-)$.
- topological spaces $\Rightarrow$ topological categories.
- categories $\Rightarrow$ strict 2-categories.
- strict $(n-1)$-categories $\Rightarrow$ strict n-categories.
- ...

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

- Objects.

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

- Objects.
- Morphisms and their compositions.

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

- Objects.
- Morphisms and their compositions.
- 2-morphisms: morphisms between morphisms,

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

- Objects.
- Morphisms and their compositions.
- 2-morphisms: morphisms between morphisms,and their vertical and horizontal compositions.

Strict 2-categories

A strict 2-category $\mathcal{C}$ consists of the following data:

- Objects.
- Morphisms and their compositions.
- 2-morphisms: morphisms between morphisms,and their vertical and horizontal compositions.

Axioms: all given by equalities. (Doctrine (2))

Strict 2-categories with one object

- $\mathcal{D}$: strict 2-category with $\mathrm{Ob}(\mathcal{D}) \simeq \{*\}$. $\Leftrightarrow$

Strict 2-categories with one object

- $\mathcal{D}$: strict 2-category with $\mathrm{Ob}(\mathcal{D}) \simeq \{*\}$. $\Leftrightarrow$
- $\mathcal{C} := \mathrm{Hom}(*, *)$: equipped with $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ strictly associative and unital.

Strict 2-categories with one object

- $\mathcal{D}$: strict 2-category with $\mathrm{Ob}(\mathcal{D}) \simeq \{*\}$. $\Leftrightarrow$
- $\mathcal{C} := \mathrm{Hom}(*, *)$: equipped with $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ strictly associative and unital.
- This is evil

  > Principle of isomorphism: *all grammatically correct properties of objects of a fixed category are to be invariant under isomorphism*.

  and impractical.

Exercise 1
In ZFC, $(X \times Y) \times Z = X \times (Y \times Z)$ or $(X \times Y) \times Z \simeq X \times (Y \times Z)$?

Strict 2-categories with one object

- $\mathcal{D}$: strict 2-category with $\mathrm{Ob}(\mathcal{D}) \simeq \{*\}$. $\Leftrightarrow$
- $\mathcal{C} := \mathrm{Hom}(*, *)$: equipped with $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ strictly associative and unital.
- This is evil

  > Principle of isomorphism: *all grammatically correct properties of objects of a fixed category are to be invariant under isomorphism*.

  and impractical.
- ~~Strict~~ monoidal category: supply associators:

$$X \otimes (Y \otimes Z) \xrightarrow{\simeq} (X \otimes Y) \otimes Z$$

subject to certain coherent conditions.

Exercise 1
In ZFC, $(X \times Y) \times Z = X \times (Y \times Z)$ *or* $(X \times Y) \times Z \simeq X \times (Y \times Z)$*?*

Weak n-categories

- ~~Doctrine (2): associativity is strict~~

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.
  - Weak 2-categories with a single object $\Leftrightarrow$ monoidal categories.

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.
  - Weak 2-categories with a single object $\Leftrightarrow$ monoidal categories.
  - $\mathrm{Hom}(a, b)$: weak $(n-1)$-category.

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.
  - Weak 2-categories with a single object $\Leftrightarrow$ monoidal categories.
  - $\mathrm{Hom}(a, b)$: weak $(n-1)$-category.
  - *Principle of isomorphism*.

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.
  - Weak 2-categories with a single object $\Leftrightarrow$ monoidal categories.
  - $\mathrm{Hom}(a, b)$: weak $(n-1)$-category.
  - *Principle of isomorphism*.
- For $n \geq 2$, we should never require two morphisms in a weak n-category to be equal.

Weak n-categories

- ~~Doctrine (2): associativity is strict~~
- Wishes for weak n-categories
  - Weak 1-categories $\Leftrightarrow$ categories.
  - Weak 2-categories with a single object $\Leftrightarrow$ monoidal categories.
  - $\mathrm{Hom}(a, b)$: weak $(n-1)$-category.
  - *Principle of isomorphism*.
- For $n \geq 2$, we should never require two morphisms in a weak n-category to be equal.
- For $n > k$, we should never require two k-morphisms in a weak n-category to be equal.

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f.$

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f$.
- Commutative diagram for 2-morphisms:

$$
\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
 & k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}
$$

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f$.
- Commutative diagram for 2-morphisms:

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
 & k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}$$

- $n \geq 3$: witnessed by an invertible 3-morphism.

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f$.
- Commutative diagram for 2-morphisms:

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
 & k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}$$

- $n \geq 3$: witnessed by an invertible 3-morphism.
- Commutative diagram for 3-morphisms.

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f$.
- Commutative diagram for 2-morphisms:

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
 & k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}$$

- $n \geq 3$: witnessed by an invertible 3-morphism.
- Commutative diagram for 3-morphisms.
- $n \geq 4$: witnessed by an invertible 4-morphism.

Nightmare of coherence data

- Associator: $h \circ (g \circ f) \xrightarrow{\simeq} (h \circ g) \circ f$.
- Commutative diagram for 2-morphisms:

$$
\begin{tikzcd}
& & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
& k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}
$$

- $n \geq 3$: witnessed by an invertible 3-morphism.
- Commutative diagram for 3-morphisms.
- $n \geq 4$: witnessed by an invertible 4-morphism.
- ...

A savior

Slogan 1 (Grothendieck's homotopy hypothesis)

$$\infty\text{-}groupoids \; = \; homotopy\ types.$$

A savior

Slogan 1 (Grothendieck's homotopy hypothesis)

$$\infty\textit{-groupoids} \;=\; \textit{homotopy types.}$$

- ∞-groupoids: morphisms and higher morphisms are all invertible.

A savior

Slogan 1 (Grothendieck's homotopy hypothesis)

$$\infty\text{-}groupoids \;=\; homotopy\ types.$$

- Coherence data for associativity can be hidden in the homotopy theory of spaces.
- ∞-groupoids: morphisms and higher morphisms are all invertible.

A strategy

- (∞, k)-categories: m-morphisms invertible for $m > k$.

A strategy

- (∞, k)-categories: m-morphisms invertible for $m > k$.
- Induction on k.

A strategy

- (∞, k)-categories: m-morphisms invertible for $m > k$.
- Induction on k.
- $(\infty, 0)$-categories$:=$ ∞-groupoids $:=$ homotopy types.

A strategy

- (∞, k)-categories: m-morphisms invertible for $m > k$.
- Induction on k.
- $(\infty, 0)$-categories$:=$ ∞-groupoids $:=$ homotopy types.
- Induction step: coherence data for associativity is already incorporated into the theory of ∞-groupoids!

Models

- Theory of $(\infty, 0)$-categories $\Leftrightarrow$ homotopy theory of topological spaces.

Models

- Theory of $(\infty, 0)$-categories $\Leftrightarrow$ homotopy theory of topological spaces.
- Theory of $(\infty, 1)$-categories $\Leftrightarrow$ homotopy theory of topological categories ?

Models

- Theory of $(\infty, 0)$-categories $\Leftrightarrow$ homotopy theory of topological spaces.
- Theory of $(\infty, 1)$-categories $\Leftrightarrow$ homotopy theory of topological categories ?
- Yes, but...

Models

- Theory of $(\infty, 0)$-categories $\Leftrightarrow$ homotopy theory of topological spaces.
- Theory of $(\infty, 1)$-categories $\Leftrightarrow$ homotopy theory of topological categories ?
- Yes, but...
- A more developed model: quasi-categories.

Models

- Theory of $(\infty, 0)$-categories $\Leftrightarrow$ homotopy theory of topological spaces.
- Theory of $(\infty, 1)$-categories $\Leftrightarrow$ homotopy theory of topological categories ?
- Yes, but...
- A more developed model: quasi-categories. ~~Doctrine (3): composition is concrete~~.

Homotopy category

- The homotopy category hTop:

Homotopy category

- The homotopy category hTop:
  - Objects=homotopy types: topological spaces.

Homotopy category

- The homotopy category hTop:
  - Objects=homotopy types: topological spaces.
  - Morphisms: homotopy classes of continuous maps.

Homotopy category

- The homotopy category hTop:
  - Objects=homotopy types: topological spaces.
  - Morphisms: homotopy classes of continuous maps.
- Homotopy equivalence: isomorphisms in hTop.

Homotopy category

- The homotopy category hTop:
  - Objects=homotopy types: topological spaces.
  - Morphisms: homotopy classes of continuous maps.
- Homotopy equivalence: isomorphisms in hTop.
- hTop: inverting homotopy equivalences in Top.

Homotopy category

- The homotopy category hTop:
  - Objects=homotopy types: topological spaces.
  - Morphisms: homotopy classes of continuous maps.
- Homotopy equivalence: isomorphisms in hTop.
- hTop: inverting homotopy equivalences in Top.
- $\mathrm{hTop}_{\leq n}$: full subcategory of homotopy n-types: $\pi_k \simeq 0$, $k > n$.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1} X$:

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1}X$:
  - Objects: points.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1} X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1} X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

Exercise 2

Check the axiom of associativity for the above construction.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1} X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

Exercise 2

Check the axiom of associativity for the above construction.

- $\pi_{\leq 1} X$ is indeed a groupoid.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1}X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

Exercise 2

Check the axiom of associativity for the above construction.

- $\pi_{\leq 1}X$ is indeed a groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 1}X)/\simeq$.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1}X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

Exercise 2

Check the axiom of associativity for the above construction.

- $\pi_{\leq 1}X$ is indeed a groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 1}X)/\simeq$.
- $\pi_1(X, x) \simeq \mathrm{Hom}(x, x)$.

fundamental groupoid

- The fundamental groupoid $\pi_{\leq 1}X$:
  - Objects: points.
  - Morphisms: homotopy classes of pathes.

Exercise 2

Check the axiom of associativity for the above construction.

- $\pi_{\leq 1}X$ is indeed a groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 1}X)/\simeq$.
- $\pi_1(X, x) \simeq \mathrm{Hom}(x, x)$.

Exercise 3

Show $\pi_{\leq 1} : \mathrm{hTop}_{\leq 1} \to \mathrm{Grpd}$ *is an equivalence.*

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2} X$:
  - Objects: points.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 2}X)/\simeq$.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 2}X)/\simeq$.
- $\pi_1(X, x) \simeq \mathrm{Hom}(x, x)/\simeq$.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 2}X)/\simeq$.
- $\pi_1(X,x) \simeq \mathrm{Hom}(x,x)/\simeq$.
- $\pi_2(X,x) \simeq \mathrm{Hom}(\mathrm{id}_x, \mathrm{id}_x)$.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 2}X)/\simeq$.
- $\pi_1(X, x) \simeq \mathrm{Hom}(x, x)/\simeq$.
- $\pi_2(X, x) \simeq \mathrm{Hom}(\mathrm{id}_x, \mathrm{id}_x)$.

Exercise 4

Describe the action of $\pi_1(X, x)$ on $\pi_2(X, x)$ in terms of $\pi_{\leq 2}X$.

fundamental 2-groupoid

- The fundamental 2-groupoid $\pi_{\leq 2}X$:
  - Objects: points.
  - Morphisms: pathes.
  - 2-morphisms: homotopy classes of homotopies between pathes.
- Put associativity aside for a moment.
- $\pi_{\leq 2}X$ is indeed a 2-groupoid.
- $\pi_0(X) \simeq \mathrm{Ob}(\pi_{\leq 2}X)/\simeq$.
- $\pi_1(X, x) \simeq \mathrm{Hom}(x, x)/\simeq$.
- $\pi_2(X, x) \simeq \mathrm{Hom}(\mathrm{id}_x, \mathrm{id}_x)$.

Exercise 4

Describe the action of $\pi_1(X, x)$ on $\pi_2(X, x)$ in terms of $\pi_{\leq 2}X$.

Exercise 5

Can we define morphisms as homotopy classes of pathes?

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases}$$

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases}$$

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{2}$$

- Homotopic but not equal.

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{2}$$

- Homotopic but not equal.
- Can you find a canonical homotopy?

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{2}$$

- Homotopic but not equal.
- Can you find a canonical homotopy?
- Homotopy class of homotopies (1) $\to$ (2) is canonical.

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{2}$$

- Homotopic but not equal.
- Can you find a canonical homotopy?
- Homotopy class of homotopies (1) $\to$ (2) is canonical. $\Rightarrow$
- $\pi_{\leq 2}X$ is a weak 2-category.

$\pi_{\leq 2}$ is not strict

- Pathes: f, g, h.

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{2}$$

- Homotopic but not equal.
- Can you find a canonical homotopy?
- Homotopy class of homotopies (1) $\to$ (2) is canonical. $\Rightarrow$
- $\pi_{\leq 2}X$ is a weak 2-category.

Exercise 6

Challenge: show $\pi_{\leq 2} : \mathrm{hTop}_{\leq 2} \to \text{2-Grpd}$ *is an equivalence.*

Nightmare of homotopy coherence data

- $\pi_{\leq n} X$, $n \geq 3$.

Nightmare of homotopy coherence data

- $\pi_{\leq n} X$, $n \geq 3$.
- 2-morphism: ~~homotopy classes of~~ homotopies between pathes.

Nightmare of homotopy coherence data

- $\pi_{\leq n} X$, $n \geq 3$.
- 2-morphism: ~~homotopy classes of~~ homotopies between pathes.
- Make a non-canonical choice for homotopy (1) $\to$ (2).

Nightmare of homotopy coherence data

- $\pi_{\leq n} X$, $n \geq 3$.
- 2-morphism: ~~homotopy classes of~~ homotopies between pathes.
- Make a non-canonical choice for homotopy (1) $\to$ (2).
- Fit the choice into the coherent data of associativity

Nightmare of homotopy coherence data

- $\pi_{\leq n}X$, $n \geq 3$.
- 2-morphism: ~~homotopy classes of~~ homotopies between pathes.
- Make a non-canonical choice for homotopy (1) $\to$ (2).
- Fit the choice into the coherent data of associativity: choose homotopies between homotopies between homotopies.

Nightmare of homotopy coherence data

- $\pi_{\leq n}X$, $n \geq 3$.
- 2-morphism: ~~homotopy classes of~~ homotopies between pathes.
- Make a non-canonical choice for homotopy (1) $\to$ (2).
- Fit the choice into the coherent data of associativity: choose homotopies between homotopies between homotopies.
- ...

In retrospect

- Two impossible tasks:

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.
- Grothendieck's homotopy hypothesis: same task

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.
- Grothendieck's homotopy hypothesis: same task; do neither.

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n} X$ a weak n-groupoid.
- Grothendieck's homotopy hypothesis: same task; do neither.

Slogan 2

$$n\text{-groupoids} \;=\; \text{homotopy } n\text{-types.}$$

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.
- Grothendieck's homotopy hypothesis: same task; do neither.

Slogan 2

$$n\textit{-groupoids} \;=\; \textit{homotopy } n\textit{-types}.$$

$$\infty\textit{-groupoids} \;=\; \textit{homotopy types}.$$

In retrospect

- Two impossible tasks:
  - Give a combinatorial definition of weak n-groupoids;
  - Supply homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.
- Grothendieck's homotopy hypothesis: same task; do neither.

Slogan 2

$$n\text{-}groupoids \;=\; homotopy\ n\text{-}types.$$

$$\infty\text{-}groupoids \;=\; homotopy\ types.$$

$$\infty\text{-}category\ theory \;=\; category\ theory \;+\; homotopy\ theory.$$

Thank you!