Geometrization of local class field theory

- E local field $\mathcal{O}_E/\pi = \mathbb{F}_q$.

<u>Thm</u> (Local class field theory) $[F:E] < \infty$

$$\begin{array}{ccc} G_{F/E}^{ab} & \xrightarrow{\cong} & E^\times / \mathrm{Norm}(F^\times) \\ \| & & \| \\ H^{-2}_{Tate}(G_{F/E}, \mathbb{Z}) & \xrightarrow{\cong} & H^0_{Tate}(G_{F/E}, \mathbb{G}_m) \end{array}$$

is given by cup product with

$$u_{F/E} \in H^2_{Tate}(G_{F/E}, \mathbb{G}_m)$$

<u>Cor</u>:

$$\begin{tikzcd} G_E^{ab} \arrow[r, "\sim"] & \widehat{E^\times} \arrow[r] & \hat{\mathbb{Z}} \\ W_E^{ab} \arrow[u] \arrow[r, "\simeq"] & E^\times \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \mathbb{Z} \arrow[u] \end{tikzcd}$$

Explicit theory

"formal complex multiplication".

<u>**Thm (Lubin–Tate)**</u>

For $f \in \mathbb{Z}_p[\![X]\!]$ s.t.

$$f \equiv X^p \mod p$$

$$\equiv pX \mod X^2.$$

$$(f = X^p + p(X + \cdots).)$$

$\exists$ unique comm. formal grp law over $\mathbb{Z}_p$

($F_f \in \mathbb{Z}_p[\![X_1, X_2]\!] \leadsto \operatorname{Spf} \mathbb{Z}_p[\![X]\!]$ is a formal grp)

Such that $[p] = f$.

($[p] \in \mathbb{Z}_p[\![X]\!] \leadsto \operatorname{Spf}\mathbb{Z}_p[\![X]\!] \xrightarrow{p} \operatorname{Spf}\mathbb{Z}_p[\![X]\!]$

Thm (Lubin-Tate)

For $f \in \mathcal{O}_F[[X]]$ s.t.

$$f \equiv X^q \mod \pi$$
$$\equiv \pi X \mod X^2.$$

$\exists$ unique formal $\mathcal{O}_F$-module law over $\mathcal{O}_F$ such that

$[\pi] = f.$

Rmk: The underlying formal $\mathcal{O}_F$-module

$\mathcal{G}_{LT}$ does not depend on f.

For f_1, f_2, $\exists$ unique

$$\operatorname{Spf} \mathcal{O}_F[[X]] \xrightarrow{\sim} \operatorname{Spf} \mathcal{O}_F[[X]]$$

intertwining the two laws

For $R/\mathcal{O}_E$, $\mathcal{G}_{LT}(R) \in \mathcal{O}_E\text{-mod}$

(abstract $\mathcal{O}_E$-module)

<u>Lem</u>: $\mathcal{G}_{LT}(R) = R^{\circ\circ}$ as set

(top. nil. elements in R)

<u>Prf</u>: $\text{LHS} = \text{Maps}_{\mathcal{O}_E}(\text{Spf}\, R, \text{Spf}\, \mathcal{O}_E[\![X]\!])$

$= \text{Maps}_{\mathcal{O}_E}(\mathcal{O}_E[\![X]\!], R)$

$= R^{\circ\circ}$ $\square$

$$\begin{array}{ccccc} A := \mathcal{G}_{LT}(\widehat{\overline{E}}) & \xrightarrow{\sim} & \mathfrak{m}_{\widehat{\overline{E}}} & \subset & \widehat{\overline{E}} \\ \downarrow \pi & & \downarrow f. & & \\ \mathcal{G}_{LT}(\overline{E}) & \longrightarrow & \mathfrak{m}_{\overline{E}} & & \end{array}$$

Thm (Lubin–Tate)

1) $\underline{A}[\pi^n] \simeq \pi^{-n}\mathcal{O}_E/\mathcal{O}_E$

as $\mathcal{O}_E$-module.

2) $E_n = E(A[\pi^n])$

E_n/E is totally ramified,

$G_{E_n/E} \simeq (\mathcal{O}_E/\pi^n\mathcal{O}_E)^\times$

$\pi \in \mathrm{Norm}(E_n^\times)$

Pf: WLOG $f = x^q + \pi x$.

$E_n = E(\text{roots of } \underbrace{f\circ f\circ\cdots\circ f}_{n})$

$\left(E_1 = E[x]/(x^{q-1}+\pi) \right)$. $\square$

Cor: $E_\infty = \bigcup E_n$
$\cap$
E_∞^{un} is the maximal abelian extension of E.

$$\begin{tikzcd} G_{E_\infty/E} \arrow[r, phantom, "\simeq"] \arrow[d] & \mathcal{O}_E^\times \arrow[dd] \\ G_{E_\infty^{un}/E} \arrow[d, phantom, "\shortparallel"] & \\ G_E^{ab} \arrow[r, "\sim"] & E^\times \end{tikzcd}$$

Rmk: E_∞ is perfectoid.

Proof: Need $\mathcal{O}_{E_\infty}/p \xrightarrow{\text{Frob}} \mathcal{O}_{E_\infty}/p$

$\Leftarrow f \equiv x^q \mod \pi$
$\mod p$

Thm:

$$\left(B_C(\mathcal{O}(1))\setminus\{0\}\right)/\underline{E}^\times \simeq \mathrm{Div}^1$$

Rmk:

$$\begin{array}{ccc} \mathrm{Div}^1 & \longrightarrow & [*/\underline{E}^\times]_{\mathbb{F}_q} \\ \parallel & & \\ \mathrm{Spd}\, E/\varphi & & \end{array}$$

$$\mathrm{Div}^1 \times \mathrm{Spa}\,\bar{\mathbb{F}}_q \longrightarrow [*/\underline{E}^\times]_{\bar{\mathbb{F}}_q}$$

$$\pi_1(\mathrm{RHS}) \simeq E^\times$$

$$\pi_1(\mathrm{Div}^1 \times \mathrm{Spa}\,\bar{\mathbb{F}}_q) = \pi_1(\mathrm{Spd}\,\breve{E}^{un}/\varphi^{\mathbb{Z}})$$
$$= W_E$$

$$(\mathrm{Spd}\,\breve{E}^{un})_{\text{pro-ét}} = (\mathrm{Spa}\,\breve{E}^{un})_{\text{pro-ét}}$$

$$= I_E\text{-sets }(\ldots\ldots)$$

$$1 \to I_E \to \pi_1(\operatorname{Spd} \breve{E}^{un}/\varphi) \to \mathbb{Z} \to 1$$

$$\begin{matrix} \pi_1(\operatorname{Spd} \breve{E}^{un}/\varphi) \\ \wr\!\! \text{I} \\ W_E \end{matrix}$$

$$W_E \longrightarrow E^\times.$$

will be Artin reciprocity map.

Recall:

$BC(\mathcal{O}(1))$: sheaf on $\mathrm{Perfd}_{\mathbb{F}_q}$

$$\begin{matrix} \cup \\ S = \operatorname{Spa}(R, R^+) \\ \varpi \in R^+ \end{matrix}$$

$$BC(\mathcal{O}(1))(S) := H^0(X_S, \mathcal{O}(1))$$

$$= H^0(Y_S, \mathcal{O})^{\varphi = \pi}$$

$$\left(\begin{cases} Y_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi]\pi = 0\} \\ X_S = Y_S / \varphi \end{cases} \right)$$

Calculate $H^0(Y_S, \mathcal{O})^{\varphi = \pi}$

- equal char case

$$E = \mathbb{F}_q((t))$$

$$A_{\inf, R^+} := W_{\mathcal{O}_E}(R^+) = R^+((t))$$

$$Y_S = \underline{\mathbb{D}}_S^*$$

$$H^0(Y_S, \mathcal{O}_{Y_S}) = \left\{ \sum_{n \in \mathbb{Z}} r_n \pi^n \,\middle|\, \begin{array}{c} r_n \in R \\ \cdots\cdots \end{array} \right\}$$

$$\varphi = \pi \iff \sum r_n^q \pi^n = \sum r_n \pi^{n+1}$$

$$\iff r_n = r_{n+1}^q$$

$$H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi = \pi} \xrightarrow{\sim} R^{\circ\circ}$$

$$\sum r_n \pi^n \longmapsto r_0$$

as $\underline{\underline{\text{sets}}}$!

$$\circlearrowright$$

$$\begin{array}{ccc} H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi = 1} & & r \\ \uparrow & & \uparrow \\ E & & r \end{array}$$

In general

$$R^{\circ\circ} \longrightarrow H^0(Y_S, \mathcal{O}_{Y_S})^{\varphi=\pi}$$

$$a \longmapsto \sum_n \frac{[a^{q^n}]}{\pi^n}$$

$$Y_S \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$$

$$[\varpi]\pi \neq 0$$

$$\left|\frac{\pi}{[\varpi]}\right| \in (0,\infty) = \bigcup [a,b]$$

$$Y_{[a,b]} \hookrightarrow \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+).$$

rational subset

Ex: $Y_{[\frac{1}{k},k]} = \mathrm{Spa}(B_{[\frac{1}{k},k]}, B^+_{[\frac{1}{k},k]})$

$$B^+_{[\frac{1}{k},k]} = A_{\inf}\left[\frac{\pi^k}{[\varpi]}, \frac{[\varpi]^k}{\pi}\right]^\wedge$$

$$B_{[\frac{1}{k},k]} = B^+[\pi^{-1}]$$

$$Y = \bigcup Y_{[\frac{1}{k},k]}$$

$$H^0(Y,\mathcal{O}) = \lim B_{[\frac{1}{k},k]}$$

<u>Rmk</u>: Unknown:

$$g \in H^0(Y_S, \mathcal{O}_{Y_S})$$

admits $\quad \sum \frac{[r_n]}{\pi^n}$

$$R^{\circ\circ} \longrightarrow \boxed{H^0(Y_S, \mathcal{O}_S)}^{\varphi=p}$$

$$a \longmapsto \sum_{n\in\mathbb{Z}} \frac{[a^{p^n}]}{\pi^n}$$

<u>Ex</u>: $E = \mathbb{Q}_p$ $\qquad S = \mathrm{Spa}(C, C^+)$

$$\begin{tikzcd} & 1+\mathfrak{m}_C \arrow[dr, "\log[\ ]"] & \\ \underline{\mathfrak{m}_C} \arrow[ur, dotted, "\cancel{\exp}\ E"] \arrow[rr] & & B^{\varphi=p} \end{tikzcd}$$

$\oplus$

$$\sum \frac{[(a\oplus b)^{p^n}]}{p^n} = \sum \frac{[a^{p^n}]}{p^n} + \sum \frac{[b^{p^n}]}{p^n}$$

$$\log[x] := \log([x]-1+1)$$

$$= 1-([x]-1)+\frac{([x]-1)^2}{2}-\cdots$$

$$\begin{array}{ccc} \widehat{M}_c & \overset{E}{\dashrightarrow} & 1+\widehat{M}_c \\ \oplus & & * \end{array}$$

E: Artin-Hasse exponential.

$$E(x) = \exp\left(x+\frac{x^p}{p}+\frac{x^{p^2}}{p^2}+\cdots\right)$$

$$\exp(x) = \prod_{d>0}\left(\frac{1}{1-x^d}\right)^{\frac{\mu(d)}{d}}$$

$\mu(d)$ Mobius function

$$E(x) = \prod_{(d,p)=1} \left(\frac{1}{1-x^d} \right)^{\frac{\mu(d)}{d}}.$$

$$\begin{tikzcd}
 & 1+m_C \arrow[dr, "\log[\,]"] & \\
m_C \arrow[ur, "E"] \arrow[rr, "a \,\mapsto\, \sum \frac{[a^{p^n}]}{p^n}"'] & & B^{\varphi=p}
\end{tikzcd}$$

$$\boxed{(m_C, \oplus)} \longrightarrow \boxed{(m_C, +)} \xrightarrow{\exp} (1+m_C, \times)$$

$$x \longmapsto \sum_{n\geq 0} \frac{x^{p^n}}{p^n} = \log_C$$

Lem: Consider $\hat{\mathbb{G}}_{a,E}$ $\quad E[\![X_1, X_2]\!]$

$$\Downarrow$$

$$X_1 + X_2.$$

$$\operatorname{Spf} E[\![X]\!] \longrightarrow \underline{\operatorname{Spf} E[\![X]\!]},$$

$$\sum_{n \geq 0} \frac{X^{q^n}}{\pi^n} \longleftarrow\!\!\mapsto X$$

$\rightsquigarrow$ formal grp law

$$F_{LT} \in E[\![X_1, X_2]\!]$$

$$F_{LT} \in \mathcal{O}_E[\![X_1, X_2]\!].$$

$\square$

Def: "The" Lubin-Tate formal grp law is defined to be given by FLT.

$$\mathcal{G}_{LT} = \mathrm{Spf}\, \mathcal{O}_E[\![X]\!]$$

s.t.

$$\log : \underline{\mathcal{G}_{LT,E}} \xrightarrow{\sim} \hat{\mathbb{G}}_{a,E}$$

$$\sum_{n \geq 0} \frac{X^{q^n}}{\pi^n} \longleftarrow\!\shortmid X$$

$$\hat{G}_{LT} = \lim\left(\cdots \xrightarrow{\pi} G_{LT} \xrightarrow{\pi} G_{LT} \right)$$

$$\left(\mathcal{O}_E[\![x]\!] \xrightarrow{x \mapsto [\pi]} \mathcal{O}_E[\![x]\!] \to \cdots \right)$$

Spf

$\cong$

$\hat{G}_{LT}$

$A \qquad \mathcal{O}_E$

$$\hat{G}_{LT}(A) \to G_{LT}(A)$$

$$\cong$$

$$\mathrm{Map}_{\mathcal{O}_E}(\mathrm{Spf}\,A,\ G_{LT})$$

$$\cong$$

$$\mathrm{Map}_{\mathcal{O}_E}(\mathrm{Spf}\,A,\ G_{LT})$$

$$\downarrow$$

$$\mathrm{Map}_E\left(\mathrm{Spa}\,A \times_{\mathrm{Spa}\,\mathcal{O}_E} \mathrm{Spa}\,E,\ \underline{G_{LT,\eta}}\right)$$

$$\downarrow \qquad \log = \sum_{n\geq 1} \frac{x^{q^n}}{\pi^n}$$

$$\mathrm{Map}_E(\mathrm{Spf} A \times_{\mathrm{Spf} \mathcal{O}_0} \mathrm{Spf} \bar{E},\ \hat{G}_{a,g})$$

$$\downarrow$$

$$H^0((\mathrm{Spf} A)_g, \mathcal{O})$$

<u>Cont</u>:

$$\hat{GL}_I(A) \longrightarrow GL_I(A) \longrightarrow H^0((\mathrm{Spf} A)_g, \mathcal{O})$$

$$R^{\#, +}[\pi^{-1}]$$

Thm (Prop II 2.2) "$R^\#$" of R

$$\begin{tikzcd}
\hat{G}_{LT}(R^+) \arrow[d, "\cong"] & \hat{G}_{LT}(A_{inf}(R^+)) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \hat{G}_{LT}(R^{\#,+}) \arrow[d] \\
G_{LT}(R^+) \arrow[d] \arrow[dr, dashed] & G_{LT}(A_{inf}(R^+)) \arrow[l] \arrow[r] \arrow[d] & G_{LT}(R^{\#,+}) \arrow[d] \\
0 & P(Y_S, \mathcal{O}) \arrow[l] \arrow[r] \arrow[d] & R^\# \\
 & P(Y_S, \mathcal{O}) &
\end{tikzcd}$$

$G_{LT}(R^+) = R^{0,0}$

1) $\cong \quad \cong \quad \cong$

2). $G_{LT}(R^+) \xrightarrow{\cong} P(Y_S, \mathcal{O})^{\varphi = \pi}$

3) $R^{00} \longrightarrow (P(Y_S, \mathcal{O}))^{\varphi = \pi}$

$x \longmapsto \sum \frac{[x^{q^n}]}{\pi^n}$

Cor:

$$BC(\mathcal{O}(1)) \simeq \operatorname{Spd} \mathbb{F}_q [\![x^{1/p^\infty}]\!]$$

Lem $\hat{G}_{LT} = \operatorname{Spa} \mathcal{O}_E [\![x^{1/p^\infty}]\!]$

Pf: $\left(\mathcal{O}_E [\![x_0]\!] \xrightarrow{[\pi]} \mathcal{O}_E [\![x_1]\!] \to \cdots \right)$

$$\wr\!\!\!\downarrow \qquad\qquad [\pi] = x^q \text{ mod } \pi$$

$$\left(\mathcal{O}_E [\![x^{1/p^\infty}]\!] \right)$$

$$x \equiv x_n^{q^n} \quad \text{mod } \pi^n$$

$$x = \lim_{n \to \infty} x_n^{q^n}$$

$\square$

Prf of Cor.:

$$\underline{BC(\mathcal{O}(1))}(S) = H^0(Y_S, \mathcal{O})^{\varphi = \mathrm{id}}$$

$$\overset{\text{Thm}}{=} \underset{\sim}{R}^{00}$$

$$\cong \mathrm{Mor}_S(S, \mathrm{Spd}\,\mathbb{F}_q[[x^{1/p^\infty}]])$$

$$= \mathrm{Mor}_S(\mathbb{F}_q[[x^{1/p^\infty}]], R^+)$$

$\square$

Lem:

$$0 \to \mathcal{G}_{LT,\eta}[\pi^\infty] \to \mathcal{G}_{LT,\eta} \twoheadrightarrow \hat{\mathbb{G}}_{a,\eta} \to 0$$

$$\log = \sum_{n \geq 0} \frac{x^{q^n}}{\pi^n}$$

$$G_{LT,g}[\pi^\infty] = (G_{LT}[\pi^\infty])_g$$

$$= (\cup \underline{G_{LT}[\pi^n]})_g$$

$$G_{LT}[\pi^n] = \mathrm{Spf}\,(\underline{O_E[[X]] / [\pi^n]})$$

$$= \mathrm{Spf}\ A_n$$

$$\cdots A_{n+1} \xrightarrow{[\frac{1}{\pi}]} A_n \xrightarrow{[\frac{1}{\pi}]} A_{n-1} \xrightarrow{[\frac{1}{\pi}]} \cdots$$

$$[\pi] = X^q + \pi(X + \cdots)$$

$$= \cancel{X}(\underline{X^{q-1} + \pi + \cdots})$$

$$\underbrace{(\mathrm{Spf}\,A_n)_g = (\mathrm{Spf}\,A_{n-1})_g}\times \mathrm{Spa}\,E_n$$

Lem.

$$1 \to G_{LT,y}[\pi^\infty] \to G_{LT,y} \to \hat{G}_{a,y} \to 1$$

$$\| $$

$$\coprod_n \operatorname{Spa} E_n.$$

..

Cnstr:

$$\begin{tikzcd} \operatorname{Spa} \mathcal{O}_{\tilde{E}^\infty} \arrow[r] & T_\pi \hat{G}_{LT} \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \hat{G}_{LT} \arrow[d] \\ & 1 \arrow[r] & G_{LT} \end{tikzcd}$$

$$\begin{tikzcd} T_\pi \hat{G}_{LT} \simeq (\cdots & G_{LT}[\pi^2] \arrow[r] & G_{LT}[\pi] \arrow[r] & 1) \\ \operatorname{Spa} \mathcal{O}_{\tilde{E}^\infty} \simeq (\arrow[u] & \operatorname{Spa} \mathcal{O}_{E_2} \arrow[u] \arrow[r] & \operatorname{Spa} \mathcal{O}_{E_1} \arrow[u] \arrow[r] & 1) \end{tikzcd}$$

$$\begin{tikzcd}
\alpha : \operatorname{Spa} \mathcal{O}_{E_\infty} \arrow[r] \arrow[d] & \widetilde{G}_{LT} \arrow[d] \\
* \arrow[r] & G_{LT}
\end{tikzcd}$$

$$\begin{tikzcd}
\widetilde{G}_{LT}(A_{\inf}) \arrow[r] \arrow[d] & \widetilde{G}_{LT}(R^{\#,+}) \arrow[d] \\
G_{LT}(A_{\inf}) \arrow[r] \arrow[d] & G_{LT}(R^{\#,+}) \arrow[d] \\
H^0(Y_{S}, \mathcal{O}) \arrow[r] & R^{\#}
\end{tikzcd}$$

If $R^{\#}$ is defined over E_∞

| $\operatorname{Spa} R^{\#,+} \longrightarrow \widetilde{G}_{LT}$

$$\downarrow \qquad \nearrow 2.$$

$$\mathrm{Spa}\, E_\infty$$

<u>Constr</u>:

If $S \to \mathrm{Spd}\, E_\infty$

$(S^\# \longrightarrow \mathrm{Spa}\, E_\infty)$,

then obtain a section

$$\mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{X_S}(1)$$

vanishes at $S^\#$.

<u>Prop</u> (II.2.3) If $S \to \mathrm{Spd}\, E_\infty$

$$0 \to \mathcal{O}_{X_S} \longrightarrow \mathcal{O}_{X_S}(1) \longrightarrow \mathcal{O}_{S^\#} \to 0$$

$$\left(\begin{array}{l} \mathcal{O}_{X_S} \hookrightarrow I(1) \\ \quad I \text{ is the ideal defn } S^\# \end{array} \right)_{/\Omega}$$

<u>Thm:</u> $(B_C(\mathcal{O}(1))\setminus 0)/\underline{E}^\times \cong \mathrm{Div}^1$

<u>Pmt</u>: $B_C(\mathcal{O}(1)) \cong \mathrm{Spd}\, \overline{\mathbb{F}}_q [\![x^{1/p^\infty}]\!]$

$B_C(\mathcal{O}(1))\setminus 0 \cong \mathrm{Spd}\, \overline{\mathbb{F}}_q (\!(x^{1/p^\infty})\!)$

$= \mathrm{Spd}\, E_\infty$

$$\begin{tikzcd}
B_C(\mathcal{O}(1))\setminus 0 \arrow[rr] \arrow[d, "\cong"] & & \mathrm{Div}^1 \arrow[d, "\cong"] \\
\mathrm{Spd}\, E_\infty \arrow[r] & \mathrm{Spd}\, \breve{E} \arrow[r] & \mathrm{Spd}\, \breve{E}/\varphi
\end{tikzcd}$$

$\underline{\mathcal{O}_E^\times}$-torsor

$\underline{E}^\times$-torsor

we want a log such that

$$\underline{\log}(x_0) = \sum_{n\in\mathbb{Z}} \frac{x^{q^n}}{\pi^n}$$

$$\operatorname{colim}\ \mathcal{O}_E[[X_0]] \xrightarrow{[\pi]} \mathcal{O}_E[[X_1]] \to \cdots$$

is

$$\mathcal{O}_E[[X]]$$

$$X = \lim_n X_n^{q^n}$$

$X \equiv X_n^{q^n} \mod \pi^n$

$$\log := \sum_{n \geq 0} \frac{x^{n+1}}{\pi^{n}}$$

$$\log(x_0) \qquad \log(x)$$

$$\log x_0 = \log(x_1^{t})$$
