Review:

For $\mathcal{A}$ is an abelian category, an object I of $\mathcal{A}$ is injective if the functor $\mathrm{Hom}(\cdot, I)$ is exact, which is equivalent to for any monomorphism $i: M \to N$ and $f: M \to I$, there exists $g: N \to I$ s.t. $f = g \circ i$:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"'] & I \\ N \arrow[ur, dashed] & \end{tikzcd}$$

Say $\mathcal{A}$ has enough injectives if for any object M, there exists a monomorphism from M into an injective object I.

For such $\mathcal{A}$ and a left exact functor f from $\mathcal{A}$ into another abelian category $\mathcal{B}$, then there is a sequence of functors called right derived functors: $R^i f: \mathcal{A} \to \mathcal{B}$, $i \geq 0$, s.t.

(a). $R^0 f = f$

(b). $R^i f(I) = 0$ for any injective object I and $i > 0$.

(c). For any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$, one has a long exact sequence $\cdots \to R^i f(M) \to R^i f(M'') \xrightarrow{\partial^i} R^{i+1} f(M') \to R^{i+1} f(M) \to \cdots$, where ∂^i are called connected morphism.

(d). The association in (c) of the long exact sequence to the short exact sequence is functorial.

M in $\mathcal{A}$ is f-acyclic if $R^i f(M) = 0$.

If $0 \to M \to N^0 \to N^1 \to N^2 \to \cdots$ is a resolution of M by f-acyclic objects N^i, then $R^i f(M)$ are canonically isomorphic to the cohomology objects of $0 \to fM \to fN^0 \to fN^1 \to fN^2 \to \cdots$

Lemma. Let $f: \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and $\mathcal{A}$ has enough injectives. Let T be a class s.t.

(a). $\forall$ object in $\mathcal{A}$ is a subobject of T

(b). Any direct summand of an object in T is also in T.

(c). $\forall$ Exact sequence $0 \to A' \to A \to A'' \to 0$ in T, if A', A are inside T, then $A'' \in T$ and $0 \to fA' \to fA \to fA'' \to 0$ is exact.

Then all injectives are in T and all elements of T are f-acyclic. Thus $R^i f$ can be computed using resolutions by objects in T.

Example: All flasque sheaves in $Sh(X)$ forms such a set where $f = \Gamma(X, -)$. (See [Har] Ex 2.1.16).

Also all injective objects.

r-th page:

$$\begin{tikzcd}
E_r^{p-r,q+r-1} \arrow[dr] & & & \\
& E_r^{pq} \arrow[rr, "d_{r+1}^{pq}"] \arrow[dr, "d_r^{pq}"] & & E^{p+r+1,q-r} \arrow[d] \\
& & E_r^{p+r,q-r+1} \arrow[dr, "d_r^{p+r,q-r+1}"] & \\
& & & E_r^{p+2r,q-2r+2}
\end{tikzcd}$$

Im / ker gives next page.

Spectral sequence:

A spectral sequence consists the following data

(a). E_r^{pq}, $p, q, r \in \mathbb{Z}$ and $r \geq 2$.

(b). $d_r^{pq}: E_r^{pq} \to E_r^{p+r,q-r+1}$

s.t. $d_r^{p+r,q-r+1} d_r^{pq} = 0$

$$Z_{r+1}(E_r^{pq}) = \ker d_r^{pq}, \quad B_{r+1}(E_r^{pq}) = \operatorname{im} d_r^{p-r,q-r+1}$$

one has $B_{r+1}(E_r^{pq}) \subset Z_{r+1}(E_r^{pq}) \subset E_r^{pq}$

(c). A family of isomorphisms: $d_r^{pq}: Z_{r+1}(E_r^{pq}) / B_{r+1}(E_r^{pq}) \cong E_{r+1}^{pq}$.

$$Z_{r+1}(E^{pq}) \to B_{r+1}(E^{p+r+1,q-r})$$

cdn. $B_\infty(E_r^{pq})$ and $Z_\infty(E_r^{pq})$ in E_r^{pq} s.t.

$$B_k(E_r^{pq}) \subset B_\infty(E_r^{pq}) \subset Z_\infty(E_r^{pq}) \subset Z_k(E_r^{pq})$$

$B_k(E_r^{pq})$ and $Z_k(E_r^{pq})$ are defined in the following sense:

For each $k \geq r+1$, define by induction on $k-(r+1)$.

When $k=r+1$, they are datas in (b).

Suppose $B_i(E_s^{pq})$ and $Z_i(E_s^{pq})$ is defined for all $0 \leq i-(s+1) < k-(r+1)$. And also, $B_k(E_{r+1}^{pq})$, $Z_k(E_{r+1}^{pq})$ are defined.

Then $\quad \alpha_r^{pq}: Z_{r+1}(E_r^{pq}) / B_{r+1}(E_r^{pq}) \cong E_{r+1}^{pq}$

$$B_k(E_r^{pq}) / B_{r+1}(E_r^{pq}) \leftarrow B_k(E_{r+1}^{pq})$$

$$Z_r(E_r^{pq}) / B_{r+1}(E_r^{pq}) \leftarrow Z_k(E_{r+1}^{pq})$$

All $B_k(E_r^{pq})$ can be defined in this way.

$$B_k(E_{r+1}^{pq}) / Z_k(E_{r+1}^{pq}) \cong B_k(E_r^{pq}) / Z_r(E_r^{pq})$$

$$\cong \cdots \cong Z_k(E_{k-1}^{pq}) / B_k(E_{k-1}^{pq}) \cong E_k^{pq}$$

And inclusion relations:

$$0 \overset{\text{defined}}{=} B_r(E_r^{pq}) \subset B_{r+1}(E_r^{pq}) \subset \cdots \subset Z_{r+2}(E_r^{pq})$$

$$\subset Z_{r+1}(E_r^{pq}) \subset Z_r(E_r^{pq}) = E_r^{pq}.$$

cdn. $B_\infty(E_2^{pq})$ and $Z_\infty(E_2^{pq}) \subset E_2^{pq}$ s.t.

$$B_k(E_2^{pq}) \subset B_\infty(E_2^{pq}) \subset Z_\infty(E_2^{pq}) \subset Z_k(E_2^{pq}),\ \forall k \geq 2,$$

Define E_∞^{pq} be quotient $Z_\infty(E_2^{pq}) / B_\infty(E_2^{pq})$

(e). H^n $(n \in \mathbb{Z})$ and a filtration

$$H^n \supset \cdots \supset F^pH^n \supset F^{p+1}H^n \supset \cdots$$

(f). $\beta^{pq}: E_\infty^{pq} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$

Spectral sequence denoted by $E_2^{pq} \Rightarrow H^{p+q}$.

Bicomplex gives a spectral sequence:

$$\begin{tikzcd}
 & \uparrow & \uparrow & \\
\rightarrow & K^{i+1,j} \arrow[r, "d_2^{i+1,j}"] & K^{i+1,j+1} \arrow[r] & \cdots \\
\rightarrow & K^{ij} \arrow[u, "d_1^{ij}"] \arrow[r, "d_2^{ij}"] & K^{i,j+1} \arrow[u, "d_1^{i,j+1}"'] \arrow[r] & \cdots \\
 & \uparrow & \uparrow &
\end{tikzcd}$$

s.t. $d_1 \circ d_1 = 0$, $d_2 \circ d_2 = 0$ and diagram commutes.

When infinite direct sum exists or $\forall n$, only finite nonzero K^{ij} in diagonal $i+j=n$.

Then complex $sK^{\cdot\cdot}$ associated to $K^{\cdot\cdot}$ is defined by

$$(sK^{\cdot\cdot})^n = \bigoplus_{i+j=n} K^{ij}$$

$d: (sK^{\cdot\cdot})^n \to (sK^{\cdot\cdot})^{n+1}$ is given by

$$dx = d_1x + (-1)^i d_2 x.$$

One can verify $d \circ d = 0$.

$H^n(K^{\cdot\cdot})$ is the n-th cohomology object $H^n(K^{\cdot\cdot})$

For each q, the q^{th} row is a complex:

$$\cdots \to K^{i,j} \xrightarrow{d_2} K^{i,j+1} \to \cdots$$

q^{th} cocycle: $Z_{II}^{q}(K^{i,\cdot}) \to Z_{II}^{q}(K^{i+1,\cdot})$

coboundary: $B_{II}^{q}(K^{i,\cdot}) \to B_{II}^{q}(K^{i+1,\cdot})$

cohomology: $H_{II}^{q}(K^{i+1,\cdot}) \to H_{II}^{q}(K^{i+1,\cdot}) \cdots$ is also a complex. Its p-th cohomology is denoted by $H_I^p H_{II}^q(K^{\cdot\cdot})$.

Define $F_I^p(sK^{\cdot\cdot})^n = \bigoplus_{i+j=n,\, i\geq p} K^{ij}$, $F^{\cdot}(sK^{\cdot\cdot})^n$ is a filtration.

$$(F_I^p(sK^{\cdot\cdot})/F_I^{p+1}(sK^{\cdot\cdot}))^n \cong K^{p,n-p}$$

$$E_{I_1}^{pq} = H^{p+q}(F_I^p(sK^{\cdot\cdot})/F_I^{p+1}(sK^{\cdot\cdot})) \cong H_{II}^{q}(K^{p,\cdot})$$

Then $E_{I_2}^{pq} = H_I H_{II}^{q}(K^{p,\cdot})$

So one gets two spectral sequence:

$$E_{I_2}^{pq} = H_I^p H_{II}^q(K^{\cdot\cdot}) \Rightarrow H^{p+q}(K^{\cdot\cdot}).$$

And $E_{II_2}^{pq} = H_{II}^p H_I^q(K^{\cdot\cdot}) \Rightarrow H^{p+q}(K^{\cdot\cdot})$

Here we construct spectral sequence from the filtered complex $F_I^p(sK^{\cdot\cdot})^n$.

Constructing spectral sequence from a given complex:

$$K^{\cdot} = F^{-\infty}K^{\cdot} \cdots \supset F^pK^{\cdot} \supset F^{p+1}K^{\cdot} \supset \cdots \supset F^{\infty}K^{\cdot} = 0.$$

Let $Z_r^{pq} = \ker(H^{p+q}(F^pK^{\cdot}/F^{p+1}K^{\cdot}) \to H^{p+q+1}(F^{p+1}K^{\cdot}/F^{p+r}K^{\cdot}))$

$= \operatorname{im}(H^{p+q}(F^pK^{\cdot}/F^{p+r}K^{\cdot}) \to H^{p+q}(F^pK^{\cdot}/F^{p+1}K^{\cdot}))$

and $B_r^{pq} = \operatorname{im}(H^{p+q-1}(F^{p-r+1}K^{\cdot}/F^pK^{\cdot}) \to H^{p+q}(F^pK^{\cdot}/F^{p+1}K^{\cdot}))$.

$Z_\infty^{pq} = \operatorname{im}(H^{p+q}(F^pK^{\cdot}) \to H^{p+q}(F^pK^{\cdot}/F^{p+1}K^{\cdot}))$

$B_\infty^{pq} = \operatorname{im}(H^{p+q-1}(K^{\cdot}/F^pK^{\cdot}) \to H^{p+q}(F^pK^{\cdot}/F^{p+1}K^{\cdot}))$

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}K^\cdot/F^{p+r}K^\cdot \arrow[r] & F^{p}K^\cdot/F^{p+r}K^\cdot \arrow[r] & F^{p}K^\cdot/F^{p+1}K^\cdot \arrow[r] & 0 \\
0 \arrow[r] & F^{p+1}K^\cdot/F^{p+r+1}K^\cdot \arrow[r] \arrow[u] & F^{p}K^\cdot/F^{p+r+1}K^\cdot \arrow[r] \arrow[u] & F^{p}K^\cdot/F^{p+1}K^\cdot \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

$\Rightarrow$ diagram:

$$\begin{tikzcd}
H^{p+q}(F^pK^\cdot/F^{p+r}K^\cdot) \arrow[r] & H^{p+q}(F^pK^\cdot/F^{p+1}K^\cdot) \\
H^{p+q}(F^pK^\cdot/F^{p+r+1}K^\cdot) \arrow[r] \arrow[u] & H^{p+q}(F^pK^\cdot/F^{p+1}K^\cdot) \arrow[u, equal]
\end{tikzcd}$$

Hence $Z_{r+1}^{pq} \subset Z_r^{pq}$.

Similarly, $B_r^{pq} \subset B_{r+1}^{pq}$,

$$\Rightarrow \quad B_2^{pq} \subset B_3^{pq} \subset \cdots \subset B_\infty^{pq}$$

$$Z_\infty^{pq} \subset \cdots \subset Z_3^{pq} \subset Z_2^{pq}$$

Consider

$$\begin{tikzcd}
0 \arrow[r] & F^pK^\cdot \arrow[r] & K^\cdot \arrow[r] & K^\cdot/F^pK^\cdot \arrow[r] & 0 \\
0 \arrow[r] & F^pK^\cdot/F^{p+1}K^\cdot \arrow[r] & K^\cdot/F^{p+1}K^\cdot \arrow[r] & K^\cdot/F^pK^\cdot \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

gives

$$\begin{tikzcd}
H^{p+q-1}(K^\cdot/F^pK^\cdot/F^pK^\cdot) \arrow[r] \arrow[d, equal] & H^{p+q}(F^pK^\cdot) \arrow[d] \\
H^{p+q-1}(K^\cdot/F^pK^\cdot) \arrow[r] & H^{p+q}(F^pK^\cdot/F^{p+1}K^\cdot)
\end{tikzcd}$$

$\Rightarrow$ $B_\infty^{pq} \subset Z_\infty^{pq}$.

Hence $B_2^{pq} \subset \cdots \subset B_\infty^{pq} \subset Z_\infty^{pq} \subset \cdots \subset Z_2^{pq}$.

For $r \geq 2$. Let $E_r^{pq} = Z_r^{pq}/B_r^{pq}$, $E_\infty^{pq} = Z_\infty^{pq}/B_\infty^{pq}$

One can prove $Z_r^{pq}/Z_{r+1}^{pq} \cong B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}$

and define d_r^{pq} to be $Z_r^{pq}/B_r^{pq} \to Z_r^{pq}/Z_{r+1}^{pq} \cong B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \to Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$

Other data: $F^pH^n = \mathrm{im}(H^n(F^pK^\cdot) \to H^n(K^\cdot))$

similarly, $F^\cdot H^n$ is a filtration.

One can prove that $E_\infty^{pq} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$.

Let $E_1^{pq} = H^{p+q}(F^pK^\cdot/F^{p+1}K^\cdot)$, and $d_1^{pq}: E_1^{pq} \to E_1^{p+1,q}$ to be the connecting morphism: $H^{p+q}(F^pK^\cdot/F^{p+1}K^\cdot) \to H^{p+q+1}(F^{p+1}K^\cdot/F^{p+2}K^\cdot)$.

this can be seen as the first page of $E_2^{pq} \Rightarrow H^{p+q}$

Particularly, when for $\forall n$. $\exists$ $f(n), g(n)$ s.t. $F^pK^n = K^n$, $\forall p \leq f(n)$ and $F^pK^n = 0$, $\forall p \geq g(n)$.

the spectral sequence is biregular.

Now, given a bicomplex, we can construct two spectral sequence.

We have the following:

biregular iff (a). For $\forall p, q$, $\exists k$, s.t. $B_\infty(E_2^{p,q}) = B_k(E_2^{p,q})$, $Z_\infty(E_2^{p,q}) = Z_k(E_2^{p,q})$

(b). For $\forall n$, $\exists p, p'$, s.t. $F^pH^n = 0$, $F^{p'}H^n = H^n$.

If there exists p_0, q_0 s.t. $K^{p,q} \neq 0$ only when $p \geq p_0$ and $q \geq q_0$, then both spectral sequences are biregular.

Prop. Biregular Spectral sequence $E_r^{p,q} \Rightarrow H^{p+q}$, if $\exists r \geq 2$ and q_0 s.t. $E_r^{p,q} = 0$ whenever $q \neq q_0$, then $E_r^{n-q_0, q_0} \cong H^n$, $\forall n$.

Cartan-Eilenberg Resolution:

Let $K^\cdot$ be a complex on an abelian category with enough injectives. then $\exists$ bicomplex $I^{\cdot\cdot}$ s.t. $\forall q$, $I^{\cdot,q}$, $Z_{II}(I^{\cdot\cdot})$, $B_{II}^q(I^{\cdot\cdot})$ and $H_{II}^q(I^{\cdot\cdot})$ are injective resolutions of K^q, $Z^q(K^\cdot)$, $B^q(K^\cdot)$, and $H^q(K^\cdot)$ respectively, and for each i,

$$0 \to B_{II}^q(I^{i,\cdot}) \to Z_{II}^q(I^{i,\cdot}) \to H_{II}^q(I^{i,\cdot}) \to 0$$

$$0 \to Z_{II}^q(I^{i,\cdot}) \to I^{i,q} \to B_{II}^{q+1}(I^{i,\cdot}) \to 0$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots \to & & I^{1,q} & \to & I^{1,q+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots \to & & I^{0,q} & \to & I^{0,q+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots \to & & K^q & \to & K^{q+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

pf: Construct

injective resolution of B and H

$$\begin{array}{ccccccccc}
0 & \to & I_{B^q}^\cdot & \to \exists & I_{Z^q}^\cdot \text{ s.t.} & \to & I_{H^q}^\cdot & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & B^q(K^\cdot) & \to & Z^q(K^\cdot) & \to & H^q(K^\cdot) & \to & 0
\end{array}$$

and it splits that $I_{Z^q}^\cdot = I_{B^q}^\cdot \oplus I_{H^q}^\cdot$

Then construct $I^{\cdot,q}$ an injective resolution of K^q.

$$\begin{array}{ccccccccc}
0 & \to & I_{Z^q}^\cdot & \to & I^{\cdot,q} & \to & I_{B^{q+1}}^\cdot & \to 0 & \text{exact} \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Z^q & \to & K^q & \to & B^{q+1} & \to 0 &
\end{array}$$

$I^{\cdot,q} = I_Z^{q} \oplus I_B^{q+1}$ $\quad$ $I^{\cdot,q} \to I^{\cdot,q+1}$ is defined by

$$\underset{\text{epi}}{I^{\cdot,q} \xrightarrow{d} I_B^{q+1}} \to \underbrace{I_Z^{q+1} \to I^{\cdot,q}}_{\text{mono}}$$

$I_Z^{q} = I_{Z^{q,r}} \oplus I_H^{q}$.

Then $B^q_{II}(I^{\cdot,\cdot}) = I_B^q$, and $Z^q_{II}(I^{\cdot,\cdot}) = I_Z^q$,

$H^q_{II}(I^{\cdot,\cdot}) = I_H^q$. One sees that they satisfies our needs.

Grothendieck Spectral Sequence:

$F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{C}$ are two additive and left exact functors between two abelian categories. $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and F takes injectives to G-acyclic objects. Then for each object A of $\mathcal{A}$, there is a spectral sequence:

$$E_2^{pq} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A)$$

Proof: Let $I^\cdot$ be an injective resolution of A, and let $J^{\cdot\cdot}$ be a Cartan–Eilenberg resolution of $F(I^\cdot)$.

$F(I^\cdot)$ are acyclic.

We have short exact sequences

$$0 \to B^q_{II}(J^{\cdot,\cdot}) \to Z^q_{II}(J^{\cdot,\cdot}) \to H^q_{II}(J^{\cdot,\cdot}) \to 0$$

$$0 \to Z^q_{II}(J^{\cdot,\cdot}) \to J^{\cdot,q} \to B^{q+1}_{II}(J^{\cdot,\cdot}) \to 0$$

split.

Hence one has exact sequences:

$$0 \to G(B^q_{II}(J^{\cdot,\cdot})) \to G(Z^q_{II}(J^{\cdot,\cdot})) \to G(H^q_{II}(J^{\cdot,\cdot})) \to 0$$

$$0 \to G(Z^q_{II}(J^{\cdot,\cdot})) \to G(J^{\cdot,q}) \to G(B^{q+1}_{II}(J^{\cdot,\cdot})) \to 0$$

thus $Z^q_{II}(G(J^{\cdot,\cdot})) = G(Z_{II}(J^{\cdot,\cdot}))$,

$B^q_{II}(G(J^{\cdot,\cdot})) = G(Z_{II}(J^{\cdot,\cdot}))$

$H^q_{II}(G(J^{\cdot,\cdot})) = G(H^q_{II}(J^{\cdot,\cdot}))$, Consider the bicomplex $G(J^{\cdot\cdot})$.

(Recall II stands for row)

$$\begin{array}{ccccccccc}
 & & & & \uparrow & & \uparrow & & \\
 & & & & J^{1,0} & \to & (J^{1,1}) & \to & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & J^{0,0} & \to & (J^{0,1}) & \to & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
0 & \to & F(A) & \to & F(I^0) & \to & F(I^1) & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $H^q_{\mathrm{II}}(J^{\cdot\cdot})$ is an injective resolution of $H^q(F(I^\cdot)) \cong R^qF(A)$,

we have $H^p_{\mathrm{I}} H^q_{\mathrm{II}} \cong H^p_{\mathrm{I}}(G(H^q_{\mathrm{II}}(J^{\cdot\cdot}))) \cong H^p(GR^qF(A)) = R^pG(R^qF(A))$

and thus a spectral sequence $R^pG(R^qF(A))$.

Consider $H^q_{\mathrm{I}}(G(J^{\cdot,p}))$, since $J^{\cdot,p}$ is an injective resolution of $F(I^p)$ which is G-acyclic, we have $H^q_{\mathrm{I}}(J^{\cdot,j}) = \begin{cases} 0 & q>0 \\ GF(I^p), & q=0 \end{cases}$

thus $H^p_{\mathrm{II}} H^q_{\mathrm{I}}(G(J^{\cdot\cdot})) = \begin{cases} 0 & q>0 \\ H^p(GF(I^\cdot)) = R^p(GF)(A), & q=0. \end{cases}$

We has a spectral sequence $H^p_{\mathrm{II}} H^q_{\mathrm{I}}(G(J^{\cdot\cdot})) \Rightarrow H^{p+q}(G(J^{\cdot\cdot}))$,

in the first page, only the zeroth column is nonzero, thus by the prop before, $H^n_{\mathrm{II}} H^0_{\mathrm{I}} \cong H^n(G(J^{\cdot\cdot}))$. thus together.

$$\underset{\displaystyle R^p(GF)(A)}{\underset{\parallel}{H^n_{\mathrm{II}} H^0_{\mathrm{I}}}}$$

we have a spectral sequence $(R^pG)(R^qF(A)) \Rightarrow R^{p+q}(FG)(A)$.

Example: $H^p(Y, R^qf_*F) \Rightarrow H^{p+q}(X,F)$, $f: X\to Y$, $g: Y \to Z$ morphisms

$R^pg_*R^qf_*F \Rightarrow R^{p+q}(gf)_*F$.

Use f_* maps injectives to injectives. this follows from it has a left adjoint functor f^{-1}. $\mathrm{Hom}(G, f^*I) \cong \mathrm{Hom}(f^{-1}G, I)$,

I injective $\Rightarrow$ f^*I injective.

Lemma. $S(X_E)$ has enough injectives (isn't necessarily étale site)

Def. (a). The functor $T(X, -) : S(X_E) \to Ab$ with $T(X, F) = F(X)$

is left exact and its right derived functors are written as:

$$R^i T(X, -) = H^i(X, -) = H^i(X_E, -).$$

(b). Right derived functors of $F \Rightarrow F(U)$, where $U \to X$ in C/X:

$$H^i(U, F) (= H^i(U, F|_U) \text{ derived of } T(U, |_U))$$

(c). $i: S(X_E) \to P(X_E)$ is left exact, right derived functors

$\underline{H}^i(X_E, F)$ or $\underline{H}^i(F)$.

(d). $Hom_S(F_0, -)$ is left exact for $F_0 \in S(X_E)$.

right derived functors: $Ext^i_S(F_0, -)$

(e). Internal Hom: $\underline{Hom}(F_0, F_1)(U) = Hom_{S(U_E)}(F_0|_U, F_1|_U)$

is a sheaf. Fix F_0. $\underline{Hom}(F_0, -)$ is left exact,

right derived functors: $\underline{Ext}^i(F_0, F)$.

(f). For any continuous $\pi: X'_E \to X_E$, the right derived functors

$R^i\pi_*$ are defined. $R^i\pi_* F$ are called the higher direct images.

Relations: $T(X, -) \cong Hom(\mathbb{Z}, -)$

$\uparrow$ constant sheaves

Then $H^i(X, -) \cong Ext^i(\mathbb{Z}, -)$

$\underline{H}^i(F)$ = presheaf: $U \to H^i(U, F)$

These functors in the case $X = \operatorname{spec} k$.

Then $S(X_{et}) \simeq G\text{-mod}$ where $G = \operatorname{Gal}(k^{sep}/k)$.

(a). If F corresponds to G-mod M, then $T(X, F) = M^G$,

then $H^i(X, F) = H^i(G, M) = H^i(k, M)$ is the Galois cohomology.

(b). F_0, F_1 corresponds to M_0 and M_1 then

$$\operatorname{Hom}(F_0, F_1) = \operatorname{Hom}_G(M_0, M_1),$$

(c). $\underline{\operatorname{Hom}}(F_0, F_1)$ corresponds to the G-mod

$\bigcup_H \operatorname{Hom}_H(M_0, M_1)$. where H runs through normal subgroups of G.

G acts on $\operatorname{Hom}(M_0, M_1)$ by $g(f) = gfg^{-1}$,

one sees that indeed $T(\operatorname{spec} k, \underline{\operatorname{Hom}}(F_0, F_1)) = \operatorname{Hom}_G(M_0, M_1)$.

Flabby sheaves on a topological space X: All restriction maps are surjective.

on a site $C(C/X)_E$: $H^i(U, F) = 0$ for all U in C/X and all $i > 0$.

Lemma. For any $\pi: U \to X$ in C/X, the functor $\pi^* = (F \mapsto F|U)$ takes injective sheaves to injective sheaves.

Pf: π^* has a left adjoint $\pi_!: \underline{S}(U) \to \underline{S}(X)$ which is exact.

Cor. $\forall$ sheaf on X_E, $U \to X$ in C/X, $H^i(U, F)$ and $H^i(U_E, F|U)$ are canonically isomorphic.

Pf: $F \to F|U$ is exact, $\Rightarrow$ for an injective resolution $0 \to F \to I_0 \to I_1 \to \cdots$, $\quad 0 \to F|U \to I_0|U \to I_1|U \to \cdots$ is also an injective resolution.

Also, $F \mapsto F|U$ sends flabby sheaves in $S(X_E)$ to flabby sheaves in $S(U_E)$.

And $\underline{H}^i(F) =$ Presheaf: $U \mapsto H^i(U_E, F|U)$.

Higher direct images: $\pi: X'_{E'} \to X_E$ continuous. for $F \in S(X'_{E'})$,

$$R^i\pi_* F = a\,\pi_p(\underline{H}^i(F)).$$ i.e.

the sheaf associated to the presheaf $U \mapsto H^i(U', F|U')$, where $U' = U \times_X X'$.

Pf: $\pi_* = a\,\pi_p\, i$, where i is the inclusion $S(X') \hookrightarrow P(X')$.

Take an injective resolution $0 \to F \to I^0 \to I^1 \to \cdots$

Then $R^i\pi_* F$ is the i^{th} cohomology group of the complex

$$0 \to \pi_* F \to a\pi_p(iI^0) \to a\pi_p(iI^1) \to \cdots$$

Since a, π_p are exact, $\Rightarrow$

$$R^i\pi_* F = H^i(a\pi_p(iI^\cdot)) = a\pi_p(H^i(iI^\cdot)) = a\pi_p(\underline{H}^i(F)),$$

$\pi_p(\underline{H}^i(F))$ is the presheaf $U \mapsto H^i(U', F|U')$, where $U' = U \times_X X'$

Cor. If F is flabby, then $H^i(U', F|U') = 0$ for all U. $\Rightarrow$ $R^i\pi_* F = 0$. Thus one can use flabby resolution to compute $R^i\pi_*$.

Thm. (a). (Leray spectral sequence). For any continuous $\pi: (C'/X')_{E'} \to (C/X)_E$, there is a spectral sequence

$$H^p(X_E, R^q\pi_* F) \Rightarrow H^{p+q}(X'_{E'}, F)$$

(b). For any continuous morphisms $X''_{E''} \xrightarrow{\pi'} X'_{E'} \xrightarrow{\pi} X_E$, there is a spectral sequence

$$(R^p\pi_*)(R^q\pi'_*)F \Rightarrow R^{p+q}(\pi\pi')_* F.$$

Pf: Recall flabby sheaves are acyclic for the global section functor and the direct image functor, these are consequences of the following fact:

For continuous π: π_* maps flabby sheaves $\to$ flabby sheaves.

Remark: Usually, π^* is exact, (when finite inverse limits exists e.g. $C/X = LFT/X$, $(Et)/X$ or $(Zar)/X$, π is in C/X and $C'/X' = (C/X)/X'$), then π_* perserves injectives.

Local-global spectral sequence for Exts:

For any F_1 and F_2 on X_E, there is a spectral sequence:

$$H^p(X_E, \underline{Ext}^q(F_1, F_2)) \Rightarrow Ext^{p+q}(F_1, F_2).$$

This follows from: if F_2 is injective, then $\underline{Hom}(F_1, F_2)$ is flabby. thus $\Gamma(X,-)$-acyclic.

Thm. $\pi: Y \to X$ be quasi-compact morphism; Let F be a sheaf on Y_{et}. Geometric point $\bar{x} \to X$ s.t. $k(\bar{x})$ = sep. closure of $k(x)$.
Let $\tilde{X} = \mathrm{spec}\, \mathcal{O}_{X,\bar{x}}$, (Recall it's local ring in the etale sense, strict henselization of $\bar{x}$). $\tilde{Y} = Y \times_X \tilde{X}$, and $\tilde{F}$ be the inverse image of F on $\tilde{Y}$.

$$\begin{tikzcd} Y \arrow[d, "\pi"'] & \tilde{Y} \arrow[l] \arrow[d] \\ X & \tilde{X} \arrow[l] \end{tikzcd}$$

Then $R^p\pi_*(F)_{\bar{x}} \cong H^p(\tilde{Y}, \tilde{F})$.

Pf: $R^p\pi_*(F)_{\bar{x}} = \varinjlim H^p(U \times_X Y, F|_{U \times Y})$ where limit takes over affine U s.t. $\tilde{X} = \varprojlim U$.

$$\tilde{Y} = (\varprojlim U) \times Y = \varprojlim (U \times Y)$$

and transition morphisms in the limit are affine morphisms then use Lemma III 1.16,

$$\varinjlim H^p(U \times_X Y, F|_{U \times Y}) = H^p(\varprojlim U \times_X Y, F|_{U \times Y})$$

$$= H^p(\tilde{Y}, \tilde{F}).$$

Remark: When π is proper, $(R^p\pi_* F)_{\bar{x}} \cong H^p(Y_{\bar{x}}, F|_{Y_{\bar{x}}})$.

For each $x \in X$. assign $u_x: \bar{x} \to X$.

Let $X' = \coprod_{\bar{x}} \bar{x}$ over X. $u: X' \to X$.

then $u^* F$ is a sheaf on X', which restriction at $\bar{x}$ is just $u_x^* F$.

Any sheaf on X' is flabby $\Rightarrow$ $u_* u^* F$ is flabby.

$F \to u_* u^* F$ is injective?

Consider stalk.

$$(u^* F)_{\bar{x}} = F_{\bar{x}}.$$

$$\forall\ U \to X, \quad u_p u^* F(U) = u^* F(U \times_X X')$$

$$= u^* F\left(\coprod_{\bar{x} \in U} \bar{x}\right) = \coprod_{\bar{x} \in U} F_{\bar{x}}.$$

the map $F \to u^* u_* F$ can be just defined as

$$F \to \text{presheaf } u_p u^* F \to \text{sheaf } u_* u^* F.$$

taking s to $\prod_{\bar{x}} s_{\bar{x}}$, injective

Godement resolution:

(i) $C^0(F) = u_* u^*(F)$, $\quad \varepsilon: F \to C^0(F)$
$\qquad \qquad \qquad \qquad \quad \| \; \prod_{x \in X} u_{x*}(F_x)$

(ii) $C^1(F) = C^0(\mathrm{Coker}(\varepsilon))$, $\quad d^0: C^0(F) \to C^1(F)$

(iii) $C^i(F) = C^0(\mathrm{Coker}(d^{i-2}))$

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

$\forall F$ on X_{et}. $i_* i^! F$ is the largest subsheaf of F that is zero outside Z.

$\Gamma(X, i_* i^! F) = \Gamma(Z, i^! F) = \ker(F(X) \to F(U))$

is called the group of sections of F with support in Z, denoted by $\Gamma_Z(X, F)$. and derived functors $H^p_Z(X, F)$.

Prop: For any F on X_{et}, there is a long exact sequence

$0 \to (i^! F)(Z) \to F(X) \to F(U) \to \cdots \to H^p(X, F) \to H^p(U, F) \to$

$H^{p+1}_Z(X, F) \to \cdots$

Pf: $\quad 0 \to j_! j^* F \to F \to i_* i^* F \to 0$

Let F be the constant sheaf $\mathbb{Z}$. then

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z}_U & \to & \mathbb{Z} & \to & \mathbb{Z}_Z & \to 0 \\ & \parallel & & & & \parallel & \\ & j_! j^* \mathbb{Z} & & & & i_* i^* \mathbb{Z} & \end{array}$$

Then $\cdots \to \mathrm{Ext}^p(\mathbb{Z}, F) \to \mathrm{Ext}^p(\mathbb{Z}_U, F) \to \mathrm{Ext}^{p+1}(\mathbb{Z}_Z, F) \to \cdots$

since $\mathrm{Ext}^p(\mathbb{Z}, F) = H^p(X, F)$

and $\mathrm{Hom}_{S(X)}(\mathbb{Z}_U, F) \approx \mathrm{Hom}_{S(U)}(\mathbb{Z}, j^* F) = \Gamma(U, F|_U)$

$\Rightarrow \mathrm{Ext}^p(\mathbb{Z}_U, F) \approx H^p(U, F|_U)$ (j^* is exact and persenves injectives)

and $\mathrm{Hom}_{S(X)}(\mathbb{Z}_Z, F) \approx \mathrm{Hom}_{S(Z)}(\mathbb{Z}, i^! F) \approx H^0_Z(X, F)$

$\Rightarrow \mathrm{Ext}^p(\mathbb{Z}_Z, F) \approx H^p_Z(X, F)$

we get the desired exact sequence.

Prop: (Excision) Let $Z \subset X$ and $Z' \subset X'$ be closed subschemes, and let $\pi: X' \to X$ be an etale morphism s.t. $\pi|_{Z'}: Z' \xrightarrow{\approx} Z$ and $\pi(X - Z') \subset X - Z$. Then $H^p_Z(X, F) \approx H^p_{Z'}(X', \pi^* F)$ for all $p \geq 0$.

Pf: π^* is exact and preserves injectives. It suffices to prove for $p=0$.

Chase diagram:

$$0 \to H^0_z(X,F) \to \Gamma(X,F) \to \Gamma(U,F)$$

$$0 \to H^0_z(X',F|X') \to \Gamma(X',F) \to \Gamma(U',F)$$

Coro. Let z be a closed point of X. Then $H^p_z(X,F) \simeq H_z^p(\mathrm{Spec}\,\mathcal{O}^h_{X,z}, F)$ for any sheaf on X_{et}.

Pf: $H^p_z(X,F) = H^p_z(Y,F)$ for any étale neighborhood.

Take limit, then $H_z^p(X,F) = \varinjlim H^p_y(Y,F) = H^p_z(\mathrm{Spec}\,\mathcal{O}^h_{X,z}, F)$

Cohomology groups with compact support:

$$\Gamma_c(X,F) = \bigcup \ker(\Gamma(X,F) \to \Gamma(X-Z,F)) \text{ is left exact}$$

Z takes over complete subvarieties.

It is tempting to define cohomology groups as derived functors, but it's uninteresting, e.g. X is affine, then $\Gamma_c(X,F) = \bigoplus H^0_x(X,F)$ and $R^p\Gamma_c(F) = \bigoplus_x H^p_x(X,F)$.

Instead, assume $j: X \overset{\text{open immersion}}{\hookrightarrow} \bar{X}$, where $\bar{X}$ is complete, we define

$$H_c^p(X,F) = H^p(\bar{X}, j_! X)$$

(Recall $j_!$ is exact but doesn't preserve injectives).

Then (a). $H^0_c(X,F) = \Gamma_c(X,F)$

(b). $H_c^p(X,-)$ form a δ-functor

(c). $\forall$ complete subvariety Z of X, there is a canonical morphism of δ-functors $H^p_Z(X,-) \to H^p_c(X,-)$

Remark: if $X = X_0 \supset X_1 \supset \cdots \supset X_r \neq \phi$ is a sequence of closed subschemes of X, there is a spectral sequence,

$$E_1^{q,p} = H_c^{p+q}(X_p - X_{p+1}, F) \Rightarrow H_c^{p+q}(X, F).$$

Čech Cohomology

Let $\mathcal{U} = \{ U_i \xrightarrow{\varphi_i} X \}_{i\in I}$ be a covering of X.

$\forall (i_0, \cdots, i_p) \in I^{p+1}$, let $U_{i_0\cdots i_p} = U_{i_0} \times_X \cdots \times_X U_{i_p}$.

Denote $P(U_{i_0\cdots \hat{i}_j \cdots i_p}) \to P(U_{i_0\cdots i_p})$ by res_j.

Čech complex: $C^p(\mathcal{U}, P) = \prod_{I^{p+1}} P(U_{i_0\cdots i_p})$, $d: C^p \to C^{p+1}$

$$(d^p s)_{i_0\cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \mathrm{res}_j (s_{i_0\cdots \hat{i}_j \cdots i_{p+1}}).$$

Cohomology groups $\check{H}^p(\mathcal{U}, P)$.

Notice $\check{H}^0(\mathcal{U}, P) = \ker (\prod P(U_i) \to \prod P(U_{ij}))$,

then $P(X) \to \check{H}^0(\mathcal{U}, P)$ is an iso. when P is a sheaf.

Properties: $\forall$ refinement $\mathcal{V}$ of $\mathcal{U}$, there is a map

$\rho(\mathcal{V}, \mathcal{U}) : \check{H}^p(\mathcal{U}, P) \to \check{H}^p(\mathcal{V}, P)$

depends only on the coverings $\mathcal{V}$ and $\mathcal{U}$.

Also, if $\mathcal{W}$ is a refinement of $\mathcal{V}$, then

$$\rho(\mathcal{V}, \mathcal{U})\, \rho(\mathcal{W}, \mathcal{V}) = \rho(\mathcal{W}, \mathcal{U}).$$

One can define Čech cohomology groups

$$\check{H}^p(X_E, P) = \varinjlim \check{H}^p(\mathcal{U}, P)$$

Also, we can define $\check{H}^p(U, P) = \varinjlim \check{H}^p(\mathcal{U}/U, P)$

Then we get a presheaf: $\underline{\check{H}}^p(P): U \mapsto \check{H}^p(U, P)$

It should be noticed that Čech cohomology on a site cannot be computed using alternating cochains unless $U_i \to X$ are universal monomorphisms.

Eg. When $X = \operatorname{spec} A$, and a cover $\operatorname{spec} B \to \operatorname{spec} A$, the Čech complex is $0 \to B^* \to (B \otimes_A B)^* \to (B \otimes_A B \otimes_A B)^* \to \cdots$

Prop: For any presheaves $0 \to P' \to P \to P'' \to 0$, we have long exact sequences:

$$0 \to \check{H}^0(\mathcal{U}, P') \to \cdots \to \check{H}^p(\mathcal{U}, P) \to \check{H}^p(\mathcal{U}, P'') \to \check{H}^{p+1}(\mathcal{U}, P') \to \cdots$$

and

$$0 \to \check{H}^0(U, P') \to \cdots \to \check{H}^p(U, P) \to \check{H}^p(U, P'') \to \check{H}^{p+1}(U, P') \to \cdots$$

Also, $\check{H}^p(\mathcal{U}/U, -)$ are derived functors of $\check{H}^0(\mathcal{U}/U, -): P(X_E) \to Ab$ in $P(X)$

Lemma. $\check{H}^p(\mathcal{U}/U, P) = 0$ for $p > 0$, P injective. (Turn $P(U)$ into $\mathrm{Hom}(\mathbb{Z}, P)$)

Coro. Čech cohomology agrees with derived functor cohomology $H^p(X, -)$ on sheaves if to every SES of sheaves, there is a functorially associated LES of Čech cohomology groups.

Eg. This is true when for any surjection $F \to F''$ of sheaves, the map $\varinjlim (\prod F(U_{i_0 \cdots i_p})) \to \prod F''(U_{i_0 \cdots i_p})$ is surjective.

Pf: Necessity is obvious.

Sufficiency: Facts: ① Injectives in $Sh(X)$ are injective in $P(X)$.

② $H^0(X, F) = \check{H}^0(X, F)$

If for k, $H^p(X, F) = \check{H}^p(X, F)$, $\forall$ sheaf F, $p \le k$.

Then for F, $0 \to F \to I \to G \to 0$

$$\Rightarrow \quad H^{p+1}(X, F) \cong H^p(X, G) \cong \check{H}^p(X, G) \cong \check{H}^{p+1}(X, F).$$

Prop: Let $U \to X$ in C/X, $\mathcal{U}$ a covering, and F a sheaf on X_E. There are spectral sequences:

$$\check{H}^p(\mathcal{U}/U, \underline{H}^q(F)) \Rightarrow H^n(U, F)$$
$$\check{H}^p(U, \underline{H}^q(F)) \Rightarrow H^n(U, F)$$

Coro. There is a spectral sequence:

$$\check{H}^p(X, \underline{H}^q(F)) \Rightarrow H^n(X,F)$$

Prop. $\check{H}^0(X, \underline{H}^q(F)) = 0$ for $q>0$, i.e. $\check{H}^0(\mathcal{U}, \underline{H}^q(F)) = 0$ for $q>0$, $\forall \mathcal{U}$.

i.e. 0 column in the first page is zero.

Pf: $F \to I^\cdot$ injective resolution of F

$\underline{H}^q(F) = H^q(i(I^\cdot))$

$\Rightarrow a(\underline{H}^q(F)) = H^q(a\, i(I^\cdot)) = 0.$

Since $\check{H}^0(\underline{H}^q(F))$ is actually a kernel of $a\underline{H}^q(F)$ to

Remark: Can be seen as $\forall s \in H^q(U,F)$, $q>0$, $\exists \mathcal{U}$, $U_i \to U$, s.t. $s = 0$ in $H^q(U_i, F)$, $\forall i$.

Coro. $\check{H}^0(U,F) \cong H^0(U,F)$ and $0 \to \check{H}^2(U,F) \to H^2(U,F) \to \check{H}^1(U, \underline{H}^1(F)) \to \check{H}^3(U,F) \to H^3(U,F)$.

$\check{H}^1(U,F) \cong H^1(U,F)$.

Pf: We have spectral sequence $\check{H}^p(U, \underline{H}^q(F)) \Rightarrow H^n(U,F)$

Lemma. Let E be biregular. If $E_2^{pq}=0$ when $p<0$ or $q<0$ or $0<q<n$, then $E_2^{i,0} \cong H^i$ for all $i<n$ and

$$0 \to E_2^{n,0} \to H^n \to E_2^{0,n} \to E_2^{n+1,0} \to H^{n+1}$$

In particular, if $E_2^{pq}=0$ when $p<0$ or $q<0$, then $E_2^{0,0} \cong H^0$ and $0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2$.

Prop: F a sheaf on X_E. TFAE:

(a). F is flabby.

(b). $\check{H}^q(\mathcal{U}/U, F) = 0$, $q>0$ for any $U \to X$, and covering $\mathcal{U}$ over U.

(c). $\check{H}^q(U,F) = 0$, $q>0$, for any $U \to X$

Pf. (a)⇒(b) F flabby $\Rightarrow \underline{H}^q(F) = 0$

Consider $\check{H}^p(\mathcal{U}/U, \underline{H}^q(F)) \Rightarrow H^n(U,F)$

When $q>0$, $\underline{H}^q(F)=0$, $\Rightarrow E_2^{pq} \neq 0$ only when $q=0$.

Thus for $n \in \mathbb{Z}$, $E_2^{n,0} \cong H^n$. $\Rightarrow$ $\check{H}^p(U/U, F) \cong \underline{H}^n(F) = 0$.

(b) $\Rightarrow$ (c) Pass to direct limit.

(c) $\Rightarrow$ (a). Consider $\check{H}^r(U, \underline{H}^s(F)) \Rightarrow H^n(U, F)$.

Since $\check{H}^0(U, F) \xrightarrow{\sim} H^0(U, F)$. we have $H^0(U, F) = 0$.

Also. $H^1(U, F) \xleftarrow{\sim} \check{H}^1(U, F)$ is zero.

Consider $0 \to \check{H}^2(U, F) \to H^2(U, F) \to \check{H}^1(U, \underline{H}^1(F)) \to \cdots$

We have $\underline{H}^1(F) = 0$ since $H^1(U, F) = 0$ for all U.

Thus $H^2(U, F) = \check{H}^2(U, F) = 0$ for all U.

By induction, $H^n(U, F) = \check{H}^n(U, F) = 0$ for all U and n.

Thus F is flabby.

Coro. (a). If F is flabby, so is $F|U$ for any $U \to X$.

(b). $\pi: X_E' \to X_E$, F' on $Sh(X_E')$ flabby, then $\pi_* F'$ is flabby.

(c). If F is injective, then $\underline{Hom}(F_0, F)$ is flabby for any sheaf F_0.

Pf: (a). ✓.

(b). For any $U \to X$, take a covering $(U_i \to U)$.

Then $U_i' \to U'$ is also a covering, where $U_i' = U \times_X X'$.

And $(U'_{i_0 \cdots i_p}) = (U_{i_0 \cdots i_p})'$.

Then the two complexes: $C^\cdot(U', F')$ and $C^\cdot(U, \pi_* F')$ are isomorphic.

Since $(\pi_* F')(U_{i_0 \cdots i_p}) = F'((U_{i_0 \cdots i_p})')$.

Note that $\pi_* = \pi_p$.

(c). key: $C^\cdot(U, \underline{Hom}(F_0, F)) \xleftarrow{\sim} Hom(F_0, \underline{Hom}(\mathbb{Z}, F^\cdot))$.

$\downarrow$ exact complexes of injective sheaves.