LECTURE 7

Last time, we defined limits and colimits in an ∞-category. In this lecture, we study functorial properties of them.

1. Functorial in u

Construction 1.1. *Let K be a simplicial set and C be an ∞-category. For a morphism $\alpha : u_0 \to u_1$ in $\mathsf{Fun}(K, \mathsf{C})$, we have a canonical functor*

$$\mathsf{C}_{/u_0} \to \mathsf{C}_{/u_1}$$

compatible with the forgetful functors to C (see [Lecture 6, Proposition-Construction 3.15]). Using this functor, we obtain a canonical lifting of $\lim u_0 \in \mathsf{C}$ to $\mathsf{C}_{/u_1}$. By the definition of final objects, there is an essentially unique morphism

$$\lim u_0 \to \lim u_1$$

compatible with their liftings to $\mathsf{C}_{/u_1}$.

Dually, there is an essentially unique morphism

$$\mathsf{colim}\, u_0 \to \mathsf{colim}\, u_1$$

compatible with their liftings to $\mathsf{C}_{u_1/}$.

By loc.cit.*, these morphisms are invertible if α is so.*

1.2. In future lectures, we will update the above construction to a functor

$$\lim : \mathsf{Fun}(K, \mathsf{C})' \to \mathsf{C},$$

where $\mathsf{Fun}(K, \mathsf{C})' \subset \mathsf{Fun}(K, \mathsf{C})$ is the full sub-∞-category consisting of diagrams $u : K \to \mathsf{C}$ such that $\lim u$ exists.

2. Functorial in C

Construction 2.1. *Let $F : \mathsf{C} \to \mathsf{C}'$ be a functor between ∞-categories, and $u : K \to \mathsf{C}$ be a diagram. We have an obvious functor*

$$\mathsf{C}_{/u} \to \mathsf{C}'_{/F \circ u}$$

compatible with F via the forgetful functors. Using this functor, we obtain a canonical lifting of $F(\lim u) \in \mathsf{C}'$ to $\mathsf{C}'_{/F \circ u}$. By the definition of final objects, there is an essentially unique morphism

$$F(\lim u) \to \lim (F \circ u)$$

compatible with their liftings to $\mathsf{C}'_{/F \circ u}$.

Dually, there is an essentially unique morphism

$$\mathsf{colim}\, (F \circ u) \to F(\mathsf{colim}\, u)$$

compatible with their liftings to $\mathsf{C}'_{F \circ u/}$.

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Definition 2.2. *Let $F : \mathsf{C} \to \mathsf{C}'$ be a functor between ∞-categories. We say a limit diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}$ is **preserved by** F if the composition $F \circ \overline{u} : K^{\triangleleft} \to \mathsf{C}'$ is a limit diagram. We also say $\lim u$ is **preserved by** F.*

Exercise 2.3. *Show that F preserves $\lim u$ iff $F(\lim u) \to \lim (F \circ u)$ is an isomorphism.*

Exercise 2.4. *Show that if two functors $F, G : \mathsf{C} \to \mathsf{C}'$ are equivalent to each other, then a limit is preserved by F iff it is preserved by G.*

Exercise 2.5. *An invertible functor preserves all existing limits.*

2.6. Although we have not yet introduced adjoint functors between ∞-categories, let us record the following result:

Theorem 2.7. *Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be an adjunction between ∞-categories. Then*

(1) *The functor F preserves all existing colimits.*
(2) *The functor G preserves all existing limits.*

Definition 2.8. *Let $F_i : \mathsf{C} \to \mathsf{D}_i$ be a collection of functors between ∞-categories. For a diagram $u : K \to \mathsf{C}$, we say $\{F_i\}$ **detect the limit of** u if an extended diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}$ is a limit diagram whenever $F_i \circ \overline{u} : K^{\triangleleft} \to \mathsf{D}_i$ are limit diagrams.*

Warning 2.9. *In the above definition, we do* not *assume $\lim u$* a priori *exists.*

2.10. Although we have not defined representable functors, let us record the following result:

Theorem 2.11. *Representable functors preserve and detect all limits.*

Remark 2.12. *Let us restate the theorem in more explicit words. In future lectures, we will construct a canonical functor*[1]

$$\mathsf{Maps}(-,-) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Grpd}_{\infty}.$$

Then the theorem says:

(1) *A diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}$ is a limit diagram iff for any object $x \in \mathsf{C}$, the covariant functor $\mathsf{Maps}(x,-)$ sends it to a limit diagram in Grpd_{∞}.*
(2) *A diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}^{\mathsf{op}}$ is a limit diagram iff for any object $x \in \mathsf{C}$, the contravariant functor $\mathsf{Maps}(-,x)$ sends it to a limit diagram in Grpd_{∞}.*

Warning 2.13. *Similar claims would be false if Grpd_{∞} is replaced by hGrpd_{∞}.*

Theorem 2.14 (Ker.02X9)**.** *Let C be an ∞-category and B be any simplicial set. Then (co)limits in $\mathsf{Fun}(B, \mathsf{C})$ can be calculated pointwisely.*

Remark 2.15. *The precise meaning of the above theorem is the following. Let $u : K \to \mathsf{Fun}(B, \mathsf{C})$ be any diagram. Suppose for any 0-simplex b in B, the diagram*

$$u|_b : K \to \mathsf{Fun}(B, \mathsf{C}) \xrightarrow{\mathsf{ev}_b} \mathsf{C}$$

admits a limit, then $\lim u$ exists and is preserved and detected by the collection of functors $\mathsf{ev}_b : \mathsf{Fun}(B, \mathsf{C}) \to \mathsf{C}$. In particular, we have

$$(\lim u)|_b \xrightarrow{\simeq} \lim u|_b.$$

Note that this claim is not trivial because the construction of a functor $B \to \mathsf{C}$, $b \mapsto \lim u|_b$ is not. See §1.

[1]We only constructed a functor $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{hGrpd}_{\infty}$ in [Lecture 4, §10].

Proposition 2.16. *A fully faithful functor detects all (co)limits.*

Exercise 2.17. *The proof of the above proposition is sketched as the following exercise. Let $F : \mathsf{C} \to \mathsf{C}'$ be a functor. Prove:*

(1) *The functor F is fully faithful iff it can be represented by a monomorphism $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ between quasi-categories such that an n-simplex of $\mathcal{C}'$ is contained in the image of $\mathcal{F}$ iff all the vertices of this simplex have the same property.*
(2) *If F is fully fatifhul, then $\mathsf{Fun}(K, \mathsf{C}) \to \mathsf{Fun}(K, \mathsf{C}')$ is fully faithful for any simplicial set K.*
(3) *If F is fully faithful, then $\mathsf{C}_{/u} \to \mathsf{C}'_{/F\circ u}$ is fully faithful for any diagram $u : K \to \mathsf{C}$.*
(4) *If F is fully faithful, then it detects limits.*

Warning 2.18. *Fully faithful functors may not* perserve *limits. Example:* $[0] \xrightarrow{0} [1]$ *does not preserve empty limits, i.e. final objects.*

3. Functorial in K

Construction 3.1. *Let $v : K' \to K$ be a morphism in Set_Δ and $u : K \to \mathsf{C}$ be a diagram in an ∞-category C. We have a restriction functor*

$$\mathsf{C}_{/u} \to \mathsf{C}_{/u\circ v}$$

compatible with the forgetful functors to C. Using this functor, we obtain a canonical lifting of $\lim_K u \in \mathsf{C}$ to $\mathsf{C}_{/u\circ v}$. By the definition of final objects, there is an essentially unique morphism

$$\lim_K u \to \lim_{K'} (u \circ v),$$

compatible with their liftings to $\mathsf{C}_{/u\circ v}$.

Dually, there is an essentially unique morphism

$$\operatorname*{colim}_{K'} (u \circ v) \to \operatorname*{colim}_K u$$

compatible with their liftings to $\mathsf{C}_{u\circ v/}$.

Definition 3.2. *Let $v : K' \to K$ be a morphism in Set_Δ. We say v is **initial**[2] if for any ∞-category C and any diagram $u : K \to \mathsf{C}$, the restriction functor $\mathsf{C}_{/u} \to \mathsf{C}_{/u\circ v}$ is an equivalence between ∞-categories[3].*

*Dually, we say $v : K' \to K$ is **final**[4] if for any ∞-category C and any diagram $u : K \to \mathsf{C}$, the restriction functor $\mathsf{C}_{u/} \to \mathsf{C}_{u\circ v/}$ is an equivalence between ∞-categories.*

3.3. It is clear that if v is initial, then $\lim_K u \to \lim_{K'} (u \circ v)$ is invertible for any diagram $u : K \to \mathsf{C}$. In fact, the converse is also true (Proposition 3.12).

3.4. Note that $v : K' \to K$ is initial iff $v^{\mathsf{op}} : (K')^{\mathsf{op}} \to K^{\mathsf{op}}$ is final. Hence in below, we focus on initial morphisms.

Proposition 3.5 (Ker.02NN, 02NP)**.** *For morphisms in Set_Δ, being initial is invariant under equivalences.*

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[2]Other terminologies in the literature: *op-cofinal*, *left cofinal*.
[3]This definition is equivalent to that in the works of Joyal and Lurie (see Ker.02N1) by Ker.02NR.
[4]Other terminologies in the literature: *cofinal*, *right cofinal*.

3.6. As a result, we obtain the notion of **initial functors between ∞-categories**.

Warning 3.7. *An initial functor between ordinary categories*[5] *may fail to be an initial functor between ∞-categories.*

Proposition 3.8 (Ker.043E). *A functor $x : \Delta^0 \to K$ between quasi-categories is initial iff x is an initial object in K.*

Warning 3.9. *In general,*

- *An initial morphism $v : K' \to K$ is* not *an initial object in $\mathsf{Fun}(K', K)$. Example: $\mathsf{Id} : \Delta^1 \to \Delta^1$.*
- *An initial object in $\mathsf{Fun}(K', K)$ is* not *an initial morphism $v : K' \to K$. Example: $K' \to \Delta^0$ such that K' is not weakly contractible. See Proposition 3.19 below.*

3.10. As an application, we obtain the following useful criterion for limits.

Proposition-Construction 3.11. *Let $\overline{u} : K^{\triangleleft} \to \mathsf{C}$ be a diagram in an ∞-category C. Write $u := \overline{u}|_K$. Consider the restriction functors*

$$\mathsf{C}_{/\overline{u}(*)} \leftarrow \mathsf{C}_{/\overline{u}} \to \mathsf{C}_{/u}.$$

(1) *The functor $\mathsf{C}_{/\overline{u}(*)} \leftarrow \mathsf{C}_{/\overline{u}}$ is an equivalence.*
(2) *The functor $\mathsf{C}_{/\overline{u}} \to \mathsf{C}_{/u}$ is an equivalence iff $\overline{u}$ is a limit diagram.*

In particular, we obtain an equivalence $\mathsf{C}_{/\lim u} \xrightarrow{\simeq} \mathsf{C}_{/u}$ whenever $\lim u$ exists.

Proof. (1) follows from the fact that the apex $\Delta^0 \to K^{\triangleleft}$ is initial (Proposition 3.8). It remains to prove (2). By [Lecture 6, Exercise 3.8], we have a canonical equivalence $\mathsf{C}_{/\overline{u}} \simeq (\mathsf{C}_{/u})_{/\overline{u}(*)}$. Now the claim follows from [Lecture 6, Proposition 4.2]. □

Proposition 3.12. *Let $v : K' \to K$ be a morphism in Set_Δ. The following conditions are equivalence:*

(1) *The morphism v is initial.*
(2) *For any diagram $u : K \to \mathsf{C}$ in an ∞-category C, we have:*
   – *The limit of u exists iff the limit of $u \circ v$ exists;*
   – *The morphism $\lim_K u \to \lim_{K'} (u \circ v)$ is invertible.*

Proof. (1)⇒(2) is obvious. Now suppose v satisfies (2).

Let C be any ∞-category, we need to show $\mathsf{C}_{/u} \to \mathsf{C}_{/u \circ v}$ is an equivalence. In future lectures, we will show that there exists a fully faithful functor $\mathsf{C} \to \mathsf{D}$ such that D admits all K-indexed limits[6]. Consider the composition $w : K \xrightarrow{u} \mathsf{C} \to \mathsf{D}$. We claim

- If $\mathsf{D}_{/w} \to \mathsf{D}_{/w \circ v}$ is invertible, then so is $\mathsf{C}_{/u} \to \mathsf{C}_{/u \circ v}$.

Indeed, under this assumption, we have a commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/u} \arrow[r] \arrow[d] & \mathsf{C}_{/u \circ v} \arrow[d] \\ \mathsf{D}_{/w} \arrow[r, "\simeq"'] & \mathsf{D}_{/w \circ v} \end{tikzcd}$$

[5] See [Sta24, Tag 09WN] for what this means.
[6] The Yoneda embedding $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Grpd}_\infty)^{\mathsf{op}}$ is such a choice as long as K and C are small.

By Exercise 2.17(3), the vertical functors are fully faithful, and it is easy to identify their essential images via the bottom equivalence. It follows formally that the top functor is an equivalence as desired.

Hence we only need to show $\mathsf{D}_{/w} \to \mathsf{D}_{/w\circ v}$ is invertible. In other words, we can assume C admits all K-indexed limits. Using Proposition-Construction 3.11, it is easy to construct a commutative diagram

$$\begin{tikzcd} \mathsf{C}_{/\mathsf{lim}\, u} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathsf{C}_{/\mathsf{lim}\,(u\circ v)} \arrow[d, "\simeq"] \\ \mathsf{C}_{/u} \arrow[r] & \mathsf{C}_{/u\circ v}, \end{tikzcd}$$

which implies the desired claim. □

Exercise 3.13. *Let $I \xrightarrow{f} J \xrightarrow{g} K$ be morphisms in Set_Δ. Suppose f is initial. Then g is initial iff $g\circ f$ is initial.*

Warning 3.14. *The collection of initial morphisms does not have the 2-out-of-3 property.*

Exercise 3.15. *Consider the following diagram in Set_Δ*

$$\begin{tikzcd} & \Delta^1 \arrow[dr] & \\ \Delta^0 \arrow[ur, "1"] \arrow[rr] & & \Delta^0. \end{tikzcd}$$

Show that exactly *two of the three morphisms are initial.*

Proposition 3.16 (Ker.02NK)**.** *The collection of final (resp. initial) morphisms is closed under finite products.*

Remark 3.17. *The above proposition can be established once we have the* ***distribution law of limits****, which says for a digram $u : J\times K \to \mathsf{C}$,*

$$\mathsf{lim}_{J\times K}\, u \xrightarrow{\simeq} \mathsf{lim}_{j\in J}\,[\mathsf{lim}_K\, u(j,-)].$$

However, this claim is not trivial because the construction of a functor $J \to \mathsf{C}$, $j \mapsto \mathsf{lim}_K\, u(j,-)$ is not. See §1.

3.18. [Lecture 6, Proposition 2.10 and Proposition 2.12] have the following generalization:

Proposition 3.19 (Ker.02N5)**.** *Let $v : K' \to K$ be a morphism in Set_Δ.*

1. *If K is a Kan complex, then v is initial (resp. final) iff it is a weak homotopy equivalence.*
2. *If v is initial (resp. final), then v is a weak homotopy equivalence.*

Exercise 3.20. *Let K be a weakly contractible simplical set and C be an ∞-category. For an object $x \in \mathsf{C}$, consider the constant diagram $K \to \mathsf{C}$ with value x. Show that:*

$$\mathsf{colim}_K\, x \to x \to \mathsf{lim}_K\, x$$

are isomorphisms.

Exercise 3.21. *Show that for* $\mathsf{C} := \mathsf{Grpd}_\infty$*, the claim of the above exercise would be false for general* K*.*

Proposition 3.22 (Ker.02N9)**.** *If a functor* $F : \mathsf{C} \to \mathsf{D}$ *exhibits* D *as the* ∞*-categorical localization of* C *with respect to a collection of morphisms in* C*, then* F *is both initial and final.*

Exercise 3.23. *Let* $u : K \to \mathsf{C}$ *be a diagram in an* ∞*-category* C *such that its image is contained in* $\mathsf{C}^{\simeq}$*. Suppose* K *is weakly contractible, show that for any* $i \in K$

$$u(i) \to \operatorname*{colim}_K u,\ \lim_K u \to u(i)$$

are isomorphisms.

3.24. Let us also record the following result:

Theorem 3.25. *Let* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be an adjunction between* ∞*-categories. Then*

(1) *The functor* F *is initial.*
(2) *The functor* G *is final.*

4. Quillen's Theorem A

The following result, known as Quillen's Theorem A for ∞-categories, was first proved by Joyal. We will prove it in future lectures.

Theorem 4.1. *Let* $v : K' \to K$ *be a morphism between simplicial sets. Suppose* K *is a quasi-category, then:*

(1) v *is initial iff for any object* $x \in K$*, the fiber product* $K' \times_K K_{/x}$ *in* Set_Δ *is weakly contractible.*
(2) v *is final iff for any object* $x \in K$*, the fiber product* $K' \times_K K_{x/}$ *in* Set_Δ *is weakly contractible.*

Remark 4.2. *When* $K' = \Delta^0$*, the theorem reduces to Proposition 3.8. This follows from the fact that* $\mathsf{Hom}^{\mathsf{L}}_K(x,y) := \{y\} \times_K K_{x/}$ *is a model of the* ∞*-groupoid* $\mathsf{Maps}_K(x,y)$*.*

Remark 4.3. *When* $K = \Delta^0$*, the theorem reduces to Proposition 3.19(1).*

Remark 4.4. *When* K' *is also a quasi-category, the fiber product* $K' \times_K K_{/x}$*, which is taken in the* ordinary *category* Set_Δ*, also calculates the corresponding fiber product in the* quasi-category $\mathcal{QC}at$ *(see [Lecture 6, Footnote 4]). Hence we have an* ∞*-categorical version of the above theorem: for a functor* $v : \mathsf{K}' \to \mathsf{K}$ *between* ∞*-categories.*

(1) v *is initial iff for any object* $x \in \mathsf{K}$*, the fiber product* $\mathsf{K}' \times_{\mathsf{K}} \mathsf{K}_{/x}$ *is weakly contractible.*
(2) v *is final iff for any object* $x \in \mathsf{K}$*, the fiber product* $\mathsf{K}' \times_{\mathsf{K}} \mathsf{K}_{x/}$ *is weakly contractible.*

Remark 4.5. *In future lectures, we will show that*

(1) *A functor* $v : \mathsf{K}' \to \mathsf{K}$ *has a right adjoint iff for any object* $x \in \mathsf{K}$*, the fiber product* $\mathsf{K}' \times_{\mathsf{K}} \mathsf{K}_{/x}$ *has a final object.*
(2) *A functor* $v : \mathsf{K}' \to \mathsf{K}$ *has a left adjoint iff for any object* $x \in \mathsf{K}$*, the fiber product* $\mathsf{K}' \times_{\mathsf{K}} \mathsf{K}_{x/}$ *has an initial object.*

Hence Theorem 3.25 can be deduced from Theorem 4.1 and [Lecture 6, Proposition 2.12].

Exercise 4.6. *Let Δ_{inj} be the subcategory of Δ defined by:*

- *Objects are $[n]$ for $n \geq 0$;*
- *Morphisms are* injective *nondecreasing maps $[m] \to [n]$.*

Show that $\mathsf{N}_\bullet(\Delta_{\mathsf{inj}}) \to \mathsf{N}_\bullet(\Delta)$ is initial.

Appendix A. Limits preserved by all functors

A.1. Certain limits are preserved by all functors.

Construction A.2. *Consider the ordinary category Idem defined by:*

- *There is an unique object $*$;*
- $\mathsf{Hom}(*,*) \coloneqq \{\mathsf{id}, e\}$*, with $e \circ e = e$.*

Exercise A.3. *Find all the non-degenerate simplexes of the nerve of Idem.*

Exercise A.4. *Prove the following theorem:*

Theorem A.5. *Limits and colimits indexed by Idem are preserved by any functor between ∞-categories.*

Construction A.6. *Let Δ_{aug} be the ordinary category defined by:*

- *Objects are $[n]$ for $n \geq -1$, where $[-1] \coloneqq \varnothing$.*
- *Morphisms are nondecreasing maps $[m] \to [n]$.*

Let Δ_{split} be the ordinary category defined by:

- *Objects are $[n] \sqcup \{-\infty\}$ for $n \geq -1$.*
- *Morphisms are nondecreasing maps $[m] \sqcup \{-\infty\} \to [n] \sqcup \{-\infty\}$ that send $-\infty$ to $-\infty$.*

We have obvious functors

$$\Delta \to \Delta_{\mathsf{aug}} \to \Delta_{\mathsf{split}}.$$

Exercise A.7. *Show that Δ_{aug} can be identified with $\Delta^{\triangleleft}$.*

Theorem A.8. *Show that if a diagram $u : \Delta^{\triangleleft} \to \mathsf{C}$ can be extended to a diagram $u' : \Delta_{\mathsf{split}} \to \mathsf{C}$, then u is a limit diagram. In this case, we say the limit is a* ***split cosimplicial limit****.*

Exercise A.9. *Deduce that split cosimplicial limits are preserved by any functor.*

A.10. **Suggested readings.** Ker.03Y9, HA.4.7.2.

Appendix B. Constant (co)limits in Grpd_∞

Exercise B.1. *Let $K \in \mathsf{Set}_\Delta$ and $\underline{X} : K \to \mathcal{K}an$ be a diagram with constant value $X \in \mathcal{K}an$. Can you extend it to a diagram $K^{\triangleleft} \to \mathcal{K}an$ such that the value of the apex is $\mathsf{Fun}(K, X)$?*

Exercise B.2. *Let* $K \in \mathsf{Set}_\Delta$ *and* $X, Y \in \mathcal{K}an$. *Consider the constant functors* $\underline{X}, \underline{Y} \in \mathsf{Fun}(K, \mathcal{K}an)$. *Construct the following weak equivalences between Kan complexes:*

$$\mathsf{Fun}(Y, \mathsf{Fun}(K, X)) \simeq \mathsf{Fun}(K, \mathsf{Fun}(Y, X)) \simeq \mathsf{Fun}(K, \mathsf{Hom}^{\mathsf{L}}_{\mathcal{K}an}(Y, X)) \simeq$$
$$\simeq \mathsf{Fun}(K, \mathsf{Hom}^{\mathsf{B}}_{\mathcal{K}an}(Y, X)) \simeq \mathsf{Hom}^{\mathsf{B}}_{\mathsf{Fun}(K, \mathcal{K}an)}(\underline{Y}, \underline{X}).$$

Deduce that

$$\lim_K \underline{X} \simeq \mathsf{Fun}(K, X).$$

Exercise B.3. *Let* $K \in \mathsf{Set}_\Delta$ *and* $\underline{X} : K \to \mathcal{K}an$ *be the constant functor with value* $X \in \mathcal{K}an$. *Suppose* K *is a Kan complex, prove that*

$$\operatorname{colim}_K \underline{X} \simeq K \times X.$$

References

[Sta24] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2024.