THE FARGUES–FONTAINE CURVE

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Let p be a prime. The Fargues–Fontaine curve is a geometrization of the field $\mathbb{Q}_p$. It takes as input a perfectoid field C of characteristic p, and in its schematic incarnation, assumes the body of a regular Noetherian scheme of Krull dimension 1. This scheme $\mathrm{X_C}$ lies over $\mathrm{Spec}(\mathbb{Q}_p)$, although it is more profitable to think of $\mathrm{X_C}$ as a relative curve parametrized by C. Its finite étale covers correspond bijectively to those of $\mathrm{Spec}(\mathbb{Q}_p)$. These notes serve the modest purpose of constructing $\mathrm{X_C}$.

1. Tilts and untilts

1.1. **Tilting.**

1.1.1. Let A be a $\mathbb{Z}_p$-algebra. Define $\mathrm{A}^\flat := \lim_{x \mapsto x^p}(\mathrm{A}/p)$. The construction $(-)^\flat$ defines a functor from $\mathbb{Z}_p$-algebras to $\mathbb{F}_p$-algebras, called *tilting*. Note that this construction only produces perfect $\mathbb{F}_p$-algebras. (Indeed, given $(x_i)_{i \geq 0} \in \mathrm{A}^\flat$ so $x_{i+1}^p = x_i$, if its pth power vanishes then all $x_i = 0$, and $(x_i)_{i\geq 0}$ admits a pth power root by "shifting to the right.") In fact, one may view $(-)^\flat$ as the composition of the modulo p functor with the right adjoint to the forgetful functor from perfect $\mathbb{F}_p$-algebras to all $\mathbb{F}_p$-algebras (i.e., inverse limit along Frobenius).

On the other hand, starting with an $\mathbb{F}_p$-algebra B, one may functorially assign a $\mathbb{Z}_p$-algebra W(B) of (p-typical) Witt vectors. Note that this construction only produces p-adically complete $\mathbb{Z}_p$-algebras. The next Lemma tells us that when restricted to these categories, the functors $(-)^\flat$ and W determine one another.

Lemma 1.1.2. *The functors* $(-)^\flat$ *and* W *define an adjunction:*

$$\mathrm{W} : \{\text{perfect } \mathbb{F}_p\text{-algebras}\} \rightleftarrows \{p\text{-adically complete } \mathbb{Z}_p\text{-algebras}\} : (-)^\flat. \tag{1.1}$$

Proof. Suppose that A is a p-adically complete $\mathbb{Z}_p$-algebra and B is a perfect $\mathbb{F}_p$-algebra. Then any morphism $\mathrm{W(B)} \to \mathrm{A}$ determines a morphism $\mathrm{B} \to \mathrm{A}/p$ by modulo p, and since B is perfect, the latter is equivalent to a morphism $\mathrm{B} \to \lim_{x\mapsto x^p}(\mathrm{A}/p)$.

It remains to show that any morphism $\mathrm{B} \to \mathrm{A}/p$ lifts uniquely to a morphism $\mathrm{W(B)} \to \mathrm{A}$. Since A is p-adically complete, it suffices to construct unique liftings for each $n \geq 0$:

$$\begin{tikzcd} \mathrm{A}/p^n & \mathrm{W}_{n+1}(\mathrm{B}) \arrow[l] \arrow[dl, dashed] \\ \mathrm{A}/p^{n+1} \arrow[u] & \mathbb{Z}/p^{n+1} \arrow[l] \arrow[u] \end{tikzcd} \tag{1.2}$$

It is enough to prove that the cotangent complex $\mathbb{L}_{\mathrm{W}_{n+1}(\mathrm{B})/(\mathbb{Z}/p^{n+1})}$ vanishes for all $n \geq 0$. Since $\mathbb{Z}/p^{n+1}$ is an extension of $\mathbb{Z}/p$ by $\mathbb{Z}/p^n$ and the cotangent complex is functorial with

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respect to change of the base ring, we reduce to the case $n = 0$. To prove $\mathbb{L}_{\mathrm{B}/\mathbb{F}_p} = 0$, we make use of the hypothesis that B is perfect, i.e., the pth power Frobenius $\varphi : \mathrm{B} \to \mathrm{B}$ is an isomorphism. This implies that the natural map:

$$\varphi^* \mathbb{L}_{\mathrm{B}/\mathbb{F}_p} \to \mathbb{L}_{\mathrm{B}/\mathbb{F}_p} \tag{1.3}$$

is a quasi-isomorphism. On the other hand, by replacing B with a simplicial resolution $\mathrm{P}^\bullet$ consisting of free $\mathbb{F}_p$-algebras, the map (1.3) is defined by $\varphi^*\Omega_{\mathrm{P}^i/\mathbb{F}_p} \to \Omega_{\mathrm{P}^i/\mathbb{F}_p}$ which is zero by the rule of calculus. This implies $\mathbb{L}_{\mathrm{B}/\mathbb{F}_p} \cong 0$. □

Remark 1.1.3. The unit of this adjunction $\mathrm{B} \to \mathrm{W}(\mathrm{B})^\flat$ is an isomorphism. In particular, perfect $\mathbb{F}_p$-algebras form a full subcategory of p-adically complete $\mathbb{Z}_p$-algebras.

1.1.4. Let us describe more explicitly the counit of this adjunction:

$$\theta : \mathrm{W}(\mathrm{A}^\flat) \to \mathrm{A}, \tag{1.4}$$

for a p-adically complete $\mathbb{Z}_p$-algebra A. From the proof above, we have seen that θ is the unique morphism that makes the following diagram commute:

$$\begin{tikzcd} \mathrm{W}(\mathrm{A}^\flat) \arrow[r, "\theta"] \arrow[d] & \mathrm{A} \arrow[d] \\ \mathrm{A}^\flat \arrow[r, "\mathrm{ev}_0"] & \mathrm{A}/p \end{tikzcd} \tag{1.5}$$

Here, ev_0 denotes the map sending $(x_i)_{i\geq 0} \in \mathrm{A}^\flat$ to x_0.

1.1.5. We shall see that θ also commutes with natural multiplicative maps $[-] : \mathrm{A}^\flat \to \mathrm{W}(\mathrm{A}^\flat)$ and $(-)^\sharp : \mathrm{A}^\flat \to \mathrm{A}$, which we will define. Let us recall that the reduction modulo p map defines a multiplicative bijection for any p-adically complete $\mathbb{Z}_p$-algebra A:

$$\lim_{x\mapsto x^p}(\mathrm{A}) \xrightarrow{\sim} \lim_{x\mapsto x^p}(\mathrm{A}/p). \tag{1.6}$$

Let $(-)^\sharp : \mathrm{A}^\flat \to \mathrm{A}$ denote the multiplicative map sending $(x_i)_{i\geq 0} \in \lim_{x\mapsto x^p}(\mathrm{A})$ to x_0. If we view $(x_i)_{i\geq 0}$ by its image under (1.6), i.e., as $x = (\bar{x}_i)_{i\geq 0}$ where $\bar{x}_i$ is the image of x_i in A/p, then how do we extract $x^\sharp \in \mathrm{A}$?

If we lift $\bar{x}_0$ to A, the result differs from x_0 by some element of $p\mathrm{A}$. If we lift $\bar{x}_1$ to A and then raise it to power p, this differs from x_0 by some element of $p^2\mathrm{A}$. Continuing like this, we see that x_0 is given by $\lim_{n\to\infty}(x'_n)^{p^n}$ where $x'_n \in \mathrm{A}$ is an arbitrary lift of $\bar{x}_n$. Said differently, each $(x'_n)^{p^n}$ comes by taking the unique (p^n)th root of $x \in \mathrm{A}^\flat$, projecting it to A/p, and then raising to the (p^n)th power.

By the commutativity of (1.5), we could have alternatively taken an arbitrary lift of x^{1/p^n} to $\mathrm{W}(\mathrm{A}^\flat)$, raise it to the power p^n, take the limit as $n \to \infty$, and send the result to A via θ. This shows that:

$$\theta([x]) = x^\sharp,$$

where $[x]$ denotes the Teichmüller lift of $x \in \mathrm{A}^\flat$.

Remark 1.1.6. The above way of extracting the element $x_0 \in \mathrm{A}$ from $x \in \lim_{x\mapsto x^p}(\mathrm{A}/p)$ can be repeated to extract x_1, x_2, etc., and constitutes a proof that (1.6) is a bijection.

1.1.7. Let B be a perfect $\mathbb{F}_p$-algebra. Elements of W(B) can be uniquely represented as a formal power series:

$$[c_0] + [c_1]p + [c_2]p^2 + \cdots \tag{1.7}$$

where $c_i \in \mathrm{B}$ and $[c_i]$ denotes its Teichmüller lift. Indeed, given $x \in \mathrm{W(B)}$, its projection to B is denoted by c_0. Then $x - [c_0]$ lies in $p\mathrm{W(B)}$, so it is of the form px_1 for a unique $x_1 \in \mathrm{W(B)}$. We let c_1 be the projection of x_1 in B, etc. The counit map θ, being a ring homomorphism, sends (1.7) to $\sum_{n\geq 0} c_n^\sharp p^n$.

Remark 1.1.8. In particular, θ is also uniquely characterized by the property of commuting $[-]$ with $(-)^\sharp$ and being continuous. Note that because $(-)^\sharp$ is surjective modulo p, the map θ is surjective.

1.1.9. The Frobenius endomorphism φ_{B} on B induces an endomorphism of W(B) by functoriality. On power series representations, it is given by:

$$\varphi_{\mathrm{B}}(\sum_{n\geq 0}[c_n]p^n) = \sum_{n\geq 0}[c_n^p]p^n.$$

1.2. **Perfectoid fields.**

1.2.1. Recall that a topological field K is called *perfectoid* if its topology is induced from a non-archimedean absolute value $|\cdot|_{\mathrm{K}} : \mathrm{K} \to \mathbb{R}^{\geq 0}$ which is complete and satisfies:

(1) there exists some element $\varpi \in \mathrm{K}$ with $|p|_{\mathrm{K}} < |\varpi|_{\mathrm{K}} < 1$;
(2) the Frobenius $\varphi : \mathcal{O}_{\mathrm{K}}/p \to \mathcal{O}_{\mathrm{K}}/p$ is surjective (where $\mathcal{O}_{\mathrm{K}}$ denotes the subring of elements with absolute value ≤ 1).

We will often view a perfectoid field K as being equipped with the absolute value $|\cdot|_{\mathrm{K}}$. Sometimes we will even fix the "pseudo-uniformizer" ϖ as in (1). However, the data defining a perfectoid field only involve its topology.

Remark 1.2.2. The subring $\mathcal{O}_{\mathrm{K}}$ and its maximal ideal $\mathfrak{m}_{\mathrm{K}}$ can be characterized purely in terms of the topology on K: $\mathcal{O}_{\mathrm{K}}$ is the subring of power bounded elements and $\mathfrak{m}_{\mathrm{K}}$ is the subset of topologically nilpotent elements.

Remark 1.2.3. If K has characteristic p, then condition (1) says that there exists $\varpi \in \mathrm{K}$ with $0 < |\varpi|_{\mathrm{K}} < 1$ and condition (2) says that K is perfect. (It rules out finite fields with discrete topology.)

1.2.4. Given a perfectoid field K, we may consider $\mathcal{O}_{\mathrm{K}}^\flat$ and the absolute value $|\cdot|_{\mathrm{K}}$ induces an absolute value $|\cdot|_{\mathrm{K}^\flat}$ on $\mathcal{O}_{\mathrm{K}}^\flat$ given by $|x|_{\mathrm{K}^\flat} := |x^\sharp|_{\mathrm{K}}$.

Lemma 1.2.5 (Useful lemma)**.** *Let* K *be a perfectoid field. Then* $|\cdot|_{\mathrm{K}} : \mathcal{O}_{\mathrm{K}} \to \mathbb{R}^{\geq 0}$ *and* $|\cdot|_{\mathrm{K}^\flat} : \mathcal{O}_{\mathrm{K}^\flat} \to \mathbb{R}^{\geq 0}$ *have the same image.*

Proof. Since $|x|_{\mathrm{K}^\flat} = |x^\sharp|_{\mathrm{K}}$, the image of $|\cdot|_{\mathrm{K}^\flat}$ is contained in the image of $|\cdot|_{\mathrm{K}}$. To prove the converse, we note that $(-)^\sharp : \mathcal{O}_{\mathrm{K}}^\flat \to \mathcal{O}_{\mathrm{K}}$ is surjective modulo p. Hence given any $x \in \mathcal{O}_{\mathrm{K}}$ we may find $y \in \mathcal{O}_{\mathrm{K}}^\flat$ with $y^\sharp \in x + p\mathcal{O}_{\mathrm{K}}$. If $|p|_{\mathrm{K}} < |x|_{\mathrm{K}}$, this shows that $|y|_{\mathrm{K}^\flat} = |x|_{\mathrm{K}}$. Applying to x being the pseudo-uniformizer in K, it tells us that we have some $\varpi \in \mathcal{O}_{\mathrm{K}}^\flat$ with $|p|_{\mathrm{K}} < |\varpi|_{\mathrm{K}^\flat} < 1$.

On the other hand, if $|x|_{\mathrm{K}} \leq |p|_{\mathrm{K}}$, then $x = (\varpi^\sharp)^n x_1$ for some $x_1 \in \mathcal{O}_{\mathrm{K}}$ with $|p|_{\mathrm{K}} < |x_1|_{\mathrm{K}} < 1$ and $n \geq 0$. Since both $\varpi^\sharp$ and x_1 have values achieved by elements of $\mathcal{O}_{\mathrm{K}}^\flat$, the same must hold for x. □

Remark 1.2.6. By the way, you can prove that the image is dense in $\mathbb{R}^{\geq 0}$. Some people use this apparently stronger axiom instead of the existence of pseudo-uniformizer in the definition of a perfectoid field.

Proposition 1.2.7. *Let* K *be a perfectoid field. Then the fraction field of* $\mathcal{O}_{\mathrm{K}}^{\flat}$ *is a perfectoid field of characteristic* p *with topology defined by* $|\cdot|_{\mathrm{K}^\flat}$.

Sketch of proof. It is clear that any $x \in \mathcal{O}_{\mathrm{K}}^{\flat}$ with $|x|_{\mathrm{K}^\flat} = 0$ is zero, so $\mathcal{O}_{\mathrm{K}}^{\flat}$ is an integral domain. To find an element $\varpi \in \mathcal{O}_{\mathrm{K}}^{\flat}$ with $0 < |\varpi|_{\mathrm{K}^\flat} < 1$, we simply apply Lemma 1.2.5. In fact, we might as well take ϖ with $|\varpi|_{\mathrm{K}^\flat} = |p|_{\mathrm{K}}$. The fact that the Frobenius is surjective on $\mathcal{O}_{\mathrm{K}}^{\flat}/p$ follows from the construction.

We claim that $\mathcal{O}_{\mathrm{K}}^{\flat}[\frac{1}{\varpi}]$ is already a field. Indeed, the multiplicative monoid underlying $\mathcal{O}_{\mathrm{K}}^{\flat}$ is isomorphic to $\lim_{x\mapsto x^p}(\mathcal{O}_{\mathrm{K}})$ so the multiplicative monoid underlying $\mathcal{O}_{\mathrm{K}}^{\flat}[\frac{1}{\varpi}]$ is isomorphic to $\lim_{x\mapsto x^p}(\mathrm{K})$. It follows that every nonzero element has a multiplicative inverse. The fact that $\mathcal{O}_{\mathrm{K}}^{\flat}[\frac{1}{\varpi}]$ is complete with respect to $|\cdot|_{\mathrm{K}^\flat}$ follows from the ϖ-adic completeness of $\mathcal{O}_{\mathrm{K}}^{\flat}$, which can be proved by identifying $\lim_n(\mathcal{O}_{\mathrm{K}}^{\flat}/\varpi^n)$ with $\lim_{x\mapsto x^p}(\mathcal{O}_{\mathrm{K}}/p)$. ☐

By abuse of notation (and language), we denote the fraction field of $\mathcal{O}_{\mathrm{K}}^{\flat}$ by $\mathrm{K}^\flat$ and call it the *tilting* of K. The absolute value $|\cdot|_{\mathrm{K}^\flat}$ extends to $\mathrm{K}^\flat$. Note that for each $\varpi \in \mathcal{O}_{\mathrm{K}}^{\flat}$ with $0 < |\varpi|_{\mathrm{K}^\flat} < 1$, the field $\mathrm{K}^\flat$ can be identified with $\mathcal{O}_{\mathrm{K}}^{\flat}[\frac{1}{\varpi}]$.

Proposition 1.2.8. *Let* K *be a perfectoid field and* $\mathrm{K}^\flat$ *be its tilt. Then tilting defines an equivalence of categories:*

$$\{\textit{perfectoid fields containing } \mathrm{K}\} \cong \{\textit{perfectoid fields containing } \mathrm{K}^\flat\}.$$

Proof. Let us read off the inverse functor from the adjunction of Lemma 1.1.2. Indeed, given finite extensions $\mathrm{K} \subset \mathrm{L}$ and $\mathrm{K}^\flat \subset \mathrm{C}$, maps of $\mathrm{K}^\flat$-extensions $\mathrm{C} \to \mathrm{L}^\flat$ are equivalent to $\mathcal{O}_{\mathrm{K}}^{\flat}$-algebra maps $\mathcal{O}_{\mathrm{C}} \to \mathcal{O}_{\mathrm{L}}^{\flat}$, which are equivalent to $\mathcal{O}_{\mathrm{K}}$-algebra maps $\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}} \to \mathcal{O}_{\mathrm{L}}$. If we know that $\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}}$ is the ring of integers in a perfectoid field, then the inverse functor must send C to the fraction field of $\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}}$.

Let us justify the statement about $\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}}$. This will use Lemma 2.1.4 below. Indeed, the kernel of $\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat}) \to \mathcal{O}_{\mathrm{K}}$ is generated by a distinguished element ξ. It suffices to show that the image of ξ in $\mathrm{W}(\mathcal{O}_{\mathrm{C}})$ is again distinguished. This follows from that fact that $\mathcal{O}_{\mathrm{K}}^{\flat} \to \mathcal{O}_{\mathrm{C}}$ preserves elements of absolute value in $(0,1)$. In particular, this also shows that for any $\varpi \in \mathcal{O}_{\mathrm{K}}^{\flat}$ with $0 < |\varpi|_{\mathrm{K}^\flat} < 1$, we have:

$$\mathrm{Frac}(\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}}) \cong \mathrm{W}(\mathcal{O}_{\mathrm{K}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}}[\frac{1}{\varpi}] \cong \mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathrm{K}.$$

Next, we shall prove that the unit and counit are both isomorphisms. Suppose $\mathrm{K} \subset \mathrm{L}$ is an extension of perfectoid fields, we argue that the canonical map $\mathrm{W}(\mathcal{O}_{\mathrm{L}}^{\flat}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}} \to \mathcal{O}_{\mathrm{L}}$ is bijective. By the argument above, we know that the source is the ring of integers in an untilt of $\mathcal{O}_{\mathrm{L}}^{\flat}$, hence of Krull dimension 1. But the map is also surjective, so it must be an isomorphism.

In the other direction, suppose $\mathrm{K}^\flat \subset \mathrm{C}$ is an extension of perfectoid fields, we argue that the canonical map $\mathcal{O}_{\mathrm{C}} \to (\mathrm{W}(\mathcal{O}_{\mathrm{C}}) \otimes_{\mathrm{W}(\mathcal{O}_{\mathrm{K}}^{\flat})} \mathcal{O}_{\mathrm{K}})^\flat$ is bijective. This follows from the fact that any quotient of $\mathrm{W}(\mathcal{O}_{\mathrm{C}})$ by a distinguished element is an untilt of C (Lemma 2.1.4). ☐

Remark 1.2.9. There is also a version of Proposition 1.2.8 which says that perfectoid spaces over K are equivalent to perfectoid spaces over $\mathrm{K}^\flat$.

Proposition 1.2.8 also preserves finite extensions and their degrees. In fact, finite extensions of perfectoid fields are automatically perfectoid, so the equivalence induces an isomorphism $\mathrm{Gal}(\bar{\mathrm{K}}/\mathrm{K}) \cong \mathrm{Gal}(\bar{\mathrm{K}}^\flat/\bar{\mathrm{K}})$.

1.3. **Untilts.**

1.3.1. Even though Proposition 1.2.8 is known as the "tilting equivalence", it is not quite an equivalence on the nose: fixing the base K is crucial for the theorem to hold. Indeed, if we were to take a non-perfectoid field such as $\mathbb{Q}_p$ (whose "tilt" is $\mathbb{F}_p$), the functor:

$$(-)^\flat : \{\text{perfectoid fields containing } \mathbb{Q}_p\} \to \{\text{perfectoid fields containing } \mathbb{F}_p\} \tag{1.8}$$

is far from being an equivalence. Of course, you would expect this since $\mathbb{Q}_p$ is a lot more complex than $\mathbb{F}_p$. (In particular, they have distinct Galois groups.)

However, Proposition 1.2.8 suggests that the relative complexity of the two categories in (1.8) only occurs in the fibral direction: once an "untilt" of $\mathrm{K}^\flat$ is fixed, the perfectoid fields containing it encode no more information than perfectoid fields containing the given $\mathrm{K}^\flat$. This suggests that studying the fibers of (1.8) actually corresponds to studying $\mathbb{Q}_p$ itself.

1.3.2. Suppose that C is a perfectoid field of characteristic p. An *untilt* of C is a pair (K, ι) where K is a perfectoid field and ι is an isomorphism $\mathrm{C} \cong \mathrm{K}^\flat$ of perfectoid fields (i.e., a continuous isomorphism). In particular, ι is equivalent to a continuous isomorphism:

$$\iota : \mathcal{O}_{\mathrm{C}} \cong \mathcal{O}_{\mathrm{K}}^\flat. \tag{1.9}$$

For an element ϖ of C with $|\varpi|_{\mathrm{C}} = |p|_{\mathrm{K}}$ (i.e., $\varpi^\sharp$ differs from p by a unit), this isomorphism is even equivalent to just an isomorphism $\mathcal{O}_{\mathrm{C}}/\varpi \cong \mathcal{O}_{\mathrm{K}}/p$ (because $\mathcal{O}_{\mathrm{C}} \cong \lim_{x \mapsto x^p}(\mathcal{O}_{\mathrm{C}}/\varpi)$). However, I think that (1.9) is still the best way to view ι.

1.3.3. There is a self-map φ on the set of isomorphism classes of untilts defined by sending (K, ι) to $(\mathrm{K}, \iota \circ \varphi_{\mathrm{C}})$, where φ_{C} denotes the Frobenius endomorphism $x \mapsto x^p$ on C. Note that there is a unique characteristic-p untilt of C given by $(\mathrm{C}, \mathrm{id})$. It is fixed by φ since φ_{C} is an automorphism of C.

Remark 1.3.4. The category of untilts of C has no nontrivial automorphisms. This is clear for the characteristic-p untilt. Suppose now that (K, ι) is a characteristic-0 untilt of C. Given $\psi : \mathrm{K} \to \mathrm{K}$ which induces the identity map on $\mathrm{K}^\flat$, the restriction of ψ to $\mathcal{O}_{\mathrm{K}}$ lifts the reduction map $\mathcal{O}_{\mathrm{K}} \to \mathcal{O}_{\mathrm{K}}/p$. Since $\mathrm{W}(\mathcal{O}_{\mathrm{K}}^\flat) \to \mathcal{O}_{\mathrm{K}}$ is surjective and the composition $\psi \circ \theta : \mathrm{W}(\mathcal{O}_{\mathrm{K}}^\flat) \to \mathcal{O}_{\mathrm{K}}$ lifts $\mathcal{O}_{\mathrm{K}}^\flat \to \mathcal{O}_{\mathrm{K}}/p$, we must have $\psi = \mathrm{id}$ (see (1.5)). In other words, the category of untilts of C is equivalent to a set, and we will view it as such.

2. Functions on the set of untilts

2.1. **The ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p}, \frac{1}{[\varpi]}]$.**

2.1.1. From now on, we fix a perfectoid field C of characteristic p. Write $\mathbf{A}_{\mathrm{inf}} := \mathrm{W}(\mathcal{O}_{\mathrm{C}})$. According to §1.1.7, elements of $\mathbf{A}_{\mathrm{inf}}$ can be uniquely represented by power series $\sum_{n\geq 0}[c_n]p^n$ where $c_n \in \mathcal{O}_{\mathrm{C}}$. Suppose (K, ι) is an untilt of C. Then there is a surjective map:

$$\mathbf{A}_{\mathrm{inf}} \cong \mathrm{W}(\mathcal{O}_{\mathrm{K}}^\flat) \to \mathcal{O}_{\mathrm{K}}, \tag{2.1}$$

where the first isomorphism is induced from ι and the second map is the counit θ of (1.4). In particular, we see that (2.1) sends $\sum_{n\geq 0}[c_n]p^n$ to $\sum_{n\geq 0} c_n^\sharp p^n$. Here, the elements $c_n^\sharp \in \mathcal{O}_{\mathrm{K}}$

are defined using ι. Clearly, the Frobenius endomorphism φ_{C} on $\mathbf{A}_{\mathrm{inf}}$ makes the following diagram commute:

$$\begin{tikzcd} \mathbf{A}_{\mathrm{inf}} \arrow[r, "\iota\circ\varphi_{\mathrm{C}}"] \arrow[d, "\varphi_{\mathrm{C}}"] & \mathcal{O}_{\mathrm{K}} \arrow[d, "\cong"] \\ \mathbf{A}_{\mathrm{inf}} \arrow[r, "\iota"] & \mathcal{O}_{\mathrm{K}} \end{tikzcd}$$

Remark 2.1.2. We have now arrived at a familiar paradigm in algebraic geometry: $\mathbf{A}_{\mathrm{inf}}$ looks like the ring of functions on the set of untilts of C. For every untilt (K, ι), one may restrict $f \in \mathbf{A}_{\mathrm{inf}}$ to its "local ring" $\mathcal{O}_{\mathrm{K}}$. The Frobenius endomorphism φ_{C} corresponds to pulling back functions along the self-map $(\mathrm{K}, \iota) \mapsto (\mathrm{K}, \iota \circ \varphi_{\mathrm{C}})$ on the set of untilts.

Since elements of $\mathbf{A}_{\mathrm{inf}}$ can be viewed as formal power series with coefficients in $\mathcal{O}_{\mathrm{C}}$, there is a close analogy between $\mathbf{A}_{\mathrm{inf}}$ and the ring of holomorphic functions on the open unit disc $\mathbb{D}$ whose Taylor series has coefficients c_n of norm $|c_n| \le 1$. This in turn suggests us to think of untilts as the open unit disc $\mathbb{D}$ itself.

2.1.3. *Digression.* Let us discuss how to obtain maps like (2.1) out of the ring $\mathbf{A}_{\mathrm{inf}}$. We call an element $\xi \in \mathbf{A}_{\mathrm{inf}}$ *distinguished* if its Teichmüller expansion $\xi = \sum_{n\ge 0}[c_0]p^n$ satisfies $|c_0|_{\mathrm{C}} < 1$ and $|c_1|_{\mathrm{C}} = 1$. In particular, the image of ξ in $\mathcal{O}_{\mathrm{C}}$ is an element ϖ with $|\varpi|_{\mathrm{C}} < 1$. For such an element ξ, we consider the quotient:

$$0 \to (\xi) \to \mathbf{A}_{\mathrm{inf}} \to \mathbf{A}_{\mathrm{inf}}/\xi \to 0.$$

Then $\mathbf{A}_{\mathrm{inf}}/(\xi, p)$ is equipped with an isomorphism with $\mathcal{O}_{\mathrm{C}}/\varpi$. Hence $(\mathbf{A}_{\mathrm{inf}}/\xi)^\flat \cong \mathcal{O}_{\mathrm{C}}$, so the fraction field of $\mathcal{O}_{\mathrm{K}} := \mathbf{A}_{\mathrm{inf}}/\xi$ defines an untilt of C.

Lemma 2.1.4. *The above construction defines a bijection between ideals of* $\mathbf{A}_{\mathrm{inf}}$ *generated by distinguished elements and untilts of* C*.*

Proof. The mapping from ideals of $\mathbf{A}_{\mathrm{inf}}$ generated by distinguished elements to untilts of C has been defined in §2.1.3. To define the map in the other direction, we need to show that for every untilt (K, ι), the kernel of (2.1) is generated by a distinguished element.

Let us first show that the kernel contains a distinguished element. Consider an element $\varpi \in \mathcal{O}_{\mathrm{C}}$ with $|\varpi|_{\mathrm{C}} = |p|_{\mathrm{K}}$ (Lemma 1.2.5). Then $\varpi^\sharp = \bar{x}p$ for some unit $\bar{x}$ in $\mathcal{O}_{\mathrm{K}}$. Lift $\bar{x}$ to $x \in \mathbf{A}_{\mathrm{inf}}$. Then $[\varpi] - xp$ belongs to the kernel and is distinguished. (One uses that x is necessarily invertible, as can be seen from the constant term in its Teichmüller expansion.) Knowing that $\xi := [\varpi] - xp$ is distinguished, we obtain a factorization of the map (2.1):

$$\mathbf{A}_{\mathrm{inf}} \to \mathbf{A}_{\mathrm{inf}}/\xi \to \mathcal{O}_{\mathrm{K}}.$$

Since $\mathbf{A}_{\mathrm{inf}}/\xi$ is the ring of integers in an untilt of C, hence of Krull dimension 1, the second map $\mathbf{A}_{\mathrm{inf}}/\xi \to \mathcal{O}_{\mathrm{K}}$ must be bijective. □

Remark 2.1.5. The unique characteristic-p untilt corresponds to the ideal (p) which is generated by any distinguished element with $c_0 = 0$.

2.1.6. If we go back to the heuristics that elements of $\mathbf{A}_{\mathrm{inf}}$ define holomorphic functions on the open unit disk, it appears quite unnatural that we only consider power series $\sum_{n\ge 0}[c_n]p^n$ whose coefficients c_n have norm ≤ 1. We could enlarge the ring to include power series with coefficients $c_n \in \mathrm{C}$ but $\{|c_n|\}_{n\ge 0}$ remains bounded. It turns out that this amounts to localizing $\mathbf{A}_{\mathrm{inf}}$ at the Teichmüller lift of some pseudouniformizer ϖ (i.e., an element of C with $0 < |\varpi|_{\mathrm{C}} < 1$).

Lemma 2.1.7. *The natural inclusion* $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\varpi]}] \to \mathrm{W}(\mathrm{C})$ *has image given by formal power series* $\sum_{n\geq 0}[c_n]p^n$ *where* $\{|c_n|\}_{n\geq 0}$ *is bounded.*

The evaluation map (2.1) for an untilt (K,ι) now extends to a map $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\varpi]}] \to \mathrm{K}$. Indeed, this is because the image of $[\varpi]$ in $\mathcal{O}_{\mathrm{K}}$ is $\varpi^{\sharp}$, which is nonzero.

2.1.8. Let us fix an absolute value $|\cdot|_{\mathrm{C}}$ on C defining its topology. Then any untilt (K,ι) inherits an absolute value $|\cdot|_{\mathrm{K}}$ from the isomorphism $\iota : \mathcal{O}_{\mathrm{C}} \cong \mathcal{O}_{\mathrm{K}}^{\flat}$ and the multiplicative map $(-)^{\sharp} : \mathcal{O}_{\mathrm{K}}^{\flat} \to \mathcal{O}_{\mathrm{K}}$. This absolute value defines a "radius function" on the set of untilts, sending (K,ι) to $|p|_{\mathrm{K}}$.

There is a unique point in this set whose radius is 0. It is the characteristic-p untilt (C,id) of C. Suppose that we throw this point away, which amounts to considering the localization $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$. This ring consists of formal Laurent series $\sum_{n\gg-\infty}[c_n]p^n$ where $\{|c_n|\}_{n\in\mathbb{Z}}$ is bounded. Then any characteristic-0 untilt (K,ι) defines a map:

$$\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}] \to \mathrm{K}. \tag{2.2}$$

From now on, we shall let Y denote the set of isomorphism classes of characteristic-0 untilts of C. For $y \in \mathrm{Y}$, we denote the image of $f \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ under (2.2) by $f(y)$.

2.1.9. *Still not enough.* The ring $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ is our first candidate for the "ring of functions" on Y. However, it creates a problem when we try to construct $\mathrm{X} := \mathrm{Y}/\varphi^{\mathbb{Z}}$. To see how this problem arises, let us consider the analogous problem in ordinary complex analysis.

The basic object is the punctured unit disk $\mathring{\mathbb{D}}$ with an endomorphism $\varphi : \mathring{\mathbb{D}} \to \mathring{\mathbb{D}}$, $z \mapsto z^p$. In order to build $\mathring{\mathbb{D}}/\varphi^{\mathbb{Z}}$ as a projective variety we need to find a line bundle $\mathcal{O}(1)$ which is globally generated, so $\mathring{\mathbb{D}}/\varphi^{\mathbb{Z}}$ may be realized as $\mathrm{Proj}(\bigoplus_{n\geq 0}\mathrm{H}^0(\mathcal{O}(n)))$. A natural candidate for the graded ring $\bigoplus_{n\geq 0}\mathrm{H}^0(\mathcal{O}(n))$ is the following one: the space $\mathrm{H}^0(\mathcal{O}(n))$ consists of functions $f : \mathring{\mathbb{D}} \to \mathbb{C}$ such that $f(z^p) = p^n f(z)$. The simplest such function is $\log(z)^n$, which is *not* meromorphic at $z = 0$.

To translate this heuristics into our context, consider elements of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ on which φ_{C} acts by multiplication by p. Writing such an element as a Laurent series $\sum_{n\gg-\infty}[c_n]p^n$, we must have $c_n^p = c_{n-1}$. Since φ_{C} is an isomorphism and $c_n = 0$ for $n \ll 0$, we see that all c_n must vanish. This suggests that we must enlarge $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ to include "holomorphic functions which are not meromorphic at 0".

Remark 2.1.10. Elements of $\mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ on which φ_{C} acts as identity are equivalent to $\sum_{n\gg-\infty}[c_n]p^n$ where $c_n \in \mathbb{F}_p$. This is nothing but $\mathbb{Q}_p = \mathrm{W}(\mathbb{F}_p)[\frac{1}{p}]$.

2.2. **The true ring of functions on** Y.

2.2.1. For a real number $0 < \rho < 1$, define the *Gauss norm* of an element $f \in \mathbf{A}_{\mathrm{inf}}[\frac{1}{p},\frac{1}{[\varpi]}]$ as $|f|_{\rho} := \sup\{|c_n|_{\mathrm{C}} \cdot \rho^n\}$, where $\sum_{n\gg-\infty}[c_n]p^n$ is the Laurent series representation of f. Since $\{|c_n|_{\mathrm{C}}\}_{n\in\mathbb{Z}}$ is bounded, the supremum is achieved by finitely many n.

Remark 2.2.2. The Gauss norm bounds the value of $f(y)$ over untilts y with radius ρ. More precisely, say $y = (\mathrm{K},\iota)$ with induced absolute value $|\cdot|_{\mathrm{K}} : \mathrm{K} \to \mathbb{R}^{\geq 0}$ such that

$|p|_{\mathrm{K}} = \rho$. Then there holds:

$$|f(y)|_{\mathrm{K}} = |\sum_{n \gg -\infty} c_n^{\sharp} p^n|_{\mathrm{K}} \leq \sup\{|c_n^{\sharp}|_{\mathrm{K}} \cdot |p|_{\mathrm{K}}^n\} = |f|_{\rho}.$$

This bound tells us something else: given any sequence $f_n \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ which is Cauchy under $|\cdot|_{\rho}$, the sequene $f_n(y) \in \mathrm{K}$ is Cauchy under $|\cdot|_{\mathrm{K}}$, which has a limit in K. In other words, the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at the Gauss norm $|\cdot|_{\rho}$ yields functions which are still well-defined on points $y \in \mathrm{Y}$ of radius ρ.

2.2.3. The ring B is defined to be the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at all the Gauss norms $|\cdot|_{\rho}$ for $0 < \rho < 1$. Namely, it is the universal topological $\mathbb{Q}_p$-vector space receiving a morphism $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \to \mathrm{B}$, such that any sequence $a_n \in \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ which is Cauchy with respect to all the norms $|\cdot|_{\rho}$ $(0 < \rho < 1)$ has a limit in B.

It follows from the universal property that B has the structure of a topological $\mathbb{Q}_p$-algebra. Each $|\cdot|_{\rho} : \mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \to \mathbb{R}^{\geq 0}$ extends uniquely to a continuous map $|\cdot|_{\rho} : \mathrm{B} \to \mathbb{R}^{\geq 0}$. By Remark 2.2.2, for every characteristic-0 untilt $y = (\mathrm{K}, \iota)$ of C, the evaluation map (2.2) extends to:

$$\mathrm{B} \to \mathrm{K}, \quad f \mapsto f(y). \tag{2.3}$$

2.2.4. To get a feeling about the elements of B, we shall consider infinite Laurent series $f = \sum_{n \in \mathbb{Z}} [c_n] p^n$ with $c_n \in \mathrm{C}$. Such an object can be viewed as a sequence of elements in $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$, so it makes sense to ask: under what condition does it converge in B? It turns out that f converges in B if and only if:

(1) $\limsup_{n \geq 0} |c_n|_{\mathrm{C}}^{1/n} \leq 1$;
(2) $\lim_{n \to -\infty} |c_n|_{\mathrm{C}}^{-1/n} = 0$.

Indeed, these follow from that fact that f converges under $|\cdot|_{\rho}$ if and only if:

$$\lim_{n \to \infty} (|c_n|_{\mathrm{C}} \cdot \rho^n) = \lim_{n \to -\infty} (|c_n|_{\mathrm{C}} \cdot \rho^n) = 0.$$

Remark 2.2.5. Let $f = \sum_{n \in \mathbb{Z}} c_n z^n$ where $c_n \in \mathbb{C}$. Then the two conditions above are equivalent to the fact that f is the Laurent series of a holomorphic function on the open punctured unit disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

Remark 2.2.6. In contrast with $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$, elements in B *cannot* be uniquely represented by formal Laurent series $f = \sum_{n \in \mathbb{Z}} [c_n] p^n$. (Example?)

2.2.7. One can rewrite B in a more finitary manner. For $0 < a \leq b < 1$, we let $\mathrm{B}_{[a,b]}$ denote the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at Gauss norms $|\cdot|_{\rho}$ for $a \leq \rho \leq b$. Then $\mathrm{B}_{[a,b]}$ is isomorphic to the completion of $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at just the pair of Gauss norms $|\cdot|_a$ and $|\cdot|_b$. Indeed, this is because a sequence f_n which converges to f under both $|\cdot|_a$ and $|\cdot|_b$ also converges to f in $|\cdot|_{\rho}$ for all $a \leq \rho \leq b$. On the other hand, the universal property of B supplies a continuous homomorphism $\mathrm{B} \to \mathrm{B}_{[a,b]}$. One checks that the natural map below is an isomorphism of topological $\mathbb{Q}_p$-algebras:

$$\mathrm{B} \xrightarrow{\sim} \lim_{[a,b] \subset (0,1)} \mathrm{B}_{[a,b]}.$$

On the other hand, $\mathrm{B}_{[a,b]}$ can be constructed completely algebraically when a, b belong to the image of $|\cdot|_{\mathrm{C}} : \mathrm{C} \to \mathbb{R}^{\geq 0}$.

Lemma 2.2.8. *Suppose $\pi_a, \pi_b \in \mathcal{O}_{\mathrm{C}}$ with $a = |\pi_a|_{\mathrm{C}}$ and $b = |\pi_b|_{\mathrm{C}}$. Then $\mathrm{B}_{[a,b]}$ is canonically isomorphic to the p-adic completion of $\mathbf{A}_{\inf}[\frac{[\pi_a]}{p}, \frac{p}{[\pi_b]}]$.*

3. The Fargues–Fontaine curve

3.1. The definition.

3.1.1. Let C be a perfectoid field of characteristic p. We have constructed a topological $\mathbb{Q}_p$-algebra B by completing the ring $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at Gauss norms $|\cdot|_\rho$ for all $0 < \rho < 1$. The Frobenius $\varphi_{\mathrm{C}} : \mathrm{C} \to \mathrm{C}$, $x \mapsto x^p$ induces an endomorphism $\varphi : \mathrm{B} \to \mathrm{B}$ by functoriality of the construction. For $n \in \mathbb{Z}$, let $\mathrm{B}^{\varphi = p^n}$ denote the subring of elements $f \in \mathrm{B}$ with $\varphi(f) = p^n f$. The *Fargues–Fontaine curve* X_{C} is defined to be:

$$\mathrm{X}_{\mathrm{C}} := \mathrm{Proj}(\bigoplus_{n \geq 0} \mathrm{B}^{\varphi = p^n}).$$

For a finite extension $\mathbb{Q}_p \subset \mathrm{E}$, we define $\mathrm{X}_{\mathrm{C},\mathrm{E}}$ to be $\mathrm{X}_{\mathrm{C}} \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spec}(\mathrm{E})$.

Remark 3.1.2. The Fargues–Fontaine curve X_{C} (or $\mathrm{X}_{\mathrm{C},\mathrm{E}}$) should be viewed as a relative curve over C which "geometrizes" $\mathrm{Spec}(\mathbb{Q}_p)$ (resp. $\mathrm{Spec}(\mathrm{E})$). More generally, one can use perfectoid rings $(\mathrm{A}, \mathrm{A}^+)$ in characteristic p as test objects and construct a version of the Fargues–Fontaine curve $\mathrm{X}_{\mathrm{S},\mathrm{E}}$ for $\mathrm{S} = \mathrm{Spa}(\mathrm{A}, \mathrm{A}^+)$.

I learned from Lin Chen that there is no terminal object in the category of perfectoid spaces in characteristic p, and consequently no "absolute Fargues–Fontaine curve."

3.2. The logarithm.

3.2.1. Following the heuristics of §2.1.9, we want to construct particular sections of $\mathrm{B}^{\varphi = p}$ which play the role of $\log(z)$. This is indeed possible, and will take the form of a map:

$$1 + \mathfrak{m}_{\mathrm{C}} \to \mathrm{B}, \quad x \mapsto \log([x]), \tag{3.1}$$

which takes multiplication to addition.

To construct (3.1), we recall that $\log([x])$ can be represented by a power series:

$$\log([x]) = \sum_{n > 0} \frac{(-1)^{n+1}}{n} ([x] - 1)^n, \tag{3.2}$$

which may be viewed as a sequence of elements in the $\mathbb{Q}_p$-algebra $\mathbf{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$. By definition of the ring B, the following Lemma suffices to define $\log([x])$.

Lemma 3.2.2. *The power series* (3.2) *converges in the Gauss norm $|\cdot|_\rho$ for all $0 < \rho < 1$.*

Proof. Let us first show that $|[x] - 1|_\rho < 1$. Indeed, consider the Teichmüller expansion of $[x] - 1 \in \mathbf{A}_{\inf}$ given by $\sum_{n \geq 0} [c_n] p^n$. We have $c_n \in \mathcal{O}_{\mathrm{C}}$ so $|c_n|_{\mathrm{C}} \leq 1$. On the other hand, $c_0 = x - 1$ which belongs to $\mathfrak{m}_{\mathrm{C}}$, so $|c_0|_{\mathrm{C}} < 1$. These observations combine to give:

$$|[x] - 1|_\rho = \sup\{|c_n|_{\mathrm{C}} \cdot \rho^n\}_{n \geq 0} < 1.$$

Next, it suffices to prove that the Gauss norm of $\frac{(-1)^{n+1}}{n}([x] - 1)^n$ goes to zero as $n \to \infty$. Writing $\alpha = |[x] - 1|_\rho$, this norm is given by:

$$|\frac{(-1)^{n+1}}{n}|_\rho \alpha^n = \frac{1}{|n|_\rho} \alpha^n. \tag{3.3}$$

Let us recall that the restriction of $|\cdot|_\rho$ to $\mathbb{Q}_p \subset \mathbf{A}_{\mathrm{inf}}$ is given by $\rho^{v_p(\cdot)}$. Hence $\frac{1}{|n|_\rho} = (\rho^{-1})^{v_p(n)} \le (\rho^{-1})^{\log_p(n)}$. (The equality is achieved when n is a power of p.) This implies that (3.3) is bounded above by $(\rho^{-1})^{\log_p(n)}\alpha^n$, which tends to zero as $n \to \infty$. $\square$

3.2.3. Let us note that $\log([x])$ belongs to the eigenspace $\mathrm{B}^{\varphi=p}$. Indeed, it follows from the power series presentation (3.2) that $\varphi(\log([x])) = \log([x]^p) = p\log([x])$. The logarithm function allows us to formulate a structure theorem about the graded ring $\bigoplus_{n\in\mathbb{Z}} \mathrm{B}^{\varphi=p^n}$.

Theorem 3.2.4. *The following statements hold true:*

(1) for $n < 0$, the eigenspace $\mathrm{B}^{\varphi=p^n}$ vanishes;
(2) for $n = 0$, the natural map $\mathbb{Q}_p \to \mathrm{B}^{\varphi=1}$ is bijective;
(3) for $n > 0$ and C algebraically closed, every nonzero element $f \in \mathrm{B}^{\varphi=p^n}$ factors as a product:

$$f = \lambda \log([x_1]) \cdots \log([x_n]), \quad \lambda \in \mathbb{Q}_p^\times \text{ and } x_i \in 1 + \mathfrak{m}_\mathrm{C}, \tag{3.4}$$

which is unique up to reordering and multiplication by elements of $\mathbb{Q}_p^\times$.

The proof of Theorem 3.2.4 is quite involved and will be explained later. It contains, among other things, the statement that $(\log([x]))$ is a prime ideal in $\bigoplus_{n\ge 0} \mathrm{B}^{\varphi=p^n}$ for any $x \in 1+\mathfrak{m}_\mathrm{C}$ (assuming that C is algebraically closed).

Remark 3.2.5. For $x \in 1 + \mathfrak{m}_\mathrm{C}$ and $a \in \mathbb{Q}_p^\times$, the exponent x^a is a well-defined element in $1 + \mathfrak{m}_\mathrm{C}$. Indeed, for $a \in \mathbb{Z}_p^\times$, we may write $x^a := \lim_n x^{a_n}$ where a_n is the reduction of a in $\mathbb{Z}/p^n$. Ambiguities like x^{p^n} converge to 1 since $x \in 1 + \mathfrak{m}_\mathrm{C}$. This operation extends to $\mathbb{Q}_p^\times$. Note that $\log([x^a]) = a\log([x])$.

3.2.6. When C is algebraically closed, the dictionary between distinguished elements and untilts (Lemma 2.1.4) can be enhanced as follows. There is a commutative diagram:

$$\begin{tikzcd}
\{\text{ideals of } \mathbf{A}_{\mathrm{inf}} \text{ generated by } \xi\} \arrow[r, "\cong"] \arrow[d] & \{\text{characteristic-0 untilts}\} \arrow[d] \\
\{\text{ideals of } \bigoplus_{n\ge 0} \mathrm{B}^{\varphi=p^n} \text{ generated by } \log([x])\} \arrow[r, "\cong"] \arrow[d, "\cong"] & \{\text{characteristic-0 untilts}\}/\varphi^{\mathbb{Z}} \\
\{\text{closed points of } \mathrm{X}_\mathrm{C}\} &
\end{tikzcd}$$

In this diagram, ξ stands for elements of the form $\sum_{n\ge 0}[c_n]p^n \in \mathbf{A}_{\mathrm{inf}}$ where $0 < |c_0| < 1$ and $|c_1| = 1$. The expression $\log([x])$ stands for elements of B defined by $x \in 1 + \mathfrak{m}_\mathrm{C}$.

Like Theorem 3.2.4, constructing this commutative diagram is nontrivial business. The essential ingredient for doing both is a well-behaved notion of "divisors" for elements of B. In particular, the element $\log([x])$ has a divisor with only simple zeros $\sum_{n\in\mathbb{Z}} \varphi^n(y)$, for $y \in \mathrm{Y}$ the untilt corresponding to the distinguished element:

$$\xi := 1 + [x^{\frac{1}{p}}] + \cdots + [x^{\frac{p-1}{p}}].$$

The definition of divisors in turn relies on a version of the completed local rings on Y: this is Fontaine's period ring $\mathrm{B}^+_{\mathrm{dR}}(y)$ for $y \in \mathrm{Y}$.

3.2.7. One can use the aforementioned facts to prove the following structure theorem about the Fargues–Fontaine curve $\mathrm{X_C}$.

Corollary 3.2.8. *Suppose* C *is algebraically closed. Then:*

(1) *the closed points of* $\mathrm{X_C}$ *are in bijection with* φ*-orbits of characteristic-*0 *untilts of* C*;*
(2) *the scheme* $\mathrm{X_C}$ *is regular, Noetherian, and of Krull dimension* 1.

Proof. Let $\mathfrak{p} \neq (0)$ be a homogeneous prime ideal of $\bigoplus_{n\geq 0} \mathrm{B}^{\varphi=p^n}$. Then $\mathfrak{p}$ contains some nonzero homogeneous element f in degree ≥ 1 which admits a product decomposition (3.4). Being prime, $\mathfrak{p}$ contains one of the factors, say $\log([x])$ for some $x \in 1+\mathfrak{m}_\mathrm{C}$. Then we must have $\mathfrak{p} = \log([x])$ since $\log([x])$ already defines a closed point.

This shows, in particular, that every point of $\mathrm{X_C}$ besides the generic point is closed. The corresponding prime ideals are all monogenic, which implies that $\mathrm{X_C}$ is Noetherian, regular, and of Krull dimension 1. □

3.3. Divisors and $\mathrm{B}^+_{\mathrm{dR}}$.

3.3.1. Suppose that $y = (\mathrm{K}, \iota)$ is an untilt of C. Let $\xi \in \mathbf{A}_{\mathrm{inf}}$ be a distinguished element it corresponds to under Lemma 2.1.4. Fix a pseudo-uniformizer ϖ of C. Define:

$$\mathrm{B}^+_{\mathrm{dR}}(y) := \lim_n((\mathbf{A}_{\mathrm{inf}}/\xi^n)[\frac{1}{[\varpi]}]).$$

According to the heuristics of viewing Y as the unit disc and $\mathbf{A}_{\mathrm{inf}}[\frac{1}{[\varpi]}]$ as holomorphic functions, the element ξ corresponds to a local coordinate at y. Hence $\mathrm{B}^+_{\mathrm{dR}}(y)$ can be viewed as the formal completion at y. The following result justifies this heuristics.

Lemma 3.3.2. *The ring* $\mathrm{B}^+_{\mathrm{dR}}(y)$ *is a discrete valuation ring whose maximal ideal is generated by the image of* ξ*. Its residue field* $\mathrm{B}^+_{\mathrm{dR}}(y)/\xi$ *is canonically isomorphic to* K.

Remark 3.3.3. If y is the unique characteristic-p untilt of C, then we may take $\xi = p$ so $(\mathbf{A}_{\mathrm{inf}}/\xi^n)[\frac{1}{[\varpi]}]$ is isomorphic to $\mathrm{W}_n(\mathcal{O}_\mathrm{C})[\frac{1}{[\varpi]}] \cong \mathrm{W}_n(\mathrm{C})$. We thus find $\mathrm{B}^+_{\mathrm{dR}}(y) \cong \mathrm{W}(\mathrm{C})$.

If y is a characteristic-0 untilt of C, then $\mathrm{B}^+_{\mathrm{dR}}(y)$ is non-canonically isomorphic to $\mathrm{K}[\![\xi]\!]$ by Cohen's structure theorem.

3.3.4. The rings $\mathrm{B}^+_{\mathrm{dR}}(y)$ allow us to define a notion of divisor associated to every nonzero $f \in \mathrm{B}$. Namely, for every characteristic-0 untilt y, the evaluation map (2.3) extends to a commutative diagram:

$$\begin{tikzcd} \mathrm{B} \arrow[r, "\mathrm{res}_y"] \arrow[dr, "\mathrm{ev}_y"'] & \mathrm{B}^+_{\mathrm{dR}}(y) \arrow[d] \\ & \mathrm{K} \end{tikzcd}$$

We write $\mathrm{ord}_y(f)$ for the valuation of f in $\mathrm{B}^+_{\mathrm{dR}}(y)$. Then $\mathrm{Div}(f)$ is defined to be the formal sum $\sum_{y\in \mathrm{Y}} \mathrm{ord}_y(f) \cdot y$.