LECTURE 5

In this lecture, we define the quasi-category $\mathcal{QC}at$ of small quasi-categories, which models the ∞-category Cat_∞ of small ∞-categories.

1. Idea of the construction

1.1. Last time, for quasi-categories $\mathcal{C}$ and $\mathcal{D}$, we constructed the quasi-category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ of functors between them. Let $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$ be the core Kan complex of this quasi-category, which can be viewed as obtained from $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ by discarding all non-invertible natural transformations.

1.2. Using these Kan complexes, we can enrich the *ordinary* cateogry QCat over Set_Δ, and obtain a simplicial (enriched) category $\mathbb{QC}\mathbb{at}$. The homotopy hypothesis suggests $\mathbb{QC}\mathbb{at}$ is already a model for Cat_∞.

1.3. To translate this into rigorous mathematics, we will construct a Quillen equivalence between model categories

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

between the Joyal model category of simplicial sets and the standard model category of small simplicial categories. The simplicial category $\mathbb{QC}\mathbb{at}$ is a fibrant object in[1] Cat_Δ because it is enriched over fibrant objects in Set_Δ. The Joyal model structure is designed such that (bi)fibrant objects are exactly quasi-categories. Therefore

$$\mathcal{QC}at \stackrel{\text{def}}{=} \mathfrak{N}_\bullet(\mathbb{QC}\mathbb{at})$$

is a fibrant object in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, i.e., a quasi-category.

2. Homotopy category of quasi-categories

2.1. The *homotopy* category of the desired quasi-category $\mathcal{QC}at$, in the sense of [Lecture 4, Definition 4.2], can be easily defined. It will also be the homotopy category of the desired model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, in the sense of [Lecture 2, Definition 2.19]. It models the homotopy category hCat_∞ of the desired ∞-category Cat_∞.

Definition 2.2. *Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be functors between quasi-categories. We call a morphism $\alpha : F_1 \to F_2$ in the quasi-category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ a* ***natural transformation*** *from F_1 to F_2.*

We say α is ***invertible****, or is an* ***equivalence****, if it is an isomorphism in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.*

Date: Sep. 27, 2024.

[1]Strictly speaking, $\mathbb{QC}\mathbb{at}$ is not an object in Cat_Δ because it is not small. Hence we need to replace the *large* model category Cat_Δ of *small* simplicial categories by the *very large* model category $\widetilde{\mathsf{Cat}}_\Delta$ of *large* simplicial categories, and similarly replace $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ by the *very large* Joyal model category $\widetilde{\mathsf{Set}}_\Delta^{\mathsf{Joyal}}$ of large simplicial sets. Here one can use *Grothendieck universes* to give precise meanings to the above size conditions. We will ignore these set-theoretic issues until we encounter really problems about them.

Theorem 2.3 (Ker.01DK)**.** *Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be functors between quasi-categories. A natural transformation $\alpha : F_1 \to F_2$ is invertible iff its value at any object $x \in \mathcal{C}$ is an isomorphism, i.e. $\alpha(x) : F_1(x) \xrightarrow{\simeq} F_2(x)$.*

Exercise 2.4. *Show that the following construction defines an ordinary category $\mathsf{h}\mathcal{QC}at$:*

- *Objects of $\mathsf{h}\mathcal{QC}at$ are small quasi-categories;*
- *Morphisms $\mathsf{h}\mathcal{QC}at$ are equivalence classes of functors between quasi-categories:*
$$\mathsf{Hom}_{\mathsf{h}\mathcal{QC}at}(\mathcal{C},\mathcal{D}) \overset{\text{def}}{=} \pi_0(\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq})$$
- *Composition is given by $[G] \circ [F] \overset{\text{def}}{=} [G \circ F]$.*

3. Equivalences between quasi-categories

3.1. Note that we have a functor $\mathsf{QCat} \to \mathsf{h}\mathcal{QC}at$. Similar to [Lecture 4, Definition 5.1], we make the following definition.

Definition 3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories, i.e. a morphism in QCat. We say F is an **equivalence** if its image $[F]$ in $\mathsf{h}\mathcal{QC}at$ is an isomorphism.*

Remark 3.3. *In other words, $F : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there exists a functor $G : \mathcal{D} \to \mathcal{X}$ such that $G \circ F$ is equivalent to $\mathsf{Id}_{\mathcal{C}}$ and $F \circ G$ is equivalent to $\mathsf{Id}_{\mathcal{D}}$.*

Theorem 3.4 (Ker.01JX)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories. Then F is an equivalence iff*

1. *F is **fully faithful**: for any objects $x, y \in \mathcal{C}$, the functor $\mathsf{Maps}_{\mathcal{C}}(x,y) \to \mathsf{Maps}_{\mathcal{D}}(Fx, Fy)$ is an equivalence between ∞-groupoids.*
2. *F is **essentially surjective**: the map $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ is surjective. In other words, for any object $d \in \mathcal{D}$, there exist $c \in \mathcal{C}$ and an isomorphism $F(c) \xrightarrow{\simeq} d$.*

Exercise 3.5. *Being essentially surjective can be checked on the level of homotopy categories. Namely, $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective iff $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$ is so.*

Exercise 3.6. *Being fully faithful* cannot *be checked on the level of homotopy categories.*

Remark 3.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories. So far we have at least four equivalence-like conditions on F:*

(a) *F is an* isomorphism *in Set_Δ.*
(b) *F is an* equivalence *between quasi-categories.*
(c) *F is a* weak homotopy equivalence *in the Kan–Quillen model category $\mathsf{Set}_\Delta^{\mathsf{KQ}}$.*
(d) *F induces an* equivalence between ordinary categorie*: $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$.*

The relations between these notions are

$$\begin{tikzcd} (a) \arrow[r, Rightarrow] & (b) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (c) \\ & (d). & \end{tikzcd}$$

Exercise 3.8. *The above implications are not inverible.*

1. $\mathsf{N}_\bullet(\{\bullet \rightleftarrows \bullet\}) \to \Delta^0$ *satisfies (b) but not (a).*
2. $\Delta^1 \to \Delta^0$ *satisfies (c) but not (b) or (d).*
3. $\mathsf{Sing}(S^2) \to \Delta^0$ *satisfies (d) but not (b) or (c).*

Remark 3.9. *Let* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *be the functor between* ∞*-categories modeled by* $F : \mathcal{C} \to \mathcal{D}$*. The above conditions tranlate into:*

(a) *This condition does not make sense because it is evil.*
(b) F *is an* equivalence *between* ∞*-categories.*
(c) F becomes an equivalence beween ∞-groupoids *after formally inverting all morphisms in* C *and* D*.*
(d) F becomes an equivalence between ordinary categories *after formally forcing all non-invertible higher morphisms in* C *and* D *to be identities.*

Remark 3.10. *Let* C *be an* ∞*-category. We denote* $\mathsf{C}^\sharp$ *the* ∞*-groupoid obtained from* C *by formally inverting all morphisms. Note that this is* not *the core* ∞*-groupoid* $\mathsf{C}^\simeq$*, which is obtained by* discarding *all non-invertible morphisms (see [Lecture 4, Exercise 5.3]). We have*

$$\mathsf{C}^\simeq \to \mathsf{C} \to \mathsf{C}^\sharp.$$

4. Joyal model structure

4.1. Morphisms in the class (W) of $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ will be called *categorical equivalences*[2] between simplicial sets. When restricted to bifibrant objects, i.e. quasi-categories, they are supposed to model equivalences between the underlying ∞-categories. In other words:

> *Categorical equivalences between quasi-categories should be equivalences, in the sense of Definition 3.2.*

Hence the main task is to define categorical equivalences between *general* simplicial sets.

4.2. Recall when defining the classical model structure on Set_Δ (see [Lecture 3, Theorem-Definition 6.2]), we declare weak homotopy equivalences between simplicial sets to be maps $f : X \to Y$ such that $|f| : |X| \to |Y|$ are weak homotopy equivalences in Top.

It is possible to define categorical equivalences in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ in a similar manner. Namely, we can first define weak equivalences in Cat_Δ and then transfer them via the functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$[3]. However, I find the construction of the functor $\mathfrak{C}$ not intuitive enough to provide actual feelings about categorical equivalences. Therefore we will follow Joyal's definition.

---

[2]Joyal called them *weak categorical equivalences*. We follow Lurie's terminology.

[3]This was the approach adopted in Lurie's HTT.2.2.5, which differs from that in Joyal's works (see [Joy08]). See HTT.2.2.5.9 and HTT.2.2.5.10 for more information.

4.3. Joyal's definition is motivated by the following observation:

Exercise 4.4. *Let $\mathcal{C}$ be a model category such that any object is cofibrant. Then a morphism $f : X \to Y$ belongs to (W) iff for any fibrant object Z, the map*

$$(\mathsf{Hom}_{\mathcal{C}}(Y, Z)/\sim) \to (\mathsf{Hom}_{\mathcal{C}}(X, Z)/\sim)$$

is bijective. Here "$\sim$" stands for the homotopy relation between morphisms, see [Lecture 2, Proposition-Definition 2.18].

Hint: Yoneda lemma; the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ detects all weak equivalences.

Proposition-Definition 4.5. *Let $f : X \to Y$ be a morphism in Set_{Δ}. The following conditions are equivalent:*

(1) *For any quasi-category Z, the map*

$$\pi_0(\mathsf{Fun}(Y, Z)^{\simeq}) \to \pi_0(\mathsf{Fun}(X, Z)^{\simeq})$$

is a bijection between sets.

(2) *For any quasi-category Z, the functor*

$$\mathsf{Fun}(Y, Z)^{\simeq} \to \mathsf{Fun}(X, Z)^{\simeq}$$

is a weak equivalence between Kan complexes.

(3) *For any quasi-category Z, the functor*

$$\mathsf{Fun}(Y, Z) \to \mathsf{Fun}(X, Z)$$

is an equivalence between quasi-categories.

We say f is a ***categorical equivalence*** *if it satisfies the above conditions.*

Proposition 4.6 ([Joy08, Corollary 2.29])**.** *Any inner horn inclusion $\Lambda^n_i \to \Delta^n$, $0 < i < n$ is a categorical equivalence.*

Exercise 4.7. *Convince yourself the above claim does* not *follow immediately from the definition of quasi-categories.*

Exercise 4.8. *Categorical equivalences are weak homotopy equivalences. Hint: [Lecture 3, §6.5].*

Exercise 4.9 (Ker.01EG)**.** *Acyclic Kan fibrations are categorical equivalences.*

Theorem-Definition 4.10 ([Joy08, Theorem 6.12])**.** *There exists a model structure on Set_{Δ} given by*

- (W) *class of categorical equivalences;*
- (C) *class of monomorphisms;*
- (F) *class of morphisms satisfying the right lifting property against $(C \cap W)$.*

We call it the ***Joyal model structure*** *on Set_{Δ}. and denote this model category by $\mathsf{Set}_{\Delta}^{\mathsf{Joyal}}$.*

Fibrant objects in this model category are exactly quasi-categories.

4.11. The following result is tautological:

Exercise 4.12. *The functor* $\mathsf{QCat} \to \mathsf{Set}_\Delta$ *induces an equivalence*[4] $\mathsf{h}\mathcal{QC}at \to \mathsf{hSet}_\Delta^{\mathsf{Joyal}}$*. In particular,* $\mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}]$ *also models* hCat_∞*.*

Definition 4.13. *Fibrations in* $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ *are called* ***categorical fibrations****.*

Proposition 4.14. *The identity functor gives a Quillen adjunction*

$$\mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}.$$

Exercise 4.15. *Let* $\mathbb{L} : \mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}] \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}[W^{-1}] : \mathbb{R}$ *be the derived adjunction of the above Quillen adjunction. Intepret it as an adjunction* $\mathsf{hCat}_\infty \rightleftarrows \mathsf{hGrpd}_\infty$*.*

Remark 4.16. *By definition, categorical fibrations* $p : X \to Y$ *in* $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ *satisfy the right lifting properties against all inner horn inclusions. Morphisms satisfy these properties are called* ***inner fibrations****. Note that when* $Y = \Delta^0$*, inner fibrations over* Y *are categorical fibrations. However, the inclusion*

$$\textit{inner fibrations} \subset \textit{categorical fibrations}$$

is strict in general. For more information, see HTT.2.4.6.5.

5. Simplicial categories

5.1. In this section, we define a model structure on the category Cat_Δ of small simplicial categories.

5.2. Weak equivalences between simplicial categories are defined similarly as weak equivalences between topological categories. See [Lecture 2, §3].

Definition 5.3. *Let* $\mathcal{C}$ *be a simplicial category. Its* ***homotopy category*** $\pi_0\mathcal{C}$ *is defined by*

$$\mathsf{Ob}(\pi_0\mathcal{C}) := \mathsf{Ob}(\mathcal{C}),\ \mathsf{Hom}_{\pi_0\mathcal{C}}(x,y) := \pi_0(|\mathsf{Hom}_\mathcal{C}(x,y)|).$$

Definition 5.4. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between simplicial categories. We say* F *is a* ***weak equivalence*** *if:*

- *It induces an equivalence* $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$*.*
- *The morphism* $\mathsf{Hom}_\mathcal{C}(x,y) \to \mathsf{Hom}_\mathcal{D}(Fx,Fy)$ *is a weak equivalence in* $\mathsf{Set}_\Delta^{\mathsf{KQ}}$ *for any* $x,y \in \mathsf{Ob}(\mathcal{C})$*.*

Theorem 5.5 (HTT.A.3.2)**.** *There exists a canonical model structure on* Cat_Δ *such that:*

- *Weak equivalences are as in Defintion 5.4.*
- *Fibrant objects are exactly simplicial categories* $\mathcal{C}$ *such that* $\mathsf{Hom}_\mathcal{C}(-,-)$ *are Kan complexes.*

We call it the ***classical*** *or* ***Bergner model structure*** *on* Cat_Δ*.*

Remark 5.6. *It is not easy to describe fibrations and cofibrant objects in* Cat_Δ*, hence we will not do it. Note that [Lecture 2, Exercise 3.6] has a simplicial analogue, which suggests many naturally defined simplicial categories are not cofibrant.*

[4] In fact, it is an *isomorphism*

6. Quasi-categories and simplicial categories

6.1. In this section, we construct the desired Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet.$$

As in the construction of

$$|-| : \mathsf{Set}_\Delta^{\mathsf{KQ}} \rightleftarrows \mathsf{Top} : \mathsf{Sing},$$

we only need to find the correct definition for the restriction $\mathfrak{C}|_\Delta : \Delta \to \mathsf{Cat}_\Delta$ along the Yoneda embedding $\Delta \to \mathsf{Set}_\Delta \overset{\text{def}}{=} \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$. The entire functor $\mathfrak{C}$ will be the unique (up to unique equivalence) colimit-preserving functor that extends $\mathfrak{C}|_\Delta$, and the functor $\mathfrak{N}_\bullet$ will be its right adjoint.

6.2. In other words, we need to construct simplicial categories $\mathfrak{C}[\Delta^n] \in \mathsf{Cat}_\Delta$ equipped with face and degeneracy operators. To motivate this construction, we look at the following question:

Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. What is a functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$?

Since we *expect* the model category Cat_Δ to model Cat_∞, and since $\mathfrak{C}[\Delta^n]$ is cofibrant (because $\mathfrak{C}$ is expected to be a left Quillen functor), we obtain

A functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$ should model a functor $[n] \to \mathsf{C}$ between corresponding ∞-categories.

6.3. For instance, for $n = 2$, it should be an invertible 2-morphism witnessing the following commutative diagram in C:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z. \end{tikzcd}$$

Note that in the siplicial category $\mathcal{C}$, the composition is *concrete*: for 0-simplexes $f \in \mathsf{Hom}_\mathcal{C}(x,y)_0$ and $g \in \mathsf{Hom}_\mathcal{C}(y,z)_0$, we have a well-defined simplex $g \circ f \in \mathsf{Hom}_\mathcal{C}(x,z)_0$. Hence the above invertible 2-morpihsm corresponds to a 1-simplex $g \circ f \to h$[5] in $\mathsf{Hom}_\mathcal{C}(x,z)_1$.

Therefore we should define $\mathfrak{C}[\Delta^2]$ as

- It has three objects, labelled by $0, 1$ and 2.
- The morphism simplicial sets are

$$\mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(i,j) \overset{\text{def}}{=} \begin{cases} \varnothing & \text{if } i > j \\ \Delta^1 & \text{if } i = 0, j = 2 \\ \Delta^0 & \text{if otherwise} \end{cases}$$

- The only non-obvious composition is

$$\mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,1) \times \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(1,2) \to \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,2)$$
$$\Delta^0 \times \Delta^0 \to \Delta^1,$$

which is given by the 0-simplex $0 \in \Delta^1$.

[5] One may also use $h \to g \circ f$. Thanks to the fibrant assumption about $\mathsf{Hom}_\mathcal{C}(x,z)$, the actual data are the same. But the obtained functor $\mathfrak{C}$ will differ by the involution $\mathsf{Set}_\Delta \to \mathsf{Set}_\Delta, X \mapsto X^{\mathsf{op}}$. Here we follow the conventions in Kerodon (see Ker.00KN), which is the opposite to HTT (see HTT.1.1.5.1.).

6.4. The above research can be conducted for general n. Namely, for any ∞-category, we *know* the correct meaning of a functor $[n] \to \mathsf{C}$, thanks to the fact that $[n]$ comes from a linearly ordered set[6]. For instance, 0-simplex in $\mathsf{Hom}_{\mathfrak{C}[\Delta^n]}(0,n)$ should be a subset $I \subset [n]$ containing 0 and n; and each chain $I_0 \supset \cdots \supset I_m$ corresponds to a m-simplex.

In formula, we define:

Definition 6.5. *Let J be any linearly ordered set. We define a simplicial category $\mathfrak{C}[\Delta^J]$ as follows:*

- *Objects are elements in J;*
- $$\mathsf{Hom}_{\mathfrak{C}[\Delta^J]}(x,y) \overset{\text{def}}{=\!=} \begin{cases} \varnothing & \text{if } x > y \\ \mathsf{N}_\bullet(P_{x,y}) & \text{if } x \le y \end{cases}$$
  where $P_{x,y} \coloneqq \{I \subset [x,y] \mid x,y \in I\}$ is equipped with the partial order $\supset$, and $\mathsf{N}_\bullet(P_{x,y})$ is the nerve of the corresponding category.
- *Morphisms are induced by*
  $$(I \subset [x,y], I' \subset [y,z]) \mapsto (I \cup I' \subset [x,z])$$

6.6. The above construction is functorial in J and therefore we have a functor

$$\Delta \to \mathsf{Cat}_\Delta,\ [n] \mapsto \mathfrak{C}[\Delta^n] \overset{\text{def}}{=\!=} \mathfrak{C}[\Delta^{[n]}]$$

Note that $\mathfrak{C}[\Delta^n]$ is *not* fibrant. This can already be seen when $n = 2$.

Let

$$\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$$

be the essentially unique colimit-preserving functor that extends the above functor.

Exercise 6.7. *What is $\mathfrak{C}[\Lambda^2_1]$?*

Theorem-Definition 6.8 (HTT.2.2.5.1, HTT.2.2.5.8)**.** *The functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$ has a right adjoint, called the **simplicial nerve functor***

$$\mathfrak{N}_\bullet : \mathsf{Cat}_\Delta \to \mathsf{Set}_\Delta.$$

The adjoint pair

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

is a Quillen equivalence.

6.9. Note that $\mathfrak{C}[\Delta^0] \simeq \{*\}$. Hence we may

- identify X_0 with $\mathsf{Ob}(\mathfrak{C}(X))$ for any simplicial set X;
- identify $\mathsf{Ob}(\mathcal{C})$ with $\mathfrak{N}_\bullet(\mathcal{C})_0$ for any simplicial category $\mathcal{C}$.

From now on, we will abuse notations by viewing them as the *same* sets.

[6]On the other hand, homotopy coherent commutative diagrams discussed in [Lecture 1, Section 3] are beyond our reach, because they are not given by (partially) ordered set.

6.10. To justify corresponding bifibrant objects in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ and Cat_Δ indeed model the same ∞-category, we need the following results.

Theorem 6.11 (Ker.01LE)**.** *Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical weak homotopy equivalence between Kan complexes*

$$\mathsf{Hom}_{\mathcal{C}}(x,y) \xrightarrow{\simeq} \mathsf{Hom}^{\mathsf{L}}_{\mathfrak{N}_\bullet(\mathcal{C})}(x,y).$$

Corollary 6.12. *Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical equivalence $\mathsf{h}\mathcal{C} \xrightarrow{\simeq} \mathsf{h}\mathfrak{N}_\bullet(\mathcal{C})$*[7]*.*

7. The quasi-category of quasi-categories

7.1. This is a verbatim of §1.

Definition 7.2. *Let $\mathbb{QCat}$ be the simplicial category defined by:*

- *Objects are quasi-categories;*
- *For quasi-categories $\mathcal{C}$ and $\mathcal{D}$,*

$$\mathsf{Hom}_{\mathbb{QCat}}(\mathcal{C},\mathcal{D}) := \mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}.$$

- *The composition*

$$\mathsf{Fun}(\mathcal{C},\mathcal{D}) \times \mathsf{Fun}(\mathcal{D},\mathcal{E}) \to \mathsf{Fun}(\mathcal{C},\mathcal{E})$$

  is given by the unversal property of $\mathsf{Fun}(-,-)$.

Definition 7.3. *The* ***quasi-category of small quasi-categories*** *is defined to be*

$$\mathcal{QC}at := \mathfrak{N}_\bullet(\mathbb{QCat}).$$

Let Cat_∞ be the ∞-category modelled by it[8]*. We call it the* ***∞-category of small ∞-categories****.*

Exercise 7.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be quasi-categories. There is a canonical weak homotopy equivalence between Kan complexes*

$$\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq} \xrightarrow{\simeq} \mathsf{Hom}^{\mathsf{L}}_{\mathcal{QC}at}(\mathcal{C},\mathcal{D}).$$

In other words, $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$ models the ∞-groupoids $\mathsf{Maps}_{\mathsf{Cat}_\infty}(\mathsf{C},\mathsf{D})$ as desired.

Variant 7.5. *Similarly, we define the* ***quasi-category of small Kan complexes***

$$\mathcal{K}an := \mathfrak{N}_\bullet(\mathbb{Kan}),$$

where $\mathbb{Kan} \subset \mathbb{QCat}$ is the full simplicial subcategory consisting of Kan complexes. As before, the quasi-category $\mathcal{K}an$ models the ∞-category Grpd_∞ of small ∞-groupoids.

Exercise 7.6. *Show that the homotopy category of $\mathcal{QC}at$ is indeed equivalent to $\mathsf{h}\mathcal{QC}at$ in Exercise 2.4.*

8. Commutative diagrams in simplicial categories

8.1. By previous discussion, in infinite category theory, for any simplicial set S, the *correct* notion of an S-indexed commutative diagram in a simplicial category $\mathcal{C}$ is a functor $\mathfrak{C}[S] \to \mathcal{C}$. We call such diagrams an S-indexed **homotopy coherent commutative diagram** in $\mathcal{C}$.

[7] In fact, this result is much more elementary. See Ker.00M4

[8] In rigorous mathematical words, we should *define* ∞-categories to be quasi-categories. However, I do want the readers to view quasi-categories as models for ∞-categories.

8.2. In most cases, the shape of the diagram is an ordinary category $\mathcal{D}$, we also abuse language and call a functor $\mathfrak{C}[\mathsf{N}_\bullet(\mathcal{D})] \to \mathcal{C}$ a $\mathcal{D}$-indexed **homotopy coherent commutative diagram** in $\mathcal{C}$. This is the notion that behaves well under standard infinite categorical constructions.

One can also ask for a diagram that **commutes on the nose**, which means a functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is viewed as enriched over discrete simplicial sets. In practice, it is hard to construct such diagrams, but most *non-formal* calculations eventually lead people to this realm[9].

One can also ask for a diagram that **commutes up to homotopy**, which means a functor $\mathcal{D} \to \mathsf{h}\mathcal{C}$. This is essentially a notion in ordinary category theory.

8.3. We have:

$$\text{on the nose} \;\Rightarrow\; \text{homotopy-coherently} \;\Rightarrow\; \text{up to homotopy}\ .$$

Exercise 8.4. *Apply the above discussion to $\mathcal{C} := \mathbb{QC}\mathrm{at}$ and obtain three notions of commutative diagram of quasi-categories.*

9. What is an ∞-categories?

9.1. In these notes, we take the following perspective:

- *An ∞-category is* modelled *by a quasi-category, and any equivalence between two quasi-categories* supplies *an identification of the ∞-categories modelled by them.*

Note however that such *identifications* are not unique, and there can be *identifications between identifications*, etc..

In particular, we do *not* assign a specific quasi-category to an ∞-category[10]. For instance, it makes no sense to talk about the *set* of objects in an ∞-category. Rather, one can talk about the ∞*-groupoid* of objects (because ∞-groupoids are defined under the same philosophy[11]), and therefore the *set* of equivalent classes of objects.

9.2. To describe the above philosophy in more rigorous words, we make the following definition.

Definition 9.3. *Let $\mathcal{C}$ be a quasi-category and W be a collection of morphisms in $\mathcal{C}$. We say a functor $\mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a **quasi-categorical localization** of $\mathcal{C}$ with respect to W if for any quasi-category $\mathcal{E}$, the functor*

$$\mathsf{Fun}(\mathcal{D}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{E})$$

is fully faithful and the essential image contains exactly functors $\mathcal{C} \to \mathcal{E}$ that send morphisms in W to isomorphisms.

[9] My personal experience: *genuine* non-formal calculations are very few.

[10] Note however that in Lurie's book, an ∞-category is just a quasi-category.

[11] See [Lecture 4, Remark 8.4].

9.4. In futuer lectures, we will show quasi-categorical localizations are essentially unique, i.e., unique up to a contractible space of choices. We will use $\mathcal{C}[W^{-1}]$ to denote *the* quasi-categorical localization of $\mathcal{C}$ with respect to W.

By Theorem A.4, we have a canonical equivalence

$$\mathsf{N}_\bullet(\mathsf{QCat})[W^{-1}] \xrightarrow{\simeq} \mathcal{QC}at,$$

where W is the collection of categorical equivalences.

9.5. Now we make the following definition:

- *An ∞-category is an object in* the quasi-categorical localization *of* $\mathsf{N}_\bullet(\mathsf{QCat})[W^{-1}]$ *with respect to categorical equivalences.*
- For a quasi-category $\mathcal{C} \in \mathsf{QCat}$, the ∞*-category modelled by* $\mathcal{C}$ is its image under $\mathsf{N}_\bullet(\mathsf{QCat}) \to \mathsf{N}_\bullet(\mathsf{QCat})[W^{-1}]$.

In above, we do *not* make a specific choice for $\mathsf{N}_\bullet(\mathsf{QCat})[W^{-1}]$, and any statements about ∞-categories in these notes will not depend on such a choice. In particular, we do *not* view ∞-categories as 0-simplexes in $\mathcal{QC}at$.

Modulo the problem in Footnote 1, $\mathsf{N}_\bullet(\mathsf{QCat})[W^{-1}]$ itself models an ∞-category, which is called the **∞-category of (small) ∞-categories**, and is denoted by Cat_∞.

9.6. In practice, the above philosophy results in the following principle. To make definitions or constructions about ∞-categories, we can either:

(i) Make them via quasi-categories and show the results are invariant under categorical equivalences.
(ii) Combine existing definitions or constructions about ∞-categories, without mentioning quasi-categories.

When both methods are available, we prefer the second one because it allows us to translate the definitions and constructions to other approaches to infinite category theory.

Appendix A. QCat vs. $\mathcal{QC}at$

Exercise A.1. *Let $\mathcal{C}$ be an ordinary category. Construct an isomorphism $\mathsf{N}_\bullet(\mathcal{C}) \simeq \mathfrak{N}_\bullet(\mathcal{C})$, where in the RHS we view $\mathcal{C}$ as enriched over discrete simplicial sets.*

Exercise A.2. *Construct a functor $\mathsf{N}_\bullet(\mathsf{QCat}) \to \mathcal{QC}at$ that models $\mathsf{QCat} \to \mathsf{Cat}_\infty$.*

Exercise A.3. *Prove the following theorem:*

Theorem A.4. *The functor $\mathsf{N}_\bullet(\mathsf{QCat}) \to \mathcal{QC}at$ exhibits $\mathcal{QC}at$ as the* quasi-categorical localization *of $\mathsf{N}_\bullet(\mathsf{QCat})$ with respect to the class of categorical equivalences.*

A.5. **Suggested readings.** [DK80c], [DK80a], [DK80b] (original) and [Hin16] (more recent).

References

[DK80a] William G Dwyer and Daniel M Kan. Calculating simplicial localizations. *Journal of Pure and Applied Algebra*, 18(1):17–35, 1980.

[DK80b] William G Dwyer and Daniel M Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.

[DK80c] William G Dwyer and Daniel M Kan. Simplicial localizations of categories. *Journal of pure and applied algebra*, 17(3):267–284, 1980.

[Hin16] Vladimir Hinich. Dwyer–kan localization revisited. *Homology, Homotopy and Applications*, 18(1):27–48, 2016.

[Joy08] André Joyal. The theory of quasi-categories and its applications. 2008.