LECTURE 3

The goal of this lecture is to introduce a combinatorial approach to the homotopy theory of topological spaces, known as **Kan–Quillen model category of simplicial sets**.

1. Simplicial sets

Definition 1.1. *For $n \in \mathbb{Z}_{\geq 0}$, let*

$$[n] \stackrel{\text{def}}{=} \{0 < 1 < \cdots < n\}$$

be the totally ordered set with $(n+1)$ objects. We view it as a category in the standard way.

Definition 1.2. *The **simplex category** Δ is the full subcategory of Cat consisting of $[n] \in \mathsf{Cat}, n \in \mathbb{Z}_{\geq 0}$, i.e.,*

$$\mathsf{Hom}_\Delta([m],[n]) = \{\textit{nondecreasing functions } [m] \to [n]\}.$$

Let Δ_{inj} and Δ_{surj} be the subcategories of Δ consisting of injective and surjective morphisms respectively.

1.3. Let $\mathcal{C}$ be a category. We have

$$\mathsf{Hom}_{\mathsf{Cat}}([n],\mathcal{C}) \simeq \{\text{chains in } \mathcal{C} \text{ of length } n\}.$$

The ubiquitous role of the simplex category Δ in category theory can be explained by the following result.

Proposition 1.4. *The functor*

$$\mathsf{Cat} \to \mathsf{Fun}(\Delta^{\mathsf{op}},\mathsf{Set}),\ \mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}}(-,\mathcal{C}) \tag{1.1}$$

is fully faithful.

Definition 1.5. *Let $\mathcal{D}$ be a category.*

- *A **simplicial object of** $\mathcal{D}$ is a functor $\Delta^{\mathsf{op}} \to \mathcal{D}$.*
- *A **cosimplicial object of** $\mathcal{D}$ is a functor $\Delta \to \mathcal{D}$.*

1.6. Let X be a simplicial object, we often write $X_n \stackrel{\text{def}}{=} X([n])$, and therefore denote this simplicial object also by $X_\bullet$. Similarly, a cosimplicial object is often denoted by $Y^\bullet$.

Definition 1.7. *Write*

$$\mathsf{Set}_\Delta \stackrel{\text{def}}{=} \mathsf{Fun}(\Delta^{\mathsf{op}},\mathsf{Set})$$

*for the category of **simplicial sets**.*

Example 1.8. *The representable functor*

$$\mathsf{Hom}_\Delta(-,[n]) : \Delta^{\mathsf{op}} \to \mathsf{Set}$$

*defines a simplicial set Δ^n, called the n-**simplex**.*

Date: Sep. 20, 2024.

1.9. Let X be a simplicial set. By the Yoneda lemma, we have

$$\mathsf{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, X) \simeq X_n.$$

This motivates the following definition:

Definition 1.10. *Let $X_\bullet$ be a simplicial set. An element $x \in X_n$ is called an **n-simplex of** X.*

Definition 1.11. *Let*

$$\mathsf{N}_\bullet(-) : \mathsf{Cat} \to \mathsf{Set}_\Delta$$

be the functor (1.1)*. For a category $\mathcal{C}$, the simplicial set $\mathsf{N}_\bullet(\mathcal{C})$ is called the **nerve of** $\mathcal{C}$.*

1.12. Proposition 1.4 says the theory of categories can be embedded into the theory of simplicial sets via the construction $\mathcal{C} \mapsto \mathsf{N}_\bullet(\mathcal{C})$. Therefore,

Slogan 1.13. *Simplicial sets generalize* categories.

2. Faces and degeneracies

Definition 2.1. *For $n > 0$, let*

$$\delta_n^i : [n-1] \to [n]$$

be the unique functor such that $i \in [n]$ is not in the image.

*Let X be a simplicial object in a category $\mathcal{D}$. The **i-th face operator** on X_n is the morphism*

$$d_i^n \overset{\text{def}}{=} X(\delta_n^i) : X_n \to X_{n-1}.$$

Definition 2.2. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. The $(n-1)$-simplex $d_i^n(x)$ is called the **i-th face** of x.*

*More generally, for any injective functor $\iota : [m] \to [n]$, the m-simplex $X(\iota)(x)$ is called the **ι-face** of x.*

Definition 2.3. *For $n \geq 0$, let*

$$\sigma_n^i : [n+1] \to [n]$$

be the unique surjective functor that is constant on $\{i, i+1\}$.

*Let X be a simplicial object in a category $\mathcal{D}$. The **i-th degeneracy operator** on X_n is the morphism*

$$s_i^n \overset{\text{def}}{=} X(\sigma_n^i) : X_n \to X_{n+1}.$$

Definition 2.4. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. The $(n+1)$-simplex $s_i^n(x)$ is called the **i-th degeneracy** of x.*

*More generally, for any surjective functor $\pi : [m] \to [n]$, the m-simplex $X(\pi)(x)$ is called the **π-degeneracy** of x.*

Definition 2.5. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. We say x is **non-degenerate** if it is not a degeneracy of any m-simplex with $m < n$.*

2.6. The proof of the following result is elementary and left to the readers.

Lemma 2.7. *Any degenerate simplex is a π-degeneracy of some non-degenerate simplex x, and the pair (π, x) is unique.*

Exercise 2.8. *Any morphism in Δ is equal to a composition of δ's and σ's.*

Remark 2.9. *It follows that knowing a simplicial object $X_\bullet$ is equivalent to knowing objects X_n and morphisms d_i^n, s_i^n satisfying certain relations. These relations, known as* ***simplicial identities****, can be written down explicitly:*

$$d_i \circ d_j = d_{j-1} \circ d_i, \ \ \text{if } i < j;$$

$$s_i \circ s_j = s_j \circ s_{i-1}, \ \ \text{if } i > j;$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & \text{if } i < j \\ \mathsf{id}, & \text{if } i = j, j+1 \\ s_j \circ d_{i-1}, & \text{if } i > j+1. \end{cases}$$

Here we omit the superscipts from the notations.

2.10. We can depict the face and degeneracy morphisms as a diagram

$$X_0 \leftleftarrows X_1 \; \cdots \; X_2 \cdots.$$

Sometimes people omit the degeneracy morphisms, and use

$$X_0 \leftleftarrows X_1 \Lleftarrow X_2 \cdots.$$

to indicate a simplicial object, especially when they study the *colimit* of this diagram[1].

3. Dimension and skeletons

Definition 3.1. *Let X be a simplicial set and $k \in \mathbb{Z}$, we say X has* ***dimension*** $\boldsymbol{\leq k}$*, or $\mathsf{dim}(X) \leq k$, if every n-simplex of X is degenerate for $n > k$.*

Example 3.2. $\mathsf{dim}(\Delta^n) = n$.

Definition 3.3. *Let X be a simplicial set and $k \leq \mathbb{Z}$. The* ***k-skeleton*** $\mathsf{sk}_k(X)$ *of X is the largest simplicial subset of X with dimension $\leq k$.*

4. Examples

4.1. We first introduce a standard way to draw a simplicial set $X_\bullet$, which is also how people actually think about them.

- Only non-degenerate simplexes are drawn. Degenerate simplexes "collapse" onto the non-degenerate ones that correspond to them in the sense of Lemma 2.7.

(0) For each $v \in X_0$, draw a vertex labelled by v.
(1) For each non-degenerate $e \in X_1$, draw an arrow labelled by e from $d_0^1(e)$ to $d_1^1(e)$.
(n) For each non-degenerate $\sigma \in X_n$, draw a *filled* n-simplex labelled by σ, with boundary given by $d_0^n(\sigma), d_1^n(\sigma), \ldots, d_n^n(\sigma)$[2].

[1] In future lectures, we will show that the degeneracy morphisms do not affect the ∞-colimit/homotopy colimit of a simplicial object.

[2] Note that these faces can be degenerate.

- When necessary, put symbols inside the simplexes to indicate the order of its vertices.

Exercise 4.2. *Find all the non-degenerate simplexes in* Δ^n.

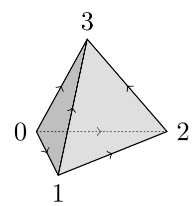

FIGURE 1. The 3-simplex Δ^3

Exercise 4.3. *Let* $\Delta^1 \times \Delta^1$ *be the product taken in* Set_Δ. *Find all the non-degenerate simplexes in* $\Delta^1 \times \Delta^1$.

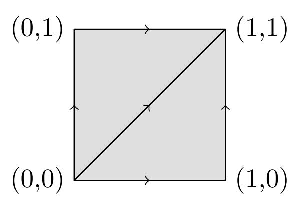

FIGURE 2. The product $\Delta^1 \times \Delta^1$

Exercise 4.4. *In the above example, what would happen if we disallow degeneracies in the definition of simplicial sets, i.e., replacing* Δ *with* Δ_{inj}?

Example 4.5. *Let*

$$\partial\Delta^n \stackrel{\text{def}}{=} \mathsf{sk}_{n-1}(\Delta^n)$$

be the $(n-1)$*-skeleton of* Δ^n. *We call it the* ***boundary of*** Δ^n.

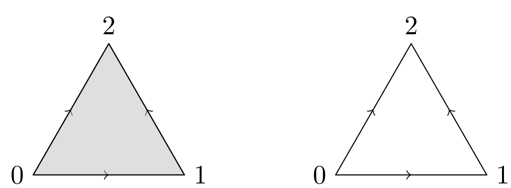

FIGURE 3. The 2-simplex Δ^2 and its boundary $\partial\Delta^2$

Example 4.6. *Let* Λ^n_i *be the largest simplicial subset of* Δ^n *that does not contain the* i*-th face of the unique non-degenerate* n*-simplex. We call it the* ***i-th horn of of*** Δ^n.

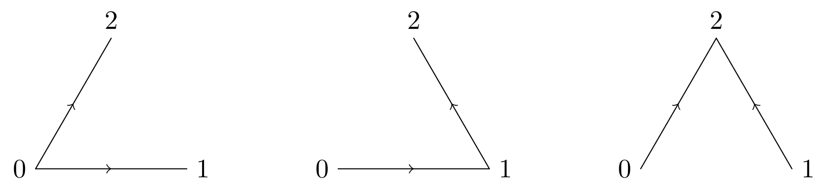

FIGURE 4. The horns Λ^2_0, Λ^2_1 and Λ^2_2

4.7. The readers might entertain themselves with the following exercise to check their understanding about the definitions.

Exercise 4.8. *Classify all simplicial quotient sets of Δ^2, i.e., simplicial sets X equipped with an epimorphism $\Delta^2 \to X$ in Set_Δ. Hint: there are 31 of them.*

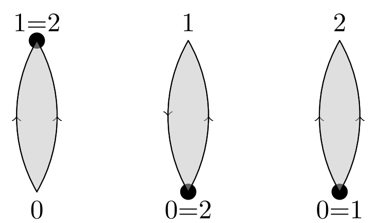

FIGURE 5. Some quotients of Δ^2

5. Simplicial sets and topological spaces

5.1. The simplex category Δ can be realized as a subcategory of Top as follows

Construction 5.2. *Let*

$$|\Delta^n| \overset{\text{def}}{=\!=} \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \,|\, x_0 + \cdots + x_n = 1\}$$

be the ***standard (topological)*** n***-simplex****. We have a functor*

$$\Delta \to \mathsf{Top}, [n] \mapsto |\Delta^n| \tag{5.1}$$

sending a functor $f : [m] \to [n]$ to the continuous map

$$|\Delta^m| \to |\Delta^n|, (y_0, \cdots, y_m) \mapsto (\sum_{j \in f^{-1}(0)} y_j, \cdots, \sum_{j \in f^{-1}(n)} y_j).$$

Proposition 5.3. *We view Δ as a full subcategory of Set_Δ via the Yoneda embedding. Then the functor* (5.1) *can be extended to a colimit-preserving functor*

$$|-| : \mathsf{Set}_\Delta \to \mathsf{Top},$$

which is unique up to a unique equivalence*. We call it the* ***geometric realization functor.***

This functor admits a right adjoint given by

$$\mathsf{Sing} : \mathsf{Top} \to \mathsf{Set}_\Delta, X \mapsto \mathsf{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X).$$

We call it the ***singlular simplicial complex***[3] ***functor****.*

[3]A better name would be singular simplicial *set* functor because we allow degeneracies.

5.4. The adjoint pair
$$|-|:\mathsf{Set}_\Delta \rightleftarrows \mathsf{Top}:\mathsf{Sing}$$
belongs to the following paradigm in category theory.

Exercise 5.5. *Let $\mathcal{C}_0$ be a small category. Define the* ***category of presheaves on*** *$\mathcal{C}_0$ to be*
$$\mathsf{PShv}(\mathcal{C}_0) \overset{\text{def}}{=\!=} \mathsf{Fun}(\mathcal{C}_0^{\mathsf{op}}, \mathsf{Set}).$$

(1) *Prove that $\mathsf{PShv}(\mathcal{C}_0)$ is the category* ***freely generated under small colimits*** *by $\mathcal{C}_0$. In other words, for any category $\mathcal{D}$ containing all small colimits, the Yoneda embedding $\mathcal{C}_0 \to \mathsf{PShv}(\mathcal{C}_0)$ induces an equivalence*
$$\mathsf{LFun}(\mathsf{PShv}(\mathcal{C}_0), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}_0, \mathcal{D}),$$
where $\mathsf{LFun}(-,-) \subset \mathsf{Fun}(-,-)$ consists of colimit-preserving functors.

(2) *Let $F : \mathsf{PShv}(\mathcal{C}_0) \to \mathcal{D}$ be a colimit-preserving functor extending $F_0 : \mathcal{C}_0 \to \mathcal{D}$. Prove that F admits a right adjoint given by*
$$G : \mathcal{D} \to \mathsf{PShv}(\mathcal{C}_0),\ d \mapsto \mathsf{Hom}_{\mathcal{D}}(F_0(-), d).$$

Exercise 5.6. *Challenge: construct an adjoint pair $L : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Cat} : R$ by applying the above paradigm to the functor $\Delta \to \mathsf{Cat}$. Describe the images of the simplicial sets in §4 under the functor.*

5.7. Unlike the nerve functor $\mathsf{N}_\bullet : \mathsf{Cat} \to \mathsf{Set}_\Delta$, the functor $\mathsf{Sing} : \mathsf{Top} \to \mathsf{Set}_\Delta$ is not fully faithful. Hence we cannot embed the theory of topological spaces into the theory of simplicial sets. Nevertheless, the following result, established by Quillen in the 1960s (see [Qui67]), says the *homotopy theories* of them are the same.

Theorem 5.8 (Quillen)**.** *The adjoint pair*
$$|-|:\mathsf{Set}_\Delta \rightleftarrows \mathsf{Top}:\mathsf{Sing}$$
induces an equivalence between the homotopy theories of topological spaces and simplicial sets.

6. Classical model structure on Set_Δ

6.1. To explain Quillen's result, we need first define a model structure on Set_Δ.

Theorem-Definition 6.2. *There exists a model structure on Set_Δ given by*

(W) *A* ***weak homotopy equivalence*** *is a morphism $f : X \to Y$ such that $|f| : |X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence.*

(C) *A* ***cofibration*** *is a monomorphism.*

(F) *A* ***Kan fibration*** *is a morphism $f : X \to Y$ that has the right lifting property against all horn inclusions $\Lambda^n_i \to \Delta^n$ for $0 \le i \le n$ and $n \ge 2$:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

We call it the ***classical****, or* ***Kan–Quillen*****,** ***model structure*** *on Set_Δ.*

Fibrant objects in this model category are called ***Kan complexes****. Let $\mathsf{Kan} \subset \mathsf{Set}_\Delta$ be the full subcategory consisting of Kan complexes.*

Exercise 6.3. *Find a fibrant replacement of Δ^1, i.e., a weak homotopy equivalence $\Delta^1 \to X$ such that X is a Kan complex.*

Exercise 6.4. *Find a fibrant replacement of $\partial\Delta^2$.*

Remark 6.5. *It is possible to define weak homotopy equivalences between simplicial sets without appealing to the geometric realization functor. Namely, we define:*

(W) *A weak homotopy equivalence is a morphism $f : X \to Y$ such that for any Kan complex Z, the map*

$$[Y, Z] \to [X, Z]$$

is a bijection, where the set $[X, Z]$ is the coequalizer of

$$\mathsf{Hom}_{\mathsf{Set}_\Delta}(X \times \Delta^1, Z) \rightrightarrows \mathsf{Hom}_{\mathsf{Set}_\Delta}(X, Z).$$

These two definitions are a posteriori *equivalent because of Quillen's Theorem 5.8 (see Theorem 8.1 for its precise form) and the Yoneda lemma. However, choosing one definition rather than the other can significantly affect how one checks the axioms of model categories and proves Quillen's theorem. For the purpose of this course, these proofs are blackboxed, and the readers are encouraged to treat* both *statements as the definition.*

7. Equivalence between homotopy theories

7.1. In this section we explain the meaning of an equivalence between two homotopy theories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair between model categories[4]. We need to answer the following question:

> *When does this adjoint pair induce an equivalence between the homotopy theories underlying $\mathcal{C}$ and $\mathcal{D}$?*

The answer would be

- The adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ should induce an adjoint pair
$$F' : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : G'$$
such that F' and G' are inverse[5] to each other.

However, to make sense of this, we need to articulate the definition of the functors F' and G'. A touchstone for such a definition is the following example from homological algebra.

Example 7.2. *Let $\mathcal{A}_i$ be abelian categories such that the projective model structures on $\mathsf{Ch}^{\leq 0}(\mathcal{A}_i)$ is well-defined (see [Lecture 2, Example 2.7]). We have*

$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_i)[W^{-1}] \simeq \mathsf{D}^{\leq 0}(\mathcal{A}_i),$$

where $\mathsf{D}^{\leq 0}(\mathcal{A}_i)$ is the connective (= non-positive) part of the derived category of $\mathcal{A}_i$. Let $F_0 : \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor and consider the functor

$$F : \mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \to \mathsf{Ch}^{\leq 0}(\mathcal{A}_2)$$

[4] We do not require F or G to preserve (W) or (C) or (F).

[5] The classical terminology would be *quasi-inverse*, i.e., we require $\mathsf{Id} \to G' \circ F'$ and $F' \circ G' \to \mathsf{Id}$ to be equivalences rather than equalities. The latter requirement violates the principle of equivalence hence does not make sense in higher category theory. Therefore we omit the prefix *quasi*.

*induced by F_0. Then we want F' to be the **left derived functor** $\mathbb{L}F$, which can be calculated by*

$$\mathbb{L}F(M^\bullet) \simeq F(P^\bullet),$$

where $P^\bullet \to M^\bullet$ is a cofibrant replacement, a.k.a. a projective resolution, of $M^\bullet$.

One needs additional assumptions on the functor F_0 to guarantee that $F(P^\bullet) \in \mathsf{D}^{\leq 0}(\mathcal{A}_2)$ does not depend on the choice of $P^\bullet$.

Example 7.3. *Dually, for the* injective *model categories $\mathsf{Ch}^{\geq 0}(\mathcal{A}_i)$, we want to recover the definition of **right derived functors**.*

7.4. Note that in the above example, the functor F does not preserve quasi-isomorphisms (because F_0 is not exact). Hence in the general setting of §7.1, we should not ask F or G to preserve weak equivalences between *all* objects. In particular, we *cannot* expect the following diagram to commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, "F'"] \\ \mathcal{D} \arrow[r] & \mathcal{D}[W^{-1}] \end{tikzcd}$$

Nevertheless, in homological algebra, derived functors provide best approximations to such a commutative square. To explain what this means, we need some definitions.

Definition 7.5. *Let $\pi : \mathcal{C} \to \mathcal{C}'$ be a functor between categories. For any category $\mathcal{E}$, we have a functor*

$$\pi^* : \mathsf{Fun}(\mathcal{C}', \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{E})$$

*given by precomposing with π. The left (resp. right) adjoint of this functor, when exists, is called the **left** (resp. **right**) **Kan extension** along π, and is denoted by*

$$\mathsf{LKE}_\pi : \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}', \mathcal{E}),$$
$$\mathsf{RKE}_\pi : \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}', \mathcal{E}).$$

Exercise 7.6. *For a functor $f : \mathcal{C} \to \mathcal{E}$, find the universal properties that characterize the functors $\mathsf{LKE}_\pi f$ and $\mathsf{RKE}_\pi f$. Hint:*

$$\begin{tikzcd} & \mathcal{C}' \arrow[dr, dotted, "\mathsf{LKE}_\pi f"] & \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[rr, "f"'] & & \mathcal{D}, \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}' \arrow[dr, dotted, "\mathsf{RKE}_\pi f"] & \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[rr, "f"'] & & \mathcal{D} \end{tikzcd}$$

are closest to be commutative.

Example 7.7. *Let $F_0 : \mathcal{A}_1 \to \mathcal{A}_2$ be as in Example 7.2. Then the left derived functor*

$$\mathbb{L}F : \mathsf{D}^{\leq 0}(\mathcal{A}_1) \to \mathsf{D}^{\leq 0}(\mathcal{A}_2)$$

is defined as the right *Kan extension*[6] *of the functor*

$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \xrightarrow{F} \mathsf{Ch}^{\leq 0}(\mathcal{A}_2) \xrightarrow{\pi_2} \mathsf{D}^{\leq 0}(\mathcal{A}_2)$$

[6]This reversal of handedness is unfortunate but unavoidable.

along

$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \xrightarrow{\pi_1} \mathsf{D}^{\leq 0}(\mathcal{A}_1).$$

In diagram:

(7.1)
$$\begin{tikzcd} \mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \arrow[r, "\pi_1"] \arrow[d, "F"'] & \mathsf{D}^{\leq 0}(\mathcal{A}_1) \arrow[d, dotted, "\mathbb{L}F \overset{\mathrm{def}}{=} \mathsf{RKE}_{\pi_1}(\pi_2 \circ F)"] \arrow[dl, Rightarrow, dotted] \\ \mathsf{Ch}^{\leq 0}(\mathcal{A}_2) \arrow[r, "\pi_2"'] & \mathsf{D}^{\leq 0}(\mathcal{A}_2). \end{tikzcd}$$

Similarly, the right derived functor is defined as a left *Kan extension.*

Exercise 7.8. *Convince yourself that left derived functor should be a right Kan extension rather than a left one by evaluating* (7.1) *on a complex* $M^\bullet$*.*

7.9. Motivated by the above, we can define derived functors in *homotopical* algebra.

Definition 7.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a (plain) functor between model categories. The* ***left derived functor*** *of F:*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \to \mathcal{D}[W^{-1}]$$

is defined to be the following right *Kan extension*

(7.2)
$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dotted, "\mathbb{L}F \overset{\mathrm{def}}{=} \mathsf{RKE}"] \arrow[dl, Rightarrow, dotted] \\ \mathcal{D} \arrow[r] & \mathcal{D}[W^{-1}]. \end{tikzcd}$$

Similarly, the ***right derived functor*** *is defined as a* left *Kan extension.*

Remark 7.11. *We can abuse notation and write the natural transformation* (7.2) *as morphisms*

$$\mathbb{L}F(X) \to F(X) \text{ in } \mathcal{D}[W^{-1}]$$

that are functorial for X in $\mathcal{C}$.

Similarly, for a functor $G : \mathcal{D} \to \mathcal{C}$, we have

$$G(Y) \to \mathbb{R}G(Y) \text{ in } \mathcal{C}[W^{-1}]$$

that are functorial for Y in $\mathcal{D}$.

7.12. Note that the definition of derived functors does not use the classes of cofibrations and fibrations. The following result provides a convenient tool to calculate derived functors using these morphisms. See e.g. [DS95, §9].

Theorem 7.13. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between model categories.*

(1) *Suppose F sends weak equivalences between cofibrant objects to weak equivalences. Then the left derived functor $\mathbb{L}F$ exists. Moreover, for any cofibrant object X in $\mathcal{C}$, we have $\mathbb{L}F(X) \xrightarrow{\simeq} F(X)$.*
(2) *Suppose G sends weak equivalences between fibrant objects to weak equivalences. Then the right derived functor $\mathbb{R}G$ exists. Moreover, for any fibrant object Y in $\mathcal{D}$, we have $G(Y) \xrightarrow{\simeq} \mathbb{R}G(Y)$.*

Exercise 7.14. *Explain how to calculate the left/right derived functor using cofibrant/fibrant replacements. What do you get for Example 7.2?*

7.15. Motivated by the above, we make the following definition.

Proposition-Definition 7.16. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair between model categories. The following conditions are equivalent:*

(i) *F preserves cofibrations and acyclic cofibrations;*
(ii) *G preserves fibrations and acyclic fibrations;*
(iii) *F preserves cofibrations and G preserves fibrations;*
(iv) *F preserves acyclic cofibrations and G preserves acyclic fibrations.*

When these conditions hold, we say $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a ***Quillen adjunction****, and call F (resp. G) a* ***left*** *(resp.* ***right****)* ***Quillen functor****.*

Exercise 7.17. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. Show that conditions in Theorem 7.13 hold.*

Proposition 7.18. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. We have a natural adjunction*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : \mathbb{R}G.$$

7.19. We are finally ready to give a precise definition to equivalences between homotopy theories.

Definition 7.20. *A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is called a* ***Quillen equivalence*** *if the induced adjunction*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : \mathbb{R}G$$

is an equivalence between categories.

8. Conclusion

Theorem 8.1 (Quillen)**.** *The adjoint pair*

$$|-| : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Top} : \mathsf{Sing}$$

is a Quillen equivalence between the classical model structures on both sides.

8.2. The above theorem, together with the Homotopy Hypothesis, justify the following definition.

Definition 8.3. *Write*

$$\mathsf{hGrpd}_\infty \overset{\text{def}}{=} \mathsf{Set}_\Delta[W^{-1}],\ \mathsf{hSpc} \overset{\text{def}}{=} \mathsf{Set}_\Delta[W^{-1}]$$

and call them respectively the ***homotopy category of ∞-groupoids*** *and the* ***homotopy category of spaces***[7].

Remark 8.4. *In the above definition, we use $\mathsf{Set}_\Delta[W^{-1}]$ rather than hSet_Δ for two reasons.*

The first reason is a practical one. We do not want to assign a specific Kan complex to each ∞-groupoid. This would be inconvenient because many constructions about Kan complexes are well-defined only up to canonical weak equivalences[8].

[7] We introduce the new notation hSpc rather than use hTop because in future lectures, we will construct the ∞-*category* Spc of spaces, which needs to be distinguished from the *ordinary category* Top of topological spaces.

[8] Compare with: we do not want to assign a specific complex of projective objects to each object in $\mathsf{D}^{\leq 0}(\mathcal{A})$.

The second reason is an aesthetics one. The localization category $\mathsf{Set}_\Delta[W^{-1}]$ *can be updated to an* ∞*-category in a* model-independent *way, because the universal property defining it makes sense for* ∞*-categories. This provides a construction of the* ∞-category Grpd_∞ of small ∞-groupoids*. On the other hand, any known update of* hSet_Δ *to an* ∞*-category depends on a choice of models for* ∞*-categories.*

8.5. In future lectures, we will construct a Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

where

- $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ is **Joyal model structure** on the category of simplicial sets;
- Cat_Δ is the category of small **simplicial categories**, i.e., Set_Δ-enriched categories. We will equip it with the model structure induced from the *classical* model structure on Set_Δ.

This will identify the homotopy theories underlying these model categories.

On the other hand, Quillen's Theorem 8.1 implies the homotopy theories of Cat_Δ and $\mathsf{Cat}_{\mathsf{Top}}$ are equivalent. Combining with the Homotopy Hypothesis ([Lecture 2, Slogan 0.1]), we will obtain strong evidences for the following:

theory of $(\infty, 1)$*-categories = homotopy theory underlying* $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.

Appendix A. More on Quillen adjunctions

Exercise A.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be model categories. Suppose we have an adjoint pair* $F' : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : G'$*, is it always possible to lift it to a Quillen adjunction? How about Quillen equivalences?*

Appendix B. Quillen's cotangent complex

Exercise B.1. *People say:*

> The cotangent complex functor is the left derived functor of the Kähler differentials functor.

Make sense of this statement using derived functors for model categories.

References

[DS95] William G Dwyer and Jan Spalinski. Homotopy theories and model categories. *Handbook of algebraic topology*, 73(126):21, 1995.

[Qui67] Daniel G Quillen. Homotopical algebra. *Lecture Notes in Mathematics*, 1967.