LECTURE 8

In this lecture, we use model categories to compute limits and colimits.

1. ∞-limits as homotopy limits

1.1. Recall we have a Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_{\Delta}^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_{\Delta} : \mathfrak{N}_{\bullet}.$$

In this section, we will expain the following result:

Theorem 1.2 (HTT.4.2.4.1)**.** *Let* $\mathbb{C}$ *be a* combinatorial simplicial model category*. Then for a small fibrant simplicial category* $\mathbb{J}$*, a diagram*

$$\mathbb{J}^{\triangleleft} \to \mathbb{C}^{\circ}$$

is a homotopy limit diagram *in* $\mathbb{C}$ *iff the corresponding diagram*

$$\mathfrak{N}_{\bullet}(\mathbb{J})^{\triangleleft} \to \mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$$

is a limit diagram in the quasi-category $\mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$*.*

Variant 1.3. *Dually, a diagram* $\mathbb{J}^{\triangleright} \to \mathbb{C}^{\circ}$ *is a* homotopy colimit diagram *iff* $\mathfrak{N}_{\bullet}(\mathbb{J})^{\triangleright} \to \mathfrak{N}_{\bullet}(\mathbb{C}^{\circ})$ *is a limit diagram.*

1.4. We will soon give the precise definitions of the undefined notions in Theorem 1.2. For now, let us be satisfied by the following informal words.

A *simplicial model category* is a model category $\mathbb{C}$ equipped with a compatible simplicial enrichment. Here the compatibility condition guarantees

(1) There is a canonical equivalence

$$\mathsf{h}\mathbb{C} \simeq \pi_0\mathbb{C},$$

where $\mathsf{h}\mathbb{C}$ is the homotopy category of the model category $\mathbb{C}$ (see [Lecture 2, Definition 2.20]), while $\pi_0\mathbb{C}$ is the homotopy category of the simplicial category $\mathbb{C}$ (see [Lecture 5, Definition 5.3]).

(2) For bifibrant objects $x, y \in \mathbb{C}$, the simplicial set $\mathsf{Hom}_{\mathbb{C}}(x, y)$ is a Kan complex. In other words, $\mathbb{C}^{\circ}$ is a fibrant object in Cat_{Δ}.

For a model category, being combinatorial is a technical set-theoretical size condition, which can be ignored for now.

For a simplicial model category $\mathbb{C}$, the *homotopy limit* of a diagram $u : \mathbb{J} \to \mathbb{C}$ is the value of the right derived functor of the naive limit functor. In other words,

$$\mathsf{holim}\, u := \mathbb{R}\mathsf{lim}(u),$$

where the functor lim is the right adjoint in a Quillen adjunction

$$\mathsf{const} : \mathbb{C} \rightleftarrows \mathsf{Fun}(\mathbb{J}, \mathbb{C}) : \mathsf{lim}.$$

Here $\mathsf{Fun}(\mathbb{J}, \mathbb{C})$ is the category of simplicial enriched functors from $\mathbb{J}$ to $\mathbb{C}$, equipped with a suitable model structure induced by the model structure on $\mathbb{C}$.

Date: Oct. 15, 2024.

Dually, the *homotopy colimit* of a diagram $u : \mathbb{J} \to \mathbb{C}$ is the value of the left derived functor of the naive colimit functor, i.e.,

$$\mathsf{hocolim}\, u := \mathbb{L}\mathsf{colim}(u),$$

1.5. Recall that Set_Δ is Cartesian closed and therefore has a natural simplicial enrichment given by $\mathsf{Fun}(-,-)$. This enrichment is compatible with the Kan–Quillen model structure. We denote the obtained simplicial model category by $\mathbb{Set}_\Delta^{\mathsf{KQ}}$. In Theorem 1.2, we can take $\mathbb{C} := \mathbb{Set}_\Delta^{\mathsf{KQ}}$. Note that $\mathbb{C}^\circ = \mathbb{Kan}$ is the simplicial category defined in [Lecture 5, §7]. We obtain:

Corollary 1.6. *Let $\mathbb{J}$ be a small fibrant simplicial category. Then the homotopy (co)limit of a $\mathbb{J}$-indexed diagram in*

$$\mathbb{Kan} \subset \mathbb{Set}_\Delta^{\mathsf{KQ}}$$

calculates the (co)limit of the corresponding $\mathfrak{N}_\bullet(\mathbb{J})$-indexed diagram in the quasi-categories

$$\mathcal{K}an := \mathfrak{N}_\bullet(\mathbb{Kan}).$$

Remark 1.7. *Note that the latter models (co)limits in the ∞-category Grpd_∞, which essentially control (co)limits in any ∞-category ([Lecture 7, Theorem 2.11]).*

Remark 1.8. *The above corollary provides a model-categorical algorithm to calculate small (co)limits of ∞-groupoids. Let us take the second case as an example. Let $u : K \to \mathsf{Grpd}_\infty$ be a small diagram with $K \in \mathsf{Set}_\Delta$.*

(i) *Choose a weak equivalence $\mathfrak{C}(K) \to \mathbb{J}$ in Cat_Δ.*
(ii) *Find a functor $w : \mathbb{J} \to \mathbb{Kan}$ such that the composition*

$$K \to \mathfrak{N}_\bullet(\mathbb{J}) \to \mathfrak{N}_\bullet(\mathbb{Kan}) =: \mathcal{K}an$$

represents u. By HTT.4.2.4.4, such w always exists.

(iii) *View w as an object in the model category $\mathsf{Fun}(\mathbb{J}, \mathbb{Set}_\Delta^{\mathsf{KQ}})$. Calculate the derived (co)limits*

$$\mathbb{R}\mathsf{lim}(w),\ \mathbb{L}\mathsf{colim}(w) \in \mathsf{Set}_\Delta^{\mathsf{KQ}}[W^{-1}]$$

by finding a (co)fibrant replacement of w.

By Corollary 1.6, the obtained objects in $\mathsf{Set}_\Delta^{\mathsf{KQ}}[W^{-1}] \simeq \mathsf{hGrpd}_\infty$ are canonically isomorphic to $\mathsf{lim}\, u$ and $\mathsf{colim}\, u$.

Exercise 1.9. *Prove the functor w in Step (ii) exists in the case when $\mathfrak{C}(K) \to \mathbb{J}$ is a cofibration.*

Remark 1.10. *In Remark 1.8, when $K = \mathsf{N}_\bullet(\mathsf{J})$ is the nerve of an ordinary category, we can take $\mathbb{J} := \mathsf{J}$, viewed as a simplicial category with* discrete *enrichment. Then $\mathsf{Fun}(\mathbb{J}, \mathbb{Set}_\Delta^{\mathsf{KQ}})$ is just the category of functors between the ordinary categories $\mathsf{J} \to \mathsf{Set}_\Delta$, and the underived functor lim is the limit functor for ordinary categories.*

In fact, the above essentially covers all the cases because for any simplicial set K, there exists an initial morphism $\mathsf{N}_\bullet(\mathsf{J}) \to K$ such that J is an ordinary category or even a partially ordered set. See HTT.4.2.3.14.

Remark 1.11. *In Remark 1.8, the obtained objects are contained in the homotopy category* hGrpd_∞ *rather than in* Grpd_∞*. In other words, the above algorithm only calculates* $\lim u$ up to homotopy. *In particular, it cannot produce the entire limit diagram, nor the canonical lifting of* $\lim u$ *in* $(\mathsf{Grpd}_\infty)_{/u}$.

To remedy this, one can try to find an extended diagram $\overline{w} : \mathbb{J}^{\triangleleft} \to \mathbb{K}\mathrm{an}$ *that* ***exhibits*** $\overline{w}(*)$ ***as the homotopy limit*** *of* w *(see Definition 2.18 below). Then Theorem 1.2 says the corresponding diagram*

$$\mathfrak{N}_\bullet(\mathbb{J})^{\triangleleft} \to \mathfrak{N}(\mathbb{K}\mathrm{an}) =: \mathcal{K}an$$

is also a limit diagram. Now restriction along the categorical equivalence $K^{\triangleleft} \to \mathfrak{N}_\bullet(\mathbb{J})^{\triangleleft}$ *produces a limit diagram extending* $v : K \to \mathcal{K}an$.

For general w*, the above extension* $\overline{w}$ *may not exsit*[1]*. Nevertheless, we can replace* w *by any fibrant replacement of it because they represent the same diagram in* Grpd_∞*. Under this additional assumption, such extension* $\overline{w}$ *exists and is unique up to unique equivalence, because it has to be the limit diagram extending* w*.*

Remark 1.12. *In future lectures, we will see that any* presentable ∞-category *can be realized as* $\mathfrak{N}_\bullet(\mathbb{C}^\circ)$ *for some combinatorial simplicial model category* $\mathbb{C}$*. This provides a model-categorical algorithm to calculate small (co)limits in any presentable* ∞*-category.*

2. Definition of homotopy limits

2.1. In this section, we give the precise definitions for the notions used in Theorem 1.2.

Definition 2.2. *Let* $\mathbb{C}$ *be a simplicial category.*

(1) *We say* $\mathbb{C}$ *is* ***tensored over*** Set_Δ *if for any* $S \in \mathsf{Set}_\Delta$ *and* $X \in \mathbb{C}$*, the functor*

$$\mathsf{Fun}(S, \mathsf{Hom}_{\mathbb{C}}(X, -)) : \mathbb{C} \to \mathsf{Set}_\Delta$$

is represented by an object in $\mathbb{C}$*, which we denote by* $S \otimes X$*.*

(2) *We say* $\mathbb{C}$ *is* ***cotensored over*** Set_Δ *if for any* $S \in \mathsf{Set}_\Delta$ *and* $Y \in \mathbb{C}$*, the functor*

$$\mathsf{Fun}(S, \mathsf{Hom}_{\mathbb{C}}(-, Y)) : \mathbb{C}^{\mathsf{op}} \to \mathsf{Set}_\Delta$$

is represented by an object in $\mathbb{C}$*, which we denote by* $\mathsf{Fun}(S, Y)$*.*

Proposition-Definition 2.3 (HTT.A.3.1.5, A.3.1.6)**.** *Let* $\mathbb{C}$ *be a model category equipped with a simplicial enrichment such that it is both tensored and cotensored over* Set_Δ*. Then the following conditions are equivalent:*

(i) *Given any cofibration* $j : X \to X'$ *and any fibration* $k : Y \to Y'$ *in* $\mathbb{C}$*, the morphism*

$$\mathsf{Hom}_{\mathbb{C}}(X', Y) \to \mathsf{Hom}_{\mathbb{C}}(X', Y') \underset{\mathsf{Hom}_{\mathbb{C}}(X, Y')}{\times} \mathsf{Hom}_{\mathbb{C}}(X, Y)$$

is a fibration in $\mathsf{Set}_\Delta^{\mathsf{KQ}}$*, which is a weak equivalence if either* j *or* k *is so.*

---

[1]For example, consider $\mathsf{Sing}_\bullet(\{0\} \to [0,1] \leftarrow \{1\})$.

(ii) *Given any cofibrations* $i : S \to S'$ *and* $j : X \to X'$ *respectively in* $\mathsf{Set}_\Delta^{\mathsf{KQ}}$ *and* $\mathbb{C}$*, the morphism*

$$(S' \otimes X) \coprod_{S \otimes X} (S \otimes X') \to S' \otimes X'$$

is a cofibration in $\mathbb{C}$*, which is a weak equivalence if either* i *or* j *is so.*

(iii) *Given any cofibration* $i : S \to S'$ *in* $\mathsf{Set}_\Delta^{\mathsf{KQ}}$ *and any fibration* $k : Y \to Y'$ *in* $\mathbb{C}$*, the morphism*

$$\mathsf{Fun}(S', Y) \to \mathsf{Fun}(S', Y') \underset{\mathsf{Fun}(S, Y')}{\times} \mathsf{Fun}(S, Y)$$

is a fibration in $\mathbb{C}$*, which is a weak equivalence if either* i *or* k *is so.*

We say $\mathbb{C}$ *is a* ***simplicial model category*** *if the above conditions are satisfied.*

Exercise 2.4. *Let* $\mathbb{C}$ *be a simplicial model category. Show that* $\mathsf{Hom}_{\mathbb{C}}(X, Y)$ *is a Kan complex if* X *is cofibrant and* Y *is fibrant.*

Exercise 2.5. *Can you make* $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ *into a simplicial model category?*

Exercise 2.6. *Let* $\mathbb{C}$ *be a simplicial model category. Construct a canonical equivalence* $\mathsf{h}\mathbb{C} \simeq \pi_0\mathbb{C}$*.*

2.7. We also need the following technical size conditions.

Definition 2.8 (HTT.A.2.6.1)**.** *Let* $\mathbb{C}$ *be a model category. We say* $\mathbb{A}$ *is* ***combinatorial*** *if the following conditions are satisfied:*

(a) *The category* $\mathbb{C}$ *is presentable.*
(b) *As a weakly saturated* class *of morphisms,* (C) *is generated by a* set.
(c) *As a weakly saturated* class *of morphisms,* $(C \cap W)$ *is generated by a* set.

Example 2.9. *The model category* $\mathsf{Set}_\Delta^{\mathsf{KQ}}$ *equipped with the simplicial enrichment* $\mathsf{Fun}(-,-)$ *is a combinatorial simplicial model category.*

Proposition-Definition 2.10 (HTT.A.3.3.2)**.** *Let* $\mathbb{C}$ *be a combinatorial simplicial model category and* $\mathbb{J}$ *be a small simplicial category. Then there exists two combinatorial model structures on* $\mathsf{Fun}(\mathbb{J}, \mathbb{C})$*:*

(1) *The* ***projective model structure****, denoted by* $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{proj}}$*, where*
 (W) *A* ***weak equivalence*** *is a natural transformation that is a pointwise weak equivalence.*
 (F) *A* ***projective fibration*** *is a natural transformation that is a pointwise fibration.*
 (C) *The collection of* ***projective cofibrations*** *is determined by* $(C \cap W)$*.*

(2) *The* ***injective model structure****, denoted by* $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}}$*, where*
 (W) *A* ***weak equivalence*** *is a natural transformation that is a pointwise weak equivalence.*
 (C) *A* ***injective cofibration*** *is a natural transformation that is a pointwise cofibration.*
 (F) *The collection of* ***injective cofibrations*** *is determined by* $(F \cap W)$*.*

Example 2.11. *When* $\mathbb{J} = [0]$ *is the singleton, both model structures coincide with the given model structure on* $\mathbb{C}$*.*

Proposition 2.12 (HTT.A.3.3.6). *Let*

$$F : \mathbb{A} \rightleftarrows \mathbb{B} : G$$

be a Quillen adjunction between combinatorial simplicial model categories. Then for any small simplcial category $\mathbb{J}$*, it induces Quillen adjunctions*

$$F \circ - : \mathsf{Fun}(\mathbb{J}, \mathbb{A})_? \rightleftarrows \mathsf{Fun}(\mathbb{J}, \mathbb{B})_? : G \circ -,$$

where ? can be either proj *or* inj. *They are Quillen equivalences if* $F : \mathbb{A} \rightleftarrows \mathbb{B} : G$ *is so.*

Proposition 2.13 (HTT.A.3.3.7, A.3.3.8). *Let* $\iota : \mathbb{J} \to \mathbb{J}'$ *be a functor between small simplicial enriched categories. For a combinatorial simplicial model category* $\mathbb{C}$*, we have Quillen adjunctions*

$$\mathsf{LKE}_\iota : \mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{proj}} \rightleftarrows \mathsf{Fun}(\mathbb{J}', \mathbb{C})_{\mathsf{proj}} : \iota \circ -;$$

$$\iota \circ - : \mathsf{Fun}(\mathbb{J}', \mathbb{C})_{\mathsf{inj}} \rightleftarrows \mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}} : \mathsf{RKE}_\iota;$$

which are Quillen equivalences if $\iota : \mathbb{J} \to \mathbb{J}'$ *is a weak equivalence in the model category* Cat_Δ *(see [Lecture 5, Definition 5.4]).*

Definition 2.14. *We call the right derived functor of*

$$\mathsf{RKE}_\iota : \mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}} \to \mathsf{Fun}(\mathbb{J}', \mathbb{C})_{\mathsf{inj}}$$

the ***homotopy right Kan extension*** *functor, and denote it by*

$$\mathsf{hoRKE}_\iota : \mathsf{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \to \mathsf{Fun}(\mathbb{J}', \mathbb{C})[W^{-1}].$$

When $\mathbb{J}' = [0]$ *is the singleton, we obtain* ***homotopy limit*** *functor*

$$\mathsf{holim} : \mathsf{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \to \mathbb{C}[W^{-1}],$$

which is the right derived functor of

$$\mathsf{lim} : \mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}} \to \mathbb{C}.$$

Dually, we define the ***homotopy left Kan extension*** *functor and the* ***homotopy colimit*** *functor.*

Warning 2.15. *Note that*

$$\mathsf{Fun}(\mathbb{J}, \mathbb{C})[W^{-1}] \neq \mathsf{Fun}(\mathbb{J}, \mathbb{C}[W^{-1}]).$$

Hence homotopy limit is not a functorial construction about diagrams in $\mathbb{C}[W^{-1}]$.

Remark 2.16. *In fact, Theorem 1.2 implies the theory of* ∞*-limits serves as a remedy for the non-functoriality of the classical theory of homotopy limits. Namely, instead of considering diagrams into the ordinary homotopy category* $\mathbb{C}[W^{-1}]$*, one should consider diagrams into the* ∞*-category modelled by the quasi-category* $\mathfrak{N}_\bullet(\mathbb{C}^\circ)$*. In fact, the latter can be canonically identified with the quasi-categorical localization* $\mathsf{N}_\bullet(\mathbb{C})[W^{-1}]$*. See [Lecture 5, A.5] for more information.*

Construction 2.17. *Let* $\overline{w} : \mathbb{J}^{\triangleleft} \to \mathbb{C}$ *be a functor and* $w : \mathbb{J} \to \mathbb{C}$ *be its restriction. There is an obvious morphism in* $\mathsf{Fun}(\mathbb{J}, \mathbb{C})$ *from the constant functor* $\underline{\overline{w}(*)}$ *to* w*. By adjunction, we obtain a canonical morphism*

$$\overline{w}(*) \to \mathsf{lim}\, w.$$

Definition 2.18. *Let $\overline{w} : \mathbb{J}^{\triangleleft} \to \mathbb{C}$ be a functor and $w : \mathbb{J} \to \mathbb{C}$ be its restriction. We say $\overline{w}$* ***exhibits*** $\overline{w}(*)$ ***as the homotopy limit*** *of w if for any/all fibrant replacement $w \to w'$, the composition*

$$\overline{w}(*) \to \lim w \to \lim w'$$

is an isomorphism.

3. Examples of homotopy (co)limits

3.1. Throughout this section, $\mathbb{C}$ is a combinatorial simplicial model category.

Exercise 3.2. *Let $\mathbb{J}$ be a set. Show that:*

(1) *A functor $w : \mathbb{J} \to \mathbb{C}$ is fibrant in $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}}$ iff $w(j) \in \mathbb{C}$ is fibrant for any $j \in \mathbb{J}$.*
(2) *A functor $w : \mathbb{J} \to \mathbb{C}$ is cofibrant in $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{proj}}$ iff $w(j) \in \mathbb{C}$ is cofibrant for any $j \in \mathbb{J}$.*

Deduce that the homotopy (co)products of (co)fibrant objects can be calculated by the naive (co)products.

Exercise 3.3. *Let $\mathbb{J} := \{a_0 \xrightarrow{f_0} b \xleftarrow{f_1} a_1\}$ be the index category of pullbacks. Show that a functor $w : \mathbb{J} \to \mathbb{C}$ is fibrant in $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}}$ iff $w(b) \in \mathbb{C}$ is fibrant and $w(f_i)$ are fibrations. Deduce a necessary condition for a homotopy pullback diagram in $\mathbb{C}$.*

Remark 3.4. *Note that the above condition is stronger than those in [Lecture 1, Exercise A.1]. The reason is: the latter conditions are obtained by using another model structure on $\mathsf{Fun}(\mathbb{J}, \mathbb{C})$, known as the* Reedy model structure*. For more information, see HTT.A.2.9.*

Exercise 3.5. *Let $\mathbb{J} := \{a \rightrightarrows b\}$ be the index category of equalizers. Show that a functor $w : \mathbb{J} \to \mathbb{C}$ is fibrant in $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}}$ iff $w(b) \in \mathbb{C}$ is fibrant and $w(a) \to w(b \times b)$ is a fibration. Deduce a necessary condition for a homotopy equalizer diagram in $\mathbb{C}$.*

Exercise 3.6. *Let $\mathbb{J} := \{\cdots < -2 < -1 < 0\}$ be the index category of sequencial limits. Show that a functor $w : \mathbb{J} \to \mathbb{C}$ is fibrant in $\mathsf{Fun}(\mathbb{J}, \mathbb{C})_{\mathsf{inj}}$ iff $w(0) \in \mathbb{C}$ is fibrant and $w(-n) \to w(-n+1)$ is a fibration. Deduce a necessary condition for a homotopy sequencial limit diagram in $\mathbb{C}$.*

Appendix A. Marked simplicial sets

Definition A.1. *Let Set_Δ^+ be the ordinary category defined by:*

- *Objects are pairs (X, E), where X is a simplicial set and $E \subset X_1$ is a subset of 1-simplexes in X, called the set of* ***marked 1-simplexes****.*
- *A morphism from (X, E) to (X', E') is a morphism $X \to X'$ in Set_Δ^+ such that E is sent into E'.*

We call it the category of ***marked simplicial set****.*

Construction A.2. *There is a functor*

$$(-)^\flat : \mathsf{Set}_\Delta \to \mathsf{Set}_\Delta^+$$

such that the marked 1-simplexes in $X^\flat$ are given by degenerate 1-simplexes in X.

There is a functor

$$(-)^\sharp : \mathsf{Set}_\Delta \to \mathsf{Set}_\Delta^+$$

such that any 1*-simplex in* X *is a marked* 1*-simplex in* $X^\sharp$*.*

There is a functor

$$(-)^\natural : \mathsf{QCat} \to \mathsf{Set}_\Delta^+$$

such that the marked 1*-simplexes in* $X^\natural$ *are given by isomorphisms in* X*.*

Exercise A.3. *Let* X *and* Y *be objects in* Set_Δ^+*. Show that the functor*

$$\mathsf{Hom}_{\mathsf{Set}_\Delta^+}((-)^\flat \times \mathsf{X}, \mathsf{Y}) : \mathsf{Set}_\Delta^{\mathsf{op}} \to \mathsf{Set}$$

is represented by an object in Set_Δ*, which we denote by* $\mathsf{Fun}^\flat(\mathsf{X}, \mathsf{Y})$*.*

Similarly, show that the functor

$$\mathsf{Hom}_{\mathsf{Set}_\Delta^+}((-)^\sharp \times \mathsf{X}, \mathsf{Y}) : \mathsf{Set}_\Delta^{\mathsf{op}} \to \mathsf{Set}$$

is represented by an object in Set_Δ*, which we denote by* $\mathsf{Fun}^\sharp(\mathsf{X}, \mathsf{Y})$*.*

Exercise A.4. *Construct a canonical model structure on* Set_Δ^+ *such that*

- *Any morphism is a cofibration;*
- *Bifibrant objects are given by* $X^\natural$ *for* $X \in \mathsf{QCat}$*;*
- *Weak equivalence between bifibrant objects are given by categorical equivalences between quasi-categories;*
- *The model structure is compatible with the simplicial enrichment given by* $\mathsf{Fun}^\sharp(-,-)$*.*

A.5. **Suggested readings.** HTT.3.1.