

## LECTURE 5

Last time we constructed a homomorphism  $\phi : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})$  fitting into the following diagram

$$(0.1) \quad \begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\phi} & \text{Sym}(\mathfrak{t}) \\ & \searrow \chi_\lambda = \xi_{-, \lambda} & \downarrow \text{ev}_\lambda \\ & & k_\lambda, \end{array}$$

where for any  $\lambda \in \mathfrak{t}^*$ ,  $\chi_\lambda = \xi_{-, \lambda}$  is the central character of the Verma module  $M_\lambda$ . We stated the following result, which will be proved today.

**Theorem 1** (Harish-Chandra). *The homomorphism  $\phi$  induces an isomorphism*

$$\phi_{\text{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \text{Sym}(\mathfrak{t})^{W_\bullet}$$

*from  $Z(\mathfrak{g})$  to the invariance of  $\text{Sym}(\mathfrak{t})$  with respect to the dotted  $W$ -action.*

### 1. STEP 1: IMAGE IS $W_\bullet$ -INVARIANT

Let us first prove the image of  $\phi$  is indeed contained in  $\text{Sym}(\mathfrak{t})^{W_\bullet}$ . Since  $W$  is generated by simple reflections  $s_\alpha$ ,  $\alpha \in \Delta$ , we only need to show  $\phi(z) = s_\alpha \cdot \phi(z)$  for any  $z \in Z(\mathfrak{g})$ . In other words, we need to show

$$(1.1) \quad \text{ev}_\lambda(\phi(z)) = \text{ev}_\lambda(s_\alpha \cdot \phi(z))$$

for any  $\lambda \in \mathfrak{t}^*$ . In fact, we only need to prove this for a Zariski dense subset of  $\mathfrak{t}^*$ . Let us first give this dense subset. The following result is obvious after drawing a picture.

**Lemma 2.** *Let  $\alpha \in \Delta$  be a simple root. The subset  $\{\lambda \in \mathfrak{t}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}\}^1$  is a Zariski dense subset of  $\mathfrak{t}^*$ .*

**Example 3.** For  $\mathfrak{g} = \mathfrak{sl}_2$ , this subset is  $\mathbb{Z}^{\geq -1} \subset \mathbb{A}^1$ . In fact, in [Exam. 15, Lect. 4], we have used this subset to show the image of  $\phi : Z(\mathfrak{sl}_2) \rightarrow k[h]$  is invariant under  $h \mapsto -h - 2$ . The argument below is an immediate generalization.

**Lemma 4.** *Let  $\alpha \in \Delta$  be a simple root and  $\lambda \in \mathfrak{t}^*$ . Suppose  $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}$ . Then  $M_\lambda$  contains  $M_{s_{\alpha \cdot} \lambda}$  as a submodule.*

**Corollary 5.** *The image of  $\phi : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})$  is indeed contained in  $\text{Sym}(\mathfrak{t})^{W_\bullet}$ .*

*Proof.* By previous discussion, we only need to prove (1.1) for  $\alpha$  and  $\lambda$  satisfying the assumption of Lemma 4. By (0.1), the LHS and RHS of (1.1) are exactly the central character of  $M_\lambda$  and  $M_{s_{\alpha \cdot} \lambda}$ . Now Lemma 4 implies they are equal because the central character of a module is equal to the central character of any nonzero submodule of it. □

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Date: Mar 25, 2024.

<sup>1</sup>In other words,  $\lambda + \rho$  is **dominant and integral**.

*Proof of Lemma 4.* Recall

$$s_\alpha(\lambda) = \lambda - 2 \frac{(\lambda, \alpha)}{(\alpha, \alpha)} \alpha = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha.$$

It follows that

$$s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \check{\alpha} \rangle \alpha = \lambda - m\alpha$$

for  $m \in \mathbb{Z}^{\geq 0}$ . The lemma is obvious for  $m = 0$ . We assume  $m > 0$ . Then we have  $\langle \lambda, \check{\alpha} \rangle = m - 1$  because  $\langle \rho, \check{\alpha} \rangle = 1$  (see [H1, Cor. to Lem. 10.2(B)]).

Let  $f_\alpha \in \mathfrak{n}^-$  be a nonzero vector of weight  $-\alpha$ . Consider the vector  $f_\alpha^m \cdot v_\lambda$ , which is of weight  $\lambda - m\alpha = s_\alpha \cdot \lambda$ . We only need to show

$$\mathfrak{n} \cdot (f_\alpha^m \cdot v_\lambda) = 0.$$

Indeed, if this is true, the map  $k_{s_\alpha \cdot \lambda} \rightarrow M_\lambda$ ,  $c \mapsto c(f_\alpha^m \cdot v_\lambda)$  is  $\mathfrak{b}$ -linear, and thereby induces a  $\mathfrak{g}$ -linear map  $M_{s_\alpha \cdot \lambda} \rightarrow M_\lambda$ . This map is injective because as a morphism between  $U(\mathfrak{n}^-)$ -modules, it is given by  $\cdot f_\alpha^m : U(\mathfrak{n}^-) \rightarrow U(\mathfrak{n}^-)$ , which is injective by the PBW theorem.

It remains to show  $\mathfrak{n}$  annihilates  $f_\alpha^m \cdot v_\lambda$ . For each simple root  $\beta \in \Delta$ , let  $e_\beta \in \mathfrak{n}$  be a nonzero vector of weight  $\beta$ . Note that the vectors  $(e_\beta)_{\beta \in \Delta}$  generate  $\mathfrak{n}$  under Lie brackets<sup>2</sup>. Hence we only need to show  $e_\beta \cdot f_\alpha^m \cdot v_\lambda = 0$ . There are two cases:

- If  $\alpha \neq \beta$ , then  $[e_\beta, f_\alpha] = 0$ <sup>3</sup> and

$$e_\beta \cdot f_\alpha^m \cdot v_\lambda = f_\alpha^m \cdot e_\beta \cdot v_\lambda = 0$$

because  $\mathfrak{n} \cdot v_\lambda = 0$ .

- If  $\alpha = \beta$ ,  $[e_\alpha, f_\alpha] \in \mathfrak{t}$  is proportionate to  $\check{\alpha}$  (see [H1, Prop. 8.3(d)]). Rescale  $e_\alpha$ , we may assume  $[e_\alpha, f_\alpha] = \check{\alpha}$ . Then  $[\check{\alpha}, f_\alpha] = \langle -\alpha, \check{\alpha} \rangle f_\alpha = -2f_\alpha$ . Now the following calculation is essentially that in [Exe. 24, Lect. 2]. We have

$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1 \leq i \leq m} f_\alpha^{m-i} \cdot [e_\alpha, f_\alpha] \cdot f_\alpha^{i-1} \cdot v_\lambda + f_\alpha^m \cdot e_\alpha \cdot v_\lambda.$$

Recall we have  $e_\alpha \cdot v_\lambda = 0$ . Also,

$$\check{\alpha} \cdot f_\alpha^j = \sum_{1 \leq i \leq j} f_\alpha^{j-i} \cdot [\check{\alpha}, f_\alpha] \cdot f_\alpha^{i-1} + f_\alpha^j \cdot \check{\alpha} = -2j f_\alpha^j + f_\alpha^j \cdot \check{\alpha}.$$

Hence

$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1 \leq i \leq m} (-2(i-1)f_\alpha^{m-1} + f_\alpha^{m-1} \cdot \check{\alpha}) v_\lambda = (-m(m-1) + m\langle \lambda, \check{\alpha} \rangle) v_\lambda = 0$$

as desired.

□[Lemma 4]

## 2. STEP 2: FILTRATIONS

By Step 1, we have a homomorphism

$$\phi_{\text{HC}} : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})^{W^\bullet}.$$

In this step, we equip both sides with filtrations, and show  $\phi_{\text{HC}}$  is compatible with them. The punchline is the following easy fact:

**Fact 6.** *Let  $V_1$  and  $V_2$  be two vector spaces equipped with  $\mathbb{Z}^{\geq 0}$ -indexed (exhausted) filtrations. Suppose  $\varphi : V_1 \rightarrow V_2$  is a  $k$ -linear map compatible with the filtrations. Then  $\varphi$  is an isomorphism iff  $\text{gr}^\bullet \varphi : \text{gr}^\bullet V_1 \rightarrow \text{gr}^\bullet V_2$  is so.*

<sup>2</sup>Because  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{[\alpha, \beta]}$  whenever  $\alpha, \beta, \alpha + \beta \in \Phi^+$ .

<sup>3</sup>Otherwise it is a nonzero vector with weight  $\beta - \alpha$  but the latter is not a root because  $\Phi = \Phi^+ \sqcup \Phi^-$ .

**Construction 7.** The PBW filtration on  $U(\mathfrak{g})$  induces a filtration on  $Z(\mathfrak{g})$  with  $F^{\leq i} Z(\mathfrak{g}) := Z(\mathfrak{g}) \cap F^{\leq i} U(\mathfrak{g})$ . Note that  $\text{gr}^\bullet Z(\mathfrak{g})$  is a subalgebra of  $\text{gr}^\bullet U(\mathfrak{g}) \simeq \text{Sym}(\mathfrak{g})$ .

**Construction 8.** The PBW filtration on  $U(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t})$  is preserved by the dotted  $W$ -action. Hence it induces a filtration on  $U(\mathfrak{t})^{W\bullet}$ .

**Warning 9.** The dotted  $W$ -action on  $\text{Sym}(\mathfrak{t})$  does not preserve the grading. But the usual (linear)  $W$ -action does.

**Lemma 10.** The homomorphism  $\phi_{\text{HC}} : Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})^{W\bullet}$  is compatible with the above filtrations.

*Proof.* This is obvious from the description of  $\phi$  as

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t}).$$

□

### 3. STEP 3: CALCULATING THE GRADED PIECES

**Construction 11.** Recall  $\text{Sym}(\mathfrak{g})$  has a natural  $\mathfrak{g}$ -module structure constructed as follows. For any  $V \in \mathfrak{g}$ , there is a natural  $\mathfrak{g}$ -module structure on  $V^{\otimes n}$  given by

$$\mathfrak{g} \times V^{\otimes n} \rightarrow V^{\otimes n}, (x, \otimes_i v_i) \mapsto \sum_i (v_1 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

This action is compatible with the symmetric group  $\Sigma_n$ -action on  $V^{\otimes n}$  and thereby induces a  $\mathfrak{g}$ -module structure on  $\text{Sym}^n(V)$ . Taking direct sum, we obtain a  $\mathfrak{g}$ -module structure on  $\text{Sym}(V)$ .

In the case  $V = \mathfrak{g}$ , to distinguish with the multiplication structure on  $\text{Sym}(\mathfrak{g})$ , we denote this action by

$$\mathfrak{g} \times \text{Sym}(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{g}), (x, u) \mapsto \text{ad}_x(u),$$

and call it the **adjoint action**.

Note that by definition, for  $x \in \mathfrak{g}$  and  $u, v \in \text{Sym}(\mathfrak{g})$ , we have

$$\text{ad}_x(u \cdot v) = \text{ad}_x(u) \cdot v + u \cdot \text{ad}_x(v).$$

In particular, the  $\mathfrak{g}$ -invariance

$$\text{Sym}(\mathfrak{g})^{\mathfrak{g}} := \{u \in \text{Sym}(\mathfrak{g}) \mid \text{ad}_x(u) = 0 \text{ for any } x \in \mathfrak{g}\}$$

is a subalgebra of  $\text{Sym}(\mathfrak{g})$ .

**Lemma 12.** There is a unique dotted graded isomorphism making the following diagram commute

$$\begin{array}{ccc} \text{gr}^\bullet Z(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \text{Sym}(\mathfrak{g})^{\mathfrak{g}} \\ \downarrow \subset & & \downarrow \subset \\ \text{gr}^\bullet U(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \text{Sym}(\mathfrak{g}), \end{array}$$

where the bottom isomorphism is given by the PBW theorem.

To prove this lemma, we use the following exercise:

**Exercise 13.** This is **Homework 2, Problem 4**. Prove: the adjoint  $\mathfrak{g}$ -action on  $U(\mathfrak{g})$ , i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \rightarrow U(\mathfrak{g}), (x, u) \mapsto \text{ad}_x(u) = [x, u],$$

preserves each  $F^{\leq n} U(\mathfrak{g})$ , and the induced  $\mathfrak{g}$ -action on  $\text{gr}^\bullet(U(\mathfrak{g})) \simeq \text{Sym}(\mathfrak{g})$  is the adjoint action in Construction 11.

<sup>4</sup>Note that  $\text{gr}^\bullet U(\mathfrak{t})$  is also isomorphic to  $\text{Sym}(\mathfrak{t})$ . To distinguish them, we always use  $U(\mathfrak{t})$  to denote the filtered commutative ring while use  $\text{Sym}(\mathfrak{t})$  to denote the graded commutative ring.

*Proof of Lemma 12.* By the above exercise, we have a short exact sequence of *finite-dimensional*  $\mathfrak{g}$ -modules:

$$0 \rightarrow F^{\leq n-1} U(\mathfrak{g}) \rightarrow F^{\leq n} U(\mathfrak{g}) \rightarrow \mathrm{Sym}^n(\mathfrak{g}) \rightarrow 0.$$

Since  $\mathfrak{g}\text{-mod}_{\mathrm{fd}}$  is semisimple, this short exact sequence splits. Hence taking  $\mathfrak{g}$ -invariance, we obtain<sup>5</sup>

$$0 \rightarrow F^{\leq n-1} Z(\mathfrak{g}) \rightarrow F^{\leq n} Z(\mathfrak{g}) \rightarrow \mathrm{Sym}^n(\mathfrak{g})^{\mathfrak{g}} \rightarrow 0$$

This gives the desired isomorphism  $\mathrm{gr}^n Z(\mathfrak{g}) \simeq \mathrm{Sym}^n(\mathfrak{g})^{\mathfrak{g}}$ .

□[Lemma 12]

A similar proof<sup>6</sup> gives:

**Lemma 14.** *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathrm{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t})^W \\ \downarrow \subset & & \downarrow \subset \\ \mathrm{gr}^\bullet U(\mathfrak{t}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t}), \end{array}$$

where the right-top corner is the invariance for the linear  $W$ -action on  $\mathrm{Sym}(\mathfrak{t})$ .

**Corollary 15.** *There is a unique dotted graded homomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathrm{gr}^\bullet Z(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \\ \downarrow \mathrm{gr}^\bullet \phi_{\mathrm{HC}} & & \downarrow \phi_{\mathrm{cl}} \\ \mathrm{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t})^W. \end{array}$$

#### 4. STEP 4: CHEVALLEY ISOMORPHISM

It remains to show

$$\phi_{\mathrm{cl}} : \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathrm{Sym}(\mathfrak{t})^W$$

is an isomorphism. Let us first give an explicit construction of this homomorphism. We need the following characterization of  $\phi$ .

**Construction 16.** *Consider the composition*

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k.$$

With respect to the adjoint  $\mathfrak{t}$ -action, the source has weight 0. Hence the composition factors through the 0-weight subspace of the target, which is exactly  $U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t})$ . By definition, the obtained map

$$Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})$$

is just  $\phi$ .

**Construction 17.** *It follows  $\phi_{\mathrm{cl}}$  can be constructed as follows. Consider the composition*

$$(4.1) \quad \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow \mathrm{Sym}(\mathfrak{g}) \twoheadrightarrow \mathrm{Sym}(\mathfrak{g}/\mathfrak{n}).$$

*It factors through  $\mathrm{Sym}(\mathfrak{b}/\mathfrak{n}) \simeq \mathrm{Sym}(\mathfrak{t})$ . The obtained map*

$$\mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathrm{Sym}(\mathfrak{t})$$

<sup>5</sup> Warning: in general, taking invariance is only *left exact*. (Memory method: it is given by  $\mathrm{Hom}_{\mathfrak{g}}(k, -)$ .) Hence we need the existence of a splitting.

<sup>6</sup>Note that  $\mathrm{Rep}(W)_{\mathrm{fd}}$  is also semisimple

can be identified with  $\mathbf{gr}^\bullet \phi$ . Since  $\phi$  factors through  $U(\mathfrak{t})^{W^\bullet}$ , the map  $\mathbf{gr}^\bullet \phi$  factors through  $\mathbf{gr}^\bullet(U(\mathfrak{t})^{W^\bullet}) \simeq \mathbf{Sym}(\mathfrak{t})^W$ . The obtained map

$$\mathbf{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathbf{Sym}(\mathfrak{t})^W$$

is just  $\phi_{\text{cl}}$ .

*Remark 18.* The geometric meaning of the above construction is as follows.

Note that  $\mathbf{Sym}(\mathfrak{g})^{\mathfrak{g}} = \mathbf{Sym}(\mathfrak{g})^G$  because  $G$ -invariance is equal to  $\mathfrak{g}$ -invariance<sup>7</sup>. Hence (4.1) corresponds to the morphisms

$$(\mathfrak{g}/\mathfrak{n})^* \rightarrow \mathfrak{g}^* \rightarrow \mathfrak{g}^* // G.$$

Since  $\mathbf{gr}^\bullet \phi$  factors through  $\mathbf{gr}^\bullet(U(\mathfrak{t})^{W^\bullet}) \simeq \mathbf{Sym}(\mathfrak{t})^W$ . The above composition factors through  $(\mathfrak{g}/\mathfrak{n})^* \rightarrow (\mathfrak{b}/\mathfrak{n})^* \simeq \mathfrak{t}^* \rightarrow \mathfrak{t}^* // W$ . In other words, we have

$$\begin{array}{ccc} (\mathfrak{g}/\mathfrak{n})^* & \longrightarrow & \mathfrak{g}^* // G \\ \downarrow & & \uparrow \text{dotted} \\ \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^* // W \end{array}$$

such that the dotted arrow is given by  $\text{Spec}(\phi_{\text{cl}})$ .

It is convenient to get rid of the dual spaces using the Killing form. Namely,  $\text{Kil}$  induces an isomorphism  $\mathfrak{g} \simeq \mathfrak{g}^*$  compatible with the  $G$ -actions, while  $\text{Kil}|_{\mathfrak{t}}$  induces an isomorphism  $\mathfrak{t} \simeq \mathfrak{t}^*$  compatible with the  $W$ -actions<sup>8</sup>. Via the first isomorphism, the subspaces  $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$  corresponds to  $(\mathfrak{g}/\mathfrak{b})^* \subset (\mathfrak{g}/\mathfrak{n})^* \subset \mathfrak{g}$ . Then the above commutative diagram is identified with

$$(4.2) \quad \begin{array}{ccc} \mathfrak{b} & \longrightarrow & \mathfrak{g} // G \\ \downarrow & & \uparrow \text{dotted} \\ \mathfrak{t} & \longrightarrow & \mathfrak{t} // W. \end{array}$$

We abuse notations and also view the dotted arrow as  $\text{Spec}(\phi_{\text{cl}})$ .

*Remark 19.* Since the projection  $\mathfrak{b} \rightarrow \mathfrak{t}$  has a splitting  $\mathfrak{t} \rightarrow \mathfrak{b}$ . The above claim implies  $\mathfrak{t} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} // G$  factors through  $\mathfrak{t} // W$ . We can also prove this using group-theoretic method. Namely, consider the normalizer  $N_G(T)$  of  $T$  inside  $G$ . Recall we have  $W \simeq N_G(T)/T$  such that the linear  $W$ -action on  $\mathfrak{t}$  can be identified with the adjoint action of  $N_G(T)/T$ . Then the morphism  $\mathfrak{t} \rightarrow \mathfrak{g} // G$  factors through  $\mathfrak{t} // W$  because  $N_G(T)$  is a subgroup of  $G$ .

**Warning 20.** I do not know any group-theoretic proof of (4.2). This is because not  $\mathfrak{b} \rightarrow \mathfrak{g} // G$  (the quotient stack) does not factor through  $\mathfrak{t}$ : two elements in  $\mathfrak{b}$  that have the same image in  $\mathfrak{t}$  are not necessarily conjugate to each other.

**Theorem 21** (Chevalley). The homomorphism

$$\phi_{\text{cl}} : \mathbf{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathbf{Sym}(\mathfrak{t})^W$$

is an isomorphism. In other words, the natural morphism  $\mathfrak{t} // W \rightarrow \mathfrak{g} // G$  is an isomorphism.

*Proof.* As in Remark 18, we can identify  $\phi_{\text{cl}}$  with the restriction map  $\text{Fun}(\mathfrak{g})^G \rightarrow \text{Fun}(\mathfrak{t})^W$ , which is also  $\mathbf{Sym}(\mathfrak{g}^*)^G \rightarrow \mathbf{Sym}(\mathfrak{t}^*)^W$ .

This map is injective because if an adjoint-invariant function on  $\mathfrak{g}$  vanishes on  $\mathfrak{t}$ , then it vanishes on each semisimple elements. But the latter are Zariski dense in  $\mathfrak{g}$ .

<sup>7</sup>Because  $\text{Rep}(G) \rightarrow \mathfrak{g}\text{-mod}$  is fully faithful when  $G$  is connected.

<sup>8</sup>Both claims follow from the fact that  $\text{Kil}$  is invariant with respect to the adjoint  $\mathfrak{g}$ -action and thereby to the adjoint  $G$ -action. Here we use  $W \simeq N_G(T)/T$ .

To prove the surjectivity, we first find generators of  $\text{Sym}(\mathfrak{t}^*)^W$  as follows. Recall the subset  $P^+$  of **dominant integral weights**, i.e.,

$$P^+ := \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0} \text{ for all } \alpha \in \Delta\}.$$

Note that  $P^+$  spans  $\mathfrak{t}^*$ . In fact, for each  $\alpha \in \Delta$ , we can find **fundamental dominant weights**  $\omega_\alpha$  such that

$$\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha\beta}, \quad \alpha, \beta \in \Delta.$$

Then  $\{\omega_\alpha\}$  is a basis of  $\mathfrak{t}^*$  and  $P^+ = \mathbb{Z}^{\geq 0}\{\omega_\alpha\}$ . A direct calculation shows that  $\{\lambda^n \mid \lambda \in P^+\}$  span  $\text{Sym}^n(\mathfrak{t}^*)$ . Hence the sums

$$b_{\lambda,n} := \sum_{w \in W} w(\lambda^n)$$

span  $\text{Sym}(\mathfrak{t}^*)^W$ .

It remains to show each  $b_{\lambda,n}$  is contained in the image of  $\phi_{\text{cl}}$ . We need the following well-known fact.

**Theorem 22** (Weyl). *For any  $\lambda \in P^+$ , there is a unique finite-dimensional irreducible  $\mathfrak{g}$ -module  $L_\lambda$  with highest weight  $\lambda$ .*

Consider the function  $a_{\lambda,n} \in \text{Fun}(\mathfrak{g})$  defined by

$$a_{\lambda,n}(x) := \text{tr}(x^n; L_\lambda),$$

i.e., its value at any  $x \in \mathfrak{g}$  is the trace of the action of  $x^n$  on  $L_\lambda$ . It is easy to see  $a_{\lambda,n}$  is  $\mathfrak{g}$ -invariant<sup>9</sup> and thereby  $G$ -invariant.

Now the following exercise implies each  $b_{\lambda,n}$  is contained in the image of  $\phi_{\text{cl}}$ . Indeed, this follows from induction on  $\lambda$  with respect to the partial ordering  $<$ <sup>10</sup>.

*Exercise 23.* This is **Homework 2, Problem 5**. The function  $\phi_{\text{cl}}(a_{\lambda,n})$ , i.e.,  $a_{\lambda,n}|_{\mathfrak{t}}$  is given by

$$a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\text{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' < \lambda} c_{\lambda'} b_{\lambda',n},$$

where  $\text{Stab}_W(\lambda) \subset W$  is the stabilizer of the  $W$ -action at  $\lambda$  and  $c_{\lambda'}$  are certain scalars.

□

Combining all the previous discussion, we finish the proof of Theorem 1.

## 5. MORE ON THE QUOTIENT $\mathfrak{t} // W$

We state some results on  $\mathfrak{t} // W \simeq \mathfrak{t} // W_\bullet$  without proof.

**Proposition 24** ([H2, Sect. 1.10]). *The morphism  $\varpi : \mathfrak{t} \rightarrow \mathfrak{t} // W_\bullet$  is surjective, and  $W_\bullet$  acts transitively on the fiber at each closed point.*

*Remark 25.* In other words, any character  $\chi : Z(\mathfrak{g}) \rightarrow k$  is the central character  $\chi_\lambda$  for some Verma module  $V_\lambda$ , and  $\chi_\lambda = \chi_\mu$  iff  $\mu = w \cdot \lambda$  for some  $w \in W$ .

**Corollary 26** (Linkage principle). *Verma modules  $M_\lambda$  and  $M_\mu$  belong to the same block iff  $\mu = w \cdot \lambda$  for some  $w \in W$ . In particular, there are only finitely many Verma modules and irreducible objects<sup>11</sup> in each block.*

**Proposition 27** ([G, Sect. 7]). *We have*

<sup>9</sup>Recall  $L_\lambda$  is  $G_{\text{sc}}$ -integrable ([Thm. 47, Lect. 3]). Hence  $a_{\lambda,n}$  is  $G_{\text{sc}}$ -invariant, and thereby  $\mathfrak{g}$ -invariant.

<sup>10</sup>Recall  $\lambda' \leq \lambda$  iff  $\lambda - \lambda' \in \mathbb{Z}^{\geq 0}\Phi^+$ . See [Defn. 22, Lect. 2].

<sup>11</sup>Recall  $M_\lambda$  has a unique irreducible quotient  $L_\lambda$ , and all irreducible objects of  $\mathcal{O}$  are of this form. See [Thm. 25, Prop. 31, Lec. 2].

- (1) *The schemes  $\mathfrak{t} // W \simeq \mathfrak{g} // G$  are smooth.*
- (2) *The map  $\mathfrak{t} \rightarrow \mathfrak{t} // W$  is flat.*
- (3) *The map  $\mathfrak{g} \rightarrow \mathfrak{g} // G$  is flat.*

## REFERENCES

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