LECTURE 14

In this lecture, we give a brief introduction to stable homotopy theory and spectra.

From this lecture on, we use the notation

$$\mathsf{Spc} := \mathsf{Grpd}_\infty.$$

1. Stable homotopy groups

1.1. Let Top_* be the ordinary category of pointed spaces. There is an adjunction

$$\Sigma : \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* : \Omega,$$

where

- The left adjoint Σ is the **(based) suspension functor** given by
$$\Sigma X := \mathbb{S}^1 \wedge X := (\mathbb{S}^1 \times X)/((\{*\} \times X) \cup (\mathbb{S}^1 \times \{*\})).$$
- The right adjoint Ω is the **loop functor** given by
$$\Omega Y := \underline{\mathsf{Hom}}_{\mathsf{Top}_*}(\mathbb{S}^1, Y),$$
where the RHS is equipped with the *compact-open topology*.

1.2. In fact, this adjunction is compatible with Quillen's classical model structure[1]. Taking derived functors, we obtain an adjunction

$$\mathbb{L}\Sigma : \mathsf{hTop}_* \rightleftarrows \mathsf{hTop}_* : \mathbb{R}\Omega. \tag{1.1}$$

Since pointed CW complexes are bifibrant, we have

$$[\Sigma X, Y] \simeq [X, \Omega Y]$$

where $[-,-]$ is the set of homotopy classes of continuous maps.

Exercise 1.3. *Show that* $\Sigma \mathbb{S}^n \simeq \mathbb{S}^{n+1}$.

Exercise 1.4. *For* $Y \in \mathsf{Top}_*$*, there is a canonical isomorphism* $\pi_{n+1}(Y) \simeq \pi_n(\Omega Y)$ *where the group structure on the RHS is induced by the concaternation map* $\Omega Y \times \Omega Y \to \Omega Y$.

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[1]For this to be true, we have to replace Top by the category of *compactly generated topological spaces* (to make sure it is Cartesian closed). Any CW complex is compactly generated.

1.5. For pointed CW complexes X and Y, define
$$[X,Y]_{\mathsf{s}} := \operatorname*{\mathsf{colim}}_k [\Sigma^k X, \Sigma^k Y].$$

Exercise 1.6. *Show that $[X,Y]_{\mathsf{s}}$ is naturally an abelian group. Hint:*
$$[\Sigma^{k+2}X, \Sigma^{k+2}Y] \simeq [\Sigma^k X, \Omega^2\Sigma^{k+2}Y].$$

Definition 1.7. *Let Y be a pointed CW complex, the n**-th stable homotopy group** of Y is defined to be*
$$\pi_n^{\mathsf{s}}(Y) := \operatorname*{\mathsf{colim}}_k \pi_{n+k}(\Sigma^k Y).$$

Example 1.8. *The group $\pi_n(\mathbb{S}) := \pi_n^{\mathsf{s}}(\mathbb{S}^0)$ is called the n-th stable homotopy group of the sphere (spectum). Up to today, people have calculated them for $n \le 90$.*

1.9. **Stable homotopy theory** studies the stable homotopy groups of spaces, and more generally, the limit behavior of various homotopy invaraints under the suspension functor Σ^k, $k \to \infty$. In constrast, the usual homotopy theory is referred as the **unstable homotopy theory**. Our guiding philosephy is

Slogan 1.10. *Stable homotopy theory is the linearization of unstable homotopy theory:*
$$\textit{stable homotopy theory} \ = \ \textit{linear algebra in homotopy theory}\ .$$

2. Spectra

2.1. In previous lectures, we have explained the following philosephy. In order to capture all the homotopy invariant information in Top, we need to word with the ∞-category Spc of spaces rather than its homotopy 1-category $\mathsf{hSpc} \simeq \mathsf{hTop}$. Similarly, the homotopy invariant information of *pointed* spaces should be captured by the coslice ∞-category
$$\mathsf{Spc}_* := \mathsf{Spc}_{\{*\}/}.$$

It follows that the "correct" playground for *stable* homotopy theory should be an ∞-categorical *stablization* or *linearization* Spc_*. For instance, we hope for (good) objects $X, Y \in \mathsf{Spc}_*$, the corresponding mapping space in this stablized ∞-category is given by
$$\operatorname*{\mathsf{colim}}_k \mathsf{Maps}_{\mathsf{Spc}_*}(\Sigma^k X, \Sigma^k Y).$$
Let us first define the ∞-categorical version of Σ and Ω.

Definition 2.2. *We say an ∞-category C is **pointed** if it admits an object $0 \in \mathsf{C}$ which is both initial and final. We call it the **zero object** of C.*

Exercise 2.3. *Let C be an ∞-category that admits a final object $*$, show that $\mathsf{C}_{*/}$ is pointed. In particular, Spc_* is pointed.*

Definition 2.4. *Let C be a pointed ∞-category that admits finite colimits. The **suspension functor** on $\mathbb{C}$ is defined as*
$$\Sigma : \mathsf{C} \to \mathsf{C},\ X \mapsto 0 \underset{X}{\sqcup} 0.$$

Definition 2.5. *Let C be an ∞-category that admits finite limits. The **loop functor** on C is defined as*
$$\Omega : \mathsf{C} \to \mathsf{C},\ Y \mapsto * \underset{Y}{\times} *,$$
where $* \in \mathsf{C}$ is the final object.

Exercise 2.6. *Let* C *be a pointed* ∞*-category that admits both finite limits and colimits. Construct an adjunction:*

$$\Sigma : \mathsf{C} \rightleftarrows \mathsf{C} : \Omega.$$

Exercise 2.7. *For* $\mathsf{C} := \mathsf{Spc}_*$*, the above adjunction induces an adjunction for homotopy categories:*

$$\mathsf{h}\Sigma : \mathsf{hSpc}_* \rightleftarrows \mathsf{hSpc}_* : \mathsf{h}\Omega.$$

Show that this adjunction can be identified with (1.1) *via the equivalence* $\mathsf{hSpc}_* \simeq \mathsf{hTop}_*$*.*

2.8. The construction

$$\mathsf{Maps}(-,-) \mapsto \operatorname*{colim}_k \mathsf{Maps}(\Sigma^k(-), \Sigma^k(-)).$$

can be viewed as formally inverting the functor Σ.

Exercise 2.9. *Let* A *be a commutative ring and* $f \in A$ *be an element. Show that*

$$A_f \simeq \mathsf{colim}\,[A \xrightarrow{f} A \xrightarrow{f} \cdots]$$

2.10. Let C be a pointed ∞-category that admits both finite limits and colimits. Motivated by the above construction, we would like to define the stablization of C to be

$$\mathsf{colim}\,[\mathsf{C} \xrightarrow{\Sigma} \mathsf{C} \xrightarrow{\Sigma} \cdots].$$

However, we need to be careful about where this colimit is taken inside. For instance, when C is presentable, such as Spc_*, we would like to obtain a presentable ∞-category.

Exercise 2.11. *Let* C *be a pointed presentable* ∞*-category. Show that the colimit*

$$\mathsf{colim}\,[\mathsf{C} \xrightarrow{\Sigma} \mathsf{C} \xrightarrow{\Sigma} \cdots] \in \mathsf{Pr}^{\mathsf{L}}$$

corresponds to the limit

$$\lim\,[\mathsf{C} \xleftarrow{\Omega} \mathsf{C} \xleftarrow{\Omega} \cdots] \in \mathsf{Pr}^{\mathsf{R}}$$

via $\mathsf{Pr}^{\mathsf{L}} \simeq (\mathsf{Pr}^{\mathsf{R}})^{\mathsf{op}}$*.*

2.12. Recall limits in Pr^{R} can be calculated as limits in $\widehat{\mathsf{Cat}}_\infty$. This motivates the following definition.

Definition 2.13. *Let* C *be an* ∞*-category that admits finite limits. Define*

$$\mathsf{Sptr}(\mathsf{C}) := \lim\,[\mathsf{C} \xleftarrow{\Omega} \mathsf{C} \xleftarrow{\Omega} \cdots]$$

and call it the ∞*-category of* ***spectum objects*** *of* C*. We denote the evaluating morphism for the* $(k+1)$*-term by*

$$\Omega^{\infty - k} : \mathsf{Sptr}(\mathsf{C}) \to \mathsf{C}.$$

Example 2.14. *For* $\mathsf{C} := \mathsf{Spc}_*$*, write*

$$\mathsf{Sptr} := \mathsf{Sptr}(\mathsf{Spc}_*)$$

and call it the ∞*-category of* ***spectra****.*

Exercise 2.15. *Show that* $\Omega : \mathsf{C} \to \mathsf{C}$ *preserves finite limits. Deduce that* $\mathsf{Sptr}(\mathsf{C})$ *admits finite limits and the functors* $\Omega^{\infty-k}$ *preserve and detect them.*

Exercise 2.16. *Show that* $\mathsf{Sptr}(\mathsf{C})$ *is pointed.*

Exercise 2.17. *Let $\Omega_{\mathsf{Sptr(C)}}$ be the loop functor on $\mathsf{Sptr(C)}$. Show that*

$$\Omega^{\infty-k} \circ \Omega_{\mathsf{Sptr(C)}}(E) \simeq \Omega^{\infty-k+1}(E).$$

Deduce that $\Omega_{\mathsf{Sptr(C)}}$ is an equivalence. Hint:

$$\begin{tikzcd} \mathsf{C} \arrow[d, "\Omega"] & \mathsf{C} \arrow[l, "\Omega"'] \arrow[d, "\Omega"] & \cdots \arrow[l, "\Omega"'] \\ \mathsf{C} & \mathsf{C} \arrow[l, "\Omega"] & \cdots \arrow[l, "\Omega"] \end{tikzcd}$$

Exercise 2.18. *Show that*

$$\Omega^{\infty-k} : \mathsf{Sptr(Sptr(C))} \to \mathsf{Sptr(C)}.$$

is an equivalence.

Remark 2.19. *In the next lecture, we will define and study* ***stable ∞-categories****, which are exactly those pointed ∞-category admitting finite limits such that Ω is an equivalence.*

Exercise 2.20. *Show that $\mathsf{hSptr(C)}$ is an additive category. Hint:*

$$\mathsf{Maps}_{\mathsf{Sptr(C)}}(E, E') \simeq \Omega^2 \mathsf{Maps}_{\mathsf{Sptr(C)}}(E, \Sigma^2 E').$$

3. Spectra and infinite loop spaces

3.1. Informally speaking, knowing an object $X \in \mathsf{Sptr(C)}$ is equivalent to knowing the following datum

- For any $n \geq 0$, an object $X_n \in \mathsf{C}$;
- For any $n \geq 0$, an equivalence $X_n \simeq \Omega X_{n+1}$.

Here we take X_n to be $\Omega^{\infty-k} X$.

Note that X_{n+1}, equipped with the equivalence $X_n \simeq \Omega X_{n+1}$, gives a **delooping** of X_n. As a consequence, we obtain the following slogan.

Slogan 3.2. *A spectrum is a space* ***equipped*** *with infinite deloopings.*

Warning 3.3. *For a space $Y \in \mathsf{Spc}_*$, its delooping is* not unique even up to homotopy. *Hence in above, it is crucial to remember all the deloopings.*

3.4. Note that a loop space ΩZ is equipped with a homotopy coherent multiplicative structure, which makes $\pi_0(\Omega Z)$ an abstract group. In future lectures, we will rigorously define such a structure, and call it a *grouplike $\mathbb{E}_1$-structure*. Moreover, given a grouplike $\mathbb{E}_1$-space Y, there is an essentially unique connected delooping of Y, denoted by $\mathbb{B}Y$, such that $Y \simeq \Omega \mathbb{B} Y$ is compatible with the grouplike $\mathbb{E}_1$-structures.

Moreover, we will generalize the above to iterated loop spaces $\Omega^n Z$ and *grouplike $\mathbb{E}_n$-spaces*. In fact, this even works for $n = \infty$, and we will explain the following slogan.

Slogan 3.5. *A connective spectrum*[2] *is a grouplike $\mathbb{E}_\infty$-space.*

[2]We say a spectrum $E \in \mathsf{Sptr}$ is **connective** if $\pi_n E \simeq 0$ for $n < 0$. See Definition 4.7 below.

4. Spaces vs. spectra

4.1. In this section, we focus on the case when C is pointed and presentable, such as $\mathsf{C} := \mathsf{Spc}_*$. By definition, we have a colimit diagram

$$[\mathsf{C} \xrightarrow{\Sigma} \mathsf{C} \xrightarrow{\Sigma} \cdots] \to \mathsf{Sptr}(\mathsf{C}) \in \mathsf{Pr}^{\mathsf{L}}$$

and a limit diagram

$$[\mathsf{C} \xleftarrow{\Omega} \mathsf{C} \xleftarrow{\Omega} \cdots] \leftarrow \mathsf{Sptr}(\mathsf{C}) \in \mathsf{Pr}^{\mathsf{R}}.$$

It follows that we have an adjunction

$$\Sigma^{\infty-k} : \mathsf{C} \rightleftarrows \mathsf{Sptr}(\mathsf{C}) : \Omega^{\infty-k}$$

with $\Sigma^{\infty-k}$ given by the evaluating morphism for the $(k+1)$-term.

Example 4.2. *The object*

$$\mathbb{S} := \Sigma^{\infty}\mathbb{S}^0 \in \mathsf{Sptr}$$

is called the ***sphere spectrum****. It plays the role of $\mathbb{Z}$ in homotopical algebra.*

Example 4.3. *Let A be an abstract abelian group. For each n, choose an Eilenburg–Maclane space $K(A,n)$, which is characterized up to homotopy by $\pi_n K(A,n) \simeq A$ and $\pi_m K(A,n) \simeq 0$ for $m \neq n$. We can also choose weak homotopy equivalences*

$$K(A,n) \xrightarrow{\sim} \Omega K(A,n+1).$$

These choices give an object $\mathbb{H}A \in \mathsf{Sptr}$, which is well-defined up to homotopy. We call it ***an Eilenburg–Maclane spectrum*** *for A.*

Remark 4.4. *In future lectures, we will characterize $\mathbb{H}A$ up to a contractible space of choices.*

Exercise 4.5. *Let $E \in \mathsf{Sptr}(\mathsf{C})$, show that*

$$\operatorname*{colim}_k \Sigma^{\infty-k}\Omega^{\infty-k}E \xrightarrow{\simeq} E.$$

Exercise 4.6. *Suppose C is compactly generated, show that for any $X \in \mathsf{C}$ and $j \geq 0$,*

$$\operatorname*{colim}_{k \geq j} \Omega^{k-j}\Sigma^k X \xrightarrow{\simeq} \Omega^{\infty-j}\Sigma^{\infty}X.$$

Deduce that if $X \in \mathsf{C}$ is compact, then for any $Y \in \mathsf{C}$, we have

$$\mathsf{Maps}_{\mathsf{Sptr}(\mathsf{C})}(\Sigma^{\infty}X, \Sigma^{\infty}Y) \simeq \operatorname*{colim}_k \mathsf{Maps}_{\mathsf{C}}(\Sigma^k X, \Sigma^k Y).$$

Definition 4.7. *Let $E \in \mathsf{Sptr}$ be a spectrum. For any $n \in \mathbb{Z}$, we define the* ***n-th homotopy group*** *of E to be*

$$\pi_n(E) := \pi_0 \mathsf{Maps}(\mathbb{S}, \Omega^n E),$$

where $\Omega^n := \Sigma^{-n}$ for $n < 0$.

Remark 4.8. *$\pi_n(E)$ is an abelian group because hSptr is additive.*

Exercise 4.9. *For $Y \in \mathsf{Spc}_*$, show that*

$$\pi_n(\Sigma^{\infty}Y) \simeq \pi_n^{\mathsf{s}}(Y).$$

In particular, it vanishes for $n < 0$.

Remark 4.10. *The above exercise implies all the stable homotopy groups of the spheres are encoded as the usual homotopy groups of the space* $\mathsf{Maps}_{\mathsf{Sptr}}(\mathbb{S},\mathbb{S})$. *Note that this space admits a homotopy coherent multiplication structure*[3].

Exercise 4.11. *Let* $E \in \mathsf{Sptr}$ *be a spectrum. Show that* $\Omega^\infty E \simeq \{*\}$ *iff* $\pi_n E \simeq 0$ *for* $n \geq 0$.

5. Finite spectra

Exercise 5.1. *Let* $\mathsf{C} := \mathsf{Ind}(\mathsf{C}_0)$ *be the ind-completion of an essentially small pointed* ∞*-category that admits finite limits and colimits. Show that*

$$\mathsf{Sptr}(\mathsf{C}) \simeq \mathsf{Ind}(\mathsf{colim}\,[\mathsf{C}_0 \xrightarrow{\Sigma} \mathsf{C}_0 \xrightarrow{\Sigma} \mathsf{C}_0 \cdots]),$$

where the colimit is taken inside Cat_∞. *Deduce that* $\mathsf{Sptr}(\mathsf{C})$ *is compactly generated.*

Example 5.2. *For* $\mathsf{C} = \mathsf{Spc}_*$, *we can take* $\mathsf{C}_0 := \mathsf{Spc}_*^{\mathsf{fin}}$, *where* $\mathsf{Spc}^{\mathsf{fin}} \subset \mathsf{Spc}$ *is the smallest full sub-*∞*-category that contains* $*$ *and admits all finite colimits*[4]. *Write*

$$\mathsf{Sptr}^{\mathsf{fin}} := \mathsf{colim}\,[\mathsf{Spc}_*^{\mathsf{fin}} \xrightarrow{\Sigma} \mathsf{Spc}_*^{\mathsf{fin}} \xrightarrow{\Sigma} \mathsf{Spc}_*^{\mathsf{fin}} \cdots]$$

and call it the ∞*-category of* ***finite spectra***. *We obtain an equivalence*

$$\mathsf{Ind}(\mathsf{Sptr}^{\mathsf{fin}}) \simeq \mathsf{Sptr},$$

which allows us to identify $\mathsf{Sptr}^{\mathsf{fin}}$ *as a full sub-*∞*-category of* Sptr.

Remark 5.3. *In above, we can also take* C_0 *to be* $\mathsf{Spc}_*^{\mathsf{cpt}}$, *which is the idempotent completion of* $\mathsf{Spc}^{\mathsf{fin}}$. *The obtained colimit would be* $\mathsf{Sptr}^{\mathsf{cpt}}$.

Appendix A. Spectra and cohomology theories

Construction A.1. *Let* $E \in \mathsf{Sptr}$ *be a spectrum. For any* CW pair (X,Y), *define*

$$E^n(X,Y) := \pi_{-n}(\mathsf{Maps}(\Sigma^\infty(X/Y),E)).$$

Write $E^n(X) := E^n(X,\varnothing)$.

Exercise A.2. *For any CW pair* (X,Y), *construct a long exact sequence*

$$\cdots E^n(X,Y) \to E^n(X) \to E^n(Y) \to E^{n+1}(X,Y) \to E^{n+1}(X) \to E^{n+1}(Y) \to \cdots.$$

Exercise A.3. *Assign a* ***(generalized) cohomology theory*** *(on CW pairs) to a spectrum* E. *What do you get for* $E := \mathbb{H}A$ *or* $\mathbb{S}$*?*

Exercise A.4. *Show that any cohomology theory is represented (in the above sense) by a spectrum, which is unique up to homotopy.*

Warning A.5. *Nonzero morphisms between spectra could induce zero transformations between cohomology theories. Such maps are called* ***phantum maps***. *See this MathOverflow question.*

Remark A.6. *We also have similar story for homology theories. However, such construction uses the* smash products *on spectra, which we have not defined yet.*

A.7. **Suggested readings.** HA.1.4.1.

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[3]We have not yet defined what this means!

[4]An object is contained in $\mathsf{Spc}^{\mathsf{fin}}$ iff it can be represented by a finite CW complex.