LECTURE 1

The goal of this course is to give a non-technical introduction to the theory of ∞-categories, or in general, homotopy coherent mathematics. This course focuses on ideas and motivations, and hopefully serves as a guide to the foundational references in this field, including [Lur09] and [Lur12].

1. A slogan

Slogan 1.1.

$$\infty\textit{-category theory} = \textit{category theory} + \textit{homotopy theory}.$$

2. Classical category theory

2.1. Categories were introduced by Eilenberg and Maclane in 1945 among their works on algebraic topology and *homological* algebra.

Definition 2.2. *A **category** $\mathcal{C}$ consists of the following data:*

- *A* class $\mathsf{Ob}(\mathcal{C})$*, whose elements are called **objects**.*
- *For any two objects a and b, a* class $\mathsf{Hom}(a,b)$*, whose elements are called **morphisms** from a to b, denoted by $f : a \to b$.*
- *A binary operation $\circ$, called **composition of morphisms**, such that for any three objects a, b and c, we have a map*

$$\circ : \mathsf{Hom}(a,b) \times \mathsf{Hom}(b,c) \to \mathsf{Hom}(a,c).$$

The above data should satisfy two axioms:

- *Associativity: for morphisms $f : a \to b$, $g : b \to c$ and $h : c \to d$,*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

- *Identity: for any object x, there exists a morphism $\mathsf{id}_x : x \to x$, called the **identity morphism** for x, such that for any morphism $f : a \to b$, we have*

$$\mathsf{id}_b \circ f = f = f \circ \mathsf{id}_a.$$

2.3. The power of category theory is reflected in the following principle:

> *In order to study a collection of objects one should also consider suitably defined morphisms between such objects.*

People assemble their favorite mathematical entities, which are often *structured sets* into a category, and declare the morphisms to be functions that preserve these structures. Examples include Set, Grp, Ring, Top...

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2.4. We want to highlight the following doctrines in (classical) category theory:

(1) Morphisms are *discrete*: for two morphisms f and g, one can say $f = g$ or $f \neq g$, and this is the only comparison that one can make.
(2) Associativity is *strict*: $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are required to be equal, rather than equivalent in a weaker sense.
(3) Composition is *concrete*: there is no ambiguity for $g \circ f$.

In this course, we will abandon all of them.

3. Towards higher categories

3.1. We will abandon Doctrine (1) and endow $\mathsf{Hom}(a, b)$ with richer structures: they can be topological spaces or even categories themselves. The latter defines **strict 2-categories**, which have objects and morphisms, as well as morphisms between morphisms, known as **2-morphisms**. There are two types of compositions of 2-morphisms: the **vertical** one and the **horizontal** one.

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5, "\Downarrow"] \arrow[r, phantom, shift right=2.5, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

These compositions should satisfy a list of axioms of associativity and identity, which are all described via *equalities*.

By induction, one obtains the notion of strict n-categories. In the language of classical category theory, we have:

Definition 3.2. *A **strict** n**-category** is a category enriched in strict* $(n-1)$*-categories.*

3.3. However, there are very few interesting examples of strict n-categories:

Example 3.4. *A strict* 2*-category with a single object* $*$ *amounts to the data of a category* $\mathcal{C} := \mathsf{Hom}(*, *)$ *equipped with a multiplication functor* $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *which is* strictly *associative and unital. This definition is* evil[1] *and impractical*[2]*. The correct notion is that of a **monoidal category**, where instead we* supply *natural isomorphisms*

$$X \otimes (Y \otimes Z) \xrightarrow{\simeq} (X \otimes Y) \otimes Z,\ \mathbb{1} \otimes X \xrightarrow{\simeq} X \xleftarrow{\simeq} X \otimes \mathbb{1}$$

subject to certain coherent conditions.

3.5. The above example suggests we should also abandon Doctrine (2) and allow associativity to hold in a weaker sense. This leads to the concept, but not a definition, of **weak n-categories** or even **weak ω-categories** when $n = \infty$.

We have at least the following wishes in a definition of weak n-categories:

- Weak 1-categories are just categories.
- Weak 2-category with a single object amounts to the data of a monoidal category.
- For any two objects a and b in a weak n-category, $\mathsf{Hom}(a, b)$ should be a weak $(n-1)$-category.
- Its definition should satisfy the *principle of isomorphism.*

[1]Principle of isomorphism: *all grammatically correct properties of objects of a fixed category are to be invariant under isomorphism.*

[2]Even for sets, $(X \times Y) \times Z = X \times (Y \times Z)$ does not make sense in ZF.

Combining the last two wishes, we obtain:

> *For $n \geq 2$, we should never require two morphisms in a weak n-category to be equal.*

Then by induction,

> *For $n > k$, we should never require two k-morphisms in a weak n-category to be equal.*

These innocuous wishes would lead to a combinatorial nightmare.

3.6. Let f, g and h be composable morphisms in a weak n-category. By previous discussion, in the axioms of associativities, the compositions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ should be *equivalent* rather than *equal*.

However, as suggested by the definition of monoidal categories, this equivalence must be viewed as a *structure* rather than a *property*: we need to *supply* an invertible 2-morphism from $h \circ (g \circ f)$ to $(h \circ g) \circ f$, called the **associator**, such that the following diagram commutes:

$$\begin{tikzcd}
 & & (kh)(gf) \arrow[drr] & & \\
k(h(gf)) \arrow[urr] \arrow[dr] & & & & ((kh)g)f \\
 & k((hg)f) \arrow[rr] & & (k(hg))f. \arrow[ur] &
\end{tikzcd}$$

However, when $n \geq 3$, even this commutativity of 2-morphisms should be understood as equivalence rather than equality, and should be *witnessed* by an invertible 3-morphism, say, from the clockwise arch to the counterclockwise one. These 3-morphisms themselves should make a certain diagram commute, which is again witnessed by an invertible 4-morphisms if $n \geq 4$...

Even worse, we also need to treat axioms of associativity for composition of higher morphisms, and there are various types of them. Reminder: 2-morphisms can be composed both **vertically** and **horizontally**, and the latter interacts with composition of 1-morphisms.

This endless list of associativity, known as the **coherence data**, soon become impossible to write down and difficult to work with.

3.7. This painful pursuit of defining weak n-categories combinatorially was started by Bénabou in 1967 and probably terminated around early 2000s.

3.8. What saves higher-categorists (and this course) is the following **homotopy hypothesis** proposed in Grothendieck's *pursuing stacks*, written around 1983:

Slogan 3.9. *The theory of ∞-groupoids should be the same as the homotopy theory of spaces.*

3.10. Here **∞-groupoids** mean weak ω-categories whose morphisms and higher morphisms are all invertible. In general, **(n,k)-categories** mean weak n-categories whose m-morphisms are invertible for $m > k$. When $n = \infty$, weak ∞-categories in above mean weak ω-categories[3].

The insight is: the coherence data for associativity, which is combinatorially formidable, can be *hidden away* in the homotopy theory of spaces.

This gives one approach to develop the theory of higher categories: we start with *declaring* ∞-groupoids, or $(\infty,0)$-categories, to be homotopy types of spaces, and inductively define (∞,k)-categories. In this induction step, the coherence data for associativity, which is about *invertible* higher morphisms, has already been tamed by the theory of ∞-groupoids.

3.11. One may ask: if the theory of $(\infty,0)$-categories is the same as homotopy theory of topological spaces, should the theory of $(\infty,1)$-categories be the same as homotopy theory of **topological categories**, i.e., categories enriched in topological spaces?

The answer is: yes, but we have to first understand the meaning of the latter. This requires Quillen's *abstract homotopy theory*, known as *model categories*, which will be the content of the next lecture.

Nevertheless, topological categories are not the most convenient model of $(\infty,1)$-categories, at least in certain interesting applications of the latter. The current most developed model, thanks to Lurie's books, is **quasi-categories**, wihch, as we alluded, abandon Doctrine (3).

4. Homotopy hypothesis

4.1. Homotopy theory dates back to the works of Poincaré on fundamental groups starting from 1895.

Definition 4.2. *Let X, Y be topological spaces, and f, $g : X \to Y$ be continuous functions. A* ***homotopy*** *between f and g is a continuous function $H : X \times [0,1] \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. We say f and g are* ***homotopic*** *if there exists a homotopy between them.*

Let X, Y be topological spaces. A ***homotopy equivalence*** *between X and Y is a pair of continuous functions $p : X \to Y$ and $q : Y \to X$ such that $q \circ p$ is homotopic to id_X and $p \circ q$ is homotopic to id_Y.*

4.3. Classical homotopy theory focuses on information about topological spaces that are invariant under homotopy equivalence, such as their homotopy groups. Such information can be encoded into a category.

Construction 4.4. *The* ***homotopy category*** hTop *is defined as follows:*

- *Objects are nice topological spaces, such as CW complexes.*
- *Morphisms are homotopy classes of continuous maps, with composition of morphisms induced by composition of continuous maps.*

Variant 4.5. *Alternatively, we can consider* pointed *spaces, i.e. spaces equipped with a base point, we obtain a category denoted by* hTop_*.

[3] $(\infty,1)$-categories are often just called ∞-categories.

4.6. By design, homotopy equivalences are exactly equivalences in hTop. In fact, hTop can be obtained from the category of *nice* topological spaces by inverting homotopy equivalences. Alternatively, it can be obtained from Top, the category of *all* topological spaces, by inverting *weak* homotopy equivalences[4].

Definition 4.7. *A* ***homotopy type*** *is an object X in hTop. We say X is a* ***homotopy n-type*** *if for any base point, $\pi_k(X,x) \simeq 0$ for $k > n$.*

Let $\mathsf{hTop}_{\leq n} \subset \mathsf{hTop}$ be the full subcategory of homotopy n-types.

4.8. It turns out the information of $\pi_0(X)$ and $\pi_1(X)$ can be captured by a category associated to X.

Construction 4.9. *Let X be a topological space. The* ***fundamental groupoid*** *$\pi_{\leq 1}X$ of X is a category defined as follows:*

- *Objects are points of X;*
- *Morphisms are* homotopy classes *of pathes in X, with induced by concatenation of intervals.*

Exercise 4.10. *Check the axiom of associativity for the above construction.*

4.11. It is easy to see:

- The fundamental groupoid $\pi_{\leq 1}X$ is indeed a groupoid, i.e., all the morphisms are invertible.
- The set $\pi_0(X)$ can be identified with the set of isomorphism classes of objects in $\pi_{\leq 1}X$.
- There is a natural group homomorphism $\pi_1(X,x) \simeq \mathsf{Hom}(x,x)$, where x in the RHS is viewed as an object in $\pi_{\leq 1}X$.

Exercise 4.12. *Show that $\pi_{\leq 1}$ defines an equivalence from $\mathsf{hTop}_{\leq 1}$ to the category hGrpd of small groupoids, where morphisms are given by* equivalence classes *of functors. Hint: Eilenberg–MacLane spaces.*

4.13. Encouraged by the above, one may attempt to construct a 2-category $\pi_{\leq 2}X$ as follows:

- Objects are points of X;
- Morphisms are pathes in X, with composition induced by concatenation of intervals.
- 2-morphisms are homotopy classes of homotopies between pathes in X, with composition induced by concatenation of squares.

If this definition is possible, note that

- All the morphisms and 2-morphisms are invertible.
- The set $\pi_0(X)$ can be identified with the set of isomorphism classes of objects in $\pi_{\leq 2}X$.
- The group $\pi_1(X,x)$ can be identified with the set of isomorphism classes of objects in $\mathsf{Hom}(x,x)$.
- There is a natural group homomorphism $\pi_2(X,x) \simeq \mathsf{Hom}(\mathsf{id}_x, \mathsf{id}_x)$, where id_x in the RHS is viewed as an object in $\mathsf{Hom}(x,x)$.

[4]A continuous map is a weak homotopy equivalence if it induces isomorphisms between π_k's. By a theorem of Whitehead, weak homotopy equivalences between CW complexes are homotopy equivalences.

Exercise 4.14. *Describe the action of $\pi_1(X,x)$ on $\pi_2(X,x)$ in terms of $\pi_{\leq 2}X$.*

Exercise 4.15. *In the above definition of $\pi_{\leq 2}X$, can we still define morphisms as* homotopy classes *of pathes in X? Convince yourself that then 2-morphisms will not be well-defined.*

If you get stuck, try the following: in the definition of $\pi_{\leq 1}X$, can we define objects as homotopy classes *of points, a.k.a. connected components of X?*

4.16. Note that $\pi_{\leq 2}X$ cannot be strict: for composable pathes f, g, $h : [0,1] \to X$, the compositions

$$\big(h \circ (g \circ f)\big)(t) = \begin{cases} f(4t) & \text{for} \quad t \in [0, 1/4] \\ g(4t-1) & \text{for} \quad t \in [1/4, 1/2] \\ h(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases} \tag{4.1}$$

$$\big((h \circ g) \circ f\big)(t) = \begin{cases} f(2t) & \text{for} \quad t \in [0, 1/2] \\ g(4t-2) & \text{for} \quad t \in [1/2, 3/4] \\ h(4t-3) & \text{for} \quad t \in [3/4, 1] \end{cases} \tag{4.2}$$

are homotopic but not equal. Moreover, it is hard to find a *natural* homotopy between them, although all such homotopies are homotopic to each other, as long as their constructions work for any X.

The last statement provides the coherent data for associativity in $\pi_{\leq 2}X$. One can check that $\pi_{\leq 2}X$ is indeed a weak 2-category.

Challenge 4.17. *Show that $\pi_{\leq 2}$ defines an equivalence from $\mathsf{hTop}_{\leq 2}$ to the category hGrpd_2 of small weak 2-groupoids, where morphisms are given by* equivalence classes *of functors.*

4.18. For $n \geq 3$, to construct $\pi_{\leq n}X$, according to Exercise 4.15, 2-morphism would be homotopies between pathes rather than the homotopy classes of such homotopies. Hence we have to make a *choice* of homotopy from (4.1) to (4.2), as long as the theory of weak n-categories is developed combinatorially. Such choice is unnatural, and we have to fit them into the coherent data of associativity in the definition of weak n-categories. As n in $\pi_{\leq n}X$ grows, such **homotopy coherent data** become formidable.

4.19. Heuristically, we have two impossible tasks:

- To give a combinatorial definition of (weak) n-groupoids;
- To verify, or rather, provide homotopy coherent data to make $\pi_{\leq n}X$ a weak n-groupoid.

Grothendieck's homotopy hypothesis says these are actually the same task, and we should do neither.

Slogan 4.20.

$$n\text{-}\textit{groupoids} \;=\; \textit{homotopy } n\text{-}\textit{types};$$

$$\infty\text{-}\textit{groupoids} \;=\; \textit{homotopy types.}$$

4.21. This reunion of category theory and homotopy theory, which can even date back to Kan's work in the 1950s, is the guiding philosophy of this course.

Appendix A. More on strict vs. weak

Exercise A.1. *Prove any weak* 2*-category is equivalent to a strict* 2*-category.*

Exercise A.2. *Show that:*

(1) *Knowing a weak* 3*-category* $\mathcal{C}$ *with a single object and a single morphism is the same as knowing a braided monoidal category* $\mathcal{D}$*.*
(2) *In above,* $\mathcal{C}$ *is equivalent to a strict* 3*-category iff* $\mathcal{D}$ *is symmetric monoidal.*

Exercise A.3. *Prove* $\pi_{\leq 3}S^2$ *is not equivalent to a strict* 3*-groupoid.*

References

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.
[Lur12] Jacob Lurie. Higher algebra, 2012.