

LECTURE 5

In this lecture, we define the quasi-category \mathcal{QCat} of small quasi-categories, which models the ∞ -category \mathbf{Cat}_∞ of small ∞ -categories.

1. IDEA OF THE CONSTRUCTION

1.1. Last time, for quasi-categories \mathcal{C} and \mathcal{D} , we constructed the quasi-category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ of functors between them. Let $\mathbf{Fun}(\mathcal{C}, \mathcal{D})^\simeq$ be the core Kan complex of this quasi-category, which can be viewed as obtained from $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ by discarding all non-invertible natural transformations.

1.2. Using these Kan complexes, we can enrich the *ordinary* category \mathbf{QCat} over \mathbf{Set}_Δ , and obtain a simplicial (enriched) category \mathbf{QCot} . The homotopy hypothesis suggests \mathbf{QCot} is already a model for \mathbf{Cat}_∞ .

1.3. To translate this into rigorous mathematics, we will construct a Quillen equivalence between model categories

$$\mathfrak{C} : \mathbf{Set}_\Delta^{\mathrm{Joyal}} \rightleftarrows \mathbf{Cat}_\Delta : \mathfrak{N}_\bullet$$

between the Joyal model category of simplicial sets and the standard model category of small simplicial categories. The simplicial category \mathbf{QCot} is a fibrant object in¹ \mathbf{Cat}_Δ because it is enriched over fibrant objects in \mathbf{Set}_Δ . The Joyal model structure is designed such that (bi)fibrant objects are exactly quasi-categories. Therefore

$$\mathcal{QCat} \stackrel{\mathrm{def}}{=} \mathfrak{N}_\bullet(\mathbf{QCot})$$

is a fibrant object in $\mathbf{Set}_\Delta^{\mathrm{Joyal}}$, i.e., a quasi-category.

2. HOMOTOPY CATEGORY OF QUASI-CATEGORIES

2.1. The *homotopy* category of the desired quasi-category \mathcal{QCat} , in the sense of [Lecture 4, Definition 4.2], can be easily defined. It will also be the homotopy category of the desired model category $\mathbf{Set}_\Delta^{\mathrm{Joyal}}$, in the sense of [Lecture 2, Definition 2.19]. It models the homotopy category \mathbf{hCat}_∞ of the desired ∞ -category \mathbf{Cat}_∞ .

Definition 2.2. Let $F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}$ be functors between quasi-categories. We call a morphism $\alpha : F_1 \rightarrow F_2$ in the quasi-category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ a **natural transformation** from F_1 to F_2 .

We say α is **invertible**, or is an **equivalence**, if it is an isomorphism in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$.

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¹Strictly speaking, \mathbf{QCot} is not an object in \mathbf{Cat}_Δ because it is not small. Hence we need to replace the *large* model category \mathbf{Cat}_Δ of *small* simplicial categories by the *very large* model category $\widetilde{\mathbf{Cat}}_\Delta$ of *large* simplicial categories, and similarly replace $\mathbf{Set}_\Delta^{\mathrm{Joyal}}$ by the *very large* Joyal model category $\widetilde{\mathbf{Set}}_\Delta^{\mathrm{Joyal}}$ of *large* simplicial sets. Here one can use *Grothendieck universes* to give precise meanings to the above size conditions. We will ignore these set-theoretic issues until we encounter really problems about them.

Theorem 2.3 (Ker.01DK). *Let $F_1, F_2 : \mathcal{C} \rightarrow \mathcal{D}$ be functors between quasi-categories. A natural transformation $\alpha : F_1 \rightarrow F_2$ is invertible iff its value at any object $x \in \mathcal{C}$ is an isomorphism, i.e. $\alpha(x) : F_1(x) \xrightarrow{\sim} F_2(x)$.*

Exercise 2.4. *Show that the following construction defines an ordinary category \mathbf{hQCat} :*

- Objects of \mathbf{hQCat} are small quasi-categories;
- Morphisms \mathbf{hQCat} are equivalence classes of functors between quasi-categories:

$$\mathrm{Hom}_{\mathbf{hQCat}}(\mathcal{C}, \mathcal{D}) \stackrel{\mathrm{def}}{=} \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$$

- Composition is given by $[G] \circ [F] \stackrel{\mathrm{def}}{=} [G \circ F]$.

3. EQUIVALENCES BETWEEN QUASI-CATEGORIES

3.1. Note that we have a functor $\mathbf{QCat} \rightarrow \mathbf{hQCat}$. Similar to [Lecture 4, Definition 5.1], we make the following definition.

Definition 3.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between quasi-categories, i.e. a morphism in \mathbf{QCat} . We say F is an **equivalence** if its image $[F]$ in \mathbf{hQCat} is an isomorphism.*

Remark 3.3. *In other words, $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence iff there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F$ is equivalent to $\mathrm{Id}_{\mathcal{C}}$ and $F \circ G$ is equivalent to $\mathrm{Id}_{\mathcal{D}}$.*

Theorem 3.4 (Ker.01JX). *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between quasi-categories. Then F is an equivalence iff*

- (1) F is **fully faithful**: for any objects $x, y \in \mathcal{C}$, the functor $\mathrm{Maps}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Maps}_{\mathcal{D}}(Fx, Fy)$ is an equivalence between ∞ -groupoids.
- (2) F is **essentially surjective**: the map $\pi_0(\mathcal{C}^{\simeq}) \rightarrow \pi_0(\mathcal{D}^{\simeq})$ is surjective. In other words, for any object $d \in \mathcal{D}$, there exist $c \in \mathcal{C}$ and an isomorphism $F(c) \xrightarrow{\sim} d$.

Exercise 3.5. *Being essentially surjective can be checked on the level of homotopy categories. Namely, $F : \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective iff $\mathbf{h}F : \mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{D}$ is so.*

Exercise 3.6. *Being fully faithful cannot be checked on the level of homotopy categories.*

Remark 3.7. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between quasi-categories. So far we have at least four equivalence-like conditions on F :*

- (a) F is an isomorphism in \mathbf{Set}_{Δ} .
- (b) F is an equivalence between quasi-categories.
- (c) F is a weak homotopy equivalence in the Kan–Quillen model category $\mathbf{Set}_{\Delta}^{\mathrm{KQ}}$.
- (d) F induces an equivalence between ordinary categories: $\mathbf{h}F : \mathbf{h}\mathcal{C} \rightarrow \mathbf{h}\mathcal{D}$.

The relations between these notions are

$$\begin{array}{ccccc} (a) & \implies & (b) & \implies & (c) \\ & & \Downarrow & & \\ & & (d) & & \end{array}$$

Exercise 3.8. *The above implications are not invertible.*

- (1) $N_\bullet(\{\bullet \rightrightarrows \bullet\}) \rightarrow \Delta^0$ satisfies (b) but not (a).
- (2) $\Delta^1 \rightarrow \Delta^0$ satisfies (c) but not (b) or (d).
- (3) $\text{Sing}(S^2) \rightarrow \Delta^0$ satisfies (d) but not (b) or (c).

Remark 3.9. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be the functor between ∞ -categories modeled by $F : \mathcal{C} \rightarrow \mathcal{D}$. The above conditions translate into:*

- (a) *This condition does not make sense because it is evil.*
- (b) *F is an equivalence between ∞ -categories.*
- (c) *F becomes an equivalence between ∞ -groupoids after formally inverting all morphisms in \mathcal{C} and \mathcal{D} .*
- (d) *F becomes an equivalence between ordinary categories after formally forcing all non-invertible higher morphisms in \mathcal{C} and \mathcal{D} to be identities.*

Remark 3.10. *Let \mathcal{C} be an ∞ -category. We denote \mathcal{C}^\sharp the ∞ -groupoid obtained from \mathcal{C} by formally inverting all morphisms. Note that this is not the core ∞ -groupoid \mathcal{C}^\simeq , which is obtained by discarding all non-invertible morphisms (see [Lecture 4, Exercise 5.3]). We have*

$$\mathcal{C}^\simeq \rightarrow \mathcal{C} \rightarrow \mathcal{C}^\sharp.$$

4. JOYAL MODEL STRUCTURE

4.1. Morphisms in the class (W) of $\text{Set}_\Delta^{\text{Joyal}}$ will be called *categorical equivalences*² between simplicial sets. When restricted to bifibrant objects, i.e. quasi-categories, they are supposed to model equivalences between the underlying ∞ -categories. In other words:

Categorical equivalences between quasi-categories should be equivalences, in the sense of Definition 3.2.

Hence the main task is to define categorical equivalences between *general* simplicial sets.

4.2. Recall when defining the classical model structure on Set_Δ (see [Lecture 3, Theorem-Definition 6.2]), we declare weak homotopy equivalences between simplicial sets to be maps $f : X \rightarrow Y$ such that $|f| : |X| \rightarrow |Y|$ are weak homotopy equivalences in \mathbf{Top} .

It is possible to define categorical equivalences in $\text{Set}_\Delta^{\text{Joyal}}$ in a similar manner. Namely, we can first define weak equivalences in Cat_Δ and then transfer them via the functor $\mathfrak{C} : \text{Set}_\Delta \rightarrow \text{Cat}_\Delta$ ³. However, I find the construction of the functor \mathfrak{C} not intuitive enough to provide actual feelings about categorical equivalences. Therefore we will follow Joyal's definition.

²Joyal called them *weak categorical equivalences*. We follow Lurie's terminology.

³This was the approach adopted in Lurie's HTT.2.2.5, which differs from that in Joyal's works (see [Joy08]). See HTT.2.2.5.9 and HTT.2.2.5.10 for more information.

4.3. Joyal's definition is motivated by the following observation:

Exercise 4.4. Let \mathcal{C} be a model category such that any object is cofibrant. Then a morphism $f : X \rightarrow Y$ belongs to (W) iff for any fibrant object Z , the map

$$(\mathrm{Hom}_{\mathcal{C}}(Y, Z)/\sim) \rightarrow (\mathrm{Hom}_{\mathcal{C}}(X, Z)/\sim)$$

is bijective. Here “ \sim ” stands for the homotopy relation between morphisms, see [Lecture 2, Proposition-Definition 2.18].

Hint: Yoneda lemma; the functor $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ detects all weak equivalences.

Proposition-Definition 4.5. Let $f : X \rightarrow Y$ be a morphism in Set_{Δ} . The following conditions are equivalent:

- (1) For any quasi-category Z , the map

$$\pi_0(\mathrm{Fun}(Y, Z)^{\simeq}) \rightarrow \pi_0(\mathrm{Fun}(X, Z)^{\simeq})$$

is a bijection between sets.

- (2) For any quasi-category Z , the functor

$$\mathrm{Fun}(Y, Z)^{\simeq} \rightarrow \mathrm{Fun}(X, Z)^{\simeq}$$

is a weak equivalence between Kan complexes.

- (3) For any quasi-category Z , the functor

$$\mathrm{Fun}(Y, Z) \rightarrow \mathrm{Fun}(X, Z)$$

is an equivalence between quasi-categories.

We say f is a **categorical equivalence** if it satisfies the above conditions.

Proposition 4.6 ([Joy08, Corollary 2.29]). Any inner horn inclusion $\Lambda_i^n \rightarrow \Delta^n$, $0 < i < n$ is a categorical equivalence.

Exercise 4.7. Convince yourself the above claim does not follow immediately from the definition of quasi-categories.

Exercise 4.8. Categorical equivalences are weak homotopy equivalences. *Hint: [Lecture 3, §6.5].*

Exercise 4.9 (Ker.01EG). Acyclic Kan fibrations are categorical equivalences.

Theorem-Definition 4.10 ([Joy08, Theorem 6.12]). There exists a model structure on Set_{Δ} given by

- (W) class of categorical equivalences;
- (C) class of monomorphisms;
- (F) class of morphisms satisfying the right lifting property against $(C \cap W)$.

We call it the **Joyal model structure** on Set_{Δ} . and denote this model category by $\mathrm{Set}_{\Delta}^{\mathrm{Joyal}}$.

Fibrant objects in this model category are exactly quasi-categories.

4.11. The following result is tautological:

Exercise 4.12. The functor $\mathbf{QCat} \rightarrow \mathbf{Set}_\Delta$ induces an equivalence⁴ $\mathbf{hQCat} \rightarrow \mathbf{hSet}_\Delta^{\text{Joyal}}$. In particular, $\mathbf{Set}_\Delta^{\text{Joyal}}[W^{-1}]$ also models \mathbf{hCat}_∞ .

Definition 4.13. Fibrations in $\mathbf{Set}_\Delta^{\text{Joyal}}$ are called **categorical fibrations**.

Proposition 4.14. The identity functor gives a Quillen adjunction

$$\mathbf{Set}_\Delta^{\text{Joyal}} \rightleftarrows \mathbf{Set}_\Delta^{\text{KQ}}.$$

Exercise 4.15. Let $\mathbb{L} : \mathbf{Set}_\Delta^{\text{Joyal}}[W^{-1}] \rightleftarrows \mathbf{Set}_\Delta^{\text{KQ}}[W^{-1}] : \mathbb{R}$ be the derived adjunction of the above Quillen adjunction. Interpret it as an adjunction $\mathbf{hCat}_\infty \rightleftarrows \mathbf{hGrpd}_\infty$.

Remark 4.16. By definition, categorical fibrations $p : X \rightarrow Y$ in $\mathbf{Set}_\Delta^{\text{Joyal}}$ satisfy the right lifting properties against all inner horn inclusions. Morphisms satisfying these properties are called **inner fibrations**. Note that when $Y = \Delta^0$, inner fibrations over Y are categorical fibrations. However, the inclusion

$$\text{inner fibrations} \subset \text{categorical fibrations}$$

is strict in general. For more information, see HTT.2.4.6.5.

5. SIMPLICIAL CATEGORIES

5.1. In this section, we define a model structure on the category \mathbf{Cat}_Δ of small simplicial categories.

5.2. Weak equivalences between simplicial categories are defined similarly as weak equivalences between topological categories. See [Lecture 2, §3].

Definition 5.3. Let \mathcal{C} be a simplicial category. Its **homotopy category** $\pi_0\mathcal{C}$ is defined by

$$\text{Ob}(\pi_0\mathcal{C}) := \text{Ob}(\mathcal{C}), \quad \text{Hom}_{\pi_0\mathcal{C}}(x, y) := \pi_0(|\text{Hom}_{\mathcal{C}}(x, y)|).$$

Definition 5.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between simplicial categories. We say F is a **weak equivalence** if:

- It induces an equivalence $\pi_0 F : \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$.
- The morphism $\text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence in $\mathbf{Set}_\Delta^{\text{KQ}}$ for any $x, y \in \text{Ob}(\mathcal{C})$.

Theorem 5.5 (HTT.A.3.2). There exists a canonical model structure on \mathbf{Cat}_Δ such that:

- Weak equivalences are as in Definition 5.4.
- Fibrant objects are exactly simplicial categories \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(-, -)$ are Kan complexes.

We call it the **classical** or **Bergner model structure** on \mathbf{Cat}_Δ .

Remark 5.6. It is not easy to describe fibrations and cofibrant objects in \mathbf{Cat}_Δ , hence we will not do it. Note that [Lecture 2, Exercise 3.6] has a simplicial analogue, which suggests many naturally defined simplicial categories are not cofibrant.

⁴In fact, it is an isomorphism

6. QUASI-CATEGORIES AND SIMPLICIAL CATEGORIES

6.1. In this section, we construct the desired Quillen equivalence

$$\mathfrak{C} : \mathbf{Set}_{\Delta}^{\text{Joyal}} \rightleftarrows \mathbf{Cat}_{\Delta} : \mathfrak{N}_{\bullet}.$$

As in the construction of

$$|-| : \mathbf{Set}_{\Delta}^{\text{KQ}} \rightleftarrows \mathbf{Top} : \mathbf{Sing},$$

we only need to find the correct definition for the restriction $\mathfrak{C}|_{\Delta} : \Delta \rightarrow \mathbf{Cat}_{\Delta}$ along the Yoneda embedding $\Delta \rightarrow \mathbf{Set}_{\Delta} \stackrel{\text{def}}{=} \mathbf{Fun}(\Delta^{\text{op}}, \mathbf{Set})$. The entire functor \mathfrak{C} will be the unique (up to unique equivalence) colimit-preserving functor that extends $\mathfrak{C}|_{\Delta}$, and the functor \mathfrak{N}_{\bullet} will be its right adjoint.

6.2. In other words, we need to construct simplicial categories $\mathfrak{C}[\Delta^n] \in \mathbf{Cat}_{\Delta}$ equipped with face and degeneracy operators. To motivate this construction, we look at the following question:

Let $\mathcal{C} \in \mathbf{Cat}_{\Delta}$ be a fibrant simplicial category. What is a functor $\mathfrak{C}[\Delta^n] \rightarrow \mathcal{C}$?

Since we *expect* the model category \mathbf{Cat}_{Δ} to model \mathbf{Cat}_{∞} , and since $\mathfrak{C}[\Delta^n]$ is cofibrant (because \mathfrak{C} is expected to be a left Quillen functor), we obtain

A functor $\mathfrak{C}[\Delta^n] \rightarrow \mathcal{C}$ should model a functor $[n] \rightarrow \mathcal{C}$ between corresponding ∞ -categories.

6.3. For instance, for $n = 2$, it should be an invertible 2-morphism witnessing the following commutative diagram in \mathcal{C} :

$$\begin{array}{ccc} & y & \\ f \nearrow & & \searrow g \\ x & \xrightarrow{h} & z. \end{array}$$

Note that in the simplicial category \mathcal{C} , the composition is *concrete*: for 0-simplexes $f \in \mathbf{Hom}_{\mathcal{C}}(x, y)_0$ and $g \in \mathbf{Hom}_{\mathcal{C}}(y, z)_0$, we have a well-defined simplex $g \circ f \in \mathbf{Hom}_{\mathcal{C}}(x, z)_0$. Hence the above invertible 2-morphism corresponds to a 1-simplex $g \circ f \rightarrow h^5$ in $\mathbf{Hom}_{\mathcal{C}}(x, z)_1$.

Therefore we should define $\mathfrak{C}[\Delta^2]$ as

- It has three objects, labelled by 0, 1 and 2.
- The morphism simplicial sets are

$$\mathbf{Hom}_{\mathfrak{C}[\Delta^2]}(i, j) \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } i > j \\ \Delta^1 & \text{if } i = 0, j = 2 \\ \Delta^0 & \text{if otherwise} \end{cases}$$

- The only non-obvious composition is

$$\begin{aligned} \mathbf{Hom}_{\mathfrak{C}[\Delta^2]}(0, 1) \times \mathbf{Hom}_{\mathfrak{C}[\Delta^2]}(1, 2) &\rightarrow \mathbf{Hom}_{\mathfrak{C}[\Delta^2]}(0, 2) \\ \Delta^0 \times \Delta^0 &\rightarrow \Delta^1, \end{aligned}$$

which is given by the 0-simplex $0 \in \Delta^1$.

⁵One may also use $h \rightarrow g \circ f$. Thanks to the fibrant assumption about $\mathbf{Hom}_{\mathcal{C}}(x, z)$, the actual data are the same. But the obtained functor \mathfrak{C} will differ by the involution $\mathbf{Set}_{\Delta} \rightarrow \mathbf{Set}_{\Delta}, X \mapsto X^{\text{op}}$. Here we follow the conventions in Kerodon (see Ker.00KN), which is the opposite to HTT (see HTT.1.1.5.1.).

6.4. The above research can be conducted for general n . Namely, for any ∞ -category, we *know* the correct meaning of a functor $[n] \rightarrow \mathcal{C}$, thanks to the fact that $[n]$ comes from a linearly ordered set⁶. For instance, 0-simplex in $\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0, n)$ should be a subset $I \subset [n]$ containing 0 and n ; and each chain $I_0 \supset \cdots \supset I_m$ corresponds to a m -simplex.

In formula, we define:

Definition 6.5. *Let J be any linearly ordered set. We define a simplicial category $\mathfrak{C}[\Delta^J]$ as follows:*

- *Objects are elements in J ;*
-

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(x, y) \stackrel{\mathrm{def}}{=} \begin{cases} \emptyset & \text{if } x > y \\ \mathbf{N}_\bullet(P_{x,y}) & \text{if } x \leq y \end{cases}$$

where $P_{x,y} := \{I \subset [x, y] \mid x, y \in I\}$ is equipped with the partial order \supset , and $\mathbf{N}_\bullet(P_{x,y})$ is the nerve of the corresponding category.

- *Morphisms are induced by*

$$(I \subset [x, y], I' \subset [y, z]) \mapsto (I \cup I' \subset [x, z])$$

6.6. The above construction is functorial in J and therefore we have a functor

$$\Delta \rightarrow \mathrm{Cat}_\Delta, [n] \mapsto \mathfrak{C}[\Delta^n] \stackrel{\mathrm{def}}{=} \mathfrak{C}[\Delta^{[n]}]$$

Note that $\mathfrak{C}[\Delta^n]$ is *not* fibrant. This can already be seen when $n = 2$.

Let

$$\mathfrak{C} : \mathrm{Set}_\Delta \rightarrow \mathrm{Cat}_\Delta$$

be the essentially unique colimit-preserving functor that extends the above functor.

Exercise 6.7. *What is $\mathfrak{C}[\Lambda_1^2]$?*

Theorem-Definition 6.8 (HTT.2.2.5.1, HTT.2.2.5.8). *The functor $\mathfrak{C} : \mathrm{Set}_\Delta \rightarrow \mathrm{Cat}_\Delta$ has a right adjoint, called the **simplicial nerve functor***

$$\mathfrak{N}_\bullet : \mathrm{Cat}_\Delta \rightarrow \mathrm{Set}_\Delta.$$

The adjoint pair

$$\mathfrak{C} : \mathrm{Set}_\Delta^{\mathrm{Joyal}} \rightleftarrows \mathrm{Cat}_\Delta : \mathfrak{N}_\bullet$$

is a Quillen equivalence.

6.9. Note that $\mathfrak{C}[\Delta^0] \simeq \{*\}$. Hence we may

- identify X_0 with $\mathrm{Ob}(\mathfrak{C}(X))$ for any simplicial set X ;
- identify $\mathrm{Ob}(\mathcal{C})$ with $\mathfrak{N}_\bullet(\mathcal{C})_0$ for any simplicial category \mathcal{C} .

From now on, we will abuse notations by viewing them as the *same* sets.

⁶On the other hand, homotopy coherent commutative diagrams discussed in [Lecture 1, Section 3] are beyond our reach, because they are not given by (partially) ordered set.

6.10. To justify corresponding bifibrant objects in $\mathbf{Set}_\Delta^{\text{Joyal}}$ and \mathbf{Cat}_Δ indeed model the same ∞ -category, we need the following results.

Theorem 6.11 (Ker.01LE). *Let $\mathcal{C} \in \mathbf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical weak homotopy equivalence between Kan complexes*

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{N}_\bullet(\mathcal{C})}^{\mathbf{L}}(x, y).$$

Corollary 6.12. *Let $\mathcal{C} \in \mathbf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical equivalence $\mathrm{h}\mathcal{C} \xrightarrow{\sim} \mathrm{h}\mathfrak{N}_\bullet(\mathcal{C})$ ⁷.*

7. THE QUASI-CATEGORY OF QUASI-CATEGORIES

7.1. This is a verbatim of §1.

Definition 7.2. *Let \mathbf{QCat} be the simplicial category defined by:*

- *Objects are quasi-categories;*
- *For quasi-categories \mathcal{C} and \mathcal{D} ,*

$$\mathrm{Hom}_{\mathbf{QCat}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}(\mathcal{C}, \mathcal{D})^\sim.$$

- *The composition*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is given by the universal property of $\mathrm{Fun}(-, -)$.

Definition 7.3. *The **quasi-category of small quasi-categories** is defined to be*

$$\mathcal{QCat} := \mathfrak{N}_\bullet(\mathbf{QCat}).$$

*Let \mathbf{Cat}_∞ be the ∞ -category modelled by it⁸. We call it the **∞ -category of small ∞ -categories**.*

Exercise 7.4. *Let \mathcal{C} and \mathcal{D} be quasi-categories. There is a canonical weak homotopy equivalence between Kan complexes*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\sim \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{QCat}}^{\mathbf{L}}(\mathcal{C}, \mathcal{D}).$$

In other words, $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\sim$ models the ∞ -groupoids $\mathrm{Maps}_{\mathbf{Cat}_\infty}(\mathcal{C}, \mathcal{D})$ as desired.

Variant 7.5. *Similarly, we define the **quasi-category of small Kan complexes***

$$\mathcal{Kan} := \mathfrak{N}_\bullet(\mathbf{Kan}),$$

where $\mathbf{Kan} \subset \mathbf{QCat}$ is the full simplicial subcategory consisting of Kan complexes. As before, the quasi-category \mathcal{Kan} models the ∞ -category \mathbf{Grpd}_∞ of small ∞ -groupoids.

Exercise 7.6. *Show that the homotopy category of \mathcal{QCat} is indeed equivalent to $\mathrm{h}\mathcal{QCat}$ in Exercise 2.4.*

8. COMMUTATIVE DIAGRAMS IN SIMPLICIAL CATEGORIES

8.1. By previous discussion, in infinite category theory, for any simplicial set S , the *correct* notion of an S -indexed commutative diagram in a simplicial category \mathcal{C} is a functor $\mathfrak{C}[S] \rightarrow \mathcal{C}$. We call such diagrams an S -indexed **homotopy coherent commutative diagram** in \mathcal{C} .

⁷In fact, this result is much more elementary. See Ker.00M4

⁸In rigorous mathematical words, we should *define* ∞ -categories to be quasi-categories. However, I do want the readers to view quasi-categories as models for ∞ -categories.

8.2. In most cases, the shape of the diagram is an ordinary category \mathcal{D} , we also abuse language and call a functor $\mathcal{C}[\mathbf{N}_\bullet(\mathcal{D})] \rightarrow \mathcal{C}$ a \mathcal{D} -indexed **homotopy coherent commutative diagram** in \mathcal{C} . This is the notion that behaves well under standard infinite categorical constructions.

One can also ask for a diagram that **commutes on the nose**, which means a functor $\mathcal{D} \rightarrow \mathcal{C}$, where \mathcal{D} is viewed as enriched over discrete simplicial sets. In practice, it is hard to construct such diagrams, but most *non-formal* calculations eventually lead people to this realm⁹.

One can also ask for a diagram that **commutes up to homotopy**, which means a functor $\mathcal{D} \rightarrow \mathbf{h}\mathcal{C}$. This is essentially a notion in ordinary category theory.

8.3. We have:

$$\text{on the nose} \Rightarrow \text{homotopy-coherently} \Rightarrow \text{up to homotopy}.$$

Exercise 8.4. Apply the above discussion to $\mathcal{C} := \mathbf{QCob}$ and obtain three notions of commutative diagram of quasi-categories.

9. WHAT IS AN ∞ -CATEGORIES?

9.1. In these notes, we take the following perspective:

- An ∞ -category is modelled by a quasi-category, and any equivalence between two quasi-categories supplies an identification of the ∞ -categories modelled by them.

Note however that such *identifications* are not unique, and there can be *identifications between identifications*, etc..

In particular, we do *not* assign a specific quasi-category to an ∞ -category¹⁰. For instance, it makes no sense to talk about the *set* of objects in an ∞ -category. Rather, one can talk about the ∞ -groupoid of objects (because ∞ -groupoids are defined under the same philosophy¹¹), and therefore the *set* of equivalent classes of objects.

9.2. To describe the above philosophy in more rigorous words, we make the following definition.

Definition 9.3. Let \mathcal{C} be a quasi-category and W be a collection of morphisms in \mathcal{C} . We say a functor $\mathcal{C} \rightarrow \mathcal{D}$ exhibits \mathcal{D} as a **quasi-categorical localization** of \mathcal{C} with respect to W if for any quasi-category \mathcal{E} , the functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is fully faithful and the essential image contains exactly functors $\mathcal{C} \rightarrow \mathcal{E}$ that send morphisms in W to isomorphisms.

⁹My personal experience: *genuine* non-formal calculations are very few.

¹⁰Note however that in Lurie's book, an ∞ -category is just a quasi-category.

¹¹See [Lecture 4, Remark 8.4].

9.4. One can show that the quasi-categorical localizations are essentially unique, i.e., unique up to a contractible space of choices (see HTT.3.1). We will use $\mathcal{C}[W^{-1}]$ to denote *the* quasi-categorical localization of \mathcal{C} with respect to W .

By Theorem A.3, we have a canonical equivalence

$$\mathbf{N}_\bullet(\mathbf{QCat})[W^{-1}] \xrightarrow{\simeq} \mathcal{QCat},$$

where W is the collection of categorical equivalences.

9.5. Now we make the following definition:

- An ∞ -category is an object in the quasi-categorical localization of $\mathbf{N}_\bullet(\mathbf{QCat})[W^{-1}]$ with respect to categorical equivalences.
- For a quasi-category $\mathcal{C} \in \mathbf{QCat}$, the ∞ -category modelled by \mathcal{C} is its image under $\mathbf{N}_\bullet(\mathbf{QCat}) \rightarrow \mathbf{N}_\bullet(\mathbf{QCat})[W^{-1}]$.

In above, we do *not* make a specific choice for $\mathbf{N}_\bullet(\mathbf{QCat})[W^{-1}]$, and any statements about ∞ -categories in these notes will not depend on such a choice. In particular, we do *not* view ∞ -categories as 0-simplices in \mathcal{QCat} .

Modulo the problem in Footnote 1, $\mathbf{N}_\bullet(\mathbf{QCat})[W^{-1}]$ itself models an ∞ -category, which is called the **∞ -category of (small) ∞ -categories**, and is denoted by \mathbf{Cat}_∞ .

9.6. In practice, the above philosophy results in the following principle. To make definitions or constructions about ∞ -categories, we can either:

- (i) Make them via quasi-categories and show the results are invariant under categorical equivalences.
- (ii) Combine existing definitions or constructions about ∞ -categories, without mentioning quasi-categories.

When both methods are available, we prefer the second one because it allows us to translate the definitions and constructions to other approaches to infinite category theory.

APPENDIX A. \mathbf{QCat} vs. \mathcal{QCat}

Exercise A.1. Let \mathcal{C} be an ordinary category. Construct an isomorphism $\mathbf{N}_\bullet(\mathcal{C}) \simeq \mathfrak{N}_\bullet(\mathcal{C})$, where in the RHS we view \mathcal{C} as enriched over discrete simplicial sets.

Exercise A.2. Construct a functor $\mathbf{N}_\bullet(\mathbf{QCat}) \rightarrow \mathcal{QCat}$ that models $\mathbf{QCat} \rightarrow \mathbf{Cat}_\infty$.

Theorem A.3. The functor $\mathbf{N}_\bullet(\mathbf{QCat}) \rightarrow \mathcal{QCat}$ exhibits \mathcal{QCat} as the quasi-categorical localization of $\mathbf{N}_\bullet(\mathbf{QCat})$ with respect to the class of categorical equivalences.

A.4. Suggested readings. [DK80c], [DK80a], [DK80b] (original) and [Hin16] (more recent).

REFERENCES

- [DK80a] William G Dwyer and Daniel M Kan. Calculating simplicial localizations. *Journal of Pure and Applied Algebra*, 18(1):17–35, 1980.
- [DK80b] William G Dwyer and Daniel M Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.
- [DK80c] William G Dwyer and Daniel M Kan. Simplicial localizations of categories. *Journal of pure and applied algebra*, 17(3):267–284, 1980.
- [Hin16] Vladimir Hinich. Dwyer–kan localization revisited. *Homology, Homotopy and Applications*, 18(1):27–48, 2016.

[Joy08] André Joyal. The theory of quasi-categories and its applications. 2008.