LECTURE 15

In this lecture, we define and study *stable ∞-categories*, which are exactly ∞-categories of the form $\mathsf{Sptr}(\mathsf{C})$.

1. Stability

Definition 1.1. *Let C be a pointed ∞-category. A* ***triangle*** *in C is a diagram $\Delta^1 \times \Delta^1 \to \mathsf{C}$ depicted as*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{1.1}$$

where $0 \in \mathsf{C}$ is the zero object. We say such a triangle is a ***fiber sequence*** *if it is a pullback square, and a* ***cofiber sequence*** *if it is a pushout square.*

For a morphism $f : X \to Y$, a ***cofiber of*** *f is a fiber sequence* (1.1). *Dually, for a morphism $g : Y \to Z$, a* ***fiber of*** *g is a cofiber sequence* (1.1).

1.2. We often abuse notation and write a triangle as $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Warning 1.3. *The datum of a triangle* (1.1) *is* not *determined by the chain $X \xrightarrow{f} Y \xrightarrow{g} Z$, even up to homotopy. Indeed, knowing such a triangle is equivalent to knowing a null-homotopy of $g \circ f$, which is not unique even up to homotopy.*

1.4. Note however that a fiber sequence (1.1) is essentially uniquely determined by the morphism g. Dually, a cofiber sequence (1.1) is essentially uniquely determined by the morphism f. Hence we can use the notations

$$\mathsf{Fib}(g),\ \mathsf{Cofib}(f) \in \mathsf{C}$$

as long as we incorporate (1.1) as data in their definitions.

Definition 1.5. *An ∞-category C is* ***stable*** *if it satisfies the following conditions:*

- *it is pointed*
- *any morphism in C admits a fiber and a cofiber*
- *a triangle in C is a fiber sequence iff it is a cofiber sequence.*

1.6. For stable ∞-categories, we can use the words **fiber-cofiber sequences**.

Exercise 1.7. *Find all ordinary categories that are stable when viewed as ∞-categories.*

Proposition 1.8. *Let C be a pointed ∞-category admitting both finite limits and colimits. Then the following conditions are equivalent.*

(i) *The functor $\Sigma : \mathsf{C} \to \mathsf{C}$ is fully faithful.*
(ii) *Any cofiber sequence in C of the form $X \to 0 \to Z$ is also a fiber sequence.*

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(iii) *Any cofiber sequence in* C *is a fiber sequence.*
(iv) *Any pushout square in* C *is a pullback square.*

Corollary 1.9. *Let* C *be a pointed* ∞*-category admitting both finite limits and colimits. Then the following conditions are equivalent.*

(i) *The functors* $\Sigma : \mathsf{C} \rightleftarrows \mathsf{C} : \Omega$ *are equivalences.*
(ii) *A triangle in* C *of the form* $X \to 0 \to Z$ *is a cofiber sequence iff it is a fiber sequence.*
(iii) *A triangle in* C *is a cofiber sequence iff it is a fiber sequence.*
(iv) *A square in* C *is a pushout iff it is a pullback.*

Proof of Proposition 1.8. The implications (ii)⇐(iii)⇐(iv) are obvious. It remains to show (i)⇒(iv). Suppose Σ is fully faithful. Since Ω is a right adjoint of Σ, we have $\mathsf{Id}_{\mathsf{C}} \simeq \Omega \circ \Sigma$. For a pushout square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z, \end{tikzcd}$$

we need to show $W \to X \times_Z Y$ is invertible. Consider the following commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[dd] \\ Y \arrow[r] \arrow[d] & Z \arrow[dr, dotted] & \\ 0 \arrow[rr] & & \Sigma W. \end{tikzcd}$$

Here both the inner and outer squares are pushout squares, hence there exists an essentially unique dotted arrow $Z \to \Sigma W$ making the above diagram commute. By functoriality of pullbacks, we obtain a morphism $X \times_Z Y \to 0 \times_{\Sigma W} 0 \simeq \Omega\Sigma W$ fitting into the following commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\simeq"] & X \times_Z Y \arrow[d, "\simeq"] \arrow[dl, dotted] \\ \Omega\Sigma W \arrow[r] & \Omega\Sigma X \times_{\Omega\Sigma Z} \Omega\Sigma Y, \end{tikzcd}$$

where the vertical morphisms are isomorphisms because of $\mathsf{Id}_{\mathsf{C}} \simeq \Omega \circ \Sigma$. By the 2-out-of-6 property of isomorphisms, we obtain $X \xrightarrow{\simeq} X \times_Z Y$ as desired. □

Proposition 1.10. *Let* C *be a pointed* ∞*-category. The following conditions are equivalent.*

(a) *The* ∞*-category* C *is stable.*
(b) *The* ∞*-category* C^{op} *is stable.*
(c) *The* ∞*-category* C *admits finite colimits and* $\Sigma : \mathsf{C} \to \mathsf{C}$ *is an equivalence.*
(d) *The* ∞*-category* C *admits finite limits and* $\Omega : \mathsf{C} \to \mathsf{C}$ *is an equivalence.*
(e) *The* ∞*-category* C *admits finite colimits and limits, and a square in* C *is a pushout square iff it is a pullback square.*

(f) *The ∞-category C admits finite limits and $\Omega^\infty : \mathsf{Sptr}(\mathsf{C}) \to \mathsf{C}$ is an equivalence.*

Proof. The equivalence (a)$\Leftrightarrow$(b) is obvious. The equivalence (d)$\Leftrightarrow$(f) was proved last time. It remains to show (c)$\Leftrightarrow$(a)$\Leftrightarrow$(e) because (d)$\Leftrightarrow$(b) would follow by passing to the opposite ∞-category.

Suppose C admits finite colimits and limits, then (c)$\Leftrightarrow$(a)$\Leftrightarrow$(e) follow from (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) in Proposition 1.8 (and its dual version). Hence it remains to show (a) or (c) implies C admits finite colimits and limits.

For (a), we only need to show a stable ∞-category admits coequalizers. This follows from Exercise 2.13 below.

For (c), we can use $\iota : \mathsf{C} \to \mathsf{Ind}(\mathsf{C})$ to embed C into a presentable ∞-category. Note that $\mathsf{Ind}(\mathsf{C})$ is pointed because ι perserves and detects both finite colimits and limits. Moreover, $\Sigma_{\mathsf{Ind}(\mathsf{C})}$ can be identified with $\mathsf{Ind}(\Sigma_{\mathsf{C}}) : \mathsf{Ind}(\mathsf{C}) \to \mathsf{Ind}(\mathsf{C})$ and thereby is also an equivalence. It follows from the previous discussion that $\mathsf{Ind}(\mathsf{C})$ satisfies all the properties in the proposition. In particular, any pushout square in $\mathsf{Ind}(\mathsf{C})$ is a pullback square. Since ι perserves and detects both finite colimits and limits, we see the same holds for C. In particular, it admits pullbacks and therefore all finite limits as desired. $\square$

Corollary 1.11. *Let C be a pointed ∞-category that admits finite limits. The ∞-category $\mathsf{Sptr}(\mathsf{C})$ is stable.*

Exercise 1.12. *Let C be a stable ∞-category. Then $f : X \to Y$ is an isomoprhism iff $\mathsf{Fib}(f)$ is a zero object iff $\mathsf{Cofib}(f)$ is a zero object.*

Exercise 1.13. *Let $f : X \to Y$ be a morphism in a stable ∞-category C. Show that $\Sigma\mathsf{Fib}(f) \simeq \mathsf{Cofib}(f)$.*

2. Homotopy category of stable ∞-category

2.1. In this section, let C be a stable ∞-category.

Exercise 2.2. *Show that the canonical morphism $X \sqcup Y \to X \times Y$ is invertible. Hint:*

$$\begin{tikzcd} X \sqcup Y \arrow[r] \arrow[d] & X \sqcup 0 \arrow[d] \\ 0 \sqcup Y \arrow[r] & 0 \sqcup 0 \end{tikzcd}$$

is a pushout square.

2.3. Since there is a canonical equivalence between $X \sqcup Y$ and $X \times Y$, we use $X \oplus Y$ to denote both of them.

Exercise 2.4. *Let $f, g : X \rightrightarrows Y$ be two morphisms. Show that the composition*

$$X \to X \oplus X \xrightarrow{(f,g)} Y \oplus Y \to Y$$

gives a well-defined binary operator on $\pi_0\mathsf{Maps}_{\mathsf{C}}(X, Y)$. We denote the above composition by $f + g$.

Exercise 2.5. *Let $f, g : X \rightrightarrows Y$ be two morphisms. Show that the above binary operator coincides with the addition operator on the abelian group*

$$\pi_0\mathsf{Maps}_{\mathsf{C}}(X, Y) \simeq \pi_0\mathsf{Maps}_{\mathsf{C}}(X, \Omega^2\Sigma^2 Y) \simeq \pi_0\Omega^2\mathsf{Maps}_{\mathsf{C}}(X, \Sigma^2 Y) \simeq \pi_2\mathsf{Maps}_{\mathsf{C}}(X, \Sigma^2 Y).$$

Exercise 2.6. *Let $\sigma : \Delta^1 \times \Delta^1 \to \mathsf{C}$ be a diagram of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$

and σ' be its transpose. By the universal property of pushouts, σ induces a morphism $\eta_\sigma : X \to \Omega\Sigma Y \simeq Y$, which is an well-defined element in $\pi_0\mathsf{Hom}_{\mathsf{C}}(X,Y)$. Show that $\eta_\sigma + \eta_{\sigma'} = 0$.

Corollary 2.7. *The homotopy category hC is an additive category.*

2.8. From now on, we write $X[n] := \Sigma^n X$, where for $n < 0$ we take $\Sigma^n := \Omega^{-2}$. Note that these objects are well-defined up to homotopy.

Definition 2.9. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a chain in hC. We say it is a* ***distinguished triangle in*** hC *if there exists a diagram $\Delta^1 \times \Delta^2 \to \mathsf{C}$ depicted as*

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\widetilde{g}"] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] & W \end{tikzcd}$$

such that

- *the objects 0 and $0'$ are zero objects*
- *the morphisms $\widetilde{f}$ and $\widetilde{g}$ lift f and g respectively*
- *the outer square is a fiber-cofiber sequence*
- *the composition $Z \to W \xrightarrow{\simeq} X[1]$ (which is well-defined up to homotopy) lifts h.*

Theorem 2.10 (HA.1.1.2.14)**.** *The above choice of the translation functor and the distinguished triangles makes hC a triangulated category.*

Exercise 2.11. *What would happen if we use diagrams of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0' \arrow[r] & W \end{tikzcd}$$

to define distinguished triangles in hC?

Exercise 2.12. *For $X \in \mathsf{C}$, construct a fiber-cofiber sequence*

$$X \xrightarrow{(\mathsf{id},-\mathsf{id})} X \oplus X \xrightarrow{(\mathsf{id},\mathsf{id})} X.$$

Exercise 2.13. *Show that the coequalizer of $f, g : X \rightrightarrows Y$ is canonically equivalent to* $\mathsf{Cofib}(f-g)$. *Hint:*

$$\begin{tikzcd} X \arrow[r, "(\mathsf{id},-\mathsf{id})"] \arrow[d] & X \oplus X \arrow[r, "(f,g)"] \arrow[d, "(\mathsf{id},\mathsf{id})"] & Y \arrow[d] \\ 0 \arrow[r] & X \arrow[r] & \bullet \end{tikzcd}$$

3. Mapping spectra

Construction 3.1. *Let* C *be a stable* ∞*-category. For* $X, Y \in \mathsf{C}$*, define*

$$\underline{\mathsf{Hom}}(X,Y)_n := \mathsf{Maps}(X, Y[n]).$$

Note that for any $n \geq 0$*, we have an isomorphism*

$$\alpha_n : \mathsf{Maps}(X, Y[n]) \simeq \mathsf{Maps}(X, \Omega Y[n+1]) \simeq \Omega\mathsf{Maps}(X, Y[n+1]).$$

Let $\underline{\mathsf{Hom}}(X,Y) \in \mathsf{Sptr}$ *be the spectrum given by the spaces* $\{\underline{\mathsf{Hom}}(X,Y)_n\}$ *and the isomorphisms* α_n*. We call it the* ***mapping spectrum*** *between* X *and* Y*.*

Remark 3.2. *The above spectrum* $\underline{\mathsf{Hom}}(X,Y)$ *is well-defined up to homotopy. In future lectures, we will equip* Sptr *with a symmetric monoidal structure such that any stable* ∞*-category* C *is canonically enriched over* Sptr*.*

Definition 3.3. *Let* C *be a stable* ∞*-category. For* $X, Y \in \mathsf{C}$*, define*

$$\mathsf{Ext}^n(X,Y) := \pi_0\underline{\mathsf{Hom}}(X,Y)_n \simeq \pi_0\mathsf{Maps}(X, Y[n])$$

and call it the n*-th extension group between* X *and* Y*.*

3.4. Note that the extension groups only depend on the images of X and Y in the triangulated category hC. It is well-known that for any distinguished triangle $Y_0 \to Y_1 \to Y_2 \to Y_0[1]$ in hC, we have a long exact sequence

$$\cdots \to \mathsf{Ext}^n(X, Y_0) \to \mathsf{Ext}^n(X, Y_1) \to \mathsf{Ext}^n(X, Y_2) \to \mathsf{Ext}^{n+1}(X, Y_0) \to \cdots.$$

Exercise 3.5. *Let* C *be a stable* ∞*-category that admits small colimits and* $X \in \mathsf{C}$ *be an object. Show that the following are equivalent:*

- *The object* X *is compact, i.e.,* $\mathsf{Maps}_{\mathsf{C}}(X,-) : \mathsf{C} \to \mathsf{Spc}$ *preserves small filtered colimits.*
- *The functor* $\underline{\mathsf{Hom}}(X,-) : \mathsf{C} \to \mathsf{Sptr}$ *preserves small colimits.*
- *The functor* $\underline{\mathsf{Hom}}(X,-) : \mathsf{C} \to \mathsf{Sptr}$ *preserves small coproducts.*

Warning 3.6. *The functor* $\mathsf{Maps}_{\mathsf{C}}(X,-)$ *in above does not preserve general finite colimits, such as suspensions.*

4. Exact functors

4.1. Proposition 1.10 and Exercise 2.13 imply the following result.

Proposition-Definition 4.2. *Let* $F : \mathsf{C} \to \mathsf{C}'$ *be a functor between stable* ∞*-category. The following conditions are equivalent.*

- *The functor* F *preserves zero objects and fiber-cofiber sequence.*
- *The functor* F *is left exact, i.e., preserves finite limits.*
- *The functor* F *is right exact, i.e., preserves finite colimits.*

We say F *is* ***exact*** *if it satisfies the above conditions.*

Exercise 4.3. *Let $F : \mathsf{C} \to \mathsf{C}'$ be an exact functor between stable ∞-category. Show that $\mathsf{h}F : \mathsf{hC} \to \mathsf{hC}'$ has a natural structure of an exact functor*[1] *between triangulated categories.*

Definition 4.4. *Let $\mathsf{Cat}_\infty^{\mathsf{ex}} \subseteq \mathsf{Cat}_\infty$ be the sub-∞-category of small stable ∞-categories and exact functors between them.*

Exercise 4.5. *Let C be a stable ∞-category. Show that Σ and Ω are exact. What are the triangulated functors induced by them?*

Exercise 4.6. *Let $F : \mathsf{C} \to \mathsf{C}'$ be an exact functor between stable ∞-categories. Show that F is conservative iff F detects zero objects.*

5. Closure properties

Exercise 5.1. *Let C be a stable ∞-category and K be a simplicial set. Show that $\mathsf{Fun}(K, \mathsf{C})$ is stable.*

Exercise 5.2. *Let C and D be stable ∞-categories. Show that $\mathsf{Fun}_{\mathsf{ex}}(\mathsf{C}, \mathsf{D})$ is stable.*

Exercise 5.3. *Let C and D be presentable ∞-categories such that D is stable. Show that $\mathsf{LFun}(\mathsf{C}, \mathsf{D})$ is stable.*

Exercise 5.4. *Let C be a small stable ∞-category, then $\mathsf{Ind}(\mathsf{C})$ is stable.*

Exercise 5.5. *Let C be a stable ∞-category, then $\mathsf{C}^{\mathsf{cpt}}$ is stable. In particular, $\mathsf{Sptr}^{\mathsf{fin}}$ is stable.*

Exercise 5.6. *Let C be a stable ∞-category, show that the idempotent completion of C is also stable.*

Exercise 5.7. *Let C be a stable ∞-category, is $\mathsf{PShv}(\mathsf{C})$ stable?*

Theorem 5.8. *The ∞-cateogry $\mathsf{Cat}_\infty^{\mathsf{ex}}$ admits small limits and the inclusion $\mathsf{Cat}_\infty^{\mathsf{ex}} \to \mathsf{Cat}_\infty$ preserves and detects small limits.*

Sketch. We only need to show for any small diagram $K \to \mathsf{Cat}_\infty$, $i \mapsto \mathsf{C}_i$ such that each C_i is stable and each connecting functor $\mathsf{C}_i \to \mathsf{C}_j$ is exact, we have

- the limit ∞-category $\mathsf{C} := \mathsf{lim}_i \, \mathsf{C}_i$ is stable
- the evaluating functors $\mathsf{C} \to \mathsf{C}_i$ are exact.

Both claims can be checked using the explicit description of objects and mapping spaces in C. □

5.9. Similarly, one can prove the following result.

Theorem 5.10. *The ∞-cateogry $\mathsf{Cat}_\infty^{\mathsf{ex}}$ admits small filtered colimits and the inclusion $\mathsf{Cat}_\infty^{\mathsf{ex}} \to \mathsf{Cat}_\infty$ preserves and detects small filtered colimits.*

[1] Also known as a triangulated functor. Warning: being a triangulated functor is a structure rather than property.

6. A universal property of Sptr

6.1. The following result implies Sptr is the stable ∞-category freely generated by one object under small colimits.

Exercise 6.2. *Let D be a presentable stable ∞-category. Show that evaluating at $\mathbb{S} \in \mathsf{Sptr}$ induces an equivalence*

$$\mathsf{LFun}(\mathsf{Sptr}, \mathsf{D}) \xrightarrow{\simeq} \mathsf{D}.$$

Hint: show $\mathsf{RFun}(\mathsf{D}, \mathsf{Sptr}) \xrightarrow{\Omega^\infty \circ -} \mathsf{RFun}(\mathsf{D}, \mathsf{Spc})$ *is an equivalence.*

Exercise 6.3. *Let D be a presentable stable ∞-category. Show that evaluating at $\mathbb{S} \in \mathsf{Sptr}^{\mathsf{fin}}$ induces an equivalence*

$$\mathsf{Fun}_{\mathsf{ex}}(\mathsf{Sptr}^{\mathsf{fin}}, \mathsf{D}) \xrightarrow{\simeq} \mathsf{D}.$$

Appendix A. Triangulated categories without models

A.1. There are triangulated categories that are not the homotopy category of any stable ∞-category.

A.2. There are exact functors between homotopy categories of stable ∞-categories that do not come from exact functors between the stable ∞-categories.

A.3. **Suggested readings.** [MSS07].

References

[MSS07] Fernando Muro, Stefan Schwede, and Neil Strickland. Triangulated categories without models. *Inventiones mathematicae*, 170(2):231–241, 2007.