

SEMINAR NOTES ON  
*GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE*

LSG

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## 1. TALK I BY LIN: THE BIG PICTURE

... once we could merely *formulate* Fargues' conjecture, enough mathinery is available to apply Lafforgue's ideas to get the "automorphic-to-Galois" direction...

[FS21, Sect. I.11]

## 1.1. What is the local Langlands correspondence?

**Notation 1.1.1.** We fix the following notations:

- $E$  is a local field (e.g. ...  $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t))$ )<sup>1</sup>;
- $G$  is a split reductive group over  $E$ ;
- $\hat{G}$  is the Langlands dual group over  $\mathbb{Z}$ ;
- $W_E$  is the Weil group for  $E$ .

**Conjecture 1.1.2.** There is a canonical map between sets:

$$\{\text{irreducible objects } \text{Rep}_{\mathbb{C}}(G(E))\} \rightarrow \{W_E \rightarrow \hat{G}(\mathbb{C})\}, \pi \mapsto \varphi_\pi$$

subject to certain compatibilities.

**Remark 1.1.3.** Several remarks are in order.

- (0) In general this is not a bijection.
- (1) From easy to hard: archimedean, char  $p$  nonarchimedean, char 0 nonarchimedean. E.g., the conjecture is unknown for  $E = \mathbb{Q}_p$  and general reductive group.
- (2)  $G(E)$  is a topological group and we require  $\pi$  to be smooth, i.e., any vector is fixed by some compact open subgroup.
- (3)  $W_E$  is a modification of  $\text{Gal}(\overline{E}/E)$ .  $W_{\mathbb{C}} = \mathbb{C}^\times$ ,  $W_{\mathbb{R}}$  is the canonical extension of  $\text{Gal}(\mathbb{C}/\mathbb{R})$  by  $\mathbb{C}^\times$ . For nonarchimedean  $E$  with residue field  $k = \mathbb{F}_q$ ,

$$\begin{array}{ccccccc} 1 & \longrightarrow & I_E & \longrightarrow & W_E & \longrightarrow & \mathbb{Z} \longrightarrow 1 \\ & & \downarrow \simeq & & \downarrow \subset & \lrcorner & \downarrow \xrightarrow{1 \mapsto \text{Frob}} \\ 1 & \longrightarrow & I_E & \longrightarrow & \text{Gal}(\overline{E}/E) & \longrightarrow & \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \longrightarrow 1. \end{array}$$

The necessity of  $W_E$  instead of  $\text{Gal}(\overline{E}/E)$  can be already seen in local class field theory.

- (4)  $W_E$  is a topological group and we require  $\varphi_\pi$  to be continuous. However, it is not equipped with the subspace topology from  $\text{Gal}(\overline{E}/E)$ . Instead,  $I_E$  has the subspace topology from  $\text{Gal}(\overline{E}/E)$ , which is pro-finite, and is forced to be open in  $W_E$ , which is therefore locally pro-finite.
- (5)  $\varphi_\pi$  is the so-called  $L$ -parameter of  $\pi$ .
- (6) Part of the compatible requirements are about  $L$ -functions and  $\epsilon$ -factors. Lin knows nothing about them. Maybe someone can explain later?

## 1.2. What is geometrization?

**Answer 1.2.1.** Do global geometric Langlands on the Fargues–Fontaine curve, which behaves like a genus 0 curve in nonarchimedean geometry.

To explain what this mean, we need some basic notions in analytic geometry.

**Notation 1.2.2.** From now on, we restrict to the case  $E = \mathbb{Q}_p$ . There is no essential difference for general  $E$ , and the char  $p$  case is even easier.

**Analogy 1.2.3.** Tutorials on analytic geometry will be provided by Lin and Yuchen in the next weeks. For now, let us be satisfied by the following:

<sup>1</sup>The colors are chosen to be compatible with [SW20, Figure 12.1].

	<i>algebraic geometry</i>	<i>analytic geometry</i>
<i>affine</i>	$\mathrm{Spec} R, R \in \mathrm{CAlg}$	$\mathrm{Spa}(R, R^+)$ , ( $R, R^+$ ) is a Huber pair: $R^+ \subset R \in \mathrm{CAlg}(\mathrm{Top})$ satisfying certain conditions
<i>globalization</i>	scheme as locally ringed spaces	(pre-)adic space as locally topologically ringed spaces
<i>point</i>	$\mathrm{Spec} K, K \text{ is a field}$	$\mathrm{Spa}(K, K^+)$ , ( $K, K^+$ ) is an affinoid field analytic if $K$ is nondiscrete nonanalytic if $K$ is discrete

**Notation 1.2.4.** In most cases, people make the canonical choice  $R^+ := R^\circ$  being the subring of power-bounded elements and write  $\mathrm{Spa} R := \mathrm{Spa}(R, R^\circ)$ .

**Remark 1.2.5.** Among all the (pre-)adic spaces, there is a class of objects, called perfectoid spaces, that are well-adapted to connect char 0 and char  $p$ . Affine perfectoid spaces are given by  $\mathrm{Spa}(R, R^+)$  such that  $R$  is a perfectoid ring. Basic examples of perfectoid rings include  $\mathbb{F}_p((t^{1/p^\infty}))$ , which is the completion of  $\bigcup_n \mathbb{F}_p((t))(t^{1/p^n})$ , and  $\mathbb{Q}_p^{\mathrm{cycl}}$ , which is the completion of  $\mathbb{Q}_p(\mu_{p^\infty})$ . Any perfectoid ring is defined over  $\mathbb{Z}_p$  although the latter itself is not a perfectoid ring.

Personally I think the following result (together with the *tilting equivalence* to be explained later) is the root of all the magic:

**Theorem 1.2.6.** There is no final object in  $\mathrm{Perfd}$ , however, products exist in the category  $\mathrm{Perfd}_p$  of perfectoid spaces of characteristic  $p$ .

**Analogy 1.2.7.** Sanath will talk about the details about FF curves. For now, let us be satisfied by the following:

	<i>Global Geometric Langlands for functional field</i>	<i>Geometrized Local Langlands</i>
<i>geometry</i>	algebraic geometry over $\mathbb{F}_p$	“perfectoid geometry”
<i>test objects</i>	schemes $S \in \mathrm{Sch}/\mathbb{F}_p$	char $p$ perfectoid space $S \in \mathrm{Perfd}_p$
<i>final test object</i>	$\mathrm{Spec} \mathbb{F}_p$	<b>not exist</b> $\mathrm{Spa} \mathbb{F}_p \notin \mathrm{Perfd}_p$
<i>spaces</i>	prestacks $\supset$ fpqc-stacks $\supset$ algebraic spaces	prestacks $\supset$ v-stacks $\supset$ diamonds
<i>absolute curve</i>	$X$ over $\mathbb{F}_p$	<b>not exist/sci-fi</b>
<i>relative curve</i>	$X_S := S \times_{\mathbb{F}_p} X$	$\mathcal{X}_S := \mathcal{Y}_S / \mathrm{Frob}_S := (S \dot{\times} \mathrm{Spa} \mathbb{Q}_p) / \mathrm{Frob}_S$

**Question 1.2.8.** Wait, how dare you multiply a char 0 object  $\mathrm{Spa} \mathbb{Q}_p$  with a char  $p$  object  $S$ !

**Warning 1.2.9.** There is a dot over the product sign in the notation  $S \dot{\times} \mathrm{Spa} \mathbb{Q}_p$ , which means it is not a product, at least not naively. For example,  $\mathrm{Spa}(R, R^+) \dot{\times} \mathrm{Spa} \mathbb{Q}_p$  is an open subspace of  $\mathrm{Spa} W(R^+)$ , where  $W(R^+)$  is the ring of  $p$ -Witt vectors in  $R^+$ . In fact, it is the open subspace where the functions  $p, [\varpi] \in W(R^+)$  are invertible, where  $\varpi \in R^+$  is a pseudo-uniformizer. Whatever this means, we see  $S \dot{\times} \mathrm{Spa} \mathbb{Q}_p$  is of char 0.

**Question 1.2.10.** Wait, if  $\mathcal{X}_S$  is of char 0, is it okay to study it using char  $p$  test objects? For example, when talking about Hecke modifications, you need a notion of Cartier divisors of  $\mathcal{X}_S$  relative to the base, but where is your base? It can't be  $S$  or  $S/\mathrm{Frob}_S$  because they are of char  $p$ .

**Answer 1.2.11.** No, at least not in the naive way. The correct way to relate  $\mathcal{X}_S$  to char  $p$  objects is via its associated diamond  $(\mathcal{X}_S)^\diamond$ , which we will explain now.

**Construction 1.2.12** ([FS21, Sect. 6.2]). For any commutative ring  $R$ , the  $(p)$ -tilt of  $R$  is

$$R^\flat := \lim(\dots \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} R).$$

A priori this is only a multiplicative monoid. If  $R$  is equipped with a good enough complete topology, such as a perfectoid ring, then one can define a ring structure where the addition law is

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) := (z^{(0)}, z^{(1)}, \dots),$$

where

$$z^{(i)} := \lim_{n \rightarrow \infty} (x^{(i+n)} + y^{(i+n)})^{p^n}.$$

In particular, when  $R$  is a perfectoid ring, we obtain a char  $p$  perfectoid ring  $R^\flat$ . We say  $R$  is a untilt of  $R^\flat$ .

We define

$$\text{Spd}(R, R^+) := \text{Spa}(R, R^+)^\flat.$$

Using gluing, we can define  $X^\flat$  for any perfectoid space  $X$ .

**Remark 1.2.13.** For perfectoid ring  $R$ , we have  $(R^\flat)^\circ \simeq (R^\circ)^\flat$ .

**Example 1.2.14.** Any char  $p$  perfectoid ring  $R$  is the tilt of itself, and is the only char  $p$  untilt of itself. But there are char 0 untilts.

**Example 1.2.15.** The tilt of  $\mathbb{Q}_p^{\text{cycl}}$  is  $\mathbb{F}_p((t^{1/p^\infty}))$ .

**Theorem 1.2.16** (Tilting Equivalence). For any perfectoid space  $X$ , the functor  $Y \mapsto Y^\flat$  induces an equivalence between the categories of perfectoid spaces over  $X$  and  $X^\flat$ . This equivalence preserves (finite) étale covers.

**Definition 1.2.17.** For any pre-adic space  $X$ , define  $X^\diamond$  to be the prestack

$$X^\diamond : \text{Perfd}_p^{\text{op}} \rightarrow \text{Set}, S \mapsto \bigsqcup_{S^\sharp \in \text{Untilt}(S)} \text{Maps}(S^\sharp, X).$$

**Remark 1.2.18.** In fact, Lin thinks the following is correct. Consider

$$\text{PreStk} := \text{Func}(\text{Perfd}^{\text{op}}, \text{Set}), \text{PreStk}_p := \text{Func}(\text{Perfd}_p^{\text{op}}, \text{Set})$$

and define

$$\diamond : \text{PreStk} \rightarrow \text{PreStk}_p$$

to be the unique colimit-preserving functor extending  $\flat$ . Then when restricted to pre-adic spaces, one recovers the above definition.

**Theorem 1.2.19.** For pre-adic space  $X$ , the underlying topological spaces of  $X$  and  $X^\diamond$  are canonically homeomorphic.

**Remark 1.2.20.** In fact, we have the following slogan: “ $\diamond$  only remembers topological information.”

**Example 1.2.21.**  $\text{Spd } \mathbb{Z}_p := (\text{Spa } \mathbb{Z}_p)^\diamond$  classifies all untilts;  $\text{Spd } \mathbb{Q}_p := (\text{Spa } \mathbb{Q}_p)^\diamond$  classifies all char 0 untilts.

**Example 1.2.22.** If  $X$  is already a perfectoid space, then  $X^\diamond \simeq X^\flat$  by the tilting equivalence.

**Example 1.2.23.** For char  $p$  pre-adic space  $X$ , the functor  $X \mapsto X^\diamond$  is just

$$\text{PreAdic}_p \rightarrow \text{Func}(\text{PreAdic}_p^{\text{op}}, \text{Set}) \rightarrow \text{Func}(\text{Perfd}_p^{\text{op}}, \text{Set}).$$

This is because only char  $p$  untilts  $S^\sharp$  can map to  $X$ .

**Example 1.2.24.** By the tilting equivalence, if  $X$  is the quotient of  $R \rightrightarrows Y$  of perfectoid spaces connected by pro-étale maps, then  $X^\diamond$  is the quotient of  $R^\flat \rightrightarrows Y^\flat$ . Essentially, diamonds are defined to be such quotients. In fact, any analytic pre-adic space, which means all its residue fields are not discrete, over  $\mathbb{Z}_p$  can be written as such a quotient.

**Remark 1.2.25.** Yifei will talk about the pro-étale topology and explain why it is powerful.

**Example 1.2.26.** Unfortunately/fortunately,  $\mathrm{Spa} \mathbb{Q}_p$  is not a perfectoid space but it has a perfectoid pro-étale cover  $\mathrm{Spa} \mathbb{Q}_p^{\mathrm{cycl}} \rightarrow \mathrm{Spa} \mathbb{Q}_p$  whose Galois group is  $\mathbb{Z}_p^\times$ . Hence

$$\mathrm{Spd} \mathbb{Q}_p \simeq \mathrm{Spd} \mathbb{Q}_p^{\mathrm{cycl}} / \underline{\mathbb{Z}_p^\times} \simeq \mathrm{Spa} \mathbb{F}_p((t^{1/p^\infty})) / \underline{\mathbb{Z}_p^\times},$$

where  $\underline{\mathbb{Z}_p^\times}$  is the discrete group diamond.

**Theorem 1.2.27.** For any char  $p$  perfectoid space  $S$ , we have

$$(S \dot{\times} \mathrm{Spa} \mathbb{Q}_p)^\diamond \simeq S \times \mathrm{Spd} \mathbb{Q}_p.$$

**Warning 1.2.28.** This is not formal. But Lin thinks  $\dot{\times}$  was, or should have been, designed to make this correct.

**Remark 1.2.29.** For a char  $p$  perfectoid space  $S$ , a map  $S \rightarrow \mathrm{Spd} \mathbb{Q}_p$  provides a char 0 untilt  $S^\sharp$ , which will provide a closed immersion  $S^\sharp \rightarrow \mathcal{Y}_S := S \dot{\times} \mathrm{Spa} \mathbb{Q}_p$  once we know the precise definition of the target. This is a Cartier divisor and so is the composition  $S^\sharp \rightarrow \mathcal{X}_S$ . Also, the latter only depends on the composition  $S \rightarrow \mathrm{Spd} \mathbb{Q}_p \rightarrow \mathrm{Spd} \mathbb{Q}_p / \mathrm{Frob}$ . This suggests  $\mathrm{Spd} \mathbb{Q}_p / \mathrm{Frob}$  should be the moduli prestack of Cartier divisors on FF curves.

In fact, as we have seen (or will see) in geometric Langlands, we use  $\overline{\mathbb{F}_p}$ -points on the curve to define Hecke modifications. Hence we should restrict our attention to  $S$  defined over  $\overline{\mathbb{F}_p}$  rather than  $\mathbb{F}_p$ . The effect is to change  $\mathrm{Spd} \mathbb{Q}_p / \mathrm{Frob}$  to  $\mathrm{Spd} \mathbb{Q}_p^{\mathrm{ur}} / \mathrm{Frob}$ , where  $\mathbb{Q}_p^{\mathrm{ur}} = \mathrm{Frac}(W(\overline{\mathbb{F}_p}))$  is the maximal unramified extension of  $\mathbb{Q}_p$ .

We can finally define the moduli of divisors on FF curves:

**Definition 1.2.30.** The moduli diamond of degree 1 closed Cartier divisors on FF curves is

$$\mathrm{Div}^1 := \mathrm{Spd} \mathbb{Q}_p^{\mathrm{ur}} / \mathrm{Frob}.$$

**Warning 1.2.31.** Unlike in algebraic geometry,  $\mathrm{Div}^1$  is not the curve itself. They live in different characteristics. In fact, we do not have the absolute FF curve.

**Remark 1.2.32.** An amazing thing is this tilting/untilting game allows us to consider products of the “curve”, or in fact, consider  $\mathrm{Div}^1 \times \cdots \times \mathrm{Div}^1$ . Unlike the self product of  $\mathrm{Spa} \mathbb{Q}_p$ , this product, which is taken in the category of diamonds, is not boring. Lin thinks this is essentially because  $\mathrm{Perfd}_p$  lacks a final object but has products (which is false for  $\mathrm{Perfd}$ ).

**1.3. Why Fargues–Fontaine curve?** Please read [FS21, Sect. I.11] (titled “The origin of the ideas”). For now, let me explain how the automorphic and Galois sides naturally appear in this geometrization picture.

For the Galois side:

**Theorem 1.3.1.**  $\pi_1(\mathrm{Div}^1) \simeq W_E$ .

**Remark 1.3.2.** Heuristically this follows from the definition  $\mathrm{Div}^1 := \mathrm{Spd} \mathbb{Q}_p^{\mathrm{ur}} / \mathrm{Frob}$ . Indeed,  $W_E$  is the extension of  $\mathbb{Z}\langle \mathrm{Frob} \rangle$  by  $\mathrm{Gal}(\overline{\mathbb{Q}_p} / \mathbb{Q}_p^{\mathrm{ur}})$ .

**Question 1.3.3.** Wait, didn't you say FF curves behaved like genus 0 curves?

**Answer 1.3.4.** Yes. But  $\mathcal{X}_C$  is not defined over  $C$  or any algebraically closed field. Instead, we have:

**Theorem 1.3.5.**  $\Gamma(\mathcal{X}_C, \mathcal{O}) \simeq \mathbb{Q}_p$ .

For the automorphic side, we consider the v-stack  $\mathrm{Bun}_G$  whose values  $\mathrm{Bun}_G(S)$  classify  $G$ -torsors on  $\mathcal{X}_S$ . Taeuk will explain the precise meaning of the following:

**Theorem 1.3.6.**  $\mathrm{Bun}_G$  has a stratification labelled by the poset  $B(G)$  such that each stratum is of the form  $\ast/H$ , where  $H$  is a group diamond which is an extension of a discrete group  $\underline{M}(\mathbb{Q}_p)$  by a unipotent group, where  $M$  is an inner form of a Levi subgroup of  $G$ .

**Remark 1.3.7.** *This is another example where FF curves behave like genus 0.*

**Corollary 1.3.8.** *For any  $\mathbb{Z}_l$ -algebra  $\Lambda$ , the category  $D(\text{Bun}_G, \Lambda)$  can be glued from categories  $\text{Rep}_\Lambda(M(\mathbb{Q}_p))$  for  $M$  being inner forms of Levi subgroups of  $G$ .*

Yifei will explain how to define  $D(-, \Lambda)$  and play with them.

#### 1.4. What can be translated from geometric Langlands?

**Answer 1.4.1.** *Essentially any pure geometric constructions in Geometric Landlands can be or at least should be translated. Things already done in [FS21]: geometric Satake, Lafforgue's automorphic-to-Galois construction via shitukas, formulation of categorical Langlands conjecture, the spectral action...*

**1.5. What else?** The story is not complete without talking about  $p$ -adic Hodge theory. After all, FF curves were born during the study of Fontaine's period rings. If people want, we can make a digression to it.

## 2. TALK 2 BY LIN: ADIC SPACES

... we should remember that valuations played a central role in Zariski's approach from the late 1930's, building up algebraic geometry by algebraic means. Later this role was reduced by Grothendieck and others in favour of prime ideals... Recently, in the Ragsquad<sup>a</sup> seminar and elsewhere, we experienced a revived interest in Zariski's work.

<sup>a</sup>Recent advances in real algebraic geometry and quadratic forms.

[HK94]

**2.1. Why adic spaces: a history of nonarchimedean geometry.** Huber introduces adic spaces as a generalization of formal schemes and rigid analytic varieties, which is also the title of [Hub94].

We are familiar with formal schemes:

**Definition 2.1.1.** *An Noetherian adic ring  $A$  is a Noetherian ring equipped with the  $I$ -adic topology for certain ideal  $I \subset A$ . For a Noetherian adic ring, define an affine formal scheme  $\mathrm{Spf} A := \mathrm{colim}_n \mathrm{Spec} A/I^n$  which only depends on the completion of  $A$ . One can realize  $\mathrm{Spf} A$  as a locally topologically ringed space by taking the colimits in the category of such spaces.*

**Example 2.1.2.**  $\mathrm{Spf} \mathbb{Z}_p$  only has one point. There is an obvious map  $\mathrm{Spec} \mathbb{F}_p \rightarrow \mathrm{Spf} \mathbb{Z}_p$ , but there is no map from  $\mathrm{Spec} \mathbb{Q}_p$  to it. Note that  $\mathrm{Spec} \mathbb{Q}_p$  can not detect the topology of  $\mathbb{Q}_p$ .

We are less familiar with Tate's rigid analytic geometry. Below is a briefing of it.

**Definition 2.1.3.** *A nonarchimedean field is a field  $K$  complete with respect to a nontrivial nonarchimedean valuation<sup>2</sup>  $K \rightarrow 0 \cup \mathbb{R}^+$ .*

*The corresponding valuation ring is denoted by  $K^\circ := \{x \in K \mid |x| \leq 1\}$ .*

*A pseudo-uniformizer  $\varpi$  in  $K$  is a topologically nilpotent unit in  $K$ , i.e.,  $0 < |\varpi| < 1$ . Such element always exists because we assumed the valuation is nontrivial.*

**Example 2.1.4.**  $\mathbb{Q}_p$  is a nonarchimedean field and  $p$  is a (pseudo-)uniformizer.

**Example 2.1.5.** Let  $\mathbb{C}_p$  be the completion of  $\overline{\mathbb{Q}_p}$ . Then  $\mathbb{C}_p$  is an algebraically closed nonarchimedean field and  $p$  is a pseudo-uniformizer.

Roughly speaking, Tate's rigid geometry over a nonarchimedean field  $K$  is the geometry where the topology of  $K$  is remembered.

**Remark 2.1.6.** *Let us try to invent rigid geometry. For simplicity, we assume  $K = \overline{K}$ . Suppose we can define the affine line  $(\mathbb{A}_K^1)^{\mathrm{rig}}$  over  $K$  such that its points correspond to elements in  $K$ . What should be the open subvariety of  $(\mathbb{A}_K^1)^{\mathrm{rig}}$ ? Of course, we want the usual Zariski opens. But we also want the subset corresponding to  $\{x \in K \mid |x| \leq c\} \subset K$  to be open for any  $c \in \mathbb{R}^+$  because the latter is open in  $K$ . Naively, we want  $K[T]$  to be the ring of functions on  $(\mathbb{A}_K^1)^{\mathrm{rig}}$ , then what should be the ring of functions on  $\{x \in K \mid |x| \leq 1\} \subset K$ ? The answer is the Tate algebra  $K\langle T \rangle$ .*

**Definition 2.1.7.** *For any topological ring  $R$ , define the Tate algebra*

$$R\langle T \rangle := \left\{ f = \sum a_i T^i \in R[[T]] \mid \lim a_i = 0 \right\} \subset R[[T]].$$

*More generally, we define  $R\langle T_1, \dots, T_n \rangle$ , and we have  $R\langle T_1, \dots, T_m \rangle \langle T_{m+1}, \dots, T_n \rangle = R\langle T_1, \dots, T_n \rangle$ .*

*A topologically finitely presented (or tfp)  $R$ -affinoid algebra is a topological  $R$ -algebra  $A$  that is isomorphic to a quotient of some  $R\langle T_1, \dots, T_n \rangle$  by a finitely generated ideal<sup>3</sup>*

**Remark 2.1.8.** *If the topology on  $R$  is given by a norm  $|\cdot|$ , then  $R\langle T \rangle$  is the completion of  $R[T]$  with respect to the Gauss norm  $|\sum a_i T^i| := \max |a_i|$ .*

<sup>2</sup>We always write valuation groups multiplicatively.

<sup>3</sup>For nonarchimedean field  $K$ ,  $K\langle T_1, \dots, T_n \rangle$  is Noetherian. See [Con08, Theorem 1.1.5]. But there is no proof in *loc.cit.*. Conrad only mentioned using Weierstrass Preparation techniques. Also, when  $K^\circ$  is Noetherian (or equivalently when the valuation is discrete),  $K^\circ\langle T_1, \dots, T_n \rangle$  and  $K\langle T_1, \dots, T_n \rangle$  are Noetherian by Example 2.1.9 below.

**Example 2.1.9.**  $K^\circ\langle T \rangle$  is the  $\varpi$ -adic completion of  $K^\circ[T]$ .  $K\langle T \rangle = K^\circ\langle T \rangle[\varpi^{-1}]$ .

**Remark 2.1.10.** For an algebraically closed nonarchimedean field  $K$ , the maximal spectrum  $\text{MaxSpec } K\langle T \rangle$  is  $\{(T-x) \mid |x| \leq 1\}$ . Therefore I suggest to view anything inside  $\langle - \rangle$  as being forced to have norm  $\leq 1$ .

**Definition 2.1.11.** For a nonarchimedean field  $K$ , a  $K$ -affinoid algebra is a topological  $K$ -algebra  $A$  that is isomorphic to a quotient of some  $K\langle T_1, \dots, T_n \rangle$ .

**Remark 2.1.12.** For a  $K$ -affinoid algebra, we want to define a topology on  $\text{MaxSpec } A$ . As explained before, there are two kinds of basic opens: the Zariski ones  $\{x \in \text{MaxSpec } A \mid g(x) \neq 0\}$  for  $g \in A$ , and the disk-like ones, or Weierstrass ones  $\{x \in \text{MaxSpec } A \mid |f(x)| \leq 1\}$  for  $f \in A$ .

However, we do not want to use the topology generated by them. Because otherwise the topology on the closed unit disk  $\text{MaxSpec } K\langle T \rangle$  is not connected (in the case  $K = \mathbb{Q}_p$ , this topology would be the totally disconnected space  $\mathbb{Z}_p$ ).

At first glance, this seems like a dilemma: we want the topology on  $\text{MaxSpec } K\langle T \rangle$  to resemble that of  $\mathbb{Z}_p$  but also behave like a disk. How can this be possible?

The key is to use Grothendieck topology instead of topology.

**Definition 2.1.13.** For a  $K$ -affinoid algebra  $A$ , and elements  $f_1, \dots, f_n, g \in A$  which generate the unit ideal, define

$$U\left(\frac{f_1, \dots, f_n}{g}\right) := \text{MaxSpec } A\langle f_1/g, \dots, f_n/g \rangle := \{x \in \text{MaxSpec } A \mid |f_i(x)| \leq |g(x)| \neq 0\}$$

and call it a rational domain of  $\text{MaxSpec } A$ .

The Tate topology on  $\text{MaxSpec } A$  is the “fpqc” Grothendieck topology: opens are given by rational domains and coverings are required to be finite.

There is an obvious definition of the structure sheaf on this site and we obtain a locally topologically ringed topoi, which is called the affinoid space  $\text{Sp}(A)$  associated to  $A$ . Such notion can be globalized to define rigid spaces.

**Remark 2.1.14.** Several remarks are in order.

- Taking  $f_i = 0$ , we get the Zariski opens; taking  $g = 1$ , we get the Weierstrass opens.
- More precisely,  $A\langle f_1/g, \dots, f_n/g \rangle := A\langle T_1, \dots, T_n \rangle / (gT_1 - f_1, \dots, gT_n - f_n)$ .
- Some authors use the condition  $(f_1, \dots, f_n) = A$  instead of  $(f_1, \dots, f_n, g) = A$ . There is no difference because we can add  $f_0 := g$ .
- The condition  $|g(x)| \neq 0$  is in fact redundant: if  $g(x) = 0$ , then  $f_i = 0$  and  $1 = 0$  because  $(f_i, g) = A$ . We keep it because later in the definition of rational domains of adic spaces, we no longer require  $f_1, \dots, f_n = A$ .
- But we certainly need the condition  $|g(x)| \neq 0$  or  $(f_1, \dots, f_n) = A$ . Otherwise taking  $g = 0$  we will get Zariski closed subset.
- On the other hand, if we keep the condition  $|g(x)| \neq 0$  and drop the condition  $(f_1, \dots, f_n, g) = A$ , we still get an open subset  $\{x \in \text{MaxSpec } A \mid |f_i(x)| \leq |g(x)| \neq 0\}$  because it is the union of  $\text{MaxSpec } A\langle f_1/g, \dots, f_n/g, \varpi^m/g \rangle$  for large enough  $m$ . Note that this is indeed a cover in the Tate topology because when restricted to any rational domain inside the target, the above union stabilizes. But  $\{x \in \text{MaxSpec } A \mid |f_i(x)| \leq |g(x)| \neq 0\}$  itself is not quasi-compact. Hence if we want a well-behaved Grothendieck topology, we need some condition on  $f_i$ .
- The reason to use non-strict inequalities instead of strict inequalities is similar: strict inequalities still define open subset of  $\text{MaxSpec } A$ , but they are not quasi-compact.
- In the Tate topology,  $\text{MaxSpec } K\langle T \rangle$  is connected. Lin does not want to prove it now<sup>4</sup>, but we can see the naive way to disconnect it does not work: the loci given by  $|T| \geq 1$  and  $|T| < 1$  no longer cover  $\text{MaxSpec } K\langle T \rangle$  because it has no finite refinement by rational domains.

<sup>4</sup>Lin thinks using sheaf theory to prove connectedness is better than point-set topology and believes we will learn such a proof in the future. Nevertheless, see [Con08, Exercise 2.4.9].



**Remark 2.1.15.** *Several shortcomings of Tate’s rigid geometry: all of these are correct but seem hopeless to prove using methods of rigid geometry:*

- Flat morphisms are open under some finiteness assumptions.
- Flatness is preserved under base change along field extensions.
- Fppc descents for rigid spaces, maps between them, coherent sheaves on them.
- There is a robust theory of étale sheaves on rigid spaces.
- ...

*Roughly speaking, these shortcomings lead to the following evolution of theories: Raynaud’s formal models, Berthelot’s formal models, Berkovich spaces, Huber’s adic spaces<sup>5</sup>...*

**Remark 2.1.16.** *We will not spend more time reviewing the theories before Huber than the following sketches.*

- Raynaud’s theory realize qcqs rigid spaces over  $K$  as “generic fibers”<sup>6</sup> of tfp formal schemes flat over  $K^\circ$  (along the map  $\cdot^7$ ). Such realizations are unique up to (formal) blow-ups in the special fiber  $\mathrm{Spec} \kappa(K^\circ)$ .
- Berthelot’s theory allows taking generic fibers for formal schemes of formally finite type over  $K^\circ$ . In the affine case, this means we allow quotients of  $K^\circ[[T_1, \dots, T_m]](T_{m+1}, \dots, T_n)$ . But the construction is very indirect: the generic fiber of  $\mathrm{Spf} K^\circ[[T]]$  is not even affine.
- Berkovich’s theory studies multiplicative seminorms (or equivalently, continuous valuations into  $0 \cup \mathbb{R}^+$ ) of  $K$ -affinoid algebras. This is motivated by the fact that any point of  $\mathfrak{m} \in \mathrm{Sp} A$  defines such a seminorm via  $A \rightarrow A/\mathfrak{m} \xrightarrow{|\cdot|} 0 \cup \mathbb{R}^+$ .
- Huber’s theory revives Zariski’s work and studies all the continuous valuations of (nice) topological algebras  $A$ . Such valuations will be points of a locally topologically ringed space<sup>8</sup>  $\mathrm{Spa} A$ , the associated adic space. There are several amazing things about adic spaces:
  - It has a underlying topological space rather than a topoi.
  - It uniformizes formal schemes and rigid spaces in the same framework. Now the taking generic fiber construction can be done naively. For example  $\mathrm{Spa} \mathbb{Z}_p$  now has two points.
  - For a  $K$ -affinoid algebra  $A$ , the underlying set  $|\mathrm{Spa} A|$  is bijective to the set of all points of the topoi  $\mathrm{Sp} A$ .
  - For a  $K$ -affinoid algebra  $A$ ,  $\mathrm{Spa} A$  is the inverse limit of all the Raynaud’s formal models (viewed as adic spaces).
  - It is well-behaved even for infinite type  $K^\circ$ -algebras, which in particular allows perfectoid geometry.
- Clausen–Scholze’s theory want to replace “topological space” by “sheaves on the site of pro-finite sets”, or equivalently “pro-étale sheaves on fields” and rewrite everything...

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<sup>5</sup>Maybe Clausen–Scholze’s condensed mathematics should be the next one.

<sup>6</sup>There is map  $\mathrm{Sp} K \rightarrow \mathrm{Spf} K^\circ$  between locally topologically ringed topos. For a tfp formal scheme  $X$ , there is a canonical morphism from  $X$  to its Raynaud’s generic fiber  $X_K$ . But Lin does not know if  $X_K$  is the fiber product  $X \times_{\mathrm{Spf} K^\circ} \mathrm{Sp} K$ .

<sup>7</sup>There is an obvious way to define such generic fibers for  $\mathrm{Spf} K^\circ(T_1, \dots, T_n)$ . One extends this construction to any flat tfp affinoid  $K^\circ$ -algebras  $R$  by showing it does not depend on presentations of  $R$  as a quotient of Tate algebras. One extends it to global case (i.e., to *admissible* formal schemes) by showing it is compatible with localizations.

<sup>8</sup> $\mathrm{Spa} A$  in fact has more structure: for any point  $x$  of it, we have a continuous valuation of the local ring  $\mathcal{O}_{X,x}$ .