LECTURE 1

The main goal of this course is to study representations of semisimple Lie algebras via geometric methods. We restrict ourselves to the case when the base field k is algebraically closed and of characteristic 0, such as the field $\mathbb{C}$ of complex numbers.

1. Semisimple Lie Algebras

This is just a quick review of the definitions about finite-dimensional semisimple Lie algebras. See [Hum, Chapter 0] for the abc's and [Ser] for a thorough textbook.

Definition 1. A **Lie algebra** (over k) is a vector space $\mathfrak{g}$ equipped with a binary operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, such that:

- The Lie bracket is **bilinear**, i.e., factors as $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.
- The Lie bracket is **alternating**: $[x,x] = 0$.
- The **Jacobi identity** holds: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A **Lie algebra homomorphism** between them is a k-linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ commuting with Lie brackets, i.e., $f([x,y]) = [f(x), f(y)]$.

This defines a category Lie_k of Lie algebras.

Example 2. Any vector space V is equipped with a trivial Lie bracket: $[x,y] = 0$. Such Lie algebras are called **abelian Lie algebras**.

Example 3. Let A be an associative algebra. Then the underlying vector space has a natural Lie algebra structure with Lie bracket given by $[x,y] := xy - yx$. This defines a functor $\mathsf{oblv} : \mathsf{Alg}_k \to \mathsf{Lie}_k$ from the category of associative algebras to that of Lie algebras.

Example 4. Let V be a vector space and $\mathfrak{gl}(V)$ be the vector space of endomorphisms of V. By Example 3, $\mathfrak{gl}(V)$ is naturally a Lie algebra with Lie bracket given by $[f,g] = f \circ g - g \circ f$. This is the **general linear Lie algebra** of V.

If V is finite-dimensional, let $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ be the subspace of endomorphisms f such that the trace $\mathsf{tr}(f) = 0$.

When $V = k^{\oplus n}$, we write $\mathfrak{gl}_n := \mathfrak{gl}(V)$, $\mathfrak{sl}_n := \mathfrak{sl}(V)$. Note that $\mathfrak{gl}_n$ (resp. $\mathfrak{sl}_n$) can be identified with the space of $n \times n$ matrices (resp. whose traces are zero).

Fact 5. *We have* $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$.

Definition 6. Let $\mathfrak{g}$ be a Lie algebra. A **representation** of $\mathfrak{g}$, or **$\mathfrak{g}$-module**, is a vector space V equipped with a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. In other words, there is a bilinear map $(- \cdot -) : \mathfrak{g} \times V \to V$ such that $[x,y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

Let V_1 and V_2 be representations of $\mathfrak{g}$. A $\mathfrak{g}$-linear map between them is a k-linear map $f : V_1 \to V_2$ such that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \times V_1 \arrow[r] \arrow[d, "\mathsf{Id} \times f"'] & V_1 \arrow[d, "f"] \\ \mathfrak{g} \times V_2 \arrow[r] & V_2. \end{tikzcd}$$

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This defines a category $\mathfrak{g}-\mathsf{mod}$ of representations of $\mathfrak{g}$.

Fact 7. *The category $\mathfrak{g}-\mathsf{mod}$ is an abelian category. The forgetful functor $\mathfrak{g}-\mathsf{mod} \to \mathsf{Vect}_k$ is exact.*

Example 8. Let $\mathfrak{g}$ be a Lie algebra. The map $\mathsf{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathsf{ad}_x := [x, -]$ defines a $\mathfrak{g}$-module structure on $\mathfrak{g}$ itself. This is called the **adjoint representation**.

Definition 9. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and $x \in \mathfrak{g}$ be an element. We say x is **semisimple** (resp. **nilpotent**) if the corresponding endomorphism $\mathsf{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is so.

Definition 10. Let $\mathfrak{g}$ be a Lie algebra. An **ideal** $\mathfrak{a} \subset \mathfrak{g}$ is a sub-representation of the adjoint representation. In other words, we require $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.

Remark 11. Note that an ideal $\mathfrak{a}$ is also a Lie subalgebra.

Example 12. Let $\mathfrak{g}$ be a Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ is an ideal. We call it the **derived Lie algebra** of $\mathfrak{g}$.

Definition 13. Let $\mathfrak{g}$ be a Lie algebra. We say $\mathfrak{g}$ is **simple** if:

- It is not abelian;
- The adjoint representation is simple (a.k.a. irreducible), i.e., $\mathfrak{g}$ has no ideal other than 0 and itself.

Example 14. The Lie algebra $\mathfrak{gl}_n$ is not simple because $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$ is a proper ideal of it. The Lie algebra $\mathfrak{sl}_n$ is simple for $n \geq 2$.

Remark 15. Finite-dimensional simple Lie algebras (over k) are fully classified. A similar classification for infinite-dimensional simple Lie algebras seems to be hopeless.

Definition 16. Let $\mathfrak{g}_i$, $i \in I$ be Lie algebras indexed by a set I. The direct sum $\oplus \mathfrak{g}_i$ of the underlying vector spaces has a natural Lie bracket given by $[(x_i)_{i \in I}, (y_i)_{i \in I}] := ([x_i, y_i])_{i \in I}$. The obtained Lie algebra is called the **direct sum** of the Lie algebras $\mathfrak{g}_i$.

Warning 17. *The direct sum $\oplus \mathfrak{g}_i$ is* not *the coproduct in the category Lie_k. Instead, if I is a finite set, then it is the* product *of $\mathfrak{g}_i$ in this category.*

Remark 18. Representation theory for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can be obtained from those for $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in a non-trivial mechanism[1].

Definition 19. Let $\mathfrak{g}$ be a Lie algebra. We say $\mathfrak{g}$ is **semisimple** if it is a direct sum of simple Lie algebras.

Remark 20. The zero Lie algebra 0 is semisimple but not simple.

The main goal of this course is to study representations of finite-dimensional semisimple Lie algebras.

Convension 21. *From now on, unless otherwise stated, Lie algebras are assumed to be finite-dimensional.*

Exercise 22. This is not a homework!

(1) Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. Show $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
- The opposite statement is generally *not* true. Below is a counterexample. Let $\mathfrak{h}$ be a simple Lie algebra and V be a nontrivial simple $\mathfrak{h}$-module. Define a bracket on the vector space by the formula $\mathfrak{h} \oplus V$ by $[(x, u), (y, v)] := ([x, y], x \cdot v - y \cdot u)$.

[1] The abelian category $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)-\mathsf{mod}$ is the *tensor product* of $\mathfrak{g}_1-\mathsf{mod}$ and $\mathfrak{g}_2-\mathsf{mod}$.

(2) Show this bracket defines a Lie algebra structure on $\mathfrak{h} \oplus V$. We denote this Lie algebra by $\mathfrak{h} \ltimes V$.
(3) Show $[\mathfrak{h} \ltimes V, \mathfrak{h} \ltimes V] = \mathfrak{h} \ltimes V$ but $\mathfrak{h} \ltimes V$ is not semisimple.

2. Root Space Decomposition

Convension 23. *From now on, unless otherwise stated, $\mathfrak{g}$ means a finite-dimensional semisimple Lie algebra.*

Definition 24. A **Cartan subalgebra** $\mathfrak{t}$ of $\mathfrak{g}$ is a maximal abelian subalgebra of it consisting of semisimple elements.

Warning 25. *A maximal abelian subalgebra of $\mathfrak{g}$ might not be a Cartan subalgebra. For example, $\mathfrak{sl}_2$ contains a maximal abelian subalgebra spanned by $e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ but e is not semisimple*[2].

Warning 26. *Cartan subalgebras for general finite-dimensional Lie algebras are defined in a different way and they are not abelian in general. That definition is equivalent to the above one if $\mathfrak{g}$ is semisimple.*

Theorem 27. *Cartan subalgebras of $\mathfrak{g}$ have a same dimension, which is called the **(semisimple) rank** of $\mathfrak{g}$*[3].

Example 28. The rank of $\mathfrak{sl}_n$ is $n-1$. One Cartan subalgebra of it is the subspace of diagonal matrices.

Notation 29. *From now on, we fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Let $\mathfrak{t}^* := \mathsf{Hom}(\mathfrak{t}, k)$ be the dual vector space of $\mathfrak{t}$. For any $\alpha \in \mathfrak{t}^*$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the α-eigenspace for the adjoint $\mathfrak{t}$-action on $\mathfrak{g}$, i.e.,*

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{t}\}.$$

Remark 30. Note that $\mathfrak{g}_0 = \mathfrak{t}$ (because $\mathfrak{t}$ is maximal) and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (because of the Jacobi identity).

Proposition 31 (Root Space Decomposition[4])**.** *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra with a fixed Cartan subalgebra $\mathfrak{t}$. Then we have a direct sum decomposition*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi \subset \mathfrak{t}^* \setminus 0$ is the finite set containing those nonzero α such that $\mathfrak{g}_\alpha$ is nonempty. Moreover, if $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $\mathfrak{g}_\alpha$ is 1-dimensional.

Proposition 32. *There exists a (non-unique) subset $\Phi^+ \subset \Phi$ such that:*

- *We have a disjoint decomposition $\Phi = \Phi^+ \sqcup -\Phi^+$;*
- *If $\alpha, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi$*[5]*, then $\alpha + \beta \in \Phi^+$.*

Notation 33. *From now on, we fix such a subset Φ^+. Write $\Phi^- = -\Phi^+$.*

Definition 34. (For above choices), elements in Φ are called **roots** of $\mathfrak{g}$. Elements in Φ^+ (resp. Φ^-) are called **positive roots** (resp. **negative roots**). For $\alpha \in \Phi^+$, we say α is a **(positive) simple root** if it cannot be written as the sum of two positive roots. Let $\Delta \subset \Phi^+$ be the subset of simple roots.

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[2] I made this mistake during the lecture.

[3] In fact, Cartan subalgebras are all conjugate to each other by a (non-unique) element of the corresponding Lie group G of $\mathfrak{g}$.

[4] Some authors, including Humphreys, prefer the name *Cartan decomposition*. But there is a completely different Cartan decomposition in the study of real Lie algerbas.

[5] I forgot to mention this condition in the class.

Proposition 35. *The subset $\Delta \subset \mathfrak{t}^*$ is a basis. In particular, any positive root can be uniquely written as a linear combination of simple roots with non-negative coefficients.*

Definition 36. Define
$$\mathfrak{b} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \ \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$
which are Lie subalgebras of $\mathfrak{g}$. We call $\mathfrak{b}$ the **Borel subalgebra** of $\mathfrak{g}$ (that corresponds to the choice of Φ^+) and $\mathfrak{n}$ the **nilpotent radical** of $\mathfrak{b}$.

Remark 37. In general, a Borel subalgebra $\mathfrak{b}$ of any Lie algebra $\mathfrak{g}$ is defined to be a maximal *solvable* subalgebra of it. Here solvable means the sequence $D^1(\mathfrak{b}) := \mathfrak{b}$, $D^{n+1}(\mathfrak{b}) := [D^n(\mathfrak{b}), D^n(\mathfrak{b})]$ satisfies $D^n(\mathfrak{b}) = 0$ for $n >> 0$. It is known that all Borel subalgebras are conjugate to each other.

The subalgebra $\mathfrak{n} \subset \mathfrak{b}$ is called the nilpotent radical because it contains exactly nilpotent elements in $\mathfrak{b}$, i.e., those elements x such that $(\mathsf{ad}_x)^{\circ n} = 0$ for $n >> 0$.

Note that we have $\mathfrak{t} \simeq \mathfrak{b}/\mathfrak{n}$.

Exercise 38. This is not a homework! For $\mathfrak{g} = \mathfrak{sl}_n$ and its standard Cartan subalgebra (Example 28).

(1) Find an explicit description of Φ and $\mathfrak{g}_\alpha$.
(2) Show there is a unique choice of Φ^+ such that the corresponding $\mathfrak{b}$ is the subspace of upper triangulated matrices.
(3) For the choice of Φ^+ in (2), find all the simple roots and write each root as a linear combination of these simple roots.

3. Root system

Definition 39. Let E be a finite-dimensional Euclidean space and $\Phi \subset E$ be a finite subset such that $0 \notin \Phi$. We say (E, Φ) is a **root system** if the following is satisfied:

- The subset Φ spans E;
- For any $\alpha \in \Phi$, $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$;
- For $\alpha, \beta \in \Phi$, the number $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer;
- The subset Φ is closed under reflection along any $\alpha \in \Phi$, i.e., for $\alpha, \beta \in \Phi$, the element $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is contained in Φ.

Definition 40. Let (E, Φ) be a root system. The **dual root system** is defined to be $(E^*, \check{\Phi})$, where E^* is the dual Euclidean space of E and $\check{\Phi}$ consists of those $\check{\alpha}$ for $\alpha \in \Phi$ defined by $\check{\alpha}(-) = 2\frac{(-,\alpha)}{(\alpha,\alpha)}$.

Exercise 41. This is not a homework! Show the double-dual of a root system is itself.

Let us return to the notations in the last section. Let $E_{\mathbb{Q}} := \mathbb{Q}\Phi$ be the $\mathbb{Q}$-vector space spaned by Φ (such that we have $E_{\mathbb{Q}} \otimes_{\mathbb{Q}} k \simeq \mathfrak{t}^*$). Write $E := E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$[6]We are going to show (E, Φ) is a root system. For this purpose, we need to define an inner product on E.

Definition 42. Let $\mathfrak{g}$ be any finite-dimensional Lie algebra. The **Killing form** on $\mathfrak{g}$ is the bilinear form $\mathsf{Kil} : \mathfrak{g} \times \mathfrak{g} \to k$, $\mathsf{Kil}(x,y) := \mathsf{tr}(\mathsf{ad}(x) \circ \mathsf{ad}(y))$.

Proposition 43. *The Killing form is symmetric and* (ad-)*invariant, i.e.,*

- *For* $x, y \in \mathfrak{g}$, $\mathsf{Kil}(x,y) = \mathsf{Kil}(y,x)$;
- *For* $x, y, z \in \mathfrak{g}$, $\mathsf{Kil}(\mathsf{ad}_z(x), y) + \mathsf{Kil}(x, \mathsf{ad}_z(y)) = 0$.

[6] In the lecture, I wrote $E \otimes_{\mathbb{R}} k \simeq \mathfrak{t}^*$ which is only valid if we are given a homomorphism $\mathbb{R} \to k$.

Proposition 44. *If $\mathfrak{g}$ is simple, then any symmetric invariant bilinear form on $\mathfrak{g}$ is of the form $c\mathsf{Kil}$ for $c \in k$.*

Warning 45. *The similar claim is false if k is not algebraically closed.*

Theorem 46 (Cartan–Killing Criterion)**.** *The Lie algebra $\mathfrak{g}$ is semisimple iff its Killing form is non-degenerate. Moreover, in this case, the restriction of Kil on $\mathfrak{t}$ is also non-degenerate.*

Construction 47. *Since $\mathsf{Kil}|_{\mathfrak{t}}$ is non-degenerate, it induces an isomorphism $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$ sending x to the unique element x^* such that $x^*(-) = \mathsf{Kil}(x,-)$. Consider the inverse $\mathfrak{t}^* \xrightarrow{\sim} \mathfrak{t}$ of this isomorphism, which also corresponds to a non-degenerate bilinear form on $\mathfrak{t}^*$.*

Lemma 48. *The restriction of the above bilinear form on $E_{\mathbb{Q}} \subset \mathfrak{t}^*$ is $\mathbb{Q}$-valued and positive-definite. In particular, it induced a inner product on $E = E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.*

Convension 49. *From now on, we always view E as an Eucilidean space via the above inner product.*

Theorem 50. *The pair (E, Φ) defined above is a root system.*

Note that for any $\check{\alpha} \in \check{\Phi}$, viewed as an element in $E^* = \mathsf{Hom}(E, \mathbb{R})$, its restriction on $E_{\mathbb{Q}} \subset E$ is $\mathbb{Q}$-valued. It follows that $\check{\Phi}$ is contained in $E^*_{\mathbb{Q}}$ (via the identification $E^*_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \simeq E^*$). Hence we can also view $\check{\Phi}$ as a subset of $\mathfrak{t} \simeq E^*_{\mathbb{Q}} \otimes_{\mathbb{Q}} k$.

Definition 51. For any root $\alpha \in \Phi$, define the correponding **coroot** to be $\check{\alpha} \in \check{\Phi} \subset \mathfrak{t}$.

Remark 52. There is a (unique if stated properly) semisimple Lie algebra corresponding to the dual root system $(E^*, \check{\Phi})$, known as the *Langlands dual* Lie algebra $\check{\mathfrak{g}}$ of $\mathfrak{g}$.

References

[Hum] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[Ser] Serre, Jean-Pierre. Complex semisimple Lie algebras. Springer Science & Business Media, 2000.