LECTURE 5

Last time we constructed a homomorphism $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{t})$ fitting into the following diagram

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\phi"] \arrow[dr, "\chi_\lambda = \xi_{-,\lambda}"'] & \mathsf{Sym}(\mathfrak{t}) \arrow[d, "\mathsf{ev}_\lambda"] \\ & k_\lambda, \end{tikzcd} \tag{0.1}$$

where for any $\lambda \in \mathfrak{t}^*$, $\chi_\lambda = \xi_{-,\lambda}$ is the central character of the Verma module M_λ. We stated the following result, which will be proved today.

Theorem 1 (Harish-Chandra)**.** *The homomorphism ϕ induces an isomorphism*

$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \mathsf{Sym}(\mathfrak{t})^{W_\bullet}$$

from $Z(\mathfrak{g})$ to the invariance of $\mathsf{Sym}(\mathfrak{t})$ with respect to the dot W-action.

1. Step 1: image is $W_\bullet$-invariant

Let us first prove the image of ϕ is indeed contained in $\mathsf{Sym}(\mathfrak{t})^{W_\bullet}$. Since W is generated by simple reflections s_α, $\alpha \in \Delta$, we only need to show $\phi(z) = s_\alpha \cdot \phi(z)$ for any $z \in Z(\mathfrak{g})$. In other words, we need to show

$$\mathsf{ev}_\lambda(\phi(z)) = \mathsf{ev}_\lambda(s_\alpha \cdot \phi(z)) \tag{1.1}$$

for any $\lambda \in \mathfrak{t}^*$. In fact, we only need to prove this for a Zariski dense subset of $\mathfrak{t}^*$. Let us first give this dense subset. The following result is obvious after drawing a picture.

Lemma 2. *Let $\alpha \in \Delta$ be a simple root. The subset $\{\lambda \in \mathfrak{t}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}\}$ is a Zariski dense subset of $\mathfrak{t}^*$.*

Example 3. For $\mathfrak{g} = \mathfrak{sl}_2$, this subset is $\mathbb{Z}^{\geq -1} \subset \mathbb{A}^1$. In fact, in [Exam. 15, Lect. 4], we have used this subset to show the image of $\phi : Z(\mathfrak{sl}_2) \to k[h]$ is invariant under $h \mapsto -h-2$. The argument below is an immediate generalization.

Lemma 4. *Let $\alpha \in \Delta$ be a simple root and $\lambda \in \mathfrak{t}^*$. Suppose $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}$. Then M_λ contains $M_{s_\alpha \cdot \lambda}$ as a submodule.*

Corollary 5. *The image of $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{t})$ is indeed contained in $\mathsf{Sym}(\mathfrak{t})^{W_\bullet}$.*

Proof. By previous discussion, we only need to prove (1.1) for α and λ satisfying the assumption of Lemma 4. By (0.1), the LHS and RHS of (1.1) are exactly the central character of M_λ and $M_{s_\alpha \cdot \lambda}$. Now Lemma 4 implies they are equal because the central character of a module is equal to the central character of any nonzero submodule of it. $\square$

Date: Mar 25, 2024.

Proof of Lemma 4. Recall
$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha = \lambda - \langle\lambda,\check{\alpha}\rangle\alpha.$$
It follows that
$$s_\alpha \cdot \lambda = \lambda - \langle\lambda+\rho,\check{\alpha}\rangle\alpha = \lambda - m\alpha$$
for $m \in \mathbb{Z}^{\geq 0}$. The lemma is obvious for $m = 0$. We assume $m > 0$. Then we have $\langle\lambda,\check{\alpha}\rangle = m-1$ because $\langle\rho,\check{\alpha}\rangle = 1$ (see [H1, Cor. to Lem. 10.2(B)]).

Let $f_\alpha \in \mathfrak{n}^-$ be a nonzero vector of weight $-\alpha$. Consider the vector $f_\alpha^m \cdot v_\lambda$, which is of weight $\lambda - m\alpha = s_\alpha \cdot \lambda$. We only need to show
$$\mathfrak{n} \cdot (f_\alpha^m \cdot v_\lambda) = 0.$$
Indeed, if this is true, the map $k_{s_\alpha\cdot\lambda} \to M_\lambda$, $c \mapsto c(f_\alpha^m \cdot v_\lambda)$ is $\mathfrak{b}$-linear, and thereby induces a $\mathfrak{g}$-linear map $M_{s_\alpha\cdot\lambda} \to M_\lambda$. This map is injective because as a morphism between $U(\mathfrak{n}^-)$-modules, it is given by $- \cdot f_\alpha^m : U(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$, which is injective by the PBW theorem.

It remains to show $\mathfrak{n}$ annihilates $f_\alpha^m \cdot v_\lambda$. For each simple root $\beta \in \Delta$, let $e_\beta \in \mathfrak{n}$ be a nonzero vector of weight β. Note that the vectors $(e_\beta)_{\beta\in\Delta}$ generate $\mathfrak{n}$ under Lie brackets[1]. Hence we only need to show $e_\beta \cdot f_\alpha^m \cdot v_\lambda = 0$. There are two cases:

- If $\alpha \neq \beta$, then $[e_\beta, f_\alpha] = 0$[2] and
$$e_\beta \cdot f_\alpha^m \cdot v_\lambda = f_\alpha^m \cdot e_\beta \cdot v_\lambda = 0$$
because $\mathfrak{n} \cdot v_\lambda = 0$.
- If $\alpha = \beta$, $[e_\alpha, f_\alpha] \in \mathfrak{t}$ is propotionate to $\check{\alpha}$ (see [H1, Prop. 8.3(d)]). Rescale e_α, we may assume $[e_\alpha, f_\alpha] = \check{\alpha}$. Then $[\check{\alpha}, f_\alpha] = \langle -\alpha, \check{\alpha}\rangle f_\alpha = -2f_\alpha$. Now the following calculation is essentially that in [Exe. 24, Lect. 2]. We have
$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1\leq i\leq m} f_\alpha^{m-i} \cdot [e_\alpha, f_\alpha] \cdot f_\alpha^{i-1} \cdot v_\lambda + f_\alpha^m \cdot e_\alpha \cdot v_\lambda.$$
Recall we have $e_\alpha \cdot v_\lambda = 0$. Also,
$$\check{\alpha} \cdot f_\alpha^j = \sum_{1\leq i\leq j} f_\alpha^{j-i} \cdot [\check{\alpha}, f_\alpha] \cdot f_\alpha^{i-1} + f_\alpha^j \cdot \check{\alpha} = -2jf_\alpha^j + f_\alpha^j \cdot \check{\alpha}.$$
Hence
$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1\leq i\leq m} (-2(i-1)f_\alpha^{m-1} + f_\alpha^{m-1} \cdot \check{\alpha})v_\lambda = \big(-m(m-1) + m\langle\lambda,\check{\alpha}\rangle\big)v_\lambda = 0$$
as desired.

□[Lemma 4]

2. Step 2: Filtrations

By Step 1, we have a homomorphism
$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{t})^{W_\bullet}.$$
In this step, we equip both sides with filtrations, and show ϕ_{HC} is compatible with them. The punchline is the following easy fact:

[1] Because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{[\alpha,\beta]}$ whenever $\alpha, \beta, \alpha+\beta \in \Phi^+$.

[2] Otherwise it is a nonzero vector with weight $\beta - \alpha$ but the latter is not a root because $\Phi = \Phi^+ \sqcup \Phi^-$.

Fact 6. *Let V_1 and V_2 be two vector spaces equipped with $\mathbb{Z}^{\geq 0}$-indexed (exhausted) filtrations. Suppose $\varphi : V_1 \to V_2$ is a k-linear map compatible with the filtrations. Then φ is an isomorphism between* filtered *vector spaces iff $\mathsf{gr}^\bullet\varphi : \mathsf{gr}^\bullet V_1 \to \mathsf{gr}^\bullet V_2$ is an isomorphism between* graded *vector spaces.*

Construction 7. *The PBW filtration on $U(\mathfrak{g})$ induces a filtration on $Z(\mathfrak{g})$ with $\mathsf{F}^{\leq i}Z(\mathfrak{g}) := Z(\mathfrak{g}) \cap \mathsf{F}^{\leq i}U(\mathfrak{g})$. Note that $\mathsf{gr}^\bullet Z(\mathfrak{g})$ is a subalgebra of $\mathsf{gr}^\bullet U(\mathfrak{g}) \simeq \mathsf{Sym}(\mathfrak{g})$.*

Construction 8. *The PBW filtration on $U(\mathfrak{t}) \simeq \mathsf{Sym}(\mathfrak{t})$ is preserved by the dot W-action. Hence it induces a filtration on $U(\mathfrak{t})^{W\bullet}$*[3]*.*

Warning 9. *The* dot *W-action on $\mathsf{Sym}(\mathfrak{t})$ does not preserve the grading. But the usual (linear) W-action does.*

Lemma 10. *The homomorphism $\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \to U(\mathfrak{t})^{W\bullet}$ is compatible with the above filtrations.*

Proof. This is obvious from the description of ϕ as

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow k \underset{U(\mathfrak{n}^-)}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} k \simeq U(\mathfrak{t}).$$

□

3. Step 3: Calculating the graded pieces

Construction 11. *Recall $\mathsf{Sym}(\mathfrak{g})$ has a natural $\mathfrak{g}$-module structure constructed as follows. For any $V \in \mathfrak{g}$, there is a natural $\mathfrak{g}$-module structure on $V^{\otimes n}$ given by*

$$\mathfrak{g} \times V^{\otimes n} \to V^{\otimes n},\ (x, \underset{i}{\otimes} v_i) \mapsto \sum_i (v_1 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \cdots \otimes v_n).$$

This action is compatible with the symmetric group Σ_n-action on $V^{\otimes n}$ and thereby induces a $\mathfrak{g}$-module structure on $\mathsf{Sym}^n(V)$. Taking direct sum, we obtain a $\mathfrak{g}$-module structure on $\mathsf{Sym}(V)$.

In the case $V = \mathfrak{g}$, to distinguish with the multiplication structure on $\mathsf{Sym}(\mathfrak{g})$, we denote this action by

$$\mathfrak{g} \times \mathsf{Sym}(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{g}),\ (x, u) \mapsto \mathsf{ad}_x(u),$$

*and call it the **adjoint action**.*

Note that by definition, for $x \in \mathfrak{g}$ and $u, v \in \mathsf{Sym}(\mathfrak{g})$, we have

$$\mathsf{ad}_x(u \cdot v) = \mathsf{ad}_x(u) \cdot v + u \cdot \mathsf{ad}_x(v).$$

In particular, the $\mathfrak{g}$-invariance

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} := \{u \in \mathsf{Sym}(\mathfrak{g}) \,|\, \mathsf{ad}_x(u) = 0 \textit{ for any } x \in \mathfrak{g}\}$$

is a subalgebra of $\mathsf{Sym}(\mathfrak{g})$.

Lemma 12. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{tikzcd} \mathsf{gr}^\bullet Z(\mathfrak{g}) \arrow[r, dotted, "\simeq"] \arrow[d, "\subset"] & \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[d, "\subset"] \\ \mathsf{gr}^\bullet U(\mathfrak{g}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{g}), \end{tikzcd}$$

where the bottom isomorphism is given by the PBW theorem.

To prove this lemma, we use the following exercise:

[3]Note that $\mathsf{gr}^\bullet U(\mathfrak{t})$ is also isomorphic to $\mathsf{Sym}(\mathfrak{t})$. To distinguish them, we always use $U(\mathfrak{t})$ to denote the filtered commutative ring while use $\mathsf{Sym}(\mathfrak{t})$ to denote the graded commutative ring.

Exercise 13. This is Homework 2, Problem 4. Prove: the adjoint $\mathfrak{g}$-action on $U(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \to U(\mathfrak{g}),\ (x, u) \mapsto \mathsf{ad}_x(u) = [x, u],$$

preserves each $\mathsf{F}^{\leq n} U(\mathfrak{g})$, and the induced $\mathfrak{g}$-action on $\mathsf{gr}^\bullet(U(\mathfrak{g})) \simeq \mathsf{Sym}(\mathfrak{g})$ is the adjoint action in Construction 11.

Proof of Lemma 12. By the above exercise, we have a short exact sequence of *finite-dimensional* $\mathfrak{g}$-modules:

$$0 \to \mathsf{F}^{\leq n-1} U(\mathfrak{g}) \to \mathsf{F}^{\leq n} U(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g}) \to 0.$$

Since $\mathfrak{g}\text{−}\mathsf{mod}_{\mathsf{fd}}$ is semisimple, this short exact sequence splits. Hence taking $\mathfrak{g}$-invariance, we obtain[4]

$$0 \to \mathsf{F}^{\leq n-1} Z(\mathfrak{g}) \to \mathsf{F}^{\leq n} Z(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}} \to 0$$

This gives the desired isomorphism $\mathsf{gr}^n Z(\mathfrak{g}) \simeq \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}}$.

□[Lemma 12]

A similar proof[5] gives:

Lemma 14. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{tikzcd}
\mathsf{gr}^\bullet(U(\mathfrak{t})^{W_\bullet}) \arrow[r, dotted, "\simeq"] \arrow[d, "\subset"] & \mathsf{Sym}(\mathfrak{t})^W \arrow[d, "\subset"] \\
\mathsf{gr}^\bullet U(\mathfrak{t}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{t}),
\end{tikzcd}$$

where the right-top corner is the invariance for the linear *W-action on $\mathsf{Sym}(\mathfrak{t})$.*

Combining the above two lemmas, we obtain:

Corollary 15. *There is a unique dotted graded homomorphism making the following diagram commute*

$$\begin{tikzcd}
\mathsf{gr}^\bullet Z(\mathfrak{g}) \arrow[r, "\simeq"] \arrow[d, "\mathsf{gr}^\bullet \phi_{\mathsf{HC}}"] & \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \arrow[d, dotted, "\phi_{\mathsf{cl}}"] \\
\mathsf{gr}^\bullet(U(\mathfrak{t})^{W_\bullet}) \arrow[r, "\simeq"] & \mathsf{Sym}(\mathfrak{t})^W.
\end{tikzcd}$$

4. Step 4: Chevalley isomorphism

It remains to show

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^W$$

is an isomorphism. Let us first give an explicit construction of this homomorphism. We need the following characterization of ϕ.

Construction 16. *Consider the composition*

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} k.$$

With respect to the adjoint $\mathfrak{t}$-action, the source has weight 0. Hence the composition factors through the 0-weight subspace of the target, which is exactly $U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t})$. By definition, the obtained map

$$Z(\mathfrak{g}) \to U(\mathfrak{t})$$

[4] Warning: in general, taking invariance is only *left exact*. (Memory method: it is given by $\mathsf{Hom}_{\mathfrak{g}}(k, -)$.) Hence we need the existence of a splitting.

[5] Note that $\mathsf{Rep}(W)_{\mathsf{fd}}$ is also semisimple

is just ϕ.

Construction 17. *It follows* ϕ_{cl} *can be constructed as follows. Consider the composition*

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow \mathsf{Sym}(\mathfrak{g}) \twoheadrightarrow \mathsf{Sym}(\mathfrak{g}/\mathfrak{n}). \tag{4.1}$$

It factors through $\mathsf{Sym}(\mathfrak{b}/\mathfrak{n}) \simeq \mathsf{Sym}(\mathfrak{t})$. *The obtained map*

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})$$

can be identified with $\mathsf{gr}^{\bullet}\phi$. *Since* ϕ *factors through* $U(\mathfrak{t})^{W_{\bullet}}$, *the map* $\mathsf{gr}^{\bullet}\phi$ *factors through* $\mathsf{gr}^{\bullet}(U(\mathfrak{t})^{W_{\bullet}}) \simeq \mathsf{Sym}(\mathfrak{t})^{W}$. *The obtained map*

$$\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^{W}$$

is just ϕ_{cl}.

Remark 18. The geometric meaning of the above construction is as follows.

Note that $\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} = \mathsf{Sym}(\mathfrak{g})^{G}$ because G-invariance is equal to $\mathfrak{g}$-invariance[6]. Hence (4.1) corresponds to the morphisms

$$(\mathfrak{g}/\mathfrak{n})^* \to \mathfrak{g}^* \to \mathfrak{g}^*/\!/G.$$

Since $\mathsf{gr}^{\bullet}\phi$ factors through $\mathsf{gr}^{\bullet}(U(\mathfrak{t})^{W_{\bullet}}) \simeq \mathsf{Sym}(\mathfrak{t})^{W}$. The above composition factors through $(\mathfrak{g}/\mathfrak{n})^* \to (\mathfrak{b}/\mathfrak{n})^* \simeq \mathfrak{t}^* \to \mathfrak{t}^*/\!/W$. In other words, we have

$$\begin{tikzcd} (\mathfrak{g}/\mathfrak{n})^* \arrow[r] \arrow[d] & \mathfrak{g}^*/\!/G \\ \mathfrak{t}^* \arrow[r] & \mathfrak{t}^*/\!/W \arrow[u, dotted] \end{tikzcd}$$

such that the dotted arrow is given by $\mathsf{Spec}(\phi_{\mathsf{cl}})$.

It is convenient to get rid of the dual spaces using the Killing form. Namely, Kil induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ compatible with the G-actions, while $\mathsf{Kil}|_{\mathfrak{t}}$ induces an isomorphism $\mathfrak{t} \simeq \mathfrak{t}^*$ compatible with the W-actions[7]. Via the first isomorphism, the subspaces $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ corresponds to $(\mathfrak{g}/\mathfrak{b})^* \subset (\mathfrak{g}/\mathfrak{n})^* \subset \mathfrak{g}$. Then the above commutative diagram is identified with

$$\begin{tikzcd} \mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g}/\!/G \\ \mathfrak{t} \arrow[r] & \mathfrak{t}/\!/W. \arrow[u, dotted] \end{tikzcd} \tag{4.2}$$

We abuse notations and also view the dotted arrow as $\mathsf{Spec}(\phi_{\mathsf{cl}})$.

Remark 19. Since the projection $\mathfrak{b} \to \mathfrak{t}$ has a splitting $\mathfrak{t} \to \mathfrak{b}$. The above claim implies $\mathsf{Spec}(\phi_{\mathsf{cl}})$ can be characterized as the following dotted arrow

$$\begin{tikzcd} \mathfrak{g} \arrow[r] & \mathfrak{g}/\!/G \\ \mathfrak{t} \arrow[r] \arrow[u] & \mathfrak{t}/\!/W. \arrow[u, dotted] \end{tikzcd}$$

We can also prove the existence of this map using group-theoretic method. Namely, consider the normalizer $N_G(T)$ of T insider G. Recall we have $W \simeq N_G(T)/T$ such that the linear W-action

[6]Because $\mathsf{Rep}(G) \to \mathfrak{g}\text{-}\mathsf{mod}$ is fully faithful when G is connected.

[7]Both claims follow from the fact that Kil is invariant with respect to the adjoint $\mathfrak{g}$-action and thereby to the adjoint G-action. Here we use $W \simeq N_G(T)/T$.

on $\mathfrak{t}$ can be identified with the adjoint action of $N_G(T)/T$. Then the morphism $\mathfrak{t} \to \mathfrak{g} \to \mathfrak{g}/\!/G$ factors through $\mathfrak{t}/\!/W$ because $N_G(T)$ is a subgroup of G.

Warning 20. *I do not know any group-theoretic proof of* (4.2)*. This is because not $\mathfrak{b} \to \mathfrak{g}/G$ (the quotient stack) does not factor through $\mathfrak{t}$: two elements in $\mathfrak{b}$ that have the same image in $\mathfrak{t}$ are not necessarily conjugate to each other. In fact, the 0-fiber of the map $\mathfrak{g} \to \mathfrak{g}/\!/G$ contains exactly the nilpotent elements in $\mathfrak{g}$.*

Theorem 21 (Chevalley)**.** *The homomorphism*

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{t})^W$$

is an isomorphism. In other words, the natural morphism $\mathfrak{t}/\!/W \to \mathfrak{g}/\!/G$ is an isomorphism.

Proof. As in Remark 18, we can identify ϕ_{cl} with the restriction map $\mathsf{Fun}(\mathfrak{g})^G \to \mathsf{Fun}(\mathfrak{t})^W$, which is also $\mathsf{Sym}(\mathfrak{g}^*)^G \to \mathsf{Sym}(\mathfrak{t}^*)^W$.

This map is injective because if an adjoint-invariant function on $\mathfrak{g}$ vanishes on $\mathfrak{t}$, then it vanishes on each semisimple elements. But the latter are Zariski dense in $\mathfrak{g}$.

To prove the surjectivity, we first find generators of $\mathsf{Sym}(\mathfrak{t}^*)^W$ as follows. Recall the subset P^+ of **dominant integral weights**, i.e.,

$$P^+ := \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0} \text{ for all } \alpha \in \Delta\}.$$

Note that P^+ spans $\mathfrak{t}^*$. In fact, for each $\alpha \in \Delta$, we can find **fundamental dominant weights** ω_α such that

$$\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha\beta},\ \alpha, \beta \in \Delta.$$

Then $\{\omega_\alpha\}$ is a basis of $\mathfrak{t}^*$ and $P^+ = \mathbb{Z}^{\geq 0}\{\omega_\alpha\}$. A direct calculation shows that $\{\lambda^n \mid \lambda \in P^+\}$ span $\mathsf{Sym}^n(\mathfrak{t}^*)$. Hence the sums

$$b_{\lambda,n} := \sum_{w \in W} w(\lambda^n),\ \lambda \in P^+, n \geq 0$$

span $\mathsf{Sym}(\mathfrak{t}^*)^W$.

It remains to show each $b_{\lambda,n}$ is contained in the image of ϕ_{cl}. We need the following well-known fact.

Theorem 22 (Weyl)**.** *For any $\lambda \in P^+$, there is a unique finite-dimensional irreducible $\mathfrak{g}$-module L_λ with higheset weight λ.*

For $\lambda \in P^+, n \geq 0$, consider the function $a_{\lambda,n} \in \mathsf{Fun}(\mathfrak{g})$ defined by

$$a_{\lambda,n}(x) := \mathsf{tr}(x^n; L_\lambda),$$

i.e., its value at any $x \in \mathfrak{g}$ is the trace of the action of x^n on L_λ. It is easy to see $a_{\lambda,n}$ is $\mathfrak{g}$-invariant[8] and thereby G-invariant.

Now the following exercise implies each $b_{\lambda,n}$ is contained in the image of ϕ_{cl}. Indeed, this follows from induction on λ with respect to the partial ordering $\prec$[9].

Exercise 23. This is Homework 2, Problem 5. Let $\lambda \in P^+$ be a dominant integral weight and $n \geq 0$. Prove there exists scalars $c_{\lambda'} \in k$, $\lambda' \prec \lambda$ such that

$$\phi_{\mathsf{cl}}(a_{\lambda,n}) = a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\mathsf{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n},$$

where $\mathsf{Stab}_W(\lambda) \subset W$ is the stablizer of the W-action at λ.

[8] Recall L_λ is G_{sc}-integrable ([Thm. 47, Lect. 3]). Hence $a_{\lambda,n}$ is G_{sc}-invariant, and thereby $\mathfrak{g}$-invariant.

[9] Recall $\lambda' \preceq \lambda$ iff $\lambda - \lambda' \in \mathbb{Z}^{\geq 0}\Phi^+$. See [Defn. 22, Lect. 2].

$\square$

Combining all the previous discussion, we finish the proof of Theorem 1.

References

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[H2] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.