

## LECTURE 1

The main goal of this course is to study representations of semisimple Lie algebras via geometric methods. We restrict ourselves to the case when the base field  $k$  is algebraically closed and of characteristic 0, such as the field  $\mathbb{C}$  of complex numbers.

### 1. SEMISIMPLE LIE ALGEBRAS

This is just a quick review of the definitions about finite-dimensional semisimple Lie algebras. See [Hum, Chapter 0] for the abc's and [Ser] for a thorough textbook.

**Definition 1.** A **Lie algebra** (over  $k$ ) is a vector space  $\mathfrak{g}$  equipped with a binary operation  $[-, -] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ , called the **Lie bracket**, such that:

- The Lie bracket is **bilinear**, i.e., factors as  $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ .
- The Lie bracket is **alternating**:  $[x, x] = 0$ .
- The **Jacobi identity** holds:  $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ .

Let  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  be Lie algebras. A **Lie algebra homomorphism** between them is a  $k$ -linear map  $f : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$  commuting with Lie brackets, i.e.,  $f([x, y]) = [f(x), f(y)]$ .

This defines a category  $\text{Lie}_k$  of Lie algebras.

**Example 2.** Any vector space  $V$  is equipped with a trivial Lie bracket:  $[x, y] = 0$ . Such Lie algebras are called **abelian Lie algebras**.

**Example 3.** Let  $A$  be an associative algebra. Then the underlying vector space has a natural Lie algebra structure with Lie bracket given by  $[x, y] := xy - yx$ . This defines a functor  $\text{oblv} : \text{Alg}_k \rightarrow \text{Lie}_k$  from the category of associative algebras to that of Lie algebras.

**Example 4.** Let  $V$  be a vector space and  $\mathfrak{gl}(V)$  be the vector space of endomorphisms of  $V$ . By Example 3,  $\mathfrak{gl}(V)$  is naturally a Lie algebra with Lie bracket given by  $[f, g] = f \circ g - g \circ f$ . This is the **general linear Lie algebra** of  $V$ .

If  $V$  is finite-dimensional, let  $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$  be the subspace of endomorphisms  $f$  such that the trace  $\text{tr}(f) = 0$ .

When  $V = k^{\oplus n}$ , we write  $\mathfrak{gl}_n := \mathfrak{gl}(V)$ ,  $\mathfrak{sl}_n := \mathfrak{sl}(V)$ . Note that  $\mathfrak{gl}_n$  (resp.  $\mathfrak{sl}_n$ ) can be identified with the space of  $n \times n$  matrices (resp. whose traces are zero).

**Fact 5.** We have  $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ .

**Definition 6.** Let  $\mathfrak{g}$  be a Lie algebra. A **representation** of  $\mathfrak{g}$ , or  **$\mathfrak{g}$ -module**, is a vector space  $V$  equipped with a Lie algebra homomorphism  $\mathfrak{g} \rightarrow \mathfrak{gl}(V)$ . In other words, there is a bilinear map  $\cdot : \mathfrak{g} \times V \rightarrow V$  such that  $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ .

Let  $V_1$  and  $V_2$  be representations of  $\mathfrak{g}$ . A  $\mathfrak{g}$ -linear map between them is a  $k$ -linear map  $f : V_1 \rightarrow V_2$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} \times V_1 & \longrightarrow & V_1 \\ \text{Id} \times f \downarrow & & \downarrow f \\ \mathfrak{g} \times V_2 & \longrightarrow & V_2. \end{array}$$

This defines a category  $\mathfrak{g}\text{-mod}$  of representations of  $\mathfrak{g}$ .

**Fact 7.** *The category  $\mathfrak{g}\text{-mod}$  is an abelian category. The forgetful functor  $\mathfrak{g}\text{-mod} \rightarrow \text{Vect}_k$  is exact.*

**Example 8.** Let  $\mathfrak{g}$  be a Lie algebra. The map  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ ,  $x \mapsto \text{ad}_x := [x, -]$  defines a  $\mathfrak{g}$ -module structure on  $\mathfrak{g}$  itself. This is called the **adjoint representation**.

**Definition 9.** Let  $\mathfrak{g}$  be a Lie algebra. An **ideal**  $\mathfrak{a} \subset \mathfrak{g}$  is a sub-representation of the adjoint representation. In other words, we require  $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$ .

*Remark 10.* Note that an ideal  $\mathfrak{a}$  is also a Lie subalgebra.

**Example 11.** Let  $\mathfrak{g}$  be a Lie algebra, then  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$  is an ideal. We call it the **derived Lie algebra** of  $\mathfrak{g}$ .

**Definition 12.** Let  $\mathfrak{g}$  be a Lie algebra. We say  $\mathfrak{g}$  is **simple** if:

- It is not abelian;
- The adjoint representation is simple (a.k.a. irreducible), i.e.,  $\mathfrak{g}$  has no ideal other than 0 and itself.

**Example 13.** The Lie algebra  $\mathfrak{gl}_n$  is not simple because  $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$  is a proper ideal of it. The Lie algebra  $\mathfrak{sl}_n$  is simple for  $n \geq 2$ .

*Remark 14.* Finite-dimensional simple Lie algebras (over  $k$ ) are fully classified. A similar classification for infinite-dimensional simple Lie algebras seems to be hopeless.

**Definition 15.** Let  $\mathfrak{g}_i$ ,  $i \in I$  be Lie algebras indexed by a set  $I$ . The direct sum  $\oplus \mathfrak{g}_i$  of the underlying vector spaces has a natural Lie bracket given by  $[(x_i)_{i \in I}, (y_i)_{i \in I}] := ([x_i, y_i])_{i \in I}$ . The obtained Lie algebra is called the **direct sum** of the Lie algebras  $\mathfrak{g}_i$ .

**Warning 16.** *The direct sum  $\oplus \mathfrak{g}_i$  is not the coproduct in the category  $\text{Lie}_k$ . Instead, if  $I$  is a finite set, then it is the product of  $\mathfrak{g}_i$  in this category.*

*Remark 17.* Representation theory for  $\mathfrak{g}_1 \oplus \mathfrak{g}_2$  can be obtained from those for  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  in a non-trivial mechanism<sup>1</sup>.

**Definition 18.** Let  $\mathfrak{g}$  be a Lie algebra. We say  $\mathfrak{g}$  is **semisimple** if it is a direct sum of simple Lie algebras.

*Remark 19.* The zero Lie algebra 0 is semisimple but not simple.

The main goal of this course is to study representations of finite-dimensional semisimple Lie algebras.

**Convention 20.** *From now on, unless otherwise stated, Lie algebras are assumed to be finite-dimensional.*

*Exercise 21.* **This is not a homework!**

- (1) Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra. Show  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ .
  - The opposite statement is generally *not* true. Below is a counterexample. Let  $\mathfrak{h}$  be a simple Lie algebra and  $V$  be a nontrivial simple  $\mathfrak{h}\text{-mod}$ . Define a bracket on the vector space by the formula  $\mathfrak{h} \oplus V$  by  $[(x, u), (y, v)] := ([x, y], x \cdot v - y \cdot u)$ .
- (2) Show this bracket makes  $\mathfrak{h} \oplus V$  into a Lie algebra, which we denote by  $\mathfrak{h} \ltimes V$ .
- (3) Show  $[\mathfrak{h} \ltimes V, \mathfrak{h} \ltimes V] = \mathfrak{h} \ltimes V$  but  $\mathfrak{h} \ltimes V$  is not semisimple.

<sup>1</sup>The abelian category  $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)\text{-mod}$  is the *tensor product* of  $\mathfrak{g}_1\text{-mod}$  and  $\mathfrak{g}_2\text{-mod}$ .

## 2. ROOT SPACE DECOMPOSITION

**Convention 22.** From now on, unless otherwise stated,  $\mathfrak{g}$  means a finite-dimensional semisimple Lie algebra.

**Definition 23.** A **Cartan subalgebra**  $\mathfrak{t}$  of  $\mathfrak{g}$  is a maximal abelian subalgebra of it.

**Warning 24.** Cartan subalgebras for general finite-dimensional Lie algebras are defined in a different way and they are not abelian in general. That definition is equivalent to the above one if  $\mathfrak{g}$  is semisimple.

**Theorem 25.** Cartan subalgebras of  $\mathfrak{g}$  have a same dimension, which is called the **(semisimple) rank** of  $\mathfrak{g}$ . In fact, Cartan subalgebras are all conjugate to each other: for Cartan subalgebras  $\mathfrak{t}_1$  and  $\mathfrak{t}_2$ , there exists  $x \in \mathfrak{g}$  such that  $\text{ad}_x(\mathfrak{t}_1) = \mathfrak{t}_2$ .

**Example 26.** The rank of  $\mathfrak{sl}_n$  is  $n - 1$ . One Cartan subalgebra of it is the subspace of diagonal matrices.

**Notation 27.** From now on, we fix a Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ . Let  $\mathfrak{t}^* := \text{Hom}(\mathfrak{t}, k)$  be the dual vector space of  $\mathfrak{t}$ . For any  $\alpha \in \mathfrak{t}^*$ , let  $\mathfrak{g}_\alpha \subset \mathfrak{g}$  be the  $\alpha$ -eigenspace for the adjoint  $\mathfrak{t}$ -action on  $\mathfrak{g}$ , i.e.,

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{t}\}.$$

**Remark 28.** Note that  $\mathfrak{g}_0 = \mathfrak{t}$  (because  $\mathfrak{t}$  is maximal) and  $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$  (because of the Jacobi identity).

**Proposition 29** (Root Space Decomposition<sup>2</sup>). Let  $\mathfrak{g}$  be a finite-dimensional semisimple Lie algebra with a fixed Cartan subalgebra  $\mathfrak{t}$ . Then we have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where  $\Phi \subset \mathfrak{t}^* \setminus 0$  is the finite set containing those nonzero  $\alpha$  such that  $\mathfrak{g}_\alpha$  is nonempty. Moreover, if  $\alpha \in \Phi$ , then  $-\alpha \in \Phi$  and  $\mathfrak{g}_\alpha$  is 1-dimensional.

**Proposition 30.** There exists a (non-unique) subset  $\Phi^+ \subset \Phi$  such that:

- We have a disjoint decomposition  $\Phi = \Phi^+ \sqcup -\Phi^+$ ;
- If  $\alpha, \beta \in \Phi^+$  and  $\alpha + \beta \in \Phi$ , then  $\alpha + \beta \in \Phi^+$ .

**Notation 31.** From now on, we fix such a subset  $\Phi^+$ . Write  $\Phi^- = -\Phi^+$ .

**Definition 32.** (For above choices), elements in  $\Phi$  are called **roots** of  $\mathfrak{g}$ . Elements in  $\Phi^+$  (resp.  $\Phi^-$ ) are called **positive roots** (resp. **negative roots**). For  $\alpha \in \Phi^+$ , we say  $\alpha$  is a **(positive) simple root** if it cannot be written as the sum of two positive roots. Let  $\Delta \subset \Phi^+$  be the subset of simple roots.

**Proposition 33.** The subset  $\Delta \subset \mathfrak{t}^*$  is a basis. In particular, any positive root can be uniquely written as a linear combination of simple roots with non-negative coefficients.

**Definition 34.** Define

$$\mathfrak{b} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

which are Lie subalgebras of  $\mathfrak{g}$ . We call  $\mathfrak{b}$  the **Borel subalgebra** of  $\mathfrak{g}$  (that corresponds to the choice of  $\Phi^+$ ) and  $\mathfrak{n}$  the **nilpotent radical** of  $\mathfrak{b}$ .

<sup>2</sup>Some authors, including Humphreys, prefer the name *Cartan decomposition*. But there is a completely different Cartan decomposition in the study of real Lie algebras.

*Remark 35.* In general, a Borel subalgebra  $\mathfrak{b}$  of any Lie algebra  $\mathfrak{g}$  is defined to be a maximal *solvable* subalgebra of it. Here solvable means the sequence  $D^1(\mathfrak{b}) := \mathfrak{b}$ ,  $D^{n+1}(\mathfrak{b}) := [D^n(\mathfrak{b}), D^n(\mathfrak{b})]$  satisfies  $D^n(\mathfrak{b}) = 0$  for  $n \gg 0$ . It is known that all Borel subalgebras are conjugate to each other.

The subalgebra  $\mathfrak{n} \subset \mathfrak{b}$  is called the nilpotent radical because it contains exactly nilpotent elements in  $\mathfrak{b}$ , i.e., those elements  $x$  such that  $(\text{ad}_x)^n = 0$  for  $n \gg 0$ .

Note that we have  $\mathfrak{t} \simeq \mathfrak{b}/\mathfrak{n}$ .

*Exercise 36.* **This is not a homework!** For  $\mathfrak{g} = \mathfrak{sl}_n$  and its standard Cartan subalgebra (Example 26).

- (1) Find an explicit description of  $\Phi$  and  $\mathfrak{g}_\alpha$ .
- (2) Show there is a unique choice of  $\Phi^+$  such that the corresponding  $\mathfrak{b}$  is the subspace of upper triangulated matrices.
- (3) For the choice of  $\Phi^+$  in (2), find all the simple roots and write each root as a linear combination of these simple roots.

### 3. ROOT SYSTEM

**Definition 37.** Let  $E$  be a finite-dimensional Euclidean space and  $\Phi \subset E$  be a finite subset such that  $0 \notin \Phi$ . We say  $(E, \Phi)$  is a **root system** if the following is satisfied:

- The subset  $\Phi$  is a basis of  $E$ ;
- For any  $\alpha \in \Phi$ ,  $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ ;
- For  $\alpha, \beta \in \Phi$ , the number  $2\frac{(\beta, \alpha)}{(\alpha, \alpha)}$  is an integer;
- The subset  $\Phi$  is closed under reflection along any  $\alpha \in \Phi$ , i.e., for  $\alpha, \beta \in \Phi$ , the element  $\beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha$  is contained in  $\Phi$ .

**Definition 38.** Let  $(E, \Phi)$  be a root system. The **dual root system** is defined to be  $(E^*, \check{\Phi})$ , where  $E^*$  is the dual Euclidean space of  $E$  and  $\check{\Phi}$  consists of those  $\check{\alpha}$  for  $\alpha \in \Phi$  defined by  $\check{\alpha}(-) = 2\frac{(-, \alpha)}{(\alpha, \alpha)}$ .

*Exercise 39.* **This is not a homework!** Show the double-dual of a root system is itself.

Let us return to the notations in the last section. Let  $E := \mathbb{R}\Phi$  be the  $\mathbb{R}$ -vector space spanned by  $\Phi$  (such that we have  $E \otimes_{\mathbb{R}} k \simeq \mathfrak{t}^*$ ). We are going to show  $(E, \Phi)$  is a root system. For this purpose, we need to define an inner product on  $E$ .

**Definition 40.** Let  $\mathfrak{g}$  be any finite-dimensional Lie algebra. The **Killing form** on  $\mathfrak{g}$  is the bilinear form  $\text{Kil} : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ ,  $\text{Kil}(x, y) := \text{tr}(\text{ad}(x) \circ \text{ad}(y))$

**Proposition 41.** The Killing form is symmetric and (ad-)invariant, i.e.,

- For  $x, y \in \mathfrak{g}$ ,  $\text{Kil}(x, y) = \text{Kil}(y, x)$ ;
- For  $x, y, z \in \mathfrak{g}$ ,  $\text{Kil}(\text{ad}_z(x), y) + \text{Kil}(x, \text{ad}_z(y)) = 0$ .

**Proposition 42.** If  $\mathfrak{g}$  is simple, then any symmetric invariant bilinear form on  $\mathfrak{g}$  is of the form  $c\text{Kil}$  for  $c \in k$ .

**Warning 43.** The similar claim is false if  $k$  is not algebraically closed.

**Theorem 44** (Cartan–Killing Criterion). The Lie algebra  $\mathfrak{g}$  is semisimple iff its Killing form is non-degenerate. Moreover, in this case, the restriction of  $\text{Kil}$  on  $\mathfrak{t}$  is also non-degenerate.

**Construction 45.** Since  $\text{Kil}|_{\mathfrak{h}}$  is non-degenerate, it induces an isomorphism  $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$  sending  $x$  to the unique element  $x^*$  such that  $x^*(-) = \text{Kil}(x, -)$ . Consider the inverse  $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$  of this isomorphism, which also corresponds to a non-degenerate bilinear form on  $\mathfrak{h}^*$ .

**Lemma 46.** *The restriction of the above bilinear form on  $E \subset \mathfrak{h}^*$  is an inner product.*

**Conversion 47.** *From now on, we always view  $E$  as an Euclidean space via the above inner product.*

**Theorem 48.** *The pair  $(E, \Phi)$  defined above is a root system.*

Note that  $\mathfrak{h} \simeq (\mathfrak{h}^*)^* \simeq (E \otimes_{\mathbb{R}} k)^* \simeq E^* \otimes_{\mathbb{R}} k$ . Hence  $E^*$  and thereby  $\check{\Phi}$  can be viewed as a subset of  $E^*$ .

**Definition 49.** For any root  $\alpha \in \Phi$ , define the corresponding **coroot** to be  $\check{\alpha} \in \check{\Phi} \subset \mathfrak{h}$ .

*Remark 50.* There is a (unique if stated properly) semisimple Lie algebra corresponding to the dual root system  $(E^*, \check{\Phi})$ , known as the *Langlands dual* Lie algebra  $\check{\mathfrak{g}}$  of  $\mathfrak{g}$ .

#### REFERENCES

- [Hum] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category  $\mathcal{O}$ . Vol. 94. American Mathematical Soc., 2008.
- [Ser] Serre, Jean-Pierre. Complex semisimple Lie algebras. Springer Science & Business Media, 2000.