Nearby Cycles and Dualities in Geometric Langlands Program

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Abstract

In this thesis, we study nearby cycles on certain Vinberg-style degenerations in the geometric Langlands program. We relate them to various exotic dualities in this field, such as the (local and global) geometric second adjointness and the miraculous duality. We also prove the Deligne-Lusztig duality for automorphic sheaves, which was conjectured by Drinfeld-Wang and Gaitsgory.

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To my parents

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Introduction

Overview

The goal of this thesis is to study the relations between certain nearby cycles and dualities in the *Geometric Langlands Program*. The precise meaning of the following words will be given later, but roughly:

(a) We continue the study of the *nearby cycles on the Drinfeld-Lafforgue-Vinberg degeneration*, also known as the *geometric Bernstein asymptotics for unramified automorphic sheaves*. The study of such nearby cycles was initiated in Schieder's thesis (see [Sch18], [Sch16]). We call them the *global nearby cycles* for short, because they depend on a projective algebraic curve.

(b) We introduce and study a local analogue of the above construction, i.e., *the nearby cycles on the Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian*. Such interpolation was introduced by Finkelberg-Krylov-Mirković in [FKM20]. We call these nearby cycles the *local nearby cycles*, because they depend on formal disks lying on the curve, and behave well when the disks "collide". We prove the global and local nearby cycles are compatible via certain local-to-global maps.

(c) We prove a geometric analogue of *Bernstein's second adjointness* and relate it to the local nearby cycles. This *geometric second adjointness* can be reformualted as a duality, a.k.a. a perfect pairing, between the categories of *semi-infinite sheaves on the affine Grassmannian* for two opposite Borel subgroups. We prove the unit object for this pairing is given by the local nearby cycles. We also prove a global analogue of this geometric second adjointness and relate it to the global nearby cycles in a similar way.

(d) We prove the *Deligne-Lusztig duality for unramified automorphic sheaves* conjectured by Drinfeld-Wang and Gaitsgory (see [DW16], [Wan18]). The statement is similar to the various Deligne-Lusztig-style results in the literature (see [DL82], [BBK18], [GYD18], [YD19]) and relates the *miraculous duality for unramified automorphic sheaves* (see [DG16], [Gai17b]) to the *enhanced geometric Eisenstein series and constant term functors* (see [Gai15a]). The global nearby cycles play an essential role in our proof of this conjecture.

In summary, we put all these results together in this thesis to exhibit the following: the nearby cycles on the various Vinberg-style degeneration are the keys to understanding certain "exotic" dualities in the automorphic side of the Geometric Langlands Program.

Motivation: nearby cycles and the long intertwining functor

To provide motivations for this work, we first review the definition of the *long intertwining functor*. Let G be a reductive group over an algebraically closed field k of characteristic 0. For simplicity, we assume the derived group $[G,G]$ to be simply connected. Fix a pair (B,B^-) of opposite Borel subgroups of G. Let $\mathrm{Fl}_G \simeq G/B$ be the flag variety of G, and N,N^- be the unipotent radicals of B,B^- respectively. Recall the following well-known fact:

Fact 1. *The long-intertwining functor*

$$\Upsilon : \mathrm{DMod}(\mathrm{Fl}_G)^N \xrightarrow{\mathbf{oblv}^N} \mathrm{DMod}(\mathrm{Fl}_G) \xrightarrow{\mathbf{Av}_*^{N^-}} \mathrm{DMod}(\mathrm{Fl}_G)^{N^-} \tag{0.1}$$

is an equivalence.

In the above formula, $\mathrm{DMod}(\mathrm{Fl}_G)^N \subset \mathrm{DMod}(\mathrm{Fl}_G)$ is the DG-category of D-modules[1] constant along the N-orbits of Fl_G, $\mathbf{oblv}^N$ is the forgetful functor, and $\mathbf{Av}_*^{N^-}$ is the right adjoint of $\mathbf{oblv}^{N^-}$.

The DG-category $\mathrm{DMod}(\mathrm{Fl}_G)^N$ is equivalent to $\mathrm{DMod}(\mathrm{Fl}_G/N)$, the DG-category of D-modules on the quotient stack Fl_G/N (see [DG13] for the definition). The Verdier duality provides an equivalence

$$\mathrm{DMod}(\mathrm{Fl}_G/N) \simeq \mathrm{DMod}(\mathrm{Fl}_G/N)^\vee.$$

Here $\mathcal{C}^\vee$ is the dual DG-category of $\mathcal{C}$, whose definition will be reviewed later. We can reinterpret Fact 1 as:

Fact 2. *The DG-categories* $\mathrm{DMod}(\mathrm{Fl}_G)^N$ *and* $\mathrm{DMod}(\mathrm{Fl}_G)^{N^-}$ *are canonically dual to each other.*

Recall that a duality datum between two DG-categories $\mathcal{C},\mathcal{D}$ consists of a *unit (a.k.a. co-evaluation) functor* $c : \mathrm{Vect}_k \to \mathcal{C}\otimes_k\mathcal{D}$ and a *counit (a.k.a. evaluation, or pairing) functor* $e : \mathcal{D}\otimes_k\mathcal{C} \to \mathrm{Vect}_k$, where $\otimes_k$ is

[1] Although we work with D-modules, our main theorems are also valid (after minor modifications) in other sheaf-theoretic contexts listed in [Gai18, § 1.2], which we refer as the *constructible contexts*. However, in order to prove them in the constructible contexts, we need a theory of group actions on categories in these sheaf-theoretic contexts. When developing this theory, one encounters some technical issues on homotopy-coherence, which are orthogonal to the main topic of this thesis. Hence we will treat these issues in another article and use remarks in this thesis to explain the required modifications. Once the aforementioned issues are settled down, these remarks become real theorems.

the Lurie tensor product for DG-categories, and Vect_k, the DG-category of k-vector spaces, is the monoidal unit for $\otimes_k$. The pair (c,e) are required to satisfy the usual axioms.

It follows formally that the counit for the duality in Fact 2 is the following composition:

$$\mathrm{DMod}(\mathrm{Fl}_G)^{N^-}\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Fl}_G)^N \xrightarrow{\mathbf{oblv}\otimes\mathbf{oblv}} \mathrm{DMod}(\mathrm{Fl}_G)\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Fl}_G)\xrightarrow{-\overset{!}{\otimes}-}\mathrm{DMod}(\mathrm{Fl}_G)\xrightarrow{C_{\mathrm{dR}}}\mathrm{Vect}_k, \tag{0.2}$$

where $\otimes^!$ is the !-tensor product, and C_{dR} is taking the de-Rham cohomology complex.

Here is a natural quesiton:

Question 1. *What is the unit functor for the duality in Fact 2?*

Of course, the question is boring if we only want *one* formula for the unit. For example, it is the composition

$$\mathrm{Vect}_k\xrightarrow{\mathbf{unit}}\mathrm{DMod}(\mathrm{Fl}_G)^N\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Fl}_G)^N\xrightarrow{\mathbf{Id}\otimes\Upsilon^{-1}}\mathrm{DMod}(\mathrm{Fl}_G)^N\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Fl}_G)^{N^-}.$$

However, it becomes interesting when we want a more *symmetric* formula. So we restate Question 1 as

Question 2. *Can one find a symmetric formula for the unit of the duality in Fact 2?*

Let us look into the nature of the desired unit object. Tautologically we have

$$\mathrm{DMod}(\mathrm{Fl}_G)^N\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Fl}_G)^{N^-}\simeq\mathrm{DMod}(\mathrm{Fl}_G\times\mathrm{Fl}_G)^{N\times N^-}.$$

Also, knowing a continuous k-linear functor $\mathrm{Vect}_k\to\mathcal{C}$ is equivalent to knowing an object in $\mathcal{C}$. Hence the unit is essentially given by an $(N\times N^-)$-equivariant complex $\mathcal{K}$ of D-modules on $\mathrm{Fl}_G\times\mathrm{Fl}_G$. We start by asking the following question:

Question 3. *What is the support of the object $\mathcal{K}$?*

It turns out that this seemingly boring question has an interesting answer. Recall that both the N-orbits and N^--orbits on Fl_G are labeled by the Weyl group W. For $w\in W$, let Δ^w and $\Delta^{w,-}$ respectively be the !-extensions of the IC D-modules on the orbits NwB/B and N^-wB/B. It follows formally that we have

$$\mathrm{Hom}(\Delta^{w_1}\boxtimes\Delta^{w_2,-},\mathcal{K})\simeq\mathrm{Hom}(\Delta^{w_2,-},\mathbb{D}^{\mathrm{Ver}}\circ\Upsilon(\Delta^{w_1})), \tag{0.3}$$

where

$$\mathbb{D}^{\mathrm{Ver}}:\mathrm{DMod}_{\mathrm{coh}}(\mathrm{Fl}_G)\simeq\mathrm{DMod}_{\mathrm{coh}}(\mathrm{Fl}_G)^{\mathrm{op}}$$

is the contravariant Verdier duality functor. It's well-known that $\mathbb{D}^{\mathrm{Ver}} \circ \Upsilon(\Delta^w) \simeq \Delta^{w,-}$. Hence (0.3) is nonzero only if $N^- w_2 B/B$ is contained in the closure of $N^- w_1 B/B$, i.e. only if $w_1 \leq w_2$, where "$\leq$" is the Bruhat order. Therefore $\mathcal{K}$ is supported on the closure of

$$\coprod_{w \in W} (N \times N^-)(w \times w)(B \times B)/(B \times B). \tag{0.4}$$

The disjoint union (0.4) has a more geometric incarnation. To describe it, let us choose a regular dominant co-character $\mathbb{G}_m \to T$, the adjoint action of T on G induces a $\mathbb{G}_m$-action on Fl_G. The attractor, repeller, fixed loci (see [DG14] for the definitions) of this action are

Hence (0.4) is identified with the 0-fiber of the *Drinfeld-Gaitsgory interpolation* $\widetilde{\mathrm{Fl}_G} \to \mathbb{A}^1$ for this action, which is a canonical $\mathbb{A}^1$-degeneration from Fl_G to the fiber product $\mathrm{Fl}_G^{\mathrm{att}} \times_{\mathrm{Fl}_G^{\mathrm{fix}}} \mathrm{Fl}_G^{\mathrm{rep}}$ (see [DG14] for its definition).

An important property of this interpolation is that there is a locally closed embedding

$$\widetilde{\mathrm{Fl}_G} \hookrightarrow \mathrm{Fl}_G \times \mathrm{Fl}_G \times \mathbb{A}^1, \tag{0.5}$$

defined over $\mathbb{A}^1$, such that its 1-fiber is the diagonal embedding $\mathrm{Fl}_G \hookrightarrow \mathrm{Fl}_G \times \mathrm{Fl}_G$, while its 0-fiber is the obvious embedding of $\mathrm{Fl}_G^{\mathrm{att}} \times_{\mathrm{Fl}_G^{\mathrm{fix}}} \mathrm{Fl}_G^{\mathrm{rep}} \hookrightarrow \mathrm{Fl}_G \times \mathrm{Fl}_G$. This motivates the following guess:

Guess 1. *Consider the trivial family $\mathrm{Fl}_G \times \mathrm{Fl}_G \times \mathbb{A}^1 \to \mathbb{A}^1$. Up to a cohomological shift, $\mathcal{K}$ is canonically isomorphic to the nearby cycles of the constant D-module supported on $\widetilde{\mathrm{Fl}_G} \times_{\mathbb{A}^1} \mathbb{G}_m$.*

The guess is in fact correct. For example, it can be proved using [BFO12, Theorem 6.1] and the localization theory[2]. We will not provide this argument in this thesis. Instead, we will prove a version of the claim for the loop group $G((t))$, and our method can be applied to the finite type case as well. This is the work (c) in the overview. In fact, it was in this work that we first encountered the local nearby cycles, even before our awareness of Schieder's calculation of the global ones.

Inv-inv duality

Consider the loop group $G((t))$ of G. Let $\mathrm{Gr}_G \simeq G((t))/G[[t]]$ be the affine Grassmannian[3]. Let P be a standard parabolic subgroup and P^- be its opposite parabolic subgroup. Let U, U^- respectively be the unipotent radical of P, P^-, and $M := P \cap P^-$ be the Levi subgroup. Consider the DG-category $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$ of D-modules on Gr_G constant along the $U((t))$-orbits. We will prove the following affine analogue of Fact 2:

[2] We are grateful to Yuchen Fu for pointing out this to us.

[3] Recall Gr_G classifies G-torsors on the disk $\mathrm{Spec}\, k[[t]]$ equipped with a trivialization on the punctured disk $\mathrm{Spec}\, k((t))$.

Theorem A. *The DG-categories* $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$ *and* $\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}$ *are compactly generated and dual to each other, with the counit functor given by*

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}\xrightarrow{\mathbf{oblv}\otimes\mathbf{oblv}}\mathrm{DMod}(\mathrm{Gr}_G)\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Gr}_G)\xrightarrow{-\otimes^!-}\mathrm{DMod}(\mathrm{Gr}_G)\xrightarrow{C_{\mathrm{dR}}}\mathrm{Vect}_k\,.$$

We refer this duality as the *inv-inv duality*. As in Question 1, we are interested in the unit of this duality, which is given by an object in [4]

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}\underset{k}{\otimes}\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}\simeq\mathrm{DMod}(\mathrm{Gr}_G\times\mathrm{Gr}_G)^{U((t))\times U^-((t))}.$$

On the other hand, choose a dominant co-character $\gamma:\mathbb{G}_m\to T$ that is regular with repect to P. The adjoint action of T on G induces a $\mathbb{G}_m$-action on Gr_G. Consider the corresponding Drinfeld-Gaitsgory interpolation $\widetilde{\mathrm{Gr}}_G^\gamma$ and the canonical embedding

$$\widetilde{\mathrm{Gr}}_G^\gamma\hookrightarrow\mathrm{Gr}_G\times\mathrm{Gr}_G\times\mathbb{A}^1.$$

We will prove the following theorem:

Theorem B. *Consider the trivial family* $\mathrm{Gr}_G\times\mathrm{Gr}_G\times\mathbb{A}^1\to\mathbb{A}^1$. *The unit object of the duality in Theorem A is canonically isomorphic, up to a cohomological shift, to the nearby cycles of the dualizing D-module supported on* $\widetilde{\mathrm{Gr}}_G^\gamma\times_{\mathbb{A}^1}\mathbb{G}_m$.

In general, knowing a duality between compactly generated DG-categories $\mathcal{C}$ and $\mathcal{D}$ is the same as knowing a contravariant equivalence $\mathcal{C}^c\simeq\mathcal{D}^{c,\mathrm{op}}$ between their subcategories of compact objects. Let us provide more information about the corresponding contravariant equivalence

$$\mathbb{D}:(\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))})^c\simeq(\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))})^{c,\mathrm{op}}.$$

Consider the closed embedding $\mathrm{Gr}_M\to\mathrm{Gr}_G$ and the composition

$$\iota_M^!:\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}\xrightarrow{\mathbf{oblv}}\mathrm{DMod}(\mathrm{Gr}_G)\xrightarrow{!\text{-pull}}\mathrm{DMod}(\mathrm{Gr}_M).$$

We will prove $\iota_M^!$ is conservative and has a left adjoint $\iota_{M,!}$. Moreover, $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$ can be "glued" [5]

[4] The equivalence below is not as trivial as it seems, whose proof can be found in the main text.

[5] As suggested by the notation, we encourage the reader to view $\iota_M^!$ as the !-pullback functor along $\iota_M:\mathrm{Gr}_M\to\mathrm{Gr}_G/U((t))$. Then this "gluing result" reflects the fact that ι_M is bijective on k-points.

from the categories $\mathrm{DMod}(\mathrm{Gr}_M^\lambda)$, where Gr_M^λ are the connected components of Gr_M. Replacing P by P^-, we also have an adjoint pair

$$\iota^-_{M,!} : \mathrm{DMod}(\mathrm{Gr}_M) \rightleftharpoons \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))} : \iota^{-,!}_M.$$

Finally, since Gr_M is an ind-finite type indscheme, the Verdier duality provides

$$\mathbb{D} : \mathrm{DMod}(\mathrm{Gr}_M)^c \simeq \mathrm{DMod}(\mathrm{Gr}_M)^{c,\mathrm{op}}.$$

We will prove

Theorem C. *Via the above dualities, the functors $\iota_{M,!}$ and $\iota^-_{M,!}$ are conjugate to each other. In other words, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_M)^c \arrow[r, "\simeq", "\mathbb{D}"'] \arrow[d, "\iota_{M,!}"] & \mathrm{DMod}(\mathrm{Gr}_M)^{c,\mathrm{op}} \arrow[d, "\iota^-_{M,!}"] \\
(\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))})^c \arrow[r, "\simeq", "\mathbb{D}"'] & (\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))})^{c,\mathrm{op}}.
\end{tikzcd}$$

Geometric second adjointness

It's well-known that the naive long-intertwining functor

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \xrightarrow{\mathbf{oblv}^{U((t))}} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{\mathbf{Av}_*^{U^-((t))}} \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}$$

is the zero functor. This is essentially due to the infinite dimension of $U((t))$. However, we will deduce from Theorem A the following theorem:

Theorem D. *The functor*

$$\Upsilon : \mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \xrightarrow{\mathbf{oblv}^{U((t))}} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{\mathbf{pr}_{U^-((t))}} \mathrm{DMod}(\mathrm{Gr}_G)_{U^-((t))} \tag{0.6}$$

is an equivalence.

In the above formula, $\mathrm{DMod}(\mathrm{Gr}_G)_{U^-((t))}$ is the *coinvariance category* for the $U^-((t))$-action on Gr_G. It can be defined as the localization of $\mathrm{DMod}(\mathrm{Gr}_G)$ that kills the kernels of $\mathbf{Av}_*^{\mathcal{N}}$ for all group subschemes $\mathcal{N}$ of $U^-((t))$.

We refer the above theorem as the (unramified) geometric second adjointness for the following reason. It was conjectured by S. Raskin in the unpublished notes [Ras17] that for any DG-category $\mathcal{C}$ equipped with a

$G((t))$-action, the composition

$$\mathcal{C}^{U((t))} \xrightarrow{\mathbf{oblv}^{U((t))}} \mathcal{C} \xrightarrow{\mathbf{pr}_{U^-((t))}} \mathcal{C}_{U^-((t))}$$

is an equivalence. He explained that this conjecture can be viewed as a categorification of Bernstein's second adjointness. Our theorem, which is the case $\mathcal{C} = \mathrm{DMod}(\mathrm{Gr}_G)$, can be viewed as the unramified part of this conjecture[6].

In the special case when $P = B$, Theorem D can be deduced from a result of S. Raskin, which says that (0.6) becomes an equivalence if we further take $T[[t]]$-invariance. We will sketch this reduction in the main text. However, our proof of the Theorem D is independent to Raskin's result. Moreover, for general parabolic subgroups, to the best of our knowledge, Theorem D is *not* a direct consequence of any known results.

Local nearby cycles

Theorem B motivates us to study the nearby cycles mentioned in its statement, which we denote by $\Psi_\gamma \in \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$. By Theorem B, it depends only on P rather than γ. We summarize known results about Ψ_γ as follows.

- Let r be the semi-simple rank of G. [FKM20] defined the *Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian*[7] VinGr_G. There is a canonical closed embedding

  $$\mathrm{VinGr}_G \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^r,$$

  which is a multi-variable degeneration of the diagonal embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$. The co-character γ chosen before extends to a map $\mathbb{A}^1 \to \mathbb{A}^r$. Let

  $$\mathrm{VinGr}_G^\gamma \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$$

  be the sub-degeneration obtained by pullback along this map.

  We will see that $\mathrm{VinGr}_G^\gamma \times_{\mathbb{G}^1} \mathbb{G}_m$ is isomorphic to $\widetilde{\mathrm{Gr}}_G^\gamma \times_{\mathbb{A}^1} \mathbb{G}_m$ as closed sub-indscheme of $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Hence Ψ_γ is supported on the 0-fiber $\mathrm{VinGr}_G|_{C_P}$ of VinGr_G^γ, and can also be calculated as the nearby

[6] However, D. Yang told us he found a counter-example for this conjecture recently.

[7] The definition will be reviewed in the main text of this thesis. Also, in the case $G = \mathrm{SL}_2$, if we fix an embedding $\mathrm{Spec}\, k[[t]] \to X$ of the disk into a smooth projective curve X, then there is a useful moduli description of VinGr_G in Footnote 9 below.

cycles of the dualizing D-module on VinGr_G^γ. Therefore, we abuse notation by writing

$$\Psi_\gamma \in \mathrm{DMod}(\mathrm{VinGr}_G|_{C_P}).$$

- We will prove Ψ_γ is constant along the $(U((t)) \times U^-((t)))$-orbits of $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Also, roughly speaking, we will prove the $*$-restriction (resp. !-restriction) of Ψ_γ along any $U((t))$-orbit is a $*$-extension (resp. !-extension) from a certain $U^-((t))$-orbit.

- We will show Ψ_γ has a canonical equivariant structure for the diagonal $M[[t]]$-action on $\mathrm{Gr}_G \times \mathrm{Gr}_G$.

- By the definition of nearby cycles, Ψ_γ carries a monodromy endomorphism. We will prove that this endomorphism is locally unipotent.

- Let X be a connected projective smooth curve over k. For non-empty finite set I, consider the *Beillinson-Drinfeld Grassmannian* $\mathrm{Gr}_{G,I} \coloneqq \mathrm{Gr}_{G,X^I}$ and the similarly defined nearby cycles $\Psi_{\gamma,I}$. By [FKM20], we also have a relative version[8] [9] $\mathrm{VinGr}_{G,I}$ of VinGr_G. As before $\Psi_{\gamma,I}$ can also be calculated as the nearby cycles of the dualizing D-module on $\mathrm{VinGr}_{G,I}^\gamma$.

  We will prove that the assignment $I \rightsquigarrow \Psi_{\gamma,I}$ factorizes. In other words, Ψ_γ can be upgraded to a *factorization algebra* in the factorization category $\mathrm{DMod}(\mathrm{VinGr}_G|_{C_P}) \subset \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ in the sense of [Ras15a]. Moreover, the theorems on the inv-inv duality are also valid in this factorization setting.

Global nearby cycles

Now we want to study the global analogue of the previous story. Let X be a fixed connected projective smooth curve over k. We have the following analogy between local and global objects:

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[8] In particular, for any embedding $\mathrm{Spec}\, k[[t]] \to X$ centered at a closed point $x \in X$, there is a canonical isomorphism between $\mathrm{VinGr}_{G,x} \coloneqq \mathrm{VinGr}_{G,X}|_x$ and VinGr_G.

[9] In the case $G = \mathrm{SL}_2$, the indscheme $\mathrm{VinGr}_{G,x}$ classifies $\mathcal{O} \dashrightarrow \mathcal{E}_1 \xrightarrow{\phi} \mathcal{E}_2 \dashrightarrow \mathcal{O}$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are rank 2 vector bundles on X equipped with trivialized determinant line bundles, and $\mathcal{E}_1 \to \mathcal{E}_2$ (resp. $\mathcal{O} \dashrightarrow \mathcal{E}_1$, $\mathcal{E}_2 \dashrightarrow \mathcal{O}$) is a map (resp. *rational* map that is regular on $X - x$) between *coherent $\mathcal{O}$-modules* such that the composition $\mathcal{O} \dashrightarrow \mathcal{O}$ is the identity map (and in particular regular). The projection $\mathrm{VinGr}_{G,x} \to \mathbb{A}^1$ sends the above data to $\det(\phi) \in \Gamma(X,\mathcal{O}) = k$.
Note that the above data uniquely determine a trivialization of $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) on $X-x$, hence determine two points of $\mathrm{Gr}_{G,x}$. We warn the reader that the projection $\mathrm{VinGr}_{G,x} \to \mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,x}$ sends the above data to $(\mathcal{E}_2, \mathcal{E}_1)$ rather than $(\mathcal{E}_1, \mathcal{E}_2)$. The reason for this convention will be explained in the main text.

	local	global
Ambient space	Gr_G	Bun_G
Parabolic category	$\mathbf{oblv}: \mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \to \mathrm{DMod}(\mathrm{Gr}_G)$	$\mathrm{Eis}^{\mathrm{enh}}_{P\to G}: \mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_G)$
Levi category	$\iota_{M,!}: \mathrm{DMod}(\mathrm{Gr}_M) \rightleftharpoons \mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}: \iota_M^!$	$\iota_{M,!}: \mathrm{DMod}(\mathrm{Bun}_M) \rightleftharpoons \mathrm{I}(G,P): \iota_M^!$
Inv-inv duality	$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \longleftrightarrow \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}$	$\mathrm{I}(G,P) \longleftrightarrow \mathrm{I}(G,P^-)$
Vinberg family	$\mathrm{VinGr}_G \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^r$	$\mathrm{VinBun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathbb{A}^r$
$\mathbb{A}^1$-sub-family	$\mathrm{VinGr}_G^\gamma \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$	$\mathrm{VinBun}_G^\gamma \to \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$.

- The stack Bun_G classifies G-torsors on X. Recall $\mathrm{Bun}_G(k) \simeq G(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$, where F is the field of rational functions on X, and $\mathbb{A}_F$ (resp. $\mathbb{O}_F$) is the ring of adeles (resp. integral adeles) of F.

- The DG-category $\mathrm{DMod}(\mathrm{Bun}_G)$ is the category of *unramified automorphic D-modules*. By [DG15], [DG16] and [Gai17b], it is compactly generated and canonically self dual. This self duality is called the *miraculous duality*[10].

- The (pre)stack $\mathrm{Bun}_G^{P\text{-gen}}$ classifies G-torsors on X equipped with a generic P-reduction. Note that $\mathrm{Bun}_G^{P\text{-gen}}(k) \simeq P(F)\backslash G(\mathbb{A}_F)/G(\mathbb{O}_F)$.

- The DG-category $\mathrm{I}(G,P)$ is the full subcategory of $\mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$ containing objects that are equivariant with respect to the $U(\mathbb{A}_F)$-action (see [Gai15a, § 6]). Alternatively, it is defined to fit into the following *Cartesian* square [11]:

$$\begin{tikzcd} \mathrm{I}(G,P) \arrow[r, "\subset"] \arrow[d, "\iota_M^!"] & \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}) \arrow[d, "\iota_P^!"] \\ \mathrm{DMod}(\mathrm{Bun}_M) \arrow[r, "\mathfrak{q}_P^*"] & \mathrm{DMod}(\mathrm{Bun}_P), \end{tikzcd}$$

  where $\iota_P: \mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}$ and $\mathfrak{q}_P: \mathrm{Bun}_P \to \mathrm{Bun}_M$ are the canonical maps. We will prove $\iota_M^!$ is conservative and has a left adjoint $\iota_{M,!}$. Moreover, $\mathrm{I}(G,P)$ can be "glued" from the categories $\mathrm{DMod}(\mathrm{Bun}_M^\lambda)$, where Bun_M^λ are the connected components of Bun_M. In particular, $\mathrm{I}(G,P)$ is compactly generated.

- The *enhanced Eisenstein series functor* $\mathrm{Eis}^{\mathrm{enh}}_{P\to G}$ is the !-pushforward functor along the map $\mathrm{Bun}_G^{P\text{-gen}} \to \mathrm{Bun}_G$. Part of this statement says such !-pushforward is well-defined on $\mathrm{I}(G,P)$. We will prove $\mathrm{Eis}^{\mathrm{enh}}_{P\to G}$ has a continous right adjoint $\mathrm{CT}^{\mathrm{enh}}_{G\to P}$.

[10] We warn that this miraculous duality is a highly non-trivial theorem. It does *not* follow from the Verdier duality for coherent D-modules because Bun_G is not quasi-compact. Instead, the unit object for this duality is given by $\Delta_!(k)$, the !-pushforward of the constant D-module along the diagonal map $\Delta: \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$. We will discuss this duality in more details later.

[11] As in Footnote 5, we encourage the reader to view $\iota_M^!$ as the !-pullback functor along $\iota_M: \mathrm{Bun}_M \to \mathrm{Bun}_G^{P\text{-gen}}/U(\mathbb{A}_F)$.

- The stack VinBun_G is the *Drinfeld-Lafforgue-Vinberg degeneration* defined in [Sch18] and [Sch16][12].

- The 0-fiber of $\mathrm{VinBun}_G^\gamma \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathbb{A}^1$, which we denote by $\mathrm{VinBun}_G|_{C_P} \to \mathrm{Bun}_G \times \mathrm{Bun}_G$, canonically factors through[13] $\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}$.

Consider the *global nearby cycles* $\Psi_{\gamma,\mathrm{glob}} \in \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})$, which were firstly studied by Schieder in his thesis. We will prove the following global analogue of Theorem A, Theorem B and Theorem C.

Theorem E. *The DG-categories* $\mathrm{I}(G,P)$ *and* $\mathrm{I}(G,P^-)$ *are canonically dual to each other, such that:*

- *Via this duality and the miraculous duality on* Bun_M*, the functors* $\iota_{M,!} : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P)$ *and* $\iota_{M,!}^- : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P^-)$ *are conjugate to each other.*

- *Via this duality and the miraculous duality on* Bun_G*, the functors* $\mathrm{Eis}_{P\to G}^{\mathrm{enh}} : \mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_G)$ *and* $\mathrm{Eis}_{P^-\to G}^{\mathrm{enh}} : \mathrm{I}(G,P^-) \to \mathrm{DMod}(\mathrm{Bun}_G)$ *are conjugate to each other.*

- *The unit for this duality, which is an object in*[14]

$$\mathrm{I}(G,P) \underset{k}{\otimes} \mathrm{I}(G,P^-) \simeq \mathrm{I}(G\times G, P\times P^-) \subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}),$$

  is isomorphic, up to a cohomological shift, to the !-pushforward of $\Psi_{\gamma,\mathrm{glob}}$ *along* $\mathrm{VinBun}_G|_{C_P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}$.

Let $x \in X$ be a closed point. Consider the local-to-global map $\mathrm{Gr}_{G,x} \to \mathrm{Bun}_G$. We will construct a Vinberg-degeneration for it, i.e., a commutative diagram

$$\begin{tikzcd} \mathrm{VinGr}_{G,x} \arrow[r] \arrow[d, "\pi_x"] & \mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,x} \times \mathbb{A}^r \arrow[d] \\ \mathrm{VinBun}_G \arrow[r] & \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathbb{A}^r. \end{tikzcd}$$

Let $\pi_x^\gamma : \mathrm{VinGr}_{G,x}^\gamma \to \mathrm{VinBun}_G^\gamma$ be the restriction of π_x to the $\mathbb{A}^1$-sub-families. We will prove

Theorem F. *The canonical map* $\Psi_{\gamma,x} \to \pi_x^{\gamma,!}(\Psi_{\gamma,\mathrm{glob}})$ *is an isomorphism.*

We will also prove the factorization version of the above theorem.

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[12] In the case $G = \mathrm{SL}_2$, VinBun_G classifies $\phi : \mathcal{E}_1 \to \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are rank 2 vector bundles on X with trivialized determinant line bundles, and ϕ is a *nonzero* map between *coherent* $\mathcal{O}$*-modules*. The projection $\mathrm{VinBun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times \mathbb{A}^1$ is defined similarly as in Footnote 9.

[13] In the case $G = \mathrm{SL}_2$, $\mathrm{VinBun}_G|_{C_P}$ classifies $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ as in Footnote 12 whose rank is equal to 1. Note that $\mathrm{Im}(\phi)$ is a generic line bundle on X. Hence $\mathcal{E}_1 \twoheadrightarrow \mathrm{Im}(\phi)$ (resp. $\mathrm{Im}(\phi) \hookrightarrow \mathcal{E}_2$) provide the desired generic B^--reduction (resp. B-reduction).

[14] As in Footnote 4, the equivalence below is not trivial and will be proved in the main text.

Deligne-Lusztig duality

We are going to explain the relation between the global nearby cycles $\Psi_{\gamma,\mathrm{glob}}$ and the Deligne-Lusztig duality on Bun_G. This is work (d) in the overview. Let us first give a quick introduction to the history of Deligne-Lusztig-style results in the literature.

The following pattern has been observed in several representation-theoretic contexts: The composition of two different duality functors on the category $\mathcal{C}_G$ attached to a reductive group G is isomorphic to a Deligne-Lusztig functor, given by a complex indexed by standard parabolic subgroups P of G, whose terms are compositions

$$\mathcal{C}_G \overset{\mathrm{CT}_P}{\to} \mathcal{C}_M \overset{\mathrm{Eis}_P}{\to} \mathcal{C}_G,$$

where

- $\mathcal{C}_M$ is the category attached to the Levi quotient group M;
- CT_P and Eis_P are adjoint functors connecting $\mathcal{C}_G$ and $\mathcal{C}_M$.

Notable examples include

- The work of Bernstein-Bezrukavnikov-Kazhdan [BBK18], where $\mathcal{C}_G$ is the category of representations of the group $G(K)$, where K is a non-Archimedian local field.
- The work of Yom Din [YD19], where $\mathcal{C}_G = \mathrm{DMod}(G/\mathrm{Ad}(G))$ is the category of character D-modules.
- The work of Drinfeld-Wang [DW16], [Wan18], where $\mathcal{C}_G$ is the space of automorphic functions for the group G. Note that this example is actually one categorical level down from the above partern (i.e., one needs to replace "categories" by "spaces" and "functors" by "operators").

In this thesis we establish yet another incarnation of this pattern. Namely, we take $\mathcal{C}_G \coloneqq \mathrm{DMod}(\mathrm{Bun}_G)$ and $\mathcal{C}_M \coloneqq \mathrm{I}(G,P)$. This can be viewed as directly categorifying that of Drinfeld-Wang. It is also closely connected to that of Yom Din because the category of character D-modules can be regarded as a genus 0 version of the automorphic category.

To give the statement of our theorem, we need to review the Drinfeld-Gaitsgory pseudo-identity functors in [DG13]. Let Y be a QCA algebraic stack[15] in the sense of *loc.cit.*. Verdier duality provides a contravariant equivalence

$$\mathbb{D} : \mathrm{DMod}(Y)^c \simeq \mathrm{DMod}(Y)^{c,\mathrm{op}} \tag{0.7}$$

[15] This means Y is a quasi-compact algebraic stack whose automorphism groups of geometric points are affine.

and therefore an equivalence

$$\text{Ps-Id}_{Y,\text{naive}} : \text{DMod}(Y)^\vee \simeq \text{DMod}(Y),$$

where $\text{DMod}(Y)^\vee$ is the *dual DG-category* of $\text{DMod}(Y)$. On the other hand, we have *the product formula*:

$$\text{DMod}(Y \times Y) \simeq \text{DMod}(Y) \underset{k}{\otimes} \text{DMod}(Y).$$

The RHS can be identified with $\text{LFun}_k(\text{DMod}(Y)^\vee, \text{DMod}(Y))$, i.e., the category of k-linear colimit-preserving functors $\text{DMod}(Y)^\vee \to \text{DMod}(Y)$. Hence the object $\Delta_!(k_Y) \in \text{DMod}(Y \times Y)$ provides a functor

$$\text{Ps-Id}_{Y,!} : \text{DMod}(Y)^\vee \to \text{DMod}(Y).$$

In general, the functor $\text{Ps-Id}_{Y,!}$ is not an equivalence. We say Y is *miraculous* if it is an equivalence. For example, the quotient stack $G/\operatorname{Ad}(G)$ is miraculous by [YD19].

Now suppose Y is only *locally* QCA. Then compact objects in $\text{DMod}(Y)$ are no longer coherent D-modules, and we do not have the equivalence (0.7). Nevertheless, in the special case $Y = \text{Bun}_G$, the product formula remains correct by [DG15, Remark 2.2.9]. Hence as before, we have equivalences

$$\begin{aligned} \text{LFun}_k(\text{DMod}(\text{Bun}_G)^\vee, \text{DMod}(\text{Bun}_G)) &\simeq \text{DMod}(\text{Bun}_G) \underset{k}{\otimes} (\text{DMod}(\text{Bun}_G)^\vee)^\vee \simeq \\ &\simeq \text{DMod}(\text{Bun}_G) \underset{k}{\otimes} \text{DMod}(\text{Bun}_G) \simeq \text{DMod}(\text{Bun}_G \times \text{Bun}_G), \end{aligned} \tag{0.8}$$

and we use the objects $\Delta_*(\omega_{\text{Bun}_G})$, $\Delta_!(k_{\text{Bun}_G})$ in the RHS to define *functors*

$$\text{Ps-Id}_{\text{Bun}_G,\text{naive}}, \text{Ps-Id}_{\text{Bun}_G,!} : \text{DMod}(\text{Bun}_G)^\vee \to \text{DMod}(\text{Bun}_G).$$

From now on, we write them just as $\text{Ps-Id}_{\text{naive}}$, $\text{Ps-Id}_!$.

We emphasize again that, unlike the quasi-compact case, the functor $\text{Ps-Id}_{\text{naive}}$ is *not* an equivalence. On the contrary, the main theorem of [Gai17b] says:

Fact 3. *The functor* $\text{Ps-Id}_! : \text{DMod}(\text{Bun}_G)^\vee \to \text{DMod}(\text{Bun}_G)$ *is an equivalence, i.e.,* Bun_G *is miraculous.*

This result is highly non-trivial and was proved by an indirect method. In particular, there was no description for the inverse functor $\text{Ps-Id}_!^{-1}$ in *loc.cit.*. The following theorem of us will supply such a description:

Theorem G. *Up to cohomological shifts, the endofunctor*

$$\text{Ps-Id}_{\text{naive}} \circ \text{Ps-Id}_!^{-1} : \text{DMod}(\text{Bun}_G) \to \text{DMod}(\text{Bun}_G)^\vee \to \text{DMod}(\text{Bun}_G)$$

can be "glued"[16] *from the functors*

$$\text{Eis}^{\text{enh}}_{P\to G} \circ \text{CT}^{\text{enh}}_{G\to P} : \text{DMod}(\text{Bun}_G) \to \text{I}(G,P) \to \text{DMod}(\text{Bun}_G)$$

for all the stardard parabolic subgroups P.

We also relate the global nearby cycles $\Psi_{\gamma,\text{glob}}$ to the Deligne-Lusztig duality in the following result:

Theorem H. *Let P be a stardard parabolic subgroup and choose γ accordingly. Then up to a cohomological shift, the endofunctor $\text{Eis}^{\text{enh}}_{P\to G} \circ \text{CT}^{\text{enh}}_{G\to P}$ is calculated by the correspondence*

$$\text{Bun}_G \overset{\text{pr}_1}{\leftarrow} \text{VinBun}_G|_{C_P} \overset{\text{pr}_2}{\to} \text{Bun}_G$$

and the kernel [17] $\Psi_{\gamma,\text{glob}} \in \text{DMod}(\text{VinBun}_G|_{C_P})$.

Organization of this thesis

- We review some basic geometric and categorical constructions in Part I. Some technical proofs are supplied in Appendix A, Appendix B, Appendix F and Appendix G.
- We introduce and study the local nearby cycles in Part II.
- We introduce and study the global nearby cycles in Part III.
- We review the theory of D-modules on prestacks in Appendix C.
- We review the theory of categories acted by groups in Appendix D.
- We review Braden's theorem and the contraction principle in Appendix E.

The main theorems in the overview are reformulated in the main text.

- Theorem A and Theorem B are summarized as Corollary 3.2.4.

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[16] The precise statement will be given in the main text.

[17] This means the endofunctor is given by $\mathcal{F} \mapsto \text{pr}_{1,!}(\text{pr}_2^*(\mathcal{F}) \otimes^* (\Psi_{\gamma,\text{glob}}))$ modulo well-definedness of the formula. The precise statement will be given in the main text.

- Theorem C is Corollary 3.2.8.
- Theorem D is Corollary 3.3.1.
- Theorem E is Theorem 5.3.5.
- Theorem F is Theorem 5.1.7.
- Theorem G is Theorem 2.4.2.
- Theorem H is Theorem 5.2.6.

Conventions

We summarize our main conventions as below.

Convention 1 (Categories)**.** Unless otherwise stated, a *category* means an $(\infty, 1)$-category in the sense of [Lur09]. Consequently, a $(1,1)$-category is refered to an *ordinary category.* We use same symbols to denote an ordinary category and its simplicial nerve. The reader can distinguish them according to the context.

For two objects $c_1, c_2 \in C$ in a category C, we write $\mathrm{Maps}_C(c_1, c_2)$ for the *mapping space* between them, which is an object in the homotopy category of spaces. We omit the subscript C if there is no ambiguity.

When saying there exists an *unique* object satisfying certain properties in a category, we always mean *unique up to a contractible space of choices.*

A functor $F : C \to D$ is *fully faithful* (resp. 1-*fully faithful*) if it induces isomorphisms (resp. monomorphisms) on mapping spaces.

To avoid awkward language, we ignore all set-theoretical difficulties in category theory. Nevertheless, we do not do anything illegal like applying the adjoint functor theorem to non-accessible categories.

Convention 2 (2-categories)**.** We also need the theory of $(\infty, 2)$-categories developed in [GR17a]. For two objects $c_1, c_2 \in \mathbb{S}$ in an $(\infty, 2)$-category $\mathbb{S}$, we write $\mathbf{Maps}_{\mathbb{S}}(c_1, c_2)$ for the *mapping* $(\infty, 1)$*-category* between them.

Convention 3 (Compositions)**.** Let C be a 2-category. Let $f, f', f'' : c_1 \to c_2$ and $g, g' : c_2 \to c_3$ be morphisms in C. Let $\alpha : f \to f'$, $\alpha' : f' \to f''$ and $\beta : g \to g'$ be 2-morphisms in C. We follow the stardard conventions in the category theory:

- The composition of f and g is denoted by $g \circ f : c_1 \to c_3$;
- The vertical composition of α and α' is denoted by $\alpha' \circ \alpha : f \to f''$;

- The horizontal composition of α and β is denoted by $\beta \bigstar \alpha : g \circ f \to g' \circ f'$.

Note that these compositions are actually well-defined up to a contractible space of choices.

We use similar symbols to denote the compositions of functors, vertical composition of natural transformations and horizontal composition of natural transformations.

Convention 4 (Algebraic geometry)**.** Unless otherwise stated, all algebro-geometric objects are defined over a fixed algebraically closed ground field k of characteristic 0, and are classical (i.e. non-derived).

A *prestack* is a contravariant functor

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Groupoids}$$

from the ordinary category of affine schemes to the category of groupoids[18].

A prestack $\mathcal{Y}$ is *reduced* if it is the left Kan extension of its restriction along $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{red}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, where $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{red}}$ is the category of reduced affine schemes. A map $\mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks is called a *nil-isomorphism* if its value on any *reduced* affine test scheme is an isomorphism.

A prestack $\mathcal{Y}$ is called *locally of finite type* or *lft* if it is the left Kan extension of its restriction along $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, where $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is the category of finite type affine schemes. For the reader's convenience, we usually denote general prestacks by mathcal fonts (e.g. $\mathcal{Y}$), and leave usual fonts (e.g. Y) for lft prestacks.

A *lft algebraic stack* is a lft 1-Artin stack in the sense of [GR17a, Chapter 2, § 4.1]. All lft algerbraic stacks in this thesis (are assumed to or can be shown to) have affine diagonals. In particular, as prestacks, they satisfy fpqc descent.

An *indscheme* is a prestack isomorphic to a filtered colimit of schemes connected by closed embeddings. All indschemes in this thesis are (assumed to or can be shown to be) isomorphic to a filtered colimit of *quasi-compact quasi-separated* schemes connected by closed embeddings. In particular, they are indschemes in the sense of [GR14].

Convention 5 (Affine line)**.** For a prestack $\mathcal{Y}$ over $\mathbb{A}^1$, we write $\mathring{\mathcal{Y}}$ (resp. $\mathcal{Y}_0$) for the base-change $\mathcal{Y} \times_{\mathbb{A}^1} \mathbb{G}_m$ (resp. $\mathcal{Y} \times_{\mathbb{A}^1} 0$), and $j : \mathring{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ (resp. $i : \mathcal{Y}_0 \hookrightarrow \mathcal{Y}$) for the corresponding schematic open (resp. closed) embedding.

Convention 6 (Curves and disks)**.** We fix a connected smooth projective curve X. For a positive integer n, we write $X^{(n)}$ for its n-th symmetric product.

We write $\mathcal{D} := \mathrm{Spf}\, k[[t]]$ for the *formal disk*, $\mathcal{D}' := \mathrm{Spec}\, k[[t]]$ for the *adic disk*, and $\mathcal{D}^\times := \mathrm{Spec}\, k((t))$ for

[18] All the prestacks in this thesis would actually have *ordinary* groupoids as values.

the *punctured disk*. For a closed point x on X, we have similarly defined pretacks $\mathcal{D}_x$, $\mathcal{D}'_x$ and $\mathcal{D}^\times_x$, which are non-canonically isomorphic to $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}^\times$.

Generally, for an affine test scheme S and an *affine* closed subscheme $\Gamma \hookrightarrow X \times S$, we write $\mathcal{D}_\Gamma$ for the formal completion of Γ inside $X \times S$. We write $\mathcal{D}'_\Gamma$ for the schematic approximation[19] of $\mathcal{D}_\Gamma$. We write $\mathcal{D}^\times_\Gamma$ for the open subscheme $\mathcal{D}'_\Gamma - \Gamma$. We have maps

$$\begin{tikzcd} \mathcal{D}^\times_\Gamma \arrow[r] & \mathcal{D}'_\Gamma \arrow[d] & \mathcal{D}_\Gamma \arrow[l] \\ & X \times S. & \end{tikzcd}$$

Convention 7 (Loops and arcs)**.** For a prestack $\mathcal{Y}$, we write $\mathcal{L}\mathcal{Y}$ (resp. $\mathcal{L}^+\mathcal{Y}$) for its *loop prestack (resp. arc prestack)* defined as follows. For an affine test scheme $S := \operatorname{Spec} R$, the groupoid $\mathcal{L}\mathcal{Y}(S)$ (resp. $\mathcal{L}^+\mathcal{Y}(S)$) classifies maps $\operatorname{Spec} R((t)) \to \mathcal{Y}$ (resp. $\operatorname{Spf} R[[t]] \to \mathcal{Y}$).

Similarly, for a non-empty finite set I, we write $\mathcal{L}\mathcal{Y}_I$ (resp. $\mathcal{L}^+\mathcal{Y}_I$) for the *loop prestack (resp. arc prestack) relative to* X^I. For an affine test scheme S, the groupoid $\mathcal{L}\mathcal{Y}_I(S)$ (resp. $\mathcal{L}^+\mathcal{Y}_I(S)$) classifies

(i) maps $x_i : S \to X$ labeled by I, and

(ii) a map $\mathcal{D}^\times_\Gamma \to \mathcal{Y}$ (resp. $\mathcal{D}_\Gamma \to \mathcal{Y}$), where $\Gamma \hookrightarrow X \times S$ is the schema-theoretic sum of the graphs of x_i.

Convention 8 (DG-categories)**.** We study *DG-categories* over k. Unless otherwise stated, DG-categories are assumed to be *cocomplete* (i.e., containing small colimits), and functors between them are assumed to be k-linear and *continuous* (i.e. preserving small colimits). The $(\infty, 1)$-category formed by them is denoted by DGCat. The corresponding $(\infty, 2)$-category is denoted by **DGCat**.

DGCat carries a closed symmetric monoidal structrue, known as the *Lurie tensor product* $\otimes_k$. The unit object for it is the DG-category Vect_k of k-vector spaces. For $\mathcal{C}, \mathcal{D} \in \mathrm{DGCat}$, we write $\mathrm{LFun}_k(\mathcal{C}, \mathcal{D})$ for the object in DGCat characterized by the universal property

$$\mathrm{Maps}(\mathcal{E}, \mathrm{LFun}_k(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Maps}(\mathcal{E} \underset{k}{\otimes} \mathcal{C}, \mathcal{D}).$$

We omit the subscript "k" if it is clear from the contexts.

Let $\mathcal{M}$ be a DG-category, we write $\mathcal{M}^c$ for its full subcategory consisting of compact objects, which is a *non-cocomplete DG-category.*

A DG-category $\mathcal{M}$ is *dualizable* if it is a dualizable object in DGCat. We write $\mathcal{M}^\vee$ for its dual DG-category, which is canonically equivalent to $\mathrm{LFun}_k(\mathcal{M}, \mathrm{Vect}_k)$. It is well-known that $\mathcal{M}$ is dualizable if it is

[19] $\mathcal{D}_\Gamma$ is an ind-affine indscheme. Its schematic approximation is $\operatorname{Spec} A$, where A is the topological ring of functions on $\mathcal{D}_\Gamma$.

compactly generated, and there is a canonical identification $\mathcal{M}^\vee \simeq \mathrm{Ind}(\mathcal{M}^{c,\mathrm{op}})$.

Convention 9 (D-modules)**.** Let Y be a finite type scheme. We write $\mathrm{Dmod}(Y)$ for the DG-category of D-modules on Y. We write ω_Y for the dualizing D-module and k_Y for the constant D-module on Y.

We will see in Appendix C, for a general prestack $\mathcal{Y}$, there are two theories of D-modules on it: $\mathrm{DMod}^!(\mathcal{Y})$ and $\mathrm{DMod}^*(\mathcal{Y})$. In the other part of this thesis, when we writing $\mathrm{DMod}(\mathcal{Y})$, we always mean the !-theory $\mathrm{DMod}^!(\mathcal{Y})$.

Convention 10 (Reductive groups)**.** We fix a connected reductive group G. For simplicity, we assume $[G,G]$ to be simply connected[20].

We fix a pair of *opposite Borel subgroups* (B,B^-) of it, therefore a *Cartan subgroup* T. We write Z_G for the center of G and $T_{\mathrm{ad}} \coloneqq T/Z_G$ for the *adjoint torus*.

We write $r \coloneqq r_G$ for the *semi-simple rank* of G, $\mathcal{I}$ for the *set of vertices in the Dynkin diagram* of G, Λ_G (resp. $\check{\Lambda}_G$) for the *coweight (resp. weight) lattice*, and $\Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ for the sub-monoid spanned by all positive simple co-roots $(\alpha_i)_{i\in\mathcal{I}}$.

We write Par for the partially ordered set of standard parabolic subgroups of G. We write $\mathrm{Par}' = \mathrm{Par} - \{G\}$. We view them as categories in the standard way.

We often use P to denote a *standard parabolic subgroup* of G (i.e. a parabolic subgroup containing B). We write P^- for the corresponding *standard opposite parabolic subgroup* and $M \coloneqq P \cap P^-$ for the *Levi subgroup*. We often use $\mathcal{J} \subset \mathcal{I}$ to denote the subset of vertices in the Dynkin diagram of M. We write U (resp. U^-) for the *unipotent radical* of P (resp. P^-). When we need to use a different parabolic subgroup, we often denote it by Q and its Levi quotient group by L.

We write $\Lambda_{G,P}$ for the quotient of Λ by the $\mathbb{Z}$-span of $(\alpha_i)_{i\in\mathcal{J}}$, and $\Lambda_{G,P}^{\mathrm{pos}}$ for the image of Λ_G^{pos} in $\Lambda_{G,P}$. The monoid $\Lambda_{G,P}^{\mathrm{pos}}$ defines a partial order $\le_P$ on $\Lambda_{G,P}$. We omit the subscript "P" if it is clear from the contexts.

Convention 11 (Colored divisors)**.** Each $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$ can be uniquely written as the image of $\sum_{i\in\mathcal{I}-\mathcal{I}_M} n_i\alpha_i$ for $n_i \in \mathbb{Z}^{\ge 0}$. We define the *configuration space* $X^\theta \coloneqq \prod_{i\in\mathcal{I}} X^{(n_i)}$, whose S-points are $\Lambda_{G,P}^{\mathrm{pos}}$*-valued (relative Cartier) divisors* on X_S. We write $X_{G,P}^{\mathrm{pos}}$ for the disjoint union of all $X^\theta, \theta \in \Lambda_{G,P}^{\mathrm{pos}}$, and omit the subscript if it is clear from the context.

For $\theta_i \in \Lambda_{G,P}^{\mathrm{pos}}, 1 \le i \le n$, we write $(\prod_{i=1}^n X^{\theta_i})_{\mathrm{disj}}$ for the open subscheme of $\prod_{i=1}^n X^{\theta_i}$ classifying those n-tuples of divisors $(D_1,\cdots,D_n)$ with disjoint supports. For a prestack $\mathcal{Y}$ over $\prod_{i=1}^n X^{\theta_i}$, we write $\mathcal{Y}_{\mathrm{disj}}$ for its base-change to this open subscheme.

[20]Such assumption was made in many references that we cite, but we do not know if our results and proofs really depend on it.

Convention 12 (Semi-group completion)**.** The collection of simple positive roots of G provides an identification $T_{\mathrm{ad}} \simeq \mathbb{G}_m^{\mathcal{I}} \coloneqq \prod_{i\in\mathcal{I}} \mathbb{G}_m$. We define $T_{\mathrm{ad}}^+ \coloneqq \mathbb{A}^{\mathcal{I}} \supset \mathbb{G}_m^{\mathcal{I}} \simeq T_{\mathrm{ad}}$, which is a semi-group completion of the adjoint torus T_{ad}.

Consider the coordinate stratification of the affine space T_{ad}^+. The set of strata can be identified with the underlying set of Par. Moreover, the scheme T_{ad}^+ is stratified by the *poset* Par. We will use $T_{\mathrm{ad},P}^+$, $T_{\mathrm{ad},\geq P}^+$ and $T_{\mathrm{ad},\leq P}^+$ to denote the corresponding subschemes.

Write C_P for the unique point in $T_{\mathrm{ad},P}^+$ whose coordinates are either 0 or 1. In particular C_B is the zero element in T_{ad}^+ and C_G is the unit element. We use the same symbols to denote the images of these points in the quotient stack T_{ad}^+/T.

Consider the homomorphism $Z_M/Z_G \to T_{\mathrm{ad}}$. Let[21] $T_{\mathrm{ad},\geq C_P}^+$ be its closure in T_{ad}^+. Note that it is a sub-semi-group of $T_{\mathrm{ad},\geq P}^+$ that contains C_P as an idempotent element.

Let $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. Note that there exists a unique morphism of monoids $\overline{\gamma} : \mathbb{A}^1 \to T_{\mathrm{ad}}^+$ extending the obvious map $\mathbb{G}_m \to Z_M \hookrightarrow T \twoheadrightarrow T_{\mathrm{ad}}$. For any prestack $\mathcal{Y}$ over T_{ad}, we often write

$$\mathcal{Y}^\gamma \coloneqq \mathcal{Y} \underset{T_{\mathrm{ad}}^+,\overline{\gamma}}{\times} \mathbb{A}^1.$$

[21]It was denoted by $T_{\mathrm{ad},\geq P,\mathrm{strict}}^+$ in [Sch16].

Part I

The Main Players

Chapter 1

The geometric players

In this chapter, we review the basic properties of our main geometric players VinGr_G and VinBun_G. For references, see [Wan18], [Sch16], [FKM20] and [DG16]. We also put some techinical proofs in Appendix A

1.1 The semi-group Vin_G

1.1.1. The *Vinberg semi-group* Vin_G is an affine normal semi-group equipped with a flat semi-group homomorphism to T_{ad}^+. Its open subgroup of invertible elements is isomorphic to $G_{\mathrm{enh}} := (G \times T)/Z_G$, where Z_G acts on $G \times T$ anti-diagonally. Its fiber at C_P is canonoically isomorphic to

$$\mathrm{Vin}_G|_{C_P} \simeq \overline{(G/U \times G/U^-)/M},$$

where the RHS is the affine closure of $(G/U \times G/U^-)/M$ [22], where M acts diagonally on $G/U^- \times G/U$ by *right* multiplication.

The $(G_{\mathrm{enh}}, G_{\mathrm{enh}})$-action on Vin_G induces a (G,G)-action on Vin_G via the embedding $G \simeq (G \times Z_G)/Z_G \hookrightarrow G_{\mathrm{enh}}$, which preserves the projection $\mathrm{Vin}_G \to T_{\mathrm{ad}}^+$. On the fiber $\mathrm{Vin}_G|_{C_P}$, this action extends the left multiplication action of $G \times G$ on $(G/U \times G/U^-)/M$.

The embedding $T \simeq (Z_G \times T)/Z_G \hookrightarrow G_{\mathrm{enh}}$ is contained in the center of the semi-group Vin_G. In particular, there is a canonical T-action on Vin_G, which we refer as the T_{cent}-action. By definition, the T_{cent}-action commutes with the (G,G)-action.

There is a canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$, which is also a semi-group homomorphism. Its restriction on

[22] This scheme is strongly quasi-affine in the sense of [BG02, Subsection 1.1].

$T_{\mathrm{ad}} := T/Z_G$ is given by

$$T/Z_G \to (G \times T)/Z_G,\ t \mapsto (t^{-1}, t).$$

The $(G\times G)$-orbit of the section $\mathfrak{s}$ is an open subscheme of Vin_G, known as the *defect-free locus* ${}_0\mathrm{Vin}_G$. We have:

$$(G \times T)/Z_G \simeq \mathrm{Vin}_G \underset{T_{\mathrm{ad}}^+}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{Vin}_G \subset \mathrm{Vin}_G\,. \tag{1.1}$$

The fiber ${}_0\mathrm{Vin}_G|_{C_P}$ is given by $(G/U \times G/U^-)/M$, and the canonical section intersects it at the point $(1,1)$.

Example 1.1.2. When $G = \mathrm{SL}_2$, the base T_{ad}^+ is isomorphic to $\mathbb{A}^1$. The semi-group Vin_G is isomorphic to the monoid $\mathrm{M}_{2,2}$ of 2×2 matrices. The projection $\mathrm{Vin}_G \to \mathbb{A}^1$ is given by the determinant function. The canonical section is $\mathbb{A}^1 \to \mathrm{M}_{2,2}$, $t \mapsto \mathrm{diag}(1,t)$. The action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $\mathrm{M}_{2,2}$ is given by $(g_1, g_2)\cdot A = g_1 A g_2^{-1}$.

Warning 1.1.3. There is no consensus convention for the order of the two G-actions on Vin_G in the literature. Even worse, this order is *not* self-consistent in either [Sch16] [23] or [FKM20] [24].

In this paper, we use the order in [Wan17] and [Wan18]. We ask the reader to keep an eye on this issue when we cite other references.

Fact 1.1.4. *Consider the $(G\times G)$-action on Vin_G. Note that it preserves the fibers of $\mathrm{Vin}_G \to T_{\mathrm{ad}}^+$. We write $\widetilde{G}$ for the corresponding stabilizer of the canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$. We have the following facts[25]:*

(1) $\widetilde{G}$ is a closed subgroup of $G \times G \times T_{\mathrm{ad}}^+$ (relative to T_{ad}^+), whose fiber at C_P is

$$\widetilde{G}_{C_P} \simeq P \underset{M}{\times} P^-.$$

(2) By [DG16, Corollary D.5.4], $\widetilde{G}$ is smooth over T_{ad}^+, and we have

$${}_0\mathrm{Vin}_G \simeq (G\times G\times T_{\mathrm{ad}}^+)/\widetilde{G},\ G\backslash {}_0\mathrm{Vin}_G/G \simeq \mathbb{B}\widetilde{G}, \tag{1.2}$$

where $\mathbb{B}$ means taking relative classifying stack.

(3) By (2), the canonical T-action on Vin_G (which commutes with the $G\times G$-action) induces a diagram between group actions:

$$(\mathrm{pt} \curvearrowleft \mathbb{B}G \times \mathbb{B}G) \leftarrow (T \curvearrowleft \mathbb{B}\widetilde{G}) \to (T \curvearrowleft T_{\mathrm{ad}}^+).$$

[23] [Sch16, Lemma 2.1.11] and [Sch16, § 6.1.2] are not consistent.

[24] [FKM20, Remark 3.14] and [FKM20, § 3.2.7] are not consistent.

[25] (1) and (2) are well-known, (4) and (5) follow from the canonical identification $\mathrm{Vin}_G \times_{T_{\mathrm{ad}}^+} T_{\mathrm{ad}} \simeq (G\times T)/Z_G$.

(4) $\widetilde{G}$ contains the locally closed subscheme

$$\Gamma : G \times T_{\mathrm{ad}} \to G \times G \times T_{\mathrm{ad}}^{+},\ (g,t) \mapsto (g, \mathrm{Ad}_t(g), t)$$

(5) $\widetilde{G}$ is preserved by the action

$$(T_{\mathrm{ad}} \times T_{\mathrm{ad}}) \curvearrowright (G \times G \times T_{\mathrm{ad}}^{+}),\ (t_1,t_2)\cdot(g_1,g_2,s) \mapsto (\mathrm{Ad}_{t_1^{-1}}(g_1), \mathrm{Ad}_{t_2^{-1}}(g_2), t_1 \cdot s \cdot t_2^{-1}).$$

Warning 1.1.5. The canonical T-action on Vin_G does not induce a T-action on $\widetilde{G}$ because this action does not preserve the canonical section $\mathfrak{s} : T_{\mathrm{ad}}^{+} \to \widetilde{G}$.

The following result generalizes [DG16, Proposition D.6.4]:

Lemma 1.1.6. *$\widetilde{G}$ is isomorphic to the closure of the locally closed embedding*

$$\Gamma : G \times T_{\mathrm{ad}} \to G \times G \times T_{\mathrm{ad}}^{+},\ (g,t) \mapsto (g, \mathrm{Ad}_t(g), t).$$

Proof. Let $\overline{\Gamma}$ be the desired closure. Hence we obtain a closed embedding $\overline{\Gamma} \to \widetilde{G}$. Since $\widetilde{G}$ is reduced, it remains to show $\overline{\Gamma} \to \widetilde{G}$ is surjective. Note that $\overline{\Gamma}$ is also preserved by the action in Fact 1.1.4(5). Hence we only need to check the fiber of $\overline{\Gamma} \to \widetilde{G}$ at $C_P \in T_{\mathrm{ad}}^{+}$ is surjective. Then we are done by [DG16, Proposition D.6.4].

□[Lemma 1.1.6]

More results on $\widetilde{G}$ can be found in § A.1.

1.2 The Drinfeld-Gaitsgory interpolation

The stablizer $\widetilde{G}$ is closely related to the *Drinfeld-Gaitsgory interpolation* constructed in [Dri13] and [DG14], which we review in this section.

Definition 1.2.1. Let Z be any lft prestack equipped with a $\mathbb{G}_m$-action. Consider the canonical $\mathbb{G}_m$-actions on $\mathbb{A}^1$ and $\mathbb{A}^1_{-} := \mathbb{P}^1 - \{\infty\}$. We define the *attractor*, *repeller*, and *fixed loci* for Z respectively by:

$$Z^{\mathrm{att}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z),\ Z^{\mathrm{rep}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1_{-}, Z),\ Z^{\mathrm{fix}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathrm{pt}, Z),$$

where $\mathbf{Maps}^{\mathbb{G}_m}(W,Z)$ is the lft prestacks classifies $\mathbb{G}_m$-equivariant maps $W \to Z$.

Construction 1.2.2. By construction, we have maps

$$p^+ : Z^{\mathrm{att}} \to Z,\ i^+ : Z^{\mathrm{fix}} \to Z^{\mathrm{att}},\ q^+ : Z^{\mathrm{att}} \to Z^{\mathrm{fix}}$$

induced respectively by the $\mathbb{G}_m$-equivariant maps $\mathbb{G}_m \to \mathbb{A}^1$, $\mathbb{A}^1 \to \mathrm{pt}$, $\mathrm{pt} \xrightarrow{0} \mathbb{A}^1$. We also have similar maps p^-, i^-, q^- for the repeller locus. Note that i^+ (resp. i^-) is a right inverse for q^+ (resp. q^-). We also have $p^+ \circ i^+ \simeq p^- \circ i^-$.

Example 1.2.3. Let P be a standard parabolic subgroup of G and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. The adjoint action of G on itself induces a $\mathbb{G}_m$-action on G. We have $G^{\gamma,\mathrm{att}} \simeq P$, $G^{\gamma,\mathrm{rep}} \simeq P^-$ and $G^{\gamma,\mathrm{fix}} \simeq M$.

1.2.4. Let Z be any finite type scheme acted by $\mathbb{G}_m$. [DG14, § 2.2.1] constructed the *Drinfeld-Gaitsgory interpolation*

$$\widetilde{Z} \to Z \times Z \times \mathbb{A}^1,$$

where $\widetilde{Z}$ is a finite type scheme. The $\mathbb{G}_m$-locus $\widetilde{Z} \times_{\mathbb{A}^1} \mathbb{G}_m$ is canonically isomorphic to the graph of the $\mathbb{G}_m$-action, i.e., the image of the map

$$\mathbb{G}_m \times Z \to Z \times Z \times \mathbb{G}_m,\ (s,z) \mapsto (z, s\cdot z, s).$$

The 0-fiber $\widetilde{Z} \times_{\mathbb{A}^1} 0$ is canonically isomorphic to $Z^{\mathrm{att}} \times_{Z^{\mathrm{fix}}} Z^{\mathrm{rep}}$.

Moreover, by [DG14, § 2.5.11], the map $\widetilde{Z} \to Z \times Z \times \mathbb{A}^1$ is a locally closed embedding if we assume:

(♣) Z admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space $\mathbb{P}(V)$, where $\mathbb{G}_m$-acts linearly on V.

Remark 1.2.5. The construction $Z \rightsquigarrow \widetilde{Z}$ is functorial in Z and is compatible with Cartesian products.

Example 1.2.6. The $\mathbb{G}_m$-action on G in Example 1.2.3 satisfies condition (♣). Indeed, using a faithful representation $G \to \mathrm{GL}_n$, we reduce the claim to the case $G = \mathrm{GL}_n$, which is obvious.

Notation 1.2.7. We denote the Drinfeld-Gaitsgory interpolation for the action in Example 1.2.3 by $\widetilde{G}^\gamma$. Using [DG16, Proposition D.6.4], it is easy to see this notation is compatible with Convention 12. In other words, we have

$$\widetilde{G}^\gamma \simeq \widetilde{G} \underset{T^+_{\mathrm{ad}},\overline{\gamma}}{\times} \mathbb{A}^1,$$

where $\widetilde{G}$ is the stablizer group studied in Appendix A.1.

In the previous example, the adjoint action of G on itself induces a G-action on $\mathrm{Gr}_{G,I}$. Hence we obtain a $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$. The following result is a folklore[26]

Proposition 1.2.8. *We have canonical isomorphisms*

$$\mathrm{Gr}_{G,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}},\ \mathrm{Gr}_{P^-,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}},\ \mathrm{Gr}_{M,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{fix}}$$

defined over $\mathrm{Gr}_{G,I}$*. Moreover, they fit into the following commutative diagrams*

$$\begin{tikzcd} \mathrm{Gr}_{P,I} \arrow[r] \arrow[d] & \mathrm{Gr}_{M,I} \arrow[d] & \mathrm{Gr}_{P^-,I} \arrow[l] \arrow[d] \\ \mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}} \arrow[r] & \mathrm{Gr}_{G,I}^{\gamma,\mathrm{fix}} & \mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}} \arrow[l] \end{tikzcd}$$

1.3 The stack VinBun_G

In this section, we review the *Drinfeld-Lafforgue-Vinberg degeneration* VinBun_G.

Definition 1.3.1. Let Y be a lft algebraic stack (see Convention 4). We write $\mathbf{Maps}(X, Y)$ for the prestack classifying maps $X \to Y$.

Let $V \subset Y$ be a schematic open embedding. We write $\mathbf{Maps}_{\mathrm{gen}}(X, Y \supset V)$ for the prestack whose value on an affine test scheme S is the groupoid of maps $\alpha : X \times S \to Y$ such that the open subscheme $\alpha^{-1}(V)$ has non-empty intersections with any geometric fiber of $X \times S \to S$. Note that there is a schematic open embedding

$$\mathbf{Maps}_{\mathrm{gen}}(X, Y \supset V) \to \mathbf{Maps}(X, Y)$$

because X is projective.

Remark 1.3.2. We will generalize the above definition to any map $V \to Y$ in Definition 2.3.1.

Fact 1.3.3. *By [HR19, Theorem 1.2],* $\mathbf{Maps}(X, Y)$ *is a lft algebraic stack. It has affine diagonal if* Y *does.*

Example 1.3.4. If Y is a finite type affine scheme, then $\mathbf{Maps}(X, Y) \simeq Y$.

Example 1.3.5. The stack $\mathrm{Bun}_G := \mathbf{Maps}(X, \mathbb{B}G)$ is the moduli stack of G-torsors on X.

[26]When X is the affine line $\mathbb{A}^1$, the following result was proved in [HR18, Theorem A]. As explained in [HR18, Remark 3.18i), Footnote 3], one can deduce the general case from this special case. For completeness, we provide the details of their argument in § A.2.

Definition 1.3.6. Following [Sch16], the *Drinfeld-Lafforgue-Vinberg degeneration* of Bun_G is defined as (see Definition 1.3.1 for the notation $\mathbf{Maps}_{\mathrm{gen}}$):

$$\mathrm{VinBun}_G := \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash \mathrm{Vin}_G /G \supset G\backslash {}_0\mathrm{Vin}_G /G). \tag{1.3}$$

Definition 1.3.7. The *defect-free locus* of VinBun_G is defined as

$${}_0\mathrm{VinBun}_G := \mathbf{Maps}(X, G\backslash {}_0\mathrm{Vin}_G /G).$$

Construction 1.3.8. The maps $G\backslash \mathrm{Vin}_G /G \to T_{\mathrm{ad}}^+$ and $G\backslash \mathrm{Vin}_G /G \to G\backslash \mathrm{pt}/G$ induce a map (see Example 1.3.4):

$$\mathrm{VinBun}_G \to \mathrm{Bun}_{G\times G} \times T_{\mathrm{ad}}^+.$$

The chain (1.1) induces schematic open embeddings:

$$\mathrm{VinBun}_G \underset{T_{\mathrm{ad}}^+}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{VinBun}_G \subset \mathrm{VinBun}_G\,. \tag{1.4}$$

Remark 1.3.9. The parabolic stratification on the base T_{ad}^+ induces a *parabolic stratification* on VinBun_G. By [Wan18, (C.2)], each stratum $\mathrm{VinBun}_{G,P}$ is constant along $T_{\mathrm{ad},P}^+$. In particular, we have $\mathrm{VinBun}_{G,G} \simeq \mathrm{Bun}_G \times T_{\mathrm{ad}}$.

Example 1.3.10. When $G = \mathrm{SL}_2$, for an affine test scheme S, the groupoid $\mathrm{VinBun}_G(S)$ classifies triples $(\mathcal{E}_1, \mathcal{E}_2, \phi)$, where $\mathcal{E}_1, \mathcal{E}_2$ are rank 2 vector bundles on X_S whose determinant line bundles are trivialized, and $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is a map such that its restriction at any geometric point s of S is an injection between *quasi-coherent sheaves* on $X \times s$. Since the determinant line bundles of $\mathcal{E}_1$ and $\mathcal{E}_2$ are trivialized, we can define the determinant $\det(\phi)$, which is a function on S because X is proper. Therefore we obtain a map $\mathrm{VinBun}_G \to \mathbb{A}^1 \simeq T_{\mathrm{ad}}^+$, which is the canonical projection.

Fact 1.3.11. *Let (P, γ) be as in Example 1.2.3. There is a canonical isomorphism*

$${}_0\mathrm{VinBun}_G^\gamma \simeq \mathrm{Bun}_{\widetilde{G}^\gamma} := \mathbf{Maps}(X, \mathbb{B}\widetilde{G}^\gamma).$$

In particular, there are canonical isomorphisms

$${}_0\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_{P\times_M P^-} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}$$

defined over $\mathrm{Bun}_{G\times G}\simeq\mathrm{Bun}_G\times\mathrm{Bun}_G$.

Warning 1.3.12. The isomorphism $\mathrm{Bun}_{P\times_M P^-}\simeq\mathrm{Bun}_P\times_{\mathrm{Bun}_M}\mathrm{Bun}_{P^-}$ is due to

$$\mathbb{B}(P\underset{M}{\times}P^-)\simeq P\backslash M/P^-\simeq\mathbb{B}P\underset{\mathbb{B}M}{\times}\mathbb{B}P^-. \tag{1.5}$$

However, the map $\mathbb{B}(G_2\times_{G_1}G_3)\to\mathbb{B}G_2\times_{\mathbb{B}G_1}\mathbb{B}G_3$ is not an isomorphism in general (for example when $G_2=P$, $G_3=P^-$ and $G_1=G$).

1.4 The indscheme VinGr_G

In this section, we review the *Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian* VinGr_G.

Definition 1.4.1. Let B be a finite type affine scheme and $Y\xrightarrow{p}B$ be an algebraic stack over it. Let $f:B\to Y$ be a section of p. Let I be a non-empty finite set. We write $\mathbf{Maps}_{I,/B}(X,Y\xleftarrow{f}B)$ for the prestack whose value on an affine test scheme S is the groupoid classifying:

(1) maps $x_i:S\to X$ labelled by I,

(2) a commutative diagram[27]

$$\begin{tikzcd}(X\times S)-\cup\Gamma_{x_i} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\subset"] & S \arrow[r, "\beta"] & B \arrow[d, "f"] \\ X\times S \arrow[rr, "\alpha"] & & Y.\end{tikzcd}$$

Note that $\mathbf{Maps}_{I,/B}(X,Y\xleftarrow{f}B)$ is defined over $X^I\times B$. Using Noetherian reduction, it is easy to see it is a lft prestack.

Construction 1.4.2. Let $(B,Y\supset V,p,f)$ be a 4-tuple such that $Y\supset V$ is as in Definition 1.3.1 and (B,Y,p,f) is as in Definition 1.4.1. Suppose the section $f:B\to Y$ factors through U, then there is a natural map

$$\mathbf{Maps}_{I,/B}(X,Y\xleftarrow{f}B)\to\mathbf{Maps}_{\mathrm{gen}}(X,Y\supset V).$$

More results on $\mathbf{Maps}_{I,/B}(X,Y\xleftarrow{f}B)$ can be found in § A.3.

Example 1.4.3. We have $\mathrm{Gr}_{G,I}\simeq\mathbf{Maps}_{I,/\mathrm{pt}}(X,\mathrm{pt}/G\leftarrow\mathrm{pt})$ and Construction 1.4.2 provides the local-to-global map $\mathrm{Gr}_{G,I}\to\mathrm{Bun}_G$.

[27] In the diagram below, for fixed $\alpha:X\times S\to Y$, the desired map $\beta:S\to B$ is unique if it exists. Indeed, the map $p\circ\alpha:X\times S\to B$ must factor through a map $\beta':S\to B$ because of Example 1.3.4. Then the commutative diagram (2) forces $\beta=\beta'$.

Definition 1.4.4. Let I be a non-empty finite set. Following [FKM20], we define the *Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian* as (see Definition 1.4.1):

$$\mathrm{VinGr}_{G,I} \coloneqq \mathbf{Maps}_{I,/T^+_{\mathrm{ad}}}(X, G\backslash \mathrm{Vin}_G /G \leftarrow T^+_{\mathrm{ad}}),$$

where the map $T^+_{\mathrm{ad}} \to G\backslash \mathrm{Vin}_G /G$ is induced by the canonical section $\mathfrak{s} : T^+_{\mathrm{ad}} \to \mathrm{Vin}_G$.

The defect-free locus of $\mathrm{VinGr}_{G,I}$ is defined as:

$$_0\mathrm{VinGr}_{G,I} \coloneqq \mathbf{Maps}_{I,/T^+_{\mathrm{ad}}}(X, G\backslash {}_0\mathrm{Vin}_G /G \leftarrow T^+_{\mathrm{ad}}).$$

Construction 1.4.5. The map $G\backslash \mathrm{Vin}_G /G \to (G\backslash \mathrm{pt}/G) \times T^+_{\mathrm{ad}}$ induces a map

$$\mathrm{VinGr}_{G,I} \to \mathrm{Gr}_{G\times G,I} \times T^+_{\mathrm{ad}}.$$

By [FKM20, Lemma 3.7], this map is a schematic closed embedding. Hence $\mathrm{VinGr}_{G,I}$ is an ind-projective indscheme.

We have schematic open embeddings

$$\mathrm{VinGr}_{G,I} \underset{T^+_{\mathrm{ad}}}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{VinGr}_{G,I} \subset \mathrm{VinGr}_{G,I} . \tag{1.6}$$

Construction 1.4.6. By Construction 1.4.2, there is a *local-to-global map*

$$\pi_I : \mathrm{VinGr}_{G,I} \to \mathrm{VinBun}_G \tag{1.7}$$

fitting into the following commutative diagram

$$\begin{tikzcd}
\mathrm{VinGr}_{G,I} \arrow[r] \arrow[d] & \mathrm{VinBun}_G \arrow[d] \\
\mathrm{Gr}_{G\times G,I} \times T^+_{\mathrm{ad}} \arrow[r] & \mathrm{Bun}_{G\times G} \times T^+_{\mathrm{ad}}.
\end{tikzcd}$$

It follows from the construction that ${}_0\mathrm{VinGr}_{G,I}$ is the pre-image of ${}_0\mathrm{VinBun}_G$ under π_I.

Example 1.4.7. When $G = \mathrm{SL}_2$, for an affine test scheme S, the groupoid $\mathrm{VinGr}_{G,I}(S)$ classifies maps $x_i : S \to X$ labelled by I and $\mathcal{O} \dashrightarrow \mathcal{E}_1 \overset{\phi}{\to} \mathcal{E}_2 \dashrightarrow \mathcal{O}$, where

- $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ is an object in $\mathrm{VinBun}_G(S)$ (see Example 1.3.10);
- $\mathcal{O} \dashrightarrow \mathcal{E}_1$ (resp. $\mathcal{E}_2 \dashrightarrow \mathcal{O}$) is a *rational* map between *coherent* $\mathcal{O}$*-modules* that is regular on $X \times S - \Gamma_x$,

such that $\mathcal{O} \dashrightarrow \mathcal{O}$ is the identity map.

Remark 1.4.8. Recall that the assignment $I \rightsquigarrow \mathrm{Gr}_{G,I}$ *factorizes* in the sense of Beilinson-Drinfeld. It is known that the assignment $I \rightsquigarrow \mathrm{VinGr}_{G,I}$ *factorizes in families* over T^+_{ad}. Recall that this means we have isomorphisms

$$\mathrm{VinGr}_{G,I} \underset{X^I}{\times} X^J \simeq \mathrm{VinGr}_{G,J}, \text{ for } I \twoheadrightarrow J,$$

$$\mathrm{VinGr}_{G,I_1 \sqcup I_2} \underset{X^{I_1 \sqcup I_2}}{\times} (X^{I_1} \times X^{I_2})_{\mathrm{disj}} \simeq (\mathrm{VinGr}_{G,I_1} \underset{T^+_{\mathrm{ad}}}{\times} \mathrm{VinGr}_{G,I_1})_{\mathrm{disj}},$$

satisfying certain compatibilities.

Construction 1.4.9. Let (P,γ) be as in Example 1.2.3, we have the following degenerations of $\mathrm{Gr}_{G,I}$:

(a) The $\mathbb{A}^1$-degeneration

$$\mathrm{VinGr}^\gamma_{G,I} \coloneqq \mathrm{VinGr}_{G,I} \underset{T^+_{\mathrm{ad}},\bar{\gamma}}{\times} \mathbb{A}^1,$$

which is a closed sub-indscheme of $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$.

(b) The $\mathbb{A}^1$-degeneration

$$\mathrm{Gr}_{\widetilde{G}^\gamma,I} \coloneqq \mathbf{Maps}_{I,/\mathbb{A}^1}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{A}^1),$$

which is equipped with a canonical map

$$\mathrm{Gr}_{\widetilde{G}^\gamma,I} \to \mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \simeq \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,$$

Lemma 1.4.10. *(1) There are canonical isomorphisms*

$$_0\mathrm{VinGr}^\gamma_{G,I} \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I}$$

defined over $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$.

(2) Consider the $\mathbb{G}_m$*-action on* $\mathrm{Gr}_{G,I}$ *induced by* γ *and the graph of this action:*

$$\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{G}_m,\ (x,t) \mapsto (x, t\cdot x, t).$$

Then there are canonical isomorphisms

$$\mathrm{VinGr}^\gamma_{G,I} \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I} \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \simeq \mathrm{Gr}_{G,I} \times \mathbb{G}_m$$

defined over $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{G}_m$.

Proof. (1) follows from the $(G \times G)$-equivariant isomorphism (1.2). The first isomorphism in (2) follows from (1) and the chain (1.6). The second isomorphism in (2) follows from the isomorphism between group schemes over $\mathbb{G}_m$:

$$\widetilde{G} \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \simeq G \times \mathbb{G}_m.$$

□[Lemma 1.4.10]

Remark 1.4.11. Note that

$${}_0\mathrm{VinGr}_{G,I}|_{C_P} \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I}|_{C_P} \simeq \mathrm{Gr}_{P,I} \underset{\mathrm{Gr}_{M,I}}{\times} \mathrm{Gr}_{P^-,I}$$

is preserved by the $\mathcal{L}(U \times U^-)_I$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$.

Remark 1.4.12. In fact, one can show $\mathrm{VinGr}_{G,I}|_{C_P}$ is preserved by the above action. This is a formal consequence of the fact that the $(U \times U^-)$-action on $\mathrm{Vin}_G|_{C_P}$ fixes the canonical section $\mathfrak{s}|_{C_P} : \mathrm{pt} \to \mathrm{Vin}_G|_{C_P}$. We do not need this fact hence we do not provide the details of its proof.

Chapter 2

The categorical players

In this chapter, we review the basic properties of our categorical players $\mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}U}$, $\mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}U}$ and $\mathrm{I}(G,P)$. We also put some techinical proofs in the Appendix.

2.1 The local categories $\mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}U}$ and $\mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}U}$

We will use the theory of group actions on categories, which is reviewed in Appendix D.

Definition 2.1.1. Let I be a non-empty finite set. Consider the canonical action of the loop group $\mathcal{L}G_I$ on the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,I}$. It provides[28] an object $\mathrm{DMod}(\mathrm{Gr}_{G,I}) \in \mathcal{L}G_I\text{-mod}$. Consider the categories of invariants and coinvariants

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$$

for the $\mathcal{L}U_I$-action obtained by restriction. We write

$$\begin{aligned} \mathbf{oblv}^{\mathcal{L}U_I} &: \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I}), \\ \mathbf{pr}_{\mathcal{L}U_I} &: \mathrm{DMod}(\mathrm{Gr}_{G,I}) \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \end{aligned}$$

for the corresponding forgetful and projection functors.

Remark 2.1.2. Similar to [Ras16, Remark 2.19.1], $\mathcal{L}U_I$ is an ind-pro-unipotent group scheme. Hence by Remark D.3.1, $\mathbf{oblv}^{\mathcal{L}U_I}$ is fully faithful, and $\mathbf{pr}_{\mathcal{L}U_I}$ is a localization functor, i.e., has a fully faithful (non-continuous) right adjoint.

[28] By [Ras16, Corollary 2.13.4], $\mathcal{L}G_I$ is placid. Hence we can apply § D.4 to this action.

Remark 2.1.3. Using (D.16), it is easy to show that when P is the Borel subgroup B, our definition of $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}N_I}$ coincides with that in [Gai17a].

Our goal in this section is to describe the structures of $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$. In particular, we will describe their compact generators.

2.1.4 (Strata)**.** It is well-known (see Proposition A.4.2) that the map $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is bijective on field-value points, and the connected components of $\mathrm{Gr}_{P,I}$ induce a stratification on $\mathrm{Gr}_{G,I}$ labelled by $\Lambda_{G,P}$. For $\lambda \in \Lambda_{G,P}$, the corresponding stratum is denoted by (see Notation A.4.1)

$$_\lambda\,\mathrm{Gr}_{G,I} \coloneqq (\mathrm{Gr}_{P,I}^{\lambda})_{\mathrm{red}}.$$

By Proposition A.4.2(2), the map $_\lambda\,\mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I}$ is a schematic locally closed embedding.

Consider the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{P,I}$. Note that $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is $\mathcal{L}P_I$-equivariant. Therefore the functors $\mathbf{p}_I^{+,!}$ and $\mathbf{p}_{I,*}^+$ can be upgraded to morphisms in $\mathcal{L}P_I$-mod. Therefore they induce $\mathcal{L}M_I$-linear functors:

$$\mathbf{p}_{I,*}^{+,\mathrm{inv}} : \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \tag{2.1}$$

$$\mathbf{p}_I^{+,!,\mathrm{inv}} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}, \tag{2.2}$$

$$\mathbf{p}_{I,*,\mathrm{co}}^{+} : \mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}, \tag{2.3}$$

$$\mathbf{p}_{I,\mathrm{co}}^{+,!} : \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I}, \tag{2.4}$$

On the other hand, consider the $\mathcal{L}M_I$-equivariant map $\mathbf{q}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{M,I}$. Note that the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{P,I}$ preserves the fibers of $\mathbf{q}_I^+$. It follows formally (see (D.11)) that there are canonical $\mathcal{L}M_I$-functors

$$\mathbf{q}_I^{+,!,\mathrm{inv}} : \mathrm{DMod}(\mathrm{Gr}_{M,I}) \to \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}, \tag{2.5}$$

$$\mathbf{q}_{I,*,\mathrm{co}}^{+} : \mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{M,I}). \tag{2.6}$$

Sometimes we omit the superscripts "inv" from these notations if there is no danger of ambiguity.

Lemma 2.1.5. *Let $\mathbf{i}_I^+ : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{P,I}$ be the canonical map. We have*

(1) (c.f. [Gai17a, Proposition 1.4.2]) The functor (2.5) is an equivalence, with a quasi-inverse given by

$$\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{P,I}) \xrightarrow{\mathbf{i}_I^{+,!}} \mathrm{DMod}(\mathrm{Gr}_{M,I}). \tag{2.7}$$

(2) The functor (2.6) is an equivalence, with a quasi-inverse given by

$$\mathrm{DMod}(\mathrm{Gr}_{M,I}) \xrightarrow{\mathbf{i}^{\dagger}_{I,*}} \mathrm{DMod}(\mathrm{Gr}_{P,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I}. \tag{2.8}$$

Proof. Follows from Lemma D.4.1 and the fact that $\mathcal{L}U_I$ acts transitively along the fibers of $\mathbf{q}^{+}_I$.

□[Lemma 2.1.5]

Lemma 2.1.6. *Let* $\mathcal{F} \in \mathrm{DMod}(\mathrm{Gr}_{G,I})$. *Suppose* $\mathbf{p}^{+,!}_I(\mathcal{F}) \in \mathrm{DMod}(\mathrm{Gr}_{P,I})$ *is contained in* $\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}$, *then* $\mathcal{F}$ *is contained in* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$.

Proof. By (D.9), we can replace $\mathcal{L}U_I$ by one of its pro-smooth group subscheme U_α. It remains to prove that $\mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}^{U_\alpha}_*(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. Using Proposition A.4.2, it is easy to see $\mathbf{p}^{+,!}_I$ is conservative. Hence it remains to prove

$$\mathbf{p}^{+,!}_I \circ \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}^{U_\alpha}_*(\mathcal{F}) \to \mathbf{p}^{!}_I(\mathcal{F})$$

is an isomorphism. By [Ras16, Corollary 2.17.10], we have

$$\mathbf{p}^{+,!}_I \circ \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}^{U_\alpha}_* \simeq \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}^{U_\alpha}_* \circ \mathbf{p}^{+,!}_I.$$

On the other hand, the assumption on $\mathbf{p}^{+,!}_I(\mathcal{F})$ implies

$$\mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}^{U_\alpha}_* \circ \mathbf{p}^{+,!}_I(\mathcal{F}) \simeq \mathbf{p}^{+,!}_I(\mathcal{F}).$$

This proves the desired isomorphism.

□[Lemma 2.1.6]

The following two results are proved in Appendix F.

Lemma 2.1.7. *(c.f. [Gai17a, Proposition 1.5.3, Corollary 1.5.6])*

We have:

(1) Consider the $\mathbb{G}_m$*-action on* $\mathrm{Gr}_{G,I}$ *in Proposition 1.2.8. We have*

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I}).$$

(2) The functor

$$\iota^{!}_{M,I} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{p}^{+,!}_I} \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{M,I})$$

is conservative and has a left adjoint

$$\iota_{M,I,!}:\mathrm{DMod}(\mathrm{Gr}_{M,I})\to\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}. \tag{2.9}$$

Consequently, $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *is compactly generated by* $\iota_{M,I,!}(\mathrm{DMod}(\mathrm{Gr}_{M,I})^c)$*.*

(3) The functor (2.1) has a left adjoint[29]

$$\mathbf{p}_I^{+,*,\mathrm{inv}}:\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\to\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I},$$

which can be canonically identified with

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}}\mathrm{DMod}(\mathrm{Gr}_{G,I})\xrightarrow{\mathbf{p}_I^{-,!}}\mathrm{DMod}(\mathrm{Gr}_{P^-,I})\xrightarrow{\mathbf{q}_{I,*}^-}\mathrm{DMod}(\mathrm{Gr}_{M,I})\simeq\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}.$$

In particular, $\mathbf{p}_I^{+,*,\mathrm{inv}}$ *is* $\mathcal{L}M_I$*-linear.*

(4) The functor (2.2) has a $\mathrm{DMod}(X^I)$*-linear*[30] *left adjoint*

$$\mathbf{p}_{I,!}^{+,\mathrm{inv}}:\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\to\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}.$$

Proposition 2.1.8. *Both* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ *are compactly generated, and they are canonically dual to each other. More precisly, via this duality and the Verdier self-duality on* $\mathrm{DMod}(\mathrm{Gr}_{G,I})$*, the functors*

$$\mathbf{oblv}^{\mathcal{L}U_I}:\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\to\mathrm{DMod}(\mathrm{Gr}_{G,I})$$
$$\mathbf{pr}_{\mathcal{L}U_I}:\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}\to\mathrm{DMod}(\mathrm{Gr}_{G,I})$$

are dual to each other.

Remark 2.1.9. It follows from the proof of Proposition 2.1.8 that via the dualities $\mathrm{DMod}(\mathrm{Gr}_{M,I})\simeq\mathrm{DMod}(\mathrm{Gr}_{M,I})^\vee$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\simeq(\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})^\vee$, the functor (2.9) is conjugate to

$$\mathrm{DMod}(\mathrm{Gr}_{M,I})\overset{(2.8)}{\simeq}\mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I}\xrightarrow{\mathbf{p}^+_{I,*,\mathrm{co}}}\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}.$$

[29] We do not know whether the following stronger claim is true: the partially defined left adjoint of $\mathbf{p}^+_{I,*}$ is well-defined on $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\subset\mathrm{DMod}(\mathrm{Gr}_{G,I})$.

[30] One can actually prove it is $\mathcal{L}M_I$-linear. Also, one can prove any (right or left) lax $\mathrm{DMod}(X^I)$-linear functor is strict.

Corollary 2.1.10. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \{X^I, \mathcal{L}U_I, \mathcal{L}U_I^-\}$ be group indschemes over X^I.*

(1) We have a commutative diagram

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{H}_1} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{H}_2} \arrow[r, "\simeq"'] \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1} \otimes \mathbf{oblv}^{\mathcal{H}_2}"] & \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I})^{\mathcal{H}_1 \times \mathcal{H}_2} \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1 \times \mathcal{H}_2}"] \\
\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I}) \arrow[r, "\boxtimes", "\simeq"'] & \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}),
\end{tikzcd}$$

where all the four functors are fully faithful, and the horizontal functors are equivalences.

(2) We have a commutative diagram

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{H}_1} \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{H}_2} \arrow[r, "\simeq"'] \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1} \otimes \mathbf{oblv}^{\mathcal{H}_2}"] & \mathrm{DMod}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})^{\mathcal{H}_1 \times_{X^I} \mathcal{H}_2} \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1 \times_{X^I} \mathcal{H}_2}"] \\
\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(\mathrm{Gr}_{G,I}) \arrow[r, "\boxtimes_{X^I}", "\simeq"'] & \mathrm{DMod}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}).
\end{tikzcd} \tag{2.10}$$

where all the four functors are fully faithful, and the horizontal functors are equivalences.

Proof. Follows from Proposition 2.1.8, Lemma D.1.11 and Lemma B.3.3).

□[Corollary 2.1.10]

Remark 2.1.11. Corollary 2.1.10 is also (obviously) correct if we replace

- the invariance categories by the coinvariance categories;
- the forgetful functors **oblv** by the localization functors **pr**.

Remark 2.1.12. In the constructible contexts, we still have the commutative diagram in (1). However, the horizontal functors are no longer equivalences. Nevertheless, one can prove that the commutative diagram is right adjointable along the horizontal direction.

Remark 2.1.13. As one would expect (using Lemma D.1.7(2), Corollary 2.1.10 and Remark 2.1.11), the factorization structures on $I \rightsquigarrow \mathrm{DMod}(\mathrm{Gr}_{G,I}), \mathrm{DMod}(\mathrm{Gr}_{P,I})$ induces factorization structures on

$$I \rightsquigarrow \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}, \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}, \mathrm{DMod}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I},$$

such that the assignments of functors $I \rightsquigarrow \mathbf{oblv}^{\mathcal{L}U_I}, \mathbf{pr}_{\mathcal{L}U_I}$ are factorizable functors. Moreover, by the base-change isomorphisms, the functors in § 2.1.4 factorize.

By its proof, the equivalences in Lemma 2.1.5 factorize.

2.2 Variant: $\mathcal{L}^+M$-invariance and coinvariance

Using [Ras16, Lemma 2.5.1], the group scheme $\mathcal{L}^+M_I$ over X^I is pro-smooth. Hence by Lemma D.2.6, we have:

Corollary 2.2.1. *For any $\mathcal{C} \in \mathcal{L}^+M_I$, there is a canonical* $\mathrm{DMod}(X^I)$*-linear equivalence $\theta : \mathcal{C}_{\mathcal{L}^+M_I} \to \mathcal{C}^{\mathcal{L}^+M_I}$ such that $\mathbf{Av}_*^{\mathcal{L}^+M_I} \simeq \theta \circ \mathbf{pr}_{\mathcal{L}^+M_I}$.*

We define $\mathcal{L}U\mathcal{L}^+M_I := \mathcal{L}P_I \times_{\mathcal{L}M_I} \mathcal{L}^+M_I$. In other words, it is the relative version of $\mathcal{L}U\mathcal{L}^+M$. Similar to [Ras16, Subsection 2.19], it is a placid ind-group scheme over X^I.

Corollary 2.2.2. *We have:*

(1) There exists a canonical $\mathrm{DMod}(X^I)$*-linear equivalence*

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U\mathcal{L}^+M_I} \simeq (\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^{\mathcal{L}^+M_I}.$$

(2) There exists a canonical $\mathrm{DMod}(X^I)$*-linear equivalence*

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U\mathcal{L}^+M_I} \simeq (\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})^{\mathcal{L}^+M_I}.$$

(3) $(\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})^{\mathcal{L}^+M_I}$ *and* $(\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^{\mathcal{L}^+M_I}$ *are dual to each other in* DGCat.

Proof. Note that the sequence $\mathcal{L}U_I \to \mathcal{L}U\mathcal{L}^+M_I \to \mathcal{L}^+M_I$ has a splitting. Hence by Lemma D.5.2 and Lemma D.5.1(2), we obtain (1). We also obtain an X^I-linear equivalence

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U\mathcal{L}^+M_I} \simeq (\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})_{\mathcal{L}^+M_I}. \tag{2.11}$$

Then we obtain (2) by using Corollary 2.2.1. Now by Lemma D.2.7 and Proposition 2.1.8, the RHS of (2.11) is dualizable in DGCat, hence so is the LHS. Now we are done by Lemma D.1.10.

□[Corollary 2.2.2]

Remark 2.2.3. In fact, one can show that the categories appeared in the corollary are all compactly generated. The proof is similar to that in Appendix F and uses the well-known fact that the spherical Hecke category $\mathrm{DMod}(\mathrm{Gr}_{M,I})^{\mathcal{L}^+M_I}$ is compactly generated. Since we do not use this result, we omit the proof.

2.3 The global category $\mathrm{I}(G,P)$

Definition 2.3.1. Let Y be any lft prestack. In [Bar14, Example 2.2.4], J. Barlev constructed a lft prestack[31] $\mathbf{GMap}(X,Y)$ classifying a generic map from X to Y.

For any map between lft prestacks $Y_1 \to Y_2$, we define

$$\mathbf{Maps}_{\mathrm{gen}}(X, Y_2 \leftarrow Y_1) := \mathbf{Maps}(X,Y_2) \underset{\mathbf{GMap}(X,Y_2)}{\times} \mathbf{GMap}(X,Y_1).$$

Note that this notation is compatible with Definition 1.3.1.

Construction 2.3.2. Let K be any algebraic group and H be a subgroup of K. In [Bar14, Example 2.2.5], J. Barlev constructed a lft prestack

$$\mathrm{Bun}_K^{H\text{-gen}} := \mathbf{Maps}(X,\mathbb{B}K) \underset{\mathbf{GMap}(X,\mathbb{B}K)}{\times} \mathbf{GMap}(X,\mathbb{B}H)$$

classifying a K-torsor on X equipped with a generic reduction to H.

Remark 2.3.3. The functor $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ commutes with finite limits (see [Bar14, Remark 2.2.6]).

Construction 2.3.4. Applying the above construction to the diagram

$$(G,P) \leftarrow (P,P) \to (M,M),$$

we obtain a diagram

$$\mathrm{Bun}_G^{P\text{-gen}} \xleftarrow{\iota_P} \mathrm{Bun}_P \xrightarrow{\mathfrak{q}_P} \mathrm{Bun}_M .$$

Remark 2.3.5. The prestack $\mathrm{Bun}_G^{P\text{-gen}}$ has the same field-valued points as Bun_P.

Definition 2.3.6. The DG-category $\mathrm{I}(G,P)$ is defined as the fiber product of the following diagram:

$$\begin{tikzcd}
\mathrm{I}(G,P) \arrow[r, dotted] \arrow[d, dotted] & \mathrm{DMod}(\mathrm{Bun}_M) \arrow[d, "\mathfrak{q}_P^*"] \\
\mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}) \arrow[r, "\iota_P^!"] & \mathrm{DMod}(\mathrm{Bun}_P).
\end{tikzcd}$$

We also use the notation[32]

$$\mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})^{U(\mathbb{A}_F)} := \mathrm{I}(G,P)$$

[31] Barlev only used this notation when Y is a scheme. But the definition in *loc.cit.* is valid for any lft prestack.

[32] $\mathbb{A}_F$ is the ring of adeles of the field of rational functions on X.

and refer objects in this category as $U(\mathbb{A}_F)$*-equivariant objects*.

Since $\mathrm{Bun}_P \to \mathrm{Bun}_M$ is smooth, in this definition, we can also use $\mathfrak{q}_P^!$ instead of $\mathfrak{q}_P^*$: the resulting full subcategory $\mathrm{I}(G,P)$ does not change.

Remark 2.3.7. The above definition is equivalent to that in [Gai15a, § 6.1] by [Gai15a, Lemma 6.3.3].

Remark 2.3.8. By [Gai15a, Lemma 6.1.2], the functor $\iota_P^!$ is conservative. By [Gai15a, § 6.2.1], the functor $\mathfrak{q}_P^*$ is fully faithful. Therefore the functor

$$\mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$$

is fully faithful and the functor $\mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_M)$ is conservative. Following *loc.cit.*, we denote the last functor by

$$\iota_M^! : \mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_M).$$

The following result was claimed in [Gai15a, § 6.2.5]. We provide a proof in Appendix G.1.

Proposition 2.3.9 (Gaitsgory)**.** *The partially defined left adjoint $\iota_{P,!}$ to $\iota_P^!$ is well-defined on the essential image of $\mathfrak{q}_P^*$, and its image is contained in $\mathrm{I}(G,P)$.*

Corollary 2.3.10. *The functor $\iota_M^! : \mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_M)$ has a left adjoint*

$$\iota_{M,!} : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P).$$

Proof. By Proposition 2.3.9, the functor $\iota_{P,!} \circ \mathfrak{q}_P^*$ uniquely factors through a functor $\mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P)$, which is the desired left adjoint.

□[Corollary 2.3.10]

Remark 2.3.11. Since $\iota_M^!$ is conservative, the image of its left adjoint $\iota_{M,!}$ generates $\mathrm{I}(G,P)$. Hence $\mathrm{I}(G,P)$ is compactly generated because so is $\mathrm{DMod}(\mathrm{Bun}_M)$. Note that $\mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$ preserves compact objects because so is $\iota_{P,!} \circ \mathfrak{q}_P^*$. In particular, its right adjoint $\mathrm{Av}^{U(\mathbb{A}_F)}$ is continuous.

The following result was implicit in [Gai11]. We provide a new proof in Remark 6.4.9.

Proposition 2.3.12 (Gaitsgory)**.** *Consider defect-free relative parabolic Zastava space (see § A.5.7)*

$${}_0Z^P_{\mathrm{rel}} \coloneqq \mathbf{Maps}_{\mathrm{gen}}(X, P^- \backslash G/P \supset P^- \backslash G^{\mathrm{Bruhat}}/P)$$

and the canonical map

$$(\overleftarrow{\mathfrak{h}}_M, \overrightarrow{\mathfrak{h}}_M) : {}_0Z^P_{\mathrm{rel}} \to \mathrm{Bun}_M \times \mathrm{Bun}_M .$$

We have a canonical equivalence between endo-functors on $\mathrm{DMod}(\mathrm{Bun}_M)$*:*

$$\iota^!_M \circ \iota_{M,!} \simeq (\overleftarrow{\mathfrak{h}}_M)_! \circ (\overrightarrow{\mathfrak{h}}_M)^*.$$

In particular, the RHS is well-defined.

Remark 2.3.13. Note that the endo-functor $\iota^!_M \circ \iota_{M,!}$ has a monad structure. One can show that the corresponding natural transformation $(\overleftarrow{\mathfrak{h}}_M)_! \circ (\overrightarrow{\mathfrak{h}}_M)^* \circ (\overleftarrow{\mathfrak{h}}_M)_! \circ (\overrightarrow{\mathfrak{h}}_M)^* \to (\overleftarrow{\mathfrak{h}}_M)_! \circ (\overrightarrow{\mathfrak{h}}_M)^*$ is induced by a Verdier (co)specialization construction along Schieder's local models ${}_0Y^P_{\mathrm{rel}}$, which gives a degeneration from ${}_0Z^P_{\mathrm{rel}}$ to ${}_0Z^P_{\mathrm{rel}} \times_{\mathrm{Bun}_M} {}_0Z^P_{\mathrm{rel}}$. See (A.24). We do not prove or use the above claim in this thesis.

Construction 2.3.14. Let (G_1, P_1) and (G_2, P_2) be two pairs as above. It follows from Remark 2.3.11 that the functor

$$\mathrm{I}(G_1, P_1) \otimes \mathrm{I}(G_2, P_2) \to \mathrm{DMod}(\mathrm{Bun}_{G_1}^{P_1\text{-gen}}) \otimes \mathrm{DMod}(\mathrm{Bun}_{G_2}^{P_2\text{-gen}})$$

is fully faithful. Moreover, it is easy to see the functor

$$\boxtimes : \mathrm{DMod}(\mathrm{Bun}_{G_1}^{P_1\text{-gen}}) \otimes \mathrm{DMod}(\mathrm{Bun}_{G_2}^{P_2\text{-gen}}) \to \mathrm{DMod}(\mathrm{Bun}_{G_1}^{P_1\text{-gen}} \times \mathrm{Bun}_{G_2}^{P_2\text{-gen}})$$

restricts to a functor

$$\boxtimes : \mathrm{I}(G_1, P_1) \otimes \mathrm{I}(G_2, P_2) \to \mathrm{I}(G_1 \times G_2, P_1 \times P_2)$$

Corollary 2.3.15. *The above functor*

$$\boxtimes : \mathrm{I}(G_1, P_1) \otimes \mathrm{I}(G_2, P_2) \to \mathrm{I}(G_1 \times G_2, P_1 \times P_2)$$

is an equivalence.

Proof. We have a commutative diagram

$$\begin{tikzcd} \mathrm{I}(G_1, P_1) \otimes \mathrm{I}(G_2, P_2) \arrow[r, "\boxtimes"] \arrow[d, "\iota^!_{M_1} \otimes \iota^!_{M_2}"] & \mathrm{I}(G_1 \times G_2, P_1 \times P_2) \arrow[d, "\iota^!_{M_1 \times M_2}"] \\ \mathrm{DMod}(\mathrm{Bun}_{M_1}) \otimes \mathrm{DMod}(\mathrm{Bun}_{M_2}) \arrow[r, "\boxtimes", "\simeq"'] & \mathrm{DMod}(\mathrm{Bun}_{M_1} \times \mathrm{Bun}_{M_2}), \end{tikzcd}$$

where the bottom equivalence is due to [DG15, Remark 2.2.9]. Note that both vertical functors are monadic. Hence by Proposition 2.3.12, we only need to identify $((\overleftarrow{\mathfrak{h}}_{M_1})_! \circ (\overrightarrow{\mathfrak{h}}_{M_1})^*) \otimes ((\overleftarrow{\mathfrak{h}}_{M_2})_! \circ (\overrightarrow{\mathfrak{h}}_{M_2})^*)$ with $(\overleftarrow{\mathfrak{h}}_{M_1\times M_2})_! \circ (\overrightarrow{\mathfrak{h}}_{M_1\times M_2})^*$. But this is obvious after passing to the right adjoints[33].

□[Corollary 2.3.15]

Construction 2.3.16. Let Q be another parabolic subgroup of G that contains P. Consider the map

$$\mathfrak{p}^{\mathrm{enh}}_{P\to Q} : \mathrm{Bun}_G^{P\text{-gen}} \to \mathrm{Bun}_G^{Q\text{-gen}}$$

and the functor

$$\mathfrak{p}^{\mathrm{enh},!}_{P\to Q} : \mathrm{DMod}(\mathrm{Bun}_G^{Q\text{-gen}}) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$$

The special case (when $Q = G$) of the following result is claimed in [Gai15a, Lemma 6.3.3]. We provide a proof in Appendix G.2.

Proposition 2.3.17 (Gaitsgory)**.** *We have:*

(1) The partially defined left adjoint $\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!}$ *to* $\mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$ *is well-defined on* $\mathrm{I}(G,P) \subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$*, and sends* $\mathrm{I}(G,P)$ *into* $\mathrm{I}(G,Q)$*.*

(2) Let

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to Q} : \mathrm{I}(G,P) \to \mathrm{I}(G,Q)$$

be the functor obtained from $\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!}$*. Then* $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$ *has a* continuous *right adjoint*

$$\mathrm{CT}^{\mathrm{enh}}_{Q\to P} : \mathrm{I}(G,Q) \to \mathrm{I}(G,P).$$

Remark 2.3.18. When $Q = G$, we also denote the adjoint pair $(\mathrm{Eis}^{\mathrm{enh}}_{P\to G}, \mathrm{CT}^{\mathrm{enh}}_{G\to P})$ by $(\mathrm{Eis}^{\mathrm{enh}}_{P}, \mathrm{CT}^{\mathrm{enh}}_{P})$.

Warning 2.3.19. The functor $\mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$ does not send $\mathrm{I}(G,Q)$ into $\mathrm{I}(G,P)$. Hence the functor $\mathrm{CT}^{\mathrm{enh}}_{Q\to P}$ is isomorphic to the composition $\mathrm{Av}^{U(\mathbb{A}_F)} \circ \mathfrak{p}^{\mathrm{enh},!}_{P\to Q}$.

---

[33]More precisely, we work with individual connected component of ${}_0Z^{P_1}_{\mathrm{rel}} \times {}_0Z^{P_2}_{\mathrm{rel}} \simeq {}_0Z^{P_1\times P_2}_{\mathrm{rel}}$, which is *quasi-compact* and schematic over $\mathrm{Bun}_{M_1\times M_2}$. Hence we only need to compare the *continuous* right adjoints of the (restriction of the) above functors (on such connected component). Then we are done because $\boxtimes$ is compactible with !-pullback functors and $*$-pushforward functors.

2.4 The Deligne-Lusztig duality on Bun_G

Construction 2.4.1. Let Par be the poset of standard parabolic subgroups and Par′ be Par $-\{G\}$. We view posets as categories in a standard way. It follows formally from Proposition 2.3.17 that we have a functor

$$\mathbf{DL}:\mathrm{Par}\to\mathrm{LFun}_k(\mathrm{DMod}(\mathrm{Bun}_G),\mathrm{DMod}(\mathrm{Bun}_G)),\ P\mapsto\mathrm{Eis}^{\mathrm{enh}}_{P\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to P}$$

such that a morphism $P\to Q$ in Par is sent to the composition

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to P}\simeq\mathrm{Eis}^{\mathrm{enh}}_{Q\to G}\circ\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}\circ\mathrm{CT}^{\mathrm{enh}}_{Q\to P}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to Q}\to\mathrm{Eis}^{\mathrm{enh}}_{Q\to G}\circ\mathrm{CT}^{\mathrm{enh}}_{G\to Q}.$$

Note that $\mathbf{DL}(G)$ is the identity functor $\mathbf{Id}$ on $\mathrm{DMod}(\mathrm{Bun}_G)$.

We will prove the following result in § 5.2.

Theorem 2.4.2. *There is a canonical equivalence*

$$\mathrm{coFib}(\operatorname*{colim}_{P\in\mathrm{Par}'}\mathbf{DL}(P)\to\mathbf{DL}(G))\simeq\mathrm{Ps\text{-}Id}_{\mathrm{naive}}\circ\mathrm{Ps\text{-}Id}_!^{-1}[-2\dim(\mathrm{Bun}_G)-\dim(Z_G)]\tag{2.12}$$

in $\mathrm{LFun}_k(\mathrm{DMod}(\mathrm{Bun}_G),\mathrm{DMod}(\mathrm{Bun}_G))$.

Remark 2.4.3. Let $\mathcal{F}\in\mathrm{DMod}(\mathrm{Bun}_G)^\heartsuit$ be an object in the heart of the t-structure. If the functors $\mathbf{DL}(P)$ were t-exact (which is not true), then the value of the LHS of (2.12) on $\mathcal{F}$ can be calculated by a complex

$$\mathbf{DL}(B)(\mathcal{F})\to\cdots\to\bigoplus_{\mathrm{corank}(P)=1}\mathbf{DL}(P)(\mathcal{F})\to\mathbf{DL}(G)(\mathcal{F}).$$

Hence the LHS of (2.12) can be viewed as an ∞-categorical replacement for the Deligne-Lusztig complex.

Part II

The Local Nearby Cycles $\Psi_{\gamma,I}$

Chapter 3

Statement of results

3.1 Basic properties of the local nearby cycles

Construction 3.1.1 (Local nearby cycles)**.** Let I be a non-empty finite set, P be a standard parabolic subgroup and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. Consider the indscheme

$$Z \coloneqq \mathrm{VinGr}^{\gamma}_{G,I} \to \mathbb{A}^1$$

defined in Construction 1.4.9.

By Lemma 1.4.10(2), we have $\overset{\circ}{Z} \simeq \mathrm{Gr}_{G,I} \times \mathbb{G}_m$. Consider the corresponding nearby cycles functor

$$\Psi_{\mathrm{VinGr}^{\gamma}_{G,I}} : \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Gr}_{G,I} \times \mathbb{G}_m) \to \mathrm{DMod}(\mathrm{VinGr}_{G,I}|_{C_P}),$$

where the subscript "indhol" means the full subcategory of ind-holonomic D-modules (see § C.5 for what this means). The dualizing D-module $\omega_{\overset{\circ}{Z}}$ is ind-holonomic. Hence we obtain an object

$$\Psi_{\gamma,I} \coloneqq \Psi_{\mathrm{VinGr}^{\gamma}_{G,I}}(\omega_{\overset{\circ}{Z}}) \in \mathrm{DMod}(\mathrm{VinGr}_{G,I}|_{C_P}).$$

Recall that there is a canonical Cartesian squares

$$\begin{tikzcd}
\mathrm{VinGr}_{G,I}|_{C_P} \arrow[r, "i"] \arrow[d] & \mathrm{VinGr}^{\gamma}_{G,I} \arrow[d] & \mathrm{Gr}_{G,I} \times \mathbb{G}_m \arrow[l, "j"'] \arrow[d, "\Gamma_I(x,t)=(x,t\cdot x,t)"] \\
\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \arrow[r, "i"] & \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 & \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{G}_m, \arrow[l, "j"']
\end{tikzcd} \tag{3.1}$$

where the vertical maps are closed embeddings (see Lemma 1.4.10).

We abuse notation by viewing

$$\Psi_{\gamma,I} \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}).$$

Since taking the nearby cycles commutes with proper push-forward functors, $\Psi_{\gamma,I}$ can also be calculated as the nearby cycles sheaf of $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$ along the constant family $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \to \mathbb{A}^1$.

We will prove the following result in § 4.1.

Proposition 3.1.2. *We have:*

(1) The canonical map $\Psi^{\mathrm{un}}_{\gamma,I} \to \Psi_{\gamma,I}$ *is an isomorphism.*

(2) The object $\Psi^{\mathrm{un}}_{\gamma,I} \simeq \Psi_{\gamma,I}$ *is contained in the full subcategory* $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$. *Moreover, it can be canonically upgraded to an object in* $(\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I})^{\mathcal{L}^+ M_I,\mathrm{diag}}$.

Remark 3.1.3. It is quite possible that one can actually upgrade $\Psi_{\gamma,I}$ to an object in $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(P\times_M P^-)}$. However, because $\mathcal{L}M_I$ is not an ind-group scheme, our current techniques can not prove it.

Construction 3.1.4. Recall (see Lemma 1.4.10) that we have an identification

$$_0\mathrm{VinGr}^{\gamma}_{G,I} \simeq \mathrm{Gr}_{\widetilde{G}^{\gamma},I}$$

as locally closed sub-indscheme of

$$\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \simeq \mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1.$$

Note that the 0-fiber of $\mathrm{Gr}_{\widetilde{G}^{\gamma},I}$ is $\mathrm{Gr}_{P\times_M P^-,I}$, which is an open sub-indscheme of $\mathrm{VinGr}_{G,I}|_{C_P}$.

Consider the map $_0\mathrm{VinGr}^{\gamma}_{G,I} \to \mathbb{A}^1$. Let $_0\Psi_{\gamma,I}$ (resp. $_0\Psi^{\mathrm{un}}_{\gamma,I}$) be the full (resp. unipotent) nearby cycles sheaf of the dualizing D-module for this family.

Also consider the map $\mathbb{A}^1 \to \mathbb{A}^1$. Let Ψ_{triv} (resp. $\Psi^{\mathrm{un}}_{\mathrm{triv}}$) be the full (resp. unipotent) nearby cycles sheaf of the dualizing D-module for this family. It is well-known that $\Psi^{\mathrm{un}}_{\mathrm{triv}} \simeq \Psi_{\mathrm{triv}} \simeq k[1]$.

We will prove the following result in § 4.5.

Proposition 3.1.5. *The canonical maps*

$$_0\Psi_{\gamma,I} \to \omega \otimes \Psi_{\mathrm{triv}} \simeq \omega[1],\ _0\Psi^{\mathrm{un}}_{\gamma,I} \to \omega \otimes \Psi^{\mathrm{un}}_{\mathrm{triv}} \simeq \omega[1]$$

are isomorphisms, where ω *is the dualizing D-module on* $_0\mathrm{VinGr}_{G,I}|_{C_P}$.

Construction 3.1.6. Let $I \twoheadrightarrow J$ be a surjection between non-empty finite sets. Consider the corresponding diagonal embedding $\Delta_{J\to I} : X^J \to X^I$. For any prestack $\mathcal{Z}$ over X^I, we abuse notation by denoting the closed embedding $\mathcal{Z} \times_{X^I} X^J \to \mathcal{Z}$ by the same symbol $\Delta_{J\to I}$.

By Remark 1.4.8, the assignment $I \rightsquigarrow (\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \hookrightarrow \mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$ factorizes in family (relative to $\mathbb{G}_m$). Hence we have the base-change isomorphism:

$$\Gamma_{J,*}(\omega_{\mathrm{Gr}_{G,J} \times \mathbb{G}_m}) \simeq \Delta^!_{J\to I} \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m}),$$

which induces a morphism

$$\Psi_{\gamma,J} \to \Delta^!_{J\to I}(\Psi_{\gamma,I}).$$

We will prove the following result in § 4.6.

Proposition 3.1.7. *The above morphism $\Psi_{\gamma,J} \to \Delta^!(\Psi_{\gamma,I})$ is an isomorphism.*

Corollary 3.1.8. *The assignment*

$$I \rightsquigarrow \Psi_{\gamma,I}[-1] \in \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$$

gives a factorization algebra $\Psi[-1]_{\gamma,\mathrm{fact}}$ in the factorization category $\mathrm{DMod}(\mathrm{Gr}_{G\times G})^{\mathcal{L}(U\times U^-)}_{\mathrm{fact}}$.

Proof. By Proposition 3.1.7, the assignment $I \rightsquigarrow \Psi_{\gamma,I}[-1]$ is compatible with diagonal restrictions. It has the factorization property because of the Künneth formula for the nearby-cycles.

□[Corollary 3.1.8]

Remark 3.1.9. It follows from the proof of Proposition 3.1.2(2) that $\Psi[-1]_{\gamma,\mathrm{fact}}$ can be upgraded to a factorization algebra in the factorization category $(\mathrm{DMod}(\mathrm{Gr}_{G\times G})^{\mathcal{L}(U\times U^-)})^{\mathcal{L}^+M}_{\mathrm{fact}}$. Moreover, one can show that $\Psi[-1]_{\gamma,\mathrm{fact}}$ is a *unital* factorization algebra. We do not need or prove these results.

We will prove the following result in § 4.7.

Proposition 3.1.10. *The $(*,\mathrm{inv})$-restriction*[34] *of $\Psi_{\gamma,I}$ along the stratification* $\mathrm{Gr}_{P,I} \times_{X^I} \mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ *is a $*$-extension along the stratification* $\mathrm{Gr}_{P,I} \times_{X^I} \mathrm{Gr}_{P^-,I} \to \mathrm{Gr}_{P,I} \times_{X^I} \mathrm{Gr}_{G,I}$. *More precisely, the following objects are canonically isomorphic:*

[34] See Footnote 29.

(a) The image of $\Psi_{\gamma,I}[-1]$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} &\overset{(2.10)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \\
&\overset{\mathbf{p}_I^{+,*,\mathrm{inv}}\otimes\mathbf{Id}}{\to} \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \\
&\overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}
\end{aligned}$$

(b) The image of $\omega_{\mathrm{Gr}_{M,I}}$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{M,I}) &\overset{\Delta_*}{\to} \mathrm{DMod}(\mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{M,I}) \\
&\overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{P^-,I})^{\mathcal{L}U_I^-} \\
&\overset{\mathbf{Id}\otimes\mathbf{p}_{I,*}^+}{\to} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}.
\end{aligned}$$

Proposition 3.1.10 implies the following corollary.

Corollary 3.1.11. *The following objects are canonically isomorphic:*

(a) The image of $\Psi_{\gamma,I}[-1]$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} &\overset{(2.10)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \\
&\overset{\mathbf{p}_I^{+,*,\mathrm{inv}}\otimes\mathbf{p}_I^{-,!}}{\to} \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{P^-,I})^{\mathcal{L}U_I^-} \\
&\overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{M,I})
\end{aligned}$$

(b) The image of $\omega_{\mathrm{Gr}_{M,I}}$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{M,I}) &\overset{\Delta_*}{\to} \mathrm{DMod}(\mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{M,I}) \\
&\simeq \mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{M,I}).
\end{aligned}$$

Corollary 3.1.11 implies the following corollary.

Corollary 3.1.12. *The* $!$*-restriction of* $\Psi_{\gamma,I}$ *along the stratification* $\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{P^-,I}\to\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}$ *is a* $(!,\mathrm{inv})$*-extension along the stratification* $\mathrm{Gr}_{P,I}\times_{X^I}\mathrm{Gr}_{P^-,I}\to\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{P^-,I}$*. More precisely, the following objects are canonically isomorphic:*

(a) The image of $\Psi_{\gamma,I}[-1]$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} &\overset{(2.10)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \\
&\xrightarrow{\mathbf{Id}\otimes\mathbf{p}_I^{-,!}} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{P^-,I})^{\mathcal{L}U_I^-} \\
&\overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{M,I})
\end{aligned}$$

(b) The image of $\omega_{\mathrm{Gr}_{M,I}}$ *under the composition*

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{M,I}) &\xrightarrow{\Delta_*} \mathrm{DMod}(\mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{M,I}) \\
&\overset{(2.7)}{\simeq} \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \\
&\xrightarrow{\mathbf{p}_{I,!}^{+,\mathrm{inv}}\otimes\mathbf{Id}} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{M,I}).
\end{aligned}$$

Remark 3.1.13. By symmetry, Proposition 3.1.10 and the above two corollaries are also correct if we exchange the roles of the two factors in $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$.

Remark 3.1.14. One can actually deduce Proposition 3.1.5 from Corollary 3.1.11. More precisely, the corollary implies a canonical isomorphism ${}_0\Psi_{\gamma,I,\mathrm{Vin}} \simeq \omega[1]$, and it follows from the proof of Proposition 3.1.10 that this isomorphism is the desired one.

3.2 The local nearby cycles and the inv-inv duality

Construction 3.2.1. Write $\iota : \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \hookrightarrow \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}$ for the obvious closed embedding. Consider the object

$$\mathcal{K}_{\gamma,I} := \iota_*(\Psi_{\gamma,I}[-1]) \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}).$$

Also consider $\mathcal{K}^\sigma_{\gamma,I} := \sigma_*\mathcal{K}_{\gamma,I}$, where σ is the involution on $\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}$ given by switching the two factors.

Recall the construction of *functors given by kernels* (see § C.6):

$$F : \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}) \to \mathrm{Funct}(\mathrm{DMod}(\mathrm{Gr}_{G,I}), \mathrm{DMod}(\mathrm{Gr}_{G,I})).$$

Evaluating it at the objects $\mathcal{K}_{\gamma,I}$ and $\mathcal{K}^\sigma_{\gamma,I}$, we obtain functors

$$F_{\mathcal{K}_{\gamma,I}}, F_{\mathcal{K}^\sigma_{\gamma,I}} : \mathrm{DMod}(\mathrm{Gr}_{G,I}) \to \mathrm{DMod}(\mathrm{Gr}_{G,I}).$$

Explicitly, we have $F_{\mathcal{K}_{\gamma,I}}(-) := \mathrm{pr}_{2,*}(\mathrm{pr}_1^!(-) \otimes^! \mathcal{K}_{\gamma,I})$, where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I}$ are the projections.

We will prove the following result in § 4.8:

Theorem 3.2.2. *We have:*

(1) The functor $\mathbf{oblv}^{\mathcal{L}U_I^-} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})$ *can be obtained as the functor given by the kernel* $\mathcal{K}_{\gamma,I}$*. In other words, we have*

$$F_{\mathcal{K}_{\gamma,I}}|_{\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}} \simeq \mathbf{oblv}^{\mathcal{L}U_I^-}.$$

(2) The functor $\mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})$ *can be obtained as the functor given by the kernel* $\mathcal{K}^{\sigma}_{\gamma,I}$*. In other words, we have*

$$F_{\mathcal{K}^{\sigma}_{\gamma,I}}|_{\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}} \simeq \mathbf{oblv}^{\mathcal{L}U_I}.$$

3.2.3. By Theorem 3.1.2, the object $\Psi_{\gamma,I}$ is contained in the full subcategory

$$\mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times_{X^I} \mathcal{L}U_I^-} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}).$$

Moreover, this full subcategory can be identified with (see Corollary 2.1.10(2))

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}.$$

It follows formally (see Lemma D.1.7(3)) that the kernel $\mathcal{K}_{\gamma,I}$ is contained in the full subcategory[35]

$$\mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times \mathcal{L}U_I^-} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}).$$

Again, this full subcategory can be identified with (see Corollary 2.1.10(1))

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}.$$

Corollary 3.2.4 (Theorem A and Theorem B)**.** *We have:*

[35]The reader might have noticed that this claim is a formal consequence of Theorem 3.2.2. However, we need to prove this fact before we prove the theorem.

(1) *The categories* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$ *are dual to each other in* DGCat*, with the unit given by*

$$\mathrm{Vect} \xrightarrow{\mathcal{K}_{\gamma,I}\otimes -} \mathrm{DMod}(\mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times\mathcal{L}U_I^-} \simeq \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-},$$

and the counit given by

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}\otimes\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I}) \to \mathrm{Vect},$$

where the last functor is the counit of the Verdier self-duality[36].

(2) *The categories* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$ *are dual to each other in* $\mathrm{DMod}(X^I)$-mod*, with the unit given by*

$$\mathrm{Vect} \xrightarrow{\Psi_{\gamma,I}[-1]\otimes -} \mathrm{DMod}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U_I^-} \simeq \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\underset{\mathrm{DMod}(X^I)}{\otimes}\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-},$$

and the counit given by

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}\otimes\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I}) \to \mathrm{DMod}(X^I),$$

where the last functor is the counit[37] *of the Verdier self-duality for* $\mathrm{DMod}(\mathrm{Gr}_{G,I})$ *as a* $\mathrm{DMod}(X^I)$*-module category.*

Proof. Using Lemma C.6.1, the axioms for the dualities are given by Theorem 3.2.2.

□[Corollary 3.2.4]

Warning 3.2.5. Our proof of Theorem 3.2.2, and therefore of Corollary 3.2.4, logically depends on the dualizability results in Proposition 2.1.8.

Remark 3.2.6. In the constructible contexts, Theorem 3.2.2 remains correct, and can be proved similarly. We also have a version of Corollary 3.2.4(1). See Remark C.6.2 and Remark 2.1.12 for more details.

However, we do *not* have a version of Corollary 3.2.4(2) in the constructible contexts. For example, we do *not* even know if $\mathrm{Shv}_c(\mathrm{Gr}_{G,I})$ is self-dual as a $\mathrm{Shv}_c(S)$-module category, where Shv_c is the DG-category of complexes of constructible sheaves.

[36]It sends $\mathcal{F}\boxtimes\mathcal{G}$ to $\Gamma_{\mathrm{dR},*}(\mathcal{F}\otimes^!\mathcal{G})$.

[37]It is given by

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{\otimes^!} \mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{*\text{-pushforward}} \mathrm{DMod}(X^I).$$

Remark 3.2.7. As a by-product of Corollary 3.2.4, the object $\Psi_{\gamma,I}$ does not depend on the choice of γ (but depends on P).

Corollary 3.2.8 (Theorem C)**.** *Via the inv-inv duality and the Verdier self-duality on* $\mathrm{DMod}(\mathrm{Gr}_{M,I})$*, the functors (see Lemma 2.1.7(2))*

$$\iota_{M,I,!} : \mathrm{DMod}(\mathrm{Gr}_{M,I}) \to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I},$$
$$\iota^{-}_{M,I,!} : \mathrm{DMod}(\mathrm{Gr}_{M,I}) \to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U^{-}_I}$$

are conjugate to each other.

Proof. By the definition of conjugate functors, we only need to prove $\iota_{M,I,!}$ is the dual functor of the right adjoint $\iota^{-,!}_{M,I}$ of $\iota^{-}_{M,I,!}$ via the above dualities. Therefore we only need to prove the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Vect} \arrow[r, "\mathbf{unit}"] \arrow[d, "\mathbf{unit}"] & \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U^{-}_I} \arrow[d, "\mathbf{Id}\otimes\iota^{-}_{M,I,!}"] \\
\mathrm{DMod}(\mathrm{Gr}_{M,I}) \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \arrow[r, "\iota_{M,I,!}\otimes\mathbf{Id}"] & \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(\mathrm{Gr}_{M,I}).
\end{tikzcd}$$

But this is exactly the content of Corollary 3.1.12.

□[Corollary 3.2.8]

3.3 The inv-inv duality and the second adjointness

Combining Corollary 3.2.4 and Proposition 2.1.8, we obtain:

Corollary 3.3.1 (Theorem D)**.** *The functor*

$$\mathbf{pr}_{\mathcal{L}U^{-}_I} \circ \mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U^{-}_I}$$

is an equivalence.

Remark 3.3.2. As mentioned in the introduction, when $P = B$, Corollary 3.3.1 is an easy consequence of [Ras16, Theorem 6.2.1, Corollary 6.2.3]. For reader's convenience, we sketch this proof, which we learned from D. Gaitsgory.

Let us first consider the non-factorization version. By construction, the functor $\mathbf{pr}_{\mathcal{L}N^-} \circ \mathbf{oblv}^{\mathcal{L}N}$ is $\mathcal{L}T$-linear. Using Raskin's results, one can show it induces an equivalence:

$$(\mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}N})^{\mathcal{L}^+T} \simeq (\mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}N^-})^{\mathcal{L}^+T} \tag{3.2}$$

Using the fact that every $\mathcal{L}N$-orbit of Gr_G is stabilized by $\mathcal{L}^+T$, one can prove that the adjoint pairs

$$\mathbf{oblv}^{\mathcal{L}^+T} : (\mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}N})^{\mathcal{L}^+T} \rightleftharpoons \mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}N} : \mathbf{Av}_*^{\mathcal{L}^+T},$$
$$\mathbf{oblv}^{\mathcal{L}^+T} : (\mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}N^-})^{\mathcal{L}^+T} \rightleftharpoons \mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}N^-} : \mathbf{Av}_*^{\mathcal{L}^+T}$$

are both monadic. Then the Barr-Beck-Lurie thoerem gives the desired result.

Now consider the factorization case. We need to show

$$\mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}N_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}N_I^-}, \tag{3.3}$$

is an equivalence. Using the étale descent, we obtain the desired equivalence when I is a singleton. Also, one can show the functor (3.3) preserves compact objects. Moreover, one can show that the assignment $I \rightsquigarrow \mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I}$ factorizes. Now it is a basic fact that a factorization functor satisfying the above properties is a factorization equivalence.

Remark 3.3.3. Let us emphasize that our proof of Corollary 3.3.1 does not rely on Raskin's results. Our proof has three advantages:

- It works for general parabolics P rather than the Borel B (the monadicity in Remark 3.3.2 fails for general P);
- It automatically works for the factorization version;
- It allows us to describe an inverse of the equivalence via a geometric construction (see Corollary 3.3.4 below), which we believe is of independent interest.

Corollary 3.3.4. *We have:*

(1) The functor $F_{\mathcal{K}_{\gamma,I}}$ factors uniquely as

$$F_{\mathcal{K}_{\gamma,I}} : \mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}} \mathrm{DMod}(\mathrm{Gr}_{G,I}),$$

and the functor in the middle is inverse to

$$\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}.$$

(2) The functor $F_{\mathcal{K}^\sigma_{\gamma,I}}$ factors uniquely as

$$F_{\mathcal{K}^\sigma_{\gamma,I}} : \mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I^-}} \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{G,I}),$$

and the functor in the middle is inverse to

$$\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-}.$$

Proof. We prove (1) and obtain (2) by symmetry. By Proposition 2.1.8, $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ are dual to each other. Moreover, by Lemma D.1.10, the counit functor of this duality fits into a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \arrow[r, "\mathbf{Id}\otimes\mathbf{oblv}^{\mathcal{L}U_I}"] \arrow[d, "\mathbf{pr}_{\mathcal{L}U_I}\otimes\mathbf{Id}"] & \mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I}) \arrow[d] \\
\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \arrow[r, "\mathbf{counit}"] & \mathrm{Vect},
\end{tikzcd} \tag{3.4}$$

where the right vertical functor is the counit for the Verdier self-duality.

On the other hand, by Corollary 3.2.4(1) and (3.4), the composition

$$\mathbf{counit} \circ ((\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-}) \otimes \mathbf{Id}) : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{Vect}$$

is also the counit of a duality. Hence by uniqueness of the dual category, the functor $\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-}$ is an equivalence. Denote the inverse of this equivalence by θ.

Note that the desired factorization of $F_{\mathcal{K}_{\gamma,I}}$ is unique if it exists because $\mathbf{pr}_{\mathcal{L}U_I}$ is a localization and $\mathbf{oblv}^{\mathcal{L}U_I^-}$ is a full embedding. Hence it remains to show that $\mathbf{oblv}^{\mathcal{L}U_I^-} \circ \theta \circ \mathbf{pr}_{\mathcal{L}U_I}$ is isomorphic to $F_{\mathcal{K}_{\gamma,I}}$. By uniqueness of the dual category, the functor θ is given by the composition

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \xrightarrow{\mathbf{Id}\otimes\mathbf{unit}^{\text{inv-inv}}} \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to$$
$$\xrightarrow{\mathbf{counit}\otimes\mathbf{Id}} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-},$$

where $\mathbf{unit}^{\text{inv-inv}}$ is the unit of the duality between $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$. Now the desired claim can be checked directly using Corollary (3.2.4)(1).

□[Corollary 3.3.4]

3.4 Variant: $\mathcal{L}^+M$-equivariant version

In this section, we describe an $\mathcal{L}^+M$-equivariant version of the main theorems. Recall that $\Psi_{\gamma,I}$ can be upgraded to an object

$$\Psi_{\gamma,I} \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I,\mathrm{diag}}.$$

It follows formally (by Lemma D.6.6(1)) that the functors $F_{\mathcal{K}_{\gamma,I}}$ and $F_{\mathcal{K}^\sigma_{\gamma,I}}$ defined in § 3.2 can be upgraded to $\mathcal{L}^+M_I$-linear functors.

We will prove the following result in § 4.8.6:

Corollary 3.4.1. *The equivalences in Theorem 3.2.2 are compatible with the $\mathcal{L}^+M_I$-linear structures on $F_{\mathcal{K}_{\gamma,I}}$, $F_{\mathcal{K}^\sigma_{\gamma,I}}$, $\mathbf{oblv}^{\mathcal{L}U_I}$ and $\mathbf{oblv}^{\mathcal{L}U^-_I}$.*

3.4.2 (The inv-inv duality: equivariant version)**.** By Corollary 2.2.1, we have a canonical equivalence[38]

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I} \simeq \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}^+M_I}.$$

Moreover, by Lemma D.2.7 and Lemma D.1.10, $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}$ is canonically self-dual.

We define

$$\mathbb{D}^{\frac{\infty}{2}} := \mathbf{Av}_*^{(\mathcal{L}^+M_I,\mathrm{diag})\to(\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I)}(\Psi_{\gamma,I}[-1]),$$

where the functor

$$\mathbf{Av}_* : \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I,\mathrm{diag}} \to (\mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}$$

is the right adjoint of the obvious forgetful functor (see § D.2 for more details).

The equivariant structures on $\Psi_{\gamma,I}[-1]$ formally imply (see Proposition 3.1.2 and Lemma D.5.2) that $\mathbb{D}^{\frac{\infty}{2}}$ can be upgraded to an object in

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U^-_I})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}.$$

[38]Via this equivalence, $\mathbf{pr}_{\mathcal{L}^+M_I}$ corresponds to $\mathbf{Av}_*^{\mathcal{L}^+M_I}$

By Corollary 2.2.2, this category is canonically isomorphic to

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}.$$

The following result is a formal consequence of Corollary 3.4.1 (see Lemma D.6.6(2)):

Corollary 3.4.3. *(1)* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}$ *and* $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}$ *are dual to each other in* DGCat*, with the counit given by*

$$\mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}\otimes\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}\otimes\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I} \to \mathrm{Vect}$$

where the last functor is the counit of the self-duality of $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}$ *in* DGCat.

(2) The unit of the duality in (1) is

$$\begin{aligned}
\mathrm{Vect} &\xrightarrow{\mathbb{D}^{\frac{\infty}{2}}\otimes -} (\mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U_I^-})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I} \\
&\simeq \mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I} \\
&\to \mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \otimes \mathrm{DMod}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}.
\end{aligned}$$

Remark 3.4.4. The last functor in the above composition is induced by $\Delta_* : \mathrm{DMod}(X^I) \to \mathrm{DMod}(X^I\times X^I)$. Namely, for any $\mathcal{M},\mathcal{N} \in \mathrm{DMod}(X^I)$-mod, we have a functor

$$\mathcal{M} \underset{\mathrm{DMod}(X^I)}{\otimes} \mathcal{N} \simeq (\mathcal{M}\otimes\mathcal{N}) \underset{\mathrm{DMod}(X^I\times X^I)}{\otimes} \mathrm{DMod}(X^I) \overset{\mathbf{Id}\otimes\Delta_*}{\to} \mathcal{M}\otimes\mathcal{N}.$$

Remark 3.4.5. We also have a version of the above corollary for the corresponding duality as $\mathrm{DMod}(X^I)$-module categories. We omit it because the notation is too heavy.

Remark 3.4.6. In the constructible contexts, (1) remains correct. However, the canonical functor

$$\begin{aligned}
&\mathrm{Shv}_c(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \underset{\mathrm{Shv}_c(X^I)}{\otimes} \mathrm{Shv}_c(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I} \to \\
&\to (\mathrm{Shv}_c(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U_I^-})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}
\end{aligned}$$

is *not* an equivalence. To make (2) correct, one needs to replace the equivalence in (2) by the right adjoint of the above functor.

As before, Corollary 3.4.1 and 3.4.3 formally imply

Corollary 3.4.7. *The inverse functors in Corollary 3.3.4 are compatible with the $\mathcal{L}^+ M_I$-linear structures on those functors.*

Chapter 4

Proofs

4.1 Proof of Proposition 3.1.2

In this subsection, we prove Proposition 3.1.2. Recall we have Cartesian squares (see Lemma D.5.2 and Lemma D.5.1):

$$\begin{tikzcd}
(\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I})^{\mathcal{L}^+M_I,\mathrm{diag}} \arrow[r] \arrow[d] & \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}^+M_I,\mathrm{diag}} \arrow[d] \\
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \arrow[r] & \mathrm{DMod}(\mathrm{Gr}_{G\times G,I}),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \arrow[r] \arrow[d] & \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1} \arrow[d] \\
\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I^-,2} \arrow[r] & \mathrm{DMod}(\mathrm{Gr}_{G\times G,I}),
\end{tikzcd}$$

where the superscripts 1 (resp. 2) indicate that $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$) acts on $\mathrm{Gr}_{G\times G,I} \simeq \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ via the first (resp. second) factor.

Hence we can prove the proposition in three steps:

(i) The objects $\Psi_{\gamma,I}$ and $\Psi^{\mathrm{un}}_{\gamma,I}$ are contained in $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$ and $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I^-,2}$.

(ii) The canonical morphism $\Psi^{\mathrm{un}}_{\gamma,I} \to \Psi_{\gamma,I}$ is an isomorphism.

(iii) The object $\Psi_{\gamma,I}$ can be canonically upgraded to an object in $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}^+M_I,\mathrm{diag}}$.

4.1.1 (Proof of (i))**.** Recall the co-character γ provides a $\mathbb{G}_m$-action on G (see Example 1.2.3). Note that $U \hookrightarrow G$ is stabilized by this action. By construction, this action is compactible with the group structure on U. In particular, the corresponding Drinfeld-Gaitsgory interpolation $\widetilde{U}^\gamma$ is a group scheme over $\mathbb{A}^1$ and the canonical map $\widetilde{U}^\gamma \to U \times U \times \mathbb{A}^1$ is a group homomorphism (relative to $\mathbb{A}^1$).

Note that the above $\mathbb{G}_m$-action on U is contractive, i.e., its attractor locus is isomorphic to itself. Hence by [DG14, Proposition 1.4.5], the $\mathbb{G}_m$-action on U can be extended to an $\mathbb{A}^1$-action on U, where $\mathbb{A}^1$ is equipped with the multiplication monoid structure. Note that the fixed locus of the $\mathbb{G}_m$-action on U is $1 \hookrightarrow U$. Hence by [DG14, Proposition 2.4.4], the map $\widetilde{U}^\gamma \to U \times U \times \mathbb{A}^1$ can be identified with

$$U \times \mathbb{A}^1 \to U \times U \times \mathbb{A}^1,\ (g,t) \mapsto (g, t\cdot g, t). \tag{4.1}$$

In particular, its 1-fiber is the diagonal embedding, while its 0-fiber is the closed embedding onto the *first* U-factor.

By taking loops, we obtain from (4.1) a homomorphism between group indschemes over $X^I \times \mathbb{A}^1$

$$a : \mathcal{L}U_I \times \mathbb{A}^1 \to \mathcal{L}U_I \underset{X^I}{\times} \mathcal{L}U_I \times \mathbb{A}^1$$

such that its 1-fiber is the diagonal embedding, while its 0-fiber is the closed embedding onto the first $\mathcal{L}U_I$-factor. Similarly, we have a morphism between group indschemes over $X^I \times \mathbb{A}^1$:

$$r : \mathcal{L}U_I^- \times \mathbb{A}^1 \to \mathcal{L}U_I^- \underset{X^I}{\times} \mathcal{L}U_I^- \times \mathbb{A}^1$$

whose 1-fiber is the diagonal embedding and 0-fiber is the closed embedding onto the *second* $\mathcal{L}U_I^-$-factor. In fact, the map a (resp. r) is the Drinfeld-Gaitsgory interpolation for the $\mathbb{G}_m$-action on $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$), if we generalize the definitions in [DG14] to arbitrary prestacks.

Via the group homomorphism a and r, we have an action of $\mathcal{L}U_I \times \mathbb{A}^1$ (resp. $\mathcal{L}U_I^- \times \mathbb{A}^1$) on $\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1$ relative to $X^I \times \mathbb{A}^1$. Equivalently, we have an action of $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$) on $\mathrm{Gr}_{G\times G,I}$ relative to X^I. We use symbols "a" (resp. "r") to distinguish these actions from other ones.

Now consider the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I}$ (relative to X^I). By construction, this action is compatible with the $\mathbb{G}_m$-actions on $\mathcal{L}U_I$ (as a group indscheme) and on $\mathrm{Gr}_{G,I}$ (as a plain indscheme). This implies we have the following compatibility

$$(\mathcal{L}U_I \times \mathbb{A}^1 \xrightarrow{a} \mathcal{L}U_I \underset{X^I}{\times} \mathcal{L}U_I \times \mathbb{A}^1) \curvearrowright (\widetilde{\mathrm{Gr}}^\gamma_{G,I} \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1).$$

Hence by Lemma 1.4.10(2), the $(\mathcal{L}U_I, a)$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$ stabilizes the schematic closed embedding

$$\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \hookrightarrow \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{G}_m,\ (x,t) \mapsto (x, t\cdot x, t). \tag{4.2}$$

Note that the restricted $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I}\times\mathbb{G}_m$ is the usual one.

We also have similar results on the $(\mathcal{L}U_I^-, r)$-action on $\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{A}^1$. Now (i) is implied by the following stronger result (and its mirror version).

Lemma 4.1.2. *We have:*

(1) Both the unipotent nearby cycles functor $\Psi^{\mathrm{un}}_{\gamma,I}$ and $i^*\circ j_*$ send the category

$$\mathrm{DMod}(\mathrm{Gr}_{G\times G,I}\times\mathbb{G}_m)^{\mathcal{L}U_I,a}\bigcap\mathrm{DMod}(\mathrm{Gr}_{G\times G,I}\times\mathbb{G}_m)^{\mathrm{good}}$$

into $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$.

(2) The full nearby cycles functor $\Psi_{\gamma,I}$ sends the category

$$\mathrm{DMod}(\mathrm{Gr}_{G\times G,I}\times\mathbb{G}_m)^{\mathcal{L}U_I,a}\bigcap\mathrm{DMod}_{\mathrm{hol}}(\mathrm{Gr}_{G\times G,I}\times\mathbb{G}_m)$$

into $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$.

Proof. Write $\mathcal{L}U_I$ as a filtered colimit $\mathcal{L}U_I\simeq\mathrm{colim}_\alpha\mathcal{N}_\alpha$ of its closed pro-unipotent group subschemes. We only need to prove the lemma after replacing $\mathcal{L}U_I$ by $\mathcal{N}_\alpha$ for any α (see § D.3). Then (1) follows from Proposition D.7.1.

To prove (2), we claim we can choose the above presentation $\mathcal{L}U_I\simeq\mathrm{colim}_\alpha\mathcal{N}_\alpha$ such that for each α, we can find a presentation $(\mathrm{Gr}_{G,I})_{\mathrm{red}}\simeq\mathrm{colim}Y_\beta$ such that each Y_β is a finite type closed subscheme of $(\mathrm{Gr}_{G,I})_{\mathrm{red}}$ stabilized by $\mathcal{N}_\alpha$. Indeed, similar to [Ras16, Remark 2.19.1], we can make each $\mathcal{N}_\alpha$ conjugate to $\mathcal{L}^+U_I$. Hence we only need to find a presentation $(\mathrm{Gr}_{G,I})_{\mathrm{red}}\simeq\mathrm{colim}Y_\beta$ such that each Y_β is stabilized by $\mathcal{L}^+U_I$. Then we can choose Y_β to be the Schubert cells of $(\mathrm{Gr}_{G,I})_{\mathrm{red}}$ (which are even stabilized by $\mathcal{L}^+G_I$). This proves the claim.

For any $\mathcal{N}_\alpha$ as above, since full nearby cycles functors commute with proper pushforward functors, it suffices to prove the claim after replacing $\mathrm{Gr}_{G,I}$ by Y_β (for any β). Then the $\mathcal{N}_\alpha$-action on Y_β factors through a smooth quotient group H. We can replace $\mathcal{N}_\alpha$ by H. Then we are done by using (D.16) and the fact that taking full nearby cycles commutes with smooth pullback functors.

□[Lemma 4.1.2]

4.1.3 (Proof of (ii))**.** Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{A}^1$ given by $s\cdot(x,y,t)=(x,s\cdot y,st)$. Note that the projection $\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{A}^1\to\mathbb{A}^1$ is $\mathbb{G}_m$-equivariant. Also note that the schematic closed embedding (4.2) is stabilized by this action. Hence by Lemma C.7.12, it suffices to prove that the object $\Psi_{\gamma,I}\in\mathrm{DMod}(\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I})$ is unipotently $\mathbb{G}_m$-monodromic, where $\mathbb{G}_m$ acts on the second factor.

By (i), we have $\Psi_{\gamma,I} \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-,2}$. Then we are done because

$$\mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-,2} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um},2}$$

by Lemma 2.1.7(1) (and Corollary 2.1.10(2)). This proves (ii).

4.1.4 (Proof of (iii))**.** Note that the Drinfeld-Gaitsgory interpolation $\widetilde{M}^\gamma \times \mathbb{A}^1 \to M \times M \times \mathbb{A}^1$ is isomorphic to the diagonal embedding $M \times \mathbb{A}^1 \to M \times M \times \mathbb{A}^1$. By an argument similar to that in § 4.1.1, we see the diagonal action of $\mathcal{L}^+ M_I$ on $\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m$ stabilizes the schematic closed embedding (4.2) and the restricted $\mathcal{L}^+ M_I$-action on $\mathrm{Gr}_{G,I} \times \mathbb{G}_m$ is the usual one.

Now let $\mathcal{C}$ be the full sub-category of $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$ generated by $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m})$ under colimits and extensions. By the previous discussion, $\mathcal{C}$ is a sub-$\mathcal{L}^+ M_I$-module of $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$. By Proposition D.7.1, we obtain a canonical $\mathcal{L}^+ M_I$-linear structure on the functor $\Psi^{\mathrm{un}}_{\gamma,I} : \mathcal{C} \to \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})$. Therefore $\Psi^{\mathrm{un}}_{\gamma,I}$ induces a functor between the $\mathcal{L}^+ M_I$-invariance categories. Then we are done because $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m})$ can be naturally upgraded to an object in $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)^{\mathcal{L}^+ M_I,\mathrm{diag}}$.

□[Proposition 3.1.2]

Remark 4.1.5. In the proof of Proposition 3.1.2(2), we actually showed

$$i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m}) \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}(U\times U^-)_I}.$$

We will use this result later.

4.2 An axiomatic framework for proving taking nearby cycles commutes with pull-push functors

The essence of our proofs of Proposition 3.1.5, Proposition 3.1.7, Proposition 3.1.10 and Theorem 3.2.2 is to use Braden's theorem and the contraction principle to show taking unipotent nearby cycles commutes with certain pull-push functors. In this subsection, we give an axiomatic framework for these arguments. To do this, we need some definitions that generalize those Appendix § E.

Definition 4.2.1. Let $\alpha' := (U' \leftarrow V' \to W')$ and $\alpha := (U \leftarrow V \to W)$ be two correspondences of lft prestacks.

A *2-morphism* $\mathfrak{s} : \alpha' \to \alpha$ between them is a commutative diagram

$$\begin{tikzcd} \alpha' \arrow[d, Rightarrow, "\mathfrak{s}"] & U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ \alpha & U & V \arrow[l, "f"] \arrow[r, "g"] & W. \end{tikzcd}$$

A 2-morphism $\mathfrak{s} : \alpha' \to \alpha$ is *right quasi-Cartesian* if the right square in the above diagram is quasi-Cartesian.

Construction 4.2.2. For a right quasi-Cartesian 2-morphism as in Definition 4.2.1, (E.5) induces a natural transformation

$$f_* \circ g^! \circ r_* \to f_* \circ q_* \circ (g')^! \simeq p_* \circ f'_* \circ (g')^!.$$

Passing to left adjoints, we obtain a natural transformation

$$\mathfrak{s}^* : p^* \circ f_* \circ g^! \to f'_* \circ (g')^! \circ r^*, \tag{4.3}$$

between functors $\mathrm{Pro}(\mathrm{DMod}(W)) \to \mathrm{Pro}(\mathrm{DMod}(U'))$, which we refer as the *$*$-transformation associated to* $\mathfrak{s}$.

Example 4.2.3. Let $(\mathcal{Y}, \mathcal{Y}^0, q, i)$ be a retraction (see Definition E.1.1). The natural transformation $q_* \to i^*$ in Construction E.1.3 is the $*$-transformation associated to the following 2-morphism between correspondences:

$$\begin{tikzcd} \mathcal{Y}^0 \arrow[d, "="] & \mathcal{Y}^0 \arrow[l, "="'] \arrow[r, "="] \arrow[d, "i"] & \mathcal{Y}^0 \arrow[d, "i"] \\ \mathcal{Y}^0 & \mathcal{Y} \arrow[l, "q"] \arrow[r, "="] & \mathcal{Y}. \end{tikzcd} \tag{4.4}$$

Definition 4.2.4. Let $\mathfrak{s}$ be a right quasi-Cartesian 2-morphism as above.

(1) We say $\mathfrak{s}$ is *pro-nice* for an object $\mathcal{F} \in \mathrm{Pro}(\mathrm{DMod}(W))$ if $\mathfrak{s}^*(\mathcal{F})$ is an isomorphism.

(2) Let $T : \mathrm{Pro}(\mathrm{DMod}(U')) \to \mathcal{C}$ be any functor. We say $\mathfrak{s}$ is *T-pro-nice* for $\mathcal{F}$ if $\mathbf{Id}_T \bigstar \mathfrak{s}^*(\mathcal{F})$ is an isomorphism (see Convention 3).

(3) We say $\mathfrak{s}$ is *nice* for $\mathcal{F}$ if it is pro-nice for $\mathcal{F}$ and $\mathfrak{s}^*(\mathcal{F})$ is a morphism in $\mathrm{DMod}(U')$.

Definition 4.2.5. Let $\alpha := (U \leftarrow V \to W)$ and $\beta := (W \leftarrow \mathcal{Y} \to \mathcal{Z})$ be two correspondences of prestacks. Their *composition* is defined to be $\alpha \circ \beta := (U \leftarrow V \times_W \mathcal{Y} \to \mathcal{Z})$.

The *horizontal* and *vertical compositions of 2-morphisms* between correspondences are defined in the obvious way.

The following two lemmas can be proved by diagram chasing. We leave the details to the reader.

Lemma 4.2.6. *Let α, α' and α'' be three correspondences of prestacks. Let $\mathfrak{t} : \alpha'' \to \alpha'$ and $\mathfrak{s} : \alpha' \to \alpha$ be two 2-morphisms. We depict them as*

$$\begin{tikzcd} \alpha'' \arrow[d, Rightarrow, "\mathfrak{t}"] & U'' \arrow[d, "l"] & V'' \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[d, "m"] & W'' \arrow[d, "n"] \\ \alpha' \arrow[d, Rightarrow, "\mathfrak{s}"] & U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ \alpha & U & V \arrow[l, "f"'] \arrow[r, "g"] & W. \end{tikzcd}$$

Suppose $\mathfrak{s}$ is right quasi-Cartesian. We have:

(1) $\mathfrak{s} \circ \mathfrak{t}$ is right quasi-Cartesian iff $\mathfrak{t}$ is right quasi-Cartesian.

(2) Suppose the conditions in (1) are satisfied, then there is a canonical equivalence

$$(\mathfrak{s} \circ \mathfrak{t})^* \simeq (\mathfrak{t}^* \bigstar \mathbf{Id}_{r^*}) \circ (\mathbf{Id}_{l^*} \bigstar \mathfrak{s}^*).$$

Lemma 4.2.7. *Let α, α', β and β' be four correspondences of prestacks such that $\alpha \circ \beta$ and $\alpha' \circ \beta'$ can be defined. Let $\mathfrak{s} : \alpha' \to \alpha$ and $\mathfrak{t} : \beta' \to \beta$ be two 2-morphisms. We depict them as*

$$\begin{tikzcd} \cdot & \cdot \arrow[l, bend right, "\alpha'"'] \arrow[l, bend left, "\alpha"] \arrow[l, phantom, "\Downarrow \mathfrak{s}"] & \cdot \arrow[l, bend right, "\beta'"'] \arrow[l, bend left, "\beta"] \arrow[l, phantom, "\Downarrow \mathfrak{t}"] \end{tikzcd} \qquad \begin{tikzcd} U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] & \mathcal{Y}' \arrow[l, "d'"'] \arrow[r, "e'"] \arrow[d, "m"] & \mathcal{Z}' \arrow[d, "n"] \\ U & V \arrow[l, "f"'] \arrow[r, "g"] & W & \mathcal{Y} \arrow[l, "d"'] \arrow[r, "e"] & \mathcal{Z}. \end{tikzcd}$$

Suppose $\mathfrak{s}$ and $\mathfrak{t}$ are both right quasi-Cartesian. We have

(1) $\mathfrak{s} \bigstar \mathfrak{t}$ is right quasi-Cartesian.

(2) There is a canonical equivalence

$$(\mathfrak{s} \bigstar \mathfrak{t})^* \simeq (\mathbf{Id}_{f'_* \circ (g')^!} \bigstar \mathfrak{t}^*) \circ (\mathfrak{s}^* \bigstar \mathbf{Id}_{d_* \circ e^!}).$$

4.2.8 (Axioms)**.** Suppose we are given the following data:

- A correspondence of prestacks $\alpha := (U \xleftarrow{f} V \xrightarrow{g} W)$ over $\mathbb{A}^1$.
- Objects $\mathring{\mathcal{F}} \in \mathrm{DMod}(\mathring{W})$ and $j_*(\mathring{\mathcal{F}}) \in \mathrm{DMod}(W)$.

- An extension of α to a correspondence between Braden 4-tuples

$$\alpha_{\mathrm{ext}} := (\alpha, \alpha^+, \alpha^-, \alpha^0) : (U, U^+, U^-, U^0) \leftarrow (V, V^+, V^-, V^0) \to (W, W^+, W^-, W^0),$$

  defined over the base Braden 4-tuple $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$ (see Example E.2.11).

- A full subcategory $\mathcal{C} \subset \mathrm{DMod}(U_0)$, where as usual $U_0 := U \times_{\mathbb{A}^1} 0$.

As usual, we use the following notations:

$$\mathring{\alpha} := (\mathring{U} \xleftarrow{\mathring{f}} \mathring{V} \xrightarrow{\mathring{g}} \mathring{W}),\ \alpha_0 := (U_0 \xleftarrow{f_0} V_0 \xrightarrow{g_0} W_0)$$

Note that when restricted to 0-fibers, we obtain a correspondence between Braden 4-tuples:

$$(U_0, U_0^+, U_0^-, U_0^0) \leftarrow (V_0, V_0^+, V_0^-, V_0^0) \to (W_0, W_0^+, W_0^-, W_0^0).$$

Suppose the above data satisfy the following axioms[39] (we strongly suggest the reader to skip these axioms and proceed to § 4.2.10):

(P1) The map $V^0 \to U^0 \times_{U^+} V^+$ is a nil-isomorphism.

(P2) The map $V^- \to U^- \times_U V$ is a nil-isomorphism.

(P3) The map $V^- \to W^- \times_{W^0} V^0$ is a nil-isomorphism.

(Q) The map $V^+ \to W^+ \times_W V$ is nil-isomorphic to a schematic open embedding.

(G1) The object $\mathring{\mathcal{F}}$ is contained in $\mathrm{DMod}(\mathring{W})^{\mathrm{good}}$ (see Notation C.7.4).

(G2) The object $(\mathring{f})_* \circ (\mathring{g})^!(\mathring{\mathcal{F}})$ is contained in $\mathrm{DMod}(\mathring{U})^{\mathrm{good}}$.

(C) The following composition is conservative[40]:

$$\mathcal{C} \hookrightarrow \mathrm{DMod}(U_0) \xrightarrow{p_{U_0}^{+,*}} \mathrm{Pro}(\mathrm{DMod}(U_0^+)) \xrightarrow{i_{U_0}^{+,!}} \mathrm{Pro}(\mathrm{DMod}(U_0^0)).$$

(M) The $*$-transformation $\mathfrak{z}^*$ of $\mathfrak{z} : \alpha_0 \to \alpha$ sends $\mathcal{F}$ to a morphism contained in $\mathcal{C} \subset \mathrm{Pro}(\mathrm{DMod}(U_0))$.

(N1) The Braden 4-tuple (W, W^+, W^-, W^0) is $*$-nice for $\mathcal{F}$.

[39](P) for *pullback*; (Q) for *quasi-Cartesian*; (C) for *conservative*; (G) for *good*; (M) for *morphism*; (N) for *nice*.

[40]For instance, this condition is satisfied if $U_0^+ \to U_0$ is a *finite* stratification and $\mathcal{C}$ is the full subcategory of D-modules that are constant along each stratum.

(N2) The Braden 4-tuple (U, U^+, U^-, U^0) is $*$-nice for $f_* \circ g^!(\mathcal{F})$.

(N3) The Braden 4-tuple $(W_0, W_0^+, W_0^-, W_0^0)$ is $*$-nice for $i^*(\mathcal{F})$.

(N4) The Braden 4-tuple $(U_0, U_0^+, U_0^-, U_0^0)$ is $*$-nice for $f_{0,*} \circ g_0^! \circ i^*(\mathcal{F})$.

Then we claim that taking the unipotent nearby cycles for $\mathring{\mathcal{F}}$ commutes with !-pull-$*$-push along the correspondence α. More precisely, we have

Theorem 4.2.9. *In the above setting, there are canonical isomorphisms*

$$i^* \circ f_* \circ g^! \circ j_*(\mathring{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ i^* \circ j_*(\mathring{\mathcal{F}}), \tag{4.5}$$

$$\Psi^{\mathrm{un}} \circ (\mathring{f})_* \circ (\mathring{g})^!(\mathring{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ \Psi^{\mathrm{un}}(\mathring{\mathcal{F}}). \tag{4.6}$$

Proof. The essence of this proof is diagram chasing on a 4-cube, which we cannot draw on a paper.

By Axioms (G1) and (G2), both sides of (4.5) and (4.6) are well-defined. By (C.21), it suffices to prove the equivalence (4.5). Hence it suffices to show the morphism $\mathfrak{z}^*(\mathcal{F})$ is an isomorphism, i.e., the 2-morphism $\mathfrak{z} : \alpha_0 \to \alpha$ is nice for $\mathcal{F}$.

By Axioms (C) and (M), it suffices to prove that $\mathfrak{z}$ is $(i_{U_0}^{+,!} \circ p_{U_0}^{+,*})$-pro-nice for $\mathcal{F}$.

By Axiom (Q), the 2-morphism $\mathfrak{p}^+ : \alpha^+ \to \alpha$ is right quasi-Cartesian. Hence so is its 0-fiber $\mathfrak{p}_0^+ : \alpha_0^+ \to \alpha_0$. Consider the commutative diagram

$$\begin{tikzcd} \alpha_0^+ \arrow[r, "\mathfrak{p}_0^+"] \arrow[d, "\mathfrak{z}^+"] & \alpha_0 \arrow[d, "\mathfrak{z}"] \\ \alpha^+ \arrow[r, "\mathfrak{p}^+"] & \alpha. \end{tikzcd}$$

By Lemma 4.2.6, it suffices to prove

(1) $\mathfrak{p}_0^+$ is pro-nice for $i^*(\mathcal{F})$;

(2) $\mathfrak{z} \circ \mathfrak{p}_0^+$ is pro-nice for $\mathcal{F}$.

Note that we have $\mathfrak{z} \circ \mathfrak{p}_0^+ \simeq \mathfrak{p}^+ \circ \mathfrak{z}^+$. Also note that $\mathfrak{z}^+ : \alpha_0^+ \to \alpha^+$ is an isomorphism (because our Braden 4-tuples are defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$). Using Lemma 4.2.6 again, we see that (2) can be replaced by

(2') $\mathfrak{p}^+$ is pro-nice for $\mathcal{F}$.

It remains to prove (1) and (2'). We will use Axioms (P1)-(P3) and (N1)-(N2) to prove (2'). One can obtain (1) similarly[41] from Axioms (P1)-(P3) and (N3)-(N4).

[41] Note that the 0-fiber versions of Axioms (P1)-(P3) are implied by themselves.

Consider 2-morphisms $\mathfrak{u}$, $\mathfrak{p}^+$ and $\mathfrak{u}\bigstar\mathfrak{p}^+$ depicted as

$$\begin{tikzcd}
U^0 \arrow[d, "i_U^-"] & U^0 \arrow[l, "="'] \arrow[r, "i_U^+"] \arrow[d, "i_U^-"] & U^+ \arrow[d, "p_U^+"] & V^+ \arrow[l, "f^+"'] \arrow[r, "g^+"] \arrow[d, "p_V^+"] & W^+ \arrow[d, "p_W^+"] \\
U^- & U^- \arrow[l, "="'] \arrow[r, "p_U^-"] & U & V \arrow[l, "f"'] \arrow[r, "g"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
U^0 \arrow[d, "i_U^-"] & U^0\times_{U^+}V^+ \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "g^+\circ\mathrm{pr}_2"] \arrow[d, "(i_U^-{,}p_V^+)"] & W^+ \arrow[d, "p_W^+"] \\
U^- & U^-\times_U V \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "g\circ\mathrm{pr}_2"] & W.
\end{tikzcd}$$

$\mathfrak{u}$ $\qquad\qquad$ $\mathfrak{p}^+$ $\qquad\qquad$ $\mathfrak{u}\bigstar\mathfrak{p}^+$

By Lemma 4.2.7, it suffices to prove

(i) $\mathfrak{u}$ is pro-nice for $f_*\circ g^!(\mathcal{F})$;

(ii) $\mathfrak{u}\bigstar\mathfrak{p}^+$ is pro-nice for $\mathcal{F}$.

Note that (i) is implied by (the quasi-Cartesian part of) Axiom (N2). It remains to prove (ii). Consider 2-morphisms $\mathfrak{i}^-$, $\mathfrak{w}$ and $\mathfrak{i}^-\bigstar\mathfrak{w}$ depicted as

$$\begin{tikzcd}
U^0 \arrow[d, "i_U^-"] & V^0 \arrow[l, "f^0"'] \arrow[r, "g^0"] \arrow[d, "i_V^-"] & W^0 \arrow[d, "i_W^-"] & W^0 \arrow[l, "="'] \arrow[r, "i_W^+"] \arrow[d, "i_W^-"] & W^+ \arrow[d, "p_W^+"] \\
U^- & V^- \arrow[l, "f^-"'] \arrow[r, "g^-"] & W^- & W^- \arrow[l, "="'] \arrow[r, "p_W^-"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
U^0 \arrow[d, "i_U^-"] & V^0 \arrow[l, "f^0"'] \arrow[r, "i_W^+\circ g^0"] \arrow[d, "i_V^-"] & W^+ \arrow[d, "p_W^+"] \\
U^- & V^- \arrow[l, "f^-"'] \arrow[r, "g^-"] & W.
\end{tikzcd}$$

$\mathfrak{i}^-$ $\qquad\qquad$ $\mathfrak{w}$ $\qquad\qquad$ $\mathfrak{i}^-\bigstar\mathfrak{w}$

By Axioms (P1) and (P2), $\mathfrak{i}^-\bigstar\mathfrak{w}$ is nil-isomorphic to $\mathfrak{u}\bigstar\mathfrak{p}^+$. By Lemma 4.2.7 again, it suffices to prove

(a) $\mathfrak{w}$ is pro-nice for $\mathcal{F}$;

(b) $\mathfrak{i}^-$ is pro-nice for $p_W^{-,!}(\mathcal{F})$.

Note that (a) is implied by (the quasi-Cartesian part of) (N1). It remains to prove (b). Consider the 2-morphism (4.4) associated to the retraction (U^-, U^0). We denote it by $\mathfrak{c}_U$. Similarly we define $\mathfrak{c}_W$. By Axiom (P3), $\mathfrak{c}_U\bigstar\mathfrak{i}^-$ is nil-isomorphic[42] to $\mathbf{Id}_{\alpha^0}\bigstar\mathfrak{c}_W$. Using Lemma 4.2.7 again, we reduce (b) to (the retraction part of) Axioms (N1) and (N2) (because of Example 4.2.3).

□[Theorem 4.2.9]

4.2.10 (A special case). Suppose we are given the following data:

- A $\mathbb{G}_m$-action on $\mathbb{A}^1$ given by $s\cdot t := s^{-n}t$, where n is a negative integer;
- Three ind-finite type indschemes U, V and W acted by $\mathbb{G}_m$ as in Assumption E.0.1;

[42]We ask the reader to pardon us for not drawing these compositions.

- A correspondence $\alpha := (U \xleftarrow{f} V \xrightarrow{g} W)$ over $\mathbb{A}^1$ compatible with the $\mathbb{G}_m$-actions;
- An object $\overset{\circ}{\mathcal{F}} \in \mathrm{DMod}(\overset{\circ}{W})^{\mathbb{G}_m\text{-um}}$;
- A full subcategory $\mathcal{C} \subset \mathrm{DMod}(U_0)$.

By constuction, we can extend α to a correspondence between Braden 4-tuples:

$$\alpha_{\mathrm{ext}} : (U, U^{\mathrm{fix}}, U^{\mathrm{rep}}, U^{\mathrm{fix}}) \leftarrow (V, V^{\mathrm{att}}, V^{\mathrm{rep}}, V^{\mathrm{fix}}) \to (W, W^{\mathrm{fix}}, W^{\mathrm{rep}}, W^{\mathrm{fix}})$$

defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$ (because n is negative).

Corollary 4.2.11. *Suppose the above data satisfy Axioms (P1)-(P3), (Q), (G1), (G2), (C) and (M), then there are canonical isomorphisms*

$$i^* \circ f_* \circ g^! \circ j_*(\overset{\circ}{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ i^* \circ j_*(\overset{\circ}{\mathcal{F}}),$$
$$\Psi^{\mathrm{un}} \circ (\overset{\circ}{f})_* \circ (\overset{\circ}{g})^!(\overset{\circ}{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ \Psi^{\mathrm{un}}(\overset{\circ}{\mathcal{F}}).$$

Proof. Axioms (N1)-(N4) are automatic because of Theorem E.2.16. Then we are done by Theorem 4.2.9.

□[Corollary 4.2.11]

4.3 More on the geometric players

In this section, we study a certian $\mathbb{G}_m$-action on $\mathrm{VinGr}^{\gamma}_{G,I}$, which will allow us to use § 4.2 to prove our theorems.

Proposition 4.3.1. *Consider the canonical action $T_{\mathrm{ad}} \curvearrowright \mathrm{Gr}_{G,I}$ induced by the adjoint action $T_{\mathrm{ad}} \curvearrowright G$. Then the action*

$$(T_{\mathrm{ad}} \times T_{\mathrm{ad}}) \times (\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times T^+_{\mathrm{ad}}) \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times T^+_{\mathrm{ad}},\ (s_1, s_2) \cdot (x, y, t) := (s_1^{-1} \cdot x, s_2^{-1} \cdot y, s_1 t s_2^{-1}).$$

stabilizes both $\mathrm{VinGr}_{G,I}$ *and* ${}_0\mathrm{VinGr}_{G,I}$.

Remark 4.3.2. The claim is obvious when restricted to $T_{\mathrm{ad}} \subset T^+_{\mathrm{ad}}$.

4.3.3 (A general paradigm)**.** Proposition 4.3.1 can be proved using the Tannakian description of VinGr_G in [FKM20, § 3.1.2]. However, we prefer to prove it in an abstract way. The construction below is a refinement of that in [Wan18, Appendix C.3].

Consider the following paradigm. Let $1 \to K \to H \to Q \to 1$ be an exact sequence of affine algebraic groups. Let $Z \to B$ be a map between finite type affine schemes. Suppose we have an H-action on Z and a Q-action on B compatible in the obvious sense. Then we have a canonical Q-equivariant map $p : K\backslash Z \to B$.

Suppose we are further given a section $B \hookrightarrow Z$ to the map $Z \to B$. Then we obtain a map $f : B \to Z \to K\backslash Z$ such that $p \circ f = \mathrm{Id}_B$.

Suppose we are futher given a splitting $s : Q \hookrightarrow H$ compatible with the actions $Q \curvearrowright B$, $H \curvearrowright Z$ and the section $B \to Z$. Consider the restricted Q-action on Z. By assumption, the map $B \to Z$ is Q-equivariant. On the other hand, there is a canonical Q-equivariant structure on $Z \to K\backslash Z$ because of the splitting $s : Q \hookrightarrow H$. Hence we obtain a canonical Q-equivariant structure on $f : B \to K\backslash Z$.

Combining the above paragraphs, we obtain a Q-action on the retraction $(K\backslash Z, B, p, f)$. This construction is functorial in $B \hookrightarrow Z \to B$ in the obvious sense.

In the special case when $Z = B$ and K acts trivially on B, we obtain a Q-action on the chain $B \to K\backslash \mathrm{pt} \times B \to B$. More or less by definition, this action is also induced by the given Q-action on B and the adjoint action $Q \curvearrowright K$ provided by the section s.

Applying Construction 1.4.1 to these retractions, (using Lemma A.3.1) we obtain Q-actions on $\mathbf{Maps}_{I,/B}(X, K\backslash Z \leftarrow B)$ and $\mathbf{Maps}_{I,/B}(X, K\backslash \mathrm{pt}{\times}B \leftarrow B)$. Moreover, the map $(B \hookrightarrow Z \to B) \to (B \simeq B \simeq B)$ induces a Q-equivariant map

$$\mathbf{Maps}_{I,/B}(X, K\backslash Z \leftarrow B) \to \mathbf{Maps}_{I,/B}(X, K\backslash \mathrm{pt} \times B \leftarrow B).$$

4.3.4 (Proof of Proposition 4.3.1)**.** Let us come back to the problem. Recall we have the following exact sequence of algebraic groups $1 \to G \to G_{\mathrm{enh}} \to T_{\mathrm{ad}} \to 1$, where $G_{\mathrm{enh}} := (G \times T)/Z_G$ is the group of invertible elements in Vin_G. Also recall we have a canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$ whose restriction to T_{ad} is $T/Z_G \to (G \times T)/Z_G$, $t \mapsto (t^{-1}, t)$. Note that the corresponding T_{ad}-action on G provided by $\mathfrak{s}$ is the *inverse* of the usual adjoint action. Now applying the above paradigm to

$$(1 \to K \to H \to Q \to 1) := (1 \to G \times G \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}} \to 1)$$

$$(B \to Z \to B) := (T_{\mathrm{ad}}^+ \xrightarrow{\mathfrak{s}} \mathrm{Vin}_G \to T_{\mathrm{ad}}^+)$$

we obtain a $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-equivariant structure on the canonical map $\mathrm{VinGr}_{G,I} \to \mathrm{Gr}_{G\times G,I} \times T_{\mathrm{ad}}^+$, where $Q = (T_{\mathrm{ad}} \times T_{\mathrm{ad}})$ acts on the RHS via the usual action on $B = T_{\mathrm{ad}}^+$ and the *inverse* of the usual action on $\mathrm{Gr}_{K,I} = \mathrm{Gr}_{G\times G,I}$. This is exactly the action described in the problem. This proves the claim for $\mathrm{VinGr}_{G,I}$.

Replacing Z by ${}_0\mathrm{Vin}_G$, we obtain the claim for ${}_0\mathrm{VinGr}_{G,I}$.

$\square$[Proposition 4.3.1]

Corollary 4.3.5. *Let* $\mathbb{G}_m \curvearrowright \mathrm{Gr}_{G,I}$ *be the action in Proposition 1.2.8. Then the action*

$$\mathbb{G}_m \times (\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1) \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,\ s \cdot (x, y, t) := (s \cdot x, s^{-1} \cdot y, s^{-2} t) \tag{4.7}$$

preserves both $\mathrm{VinGr}^{\gamma}_{G,I}$ *and* ${}_0\mathrm{VinGr}^{\gamma}_{G,I}$.

Construction 4.3.6. Consider the above action $\mathbb{G}_m \curvearrowright (\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1)$. Using Proposition 1.2.8, it is easy to see the Braden 4-tuple for the action (4.7) is

$$\mathrm{Br}^{\gamma}_I := (\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1, \mathrm{Gr}_{P\times P^-,I} \times 0, \mathrm{Gr}_{P^-\times P,I} \times \mathbb{A}^1, \mathrm{Gr}_{M\times M,I} \times 0).$$

Hence by [DG14, Lemma 1.4.9(ii)], the attractor (resp. repeller, fixed) locus for the action on $\mathrm{VinGr}^{\gamma}_{G,I}$ is given by

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1)}{\times} (\mathrm{Gr}_{P\times P^-,I} \times 0), \tag{4.8}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1)}{\times} (\mathrm{Gr}_{P^-\times P,I} \times \mathbb{A}^1), \tag{4.9}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1)}{\times} (\mathrm{Gr}_{M\times M,I} \times 0). \tag{4.10}$$

We denote the corresponding Braden 4-tuple by

$$\mathrm{Br}^{\gamma}_{\mathrm{Vin},I} := (\mathrm{VinGr}^{\gamma}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}).$$

4.3.7 (An alternate description)**.** The reader is advised to skip the rest of this subsection and return when necessary.

The formulae in Construction 4.3.6 are not satisfactory because for example they do not describe[43] the canonical map $\mathbf{q}^{+}_{\mathrm{Vin},I} : \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \to \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}$. In this sub-subsection, we use mapping stacks to give an alternative description of the Braden 4-tuple $\mathrm{Br}^{\gamma}_{\mathrm{Vin},I}$. Once we have this alternative description, we exhibit how to use them to study the geometry of $\mathrm{VinGr}_{G,I}$ in the rest of this subsection.

We assume the reader is familiar with the constructions in § A.5.1-A.5.2 and § A.5.5.

[43] Of course, the map $\mathbf{q}^{+}_{\mathrm{Vin},I}$ is the unique one that is compatible with the map $\mathrm{Gr}_{P\times P^-,I} \to \mathrm{Gr}_{M\times M,I}$. But this description is not convenient in practice.

By Lemma A.3.6, we can rewrite (4.8)-(4.10) as

$$\begin{aligned}
\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{att}} &\simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, P\backslash \mathrm{Vin}_G|_{C_P}/P^- \leftarrow \mathrm{pt}), && (4.11)\\
\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} &\simeq \mathbf{Maps}_{I,/\mathbb{A}^1}(X, P^-\backslash \mathrm{Vin}_G^{\gamma}/P \leftarrow \mathbb{A}^1), && (4.12)\\
\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} &\simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, M\backslash \mathrm{Vin}_G|_{C_P}/M \leftarrow \mathrm{pt}), && (4.13)
\end{aligned}$$

where the sections are all induced by the canonical section $\mathfrak{s}: T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$. For example, the map $\mathrm{pt} \to P\backslash \mathrm{Vin}_G|_{C_P}/P^-$ is given by the composition

$$\mathrm{pt} \xrightarrow{\mathfrak{s}|_{C_P}} \mathrm{Vin}_G|_{C_P} \to P\backslash \mathrm{Vin}_G|_{C_P}/P^-.$$

Recall we have a $(P \times P^-)$-equivariant closed embedding $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$ (see § A.5.1). By definition, the canonical section $\mathfrak{s}|_{C_P}: \mathrm{pt} \to \mathrm{Vin}_G|_{C_P}$ factors through this embedding. Hence the map $\mathrm{pt} \to P\backslash \mathrm{Vin}_G|_{C_P}/P^-$ factors as

$$\mathrm{pt} \to P\backslash\overline{M}/P^- \hookrightarrow P\backslash \mathrm{Vin}_G|_{C_P}/P^-,$$

where the last map is a schematic closed embedding. By Lemma A.3.2 and (4.11), we obtain a canonical isomorphism:

$$\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{att}} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, P\backslash\overline{M}/P^- \leftarrow \mathrm{pt}). \tag{4.14}$$

Similarly we have a canonical isomorphism

$$\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, M\backslash\overline{M}/M \leftarrow \mathrm{pt}). \tag{4.15}$$

Under these descriptions, we claim the commutative diagram

$$\begin{tikzcd}
 & & \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} \\
 & \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} \arrow[ur, "="] \arrow[r, "\mathbf{i}_{\mathrm{Vin},I}^{+}"'] \arrow[d, "\mathbf{i}_{\mathrm{Vin},I}^{-}"'] \arrow[dl, "="'] & \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{att}} \arrow[u, "\mathbf{q}_{\mathrm{Vin},I}^{+}"'] \arrow[d, "\mathbf{p}_{\mathrm{Vin},I}^{+}"] \\
\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} & \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \arrow[l, "\mathbf{q}_{\mathrm{Vin},I}^{-}"] \arrow[r, "\mathbf{p}_{\mathrm{Vin},I}^{-}"'] & \mathrm{VinGr}_{G,I}^{\gamma}
\end{tikzcd} \tag{4.16}$$

is induced by a commutative diagram

$$\begin{tikzcd}
 & & (M\backslash\overline{M}/M \leftarrow \mathrm{pt}) \\
 & (M\backslash\overline{M}/M \leftarrow \mathrm{pt}) \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{sect}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{sect}}"'] & (P\backslash\overline{M}/P^- \leftarrow \mathrm{pt}) \arrow[u, "\mathbf{q}^+_{\mathrm{sect}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{sect}}"] \\
(M\backslash\overline{M}/M \leftarrow \mathrm{pt}) & (P^-\backslash \mathrm{Vin}^\gamma_G/P \leftarrow \mathbb{A}^1) \arrow[l, "\mathbf{q}^-_{\mathrm{sect}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{sect}}"'] & (G\backslash \mathrm{Vin}^\gamma_G/G \leftarrow \mathbb{A}^1),
\end{tikzcd} \tag{4.17}$$

where the only non-obvious morphism is $\mathbf{q}^-_{\mathrm{sect}}$, which is induced by the commutative diagram (A.20). Indeed, (4.16) is induced by (4.17) because the maps in (4.16) are uniquely determined by their compatibilities with the maps in the Braden 4-tuple

$$\mathrm{Br}^\gamma_I \coloneqq (\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1, \mathrm{Gr}_{P\times P^-,I}\times 0, \mathrm{Gr}_{P^-\times P,I}\times\mathbb{A}^1, \mathrm{Gr}_{M\times M,I}\times 0).$$

4.3.8 (Stratification on $\mathrm{VinGr}_{G,I}|_{C_P}$)**.** By Corollary A.4.11(1), the map

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathrm{VinGr}_{G,I}|_{C_P} \underset{\mathrm{Gr}_{G\times G,I}}{\times} \mathrm{Gr}_{P\times P^-,I} \to \mathrm{VinGr}_{G,I}|_{C_P}$$

is bijective on field-valued points. Hence the connected components of $\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I}$ provide a stratification on $\mathrm{VinGr}_{G,I}|_{C_P}$. On the other hand, [Sch16] defined a *defect stratification* on $\mathrm{VinBun}_G|_{C_P}$ (see § A.5.4 for a quick review). Let ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ be the disjoint union of all the defect strata. The following result says these two stratifications are compatible via the local-to-global-map.

Proposition 4.3.9. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Gr}_{P\times P^-,I} \arrow[d] & \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \arrow[l] \arrow[r] \arrow[d] & \mathrm{VinGr}_{G,I}|_{C_P} \arrow[d] \\
\mathrm{Bun}_{P\times P^-} & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[l] \arrow[r] & \mathrm{VinBun}_G|_{C_P}
\end{tikzcd}$$

such that its right square is Cartesian.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
(P\backslash \mathrm{pt}/P^- \leftarrow \mathrm{pt}) \arrow[d] & (P\backslash\overline{M}/P^- \leftarrow \mathrm{pt}) \arrow[l] \arrow[r] \arrow[d] & (G\backslash \mathrm{Vin}_G|_{C_P}/G \leftarrow \mathrm{pt}) \arrow[d] \\
(P\backslash \mathrm{pt}/P^- \supset P\backslash \mathrm{pt}/P^-) & (P\backslash\overline{M}/P^- \supset P\backslash M/P^-) \arrow[l] \arrow[r] & (G\backslash \mathrm{Vin}_G|_{C_P}/G \supset G\backslash{}_0\mathrm{Vin}_G|_{C_P}/G).
\end{tikzcd}$$

By Construction 1.4.2, we obtain the desired commutative diagram in the problem. It remains to show its right square is Cartesian. By Lemma A.3.7, it suffices to show the canonical map

$$\mathrm{pt} \to \mathrm{pt} \underset{(G\backslash \mathrm{Vin}_G|_{C_P}/G)}{\times} (P\backslash \overline{M}/P^-)$$

is an isomorphism. Using the Cartesian diagram (A.11), the RHS is isomorphic to

$$\mathrm{pt} \underset{(G\backslash {}_0\mathrm{Vin}_G|_{C_P}/G)}{\times} (P\backslash M/P^-).$$

Then we are done because ${}_0\mathrm{Vin}_G|_{C_P} \simeq (G\times G)/(P\times_M P^-)$.

□[Proposition 4.3.9]

Corollary 4.3.10. *Let $\lambda,\mu \in \Lambda_{G,P}$ be two elements. Then the fiber product (see Notation A.4.1)*

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \underset{\mathrm{Gr}_{P\times P^-,I}}{\times} (\mathrm{Gr}^{\lambda}_{P,I}\times \mathrm{Gr}^{\mu}_{P,I})$$

is empty unless $\lambda \le \mu$.

Proof. Using Proposition 4.3.9, it suffices to show the fiber product

$${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \underset{\mathrm{Bun}_{P\times P^-}}{\times} (\mathrm{Bun}_P^{-\lambda}\times \mathrm{Bun}_{P^-}^{-\mu})$$

is empty unless $\lambda\le\mu$. Then we are done by (A.17) and (A.15).

□[Corollary 4.3.10]

Corollary 4.3.11. *(c.f. [FKM20, Lemma 3.13]) Let ${}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I}$ be the closed sub-indscheme of $\mathrm{Gr}_{G\times G,I}$ defined in Corollary A.4.11. Then $(\mathrm{VinGr}_{G,I}|_{C_P})_{\mathrm{red}}$ is contained in ${}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I}$.*

Proof. Note that $(\mathrm{VinGr}_{G,I}|_{C_P})_{\mathrm{red}}$ is also a closed sub-indscheme of $\mathrm{Gr}_{G\times G,I}$. Hence it suffices to show the set of field values points of $\mathrm{VinGr}_{G,I}|_{C_P}$ is a subset of that of ${}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I}$. Then we are done by Corollary 4.3.10.

□[Corollary 4.3.11]

Proposition 4.3.12. *The following commutative square is Cartesian:*

$$\begin{tikzcd} \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \arrow[r] \arrow[d] & \mathrm{Gr}_{P\times P^-,I} \arrow[d] \\ \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \arrow[r] & \mathrm{Gr}_{M\times M,I}. \end{tikzcd}$$

Proof. Follows from Lemma A.3.6 and the Cartesian square

$$\begin{tikzcd} P\backslash\overline{M}/P^- \arrow[r] \arrow[d] & P\backslash \mathrm{pt}/P^- \arrow[d] \\ M\backslash\overline{M}/M \arrow[r] & M\backslash \mathrm{pt}/M. \end{tikzcd}$$

□[Proposition 4.3.12]

Remark 4.3.13. One can use Proposition 4.3.12 to prove the claim in Remark 1.4.12.

4.3.14 (Defect-free version)**.** By Proposition 4.3.1, the $\mathbb{G}_m$-action (4.7) also stabilizes ${}_0\mathrm{VinGr}^{\gamma}_{G,I} \simeq \mathrm{Gr}_{\widetilde{G}^{\gamma},I}$. Let $\mathrm{Br}^{\gamma}_{{}_0\mathrm{Vin},I}$ be the Braden 4-tuple for this restricted action.

On the other hand, there is a canonical Braden 4-tuple

$$(\mathrm{Gr}_{\widetilde{G}^{\gamma},I}, \mathrm{Gr}_{P\times_M P^-,I} \times 0, \mathrm{Gr}_{M,I} \times \mathbb{A}^1, \mathrm{Gr}_{M,I} \times 0),$$

where the only non obvious map $p^- : \mathrm{Gr}_{M,I} \times \mathbb{A}^1 \to \mathrm{Gr}_{\widetilde{G}^{\gamma},I}$ is given by the composition

$$\mathrm{Gr}_{M,I} \times \mathbb{A}^1 \simeq \mathrm{Gr}_{\widetilde{M}^{\gamma},I} \to \mathrm{Gr}_{\widetilde{G}^{\gamma},I}.$$

Proposition 4.3.15. *There is a canonical isomorphism between Braden* 4*-tuples*

$$\mathrm{Br}^{\gamma}_{{}_0\mathrm{Vin},I} \simeq (\mathrm{Gr}_{\widetilde{G}^{\gamma},I}, \mathrm{Gr}_{P\times_M P^-,I} \times 0, \mathrm{Gr}_{M,I} \times \mathbb{A}^1, \mathrm{Gr}_{M,I} \times 0).$$

Proof. The statements concerning the attractor and fixed loci follow directly from Proposition 4.3.1 because the $\mathbb{G}_m$-action on ${}_0\mathrm{VinGr}_{G,I}|_{C_P} \simeq \mathrm{Gr}_{P\times_M P^-,I}$ is contractive.

Let us calculate the repeller locus. By [DG14, Lemma 1.4.9(i)], the canonical map

$${}_0\mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \to \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \underset{\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}}{\times} {}_0\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}$$

is an isomorphism. On the other hand, we have a Cartesian square (see (A.20))

$$\begin{tikzcd} (P^-\backslash \mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G/P \leftarrow \mathbb{A}^1) \arrow[r] \arrow[d] & (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \leftarrow \mathbb{A}^1) \arrow[d, "\mathbf{q}^-_{\mathrm{sect}}"] \\ (M\backslash M/M \leftarrow \mathrm{pt}) \arrow[r] & (M\backslash\overline{M}/M \leftarrow \mathrm{pt}). \end{tikzcd}$$

Note that $P^{-}\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P \simeq M\backslash M/M \times \mathbb{A}^1$ by (A.19). Hence by Lemma A.3.6, we have an isomorphism

$$\mathrm{Gr}_{M,I} \times \mathbb{A}^1 \simeq \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \underset{\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}}{\times} {}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}.$$

This provides the desired isomorphism ${}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \simeq \mathrm{Gr}_{M,I} \times \mathbb{A}^1$. It follows from construction that this isomorphism is compatible with the natural maps in the Braden 4-tuples.

□[Proposition 4.3.15]

4.4 Two auxiliary results

In this section, we prove two results which play key technical roles in our proofs. Namely, they serve respectively as Axioms (C) and (M) in § 4.2.8.

For $\lambda \in \Lambda_{G,P}$, let ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ be the closed sub-indscheme of $\mathrm{Gr}_{G,I}$ defined in Proposition A.4.2. As explained in § F.1.3, the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I}$ preserves ${}_{\le\lambda}\mathrm{Gr}_{G,I}$. Hence we have a fully faithful functor

$$\mathrm{DMod}({}_{\le\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}.$$

Similarly, for $\delta \in \Lambda_{G,P}$, the closed subscheme ${}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I}$ of $\mathrm{Gr}_{G\times G,I}$ defined in Corollary A.4.11 is preserved by the $\mathcal{L}(U\times U^{-})_I$-action on $\mathrm{Gr}_{G\times G,I}$. Hence we have a fully faithful functor

$$\mathrm{DMod}({}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^{-})_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^{-})_I}.$$

Lemma 4.4.1. *We have:*

(1) For $\lambda \in \Lambda_{G,P}$, the following composition is conservative

$$\mathrm{DMod}({}_{\le\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{p}_I^{+,*}} \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{P,I})) \xrightarrow{\mathbf{i}_I^{+,!}} \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{M,I})). \tag{4.18}$$

(2) For $\delta \in \Lambda_{G,P}$, the following composition is conservative

$$\mathrm{DMod}({}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^{-})_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^{-})_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G\times G,I}) \to$$
$$\xrightarrow{*\text{-pullback}} \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{P\times P^{-},I})) \xrightarrow{!\text{-pullback}} \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{M\times M,I})).$$

Warning 4.4.2. We warn that (1) would be *false* if one replaces ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ by the entire $\mathrm{Gr}_{G,I}$. For example, the dualizing D-module $\omega_{\mathrm{Gr}_{G,I}}$ is sent to zero by that composition.

Proof. We will prove (1). The proof for (2) is similar.

Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$ in Proposition 1.2.8. By Lemma 2.1.7(1), Braden's theorem and the contraction principle, the composition (4.18) is isomorphic to

$$\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{p}_I^{-,!}} \mathrm{DMod}(\mathrm{Gr}_{P^-,I}) \xrightarrow{\mathbf{q}_{I,*}^-} \mathrm{DMod}(\mathrm{Gr}_{M,I}) \hookrightarrow \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{M,I})).$$

Hence by Lemma 2.1.7(3), it is also isomorphic to

$$\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{p}_I^{+,*,\mathrm{inv}}} \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \simeq \mathrm{DMod}(\mathrm{Gr}_{M,I}) \hookrightarrow \mathrm{Pro}(\mathrm{DMod}(\mathrm{Gr}_{M,I})).$$

Then we are done by Lemma F.2.2.

□[Lemma 4.4.1]

Lemma 4.4.3. *The object* $i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \in \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})$ *is contained in*

$$\mathrm{DMod}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{DMod}(\mathrm{Gr}_{G\times G,I}).$$

Proof. By Remark 4.1.5, $i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$ is contained in $\mathrm{DMod}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$. It remains to show it is also contained in $\mathrm{DMod}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,I}) \subset \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})$. By Lemma 1.4.10, the support of this object is contained in $\mathrm{VinGr}_{G,I}|_{C_P} \hookrightarrow \mathrm{Gr}_{G\times G,I}$. Hence we are done by Corollary 4.3.11.

□[Lemma 4.4.3]

4.5 Proof of Proposition 3.1.5

By Proposition 3.1.2(1) and the fact that taking (unipotent) nearby cycles commutes with open restrictions, we have ${}_0\Psi^{\mathrm{un}}_{\gamma,I,\mathrm{Vin}} \simeq {}_0\Psi_{\gamma,I,\mathrm{Vin}}$. Hence it is enough to prove the claim for the unipotent nearby cycles.

We equip ${}_0\mathrm{VinGr}^{\gamma}_{G,I}$ with the $\mathbb{G}_m$-action in § 4.3.14. We also equip $\mathbb{A}^1$ with the $\mathbb{G}_m$-action given by $s\cdot t := s^{-2}t$. Then we are done by applying Corollary 4.2.11 to

- the integer $n = -2$;
- the correspondence $({}_0\mathrm{VinGr}^{\gamma}_{G,I} \overset{=}{\leftarrow} {}_0\mathrm{VinGr}^{\gamma}_{G,I} \to \mathbb{A}^1)$;
- the object $\overset{\circ}{\mathcal{F}} := \omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}$;

- the subcategory $\mathrm{DMod}({}_0\mathrm{VinGr}_{G,I}|_{C_P})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{DMod}({}_0\mathrm{VinGr}_{G,I}|_{C_P})$ (see Remark 1.4.11).

Indeed, Axioms (P1-P3) and (Q) follows from Proposition 4.3.15. Axioms (G1) and (G2) are obvious because $\mathring{\mathcal{F}}$ is ind-holonomic. Axiom (C) follows from Lemma 4.4.1(2) and Lemma 4.3.11. Axiom (M) follows from Lemma 4.4.3.

□[Proposition 3.1.5]

4.6 Proof of Proposition 3.1.7

Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1$ and $\mathrm{Gr}_{G\times G,J}\times\mathbb{A}^1$ defined in Corollary 4.3.5. We apply Corollary 4.2.11 to

- the integer $n = -2$;
- the correspondence $(\mathrm{Gr}_{G\times G,J}\times\mathbb{A}^1 \overset{=}{\leftarrow} \mathrm{Gr}_{G\times G,J}\times\mathbb{A}^1 \to \mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1)$;
- the object $\mathring{\mathcal{F}} := \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$;
- the subcategory $\mathrm{DMod}({}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,J})^{\mathcal{L}(U\times U^-)_J} \subset \mathrm{DMod}(\mathrm{Gr}_{G\times G,J})$.

Axioms (P1-P3) and (Q) follows from Construction 4.3.6. Axioms (G1) and (G2) are obvious because $\mathring{\mathcal{F}}$ is ind-holonomic. Axiom (C) is just Lemma 4.4.1(2). Axiom (M) is just Lemma 4.4.3.

□[Proposition 3.1.7]

4.7 Proof of Proposition 3.1.10

We only need to prove the objects in (a) and (b) are isomorphic after forgetting the $\mathcal{L}U_I^-$-equivariant structure, i.e., when viewed as objects in

$$\mathrm{DMod}(\mathrm{Gr}_{M,I}) \underset{\mathrm{DMod}(X^I)}{\otimes} \mathrm{DMod}(\mathrm{Gr}_{G,I}) \simeq \mathrm{DMod}(\mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}).$$

Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{A}^1$ defined by

$$\mathbb{G}_m \times (\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}\times\mathbb{A}^1) \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}\times\mathbb{A}^1,\ s\cdot(x,y,t) := (s\cdot x, y, s^{-1}t).$$

The corresponding Braden 4-tuple is

$$(\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1, \mathrm{Gr}_{P\times G,I}\times 0, \mathrm{Gr}_{P^-\times G,I}\times\mathbb{A}^1, \mathrm{Gr}_{M\times G,I}\times 0).$$

Hence by Braden's theorem and the contraction principle, the object of (a) is isomorphic to $k \otimes_{C^\bullet(\mathbb{G}_m)} \mathcal{F}[-2]$, where $\mathcal{F}$ is the image of $\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m}$ under the composition

$$\begin{aligned}
\mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathbb{G}_m) &\xrightarrow{\Gamma_{I,*}} \mathrm{DMod}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m) \\
&\xrightarrow{j_*} \mathrm{DMod}(\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1) \\
&\xrightarrow{!\text{-pull}} \mathrm{DMod}(\mathrm{Gr}_{P^-\times G,I} \times \mathbb{A}^1) \\
&\xrightarrow{*\text{-push}} \mathrm{DMod}(\mathrm{Gr}_{M\times G,I} \times 0) \\
&\simeq \mathrm{DMod}(\mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}).
\end{aligned}$$

By the base-change isomorphisms, $\mathcal{F}$ is the $*$-pushforward of $\omega_{\mathrm{Gr}_{P^-,I} \times \mathbb{G}_m}$ along the map

$$\mathrm{Gr}_{P^-,I} \times \mathbb{G}_m \to \mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I},\ (x,s) \mapsto (\mathbf{q}_I^-(x), \mathbf{p}_I^-(s\cdot x)).$$

Note that the $\mathbb{G}_m$-action on $\mathrm{Gr}_{P^-,I}$ preserves the fibers of $\mathbf{q}_I^-$. Hence $\mathcal{F}$ is also the $*$-pushforward of $\omega_{\mathrm{Gr}_{P^-,I} \times \mathbb{G}_m}$ along the map

$$\mathrm{Gr}_{P^-,I} \times \mathbb{G}_m \to \mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I},\ (x,s) \mapsto (\mathbf{q}_I^-(x), \mathbf{p}_I^-(x)).$$

Therefore the object in (a) is isomorphic to the $*$-pushforward of $\omega_{\mathrm{Gr}_{P^-,I}}$ along the map

$$\mathrm{Gr}_{P^-,I} \to \mathrm{Gr}_{M,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I},\ x \mapsto (\mathbf{q}_I^-(x), \mathbf{p}_I^-(x)).$$

By the base-change isomorphisms, this is exactly the object in (b).

□[Proposition 3.1.10]

4.8 Proof of Theorem 3.2.2 and Corollary 3.4.1

We prove Theorem 3.2.2 (and Corollary 3.4.1) in this subsection. To simplify the notations, we denote all unipotent nearby cycles functors by Ψ^{un}. By symmetry, it is enough to prove (2).

4.8.1 (Preparation)**.** Consider the diagonal embedding

$$\Delta : \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \hookrightarrow \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,\ (x,y,t) \mapsto (x,x,y,t).$$

We have the following diagram

$$\begin{tikzcd}
& \mathrm{Gr}_{G,I} \times \mathbb{G}_m \arrow[r, "\Gamma^\sigma"] \arrow[d, "\Gamma^\sigma_I"] & \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \times \mathbb{G}_m \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbf{Id}\times\Gamma^\sigma_I"] & \mathrm{Gr}_{G,I} \\
\mathrm{Gr}_{G,I} \times \mathbb{G}_m & \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{G}_m \arrow[l, "\mathrm{pr}_{23}"'] \arrow[r, "\mathring{\Delta}"] & \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{G}_m, &
\end{tikzcd}$$

where Γ^σ and Γ^σ_I are given by the formula[44]:$(x,t) \mapsto (t\cdot x, x, t)$, the maps pr_1 and pr_{23} are the projections onto the factors indicated by the subscripts. Note that the square in this diagram is Cartesian.

We also have the following correspondence:

$$\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \overset{\mathrm{pr}_2}{\leftarrow} \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \overset{\Delta_0}{\to} \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}.$$

We claim:

(i) the functor $\Psi^{\mathrm{un}}[-1]\circ\mathrm{pr}_{23,*}\circ(\mathring{\Delta})^!\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!$ is well-defined on $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, and is isomorphic to $\mathbf{oblv}^{\mathcal{L}U_I}$.

(ii) the functor $\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!$ is well-defined, and we have

$$\mathrm{pr}_{2,*}\circ\Delta_0^!\circ\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^! \simeq F_{\mathcal{K}^\sigma}.$$

Note that these two claims translate the theorem into a statement that taking certain unipotent nearby cycles commutes with certain pull-push functors (see (4.21) below).

4.8.2 (Proof of (ii))**.** By Lemma 4.8.3 below, for any $\mathcal{G}\in\mathrm{DMod}(\mathrm{Gr}_{G,I})$, the object

$$(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!(\mathcal{G}) \simeq \mathcal{G}\boxtimes\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$$

is contained in $\mathrm{DMod}(\mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{G}_m)^{\mathrm{good}}$, and we have

$$\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!(\mathcal{G}) \simeq \Psi^{\mathrm{un}}[-1](\mathcal{G}\boxtimes\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})) \simeq \mathcal{G}\boxtimes\Psi^{\mathrm{un}}[-1]\circ\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \simeq \mathcal{G}\boxtimes\mathcal{K}^\sigma.$$

Then (ii) follows from the definition of $F_{\mathcal{K}^\sigma}$.

Lemma 4.8.3. *Let Z be an ind-finite type indscheme over $\mathbb{A}^1$, and Y be any ind-finite type indscheme. Let $\mathcal{F}\in\mathrm{DMod}(\mathring{Z})$ and $\mathcal{G}\in\mathrm{DMod}(Y)$. Suppose the !-restriction of $\mathcal{F}$ on any finite type closed subscheme of $\mathring{Z}$ is holonomic, then the object $\mathcal{G}\boxtimes\mathcal{F}$ is contained in $\mathrm{DMod}(Y\times\mathring{Z})^{\mathrm{good}}$ and we have $j_!(\mathcal{G}\boxtimes\mathcal{F})\simeq\mathcal{G}\boxtimes j_!(\mathcal{F})$.*

[44]Note that the order is different from that for Γ_I.

Proof. (Sketch) Let we first assume Y and Z to be finite type schemes. When $\mathcal{G}$ is compact (i.e. coherent), the claim follows from the Verdier duality. The general case can be obtained from this by a standard devissage argument.

□[Lemma 4.8.3]

4.8.4 (Proof of (i))**.** Consider the automorphism α on $\mathrm{Gr}_{G,I} \times \mathbb{G}_m$ given by $(x,t) \mapsto (t \cdot x, t)$. By the base-change isomorphisms, the functor in (i) is isomorphic to

$$\Psi^{\mathrm{un}} \circ \alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})[-1] \simeq k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} (i^* \circ j_* \circ \alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m}))[-2].$$

Suppose $\mathcal{G}$ is contained in $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. By Lemma 2.1.7(1), $\mathcal{G}$ is unipotently $\mathbb{G}_m$-monodromic. Therefore $\mathcal{G} \boxtimes \omega_{\mathbb{G}_m} \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathbb{G}_m)$ is unipotently $\mathbb{G}_m$-monodromic for the diagonal action, which implies $\alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m}) \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \times \mathbb{G}_m)$ is unipotently $\mathbb{G}_m$-monodromic for the $\mathbb{G}_m$-action on the second factor. Hence we can apply the contraction principle to $j_* \circ \alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})$ and obtain

$$i^* \circ j_* \circ \alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})[-2] \simeq \mathrm{pr}_{1,*} \circ j_* \circ \alpha^!(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})[-2], \tag{4.19}$$

where $\mathrm{pr}_1 : \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \to \mathrm{Gr}_{G,I}$ is the projection. In particular, the LHS of (4.19) is well-defined. Hence the functor in (i) is well-defined on $\mathcal{G}$.

By the base-change isomorphisms, the RHS of (4.19) is isomorphic to $\mathrm{act}_*(\mathcal{G} \boxtimes k_{\mathbb{G}_m})$, where $\mathrm{act} : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \to \mathrm{Gr}_{G,I}$ is the action map. It remains to prove

$$(k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} k_{\mathbb{G}_m}) \star \mathcal{G} \simeq \mathcal{G},$$

where $\star$ denotes the action $\mathrm{DMod}(\mathbb{G}_m) \curvearrowright \mathrm{DMod}(\mathrm{Gr}_{G,I})$. By [Cam18, Lemma 3.2.1], $k \otimes_{C^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m}$ is Beilinson's infinite Jordan block, hence there is a canonical map[45]

$$k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} k_{\mathbb{G}_m} \to e_*(k), \tag{4.20}$$

where $e_*(k)$ is the δ-D-module at the unit $e \in \mathbb{G}_m$. We only need to prove $- \star \mathcal{G}$ sends the above map to an isomorphism. Recall that $\mathcal{G} \in \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}}$ is generated by the image of $\mathbf{oblv}^{\mathbb{G}_m}$. Hence we can assume $\mathcal{G} = \mathbf{oblv}^{\mathcal{G}}(\mathcal{M})$ for some $\mathcal{M} \in \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m}$. Then for any

[45] Alternatively, the map below can be obtained from the following map

$$k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} e^*(k_{\mathbb{G}_m}) \simeq k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} k \to k$$

by using the adjoint pair (e^*, e_*).

$\mathcal{N} \in \mathrm{DMod}(\mathbb{G}_m)$, we have $\mathcal{N} \star \mathcal{G} \simeq \Gamma_{\mathrm{dR}}(\mathcal{N}) \otimes \mathcal{G}$. Hence we only need to show Γ_{dR} sends (4.20) to an isomorphism. But this is obvious. This proves (i).

4.8.5 (Proof of Theorem 3.2.2)**.** By (i) and (ii), it remains to prove that for any $\mathcal{G}$ contained in $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, the natural map

$$\Psi^{\mathrm{un}} \circ \mathrm{pr}_{23,*} \circ (\overset{\circ}{\Delta})^! \circ (\mathbf{Id} \times \Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!(\mathcal{G}) \to \mathrm{pr}_{2,*} \circ \Delta_0^! \circ \Psi^{\mathrm{un}} \circ (\mathbf{Id} \times \Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!(\mathcal{G}) \tag{4.21}$$

is an isomorphism[46].

Note that it is enough to prove this for a set of compact generators $\mathcal{G}$ of $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. Hence by Lemma 2.1.7(2) and (4), we can assume that $\mathcal{G}$ is supported on ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ for some $\lambda \in \Lambda_{G,P}$ (see Proposition A.4.2 for the notation).

We apply Corollary 4.2.11 to

- the integer $n = -1$;
- the correspondence

$$(U \leftarrow V \to W) \coloneqq (\mathrm{Gr}_{G,I} \times \mathbb{A}^1 \xleftarrow{\mathrm{pr}_{23}} \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \xrightarrow{\Delta} \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1),$$

  where $\mathbb{G}_m$ acts on W by $s \cdot (x,t,z,t) \coloneqq (x,y,s\cdot z, s^{-1}t)$, on V by restriction, and on U by $s \cdot (z,t) \coloneqq (s \cdot z, s^{-1}t)$.
- the object $\overset{\circ}{\mathcal{F}} \coloneqq (\mathbf{Id} \times \Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!(\mathcal{G})$;
- the subcategory $\mathrm{DMod}({}_{\le\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I})$.

Axioms (P1-P3) and (Q) can be checked directly using Proposition 1.2.8. Axioms (G1) and (G2) follow from (i) and (ii). Axiom (C) is just Lemma 4.4.1(1). It remains to check Axiom (M).

Write $\mathcal{F} \coloneqq j_*(\overset{\circ}{\mathcal{F}})$. Unwinding the definition, we only need to prove that both sides of

$$i^* \circ \mathrm{pr}_{23,*} \circ \Delta^!(\mathcal{F}) \to \mathrm{pr}_{2,*} \circ \Delta_0^! \circ i^*(\mathcal{F}) \tag{4.22}$$

are contained in the full subcategory $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, and are supported on ${}_{\le\lambda}\mathrm{Gr}_{G,I}$.

[46] Although $\Psi^{\mathrm{un}} \circ \mathrm{pr}_{23,*} \simeq \mathrm{pr}_{2,*} \circ \Psi^{\mathrm{un}}$ because pr_{23} is ind-proper, we do *not* know if the stronger claim

$$\Psi^{\mathrm{un}} \circ (\overset{\circ}{\Delta})^! \circ (\mathbf{Id} \times \Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!(\mathcal{G}) \simeq \Delta_0^! \circ \Psi^{\mathrm{un}} \circ (\mathbf{Id} \times \Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!(\mathcal{G})$$

is correct. The reason is that the support of the LHS might be the entire $\mathrm{Gr}_{G\times G,I}$ hence Axiom (M) is not satisfied (see Warning 4.4.2).

For the LHS of (4.22), in § 4.8.4, we proved that it is isomorphic to $\mathrm{act}_*(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})$. Since each stratum ${}_\mu\mathrm{Gr}_{G,I}\simeq(\mathrm{Gr}^\mu_{P,I})_{\mathrm{red}}$ is stabilized by the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$, so is ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$. Hence $\mathrm{act}_*(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})$ is supported on ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$ because $\mathcal{G}$ is so. To prove it is contained in $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, by Lemma 2.1.6, it suffices to prove that its !-pullback to $\mathrm{Gr}_{P,I}$ is contained in $\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}$. Hence it suffices to show !-pull-$*$-push along the correspondence

$$\mathrm{Gr}_{P,I}\overset{\mathrm{act}}{\leftarrow}\mathrm{Gr}_{P,I}\times\mathbb{G}_m\overset{\mathrm{pr}_1}{\to}\mathrm{Gr}_{P,I}$$

preserves the subcategory $\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\subset\mathrm{DMod}(\mathrm{Gr}_{P,I})$. However, this follows from Lemma 2.1.5(1) and the fact that the $\mathbb{G}_m$-action on $\mathrm{Gr}_{P,I}$ contracts it onto $\mathrm{Gr}_{M,I}$.

To prove that the RHS of (4.22) is contained in $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, it suffices to show that

$$i^*(\mathcal{F})\in\mathrm{DMod}(\mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I,3},$$

where 3 indicates that we are considering the $\mathcal{L}U_I$-action on the third factor. We have

$$i^*(\mathcal{F})\simeq\mathcal{G}\boxtimes i^*\circ j_*\circ\Gamma^\sigma_I(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}).$$

Hence it suffices to prove that

$$i^*\circ j_*\circ\Gamma^\sigma_I(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})\in\mathrm{DMod}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I,2},$$

or equivalently

$$i^*\circ j_*\circ\Gamma_I(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})\in\mathrm{DMod}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I,1}.$$

However, this is just Remark 4.1.5.

For the claim about the support of the RHS, by the base-change isomorphisms, it suffices to prove the following statement. If a stratum $\mathrm{Gr}^{\mu_1}_{P^-,I}\times_{X^I}\mathrm{Gr}^{\mu_2}_{P,I}$ has non-empty intersection with both $\sigma(\mathrm{VinGr}_{G,I}|_{C_P})$ and ${}_{\leq\lambda}\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}$, then $\mu_2\leq\lambda$. By Corollary 4.3.10, the first non-empty intersection implies $\mu_2\leq\mu_1$. On the other hand, the second non-empty intersection implies $\mu_1\leq\lambda$ by definition. Hence we have $\mu_2\leq\lambda$ as desired. This finishes the proof of the theorem.

□[Theorem 3.2.2]

4.8.6 (Proof of Corollary 3.4.1)**.** By (4.21), we have the following natural transformation

$$\Psi^{\mathrm{un}}\circ\mathrm{pr}_{23,*}\circ(\overset{\circ}{\Delta})^!\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!\to\mathrm{pr}_{2,*}\circ\Delta_0^!\circ\Psi^{\mathrm{un}}\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!$$

between two functors $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{DMod}(\mathrm{Gr}_{G,I})$. By Proposition D.7.1, both sides can be canonically upgraded to $\mathcal{L}^+M_I$-linear functors. It follows from construction that the above natural transformation is compatible with these $\mathcal{L}^+M_I$-linear structures.

It remains to prove that the isomorphisms in § 4.8.1(i) and (ii) are compatible with the $\mathcal{L}^+M_I$-linear structures. This is tautological for (ii) because both $\mathcal{L}^+M_I$-linear structures come from Proposition D.7.1 (see § 4.1.4). For the isomorphism in (i), unwinding the proof in § 4.8.4, it suffices to show the functor

$$k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} \mathrm{act}_*(- \boxtimes k_{\mathbb{G}_m}) \to \mathbf{Id}$$

constructed there is compatible with the $\mathcal{L}^+M_I$-linear structures. But this is obvious because it is obtained from (4.20).

□[Corollary 3.4.1]

Part III

The Global Nearby Cycles $\Psi_{\gamma,\mathrm{glob}}$

Chapter 5

Statement of results

5.1 Basic properties of the global nearby cycles

Construction 5.1.1 (Global nearby cycles)**.** Let P be a standard parabolic subgroup and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. Consider the stack

$$Z := \mathrm{VinBun}_G^{\gamma} \to \mathbb{A}^1.$$

By Remark 1.3.9, we have $\overset{\circ}{Z} \simeq \mathrm{Bun}_G \times \mathbb{G}_m$. Consider the corresponding nearby cycles functor

$$\Psi_{\mathrm{VinBun}_G^{\gamma}} : \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G \times \mathbb{G}_m) \to \mathrm{DMod}(\mathrm{VinBun}_G |_{C_P}).$$

We obtain an object

$$\Psi_{\gamma,\mathrm{glob}} := \Psi_{\mathrm{VinGr}_{G,I}^{\gamma}}(\omega_{\overset{\circ}{Z}}) \in \mathrm{DMod}(\mathrm{VinGr}_{G,I} |_{C_P}),$$

which was studied in [Sch18] and [Sch16].

The following result, which is the global analogue of Proposition 3.1.2(1), was essentially proved[47] in [Sch16, § 8.1]

Fact 5.1.2. *The canonical map $\Psi^{\mathrm{un}}_{\gamma,\mathrm{glob}} \to \Psi_{\gamma,\mathrm{glob}}$ is an isomorphism.*

To describe the global analogue of Proposition 3.1.2(2), we need the following definition.

[47] Schieder only proved for the principal case $P = B$, but the proof there works for any P. Also, he made a minor mistake on references. See Footnote 120.

Definition 5.1.3. Consider the defect stratification on $\mathrm{VinBun}_G|_{C_P}$ in § A.5.4. Recall the disjoint union of its strata is given by

$$ {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}. $$

It is well-known that the map $\mathrm{Bun}_P \to \mathrm{Bun}_M$ is *universally homological contractible*, or *UHC*[48]. In particular, the !-pullback functor

$$ \mathrm{DMod}(H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}) \to \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}) $$

is fully faithful. As before, an object $\mathcal{F} \in \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})$ is called $U(\mathbb{A}_F)$*-equivariant* if it is contained in the essential image of the above functor. We denote the full subcategory of $U(\mathbb{A}_F)$-equivariant objects as

$$ \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)} \subset \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}). $$

We define $\mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)}$ to fit into the following pullback diagram

$$ \begin{tikzcd} \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)} \arrow[r, "\subset"] \arrow[d] & \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P}) \arrow[d, "!\text{-pull}"] \\ \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)} \arrow[r, "\subset"] & \mathrm{DMod}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}). \end{tikzcd} $$

Similarly we define $\mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})^{U^-(\mathbb{A}_F)}$ and $\mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)}$. We also define the version of these sub-categories for $\overline{\mathrm{Bun}}_{G,P}$.

We will prove the following result in § 6.1.

Proposition 5.1.4. *The object $\Psi_{\gamma,\mathrm{glob}}$ is contained in*

$$ \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)} \subset \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P}). $$

Remark 5.1.5. The above proposition is a corollary of (the Verdier dual of) [Sch16, Theorem 4.3.1]. However, the proof of [Sch16, Theorem 4.3.1] implicitly used (the Verdier dual of) our proposition. Namely, what S. Schieder called the *interplay principle* only proved his theorem up to a possible twist by local systems pulled back from $\mathrm{Bun}_P \times \mathrm{Bun}_{P^-}$, and one needs the above lemma to rule out such twists[49].

For the mixed sheaf context as in [Sch16], thanks to the sheaf-function-correspondence, the proposition

[48]This means for any lft prestack $Y \to \mathrm{Bun}_M$, the !-pullback functor $\mathrm{DMod}(Y) \to \mathrm{DMod}(Y \times_{\mathrm{Bun}_M} \mathrm{Bun}_P)$ is fully faithful.

[49]See [BG06, proof of Proposition 4.4] for an analogue of this logic for the interplay principle between the Zastava spaces and $\overline{\mathrm{Bun}}_B$.

can be easily proved by showing that the stalks are constant along the fibers of ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to H_{M,G\text{-pos}}$ (a similar argument can be found in [BG02, Subsection 6.3]). However, in the D-module context, one needs more work.

Construction 5.1.6. Recall we have the following commutative diagram (see Construction 1.4.6):

$$\begin{tikzcd}
\mathrm{VinGr}_{G,I}|_{C_P} \arrow[r, "i"] \arrow[d, "\pi_I|_{C_P}"] & \mathrm{VinGr}_{G,I}^{\gamma} \arrow[d, "\pi_I^{\gamma}"] & \mathrm{Gr}_{G,I} \times \mathbb{G}_m \arrow[l, "j"'] \arrow[d] \\
\mathrm{VinBun}_G|_{C_P} \arrow[r, "i"] & \mathrm{VinBun}_G^{\gamma} & \mathrm{Bun}_G \times \mathbb{G}_m, \arrow[l, "j"']
\end{tikzcd}$$

It induces a morphism

$$\Psi_{\gamma,I} \to (\pi_I|_{C_P})^!(\Psi_{\gamma,\mathrm{glob}}).$$

We will prove the following result in § 6.15.

Theorem 5.1.7 (Theorem F)**.** *The canonical map* $\Psi_{\gamma,I} \to (\pi_I|_{C_P})^!(\Psi_{\gamma,\mathrm{glob}})$ *is an isomorphism.*

5.2 The global nearby cycles and the Deligne-Lusztig duality

In this section, we describe the relation between the global nearby cycles and Theorem 2.4.2. We also deduce the latter theorem from a result on VinBun_G.

Definition 5.2.1. The T_{cent}-action on Vin_G induces a canonical T-action on VinBun_G. We define the *Drinfeld compactification* as

$$\overline{\mathrm{Bun}}_G := \mathrm{VinBun}_G/T.$$

Construction 5.2.2. The parabolic stratification on the base T_{ad}^+ is compatible with the T-action. Hence we obtain a *parabolic stratification* on T_{ad}^+/T and therefore on $\overline{\mathrm{Bun}}_G$. Note that the T-action on VinBun_G induces a Z_M-action on $\mathrm{VinBun}_G|_{C_P}$. We have

$$\overline{\mathrm{Bun}}_{G,P} \simeq \mathrm{VinBun}_{G,P}/T \simeq (\mathrm{VinBun}_G|_{C_P})/Z_M.$$

Note that the biggest stratum is

$$\overline{\mathrm{Bun}}_G \simeq \mathrm{Bun}_G \times \mathbb{B}Z_G.$$

The following result was proved by D. Gaitsgory in [FKM20, Appendix A].

Fact 5.2.3. *The diagonal map* $\Delta : \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ *canonically factors as*

$$\mathrm{Bun}_G \xrightarrow{b} \overline{\mathrm{Bun}}_G \xrightarrow{\overline{\Delta}} \mathrm{Bun}_G \times \mathrm{Bun}_G$$

such that $\overline{\Delta}$ *is schematic and proper. Also the above map* b *is*

$$\mathrm{Bun}_G \xrightarrow{r} \mathrm{Bun}_G \times \mathbb{B}Z_G \simeq \overline{\mathrm{Bun}}_{G,G} \xrightarrow{j_G} \overline{\mathrm{Bun}}_G.$$

Construction 5.2.4. By a general construction for stacks stratified by power posets (see Corollary C.8.7), we have a canonically defined functor[50]

$$\mathbf{K} : \mathrm{Par} \to \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G), \tag{5.1}$$
$$P \mapsto i_{P,!} \circ i_P^* \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G})[\mathrm{rank}(M) - \mathrm{rank}(G)],$$

and a canonical isomorphism (see Lemma C.8.9)

$$\mathrm{coFib}(\underset{P \in \mathrm{Par}'}{\mathrm{colim}}\, \mathbf{K}(P) \to \mathbf{K}(G)) \simeq j_{G,*} \circ r_!(k_{\mathrm{Bun}_G}). \tag{5.2}$$

Consider the composition

$$\mathbf{E} : \mathrm{LFun}_k(\mathrm{DMod}(\mathrm{Bun}_G), \mathrm{DMod}(\mathrm{Bun}_G)) \to \mathrm{LFun}_k(\mathrm{DMod}(\mathrm{Bun}_G)^\vee, \mathrm{DMod}(\mathrm{Bun}_G)) \simeq \mathrm{DMod}(\mathrm{Bun}_G \times \mathrm{Bun}_G), \tag{5.3}$$

where the first functor is given by precomposition with $\mathrm{Ps\text{-}Id}_!$ and the last equivalence is (0.8). Equivalently, $\mathbf{E}$ sends an endo-functor F to

$$\mathbf{E}(F) \simeq (F \otimes \mathbf{Id}) \circ \Delta_!(k_{\mathrm{Bun}_G}),$$

where we view $F \otimes \mathbf{Id}$ as an endo-functor of $\mathrm{DMod}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \simeq \mathrm{DMod}(\mathrm{Bun}_G) \otimes_k \mathrm{DMod}(\mathrm{Bun}_G)$.

We will prove the following results in § 6.16 and § 6.4 respectively.

Proposition 5.2.5. *The* !*-pushforward functor along the composition*

$$\mathrm{VinBun}_G|_{C_P} \to \mathrm{VinBun}_G \to \overline{\mathrm{Bun}}_G$$

sends $\Psi_{\gamma,\mathrm{glob}}[-1]$ *to* $\mathbf{K}(P)[2\dim(\mathrm{Bun}_G)]$.

[50]The functor $\mathbf{K}$ is given by $\mathbf{G}^*_{r_!(k_{\mathrm{Bun}_G}),\overline{\mathrm{Bun}}_G}$, which is defined in Corollary C.8.7.

Theorem 5.2.6 (Theorem H)**.** *There is a canonical commutative diagram*

$$\begin{tikzcd} \mathrm{Par} \arrow[r, "\mathbf{DL}"] \arrow[d, "\mathbf{K}"] & \mathrm{LFun}_k(\mathrm{DMod}(\mathrm{Bun}_G), \mathrm{DMod}(\mathrm{Bun}_G)) \arrow[d, "\mathbf{E}"] \\ \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) \arrow[r, "\overline{\Delta}_!"] & \mathrm{DMod}(\mathrm{Bun}_G \times \mathrm{Bun}_G). \end{tikzcd}$$

5.2.7 (Deduction of Theorem 2.4.2 from Theorem 5.2.6)**.** The deduction below is due to D. Gaitsgory.

First note that $\mathbf{E}$ is an equivalence because Ps-Id$_!$ is. By definition,

$$\mathbf{E}^{-1}(\Delta_*(\omega_{\mathrm{Bun}_G})) \simeq \text{Ps-Id}_{\mathrm{naive}} \circ \text{Ps-Id}_!^{-1}.$$

On the other hand, as in [Gai17b, § 3.2.3], we have a canonical isomorphism

$$\Delta_*(\omega_{\mathrm{Bun}_G})[-2\dim(\mathrm{Bun}_G) - \dim(Z_G)] \simeq \overline{\Delta}_! \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G}),$$

where the cohomological shift by $[-2\dim(\mathrm{Bun}_G)]$ is due to the difference between ω_{Bun_G} and k_{Bun_G}, while that by $[-\dim(Z_G)]$ is due to the difference between r_* and $r_!$. Hence the isomorphism (5.2) implies

$$\mathrm{coFib}(\operatorname*{colim}_{P \in \mathrm{Par}'} \mathbf{E}^{-1} \circ \overline{\Delta}_! \circ \mathbf{K}(P) \to \mathbf{E}^{-1} \circ \overline{\Delta}_! \circ \mathbf{K}(G)) \simeq \text{Ps-Id}_{\mathrm{naive}} \circ \text{Ps-Id}_!^{-1}[-2\dim(\mathrm{Bun}_G) - \dim(Z_G)].$$

Then we are done because $\mathbf{E}^{-1} \circ \overline{\Delta}_! \circ \mathbf{K} \simeq \mathbf{DL}$.

□[Theorem 2.4.2]

Remark 5.2.8. As a first test for Thoerem 5.2.6, let us evaluate the above diagram at $G \in \mathrm{Par}$. By definition, $\mathbf{K}(G) \simeq j_{G,!} \circ r_!(k_{\mathrm{Bun}_G})$. Hence $\overline{\Delta}_! \circ \mathbf{K}(G) \simeq \Delta_!(k_{\mathrm{Bun}_G})$. On the other hand $\mathbf{DL}(G) \simeq \mathbf{Id}$, hence $\mathbf{E} \circ \mathbf{DL}(G) \simeq \Delta_!(k_{\mathrm{Bun}_G})$ by the definition of Ps-Id$_!$.

Remark 5.2.9. The statement of Theorem 2.4.2 depends on the miraculous duality on Bun_G (i.e., Ps-Id$_!$ is invertible) but that of Theorem 5.2.6 does not. Our proof of the latter will not depend on the miraculous duality either.

5.3 The global nearby cycles and the inv-inv duality

We will prove the following result in § 6.2.

Proposition-Construction 5.3.1. *There exist canonical maps*

$$\overline{\Delta}^{\mathrm{enh}}_{\leq P}:\overline{\mathrm{Bun}}_{G,\leq P}\to \mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G^{P^-\text{gen}}$$

that are functorial[51] *in P such that when $P=G$ we have $\overline{\Delta}^{\mathrm{enh}}_{\leq G}=\overline{\Delta}:\overline{\mathrm{Bun}}_G\to \mathrm{Bun}_G\times\mathrm{Bun}_G$. In particular, we have canonical maps*

$$\overline{\Delta}^{\mathrm{enh}}_{P}:\overline{\mathrm{Bun}}_{G,P}\to \mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G^{P^-\text{gen}}.$$

Remark 5.3.2. See Footnote 13 for the construction in the case $G=\mathrm{SL}_2$.

Construction 5.3.3. By Proposition-Construction 5.3.1, we obtain a canonical map

$$\mathrm{VinBun}_G|_{C_P}\to \mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G^{P^-\text{-gen}}.$$

Let

$$\mathcal{K}_{\gamma,\mathrm{glob}}\in \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G^{P^-\text{-gen}})$$

be the !-pushforward of $\Psi_{\gamma,\mathrm{glob}}[-1-2\dim(\mathrm{Bun}_G)]$ along this map. By Proposition 5.2.5, we also have $\mathcal{K}_{\gamma,\mathrm{glob}}\simeq \overline{\Delta}^{\mathrm{enh}}_{P,!}\circ i_P^*\circ \mathbf{K}(P)$.

We will prove the following result in § 6.3.

Proposition 5.3.4. *The object $\mathcal{K}_{\gamma,\mathrm{glob}}$ is contained in*

$$\mathrm{I}(G\times G,P\times P^-)\subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}\times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

Recall we have (see Corollary 2.3.15):

$$\mathrm{I}(G,P)\otimes \mathrm{I}(G,P^-)\simeq \mathrm{I}(G\times G,P\times P^-).$$

We will prove the following result in § 6.17.

Theorem 5.3.5 (Theorem E)**.** *The functor*

$$\mathrm{Vect}\xrightarrow{\mathcal{K}_{\gamma,\mathrm{glob}}\otimes -}\mathrm{I}(G\times G,P\times P^-)\simeq \mathrm{I}(G,P)\otimes\mathrm{I}(G,P^-)$$

[51] Note that for any $P\subset Q$, we have canonical maps $\overline{\mathrm{Bun}}_{G,\leq P}\to\overline{\mathrm{Bun}}_{G,\leq Q}$ and $\mathrm{Bun}_G^{P\text{-gen}}\times\mathrm{Bun}_G^{P^-\text{-gen}}\to \mathrm{Bun}_G^{Q\text{-gen}}\times\mathrm{Bun}_G^{Q^-\text{-gen}}$.

is the unit of a duality between $\mathrm{I}(G,P)$ *and* $\mathrm{I}(G,P^-)$. *Moreover:*

(a) Via this duality and the miraculous duality on Bun_M, *the functors* $\iota_{M,!} : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P)$ *and* $\iota^-_{M,!} : \mathrm{DMod}(\mathrm{Bun}_M) \to \mathrm{I}(G,P^-)$ *are conjugate to each other.*

(b) Via this duality and the miraculous duality on Bun_G, *the functors* $\mathrm{Eis}^{\mathrm{enh}}_{P\to G} : \mathrm{I}(G,P) \to \mathrm{DMod}(\mathrm{Bun}_G)$ *and* $\mathrm{Eis}^{\mathrm{enh}}_{P^-\to G} : \mathrm{I}(G,P^-) \to \mathrm{DMod}(\mathrm{Bun}_G)$ *are conjugate to each other.*

Chapter 6

Proofs

6.1 Proof of Proposition 5.1.4

In the proof below, we focus mainly on the geometric constructions, and omit some details about general properties of D-modules. In particular, we stop mentioning the well-definedness of certain $*$-pullbacks because our main interest is on the ind-holonomic object $\omega_{\mathrm{Bun}_G\times\mathbb{G}_m}$.

Recall we have

$$_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

Consider the maps

$$\mathbf{p}^{+}_{\mathrm{glob}} : {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to \mathrm{VinBun}_G|_{C_P},$$

$$\overleftarrow{q} : {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}},$$

$$\overrightarrow{q} : {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

By definition, we only need to show

Goal 6.1.1. The object $\mathbf{p}^{+,!}_{\mathrm{glob}} \circ i^* \circ j_*(\omega)$, is contained in the essential image of $(\overrightarrow{q})^!$.

6.1.2 (Strategy)**.** Our strategy is similar to that in [BG02, Subsection 6.3]. In particular, we study the Hecke modifications on VinBun_G.

Let x_i be distinct closed points on X and $x \hookrightarrow X$ be the union of them. We define $H^{\mathrm{d.f.}_{\infty\cdot x}}_{M,G\text{-}\mathrm{pos}}$ to be the open sub-stack of $H_{M,G\text{-}\mathrm{pos}}$ classifying maps $X \to M\backslash\overline{M}/M$ that send x into $M\backslash M/M$. The symbol "$\mathrm{d.f.}_{\infty\cdot x}$" stands for "defect-free near x". Note that when x varies, these open sub-stacks form a Zariski cover of $H_{M,G\text{-pos}}$. We define $({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty\cdot x}$ to be the pre-image of this open sub-stack for the map

$$\mathbf{q}^+_{\mathrm{glob}} : {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to H_{M,G\text{-pos}}.$$

The map $\overrightarrow{q}$ restricts to a map

$$({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty\cdot x} \to H^{\mathrm{d.f.}\infty\cdot x}_{M,G\text{-pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

Consider the Čech nerve of this map. The first two terms are

$$(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P) \underset{\mathrm{Bun}_M}{\times} H^{\mathrm{d.f.}\infty\cdot x}_{M,G\text{-pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} (\mathrm{Bun}_P) \underset{\mathrm{Bun}_M}{\times} H^{\mathrm{d.f.}\infty\cdot x}_{M,G\text{-pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}. \tag{6.1}$$

By Lemma 6.1.18 and 6.1.19 below, we only need to show $\partial_0^!(\mathcal{G})$ and $\partial_1^!(\mathcal{G})$ are isomorphic, where

$$\mathcal{G} := [\mathbf{p}^{+,!}_{\mathrm{glob}} \circ i^* \circ j_*(\omega)]|_{({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty\cdot x}}$$

is the restriction of $\mathbf{p}^{+,!}_{\mathrm{glob}} \circ i^* \circ j_*(\omega)$ on $({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty\cdot x}$.

We want to replace the factor $\mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_P$ in (6.1) by a local object that is easier to handle. Consider the Hecke ind-stack

$$H_{P,x} := \mathrm{Gr}_{P,x} \widetilde{\times} \mathrm{Bun}_P.$$

Recall that it is equipped with two projections

$$\overrightarrow{\mathfrak{h}}, \overleftarrow{\mathfrak{h}} : H_{P,x} \to \mathrm{Bun}_P.$$

Also recall we have a "diagonal" map $\Delta : \mathrm{Bun}_P \to H_{P,x}$ such that $\overrightarrow{\mathfrak{h}} \circ \Delta \simeq \overleftarrow{\mathfrak{h}} \circ \Delta \simeq \mathrm{Id}$. Hence we have a map

$$H_{P,x} \underset{H_{M,x},\Delta}{\times} \mathrm{Bun}_M \to \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P,$$

where the LHS is the moduli prestack of those Hecke modifications on P-torsors that fix the induced M-torsors. The above map is known to be UHC (it can be proved similarly as in [Gai17a, Subsection 3.5]), hence so is the map

$${}_{\mathrm{str}}H_x := (H_{P,x} \underset{H_{M,x},\Delta}{\times} \mathrm{Bun}_M) \underset{\mathrm{Bun}_M}{\times} H^{\mathrm{d.f.}\infty\cdot x}_{M,G\text{-pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-} \to (\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_P) \underset{\mathrm{Bun}_M}{\times} H^{\mathrm{d.f.}\infty\cdot x}_{M,G\text{-pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

By construction, the maps ∂_0 and ∂_1 induce two maps

$$h_0, h_1 : {}_{\mathrm{str}}H_x \to ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty\cdot x}.$$

By the above discussion, we only need to show

Goal 6.1.3. The objects $h_0^!(\mathcal{G})$ and $h_1^!(\mathcal{G})$ are isomorphic.

6.1.4 (How about VinBun_G^γ?)**.** Goal 6.1.3 suggests us to construct certain Hecke modifications on VinBun_G^γ that are compatible with the Hecke modifications on $({}_{\mathrm{str}}\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}\infty\cdot x}$ given by ${}_{\mathrm{str}}H_x$. However, there is no direct way to do this because VinBun_G^γ does not map to Bun_P. Instead, it maps to $\mathrm{Bun}_G \times \mathrm{Bun}_G$.

This suggests us to consider the Vinberg-vesion of P-structrues on G-torsors. However, we shall not use the naive candidate, i.e., the P-structures on the G-torsor given by the "left" forgetful map $\mathrm{VinBun}_G^\gamma \to \mathrm{Bun}_G$, because this notion is ill-behaved when moving along $\mathbb{A}^1$. Instead, the correct notion of the P-stuctures should behave "diagonally" on $\mathrm{VinBun}_G|_{C_G}$ and "leftly" on $\mathrm{VinBun}_G|_{C_P}$. In other words, we should consider the closed embedding

$$P \times \mathbb{A}^1 \to \widetilde{G}^\gamma \tag{6.2}$$

provided by (the mirror version of) Lemma A.1.2(2). The rest of this section is to realize the above ideas.

Notation 6.1.5. Recall the notations $\mathcal{D}'_x$ and $\mathcal{D}^\times_x$ (see Notation 6). Let $Y_1 \to Y_2$ be a map between algebraic stacks. We define

$$\mathbf{Maps}(\mathcal{D}'_x \to X, Y_1 \to Y_2)$$

to be the prestack whose value for an affine test scheme S classifies commutative squares

$$\begin{tikzcd} \mathcal{D}'_x \times S \arrow[r] \arrow[d, "\delta"] & X \times S \arrow[d, "\alpha"] \\ Y_1 \arrow[r] & Y_2. \end{tikzcd}$$

Remark 6.1.6. When Y_1 and Y_2 satisfy the condition (♠) in Remark A.6.5, for an affine test scheme S, the groupoid $\mathbf{Maps}(\mathcal{D}'_x \to X, Y_1 \to Y_2)(S)$ also classifies commutative diagrams

$$\begin{tikzcd} \mathcal{D}^\times_x \times S \arrow[r] \arrow[d] & (X - x) \times S \arrow[dd, "\alpha'"] \\ \mathcal{D}'_x \times S \arrow[d, "\delta"] & \\ Y_1 \arrow[r] & Y_2. \end{tikzcd}$$

In this appendix, we only use the notation $\mathbf{Maps}(\mathcal{D}'_x \to X, Y_1 \to Y_2)$ in the above case.

6.1.7 (P-structures)**.** The closed embedding (6.2) induces a chain

$$\mathbb{A}^1 \times \mathrm{pt}/P \to \mathbb{A}^1/\widetilde{G}^\gamma \simeq G\backslash_0 \mathrm{Vin}_G^\gamma/G \to G\backslash \mathrm{Vin}_G^\gamma/G.$$

It is easy to see the 0-fiber of the above composition factors as

$$\mathrm{pt}/P \to \mathrm{pt}/(P \underset{M}{\times} P^-) \simeq P\backslash M/P^- \to P\backslash \overline{M}/P \to G\backslash \mathrm{Vin}_G|_{C_P}/G.$$

We define[52]

$$\begin{aligned}
(\mathrm{VinBun}_G^\gamma)^{P_\infty \cdot x} &:= \mathbf{Maps}(\mathcal{D}'_x \to X, \mathbb{A}^1 \times \mathrm{pt}/P \to G\backslash \mathrm{Vin}_G^\gamma/G),\\
({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{P_\infty \cdot x} &:= \mathbf{Maps}(\mathcal{D}'_x \to X, \mathrm{pt}/P \to P\backslash \overline{M}/P),\\
(\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}\infty \cdot x} &:= \mathbf{Maps}(\mathcal{D}'_x \to X, G\backslash_0 \mathrm{Vin}_G^\gamma/G \to G\backslash \mathrm{Vin}_G^\gamma/G),\\
({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty \cdot x} &:= \mathbf{Maps}(\mathcal{D}'_x \to X, P\backslash M/P^- \to P\backslash \overline{M}/P),
\end{aligned}$$

where the symbol "P_∞" stands for "P-structure near x".

By construction, there is a commutative diagram

$$\begin{tikzcd}
({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{P_\infty \cdot x} \arrow[r] \arrow[d] & ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty \cdot x} \arrow[r, "\subset"] \arrow[d] & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[d, "\mathbf{p}^+_{\mathrm{glob}}"] \\
(\mathrm{VinBun}_G^\gamma)^{P_\infty \cdot x} \arrow[r] & (\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}\infty \cdot x} \arrow[r, "\subset"] & \mathrm{VinBun}_G^\gamma,
\end{tikzcd}$$

where the symbol "$\subset$" indicates the corresponding map is a schematic open embedding.

Lemma 6.1.8. *Locally on the smooth topology of* $({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty \cdot x}$*, the map*

$$({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{P_\infty \cdot x} \to ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty \cdot x}$$

is a trivial fibration with fibers isomorphic to $\mathcal{L}^+U_x^-$*.*

Proof. This follows from the following two facts:

- For any affine test scheme S and any $(P \times_M P^-)$-torsor $\mathcal{F}$ on $\mathcal{D}'_x \times S$, there exists an étale cover $S' \to S$ such that $\mathcal{F}$ is trivial after base-change along $S' \to S$.

[52] The definition of $({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}\infty \cdot x}$ below coincides with the previous one because of Remark 6.1.6.

- As plain schemes, $(P \times_M P^-)/P \simeq U^-$.

$\square$[Lemma 6.1.8]

6.1.9 (Hecke modifications)**.** We need to study those Hecke modifications on P-structures of VinBun_G^γ that fix the induced M-structures. The precise definition is as follows.

We temporarily write $q : \mathbb{A}^1 \times \mathrm{pt}/P \to \mathbb{A}^1 \times \mathrm{pt}/M$ for the canonical projection. We define $\mathcal{H}_x^{P_{\infty\cdot x}}$ to be the prestack whose value on an affine test scheme S classifies commuatative diagrams

$$\begin{tikzcd}
 & \mathcal{D}_x^\times \times S \arrow[dl] \arrow[dr] \arrow[rr] & & (X-x)\times S \arrow[dd, "\alpha'"] \\
\mathcal{D}'_x \times S \arrow[dr, "\delta_0"] & & \mathcal{D}'_x \times S \arrow[dl, "\delta_1"] & \\
 & \mathbb{A}^1 \times \mathrm{pt}/P \arrow[rr] & & G\backslash \mathrm{Vin}_G^\gamma/G.
\end{tikzcd}$$

such that the isomorphism

$$q \circ \delta_0|_{\mathcal{D}_x^\times \times S} \simeq q \circ \delta_1|_{\mathcal{D}_x^\times \times S}$$

given by the above diagram can be extended[53] to an isomorphism $q \circ \delta_0 \simeq q \circ \delta_1$.

By construction, we have two maps

$$h_0, h_1 : \mathcal{H}_x^{P_{\infty\cdot x}} \to (\mathrm{VinBun}_G^\gamma)^{P_{\infty\cdot x}}$$

given respectively by (δ_0, α') and (δ_1, α').

In the above definition, replacing the map $\mathbb{A}^1 \times \mathrm{pt}/P \to G\backslash \mathrm{Vin}_G^\gamma/G$ by $\mathrm{pt}/P \to P\backslash \overline{M}/P^-$ (and q by its 0-fiber), we define another prestack ${}_{\mathrm{str}}\mathcal{H}_x^{P_{\infty\cdot x}}$ equipped with two maps

$$h_0, h_1 : {}_{\mathrm{str}}\mathcal{H}_x^{P_{\infty\cdot x}} \to ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{P_{\infty\cdot x}}.$$

Lemma 6.1.10. *We have a canonical commutative diagram defined over* VinBun_G^γ*:*

$$\begin{tikzcd}
\mathcal{H}_x^{P_{\infty\cdot x}} \arrow[r, "h_0"] & (\mathrm{VinBun}_G^\gamma)^{P_{\infty\cdot x}} & \mathcal{H}_x^{P_{\infty\cdot x}} \arrow[l, "h_1"'] \\
{}_{\mathrm{str}}\mathcal{H}_x^{P_{\infty\cdot x}} \arrow[u, "p_{\mathcal{H}}"] \arrow[d, "f_{\mathcal{H}}"'] \arrow[r, "h_0"] & ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{P_{\infty\cdot x}} \arrow[u, "p"] \arrow[d, "f"'] & {}_{\mathrm{str}}\mathcal{H}_x^{P_{\infty\cdot x}} \arrow[u, "p_{\mathcal{H}}"'] \arrow[d, "f_{\mathcal{H}}"] \arrow[l, "h_1"'] \\
{}_{\mathrm{str}}H_x \arrow[r, "h_0"] & ({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P})^{\mathrm{d.f.}_{\infty\cdot x}} & {}_{\mathrm{str}}H_x, \arrow[l, "h_1"']
\end{tikzcd}$$

[53]Note that such extension is unique if it exists. Also, we can repalce $\mathbb{A}^1 \times \mathrm{pt}/M$ in the definition by pt/M because the given commutative diagram would determine a unique map $S \to \mathbb{A}^1$ such that the diagram is defined over $\mathbb{A}^1$.

such that the two lower squares are Cartesian.

Proof. The two top squares are obvious from definition. To prove the claims for the lower two squares, notice that the composition

$$\mathrm{pt}/P \to \mathrm{pt}/(P \underset{M}{\times} P^-) \simeq P\backslash M/P^- \hookrightarrow P\backslash \overline{M}/P^- \to P\backslash \mathrm{pt}$$

is isomorphic to the identity map. Therefore for a given $(P \times_M P^-)$-torsor $\mathcal{F}_{P\times_M P^-}$ on the disk $\mathcal{D}'_x$ and a given P-structure $\mathcal{F}_P^{\mathrm{sub}}$ of it, we have a canonical isomorphism

$$\mathcal{F}_P^{\mathrm{sub}} \simeq P \overset{(P\times_M P^-)}{\times} \mathcal{F}_{P\times_M P^-} =: \mathcal{F}_P^{\mathrm{ind}}.$$

Therefore a Hecke modification on $\mathcal{F}_P^{\mathrm{sub}}$ is the same as a Hecke modification on the induced P-torsor $\mathcal{F}_P^{\mathrm{ind}}$. This implies our claims by unwinding the definitions.

□[Lemma 6.1.10]

Consider the diagram

$$\begin{tikzcd} \mathcal{H}_x^{P_{\infty\cdot x}} \arrow[r, "h_0"] & (\mathrm{VinBun}_G^\gamma)^{P_{\infty\cdot x}} \arrow[d, "g"] & \mathcal{H}_x^{P_{\infty\cdot x}} \arrow[l, "h_1"'] \\ & \mathrm{VinBun}_G^\gamma, & \end{tikzcd}$$

and its fiber at C_P.

Lemma 6.1.11. *Goal 6.1.3 is implied by*

Goal 6.1.12. The objects $((g\circ h_0)|_{C_P})^!(\mathcal{M})$ and $((g\circ h_1)|_{C_P})^!(\mathcal{M})$ are isomorphic, where $\mathcal{M} := i^* \circ j_*(\omega)$.

Proof. Suppose we have an isomorphism as in the statement. Using Lemma 6.1.10 and a diagram chasing, we obtain an isomorphism

$$f_{\mathcal{H}}^! \circ h_0^!(\mathcal{G}) \simeq f_{\mathcal{H}}^! \circ h_1^!(\mathcal{G}). \tag{6.3}$$

On the other hand, by Lemma 6.1.8 and the Cartesian squares in Lemma 6.1.10, locally on the smooth topology of the target, $f_{\mathcal{H}}$ is a trivial fibration with contractible fibers. This implies $f_{\mathcal{H}}^!$ is fully faithful. Combining with the equivalence (6.3), we obtain an isomorphism $h_0^!(\mathcal{G}) \simeq h_1^!(\mathcal{G})$.

□[Lemma 6.1.11]

6.1.13 (Level structures)**.** To finish the proof, we need one last geometric construction. We define

$$(\mathrm{VinBun}_G^\gamma)^{\mathrm{level}_{\infty\cdot x}} := \mathbf{Maps}(\mathcal{D}'_x \to X, \mathbb{A}^1 \to G\backslash \mathrm{Vin}_G^\gamma/G),$$

where $\mathbb{A}^1 \to G\backslash \mathrm{Vin}_G^\gamma/G$ is induced by the canoncial section $\mathfrak{s}^\gamma : \mathbb{A}^1 \to \mathrm{Vin}_G^\gamma$. By definition, we have a chain

$$(\mathrm{VinBun}_G^\gamma)^{\mathrm{level}_{\infty\cdot x}} \to (\mathrm{VinBun}_G^\gamma)^{P_{\infty\cdot x}} \to (\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}_{\infty\cdot x}}.$$

Consider the relative jets scheme $\mathcal{L}^+_{\mathbb{A}^1}\widetilde{G}^\gamma_x$ whose value on an affine test scheme S classifies commutative diagrams

$$\begin{tikzcd} \mathcal{D}'_x \times S \arrow[r] \arrow[d] & \widetilde{G}^\gamma \arrow[d] \\ S \arrow[r, "\alpha"] & \mathbb{A}^1. \end{tikzcd}$$

It is a group scheme over $\mathbb{A}^1$. Since $\widetilde{G}^\gamma \to \mathbb{A}^1$ is smooth, a relative (to $\mathbb{A}^1$) version of [Ras16, Lemma 2.5.1] implies $\mathcal{L}^+_{\mathbb{A}^1}\widetilde{G}^\gamma_x \to \mathbb{A}^1$ is pro-smooth. Since $G\backslash_0 \mathrm{Vin}_G^\gamma/G \simeq \mathbb{A}^1/\widetilde{G}^\gamma$, there is a canonical $\mathcal{L}^+_{\mathbb{A}^1}\widetilde{G}^\gamma_x$-action on $(\mathrm{VinBun}_G^\gamma)^{\mathrm{level}_{\infty\cdot x}}$, which preserves the projection to $(\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}_{\infty\cdot x}}$.

Lemma 6.1.14. $(\mathrm{VinBun}_G^\gamma)^{\mathrm{level}_{\infty\cdot x}}$ *is an* $\mathcal{L}^+_{\mathbb{A}^1}\widetilde{G}^\gamma_x$*-torsor on* $(\mathrm{VinBun}_G^\gamma)^{\mathrm{d.f.}_{\infty\cdot x}}$*, and it is a trivial torsor locally on the smooth topology.*

Proof. It suffices to show that for any affine test scheme S over $\mathbb{A}^1$ and any (fppf) $\widetilde{G}^\gamma$-torsor $\mathcal{E}$ on $\mathcal{D}'_x \times S$, there exists an étale cover $S' \to S$ such that $\mathcal{E} \times_S S'$ is a trivial $\widetilde{G}^\gamma$-torsor on $\mathcal{D}'_x \times S'$.

Consider the restiction of $\mathcal{E}|_x$ on $x \times S \to \mathcal{D}' \times S$. Since $\widetilde{G}^\gamma \to \mathbb{A}^1$ is smooth, there exists an étale cover $S' \to S$ such that $(\mathcal{E} \times_S S')|_x$ is a trivial $\widetilde{G}^\gamma$-torsor on $x \times S'$. Since $\mathcal{E} \times_S S' \to S'$ is smooth, by the lifting property of smooth maps, $(\mathcal{E} \times_S S')|_{\mathcal{D}_x}$ is a trivial $\widetilde{G}^\gamma$-torsor on $\mathcal{D}_x \times S'$, where $\mathcal{D}_x$ is the *formal* disk.

It remain to show that a $\widetilde{G}^\gamma$-torsor on $\mathcal{D}'_x \times S$ is trivial iff its restiction on $\mathcal{D}_x \times S$ is trivial. The proof is similar to that of [Ras16, Lemma 2.12.1][54] and the only necessary modification is to show $\widetilde{G}^\gamma \to \mathbb{A}^1$ has enough vector bundle representations on $\mathbb{A}^1$. But this is obvious because any sub-representation of $\mathcal{O}_{\widetilde{G}^\gamma}$ is a flat $\mathcal{O}_{\mathbb{A}^1}$-module.

□[Lemma 6.1.14]

Lemma 6.1.15. $(\mathrm{VinBun}_G^\gamma)^{\mathrm{level}_{\infty\cdot x}}$ *is an* $\mathcal{L}^+P_x$*-torsor on* $(\mathrm{VinBun}_G^\gamma)^{P_{\infty\cdot x}}$*, and it is a trivial torsor locally on the smooth topology.*

Proof. The proof is similar to that of Lemma 6.1.14. Actually, it is much easier because $\mathcal{L}^+U_x$ is a absolute group.

□[Lemma 6.1.15]

[54]The difference is: our group scheme is relative to $\mathbb{A}^1$, while that in [Ras16] is relative to X.

Lemma 6.1.16. *Locally on the smooth topology of* $(\mathrm{VinBun}_G^{\gamma})^{P_{\infty\cdot x}}$*, both the projections*

$$h_0, h_1 : \mathcal{H}_x^{P_{\infty\cdot x}} \to (\mathrm{VinBun}_G^{\gamma})^{P_{\infty\cdot x}}$$

are isomorphic to trivial fibrations with fibers isomorphic to $\mathrm{Gr}_{U,x}$.

Proof. For an affine test scheme S over $(\mathrm{VinBun}_G^{\gamma})^{P_{\infty\cdot x}}$, let $\mathcal{F}_P$ be the corresponding P-torsor on $\mathcal{D}'_x \times S$. Replace S by an étale cover, we can assume $\mathcal{F}_P$ is trivial. Then the fiber product

$$\mathcal{H}_x^{P_{\infty\cdot x}} \underset{h_0,(\mathrm{VinBun}_G^{\gamma})^{P_{\infty\cdot x}}}{\times} S$$

classifies P-torsors $\mathcal{F}'_P$ on $\mathcal{D}'_x \times S$ equipped with an isomorphism $\mathcal{F}'_P|_{\mathcal{D}_x^{\times}\times S} \simeq \mathcal{F}_P|_{\mathcal{D}_x^{\times}\times S}$ such that the induced isomorphism on induced M-torsors can be extended to $\mathcal{D}'_x \times S$. Since $\mathcal{F}_P$ is trivial, this fiber product is isomorphic to $\mathrm{Gr}_{U,x} \times S$.

□[Lemma 6.1.16]

6.1.17 (Finish of the proof)**.** By Lemma 6.1.11, it suffices to show for any $k = 0$ or 1, the operation $i^* \circ j_*$ commutes with !-pullback functor along the composition

$$\mathcal{H}_x^{P_{\infty\cdot x}} \xrightarrow{h_k} (\mathrm{VinBun}_G^{\gamma})^{P_{\infty\cdot x}} \xrightarrow{g} \mathrm{VinBun}_G^{\gamma}.$$

The claim for the map h_k follows from Lemma 6.1.16. To prove the claim for the map g, by Lemma 6.1.15, it suffices to prove the claim for the map

$$(\mathrm{VinBun}_G^{\gamma})^{\mathrm{level}_{\infty\cdot x}} \to \mathrm{VinBun}_G^{\gamma}.$$

Then we are done by Lemma 6.1.14.

□[Proposition 5.1.4]

Lemma 6.1.18. *Let* $q : Z_1 \to Z_2$ *be a smooth, safe and UHC map. Let* $Z'_2 \to Z_2$ *be a Zariski cover and* $q' : Z'_1 \to Z'_2$ *be the base-change of* q*. Then an object* $\mathcal{G} \in \mathrm{DMod}(Z_1)$ *is contained in the essential image of* $q^!$ *iff its* !*-pullback in* $\mathrm{DMod}(Z'_1)$ *is contained in the essential image of* $(q')^!$.

Proof. Follows from the Zariski descent of D-modules and the fact that $q^!$ is fully faithful.

□[Lemma 6.1.18]

Lemma 6.1.19. *Let $q : Z_1 \to Z_2$ be a smooth, safe and UHC map. Consider the projections*

$$\mathrm{pr}_1, \mathrm{pr}_2 : Z_1 \underset{Z_2}{\times} Z_1 \to Z_2.$$

Then an object $\mathcal{G} \in \mathrm{DMod}(Z_1)$ is contained in the essential image of $q^!$ iff $\mathrm{pr}_1^!(\mathcal{G})$ is isomorphic to $\mathrm{pr}_2^!(\mathcal{G})$.

Proof. The "only if" part is trivial. Now suppose we have an isomorphism $\mathrm{pr}_1^!(\mathcal{G}) \simeq \mathrm{pr}_2^!(\mathcal{G})$. It follows from definitions that pr_1 and pr_2 are also smooth, safe and UHC. Hence we have

$$\mathcal{G} \simeq (\mathrm{pr}_1^!)^R \circ \mathrm{pr}_1^!(\mathcal{G}) \simeq (\mathrm{pr}_1^!)^R \circ \mathrm{pr}_2^!(\mathcal{G}) \simeq q^! \circ (q^!)^R$$

as desired, where the last isomorphism is the base-change isomorphism.

□[Lemma 6.1.19]

In the above proof of Proposition 5.1.4, we can

- replace VinBun_G^γ by $\overline{\mathrm{Bun}}_{G,\geq P}$,
- replace $\widetilde{G}^\gamma$ by $\widetilde{G}_{\geq C_P}$,
- replace $i^* \circ j_* : \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G \times \mathbb{G}_m) \to \mathrm{DMod}(\mathrm{VinBun}_G|_{C_P})$ by $i_P^* \circ j_{G,*} : \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,G}) \to \mathrm{DMod}(\overline{\mathrm{Bun}}_{G,P})$,

and obtain the following result:

Variant 6.1.20. *The objects*

$$i_P^* \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G}), \text{ and } i_P^! \circ j_{G,!} \circ r_!(k_{\mathrm{Bun}_G})$$

are contained in

$$\mathrm{DMod}(\overline{\mathrm{Bun}}_{G,P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)} \subset \mathrm{DMod}(\overline{\mathrm{Bun}}_{G,P}).$$

6.2 Proof of Proposition-Construction 5.3.1

By definition, we have $\overline{\mathrm{Bun}}_{G,\leq P} \simeq \mathrm{VinBun}_{G,\leq P}/T$ and

$$\mathrm{VinBun}_{G,\leq P} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash \mathrm{Vin}_{G,\leq P}/G \supset G\backslash_0 \mathrm{Vin}_{G,\leq P}/G).$$

By Fact 1.1.4(2), we have

$$G\backslash {}_0\mathrm{Vin}_{G,\le P}/G \simeq \mathbb{B}\widetilde{G}_{\le P}, \tag{6.4}$$

where $\widetilde{G}_{\le P}$ is a subgroup scheme of $G\times G\times T^+_{\mathrm{ad},\le P}\to T^+_{\mathrm{ad},\le P}$.

By Lemma A.1.1, the map $\mathbb{B}\widetilde{G}_{\le P}\to \mathbb{B}G\times\mathbb{B}G$ factors as

$$\mathbb{B}\widetilde{G}_{\le P}\to \mathbb{B}P\times\mathbb{B}P^-\to\mathbb{B}G\times\mathbb{B}G. \tag{6.5}$$

Also, the maps $\mathbb{B}\widetilde{G}_{\le P}\to\mathbb{B}P\times\mathbb{B}P^-$ are functorial in P. Now we have the following commutative diagram of algebraic stacks:

$$\begin{tikzcd} G\backslash \mathrm{Vin}_{G,\le P}/G \arrow[d] & G\backslash {}_0\mathrm{Vin}_{G,\le P}/G \arrow[l] \arrow[d, "(6.5)\circ(6.4)"] \\ \mathbb{B}G\times\mathbb{B}G & \mathbb{B}P\times\mathbb{B}P^-. \arrow[l] \end{tikzcd} \tag{6.6}$$

Taking $\mathbf{Maps}_{\mathrm{gen}}(X,-)$, we obtain canonical maps

$$\mathrm{VinBun}_{G,\le P}\to \mathrm{Bun}_G^{P\text{-gen}}\times\mathrm{Bun}_G^{P^-\text{gen}}$$

functorial in P. To finish the construction, we only need to show:

Lemma 6.2.1. *The map $G\backslash {}_0\mathrm{Vin}_{G,\le P}/G\to\mathbb{B}P\times\mathbb{B}P^-$ constructed above can be uniquely lifted to a morphism*

$$(T \curvearrowright G\backslash {}_0\mathrm{Vin}_{G,\le P}/G)\to(\mathrm{pt}\curvearrowright \mathbb{B}P\times\mathbb{B}P^-)$$

fitting into the following commutative diagram

$$\begin{tikzcd} (T \curvearrowright G\backslash \mathrm{Vin}_{G,\le P}/G) \arrow[d] & (T \curvearrowright G\backslash {}_0\mathrm{Vin}_{G,\le P}/G) \arrow[l] \arrow[d] \\ (\mathrm{pt}\curvearrowright\mathbb{B}G\times\mathbb{B}G) & (\mathrm{pt}\curvearrowright\mathbb{B}P\times\mathbb{B}P^-). \arrow[l] \end{tikzcd}$$

Proof. The uniqueness follows from the fact that $\mathbb{B}P\times\mathbb{B}P^-\to\mathbb{B}G\times\mathbb{B}G$ is schematic. It remains to prove the existence.

The map $G\backslash {}_0\mathrm{Vin}_{G,\le P}/G\to\mathbb{B}P\times\mathbb{B}P^-$ induces a $(G\times G)$-equivariant map

$${}_0\mathrm{Vin}_{G,\le P}\to G/P\times G/P^-. \tag{6.7}$$

We only need to show the T-action on ${}_0\mathrm{Vin}_{G,\le P}$ preserves the fibers of this map.

Recall that any closed point in ${}_0\mathrm{Vin}_{G,\le P}$ is of the form $g_1\cdot\mathfrak{s}(s)\cdot g_2^{-1}$ where g_1 and g_2 are closed points of G, s is a closed point of $T^+_{\mathrm{ad},\leq P}$ and $\mathfrak{s}$ is the canonical section. Unwinding the definitions, the map (6.7) sends this point to (g_1, g_2). Now consider the canonical T-action on Vin_G. It follows from definition that a closed point t of T sends the point $\mathfrak{s}(s)$ to the point $\iota(t)\cdot\mathfrak{s}(ts)$, where $\iota: T \hookrightarrow G$ is the embedding. Since the T-action commutes with the $(G\times G)$-action, the element t sends $g_1\cdot\mathfrak{s}(s)\cdot g_2^{-1}$ to $g_1\iota(t)\cdot\mathfrak{s}(s)\cdot g_2^{-1}$. This makes the desired claim manifest.

□[Lemma 6.2.1]

□[Proposition-Construction 5.3.1]

6.3 Proof of Proposition 5.3.4

Proposition 5.3.4 follows from the following two lemmas and Proposition 5.1.4.

Lemma 6.3.1. *The !-pushforward functors*

$$\begin{tikzcd}
\mathrm{DMod}_{\mathrm{indhol}}({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}) \arrow[r] \arrow[d] & \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{VinBun}_G|_{C_P}) \arrow[d] \\
\mathrm{DMod}_{\mathrm{indhol}}({}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P}) \arrow[r] & \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})
\end{tikzcd}$$

preserve $(U(\mathbb{A}_F)\times U^-(\mathbb{A}_F))$-equivariant objects.

Proof. The claim for the left functor follows from definitions. It is clear that the claim for the bottom and right functor follows from that for the top functor.

By [Sch16, § 3.3.2], the map $f: {}_{\mathrm{str}}\mathrm{VinBun}_{G,C_P} \to \mathrm{VinBun}_{G,C_P}$ factors as

$${}_{\mathrm{str}}\mathrm{VinBun}_{G,C_P} \xrightarrow{j} \widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-} \xrightarrow{\overline{f}} \mathrm{VinBun}_{G,C_P}$$

such that j is a schematic open embedding and $\overline{f}$ is proper on each connected component. Recall that $\widetilde{\mathrm{Bun}}_P$ also has a defect stratification with

$${}_{\mathrm{str}}\widetilde{\mathrm{Bun}}_P \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}}.$$

We define $\mathrm{DMod}({}_{\mathrm{str}}\widetilde{\mathrm{Bun}}_P)^{U(\mathbb{A}_F)}$ to be the full subcategory of $\mathrm{DMod}({}_{\mathrm{str}}\widetilde{\mathrm{Bun}}_P)$ consisting of objects that are !-pullbacks from $\mathrm{DMod}(H_{M,G\text{-}\mathrm{pos}})$. We define $\mathrm{DMod}(\widetilde{\mathrm{Bun}}_P)^{U(\mathbb{A}_F)}$ similarly as before. We also define

$$\mathrm{DMod}(\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)}.$$

We claim the functor $\overline{f}_!$ preserves $(U(\mathbb{A}_F)\times U^-(\mathbb{A}_F))$-equivariant objects. To prove the claim, we use the fact that $\overline{f}$ is compatible with the defect stratifications. In other words, we have

$$
\begin{aligned}
&{}_{\mathrm{str}}\mathrm{VinBun}_{G,C_P} \underset{\mathrm{VinBun}_{G,C_P}}{\times} (\widetilde{\mathrm{Bun}}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \widetilde{\mathrm{Bun}}_{P^-}) \simeq \\
&\simeq (\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}}) \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}),
\end{aligned}
$$

such that the porjection from the RHS to

$$
{}_{\mathrm{str}}\mathrm{VinBun}_{G,C_P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}
$$

is induced by a canonical map (the composition map)

$$
H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \to H_{M,G\text{-}\mathrm{pos}}.
$$

Then the claim follows from the base-change isomorphisms (which exist because $\overline{f}$ is proper on each connected component).

It remains to show $j_!$ preserves $(U(\mathbb{A}_F)\times U^-(\mathbb{A}_F))$-equivariant objects. Using the base-change isomorphism, it suffices to show that the !-pushforward functor

$$
\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_P) \to \mathrm{DMod}_{\mathrm{indhol}}(\widetilde{\mathrm{Bun}}_P)
$$

preserves $U(\mathbb{A}_F)$-equivariant object. However, this is well-known and was proved in § G.1.

$\square$[Lemma 6.3.1]

Lemma 6.3.2. *The functor*

$$
\overline{\Delta}^{\mathrm{enh}}_{P,!} : \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P}) \to \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}})
$$

sends objects in $\mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)}$ *to objects in* $\mathrm{I}(G\times G, P\times P^-)$.

Proof. Lemma 6.3.1 formally implies $\mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)}$ is generated under colimits and extensions by the image of the !-pushforward funnctor

$$
\mathrm{DMod}_{\mathrm{indhol}}({}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)} \to \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})^{U(\mathbb{A}_F)\times U^-(\mathbb{A}_F)}.
$$

Hence it suffices to show the !-pushforward along

$$
{}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G
$$

preserves $(U(\mathbb{A}_F)\times U^-(\mathbb{A}_F))$-equivariant objects. Unwinding the definitions, this map is isomorphic to

$$
\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-pos}}/Z_M) \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-} \xrightarrow{a} \mathrm{Bun}_P \times \mathrm{Bun}_{P^-} \xrightarrow{b} \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}.
$$

By the base-change isomorphism, $a_!$ preserves $(U(\mathbb{A}_F)\times U^-(\mathbb{A}_F))$-equivariant objects. By Proposition 2.3.17(1), so is $b_!$.

□[Lemma 6.3.2]

□[Proposition 5.3.4]

6.4 Outline of the proof of Theorem 5.2.6

In this section, we reduce Theorem 5.2.6 to a series of partial results, which will be proved in later sections.

Step 1: constructing the natural transformation

The first step is to construct a natural transformation from $\overline{\Delta}_! \circ \mathbf{K}$ to $\mathbf{E}\circ\mathbf{DL}$. Let us first explain how to construct the morphism

$$
\overline{\Delta}_! \circ \mathbf{K}(P) \to \mathbf{E}\circ\mathbf{DL}(P). \tag{6.8}
$$

For $P \in \mathrm{Par}$, let $\overline{\mathrm{Bun}}_{G,P}$ be the P-stratum of $\overline{\mathrm{Bun}}_G$. By Proposition-Construction 5.3.1, we have the following commutative diagram

$$
\begin{tikzcd}
\overline{\mathrm{Bun}}_{G,P} \arrow[r, "\overline{\Delta}_P^{\mathrm{enh},l}"] \arrow[d, "i_P"] & \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \arrow[d, "\mathfrak{p}^{\mathrm{enh}}_{P\times G\to G\times G}"] \\
\overline{\mathrm{Bun}}_G \arrow[r, "\overline{\Delta}"] & \mathrm{Bun}_G \times \mathrm{Bun}_G .
\end{tikzcd} \tag{6.9}
$$

Consider the object

$$
\mathcal{F}_P := \overline{\Delta}^{\mathrm{enh},l}_{P,!} \circ i_P^! \circ \mathbf{K}(P) \in \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G). \tag{6.10}
$$

Note that

$$
\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathfrak{p}^{\mathrm{enh}}_{P\times G\to G\times G,!}(\mathcal{F}_P) \tag{6.11}
$$

because $\mathbf{K}(P) \simeq i_{P,!}\circ i_P^!(\mathbf{K}(P))$.

The proof of the following result is similar to[55] that of Proposition 5.3.4.

Variant 6.4.1. *The object $\mathcal{F}_P$ is contained in the full subcategory*

$$\mathrm{I}(G \times G, P \times G) \subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

By (6.11), we have

$$\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}_P). \tag{6.12}$$

Hence by functoriality of the LHS, we obtain a morphism

$$\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}_P) \to \mathcal{F}_G. \tag{6.13}$$

By adjunction, we have a morphism

$$\theta_P : \mathcal{F}_P \to \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\mathcal{F}_G). \tag{6.14}$$

Note that $\mathcal{F}_G \simeq \Delta_!(k_{\mathrm{Bun}_G})$.

On the other hand, it is easy to see[56]

$$\mathbf{E} \circ \mathbf{DL}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\Delta_!(k_{\mathrm{Bun}_G})). \tag{6.15}$$

Now we declare the morphism (6.8) to be given by

$$\overline{\Delta}_! \circ \mathbf{K}(P) \simeq \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}_P) \overset{\mathrm{Eis}^{\mathrm{enh}}(\theta_P)}{\to} \mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}(\mathcal{F}_G) \simeq \mathbf{E} \circ \mathbf{DL}(P). \tag{6.16}$$

In order to obtain the desired natural transformation, recall Proposition-Construction 5.3.1 provides a map

$$\overline{\Delta}^{\mathrm{enh},l}_{\leq P} : \overline{\mathrm{Bun}}_{G,\leq P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G$$

extending $\overline{\Delta}^{\mathrm{enh},l}_P$. By definition, we also have

$$\mathcal{F}_P \simeq \overline{\Delta}^{\mathrm{enh},l}_{\leq P,!} \circ i^!_{\leq P} \circ \mathbf{K}(P).$$

[55]We use Variant 6.1.20 instead of Proposition 5.1.4.

[56]One needs to use Lemma 6.5.1.

Using this, it is pure formal to show that the morphisms (6.16) constructed above is functorial in P. Namely, in § 6.5, we will use the theory of (co)Cartesian fibrations to prove:

Proposition-Construction 6.4.2. *There exists a canonical natural transformation* $\overline{\Delta}_! \circ \mathbf{K} \to \mathbf{E} \circ \mathbf{DL}$ *whose value at* $P \in \mathrm{Par}$ *is equivalent to the morphism (6.16).*

Step 2: translating by the second adjointness

After obtaining the natural transformation, we only need to show its value at each $P \in \mathrm{Par}$ is invertible. From this step on, we fix such a standard parabolic P.

Unwinding the definitions, we only need

Goal 6.4.3. To show the map (6.14)

$$\theta_P : \mathcal{F}_P \to \mathrm{CT}^{\mathrm{enh}}_{G\times G \to P\times G}(\mathcal{F}_G)$$

is invertible.

Recall (see Remark 2.3.8) that the functor

$$\iota^!_{M\times G} : \mathrm{I}(G\times G, P\times G) \to \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$$

is conservative. Hence we only need to show the map $\iota^!_{M\times G}(\theta_P)$ is invertible. By definition, $\iota^!_{M\times G}$ is isomorphic to

$$\mathrm{I}(G\times G, P\times G) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \xrightarrow{\iota^!_{P\times G}} \mathrm{DMod}(\mathrm{Bun}_P \times \mathrm{Bun}_G) \xrightarrow{\mathfrak{q}_{P\times G,*}} \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G).$$

We denote the composition of the latter two functors by

$$\mathrm{CT}^{\mathrm{gen}}_{P\times G,*} : \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \xrightarrow{\iota^!_{P\times G}} \mathrm{DMod}(\mathrm{Bun}_P \times \mathrm{Bun}_G) \xrightarrow{\mathfrak{q}_{P\times G,*}} \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G).$$

Then the source of $\iota^!_{M\times G}(\theta_P)$ is isomorphic to $\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P)$.

On the other hand, the functor $\iota^!_{M\times G} \circ \mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}$ is isomorphic to the usual geometric constant term functor

$$\mathrm{CT}_{P\times G,*} : \mathrm{DMod}(\mathrm{Bun}_G \times \mathrm{Bun}_G) \to \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$$

(as can be seen by passing to left adjoints). Hence the target of $\iota^!_{M\times G}(\theta_P)$ is isomorphic to $\mathrm{CT}_{P\times G,*}(\mathcal{F}_G)$.

Let

$$\gamma_P : \mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P) \to \mathrm{CT}_{P\times G,*}(\mathcal{F}_G). \tag{6.17}$$

be the morphism obtained from $\iota^!_{M\times G}(\theta_P)$ via the above isomorphisms. Then we have reduced Theorem 5.2.6 to

Goal 6.4.4. To show γ_P is invertible.

Recall that the main theorem of [DG16] says that when restricted to each connected component $\mathrm{Bun}_{M,\lambda}$ of Bun_M, the functor

$$\mathrm{CT}_{P,*,\lambda} : \mathrm{DMod}(\mathrm{Bun}_G) \xrightarrow{!\text{-pull}} \mathrm{DMod}(\mathrm{Bun}_{P,\lambda}) \xrightarrow{*\text{-push}} \mathrm{DMod}(\mathrm{Bun}_{M,\lambda})$$

is canonically left adjoint to

$$\mathrm{Eis}_{P^-,*,\lambda} : \mathrm{DMod}(\mathrm{Bun}_{M,\lambda}) \xrightarrow{!\text{-pull}} \mathrm{DMod}(\mathrm{Bun}_{P^-,\lambda}) \xrightarrow{*\text{-push}} \mathrm{DMod}(\mathrm{Bun}_G).$$

In particular, the functor $\mathrm{CT}_{P,*} \simeq \bigoplus \mathrm{CT}_{P,*,\lambda}$ preserves ind-holonomic objects and its restriction to $\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G)$ is canonically isomorphic to[57]

$${}'\mathrm{CT}_{P^-,!} : \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G) \xrightarrow{\mathfrak{p}^{-,*}} \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_{P^-}) \xrightarrow{\mathfrak{q}^-_!} \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_M).$$

Hence we obtain a canonical isomorphism

$$\mathrm{CT}_{P\times G,*}(\mathcal{F}_G) \simeq {}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G).$$

Now there is a similar story when we replace Bun_G by $\mathrm{Bun}_G^{P\text{-gen}}$. To state it, we need to review the counit natural transformation for the adjoint pair $(\mathrm{CT}^\lambda_{P,*}, \mathrm{Eis}^\lambda_{P^-,*})$. In *loc.cit.*, the authors considered the correspondences

$$\alpha^+_{P,\lambda} : (\mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P,\lambda} \to \mathrm{Bun}_G),\ \alpha^-_{P,\lambda} : (\mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_{M,\lambda})$$

and a canonical 2-morphism in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$ (see § C.1 and § C.2 for what this means):

$$\alpha^+_{P,\lambda} \circ \alpha^-_{P,\lambda} \to \mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}.$$

[57] We use the notation ${}'\mathrm{CT}_{P^-,!}$ because $\mathrm{CT}_{P^-,!}$ was used in *loc.cit.* to denote the corresponding functor for all the D-modules.

Explicitly, this 2-morphism is given by the schematic open embedding

$$\mathrm{Bun}_{M,\lambda} \to \mathrm{Bun}_{P,\lambda} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-,\lambda}.$$

Then the counit natural transformation is given by[58]

$$\mathrm{CT}^{\lambda}_{P,*} \circ \mathrm{Eis}^{\lambda}_{P^-,*} \simeq \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^+_{P,\lambda}) \circ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^-_{P,\lambda}) \to \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}) \simeq \mathbf{Id}.$$

Motivated by this construction, we prove the following two results in § 6.6 and § 6.8.

Lemma 6.4.5. *We have:*

(1) The correspondences

$$\begin{aligned} \alpha^{+,\mathrm{gen}}_{P,\lambda} &: (\mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P,\lambda} \to \mathrm{Bun}_G^{P\text{-gen}}), \\ \alpha^{-,\mathrm{gen}}_{P,\lambda} &: (\mathrm{Bun}_G^{P\text{-gen}} \leftarrow \mathrm{Bun}^{M\text{-gen}}_{P^-,\lambda} \to \mathrm{Bun}_{M,\lambda}) \end{aligned}$$

are morphisms in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$. *In fact, the first leftward map is* safe *(by [DG16, Footnote 2]) and the second leftward map is quasi-compact and schematic.*

(2) There is a 2-morphism $\alpha^{+,\mathrm{gen}}_{P,\lambda} \circ \alpha^{-,\mathrm{gen}}_{P,\lambda} \to \mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}$ *given by the canonical map*

$$\mathrm{Bun}_{M,\lambda} \to \mathrm{Bun}_{P,\lambda} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}^{M\text{-gen}}_{P^-,\lambda}.$$

In other words, this map is a schematic open embedding[59].

Theorem 6.4.6. *The natural transformation*

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{+,\mathrm{gen}}_{P,\lambda}) \circ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{-,\mathrm{gen}}_{P,\lambda}) \to \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}) \simeq \mathbf{Id}_{\mathrm{DMod}(\mathrm{Bun}_{M,\lambda})}$$

is the counit natural transformation for an adjoint pair

$$(\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{+,\mathrm{gen}}_{P,\lambda}), \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{-,\mathrm{gen}}_{P,\lambda})).$$

As before, the above theorem implies that $\mathrm{CT}^{\mathrm{gen}}_{P\times G,*}$ preserves ind-holonomic objects and its restriction

[58]See Construction C.2.13 for the notation $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}$.

[59]In fact, $\mathrm{Bun}_P \times_{\mathrm{Bun}_G^{P\text{-gen}}} \mathrm{Bun}^{M\text{-gen}}_{P^-}$ is the open Zastava space in the literature. See Remark 6.4.9 below.

to $\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$ is canonically isomorphic to

$$'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!} : \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \overset{*\text{-pull}}{\to} \mathrm{DMod}(\mathrm{Bun}_{P^-}^{M\text{-gen}} \times \mathrm{Bun}_G) \overset{!\text{-push}}{\to} \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G).$$

Hence the morphism (6.17) is equivalent to a certain morphism

$$'\gamma_P : {}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}(\mathcal{F}_P) \to {}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G).$$

Hence we have reduced the main theorem to the following problem:

Goal 6.4.7. To show $'\gamma_P$ is invertible.

Remark 6.4.8. It is easier to study $'\gamma_P$ than γ_P because we can use the base-change isomorphims.

Remark 6.4.9. Note that we have

$$\begin{aligned}\mathrm{Bun}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_{P^-}^{M\text{-gen}} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P \underset{\mathbb{B}G}{\times} \mathbb{B}P^- \leftarrow \mathbb{B}P \underset{\mathbb{B}P}{\times} \mathbb{B}M) \simeq \\ \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash G/P^- \supset P\backslash G^{\mathrm{Bruhat}}/P^-) \simeq {}_0Z^P_{\mathrm{rel}}.\end{aligned}$$

Using this, it is easy to deduce Proposition 2.3.12 from Theorem 6.4.6.

Step 3: diagram chasing

Using the base-change isomorphisms, and using the facts that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G$, one can simplify the source and target of $'\gamma_P$. Let us state the result directly[60] Consider the correspondences

$$\begin{aligned}\beta_P : (\mathrm{Bun}_M \times \mathrm{Bun}_G \quad &\leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \times_{\mathrm{Bun}_G^{P\text{-gen}}} \overline{\mathrm{Bun}}_{G,P} \to \quad \overline{\mathrm{Bun}}_G) \\ \beta_G : (\mathrm{Bun}_M \times \mathrm{Bun}_G \quad &\leftarrow \mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,G} \to \quad \overline{\mathrm{Bun}}_G),\end{aligned}$$

where the left arm of β_P is given by

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_M \times \mathrm{Bun}_G \times \mathrm{Bun}_G \overset{\mathrm{pr}_{13}}{\to} \mathrm{Bun}_M \times \mathrm{Bun}_G\,.$$

---

[60]The result below only serves as motivation and will be incorporated into Lemma 6.4.11.

Then the base-change isomorphisms provide[61]

$$
\begin{aligned}
{}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}(\mathcal{F}_P) &\simeq (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta_P)\circ \mathbf{K}(P),\\
{}'\mathrm{CT}_{P^-\times G,!}(\mathcal{F}_G) &\simeq (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta_G)\circ \mathbf{K}(G).
\end{aligned}
$$

This motivates the following construction (see § 6.9):

Proposition-Construction 6.4.10. *There exists an open substack*[62]

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\subset \mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P}$$

such that the parameterized correspondence

$$
\begin{tikzcd}
\beta: & \mathrm{Bun}_M\times\mathrm{Bun}_G & (\mathrm{Bun}_{P^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \arrow[l] \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_G \\
& & T^+_{\mathrm{ad},\geq P}/T &
\end{tikzcd}
$$

captures β_P (resp. β_G) as its restriction to the P-stratum (resp. G-stratum) of $T^+_{\mathrm{ad},\geq P}/T$.

Using the fact that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P}\to\overline{\mathrm{Bun}}_G$ again, we obtain isomorphisms

$$
\begin{aligned}
(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta_P)\circ \mathbf{K}(P) &\simeq (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ \mathbf{K}(P),\\
(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta_G)\circ \mathbf{K}(G) &\simeq (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ \mathbf{K}(G).
\end{aligned}
$$

We will prove the following result in § 6.10 by a routine diagram chasing:

Lemma 6.4.11. *The morphisms γ_P and ${}'\gamma_P$ are equivalent to the morphism*

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ \mathbf{K}(P)\to(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ \mathbf{K}(G)$$

given by the functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ\mathbf{K}$.

Hence we have reduced the main theorem to the following problem:

Goal 6.4.12. To show the functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)\circ\mathbf{K}$ sends the arrow $P\to G$ to an isomorphism.

[61] See § C.5.7 for the notation $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}$.

[62] In the case $G=\mathrm{SL}_2$, recall that $\mathrm{Bun}_{B^-}\times_{\mathrm{Bun}_G}\overline{\mathrm{Bun}}_G$ classifies certain chains $\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{L}_2$. Then the desired open substack classifies those chains such that the restriction of $\mathcal{E}_1\to\mathcal{L}_2$ at any geometric point of S is nonzero.

Step 4: restoring the symmetry

In § 6.11, we will show

Proposition-Construction 6.4.13. *There exists a canonical factorization of the map*[63]

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G$$

via $\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}$.

In particular we obtain a correspondence

$$\beta' : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow (\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \overline{\mathrm{Bun}}_G) \tag{6.18}$$

and we only need to show $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}$ sends the arrow $P \to G$ to an isomorphism.

The following result will be proved in § 6.12:

Proposition 6.4.14. *The objects* $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}(P)$ *and* $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}(G)$ *are both contained in the full subcategory*

$$\mathrm{I}(M \times G, M \times P^-) \subset \mathrm{DMod}(\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

Consider the correspondence

$$\delta : (\mathrm{Bun}_M \times \mathrm{Bun}_M \leftarrow \mathrm{Bun}_M \times \mathrm{Bun}_P^{M\text{-gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}) \tag{6.19}$$

and the functor

$$'\mathrm{CT}^{\mathrm{gen}}_{M\times P,!} := (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta).$$

Similar to Step 2, we can use "the second adjointness" to reduce the main theorem to the following problem:

Goal 6.4.15. To show the functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta \circ \beta') \circ \mathbf{K}$ sends the arrow $P \to G$ to an isomorphism.

[63]In the case $G = \mathrm{SL}_2$, the map $(\mathrm{Bun}_{B^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_G)^{\mathrm{gen}} \to \mathrm{Bun}_G^{B^-\text{-gen}}$ sends a chain $\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{L}_2$ in Footnote 62 to the generic B^--reduction provided by the map $\mathcal{E}_1 \to \mathcal{L}_2$.

Step 5: calculating via the local models

Now comes the critical observation. Consider Schieder's (relative) local model[64] (see § A.5.6):

$$Y^P_{\mathrm{rel}} := \mathbf{Maps}_{\mathrm{gen}}(X, P^- \backslash \mathrm{Vin}_{G,\geq C_P}/P \supset P^- \backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/P).$$

The T-action on $G\backslash \mathrm{Vin}_{G,\geq P}/G$ induces a Z_M-action on $P^- \backslash \mathrm{Vin}_{G,\geq C_P}/P$. It is known that it induces a Z_M-action on Y^P_{rel}. Note that we have a morphism between stacks equipped with group actions:

$$(\mathrm{pt} \curvearrowright \mathrm{Bun}_M \times \mathrm{Bun}_M) \leftarrow (Z_M \curvearrowright Y^P_{\mathrm{rel}}) \to (T \curvearrowright \mathrm{VinBun}_{G,\geq P}).$$

The following result is proved in § 6.11.

Lemma 6.4.16. *The composition $\delta \circ \beta'$ is isomorphic to*

$$\mathrm{Bun}_M \times \mathrm{Bun}_M \leftarrow Y^P_{\mathrm{rel}}/Z_M \to \overline{\mathrm{Bun}}_G.$$

It is known (see Construction A.5.8) that $Y^P_{\mathrm{rel}}/Z_M \to \mathrm{Bun}_M \times \mathrm{Bun}_M$ factors via $H_{M,G\text{-}\mathrm{pos}}/Z_M$, where $H_{M,G\text{-}\mathrm{pos}}$ is the G-positive part of Hecke stack for M-torsors[65]:

$$H_{M,G\text{-}\mathrm{pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash \overline{M}/M \supset M\backslash M/M).$$

Consider the correspondence

$$\psi_P : H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow Y^P_{\mathrm{rel}}/Z_M \to \overline{\mathrm{Bun}}_G.$$

We have reduced the main theorem to

Goal 6.4.17. The functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P) \circ \mathbf{K}$ sends the arrow $P \to G$ to an isomorphism.

We will prove a stronger result:

Goal 6.4.18. For any $Q \in \mathrm{Par}_{\geq P}$, the functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P) \circ \mathbf{K}$ sends the arrow $Q \to G$ to an isomorphism.

We prove this by induction on the relative rank between Q and G. When $Q = G$, there is nothing to prove. Hence we assume $Q \neq G$ and assume the above claim is correct for any Q' strictly greater than Q.

[64] In the case $G = \mathrm{SL}_2$, Y^B_{rel} classifies chains $\mathcal{L}_1 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{L}_2$ where $\mathcal{L}_1 \to \mathcal{E}_1$, $\mathcal{E}_1 \to \mathcal{E}_2$ and $\mathcal{E}_2 \to \mathcal{L}_2$ are respectively S-points of Bun_B, VinBun_G and Bun_{B^-} such that the restriction of $\mathcal{L}_1 \to \mathcal{L}_2$ at any geometric point of S is nonzero.

[65] In the case $G = \mathrm{SL}_2$, $H_{T,G\text{-}\mathrm{pos}}$ classifies morphisms between line bundles $\mathcal{L}_1 \to \mathcal{L}_2$ whose restriction at any geometric point of S is nonzero. The map $Y^P_{\mathrm{rel}} \to H_{M,G\text{-}\mathrm{pos}}$ sends the chain $\mathcal{L}_1 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{L}_2$ in Footnote 64 to $\mathcal{L}_1 \to \mathcal{L}_2$.

Let L be the Levi subgroup of Q.

Consider the object

$$\mathcal{D}_Q := \mathrm{coFib}(\operatorname*{colim}_{Q'\in \mathrm{Par}'\cap \mathrm{Par}_{\geq Q}} \mathbf{K}(Q') \to \mathbf{K}(G))$$

We claim

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)(\mathcal{D}_Q) \simeq 0. \tag{6.20}$$

Let us execute the induction step using this claim. Note that the category $\mathrm{Par}'\cap \mathrm{Par}_{\geq Q}$ is weakly contractible, hence

$$\mathcal{D}_Q := \operatorname*{colim}_{Q'\in \mathrm{Par}'\cap \mathrm{Par}_{\geq Q}} \mathrm{coFib}(\mathbf{K}(Q') \to \mathbf{K}(G)).$$

By induction hypothesis, the functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)$ sends $\mathrm{coFib}(\mathbf{K}(Q') \to \mathbf{K}(G))$ to 0 unless $Q' = Q$. Hence $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)(\mathcal{D}_Q)$ is isomorphic to[66]

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)(\mathrm{coFib}(\mathbf{K}(Q) \to \mathbf{K}(G))[\mathrm{rank}(G) - \mathrm{rank}(L) + 1].$$

Then the claim implies $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)$ sends $\mathbf{K}(Q) \to \mathbf{K}(G)$ to an isomorphism as desired.

It remains to prove (6.20). Consider the maps

$$\overline{\mathrm{Bun}}_{G,G} \overset{j_{G,\geq Q}}{\to} \overline{\mathrm{Bun}}_{G,\geq Q} \overset{j_{\geq Q}}{\to} \overline{\mathrm{Bun}}_G.$$

By Lemma C.8.9, we have $\mathcal{D}_Q \simeq j_{\geq Q,!}(\mathcal{F})$, where

$$\mathcal{F} := (j_{G,\geq Q})_* \circ r_!(k_{\mathrm{Bun}_G}).$$

Hence by the base-change isomorphism, $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_P)(\mathcal{D}_Q)$ is isomorphic to $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_{P,\geq Q})(\mathcal{F})$, where

$$\psi_{P,\geq Q} : (H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \to \overline{\mathrm{Bun}}_{G,\geq Q})$$

and $(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q}$ is the open substack of Y^P_{rel}/Z_M containing those Q'-strata with $Q' \supset Q$. The following construction will be provided in § 6.13:

[66] We use the following formal fact. Let I be an index category obtained by removing the final object from $[1]^r$ ($r \geq 1$). Let C be any stable category. Suppose $F : I \to C$ is a functor such that $F(x) \simeq 0$ unless x is the initial object i_0. Then $\mathrm{colim} F \simeq F(i_0)[r-1]$. This fact can be proven by induction on r.

Proposition-Construction 6.4.19. *The correspondence $\psi_{P,\geq Q}$ is isomorphic to the composition of*

$$\psi_{Q,\geq Q} : (H_{L,G\text{-pos}}/Z_L \leftarrow Y^Q_{\mathrm{rel}}/Z_L \rightarrow \overline{\mathrm{Bun}}_{G,\geq Q})$$

by a certain correspondence from $H_{L,G\text{-pos}}/Z_L$ to $H_{M,G\text{-pos}}/Z_M$.

Therefore we only need to show $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_{Q,\geq Q})(\mathcal{F}) \simeq 0$. We will prove the following stronger claim: for any $Q \in \mathrm{Par}$, we have

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_{Q,\geq Q}) \circ (j_{G,\geq Q})_* \simeq 0.$$

To finish the proof, we need one more geometric input. Recall the *defect stratification* on $\overline{\mathrm{Bun}}_{G,P}$ (see § A.5.4):

$${}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} (H_{M,G\text{-pos}}/Z_M) \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}.$$

Consider the diagram

$$\begin{array}{ccccc} H_{M,G\text{-pos}}/Z_M & \overset{\mathfrak{q}^+_{P,\mathrm{Vin}}}{\leftarrow} & {}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P} & \overset{\mathfrak{p}^+_{P,\mathrm{Vin}}}{\rightarrow} & \overline{\mathrm{Bun}}_{G,\geq P}, \\ H_{M,G\text{-pos}}/Z_M & \overset{\mathfrak{q}^-_{P,\mathrm{Vin}}}{\leftarrow} & Y^P_{\mathrm{rel}}/Z_M & \overset{\mathfrak{p}^-_{P,\mathrm{Vin}}}{\rightarrow} & \overline{\mathrm{Bun}}_{G,\geq P}. \end{array}$$

In § 6.14, we will prove the following “second-adjointness-style” result:

Theorem 6.4.20. *The functor*

$$\mathfrak{q}^{\mp}_{P,\mathrm{Vin},!} \circ \mathfrak{p}^{\mp,*}_{P,\mathrm{Vin}} : \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\geq P}) \rightarrow \mathrm{DMod}_{\mathrm{indhol}}(H_{M,G\text{-pos}}/Z_M)$$

is isomorphic to the restriction of the functor

$$\mathfrak{q}^{\pm}_{P,\mathrm{Vin},*} \circ \mathfrak{p}^{\pm,!}_{P,\mathrm{Vin}} : \mathrm{DMod}(\overline{\mathrm{Bun}}_{G,\geq P}) \rightarrow \mathrm{DMod}(H_{M,G\text{-pos}}/Z_M).$$

Now the Q-version of Theorem 6.4.20 says

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_{Q,\geq Q}) \simeq \mathfrak{q}^+_{Q,\mathrm{Vin},*} \circ \mathfrak{p}^{+,!}_{Q,\mathrm{Vin}}.$$

Hence we have

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\psi_{Q,\geq Q}) \circ (j_{G,\geq Q})_* \simeq \mathfrak{q}^+_{Q,\mathrm{Vin},*} \circ \mathfrak{p}^{+,!}_{Q,\mathrm{Vin}} \circ (j_{G,\geq Q})_*.$$

Note that ${}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,Q}$ and $\overline{\mathrm{Bun}}_{G,G}$ have empty intersection (because $Q \neq G$). Hence $\mathfrak{p}^{+,!}_{Q,\mathrm{Vin}} \circ (j_{G,\geq Q})_* \simeq 0$. This finishes the proof.

□[Theorem 5.2.6]

Remark 6.4.21. In the case $G = \mathrm{SL}_2$, one can use Theorem 6.4.20 to give a quicker proof of Goal 6.4.17. Namely, using the theorem, we only need to show $\mathfrak{q}^+_{B,\mathrm{Vin},*} \circ \mathfrak{p}^{+,!}_{B,\mathrm{Vin}} \circ \mathbf{K}$ sends the arrow $B \to G$ to an isomorphism. Recall that $\mathfrak{p}^+_{B,\mathrm{Vin}}$ factors through

$$i_B : \overline{\mathrm{Bun}}_{G,B} \to \overline{\mathrm{Bun}}_G.$$

Hence we only need to show $i^!_B \circ \mathbf{K}$ sends $B \to G$ to an isomorphism. However, this is obvious because the image of this arrow is the canonical map (see Remark C.8.8):

$$i^*_B \circ j_{G,*}(r_!(k_{\mathrm{Bun}_G}))[-1] \to i^!_B \circ j_{G,!}(r_!(k_{\mathrm{Bun}_G})),$$

which is an isomorphism because i_B and j_G are complementary to each other.

6.5 Proof of Proposition-Construction 6.4.2

Proposition 2.3.17 provides a functor

$$\mathrm{I}(G,-) : \mathrm{Par} \to \mathrm{DGCat}$$

that sends an arrow $P \to Q$ to the functor $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$. Hence we also have a functor

$$\mathrm{I}(G \times G, - \times G) : \mathrm{Par} \to \mathrm{DGCat} \tag{6.21}$$

that sends an arrow $P \to Q$ to the functor $\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to Q\times G}$.

Lemma 6.5.1. *The functor (6.21) is canonically isomorphic to the functor*

$$\mathrm{Par} \to \mathrm{DGCat},\ P \mapsto \mathrm{I}(G,P) \otimes \mathrm{DMod}(\mathrm{Bun}_G).$$

Proof. By the proof of [DG13, Corollary 2.3.4], the canonical functor $\mathrm{DMod}(Y)\otimes\mathrm{DMod}(\mathrm{Bun}_G) \to \mathrm{DMod}(Y\times \mathrm{Bun}_G)$ is an equivalence for any lft prestack Y. Then the lemma follows from definitions.

□[Lemma 6.5.1]

Let $\widetilde{\mathbf{I}} \to \mathrm{Par}$ be the presentable fibration[67] classifying the functor (6.21). Note that Par has a final object G, and the fiber of $\widetilde{\mathbf{I}}$ at this object is $\widetilde{\mathbf{I}}_G := \mathrm{DMod}(\mathrm{Bun}_G\times\mathrm{Bun}_G)$. Consider the trivial fibration $\widetilde{\mathbf{I}}_G \times \mathrm{Par} \to \mathrm{Par}$. It follows formally that we have an adjoint pair

$$\mathrm{Eis}^{\mathrm{enh}} : \widetilde{\mathbf{I}} \rightleftharpoons \widetilde{\mathbf{I}}_G \times \mathrm{Par} : \mathrm{CT}^{\mathrm{enh}},$$

where $\mathrm{Eis}^{\mathrm{enh}}$ (resp. $\mathrm{CT}^{\mathrm{enh}}$) preserves co-Cartesian (resp. Cartesian) arrows and its fiber at $P \in \mathrm{Par}$ is $\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}$ (resp. $\mathrm{CT}^{\mathrm{enh}}_{G\times G\to P\times G}$). Using Lemma 6.5.1, the functor $\mathbf{E}\circ\mathbf{DL}$ is isomorphic to

$$\mathrm{Par} \xrightarrow{(\Delta_!(k_{\mathrm{Bun}_G}),-)} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{CT}^{\mathrm{enh}}} \widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

Denote the composition of the first two functors by $\mathbf{S}_{\mathrm{CT}} : \mathrm{Par} \to \widetilde{\mathbf{I}}$. Note that it is the unique Cartesian secton whose value at $G \in \mathrm{Par}$ is $\Delta_!(k_{\mathrm{Bun}_G}) \in \widetilde{\mathbf{I}}_G$.

We also have a functor

$$\mathrm{Par}^{\mathrm{op}} \to \mathrm{DGCat},\ P \mapsto \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$$

that sends an arrow to the corresponding !-pullback functor. Let $\mathcal{D}_{\mathrm{gen}} \to \mathrm{Par}$ be the corresponding *Cartesian* fibration. By Proposition 2.3.17(1), we have a fully faithful functor $\widetilde{\mathbf{I}} \to \mathcal{D}_{\mathrm{gen}}$ that preserves *co-Cartesian* arrows (although $\mathcal{D}_{\mathrm{gen}}$ is not a co-Cartesian fibration).

On the other hand, consider the functor

$$\mathrm{Par} \to \mathrm{DGCat},\ P \mapsto \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\le P})$$

that sends an arrow to the corresponding !-extension functor. Let $\overline{\mathcal{D}} \to \mathrm{Par}$ be the presentable fibration classifying this functor. We have a fully faithful functor

$$\overline{\mathcal{D}} \to \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) \times \mathrm{Par}$$

[67] A presentable fibration is both a Cartesian fibration and a coCartesian fibration whose fibers are presentable $(\infty,1)$-categories. See [Lur09, Definition 5.5.3.2].

whose fiber at $P \in \mathrm{Par}$ is the corresponding !-extension functor. The graph of the functor $\mathbf{K}$:

$$\mathrm{Par} \to \mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_G) \times \mathrm{Par},\ P \mapsto (\mathbf{K}(P), P)$$

is contained in the above full subcategory $\overline{\mathcal{D}}$. Hence we obtain a section $\mathbf{S}_{\mathbf{K}} : \mathrm{Par} \to \overline{\mathcal{D}}$ to the projection $\overline{\mathcal{D}} \to \mathrm{Par}$.

By Proposition-Construction 5.3.1, we also have functorial maps

$$\overline{\mathbf{\Delta}}^{\mathrm{enh},l}_{\leq P} : \overline{\mathrm{Bun}}_{G,\leq P} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G\,.$$

Hence there is a canonical functor

$$\overline{\mathcal{D}} \to \mathcal{D}_{\mathrm{gen}}$$

that preserves co-Cartesian arrows such that its fiber at $P \in \mathrm{Par}$ is the composition

$$\mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,\leq P}) \overset{\overline{\mathbf{\Delta}}^{\mathrm{enh},l}_{\leq P,!}}{\to} \mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G).$$

By construction, the composition

$$\mathrm{Par} \overset{\mathbf{S}_{\mathbf{K}}}{\to} \overline{\mathcal{D}} \to \mathcal{D}_{\mathrm{gen}}$$

sends P to $\mathcal{F}_P$, viewed as an object in $\mathcal{D}_{\mathrm{gen}}$ over $P \in \mathrm{Par}$. Hence by Variant 6.4.1, this functor factors through the full subcategory $\widetilde{\mathbf{I}} \subset \mathcal{D}_{\mathrm{gen}}$. Let $\mathbf{S}'_{\mathbf{K}} : \mathrm{Par} \to \widetilde{\mathbf{I}}$ be the corresponding functor. By constuction, $\overline{\Delta}_! \circ \mathbf{K}$ is isomorphic to the composition

$$\mathrm{Par} \overset{\mathbf{S}'_{\mathbf{K}}}{\to} \widetilde{\mathbf{I}} \overset{\mathrm{Eis}^{\mathrm{enh}}}{\to} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \overset{\mathrm{pr}}{\to} \widetilde{\mathbf{I}}_G.$$

In summary, we have obtained two sections $\mathbf{S}_{\mathrm{CT}}$ and $\mathbf{S}'_{\mathbf{K}}$ to the Cartesian fibration $\widetilde{\mathbf{I}} \to \mathrm{Par}$ such that $\overline{\Delta}_! \circ \mathbf{K}$ and $\mathbf{E} \circ \mathbf{DL}$ are obtained respectively by composing them with

$$\widetilde{\mathbf{I}} \overset{\mathrm{Eis}^{\mathrm{enh}}}{\to} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \overset{\mathrm{pr}}{\to} \widetilde{\mathbf{I}}_G.$$

Now the identification $\mathcal{F}_G \simeq \Delta_!(k_{\mathrm{Bun}_G})$ provides an isomorphism $\mathbf{S}'_{\mathbf{K}}(G) \simeq \mathbf{S}_{\mathrm{CT}}(G)$. Since $G \in \mathrm{Par}$ is the final object and since $\mathbf{S}_{\mathrm{CT}}$ is a Cartesian section, we obtain a natural transformation $\mathbf{S}'_{\mathbf{K}} \to \mathbf{S}_{\mathrm{CT}}$ whose value at $P \in \mathrm{Par}$ is the unique arrow $\mathbf{S}'_{\mathbf{K}}(P) \to \mathbf{S}_{\mathrm{CT}}(P)$ fitting into the following commutative diagram

$$\begin{tikzcd} \mathbf{S}'_{\mathbf{K}}(P) \arrow[r] \arrow[d] & \mathbf{S}_{\mathrm{CT}}(P) \arrow[d] \\ \mathbf{S}'_{\mathbf{K}}(G) \arrow[r, "\simeq"] & \mathbf{S}_{\mathrm{CT}}(G). \end{tikzcd}$$

By construction, when viewed as a morphism in $\widetilde{\mathbf{I}}_P \simeq \mathrm{I}(G \times G, P \times G)$, the arrow $\mathbf{S}'_{\mathbf{K}}(P) \to \mathbf{S}_{\mathrm{CT}}(P)$ is equivalent to (6.14). Now the desired natural transformation $\overline{\Delta}_! \circ \mathbf{K} \to \mathbf{E} \circ \mathbf{DL}$ is given by composing the above natural transformation $\mathbf{S}'_{\mathbf{K}} \to \mathbf{S}_{\mathrm{CT}}$ with the functor

$$\widetilde{\mathbf{I}} \xrightarrow{\mathrm{Eis}^{\mathrm{enh}}} \widetilde{\mathbf{I}}_G \times \mathrm{Par} \xrightarrow{\mathrm{pr}} \widetilde{\mathbf{I}}_G.$$

$\square$[Proposition-Construction 6.4.2]

6.6 Proof of Lemma 6.4.5

By definition, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_{P^-}^{M\text{-gen}} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P\text{-gen}} \arrow[d] \\ \mathrm{Bun}_{P^-} \arrow[r] & \mathrm{Bun}_G . \end{tikzcd}$$

We claim it induces a schematic open embedding

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{P^-} .$$

Indeed, the RHS is isomorphic to $\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-)$ and the above map is isomorphic to the map

$$\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}M) \to \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P^- \leftarrow \mathbb{B}P \underset{\mathbb{B}G}{\times} \mathbb{B}P^-)$$

induced by the map $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$. Then the claim follows from the fact that $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$ is a schematic open embedding.

Now (1) follows from the above claim and the well-known fact that $\mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_G$ is quasi-compact and schematic.

To prove (2), we only need to show

$$\mathrm{Bun}_M \to \mathrm{Bun}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \mathrm{Bun}_{P^-}^{M\text{-gen}}$$

is a schematic open embedding. As before, this follows from the fact that it is isomorphic to

$$\mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}M \leftarrow \mathbb{B}M) \to \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}P \underset{\mathbb{B}G}{\times} \mathbb{B}P^- \leftarrow \mathbb{B}M)$$

and the fact that $\mathbb{B}M \to \mathbb{B}P \times_{\mathbb{B}G} \mathbb{B}P^-$ is a schematic open embedding.

□[Lemma 6.4.5]

6.7 Recollection: Drinfeld's framework

In [Dri13, Appendix C], Drinfeld set up a general framework to prove results like Theorem 6.4.6. We review this framework in this subsection. In fact, we slightly generalize it to the case of lft prestacks.

Definition 6.7.1. We equip the category $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ with the Cartesian symmetric monoidal structure. Recall the notion of *enriched categories*. Following *loc.cit.*, we define a category $\mathbf{P}_{\mathbb{A}^1}$ enriched in $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$ as follows:

- It has two objects: the "big" one $\mathbf{b}$ and the "small" one $\mathbf{s}$.
- The mapping scheme $\mathbf{Hom}_{\mathbf{P}_{\mathbb{A}^1}}(\mathbf{b}, \mathbf{b})$ is defined to be $\mathbb{A}^1$. The other three mapping schemes are defined to be pt, viewed as the zero point in $\mathbb{A}^1$. The composition laws are all induced by the semi-group structure on $\mathbb{A}^1$.

The unique morphism $\mathbf{s} \to \mathbf{b}$ is denoted by α^+ and the unique morphism $\mathbf{b} \to \mathbf{s}$ is denoted by α^-.

Definition 6.7.2. We equip $\mathrm{AlgStk}_{\mathrm{ft,ad}}$ (see Corollary C.2.4) with the Cartesian symmetric monoidal structure. We define a category[68] **Dri** enriched in $\mathrm{AlgStk}_{\mathrm{ft,ad}}$ by replacing $\mathbb{A}^1$ in Definition 6.7.1 by the quotient stack $\mathbb{A}^1/\mathbb{G}_m$, and the zero map $\mathrm{pt} \to \mathbb{A}^1$ by the map $\mathbb{B}\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ obtained by taking quotients.

Note that there is an obvious functor $\mathbf{P}_{\mathbb{A}^1} \to \mathbf{Dri}$. We use the same symbols α^+ and α^- to denote the corresponding morphisms in **Dri**.

Definition 6.7.3. Let $\mathcal{O}$ be a monoidal $(\infty,1)$-category, $\mathcal{A}$ be a category enriched in $\mathcal{O}$ and $\mathcal{C}$ be a module $(\infty,2)$-category of $\mathcal{O}$. As explained in [Dri13, § C.13.1], there is a notion of *weakly $\mathcal{O}$-enriched (unital)*

[68] It was denoted by $\mathbf{P}_{\mathbb{A}^1}/\mathbb{B}\mathbb{G}_m$ in [Dri13].

right-lax functors[69] from $\mathcal{A}$ to $\mathcal{C}$. We will review its explicit meaning later in our particular examples. For now, let us give the formal definition.

We assume $\mathcal{O}$ is small. Consider the $(\infty,1)$-category $\mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$ equipped with the Day convolution monoidal structure (see [Lur12, § 2.2.6]). Then $\mathcal{C}$ has a canonical $\mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$-enriched structure such that for any $x,y\in\mathcal{C}$, the object

$$\mathbf{Hom}_{\mathcal{C}}(x,y)\in \mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat})$$

is the functor $o\mapsto \mathbf{Maps}_{\mathcal{C}}(o\otimes x,y)$.

On the other hand, there is a canonical *right-lax* monoidal structure on the Yoneda functor

$$\mathcal{O}\to \mathrm{Funct}(\mathcal{O}^{\mathrm{op}},(\infty,1)\text{-Cat}).$$

Then a weakly $\mathcal{O}$-enriched functor (resp. right-lax functor) $F:\mathcal{A}\dashrightarrow\mathcal{C}$ is defined to be a functor (resp. right-lax functor) F that intertwines the enrichment via the above right-lax monoidal functor.

Notation 6.7.4. Consider the $(3,2)$-category $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$. We equip it with the obvious $\mathrm{AlgStk}_{\mathrm{ft,ad}}$-action.

A *Drinfeld pre-input* is a weakly $\mathrm{AlgStk}_{\mathrm{ft,ad}}$-enriched right-lax functor $F:\mathbf{P}_{\mathbb{A}^1}\dashrightarrow\mathbf{Corr}$ such that it is strict at the composition $\alpha^+\circ\alpha^-$, i.e., the 2-morphism $F(\alpha^+)\circ F(\alpha^-)\to F(\alpha^+\circ\alpha^-)$ is invertible.

A *Drinfeld input* is a weakly $\mathrm{AlgStk}_{\mathrm{ft,ad}}$-enriched right-lax functor $F^\sharp:\mathbf{Dri}\dashrightarrow\mathbf{Corr}$ such that the composition $\mathbf{P}_{\mathbb{A}^1}\to\mathbf{Dri}\dashrightarrow\mathbf{Corr}$ is a Drinfeld pre-input.

Remark 6.7.5. Unwinding the definitions, a Drinfeld pre-input provides

- Two lft prestacks $Z:=F(\mathbf{b})$ and $Z^0:=F(\mathbf{s})$;
- Two correspondences
$$F(\alpha^+):(Z\xleftarrow{p^+}Z^+\xrightarrow{q^+}Z^0),\ F(\alpha^-):(Z^0\xleftarrow{q^-}Z^-\xrightarrow{p^-}Z)$$
whose left arms are QCAD maps;
- An $\mathbb{A}^1$-family of correspondences:
$$\begin{tikzcd} Z & \widetilde{Z} \arrow[l] \arrow[r] \arrow[d] & Z; \\ & \mathbb{A}^1 & \end{tikzcd}$$

[69]It was called just by *lax functors* in *loc.cit.*.

given by $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times F(\mathbf{b}) \to F(\mathbf{b})$;

- Isomorphisms

$$Z^+ \underset{Z^0}{\times} Z^- \simeq \widetilde{Z} \underset{\mathbb{A}^1}{\times} 0,\ Z \simeq \widetilde{Z} \underset{\mathbb{A}^1}{\times} 1$$

  defined over $Z \times Z$, given respectively by the invertible 2-morphism $F(\alpha^+) \circ F(\alpha^-) \to F(\alpha^+ \circ \alpha^-)$ and $\mathrm{Id}_{F(\mathbf{b})} \simeq F(\mathrm{Id}_{\mathbf{b}})$

- An open embedding

$$j : Z^0 \to Z^- \underset{Z}{\times} Z^+$$

  defined over $Z^0 \times Z^0$, given by the lax composition law for $\mathbf{s} \leftarrow \mathbf{b} \leftarrow \mathbf{s}$;

- Open embeddings

$$Z^+ \times \mathbb{A}^1 \to \widetilde{Z} \underset{Z}{\times} Z^+,\ Z^- \times \mathbb{A}^1 \to Z^- \underset{Z}{\times} \widetilde{Z},$$

  defined respectively over $Z \times Z^0 \times \mathbb{A}^1$ and $Z^0 \times Z \times \mathbb{A}^1$, given respectively by the lax composition laws for $\mathbf{b} \leftarrow \mathbf{b} \leftarrow \mathbf{s}$ and $\mathbf{s} \leftarrow \mathbf{b} \leftarrow \mathbf{b}$;

- An open embedding[70]

$$\widetilde{Z} \underset{\mathbb{A}^1}{\times} \mathbb{A}^2 \to \widetilde{Z} \underset{Z}{\times} \widetilde{Z}$$

  defined over $Z \times Z \times \mathbb{A}^2$, given by the lax composition law for $\mathbf{b} \leftarrow \mathbf{b} \leftarrow \mathbf{b}$.

- Some higher compatibilities.

Example 6.7.6. For any finite type scheme Z equipped with a $\mathbb{G}_m$-action, [Dri13] constructed a Drinfeld pre-input such that Z^+, Z^- and Z^0 are respectively the attractor, repeller and fixed loci of Z. Also, $\widetilde{Z}$ is the so-called Drinfeld-Gaitsgory interpolation, which is an $\mathbb{A}^1$-degeneration from Z to $Z^{\mathrm{att}} \times_{Z^{\mathrm{fix}}} Z^{\mathrm{rep}}$. Moreover, this construction is functorial in Z and compatible with Cartesian products.

When Z is affine, the corresponding right-lax functor $\mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}$ is strict. In particular, we obtain a functor $\mathbf{P}_{\mathbb{A}^1} \to \mathrm{Corr}$.

It was also shown in *loc.cit.* that there is a canonical Drinfeld input with $F^\sharp(\mathbf{b}) = Z/\mathbb{G}_m$ and $F^\sharp(\mathbf{s}) = Z^{\mathrm{fix}}/\mathbb{G}_m$

Construction 6.7.7. Let $F^\sharp : \mathbf{Dri} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$ be a Drinfeld input and F be the corresponding Drinfeld pre-input. We use the notations in Remark 6.7.5. Consider the composition

$$\mathbf{P}_{\mathbb{A}^1} \overset{F}{\dashrightarrow} \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}} \overset{\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}}{\to} \mathbf{DGCat}.$$

[70]The map $\mathbb{A}^2 \to \mathbb{A}^1$ in the formula is the multiplication map.

By construction, it sends α^+ and α^- respectively to the functors

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} \circ F(\alpha^+) \simeq p^+_\blacktriangle \circ q^{+,!},\ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} \circ F(\alpha^-) \simeq q^-_\blacktriangle \circ p^{-,!}.$$

The 2-morphism: $F(\alpha^-) \circ F(\alpha^+) \to F(\alpha^- \circ \alpha^+) = F(\mathrm{Id}_{\mathbf{s}})$ gives a natural transformation[71]

$$q^-_\blacktriangle \circ p^{-,!} \circ p^+_\blacktriangle \circ q^{+,!} \to \mathbf{Id}_{\mathrm{DMod}(Z^0)}. \tag{6.22}$$

The following result was proved in [Dri13, Appendix C]. For the reader's convenience, we translate Drinfeld's proof into our notations.

Theorem 6.7.8 (Drinfeld)**.** *In the above setting, there is a canonical adjoint pair*

$$q^-_\blacktriangle \circ p^{-,!} : \mathrm{DMod}(Z) \rightleftharpoons \mathrm{DMod}(Z^0) : p^+_\blacktriangle \circ q^{+,!}$$

with the counit adjunction natrual transformation given by (6.22).

Remark 6.7.9. The unit adjunction is given by a specialization construction along $\widetilde{Z} \to \mathbb{A}^1$. We do not need it in this thesis.

Remark 6.7.10. More precisely, *loc.cit.* focused on the problem of reproving the Braden's theorem ([Bra03]) using the Drinfeld input in Example 6.7.6. However, the proof there works for any Drinfeld input.

Proof. Consider the functor

$$\mathrm{DMod}_{\blacktriangle\text{-push}} : \mathrm{AlgStk}_{\mathrm{ft,ad}} \to \mathrm{DGCat}$$

sending Y to $\mathrm{DMod}(Y)$ and a map f to $f_\blacktriangle$. It has a canonical symmetric monoidal structure because of the product formula. Moreover, this symmetric monoidal structure is compatible with[72] the functor

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} : \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}} \to \mathbf{DGCat}$$

and the actions $\mathrm{AlgStk}_{\mathrm{ft,ad}} \curvearrowright \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}}$, $\mathrm{DGCat} \curvearrowright \mathbf{DGCat}$. In particular, $\mathrm{AlgStk}_{\mathrm{ft,ad}}$ acts

[71] Explicitly, the LHS is the !-pull-▲-push along $Z^0 \leftarrow Z^- \times_Z Z^+ \to Z^0$, while the RHS is that along $Z^0 \leftarrow Z^0 \to Z^0$. The desired natural transformation is induced by the adjoint pair $(j^!, j_\blacktriangle)$ for the open embedding $j : Z^0 \to Z^- \times_Z Z^+$.

[72] This is because the product formular $\mathrm{DMod}(Y_1) \otimes \mathrm{DMod}(Y_2) \simeq \mathrm{DMod}(Y_1 \times Y_2)$ holds as long as one of Y_1 and Y_2 is a QCA algebraic stack, by [DG13, Corollary 8.3.4].

on **DGCat** and the composition

$$\mathbf{Dri} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}} \to \mathbf{DGCat} \tag{6.23}$$

is a right-lax functor. Let $\mathbf{Dri}'$ be the DGCat-enriched category induced from **Dri** along the symmetric monoidal functor $\mathrm{DMod}_{\blacktriangle\text{-push}}$. It follows formally that (6.23) corresponds to a right-lax functor[73]

$$\mathbf{Dri}' \dashrightarrow \mathbf{DGCat}. \tag{6.24}$$

Note that the action DGCat $\curvearrowright$ **DGCat** induces[74] a DGCat-enriched structrue on **DGCat**. Hence (6.24) can also be viewed as a right-lax functor between DGCat-enriched categoies. In particular, it is a right-lax functor between the underlying $(\infty,2)$-categories.

Let us look into the structure of the underlying $(\infty,2)$-category of $\mathbf{Dri}'$. It contains two objects $\mathbf{b}$ and $\mathbf{s}$. The morphism $(\infty,1)$-category $\mathbf{Maps}_{\mathbf{Dri}'}(\mathbf{b},\mathbf{b})$ is given by $\mathrm{DMod}(\mathbb{A}^1/\mathbb{G}_m)$, while the other three morphism $(\infty,1)$-categories are given by $\mathrm{DMod}(\mathbb{B}\mathbb{G}_m)$. The composition laws are induced by the symmetric monoidal structure on $\mathrm{DMod}(\mathbb{A}^1/\mathbb{G}_m)$, where we view $\mathrm{DMod}(\mathbb{B}\mathbb{G}_m)$ as an ideal of $\mathrm{DMod}(\mathbb{A}^1/\mathbb{G}_m)$ via the closed embedding $\mathbb{B}\mathbb{G}_m \overset{0}{\hookrightarrow} \mathbb{A}^1/\mathbb{G}_m$.

Now the symmetric monoidal specialization functor[75]

$$\mathrm{sp} : [1]^{\mathrm{op}} \to \mathrm{DMod}(\mathbb{A}^1/\mathbb{G}_m)$$

provides a functor

$$\mathbf{P}_{1\to 0} \to \mathbf{Dri}',$$

where $\mathbf{P}_{1\to 0}$ is the (strict) 2-category obtained from $\mathbf{Dri}'$ by replacing $\mathrm{DMod}(\mathbb{A}^1/\mathbb{G}_m) \supset \mathrm{DMod}(\mathbb{B}\mathbb{G}_m)$ by $[1]^{\mathrm{op}} \supset \{0\}$. In other words, there are morphisms $\alpha^+ : \mathbf{s} \to \mathbf{b}$, $\alpha^- : \mathbf{b} \to \mathbf{s}$, $\mathrm{Id}_{\mathbf{s}} : \mathbf{s} \to \mathbf{s}$, $\mathrm{Id}_{\mathbf{b}} : \mathbf{b} \to \mathbf{b}$ and $f_0 : \mathbf{b} \to \mathbf{b}$ and a 2-morphism $\mathrm{Id}_{\mathbf{b}} \to f_0$. The only non-obvious composition law is $\alpha^+ \circ \alpha^- = f_0$.

Combining the previous discussion, we obtain a right-lax functor

$$H : \mathbf{P}_{1\to 0} \dashrightarrow \mathbf{DGCat}.$$

[73] It is the categorification of the following well-known fact. Let $A \to B$ be a homomorphism between algebras. For an A-module M and a B-module N, knowing an A-linear map $M \to N$ is the same as knowing a B-linear map $B \otimes_A M \to N$.

[74] An action $\mathcal{O} \curvearrowright \mathcal{C}$ induces an $\mathcal{O}$-enriched structre on $\mathcal{C}$ if for any $x, y \in \mathcal{C}$, there exists an object $\underline{\mathrm{Hom}}_{\mathcal{C}}(x,y)$ such that $\mathrm{Hom}_{\mathcal{C}}(-\otimes x, y) \simeq \mathrm{Hom}_{\mathcal{O}}(-, \underline{\mathrm{Hom}}_{\mathcal{C}}(x,y))$.

[75] This functor sends $0, 1 \in [1]$ to $s_{\blacktriangle}(k), b_{\blacktriangle}(k)$, where $s : \mathrm{pt} \xrightarrow{0} \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$ and $s : \mathrm{pt} \xrightarrow{1} \mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m$.

Moreover, by assumption, the 2-morphism $H(\alpha^+) \circ H(\alpha^-) \to H(\alpha^+ \circ \alpha^-)$ is invertible. One can check immediately that any such right-lax functor provides an adjoint pair

$$H(\alpha^-) : H(\mathbf{b}) \rightleftharpoons H(\mathbf{s}) : H(\alpha^+)$$

with the adjunction natural transformations given by

$$H(\alpha^-) \circ H(\alpha^+) \to H(\alpha^- \circ \alpha^+) \simeq H(\mathrm{Id}_{\mathbf{s}}) \simeq \mathbf{Id}_{H(\mathbf{s})},$$
$$\mathbf{Id}_{H(\mathbf{b})} \simeq H(\mathrm{Id}_{\mathbf{b}}) \to H(f_0) \simeq H(\alpha^+ \circ \alpha^-) \simeq H(\alpha^+) \circ H(\alpha^-).$$

Finally, it follows from construction that the above data are exactly those described in Theorem 6.7.8.

□[Theorem 6.7.8]

6.8 Proof of Theorem 6.4.6

We will use Drinfeld's framework reviewed in § 6.7. We first deduce the theorem from the following result:

Proposition-Construction 6.8.1. *There exists a canonical Drinfeld input*

$$F^\sharp : \mathbf{Dri} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$$

such that[76] *it sends α^+ and α^- respectively to*

$$\mathrm{Bun}_G^{P^-\text{-gen}}/\mathbb{G}_m \leftarrow \mathrm{Bun}_{P,\lambda}^{M\text{-gen}}/\mathbb{G}_m \to \mathrm{Bun}_{M,\lambda}/\mathbb{G}_m,$$
$$\mathrm{Bun}_{M,\lambda}/\mathbb{G}_m \leftarrow \mathrm{Bun}_{P^-,\lambda}/\mathbb{G}_m \to \mathrm{Bun}_G^{P^-\text{-gen}}/\mathbb{G}_m.$$

6.8.2 (Deducing Theorem 6.4.6)**.** We use the mirror version of Proposition-Construction 6.8.1 by exchanging P and P^-. Using Theorem 6.7.8, we obtain the version of Theorem 6.4.6 after replacing the relevant stacks by their $\mathbb{G}_m$-quotients. The same proof of [DG14, Theorem 3.4.3] implies we obtain canonical adjunctions

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{+,\mathrm{gen}}_{P,\lambda}) : \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbb{G}_m\text{-mon}} \rightleftharpoons \mathrm{DMod}(\mathrm{Bun}_M) : \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\alpha^{-,\mathrm{gen}}_{P,\lambda}),$$

where

$$\mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})^{\mathbb{G}_m\text{-mon}} \subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}})$$

[76]We also require that the 2-morphism $F^\sharp(\alpha^+) \circ F^\sharp(\alpha^-) \to F^\sharp(\mathrm{Id}_{\mathbf{s}})$ is given by the obvious open embedding.

is the full subcategory generated by the essential image of the !-pullback functor

$$\mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}/\mathbb{G}_m) \to \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}}).$$

Then we are done because the $\mathbb{G}_m$-action on $\mathrm{Bun}_G^{P\text{-gen}}$ can be trivialized.

□[Theorem 6.4.6]

It remains to construct the Drinfeld input in Proposition-Construction 6.8.1.

Notation 6.8.3. Let $\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}$ be the category of group schemes $H \to S$ with H and S being finite type affine schemes. Consider its arrow category $\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}})$. We equip the category

$$\mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}))_{\mathrm{all},\mathrm{all}}$$

with the obvious $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-action.

Construction 6.8.4. Recall the actions $\mathbb{G}_m \curvearrowright G$ and $\mathbb{G}_m \curvearrowright P$ in Example 1.2.3. The corresponding attractor, repeller and fixed loci are:

$$G^{\mathrm{att},\gamma} = P,\ G^{\mathrm{rep},\gamma} = P^-,\ G^{\mathrm{fix},\gamma} = M,\ (P^-)^{\mathrm{att},\gamma} = M,\ (P^-)^{\mathrm{rep},\gamma} = P^-,\ (P^-)^{\mathrm{fix},\gamma} = M.$$

Using Example 6.7.6, we obtain a weakly $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-enriched functor

$$\Theta_{P^-\to G} : \mathbf{P}_{\mathbb{A}^1} \to \mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}))_{\mathrm{all},\mathrm{all}}$$

sending α^+ and α^- respectively to

$$(P^- \to G) \leftarrow (M \to P) \to (M \to M),\ (M \to M) \leftarrow (P^- \to P^-) \to (P^- \to G).$$

Remark 6.8.5. By construction, $\mathrm{Hom}(\mathbf{b},\mathbf{b})\times\Theta_{P^-\to G}(\mathbf{b}) \to \Theta_{P^-\to G}(\mathbf{b})$ corresponds to the following diagram

$$\begin{tikzcd}
(P^- \to G) & (\widetilde{P^-}^{\gamma} \to \widetilde{G}^{\gamma}) \arrow[l] \arrow[r] \arrow[d] & (P^- \to G); \\
& \mathbb{A}^1, &
\end{tikzcd}$$

where $\widetilde{G}^{\gamma}$ (resp. $\widetilde{P^-}^{\gamma}$) is the Drinfeld-Gaitsgory interpolation for the action $\mathbb{G}_m \curvearrowright G$ (resp. $\mathbb{G}_m \curvearrowright P$). Note that we have

$$\widetilde{G}^{\gamma} \simeq \widetilde{G} \underset{T_{\mathrm{ad}}^{+},\overline{\gamma}}{\times} \mathbb{A}^1,\ \widetilde{P^{-}}^{\gamma} \simeq P^{-} \underset{G}{\times} \widetilde{G}^{\gamma}.$$

Construction 6.8.6. Consider the functor

$$\mathbb{B} : \mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}} \to \mathrm{AlgStk}_{\mathrm{lft}},\ (H \to S) \mapsto \mathbb{B}H,$$

where $\mathbb{B}H \coloneqq S/H$ is the quotient stack. Similarly we have a functor $\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}) \to \mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}})$. This functor does not commute with fiber products, hence we only have a *right-lax* functor

$$\mathrm{Corr}(\mathrm{Arr}(\mathrm{Grp}_{\mathrm{ft}}^{\mathrm{aff}}))_{\mathrm{all,all}} \dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{all,2\text{-}op}}.$$

This right-lax functor has a canonical $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-linear structure. Hence by composing with $\Theta_{P^{-}\to G}$, we obtain a weakly $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-enriched right-lax functor

$$\Theta_{\mathbb{B}P^{-}\to\mathbb{B}G} : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{all,2\text{-}op}}.$$

Definition 6.8.7. A morphism $(Y_1 \to Y_2) \to (Y_1' \to Y_2')$ in $\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}})$ is called an *open embedding* if both $Y_1 \to Y_1'$ and $Y_2 \to Y_2'$ are schematic open embeddings.

Lemma 6.8.8. *The right-lax functor $\Theta_{\mathbb{B}P^{-}\to\mathbb{B}G}$ factors through $\mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}$ and is strict at the composition $\alpha^{+} \circ \alpha^{-}$.*

Proof. Consider the two forgetful functors $\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}) \to \mathrm{AlgStk}_{\mathrm{lft}}$, $(Y_1 \to Y_2) \mapsto Y_i$. We only need to prove the similar claims after applying these forgetful functors. Those claims for the first forgetful functor are obvious (because $\widetilde{P^{-}}^{\gamma} \simeq P^{-} \times \mathbb{A}^1$). It remains to prove those for the second forgetful functor.

To prove the claim on strictness, we only need to check $\mathbb{B}(P \times_M P^{-}) \to \mathbb{B}P \times_{\mathbb{B}M} \mathbb{B}P^{-}$ is an isomorphism. But this is obvious.

To prove the claim on openness, we only need to check that the following four maps are schematic open embeddings:

$$\mathbb{B}(P^{-} \underset{G}{\times} \widetilde{G}^{\gamma}) \to \mathbb{B}P^{-} \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^{\gamma}, \quad \mathbb{B}(\widetilde{G}^{\gamma} \times_G P) \to \mathbb{B}\widetilde{G}^{\gamma} \times_{\mathbb{B}G} \mathbb{B}P,$$
$$\mathbb{B}(\widetilde{G}^{\gamma} \underset{G}{\times} \widetilde{G}^{\gamma}) \to \mathbb{B}\widetilde{G}^{\gamma} \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^{\gamma} \quad \mathbb{B}(P^{-} \times_G P) \to \mathbb{B}P^{-} \times_{\mathbb{B}G} \mathbb{B}P$$

The claim for the last one is obvious. The claims for the first two maps follows from Corollary A.1.5. The proof for the third one is similar. Namely, consider the action

$$(G \times G \times G) \curvearrowright {}_0\mathrm{Vin}_G^\gamma \times {}_0\mathrm{Vin}_G^\gamma,\ (g_1, g_2, g_3) \cdot (x_1, x_2) \mapsto (g_1 x_1 g_2^{-1}, g_2 x_2 g_3^{-1}).$$

Its stablizer for the canonical section is the group scheme $\widetilde{G}^\gamma \times_G \widetilde{G}^\gamma$. We only need to prove the similar version of Lemma A.1.4, i.e., to show

$$(G \times G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{G}{\times} \widetilde{G}^\gamma) \to {}_0\mathrm{Vin}_G^\gamma \times {}_0\mathrm{Vin}_G^\gamma$$

is an open embedding. As before, we only need to show the LHS is smooth. Now the functor $\Theta_{P^- \to G}$ provides an isomorphism $\widetilde{G}^\gamma \times_G \widetilde{G}^\gamma \simeq \widetilde{G}^\gamma \times_{\mathbb{A}^1} \mathbb{A}^2$ covering the map

$$\mathrm{pr}_{13} \times \mathrm{Id}_{\mathbb{A}^2} : (G \times G \times G) \times \mathbb{A}^2 \to (G \times G) \times \mathbb{A}^2.$$

Hence we have a canonical map

$$(G \times G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{G}{\times} \widetilde{G}^\gamma) \to (G \times G \times \mathbb{A}^2)/(\widetilde{G}^\gamma \underset{\mathbb{A}^1}{\times} \mathbb{A}^2) \simeq {}_0\mathrm{Vin}_G^\gamma \underset{\mathbb{A}^1}{\times} \mathbb{A}^2.$$

Then we are done because it is a smooth map to a smooth scheme.

□[Lemma 6.8.8]

Construction 6.8.9. Consider the functor

$$\mathbf{Maps}_{\mathrm{gen}}(X, -) : \mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}) \to \mathrm{PreStk}_{\mathrm{lft}},\ (Y_1 \to Y_2) \mapsto \mathbf{Maps}_{\mathrm{gen}}(X, Y_1 \leftarrow Y_2).$$

It is easy to see that it sends open embeddings to schematic open embeddings. Hence we obtain a functor

$$\mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}} \to \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}.$$

This functor has a canonical $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-linear structure[77]. Hence by composing with $\Theta_{\mathbb{B}P^- \to \mathbb{B}G}$, we obtain a weakly $\mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$-enriched right-lax functor

$$\Theta : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}$$

[77]This is because for affine schemes Y, we have $\mathbf{Maps}(X, Y) \simeq Y$.

that is strict at the composition $\alpha^+ \circ \alpha^-$.

Remark 6.8.10. Explicitly, we have:

- The right-lax functor Θ sends α^+ and α^- respectively to

$$\mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_P^{M\text{-gen}} \to \mathrm{Bun}_M,\ \mathrm{Bun}_M \leftarrow \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G^{P^-\text{-gen}}.$$

- The map $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta(\mathbf{b}) \to \Theta(\mathbf{b})$ is provided by the $\mathbb{A}^1$-family of correspondences:

$$\begin{tikzcd} \mathrm{Bun}_G^{P^-\text{-gen}} & \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{B}\widetilde{P^-}^\gamma) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P^-\text{-gen}}. \\ & \mathbb{A}^1 & \end{tikzcd}$$

Construction 6.8.11. We write

$$\begin{aligned} \mathrm{VinBun}_G^{P^-\text{-gen},\gamma} &:= \mathbf{Maps}_{\mathrm{gen}}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{B}\widetilde{P^-}^\gamma), \\ \mathrm{VinBun}_G^\gamma &:= \mathbf{Maps}(X, \mathbb{B}\widetilde{G}^\gamma). \end{aligned}$$

There is a canonical map

$$\mathrm{VinBun}_G^{P^-\text{-gen},\gamma} \to \mathrm{Bun}_G^{P^-\text{-gen}} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_G^\gamma$$

induced by the map

$$\mathbb{B}\widetilde{P^-}^\gamma \simeq \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}^\gamma) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}^\gamma.$$

By Corollary A.1.5, these maps are schematic open embeddings.

Construction 6.8.12. Recall that $\mathrm{VinBun}_{G,C_P} \simeq \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_{P^-}$. Hence there is a unique open substack $\mathrm{VinBun}_{G,\lambda}^\gamma$ of VinBun_G^γ obtained by removing all the connected components

$$\mathrm{Bun}_{P,\mu} \underset{\mathrm{Bun}_{M,\mu}}{\times} \mathrm{Bun}_{P^-,\mu} \subset \mathrm{VinBun}_{G,C_P}$$

with $\mu \neq \lambda$ from its 0-fiber. Let $\mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma}$ be the corresponding open sub-prestack. It is easy to see we can modify Θ to obtain

$$\Theta_\lambda : \mathbf{P}_{\mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,all}}^{\mathrm{open,2\text{-}op}}$$

such that

- It sends α^+ and α^- respectively to

$$\mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P,\lambda}^{M\text{-gen}} \to \mathrm{Bun}_{M,\lambda},\ \mathrm{Bun}_{M,\lambda} \leftarrow \mathrm{Bun}_{P^-,\lambda} \to \mathrm{Bun}_G^{P^-\text{-gen}}.$$

- The map $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta_\lambda(\mathbf{b}) \to \Theta_\lambda(\mathbf{b})$ is provided by the $\mathbb{A}^1$-family of correspondences:

$$\begin{tikzcd} \mathrm{Bun}_G^{P^-\text{-gen}} & \mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Bun}_G^{P^-\text{-gen}}. \\ & \mathbb{A}^1 & \end{tikzcd}$$

- The other data are induced from Θ.

Lemma 6.8.13. *The right-lax functor Θ_λ factors through* $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD,all}}$.

Proof. We only need to check all the three left arms in the above three correspondences are QCAD. The claims for the first two arms are just (the mirror version of) Lemma 6.4.5(1). To prove the claim for the third arm, using the open embedding in Construction 6.8.11, we only need to show $\mathrm{Bun}_G \leftarrow \mathrm{VinBun}_{G,\lambda}^{\gamma}$ is QCAD. Note that VinBun_G has an affine diagonal. Hence we only need to show $\mathrm{VinBun}_{G,\lambda}^{\gamma} \to \mathrm{Bun}_G$ is quasi-compact. Then we are done because both the $\mathbb{G}_m$-locus and the 0-fiber of $\mathrm{VinBun}_{G,\lambda}^{\gamma}$ is quasi-compact over Bun_G.

□[Lemma 6.8.13]

We are going to obtain a Drinfeld input from Θ_λ by taking quotients for the torus actions. We first introduce some notations.

Notation 6.8.14. Let $\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}}$ be the category whose objects are $(H \curvearrowright Y)$, where H is an affine algberaic group and $Y \in \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$. We equip $\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}}$ with the Cartesian symmetric monoidal structure. Note that the monoidal unit for it is $(\mathrm{pt} \curvearrowright \mathrm{pt})$. Also note that there is a symmetric monoidal forget functor $\mathbf{oblv}_{\mathrm{Act}}$: $\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}} \to \mathrm{Sch}_{\mathrm{ft}}^{\mathrm{aff}}$.

As in Definition 6.7.1, we define a category $\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1}$ enriched in $\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}}$ such that

$$\mathbf{Hom}_{\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1}}(\mathbf{b},\mathbf{b}) = (\mathbb{G}_m \curvearrowright \mathbb{A}^1)$$

and the other three mapping objects are $(\mathbb{G}_m \curvearrowright 0)$. We use the same symbols α^+ and α^- to the canonical morphisms.

Note that $\mathbf{P}_{\mathbb{A}^1}$ can be obtained from it by the procudure of *changing of enrichment* along $\mathbf{oblv}_{\mathrm{Act}}$. In particular, there is a forgetful functor $\mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1} \to \mathbf{P}_{\mathbb{A}^1}$ that intertwines the enrichment via $\mathbf{oblv}_{\mathrm{Act}}$.

Let $\mathrm{ActPreStk}_{\mathrm{lft}}$ be the similarly defined category. A morphism $(H \curvearrowright Y_1) \to (H_2 \curvearrowright Y_2)$ is said to be an open embedding if $H_1 \simeq H_2$ and $Y_1 \to Y_2$ is a schemtaic open embedding. It is said to be QCAD if $Y_1 \to Y_2$ is QCAD.

Construction 6.8.15. (c.f. [Dri13, § C.13.4])

In the previous connstructions of Θ_λ, we ignored the various $\mathbb{G}_m$-actions. If we keep tracking them, we can obtain a weakly $\mathrm{ActSch}_{\mathrm{ft}}^{\mathrm{aff}}$-enriched right-lax functor

$$\Theta_\lambda^{\mathrm{Act}} : \mathbf{P}_{\mathbb{G}_m \curvearrowright \mathbb{A}^1} \dashrightarrow \mathbf{Corr}(\mathrm{ActPreStk}_{\mathrm{lft}})_{\mathrm{QCAD,all}}^{\mathrm{open,2\text{-}op}}$$

such that

- It sends α^+ and α^- respectively to

$$(\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}) \leftarrow (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{P,\lambda}^{M\text{-gen}}) \to (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{M,\lambda}),$$
$$(\mathbb{G}_m \curvearrowright \mathrm{Bun}_{M,\lambda}) \leftarrow (\mathbb{G}_m \curvearrowright \mathrm{Bun}_{P^-,\lambda}) \to (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}).$$

- The map $\mathrm{Hom}(\mathbf{b},\mathbf{b}) \times \Theta_\lambda^{\mathrm{Act}}(\mathbf{b}) \to \Theta_\lambda^{\mathrm{Act}}(\mathbf{b})$ is provided by the diagram:

$$\begin{tikzcd} (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}) & (\mathbb{G}_m \times \mathbb{G}_m \curvearrowright \mathrm{VinBun}_{G,\lambda}^{P^-\text{-gen},\gamma}) \arrow[l] \arrow[r] \arrow[d] & (\mathbb{G}_m \curvearrowright \mathrm{Bun}_G^{P^-\text{-gen}}). , \\ & (\mathbb{G}_m \curvearrowright \mathbb{A}^1) & \end{tikzcd}$$

  which is induced by the canonical morphism

$$(\mathbb{G}_m \times \mathbb{G}_m \curvearrowright \widetilde{Z}) \to (\mathbb{G}_m \curvearrowright Z) \times (\mathbb{G}_m \curvearrowright Z) \times (\mathbb{G}_m \curvearrowright \mathbb{A}^1)$$

  that exists for any Drinfeld-Gaitsgory interpolation $\widetilde{Z}$ (see [DG14, § 2.2.3]).

- It is compatible with Θ_λ via the forgetful functors.

Then as in [Dri13, Footnote 41], we obtain the desired Drinfeld input by passing to quotients and changing enrichment.

□[Proposition-Construction 6.8.1]

6.9 Proof of Proposition-Construction 6.4.10

By definition, we have

$$\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \supset P^-\backslash_0 \mathrm{Vin}_{G,\geq C_P}/G).$$

Note that

$$P^-\backslash_0 \mathrm{Vin}_{G,\geq C_P}/G \simeq \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P},$$

where $\widetilde{G}_{\geq C_P} := \widetilde{G} \times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P}$. By Corollary A.1.5, the canonical map

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P} \simeq P^-\backslash_0 \mathrm{Vin}_{G,\geq C_P}/G$$

is a schematic open embedding. We define

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} := \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/G \leftarrow \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P})).$$

Then we have a canonical schematic open embedding

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P}.$$

As in the proof of Lemma 6.2.1, a direct calculation shows that the canonical Z_M-action on ${}_0\mathrm{Vin}_{G,\geq C_P}$ preserves the open substack

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{(P^-\backslash_0 \mathrm{Vin}_{G,\geq C_P}/G)}{\times} {}_0\mathrm{Vin}_{G,\geq C_P}.$$

Hence it makes sense to define

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} := (\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}}/Z_M.$$

It is obvious that the restriction of

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,\geq P}$$

to the G-stratum is an isomorphism. It remains to identify its restriction to the P-stratum with the map

$$\mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{G,P}.$$

Unwinding the definitions, we only need to identify the C_P-fiber of the open embedding

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P}$$

with the C_P-fiber of the map

$$\mathbb{B}M \underset{\mathbb{B}P}{\times} \mathbb{B}\widetilde{G}_{\leq P} \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\leq P}.$$

However, this follows from $\widetilde{G}|_{C_P} \simeq P \times_M P^-$.

□[Proposition-Construction 6.4.10]

6.10 Proof of Lemma 6.4.11

We will introduce many temporary notations in this subsection. When we use an english letter, like c, to denote a correspondence, or when we use a letter of plain font, like K, to denote a D-module, it means such notations are only used in this subsection.

6.10.1 (The arrow γ_P)**.** We first give the following tautological description of

$$\gamma_P : \mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\mathcal{F}_P) \to \mathrm{CT}_{P\times G,*}(\mathcal{F}_G).$$

Recall the morphism (6.13):

$$\mathrm{Eis}^{\mathrm{enh}}_{P\times G\to G\times G}(\mathcal{F}_P) \to \mathcal{F}_G.$$

Its underlying morphism in $\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_G \times \mathrm{Bun}_G)$ is a map

$$\vartheta_P : \mathfrak{p}^{\mathrm{enh}}_{P\times G,!}(\mathcal{F}_P) \to \mathcal{F}_G,$$

which by adjunction induces a morphism

$$\theta_P : \mathcal{F}_P \to \mathfrak{p}^{\mathrm{enh},!}_{P\times G}(\mathcal{F}_G).$$

Then we have

$$\gamma_P \simeq \mathrm{CT}^{\mathrm{gen}}_{P\times G,*}(\theta_P).$$

Note that we indeed have $\mathrm{CT}_{P\times G,*} \simeq \mathrm{CT}^{\mathrm{gen}}_{P\times G,*} \circ \mathfrak{p}^{\mathrm{enh},!}_{P\times G}$.

6.10.2. Next, we give a more convenient description for the second adjointness, when restricted to ind-holonomic objects.

Let $'\mathrm{CT}_{P,*}$ be the restriction of $\mathrm{CT}_{P,*}$ to the full subcategory of ind-holonomic objects. Recall that the natural transformation $'\mathrm{CT}_{P,*} \simeq {}'\mathrm{CT}_{P^-,!}$ is obtained as follows. We apply $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}$ to the 2-morphism

$$\alpha^+_{P,\lambda} \circ \alpha^-_{P,\lambda} \to \mathrm{Id}_{\mathrm{Bun}_{M,\lambda}}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}}$ and obtain a natural transformation

$$'\mathrm{CT}_{P,*,\lambda} \circ ('\mathrm{CT}_{P^-,!,\lambda})^R \to \mathbf{Id}_{\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_M)}.$$

Then we obtain the natural transfomation $'\mathrm{CT}_{P,*,\lambda} \to {}'\mathrm{CT}_{P^-,!,\lambda}$ by using adjunctions. Equivalently, we have the left adjoint version of the above pircture. Namely, we start from the 2-morphism[78]

$$\mathrm{Id}_{\mathrm{Bun}_M} \to (\alpha^-_P)^{\mathrm{rev}} \circ (\alpha^+_P)^{\mathrm{rev}}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all},\mathrm{Stacky}}$, and use $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}$ to obtain a natural transformation

$$\mathbf{Id} \to {}'\mathrm{CT}_{P^-,!} \circ ('\mathrm{CT}_{P,*})^L.$$

Then we can obtain the same natural transfomation $'\mathrm{CT}_{P,*} \to {}'\mathrm{CT}_{P^-,!}$ by using adjunctions.

The advantage is: if we use left functors, we can work with all the connected components simultaneously.

Similarly, the natural transformation of $'\mathrm{CT}_{P\times G,*} \simeq {}'\mathrm{CT}_{P^-\times G,!}$ can be obtained by the same procedure from the correspondences

$$\begin{aligned} c^+ &: (\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_G \to \mathrm{Bun}_M \times \mathrm{Bun}_G), \\ c^- &: (\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-} \times \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G), \end{aligned}$$

[78]The superscript "rev" means exchanging the two arms of a correspondence.

and the 2-morphism

$$\mathrm{Id}_{(\mathrm{Bun}_M \times \mathrm{Bun}_G)} \to c^- \circ c^+ \tag{6.25}$$

in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all},\mathrm{Stacky}}$.

Similarly, the natural transformation of $'\mathrm{CT}^{\mathrm{gen}}_{P\times G,*} \simeq {}'\mathrm{CT}^{\mathrm{gen}}_{P^-\times G,!}$ can be obtained by the same procedure from the correspondences

$$\begin{aligned} c^{+,\mathrm{gen}} \quad &: \quad (\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_G \to \mathrm{Bun}_M \times \mathrm{Bun}_G), \\ c^{-,\mathrm{gen}} \quad &: \quad (\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \times \mathrm{Bun}_G \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G). \end{aligned}$$

and the 2-morphism

$$\mathrm{Id}_{(\mathrm{Bun}_M \times \mathrm{Bun}_G)} \to c^{-,\mathrm{gen}} \circ c^{+,\mathrm{gen}}, \tag{6.26}$$

where in $\mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all},\mathrm{Stacky}}$.

Notation 6.10.3. We introduce the following shorthands: for a correspondence c (in english letter), we use the symbol $\mathbf{c}$ to denote the corresponding functor $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(c)$. These shorthands are only used in this subsection.

6.10.4. Using the above shorthands, the results in § 6.10.2 are translated as below. The 2-morphisms (6.25) and (6.26) induce natural transformations[79]

$$\mu : \mathbf{Id} \to \mathbf{c}^- \circ \mathbf{c}^+, \ \mu^{\mathrm{gen}} : \mathbf{Id} \to \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}}$$

such that the following compositions are isomorphisms

$$(\mathbf{c}^{+,\mathrm{gen}})^R \overset{\mu^{\mathrm{gen}}}{\to} \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}} \circ (\mathbf{c}^{+,\mathrm{gen}})^R \overset{\mathbf{counit}}{\to} \mathbf{c}^{-,\mathrm{gen}} \tag{6.27}$$

$$(\mathbf{c}^+)^R \overset{\mu}{\to} \mathbf{c}^- \circ \mathbf{c}^+ \circ (\mathbf{c}^+)^R \overset{\mathbf{counit}}{\to} \mathbf{c}^-. \tag{6.28}$$

6.10.5. Consider the map $\mathfrak{p}^{\mathrm{enh}}_{P\times G} : \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$. Let

$$p : (\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \overset{=}{\to} \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G)$$

be the corresponding correspondence. Note that we have $\mathbf{p} \simeq \mathfrak{p}^{\mathrm{enh}}_{P\times G,!}$.

[79]The functor $\mathbf{Id}$ below is the identity functor for $\mathrm{DMod}_{\mathrm{indhol}}(\mathrm{Bun}_M \times \mathrm{Bun}_G)$.

By definition, we have $c^+ \simeq p \circ c^{+,\mathrm{gen}}$, which provides

$$\mathbf{c}^+ \simeq \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}}.$$

Also, it is easy to see that the canonical map $\mathrm{Bun}_{P^-}^{M\text{-gen}} \to \mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^{P\text{-gen}}$ is a schematic open embedding. Hence we also have a 2-morphism $c^{-,\mathrm{gen}} \to c^- \circ p$, which provides

$$\nu : \mathbf{c}^{-,\mathrm{gen}} \to \mathbf{c}^- \circ \mathbf{p}.$$

By construction, the 2-morphism (6.25) is equivalent to the composition

$$\mathrm{Id}_{(\mathrm{Bun}_M \times \mathrm{Bun}_G)} \overset{(6.26)}{\to} c^{-,\mathrm{gen}} \circ c^{+,\mathrm{gen}} \to c^- \circ p \circ c^{+,\mathrm{gen}} \simeq c^- \circ c^+.$$

Hence μ is isomorphic to

$$\mathbf{Id} \overset{\mu^{\mathrm{gen}}}{\to} \mathbf{c}^{-,\mathrm{gen}} \circ \mathbf{c}^{+,\mathrm{gen}} \overset{\nu(\mathbf{c}^{+,\mathrm{gen}})}{\to} \mathbf{c}^- \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}} \simeq \mathbf{c}^- \circ \mathbf{c}^+. \tag{6.29}$$

Lemma 6.10.6. *The arrow γ_P is equivalent to the composition*

$$(\mathbf{c}^{-,\mathrm{gen}})(\mathcal{F}_P) \overset{\nu}{\to} \mathbf{c}^- \circ \mathbf{p}(\mathcal{F}_P) \overset{\mathbf{c}^-(\vartheta_P)}{\to} \mathbf{c}^-(\mathcal{F}_G).$$

Proof. By definition, the arrow $\theta_P : \mathcal{F}_P \to \mathbf{p}^R(\mathcal{F}_G)$ is isomorphic to

$$\mathcal{F}_P \overset{\mathbf{unit}}{\to} \mathbf{p}^R \circ \mathbf{p}(\mathcal{F}_P) \overset{\mathbf{p}^R(\vartheta_P)}{\to} \mathbf{p}^R(\mathcal{F}_G).$$

Hence by definition, γ_P is isomorphic to

$$(\mathbf{c}^{+,\mathrm{gen}})^R(\mathcal{F}_P) \overset{\mathbf{unit}}{\to} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p}(\mathcal{F}_P) \simeq (\mathbf{c}^+)^R \circ \mathbf{p}(\mathcal{F}_P) \overset{(\mathbf{c}^+)^R(\vartheta_P)}{\to} (\mathbf{c}^+)^R(\mathcal{F}_G).$$

Hence we only need to show the following diagram of functors commute

$$\begin{tikzcd}
(\mathbf{c}^{+,\mathrm{gen}})^R \arrow[r, "\mathbf{unit}"] \arrow[d, "(6.27)"', "\simeq"] & (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p} \arrow[r, "\simeq"] & (\mathbf{c}^+)^R \circ \mathbf{p} \arrow[d, "(6.28)"', "\simeq"] \\
\mathbf{c}^{-,\mathrm{gen}} \arrow[rr, "\nu"] & & \mathbf{c}^- \circ \mathbf{p}.
\end{tikzcd} \tag{6.30}$$

Note that we have

$$\mathrm{Maps}((\mathbf{c}^{+,\mathrm{gen}})^R, \mathbf{c}^- \circ \mathbf{p}) \simeq \mathrm{Maps}(\mathbf{Id}, \mathbf{c}^- \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}}) \simeq \mathrm{Maps}(\mathbf{Id}, \mathbf{c}^- \circ \mathbf{c}^+).$$

Via this isomorphism, the top arc in (6.30), which is a point of the LHS, is given by the following point of the RHS:

$$\begin{aligned}\mathbf{Id} \overset{\mathbf{unit}}{\to} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{c}^{+,\mathrm{gen}} \overset{\mathbf{unit}}{\to} (\mathbf{c}^{+,\mathrm{gen}})^R \circ \mathbf{p}^R \circ \mathbf{p} \circ \mathbf{c}^{+,\mathrm{gen}} \simeq (\mathbf{c}^+)^R \circ \mathbf{c}^+ \to \\ \overset{\mu}{\to} \mathbf{c}^- \circ \mathbf{c}^+ \circ (\mathbf{c}^+)^R \circ \mathbf{c}^+ \overset{\mathbf{counit}}{\to} \mathbf{c}^- \circ \mathbf{c}^+.\end{aligned}$$

The first row in the above composition is just $\mathbf{unit} : \mathbf{Id} \to (\mathbf{c}^+)^R \circ \mathbf{c}^+$. Hence this composition is isomorphic to

$$\mathbf{Id} \overset{\mu}{\to} \mathbf{c}^- \circ \mathbf{c}^+ \overset{\mathbf{unit}}{\to} \mathbf{c}^- \circ \mathbf{c}^+ \circ (\mathbf{c}^+)^R \circ \mathbf{c}^+ \overset{\mathbf{counit}}{\to} \mathbf{c}^- \circ \mathbf{c}^+,$$

which is just $\mathbf{Id} \to \mathbf{c}^- \circ \mathbf{c}^+$ by the axioms for **unit** and **counit**.

Similarly, one shows that the bottom arc corresponds to natural transformation (6.29). Then we are done by the discussion above the lemma.

□[Lemma 6.10.6]

6.10.7 (Finish of the proof)**.** We give temporary labels to the following correspondences

$$\begin{array}{rcl} i : (\overline{\mathrm{Bun}}_{G,\geq P} \leftarrow & \overline{\mathrm{Bun}}_{G,P} & \overset{=}{\to} \overline{\mathrm{Bun}}_{G,P}), \\ d^{\mathrm{gen}} : (\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G \leftarrow & \overline{\mathrm{Bun}}_{G,P} & \overset{=}{\to} \overline{\mathrm{Bun}}_{G,P}), \\ d : (\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow & \overline{\mathrm{Bun}}_{G,\geq P} & \overset{=}{\to} \overline{\mathrm{Bun}}_{G,\geq P}), \\ j : (\overline{\mathrm{Bun}}_{G,\geq P} \overset{=}{\leftarrow} & \overline{\mathrm{Bun}}_{G,\geq P} & \to \overline{\mathrm{Bun}}_G), \\ b : (\mathrm{Bun}_M \times \mathrm{Bun}_G \leftarrow & (\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} & \to \overline{\mathrm{Bun}}_{G,\geq P}). \end{array}$$

Note that we have an obvious isomorphism $\beta \simeq b \circ j$, hence

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta) \simeq \mathbf{b} \circ \mathbf{j}.$$

We have an isomorphism $p \circ d^{\mathrm{gen}} \simeq d \circ i$ because both sides are just

$$\mathrm{Bun}_G \times \mathrm{Bun}_G \leftarrow \overline{\mathrm{Bun}}_{G,P} \overset{=}{\to} \overline{\mathrm{Bun}}_{G,P}.$$

Hence $\mathbf{d}\circ\mathbf{d}^{\mathrm{gen}} \simeq \mathbf{d}\circ\mathbf{i}$. We have an isomorphism $b\circ i \simeq c^{-,\mathrm{gen}}\circ d^{\mathrm{gen}}$ because both sides are just

$$\mathrm{Bun}_M\times\mathrm{Bun}_G \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P}.$$

Hence $\mathbf{b}\circ\mathbf{i}\simeq \mathbf{c}^{-,\mathrm{gen}}\circ\mathbf{d}^{\mathrm{gen}}$. We have a 2-morphism $b\to c^-\circ d$ induced by the canonical open embedding

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \subset \mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P}.$$

Hence we have a natural transformation

$$\xi:\mathbf{b}\to\mathbf{c}^-\circ\mathbf{d}.$$

Moreover, the 2-morphism

$$b\circ i\simeq c^{-,\mathrm{gen}}\circ d^{\mathrm{gen}}\to c^-\circ p\circ d^{\mathrm{gen}}\simeq c^-\circ d\circ i$$

is isomorphic to the 2-morphism induced from $b\to c^-\circ d$. Hence we have the following commutative diagram of functors

$$\begin{tikzcd}
\mathbf{b}\circ\mathbf{i} \arrow[r, "\xi(\mathbf{i})"] \arrow[d, "\simeq"] & \mathbf{c}^-\circ\mathbf{d}\circ\mathbf{i} \arrow[d, "\simeq"] \\
\mathbf{c}^{-,\mathrm{gen}}\circ\mathbf{d}^{\mathrm{gen}} \arrow[r, "\nu(\mathbf{d}^{\mathrm{gen}})"] & \mathbf{c}^-\circ\mathbf{p}\circ\mathbf{d}^{\mathrm{gen}}.
\end{tikzcd} \tag{6.31}$$

After these preparations, we are ready to finish the proof. Recall that $\mathbf{K}(P)$ is a !-extension along $\overline{\mathrm{Bun}}_{G,P}\to\overline{\mathrm{Bun}}_G$. Let K_1 be the corresponding object in $\mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P})$. We also write $K_2 := j^*_{\geq P}(\mathbf{K}(G))$, where $j^*_{\geq P}:\overline{\mathrm{Bun}}_{G,\geq P}\to\overline{\mathrm{Bun}}_G$ is the open embedding. The morphism $\mathbf{K}(P)\to\mathbf{K}(G)$ is sent by $\mathbf{j}=j^*_{\geq P}$ to a morphism

$$\eta:\mathbf{i}(K_1)\to K_2.$$

It follows from definition that the arrow $\vartheta_P:\mathbf{p}(\mathcal{F}_P)\to\mathcal{F}_G$ is equivalent to

$$\mathbf{p}\circ\mathbf{d}^{\mathrm{gen}}(K_1)\simeq\mathbf{d}\circ\mathbf{i}(K_1)\overset{\mathbf{d}(\eta)}{\to}\mathbf{d}(K_2),$$

where $\mathcal{F}_P\simeq\mathbf{d}^{\mathrm{gen}}(K_1)$ and $\mathcal{F}_G\simeq\mathbf{d}(K_2)$. Hence by Lemma 6.10.6, the arrow γ_P is equivalent to

$$\mathbf{c}^{-,\mathrm{gen}}\circ\mathbf{d}^{\mathrm{gen}}(K_1)\overset{\nu(\mathbf{d}^{\mathrm{gen}})}{\to}\mathbf{c}^-\circ\mathbf{p}\circ\mathbf{d}^{\mathrm{gen}}(K_1)\simeq\mathbf{c}^-\circ\mathbf{d}\circ\mathbf{i}(K_1)\overset{\mathbf{c}^-\circ\mathbf{d}(\eta)}{\to}\mathbf{c}^-\circ\mathbf{d}(K_2).$$

By (6.31), this arrow is equivalent to

$$\mathbf{b}\circ\mathbf{i}(K_1)\overset{\xi(\mathbf{i}(K_1))}{\to}\mathbf{c}^-\circ\mathbf{d}\circ\mathbf{i}(K_1)\overset{\mathbf{c}^-\circ\mathbf{d}(\eta)}{\to}\mathbf{c}^-\circ\mathbf{d}(K_2),$$

or equivalently

$$\mathbf{b}\circ\mathbf{i}(K_1)\overset{\mathbf{b}(\eta)}{\to}\mathbf{b}(K_2)\overset{\xi(K_2)}{\to}\mathbf{c}^-\circ\mathbf{d}(K_2).$$

We claim $\xi(K_2)$ is invertible. Indeed, this is because K_2 is a !-extension from the G-stratum, and the open embedding

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\subset\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P}$$

is an isomorphism when restricted to the G-stratum. Hence γ_P is equivalent to $\mathbf{b}(\eta)$, which by definition is the image of $\mathbf{K}(P)\to\mathbf{K}(G)$ under $\mathbf{b}\circ\mathbf{j}\simeq(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta)$.

□[Lemma 6.4.11]

6.11 Proof of Proposition-Construction 6.4.13 and Lemma 6.4.16

Goal 6.11.1. Construct a canonical factorization of the map

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\to\mathrm{Bun}_M\times\mathrm{Bun}_G$$

through $\mathrm{Bun}_M\times\mathrm{Bun}_G^{P^-\text{-gen}}$ such that we have an isomorphism

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}}\underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times}\mathrm{Bun}_P^{M\text{-gen}}\simeq Y^P_{\mathrm{rel}}/Z_M \tag{6.32}$$

defined over $\mathrm{Bun}_M\times\overline{\mathrm{Bun}}_G\times\mathrm{Bun}_M$.

The proof below is similar to that in § 6.2. Hence we omit some details.

Recall in § 6.9, we defined

$$(\mathrm{Bun}_{P^-}\underset{\mathrm{Bun}_G}{\times}\mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}}:=\mathbf{Maps}_{\mathrm{gen}}(X,P^-\backslash\mathrm{Vin}_{G,\geq C_P}/G\leftarrow\mathbb{B}(P^-\underset{G}{\times}\widetilde{G}_{\geq C_P})).$$

By Lemma A.1.2, the *right* projection map $P^-\times_G\widetilde{G}_{\geq C_P}\to G$ factors through P^-. Hence we obtain the following commutative diagram of algebraic stacks

$$\begin{tikzcd}
P^-\backslash\mathrm{Vin}_{G,\geq C_P}/G \arrow[d] & \mathbb{B}(P^-\times_G\widetilde{G}_{\geq C_P}) \arrow[l] \arrow[d] \\
\mathbb{B}M\times\mathbb{B}G & \mathbb{B}M\times\mathbb{B}P^- \arrow[l]
\end{tikzcd}$$

Taking $\mathbf{Maps}_{\mathrm{gen}}(X,-)$, we obtain a canonical map

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P\text{-gen}}.$$

To obtain the map $(\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} \to \mathrm{Bun}_M \times \mathrm{Bun}_G$, as before, we can show that the map $\mathbb{B}(P^- \times_G \widetilde{G}_{\geq C_P}) \to \mathbb{B}M \times \mathbb{B}P^-$ can be uniquely lifted to a morphism

$$(Z_M \curvearrowright \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P})) \to (\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}P^-)$$

fitting into the diagram

$$\begin{tikzcd}
(Z_M \curvearrowright P^- \backslash \mathrm{Vin}_{G,\geq C_P}/G) \arrow[d] & (Z_M \curvearrowright \mathbb{B}(P^- \times_G \widetilde{G}_{\geq C_P})) \arrow[l] \arrow[d] \\
(\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}G) & (\mathrm{pt} \curvearrowright \mathbb{B}M \times \mathbb{B}P^-). \arrow[l]
\end{tikzcd}$$

It remains to compare both sides of (6.32). By construction,

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times} \mathrm{Bun}_P^{M\text{-gen}}$$

is isomorphic to the image of

$$P^- \backslash \mathrm{Vin}_{G,\geq C_P}/P \leftarrow \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{\mathbb{B}P^-}{\times} \mathbb{B}M$$

under the functor $\mathbf{Maps}_{\mathrm{gen}}(X,-)$. Using Lemma A.1.2(1), the canonical map

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{P^-}{\times} M) \to \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \underset{\mathbb{B}P^-}{\times} \mathbb{B}M$$

is an isomorphism. Also, the LHS is just

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{P^-}{\times} M) \simeq \mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \underset{G}{\times} P) \simeq P^- \backslash \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}/P^-.$$

Hence we obtain a Z_M-equivariant isomorphism

$$(\mathrm{Bun}_{P^-} \underset{\mathrm{Bun}_G}{\times} \mathrm{VinBun}_{G,\geq C_P})^{\mathrm{gen}} \underset{\mathrm{Bun}_G^{P^-\text{-gen}}}{\times} \mathrm{Bun}_P^{M\text{-gen}} \simeq Y^P_{\mathrm{rel}}.$$

It follows from construction that it is defined over $\mathrm{Bun}_M \times \mathrm{VinBun}_G \times \mathrm{Bun}_M$. Then we obtain the isomorphism (6.32) by taking quotients for the Z_M-actions.

□[Proposition-Construction 6.4.13 and Lemma 6.4.16]

6.12 Proof of Proposition 6.4.14

We first prove the claim for the second object. Using the base-change isomorphisms, it is easy to see $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}(G)$ is isomorphic to the image of $k_{\mathrm{Bun}_{P^-}}$ under the !-pushforward functor along

$$\mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}.$$

This map has a factorization

$$\mathrm{Bun}_{P^-} \xrightarrow{f} \mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}.$$

It is clear that $f_!(k_{\mathrm{Bun}_{P^-}})$ is $U^-(\mathbb{A}_F)$-equivariant, i.e., is a $*$-pullback along

$$\mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_M.$$

Then we are done by applying Proposition 2.3.17(1) to the reductive group $M \times G$ and the parabolic subgroup $M \times P^-$.

Now we prove the claim for the first object. Consider the restriction of β on the P-stratum:

$$\beta'_P : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,P}).$$

Since $\mathbf{K}(P)$ is a !-extension along $i_P : \overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G$,

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}(P) \simeq (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta'_P) \circ i_P^*(\mathbf{K}(P)).$$

It follows from construction that β'_P is isomorphic to the composition of

$$\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}} \xleftarrow{\overline{\mathbf{\Delta}}_P^{\mathrm{enh}}} \overline{\mathrm{Bun}}_{G,P} \xrightarrow{=} \overline{\mathrm{Bun}}_{G,P}$$

and

$$\delta^- : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_{P^-}^{M\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}),$$

where the map $\overline{\mathbf{\Delta}}_P^{\mathrm{enh}}$ is provided by Proposition-Construction 5.3.1. Hence we only need to show

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta^-) \circ \overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P)) \tag{6.33}$$

is contained in $\mathrm{I}(M \times G, M \times P^-)$. In other words, we need to show its !-pullback along

$$\iota_{M\times P^-} : \mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}}$$

is $U^-(\mathbb{A}_F)$-equivariant.

Consider the correspondence

$$\delta^+ : (\mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_G^{P^-\text{-gen}} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

As before, $(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta^-)$ is isomorphic to the restriction of $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\delta^+)$. Hence we can rewrite (6.33) as $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\delta^+) \circ \overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P))$.

Consider the correspondence

$$e : (\mathrm{Bun}_M \times \mathrm{Bun}_{P^-} \leftarrow \mathrm{Bun}_P \times \mathrm{Bun}_{P^-} \to \mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

By the base-change isomorphisms, the functor $\iota_{M\times P^-}^! \circ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\delta^+)$ is just $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(e)$. Hence we only need to show $\overline{\mathbf{\Delta}}_{P,!}^{\mathrm{enh}} \circ i_P^*(\mathbf{K}(P))$ is contained in the full subcategory

$$\mathrm{I}(G \times G, P \times P^-) \subset \mathrm{DMod}(\mathrm{Bun}_G^{P\text{-gen}} \times \mathrm{Bun}_G^{P^-\text{-gen}}).$$

Now this can be proved similarly to Variant 6.4.1.

□[Proposition 6.4.14]

6.13 Proof of Proposition-Construction 6.4.19

Notation 6.13.1. Recall L is the Levi subgroup of Q. We write Z_L for the center of L. Let $P_L = P \cap L$ and $P_L^- = P^- \cap L$ be the parabolic subgroups of L corresponding to P and P^-. Let $L^{P\text{-Bruhat}}$ be the open Bruhat cell $P_L^- P_L$ in L.

Notation 6.13.2. The projection map

$$Y^P_{\mathrm{rel}}/Z_M \to T^+_{\mathrm{ad},\geq C_P}/Z_M \simeq T^+_{\mathrm{ad},\geq P}/T$$

induces a stratification on Y^P_{rel}/Z_M labelled by the poset $\mathrm{Par}_{\geq P}$. As usual, for $Q \in \mathrm{Par}_{\geq P}$ we use the notation:

$$(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \coloneqq (Y^P_{\mathrm{rel}}/Z_M) \underset{T^+_{\mathrm{ad},\geq P}/T}{\times} (T^+_{\mathrm{ad},\geq Q}/T).$$

The stack

$$Y^P_{\mathrm{rel},\geq C_Q} \coloneqq Y^P_{\mathrm{rel}} \underset{T^+_{\mathrm{ad},\geq C_P}}{\times} T^+_{\mathrm{ad},\geq C_Q}$$

inherits a Z_L-action from the Z_M-action on Y^P_{rel}. Note that we have a canonical isomorphism

$$Y^P_{\mathrm{rel},\geq C_Q}/Z_L \simeq (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q}.$$

Construction 6.13.3. By construction, we have

$$Y^P_{\mathrm{rel},\geq C_Q} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \supset P^-\backslash \mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}/P),$$

where

$$\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q} \coloneqq \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P} \cap \mathrm{Vin}_{G,\geq C_Q}.$$

Note that the open locus $\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}$ is contained in $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_Q}$. Indeed, the former is the $(P^- \times P)$-orbit of the canonical section, while the later is the $(Q^- \times Q)$-orbit. Hence the map

$$P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \to Q^-\backslash \mathrm{Vin}_{G,\geq C_Q}/Q$$

induces a Z_L-equivariant map $Y^P_{\mathrm{rel},\geq C_Q} \to Y^Q_{\mathrm{rel}}$. Hence we obtain a canonical map

$$\pi_{P,Q} : (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \simeq Y^P_{\mathrm{rel},\geq C_Q}/Z_L \to Y^Q_{\mathrm{rel}}/Z_L.$$

By construction, we have the following commutative diagram

$$\begin{tikzcd}
Y^P_{\mathrm{rel}}/Z_M \arrow[d, "\mathfrak{p}^-_{P,\mathrm{Vin}}"] & (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \arrow[l, "\supset"'] \arrow[r, "\pi_{P,Q}"] \arrow[d] & Y^Q_{\mathrm{rel}}/Z_L \arrow[d, "\mathfrak{p}^-_{Q,\mathrm{Vin}}"] \\
\overline{\mathrm{Bun}}_{G,\geq P} & \overline{\mathrm{Bun}}_{G,\geq Q} \arrow[l, "\supset"'] & \overline{\mathrm{Bun}}_{G,\geq Q}, \arrow[l, "="']
\end{tikzcd} \tag{6.34}$$

where the left square is Cartesian.

Proposition-Construction 6.13.4. *Consider the lft algebraic stack*[80]

$$W_{P,Q} := \mathbf{Maps}_{\mathrm{gen}}(X, P_L^-\backslash\overline{L}/P_L \supset P_L^-\backslash L^{P\text{-}\mathrm{Bruhat}}/P_L).$$

Then there exists a canonical commutative diagram

$$\begin{tikzcd} Y^P_{\mathrm{rel}}/Z_M \arrow[d, "\mathfrak{q}^-_{P,\mathrm{Vin}}"] & (Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \arrow[l, "\supset"'] \arrow[r, "\pi_{P,Q}"] \arrow[d] & Y^Q_{\mathrm{rel}}/Z_L \arrow[d, "\mathfrak{q}^-_{Q,\mathrm{Vin}}"] \\ H_{M,G\text{-}\mathrm{pos}}/Z_M & W_{P,Q}/Z_L \arrow[l] \arrow[r] & H_{L,G\text{-}\mathrm{pos}}/Z_L \end{tikzcd} \tag{6.35}$$

such that the right square in it is Cartesian.

Proof. Via the canonical identification

$$P_L^-\backslash\overline{L}/P_L \simeq \mathbb{B}P_L^- \underset{\mathbb{B}L}{\times} (L\backslash\overline{L}/L) \underset{\mathbb{B}L}{\times} \mathbb{B}P_L,$$

the open substack $P_L^-\backslash L^{\mathrm{Bruhat}}/P_L$ of the LHS is contained in the open substack $\mathbb{B}P_L^- \times_{\mathbb{B}L} \mathbb{B}L \times_{\mathbb{B}L} \mathbb{B}P_L$ of the RHS. Hence we obtain a Z_L-equivariant schematic open embedding

$$W_{P,Q} \to \mathrm{Bun}_{P_L^-} \underset{\mathrm{Bun}_L}{\times} H_{L,G\text{-}\mathrm{pos}} \underset{\mathrm{Bun}_L}{\times} \mathrm{Bun}_{P_L}.$$

In particular, we obtain a canonical map

$$W_{P,Q}/Z_L \to H_{L,G\text{-}\mathrm{pos}}/Z_L. \tag{6.36}$$

As explained in Construction A.5.8, the map

$$\mathrm{Vin}_{G,\geq C_Q} \to \mathrm{Vin}_{G,C_Q},\ x \mapsto \mathfrak{s}(C_Q)\cdot x \cdot \mathfrak{s}(C_Q)$$

factors through $\overline{L}$. It is easy to see the obtained map $\mathrm{Vin}_{G,\geq C_Q} \to \overline{L}$ intertwines the actions of $Q^- \times Q \to L \times L$ and is Z_L-equivariant[81]. Moreover, the map

$$P^-\backslash \mathrm{Vin}_{G,\geq C_Q}/P \to P_L^-\backslash\overline{L}/P_L$$

[80] When $Q = G$, $W_{P,G}$ is just the open Zastava stack. When $Q = P$, $W_{P,P}$ is $H_{M,G\text{-}\mathrm{pos}}$.

[81] This Z_L-action on $\mathrm{Vin}_{G,\geq C_Q}$ is induced by the canonical T-action on Vin_G.

sends the P-Bruhat cell to the P-Bruhat cell. Hence we obtain a Z_L-linear map $Y^P_{\mathrm{rel},\geq C_Q} \to W_{P,Q}$. By taking quotient, we obtain a canonical map

$$(Y^P_{\mathrm{rel}}/Z_M)_{\geq Q} \simeq Y^P_{\mathrm{rel},\geq C_Q}/Z_L \to W_{P,Q}/Z_L. \tag{6.37}$$

Note that we have $\mathfrak{s}(C_P)\cdot x\cdot \mathfrak{s}(C_P) = \mathfrak{s}(C_P)\cdot\mathfrak{s}(C_Q)\cdot x\cdot\mathfrak{s}(C_Q)\cdot\mathfrak{s}(C_P)$ for $x \in \mathrm{Vin}_{G,\geq C_Q}$. Hence the composition

$$\mathrm{Vin}_{G,\geq C_Q} \to \mathrm{Vin}_{G,\geq C_P} \to \overline{M}$$

factors through $\overline{L}$. Since the above composition intertwines the action of $P^- \times P \to M\times M$ and is Z_L-equivariant, the obtained map $\overline{L}\to\overline{M}$ intertwines the actions of $P_L^-\times P_L \to M\times M$ and is Z_L-equivariant. Moreover, the map

$$P_L^-\backslash \overline{L}/P_L \to M\backslash\overline{M}/M$$

sends the P-Bruhat cell into $M\backslash M/M$. Hence we obtain a canonical map

$$W_{P,Q}/Z_L \to H_{M,G\text{-}\mathrm{pos}}/Z_L \to H_{M,G\text{-}\mathrm{pos}}/Z_M. \tag{6.38}$$

It follows from constructions that the above maps (6.36), (6.37) and (6.38) fit into a commutative diagram (6.35). It remains to show its right square is Cartesian. We only need to show the canonical maps

$$\begin{array}{rcl} P^-\backslash\,\mathrm{Vin}_{G,\geq C_Q}/P & \to & (Q^-\backslash\,\mathrm{Vin}_{G,\geq C_Q}/Q) \underset{(L\backslash\overline{L}/L)}{\times} (P_L^-\backslash\overline{L}/P_L), \\ P^-\backslash\,\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}/P & \to & (Q^-\backslash\,\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_Q}/Q) \underset{(L\backslash L/L)}{\times} (P_L^-\backslash L^{P\text{-}\mathrm{Bruhat}}/P_L) \end{array}$$

are isomorphisms. To prove the claim for the first map, we only need to show $\mathbb{B}P \simeq \mathbb{B}Q\times_{\mathbb{B}L}\mathbb{B}P_L$, but this follows from the fact that $Q\to L$ is surjective. The claim for the second map follows from the fact that the canonical maps

$$\begin{array}{rcl} P^-\backslash\,\mathrm{Vin}^{P\text{-}\mathrm{Bruhat}}_{G,\geq C_Q}/P & \to & M\backslash M/M\times T_{\mathrm{ad},\geq C_Q}, \\ Q^-\backslash\,\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_Q}/Q & \to & L\backslash L/L\times T_{\mathrm{ad},\geq C_Q}, \\ P_L^-\backslash L^{P\text{-}\mathrm{Bruhat}}/P_L & \to & M\backslash M/M. \end{array}$$

are all isomorphisms.

□[Proposition-Construction 6.13.4]

6.13.5 (Finish of the proof)**.** The correspondence from $H_{L,G\text{-pos}}/Z_L$ to $H_{M,G\text{-pos}}/Z_M$ is

$$H_{M,G\text{-pos}}/Z_M \leftarrow W_{P,Q}/Z_L \to H_{L,G\text{-pos}}/Z_L.$$

It satsifies the requirement because of (6.34) and (6.35).

□[Proposition-Construction 6.4.19]

6.14 Proof of Theorem 6.4.20

The proof is similar to that of Theorem 6.4.6 hence we omit some details.

Using the homomorphism

$$\mathbb{G}_m \xrightarrow{\gamma} Z_M \to T_{\mathrm{ad}} \xrightarrow{t\mapsto(t^{-1},t)} T_{\mathrm{ad}} \times T_{\mathrm{ad}},$$

we obtain a $\mathbb{G}_m$-action on $G \times G$, whose attractor, repellor and fixed loci are respectively given by $P^- \times P$, $P \times P^-$ and $M \times M$.

On the other hand, consider the action

$$\mathbb{G}_m \times \mathrm{Vin}_{G,\geq C_P},\ (s,x) \mapsto \mathfrak{s}(\overline{\gamma}(s)) \cdot x \cdot \mathfrak{s}(\overline{\gamma}(s)).$$

This action can actually be extended to an $\mathbb{A}^1$-action using the same formula. Hence its attractor, repellor and fixed loci are respectively given by $\mathrm{Vin}_{G,\geq C_P}$, $\overline{M}$ and $\overline{M}$. Also, the attractor, repellor and fixed loci for the restricted action on $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$ are respectively given by $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$, M and M.

We claim the above $\mathbb{G}_m$-actions are compatible with the action $G \times G \curvearrowright \mathrm{Vin}_{G,\geq C_P}$. Indeed, one only need to prove this claim for the restricted actions on $\mathrm{Vin}_{G,\geq C_P} \times_{T^+_{\mathrm{ad}}} T_{\mathrm{ad}}$, which can be checked directly (see Lemma 6.14.2 below). As a corollary of this claim, we obtain an action (relative to $\mathbb{A}^1$) of the Drinfeld-Gaitsgory interpolation for $G \times G$ on that for $\mathrm{Vin}_{G,\geq C_P}$.

Let $(\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}}$ be the category defined similarly as $\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}}$ (see Notation 6.8.14) but we replace "algebraic groups" by "affine group schemes over an affine base scheme". In other words, its objects are $(H \curvearrowright Y)_{/S}$, where S is an affine scheme, $H \to S$ is an affine group scheme and $Y \to S$ is an affine scheme equipped with an H-action. There is an obvious $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-action on $(\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}}$. By the previous discussion,

$$(G \times G \curvearrowright \mathrm{Vin}_{G,\geq C_P})_{/\mathrm{pt}}.$$

is a $\mathbb{G}_m$-module object. Then Example 6.7.6 provides a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched functor

$$\Theta_{(G\times G\curvearrowright \mathrm{Vin}_{G,\geq C_P})}:\mathbf{P}_{\mathbb{A}^1}\to \mathrm{Corr}((\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}})_{\mathrm{all,all}},$$

sending α^+ and α^- respectively to

$$\begin{array}{ccccc}(G\times G\curvearrowright \mathrm{Vin}_{G,\geq C_P}) & \leftarrow (P^-\times P\curvearrowright \mathrm{Vin}_{G,\geq C_P})\to & (M\times M\curvearrowright \overline{M}),\\ (M\times M\curvearrowright \overline{M}) & \leftarrow (P\times P^-\curvearrowright \overline{M})\to & (G\times G\curvearrowright \mathrm{Vin}_{G,\geq C_P}).\end{array}$$

Passing to quotients, we obtain a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\Theta_{(G\backslash \mathrm{Vin}_{G,\geq C_P}/G)}:\mathbf{P}_{\mathbb{A}^1}\dashrightarrow \mathbf{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})^{\mathrm{all},2\text{-op}}_{\mathrm{all,all}}.$$

It is easy to see it is strict at the composition $\alpha^+\circ\alpha^-$. Moreover, we claim it factors through $\mathbf{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all,all}}$. To prove the claim, one first proves Fact 6.14.1 below, then uses it to deduce the desired claim from Lemma 6.8.8.

In the previous construction, we ignored the open Bruhat cell. If we keep tracking it, we would obtain a certain weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\mathbf{P}_{\mathbb{A}^1}\dashrightarrow \mathbf{Corr}(\mathrm{Arr}(\mathrm{AlgStk}_{\mathrm{lft}}))^{\mathrm{open},2\text{-op}}_{\mathrm{all,all}}.$$

By taking $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ for it, we obtain a weakly $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$-enriched right-lax functor

$$\Theta:\mathbf{P}_{\mathbb{A}^1}\dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all,all}}$$

sending α^+ and α^- respectively to

$$\begin{array}{ccc}\mathrm{VinBun}_{G,\geq C_P} & \leftarrow Y^P_{\mathrm{rel}}\to & H_{M,G\text{-pos}},\\ H_{M,G\text{-pos}} & \leftarrow {}_{\mathrm{str}}\mathrm{VinBun}_{G,C_P}\to & \mathrm{VinBun}_{G,\geq C_P}.\end{array}$$

Also, Θ is strict at the composition $\alpha^+\circ\alpha^-$.

As before, we can restrict to each connected component $H^{\lambda,\mu}_{M,G\text{-pos}}$ of $H^{\lambda,\mu}_{M,G\text{-pos}}$ and obtain a Drinfeld pre-input

$$\Theta_{\lambda,\mu}:\mathbf{P}_{\mathbb{A}^1}\dashrightarrow \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{safe,safe}}.$$

In fact, the right arms of the relevant correspondences are schematic.

Also, by taking quotients for the $\mathbb{G}_m$-actions, we can obtain a Drinfeld input sending α^+ and α^- respectively to

$$\begin{array}{ccc} \mathrm{VinBun}_{G,\geq C_P}/\mathbb{G}_m & \leftarrow Y^{P,\lambda,\mu}_{\mathrm{rel}}/\mathbb{G}_m \rightarrow & H^{\lambda,\mu}_{M,G\text{-pos}}/\mathbb{G}_m, \\ H^{\lambda,\mu}_{M,G\text{-pos}}/\mathbb{G}_m & \leftarrow {}_{\mathrm{str}}\mathrm{VinBun}^{\lambda,\mu}_{G,C_P}/\mathbb{G}_m \rightarrow & \mathrm{VinBun}_{G,\geq C_P}/\mathbb{G}_m. \end{array}$$

By Lemma 6.14.3 below, we see that the above $\mathbb{G}_m$-action on $\mathrm{VinBun}_{G,\geq C_P}$ can be obtained from the canonical Z_M-actions by restriction along $2\gamma : \mathbb{G}_m \to Z_M$. Hence Theorem 6.7.8 implies $\mathfrak{q}^+_{P,\mathrm{Vin},*} \circ \mathfrak{p}^{+,!}_{P,\mathrm{Vin}}$ is left adjoint to

$$\prod_{\lambda,\mu} \mathfrak{p}^{-,\lambda,\mu}_{P,\mathrm{Vin},*} \circ \mathfrak{q}^{-,\lambda,\mu,!}_{P,\mathrm{Vin}}.$$

Note that the above functor is also the right adjoint of $\mathfrak{q}^-_{P,\mathrm{Vin},!} \circ \mathfrak{p}^{-,*}_{P,\mathrm{Vin}}$. Hence we obtain

$$\mathfrak{q}^+_{P,\mathrm{Vin},*} \circ \mathfrak{p}^{+,!}_{P,\mathrm{Vin}} \simeq \mathfrak{q}^-_{P,\mathrm{Vin},!} \circ \mathfrak{p}^{-,*}_{P,\mathrm{Vin}}.$$

The equivalence $\mathfrak{q}^-_{P,\mathrm{Vin},*} \circ \mathfrak{p}^{-,!}_{P,\mathrm{Vin}} \simeq \mathfrak{q}^+_{P,\mathrm{Vin},!} \circ \mathfrak{p}^{+,*}_{P,\mathrm{Vin}}$ can be obtained by exchanging the roles of α^+ and α^-.

□[Theorem 6.4.20]

Fact 6.14.1. *For a diagram*

$$(H_1 \curvearrowright Y_1)_{/S_1} \to (H_2 \curvearrowright Y_2)_{/S_2} \leftarrow (H_3 \curvearrowright Y_3)_{/S_3}$$

in $(\mathrm{ActSch}^{\mathrm{aff}}_{\mathrm{ft}})_{\mathrm{rel}}$*, if* H_1*,* H_2*,* H_3 *and* $H_1 \times_{H_2} H_3$ *are all flat over their base schemes, then the following square is Cartesian*

$$\begin{tikzcd} (Y_1 \times_{Y_2} Y_3)/(H_1 \times_{H_2} H_3) \arrow[r] \arrow[d] & (Y_1/H_1) \times_{(Y_2/H_2)} (Y_3/H_3) \arrow[d] \\ \mathbb{B}(H_1 \times_{H_2} H_3) \arrow[r] & \mathbb{B}H_1 \times_{\mathbb{B}H_2} \mathbb{B}H_3. \end{tikzcd}$$

Lemma 6.14.2. *Consider the actions*

$$\begin{array}{ll} T_{\mathrm{ad}} \curvearrowright \mathrm{Vin}_G, & t \cdot x := \mathfrak{s}(t) \cdot x \cdot \mathfrak{s}(t), \\ T_{\mathrm{ad}} \curvearrowright (G \times \mathrm{Vin}_G \times G), & t \cdot (g_1, x, g_2) := (\mathrm{Ad}_{t^{-1}}(g_1), \mathfrak{s}(t) \cdot x \cdot \mathfrak{s}(t), \mathrm{Ad}_t(g_2)). \end{array}$$

The map

$$G \times \mathrm{Vin}_G \times G \to \mathrm{Vin}_G, (g_1, x, g_2) \mapsto g_1 \cdot x \cdot g_2^{-1}$$

is equivariant for these actions.

Proof. We only need to prove the lemma after restricting to the subgroup of invertible elements in Vin_G, which is given by $G_{\mathrm{enh}} \coloneqq (G \times T)/Z_G$. Then we are done by a direct calculation. (Recall that the canonical section $T/Z_G \to (G \times T)/Z_G$ is given by $t \mapsto (t^{-1}, t)$).

□[Lemma 6.14.2]

Lemma 6.14.3. *Consider the following two T-actions on $G \backslash \mathrm{Vin}_G / G$:*

(i) The action provided by Lemma 6.14.2 via the homomorphism $T \to T_{\mathrm{ad}}$.

(ii) The one obtained from the canonical T-action on Vin_G, which commutes with the $(G \times G)$-action.

The action in (i) is isomorphic to the square of the action in (ii).

Proof. Recall that the subgroup of invertible elements in Vin_G is isomorphic to $G_{\mathrm{enh}} \coloneqq (G \times T)/Z_G$. We have a short exact sequence $1 \to G \to G_{\mathrm{enh}} \to T_{\mathrm{ad}} \to 1$. The canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$ provides a splitting to the above sequence. Explicitly, this splitting is given by $t \mapsto (t^{-1}, t)$. Note that the corresponding T_{ad} on G is the *inverse* of the usual adjoint action.

Consider the sequence:

$$1 \to G \times G \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}} \to 1.$$

Recall that the $(G \times G)$-action on Vin_G is defined to be the restriction of the $(G_{\mathrm{enh}} \times G_{\mathrm{enh}})$-action on Vin_G. Hence the quotient stack $G \backslash \mathrm{Vin}_G / G$ inherits a $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-action. By the last paragraph, the action in (i) is obtained from this $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-action by restriction along the homomorphism

$$a : T \to T_{\mathrm{ad}} \times T_{\mathrm{ad}},\ t \mapsto (t, t^{-1}). \tag{6.39}$$

On the other hand, consider the center $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$ of $G_{\mathrm{enh}} \times G_{\mathrm{enh}}$. Then $G_{\mathrm{enh}} \times G_{\mathrm{enh}}$-action on Vin_G induces a $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$-action on $G \backslash \mathrm{Vin}_G / G$. By construction, this action factors through the homomorphism

$$q : Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \to Z(G_{\mathrm{enh}}),\ (s_1, s_2) \mapsto s_1 s_2^{-1}.$$

In summary, we obtain compatible actions on $G \backslash \mathrm{Vin}_G / G$ by

$$Z(G_{\mathrm{enh}}) \xleftarrow{q} Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \xrightarrow{p} T_{\mathrm{ad}} \times T_{\mathrm{ad}},$$

where p is the composition $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}}) \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}}$.

Recall that the homomorphism $T \to (G \times T)/Z_G, t \mapsto (1,t)$ provides an isomorphism between $T \simeq Z(G_{\mathrm{enh}})$ and the canonical T-action on Vin_G is defined by using this identification. Hence the square of the action in (ii) can be obtained from the $Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$-action via the homomorphism

$$T \simeq Z(G_{\mathrm{enh}}) \overset{s \mapsto (s,s^{-1})}{\to} Z(G_{\mathrm{enh}}) \times Z(G_{\mathrm{enh}})$$

(because its composition with q is the square map). Then we are done because the composition of this map by p is equal to a.

□[Lemma 6.14.3]

Variant 6.14.4. *Theorem 6.4.20 is also correct if we replace the diagram*

$$\overline{\mathrm{Bun}}_{G,\geq P} \leftarrow Y^P_{\mathrm{rel}}/Z_M \to H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow {}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_{G,\geq P}$$

by the corresponding γ-version:

$$\mathrm{VinBun}^{\gamma}_G/\mathbb{G}_m \leftarrow Y^{P,\gamma}_{\mathrm{rel}}/\mathbb{G}_m \to H_{M,G\text{-}\mathrm{pos}}/\mathbb{G}_m \leftarrow {}_{\mathrm{str}}\mathrm{VinBun}^{\gamma}_G/\mathbb{G}_m \to \mathrm{VinBun}^{\gamma}_G/\mathbb{G}_m.$$

The proof is similar.

6.15 Proof of Theorem 5.1.7

In this section, we prove Theorem 5.1.7. We want to apply Theorem 4.2.9 to the correspondence

$$\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \leftarrow \mathrm{VinGr}^{\gamma}_{G,I} \overset{\pi_I}{\to} \mathrm{VinBun}^{\gamma}_G. \tag{6.40}$$

The Braden 4-tuples for $\mathrm{Gr}_{G\times G,I}$ and $\mathrm{VinGr}_{G,I}$ are provided by Construction 4.3.6. The only missing ingredient is a suitable Braden 4-tuple $\mathrm{Br}^{\gamma}_{\mathrm{glob}}$ for $\mathrm{VinBun}^{\gamma}_G$, which we propose to be

$$(\mathrm{VinBun}^{\gamma}_G, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y^{P,\gamma}_{\mathrm{rel}}, H_{M,G\text{-}\mathrm{pos}}).$$

Construction 6.15.1. Recall that

- ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ is the disjoint union of the *defect strata* of $\mathrm{VinBun}_G|_{C_P}$ (see § A.5.4):

$$ {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} := \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash \mathrm{Vin}_G^\gamma/G \supset G\backslash_0 \mathrm{Vin}_G^\gamma/G); $$

- $Y_{\mathrm{rel}}^{P,\gamma}$ is the *Schieder's local model* for VinBun_G^γ (see § A.5.6):

$$ Y_{\mathrm{rel}}^{P,\gamma} := \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_G^\gamma/P \supset P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P); $$

- $H_{M,G\text{-}\mathrm{pos}}$ is the *G-position Hecke stack* for Bun_M (see § A.5.3):

$$ H_{M,G\text{-}\mathrm{pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M). $$

By (A.20), we have the following commutative diagram (c.f. (4.17))

$$ \begin{tikzcd} & & (M\backslash\overline{M}/M \supset M\backslash M/M) \\ & (M\backslash\overline{M}/M \supset M\backslash M/M) \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{pair}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{pair}}"'] & (P\backslash\overline{M}/P^- \supset P\backslash M/P^-) \arrow[u, "\mathbf{q}^+_{\mathrm{pair}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{pair}}"] \\ (M\backslash\overline{M}/M \supset M\backslash M/M) & (P^-\backslash \mathrm{Vin}_G^\gamma/P \supset P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P) \arrow[l, "\mathbf{q}^-_{\mathrm{pair}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{pair}}"'] & (G\backslash \mathrm{Vin}_G^\gamma/G \supset G\backslash_0 \mathrm{Vin}_G^\gamma/G). \end{tikzcd} \tag{6.41} $$

It induces a commutative diagram

$$ \begin{tikzcd} & & H_{M,G\text{-}\mathrm{pos}} \\ & H_{M,G\text{-}\mathrm{pos}} \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{glob}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{glob}}"'] & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[u, "\mathbf{q}^+_{\mathrm{glob}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{glob}}"] \\ H_{M,G\text{-}\mathrm{pos}} & Y_{\mathrm{rel}}^{P,\gamma} \arrow[l, "\mathbf{q}^-_{\mathrm{glob}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{glob}}"'] & \mathrm{VinBun}_G^\gamma. \end{tikzcd} \tag{6.42} $$

Proposition-Definition 6.15.2. *The above commutative square defines a Braden 4-tuple (see Definition E.2.9):*

$$ (\mathrm{VinBun}_G^\gamma, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y_{\mathrm{rel}}^{P,\gamma}, H_{M,G\text{-}\mathrm{pos}}), $$

such that $\mathbf{i}^-_{\mathrm{glob}}$, $\mathbf{p}^+_{\mathrm{glob}}$ and $\mathbf{q}^-_{\mathrm{glob}}$ are ind-finite type ind-schematic.

We call it the global Braden 4-tuple $\mathrm{Br}^\gamma_{\mathrm{glob}}$.

Proof. To show $(\mathrm{VinBun}_G^\gamma, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y_{\mathrm{rel}}^{P,\gamma}, H_{M,G\text{-}\mathrm{pos}})$ defines a Braden 4-tuple, we only need to show that the square in (6.42) is quasi-Cartesian. This follows from Lemma A.3.5(1) and the schematic open embedding

$$\mathrm{pt}/M \to (\mathrm{pt}/P) \underset{(\mathrm{pt}/G)}{\times} (\mathrm{pt}/P).$$

The map $\mathbf{p}^+_{\mathrm{glob}}$ is ind-finite type ind-schematic because its restriction to each connected component is a schematic locally closed embedding (see [Sch16, Proposition 3.3.2(a)]). Hence $\mathbf{i}^-_{\mathrm{glob}}$ is also ind-finite type ind-schematic because the square in (6.42) is quasi-Cartesian.

It remains to show $\mathbf{q}^-_{\mathrm{glob}}$ is ind-finite type ind-schematic. We claim it is affine and of finite type. By Lemma A.6.4, we only need to prove the similar claim for $Y^{P,\gamma} \to \mathrm{Gr}_{M,G\text{-}\mathrm{pos}}$. However, this follows from [Sch16, Lemma 6.5.6] and [DG14, Theorem 1.5.2(2)].

□[Proposition-Definition 6.15.2]

Proposition-Construction 6.15.3. *The correspondence*

$$\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \leftarrow \mathrm{VinGr}^{\gamma}_{G,I} \xrightarrow{\pi_I} \mathrm{VinBun}^{\gamma}_G$$

can be extended to a correspondence between Braden 4*-tuples*

$$\mathrm{Br}^{\gamma}_I \leftarrow \mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \to \mathrm{Br}^{\gamma}_{\mathrm{glob}}$$

defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$. *Moreover, this extension satisfies Axioms (P1)-(P3) and (Q) in § 4.2.8.*

Proof. The morphism $\mathrm{Br}^{\gamma}_I \leftarrow \mathrm{Br}^{\gamma}_{\mathrm{Vin},I}$ was constructed in Construction 4.3.6. The morphism $\mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \to \mathrm{Br}^{\gamma}_{\mathrm{glob}}$ is induced by the obvious morphism from the diagram (4.17) to (6.41) (see Construction 1.4.2).

Axioms (P1)-(P2) follow from the calculation in Construction 4.3.6. Axiom (Q) follows from Proposition 4.3.9. It remains to verify Axiom (P3). In other words, we only need to show the commutative diagram

$$\begin{tikzcd} \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \arrow[r] \arrow[d] & \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \arrow[d] \\ Y^{P,\gamma}_{\mathrm{rel}} \arrow[r] & H_{M,G\text{-}\mathrm{pos}} \end{tikzcd}$$

is Cartesian. Recall it is obtained by applying Construction 1.4.2 to the following commutative diagram

$$\begin{tikzcd} (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \leftarrow \mathbb{A}^1) \arrow[r, "\mathbf{q}^-_{\mathrm{sect}}"] \arrow[d] & (M\backslash \overline{M}/M \leftarrow \mathrm{pt}) \arrow[d] \\ (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \supset P^-\backslash \mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G/P) \arrow[r, "\mathbf{q}^-_{\mathrm{pair}}"] & (M\backslash \overline{M}/M \supset M\backslash M/M). \end{tikzcd}$$

By Lemma A.3.7, it suffices to show the canonical map

$$\mathbb{A}^1 \to \mathrm{pt} \underset{(M\backslash \overline{M}/M)}{\times} (P^- \backslash \mathrm{Vin}_G^\gamma / P)$$

is an isomorphism. Using the Cartesian diagram (A.20), the RHS is isomorphic to

$$\mathrm{pt} \underset{(M\backslash M/M)}{\times} (P^- \backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}} / P).$$

Then we are done by the $(M \times M)$-equivariant isomorphism (A.19).

□Proposition-Construction 6.15.3

6.15.4 (Finish of the proof)**.** We apply Theorem 4.2.9 to

- the correspondence $\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \leftarrow \mathrm{VinGr}_{G,I}^\gamma \xrightarrow{\pi_I} \mathrm{VinBun}_G^\gamma$;
- the object $\mathring{\mathcal{F}} \coloneqq \omega_{\mathrm{Bun}_G \times \mathbb{G}_m}$;
- the correspondence between Braden 4-tuples $\mathrm{Br}_I^\gamma \leftarrow \mathrm{Br}_{\mathrm{Vin},I}^\gamma \to \mathrm{Br}_{\mathrm{glob}}^\gamma$ defined in Proposition-Construction 6.15.3;
- the subcategory $\mathrm{DMod}({}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{DMod}(\mathrm{Gr}_{G\times G,I})$.

The Axioms (P1)-(P3) and (Q) are verified in Proposition-Construction 6.15.3. Axioms (G1) and (G2) are obvious because $\mathring{\mathcal{F}}$ is ind-holonomic. Axiom (C) is just Lemma 4.4.1(2). Axiom (M) is just Proposition 5.1.4 and Lemma 4.4.3. Axioms (N1) and (N3) follow from Variant 6.14.4 and the contraction principle. Axioms (N2), (N4) follow from Braden's theorem and the contraction principle.

□[Theorem 5.1.7]

6.16 Proof of Proposition 5.2.5

We have the following Cartesian squares

$$\begin{tikzcd}
\mathrm{VinBun}_G|_{C_P} \arrow[r, "i"] \arrow[d, "\pi_0"] & \mathrm{VinBun}_G^\gamma \arrow[d, "\pi"] & \mathrm{Bun}_G \times \mathbb{G}_m \arrow[l, "j"'] \arrow[d, "\mathring{\pi}"] \\
\mathrm{VinBun}_G|_{C_P}/\mathbb{G}_m \arrow[r, "i/\mathbb{G}_m"] \arrow[d, "s"] & \mathrm{VinBun}_G^\gamma/\mathbb{G}_m \arrow[d] & \mathrm{Bun}_G \arrow[l, "j/\mathbb{G}_m"'] \arrow[d, "r"] \\
\overline{\mathrm{Bun}}_{G,P} \arrow[r, "i_P"] & \overline{\mathrm{Bun}}_G & \overline{\mathrm{Bun}}_{G,G}, \arrow[l, "j_G"']
\end{tikzcd}$$

defined over $\mathbb{A}^1 \to \mathbb{A}^1/\mathbb{G}_m \to T_{\mathrm{ad}}^+/T$. By (the Verdier conjugate of) Lemma C.7.13 and Fact 5.1.2, we have

$$\pi_{0,!}(\Psi_{\gamma,\mathrm{glob}})[-1] \simeq (i/\mathbb{G}_m)^! \circ (j/\mathbb{G}_m)_!(\omega_{\mathrm{Bun}_G}).$$

The second and third layers provide a canonical map

$$s_! \circ (i/\mathbb{G}_m)^! \circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G}) \to i_P^! \circ j_{G,!} \circ r_!(k_{\mathrm{Bun}_G}). \tag{6.43}$$

And finally, we have a canonical map $i_P^! \circ \mathbf{K}(P) \to i_P^! \circ \mathbf{K}(G)$, i.e.,

$$i_P^* \circ j_{G,*} \circ r_!(k_{\mathrm{Bun}_G})[\mathrm{rank}(M) - \mathrm{rank}(G)] \to i_P^! \circ j_{G,!} \circ r_!(k_{\mathrm{Bun}_G}). \tag{6.44}$$

Hence to prove Proposition 5.2.5, we only need to prove the above two maps are isomorphisms.

We first note that all the three objects in (6.43) and (6.44) are $(U(\mathbb{A}_F) \times U^-(\mathbb{A}_F))$-equivariant. Indeed, these are the contents of Proposition 5.1.4 and Variant 6.1.20. Now consider the correspondence

$$\phi_P : H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow {}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P} \to \overline{\mathrm{Bun}}_G$$

and its base-change

$$\phi_P^\gamma : H_{M,G\text{-}\mathrm{pos}}/\mathbb{G}_m \leftarrow {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}/\mathbb{G}_m \to \mathrm{VinBun}_G^\gamma/\mathbb{G}_m.$$

Let $\phi_{P,=P}$ and $\phi_{P,=P}^\gamma$ be the corresponding correspondences on the P-strata. We only need to show $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P})$, i.e., the functor

$$\mathrm{DMod}_{\mathrm{indhol}}(\overline{\mathrm{Bun}}_{G,P}) \xrightarrow{!\text{-pull}} \mathrm{DMod}_{\mathrm{indhol}}({}_{\mathrm{str}}\overline{\mathrm{Bun}}_{G,P}) \xrightarrow{*\text{-push}} \mathrm{DMod}_{\mathrm{indhol}}(H_{M,G\text{-}\mathrm{pos}}/Z_M)$$

sends (6.43) and (6.44) to isomorphisms.

By Theorem 6.4.20, the functor

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P}) \circ i_P^! \simeq \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_P)$$

is isomorphic to $\mathrm{DMod}_{!\text{-push},*\text{-pull}}(\psi_P)$, where

$$\psi_P : H_{M,G\text{-}\mathrm{pos}}/Z_M \leftarrow Y_{\mathrm{rel}}^P/Z_M \to \overline{\mathrm{Bun}}_G.$$

By Goal 6.4.17 (which we have already proved), this functor sends $\mathbf{K}(P)\to\mathbf{K}(G)$ to an isomorphism. Hence $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P})$ sends (6.44) to an isomorphism as desired.

It remains to show $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P})$ sends (6.43) to an isomorphism. By the last paragraph, the RHS is sent to

$$\mathrm{DMod}_{!\text{-push},*\text{-pull}}(\psi_P)\circ j_{G,!}\circ r_!(k_{\mathrm{Bun}_G}).$$

Let $t: H_{M,G\text{-pos}}/\mathbb{G}_m \to H_{M,G\text{-pos}}/Z_M$ be the canonical projection. Then the LHS is sent to

$$\begin{aligned}
& \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P})\circ s_!\circ (i/\mathbb{G}_m)^!\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G})\\
\simeq\ & \mathrm{DMod}_{!\text{-push},*\text{-pull}}(\psi_{P,=P})\circ s_!\circ (i/\mathbb{G}_m)^!\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G})\\
\simeq\ & t_!\circ \mathrm{DMod}_{!\text{-push},*\text{-pull}}(\psi^{\gamma}_{P,=P})\circ (i/\mathbb{G}_m)^!\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G})\\
\simeq\ & t_!\circ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi^{\gamma}_{P,=P})\circ (i/\mathbb{G}_m)^!\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G})\\
\simeq\ & t_!\circ \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi^{\gamma}_{P})\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G})\\
\simeq\ & t_!\circ \mathrm{DMod}_{!\text{-push},*\text{-pull}}(\psi^{\gamma}_{P})\circ (j/\mathbb{G}_m)_!(k_{\mathrm{Bun}_G}),
\end{aligned}$$

where

- the first equivalence is by Theorem 6.4.20;
- the second equivalence is due to $\psi_{P,=P}\circ s\simeq t\circ\psi^{\gamma}_{P,=P}$ where we view s and t as vertical correspondences;
- the third equivalence is by Variant 6.14.4;
- the fourth equivalence is due to $\phi^{\gamma}_{P,=P}\circ(i/\mathbb{G}_m)\simeq\phi^{\gamma}_P$ where we view $(i/\mathbb{G}_m)$ as a horizontal correspondence;
- the fifth equivalence is by Variant 6.14.4.

Then we are done[82] because we have $\psi_P\circ j_G\circ r\simeq t\circ\psi^{\gamma}_P\circ(j/\mathbb{G}_m)$. Indeed, they are both given by

$$H_{M,G\text{-pos}}/Z_M \leftarrow Y^{\mathrm{rel}}_P|_{C_G}\to \mathrm{Bun}_G\,.$$

□[Proposition 5.2.5]

[82]One also need to check $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}(\phi_{P,=P})$ sends (6.43) to *the* isomorphism obtained from $\psi_P\circ j_G\circ r\simeq t\circ\psi^{\gamma}_P\circ(j/\mathbb{G}_m)$. This can be done by a boring diagram-chasing, very similar to that in the proof of Theorem 4.2.9. In fact, in the statement of Theorem 4.2.9, we can replace the base Braden 4-tuple $(\mathbb{A}^1,0,\mathbb{A}^1,0)$ by any Braden 4-tuple, such as $(T^+_{\mathrm{ad},\geq P}/T, T^+_{\mathrm{ad},P}/T, T^+_{\mathrm{ad},\geq P}/T, T^+_{\mathrm{ad},P}/T)$, and obtain an equivalence (4.5) essentially by the same proof. Then we obtain the equivalence (6.44) as a special case.

6.17 Proof of Theorem 5.3.5

As we will soon see, the main ingredients for the proof of Theorem 5.3.5 have already been provided in § 6.4 when we proved the Deligne-Lusztig duality.

Recall we have monadic adjoint pairs

$$\iota_{M,!} : \mathrm{DMod}(\mathrm{Bun}_M) \rightleftharpoons \mathrm{I}(G,P) : \iota_M^!$$
$$\iota_{M,!}^- : \mathrm{DMod}(\mathrm{Bun}_M) \rightleftharpoons \mathrm{I}(G,P^-) : \iota_M^{-,!}$$

We first deduce Theorem 5.3.5 from the following result:

Proposition 6.17.1. *There exist canonical isomorphisms*

$$(\mathbf{Id} \otimes \iota_{M,!}^-)(\Delta_!(k_{\mathrm{Bun}_M})) \simeq (\iota_M^! \otimes \mathbf{Id})(\mathcal{K}_{\gamma,\mathrm{glob}}) \tag{6.45}$$
$$(\iota_{M,!} \otimes \mathbf{Id})(\Delta_!(k_{\mathrm{Bun}_M})) \simeq (\mathbf{Id} \otimes \iota_M^{-,!})(\mathcal{K}_{\gamma,\mathrm{glob}}) \tag{6.46}$$

such that the two morphisms

$$\Delta_!(k_{\mathrm{Bun}_M}) \to (\iota_M^! \otimes \iota_M^{-,!})\mathcal{K}_{\gamma,\mathrm{glob}} \tag{6.47}$$

induced by them are equivalent to each other.

6.17.2 (Deducing Theorem 5.3.5)**.** Recall that $\mathrm{I}(G,P^-)$ is dualizable. Hence $\mathcal{K}_{\gamma,\mathrm{glob}} \in \mathrm{I}(G,P) \otimes \mathrm{I}(G,P^-)$ induces a functor $\Theta : \mathrm{I}(G,P^-)^\vee \to \mathrm{I}(G,P)$. Via the identification

$$\mathrm{DMod}(\mathrm{Bun}_M) \otimes \mathrm{I}(G,P^-) \simeq \mathrm{LFun}_k(\mathrm{I}(G,P^-), \mathrm{DMod}(\mathrm{Bun}_M)),$$

the isomorphism (6.45)

$$\begin{tikzcd} \mathrm{DMod}(\mathrm{Bun}_M)^\vee \arrow[d, "\text{Ps-Id}_!"] & \mathrm{I}(G,P^-)^\vee \arrow[l, "(\iota_{M,!}^-)^\vee"'] \arrow[d, "\Theta"] \\ \mathrm{DMod}(\mathrm{Bun}_M) & \mathrm{I}(G,P) \arrow[l, "\iota_M^!"] \end{tikzcd} \tag{6.48}$$

Similarly, (6.45) provides a commutative diagram

$$\begin{tikzcd} \mathrm{DMod}(\mathrm{Bun}_M)^\vee \arrow[r, "(\iota_M^{-,!})^\vee"] \arrow[d, "\text{Ps-Id}_!"] & \mathrm{I}(G,P^-)^\vee \arrow[d, "\Theta"] \\ \mathrm{DMod}(\mathrm{Bun}_M) \arrow[r, "\iota_{M,!}"] & \mathrm{I}(G,P). \end{tikzcd} \tag{6.49}$$

Moreover, by the last requirement in Proposition 6.17.1, the square (6.49) is obtained from (6.48) by passing to left adjoints along the horizontal direction.

Recall that $\mathrm{DMod}(\mathrm{Bun}_M)$ is compactly generated and $(\iota_{M,!}, \iota_M^!)$ is a monadic adjoint pair. It follows formally that the adjoint pair $((\iota_M^{-,!})^\vee, (\iota_{M,!}^-)^\vee)$ is also monadic. Now by the Barr-Beck-Lurie theorem, the invertibility of $\text{Ps-Id}_!$ implies Θ is an equivalence. In other words, $\mathcal{K}_{\gamma,\mathrm{glob}}$ is the unit for a duality between $\mathrm{I}(G,P)$ and $\mathrm{I}(G,P^-)$. Note that by (6.48), this duality satisfies (a).

To prove the above duality satisfies (b), we only need to prove

$$(\mathbf{Id} \otimes \mathrm{Eis}^{\mathrm{enh}}_{P^- \to G})(\mathcal{K}_{\gamma,\mathrm{glob}}) \simeq (\mathrm{CT}^{\mathrm{enh}}_{G\to P} \otimes \mathbf{Id})(\Delta_!(k_{\mathrm{Bun}_G})).$$

Unwinding the definitions, the LHS is the object $\mathcal{F}_P$ defined in (6.10) and the RHS is $\mathrm{CT}^{\mathrm{enh}}_{G\times G \to P\times G}(\mathcal{F}_G)$. Now the desired isomorphism is given by Goal 6.4.3 (which we have already proved).

$\square$[Theorem 5.3.5]

6.17.3 (Proof of Proposition 6.17.1)**.** Recall the parameterized correspondence (see (6.18))

$$\beta' : \qquad \begin{array}{ccccc} \mathrm{Bun}_M \times \mathrm{Bun}_G^{P^-\text{-gen}} & \longleftarrow & (\mathrm{Bun}_{P^-} \times_{\mathrm{Bun}_G} \overline{\mathrm{Bun}}_{G,\geq P})^{\mathrm{gen}} & \longrightarrow & \overline{\mathrm{Bun}}_G \\ & & \downarrow & & \\ & & T^+_{\mathrm{ad},\geq P}/T. & & \end{array}$$

In § 6.4, we have already showed the functor

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\beta') \circ \mathbf{K}$$

sends $P \to G$ to an isomorphism, and its value on P and G are exactly the RHS and the LHS of (6.45). Hence we obtain *an* isomorphism

$$(\mathbf{Id} \otimes \iota^-_{M,!})(\Delta_!(k_{\mathrm{Bun}_M})) \simeq (\iota_M^! \otimes \mathbf{Id})(\mathcal{K}_{\gamma,\mathrm{glob}}).$$

Now we describe the morphism (6.45) induced by the above isomorphism. By definition, it is the composition of

$$(\mathbf{Id} \otimes (\iota_M^{-,!} \circ \iota^-_{M,!}))(\Delta_!(k_{\mathrm{Bun}_M})) \simeq (\iota_M^! \otimes \iota_M^{-,!})(\mathcal{K}_{\gamma,\mathrm{glob}}) \tag{6.50}$$

and

$$\Delta_!(k_{\mathrm{Bun}_M}) \to (\mathbf{Id} \otimes (\iota_M^{-,!} \circ \iota^-_{M,!}))(\Delta_!(k_{\mathrm{Bun}_M})) \tag{6.51}$$

Recall the correspondence (see Lemma 6.4.16)

$$\delta \circ \beta' : \mathrm{Bun}_M \times \mathrm{Bun}_M \leftarrow Y^P_{\mathrm{rel}}/Z_M \to \overline{\mathrm{Bun}}_G.$$

In § 6.4, we have already showed that (6.50) is obtained by applying the functor

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta \circ \beta') \circ \mathbf{K}$$

to $G \leftarrow P$. On the other hand, recall there is a schematic open embedding[83]

$$\mathbf{Maps}(X, P^- \backslash (\mathrm{Vin}^{\mathrm{Bruhat}}_{G,G})/P)/Z_M \to Y^P_{\mathrm{rel}}/Z_M$$

inducing a 2-morphism from

$$\xi : \mathrm{Bun}_M \times \mathrm{Bun}_M \leftarrow \mathbf{Maps}(X, P^- \backslash (\mathrm{Vin}^{\mathrm{Bruhat}}_{G,G})/P)/Z_M \to \overline{\mathrm{Bun}}_G$$

to $\delta \circ \beta'$. It is easy to see (6.51) is obtained by applying the natural transformation

$$(\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\xi) \to (\mathrm{DMod}_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}(\delta \circ \beta')$$

to $\mathbf{K}(G)$. Note that the above descriptions of (6.50) and (6.51) are both symmetric. Hence we can exchange the role of P and P^-, and obtain an equivalence (6.46) inducing the same (6.47).

□[Proposition 6.17.1]

Remark 6.17.4. One can actually use the above method to prove that the local nearby cycles induce a duality between $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U^-_I}$, i.e., prove the claim on the unit in Corollary 3.2.4. However, we can not use this method to prove the claim on the counit.

[83] This is just $\mathrm{Bun}_M \times \mathbb{B}Z_M$.

Appendix

Appendix A

Geometric miscellanea

A.1 The stablizer group scheme $\widetilde{G}$

In this section, we prove some technical results on $\widetilde{G}$, which was defined in Fact 1.1.4.

Lemma A.1.1. *The closed subscheme*

$$\widetilde{G}_{\leq P} := \widetilde{G} \underset{T^+_{\mathrm{ad}}}{\times} T^+_{\mathrm{ad},\leq P} \hookrightarrow G \times G \times T^+_{\mathrm{ad},\leq P}$$

is contained in $P \times P^- \times T^+_{\mathrm{ad},\leq P}$.

Proof. Using the action in Fact 1.1.4(5), we only need to show $\widetilde{G}_{C_Q}$ is contained in $P \times P^-$ for any $Q \subset P$. But this is obvious.

□[Lemma A.1.1]

Lemma A.1.2. *We write* $\widetilde{G}_{\geq C_P} := \widetilde{G} \times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P}$. *We have:*

(1) The closed subscheme

$$P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \hookrightarrow P^- \times G \times T^+_{\mathrm{ad},\geq C_P}$$

is contained in $P^- \times P^- \times T^+_{\mathrm{ad},\geq C_P}$.

(2) The composition

$$P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \to P^- \times P^- \times T^+_{\mathrm{ad},\geq C_P} \xrightarrow{\mathrm{pr}_{23}} P^- \times T^+_{\mathrm{ad},\geq C_P} \tag{A.1}$$

is an isomorphism, where the first map is obtained by (1).

Warning A.1.3. The similar statement for pr_{13} is false.

Proof. We first prove (1). Using the action in Fact 1.1.4(5), we only need to check the similar claim at any $C_{P'} \in T^+_{\mathrm{ad},\geq C_P}$. But this is obvious.

Similarly, it is easy to see (A.1) induces isomorphisms between fibers at any closed point of $T^+_{\mathrm{ad},\geq C_P}$. To prove (2), we only need to show $P^- \times_G \widetilde{G} \times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P}$ is smooth over $T^+_{\mathrm{ad},\geq C_P}$.

We claim $P^- \times G \times T^+_{\mathrm{ad},\geq C_P}$ and $\widetilde{G}_{\geq C_P}$ are transversal in $G \times G \times T^+_{\mathrm{ad},\geq C_P}$. Indeed, by the last paragraph, the dimension of any irreducible component of their intersection is at most $\dim(P^-) + \dim(T^+_{\mathrm{ad},\geq C_P})$. But this number is equal to

$$\dim(P^- \times G \times T^+_{\mathrm{ad},\geq C_P}) + \dim(\widetilde{G}_{\geq C_P}) - \dim(G \times G \times T^+_{\mathrm{ad},\geq C_P}).$$

This proves the transversity. In particular, we obtain that $P^- \times_G \widetilde{G} \times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P}$ is smooth.

It remains to show $f : P^- \times_G \widetilde{G} \times_{T^+_{\mathrm{ad}}} T^+_{\mathrm{ad},\geq C_P} \to T^+_{\mathrm{ad},\geq C_P}$ induces surjection between tangent spaces. Note that the fibers of this map is smooth and of dimension $\dim(P^-)$. Hence at any closed point x of the source, we have

$$\dim(\ker(df_x)) = \dim(P^-) = \dim(P^- \underset{G}{\times} \widetilde{G} \underset{T^+_{\mathrm{ad}}}{\times} T^+_{\mathrm{ad},\geq C_P}) - \dim(T^+_{\mathrm{ad},\geq C_P}).$$

This implies df_x is surjective.

□[Lemma A.1.2]

Lemma A.1.4. *Consider the $(P^- \times G)$-action on $\mathrm{Vin}_{G,\geq C_P}$. Its stablizer for the canonical section is $P^- \times_G \widetilde{G}_{\geq C_P}$. Then the canonical map*

$$(P^- \times G \times T^+_{\mathrm{ad},\geq C_P})/(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to {}_0\mathrm{Vin}_{G,\geq C_P} \tag{A.2}$$

induced by this action is an open embedding.

Proof. We claim the LHS is a smooth scheme. By Lemma A.1.2, there is a canonical isomorphism

$$P^- \underset{G}{\times} \widetilde{G}_{\geq C_P} \simeq P^- \times T^+_{\mathrm{ad},\geq C_P} \tag{A.3}$$

between group schemes over $T^+_{\mathrm{ad},\geq C_P}$. Moreover, the projection map $P^- \times G \times T^+_{\mathrm{ad},\geq C_P} \to G \times T^+_{\mathrm{ad},\geq C_P}$ intertwines the actions of (A.3). Hence we obtain a canonical map

$$(P^- \times G \times T^+_{\mathrm{ad},\geq C_P})/(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to (G \times T^+_{\mathrm{ad},\geq C_P})/(P^- \times T^+_{\mathrm{ad},\geq C_P}) \simeq G/P^- \times T^+_{\mathrm{ad},\geq C_P}.$$

Since $P^- \times G \times T^+_{\mathrm{ad},\geq C_P} \to G \times T^+_{\mathrm{ad},\geq C_P}$ is affine and smooth, the above map is also affine and smooth. This proves the claim on smoothness. Then the lemma follows from the fact that both sides of (A.2) have the same dimenison and that this map is injective on the level of closed points.

□[Lemma A.1.4]

Corollary A.1.5. *The canonical map*

$$\mathbb{B}(P^- \underset{G}{\times} \widetilde{G}_{\geq C_P}) \to \mathbb{B}P^- \underset{\mathbb{B}G}{\times} \mathbb{B}\widetilde{G}_{\geq C_P}$$

is a schematic open embedding

Proof. Follows from Lemma A.1.4 by taking quotients for the $(P^- \times G)$-actions.

□[Corollary A.1.5]

A.2 The attractor, repeller and fixed loci of $\mathrm{Gr}_{G,I}$

In this section, we prove Proposition 1.2.8.

In this proof, we do not require X to be complete. In other words, X can be any separated smooth curve over k. Also, we write Gr_{G,X^I} for the Beilinson-Drinfeld Grassmannian (which are denoted by $\mathrm{Gr}_{G,I}$ in other parts of this thesis).

We first construct the desired maps. We do it formally. Consider the Čech nerve $\mathfrak{c}_G$ of the map $\mathrm{pt} \to \mathrm{pt}/G$. Since the $\mathbb{G}_m$-action on G is induced from the adjoint action, it induces a $\mathbb{G}_m$-action on $\mathfrak{c}_G$. This gives a $\mathbb{G}_m$ action on the *pointed* algebraic stack[84] $\mathrm{pt} \to \mathrm{pt}/G$. More or less by definition, the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,X^I} := \mathbf{Maps}_{I,/\mathrm{pt}}(X, \mathrm{pt}/G \leftarrow \mathrm{pt})$ is induced by this $\mathbb{G}_m$-action on $\mathrm{pt} \to \mathrm{pt}/G$. Now consider the restricted $\mathbb{G}_m$-action on the Čech nerve $\mathfrak{c}_P$ of $\mathrm{pt} \to \mathrm{pt}/P$. By design, it can be extended to an action by the monoid $\mathbb{A}^1$. This gives an extension of the $\mathbb{G}_m$-action on the pointed algebraic stack $\mathrm{pt} \to \mathrm{pt}/P$ to an $\mathbb{A}^1$-action, hence gives an extension of the $\mathbb{G}_m$-action on Gr_{P,X^I} to an $\mathbb{A}^1$-action. In other words, we obtain a canonical map $\mathrm{Gr}_{P,X^I} \to \mathrm{Gr}^{\gamma,\mathrm{att}}_{P,X^I}$. Then the desired map is given by

$$\mathrm{Gr}_{P,X^I} \to \mathrm{Gr}^{\gamma,\mathrm{att}}_{P,X^I} \to \mathrm{Gr}^{\gamma,\mathrm{att}}_{G,X^I}.$$

The maps for the repellor and fixed loci are constructed similarly. It follows from construction that these maps are defined over Gr_{G,X^I} and fit into the desired commutative diagram.

[84]Note that the $\mathbb{G}_m$-action on pt/G is (non-canonically) trivial, but the $\mathbb{G}_m$-action on $\mathrm{pt} \to \mathrm{pt}/G$ is not trivial. We are grateful to Yifei Zhao for teaching us this.

It remains to prove these maps are isomorphisms. We will prove

$$\theta^+_{X^I} : \mathrm{Gr}_{P,X^I} \to \mathrm{Gr}^{\gamma,\mathrm{att}}_{G,X^I}$$

is an isomorphism. The proofs for the other two isomorphisms are similar. The proof can be summarized as: the functor from the category of *universal* factorization spaces to the category of factorization spaces over $\mathbb{A}^1$ is conservative. Let us explain this in details.

For a separated smooth curve X and a closed point $x \in X$, we write $T(x, X, I)$ for the following statement:

- there exists an étale neighborhood V of $x^I \in X^I$ such that the base-change of $\theta^+_{X^I}$ along $V \to X^I$ is an isomorphism.

By the factorization property, we only need to prove $T(x, X, I)$ is true for any choice of (x, X, I). Note that by [HR18, Theorem A][85], $T(x, \mathbb{A}^1, I)$ is true. Hence it remains to prove $T(x, X, I) \Leftrightarrow T(x', X', I)$ for any étale map $p : X \to X'$ sending x to x'.

Note that the diagonal map $X \to X \times_{X'} X$ is an open and closed embedding. Hence so is the map $(X \times_{X'} X) - X \to X \times_{X'} X$. Therefore $(X \times_{X'} X) - X \to X \times X$ is a closed embedding. Let W be the complement open subscheme. We define $V \subset X^I$ to be the intersection of $\mathrm{pr}_{ij}^{-1}(W)$ for any $i \neq j \in I$, where $\mathrm{pr}_{ij} : X^I \to X^2$ is the projection onto the product of the i-th and j-th factors. Note that a closed point $(x_i)_{i \in I}$ of X^I is contained in V iff $(p(x_i) = p(x_j)) \Rightarrow (x_i = x_j)$. In particular, the point x^I is contained in V. Note that we have a chain of étale maps $V \to X^I \to (X')^I$. By [Cli19, Proposition 7.5], for any affine algebraic group[86] H, we have canonical isomorphisms

$$\mathrm{Gr}_{H,X^I} \underset{X^I}{\times} V \simeq \mathrm{Gr}_{H,(X')^I} \underset{(X')^I}{\times} V$$

defined over V. It is easy to see from its construction that this isomorphism is functorial in H. Hence we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_{P,X^I} \times_{X^I} V \arrow[r, "\simeq"] \arrow[d] & \mathrm{Gr}_{P,(X')^I} \times_{(X')^I} V \arrow[d] \\
\mathrm{Gr}^{\gamma,\mathrm{att}}_{G,X^I} \times_{X^I} V \arrow[r, "\simeq"] & \mathrm{Gr}^{\gamma,\mathrm{att}}_{G,(X')^I} \times_{(X')^I} V.
\end{tikzcd}$$

This makes $T(x, X, I) \Leftrightarrow T(x', X', I)$ manifest.

□[Proposition 1.2.8]

[85] It is easy to see that the map $\mathrm{Gr}_{P,X^I} \to \mathrm{Gr}^{\gamma,\mathrm{att}}_{G,X^I}$ constructed above coincides with that in [HR18]. However, we can get around this because both $\mathrm{Gr}_{P,X^I} \to \mathrm{Gr}_{G,X^I}$ and $\mathrm{Gr}^{\gamma,\mathrm{att}}_{G,X^I} \to \mathrm{Gr}_{G,X^I}$ are monomorphisms.

[86] [Cli19] stated the isomorphism below for reductive groups, but the proof there works for any affine algebraic group.

A.3 Mapping prestacks

In this section, we prove some techinical results on $\mathbf{Maps}_{I,/B}(X, Y \overset{f}{\leftarrow} B)$ introduced in Definition 1.4.1.

Lemma A.3.1. *Let (B, Y, p, f) be as in Definition 1.4.1. Let A be any finite type affine scheme. We have a canonical isomorphism*

$$\mathbf{Maps}_{I,/A\times B}(X, A\times Y \overset{\mathrm{Id}\times f}{\leftarrow} A\times B) \simeq A\times \mathbf{Maps}_{I,/B}(X, Y \overset{f}{\leftarrow} B).$$

Proof. Follows from Example 1.3.4.

□[Lemma A.3.1]

Lemma A.3.2. *Let B be a finite type affine scheme and $g : Y_1 \hookrightarrow Y_2$ be a schematic closed embedding between algebraic stacks over B. Let $f_1 : B \to Y_1$ be a section of $Y_1 \to B$. Let $f_2 : B \to Y_2$ be the section of $Y_2 \to B$ induced by f_1. Then we have a canonical isomorphism:*

$$\mathbf{Maps}_{I,/B}(X, Y_1 \overset{f_1}{\leftarrow} B) \simeq \mathbf{Maps}_{I,/B}(X, Y_2 \overset{f_2}{\leftarrow} B).$$

Proof. Let S be any finite type affine scheme. Let $x_i : S \to X$, $\alpha : X\times S \to Y_2$ and $\beta : S \to B$ be as in Definition 1.4.1. By Lemma A.3.3 below, the schema-theoretic closure of $(X\times S) - \cup\Gamma_{x_i}$ inside $X\times S$ is $X\times S$. Therefore the commutative diagram in Definition 1.4.1(2) forces α to factor through $Y_1 \hookrightarrow Y_2$. Then we are done because such a factorization is unique.

□[Lemma A.3.2]

Lemma A.3.3. *Let S be a finite type affine scheme and $x_i : S \to X$ be maps labelled by a finite set I. Let $\Gamma_{x_i} \hookrightarrow X\times S$ be the graph of x_i. Then the schema-theoretic closure of $(X\times S) - \cup\Gamma_{x_i}$ inside $X\times S$ is $X\times S$.*

Proof. This lemma is well-known. For the reader's convenience, we provide a proof here[87]. Let Γ be the schema-theoretic sum of the graphs of the maps x_i. Then $\Gamma \hookrightarrow X\times S$ is a relative effective Cartier divisor for $X\times S \to S$. Write $U_x : (X\times S) - \Gamma$. Let $\iota : U_x \to X\times S$ be the open embedding. We only need to show $\mathcal{O}_{X\times S} \to \iota_*(\mathcal{O}_U)$ is an injection. Note that the set-theoretic support of the kernel of this map is contained in Γ. Hence we are done by Lemma A.3.4 below.

□[Lemma A.3.3]

[87] We learn the proof below from Ziquan Yang.

Lemma A.3.4. *Let Y be any Noetherian scheme and $D \hookrightarrow Y$ be an effective Cartier divisor. Let $\mathcal{M}$ be a flat coherent $\mathcal{O}_Y$-module and $\mathcal{N}$ be a sub-module of it. Suppose the set-theoretic support of $\mathcal{N}$ is contained in D, then $\mathcal{N} = 0$.*

Proof. Let $\mathcal{I}$ be the sheaf of ideals for D. By assumption, it is invertible. Since Y is Noetherian, $\mathcal{N}$ is also a coherent $\mathcal{O}_Y$-module. Hence by assumption, there exists a positive integer n such that the map $\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} \to \mathcal{N}$ is zero. Consider the commutative square

$$\begin{tikzcd}
\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\
\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{M} \arrow[r] & \mathcal{M}.
\end{tikzcd}$$

The right vertical map is injective by assumption. Hence the left vertical map is injective because $\mathcal{I}^n$ is $\mathcal{O}_Y$-flat. The bottom map is injective because $\mathcal{M}$ is $\mathcal{O}_Y$-flat. Hence we see the top map is also injective. This forces $\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} = 0$. Then we are done because $\mathcal{I}^n$ is invertible.

□[Lemma A.3.4]

The following three lemmas can be proved by unwinding the definitions. We leave the details to the reader.

Lemma A.3.5. *Suppose we are given the following commutative diagram of schematic open embeddings between algebraic stacks:*

$$\begin{tikzcd}
(Y_1 \supset V_1) \arrow[r] \arrow[d] & (Y_2 \supset V_2) \arrow[d] \\
(Y_3 \supset V_3) \arrow[r] & (Y_4 \supset V_4).
\end{tikzcd} \tag{A.4}$$

(1) If the commutative square formed by Y_i is strictly quasi-Cartesian[88]*, then $\mathbf{Maps}_{\mathrm{gen}}(X, -)$ sends (A.4) to a strictly quasi-Cartesian square.*

(2) If the two commutative squares formed respectively by Y_i and V_i are both Cartesian, then $\mathbf{Maps}_{\mathrm{gen}}(X, -)$ sends (A.4) to a Cartesian square.

Lemma A.3.6. *Let* **Sect** *be the category of 4-tuples (B, Y, p, f) as in Definition 1.4.1. Then the functor*

$$\mathbf{Sect} \to \mathrm{PreStk}_{\mathrm{lft}},\ (B, Y, p, f) \mapsto \mathbf{Maps}_{I,/B}(X, Y \overset{f}{\leftarrow} B)$$

commutes with fiber products.

[88]Recall this means $Y_1 \to Y_2 \times_{Y_4} Y_3$ is a schematic open embedding.

Lemma A.3.7. *Let*

$$(B_1, Y_1 \supset V_1, p_1, f_1) \to (B_2, Y_2 \supset V_2, p_2, f_2)$$

be a morphism between two 4-tuples satisfy the conditions in Construction 1.4.2. Suppose the natural map $B_1 \to B_2 \times_{Y_2} Y_1$ *is an isomorphism. Then the natural commutative square*

$$\begin{tikzcd}
\mathbf{Maps}_{I,/B_1}(X, Y_1 \overset{f_1}{\leftarrow} B_1) \arrow[r] \arrow[d] & \mathbf{Maps}_{\mathrm{gen}}(X, Y_1 \supset V_1) \arrow[d] \\
\mathbf{Maps}_{I,/B_2}(X, Y_2 \overset{f_2}{\leftarrow} B_2) \arrow[r] & \mathbf{Maps}_{\mathrm{gen}}(X, Y_2 \supset V_2),
\end{tikzcd}$$

is Cartesian.

A.4 The stratification on $\mathrm{Gr}_{G,I}$ given by $\mathrm{Gr}_{P,I}$

The results in this section are folklore. However, we fail to find proofs in the literature.

Notation A.4.1. Write $A_M \coloneqq M/[M,M]$ for the abelianization of M. For $\lambda \in \Lambda_{G,P} = \mathrm{Hom}(\mathbb{G}_m, A_M)$, let $\mathrm{Bun}_{A_M}^{\lambda}$ be the connected component of Bun_{A_M} corresponding to A_M-torsors of degree λ.

Let Bun_M^{λ} (resp. Bun_P^{λ} and $\mathrm{Bun}_{P^-}^{\lambda}$) be the inverse image of $\mathrm{Bun}_{A_M}^{\lambda}$ along the projection maps.

Let $\mathrm{Gr}_{M,I}^{-\lambda}$ (resp. $\mathrm{Gr}_{P,I}^{-\lambda}$ and $\mathrm{Gr}_{P^-,I}^{-\lambda}$) be the inverse image of Bun_M^{λ} (resp. Bun_P^{λ} and $\mathrm{Bun}_{P^-}^{\lambda}$) along the local-to-global maps[89].

Proposition A.4.2. *(c.f. [Gai17a, § 1.3]) For $\lambda \in \Lambda_{G,P}$, we have*

(1) The canonical map $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is a monomorphism, and is bijective on field valued points.

(2) The canonical map $\mathbf{p}_I^{+,\lambda} : \mathrm{Gr}_{P,I}^{\lambda} \to \mathrm{Gr}_{G,I}$ is a schematic locally closed embedding.

(3) There exists a schematic closed embedding

$$_{\le\lambda}\mathrm{Gr}_{G,I} \hookrightarrow \mathrm{Gr}_{G,I}$$

such that $_{\le\lambda}\mathrm{Gr}_{G,I}$ is ind-reduced[90] *and a field valued point of $\mathrm{Gr}_{G,I}$ is contained in $_{\le\lambda}\mathrm{Gr}_{G,I}$ iff it is contained in the image of $\mathrm{Gr}_{P,I}^{\mu} \to \mathrm{Gr}_{G,I}$ for some $\mu \le \lambda$. Moreover, the canonical map*

$$\operatorname*{colim}_{\lambda \in \Lambda_{G,P}} {_{\le\lambda}\mathrm{Gr}_{G,I}} \to \mathrm{Gr}_{G,I}$$

[89] The negative signs are compatible with the conventions in the literature. Namely, via the identification $\mathrm{Gr}_M(k) \simeq M((t))/M[[t]]$, the point t^{λ} is contained in Gr_M^{λ}.

[90] Note that an ind-reduced indscheme is reduced in the sense of Convention 4. It is quite possible that the converse is also true.

is a nil-isomorphism.

(4) There exists a schematic open embedding

$$_{\geq\lambda}\operatorname{Gr}_{G,I} \to \operatorname{Gr}_{G,I}$$

such that a field valued point of $\operatorname{Gr}_{G,I}$ *is contained in* $_{\geq\lambda}\operatorname{Gr}_{G,I}$ *iff it is contained in the image of* $\operatorname{Gr}^{\mu}_{P,I} \to \operatorname{Gr}_{G,I}$ *for some* $\mu \geq \lambda$*. In particular, we have an isomorphism*

$$\operatorname*{colim}_{\lambda\in\Lambda_{G,P}} {}_{\geq\lambda}\operatorname{Gr}_{G,I} \simeq \operatorname{Gr}_{G,I}.$$

Remark A.4.3. The case $P = B$ and $I = *$ is well-studied in the literature under the name *semi-infinite orbits.*

A.4.4 (Proof of Proposition A.4.2(1))**.** We first prove (1). Note that $\mathrm{pt}/P \to \mathrm{pt}/G$ is schematic and separated. Using this, one can deduce $\mathbf{p}^+_I : \operatorname{Gr}_{P,I} \to \operatorname{Gr}_{G,I}$ is a monomorphism from Lemma A.3.3.

Recall that a field valued point $\operatorname{Spec} K \to \operatorname{Gr}_{G,I}$ corresponds to

- K-points x_i on X_K labelled by I,
- a G-torsor F_G on X_K trivialized away from x_i.

We only need to show this K-point can be lifted to a K-point of $\operatorname{Gr}_{P,I}$. Write $U_x \coloneqq X - \cup x_i$. For any representation $V \in \operatorname{Rep}(G)$, consider the map

$$(V^U)_{F_M^{\mathrm{triv}}}|_{U_x} \hookrightarrow V_{F_G^{\mathrm{triv}}}|_{U_x} \simeq V_{F_G}|_{U_x}. \tag{A.5}$$

We claim there exists a maximal sub-bundle $\mathcal{K}_V$ of V_{F_G} such that its restriction on U_x is the image of (A.5). Indeed, by Lemma A.4.5 below, there exists $n > 0$ such that (A.5) can be extended to an injection

$$(V^U)_{F_M^{\mathrm{triv}}}(-n\cdot\Gamma_x) \to V_{F_G}.$$

Consider the cokernel $\mathcal{Q}$ of this map. Since X_K is a smooth curve over K, the torsion free quotient $\mathcal{Q}^{\text{tor-free}}$ is a vector bundle. It is easy to see $\ker(V_{F_G} \to \mathcal{Q}^{\text{tor-free}})$ is the desired $\mathcal{K}_V$. This proves the claim.

Using the uniqueness of $\mathcal{K}_V$ and the Tannakian formalism, it is easy to see the injections $\mathcal{K}_V \to V_{F_G}$ give a P-reduction on F_G that is compatible with its trivialization on U_x. In other words, we obatin a K-point of $\operatorname{Gr}_{P,I}$. This proves (1).

Lemma A.4.5. *Let S be a finite type affine scheme and $x_i : S \to X$ be maps labelled by a finite set I. Let $\Gamma_x \hookrightarrow X \times S$ be the schema-theoretic sum of the graphs of x_i and $U_x := (X \times S) - \Gamma_x$ be its complement. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two flat coherent $\mathcal{O}_{X\times S}$-modules. Let $f : \mathcal{F}_1|_{U_x} \to \mathcal{F}_2|_{U_x}$ be an injection. Then there exists a positive integer n such that f can be extended to an injection $\mathcal{F}_1 \to \mathcal{F}_2(n\cdot\Gamma_x)$.*

Proof. Let $\jmath : U_x \to X \times S$ be the open embedding. For $n > 0$, consider the map

$$g_n : \mathcal{F}_2(n\cdot\Gamma_x) \to \jmath_* \circ \jmath^*(\mathcal{F}_2(n\cdot\Gamma_x)) \simeq \jmath_* \circ \jmath^*(\mathcal{F}_2).$$

Note that the set-theoretic support of its kernel is contained in Γ_x. Hence by Lemma A.3.4, this kernel is zero. In other words, g_n is injective. Moreover, the union of the images for g_n for all n is equal to $\jmath_* \circ \jmath^*(\mathcal{F}_2)$ because the divisor Γ_x is ample. Since $\mathcal{F}_1$ is coherent, there exists $n > 0$ such that the map

$$\mathcal{F}_1 \to \jmath_* \circ \jmath^*(\mathcal{F}_1) \xrightarrow{f} \jmath_* \circ \jmath^*(\mathcal{F}_2)$$

factors through $\mathcal{F}_2(n\cdot\Gamma_x)$. The resulting map $\mathcal{F}_1 \to \mathcal{F}_2(n\cdot\Gamma_x)$ is injective again because of Lemma A.3.4.

□[Lemma A.4.5]

A.4.6 (Compactification)**.** To proceed, we need to compactify the map $\mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$. Recall the Drinfeld compactification

$$\widetilde{\mathrm{Bun}}_P := \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash\overline{G/U}/M \supset G\backslash(G/U)/M)$$

defined in [BG02, § 1.3.5]. As before, we write $\widetilde{\mathrm{Bun}}_P^{\lambda}$ for the inverse image of Bun_M^{λ} along the map $\widetilde{\mathrm{Bun}}_P \to \mathrm{Bun}_M$. By [BG02, Proposition 1.3.6], the canonical map $\widetilde{\mathrm{Bun}}_P^{\lambda} \to \mathrm{Bun}_G$ is schematic and proper. In particular, the fiber product $\widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,I}$ is an ind-complete indscheme.

Let S be a finite type affine scheme. By [BG02, § 1.3.5], the set $(\widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,I})(S)$ classifies

(i) maps $x_i : S \to X$ labelled by I,

(ii) a G-torsor F_G on $X \times S$ trivialized on U_x,

(iii) an M-torsor F_M on $X \times S$,

(iv) for any $V \in \mathrm{Rep}(G)$, a map $\kappa_V : (V^U)_{F_M} \to V_{F_G}$

such that

(a) κ_V is injective and the cokernel of κ_V is $\mathcal{O}_S$-flat[91],

[91] This is equivalent to the condition that the base-change of κ_V at every geometric point of S is injective.

(b) the assignment $V \rightsquigarrow \kappa_V$ satisfies the Plücker relations (see [BG02, § 1.3.5] for what this means).

We define $\widetilde{\mathrm{Gr}}_{P,I}$ to be the subfunctor that classifies the above data with an additional condition:

(c) for any *irreducible*[92] G-representation V, the image of

$$(V^U)_{F_M}|_{U_x} \xrightarrow{\kappa_V} V_{F_G}|_{U_x} \simeq V_{F_G^{\mathrm{triv}}}|_{U_x} \tag{A.6}$$

is contained in $(V^U)_{F_M^{\mathrm{triv}}}|_{U_x}$.

Note that we have commutative diagrams

$$\begin{tikzcd} \mathrm{Gr}_{P,I} \arrow[r] \arrow[d] & \widetilde{\mathrm{Gr}}_{P,I} \arrow[r] \arrow[d] & \mathrm{Gr}_{G,I} \arrow[d] \\ \mathrm{Bun}_P \arrow[r] & \widetilde{\mathrm{Bun}}_P \arrow[r] & \mathrm{Bun}_G \end{tikzcd} \tag{A.7}$$

Lemma A.4.7. *We have:*

(1) The left square in (A.7) is Cartesian.

(2) The canoncial map $\widetilde{\mathrm{Gr}}_{P,I} \to \mathrm{Gr}_{G,I} \times_{\mathrm{Bun}_G} \widetilde{\mathrm{Bun}}_P$ *is a schematic closed embedding.*

Proof. Let S be a finite type affine test scheme. We use the notations in § A.4.6.

We first prove (1). By definition, the set $(\widetilde{\mathrm{Gr}}_{P,I} \times_{\widetilde{\mathrm{Bun}}_P} \mathrm{Bun}_P)(S)$ classifies (i)-(iv) satisfying conditions (a)-(c) and

(d) $\mathrm{coker}(\kappa_V)$ is locally free.

With condition (d), condition (c) is equivalent to

- the image of (A.6) is equal to $(V^U)_{F_M^{\mathrm{triv}}}|_{U_x}$.

This makes the desired claim manifest.

Now we prove (2). Fix a map $S \to \widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,I}$ corresponding to the data (i)-(iv) satisfying conditions (a)-(b). To simplify the notation, we write

$$\mathcal{V}_V^1 := V_{F_G},\ \mathcal{V}_V^2 := V_{F_G^{\mathrm{triv}}},\ \mathcal{K}_V^1 := (V^U)_{F_M},\ \mathcal{K}_V^2 := (V^U)_{F_M^{\mathrm{triv}}},\ \mathcal{Q}_V^2 := \mathcal{V}_V^2/\mathcal{K}_V^2.$$

Note that they are all vector bundles on $X \times S$. For $V \in \mathrm{Rep}(G)$, consider the composition

$$\mathcal{K}_V^1|_{U_x} \xrightarrow{\kappa_V} \mathcal{V}_V^1|_{U_x} \simeq \mathcal{V}_V^2|_{U_x} \to \mathcal{Q}_V^2|_{U_x}.$$

[92] We only need to consider irreducible representations because the Plücker relations force $\kappa_{V_1 \oplus V_2} = \kappa_{V_1} \oplus \kappa_{V_2}$.

By Lemma A.4.5, there exists an integer $n_V > 0$ such that the above composition can be extended to a map

$$\delta_V : \mathcal{K}_V^1 \to \mathcal{Q}_V^2(n_V \cdot \Gamma_x).$$

Now let S' be a finite type affine test scheme over S. Note that we have a short exact sequence

$$0 \to \mathcal{K}_V^2 \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'} \to \mathcal{V}_V^2 \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'} \to \mathcal{Q}_V^2 \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_{S'} \to 0$$

Hence the composition $S' \to S \to \widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,I}$ is an element in $\widetilde{\mathrm{Gr}}_{P,I}(S')$ iff for any irreducible $V \in \mathrm{Rep}(G)$,

(c_V) the restriction of the map $\delta_V \otimes \mathrm{Id} : \mathcal{K}_V^1 \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to \mathcal{Q}_V^2(n_V \cdot \Gamma_x) \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ on $U_x \times_S S'$ is zero.

However, we claim this condition is equivalent to

(c_V') the map $\delta_V \otimes \mathrm{Id}$ is zero.

Indeed, (c_V')⇒(C_V) is obvious. On the other hand, if condition (c_V) is satisfied, then the image of $\delta_V \otimes \mathrm{Id}$ is set-theoretically supported on $\Gamma_x \times_S S'$. Hence it has to be zero because of Lemma A.3.4. This proves (c_V')⇔(C_V).

By Lemma A.4.8 below, there exists a closed subscheme Z_V of S such that condition (c_V') is equivalent to

- $S' \to S$ factors through Z_V.

This implies the fiber product

$$\widetilde{\mathrm{Gr}}_{P,I} \underset{(\widetilde{\mathrm{Bun}}_P \times_{\mathrm{Bun}_G} \mathrm{Gr}_{G,I})}{\times} S$$

is isomorphic to the intersection of all the Z_V inside S. In particular, it is a closed subscheme of S as desired.

□[Lemma A.4.7]

Lemma A.4.8. *Let S be a finite type affine scheme. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a map between $\mathcal{O}_S$-flat coherent $\mathcal{O}_{X\times S}$-modules. Then there exists a closed subscheme Z of S such that for a finite type affine test scheme S' over S, the following conditions are equivalent*

- *the map $S' \to S$ factors through Z,*
- *the map $f \otimes \mathrm{Id} : \mathcal{F}_1 \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to \mathcal{F}_2 \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ is zero.*

Proof. Consider the injections $(\mathrm{Id},0):\mathcal{F}_1\to\mathcal{F}_1\oplus\mathcal{F}_2$ and $(\mathrm{Id},f):\mathcal{F}_1\to\mathcal{F}_1\oplus\mathcal{F}_2$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be their cokernels. Note that $\mathcal{Q}_1$ (resp. $\mathcal{Q}_2$) is $\mathcal{O}_S$-flat because they are both isomorphic to $\mathcal{F}_2$ (as $\mathcal{O}_{X\times S}$-modules). Hence $\mathcal{Q}_1$ (resp. $\mathcal{Q}_2$) gives two sections to the canonical map $\mathrm{Quot}_{\mathcal{F}_1\oplus\mathcal{F}_2/X\times S/S}\to S$. Recall that $\mathrm{Quot}_{\mathcal{F}_1\oplus\mathcal{F}_2/X\times S/S}$ is separated. Then the desired Z is given by the intersection of these two sections.

□[Lemma A.4.8]

A.4.9 (Proof of Proposition A.4.2(2))**.** Let $\lambda\in\Lambda_{G,P}$. Let $\widetilde{\mathrm{Gr}}^{\lambda}_{P,I}$ be the inverse image of $\widetilde{\mathrm{Bun}}_P^{-\lambda}$ along the map $\widetilde{\mathrm{Gr}}_{P,I}\to\widetilde{\mathrm{Bun}}_P$. Consider the composition $\widetilde{\mathrm{Gr}}^{\lambda}_{P,I}\to\widetilde{\mathrm{Bun}}_P^{-\lambda}\times_{\mathrm{Bun}_G}\mathrm{Gr}_{G,I}\to\mathrm{Gr}_{G,I}$. By [BG02, Proposition 1.3.6] and Lemma A.4.7(2), this map is schematic and proper. Hence we have a factorization of $\mathbf{p}_I^{+,\lambda}$:

$$\mathbf{p}_I^{+,\lambda}:\mathrm{Gr}^{\lambda}_{P,I}\to\widetilde{\mathrm{Gr}}^{\lambda}_{P,I}\to\mathrm{Gr}_{G,I},$$

such that the first map is a schematic open embedding (by Lemma A.4.7(1)) and the second map is schematic and proper. Let S be any finite type affine test scheme over $\mathrm{Gr}_{G,I}$. Consider the chain

$$(S_1\xrightarrow{f}S_2\xrightarrow{g}S):=(S\underset{\mathrm{Gr}_{G,I}}{\times}\mathrm{Gr}^{\lambda}_{P,I}\to S\underset{\mathrm{Gr}_{G,I}}{\times}\widetilde{\mathrm{Gr}}^{\lambda}_{P,I}\to S).$$

By the previous discussion, $S_1\to S_2$ is an open embedding while $S_2\to S$ is proper. Consider the open subset $V:=S-g(S_2-S_1)$ of S. We claim[93] the map $g\circ f$ factors through V.

To prove the claim, let y be a K-point of $\widetilde{\mathrm{Gr}}^{\lambda}_{P,I}$ that is not contained in $\mathrm{Gr}^{\lambda}_{P,I}$. Let z be the image of y in $\mathrm{Gr}_{G,I}$. By (1), z is contained in $\mathrm{Gr}^{\mu}_{P,I}$ for a unique $\mu\in\Lambda_{G,P}$. We only need to show $\mu\neq\lambda$. In fact, we will prove $\mu<\lambda$. Unwinding the definitions, we are given the following data

- K-points x_i on X_K labelled by I,
- a G-torsor F_G on X_K trivialized on $U_x:=X_K-\cup x_i$,
- an M-torsor F_M on X_K whose induced A_M-torsor F_{A_M} is of degree $-\lambda$,
- an M-torsor F'_M on X_K trivialized on U_x, whose induced A_M-torsor F'_{A_M} is of defree $-\mu$,
- for any $V\in\mathrm{Rep}(G)$, an injection $\kappa_V:(V^U)_{F_M}\to V_{F_G}$.
- for any $V\in\mathrm{Rep}(G)$, an injection $\kappa'_V:(V^U)_{F'_M}\to V_{F_G}$ such that $\mathrm{coker}(\kappa'_V)$ is always a vector bundle.

[93] In fact, $\widetilde{\mathrm{Gr}}_{P,I}$ is designed to make this claim correct. Also, the similar claim for the bigger compactification $\widetilde{\mathrm{Bun}}_P\times_{\mathrm{Bun}_G}\mathrm{Gr}_{G,I}$ is false.

- commutative diagrams

$$\begin{tikzcd} (V^U)_{F_M}|_{U_x} \arrow[r] \arrow[d, "\kappa_V"] & (V^U)_{F_M^{\mathrm{triv}}}|_{U_x} \arrow[d] & (V^U)_{F'_M}|_{U_x} \arrow[l, "\simeq"'] \arrow[d, "\kappa'_V"] \\ V_{F_G}|_{U_x} \arrow[r, "\simeq"] & V_{F_G^{\mathrm{triv}}}|_{U_x} & V_{F_G}|_{U_x}. \arrow[l, "\simeq"'] \end{tikzcd} \tag{A.8}$$

Consider the composition $\delta_V : (V^U)_{F_M} \xrightarrow{\kappa_V} V_{F_G} \to \mathrm{coker}(\kappa'_V)$. The diagram A.8 implies the image of δ_V is set-theoretically supported on $\cup x_i$. Hence δ_V is zero because $\mathrm{coker}(\kappa'_V)$ is a vector bundle. Hence as sub-module of V_{F_G}, we have $(V^U)_{F_M} \subset (V^U)_{F'_M}$. On the other hand, since y is not contained in $\mathrm{Gr}^{\lambda}_{P,I}$, by Lemma A.4.7(1), its image in $\widetilde{\mathrm{Bun}}_P$ is not contained in Bun_P. Hence by the defect stratification on $\widetilde{\mathrm{Bun}}_P$ (see [BFGM02, § 1.4-1.9]), there exists $V_0 \in \mathrm{Rep}(G)$ with $\dim(V_0^U) = 1$ such that $\mathrm{coker}(\kappa_{V_0})$ is not a vector bundle. This implies the inclusion $(V_0^U)_{F_M} \subset (V_0^U)_{F'_M}$ is strict. Hence the degree of F_{A_M} is smaller than the degree of F'_{A_M}. In other words, we have $\lambda \le \mu$. This proves the claim.

Using this claim, the map $g \circ f$ factors as

$$S_1 = S_1 \underset{S}{\times} V = S_2 \underset{S}{\times} V \to V \to S.$$

Note that the map $S_2 \times_S V \to V$ is proper (because $S_2 \to S$ is proper) and is a monomorphism (by (1)), hence it is a closed embedding. This proves (2).

A.4.10 (Finish the proof)**.** Let $Y \hookrightarrow \mathrm{Gr}_{G,I}$ be any finite type closed subscheme of $\mathrm{Gr}_{G,I}$. Let ${}_{\le\lambda}|Y|$ be the subset of $|Y|$ consisting of points contained in the image of $\mathrm{Gr}^{\mu}_{P,I} \to \mathrm{Gr}_{G,I}$ for some $\mu \le \lambda$. Similarly we define ${}_{\ge\lambda}|Y|$. To prove (3) and (4), it suffices to show ${}_{\le\lambda}|Y|$ (resp. ${}_{\ge\lambda}|Y|$) is a closed (resp. open) subset of $|Y|$. By (1), (2) and Noetherian induction, there are only finitely many μ such that Y has non-empty intersection with $\mathrm{Gr}^{\mu}_{P,I}$ inside $\mathrm{Gr}_{G,I}$. Hence ${}_{\le\lambda}|Y|$ and ${}_{\ge\lambda}|Y|$ are constructible subsets of $|Y|$. It remains to show ${}_{\le\lambda}|Y|$ (resp. ${}_{\ge\lambda}|Y|$) is closed under specialization (resp. generalization). However, this is clear from the proof of (1).

□[Proposition A.4.2]

Corollary A.4.11. *We have*

(1) The canonical map $\mathbf{p}^+_I : \mathrm{Gr}_{P,I} \times_{X^I} \mathrm{Gr}_{P^-,I} \to \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ *is a monomorphism, and is bijective on field valued points.*

(2) For $\theta \in \Lambda_{G,P}$*, the canonical map*

$$\coprod_{\lambda-\mu=\theta} \mathrm{Gr}^{\lambda}_{P,I} \underset{X^I}{\times} \mathrm{Gr}^{\mu}_{P^-,I} \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \tag{A.9}$$

is a schematic locally closed embedding.

(3) For $\delta \in \Lambda_{G,P}$, there exists a schematic closed embedding

$$_{\mathrm{diff}\le\delta}\,\mathrm{Gr}_{G\times G,I} \hookrightarrow \mathrm{Gr}_{G\times G,I}$$

such that $_{\mathrm{diff}\le\delta}\,\mathrm{Gr}_{G\times G,I}$ is ind-reduced and a field valued point of $\mathrm{Gr}_{G\times G,I} \simeq \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ is contained in $_{\le\delta}\,\mathrm{Gr}_{G,I}$ iff it is contained in the image of (A.9) for some $\theta \le \delta$. Moreover, the canonical map

$$\operatorname*{colim}_{\delta\in\Lambda_{G,P}}\ {}_{\mathrm{diff}\le\delta}\,\mathrm{Gr}_{G\times G,I} \to \mathrm{Gr}_{G\times G,I}$$

is a nil-isomorphism.

(4) There exists a schematic open embedding

$$_{\mathrm{diff}\ge\delta}\,\mathrm{Gr}_{G\times G,I} \hookrightarrow \mathrm{Gr}_{G\times G,I}$$

such that $_{\mathrm{diff}\ge\delta}\,\mathrm{Gr}_{G\times G,I}$ is ind-reduced and a field valued point of $\mathrm{Gr}_{G\times G,I} \simeq \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ is contained in $_{\ge\delta}\,\mathrm{Gr}_{G,I}$ iff it is contained in the image of (A.9) for some $\theta \ge \delta$. In particular, the canonical map

$$\operatorname*{colim}_{\delta\in\Lambda_{G,P}}\ {}_{\mathrm{diff}\ge\delta}\,\mathrm{Gr}_{G\times G,I} \simeq \mathrm{Gr}_{G\times G,I}\,.$$

Proof. (1) follows from Proposition A.4.2(1).

By Proposition A.4.2(2), for $\lambda, \mu \in \Lambda_{G,P}$, the canonical map

$$\mathrm{Gr}^{\lambda}_{P,I} \underset{X^I}{\times} \mathrm{Gr}^{\mu}_{P^-,I} \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \tag{A.10}$$

is a schematic locally closed embedding. Let $Y \hookrightarrow \mathrm{Gr}_{G\times G,I}$ be any finite type closed subscheme of $\mathrm{Gr}_{G\times G,I} \simeq \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$. For any $\lambda, \mu \in \Lambda_{G,P}$, let $_{\lambda,\mu}|Y|$ be the locally closed subset of $|Y|$ consisting of points contained in the image of (A.10). As in § A.4.10, there are only finitely many pairs (λ, μ) such that $_{\lambda,\mu}|Y|$ is non-empty. Hence to prove (2), it remains to show if $\mu_1 \ne \mu_2$, then the closure of $_{\mu_1+\theta,\mu_1}|Y|$ in $|Y|$ has empty intersection with $_{\mu_2+\theta,\mu_2}|Y|$. However, by Proposition A.4.2(3), the closure of $_{\mu_1+\theta,\mu_1}|Y|$ in $|Y|$ is contained in

$$\bigcup_{\lambda\le\mu_1+\theta,\mu\ge\mu_1} {}_{\lambda,\mu}|Y|.$$

This makes the desired claim manifest. This proves (2).

To prove (3) and (4), consider the similarly defined subsets $_{\mathrm{diff}\le\delta}|Y|$ and $_{\mathrm{diff}\ge\delta}|Y|$. As in § A.4.10, they are constructible. Moreover, by Proposition A.4.2(3) (resp. Proposition A.4.2(4)), ${}_{\mathrm{diff}\le\delta}|Y|$ (resp. ${}_{\mathrm{diff}\ge\delta}|Y|$) is closed under specialization (resp. generalization). Then we are done.

□[Corollary A.4.11]

A.5 The geometric objects in [Sch16]: Constructions

In this section, we review some geometric constructions in [Sch16] about VinBun_G. We personally think some proofs in [Sch16] are too concise. Hence we provide details to them in Appendix A.6.

A.5.1 (The monoid $\overline{M}$)**.** The unproven claims in this subsection can be found in [Sch16, § 3.1] and [Wan17].

Consider the closed embedding $M \simeq P/U \hookrightarrow G/U$. It is well-known that G/U is strongly quasi-affine (see e.g. [BG02, Theorem 1.1.2]). Let $\overline{M}$ be the closure of M inside $\overline{G/U}$. [Wan17, § 3] shows that $\overline{M}$ is normal and the group structure on M extends uniquely to a monoid structure on $\overline{M}$ such that its open subgroup of invertible elements is isomorphic to M.

On the other hand, by [Wan17, Theorem 4.1.4], the closed embedding

$$G/U \simeq (G/U \times P/U^-)/M \hookrightarrow (G/U \times G/U^-)/M \simeq {}_0\mathrm{Vin}_G|_{C_P}$$

extends uniquely to a closed embedding $\overline{G/U} \hookrightarrow \mathrm{Vin}_G|_{C_P}$. Hence the closed embedding[94]

$$M \to (G/U \times G/U^-)/M \simeq {}_0\mathrm{Vin}_G|_{C_P}\ m \mapsto (m,1)$$

extends uniquely to a closed embedding $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$. Moreover, $\overline{M}$ is also isomorphic to the closure of M inside $\mathrm{Vin}_G|_{C_P}$. By construction, $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$ is stabilized by the $(P \times P^-)$-action and fixed by the $(U \times U^-)$-action. Hence we have a commutative square of schemes acted by $(P \times P^-)$:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\ {}_0\mathrm{Vin}_G|_{C_P} \arrow[r] & \mathrm{Vin}_G|_{C_P}. \end{tikzcd} \tag{A.11}$$

Note that this square is Cartesian because $M \hookrightarrow {}_0\mathrm{Vin}_G|_{C_P}$ is already a closed embedding.

A.5.2 (The monoid $\overline{A_M}$)**.** The unproven claims in this subsection can be found in [Sch16, § 3.1.7].

Consider the abelianization[95] $A_M := M/[M,M] \simeq P/[P,P]$. It can be embedded into $G/[P,P]$ (which is

[94]Note that the image of $(m,1)$ and $(1,m^{-1})$ in $(G/U \times G/U^-)/M$ are equal.

[95] [Sch16] denoted it by T_M. We use the notation A_M to avoid confusions with the Cartan subgroup of M.

strongly quasi-affine). Its closure $\overline{A_M}$ inside the affine closure $\overline{G/[P,P]}$ is known to be normal. The commutative group structure on A_M extends to a commutative monoid structure on $\overline{A_M}$ whose open subgroup of invertible elements is A_M.

The projection $M \twoheadrightarrow M/[M,M]$ induces a map $\overline{M} \to \overline{A_M}$, which is $(M\times M)$-equivariant by construction. Hence we have the following commutative diagram of schemes acted by $(M \times M)$:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\ A_M \arrow[r] & \overline{A_M}, \end{tikzcd} \tag{A.12}$$

which is *Cartesian* by Lemma A.6.1.

A.5.3 (The stack $H_{M,G\text{-pos}}$)**.** The unproven claims in this subsection can be found in [Sch16, § 3.1.5] and [Wan18, Appendix A].

Recall that X^{pos} is defined as the disjoint union of X^θ for $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$. By [Sch16, § 3.1.7], we have

$$X^{\mathrm{pos}} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, A_M\backslash\overline{A_M} \supset A_M\backslash A_M),$$

where A_M acts on $\overline{A_M}$ via multiplication. Under this isomorphism, the addition map $X^{\mathrm{pos}} \times X^{\mathrm{pos}} \to X^{\mathrm{pos}}$ is induced by the *commutative* monoid structure on $\overline{A_M}$.

The *G-positive affine Grassmannian* is defined as (see § A.5.1 for the definition of $\overline{M}$)

$$\mathrm{Gr}_{M,G\text{-pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, \overline{M}/M \supset M/M),$$

where M acts on $\overline{M}$ by right multiplication. The map $\overline{M}/M \to \mathrm{pt}/M$ induces a map $\mathrm{Gr}_{M,G\text{-pos}} \to \mathrm{Bun}_M$.

By (A.12), the composition

$$\overline{M}/M \to \overline{A_M}/A_M \simeq A_M\backslash\overline{A_M} \tag{A.13}$$

sends M/M into $A_M\backslash A_M$. Hence we have a projection $\mathrm{Gr}_{M,G\text{-pos}} \to X^{\mathrm{pos}}$. We define[96]

$$\mathrm{Gr}^\theta_{M,G\text{-pos}} := \mathrm{Gr}_{M,G\text{-pos}} \underset{X^{\mathrm{pos}}}{\times} X^\theta.$$

By [Wan18, § 5.7], the definition above coincides with the definition in [BFGM02, Sub-section 1.8]. In particular, $\mathrm{Gr}^\theta_{M,G\text{-pos}}$ is represented by a scheme of finite type.

[96]Note that the last map in the composition (A.13) is induced by the group homomorphism $A_M \to A_M$, $t \mapsto t^{-1}$. Hence $\mathrm{Gr}^\theta_{M,G\text{-pos}}$ lives over $\mathrm{Bun}^{-\theta}_M$, which is compatible with the conventions in the literature.

The *G-positive Hecke stack* is defined as

$$H_{M,G\text{-}\mathrm{pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M). \tag{A.14}$$

As before, we have a projection $H_{M,G\text{-}\mathrm{pos}} \to X^{\mathrm{pos}}$ induced by the composition

$$M\backslash\overline{M}/M \to A_M\backslash\overline{A_M}/A_M \to A_M\backslash\overline{A_M},$$

where the last map is induced by the group morphism

$$A_M \times A_M \to A_M,\ (s,t) \mapsto st^{-1}.$$

The base-change of this map to X^θ is denoted by $H^\theta_{M,G\text{-}\mathrm{pos}}$. Hence we have a disjoint union decomposition

The map $M\backslash\overline{M}/M \to M\backslash\mathrm{pt}/M$ induces a map

$$\overleftarrow{\mathfrak{h}} \times \overrightarrow{\mathfrak{h}} : H_{M,G\text{-}\mathrm{pos}} \to \mathrm{Bun}_M \times \mathrm{Bun}_M .$$

Hence we obtain a disjoint union decomposition[97]

$$H_{M,G\text{-}\mathrm{pos}} = \coprod_{\theta\in\Lambda^{\mathrm{pos}}_{G,P}} H^\theta_{M,G\text{-}\mathrm{pos}} = \coprod_{\theta\in\Lambda^{\mathrm{pos}}_{G,P}} \coprod_{\lambda_1-\lambda_2=\theta} H^{\lambda_1,\lambda_2}_{M,G\text{-}\mathrm{pos}} \tag{A.15}$$

where for $\lambda_1,\lambda_2 \in \Lambda_{G,P}$, $H^{\lambda_1,\lambda_2}_{M,G\text{-}\mathrm{pos}}$ lives over the connected component $\mathrm{Bun}^{\lambda_1}_M \times \mathrm{Bun}^{\lambda_2}_M$.

Note that the fiber of $\overleftarrow{\mathfrak{h}}$ at the point $\mathcal{F}^{\mathrm{triv}}_M$ of Bun_M is $\mathrm{Gr}_{M,G\text{-}\mathrm{pos}}$.

A.5.4 (The stack ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$)**.** The unproven claims in this subsection can be found in [Sch16, § 3.2].

The *defect stratification* on $\mathrm{VinBun}_G|_{C_P}$ is a stratification labeled by $\Lambda^{\mathrm{pos}}_{G,P}$. For $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$, the corresponding stratum is

$${}_\theta\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_{P\times P^-} \underset{\mathrm{Bun}_{M\times M}}{\times} H^\theta_{M,G\text{-}\mathrm{pos}}. \tag{A.16}$$

We write ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ for the disjoint union of all the defect strata. By Lemma A.3.5(2), we have

$${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_{P\times P^-} \underset{\mathrm{Bun}_{M\times M}}{\times} H_{M,G\text{-}\mathrm{pos}} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/P^- \supset P\backslash M/P^-). \tag{A.17}$$

Recall we have a $(P\times P^-)$-equivariant closed embedding (see A.5.1) $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$, which sends M into

[97] Our labels λ_1,λ_2 below are in the opposite order against that in [Sch16] because of Warning 1.1.3. Our order is compatible with [Wan18, § 5.3].

${}_0\mathrm{Vin}_G|_{C_P}$. Hence we obtain a map

$$(P\backslash\overline{M}/P^- \supset P\backslash M/P^-) \to (G\backslash \mathrm{Vin}_G|_{C_P}/G \supset G\backslash_0\mathrm{Vin}_G|_{C_P}/G).$$

Applying $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ to it, we obtain a map

$${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \to \mathrm{VinBun}_G|_{C_P}$$

By [Sch16, Proposition 3.2.2], the connected components of the source provide a stratification for $\mathrm{VinBun}_G|_{C_P}$.

It follows from construction that the Z_M-action on $\mathrm{VinBun}_G|_{C_P}$ is compatible with the defect stratification. Hence we obtain a defect stratification on $\overline{\mathrm{Bun}}_{G,P}$.

A.5.5 (The open Bruhat cell $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$)**.** Consider the $(P^-\times P)$-action on $\mathrm{Vin}_{G,\geq C_P}$ induced from the $(G\times G)$-action on Vin_G. Also consider the canonical section (see § 1.1.1) $\mathfrak{s}_{\geq C_P}: T^+_{\mathrm{ad},\geq C_P}\to \mathrm{Vin}_{G,\geq C_P}$. By Lemma 1.1.6, the stabilizer subgroup of this section is given by

$$M\times T^+_{\mathrm{ad},\geq C_P}\hookrightarrow P^-\times P\times T^+_{\mathrm{ad},\geq C_P},\ (m,t)\mapsto (m,m,t). \tag{A.18}$$

Hence we obtain a locally closed embedding $(P^-\times P)/M\times T^+_{\mathrm{ad},\geq C_P}\hookrightarrow \mathrm{Vin}_{G,\geq C_P}$. By the dimension reason, this is an open embedding. We define the corresponding open subscheme of $\mathrm{Vin}_{G,\geq C_P}$ to be the *open Bruhat cell* $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$. By § 1.1.1, we have

$$\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}\subset {}_0\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}\subset \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}.$$

We also define

$$\mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G := \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}\underset{(T^+_{\mathrm{ad},\geq C_P},\overline{\gamma})}{\times}\mathbb{A}^1$$

according to Convention 12.

Consider the composition $(P^-\times P)/M\twoheadrightarrow (M\times M)/M\simeq M$, where the last map is given by $(a,b)\mapsto ab^{-1}$. It induces an $(M\times M)$-equivariant isomorphism

$$U^-\backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/U\simeq M\times T^+_{\mathrm{ad},\geq C_P}. \tag{A.19}$$

In particular, there is a $(P^-\times P)$-equivariant map $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}\to M$. By Lemma A.6.2, it can be uniquely extended to a map $\mathrm{Vin}_{G,\geq C_P} \to \overline{M}$ fitting into the following *Cartesian* square of schemes acted by $(P^- \times P)$:

$$\begin{tikzcd} \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}} \arrow[r] \arrow[d] & \mathrm{Vin}_{G,\geq C_P} \arrow[d] \\ M \arrow[r] & \overline{M}. \end{tikzcd} \tag{A.20}$$

Note that the composition $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P} \hookrightarrow \mathrm{Vin}_{G,\geq C_P} \to \overline{M}$ is the identity map since its restriction on M is so.

Combining the Cartesian squares (A.21) and (A.20), we obtain a *Cartesian* square of schemes acted by $(P^- \times P)$:

$$\begin{tikzcd} \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}} \arrow[r] \arrow[d] & \mathrm{Vin}_{G,\geq C_P} \arrow[d] \\ A_M \arrow[r] & \overline{A_M}. \end{tikzcd} \tag{A.21}$$

A.5.6 (Schieder's local models)**.** [Sch16, § 6.1.6] constructed what is known as *Schieder's local models* for VinBun_G, which model the singularities of VinBun_G in the same sense as how the parabolic Zastava spaces model the Drinfeld compactifications $\widetilde{\mathrm{Bun}}_P$ in [BFGM02].

The *absolute local model* is defined as

$$Y^P := \mathbf{Maps}_{\mathrm{gen}}(X, U^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \supset U^-\backslash \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}/P).$$

The *relative local model* is defined as

$$Y_{\mathrm{rel}}^P := \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \supset P^-\backslash \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}/P). \tag{A.22}$$

We similarly define the defect-free locus ${}_0Y^{P,\gamma}$ and ${}_0Y_{\mathrm{rel}}^{P,\gamma}$. We also define

$$Y^{P,\gamma} := Y^P \underset{(T_{\mathrm{ad},\geq C_P}^+, \overline{\gamma})}{\times} \mathbb{A}^1$$

and $Y_{\mathrm{rel}}^{P,\gamma}$ similarly. It is known that each connected component of Y^P is a finite type scheme.

Consider the isomorphism

$$P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \simeq (P^-\backslash \mathrm{pt}/P) \underset{(G\backslash \mathrm{pt}/G)}{\times} (G\backslash \mathrm{Vin}_{G,\geq C_P}/G).$$

Since $\mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}}$ is an open subscheme of ${}_0\mathrm{Vin}_{G,\geq C_P}$, by Lemma A.3.5(1), we obtain a schematic open embedding

$$Y_{\mathrm{rel}}^{P} \to \mathrm{VinBun}_{G,\geq C_P} \underset{\mathrm{Bun}_{G\times G}}{\times} \mathrm{Bun}_{P^- \times P}\,. \tag{A.23}$$

In particular, there is a *local-model-to-global* map

$$\mathbf{p}^{-}_{\mathrm{glob}} : Y_{\mathrm{rel}}^{P} \to \mathrm{VinBun}_{G,\geq C_P}\,.$$

A.5.7 (Zastava spaces)**.** The *defect-free relative parabolic Zastava space* in [BFGM02] is defined as

$${}_0Z_{\mathrm{rel}}^{P} := \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash G/P \supset P^-\backslash G^{\mathrm{Bruhat}}/P),$$

where $G^{\mathrm{Bruhat}} = P^-P \subset G$ is the open Bruhat cell. The map $P^-\backslash G/P \to \mathbb{B}M \times \mathbb{B}M$ induces a map

$$(\overleftarrow{\mathfrak{h}}, \overrightarrow{\mathfrak{h}}) : {}_0Z_{\mathrm{rel}}^{P} \to \mathrm{Bun}_M \times \mathrm{Bun}_M\,.$$

It follows from definitions that we have

$${}_0Y_{\mathrm{rel}}^{P}|_{C_G} \simeq Y_{\mathrm{rel}}^{P}|_{C_G} \simeq {}_0Z_{\mathrm{rel}}^{P},\ {}_0Y_{\mathrm{rel}}^{P}|_{C_P} \simeq {}_0Z_{\mathrm{rel}}^{P} \underset{(\overrightarrow{\mathfrak{h}},\mathrm{Bun}_M,\overleftarrow{\mathfrak{h}})}{\times} {}_0Z_{\mathrm{rel}}^{P} \tag{A.24}$$

Construction A.5.8. The commutative diagram (A.20) induces a map

$$P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \to M\backslash \overline{M}/M$$

sending $P^-\backslash \mathrm{Vin}_{G,C_P}^{\mathrm{Bruhat}}/P$ into $M\backslash M/M$. Hence we obtain a map

$$Y_{\mathrm{rel}}^{P} \to H_{M,G\text{-}\mathrm{pos}}$$

A.6 The geometric objects in [Sch16]: Complementary proofs

In this section, we provide proofs for some results in Appendix A.5. This appendix should not be read separately because there are no logical connections between these results.

Lemma A.6.1. *Let $f : Y \to Z$ be an affine morphism between strongly quasi-affine schemes. Suppose Y is* *integral, then the following obvious commutative diagram is Cartesian:*

$$\begin{tikzcd} Y \arrow[r, "j_Y"] \arrow[d, "f"] & \overline{Y} \arrow[d, "\overline{f}"] \\ Z \arrow[r, "j_Z"] & \overline{Z}. \end{tikzcd}$$

Proof. Let Y' be the fiber product $Z \times_{\overline{Z}} \overline{Y}$. We have a commutative diagram

$$\begin{tikzcd} Y \arrow[dr, "g"] \arrow[drr, "j_Y"] \arrow[ddr, "f"'] & & \\ & Y' \arrow[r, "q"] \arrow[d, "p"] & \overline{Y} \arrow[d, "\overline{f}"] \\ & Z \arrow[r, "j_Z"] & \overline{Z}. \end{tikzcd}$$

$\overline{f}$ is obviously affine, so is its base-change p. Since $f \simeq p \circ g$ is assumed to be affine, g is affine. On the other hand, j_Z is an open embedding, so is its base-change q. Since $j_Y \simeq q \circ g$ is an open embedding, g is an embedding. Also, since Y is integral, $\overline{Y}$ is integral. Hence its open subscheme Y' is also integral. In summary, g is an affine open embedding between integral schemes.

Since Z is strongly quasi-affine, it is quasi-affine in the sense of [Gro61, Chapter 5]. Since p is affine, by [Gro61, Proposition 5.1.10(ii)], Y' is also quasi-affine. Consider the natural map $\overline{g} : \overline{Y} \to \overline{Y'}$ between their affine closures. We claim it is an isomorphism. Indeed, the open embedding $Y' \hookrightarrow \overline{Y}$ induces a map

$$H^0(Y, \mathcal{O}_Y) \simeq H^0(\overline{Y}, \mathcal{O}_{\overline{Y}}) \xrightarrow{q^*} H^0(Y', \mathcal{O}_{Y'}),$$

which by construction is a right inverse to the map $g^* : H^0(Y', \mathcal{O}_{Y'}) \to H^0(Y, \mathcal{O}_Y)$. Hence g^* is surjective. But g is dominant and Y' is reduced, hence this map is also injective and therefore an isomorphism. This proves the claim.

Now consider the natural map $\mathcal{O}_{Y'} \to g_*(\mathcal{O}_Y)$. Since g is dominant, this map is injective. On the other hand, we proved in the last paragraph that the natural map

$$H^0(Y', \mathcal{O}_{Y'}) \to H^0(Y', g_*(\mathcal{O}_Y)) \simeq H^0(Y, \mathcal{O}_Y)$$

is an isomorphism. Since Y' is quasi-affine, by [Gro61, Proposition 5.1.2(e)], any quasi-coherent $\mathcal{O}_{Y'}$-module is generated by its global sections. Hence $\mathcal{O}_{Y'} \to g_*(\mathcal{O}_Y)$ is also surjective and therefore an isomorphism. Since g is affine, this means g is an isomorphism.

□[Lemma A.6.1]

Lemma A.6.2. *There is a unique map* $\mathrm{Vin}_{G,\geq C_P} \to \overline{M}$ *extending the canonical map* $\mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}} \to M$*. Moreover, the inverse image of* $M \subset \overline{M}$ *along this map is* $\mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}} \subset \mathrm{Vin}_{G,\geq C_P}$*.*

Remark A.6.3. In the case $P = B$, [Sch17, Lemma 4.1.3] proved the first claim by showing

$$\overline{M} \times T_{\mathrm{ad},\geq C_P}^{+} \simeq \mathrm{Vin}_{G,\geq C_P}^{\mathrm{Bruhat}} /\!/ (U^- \times U),$$

where the RHS is the GIT quotient. The second claim was also stated in [Sch17, Lemma 4.1.3]. However, we do *not* think Schieder actually proved it. Therefore we provide a proof as below.

Proof. Recall $G_{\mathrm{enh}} := (G \times T)/Z_G$ is the group of invertible elements in Vin_G. Note that we have a short exact sequence of algebraic groups

$$1 \to G \to G_{\mathrm{enh}} \to T_{\mathrm{ad}} \to 1.$$

The canonical section $\mathfrak{s} : T_{\mathrm{ad}}^{+} \to \mathrm{Vin}_G$ provides a splitting to the above sequence. Explicitly, this splitting is $T/Z_G \to (G \times T)/Z_G,\ t \mapsto (t^{-1}, t)$. Note that the T_{ad}-action on G given by this splitting is the inverse of the usual adjoint action.

Now consider the $(G_{\mathrm{enh}} \times G_{\mathrm{enh}})$-action on Vin_G. Using the above splitting, we obtain a $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-action on Vin_G and $G \times G$ such that the action map

$$G \times \mathrm{Vin}_G \times G \to \mathrm{Vin}_G,\ (g_1, g, g_2) \mapsto g_1 \cdot g \cdot g_2^{-1} \tag{A.25}$$

is $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-equivariant, where the T_{ad}-action on G is the inverse of the usual adjoint action[98].

By restriction along $\mathbb{G}_m \xrightarrow{\gamma} Z_M \to T_{\mathrm{ad}}$, we obtain a $(\mathbb{G}_m \times \mathbb{G}_m)$-action on Vin_G that preserves $\mathrm{Vin}_{G,\geq C_P}$. Consider the group homomorphism $\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m,\ s \mapsto (s, s^{-1})$. By restriction, we obtain a $\mathbb{G}_m$-action on $\mathrm{Vin}_{G,\geq C_P}$. Explicitly, this action is given by

$$\mathbb{G}_m \times \mathrm{Vin}_{G,\geq C_P} \to \mathrm{Vin}_{G,\geq C_P},\ (s, g) \mapsto \mathfrak{s}^{\gamma}(s) \cdot g \cdot \mathfrak{s}^{\gamma}(s),$$

where $\mathfrak{s}^{\gamma} : \mathbb{A}^1 \to \mathrm{Vin}_G^{\gamma}$ is the canonical section. In particular, this $\mathbb{G}_m$-action can be extended to an $\mathbb{A}^1$-action (because $\mathfrak{s}^{\gamma}$ is a monoid homomorphism). By [Wan17, Theorem 4.2.10], the corresponding fixed locus

$$\mathfrak{s}^{\gamma}(0) \cdot \mathrm{Vin}_{G,\geq C_P} \cdot \mathfrak{s}^{\gamma}(0) \simeq \mathfrak{s}^{\gamma}(0) \cdot \mathrm{Vin}_G|_{C_P} \cdot \mathfrak{s}^{\gamma}(0)$$

[98] Note that when $G = \mathrm{SL}_2$, the canonical section $\mathbb{A}^1 \to \mathrm{M}_{2,2}$ is given by $t \mapsto \mathrm{diag}(1, t)$. Hence our description is correct in this case.

is equal to $\overline{M}$ as closed subschemes of $\mathrm{Vin}_{G,\geq C_P}$. Hence we obtain a projection map $\mathrm{Vin}_{G,\geq C_P} \to \overline{M}$, which is left inverse to the closed embedding $\overline{M} \hookrightarrow \mathrm{Vin}_{G,\geq C_P}$.

It remains to prove the obtained map $\mathrm{Vin}_{G,\geq C_P} \to \overline{M}$ satisfies the desired properties. By construction, the action map (A.25) is compatible with the $\mathbb{G}_m$-actions, where $\mathbb{G}_m$ acts on the first G-factor by

$$\mathbb{G}_m \times G \to G,\ (s,g) \mapsto \gamma(s^{-1}) \cdot g \cdot \gamma(s),$$

and acts on the third G-factor inversely. Note that

(i) the attractor for this $\mathbb{G}_m$-action on $G \times G$ is $P^- \times P$;

(ii) this $\mathbb{G}_m$-action on $G \times G$ contracts $U^- \times U$ to the multiplicative unit.

Now by (i), the above extended $\mathbb{A}^1$-action on $\mathrm{Vin}_{G,\geq C_P}$ preserves the open Bruhat cell $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$. Note that the corresponding fixed locus is

$$\overline{M} \underset{\mathrm{Vin}_{G,\geq C_P}}{\times} \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P} \simeq M,$$

viewed as closed subschemes of $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$. Hence we obtain a projection map $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P} \to M$, which is left inverse to the canonical closed embedding. Moreover, by (ii), the $(U^- \times U)$-action on $\mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}$ preserves this projection. Hence this projection is equal to the projection mentioned in the problem. Now we are done by [DG14, Lemma 1.4.9(i)].

□[Lemma A.6.2]

Lemma A.6.4. *Let S be any finite type affine test scheme over $\mathrm{Bun}_M \times X^{\mathrm{pos}}$, then after replacing S by an étale cover, the retractions*

$$(Y^P_{\mathrm{rel}} \underset{(\mathrm{Bun}_M \times X^{\mathrm{pos}})}{\times} S,\ H_{M,G\text{-}\mathrm{pos}} \underset{(\mathrm{Bun}_M \times X^{\mathrm{pos}})}{\times} S) \text{ and } (Y^P \underset{X^{\mathrm{pos}}}{\times} S,\ \mathrm{Gr}_{M,G\text{-}\mathrm{pos}} \underset{X^{\mathrm{pos}}}{\times} S) \tag{A.26}$$

are isomorphic over $(T^+_{\mathrm{ad},\geq C_P} \times S, C_P \times S)$.

Remark A.6.5. We need to use the *Beauville-Laszlo descent theorem* to conduct a re-gluing construction. Let us first reveiw it. Let Z be an algebraic stack. Consider the following condition on Z:

(♠) For any affine test scheme S' and a relative effective Cartier divisor Γ' of $X \times S' \to S'$ that is contained in an affine open subset[99] of $X \times S'$, the following commutative diagram of groupoids is Cartesian (see

[99] We need this technical restriction because the Beauville-Laszlo descent theorem is stated for affine schemes. Alternatively, one can use the main theorem of [Sch15] which generalizes the Beauville-Laszlo descent theorem to the global case.

Notation 6):

$$\begin{tikzcd} Z(X \times S') \arrow[r] \arrow[d] & Z(X \times S' - \Gamma') \arrow[d] \\ Z(\mathcal{D}'_{\Gamma'}) \arrow[r] & Z(\mathcal{D}^{\times}_{\Gamma'}). \end{tikzcd}$$

Using the Tannakian duality, the well-known Beauville-Laszlo descent theorem for vector bundles implies pt/H satisfies the condition (♠) for any affine algebraic group H. Similarly, the Tannakian description for Vin_G in [FKM20, § 2.2.8] (resp. for $\overline{M}$ in [Wan17, § 3.3]) implies that $G\backslash \mathrm{Vin}_{G,\geq C_P}/G$ (resp. $M\backslash\overline{M}/M$) satisfies the condition (♠). Hence by taking fiber products, all the algebraic stacks in (4.17) satisfy the condition (♠).

A.6.6 (Proof of Lemma A.6.4)**.** The map $S \to \mathrm{Bun}_M \times X^{\mathrm{pos}}$ gives an M-torsor F_M on $X \times S$ and a $\Lambda^{\mathrm{pos}}_{G,P}$-valued relative Cartier divisor D on $X \times S \to S$. By forgetting the color, we obtain a relative effective Cartier divisor $\Gamma \hookrightarrow X \times S$. Replacing S by a Zariski cover, we can assume Γ is contained in an affine open subset of $X \times S$. Using Lemma A.6.7 below, we can further assume F_M is trivial on $\mathcal{D}'_{\Gamma}$. We claim under these assumptions, the two retractions in (A.26) are isomorphic over $(T^{+}_{\mathrm{ad},\geq C_P} \times S, C_P \times S)$.

Recall that the diagram

$$Y^P_{\mathrm{rel}} \to \mathrm{Bun}_M \times X^{\mathrm{pos}} \leftarrow \mathrm{Bun}_M \times Y^P$$

is obtained by applying $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ to the following commutative diagram

$$\begin{tikzcd} P^-\backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/P \arrow[r, "\simeq"] \arrow[d, "\subset"] & M\backslash \mathrm{pt} \times A_M/A_M \arrow[d, "\subset"] & M\backslash \mathrm{pt} \times U^-\backslash \mathrm{Vin}^{\mathrm{Bruhat}}_{G,\geq C_P}/P \arrow[l, "\simeq"'] \arrow[d, "\subset"] \\ P^-\backslash \mathrm{Vin}_{G,\geq C_P}/P \arrow[r] & M\backslash \mathrm{pt} \times \overline{A_M}/A_M & M\backslash \mathrm{pt} \times U^-\backslash \mathrm{Vin}_{G,\geq C_P}/P. \arrow[l] \end{tikzcd}$$

Note that the above diagram is defined over $M\backslash\mathrm{pt} \times \mathbb{A}^1$. Also note that both squares in it are Cartesian because of the Cartesian square (A.21). To simplify the notations, we write the above diagram as

$$(V_1 \simeq V_2 \simeq V_3) \subset (Z_1 \to Z_2 \leftarrow Z_3),$$

and write its base-change along $\mathrm{pt} \to M\backslash\mathrm{pt}$ as

$$(V'_1 \simeq V'_2 \simeq V'_3) \subset (Z'_1 \to Z'_2 \leftarrow Z'_3).$$

Note that there is a canonical isomorphism $Z'_1 \simeq Z'_3$ defined over Z'_2 extending the isomorphism $V'_1 \simeq V'_3$.

The given map $S \to \mathrm{Bun}_M \times X^{\mathrm{pos}}$ provides a map $\alpha : X \times S \to Z_2$. By our assumption on F_M, the composition

$$\mathcal{D}'_\Gamma \to X \times S \to Z_2 \to M\backslash\mathrm{pt}$$

factors (non-canonically) through $\mathrm{pt} \to M\backslash\mathrm{pt}$. We fix such a factorization. Hence we obtain a factorization

$$\alpha|_{\mathcal{D}'_\Gamma} : \mathcal{D}'_\Gamma \xrightarrow{\beta} Z'_2 \to Z_2.$$

This gives an isomorphism

$$\widehat{\delta} : Z_1 \underset{(Z_2,\alpha)}{\times} \mathcal{D}'_\Gamma \simeq Z'_1 \underset{(Z'_2,\beta)}{\times} \mathcal{D}'_\Gamma \simeq Z'_3 \underset{(Z'_2,\beta)}{\times} \mathcal{D}'_\Gamma \simeq Z_3 \underset{(Z_2,\alpha)}{\times} \mathcal{D}'_\Gamma$$

defined over $\mathcal{D}'_\Gamma$. On the other hand, note that by definition α sends $(X \times S) - \Gamma$ into $V_2 \subset Z_2$. Hence we have an isomorphism

$$\mathring{\delta} : Z_1 \underset{(Z_2,\alpha)}{\times} (X \times S - \Gamma) \simeq X \times S - \Gamma \simeq Z_3 \underset{(Z_2,\alpha)}{\times} (X \times S - \Gamma)$$

defined over $X \times S - \Gamma$. Moreover, the restrictions of $\widehat{\delta}$ and $\mathring{\delta}$ on $\mathcal{D}^\times_\Gamma$ are canonically isomorphic (because the isomorphism $Z'_1 \simeq Z'_3$ extends $V'_1 \simeq V'_3$).

Let S' be a finite type affine test scheme. Unwinding the definitions, the groupoid $(Y^P_{\mathrm{rel}} \times_{(\mathrm{Bun}_M \times X^{\mathrm{pos}})} S)(S')$ classifies

(i) a map $S' \to S$

(ii) a commutative diagram

$$\begin{tikzcd} X \times S' \arrow[r, "\epsilon"] \arrow[d] & Z_1 \arrow[d] \\ X \times S \arrow[r, "\alpha"] & Z_2. \end{tikzcd}$$

(Note that $\epsilon^{-1}(V_1) = \alpha^{-1}(V_2)$ automatically has non-empty intersections with any geometric fiber of $X \times S' \to S'$). Define $\Gamma' : \Gamma \times_S S'$. By assumption, Γ' is contained in an affine open subset of $X \times S'$. Since Z_1 satisfies the condition (♠), we can replace (ii) by

(ii') commutative diagrams

$$\begin{tikzcd} X \times S' - \Gamma' \arrow[r, "\mathring{\epsilon}"] \arrow[d] & Z_1 \arrow[d] \\ X \times S - \Gamma \arrow[r, "\alpha"] & Z_2 \end{tikzcd} \quad \begin{tikzcd} \mathcal{D}'_{\Gamma'} \arrow[r, "\widehat{\epsilon}"] \arrow[d] & Z_1 \arrow[d] \\ \mathcal{D}'_\Gamma \arrow[r, "\alpha"] & Z_2 \end{tikzcd} \quad \begin{tikzcd} \mathcal{D}^\times_{\Gamma'} \arrow[r, "\widehat{\mathring{\epsilon}}"] \arrow[d] & Z_1 \arrow[d] \\ \mathcal{D}^\times_\Gamma \arrow[r, "\alpha"] & Z_2 \end{tikzcd}$$

such that the third square is isomorphic to the restrictions of the first two squares.

Similarly, we can describe the groupoid $(Y^P \times_{X^{\mathrm{pos}}} S)(S')$ by replacing Z_1 by Z_3. Therefore the isomorphisms $\overset{\circ}{\delta}$ and $\widehat{\delta}$ (and their compatibility over $\mathcal{D}_\Gamma^\times$) provide an isomorphism

$$Y^P_{\mathrm{rel}} \underset{(\mathrm{Bun}_M \times X^{\mathrm{pos}})}{\times} S \simeq Y^P \underset{X^{\mathrm{pos}}}{\times} S$$

defined over S. It is also defined over $\mathbb{A}^1$ because $\overset{\circ}{\delta}$ and $\widehat{\delta}$ are defined over $\mathbb{A}^1$ by construction.

Similarly we have an isomorphism[100]

$$H_{M,G\text{-}\mathrm{pos}} \underset{(\mathrm{Bun}_M \times X^{\mathrm{pos}})}{\times} S \simeq \mathrm{Gr}_{M,G\text{-}\mathrm{pos}} \underset{X^{\mathrm{pos}}}{\times} S$$

defined over S. These two isomophisms are compatible with the structures of retractions because the above construction is functorial in Z_1 and Z_3.

□[Lemma A.6.4]

Lemma A.6.7. *Let S be any finite type affine test scheme over $\mathrm{Bun}_M \times X^\theta$, then there exists an étale covering S' satisfying the following condition*

- *Let (F'_M, D') be the object classified by the map $S' \to S \to \mathrm{Bun}_M \times X^\theta$, where F'_M is an M-torsor on $X \times S'$ and D' is a $\Lambda^{\mathrm{pos}}_{G,P}$-valued relative Cartier divisor on $X \times S \to S$. Let Γ' be the underlying relative Cartier divisor of D'. Then F'_M is trivial over $\mathcal{D}'_{\Gamma'}$ (see Notation 6).*

Proof. We prove by induction on θ. Note that the disjoint union of $(X^{\theta_1} \times X^{\theta-\theta_1})_{\mathrm{disj}}$ for all $\theta_1 < \theta$ is an étale cover of $X^\theta - X$ (the complement of the main diagonal). Hence by induction hypothesis, it remains to prove the following claim. For any closed point s of $S \times_{X^\theta} X \hookrightarrow S$, there exists an étale neighborhood S' of s satisfying the condition in the problem.

Let $x \in X^\theta$ be the image of s. By assumption, x is a closed point on the main diagnoal. By [DS95, Theorem 2], after replacing S by an étale cover S, we can assume F_M to be locally trivial in the Zariski topology of $X \times S$. Let U be an open of $X \times S$ containing (x,s) such that F_M is trivial on it. Denote its complement closed subset in $X \times S$ by Y. Note that $Y \cap \Gamma$ is a closed subset of $X \times S$. Since the projection $X \times S \to S$ is proper, the image of $Y \cap \Gamma$ is a closed subset of S. By construction, this closed subset does not contain s. We choose S' to the complement open of this closed subset. It follows from construction that it satisfies the desired property.

□[Lemma A.6.7]

[100] This time we need to use the Carteisan square (A.12).

Appendix B

Abstract miscellanea

In this appendix, we collect some abstract miscellanea. All these results, maybe except Lemma B.3.3, are known to the experts.

B.1 Colimits and limits of categories

In this subsection, we review colimits and limits in DGCat. We provide proofs only when we fail to find a good reference.

Following [Lur09], we have the following categories:

	objects	morphisms
$\mathrm{Cat}^{\mathrm{st}}$	stable categories	exact functors
$\mathrm{Pr}^L . \mathrm{Pr}^R$	presentable categrories	commuting with colimits (resp. limits)
$\mathrm{Pr}^{\mathrm{st},L}, \mathrm{Pr}^{\mathrm{st},R}$	presentable stable categories	commuting with colimits (resp. limits)
$\mathrm{DGCat}, \mathrm{DGCat}^R$	cocomplete DG-categories	commuting with colimits (resp. limits).

Passing to adjoints provides equivalences $(\mathrm{Pr}^L)^{\mathrm{op}} \simeq \mathrm{Pr}^R, (\mathrm{Pr}^{\mathrm{st},L})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{st},R}$ and $\mathrm{DGCat}^{\mathrm{op}} \simeq \mathrm{DGCat}^R$.

Lemma B.1.1. *(1) ([Lur09, Proposition 5.5.3.13, Proposition 5.5.3.18])* $\mathrm{Pr}^L \to \mathrm{Cat}$ *and* $\mathrm{Pr}^R \to \mathrm{Cat}$ *commute with limits.*

(1') Pr^L *(resp.* Pr^R*) contains all colimits and limits.*

(2) ([Lur12, Theorem 1.1.4.4]) $\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}$ *commutes with limits.*

(2') $\mathrm{Pr}^{\mathrm{st},L} \to \mathrm{Pr}^L$ *and* $\mathrm{Pr}^{\mathrm{st},R} \to \mathrm{Pr}^R$ *commute with colimits and limits.*

(3) $\mathrm{DGCat} \to \mathrm{Pr}^{\mathrm{st},L}$ *and* $\mathrm{DGCat}^R \to \mathrm{Pr}^{\mathrm{st},R}$ *commute with colimits and limits.*

Proof. (1') is obtained from (1) by $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$. (2') follows from (1), (2) and the equivalence $\mathrm{Pr}^{\mathrm{st},L} \simeq (\mathrm{Pr}^{\mathrm{st},R})^{\mathrm{op}}$. (3) is a particular case of the following general fact. Let $\mathcal{C}$ be a presentable symmetric monoidal category whose tensor products preserve colimits, and A be a commutative algebra object in $\mathcal{C}$, then the forgetful functor $A\text{-mod}(\mathcal{C}) \to \mathcal{C}$ commutes with both colimits and limits.

□[Lemma B.1.1]

Remark B.1.2. The lemma provides a description for colimits in DGCat as follows. For a diagram $F : I \to \mathrm{DGCat}$, passing to right adjoints provides a diagram $G : I^{\mathrm{op}} \to \mathrm{DGCat}^R$. Tautologically there is an equivalence $\mathrm{colim}_I F \simeq \lim_{I^{\mathrm{op}}} G$ such that the insertion functor $\mathrm{ins}_i : F(i) \to \mathrm{colim}_I F$ corresponds to the left adjoint of the evaluation functor $\mathrm{ev}_i : \lim_{I^{\mathrm{op}}} G \to G(i)$. By the lemma, the above limit can be calculated in Cat, whose objects and morphisms can be described explicitly as in [Lur09, § 3.3.3].

Lemma B.1.3. *(1) Let $F_1, F_2 : I \to \mathrm{Pr}^L$ be two diagrams, and $\alpha : F_1 \to F_2$ be a natural transformation. Suppose that for any morphism $i \to j$ in I, the commutative square*

$$\begin{tikzcd} F_1(i) \arrow[r] \arrow[d, "\alpha(i)"] & F_1(j) \arrow[d, "\alpha(j)"] \\ F_2(i) \arrow[r] & F_2(j) \end{tikzcd}$$

is left adjointable along the vertical direction, so that we have a natural transformation $\alpha^L : F_2 \to F_1$. Then we have an adjoint pair

$$\operatorname*{colim}_I \alpha^L : \operatorname*{colim}_I F_2 \rightleftharpoons \operatorname*{colim}_I F_1 : \operatorname*{colim}_I \alpha.$$

(2) Let $G_1, G_2 : I^{\mathrm{op}} \to \mathrm{Pr}^R$ be two diagrams, and $\beta : G_2 \to G_1$ be a natural transformation. Suppose that for any morphism $i \to j$ in I, the commutative square

$$\begin{tikzcd} G_1(i) & G_1(j) \arrow[l] \\ G_2(i) \arrow[u, "\beta(i)"] & G_2(j) \arrow[l] \arrow[u, "\beta(j)"] \end{tikzcd}$$

is left adjointable along the vertical direction, so that we have a natural transformation $\beta^L : G_1 \to G_2$. Then we have an adjoint pair

$$\lim_{I^{\mathrm{op}}} \beta^L : \lim_{I^{\mathrm{op}}} G_1 \rightleftharpoons \lim_{I^{\mathrm{op}}} G_2 : \lim_{I^{\mathrm{op}}} \beta.$$

Proof. (1) is obtained from (2) by passing to left adjoints. For (2), consider objects $x \in \lim_{I^{\mathrm{op}}} G_1$ and $y \in \lim_{I^{\mathrm{op}}} G_2$. Write x_i (resp. y_i) for their evaluations in $G_1(i)$ (resp. $G_2(i)$). By [Lur09, § 3.3.3], we have functorial isomorphisms

$$
\begin{aligned}
& \operatorname{Maps}(\lim_{I^{\mathrm{op}}} \beta^L(x), y) \\
\simeq\ & \lim_{I^{\mathrm{op}}} \operatorname{Maps}(\mathrm{ev}_i(\lim_{I^{\mathrm{op}}} \beta^L(x)), \mathrm{ev}_i(y)) \\
\simeq\ & \lim_{I^{\mathrm{op}}} \operatorname{Maps}(\beta(i)^L(x_i), y_i) \\
\simeq\ & \lim_{I^{\mathrm{op}}} \operatorname{Maps}(x_i, \beta(i)(y_i)) \\
\simeq\ & \lim_{I^{\mathrm{op}}} \operatorname{Maps}(\mathrm{ev}_i(x), \mathrm{ev}_i(\lim_{I^{\mathrm{op}}} \beta(y))) \\
\simeq\ & \operatorname{Maps}(x, \lim_{I^{\mathrm{op}}} \beta(y)).
\end{aligned}
$$

□[Lemma B.1.3]

Remark B.1.4. By Lemma B.1.1, the lemma remains correct if we replace Pr by $\mathrm{Pr}^{\mathrm{st}}$ or DGCat.

Lemma B.1.5. *([DG15, Corollary 1.9.4, Lemma 1.9.5]) Let* $F : I \to \mathrm{Pr}^{\mathrm{st},L}$ *(or* $F : I \to \mathrm{DGCat}$*) be a diagram such that each* $F(i)$ *is compactly generated and each functor* $F(i) \to F(j)$ *sends compact objects to compact objects, then* $\mathrm{colim}_I F$ *is compactly generated by objects of the form* $\mathrm{ins}_i(x_i)$ *with* x_i *being compact in* $F(i)$*. If* I *is further assumed to be filtered, then every compact object in* $\mathrm{colim}_I F$ *is of the above form.*

B.2 Duality

In this subsection we review the notion of duality for bimodules developed in [Lur12, Sub-section 4.6]. The unproven claims can be found in *loc.cit.*.

Let $\mathcal{C}$ be a monoidal category that admits geometric realizations such that the multiplication functor $\otimes$ preserves geometric realizations. Let A, B be two associative algebra objects in $\mathcal{C}$. We write ${}_A\mathrm{BiMod}_B(\mathcal{C})$ for the category of (A,B)-bimodules in $\mathcal{C}$.

B.2.1. For $x \in {}_A\mathrm{BiMod}_B(\mathcal{C})$ and $y \in {}_B\mathrm{BiMod}_A(\mathcal{C})$, and a (B,B)-linear map $c : B \to y \otimes_A x$ (resp. an (A,A)-linear map $e : x \otimes_B y \to A$), we say (c,e) *exhibits* x *as the right-dual of* y*, or* y *as the left-dual of* x, if the following compositions are both isomorphic to the identity maps:

$$
x \simeq x \underset{B}{\otimes} B \xrightarrow{c} x \underset{B}{\otimes} (y \underset{A}{\otimes} x) \simeq (x \underset{B}{\otimes} y) \underset{A}{\otimes} x \xrightarrow{e} A \underset{A}{\otimes} x \simeq x,
$$

$$
y \simeq B \underset{B}{\otimes} y \xrightarrow{c} (y \underset{A}{\otimes} x) \underset{B}{\otimes} y \simeq y \underset{A}{\otimes} (x \underset{B}{\otimes} y) \xrightarrow{e} y \underset{A}{\otimes} A \simeq y.
$$

We refer c (resp. e) as the *unit (resp. counit) map* for this duality.

For a fixed x (resp. y), the data (y,c,e) (resp. (x,c,e)) satisfying the above conditions is unique if it exists. Also, for fixed (x,y,c) (resp. (x,y,e)), the map e (resp. c) satisfying the above conditions is unique if exists. Hence if x (resp. y) is left-dualizable (resp. right-dualizable), we write $x^{\vee,L}$ (resp. $y^{\vee,R}$) for its left-dual (resp. left-dual) and treating (c,e) as implicit. We also write $x^{\vee,A}$ (resp. $y^{\vee,A}$) for the reason of § B.2.3 below.

B.2.2. Let (x,y,c,e) be a duality data as above. For any $m \in A\text{-mod}^l(\mathcal{C})$ and $n \in B\text{-mod}^l(\mathcal{C})$, it is easy to check that the following two compositions are quasi-inverse to each other.

$$\begin{aligned}
&\mathrm{Maps}_A(x \underset{B}{\otimes} n, m) \to \mathrm{Maps}_B(y \underset{A}{\otimes} x \underset{B}{\otimes} n, y \underset{A}{\otimes} m) \to \\
&\xrightarrow{-\circ(e\otimes \mathrm{Id})} \mathrm{Maps}_B(B \underset{B}{\otimes} n, y \underset{A}{\otimes} m) \simeq \mathrm{Maps}_B(n, y \underset{A}{\otimes} m),\\
&\mathrm{Maps}_B(n, y \underset{A}{\otimes} m) \to \mathrm{Maps}_A(x \underset{B}{\otimes} n, x \underset{B}{\otimes} y \underset{A}{\otimes} m) \to \\
&\xrightarrow{(c\otimes \mathrm{Id})\circ -} \mathrm{Maps}_A(x \underset{B}{\otimes} n, A \underset{A}{\otimes} m) \simeq \mathrm{Maps}_A(x \underset{B}{\otimes} n, m)
\end{aligned}$$

In particular, they are both isomorphisms. Similarly, for any $m \in A\text{-mod}^r(\mathcal{C})$ and $n \in B\text{-mod}^r(\mathcal{C})$, there is an isomorphism $\mathrm{Maps}_{A^{\mathrm{rev}}}(n \otimes_B y, m) \simeq \mathrm{Maps}_{B^{\mathrm{rev}}}(n, m \otimes_A x)$.

Conversely, if for given $x \in {}_A\mathrm{BiMod}_B(\mathcal{C})$ and $y \in {}_B\mathrm{BiMod}_A(\mathcal{C})$, there are functorial (in m and n) isomorphisms $\mathrm{Maps}_A(x \otimes_B n, m) \simeq \mathrm{Maps}_B(n, y \otimes_A m)$ (or $\mathrm{Maps}_{A^{\mathrm{rev}}}(n \otimes_B y, m) \simeq \mathrm{Maps}_{B^{\mathrm{rev}}}(n, m \otimes_A x)$), one can recover a duality for x and y.

B.2.3. In the special case when $B = \mathbf{1}$ is the unit object, we obtain the usual notion of duality between left A-modules and right A-modules. Moreover, by [Lur12, Proposition 4.6.2.13], an object x in ${}_A\mathrm{BiMod}_B(\mathcal{C})$ (resp. y in ${}_B\mathrm{BiMod}_A(\mathcal{C})$) is left-dualizable (resp. right-dualizable) if and only if its underlying object $\underline{x} \in A\text{-mod}^l(\mathcal{C})$ (resp. $\underline{y} \in A\text{-mod}^r(\mathcal{C})$) is left-dualizable (resp. right-dualizable) as a left (resp. right) A-module. Moreover, the underlying right (resp. left) A-module structure on $x^{\vee,L}$ (resp. $y^{\vee,R}$) is isomorphic to $\underline{x}^{\vee,L}$ (resp. $\underline{y}^{\vee,R}$).

Explicitly, the corresponding B-action maps $B \otimes \underline{x}^{\vee,L} \to \underline{x}^{\vee,L}$, $\underline{y}^{\vee,R} \otimes B \to \underline{y}^{\vee,R}$ are induced respectively by the universal properties from the action maps $\underline{x} \otimes B \to \underline{x}$, $B \otimes \underline{y} \to \underline{y}$.

The proof of the following lemma is obvious.

Lemma B.2.4. *(c.f. [Lur12, Proposition 4.6.2.13]) Let $x \in {}_A\mathrm{BiMod}_B(\mathcal{C})$ and $y \in {}_B\mathrm{BiMod}_A(\mathcal{C})$. Suppose $e : \underline{x} \otimes \underline{y} \to A$ is the counit map of a duality between $\underline{x}$ and $\underline{y}$ as A-modules. Then there is an isomorphism between the space of B-linear structures on the isomorphism $\underline{x} \simeq \underline{y}^{\vee,R}$ and the space of factorizations of e as $\underline{x} \otimes \underline{y} \to x \otimes_B y \to A$.*

B.2.5. Suppose that $\mathcal{C}$ is a symmetric monoidal category and A, B are commutative algebra objects in it. Then there is no difference between left and right modules, or left-duals and right-duals.

In the special case when $B := \mathbf{1}$, one can replace the duality data in § B.2.1 by A-linear maps $c' : A \to y \otimes_A x$ and $e' : x \otimes_A y \to A$, such that both the following compositions are isomorphic to the identity maps.

$$x \simeq x \underset{A}{\otimes} A \xrightarrow{c'} x \underset{A}{\otimes} (y \underset{A}{\otimes} x) \simeq (x \underset{A}{\otimes} y) \underset{A}{\otimes} x \xrightarrow{e'} A \underset{A}{\otimes} x \simeq x,$$
$$y \simeq A \underset{A}{\otimes} y \xrightarrow{c'} (y \underset{A}{\otimes} x) \underset{A}{\otimes} y \simeq y \underset{A}{\otimes} (x \underset{A}{\otimes} y) \xrightarrow{e'} y \underset{A}{\otimes} A \simeq y.$$

B.2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two associative algebra objects in DGCat, $\mathcal{M}$ (resp. $\mathcal{N}$) be an $(\mathcal{A}, \mathcal{B})$-bimodule (resp. a $(\mathcal{B}, \mathcal{A})$-bimodule) DG-category. If $\mathcal{M}$ and $\mathcal{N}$ are dual to each other, the universal properties can be upgraded to equivalences between categories:

$$\mathrm{Funct}_{\mathcal{A}}(\mathcal{M}, -) \simeq \mathrm{Funct}(\mathrm{Vect}, \mathcal{N} \underset{\mathcal{A}}{\otimes} -) \simeq \mathcal{N} \underset{\mathcal{A}}{\otimes} -,$$
$$\mathrm{Funct}_{\mathcal{A}^{\mathrm{rev}}}(\mathcal{N}, -) \simeq \mathrm{Funct}(\mathrm{Vect}, - \underset{\mathcal{A}}{\otimes} \mathcal{M}) \simeq - \underset{\mathcal{A}}{\otimes} \mathcal{M}.$$

Moreover, the above equivalences are $\mathcal{B}$-linear (resp. $\mathcal{B}^{\mathrm{rev}}$-linear), where $\mathcal{B}$ acts leftly (resp. rightly) on the LHS's via its right (resp. left) action on $\mathcal{M}$ (resp. $\mathcal{N}$).

Conversely, in the special case when $\mathcal{B} := \mathrm{Vect}$, given an invertible natural transformation $\mathrm{Funct}_{\mathcal{A}}(\mathcal{M}, -) \simeq \mathcal{N} \otimes_{\mathcal{A}} -$ (or $\mathrm{Funct}_{\mathcal{A}^{\mathrm{rev}}}(\mathcal{N}, -) \simeq - \otimes_{\mathcal{A}} \mathcal{M}$), one can recover a duality for $\mathcal{M}$ and $\mathcal{N}$.

Note that a priori (without the duality) the functors

$$- \underset{\mathcal{A}}{\otimes} \mathcal{M} : \mathcal{A}\text{-mod}^r \to \mathcal{B}\text{-mod}^r,\ \mathcal{N} \underset{\mathcal{A}}{\otimes} - : \mathcal{A}\text{-mod}^l \to \mathcal{B}\text{-mod}^l$$

commute with colimits, and the functors

$$\mathrm{Funct}_{\mathcal{A}}(\mathcal{M}, -) : \mathcal{A}\text{-mod}^l \to \mathcal{B}\text{-mod}^l,\ \mathrm{Funct}_{\mathcal{A}^{\mathrm{rev}}}(\mathcal{N}, -) : \mathcal{A}\text{-mod}^r \to \mathcal{B}\text{-mod}^r$$

commute with limits. Hence if $\mathcal{M}$ and $\mathcal{N}$ are dual to each other, by the universal properties, these functors commute with both colimits and limits.

B.2.7. Let $F : \mathcal{M} \to \mathcal{N}$ be a morphism in DGCat. It follows from definition that if F has a continuous right adjoint F^R, then it sends compact objects to compact objects. Moreover, the converse is also correct if we assume $\mathcal{M}$ to be compactly generated.

On the other hand, it is well-known that if $\mathcal{M}$ is compactly generated, then it is dualizable. Moreover, there is a canonical equivalence $(\mathcal{M}^\vee)^c \simeq \mathcal{M}^{c,\mathrm{op}}$.

Now suppose both $\mathcal{M}$ and $\mathcal{N}$ are compactly generated and F sends compact objects to compact objects. Then we obtain a functor $F^c : \mathcal{M}^c \to \mathcal{N}^c$ and therefore a functor $F^{c,\mathrm{op}} : \mathcal{M}^{c,\mathrm{op}} \to \mathcal{N}^{c,\mathrm{op}}$. Hence by ind-completion, we obtain a functor $F^{\mathrm{conj}} : \mathcal{M}^\vee \to \mathcal{N}^\vee$, known as the *conjugate functor* of F. On the other hand, using the universal properties (twice), we obtain a functor $F^\vee : \mathcal{N}^\vee \to \mathcal{M}^\vee$, known as the *dual functor* of F. We have:

Lemma B.2.8. *([Gai16, Lemma 1.5.3][101]) In the above setting, F^{conj} is the left adjoint of $F^\vee$. Therefore F^{conj} is isomorphic to $(F^R)^\vee$.*

B.3 Duality for module DG-categories vs. for plain DG-categories

let $\mathcal{A}$ be a monoidal DG-category which is dualizable as a plain DG-category. By § B.2.3, the dual DG-category $\mathcal{A}^\vee$ has a natural $(\mathcal{A}, \mathcal{A})$-bimodule structure. The following lemma was proved[102] in [GR17a, Chapter 1, Proposition 9.4.4].

Lemma B.3.1. *Let $\mathcal{A}$ be as above and $\mathcal{M}$ be a left-dualizable object in $\mathcal{A}$-*mod*. We have*

(1) $\mathcal{M}$ is dualizable in DGCat

(2) Suppose we have an equivalence $\varphi : \mathcal{A} \simeq \mathcal{A}^\vee$ between $(\mathcal{A}, \mathcal{A})$-bimodule DG-categories. Then we have an equivalence (depending on φ) $\mathcal{M}^{\vee,\mathcal{A}} \simeq \mathcal{M}^\vee$ in $\mathcal{A}$-mod^r.

Remark B.3.2. For a finite type scheme Y, the DG-category $(\mathrm{DMod}(Y), \otimes^!)$ of D-modules on Y satisfies the assumption of (2) thanks to the Verdier duality.

On the other hand, if $\mathcal{A}$ is rigid (see [GR17a, Chapter 1, Section 9] for what this means), the converse of Lemma B.3.1 is also correct. Unfortunately, $\mathrm{DMod}(Y)$ is *not* rigid even for nicest variety Y. Nevertheless, the lemma below shows that the converse of Lemma B.3.1 is still correct for $\mathrm{DMod}(Y)$ when Y is separated.

Lemma B.3.3. *Let Y be a separated finite type scheme, and $\mathcal{M}$ be an object in* $\mathrm{DMod}(Y)$-mod*, i.e. a* $\mathrm{DMod}(Y)$*-module DG-category. Then $\mathcal{M}$ is dualizable in* $\mathrm{DMod}(Y)$-mod *if and only if it is dualizable in* DGCat*.*

The rest of this subsection is devoted to proof of the lemma. In fact, we provide two proofs. The first (which is an overkill) uses the fact that Y_{dR} is 1-affine (see [Gai15b] for what this means), while the second (which is more elementary) uses the fact that the multiplication functor $\otimes^!$ has a fully faithful dual functor.

[101] The functor F^{conj} was denoted by F^{op} in *loc.cit.*.

[102] In *loc.cit.*, the ambient symmetric monoidal category is the category of stable presentable categories and continuous functors. However, the proof there also works for DG-categories.

B.3.4 (First proof of Lemma B.3.3)**.** By Remark B.3.2, it is enough to show that the dualizability of $\mathcal{M}$ in DGCat implies its dualizability in $\mathrm{DMod}(Y)$-mod.

By [Gai15b, Theorem 2.6.3], Y_{dR} is 1-affine. Hence by [Gai15b, Corollary 1.4.3, Proposition 1.4.5], it is enough to show that for a finite type affine test scheme S over Y, $\mathcal{M} \otimes_{\mathrm{DMod}(Y)} \mathrm{QCoh}(S)$ is dualizable in DGCat. By Lemma B.3.5 below, it is enough to show that $\mathrm{QCoh}(S)$ is dualizable in $\mathrm{DMod}(Y)$-mod.

Since $\mathrm{QCoh}(Y)$ is rigid and $\mathrm{QCoh}(S)$ is dualizable in DGCat, $\mathrm{QCoh}(S)$ is dualizable in $\mathrm{QCoh}(Y)$-mod. Hence by Lemma B.3.6 below, it is enough to show that $\mathrm{QCoh}(Y)$ is left dualizable as a $(\mathrm{DMod}(Y), \mathrm{QCoh}(Y))$-bimodule DG-category. By § B.2.3, it is enough to show that $\mathrm{QCoh}(Y)$ is dualizable in $\mathrm{DMod}(Y)$-mod. By [Gai15b, Corollary 1.4.3, Proposition 1.4.5] again, it is enough to show that for a finite type affine scheme S over Y, $\mathrm{QCoh}(Y) \otimes_{\mathrm{DMod}(Y)} \mathrm{QCoh}(S)$ is dualizable in DGCat.

Note that we have

$$\mathrm{QCoh}(Y) \underset{\mathrm{DMod}(Y)}{\otimes} \mathrm{QCoh}(S) \simeq (\mathrm{QCoh}(Y) \underset{\mathrm{DMod}(Y)}{\otimes} \mathrm{QCoh}(Y)) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(S).$$

Hence by Lemma B.3.5 below again, it is enough to show $\mathrm{QCoh}(Y) \otimes_{\mathrm{DMod}(Y)} \mathrm{QCoh}(Y)$ is dualizable in DGCat. By [Gai15b, Proposition 3.1.9], we have $\mathrm{QCoh}(Y) \otimes_{\mathrm{DMod}(Y)} \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(Y \times_{Y_{\mathrm{dR}}} Y)$. Since Y is separated, the prestack $Y \times_{Y_{\mathrm{dR}}} Y$ is the formal completion of $Y \times Y$ along its diagonal. Now we are done by [GR14, Corollary 7.2.1].

□[First proof of Lemma B.3.3]

Lemma B.3.5. *Let $\mathcal{A}$ be any monoidal DG-category, and $\mathcal{M} \in \mathcal{A}\text{-mod}^l, \mathcal{N} \in \mathcal{A}\text{-mod}^r$. Suppose $\mathcal{M}$ is dualizable in* DGCat*, and $\mathcal{N}$ is right-dualizable as a right $\mathcal{A}$-module DG-category, then $\mathcal{N} \otimes_{\mathcal{A}} \mathcal{M}$ is dualizable in* DGCat*, and its dual is canonically identified with $\mathcal{M}^\vee \otimes_{\mathcal{A}} \mathcal{N}^{\vee,\mathcal{A}}$.*

Proof. Recall that $\mathcal{M}^\vee$ is equipped with the right $\mathcal{A}$-module structure described in § B.2.3. We have

$$\mathrm{Funct}(\mathcal{N} \underset{\mathcal{A}}{\otimes} \mathcal{M}, -) \simeq \mathrm{Funct}_{\mathcal{A}^{\mathrm{op}}}(\mathcal{N}, \mathrm{Funct}(\mathcal{M}, -)) \simeq \mathrm{Funct}(\mathcal{M}, -) \underset{\mathcal{A}}{\otimes} \mathcal{N}^{\vee,\mathcal{A}} \simeq - \otimes \mathcal{M}^\vee \underset{\mathcal{A}}{\otimes} \mathcal{N}^{\vee,\mathcal{A}},$$

which provides the desired duality by § B.2.6.

□[Lemma B.3.5]

Lemma B.3.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a morphism between two monoidal DG-categories, and $\mathcal{M} \in \mathcal{B}\text{-mod}^l$. We can view $\mathcal{B}$ and $\mathcal{M}$ as objects in $\mathcal{A}\text{-mod}^l$ by restriction along F. Suppose $\mathcal{M}$ is left-dualizable as a left $\mathcal{B}$-module DG-category, and $\mathcal{B}$ is left-dualizable as a $(\mathcal{A}, \mathcal{B})$-bimodule DG-category. Then $\mathcal{M}$ is left-dualizable as a left $\mathcal{A}$-module category, and its dual is canonically identified with $\mathcal{M}^{\vee,\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{B}^{\vee,\mathcal{A}}$.*

Proof. We have

$$\mathrm{Funct}_{\mathcal{A}}(\mathcal{M},-) \simeq \mathrm{Funct}_{\mathcal{A}}(\mathcal{B} \underset{\mathcal{B}}{\otimes} \mathcal{M},-) \simeq \mathrm{Funct}_{\mathcal{B}}(\mathcal{M}, \mathrm{Funct}_{\mathcal{A}}(\mathcal{B},-)) \simeq$$

$$\mathrm{Funct}_{\mathcal{B}}(\mathcal{M}, \mathcal{B}^{\vee,\mathcal{A}} \underset{\mathcal{A}}{\otimes} -) \simeq \mathcal{M}^{\vee,\mathcal{B}} \underset{\mathcal{B}}{\otimes} \mathcal{B}^{\vee,\mathcal{A}} \underset{\mathcal{A}}{\otimes} -,$$

which provides the desired duality data by § B.2.6.

□[Lemma B.3.6]

B.3.7 (Second proof of Lemma B.3.3)**.** As before, it is enough to prove that any object $\mathcal{M} \in \mathrm{DMod}(Y)$-mod that is dualizable in DGCat is also dualizable in $\mathrm{DMod}(Y)$-mod. In this proof we construct the duality data directly.

We only use the following formal properties of $\mathcal{A} := \mathrm{DMod}(Y)$:

(i) There is an equivalence $\varphi : \mathcal{A} \simeq \mathcal{A}^{\vee}$ as $(\mathcal{A},\mathcal{A})$-bimodule DG-categories.

(ii) The compositions

$$\mathrm{Vect} \xrightarrow{c} \mathcal{A}^{\vee} \otimes \mathcal{A} \xrightarrow{\varphi^{-1}\otimes \mathbf{Id}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathbf{mult}} \mathcal{A},\ \mathrm{Vect} \xrightarrow{c} \mathcal{A} \otimes \mathcal{A}^{\vee} \xrightarrow{\mathbf{Id}\otimes\varphi^{-1}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathbf{mult}} \mathcal{A},$$

are both isomorphic to the functor $\mathbf{1} : \mathrm{Vect} \to \mathcal{A}$.

Note that the first property is given by the Verdier duality, while the second property is guaranteed by the fact that **mult** has a fully faithful dual functor.

The unit functor for the desired duality is defined as the composition $\mathrm{Vect} \to \mathcal{M}^{\vee} \otimes \mathcal{M} \to \mathcal{M}^{\vee} \otimes_{\mathcal{A}} \mathcal{M}$, where the first functor is the unit functor for the duality between $\mathcal{M}$ and $\mathcal{M}^{\vee}$ in DGCat, and the second functor is the obvious one.

Consider the functor $\mathbf{coact} : \mathcal{M} \to \mathcal{A}^{\vee} \otimes \mathcal{M}$ induced from the action functor $\mathbf{act} : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$. Recall that **coact** has a natural $\mathcal{A}$-linear structure, where $\mathcal{A}$ acts on the target via its left action on $\mathcal{A}^{\vee}$. Similarly, the right action of $\mathcal{A}$ on $\mathcal{M}^{\vee}$ gives another functor $\mathbf{coact} : \mathcal{M}^{\vee} \to \mathcal{M}^{\vee} \otimes \mathcal{A}^{\vee}$, which has a natural $\mathcal{A}^{\mathrm{rev}}$-linear structure. Moreover, by construction, we have the following canonical commutative diagram:

$$\begin{tikzcd} \mathcal{M} \otimes \mathcal{M}^{\vee} \arrow[r, "\mathbf{Id}\otimes\mathbf{coact}"] \arrow[d, "\mathbf{coact}\otimes\mathbf{Id}"] & \mathcal{M} \otimes \mathcal{M}^{\vee} \otimes \mathcal{A}^{\vee} \arrow[d, "e\otimes\mathbf{Id}"] \\ \mathcal{A}^{\vee} \otimes \mathcal{M} \otimes \mathcal{M}^{\vee} \arrow[r, "\mathbf{Id}\otimes e"] & \mathcal{A}^{\vee}. \end{tikzcd} \tag{B.1}$$

Hence the functor from the left-top corner to the right-bottom corner has a natural $(\mathcal{A},\mathcal{A})$-linear structure, which is declared to be the counit functor for the desired duality.

It remains to check the axioms for duality, which reduces to (ii) by a routine diagram chasing.

□[Second proof of Lemma B.3.3]

Remark B.3.8. We do *not* know whether Lemma B.3.3 holds in the constructible contexts because of failure of knowing (ii).

Appendix C

Theory of D-modules on prestacks

In this appendix, we review the theory of D-modules on (infinite type) prestacks. We only claim originality for results in § C.2.

Remark C.0.1. All the D-module theories considered in this appendix are insensitive to nil-isomorphisms.

C.1 The !-theory and $*$-theory on lft prestacks

C.1.1. Recall that we have a symmetric monoidal functor

$$\mathrm{DMod}: (\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat},\ Y \mapsto \mathrm{DMod}(Y),\ (f: Y_1 \to Y_2) \mapsto (f^!: \mathrm{DMod}(Y_2) \to \mathrm{DMod}(Y_1)),$$

where $\mathrm{DMod}(Y)$ is the DG-categories of D-modules on Y. The symmetric monoidal structure mentioned above is given by the equivalences $\boxtimes: \mathrm{DMod}(Y_1) \otimes \mathrm{DMod}(Y_2) \simeq \mathrm{DMod}(Y_1 \times Y_2)$, which we refer as the *product formula*. As in [Gai18, § 1.2.3], the functor $\mathrm{DMod}_{!\text{-pull}}$ encodes not only the !-pullback functors, but also the $*$-pushforward ones. Moreover, they can be extended and assembled into a 2-functor

$$\mathrm{DMod}: \mathbf{Corr}(\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{open},2\text{-op}}_{\mathrm{all},\mathrm{all}} \to \mathbf{DGCat} \tag{C.1}$$

that also encodes the base-change isomorphisms.

We refer the reader to [GR17a, Chapter 7] for the theory of categories of correspondences. Roughly speaking, for a category $\mathcal{C}$ and three classes $vert, hori, adm$ of morphisms satisfying certain properties, one can define an $(\infty, 2)$-category $\mathbf{Corr}(\mathcal{C})^{adm}_{vert,hori}$

- whose objects are objects in $\mathcal{C}$;

- whose category of morphisms $\mathbf{Maps}_{\mathbf{Corr}}(c_1, c_2)$ is the category

  – whose objects are diagrams $c_2 \leftarrow d \to c_1$ such that $c_2 \leftarrow d$ (resp. $d \to c_1$) is contained in *vert* (resp. *hori*)[103];

  – whose space of morphisms from $(c_2 \leftarrow d \to c_1)$ to $c_2 \leftarrow d' \to c_1$ is the space of maps $d \to d'$ in *adm* that are defined over c_2 and c_1;

- the composition map

$$\mathbf{Maps}_{\mathbf{Corr}}(c_1, c_2) \times \mathbf{Maps}_{\mathbf{Corr}}(c_2, c_3) \to \mathbf{Maps}_{\mathbf{Corr}}(c_1, c_3)$$

  sends $c_2 \leftarrow d \to c_1$ and $c_3 \leftarrow e \to c_2$ to

$$c_3 \leftarrow e \underset{c_2}{\times} d \to c_1.$$

When $adm = isom$ is the class of isomorphisms, we also write $\mathrm{Corr}(\mathcal{C})_{vert,hori} \coloneqq \mathbf{Corr}(\mathcal{C})^{isom}_{vert,hori}$, which is an $(\infty, 1)$-category.

Note that there are faithful functors $\mathcal{C}_{vert} \to \mathrm{Corr}(\mathcal{C})_{vert,hori}$ and $\mathcal{C}^{\mathrm{op}}_{hori} \to \mathrm{Corr}(\mathcal{C})_{vert,hori}$. Then a functor $\Phi : \mathrm{Corr}(\mathcal{C})_{vert,hori} \to \mathrm{DGCat}$ encodes the following data[104]:

- An assignment $c \in \mathcal{C} \rightsquigarrow \Phi(c) \in \mathrm{DGCat}$, which is covariant for morphisms in *vert*, contravariant for morphisms in *hori*. For $f : c_1 \to c_2$ in *vert* (resp. *hori*), we denote the corresponding functor by $\Phi(f)_{\mathrm{push}} : \Phi(c_1) \to \Phi(c_2)$ (resp. $\Phi(f)_{\mathrm{pull}} : \Phi(c_2) \to \Phi(c_1)$), and call them the pushforward functor (resp. the pullback functor) along f.

- *Base-change isomorphisms* between the pushforward functors and the pullback functors whenever they are defined.

A 2-functor $\Phi : \mathbf{Corr}(\mathcal{C})^{adm,2\text{-op}}_{vert,hori} \to \mathbf{DGCat}$ encodes the above data and

- a natural transformation $\mathrm{Id}_{\Phi(d')} \to \Phi(f)_{\mathrm{push}} \circ \Phi(f)_{\mathrm{pull}}$ for any $f : d \to d'$ in *adm*.

For the functor (C.1), the above natural transformation is provided by the adjoint pair $(f^!, f_*)$ because f is an open embedding.

---

[103]In the literature, the above diagram is often drawn as
$$\begin{tikzcd} d \arrow[r] \arrow[d] & c_1 \\ c_2 & \end{tikzcd}$$
, which is less eco-friendly than our convention.

[104]The data below should be compatible homotopy-coherently. On the other hand, if the readers do not worry about homotopy-coherence, they can ignore the appearance of **Corr** in this thesis.

C.1.2. Let $\mathrm{IndSch}_{\mathrm{ift}}$ be the category of indschemes of ind-finite type. Using [GR17a, Chapter 8, Theorem 1.1.9] and [GR17a, Chapter 9], there is a symmetric monoidal functor

$$\mathrm{DMod} : \mathrm{Corr}(\mathrm{IndSch}_{\mathrm{ift}})_{\mathrm{all,all}} \to \mathrm{DGCat} \tag{C.2}$$

extending the functor (C.1), such that

- the restriction $\mathrm{DMod}|_{(\mathrm{IndSch}_{\mathrm{ift}})^{\mathrm{op}}}$ is the right Kan extension of $\mathrm{DMod}|_{(\mathrm{Sch}_{\mathrm{ft}})^{\mathrm{op}}}$;
- the restriction $\mathrm{DMod}|_{\mathrm{IndSch}_{\mathrm{ift}}}$ is the left Kan extension of $\mathrm{DMod}|_{\mathrm{Sch}_{\mathrm{ft}}}$.

In other words, for an ind-finite type indscheme $Y = \mathrm{colim} Y_\alpha$, we have

$$\mathrm{DMod}(Y) \simeq \underset{*\text{-push}}{\mathrm{colim}}\, \mathrm{DMod}(Y_\alpha) \simeq \underset{!\text{-pull}}{\lim}\, \mathrm{DMod}(Y_\alpha).$$

C.1.3. Unlike the case of indschemes, even for lft prestacks, there are two different theories of D-modules. Let *indsch* be the class of morphisms in $\mathrm{PreStk}_{\mathrm{lft}}$ that are ind-schematic. Using [GR17b, Chapter 4, Theorem 2.1.2], there are *right-lax* symmetric monoidal functors

$$\mathrm{DMod}^! : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{indsch,all}} \to \mathrm{DGCat} \tag{C.3}$$

$$\mathrm{DMod}^* : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,indsch}} \to \mathrm{DGCat} \tag{C.4}$$

extending the functor (C.2), such that

- the restriction $\mathrm{DMod}^!|_{(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}}}$ is the right Kan extension of $\mathrm{DMod}|_{(\mathrm{IndSch}_{\mathrm{ift}})^{\mathrm{op}}}$;
- the restriction $\mathrm{DMod}^*|_{\mathrm{PreStk}_{\mathrm{lft}}}$ is the left Kan extension of $\mathrm{DMod}|_{\mathrm{IndSch}_{\mathrm{ift}}}$.

In particular, for a lft prestack $Y = \mathrm{colim} Y_\alpha$ with $Y_\alpha \in \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we have

$$\mathrm{DMod}^!(Y) \simeq \underset{!\text{-pull}}{\lim}\, \mathrm{DMod}(Y_\alpha),\ \mathrm{DMod}^*(Y) \simeq \underset{*\text{-push}}{\mathrm{colim}}\, \mathrm{DMod}(Y_\alpha).$$

Remark C.1.4. It is easy to verify that in both theories, the pullback functor along a schematic open embedding is canonically left adjoint to the pushforward functor. Hence by [GR17a, Chapter 7, § 4], there are canonically 2-functors

$$\mathrm{DMod}^! : \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open,2\text{-}op}}_{\mathrm{indsch,all}} \to \mathbf{DGCat}$$

$$\mathrm{DMod}^* : \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open,2\text{-}op}}_{\mathrm{all,indsch}} \to \mathbf{DGCat}$$

C.2 The !-theory and $*$-theory on finite type algebraic stacks with affine diagonals

As explained in § C.1, for general lft prestack Y, we have two categories, or theories, of D-modules on Y. We have already seen that they are equivalent when Y is an ind-finite type indscheme. In this section, we prove this is also correct when Y is a finite type algebraic stack with an affine diagonal.

Construction C.2.1. Let Y be a lft prestack such that the diagonal map $Y \to Y \times Y$ is quasi-compact and schematic. Then for any finite type scheme S over Y, the map $S \to Y$ is quasi-compact and schematic. Consider the $*$-pushforward functors for the $\mathrm{DMod}^!$-theory: $\mathrm{DMod}(S) \to \mathrm{DMod}^!(Y)$. By taking colimits, we obtain a canonical functor

$$\theta_Y : \mathrm{DMod}^*(Y) \simeq \operatorname*{colim}_S \mathrm{DMod}(S) \to \mathrm{DMod}^!(Y).$$

The following proposition will be proved in the next section:

Theorem C.2.2. *If Y is a finite type algebraic stack with an affine diagonal, then θ_Y is an equivalence.*

Warning C.2.3. Similar statement is *false* if Y is only locally finite type. For instance, θ_{Bun_G} is not an equivalence. In fact, it is easy to see θ_{Bun_G} is equivalent to the functor $\text{Ps-Id}_{\mathrm{Bun}_G,\mathrm{naive}}$.

Corollary C.2.4. *Let $\mathrm{AlgStk}_{\mathrm{ft,ad}}$ be the category of finite type algebraic stacks with affine diagonals and $(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch}}$ be the 1-full subcategory containing only schematic morphisms. Then there is a canonical isomorphism*

$$\mathrm{DMod}^! |_{(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch}}} \simeq \mathrm{DMod}^* |_{(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch}}}.$$

Construction C.2.5. For any morphism $f : Y_1 \to Y_2$ in $\mathrm{AlgStk}_{\mathrm{ft,ad}}$, we define the $\blacktriangle$-*pushforward functor* $f_\blacktriangle$ (for the $\mathrm{DMod}^!$-theory), a.k.a. the *renormalized de-Rham pushforward functor*, to be the unique functor fitting into the following commutative diagram:

$$\begin{tikzcd}
\mathrm{DMod}^*(Y_1) \arrow[r, "\theta_{Y_1}", "\simeq"'] \arrow[d, "f_*"] & \mathrm{DMod}^!(Y_1) \arrow[d, "f_\blacktriangle"] \\
\mathrm{DMod}^*(Y_2) \arrow[r, "\theta_{Y_2}", "\simeq"'] & \mathrm{DMod}^!(Y_2).
\end{tikzcd}$$

Remark C.2.6. In [DG13], the $\blacktriangle$-pushforward functors are defined for any maps between (finite type) *QCA algebraic stacks*[105]. It is easy to see when restricted to $\mathrm{AlgStk}_{\mathrm{ft,ad}} \subset \mathrm{AlgStk}_{\mathrm{QCA}}$, the two definitions

[105]QCA is shorthand for "quasi-compact and with affine automorphism groups".

coincide.

Warning C.2.7. We warn that $f_{\blacktriangle}$ is *not* equivalent to the usual de-Rham pushforward functor in the literature. The latter is not continuous in general. However, if f is *safe*[106], then it is shown in [DG13] that $f_{\blacktriangle}$ can be canonically identified with the usual de-Rham pushforward functor f_*. Therefore we keep the notation f_* and only use $f_{\blacktriangle}$ for non-safe map f.

Proposition-Construction C.2.8. *There exists a canonical symmetric monoidal functor*

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} : \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,all}} \to \mathrm{DGCat},\ Y \mapsto \mathrm{DMod}(Y) \tag{C.5}$$

such that

- *its restriction on* $\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch,all}}$ *is equivalent to the restriction of the* $\mathrm{DMod}^!$*-theory (C.3);*
- *its restriction on* $\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,sch}}$ *is equivalent to the restriction of the* DMod^**-theory (C.4).*

Notation C.2.9. For $Y \in \mathrm{AlgStk}_{\mathrm{ft,ad}}$, we no longer distinguish $\mathrm{DMod}^!(Y)$ and $\mathrm{DMod}^*(Y)$.

Remark C.2.10. The symmetric monoidal structure follows from [DG13, Corollary 8.3.4]. The base-change isomorphisms between $\blacktriangle$-pushforward functors and !-pullbacks have been constructed in [DG13, Proposition 9.3.12]. However, we do not know how to construct the higher compatibilities using *their* methods. On the other hand, our definition of the $\blacktriangle$-pushforward functors is more adapted to problems on homotopy-coherence. The only caveat is that we have to work with a slightly smaller class of algebraic stacks.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}^1 := \mathrm{Corr}(\mathrm{Sch}_{\mathrm{ft}})_{\mathrm{all,all}} \arrow[r, "a"] \arrow[d, "c"] & \mathcal{C}^2 := \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch,all}} \arrow[d, "d"] \\
\mathcal{C}^3 := \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,sch}} \arrow[r, "b"] & \mathcal{C}^4 := \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,all}}.
\end{tikzcd}$$

By definition,

$$\mathrm{DMod}^!|_{\mathcal{C}^2} \simeq \mathrm{RKE}_a(\mathrm{DMod}),\ \mathrm{DMod}^*|_{\mathcal{C}^3} \simeq \mathrm{LKE}_c(\mathrm{DMod}).$$

We define the desired functor to be

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} := \mathrm{LKE}_d \circ \mathrm{RKE}_a(\mathrm{DMod}).$$

[106] A QCA algebraic stack is safe if the automorphism groups are unipotent. The notion of safe morphisms is defined in the usual way. For example, the map $\mathbb{B}B \to \mathbb{B}T$ is safe, while $\mathbb{B}T \to \mathrm{pt}$ is not safe.

We first show the canonical natural transformation $\mathrm{DMod}^!|_{\mathcal{C}^2} \to \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}|_{\mathcal{C}^2}$, i.e.,

$$\mathrm{RKE}_a(\mathrm{DMod}) \to [\mathrm{LKE}_d \circ \mathrm{RKE}_a(\mathrm{DMod})] \circ d,$$

is invertible. We only need to prove for any $Y \in \mathcal{C}^2$, the insertion

$$\mathrm{DMod}^!(Y) \to \operatorname*{colim}_{(\mathcal{C}^2)_{/d(Y)}} \mathrm{DMod}^! \tag{C.6}$$

is an equivalence, where $(\mathcal{C}^2)_{/d(Y)}$ is the comma category[107]. Consider the faithful functor $v^j : \mathcal{D}^j \to \mathcal{C}^j$ given by only "remembering" the vertical arrows. For example,

$$v^2 : (\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch}} \to \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{sch,all}}.$$

We can pick $Z \in \mathcal{D}^2$ such that $Y = v^2(Z)$. Let $d_v : \mathcal{D}^2 \to \mathcal{D}^4$ be the functor induced by d. Consider the functor $(\mathcal{D}^2)_{/d_v(Z)} \to (\mathcal{C}^2)_{/d(Y)}$. We claim it is (co)final, i.e., induces isomorphisms between colimits indexed by the source and the target. To prove the claim, we only need to show it has a left adjoint, but this is obvious after unwinding the definitions[108]. Using this claim, the functor (C.6) can be replaced by

$$\mathrm{DMod}^!(Z) \to \operatorname*{colim}_{(\mathcal{D}^2)_{/d_v(Z)}} \mathrm{DMod}^!.$$

Hence we only need to show $\mathrm{DMod}^!|_{\mathcal{D}^2} \simeq [\mathrm{LKE}_{d_v}(\mathrm{DMod}^!|_{\mathcal{D}^2})] \circ d_v$. Note that the functor $b_v : \mathcal{D}^3 \to \mathcal{D}^4$ is invertible, which allows us to consider $\mathrm{DMod}^*|_{\mathcal{D}^2}$. By Corollary C.2.4, we only need to prove $\mathrm{DMod}^*|_{\mathcal{D}^2} \simeq [\mathrm{LKE}_{d_v}(\mathrm{DMod}^*|_{\mathcal{D}^2})] \circ d_v$. Recall by definition $\mathrm{DMod}^*|_{\mathcal{D}^2} \simeq \mathrm{LKE}_{a_v}(\mathrm{DMod}|_{\mathcal{D}^1})$. Hence we only need to show $\mathrm{LKE}_{a_v}(\mathrm{DMod}|_{\mathcal{D}^1}) \simeq \mathrm{LKE}_{d_v \circ a_v}(\mathrm{DMod}|_{\mathcal{D}^1}) \circ d_v$. But this is obvious because the comma categories $(\mathcal{D}^1)_{/Z}$ and $(\mathcal{D}^1)_{/d_v(Z)}$ are equivalent[109].

Note that we have already proved $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} \circ (b \circ c) \simeq \mathrm{DMod}$. Hence we have a canonical natural transformation

$$\mathrm{DMod}^*|_{\mathcal{C}^3} \simeq \mathrm{LKE}_c(\mathrm{DMod}) \to \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}} \circ b.$$

---

[107] An object in $(\mathcal{C}^2)_{/d(Y)}$ is a pair of an object $S \in \mathcal{C}^2$ and a morphism $d(S) \to d(Y)$ in $\mathcal{C}^4$.

[108] In details, an object in $(\mathcal{C}^2)_{/d(Y)}$ is a diagram $(Z \leftarrow S \to T)$ in $\mathrm{AlgStk}_{\mathrm{ft,ad}}$, and a morphism from $(Z \leftarrow S_1 \to T_1)$ to $(Z \leftarrow S_2 \to T_2)$ is a diagram $(T_2 \leftarrow R \to T_1)$ and an isomorphism $S_2 \times_{T_2} R \simeq S_1$ defined over $Z \times T_1$ such that $R \to T_2$ is *schematic*. The category $(\mathcal{D}^2)_{/d_v(Z)}$ is the *full* subcategory of $(\mathcal{C}^2)_{/d(Y)}$ consisting of diagrams $(Z \leftarrow S \to T)$ such that $S \to T$ is an isomorphism. The desired left adjoint sends a diagram $(Z \leftarrow S \to T)$ to $(Z \leftarrow S \to S)$.

[109] This is because a map from a scheme to Z is schematic.

We claim it is invertible. As before, we only need to show

$$\mathrm{LKE}_{c_v}(\mathrm{DMod}|_{\mathcal{D}^1}) \to \mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}|_{\mathcal{D}^3}$$

is invertible. Recall we have already proved

$$\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}|_{\mathcal{D}^4} \simeq \mathrm{LKE}_{d_v}(\mathrm{DMod}^!|_{\mathcal{D}^2}) \simeq \mathrm{LKE}_{d_v}(\mathrm{DMod}^*|_{\mathcal{D}^2}) \simeq \mathrm{LKE}_{d_v\circ a_v}(\mathrm{DMod}|_{\mathcal{D}^1}).$$

Then we are done because $d_v \circ a_v = c_v \circ b_v$ and b_v is invertible.

So far we have verified $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}$ has the desired restrictions. To finish the proof, we only need to show the canonical right-lax symmetric monoidal structure on it is strict. But this was proved in [DG13, Corollary 8.3.4].

□[Proposition-Construction C.2.8]

Definition C.2.11. We say an lft prestack Y is *QCAD*[110] if it is a finite type algebraic stack with an affine diagonal. We say a morphism $Y \to Z$ between lft prestacks is *QCAD* if for any finite type affine test scheme S over Z, the fiber product $S \times_Z Y$ is QCAD.

Fact C.2.12. *It is easy to verify the following facts.*

(1) The composition of two QCAD morphisms is QCAD.

(2) Let $f: Y \to Z$ be a map between lft algebraic stacks. If Y is QCAD, then f is QCAD.

Construction C.2.13. By right Kan extension along

$$\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{QCAD,all}},$$

the functor (C.5) induces a functor

$$\mathrm{DMod}^!_{\blacktriangle\text{-push},!\text{-pull}} : \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{QCAD,all}} \to \mathrm{DGCat}.$$

It is easy to see

- its restriction on $\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{ft,ad}})_{\mathrm{all,all}}$ is equivalent to $\mathrm{DMod}_{\blacktriangle\text{-push},!\text{-pull}}$;
- its restriction on $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{sch,all}}$ is equivalent to the restriction of the $\mathrm{DMod}^!$-theory.

[110] QCAD is shorthand for "quasi-compact and with an affine diagonal".

As in Remark C.1.4, we further obtain a 2-functor

$$\mathrm{DMod}^!_{\blacktriangle\text{-push},!\text{-pull}} : \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open},2\text{-op}}_{\mathrm{QCAD},\mathrm{all}} \to \mathbf{DGCat}. \tag{C.7}$$

The content of this 2-functor is:

- for any lft prestack Y, there is a DG-category $\mathrm{DMod}^!(Y)$, which is the !-theory of D-modules on Y;
- for any morphism $f : Y_1 \to Y_2$, there is a *!-pullback functor* $f^!$;
- for any QCAD morphism $f : Y_1 \to Y_2$, there is a *renormalized pushforward functor* $f_\blacktriangle$ defined in [DG13];
- there are base-change isomorphisms for these !-pullback and $\blacktriangle$-pushforward functors;
- for any schematic open embedding $f : Y_1 \to Y_2$, there is an adjoint pair $(f^!, f_\blacktriangle)$;
- there are certain higher compatibilities for the above data.

C.3 Proof of Theorem C.2.2

C.3.1. We use the theory of *sheaves of categories* developed in [Gai15b]. Let Z be any lft prestack. Recall a sheaf of categories $\mathcal{C}$ over Z is an assignment $S \rightsquigarrow \mathbf{\Gamma}(S,\mathcal{C}) \in \mathrm{QCoh}(S)\text{-mod}$ for any $S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Z}$ equipped with equivalences

$$\mathbf{\Gamma}(S_1,\mathcal{C}) \simeq \mathbf{\Gamma}(S_2,\mathcal{C}) \underset{\mathrm{QCoh}(S_2)}{\otimes} \mathrm{QCoh}(S_1), \text{ for } S_1 \to S_2 \text{ in } (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Z}$$

and certain higher compatibilities. Let $\mathrm{ShvCat}(Z)$ be the category of sheaves of categories on Z. Recall there are adjoint functors

$$\mathbf{Loc}_Z : \mathrm{QCoh}(Z)\text{-mod} \rightleftharpoons \mathrm{ShvCat}(Z) : \mathbf{\Gamma}_Z,$$

where $\mathbf{Loc}_Z(\mathcal{M})$ is the sheaf defined by

$$\mathbf{\Gamma}(S, \mathbf{Loc}_Z(\mathcal{M})) \coloneqq \mathcal{M} \underset{\mathrm{QCoh}(Z)}{\otimes} \mathrm{QCoh}(S),$$

and

$$\mathbf{\Gamma}_Z(\mathcal{C}) \coloneqq \mathbf{\Gamma}(Z,\mathcal{C}) \coloneqq \lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Z}} \mathbf{\Gamma}(S,\mathcal{C})$$

is equipped with the canonical $\mathrm{QCoh}(Z)$-module structure.

Let Y_{dR} be the de-Rham prestack of Y, i.e., $Y_{\mathrm{dR}}(S) = Y(S^{\mathrm{red}})$ for any affine test scheme S. Recall we have a symmtric monoidal equivalence $\mathrm{DMod}^!(Y_{\mathrm{dR}}) \simeq \mathrm{QCoh}(Y_{\mathrm{dR}})$.

Theorem C.3.2. *([Gai15b, Conjecture 2.6.6]) If Y is a finite type algebraic stack with an affine diagonal, then $\mathbf{Loc}_{Y_{\mathrm{dR}}}$ is fully faithful.*

We will prove Theorem C.3.2 at the end of this section. Let us first deduce Theorem C.2.2 from it. We also need the following fact:

Fact C.3.3. *([Gai15b, Proposition 3.1.9]) Let Z be a lft prestack such that $\mathbf{Loc}_Z$ is fully faithful. Then for any lft prestack Z' over Z and $T \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Z}$, the canonical functor*

$$\mathrm{QCoh}(Z') \underset{\mathrm{QCoh}(Z)}{\otimes} \mathrm{QCoh}(T) \to \mathrm{QCoh}(Z' \underset{Z}{\times} T)$$

is an equivalence.

C.3.4 (Proof of Theorem C.2.2)**.** It is easy to see $\mathrm{DMod}^*(Y) \simeq \mathrm{colim}_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} \mathrm{DMod}(S)$ has a canonical $\mathrm{DMod}^!(Y)$-module structure and the functor $\theta_Y : \mathrm{DMod}^*(Y) \to \mathrm{DMod}^!(Y)$ has a canonical $\mathrm{DMod}^!(Y)$-module. Hence by Theorem C.3.2, we only need to prove $\mathbf{Loc}_{Y_{\mathrm{dR}}}(\theta_Y)$ is invertible. In other words, for any $T \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}$, we need to prove

$$\underset{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}}{\mathrm{colim}} \mathrm{QCoh}(S_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(T) \to \mathrm{QCoh}(T)$$

is invertible. Using Fact C.3.3 twice, we have

$$\mathrm{QCoh}(S_{\mathrm{dR}}) \underset{\mathrm{QCoh}(Y_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(T) \simeq \mathrm{QCoh}(S_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} T) \simeq \mathrm{QCoh}(S_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} T_{\mathrm{dR}}) \underset{\mathrm{QCoh}(T_{\mathrm{dR}})}{\otimes} \mathrm{QCoh}(T),$$

where the second equivalence is because T_{dR} is 1-affine, i.e. $\mathbf{Loc}_Z$ is an equivalence, by [Gai15b, Theorem 2.6.4]. Hence we only need to prove

$$\underset{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}}{\mathrm{colim}} \mathrm{QCoh}(S_{\mathrm{dR}} \underset{Y_{\mathrm{dR}}}{\times} T_{\mathrm{dR}}) \to \mathrm{QCoh}(T_{\mathrm{dR}})$$

is invertible. Note that $S \times_Y T$ is affine and the above functor is just the functor

$$\underset{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}}{\mathrm{colim}} \mathrm{DMod}(S \underset{Y}{\times} T) \to \mathrm{DMod}(T)$$

induced by $*$-pushforward functors. Note that in $\mathrm{PreStk}_{\mathrm{lft}} = \mathrm{PShv}(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})$, we have $Y \simeq \mathrm{colim}_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} S$

and hence

$$T = \operatorname*{colim}_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} S \underset{Y}{\times} T,$$

where the colimit is taken in $\mathrm{PreStk}_{\mathrm{lft}}$ but actually contained in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Now we are done by the following general fact (see [Lur09, Theorem 5.1.5.6])[111]: if a colimit diagram in a (small) $(\infty,1)$-category $\mathcal{C}$ is preserved by the Yoneda embedding $\iota : \mathcal{C} \to \mathrm{PShv}(\mathcal{C})$, then it is preserved by any functor $F : \mathcal{C} \to \mathcal{D}$ as long as (small) colimit exists in $\mathcal{D}$.

□[Theorem C.2.2]

The rest of this section is devoted to the proof of Theorem C.3.2. For any $\mathcal{M} \in \mathrm{QCoh}(Y_{\mathrm{dR}})$-mod, we need to show $\mathbf{\Gamma}_{Y_{\mathrm{dR}}} \circ \mathbf{Loc}(\mathcal{M}) \to \mathcal{M}$ is an equivalence. Let $Z \to Y$ be a smooth atlas of Y and denote the Čech nerve by $Z^{\bullet}$. We first show

Lemma C.3.5. *In the above setting, for any $\mathcal{C} \in \mathrm{ShvCat}(Y_{\mathrm{dR}})$, the canonical functor*

$$\mathbf{\Gamma}(Y_{\mathrm{dR}}, \mathcal{C}) \to \mathrm{Tot}\,\mathbf{\Gamma}(Z^{\bullet}_{\mathrm{dR}}, \mathcal{C})$$

is an equivalence.

Proof. Let us first prove the claim under the assumption that Y is a finite type separated scheme. Since Y_{dR} is 1-affine, we only need to prove that for any $\mathcal{M} \in \mathrm{DMod}(Y)$-mod, the !-pullback functors induce an equivalence

$$\mathcal{M} \to \mathrm{Tot}(\mathcal{M} \underset{\mathrm{DMod}(Y)}{\otimes} \mathrm{DMod}(Z^{\bullet})). \tag{C.8}$$

By Lemma B.3.3, $\mathrm{DMod}(Z^{\bullet})$ is self-dual as $\mathrm{DMod}(Y)$-module category. Hence we have

$$\mathrm{Tot}(\mathcal{M} \underset{\mathrm{DMod}(Y)}{\otimes} \mathrm{DMod}(Z^{\bullet})) \simeq \mathrm{Tot}\,\mathrm{Funct}_{\mathrm{DMod}(Y)}(\mathrm{DMod}(Z^{\bullet}), \mathcal{M}) \simeq \mathrm{Funct}_{\mathrm{DMod}(Y)}(\mathrm{colim}\,\mathrm{DMod}(Z^{\bullet}), \mathcal{M})$$

where the connecting functors in the colimit are induced by the $*$-pushforward functors along $Z^m \to Z^n$. By [Ras15b, Variant 3.23.2], we have $\mathrm{colim}\,\mathrm{DMod}(Z^{\bullet}) \simeq \mathrm{DMod}(Y)$. Hence we obtain an equivalence

$$\mathrm{Tot}(\mathcal{M} \underset{\mathrm{DMod}(Y)}{\otimes} \mathrm{DMod}(Z^{\bullet})) \simeq \mathcal{M},$$

which is the inverse to the functor (C.8).

[111] A set-theoretic remark: we need to enlarge the universe in order to apply the following fact to $\mathcal{C} := \mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$.

Now we prove the general case. Note that $Y_{\mathrm{dR}} = \mathrm{colim}_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} S_{\mathrm{dR}}$. Applying the above special case to the cover $Z \times_Y S \to S$, we have

$$\begin{aligned}
\mathbf{\Gamma}(Y_{\mathrm{dR}},\mathcal{C}) &\simeq \lim_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} \mathbf{\Gamma}(S_{\mathrm{dR}},\mathcal{C})\\
&\simeq \lim_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} \mathrm{Tot}\,\mathbf{\Gamma}((Z^\bullet \underset{Y}{\times} S)_{\mathrm{dR}},\mathcal{C})\\
&\simeq \mathrm{Tot} \lim_{S\in(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/Y}} \mathbf{\Gamma}((Z^\bullet \underset{Y}{\times} S)_{\mathrm{dR}},\mathcal{C})\\
&\simeq \mathrm{Tot}\,\mathbf{\Gamma}(Z^\bullet_{\mathrm{dR}},\mathcal{C})
\end{aligned}$$

as desired.

□[Lemma C.3.5]

By the above lemma, to prove Theorem C.3.2, we only need to show

$$\mathcal{M} \to \mathrm{Tot}(\mathcal{M} \underset{\mathrm{DMod}^!(Y)}{\otimes} \mathrm{DMod}^!(Z^\bullet))$$

is an equivalence. To prove it, we use the theory of ind-coherent sheaves developed in [GR17b]. Consider the canonical map $\pi : Y \to Y_{\mathrm{dR}}$. Using the language of *loc.cit.*, it is an ind-inf-schematic nil-isomorphism. Hence we have a monadic adjoint pair[112]

$$\pi^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(Y) \rightleftharpoons \mathrm{IndCoh}(Y_{\mathrm{dR}}) : \pi^{!,\mathrm{IndCoh}}.$$

Recall we also have a canonical equivalence $\mathrm{DMod}^!(Y) \simeq \mathrm{IndCoh}(Y_{\mathrm{dR}})$. Via this equivalence, the above monadic pair is just

$$\mathbf{ind}_{\mathrm{DMod}(Y)} : \mathrm{IndCoh}(Y) \rightleftharpoons \mathrm{DMod}^!(Y) : \mathbf{oblv}_{\mathrm{DMod}(Y)}.$$

Note that $\mathbf{oblv}_{\mathrm{DMod}(Y)}$ has a symmetric monoidal structure, which allows us to view $\mathrm{IndCoh}(Y)$ as a $\mathrm{DMod}^!(Y)$-module such that $\mathbf{oblv}_{\mathrm{DMod}(Y)}$ is $\mathrm{DMod}^!(Y)$-linear. Also, the projection formula for ind-coherent sheaves implies the left-lax $\mathrm{DMod}^!(Y)$-linear functor $\mathbf{ind}_{\mathrm{DMod}(Y)}$ is strict[113]. Hence we have an adjoint pair

$$\mathbf{Id}\otimes\pi^{\mathrm{IndCoh}}_* : \mathcal{M} \underset{\mathrm{DMod}^!(Y)}{\otimes} \mathrm{IndCoh}(Y) \rightleftharpoons \mathcal{M} \underset{\mathrm{DMod}^!(Y)}{\otimes} \mathrm{IndCoh}(Y_{\mathrm{dR}}) : \mathbf{Id}\otimes\pi^{!,\mathrm{IndCoh}}.$$

Note that it is also a monadic pair because the left adjoint generates the target category. By a similar

[112] Since the right adjoint $\pi^{!,\mathrm{IndCoh}}$ is continuous, this monadicity is equivalent to $\pi^{!,\mathrm{IndCoh}}$ being conservative, which is also equivalent to the image of π^{IndCoh}_* generating the target.

[113] This was also proved in [DG13, Lemma 6.3.20] using more elementary methods.

argument, we also have monadic adjoint pairs

$$\mathbf{Id}\otimes\pi_*^{\bullet,\mathrm{IndCoh}}:\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Z^\bullet_{\mathrm{dR}}\underset{Y_{\mathrm{dR}}}{\times}Y)\rightleftharpoons\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{DMod}^!(Z^\bullet_{\mathrm{dR}}):\mathbf{Id}\otimes\pi^{\bullet,!,\mathrm{IndCoh}},$$

where $\pi^\bullet: Z^\bullet_{\mathrm{dR}}\times_{Y_{\mathrm{dR}}}Y\to Z^\bullet_{\mathrm{dR}}$ is the canonical projection. Moreover, the base-change formula for ind-coherent sheaves implies these monadic adjoint pairs are compatible with the simplicial structure and the augmentations. In other words, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}\otimes_{\mathrm{DMod}^!(Y)}\mathrm{IndCoh}(Y) \arrow[r] \arrow[d, shift right] & \mathrm{Tot}(\mathcal{M}\otimes_{\mathrm{DMod}^!(Y)}\mathrm{IndCoh}(Z^\bullet_{\mathrm{dR}}\times_{Y_{\mathrm{dR}}}Y)) \arrow[d, shift right] \\
\mathcal{M}\otimes_{\mathrm{DMod}^!(Y)}\mathrm{IndCoh}(Y_{\mathrm{dR}}) \arrow[r] \arrow[u, shift right] & \mathrm{Tot}(\mathcal{M}\otimes_{\mathrm{DMod}^!(Y)}\mathrm{IndCoh}(Y_{\mathrm{dR}})) \arrow[u, shift right]
\end{tikzcd}$$

such that both (vertical) adjoint pairs are monadic. Suppose we know the top functor is an equivalence. Then it is easy to see the above two monads are intertwined by this equivalence. Hence in order to prove the bottom functor is an equivalence, we only need to prove the top functor

$$\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Y)\to\mathrm{Tot}(\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Z^\bullet_{\mathrm{dR}}\underset{Y_{\mathrm{dR}}}{\times}Y))\tag{C.9}$$

is an equivalence.

Recall there is a canonical symmetric monoidal functor $\Upsilon:\mathrm{QCoh}(Y)\to\mathrm{IndCoh}(Y)$. Hence (C.9) has a canonical $\mathrm{QCoh}(Y)$-linear structure. By [Gai15b, Theorem 2.2.6], Y is 1-affine[114]. Hence the functor

$$\mathrm{QCoh}(Y)\text{-mod}\to\mathrm{QCoh}(Z)\text{-mod},\ \mathcal{N}\mapsto\mathcal{N}\underset{\mathrm{QCoh}(Y)}{\otimes}\mathrm{QCoh}(Z)\tag{C.10}$$

commutes with limits. Also, by [Gai15b, Corollary 1.5.5(a)], this functor is conservative. Hence we only need to prove

$$\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Y)\underset{\mathrm{QCoh}(Y)}{\otimes}\mathrm{QCoh}(Z)\to\mathrm{Tot}(\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Z^\bullet_{\mathrm{dR}}\underset{Y_{\mathrm{dR}}}{\times}Y)\underset{\mathrm{QCoh}(Y)}{\otimes}\mathrm{QCoh}(Z))$$

is an equivalence. By Lemma C.3.6 below, the above functor is just

$$\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Z)\to\mathrm{Tot}(\mathcal{M}\underset{\mathrm{DMod}^!(Y)}{\otimes}\mathrm{IndCoh}(Z^\bullet_{\mathrm{dR}}\underset{Y_{\mathrm{dR}}}{\times}Z)).$$

Now we are done because the above augmented cosimplicial diagram splits. In fact, the augmented simplicial

[114] Essentially this is the only place where we need to use the assumption that Y has an affine diagonal.

diagram $Z^\bullet_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Z \to Z$ already splits.

$\square$[Theorem C.3.2]

Lemma C.3.6. *In the above setting, for any lft prestack W over Y, the canonical functor*

$$\mathrm{IndCoh}(W) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(W \underset{Y}{\times} Z)$$

is an equivalence.

Proof. We have

$$\begin{aligned}
\mathrm{IndCoh}(W) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Z) &\simeq (\lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/W}} \mathrm{IndCoh}(S)) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Z) \\
&\simeq \lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/W}} (\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Z)) \\
&\simeq \lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/W}} (\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Z)) \\
&\simeq \lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/W}} (\mathrm{IndCoh}(S) \underset{\mathrm{QCoh}(S)}{\otimes} \mathrm{QCoh}(S \underset{Y}{\times} Z)) \\
&\simeq \lim_{S \in (\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})_{/W}} (\mathrm{IndCoh}(S \underset{Y}{\times} Z)) \\
&\simeq \mathrm{IndCoh}(W \underset{Y}{\times} Z),
\end{aligned}$$

where

- the first and last equivalences follow from the definition of IndCoh on lft prestacks as a right Kan extension;
- the second equivalence is because (C.10) commutes with limits;
- the third equivalence is obvious;
- the fourth equivalence follows from Fact C.3.3 and 1-affineness of Y;
- the fifth equivalence is [Gai13, Corollary 7.5.7].

$\square$[Lemma C.3.6]

C.4 The !-theory and $*$-theory on infinite type prestacks

C.4.1. Let fp be the class of morphisms in PreStk that are schematic and of finite presentation. As in [Ras15b][115], there are *right-lax* symmetric monoidal functors

$$\mathrm{DMod}^! : \mathrm{Corr}(\mathrm{PreStk})_{\mathrm{fp},\mathrm{all}} \to \mathrm{DGCat}, \tag{C.11}$$

$$\mathrm{DMod}^* : \mathrm{Corr}(\mathrm{PreStk})_{\mathrm{all},\mathrm{fp}} \to \mathrm{DGCat}, \tag{C.12}$$

such that

- $\mathrm{DMod}^!$ coincides with (C.3) when restricted to $\mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{sch},\mathrm{all}}$;
- DMod^* coincides with (C.4) when restricted to $\mathrm{Corr}(\mathrm{IndSch}_{\mathrm{ift}})_{\mathrm{all},\mathrm{fp}}$;
- $\mathrm{DMod}^!|_{\mathrm{PreStk}^{\mathrm{op}}}$ is the right Kan extension of $\mathrm{DMod}^!|_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}}$, while $\mathrm{DMod}^!|_{(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}}$ is the left Kan extension of $\mathrm{DMod}|_{(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}}}$;
- $\mathrm{DMod}^*|_{\mathrm{PreStk}}$ is the left Kan extension of $\mathrm{DMod}^*|_{\mathrm{Sch}^{\mathrm{aff}}}$, while $\mathrm{DMod}^*|_{\mathrm{Sch}^{\mathrm{aff}}}$ is the right Kan extension of $\mathrm{DMod}|_{\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}}$.

C.4.2. In summary, there are two different theories $\mathrm{DMod}^!$ and DMod^* of D-modules on prestacks, which coincide on indschemes of ind-finite type and on finite type algebraic stacks with affine diagonals. The always-existing functoriality for $\mathrm{DMod}^!$ (resp. DMod^*) is given by !-pullback (resp. $*$-pushforward) functors. Moreover, if a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is of finite presentation, we also have functors[116]

$$f_*^{\mathrm{DMod}^!} : \mathrm{DMod}^!(\mathcal{Y}_1) \to \mathrm{DMod}^!(\mathcal{Y}_2),\; f^!_{\mathrm{DMod}^*} : \mathrm{DMod}^*(\mathcal{Y}_2) \to \mathrm{DMod}^*(\mathcal{Y}_1)$$

such that

- For schematic open embeddings f, we have adjoint pairs $(f^!, f_*^{\mathrm{DMod}^!})$ and $(f^!_{\mathrm{DMod}^*}, f_*)$;
- For schematic and proper maps f, we have adjoint pairs $(f_*^{\mathrm{DMod}^!}, f^!)$ and $(f_*, f^!_{\mathrm{DMod}^*})$.

For two prestacks $\mathcal{Y}_1, \mathcal{Y}_2$, we write $\boxtimes^* : \mathrm{DMod}^*(\mathcal{Y}_1) \otimes \mathrm{DMod}^*(\mathcal{Y}_2) \to \mathrm{DMod}^*(\mathcal{Y}_1 \times \mathcal{Y}_2)$ (resp. $\boxtimes^! : \mathrm{DMod}^!(\mathcal{Y}_1) \otimes \mathrm{DMod}^!(\mathcal{Y}_2) \to \mathrm{DMod}^!(\mathcal{Y}_1 \times \mathcal{Y}_2)$) for the functors witnessing the right-lax symmetric monoidal structures mentioned before. They are not equivalences in general.

[115] [Ras15b, Subsection 6.3] only stated these functors out of categories of correspondences for indschemes. However, the constructions there work for all prestacks. In details, one can define the desired functor $\mathrm{Corr}(\mathrm{PreStk})_{\mathrm{fp},\mathrm{all}} \to \mathrm{DGCat}$ as the right Kan extension of the functor $\mathrm{DMod}^! : \mathrm{Corr}(\mathrm{Sch}_{\mathrm{qcqs}})_{\mathrm{fp},\mathrm{all}}$ (defined in [Ras15b, Subsection 3.8]) along the fully faithful functor $\mathrm{Corr}(\mathrm{Sch}_{\mathrm{qcqs}})_{\mathrm{fp},\mathrm{all}} \subset \mathrm{Corr}(\mathrm{PreStk})_{\mathrm{fp},\mathrm{all}}$. The restriction of the resulting extension to $\mathrm{PreStk}^{\mathrm{op}}$ coincides with the functor in *loc.cit.* by an obvious check of cofinality. The construction of $\mathrm{Corr}(\mathrm{PreStk})_{\mathrm{all},\mathrm{fp}} \to \mathrm{DGCat}$ is similar.

[116] They were denoted by $f_{*,!\text{-dR}}$ and $f^{\text{¡}}$ respectively in [Ras15b].

C.4.3. Write $\mathrm{IndSch}_{\mathrm{placid}}$ for the full subcategory of PreStk consisting of placid indschemes[117]. It is known that the right-lax symmetric monoidal structures on the restrictions $\mathrm{DMod}^!|_{\mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{fp,all}}}$ and $\mathrm{DMod}^*|_{\mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{all,fp}}}$ are both strict.

Let $\mathcal{Y} \in \mathrm{IndSch}_{\mathrm{placid}}$. It is known that both $\mathrm{DMod}^!(\mathcal{Y})$ and $\mathrm{DMod}^*(\mathcal{Y})$ are compactly generated hence dualizable. Moreover, there is a canonical commutative diagram

$$\begin{tikzcd}
(\mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{all,fp}})^{\mathrm{op}} \arrow[r, "(\mathrm{DMod}^*)^{\mathrm{op}}"] \arrow[d, "\varpi", "\simeq"'] & (\mathrm{DGCat}^d)^{\mathrm{op}} \arrow[d, "\mathbf{dualize}", "\simeq"'] \\
\mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{fp,all}} \arrow[r, "\mathrm{DMod}^!"] & \mathrm{DGCat}^d,
\end{tikzcd} \tag{C.13}$$

where ϖ is the canonical anti-involution whose restriction on the sets of objects is the identity map (see [GR17a, Chapter 9, Subsection 2.2]), and DGCat^d is the full subcategory of DGCat consisting of dualizable DG-categories. Also, the above diagram is compatible with the Verdier duality for D-modules on indschemes of ind-finite type.

The following lemma is put here for future reference

Lemma C.4.4. *(c.f. [Ras15b, Lemma 6.9.1(2)]) For a separated finite type scheme S, and two placid indshemes $\mathcal{Y}_1, \mathcal{Y}_2$ over S, write $\Delta' : \mathcal{Y}_1 \times_S \mathcal{Y}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ for the base-change of the diagonal map $\Delta : S \to S \times S$. Then the functor*

$$\mathrm{DMod}^*(\mathcal{Y}_1) \otimes \mathrm{DMod}^*(\mathcal{Y}_2) \overset{\boxtimes^*}{\simeq} \mathrm{DMod}^*(\mathcal{Y}_1 \times \mathcal{Y}_2) \xrightarrow{(\Delta')^!_{\mathrm{DMod}^*}} \mathrm{DMod}^*(\mathcal{Y}_1 \underset{S}{\times} \mathcal{Y}_2)$$

induces an isomorphism

$$\mathrm{DMod}^*(\mathcal{Y}_1) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}^*(\mathcal{Y}_2) \simeq \mathrm{DMod}^*(\mathcal{Y}_1 \underset{S}{\times} \mathcal{Y}_2).$$

Proof. Note that $(\Delta')^!_{\mathrm{DMod}^*}$ has a fully faithful left adjoint Δ'_*. Also note that the obvious functor $p : \mathrm{DMod}^*(\mathcal{Y}_1) \otimes \mathrm{DMod}^*(\mathcal{Y}_2) \to \mathrm{DMod}^*(\mathcal{Y}_1) \otimes_{\mathrm{DMod}(S)} \mathrm{DMod}^*(\mathcal{Y}_2)$ can be identified with

$$(\mathrm{DMod}(S) \otimes \mathrm{DMod}(S)) \otimes_{\mathrm{DMod}(S \times S)} (\mathrm{DMod}^*(\mathcal{Y}_1) \otimes \mathrm{DMod}^*(\mathcal{Y}_2)) \to$$
$$\xrightarrow{\otimes^! \otimes \mathbf{Id}} \mathrm{DMod}(S) \underset{\mathrm{DMod}(S \times S)}{\otimes} (\mathrm{DMod}^*(\mathcal{Y}_1) \otimes \mathrm{DMod}^*(\mathcal{Y}_2)).$$

[117] We refer the reader to [Ras15b, Subsection 6.8] for the notion of placid indschemes. All indschemes appear in this paper are placid.

It has a left adjoint p^L induced by the $\mathrm{DMod}(S\times S)$-linear functor

$$\mathrm{DMod}(S)\xrightarrow{\Delta_*}\mathrm{DMod}(S\times S)\simeq\mathrm{DMod}(S)\otimes\mathrm{DMod}(S).$$

By construction, the corresponding natural transformation $\mathbf{Id}\to p\circ p^L$ is an isomorphism. Hence p^L is also fully faithful. Therefore, it remains to show that the endo-functor $p^L\circ p$ is identified with the endo-functor $\Delta'_*\circ(\Delta')^!_{\mathrm{DMod}^*}$ via the equivalence $\boxtimes^*:\mathrm{DMod}^*(\mathcal{Y}_1)\otimes\mathrm{DMod}^*(\mathcal{Y}_2)\simeq\mathrm{DMod}^*(\mathcal{Y}_1\times\mathcal{Y}_2)$. However, this follows from the compatibility between exterior products and base-change isomorphisms.

□[Lemma C.4.4]

C.5 Ind-holonomic D-modules

C.5.1. For any finite type affine scheme $S\in\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we write $\mathrm{DMod}_{\mathrm{indhol}}(S)$ for the full subcategory of $\mathrm{DMod}(S)$ generated by holonomic objects (under extensions and colimits). We refer the objects in $\mathrm{DMod}_{\mathrm{indhol}}(S)$ as *ind-holonomic D-modules on S*. For any map $f:S\to T$ in $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, it is well-known that $f^!$ and f_* send ind-holonomic D-modules to ind-holonomic D-modules. Moreover, their partially defined left adjoints $f_!$ and f^* are well-defined on ind-holonomic D-modules and preserve them. Moreover, the Verdier duality induces an equivalence $\mathrm{DMod}_{\mathrm{indhol}}(S)\simeq\mathrm{DMod}_{\mathrm{indhol}}(S)^\vee$.

C.5.2. For any lft prestack Y, we write $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)$ for the full subcategory of $\mathrm{DMod}^!(Y)$ containing objects $\mathcal{F}$ such that $f^!(\mathcal{F})\in\mathrm{DMod}_{\mathrm{indhol}}(S)$ for any map $f:S\to Y$ with $S\in\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$. Equivalently, if $Y=\mathrm{colim}Y_\alpha$ with $Y_\alpha\in\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$, we have

$$\mathrm{DMod}^!_{\mathrm{indhol}}(Y)\simeq\lim_{!\text{-pull}}\mathrm{DMod}_{\mathrm{indhol}}(Y_\alpha).\tag{C.14}$$

Passing to left adjoints, we also have

$$\mathrm{DMod}^!_{\mathrm{indhol}}(Y)\simeq\operatorname*{colim}_{!\text{-push}}\mathrm{DMod}_{\mathrm{indhol}}(Y_\alpha).\tag{C.15}$$

Hence by Lemma B.1.5, $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)$ is compactly generated by $\mathrm{ins}_\alpha(\mathcal{F})$, where $\mathrm{ins}_\alpha:\mathrm{DMod}_{\mathrm{indhol}}(Y_\alpha)\to\mathrm{DMod}^!_{\mathrm{indhol}}(Y)$ is the insertion functor[118] and $\mathcal{F}$ is a compact object in $\mathrm{DMod}_{\mathrm{indhol}}(Y_\alpha)$. Also, if the index category is filtered, then the compact objects in $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)$ are exactly these objects.

[118]Note that ins_α is left adjoint to the !-pullback functor $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)\to\mathrm{DMod}_{\mathrm{indhol}}(Y_\alpha)$. We will soon see in Lemma C.5.3 that it is also the partially defined left adjoint of $\mathrm{DMod}^!(Y)\to\mathrm{DMod}(Y_\alpha)$. In particular, the inclusion $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)\to\mathrm{DMod}^!(Y)$ preserves compact objects.

An object in $\mathrm{DMod}^!_{\mathrm{indhol}}(Y)$ is called an *ind-holonomic* object in $\mathrm{DMod}^!(Y)$.

Lemma C.5.3. *Let $f : Y_1 \to Y_2$ be a map between lft prestacks. Consider the functor $f^! : \mathrm{DMod}^!(Y_2) \to \mathrm{DMod}^!(Y_1)$.*

(1) The functor $f^!$ preserves ind-holonomic objects.

(2) The partially defined left adjoint $f_!$ of $f^!$ is well-defined on $\mathrm{DMod}^!_{\mathrm{indhol}}(Y_1)$ and sends it into $\mathrm{DMod}^!_{\mathrm{indhol}}(Y_2)$.

Proof. (1) follows from definition.

By (C.15), the functor $f^! : \mathrm{DMod}^!_{\mathrm{indhol}}(Y_2) \to \mathrm{DMod}^!_{\mathrm{indhol}}(Y_1)$ has a left adjoint $f_!^{\mathrm{indhol}}$. We only need to show the desired partially defined left adjoint is given by it. It is easy to see that this is equivalent to the following claim: the commutative square

$$\begin{tikzcd}
\mathrm{DMod}^!_{\mathrm{indhol}}(Y_2) \arrow[r, "f^!"] \arrow[d, "\subset"] & \mathrm{DMod}^!_{\mathrm{indhol}}(Y_1) \arrow[d, "\subset"] \\
\mathrm{DMod}^!(Y_2) \arrow[r, "f^!"] & \mathrm{DMod}^!(Y_1)
\end{tikzcd}$$

is right adjointable along the vertical directions. In particular, since (2) is correct when Y_1 and Y_2 are affine schemes, the above claim is also correct in this case. Using (C.14) and Lemma B.1.3, we can prove the claim is also correct in the general case.

□[Lemma C.5.3]

Remark C.5.4. In particular, we have a functor

$$\mathrm{DMod}^!_{\mathrm{indhol}} : \mathrm{PreStk}_{\mathrm{lft}} \to \mathrm{DGCat} \tag{C.16}$$

sending morphisms to !-pushforward functors.

C.5.5. Using the base-change isomorphisms for the !-pullback and $*$-pushforward functors, one can show that for any quasi-compact schematic map $f : Y_1 \to Y_2$ between lft prestacks, the functor f_* preserves ind-holonomic objects. Also, using the base-change isomorphisms for the $*$-pullback and !-pushforward functors, one can show that the partially defined left adjoint f^* of f_* is well defined on $\mathrm{DMod}^!_{\mathrm{indhol}}(Y_2)$ and sends it into $\mathrm{DMod}^!_{\mathrm{indhol}}(Y_1)$.

C.5.6. For any lft *algebraic stacks*, there is a canonical equivalence

$$\mathrm{DMod}^!_{\mathrm{indhol}}(Y) := \lim_{S \in (\mathrm{AffSch}_{\mathrm{ft}})_{/Y}} \mathrm{DMod}_{\mathrm{indhol}}(S),$$

with the connecting functors given by $*$-pullbacks. This is implicit in [DG13, § 6.2.1-6.2.2]. Hence for any map $f: Y_1 \to Y_2$ between lft algebraic stacks, there is a functor

$$f^*: \mathrm{DMod}^!_{\mathrm{indhol}}(Y_2) \to \mathrm{DMod}^!_{\mathrm{indhol}}(Y_1)$$

uniquely characterized by its compatibility with the above limit diagrams. Also, when f is quasi-compact and schematic, the above definition coincides with that in § C.5.5. Moreover, there exists a canonical functor

$$\begin{aligned}(\mathrm{DMod}^!_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}: \mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})_{\mathrm{all,all}} \to \mathrm{DGCat},\ Y \mapsto \mathrm{DMod}^!_{\mathrm{indhol}}(Y),\\ (Y_2 \xleftarrow{f} Z \xrightarrow{g} Y_1) \mapsto (f_! \circ g^*: \mathrm{DMod}^!_{\mathrm{indhol}}(Y_1) \to \mathrm{DMod}^!_{\mathrm{indhol}}(Y_2)).\end{aligned} \tag{C.17}$$

C.5.7. Let *stacky* be the class of stacky morphisms in $\mathrm{PreStk}_{\mathrm{lft}}$, i.e., those morphisms $f: Y_1 \to Y_2$ such that $Y_1 \times_{Y_2} S$ is an algebraic stack for any finite type affine scheme S over Y_2.

By left Kan extension along

$$\mathrm{Corr}(\mathrm{AlgStk}_{\mathrm{lft}})_{\mathrm{all,all}} \to \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,Stacky}},$$

we obtain from (C.17) a functor

$$(\mathrm{DMod}^!_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}: \mathrm{Corr}(\mathrm{PreStk}_{\mathrm{lft}})_{\mathrm{all,Stacky}} \to \mathrm{DGCat}\,.$$

It follows from construction that its restriction on $\mathrm{PreStk}_{\mathrm{lft}}$ can be identified with (C.16).

We also have its $(\infty, 2)$-categorical enrichment

$$(\mathrm{DMod}^!_{\mathrm{indhol}})_{!\text{-push},*\text{-pull}}: \mathbf{Corr}(\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{open}}_{\mathrm{all,Stacky}} \to \mathbf{DGCat} \tag{C.18}$$

obtained by using the "no cost" extension in [GR17a, Chapter 7, § 4].

C.6 Functors given by kernels

Let S be a separated finite type scheme, and $f: Y \to S$ be an ind-finite type indscheme over it. We consider $\mathrm{DMod}(Y)$ as an object in $\mathrm{DMod}(S)$-mod, with the action functor given by $\mathcal{A} \cdot \mathcal{F} := f^!(\mathcal{A}) \otimes^! \mathcal{F}$.

Recall that $\mathrm{DMod}(Y)$ is dualizable in DGCat. By § B.2.3, $\mathrm{DMod}(Y)^\vee$ is equipped with a $\mathrm{DMod}(S)$-module DG-category structure. It follows from Lemma B.2.4 that the Verdier duality $\mathrm{DMod}(Y) \simeq$

$\mathrm{DMod}(Y)^\vee$ has a $\mathrm{DMod}(S)$-linear structure. On the other hand, by Lemma B.3.3, $\mathrm{DMod}(Y)$ is also dualizable in $\mathrm{DMod}(S)$-mod, and its dual $\mathrm{DMod}(Y)^{\vee,\mathrm{DMod}(S)}$ is identified with $\mathrm{DMod}(Y)^\vee$ by Lemma B.3.1. Therefore $\mathrm{DMod}(Y)$ is also self-dual as a $\mathrm{DMod}(S)$-module DG-category.

Let $g : Z \to S$ be another ind-finite type indscheme over S. Consider the functor

$$F_{Y\to Z} : \mathrm{DMod}(Y \underset{S}{\times} Z) \to \mathrm{Funct}_S(\mathrm{DMod}(Y), \mathrm{DMod}(Z))$$

given by $F_{Y\to Z}(\mathcal{K})(\mathcal{F}) := p_{2,*}(\mathcal{K} \otimes^! p_1^!(\mathcal{F}))$, where p_1, p_2 are the projections. The functor $F_{Y\to Z}(\mathcal{K})$ is known as the *functor given by the kernel* $\mathcal{K}$.

On the other hand, we have an equivalence (e.g. see [Ras15b, Lemma 6.9.2])

$$\boxtimes_S : \mathrm{DMod}(Y) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}(Z) \simeq \mathrm{DMod}(Y \underset{S}{\times} Z), \tag{C.19}$$

which sends $(\mathcal{F}, \mathcal{G}) \in \mathrm{DMod}(Y) \times \mathrm{DMod}(Z)$ to $p_1^!(\mathcal{F}) \otimes^! p_2^!(\mathcal{G})$. The following lemma is well-known and can be proved by unwinding the definitions.

Lemma C.6.1. *The composition*

$$\begin{aligned} \mathrm{DMod}(Y \underset{S}{\times} Z) &\xrightarrow{F_{Y\to Z}} \mathrm{Funct}_S(\mathrm{DMod}(Y), \mathrm{DMod}(Z)) \simeq \\ \simeq \mathrm{DMod}(Y)^\vee \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}(Z) &\simeq \mathrm{DMod}(Y) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}(Z) \end{aligned}$$

is inverse to $\boxtimes_S$, where the second functor is given by the universal properties of dualities, and the third functor is given by the self-duality of $\mathrm{DMod}(Y)$ *in* $\mathrm{DMod}(S)$-mod.

Remark C.6.2. In the constructible contexts, when $S = \mathrm{pt}$, the composition in the lemma is canonically isomorphic to the right adjoint of $\boxtimes$. The proof is obvious modulo homotopy-coherence. However, it becomes subtle when one is serious about such issues.

C.7 Unipotent nearby cycles

Let $f : \mathcal{Z} \to \mathbb{A}^1$ be an $\mathbb{A}^1$-family of prestacks. In this section, we review *a* definition of the unipotent nearby cycles functor for the family f. This construction recovers the same-named functor defined in [Bei87] when $\mathcal{Z}$ is a finite type scheme.

Construction C.7.1. Let $p : S \to \mathrm{pt}$ be any finite type scheme. Recall the cohomology complex of S

$$C^\bullet(S) := p_* \circ p^*(k).$$

The adjoint pair (p^*, p_*) defines a monad structure on $p_* \circ p^*$. Hence $C^\bullet(S)$ can be upgraded to an associative algebra in Vect.

The algebra $C^\bullet(S)$ acts naturally on the constant D-module $k_S := p^*(k)$. The action morphism is given by

$$C^\bullet(S) \otimes k_S \simeq p^* \circ p_* \circ p^*(k) \to p^*(k) \simeq k_S,$$

where the second morphism is given by the adjoint pair (p^*, p_*).

Construction C.7.2. Consider the case $S = \mathbb{G}_m$. The point $1 : \mathrm{pt} \hookrightarrow \mathbb{G}_m$ defines an augmentation of $C^\bullet(\mathbb{G}_m)$:

$$p_* \circ p^*(k) \to p_* \circ 1_* \circ 1^* \circ p^*(k) \simeq (p \circ 1)_* \circ (p \circ 1)^*(k) \simeq k.$$

Construction C.7.3. Let $f : \mathcal{Z} \to \mathbb{G}_m$ be a prestack over $\mathbb{G}_m$. For any $\mathcal{F} \in \mathrm{DMod}^!(\mathcal{Z})$, we have

$$\mathcal{F} \simeq f^!(k_{\mathbb{G}_m}) \otimes^! \mathcal{F}[2] \simeq f^*(k_{\mathbb{G}_m}) \overset{*}{\otimes} \mathcal{F}$$

Hence Construction C.7.1 provides a natural $C^\bullet(\mathbb{G}_m)$-action on $\mathcal{F}$. In other words, we have a canonical functor

$$\mathrm{DMod}^!(\mathcal{Z}) \to C^\bullet(\mathbb{G}_m)\text{-mod}(\mathrm{DMod}^!(\mathcal{Z}))$$

right inverse to the forgetful functor.

The above functor is functorial in $\mathcal{Z}$ in the following sense. It is compatible with $!/*$-pullback/pushforward functors whenever such functors are defined.

Notation C.7.4. Let $\mathcal{Z}$ be any prestack over $\mathbb{A}^1$. We write $\mathrm{DMod}^!(\overset{\circ}{\mathcal{Z}})^{\mathrm{good}}$ for the full subcategory of $\mathrm{DMod}^!(\overset{\circ}{\mathcal{Z}})$ consisting of objects $\mathcal{F}$ such that the partially defined left adjoint $j_!$ of $j^!$ is defined on $\mathcal{F}$. This condition is equivalent to $i^* \circ j_*(\mathcal{F})$ being defined on $\mathcal{F}$.

Definition C.7.5. Let $f : \mathcal{Z} \to \mathbb{G}_m$ be a prestack over $\mathbb{G}_m$. We define the *unipotent nearby cycles sheaf* of $\mathcal{F} \in \mathrm{DMod}^!(\overset{\circ}{\mathcal{Z}})^{\mathrm{good}}$ to be

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) := k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} i^! \circ j_!(\mathcal{F}), \tag{C.20}$$

where $C^\bullet(\mathbb{G}_m)$ acts on the RHS vis $\mathcal{F}$, and the augmentation $C^\bullet(\mathbb{G}_m)$-module is defined in Construction

C.7.2.

Fact C.7.6. *By the base-change isomorphisms, Ψ_f^{un} commutes with $*$-pushforward functors along schematic proper maps (resp. $!$-pullback functors along schematic smooth maps).*

Remark C.7.7. By the excision triangle, we also have:

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) \simeq k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} i^* \circ j_*(\mathcal{F})[-1]. \tag{C.21}$$

Remark C.7.8. When $\mathcal{Z}$ is a finite type scheme and $\mathcal{F}$ is ind-holonomic, by [Cam18, Proposition 3.1.2(1)][119], the above definition coincides with the well-known definition in [Bei87]. Essentially, this follows from the fact that $k \otimes_{C^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m} \in \mathrm{DMod}(\mathbb{G}_m)$ is the "infinite Jordan block" in *loc.cit.*. See [Cam18, Lemma 3.2.1].

Construction C.7.9. A direct calculation provides an isomorphism between augmented DG-algebras

$$\mathrm{Maps}_{C^\bullet(\mathbb{G}_m)\text{-mod}^r}(k,k) \simeq k[[t]],$$

where the RHS is contained in $\mathrm{Vect}^\heartsuit$. Hence $\Psi_f^{\mathrm{un}}(\mathcal{F})$ is equipped with an action of $k[[t]]$. The action of $t \in k[[t]]$ on $\Psi_f^{\mathrm{un}}(\mathcal{F})$ is the *monodromy endomorphism* in the literature.

By the Koszul duality, we have

$$i^* \circ j_*(\mathcal{F})[-1] \simeq i^! \circ j_!(\mathcal{F}) \simeq \mathrm{Hom}_{k[[t]]}(k, \Psi_f^{\mathrm{un}}(\mathcal{F})). \tag{C.22}$$

C.7.10 (Full nearby cycles functor)**.** Suppose Z is an indscheme of ind-finite type. Consider the category $\mathrm{DMod}_{\mathrm{indhol}}(\mathring{Z})$ of ind-holonomic D-modules on $\mathring{Z}$ (see § C.5 for the definition). It is well-known that

$$\mathrm{DMod}_{\mathrm{indhol}}(\mathring{Z}) \subset \mathrm{DMod}(\mathring{Z})^{\mathrm{good}}.$$

Hence the unipotent nearby cycles functor is always defined for ind-holonomic D-modules on $\mathring{Z}$.

On the other hand, there is also a *full nearby cycles functor*

$$\Psi_f : \mathrm{DMod}_{\mathrm{indhol}}(\mathring{Z}) \to \mathrm{DMod}(Z_0).$$

Ψ_f satisfies the same standard properties as the unipotent one. Moreover, there is a Künneth formula for the *full* nearby cycles functors (e.g. see [BB93, Lemma 5.1.1] and the remark below it), which is not shared

[119] Although [Cam18] stated the result below with the assumption that there is a $\mathbb{G}_m$-action on $\mathcal{Z}$, it was only used in the proof of [Cam18, Proposition 3.1.2(2)].

by the unipotent ones.

We have a canonical map $\Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f(\mathcal{F})$ for any ind-holonomic $\mathcal{F}$.

C.7.11. In this thesis, we are interested in the case when there is a $\mathbb{G}_m$-action on $\mathcal{Z}$ compatible with the projection $\mathcal{Z} \to \mathbb{A}^1$. We will use the following two folklore results.

Lemma C.7.12. *(See [AB09, Claim 2][120]) Suppose that Z is equipped with a $\mathbb{G}_m$-action such that it can be written as a filtered colimit of closed subschemes stabilized by $\mathbb{G}_m$, and suppose the map $f : Z \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant. Let $\mathcal{F}$ be an ind-holonomic regular D-module on $\overset{\circ}{Z}$ such that both $\mathcal{F}$ and $\Psi_f(\mathcal{F})$ are unipotently $\mathbb{G}_m$-monodromic[121]. Then the obvious map $\Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f(\mathcal{F})$ is an isomorphism.*

Lemma C.7.13. *Suppose $f : Z \to \mathbb{A}^1$ is a $\mathbb{G}_m$-equivariant map between lft algebraic stacks. Consider the Cartesian squares*

$$\begin{tikzcd}
Z_0 \arrow[r, "i"] \arrow[d, "\pi_0"] & Z \arrow[d, "\pi"] & \overset{\circ}{Z} \arrow[l, "j"'] \arrow[d, "\overset{\circ}{\pi}"] \\
Z_0/\mathbb{G}_m \arrow[r, "i/\mathbb{G}_m"] & Z/\mathbb{G}_m & \overset{\circ}{Z}/\mathbb{G}_m. \arrow[l, "j/\mathbb{G}_m"']
\end{tikzcd}$$

Then we have a canonical equivalence

$$\pi_{0,*} \circ \Psi_f^{\mathrm{un}} \circ (\overset{\circ}{\pi})^*(\mathcal{F})[1] \simeq (i/\mathbb{G}_m)^* \circ (j/\mathbb{G}_m)_*(\mathcal{F})$$

whenever the RHS is well-defined.

Proof. We first prove in the case when Z is a scheme. We identify $\mathrm{DMod}(Z_0/\mathbb{G}_m) \simeq \mathrm{DMod}(Z_0)^{\mathbb{G}_m}$ such that $\mathbf{oblv}^{\mathbb{G}_m}$ is given by π_0^*. By [Cam18, Proposition 3.2.2(2)], we have

$$\Psi_f^{\mathrm{un}} \circ (\overset{\circ}{\pi})^*(\mathcal{F})[1] \in \mathrm{DMod}(Z_0)^{\mathbb{G}_m\text{-um}} \subset \mathrm{DMod}(Z_0)$$

and the $k[[t]]$-action on it (that comes from Construction C.7.9) coincides with the $k[[t]]$-linear structure[122]

[120] An erreneous version of the lemma, which did not require $\Psi_f(\mathcal{F})$ to be unipotently $\mathbb{G}_m$-monodromic, appeared in an unpublished version of [Gai01]. This wrong claim was cited by [Sch16, Lemma 8.0.4]. Nevertheless, one can use the correct version (and the smooth descent) to obtain the result in *loc.cit.*.

[121] See Definition E.1.6.

[122] For any $\mathcal{C} \in \mathbb{G}_m\text{-mod}$, the category $\mathcal{C}^{\mathbb{G}_m\text{-um}}$ has a canonical $\mathrm{QCoh}(\mathrm{Spf}\, k[[t]])$-module structure such that

$$\mathcal{C}^{\mathbb{G}_m} \simeq \mathcal{C}^{\mathbb{G}_m\text{-um}} \underset{\mathrm{QCoh}(\mathrm{Spf}\, k[[t]])}{\otimes} \mathrm{Vect},$$

where the functor $\mathrm{QCoh}(\mathrm{Spf}\, k[[t]]) \to \mathrm{Vect}$ is the !-pullback functor $\mathrm{Hom}_{k[[t]]}(k, -)$ (i.e., the right adjoint of the $*$-pushforward functor).

on $\mathrm{DMod}(Z_0)^{\mathbb{G}_m\text{-um}}$. Hence we have an equivalence in $\mathrm{DMod}(Z_0)^{\mathbb{G}_m}$:

$$\mathbf{Av}_*^{\mathbb{G}_m}(\Psi_f^{\mathrm{un}} \circ (\mathring{\pi})^*(\mathcal{F})[1]) \simeq \mathrm{Hom}_{k[[t]]}(k, \Psi_f^{\mathrm{un}} \circ (\mathring{\pi})^*(\mathcal{F})[1]) \simeq i^* \circ j_* \circ (\mathring{\pi})^*(\mathcal{F}),$$

where the last equivalence is the Koszul duality (C.22), and the $\mathbb{G}_m$-equivariant structure on $i^* \circ j_* \circ (\mathring{\pi})^*(\mathcal{F})$ is the one induced from $\mathcal{F} \in \mathrm{DMod}(\mathring{Z})^{\mathbb{G}_m}$. Now the desired equivalence is obtained by translating the above one back into $\mathrm{DMod}(Z_0/\mathbb{G}_m)$.

The above construction is functorial in Z in the sense that it is compatible with smooth pullback functors. Hence we obtain the general case by smooth descents.

□[Lemma C.7.13]

C.8 D-modules on stacks stratified by power sets

We begin with the following definition.

Definition C.8.1. Let Y be an algebraic stack and I be a finite set. A *stratification of Y labelled by the power poset $P(I)$* is an assignment of open substacks $U_i \subset Y$ for any $i \in I$.

The above definition coincides with the usual one in the literature because of the following construction.

Construction C.8.2. For any object $J \in P(I)$, we define $i_J : Y_J \to Y$ to be the reduced locally closed substack of Y given by

$$(\bigcup_{j\in J} U_j) - (\bigcup_{i\notin J} U_i).$$

We call Y_J *the stratum labelled by J*. Note that every geometric point of Y is contained in exactly one stratum.

For any object $J \in P(I)$, there is a unique open substack $Y_{\geq J} \subset Y$ whose geometric points are exactly those contained in $\bigcup_{K\supset J} Y_K$. Similarly, we define the reduced closed substack $Y_{\leq J}$.

Note that the stratum Y_I is an open substack of Y. Hence we also write $j_I := i_I$ for this open embedding.

Also note that U_i can be recovered as $Y_{\geq\{i\}}$.

Example C.8.3. Let Y be a finite type scheme and $\{f_i\}_{i\in I}$ be regular functions on Y. Then we obtain a stratification of Y labelled by $P(I)$ with U_i given by the non-vanishing locus of f_i. In particular, the coordinate functions induce a stratification of the affine space $\mathbb{A}^I$ labelled by $P(I)$. This stratification is known as the *coordinate stratification*.

Remark C.8.4. Since the theory of D-modules is insensitive to non-reduced structures, in this paper, we also use the notations Y_J and $Y_{\le J}$ for certain infinitesimal thickening of the stacks defined above. For example, if we have a map $Y \to Z$ and a stratification of Z labelled by $P(I)$, then we obtain a stratification of Y labelled by $P(I)$ by pulling back the open substacks. We often write $Y_J := Y \times_Z Z_J$ although it is not necessarily reduced.

Definition C.8.5. Let Y be an algebraic stack stratified by a power poset $P(I)$. We define

$$\mathrm{Funct}(P(I), \mathrm{DMod}_{\mathrm{indhol}}(Y))_! \subset \mathrm{Funct}(P(I), \mathrm{DMod}_{\mathrm{indhol}}(Y))$$

to be the full subcategory consisting of those functors $F : P(I) \to \mathrm{DMod}(Y)$ such that $F(J)$ is !-extended from the stratum Y_J.

Lemma C.8.6. *Let Y be an algebraic stack stratified by a power poset $P(I)$. The functor*

$$\mathbf{C}_Y : \mathrm{Funct}(P(I), \mathrm{DMod}_{\mathrm{indhol}}(Y))_! \to \mathrm{DMod}_{\mathrm{indhol}}(Y),$$
$$F \mapsto \mathrm{coFib}(\operatorname*{colim}_{J \subsetneq I} F(J) \to F(I))$$

is an equivalence. Also, its inverse sends an object $\mathcal{F} \in \mathrm{DMod}_{\mathrm{indhol}}(Y)$ to a certain functor

$$P(I) \to \mathrm{DMod}_{\mathrm{indhol}}(Y),\ J \mapsto i_{J,!} \circ i_J^*(\mathcal{F})[|J| - |I|].$$

Proof. First note that the second claim follows from the first one because

$$i_K^*(\mathrm{coFib}(\operatorname*{colim}_{J \subsetneq I} F(J) \to F(I))) \simeq i_K^*(F(K))[|I| - |K|].$$

It remains to show $\mathbf{C}_Y$ is an equivalence. The case $I = \{*\}$ is well-known. The general case can be proved by induction as follows. Suppose $I = I^\flat \bigsqcup \{a\}$ and I is nonempty. Note that apart from the embedding $P(I^\flat) \subset P(I)$, we also have a map

$$P(I^\flat) \to P(I),\ J \mapsto J^\sharp := J \bigsqcup \{a\}.$$

The open substacks $\{U_i\}_{i \in I^\flat}$ provide a stratification of Y labelled by $P(I^\flat)$. We use the notation Z to denote the same stack Y equipped with this new stratification. For any $K \in P(I^\flat)$, the stratum Z_K inherits a stratification by $P(\{i\})$, whose big stratum is isomorphic to $Y_{K^\sharp}$ and small stratum is isomorphic to Y_K.

Consider the functor

$$A : \mathrm{Funct}(P(I), \mathrm{DMod}_{\mathrm{indhol}}(Y))_! \to \mathrm{Funct}(P(I^\flat), \mathrm{DMod}_{\mathrm{indhol}}(Z))_!, F \mapsto A(F),$$

where $A(F)(K) := \mathrm{coFib}(F(K) \to F(K^\sharp))$. Note that this is well-defined, i.e. $A(F)(K)$ is indeed a !-extension from Z_K. Moreover, A is an equivalence by the $I = \{*\}$ case of the lemma (applying to each Z_K).

Hence by induction hypothesis, $\mathbf{C}_Z \circ A$ is also an equivalence. It remains to show $\mathbf{C}_Y \simeq \mathbf{C}_Z \circ A$. Note that we have the following pushout diagram

$$\begin{tikzcd}
\mathrm{colim}_{K \subsetneq I^\flat} F(K) \arrow[r] \arrow[d] & \mathrm{colim}_{J \subsetneq I, J \neq I^\flat} F(J) \arrow[d] \\
\mathrm{colim}_{K \subset I^\flat} F(K) \arrow[r] & \mathrm{colim}_{J \subsetneq I} F(J),
\end{tikzcd}$$

which is obtained by writing the simplicial nerve of $P(I) - \{I\}$ as a pushout. By cofinality, the above diagram is equivalent to

$$\begin{tikzcd}
\mathrm{colim}_{K \subsetneq I^\flat} F(K) \arrow[r] \arrow[d] & \mathrm{colim}_{K \subsetneq I^\flat} F(K^\sharp) \arrow[d] \\
F(I^\flat) \arrow[r] & \mathrm{colim}_{J \subsetneq I} F(J).
\end{tikzcd}$$

Then we have

$$\begin{aligned}
& \mathrm{coFib}(\operatorname*{colim}_{J \subsetneq I} F(J) \to F(I)) \\
\simeq \quad & \mathrm{coFib}(\mathrm{coFib}(\operatorname*{colim}_{K \subsetneq I^\flat} F(K) \to \operatorname*{colim}_{K \subsetneq I^\flat} F(K^\sharp)) \to \mathrm{coFib}(F(I^\flat) \to F(I))) \\
\simeq \quad & \mathrm{coFib}(\operatorname*{colim}_{K \subsetneq I^\flat} (\mathrm{coFib}(F(K) \to F(K^\sharp))) \to \mathrm{coFib}(F(I^\flat) \to F(I))) \\
\simeq \quad & \mathrm{coFib}(\operatorname*{colim}_{K \subsetneq I^\flat} A(F)(K) \to A(F)(I^\flat))
\end{aligned}$$

as desired. This proves the claim.

□[Lemma C.8.6]

The above lemma implies

Corollary C.8.7. *Let Y be an algebraic stack stratified by a power poset $P(I)$. The functor*

$$\mathbf{J}_Y : \mathrm{Funct}(P(I), \mathrm{DMod}_{\mathrm{indhol}}(Y))_! \to \mathrm{DMod}_{\mathrm{indhol}}(Y_I),\ F \mapsto j_I^* \circ F(I)$$

has a right adjoint sending an object $\mathcal{F} \in \mathrm{DMod}_{\mathrm{indhol}}(Y_I)$ to a certain functor

$$\mathbf{G}^*_{\mathcal{F},Y} : P(I) \to \mathrm{DMod}_{\mathrm{indhol}}(Y),\ J \mapsto i_{J,!} \circ i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|].$$

Proof. Follows from the fact that $\mathbf{J}_Y \simeq j_I^* \circ \mathbf{C}_Y$.

□[Corollary C.8.7]

Remark C.8.8. Note that the functor $\mathbf{G}^*_{\mathcal{F},Y}$ sends the arrow $J \subset I$ to a morphism

$$i_{J,!} \circ i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|] \to i_{I,!}(\mathcal{F}).$$

Applying $i_J^!$ to this map, we obtain a map

$$i_J^* \circ j_{I,*}(\mathcal{F})[|J| - |I|] \to i_J^! \circ i_{I,!}(\mathcal{F}).$$

Note that this map is invertible if $|J| = |I| - 1$, but not for general J.

Lemma C.8.9. *Let Y be an algebraic stack stratified by a power poset $P(I)$ and $J \in P(I)$. Consider the maps*

$$Y_I \overset{j_{I,\geq J}}{\to} Y_{\geq J} \overset{j_{\geq J}}{\to} Y.$$

For any $\mathcal{F} \in \mathrm{DMod}_{\mathrm{indhol}}(Y_I)$, we have

$$\mathrm{coFib}(\underset{J \subset K \subsetneq I}{\mathrm{colim}}\, \mathbf{G}^*_{\mathcal{F},Y}(K) \to \mathbf{G}^*_{\mathcal{F},Y}(I)) \simeq (j_{\geq J})_! \circ (j_{I,\geq J})_*(\mathcal{F}).$$

Proof. The case $J = I$ follows from definition. Indeed, the LHS is given by

$$\mathbf{C}_Y \circ (\mathbf{J}_Y)^R \simeq \mathbf{C}_Y \circ (j_I^* \circ \mathbf{C}_Y)^R \simeq (j_I^*)^R \simeq j_{I,*}.$$

In the general case, note that both sides are contained in the image of the functor $(j_{\geq J})_!$. Hence we only need to show

$$\mathrm{coFib}(\underset{J \subset K \subsetneq I}{\mathrm{colim}}\, j_{\geq J}^* \circ \mathbf{G}^*_{\mathcal{F},Y}(K) \to j_{\geq J}^* \circ \mathbf{G}^*_{\mathcal{F},Y}(I)) \simeq (j_{I,\geq J})_*(\mathcal{F}).$$

Consider the open substack $Y_{\geq J}$. It inherits a stratification by the poset $P(I - J)$ with $(Y_{\geq J})_K \simeq Y_{J \sqcup K}$. Hence we also have a functor

$$\mathbf{G}^*_{\mathcal{F},Y_{\geq J}} : P(I - J) \to \mathrm{DMod}_{\mathrm{indhol}}(Y_{\geq J}).$$

It follows from construction that this functor is isomorphic to

$$P(I-J) \overset{-\sqcup J}{\to} P(I) \overset{\mathbf{G}^*_{\mathcal{F},Y}}{\to} \mathrm{DMod}_{\mathrm{indhol}}(Y) \overset{j^*_{\geq J}}{\to} \mathrm{DMod}_{\mathrm{indhol}}(Y_{\geq J}).$$

Hence we only need to show

$$\mathrm{coFib}(\operatorname*{colim}_{K \subsetneq I-J} \mathbf{G}^*_{\mathcal{F},Y_{\geq J}}(K) \to \mathbf{G}^*_{\mathcal{F},Y_{\geq J}}(I-J)) \simeq (j_{I,\geq J})_*(\mathcal{F}).$$

In other words, we have reduced the lemma to the case $J = I$.

□[Lemma C.8.9]

Appendix D

Group actions on categories

In this Appendix, we review the general framework of categories acted by *relative* placid group indschemes, which was established in [Ras16, Subsection 2.17].

D.1 Invariance and coinvariance

Let S be a separated finite type scheme and $p : \mathcal{H} \to S$ be a group indscheme over S whose underlying indscheme is placid. The symmetric monoidal structure on $\mathrm{DMod}^* : \mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{all,fp}} \to \mathrm{DGCat}$ upgrades $\mathrm{DMod}^*(\mathcal{H})$ to an augmented associative algebra object in $\mathrm{DMod}(S)\text{-mod}$. Forgetting the $\mathrm{DMod}(S)$-linearity, we obtain a monoidal DG-category $(\mathrm{DMod}^*(\mathcal{H}), \star)$, whose multiplication is given by convolutions.

Dually, the pair $\mathrm{DMod}^!(\mathcal{H})$ can be upgraded to a co-augmented co-associative coalgebra object in $\mathrm{DMod}(S)\text{-mod}$. And we obtain a co-monoidal DG-category $(\mathrm{DMod}^!(\mathcal{H}), \delta)$. By construction, it is dual to the monoidal DG-category $(\mathrm{DMod}^*(\mathcal{H}), \star)$.

Moreover, by Lemma B.3.3, B.3.1, $\mathrm{DMod}^*(\mathcal{H})$ and $\mathrm{DMod}^!(\mathcal{H})$ are dual in $\mathrm{DMod}(S)\text{-mod}$. Therefore we have:

Proposition-Definition D.1.1. *The following categories are canonically equivalent:*

(1) $(\mathrm{DMod}^*(\mathcal{H}), \star)\text{-mod}$*;*

(2) $\mathrm{DMod}^*(\mathcal{H})\text{-mod}(\mathrm{DMod}(S)\text{-mod})$*;*

(3) $(\mathrm{DMod}^!(\mathcal{H}), \delta)\text{-comod}$*;*

(4) $(\mathrm{DMod}^!(\mathcal{H}))\text{-comod}(\mathrm{DMod}(S)\text{-mod})$.

Moreover, the above equivalences are compatible with forgetful functors to DGCat *and tensoring with objects in* DGCat*.*

We define $\mathcal{H}$-mod *as any/all of the above categories, and refer it as the category of categories acted by* $\mathcal{H}$ *(relative to* S*).*

Remark D.1.2. In the constructible contexts, because of lack of Lemma B.3.3, we do not know whether $\mathrm{Shv}^!(\mathcal{H})$ can be upgraded to a coalgebra object in $\mathrm{Shv}(S)$-mod. Hence (4) does not make sense. However, (1)(2)(3) remain valid in the constructible contexts.

Remark D.1.3. As usual, $\mathcal{H}$-mod can be enriched over $\mathrm{DMod}(S)$-mod, i.e. for any $\mathcal{M},\mathcal{N} \in \mathcal{H}$-mod, we have an object

$$\mathrm{Funct}_{\mathcal{H}}(\mathcal{M},\mathcal{N}) \in \mathrm{DMod}(S)\text{-mod}$$

satisfying the following universal property:

$$\mathrm{Funct}_S(\mathcal{C}, \mathrm{Funct}_{\mathcal{H}}(\mathcal{M},\mathcal{N})) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathcal{M} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C},\mathcal{N}).$$

D.1.4 (Invariance and coinvariance)**.** Let $\mathcal{H}$ be as before. The augmentation $p_* : \mathrm{DMod}^*(\mathcal{H}) \to \mathrm{DMod}(S)$ induces a functor (the trivial action functor)

$$\mathbf{triv}_{\mathcal{H}} : \mathrm{DMod}(S)\text{-mod} \to \mathcal{H}\text{-mod},$$

which commutes with both colimits and limits. It has both a left adjoint and a right adjoint, which we refer respectively as taking *coinvariance* and *invariance*:

$$\mathbf{coinv}_{\mathcal{H}} : \mathcal{H}\text{-mod} \to \mathrm{DMod}(S)\text{-mod},\ \mathcal{C} \mapsto \mathcal{C}_{\mathcal{H}},$$
$$\mathbf{inv}_{\mathcal{H}} : \mathcal{H}\text{-mod} \to \mathrm{DMod}(S)\text{-mod},\ \mathcal{C} \mapsto \mathcal{C}^{\mathcal{H}}.$$

Explicitly, they are given by formula

$$\mathcal{C}_{\mathcal{H}} \simeq \mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C},$$
$$\mathcal{C}^{\mathcal{H}} \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C}),$$

and can be calculated via bar (resp. cobar) constructions. Note that the adjunction natural transformations for the pairs $(\mathbf{coinv}_{\mathcal{H}}, \mathbf{triv}_{\mathcal{H}})$ and $(\mathbf{triv}_{\mathcal{H}}, \mathbf{inv}_{\mathcal{H}})$ are given respectively by

$$\mathbf{pr}_{\mathcal{H}} : \mathcal{C} \simeq \mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C} \xrightarrow{p_* \otimes \mathbf{Id}} \mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C} \simeq \mathbf{triv}_{\mathcal{H}}(\mathcal{C}_{\mathcal{H}}),$$
$$\mathbf{oblv}^{\mathcal{H}} : \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C}) \xrightarrow{-\circ p_*} \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}^*(\mathcal{H}), \mathcal{C}) \simeq \mathcal{C}.$$

We abuse notation by using the same symbols to denote the functors between the underlying DG-categories.

Let $\mathcal{H} \to \mathcal{G}$ be a morphism between two group indschemes as above. The restriction functors $\mathbf{res}_{\mathcal{G}\to\mathcal{H}} : \mathcal{G}\text{-mod} \to \mathcal{H}\text{-mod}$ commutes with both colimits and limits. It has both a left adjoint $\mathbf{ind}_{\mathcal{H}\to\mathcal{G}}$ and a right adjoint $\mathbf{coind}_{\mathcal{H}\to\mathcal{G}}$ calculated by obvious formulae.

The following lemma is put here for future reference.

Lemma D.1.5. *Let $\mathcal{D} \to \mathcal{C}$ be a morphism in $\mathcal{H}$-mod. Suppose the underlying functor $\mathcal{D} \to \mathcal{C}$ is fully faithful, then the induced functor $\mathcal{D}^{\mathcal{H}} \to \mathcal{C}^{\mathcal{H}}$ is also fully faithful, and the obvious functor $\mathcal{D}^{\mathcal{H}} \to \mathcal{C}^{\mathcal{H}} \times_{\mathcal{C}} \mathcal{D}$ is an equivalence.*

Proof. It follows from the cobar construction . □[Lemma D.1.5]

D.1.6 (Change of base)**.** Let $\mathcal{H}_S \to S$ be as before and $T \to S$ be a separated finite type scheme over S. Write $\mathcal{H}_T \to T$ for the base-change of p_S. This sub-subsection is devoted to the study of the relationships between taking invariance or coinvariance in $\mathcal{H}_S$-mod and $\mathcal{H}_T$-mod.

Note that the projection map $\phi : \mathcal{H}_T \to \mathcal{H}_S$ is finitely presented, hence we have the functor $\phi^!_{\mathrm{DMod}^*} : \mathrm{DMod}^*(\mathcal{H}_S) \to \mathrm{DMod}^*(\mathcal{H}_T)$. Thanks to the symmetric monoidal structure on

$$\mathrm{DMod}^* : \mathrm{Corr}(\mathrm{IndSch}_{\mathrm{placid}})_{\mathrm{all,fp}} \to \mathrm{DGCat},$$

$\phi^!_{\mathrm{DMod}^*}$ can be upgraded to a monoidal functor. Hence we have the following commutative diagrams:

$$\begin{tikzcd}
\mathcal{H}_T\text{-mod} \arrow[r, "\mathbf{res}_{\mathcal{H}_T\to\mathcal{H}_S}"] \arrow[d, "\mathbf{res}_{\mathcal{H}_T\to T}"'] & \mathcal{H}_S\text{-mod} \arrow[d, "\mathbf{res}_{\mathcal{H}_S\to S}"] \\
\mathrm{DMod}(T)\text{-mod} \arrow[r, "\mathbf{res}_{T\to S}"] & \mathrm{DMod}(S)\text{-mod},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}_T\text{-mod} \arrow[r, "\mathbf{res}_{\mathcal{H}_T\to\mathcal{H}_S}"] & \mathcal{H}_S\text{-mod} \\
\mathrm{DMod}(T)\text{-mod} \arrow[r, "\mathbf{res}_{T\to S}"] \arrow[u, "\mathbf{triv}_{\mathcal{H}_T}"] & \mathrm{DMod}(S)\text{-mod}. \arrow[u, "\mathbf{triv}_{\mathcal{H}_S}"']
\end{tikzcd}
\tag{D.1}$$

Lemma D.1.7. *We have:*

(1) Both commutative squares in (D.1) are left adjointable along the horizontal directions. In other words,

we have commutative diagrams

$$\begin{tikzcd} \mathcal{H}_T\text{-mod} \arrow[d, "\mathbf{res}_{\mathcal{H}_T\to T}"'] & \mathcal{H}_S\text{-mod} \arrow[l, "\mathbf{ind}_{\mathcal{H}_S\to\mathcal{H}_T}"'] \arrow[d, "\mathbf{res}_{\mathcal{H}_S\to S}"] \\ \mathrm{DMod}(T)\text{-mod} & \mathrm{DMod}(S)\text{-mod}, \arrow[l, "\mathbf{ind}_{S\to T}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_T\text{-mod} & \mathcal{H}_S\text{-mod} \arrow[l, "\mathbf{ind}_{\mathcal{H}_S\to\mathcal{H}_T}"'] \\ \mathrm{DMod}(T)\text{-mod} \arrow[u, "\mathbf{triv}_{\mathcal{H}_T}"] & \mathrm{DMod}(S)\text{-mod}. \arrow[l, "\mathbf{ind}_{S\to T}"'] \arrow[u, "\mathbf{triv}_{\mathcal{H}_S}"'] \end{tikzcd}$$

(2) *The second commutative square in (1) is both left adjointable and right adjointable along the vertical directions. In other words, for any* $\mathcal{C} \in \mathcal{H}_S\text{-mod}$*, the base-change* $\mathrm{DMod}(T)\otimes_{\mathrm{DMod}(S)}\mathcal{C}$ *can be canonically upgraded to an object in* $\mathcal{H}_T\text{-mod}$ *such that there are canonical* $\mathrm{DMod}(S)$*-linear isomorphisms*

$$(\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C})_{\mathcal{H}_T} \simeq \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}_{\mathcal{H}_S},$$
$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}^{\mathcal{H}_S} \simeq (\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C})^{\mathcal{H}_T}.$$

(3) *The second commutative square in (D.1) is both left adjointable and right adjointable along the vertical directions. In other words, for any* $\mathcal{C} \in \mathcal{H}_T\text{-mod}$*, it can be viewed as an object in* $\mathcal{H}_S\text{-mod}$ *via restriction such that there are canonical* $\mathrm{DMod}(S)$*-linear isomorphisms* $\mathcal{C}_{\mathcal{H}_S} \simeq \mathcal{C}_{\mathcal{H}_T}$, $\mathcal{C}^{\mathcal{H}_T} \simeq \mathcal{C}^{\mathcal{H}_S}$.

Proof. We first prove the first commutative diagram in (1). Let $\mathcal{C} \in \mathcal{H}_S\text{-mod}$. It suffices to show that the natural functor

$$(\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}^*(\mathcal{H}_S)) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C} \to \mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C}$$

is an isomorphism. However, by [Ras16, Proposition 6.9.1], we have

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}^*(\mathcal{H}_S) \simeq \mathrm{DMod}^*(\mathcal{H}_T) \tag{D.2}$$

as desired.

The proof for the second commutative diagram in (1) is similar. In fact, it is a formal consequence of this first one, because $\mathbf{res}_{\mathcal{H}_T\to T}$ is conservative.

Now we prove (2). The left adjointability is obtained by passing to left adjoints in the second commutative square of (D.1). For the right adjointablity, let $\mathcal{C} \in \mathcal{H}_S\text{-mod}$. It suffices to show that the natural functor

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{Funct}_{\mathcal{H}_S}(\mathrm{DMod}(S), \mathcal{C}) \to \mathrm{Funct}_{\mathcal{H}_T}(\mathrm{DMod}(T), \mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C})$$

is an isomorphism. Unwinding the definitions, the above functor is the composition of functors

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{Funct}_{\mathcal{H}_S}(\mathrm{DMod}(S), \mathcal{C}) \to \mathrm{Funct}_{\mathcal{H}_S}(\mathrm{DMod}(S), \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}), \tag{D.3}$$

$$\begin{aligned} & \mathrm{Funct}_{\mathcal{H}_S}(\mathrm{DMod}(S), \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}) \\ \simeq\ & \mathrm{Funct}_{\mathcal{H}_S}(\mathrm{DMod}(S), \mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C}) \\ \simeq\ & \mathrm{Funct}_{\mathcal{H}_T}(\mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathrm{DMod}(S), \mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C}) \\ \simeq\ & \mathrm{Funct}_{\mathcal{H}_T}(\mathrm{DMod}(T), \mathrm{DMod}^*(\mathcal{H}_T) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathcal{C}), \end{aligned} \tag{D.4}$$

where the equivalences (D.4) are due to (D.2). Therefore it suffices to prove that (D.3) is an equivalence.

Rewrite (D.3) as

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \lim_{\Delta} \mathrm{Funct}_S(\mathrm{DMod}^*(\mathcal{H}_S)^{\otimes^{\bullet}_{\mathrm{DMod}(S)}}, \mathcal{C}) \to \lim_{\Delta} \mathrm{Funct}_S(\mathrm{DMod}^*(\mathcal{H}_S)^{\otimes^{\bullet}_{\mathrm{DMod}(S)}}, \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}).$$

Recall $\mathrm{DMod}(T)$ is self-dual in $\mathrm{DMod}(S)$-mod (see Lemma B.3.3). Hence $\mathrm{DMod}(T) \otimes_{\mathrm{DMod}(S)} -$ commutes with limits. Hence it remains to prove

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{Funct}_S(\mathrm{DMod}^*(\mathcal{H}_S)^{\otimes^{\bullet}_{\mathrm{DMod}(S)}}, \mathcal{C}) \simeq \mathrm{Funct}_S(\mathrm{DMod}^*(\mathcal{H}_S)^{\otimes^{\bullet}_{\mathrm{DMod}(S)}}, \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}).$$

Note that $\mathrm{DMod}^*(\mathcal{H}_S)$ is dualizable in DGCat (see § C.4.3). By Lemma B.3.1, it is also dualizable in $\mathrm{DMod}(S)$-mod. Hence it suffices to prove

$$\mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{Funct}_S(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Funct}_S(\mathcal{D}, \mathrm{DMod}(T) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C})$$

for any dualizable object $\mathcal{D} \in \mathrm{DMod}(S)$-mod. However, we have $\mathrm{Funct}_S(\mathcal{D}, -) \simeq \mathcal{D}^{\vee, \mathrm{DMod}(S)} \otimes_{\mathrm{DMod}(S)} -$, which makes the desired claim obvious.

It remains to prove (3). The right adjointability is obtained by passing to right adjoints in the second commutative square of (1). For the left adjointability, let $\mathcal{C} \in \mathcal{H}_T$-mod. It suffices to show that

$$(\mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{H}_S)}{\otimes} \mathrm{DMod}^*(\mathcal{H}_T)) \underset{\mathrm{DMod}^*(\mathcal{H}_T)}{\otimes} \mathcal{C} \to \mathrm{DMod}(T) \underset{\mathrm{DMod}^*(\mathcal{H}_T)}{\otimes} \mathcal{C}$$

is an equivalence. However, this follows from the equivalence (D.2).

□[Lemma D.1.7]

Remark D.1.8. In the constructible contexts, we can only prove the lemma when $T \to S$ is either a closed or open embedding.

D.1.9 (Duality)**.** Let $\mathcal{C} \in \mathcal{H}$-mod. Assume $\mathcal{C}$ is dualizable in DGCat. By § B.2.3, it is right-dualizable as a $(\mathrm{DMod}^*(\mathcal{H}), \mathrm{Vect})$-bimodule DG-category. We denote its right-dual by $\mathcal{C}^\vee$, which is a $(\mathrm{Vect}, \mathrm{DMod}^*(\mathcal{H}))$-bimodule DG-category, i.e. a right $\mathrm{DMod}^*(\mathcal{H})$-module DG-category.

Consider the anti-involution on $\mathcal{H}$ given by taking inverse. It induces an anti-involution $(\mathrm{DMod}^*(\mathcal{H}), \star) \simeq (\mathrm{DMod}^*(\mathcal{H}), \star)^{\mathrm{rev}}$. Hence we can also view $\mathcal{C}^\vee$ as a *left* $\mathrm{DMod}^*(\mathcal{H})$-module DG-category. In other words, $\mathcal{C}^\vee$ can be upgraded to an object in $\mathcal{H}$-mod.

Lemma D.1.10. *Suppose $\mathcal{C}_{\mathcal{H}}$ is dualizable in* DGCat*. Then we have a S-linear equivalence*

$$(\mathcal{C}_{\mathcal{H}})^\vee \simeq (\mathcal{C}^\vee)^{\mathcal{H}}.$$

Moreover, via this duality, the functors $\mathbf{pr}_{\mathcal{H}} : \mathcal{C} \to \mathcal{C}_{\mathcal{H}}$ and $\mathbf{oblv}^{\mathcal{H}} : (\mathcal{C}^\vee)^{\mathcal{H}} \to \mathcal{C}^\vee$ are dual to each other.

Proof. We have

$$\mathrm{Funct}(\mathcal{C}_{\mathcal{H}}, \mathrm{Vect}) \simeq \mathrm{Funct}(\mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C}, \mathrm{Vect}) \simeq \mathrm{Funct}_{\mathcal{H}^{\mathrm{rev}}}(\mathrm{DMod}(S), \mathrm{Funct}(\mathcal{C}, \mathrm{Vect})) \simeq$$
$$\simeq \mathrm{Funct}_{\mathcal{H}^{\mathrm{rev}}}(\mathrm{DMod}(S), \mathcal{C}^\vee) \simeq (\mathcal{C}^\vee)^{\mathcal{H}}.$$

□[Lemma D.1.10]

Lemma D.1.11. *Let $\mathcal{C} \in \mathcal{H}$-mod.*

(1) For any $\mathcal{D} \in$ DGCat, there is a canonical functor

$$\mathcal{C}^{\mathcal{H}} \otimes \mathcal{D} \to (\mathcal{C} \otimes \mathcal{D})^{\mathcal{H}}.$$

(2) For any $\mathcal{D} \in \mathrm{DMod}(S)$-mod, there is a canonical functor

$$\mathcal{C}^{\mathcal{H}} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D} \to (\mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D})^{\mathcal{H}}.$$

(3) The functors in (1) and (2) are equivalences if $\mathcal{D}$ is dualizable in DGCat*.*

(4) Suppose $\mathcal{C}$ is dualizable in DGCat*. The following statements are equivalent:*

(a) the functor in (1) is an equivalence for any $\mathcal{D} \in$ DGCat*;*

(b) the functor in (2) is an equivalence for any $\mathcal{D} \in \mathrm{DMod}(S)$*-mod;*

(c) $(\mathcal{C}^\vee)_{\mathcal{H}}$ *is dualizable in* $\mathrm{DMod}(S)$*-mod,*

(d) $(\mathcal{C}^\vee)_{\mathcal{H}}$ *is dualizable in* DGCat*.*

Proof. The functor in (2) is given by

$$\mathcal{C}^{\mathcal{H}} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D} \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C}) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D} \to \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D}) \simeq (\mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D})^{\mathcal{H}}.$$

The functor in (1) is obtained by replacing $\mathcal{D}$ in (2) by $\mathcal{D} \otimes \mathrm{DMod}(S)$.

If $\mathcal{D}$ is dualizable in $\mathrm{DMod}(S)$-mod, writing $\mathcal{E}$ for its dual, we have

$$\mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C}) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D} \simeq \mathrm{Funct}_S(\mathcal{E}, \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C})) \simeq$$
$$\simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathrm{Funct}_S(\mathcal{E}, \mathcal{C})) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D}).$$

This proves (3).

It remains to prove (4). Note that by Lemma B.3.3, B.3.1, $\mathcal{C}$ is also dualizable in $\mathrm{DMod}(S)$-mod, and the duals of $\mathcal{C}$ in these two senses are canonically identified.

By construction, we have $(b) \Rightarrow (a)$.

Suppose that (c) holds. By Lemma D.1.10, $(\mathcal{C}^\vee)_{\mathcal{H}}$ and $\mathcal{C}^{\mathcal{H}}$ are dual to each other in $\mathrm{DMod}(S)$-mod. Hence we have

$$\mathcal{C}^{\mathcal{H}} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D} \simeq \mathrm{Funct}_S((\mathcal{C}^\vee)_{\mathcal{H}}, \mathcal{D}) \simeq \mathrm{Funct}_S(\mathcal{C}^\vee \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathrm{DMod}(S), \mathcal{D}) \simeq$$
$$\simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathrm{Funct}_S(\mathcal{C}^\vee, \mathcal{D})) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D}) \simeq (\mathcal{C} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{D})^{\mathcal{H}}.$$

It follows from construction that this equivalence is the functor in (2). This proves $(c) \Rightarrow (b)$.

By Lemma B.3.3, we have $(d) \Rightarrow (c)$.

It remains to prove $(a) \to (d)$. For any testing $\mathcal{D} \in \mathrm{DGCat}$, we have

$$\mathrm{Funct}_{\mathrm{Vect}}((\mathcal{C}^\vee)_{\mathcal{H}}, \mathcal{D}) \simeq \mathrm{Funct}_{\mathrm{Vect}}(\mathcal{C}^\vee \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathrm{DMod}(S), \mathcal{D}) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathrm{Funct}_{\mathrm{Vect}}(\mathcal{C}^\vee, \mathcal{D})) \simeq$$
$$\simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C} \otimes \mathcal{D}) \simeq (\mathcal{C} \otimes \mathcal{D})^{\mathcal{H}} \simeq \mathcal{C}^{\mathcal{H}} \otimes \mathcal{D}.$$

This proves that $(\mathcal{C}^\vee)_{\mathcal{H}}$ and $\mathcal{C}^{\mathcal{H}}$ are dual to each other.

□[Lemma D.1.11]

Remark D.1.12. In the constructible contexts, we can only prove $(b) \Leftrightarrow (c) \Rightarrow (d) \Leftrightarrow (a)$.

D.2 Pro-smooth group schemes

Construction D.2.1. Suppose $p:\mathcal{H}\to S$ is a *pro-smooth group scheme*, i.e. a filtered limit of smooth affine groups schemes under smooth surjections. In the proof of [Ras16, Proposition 2.17.9], it is shown[123] that the functor p_* has a $(\mathcal{H},\mathcal{H})$-linear left adjoint $p^*:\mathrm{DMod}(S)\to\mathrm{DMod}^*(\mathcal{H})$[124].

Therefore for any $\mathcal{C}\in\mathcal{H}$-mod, the functor $\mathbf{oblv}^{\mathcal{H}}$ has a $\mathcal{H}$-linear right adjoint

$$\mathbf{Av}_*^{\mathcal{H}}:\mathcal{C}\simeq\mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}^*(\mathcal{H}),\mathcal{C})\xrightarrow{-\circ p^*}\mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S),\mathcal{C})\simeq\mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}). \tag{D.5}$$

By [Ras16, Proposition 2.17.9], the adjoint pair $(\mathbf{oblv}^{\mathcal{H}},\mathbf{Av}_*^{\mathcal{H}})$ is co-monadic.

Construction D.2.2. Similarly, the functor $\mathbf{pr}_{\mathcal{H}}$ has a $\mathcal{H}$-linear left adjoint

$$\mathbf{pr}_{\mathcal{H}}^L:\mathbf{triv}_{\mathcal{H}}(\mathcal{C}_{\mathcal{H}})\simeq\mathrm{DMod}(S)\underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes}\mathcal{C}\xrightarrow{p^*\otimes\mathbf{Id}}\mathrm{DMod}^*(\mathcal{H})\underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes}\mathcal{C}\simeq\mathcal{C}.$$

We have

Lemma D.2.3. *The adjoint pair* $(\mathbf{pr}_{\mathcal{H}}^L,\mathbf{pr}_{\mathcal{H}})$ *is co-monadic.*

Proof. Using the (co-monadic) Barr-Beck-Lurie theorem, it suffices to prove

- the functor $\mathbf{pr}_{\mathcal{H}}^L$ is conservative;
- the functor $\mathbf{pr}_{\mathcal{H}}^L$ preserves limits of $\mathbf{pr}_{\mathcal{H}}^L$-split cosimplicial objects.

We will prove the following stronger results:

(1) the endo-functor $\mathbf{pr}_{\mathcal{H}}\circ\mathbf{pr}_{\mathcal{H}}^L$ is conservative;

(2) any $\mathbf{pr}_{\mathcal{H}}^L$-split cosimplicial object in $\mathcal{C}_{\mathcal{H}}$ splits.

Define $A:=p_*\circ p^*(\omega_S)\in\mathrm{DMod}(S)$. Note that A is naturally an augmented commutative Hopf algebra object in the monoidal category $(\mathrm{DMod}(S),\otimes^!)$. Indeed, the commutative algebra structure is given by the monad $p_*\circ p^*$, and the co-associative co-algebra structure is given by the group structure on $\mathcal{H}\to S$. These two structures can be assembled to a Hopf algebra structure because the functor

$$(\mathrm{Sch}_{\text{placid over }S})^{\mathrm{op}}\to\mathrm{CommAlg}(\mathrm{DMod}(S)),\ (p:\mathcal{Y}\to S)\mapsto p_*\circ p^*(\omega_S)$$

[123] In fact, *loc.cit.* proved that $p^!:\mathrm{DMod}(S)\to\mathrm{DMod}^!(\mathcal{H})$ has a $(\mathcal{H},\mathcal{H})$-linear right adjoint. We get the desired claim by passing to duals.

[124] It is denoted by $p^{!,\mathrm{ren}}$ in [Ras15b].

can be upgraded to a symmetric monoidal functor. It follows from construction that this commutative Hopf algebra object is augmented.

Now consider the full subcategory $\mathrm{DMod}^*(\mathcal{H})^0$ of $\mathrm{DMod}^*(\mathcal{H})$ generated (under colimits and extensions) by the image of p^*. Since p^* sends compact objects to compact objects, the category $\mathrm{DMod}^*(\mathcal{H})^0$ is compactly generated, and the inclusion functor $\iota : \mathrm{DMod}^*(\mathcal{H})^0 \to \mathrm{DMod}^*(\mathcal{H})$ sends compact objects to compact objects. Hence ι has a continuous right adjoint ι^R. Consider the functor $F : \mathrm{DMod}(S) \to \mathrm{DMod}^*(\mathcal{H})^0$ obtained from p^* (such that $p^* \simeq \iota \circ F$). Note that the adjoint pair (p^*, p_*) induces an adjoint pair

$$F : \mathrm{DMod}(S) \rightleftharpoons \mathrm{DMod}^*(\mathcal{H})^0 : p_* \circ \iota,$$

which is monadic by the (monadic) Barr-Beck-Lurie theorem. Moreover, this monad is given by tensoring with the commutative algebra object $A \in \mathrm{DMod}(S)$. Hence we obtain a commutative diagram of adjoint pairs:

$$\begin{tikzcd} \mathrm{DMod}(S) \arrow[r, shift left, "p^*"] \arrow[d, shift right, "\mathbf{ind}_A"'] & \mathrm{DMod}^*(\mathcal{H}) \arrow[l, shift left, "p_*"] \arrow[d, shift left, "\iota^R"] \\ A\text{-mod}(\mathrm{DMod}(S)) \arrow[r, shift left, "\simeq"] \arrow[u, shift right, "\mathbf{oblv}_A"'] & \mathrm{DMod}^*(\mathcal{H})^0. \arrow[l, shift left, "\simeq"] \arrow[u, shift left, "\iota"] \end{tikzcd} \tag{D.6}$$

By Lemma D.2.4 below, $\mathrm{DMod}^*(\mathcal{H})^0$ is a monoidal ideal of $(\mathrm{DMod}^*(\mathcal{H}), \star)$ and the functor ι^R is monoidal. Hence all the four categories in (D.6) are naturally $(S, \mathcal{H})$-bimodule categories. We claim all the fucntors in (D.6) are naturally $(S, \mathcal{H})$-linear. The claim is obvious for $\mathbf{ind}_A$ and $\mathbf{oblv}_A$. The claim for ι and ι^R follows from Lemma D.2.4. Also, as mentioned in § D.2, p_* and p^* are naturally $(\mathcal{H}, \mathcal{H})$-linear therefore $(S, \mathcal{H})$-linear. Finally, it follows formally that the equivalence $A\text{-mod}(\mathrm{DMod}(S)) \simeq \mathrm{DMod}^*(\mathcal{H})^0$ is naturally $(S, \mathcal{H})$-linear.

Therefore we can tensor (D.6) with the object $\mathcal{C} \in \mathcal{H}\text{-mod}$ and obtain the following commutative diagram of adjoint pairs:

$$\begin{tikzcd} \mathcal{C}_{\mathcal{H}} \arrow[r, shift left, "\mathbf{pr}^L"] \arrow[d, shift right, "\mathbf{ind}_A"'] & \mathcal{C} \arrow[l, shift left, "\mathbf{pr}"] \arrow[d, shift left, "\epsilon^R"] \\ A\text{-mod}(\mathcal{C}_{\mathcal{H}}) \arrow[r, shift left, "\simeq"] \arrow[u, shift right, "\mathbf{oblv}_A"'] & \mathrm{DMod}^*(\mathcal{H})^0 \otimes_{\mathrm{DMod}^*(\mathcal{H})} \mathcal{C}. \arrow[l, shift left, "\simeq"] \arrow[u, shift left, "\epsilon"] \end{tikzcd} \tag{D.7}$$

Note that all the four categories are naturally $\mathrm{DMod}(S)$-modules and all the functors are naturally $\mathrm{DMod}(S)$-linear. Since ι is fully faithful, the unit natural transformation $\mathbf{Id} \to \iota^R \circ \iota$ is an isomorphism. Hence by construction, the unit natural transformation $\mathbf{Id} \to \epsilon^R \circ \epsilon$ is an isomorphism. Therefore ϵ is fully faitfhul.

This implies the endo-functor $\mathbf{pr} \circ \mathbf{pr}^L$ is isomorphic to the endo-functor $\mathbf{oblv}_A \circ \mathbf{ind}_A$. Note that $\mathbf{oblv}_A$ is conservative. On the other hand, $\mathbf{ind}_A$ is conservative because the augmentation $A \to \omega_S$ provides a left inverse to it. Hence $\mathbf{pr} \circ \mathbf{pr}^L$ is conservative. This proves (1).

Now let $x^\bullet$ be a $\mathbf{pr}^L$-split cosimplicial object in $\mathcal{C}_{\mathcal{H}}$. Let $y \in \mathcal{C}$ be the totalization of $\mathbf{pr}^L(x^\bullet)$. By definition, we have a split augmented cosimplicial diagram $y \to \mathbf{pr}^L(x^\bullet)$. Applying the endo-functor $\epsilon \circ \epsilon^R$ to this diagram, we obtain another split augmented cosimplicial diagram

$$\epsilon \circ \epsilon^R(y) \to \epsilon \circ \epsilon^R \circ \mathbf{pr}^L(x^\bullet).$$

However, it follows from (D.7) (and ϵ being fully faithful) that $\epsilon \circ \epsilon^R \circ \mathbf{pr}^L \simeq \mathbf{pr}^L$. Hence by uniquesness of splitting, we obtain an isomorphism $y \simeq \epsilon \circ \epsilon^R(y)$. In particular, y is contained in the essential image of ϵ. Since ϵ is fully faithful, using (D.7), we see that $x^\bullet$ is $\mathbf{ind}_A$-split. Therefore $x^\bullet$ itself splits because $\mathbf{ind}_A$ has a left inverse. This proves (2).

□[Lemma D.2.3]

Lemma D.2.4. *(1)* $\mathrm{DMod}^*(\mathcal{H})^0$ *is a monoidal ideal of the monoidal category* $(\mathrm{DMod}^*(\mathcal{H}), \star)$.

(2) The right-lax monoidal functor $\iota^R : \mathrm{DMod}^*(\mathcal{H}) \to \mathrm{DMod}^*(\mathcal{H})^0$ *(between* non-unital *monoidal categories) is strict. In particular,* $\mathrm{DMod}^*(\mathcal{H})^0$ *is a* unital *monoidal category.*

Proof. To prove (1), by symmetry, it suffices to show that $\mathrm{DMod}^*(\mathcal{H})^0$ is a left monoidal ideal of $(\mathrm{DMod}^*(\mathcal{H}), \star)$. It suffices to prove that for any $\mathcal{F} \in \mathrm{DMod}^*(\mathcal{H})$ and $\mathcal{G} \in \mathrm{DMod}(S)$, the object $\mathcal{F} \star p^*(\mathcal{G})$ is contained in $\mathrm{DMod}^*(\mathcal{H})^0$. We first claim there is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{DMod}^*(\mathcal{H} \times \mathcal{H}) \arrow[r, "!\text{-pullback}"] & \mathrm{DMod}^*(\mathcal{H} \times_S \mathcal{H}) \\
\mathrm{DMod}^*(\mathcal{H} \times S) \arrow[u, "(\mathrm{Id} \times p)^*"] \arrow[r, "!\text{-pullback}"] & \mathrm{DMod}^*(\mathcal{H}). \arrow[u, "p_1^*"']
\end{tikzcd}$$

Indeed, by [Ras15b, Example 6.12.4], after choosing a suitable dimension theory on $\mathcal{H}$ and using it to identify DMod^* with $\mathrm{DMod}^!$, all the functors in the above diagram are !-pullback functors (in the theory $\mathrm{DMod}^!$).

Using the above diagram, to prove (1), it suffices to prove that the image of

$$m_* \circ p_1^* : \mathrm{DMod}^*(\mathcal{H}) \to \mathrm{DMod}^*(\mathcal{H} \underset{S}{\times} \mathcal{H}) \to \mathrm{DMod}^*(\mathcal{H})$$

is contained in $\mathrm{DMod}^*(\mathcal{H})^0$. However, this functor is isomorphic to $p_{2,*} \circ p_1^* \simeq p^* \circ p_*$. This proves (1).

It remains to prove (2). By (1), $\mathrm{DMod}^*(\mathcal{H})^0$ is a *non-unital* monoidal category and ι is a *non-unital* monoidal functor. Recall that p_* is naturally a monoidal functor. Hence $p_* \circ \iota$ is naturally a non-unital monoidal functor. Note that $p_* \circ \iota$ is conservative because its left adjoint F generates (under colimits and extensions) the category $\mathrm{DMod}^*(\mathcal{H})^0$. Hence it remains to prove that the right-lax monoidal functor $p_* \circ \iota \circ \iota^R$ is strict. However, this right-lax monoidal functor is isomorphic to p_* by (D.6). This proves (2).

$\square$[Lemma D.2.4]

Construction D.2.5. For any pro-smooth $\mathcal{H}$, applying the adjoint pair $(\mathbf{triv}_{\mathcal{H}}^L, \mathbf{triv}_{\mathcal{H}})$ to (D.5), we obtain a S-linear functor $\theta_{\mathcal{H}} : \mathcal{C}_{\mathcal{H}} \to \mathcal{C}^{\mathcal{H}}$ such that $\mathbf{Av}_*^{\mathcal{H}} \simeq \theta_{\mathcal{H}} \circ \mathbf{pr}_{\mathcal{H}}$.

We have:

Lemma D.2.6. *The functor $\theta_{\mathcal{H}} : \mathcal{C}_{\mathcal{H}} \to \mathcal{C}^{\mathcal{H}}$ defined above is an equivalence.*

Proof. By [Ras16, Proposition 2.17.9] and Lemma D.2.3, the co-monadic adjoint pairs $(\mathbf{oblv}^{\mathcal{H}}, \mathbf{Av}_*^{\mathcal{H}})$ and $(\mathbf{pr}_{\mathcal{H}}^L, \mathbf{pr}_{\mathcal{H}})$ are both co-monadic. Hence it remains to show that the corresponding co-monads are isomorphic. Write $T := p^* \circ p_*$ for the co-monad acting on $\mathrm{DMod}^*(\mathcal{H})$. Note that T is naturally $(\mathcal{H}, \mathcal{H})$-linear. It follows from definition that the desired two co-monads are given respectively by

$$\mathcal{C} \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}^*(\mathcal{H}), \mathcal{C}) \xrightarrow{-\circ T} \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}^*(\mathcal{H}), \mathcal{C}) \simeq \mathcal{C},$$

$$\mathcal{C} \simeq \mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C}) \xrightarrow{T \otimes \mathbf{Id}} \mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C}) \simeq \mathcal{C}.$$

This makes the desired claim formal and manifest.

$\square$[Lemma D.2.6]

Lemma D.2.7. *Let $\mathcal{H} \to S$ be a pro-smooth group scheme. Suppose $\mathcal{C} \in \mathcal{H}$-mod is dualizable in DGCat. Then $\mathcal{C}_{\mathcal{H}}$ is dualizable in DGCat.*

Proof. We have:

$$\mathcal{C}_{\mathcal{H}} \otimes - \simeq (\mathcal{C} \otimes -)_{\mathcal{H}} \overset{\theta_{\mathcal{H}}}{\simeq} (\mathcal{C} \otimes -)^{\mathcal{H}} \simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathcal{C} \otimes -) \simeq$$

$$\simeq \mathrm{Funct}_{\mathcal{H}}(\mathrm{DMod}(S), \mathrm{Funct}(\mathcal{C}^\vee, -)) \simeq \mathrm{Funct}(\mathcal{C}^\vee \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathrm{DMod}(S), -).$$

Hence by §B.2.6, $\mathcal{C}_{\mathcal{H}}$ is dualizable in DGCat.

$\square$[Lemma D.2.7]

Remark D.2.8. If $\mathcal{H}$ is further assumed to be *pro-unipotent* (see [Ras16, Definition 2.18.1]), then p^* is fully faithful. Then the natural transformation $\mathbf{Id} \to \mathbf{Av}_*^{\mathcal{H}} \circ \mathbf{oblv}^{\mathcal{H}}$ is also an isomorphism. Hence $\mathbf{oblv}^{\mathcal{H}}$ is fully faithful. Similarly, the natural transformation $\mathbf{Id} \to \mathbf{pr}_{\mathcal{H}} \circ \mathbf{pr}_{\mathcal{H}}^L$ is an isomorphism. Hence $\mathbf{pr}_{\mathcal{H}}^L$ (and therefore the non-continuous functor $\mathbf{pr}_{\mathcal{H}}^R$) is fully faithful. Using these, it is easy to show

$$\mathbf{triv}_{\mathcal{H}}(\mathcal{D})_{\mathcal{H}} \simeq \mathcal{D} \simeq \mathbf{triv}_{\mathcal{H}}(\mathcal{D})^{\mathcal{H}}.$$

We warn that the same formula is *false* for general $\mathcal{H}$.

D.3 Case of ind-group schemes

Suppose that $\mathcal{H}$ is an (placid) *ind-group scheme* over S. This means we can write it as a filtered colimit of group schemes connected by closed embeddings. By construction, we have an equivalence of monoidal categories

$$\mathrm{DMod}^*(\mathcal{H}) \simeq \underset{*\text{-pushforward}}{\mathrm{colim}} \mathrm{DMod}^*(\mathcal{H}_\alpha).$$

Hence we have a

$$\mathcal{H}\text{-mod} \simeq \lim_{\mathbf{res}} \mathcal{H}_\alpha\text{-mod}.$$

It follows formally that, for any $\mathcal{C} \in \mathcal{H}$-mod, we have

$$\operatorname*{colim}_\alpha \mathbf{ind}_{\mathcal{H}_\alpha \to \mathcal{H}} \circ \mathbf{res}_{\mathcal{H} \to \mathcal{H}_\alpha}(\mathcal{C}) \simeq \mathcal{C},\ \mathcal{C} \simeq \lim_\alpha \mathbf{coind}_{\mathcal{H}_\alpha \to \mathcal{H}} \circ \mathbf{res}_{\mathcal{H} \to \mathcal{H}_\alpha}(\mathcal{C}). \tag{D.8}$$

Therefore we have

$$\mathcal{C}_{\mathcal{H}} \simeq \operatorname*{colim}_\alpha (\mathbf{res}_{\mathcal{H} \to \mathcal{H}_\alpha}(\mathcal{C}))_{\mathcal{H}_\alpha}, \mathcal{C}^{\mathcal{H}} \simeq \lim_\alpha (\mathbf{res}_{\mathcal{H} \to \mathcal{H}_\alpha}(\mathcal{C}))^{\mathcal{H}_\alpha}. \tag{D.9}$$

Remark D.3.1. If $\mathcal{H}$ is further assumed to be *ind-pro-unipotent* (i.e. each $\mathcal{H}_\alpha$ is pro-unipotent), the functors $\mathbf{oblv}^{\mathcal{H}_\alpha}$ (resp. $\mathbf{pr}_{\mathcal{H}\alpha}$) are fully faithful (resp. localization functors). Hence the functors $\mathbf{oblv}^{\mathcal{H}_\beta \to \mathcal{H}_\alpha}$ (resp. $\mathbf{pr}_{\mathcal{H}_\alpha \to \mathcal{H}_\beta}$) are fully faithful (resp. localization functors). Note that the index category in (D.9) is filtered. It follows formally that $\mathbf{oblv}^{\mathcal{H}}$ is fully faithful and $\mathbf{pr}_{\mathcal{H}}$ is a localization functor.

As before, we also have

$$\mathbf{triv}(\mathcal{D})_{\mathcal{H}} \simeq \mathcal{D} \simeq \mathbf{triv}(\mathcal{D})^{\mathcal{H}}.$$

D.4 Geometric action

Let $\mathcal{H} \to S$ be a (placid) group indscheme, and $\mathcal{Y} \to S$ be a placid indscheme equipped with an $\mathcal{H}$-action. By definition, we can upgrade $\mathrm{DMod}^*(\mathcal{Y})$ to an object in $\mathcal{H}$-mod. Explicitly, the $\mathrm{DMod}^*(\mathcal{H})$-module structure is given by

$$\mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}^*(\mathcal{Y}) \simeq \mathrm{DMod}^*(\mathcal{H} \underset{S}{\times} \mathcal{Y}) \xrightarrow{\mathrm{act}_*} \mathrm{DMod}^*(\mathcal{Y}),$$

where the first equivalence is given by Lemma C.4.4. Dually, we can upgrade $\mathrm{DMod}^!(\mathcal{Y})$ to be in $\mathcal{H}$-mod, with the $\mathrm{DMod}^!(\mathcal{H})$-comodule structure given by

$$\mathrm{DMod}^!(\mathcal{Y}) \xrightarrow{\mathrm{act}^!} \mathrm{DMod}^!(\mathcal{H} \underset{S}{\times} \mathcal{Y}) \simeq \mathrm{DMod}^!(\mathcal{H}) \underset{\mathrm{DMod}(S)}{\otimes} \mathrm{DMod}^!(\mathcal{Y}),$$

where the last equivalence is by [Ras15b, Proposition 6.9.1(2)]. By construction, the duality between $\mathrm{DMod}^!(\mathcal{Y})$ and $\mathrm{DMod}^*(\mathcal{Y})$ are compatible with the $\mathcal{H}$-module structures in the sense of § D.1.9.

Using Lemma C.4.4 and [Ras15b, Proposition 6.9.1(2)], one can write the cobar and bar constructions as

$$\mathrm{DMod}^!(\mathcal{Y})^{\mathcal{H}} \simeq \lim_{\Delta} \mathrm{DMod}^!(\mathcal{H}^{\times_S^\bullet} \underset{S}{\times} \mathcal{Y}),\ \mathrm{DMod}^*(\mathcal{Y})_{\mathcal{H}} \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{DMod}^*(\mathcal{H}^{\times_S^\bullet} \underset{S}{\times} \mathcal{Y}). \tag{D.10}$$

Suppose we have an augmented simplicial diagram (over S):

$$\mathcal{H}^{\times_S^\bullet} \underset{S}{\times} \mathcal{Y} \to \mathcal{Q},$$

where $\mathcal{Q}$ is any prestack. Using (D.10), we obtain functors

$$\mathrm{DMod}^!(\mathcal{Q}) \to \mathrm{DMod}^!(\mathcal{Y})^{\mathcal{H}},\ \mathrm{DMod}^*(\mathcal{Y})_{\mathcal{H}} \to \mathrm{DMod}^*(\mathcal{Q}). \tag{D.11}$$

We have the following technical result:

Lemma D.4.1. *In the above setting, suppose*

- $Y := \mathcal{Y}$ *and* $Q := \mathcal{Q}$ *are ind-finite type indschemes,*
- *the projection* $q : Y \to Q$ *admits a section* $s : Q \to Y$*,*
- $\mathcal{H}$ *is ind-pro-unipotent and acts transitively on the fibers of* $Y \to Q$*.*

Then the fucntors D.11 are isomorphisms.

Proof. Consider the map

$$\mathcal{H}^{\times_S^n} \underset{S}{\times} Y \to Y \underset{Q}{\times} Y^{\times_Q^n},\ (g_1, \cdots, g_n, y) \mapsto (g_1 \cdots g_n y, g_2 \cdots g_n y, \cdots, y). \tag{D.12}$$

It induces cosimplicial (resp. simplicial) functors:

$$\mathrm{DMod}(Y \underset{Q}{\times} Y^{\times_Q^\bullet}) \to \mathrm{DMod}^!(\mathcal{H}^{\times_S^\bullet} \underset{S}{\times} Y), \tag{D.13}$$

$$\mathrm{DMod}^*(\mathcal{H}^{\times_S^\bullet}\underset{S}{\times}Y)\to\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet}). \tag{D.14}$$

By assumption, (D.12) is surjective and has ind-contractible fibers, hence the functors in (D.13) are fully faithful, and the functors in (D.14) are localizations. Note that the [0]-terms of (D.13) and (D.14) are both equivalences. It follows formally that they induce equivalences

$$\lim_{\Delta}\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet})\to\lim_{\Delta}\mathrm{DMod}^!(\mathcal{H}^{\times_S^\bullet}\underset{S}{\times}Y),\ \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\,\mathrm{DMod}^*(\mathcal{H}^{\times_S^\bullet}\underset{S}{\times}Y)\to\underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\,\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet}).$$

Hence it remains to prove the following equivalences:

$$\mathrm{DMod}(Q)\simeq\lim_{\Delta}\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet}),\ \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\,\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet})\simeq\mathrm{DMod}(Q). \tag{D.15}$$

A standard argument reduces to the case when Q is an affine scheme of finite type.

Consider the base-change functor $\mathrm{DMod}(Y)\otimes_Q - : Q\text{-mod}\to\mathrm{DMod}(Y)\text{-mod}$. By the existence of the section s, the above functor has a left inverse, hence is conservative. Hence it suffices to prove (D.15) become equivalences after applying this base-change. However, since $\mathrm{DMod}(Y)$ is dualizable in Q-mod, $\mathrm{DMod}(Y)\otimes_Q -$ commutes with both colimits and limits. Hence it remains to prove

$$\mathrm{DMod}(Y)\simeq\lim_{\Delta}\mathrm{DMod}(Y)\underset{Q}{\otimes}\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet}),\ \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\,\mathrm{DMod}(Y)\underset{Q}{\otimes}\mathrm{DMod}(Y\underset{Q}{\times}Y^{\times_Q^\bullet})\simeq\mathrm{DMod}(Y).$$

Using Lemma C.4.4, it remains to prove

$$\mathrm{DMod}(Y)\simeq\lim_{\Delta}\mathrm{DMod}(Y\underset{Q}{\times}Y\underset{Q}{\times}Y^{\times_Q^\bullet}),\ \underset{\Delta^{\mathrm{op}}}{\mathrm{colim}}\,\mathrm{DMod}(Y\underset{Q}{\times}Y\underset{Q}{\times}Y^{\times_Q^\bullet})\simeq\mathrm{DMod}(Y).$$

Now we are done because the above augemented cosimplicial (resp. simplical) diagram splits.

□[Lemma D.4.1]

D.4.2 (Finite type case)**.** Let $H\to S$ be a smooth group scheme, and $Y\to S$ be an ind-finite type indscheme acted by H. Suppose further that Y can be written as a filtered colimit of finite type schemes stabilized by H connected by closed embeddings. This implies $Q:=Y/H$ exists as an ind-algebraic stack.

By construction, the identification $\mathrm{DMod}^*(Y)\simeq\mathrm{DMod}^!(Y)$ is compatible with the H-module structures. Therefore, (D.10) and smooth descent for D-modules (on finite type schemes) imply

$$\mathrm{DMod}(Y)^H\simeq\mathrm{DMod}^!(Y/H),\ \mathrm{DMod}(Y)_H\simeq\mathrm{DMod}^*(Y/H). \tag{D.16}$$

D.5 Action by quotient group

Let $\mathcal{H} \to S$ be a (placid) group indscheme, and $\mathcal{N}$ be a normal (placid) sub-group indscheme. Consider the functor $(\mathrm{Sch}^{\mathrm{aff}}_{/S})^{\mathrm{op}} \to \mathrm{Set}, T \mapsto \mathrm{Maps}_S(T, \mathcal{H})/\mathrm{Maps}_S(T, \mathcal{N})$. Suppose it is represented by a placid indscheme $\mathcal{Q}$ over S. Then $\mathcal{Q} \to S$ is a (placid) group indscheme. We refer $\mathcal{Q}$ as the *quotient group indscheme* of $\mathcal{H}$ by $\mathcal{N}$.

Consider the obvious commutative diagram

$$\begin{tikzcd} \mathcal{Q}\text{-mod} \arrow[r, "\mathbf{res}_{\mathcal{Q}\to\mathcal{H}}"] \arrow[d, "\mathbf{res}_{\mathcal{Q}\to S}"] & \mathcal{H}\text{-mod} \arrow[d, "\mathbf{res}_{\mathcal{Q}\to\mathcal{N}}"] \\ \mathrm{DMod}(S)\text{-mod} \arrow[r, "\mathbf{triv}_{\mathcal{N}}"] & \mathcal{N}\text{-mod}. \end{tikzcd} \tag{D.17}$$

We have

Lemma D.5.1. *Consider the $\mathcal{N}$-action on $\mathcal{H}$ given by left multiplication. Suppose the functor $\mathrm{DMod}^*(\mathcal{H})_{\mathcal{N}} \to \mathrm{DMod}^*(\mathcal{Q})$ (in (D.11)) is an equivalence. Then:*

(1) The commutative square (D.17) is both left adjointable and right adjointable along the horizontal directions.

(2) For any $\mathcal{C} \in \mathcal{H}$-mod, there are natural $\mathcal{Q}$-module structures on $\mathcal{C}^{\mathcal{N}}$ and $\mathcal{C}_{\mathcal{N}}$ such that $\mathcal{C}^{\mathcal{H}} \simeq (\mathcal{C}^{\mathcal{N}})^{\mathcal{Q}}$ and $\mathcal{C}_{\mathcal{H}} \simeq (\mathcal{C}_{\mathcal{N}})_{\mathcal{Q}}$.

(3) The commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\mathcal{H}} \arrow[r, "\mathbf{oblv}^{\mathcal{H}\to\mathcal{Q}}"] \arrow[d, "\mathbf{oblv}^{\mathcal{H}\to\mathcal{N}}"] & \mathcal{C}^{\mathcal{Q}} \arrow[d, "\mathbf{oblv}^{\mathcal{Q}}"] \\ \mathcal{C}^{\mathcal{N}} \arrow[r, "\mathbf{oblv}^{\mathcal{N}}"] & \mathcal{C}. \end{tikzcd}$$

is right adjointable along the vertical direction.

Proof. Note that (2) is a corollary of (1). We first prove (1). For any $\mathcal{C} \in \mathcal{H}$-mod, we have

$$\mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{N})}{\otimes} \mathcal{C} \simeq \mathrm{DMod}(S) \underset{\mathrm{DMod}^*(\mathcal{N})}{\otimes} \mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C} \simeq \mathrm{DMod}^*(\mathcal{H})_{\mathcal{N}} \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C} \simeq \mathrm{DMod}^*(\mathcal{Q}) \underset{\mathrm{DMod}^*(\mathcal{H})}{\otimes} \mathcal{C}.$$

This proves the claim on left adjointable in (1).

Consider the $\mathcal{N}$-action on $\mathcal{H}$ given by right multiplication. By symmetry, the functor $\mathrm{DMod}^*(\mathcal{H})_{\mathcal{N},r} \to$

$\mathrm{DMod}^*(\mathcal{Q})$ is also an equivalence. Hence for any $\mathcal{C} \in \mathrm{DMod}(S)$-mod, we have

$$\mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{N})}{\otimes} \mathbf{triv}_{\mathcal{N}}(\mathcal{C}) \simeq \mathrm{DMod}^*(\mathcal{H}) \underset{\mathrm{DMod}^*(\mathcal{N})}{\otimes} \mathrm{DMod}(S) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C} \simeq$$
$$\simeq \mathrm{DMod}^*(\mathcal{H})_{\mathcal{N},r} \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C} \simeq \mathrm{DMod}^*(\mathcal{Q}) \underset{\mathrm{DMod}(S)}{\otimes} \mathcal{C}.$$

This proves that (D.17) is left adjointable along the vertical directions, which implies its right adjointability along the horizontal direction (because the relevant right adjoints exist). This proves (1).

(3) follows from [Ras16, Corollary 2.17.10].

□[Lemma D.5.1]

Lemma D.5.2. *Suppose $\mathcal{H} \to \mathcal{Q}$ has a splitting $\mathcal{Q} \to \mathcal{H}$, then the assumption of Lemma D.5.1 is satisfied. Moreover:*

(1) For any $\mathcal{C} \in \mathcal{H}$-mod, the functors $\mathbf{oblv}^{\mathcal{N}} : \mathcal{C}^{\mathcal{N}} \to \mathcal{C}$ and $\mathbf{pr}_{\mathcal{N}} : \mathcal{C} \to \mathcal{C}_{\mathcal{N}}$ are $\mathcal{Q}$-linear, where the $\mathcal{Q}$-module structures on $\mathcal{C}$ is given by restriction along the splitting $\mathcal{Q} \to \mathcal{H}$.

(2) If $\mathcal{N}$ is further assumed to be ind-pro-unipotent, then for any $\mathcal{C} \in \mathcal{H}$-mod, the commutative diagram in Lemma D.5.1(3) is Cartesian. Moreover, both horizontal functors are fully faithful.

Proof. Note that the splitting provides an isomorphism between $\mathcal{H}$ and $\mathcal{N} \times_S \mathcal{Q}$ as indschemes equipped with $\mathcal{N}$-actions. Hence by [Ras15b, Proposition 6.7.1][125] we obtain an equivalence

$$\operatorname*{colim}_{\Delta^\bullet} \mathrm{DMod}^*(\mathcal{N}^{\times_S^\bullet} \underset{S}{\times} \mathcal{H}) \simeq \mathrm{DMod}^*(\mathcal{Q}).$$

By Lemma C.4.4, the above simplicial diagram can be identified with the bar construction calculating $\mathrm{DMod}^*(\mathcal{H})_{\mathcal{N}}$. This proves the desired equivalence $\mathrm{DMod}^*(\mathcal{H})_{\mathcal{N}} \simeq \mathrm{DMod}^*(\mathcal{Q})$.

Let $\mathcal{C} \in \mathcal{H}$-mod. By Lemma D.5.1, the functor $\mathbf{oblv}^{\mathcal{N}} : \mathcal{C}^{\mathcal{N}} \to \mathcal{C}$ can be upgraded to a $\mathcal{H}$-linear functor $\mathbf{res}_{\mathcal{Q}\to\mathcal{H}}(\mathcal{C}) \circ \mathbf{coind}_{\mathcal{H}\to\mathcal{Q}} \to \mathcal{C}$. The desired $\mathcal{Q}$-linear structure on $\mathbf{oblv}^{\mathcal{N}}$ is obtained by restriction along the splitting. This proves the claim for the invariance in (1). The proof for the coinvariance is similar.

It remains to prove (2). Consider the $\mathcal{Q}$-linear functor $\mathbf{oblv}^{\mathcal{N}} : \mathcal{C}^{\mathcal{N}} \to \mathcal{C}$ obtained in (1). It is fully faithful because $\mathcal{N}$ is ind-pro-unipotent. Now we are done by Lemma D.5.1(2) and Lemma D.1.5.

□[Lemma D.5.2]

[125]We apply *loc.cit.*to the case where the triple $(S, \mathcal{G}, \mathcal{P}_{\mathcal{G}})$ there is given by our $(\mathcal{Q}, \mathcal{N} \times_S \mathcal{Q}, \mathcal{H})$.

D.6 Application: functors given by kernels in equivariant settings

In this section, we generalize Lemma C.6.1 to equivariant settings. Let us point out that although the results from this section are correct in the constructible contexts with minor modifications, the statements and proofs would be much more technical. In fact, this is the main reason we choose to work in the D-module context in this paper.

Notation D.6.1. To avoid overfull boxes, in this section, we replace DMod by D.

D.6.2 (Settings)**.** Throughout this section, we fix a pro-smooth group scheme $\mathcal{H} \to S$. By Lemma D.2.6, for any $\mathcal{C} \in \mathcal{H}$-mod, there is a canonical equivalence $\theta_{\mathcal{H}} : \mathcal{C}_{\mathcal{H}} \to \mathcal{C}^{\mathcal{H}}$. Consequently, for any two ind-finite type indschemes Y, Z acted by $\mathcal{H}$, we have

- $\mathrm{D}(Y)^{\mathcal{H}}$ is self-dual both in DGCat and $\mathrm{D}(S)$-mod (by Lemma D.2.7 and Lemma B.3.3);
- a commutative diagram (by Lemma D.1.11 and (C.19))

$$\begin{tikzcd} \mathrm{D}(Y)^{\mathcal{H}} \otimes_{\mathrm{D}(S)} \mathrm{D}(Z)^{\mathcal{H}} \arrow[r, "\mathbf{Id}\otimes\mathbf{oblv}^{\mathcal{H}}"] \arrow[d, "\simeq"] & \mathrm{D}(Y)^{\mathcal{H}} \otimes_{\mathrm{D}(S)} \mathrm{D}(Z) \arrow[r, "\mathbf{oblv}^{\mathcal{H}}\otimes\mathbf{Id}"] \arrow[d, "\simeq"] & \mathrm{D}(Y) \otimes_{\mathrm{D}(S)} \mathrm{D}(Z) \arrow[d, "\simeq"] \\ \mathrm{D}(Y \times_S Z)^{\mathcal{H}\times_S\mathcal{H}} \arrow[r, "\mathbf{oblv}^{\mathcal{H}\times_S\mathcal{H}\to(\mathcal{H},1)}"] & \mathrm{D}(Y \times_S Z)^{\mathcal{H},1} \arrow[r, "\mathbf{oblv}^{\mathcal{H}}"] & \mathrm{D}(Y \times_S Z), \end{tikzcd}$$

  where $(\mathcal{H}, 1)$ indicates that $\mathcal{H}$ acts on the first factor of $Y \times_S Z$.

We shall use these results in this section without repeating the above arguments.

D.6.3 (Bi-equivariant case)**.** Consider the composition

$$\mathrm{D}(Y \underset{S}{\times} Y)^{\mathcal{H}\times_S\mathcal{H}} \simeq \mathrm{D}(Y)^{\mathcal{H}} \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Y)^{\mathcal{H}} \to \mathrm{D}(S), \tag{D.18}$$

where the last functor is the counit for the self-duality of $\mathrm{D}(Y)^{\mathcal{H}}$ in $\mathrm{D}(S)$-mod. Using it, we obtain a functor

$$F_{Y/\mathcal{H}\to Z/\mathcal{H}} : \mathrm{D}(Y \underset{S}{\times} Z)^{\mathcal{H}\times_S\mathcal{H}} \to \mathrm{Funct}_S(\mathrm{D}(Y)^{\mathcal{H}}, \mathrm{D}(Z)^{\mathcal{H}})$$

given by the composition

$$\mathrm{D}(Y)^{\mathcal{H}} \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Y \underset{S}{\times} Z)^{\mathcal{H}\times_S\mathcal{H}} \simeq \mathrm{D}(Y \underset{S}{\times} Y)^{\mathcal{H}\times_S\mathcal{H}} \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Z)^{\mathcal{H}} \xrightarrow{(D.18)\otimes\mathbf{Id}} \mathrm{D}(S) \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Z)^{\mathcal{H}} \simeq \mathrm{D}(Z)^{\mathcal{H}}.$$

As indicated by the notation, it can be considered as the functor given by kernels for the stacks $Y/\mathcal{H}$ and $Z/\mathcal{H}$.

The following lemma can be proved by unwinding the definitions.

Lemma D.6.4. *The composition*

$$\mathrm{D}(Y \underset{S}{\times} Z)^{\mathcal{H}\times_S\mathcal{H}} \xrightarrow{F_{Y/\mathcal{H}\to Z/\mathcal{H}}} \mathrm{Funct}_{\mathrm{D}(S)}(\mathrm{D}(Y)^{\mathcal{H}}, \mathrm{D}(Z)^{\mathcal{H}}) \simeq$$
$$\simeq (\mathrm{D}(Y)^{\mathcal{H}})^{\vee} \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Z)^{\mathcal{H}} \simeq \mathrm{D}(Y)^{\mathcal{H}} \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}(Z)^{\mathcal{H}}$$

is quasi-inverse to the equivalence in § D.18.

D.6.5 (Diagonal-equivariant case)**.** Let $\mathcal{C}, \mathcal{D} \in \mathcal{H}$-mod be two objects. Consider the functor induced by taking invariance:

$$\mathrm{Funct}_{\mathcal{H}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Funct}_{\mathrm{D}(S)}(\mathcal{C}^{\mathcal{H}}, \mathcal{D}^{\mathcal{H}}). \tag{D.19}$$

By definition, we have $\mathrm{Funct}_{\mathrm{D}(S)}(\mathcal{C}^{\mathcal{H}}, \mathcal{D}^{\mathcal{H}}) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}), \mathcal{D})$. Via this equivalence, the functor (D.19) is induced by $\mathbf{oblv}^{\mathcal{H}} : \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}) \to \mathcal{C}$. Recall that $\mathbf{oblv}^{\mathcal{H}}$ has a $\mathcal{H}$-linear right adjoint $\mathbf{Av}_*^{\mathcal{H}} : \mathcal{C} \to \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}})$, hence we obtain a left adjoint to (D.19)

$$\mathrm{Funct}_{\mathrm{D}(S)}(\mathcal{C}^{\mathcal{H}}, \mathcal{D}^{\mathcal{H}}) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}), \mathcal{D}) \xrightarrow{-\circ \mathbf{Av}_*^{\mathcal{H}}} \mathrm{Funct}_{\mathcal{H}}(\mathcal{C}, \mathcal{D}). \tag{D.20}$$

Explicitly, it sends an S-linear functor $\mathcal{C}^{\mathcal{H}} \to \mathcal{D}^{\mathcal{H}}$ to the composition

$$\mathcal{C} \xrightarrow{\mathbf{Av}_*^{\mathcal{H}}} \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}) \to \mathbf{triv}_{\mathcal{H}}(\mathcal{D}^{\mathcal{H}}) \xrightarrow{\mathbf{oblv}^{\mathcal{H}}} \mathcal{D}.$$

Lemma D.6.6. *We have*

(1) There is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{D}(Y \times_S Z)^{\mathcal{H}\times_S\mathcal{H}} \arrow[r, "\mathbf{oblv}^{\mathcal{H}\times_S\mathcal{H}\to(\mathcal{H},\mathrm{diag})}"] \arrow[d, "F_{Y/\mathcal{H}\to Z/\mathcal{H}}"', "\simeq"] & \mathrm{D}(Y \times_S Z)^{\mathcal{H},\mathrm{diag}} \arrow[r, "\mathbf{oblv}^{\mathcal{H}}"] \arrow[d, "F^{\mathcal{H}}_{Y\to Z}"', "\simeq"] & \mathrm{D}(Y \times_S Z) \arrow[d, "\simeq"] \\
\mathrm{Funct}_S(\mathrm{D}(Y)^{\mathcal{H}}, \mathrm{D}(Z)^{\mathcal{H}}) \arrow[r, "(D.20)"] & \mathrm{Funct}_{\mathcal{H}}(\mathrm{D}(Y), \mathrm{D}(Z)) \arrow[r] & \mathrm{Funct}_S(\mathrm{D}(Y), \mathrm{D}(Z)).
\end{tikzcd}$$

(2) Both of the commutative squares in (1) are right adjointable along the horizontal direction.

Proof. There is a canonical cocommutative Hopf algebra structure on $\mathrm{D}^*(\mathcal{H}) \in \mathrm{D}(S)$-mod, whose comultiplication is

$$\mathrm{D}^*(\mathcal{H}) \xrightarrow{\Delta_*} \mathrm{D}^*(\mathcal{H} \underset{S}{\times} \mathcal{H}) \simeq \mathrm{D}^*(\mathcal{H}) \underset{\mathrm{D}(S)}{\otimes} \mathrm{D}^*(\mathcal{H}),$$

where the last equivalence is given by Lemma C.4.4. Therefore for any $\mathcal{C}, \mathcal{D} \in \mathcal{H}$-mod, we can consider the diagonal action of $\mathcal{H}$ on $\mathcal{C} \otimes_{\mathrm{D}} (S)\mathcal{D}$. By construction, when $\mathcal{C}$ and $\mathcal{D}$ are given respectively by $\mathrm{D}(Y)$ and $\mathrm{D}(Z)$, the equivalence $\mathrm{D}(Y) \otimes_{\mathrm{D}(S)} \mathrm{D}(Z) \simeq \mathrm{D}(Y \times_S Z)$ is $\mathcal{H}$-linear.

Suppose $\mathcal{C}$ is dualizable in DGCat (and hence in $\mathrm{D}(S)$-mod by Lemma B.3.3)). Viewing $\mathcal{C}^\vee$ as an object in $\mathcal{H}$-mod as in § D.1.9, we have an equivalence

$$F^{\mathcal{H}}_{\mathcal{C}\to\mathcal{D}} : (\mathcal{C}^\vee \underset{\mathrm{D}(S)}{\otimes} \mathcal{D})^{\mathcal{H},\mathrm{diag}} \simeq \lim_{\Delta} \mathrm{Funct}_{\mathrm{D}(S)}(\mathrm{D}^*(\mathcal{H})^{\otimes^\bullet_{\mathrm{D}(S)}}, \mathcal{C}^\vee \underset{\mathrm{D}(S)}{\otimes} \mathcal{D}) \simeq$$
$$\simeq \lim_{\Delta} \mathrm{Funct}_{\mathrm{D}(S)}(\mathrm{D}^*(\mathcal{H})^{\otimes^\bullet_{\mathrm{D}(S)}} \underset{\mathrm{D}(S)}{\otimes} \mathcal{C}, \mathcal{D}) \simeq \mathrm{Funct}_{\mathcal{H}}(\mathcal{C}, \mathcal{D}),$$

where the first and last equivalences are the cobar constructions. Applying the above paradigm to $\mathrm{D}(Y)$ and $\mathrm{D}(Z)$, we obtain the right half of the desired commutative diagram.

Moreover, by functoriality of the above paradigm, we obtain the commutative diagram (note that $\mathcal{C}^{\mathcal{H}}$ is dual to $(\mathcal{C}^\vee)^{\mathcal{H}}$ in $\mathrm{D}(S)$-mod by Lemma D.2.7 and Lemma B.3.3)

$$\begin{tikzcd}
(\mathcal{C}^\vee)^{\mathcal{H}} \otimes_{\mathrm{D}(S)} \mathcal{D}^{\mathcal{H}} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (\mathbf{triv}_{\mathcal{H}}((\mathcal{C}^\vee)^{\mathcal{H}}) \otimes_{\mathrm{D}(S)} \mathcal{D})^{\mathcal{H},\mathrm{diag}} \arrow[r, "\mathbf{oblv}^{\mathcal{H}}\otimes\mathbf{Id}"] \arrow[d, "F^{\mathcal{H}}_{\mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}})\to\mathcal{D}}"', "\simeq"] & (\mathcal{C}^\vee \otimes_{\mathrm{D}(S)} \mathcal{D})^{\mathcal{H},\mathrm{diag}} \arrow[d, "F^{\mathcal{H}}_{\mathcal{C}\to\mathcal{D}}"', "\simeq"] \\
\mathrm{Funct}_{\mathrm{D}(S)}(\mathcal{C}^{\mathcal{H}}, \mathcal{D}^{\mathcal{H}}) \arrow[r, "\simeq"] & \mathrm{Funct}_{\mathcal{H}}(\mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}), \mathcal{D}) \arrow[r, "-\circ(\mathbf{oblv}^{\mathcal{H}})^\vee"] & \mathrm{Funct}_{\mathcal{H}}(\mathcal{C}, \mathcal{D}),
\end{tikzcd}$$

where $(\mathbf{oblv}^{\mathcal{H}})^\vee : \mathcal{C} \to \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}})$ is the dual functor of $\mathbf{oblv}^{\mathcal{H}} : \mathbf{triv}_{\mathcal{H}}((\mathcal{C}^\vee)^{\mathcal{H}}) \to \mathcal{C}^\vee$. By construction, it is identified with

$$\mathcal{C} \xrightarrow{\mathbf{pr}_{\mathcal{H}}} \mathbf{triv}_{\mathcal{H}}(\mathcal{C}_{\mathcal{H}}) \xrightarrow{\theta_{\mathcal{H}}} \mathbf{triv}_{\mathcal{H}}(\mathcal{C}^{\mathcal{H}}),$$

hence we have $(\mathbf{oblv}^{\mathcal{H}})^\vee \simeq \mathbf{Av}^{\mathcal{H}}_*$. Applying the above paradigm to $\mathrm{D}(Y)$ and $\mathrm{D}(Z)$, we obtain the left half of the desired commutative diagram. This proves (1).

The two commutative squares in (1) are both right adjointable along the horizontal direction because the right adjoints of the horizontal functors exist and the vertical functors are equivalences.

□[Lemma D.6.6].

Remark D.6.7. In the constructible contexts, even when $S = \mathrm{pt}$, the modifications and proofs for the lemma are subtle[126], and we do not have the energy to articulate them in this paper.

[126] For example, even the Hopf algebra structure on $\mathrm{Shv}^*_c(\mathcal{H})$ requires a homotopy-coherent justification.

D.7 Application: equivariant unipotent nearby cycles

Let $\mathcal{H} \to S$ be a pro-smooth group scheme and $\mathcal{Y} \to S$ be any placid indscheme acted by $\mathcal{H}$. Suppose $\mathcal{Y}$ admits a dimension theory[127]. Let $\mathcal{Y} \to \mathbb{A}^1 \times S$ be an $\mathcal{H}$-equivariant map, where $\mathbb{A}^1 \times S$ is equipped with the trivial $\mathcal{H}$-action. By § D.4, both $\mathrm{DMod}^!(\overset{\circ}{\mathcal{Y}})$ and $\mathrm{DMod}^!(\mathcal{Y}_0)$ are naturally objects in $\mathcal{H}$-mod. Suppose $\mathcal{C}$ is a sub-$\mathcal{H}$-module of $\mathrm{DMod}^!(\overset{\circ}{\mathcal{Y}})$ such that as a plain DG-category it is contained in $\mathrm{DMod}^!(\overset{\circ}{\mathcal{Y}})^{\mathrm{good}}$ (see Notation C.7.4). The goal of this section is to prove the folllowing result:

Proposition D.7.1. *In the above setting, the restrictions of the functors*

$$\Psi^{\mathrm{un}}, i^! \circ j_! : \mathrm{DMod}^!(\overset{\circ}{\mathcal{Y}})^{\mathrm{good}} \to \mathrm{DMod}^!(\mathcal{Y}_0)$$

on $\mathcal{C}$ have natural $\mathcal{H}$-linear structures.

Remark D.7.2. The reader can skip the proof if they are satisfied by the following two slogans: "the left adjoint of a strict linear functor is left-lax linear"; "any lax linear functor between categories with group actions is strict". However, our problem does not follow from these slogans. Namely, $j_!$ is a *partially defined* left adjoint, and $\mathcal{H} \to S$ is an *infinite dimensional* group scheme.

Warning D.7.3. In the rest of this subsction, we retract our convention of using $\otimes$ to denote the tensor product in DGCat and reclaim the notation $\otimes_k$. This is because we need to consider the tensor products of presentable categories.

Definition D.7.4. Let $\mathcal{M}_0 \xrightarrow{\iota} \mathcal{M} \xleftarrow{G} \mathcal{N}$ be a diagram in $\mathrm{Pr}^{\mathrm{st},L}$ such that ι is fully faithful. For a functor $F : \mathcal{M}_0 \to \mathcal{N}$ and a natural transformation $\alpha : \iota \to G \circ F$, we say α *exhibits F as a partially defined left adjoint to G* if for any $x \in \mathcal{M}_0$ and $y \in \mathcal{N}$, the following composition is an isomorphism.

$$\mathrm{Maps}_{\mathcal{N}}(F(x), y) \to \mathrm{Maps}_{\mathcal{M}}(G \circ F(x), G(y)) \xrightarrow{-\circ\alpha(x)} \mathrm{Maps}_{\mathcal{M}}(\iota(x), G(y)). \tag{D.21}$$

Note that such pair (F, α) is unique if it exists.

We write $G^L|_\iota : \mathcal{M}_0 \to \mathcal{N}$ for *the* partially defined left adjoint and treat the natural transformation $\iota \to G \circ G^L|_\iota$ as implicit.

Remark D.7.5. If $G^L|_\iota$ exists, then it is canonically isomorphic to the left adjoint of the (non-continuous) functor $\iota^R \circ G$.

[127] See [Ras15b, § 6.10] for what this means. For the purpose of this paper, it is enough to know that ind-finite type indschemes and placid schemes admit dimension theories.

Construction D.7.6. Suppose we have the following commutative diagram

$$
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "\iota"] \arrow[d, "S_0"] & \mathcal{M} \arrow[d, "S"] & \mathcal{N} \arrow[l, "G"'] \arrow[d, "T"] \\
\mathcal{M}_0' \arrow[r, "\iota'"] & \mathcal{M}' & \mathcal{N}', \arrow[l, "G'"']
\end{tikzcd}
\tag{D.22}
$$

such that both rows satisfy the assumption in Definition D.7.4. We warn the reader that we do not put any restrictions to the vertical functors. Suppose $G^L|_\iota$ and $(G')|^L_{\iota'}$ exist. Then there is a canonical natural transformation

$$
\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "G^L|_\iota"] \arrow[d, "S_0"'] & \mathcal{N} \arrow[d, "T"] \\
\mathcal{M}_0' \arrow[r, "(G')^L|_{\iota'}"'] \arrow[ur, Rightarrow] & \mathcal{N}',
\end{tikzcd}
\tag{D.23}
$$

whose value on $x \in \mathcal{M}_0$ is the morphism

$$
(G')|^L_{\iota'} \circ S_0(x) \to T \circ G^L|_\iota(x)
$$

corresponds via (D.21) to the composition

$$
\iota' \circ S_0(x) \simeq S \circ \iota(x) \to S \circ G \circ G^L|_\iota(x) \simeq G' \circ T \circ G^L|_\iota(x).
$$

The above natural transformation is obtained by the following steps. We first pass to right adjoints along the horizontal directions for the left square of (D.22) and obtain

$$
\begin{tikzcd}
\mathcal{M}_0 \arrow[d, "S_0"'] \arrow[dr, Rightarrow] & \mathcal{M} \arrow[l, "\iota^R"'] \arrow[d, "S"] & \mathcal{N} \arrow[l, "G"'] \arrow[d, "T"] \\
\mathcal{M}_0' & \mathcal{M}' \arrow[l, "(\iota')^R"] & \mathcal{N}'. \arrow[l, "G'"']
\end{tikzcd}
$$

Then we pass to left adjoints along the horizontal directions for the outside square in the above diagram.

Construction D.7.7. Construction D.7.6 is functorial in the following sense. Let C_1 be the category of diagrams $\mathcal{M}_0 \xrightarrow{\iota} \mathcal{M} \xleftarrow{G} \mathcal{N}$ in Cat such that

- $\mathcal{M}_0$, $\mathcal{M}$ and $\mathcal{N}$ are stable and presentable,
- ι and G are morphisms in $\mathrm{Pr}^{\mathrm{st},L}$,
- $\iota^R \circ G$ has a left adjoint.

Let C_2 be the category of presentable fibrations over Δ^1 (see [Lur12, Definition 5.5.3.2]) such that the 0-fiber and 1-fiber are both stable. Then Construction D.7.6 provides a functor

$$L : \mathrm{C}_1 \to \mathrm{C}_2,$$

which sends $\mathcal{M}_0 \xrightarrow{\iota} \mathcal{M} \xleftarrow{G} \mathcal{N}$ to the presentable fibration classifying the adjoint pair

$$G^L|_\iota : \mathcal{M}_0 \rightleftharpoons \mathcal{N} : \iota^R \circ G.$$

Let C_3 be the cateogry of diagrams $\mathcal{M}_0 \xrightarrow{F} \mathcal{N}$ in Cat such that

- $\mathcal{M}_0$, $\mathcal{N}$ are stable and presentable,
- F is in $\mathrm{Pr}^{\mathrm{st},L}$.

Then Grothendieck construction provides a 1-fully faithful functor $J : \mathrm{C}_3 \to \mathrm{C}_2$. By definition, a morphism in C_1 is sent by L into the image of J iff the corresponding natural transformation (D.23) is invertible.

Definition D.7.8. A morphism in C_1 is *left adjointable* if L sends it into the image of J.

Lemma D.7.9. *Let $\beta \to \beta'$ be a morphism in C_1 depicted as (D.22). Suppose the right square in (D.22) is right adjointable along the vertical directions, then the morphism $\beta \to \beta'$ is left adjointable.*

Proof. Diagram chasing.

□[Lemma D.7.9]

Lemma D.7.10. *Let $\beta := (\mathcal{M}_0 \xrightarrow{\iota} \mathcal{M} \xleftarrow{G} \mathcal{N})$ be an object in C_1 and $\mathcal{D}$ be an object in $\mathrm{Pr}^{\mathrm{st},L}$. Then*

(1) The diagram

$$\mathcal{D} \times \beta := (\mathcal{D} \times \mathcal{M}_0 \xrightarrow{\mathbf{Id}\times\iota} \mathcal{D} \times \mathcal{M} \xleftarrow{\mathbf{Id}\times G} \mathcal{D} \times \mathcal{N})$$

is an object in C_1, and we have canonical isomorphism

$$(\mathbf{Id} \times G)^L|_{\mathbf{Id}\times\iota} \simeq \mathbf{Id} \times G^L|_\iota. \tag{D.24}$$

(2) The diagram[128]

$$\mathrm{LFun}(\mathcal{D}, \beta) := (\mathrm{LFun}(\mathcal{D}, \mathcal{M}_0) \xrightarrow{\iota\circ-} \mathrm{LFun}(\mathcal{D}, \mathcal{M}) \xleftarrow{G\circ-} \mathrm{LFun}(\mathcal{D}, \mathcal{N}))$$

[128] $\mathrm{LFun}(-,-)$ is the inner-Hom object in $\mathrm{Pr}^{L,\mathrm{st}}$. Its objects are functors that have right adjoints.

is an object in C_1*, and the corresponding partially defined left adjoint is canonical isomorphic to*

$$\mathrm{LFun}(\mathcal{D},\mathcal{M}_0) \xrightarrow{G^L|_\iota\circ-} \mathrm{LFun}(\mathcal{D},\mathcal{N}).$$

(3) Suppose $\mathcal{D}$ *is dualizable in* $\mathrm{Pr}^{\mathrm{st},L}$*, then the diagram*

$$\mathcal{D}\otimes\beta := (\mathcal{D}\otimes\mathcal{M}_0 \overset{\mathbf{Id}\otimes\iota}{\to} \mathcal{D}\otimes\mathcal{M} \overset{\mathbf{Id}\otimes G}{\leftarrow} \mathcal{D}\otimes\mathcal{N})$$

is an object in C_1*, and we have canonical isomorphism*

$$(\mathbf{Id}\otimes G)^L|_{\mathbf{Id}\otimes\iota} \simeq \mathbf{Id}\otimes G^L|_\iota. \tag{D.25}$$

Proof. (1) is obvious. Let us first prove (2). Since ι is fully faithful, the functor $(\mathrm{LFun}(\mathcal{D},\mathcal{M}_0) \overset{\iota\circ-}{\to} \mathrm{LFun}(\mathcal{D},\mathcal{M})$ is also fully faithful. Consider the canonical natural transformation $\iota \to G^L|_\iota \circ G$. It induces a natural transformation

$$\begin{tikzcd}
\mathrm{LFun}(\mathcal{D},\mathcal{M}_0) \arrow[rr, "\iota\circ-"] \arrow[dr, "G^L|_\iota\circ-"'] & \arrow[d, phantom, "\Downarrow"] & \mathrm{LFun}(\mathcal{D},\mathcal{M}) \\
& \mathrm{LFun}(\mathcal{D},\mathcal{N}). \arrow[ur, "G\circ-"'] &
\end{tikzcd}$$

In order to prove (2), we only need to verify the axiom in Definition D.7.4. However, this can be checked directly by evaluating on objects $d\in\mathcal{D}$. This proves (2).

(3) can be obtained from (2) by using the canonical equivalence

$$\mathrm{LFun}(\mathcal{D}^\vee,-) \simeq \mathcal{D}\otimes -.$$

□[Lemma D.7.10]

Corollary D.7.11. *Let* β *be an object in* C_1 *and* $\mathcal{D}$ *be a dualizable object in* $\mathrm{Pr}^{\mathrm{st},L}$*. Then the natural morphism* $\mathcal{D}\times\beta \to \mathcal{D}\otimes\beta$ *is left adjointable.*

Proof. Follows from (D.25) and (D.24).

□[Corollary D.7.11]

Definition D.7.12. A morphism $\beta\to\beta'$ in C_1 depicted as (D.22) is *continuous* if the functors corresponding functors S_0, S and T are morphisms in $\mathrm{Pr}^{\mathrm{st},L}$.

Construction D.7.13. Let $\beta \to \beta'$ be a continuous morphism in C_1. Let $\mathcal{D}$ be a dualizable object in $\mathrm{Pr}^{\mathrm{st},L}$. Then there is a natural continuous morphism $\mathcal{D} \otimes \beta \to \mathcal{D} \otimes \beta'$ in C_1.

Corollary D.7.14. *In Construction D.7.13, suppose $\beta \to \beta'$ is left adjointable, then $\mathcal{D} \otimes \beta \to \mathcal{D} \otimes \beta'$ is left adjointable.*

Proof. Follows from (D.25).

□[Corollary D.7.14]

Construction D.7.15. Let β be an object in C_1, and $\mathcal{D}_1 \to \mathcal{D}_2$ be a morphism in $\mathrm{Pr}^{\mathrm{st},L}$ such that $\mathcal{D}_1$ and $\mathcal{D}_2$ are dualizable. Then there is a natural continuous morphism $\mathcal{D}_1 \otimes \beta \to \mathcal{D}_2 \otimes \beta$ in C_1.

Corollary D.7.16. *In Construction D.7.15, $\mathcal{D}_1 \otimes \beta \to \mathcal{D}_2 \otimes \beta$ is always left adjointable.*

Proof. Follows from (D.25).

□[Corollary D.7.14]

Construction D.7.17. Let β and β' be two objects in C_1. Let $\mathcal{D}$ be a dualizable object in $\mathrm{Pr}^{\mathrm{st},L}$. For a given *continuous* morpihsm $a : \mathcal{D} \otimes \beta \to \beta'$, we can construct the following morphism

$$b : \beta \simeq \mathrm{Sptr} \otimes \beta \xrightarrow{\mathbf{unit} \otimes \mathbf{Id}} \mathcal{D}^\vee \otimes \mathcal{D} \otimes \beta \xrightarrow{\mathbf{Id} \otimes a} \mathcal{D}^\vee \otimes \beta'.$$

We call this construction as *passing to the dual morphism.*

Lemma D.7.18. *In Construction D.7.17, suppose the dual morphism $b : \beta \to \mathcal{D}^\vee \otimes \beta'$ is left adjointable, then the original morphism $a : \mathcal{D} \otimes \beta \to \beta'$ is left adjointable.*

Proof. By the axiom of duality data, the morphism a can be recovered as the composition

$$\mathcal{D} \otimes \beta \xrightarrow{\mathbf{Id} \otimes b} \mathcal{D} \otimes \mathcal{D}' \otimes \beta' \xrightarrow{\mathbf{counit} \otimes \mathbf{Id}} \beta'.$$

Hence it suffices to show both $\mathbf{Id} \otimes b$ and $\mathbf{counit} \otimes \mathbf{Id}$ are left adjointable. The claim for $\mathbf{Id} \otimes b$ follows from Corollary D.7.14, while that for $\mathbf{counit} \otimes \mathbf{Id}$ follows from Corollary D.7.16.

□[Lemma D.7.18]

Now we prove Proposition D.7.1. We prove the result on $i^! \circ j_!$ and deduce that on Ψ^{un} from its defnition formula (C.20). It suffices to prove $j_!$ has a natural $\mathcal{H}$-linear structure.

D.7.19 (Left lax $\mathcal{H}$-linear structure)**.** We first show $j_!$ has a natural *left lax* $\mathcal{H}$-linear structure. Consider the following forgetful functors

$$\mathrm{DGCat} \to \mathrm{Pr}^{\mathrm{st},L} \to \mathrm{Cat},$$

note that they have natural right lax symmetric monoidal structures. Hence the monoidal object $(\mathrm{DMod}^*(\mathcal{H}), \star) \in \mathrm{DGCat}$ induces monoidal algebra in $\mathrm{Pr}^{\mathrm{st},L}$ and Cat, which we denote respectively by A and B. Note that the underlying categories of them are just $\mathrm{DMod}^*(\mathcal{H})$.

Let $\iota : \mathcal{C} \to \mathrm{DMod}^!(\mathring{\mathcal{Y}})$ be the fully faithful functor in the problem. We write F for the partially defined left adjoint $j_!|_\iota$ to $j^!$ (see Definition D.7.4). In other words, F is the left adjoint to the non-continuous functor $\iota^R \circ j^!$.

Both ι and $j^!$ are naturally $\mathcal{H}$-linear. Hence $\iota^R \circ j^!$ is naturally right lax B-linear. Hence F is naturally left lax B-linear. Note that $F : \mathcal{C} \to \mathrm{DMod}^!(\mathcal{Y})$ is a morphism in $\mathrm{Pr}^{\mathrm{st},L}$, and the B-module structures on $\mathcal{C}$ and $\mathrm{DMod}^!(\mathcal{Y})$ are induced by their A-module structures. Hence F is naturally left lax A-linear. Recall we have a monoidal functor in DGCat (the unit functor) $\mathrm{Vect} \to (\mathrm{DMod}^*(\mathcal{H}), \star)$, therefore a monoidal functor $(\mathrm{Vect}, \otimes) \to A$ in $\mathrm{Pr}^{\mathrm{st},L}$. Hence F is naturally left lax $(\mathrm{Vect}, \otimes)$-linear. Since $(\mathrm{Vect}, \otimes)$ is rigid, this left lax $(\mathrm{Vect}, \otimes)$-linear structure on F is strict. Therefore F can be upgraded to a left lax $(\mathrm{DMod}^*(\mathcal{H}), \star)$-linear functor in DGCat. In other words, F is a left lax $\mathcal{H}$-linear functor.

D.7.20 (Strictness)**.** It remains to show the obtained left lax $\mathcal{H}$-linear structure on F is strict. It suffices to show the left lax B-linear structure on F is strict. In other words, we need to show the natural transformation

$$\begin{tikzcd} B \times \mathcal{C} \arrow[r, "\mathbf{Id}\times F"] \arrow[d, "\mathbf{act}_B"'] & B \times \mathrm{DMod}^!(\mathcal{Y}) \arrow[d, "\mathbf{act}_B"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, Rightarrow] & \mathrm{DMod}^!(\mathcal{Y}), \end{tikzcd}$$

which is obtained by applying Construction D.7.6 to the commutative diagram

$$\begin{tikzcd} B \times \mathcal{C} \arrow[r, "\mathbf{Id}\times\iota"] \arrow[d, "\mathbf{act}_B"] & B \times \mathrm{DMod}^!(\mathring{\mathcal{Y}}) \arrow[d, "\mathbf{act}_B"] & B \times \mathrm{DMod}^!(\mathcal{Y}) \arrow[l, "\mathbf{Id}\times j^!"'] \arrow[d, "\mathbf{act}_B"] \\ \mathcal{C} \arrow[r, "\iota"] & \mathrm{DMod}^!(\mathring{\mathcal{Y}}) & \mathrm{DMod}^!(\mathcal{Y}), \arrow[l, "j^!"'] \end{tikzcd}$$

is invertible.

In the proof below, we use the notations in Construction D.7.7 and Lemma D.7.10. Note that

$$\beta := (\mathcal{C} \xrightarrow{\iota} \mathrm{DMod}^!(\mathring{\mathcal{Y}}) \xleftarrow{j^!} \mathrm{DMod}^!(\mathcal{Y}))$$

is an object in C_1. Our problem can be reformulated as showing

$$\mathbf{act}_B : B \times \beta \to \beta$$

being left adjointable. Note that $\mathbf{act}_B$ is the composition

$$B \times \beta \xrightarrow{T} A \otimes \beta \xrightarrow{\mathbf{act}_A} \beta.$$

Hence we only need to show both T and $\mathbf{act}_A$ are left adjointable.

Recall $\mathrm{DMod}^*(\mathcal{H})$ is dualizable in DGCat. Since $(\mathrm{Vect}, \otimes)$ is rigid, $\mathrm{DMod}^*(\mathcal{H})$ is also dualizable in $\mathrm{Pr}^{\mathrm{st},L}$. Hecne T is left adjointable by Corollary D.7.11.

It remains to show $\mathbf{act}_A$ is left adjointable. By Lemma D.7.18, it suffices to show the morphism

$$\mathbf{coact}_{A^\vee} : \beta \to A^\vee \otimes \beta$$

is left adjointable. By Lemma D.7.9, it suffices to show the commutative square

$$\begin{tikzcd}
\mathrm{DMod}^!(\mathcal{Y}) \arrow[d, "\mathbf{coact}"] & & \mathrm{DMod}^!(\mathring{\mathcal{Y}}) \arrow[ll, "j^!"'] \arrow[d, "\mathbf{coact}"] \\
\mathrm{DMod}^!(\mathcal{H}) \otimes \mathrm{DMod}^!(\mathring{\mathcal{Y}}) & & \mathrm{DMod}^!(\mathcal{H}) \otimes \mathrm{DMod}^!(\mathcal{Y}) \arrow[ll, "\mathbf{Id}\otimes j^!"']
\end{tikzcd}$$

is right adjointable along vertical directions. By definition, we have a factorization

$$\mathbf{coact} : \mathrm{DMod}^!(\mathcal{Y}) \xrightarrow{\mathrm{act}^!} \mathrm{DMod}^!(\mathcal{H} \underset{S}{\times} \mathcal{Y}) \xrightarrow{*\text{-pushforward}} \mathrm{DMod}^!(\mathcal{H} \times \mathcal{Y}) \simeq \mathrm{DMod}^!(\mathcal{H}) \otimes \mathrm{DMod}^!(\mathcal{Y}).$$

Note that the $*$-pushforward functor in the above composition is the left adjoint to the !-pullback functor. Hence it remains to show the commutative square

$$\begin{tikzcd}
\mathrm{DMod}^!(\mathcal{Y}) \arrow[d, "\mathrm{act}^!"] & & \mathrm{DMod}^!(\mathring{\mathcal{Y}}) \arrow[ll, "j^!"'] \arrow[d, "\mathrm{act}^!"] \\
\mathrm{DMod}^!(\mathcal{H} \times_S \mathring{\mathcal{Y}}) & & \mathrm{DMod}^!(\mathcal{H} \times_S \mathcal{Y}) \arrow[ll, "\mathbf{Id}\otimes j^!"']
\end{tikzcd}$$

is right adjointable along the vertical directions. Note that the relavant maps are placid maps between placid indschemes. Hence by [Ras15b, Proposition 6.18.1] after choosing a dimension theory on $\mathcal{Y}$, we can replace $\mathrm{DMod}^!$ in the above square by DMod^* and !-pullback functors by $*$-pullback functors. Then we are done by the usual base-change isomorphism.

□[Proposition D.7.1]

Appendix E

Braden's theorem and the contraction principle

In this appendix, we review Braden's theorem and the contraction principle. We make the following assumption:

Assumption E.0.1. Let Z be an ind-finite type indscheme. In this subsection, when discussing $\mathbb{G}_m$-actions on Z, we always assume it can be written as a filtered colimit $Z \simeq \operatorname{colim}_\alpha Z_\alpha$ with each Z_α being a finite type closed subscheme stabilized by $\mathbb{G}_m$.

Remark E.0.2. Let $\mathbb{G}_m \curvearrowright Z$ be an action as above. Using [DG14, Lemma 1.4.9(ii), Corollary 1.5.3(ii)][129], we have $Z^{\mathrm{att}} \simeq \operatorname{colim}_\alpha Z_\alpha^{\mathrm{att}}$, and it exhibits Z^{att} as an ind-finite type indscheme. Using [DG14, Proposition 1.3.4], we also have similar result for Z^{fix}.

E.1 The contraction principle

Definition E.1.1. A *retraction* consists of two lft prestacks (Y, Y^0) together with morphisms $i : Y^0 \to Y$, $q : Y \to Y^0$ and an isomorphism $q \circ i \simeq \mathrm{Id}_{Y^0}$. We abuse notation by calling (Y, Y^0) a retraction and treat the other data as implicit.

Construction E.1.2. Let $\mathbb{G}_m \curvearrowright Z$ be as in Assumption E.0.1, there are canonical retractions $(Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ and $(Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$.

[129]There is a typo in the statement of [DG14, Lemma 1.4.9]: it should be "$Y \subset Z$ be a $\mathbb{G}_m$-stable *subspace*" rather than "... open subspace".

Construction E.1.3. Let (Y, Y^0) be a retraction. We have natural transformations

$$q_* \to q_* \circ i_* \circ i^* = (q \circ i)_* \circ i^* = i^*, \tag{E.1}$$

$$i^! \to i^! \circ q^! \circ q_! = (q \circ i)^! \circ q_! = q_!. \tag{E.2}$$

between functors $\mathrm{DMod}(Y) \to \mathrm{Pro}(\mathrm{DMod}(Y^0))$ (see e.g. [DG14, Appendix A] for the definition of pro-categories). We refer them as the *contraction natual transformations*.

Remark E.1.4. In order to construct (E.1), we need to assume the $*$-pushforward functors are well-defined and continuous.

Definition E.1.5. We say a retraction (Y, Y^0) is *$*$-nice* (resp. !-nice) for an object $\mathcal{F} \in \mathrm{DMod}(Z)$ if the values of (E.1) (resp. (E.2)) on $\mathcal{F}$ are isomorphisms.

Definition E.1.6. Let Z first be a finite type scheme acted by $\mathbb{G}_m$. The category

$$\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \subset \mathrm{DMod}(Z)$$

of *unipotently $\mathbb{G}_m$-monodromic D-modules*[130] on Z is defined as the full DG-subcategory of $\mathrm{DMod}(Z)$ generated by the image of the !-pullback functor $\mathrm{DMod}(Z/\mathbb{G}_m) \to \mathrm{DMod}(Z)$.

Let Z be an ind-finite type indscheme acted by $\mathbb{G}_m$ satisfying Assumption E.0.1. We define

$$\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} := \lim_{!\text{-pullback}} \mathrm{DMod}(Z_\alpha)^{\mathbb{G}_m\text{-um}}.$$

Remark E.1.7. It is clear that the !-pullback functor $\mathrm{DMod}(Z_\beta) \to \mathrm{DMod}(Z_\alpha)$ sends unipotently $\mathbb{G}_m$-monodromic objects to unipotently $\mathbb{G}_m$-monodromic ones. Hence the above limit is well-defined. Also, a standard argument shows that it does not depend on the choice of writing Z as $\mathrm{colim}_\alpha Z_\alpha$.

By passing to left adjoints (see Remark B.1.2), we also have

$$\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \simeq \operatorname*{colim}_{*\text{-pushforward}} \mathrm{DMod}(Z_\alpha)^{\mathbb{G}_m\text{-um}}. \tag{E.3}$$

Theorem E.1.8. *(Contraction principle) Let $\mathbb{G}_m \curvearrowright Z$ be an action as in Assumption E.0.1. The retractions $(Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ and $(Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$ are both !-nice and $*$-nice for any object in $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}}$.*

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[130] [DG14] referred to them as just $\mathbb{G}_m$-monodromic D-modules. We keep the adverb *unipotently* because we need to consider other monodromies when discussing nearby-cycles.

Remark E.1.9. When Z is a finite type scheme, the contraction principle is proved in [DG15, Theorem C.5.3]. The case of ind-finite type indschemes can be formally deduced because of (E.3).

E.2 Braden's theorem

Definition E.2.1. A commutative square of lft prestacks

$$\begin{tikzcd} V' \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ V \arrow[r, "g"] & W \end{tikzcd} \tag{E.4}$$

is *strictly quasi-Cartesian* if the map $j : V' \to W' \times_W V$ is a schematic open embedding.

It is *quasi-Cartesian* if it is nil-isomorphic to a strictly quasi-Cartesian square.

Construction E.2.2. For a quasi-Cartesian square as in Definition E.2.1, we extend it to a commutative diagram

$$\begin{tikzcd} V' \arrow[dr, "j"] \arrow[drr, "g'"] \arrow[ddr, "q"'] & & \\ & W' \times_W V \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & W' \arrow[d, "r"] \\ & V \arrow[r, "g"] & W. \end{tikzcd}$$

Consider the category of D-modules on these prestacks. We have the following *base-change transformation*

$$g^! \circ r_* \simeq \mathrm{pr}_{2,*} \circ \mathrm{pr}_1^! \to \mathrm{pr}_{2,*} \circ j_* \circ j^! \circ \mathrm{pr}_1^! \simeq q_* \circ (g')^!. \tag{E.5}$$

Using the adjoint pairs

$$q^* : \mathrm{Pro}(\mathrm{DMod}(V)) \rightleftharpoons \mathrm{Pro}(\mathrm{DMod}(V')) : q_*,$$

$$r^* : \mathrm{Pro}(\mathrm{DMod}(W)) \rightleftharpoons \mathrm{Pro}(\mathrm{DMod}(W')) : r_*,$$

we obtain a natural transformation

$$q^* \circ g^! \to (g')^! \circ r^*. \tag{E.6}$$

Definition E.2.3. A quasi-Cartesian square (E.4) is *nice* for an object $\mathcal{F} \in \mathrm{DMod}(W)$ if the value of (E.6) on $\mathcal{F}$ is an isomorphism in $\mathrm{DMod}(V')$.

Warning E.2.4. One can obtain another quasi-Cartesian square from (E.4) by exchanging the positions of V and W'. However, the above definition is not preserved by this symmetry.

Construction E.2.5. Let $\mathbb{G}_m \curvearrowright Z$ be as in Assumption E.0.1. By [DG14, Proposition 1.9.4], there is canonical quasi-Cartesian diagrams

$$\begin{tikzcd} Z^{\mathrm{fix}} \arrow[r, "i^+"] \arrow[d, "i^-"] & Z^{\mathrm{att}} \arrow[d, "p^+"] \\ Z^{\mathrm{rep}} \arrow[r, "p^-"] & Z, \end{tikzcd} \qquad \begin{tikzcd} Z^{\mathrm{fix}} \arrow[r, "i^-"] \arrow[d, "i^+"] & Z^{\mathrm{rep}} \arrow[d, "p^-"] \\ Z^{\mathrm{att}} \arrow[r, "p^+"] & Z \end{tikzcd}$$

Theorem E.2.6. *(Braden) Let $\mathbb{G}_m \curvearrowright Z$ be an action as in Assumption E.0.1. The above two quasi-Cartesian diagrams are nice for any object in* $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}}$.

Remark E.2.7. When Z is a finite type scheme, Braden's theorem is proved in [Bra03] and reproved in [DG14]. The case of ind-finite type indschemes can be formally deduced because of (E.3).

Remark E.2.8. Using the contraction principle, Braden's theorem can be reformulated as the existence of a canonical adjoint pair[131]

$$q_*^{\pm} \circ p^{\pm,!} : \mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \rightleftharpoons \mathrm{DMod}(Z^{\mathrm{fix}}) : p_*^{\mp} \circ q^{\mp,!}.$$

In fact, this is how [DG14] proved Braden's theorem.

For the purpose of this paper, we also introduce the following definition:

Definition E.2.9. A *Braden 4-tuple* consists of four prestacks (Z, Z^+, Z^-, Z^0) together with

- a quasi-Cartesian square (see Definition E.2.1):

$$\begin{tikzcd} Z^0 \arrow[r, "i^+"] \arrow[d, "i^-"] & Z^+ \arrow[d, "p^+"] \\ Z^- \arrow[r, "p^-"] & Z. \end{tikzcd}$$

- morphisms $q^+ : Z^+ \to Z^0$ and $q^- : Z^- \to Z^0$ and isomorphisms $q^+ \circ i^+ \simeq \mathrm{Id}_{Z^0} \simeq q^- \circ i^-$.

We abuse notation by calling (Z, Z^+, Z^-, Z^0) a Braden 4-tuple and treat the other data as implicit.

Given a Braden 4-tuple (Z, Z^+, Z^-, Z^0), we define its *opposite Braden 4-tuple* to be (Z, Z^-, Z^+, Z^0).

Construction E.2.10. Let $\mathbb{G}_m \curvearrowright Z$ be as in Assumption E.0.1. We have a canonical Braden 4-tuple $(Z, Z^{\mathrm{att}}, Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$.

[131] Note that the image of the functor $p_*^- \circ q^{-,!} : \mathrm{DMod}(Z^{\mathrm{fix}}) \to \mathrm{DMod}(Z)$ is contained in $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}}$.

Example E.2.11. The *inverse* of the dilation $\mathbb{G}_m$-action on $\mathbb{A}^1$ induces the Braden 4-tuple

$$\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0).$$

Example E.2.12. It is well-known that the action $\mathbb{G}_m \curvearrowright \mathrm{Gr}_{G,I}$ in Proposition 1.2.8 satisfies the condition $(\diamond)$. Indeed, as in the proof of [Zhu16, Theorem 3.1.3], one can reduce to the case $G = \mathrm{GL}_n$. Then the subschemes $\mathrm{Gr}^{(N)}_{\mathrm{GL}_n,I}$ defined there are stabilized by $\mathbb{G}_m$. In particular, we obtain a canonical Braden 4-tuple $(\mathrm{Gr}_{G,I}, \mathrm{Gr}_{P,I}, \mathrm{Gr}_{P^-,I}, \mathrm{Gr}_{M,I})$.

Remark E.2.13. We will encounter Braden 4-tuples that are not obtained from Construction E.2.10 in the main text.

Definition E.2.14. For a Braden 4-tuple as in Definition E.2.9, we say it is $*$-*nice* for an object $\mathcal{F} \in \mathrm{DMod}(Z)$ if

- (i) The corresponding quasi-Cartesian square is nice for $\mathcal{F}$;
- (ii) The retraction (Z^-, Z^0) is $*$-nice for $p^{-,!} \circ \mathcal{F}$.

Remark E.2.15. We do not need the notion of !-niceness in this paper.

Then Braden's theorem and the contraction principle imply

Theorem E.2.16. *Let $\mathbb{G}_m \curvearrowright Z$ be an action as in Assumption E.0.1. Then $(Z, Z^{\mathrm{att}}, Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$ and $(Z, Z^{\mathrm{rep}}, Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ are $*$-nice for any objects in $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}}$.*

E.3 The parameterized Braden's theorem

We also need a parameterized version of Braden's theorem.

Proposition-Construction E.3.1. *Let $\mathbb{G}_m \curvearrowright Z$ be an action as in Assumption E.0.1, and $\mathcal{D}$ be any DG-category. Then the obvious functor*

$$\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \otimes \mathcal{D} \to \mathrm{DMod}(Z) \otimes \mathcal{D}$$

is fully faithful.

We define $(\mathrm{DMod}(Z) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}$ to be the essential image of the above functor.

Proof. It suffices to show that the fully faithful functor $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \to \mathrm{DMod}(Z)$ has a continuous right adjoint. Recall that both $\mathrm{DMod}(Z)^{\mathbb{G}_m} \simeq \mathrm{DMod}(Z/\mathbb{G}_m)$ and $\mathrm{DMod}(Z)$ are compactly generated, and the functor $\mathbf{oblv}^{\mathbb{G}_m}$ betweem them sends compact objects to compact objects. This formally implies that $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}}$ is compactly generated and the functor $\mathrm{DMod}(Z)^{\mathbb{G}_m\text{-um}} \to \mathrm{DMod}(Z)$ sends compact objects to compact objects. In particular, this functor has a continuous right adjoint.

□[Propsition-Construction E.3.1]

Let Z and $\mathcal{D}$ be as in Lemma-Definition E.3.1. Consider the functor

$$\mathrm{DMod}(Z^{\mathrm{fix}}) \otimes \mathcal{D} \xrightarrow{q^{-,!}\otimes \mathbf{Id}} \mathrm{DMod}(Z^{\mathrm{rep}}) \otimes \mathcal{D} \xrightarrow{p^{-}_{*}\otimes \mathbf{Id}} \mathrm{DMod}(Z) \otimes \mathcal{D}.$$

By definition, its image is contained in the full subcategory $(\mathrm{DMod}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}$. Therefore we obtain a functor

$$(p^-_* \circ q^{-,!}) \otimes \mathbf{Id} : \mathrm{DMod}(Z^{\mathrm{fix}}) \otimes \mathcal{D} \to (\mathrm{DMod}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}.$$

Remark E.2.8 implies

Theorem E.3.2. *(Parameterized Braden's theorem) There is a canonical adjoint pair*

$$(q^+_* \circ p^{+,!}) \otimes \mathbf{Id} : (\mathrm{DMod}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}} \rightleftharpoons \mathrm{DMod}(Z^{\mathrm{fix}}) \otimes \mathcal{D} : (p^-_* \circ q^{-,!}) \otimes \mathbf{Id}.$$

Remark E.3.3. We do not need the parameterized version of the contraction prinicple.

Appendix F

Compact generation of $\mathrm{DMod}(\mathrm{Gr}_G)^{\mathcal{L}U}$ and $\mathrm{DMod}(\mathrm{Gr}_G)_{\mathcal{L}U}$

The goal of this appendix is to prove Lemma 2.1.7 and Proposition 2.1.8. The proofs below owe their existences to D. Gaitsgory.

F.1 A parameterized version of Lemma 2.1.7

In this section, we prove a parameterized version of Lemma 2.1.7. We need the additional parameter to help us to deal with the coinvariance category later.

Lemma F.1.1. *Let $\mathcal{D}$ be any DG-category.*

(0) We have canonical equivalences

$$\mathrm{DMod}(\mathrm{Gr}_{M,I}) \otimes \mathcal{D} \simeq \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \simeq (\mathrm{DMod}(\mathrm{Gr}_{P,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}. \tag{F.1}$$

(1) We have[132]

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \subset (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \subset \mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D}.$$

[132]The category $(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}$ is defined in Proposition-Definition E.3.1.

(2) *The functor*

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\xrightarrow{(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}}(\mathrm{DMod}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\overset{(F.1)}{\simeq}\mathrm{DMod}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}$$

is conservative and has a left adjoint

$$\mathrm{DMod}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}\to(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}. \tag{F.2}$$

(3) *The functor*

$$(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{DMod}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\to(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

has a left adjoint canonically isomorphic to

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}}\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D}\xrightarrow{(\mathbf{q}_{I,*}^{-}\circ\mathbf{p}_I^{-,!})\otimes\mathbf{Id}}$$
$$\to\mathrm{DMod}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}\simeq\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\simeq(\mathrm{DMod}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

(4) *The functor*

$$(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\to(\mathrm{DMod}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

has a $\mathrm{DMod}(X^I)$*-linear left adjoint.*

The rest of this section is devoted to the proof of the lemma. We first note that (0) is a Corollary of Lemma D.1.11(4) and Lemma 2.1.5(2).

We recall the following well-known result:

Lemma F.1.2. *Let Y be any ind-finite type indscheme and $\mathcal{D}\in\mathrm{DGCat}$.*

(1) *Suppose Y is written as $\mathrm{colim}_{\alpha\in I}Y_\alpha$, where Y_α are closed sub-indschemes of Y. Then the natural functor*

$$\mathrm{DMod}(Y)\otimes\mathcal{D}\to\lim_{!\text{-pullback}}\mathrm{DMod}(Y_\alpha)\otimes\mathcal{D}$$

is an equivalence.

(2) *Suppose Y is written as $\mathrm{colim}_{\beta\in J}U_\beta$, where U_β are open sub-indschemes of Y and J is* filtered. *Then the natural functor*

$$\mathrm{DMod}(Y)\otimes\mathcal{D}\to\lim_{!\text{-pullback}}\mathrm{DMod}(U_\beta)\otimes\mathcal{D}$$

is an equivalence.

Proof. We first prove (1). By definition, we have

$$\mathrm{DMod}(Y) \otimes \mathcal{D} \simeq \underset{*\text{-pushforward}}{\mathrm{colim}} \mathrm{DMod}(Y_\alpha) \otimes \mathcal{D}.$$

Then we are done by passing to right adjoints.

Now let us prove (2). Write Y as the filtered colimit of its closed subschemes $Y \simeq \mathrm{colim}_{\alpha \in I} Y_\alpha$. For $\alpha \in I$ and $\beta \in J$, let Y_α^β be the intersection of Y_α with U_β (inside Y). By (1), we have

$$\mathrm{DMod}(Y) \otimes \mathcal{D} \simeq \underset{!\text{-pullback}}{\lim} \mathrm{DMod}(Y_\alpha) \otimes \mathcal{D},$$

$$\mathrm{DMod}(U_\beta) \otimes \mathcal{D} \simeq \underset{!\text{-pullback}}{\lim} \mathrm{DMod}(Y_\alpha^\beta) \otimes \mathcal{D}.$$

Hence it remains to prove that for a fixed $\alpha \in I$, the natural functor

$$\mathrm{DMod}(Y_\alpha) \otimes \mathcal{D} \to \underset{!\text{-pullback}}{\lim} \mathrm{DMod}(Y_\alpha^\beta) \otimes \mathcal{D}$$

is an isomorphism. However, this is obvious because for large enough β, the subscheme Y_α is contained inside U_β and hence $Y_\alpha^\beta \simeq Y_\alpha$.

□[Lemma F.1.2]

F.1.3 (Proof of Lemma F.1.1(1)). Recall the stratification on $\mathrm{Gr}_{G,I}$ defined in § 2.1.4 and Proposition A.4.2. Since the map $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is $\mathcal{L}U_I$-equivariant and $\mathcal{L}U_I$ is ind-reduced, the sub-indschemes ${}_\lambda \mathrm{Gr}_{G,I}$, ${}_{\leq\lambda} \mathrm{Gr}_{G,I}$ and ${}_{\geq\lambda} \mathrm{Gr}_{G,I}$ of $\mathrm{Gr}_{G,I}$ are all stabilized by the $\mathcal{L}U_I$-action.

By Proposition A.4.2(3) and Lemma F.1.2(1), we have

$$\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D} \simeq \underset{!\text{-pullback}}{\lim} \mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D}. \tag{F.3}$$

Hence

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \simeq \underset{!\text{-pullback}}{\lim} (\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$$

because taking invariance is a right adjoint.

On the other hand, we also have

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}} \simeq \underset{*\text{-pushforward}}{\mathrm{colim}} (\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}} \simeq \underset{!\text{-pullback}}{\lim} (\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}.$$

Hence to prove (1), it suffices to replace $\mathrm{Gr}_{G,I}$ by ${}_{\leq\lambda} \mathrm{Gr}_{G,I}$ (for all $\lambda \in \Lambda_{G,P}$).

Note that ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$ is the union of its open sub-indschemes ${}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}$. Moreover, it is easy to see that the relation "$\geq$" defines a *filtered* partial ordering on $\{\mu \in \Lambda_{G,P} | \mu \leq \lambda\}$. Hence by Lemma F.1.2(2), we have

$$\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D} \simeq \lim_{!\text{-pullback}} \mathrm{DMod}({}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D}. \tag{F.4}$$

Therefore

$$(\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \simeq \lim_{!\text{-pullback}} (\mathrm{DMod}({}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}. \tag{F.5}$$

On the other hand, a similar argument as in the proof of Lemma F.1.2(2) shows

$$(\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \simeq \lim_{!\text{-pullback}} (\mathrm{DMod}({}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}.$$

Hence to prove (1), it suffices to replace $\mathrm{Gr}_{G,I}$ by ${}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}$ (for all $\lambda, \mu \in \Lambda_{G,P}$ with $\mu \leq \lambda$). Note that ${}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Using induction and the excision triangle, we can further replace $\mathrm{Gr}_{G,I}$ by a single stratum ${}_{\theta}\mathrm{Gr}_{G,I} \simeq (\mathrm{Gr}^{\theta}_{P,I})_{\mathrm{red}}$. Then we are done by (0) and Lemma 2.1.5(1). This proves (1).

F.1.4 (Proof of Lemma F.1.1(3))**.** Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$. By Proposition 1.2.8, the attractor (resp. repeller, fixed) locus is $\mathrm{Gr}_{P,I}$ (resp. $\mathrm{Gr}_{P^-,I}$, $\mathrm{Gr}_{M,I}$). Applying Theorem E.3.2 to the inverse of this action, we obtain an adjoint pair

$$(\mathbf{q}^-_{I,*} \circ \mathbf{p}^{-,!}_I) \otimes \mathbf{Id} : (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \rightleftharpoons \mathrm{DMod}(\mathrm{Gr}_{M,I}) \otimes \mathcal{D} : (\mathbf{p}^+_* \circ \mathbf{q}^{+,!}) \otimes \mathbf{Id}.$$

By (0) and Lemma 2.1.5(1), the image of the above right adjoint is contained in $(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$, which itself is contained in $(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}$ by (1). Hence we can formally obtain the adjoint pair in (3) from the above adjoint pair. This proves (3).

F.1.5 (Proof of Lemma F.1.1(4))**.** To prove (4), we can replace $\mathrm{Gr}_{P,I}$ by $\mathrm{Gr}^{\lambda}_{P,I}$. Consider the following maps

$${}_{\lambda}\mathrm{Gr}_{G,I} \xrightarrow{{}_{\lambda}j} {}_{\leq\lambda}\mathrm{Gr}_{G,I} \xrightarrow{{}_{\leq\lambda}\mathbf{p}^+_I} \mathrm{Gr}_{G,I}.$$

Since ${}_{\leq\lambda}\mathbf{p}^+_I$ is a schematic closed embedding, we have an adjoint pair

$$({}_{\leq\lambda}\mathbf{p}^+_{I,*} \otimes \mathbf{Id})^{\mathrm{inv}} : (\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \rightleftharpoons (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} : ({}_{\leq\lambda}\mathbf{p}^{+,!}_I \otimes \mathbf{Id})^{\mathrm{inv}}.$$

Hence it suffices to prove that

$$({}_{\lambda}j^{!} \otimes \mathbf{Id})^{\mathrm{inv}} : (\mathrm{DMod}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \to (\mathrm{DMod}({}_{\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$$

has a strictly $\mathrm{DMod}(X^I)$-linear left adjoint. For any $\mu_1 \le \mu_2 \le \lambda$, consider the following commutative square induced by !-pullback functors:

$$\begin{tikzcd}
(\mathrm{DMod}({}_{\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \arrow[r, "="] & (\mathrm{DMod}({}_{\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \\
(\mathrm{DMod}({}_{\le\lambda,\ge\mu_1}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \arrow[u] \arrow[r] & (\mathrm{DMod}({}_{\le\lambda,\ge\mu_2}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}. \arrow[u]
\end{tikzcd}$$

Using (F.5), by Remark B.1.4, it suffices to prove this square is left-adjointable along the vertical direction and the relevant left adjoints are strictly $\mathrm{DMod}(X^I)$-linear. By the base-change isomorphism, the above square is right adjointable along the horizontal direction. Hence it suffices to prove that the vertical fucntors have strictly $\mathrm{DMod}(X^I)$-linear left adjionts. Note that ${}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Hence we are done by using (3) and the excision triangle. This proves (F.2) is well-defined and strictly $\mathrm{DMod}(X^I)$-linear.

F.1.6 (Proof of Lemma F.1.1(2))**.** It remains to prove $(\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv}}$ is conservative. We only need to prove $\mathbf{p}_I^{+,!} \otimes \mathbf{Id}$ is conservative. Suppose $y \in \mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D}$ and $\mathbf{p}_I^{+,!} \otimes \mathbf{Id}(y) \simeq 0$. We need to show $y \simeq 0$. By (F.3) and (F.4), it suffices to show the !-restriction of y to $\mathrm{DMod}({}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D}$ is zero for any $\lambda, \mu \in \Lambda_{G,P}$. Note that ${}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ contains only finite many strata. Hence we are done by using the excision triangle.

□[Lemma F.1.1]

F.2 Proof of Lemma 2.1.7 and Proposition 2.1.8

Note that Lemma 2.1.7 can be obtained[133] from Lemma F.1.1 by letting $\mathcal{D} := \mathrm{Vect}$.

The rest of this subsection is devoted to the proof of Proposition 2.1.8. Let $\mathcal{D} \in \mathrm{DGCat}$ be a test DG-category. Consider the tautological functor

$$\alpha : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \to (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}.$$

We have

[133] Of course, in order to get the *compact* generation of $\mathrm{DMod}(\mathrm{Gr}_{G,I})$, we need to use the compact generation of $\mathrm{DMod}(\mathrm{Gr}_{M,I})$.

Lemma F.2.1. *The following two commutative squares are left adjointable along horizontal diresctions.*

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \arrow[r, "\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id}"] \arrow[d, "\alpha"] & \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \arrow[d, "\beta"', "\simeq"] \\
(\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \arrow[r, "(\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv}}"] & (\mathrm{DMod}(\mathrm{Gr}_{P,I}) \otimes \mathcal{D})^{\mathcal{L}U_I},
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \arrow[r, "\mathbf{p}_{I,*}^{+,\mathrm{inv}} \otimes \mathbf{Id}"] \arrow[d, "\beta"', "\simeq"] & \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \arrow[d, "\alpha"] \\
(\mathrm{DMod}(\mathrm{Gr}_{P,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \arrow[r, "(\mathbf{p}_{I,*}^{+} \otimes \mathbf{Id})^{\mathrm{inv}}"] & (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}.
\end{tikzcd}$$

Proof. First note that β is indeed an equivalence by Lemma F.1.1(0).

The claim for the second commutative square is a corollary of Lemma F.2(3). It remains to prove the claim for the first commutative square. By Lemma F.2(4), the relevant left adjoints are well-defined.

Let x be any object in $\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D}$. It suffices to prove the canonical morphism

$$(\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv},L} \circ \beta(x) \to \alpha \circ (\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L(x) \tag{F.6}$$

is an isomorphism. Note that we have

$$\mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \simeq \coprod_{\lambda \in \Lambda_{G,P}} (\mathrm{DMod}(\mathrm{Gr}_{P,I}^{\lambda})^{\mathcal{L}U_I} \otimes \mathcal{D}).$$

Without lose of generality, we can assume x is contained in the direct summand labelled by λ.

Consider the closed embedding ${}_{\le\lambda}\mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I}$. It induces a fully faithful functor

$$(\mathrm{DMod}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \hookrightarrow (\mathrm{DMod}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}.$$

It is easy to see that both sides of (F.6) are contained in this full subcategory. Hence by Lemma F.2.2 below, it suffices to prove that the canonical map

$$(\mathbf{p}_{I,*}^{+} \otimes \mathbf{Id})^{\mathrm{inv},L} \circ (\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv},L} \circ \beta \to (\mathbf{p}_{I,*}^{+} \otimes \mathbf{Id})^{\mathrm{inv},L} \circ \alpha \circ (\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L$$

is an isomorphism. By the left adjointability of the second square, the RHS is canonically isomorphic to $\beta \circ (\mathbf{p}_{I,*}^{+,\mathrm{inv}} \otimes \mathbf{Id})^L \circ (\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L$. Then we are done because of the obvious isomorphism

$$(\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id}) \circ (\mathbf{p}_{I,*}^{+,\mathrm{inv}} \otimes \mathbf{Id}) \simeq (\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv}} \circ (\mathbf{p}_{I,*}^{+} \otimes \mathbf{Id})^{\mathrm{inv}}.$$

□[Lemma F.2.1]

Lemma F.2.2. *Let $\lambda \in \Lambda_{G,P}$. The following composition is conservative*

$$(\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \hookrightarrow (\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \xrightarrow{(\mathbf{p}^{+}_{I,*}\otimes\mathbf{Id})^{\mathrm{inv},L}} (\mathrm{DMod}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

Proof. Suppose that $y \in (\mathrm{DMod}({}_{\leq\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$ is sent to zero by the above composition. We need to show that $y \simeq 0$. By (F.5), it suffices to prove that the !-restrictions of y to $(\mathrm{DMod}({}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$ is zero for any $\mu \leq \lambda$. Note that these !-restrictions are equal to $*$-restrictions. Also note that ${}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Hence we are done by using induction and the excision triangle.

□[Lemma F.2.2]

Lemma F.2.3. *Let $\mathcal{D}$ be any DG-category. The tautological functor*

$$\alpha : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D} \to (\mathrm{DMod}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

is an isomorphism.

Proof. By Lemma F.1.1(2)(4) and Lemma F.2.1, the image of α generates the target under colimits and extensions. It remains to prove that α is fully faithful, which can be proved by diagram chasing with help of Lemma F.2.1. We exhibit it as follows.

Let $y \in \mathrm{DMod}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}$ and $z \in \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D}$. We have

$$\begin{aligned}
& \mathrm{Maps}((\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L(y),z)\\
\simeq\ & \mathrm{Maps}(y,(\mathbf{p}_I^{+!,\mathrm{inv}}\otimes\mathbf{Id})(z))\\
\simeq\ & \mathrm{Maps}(\beta(y),\beta\circ(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})(z))\\
\simeq\ & \mathrm{Maps}(\beta(y),(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}\circ\alpha(z))\\
\simeq\ & \mathrm{Maps}((\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv},L}\circ\beta(y),\alpha(z))\\
\simeq\ & \mathrm{Maps}(\alpha\circ(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L(y),\alpha(z)).
\end{aligned}$$

Then we are done because the category $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D}$ is generated under colimits and extensions by $(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L(y)$.

□[Lemma F.2.3]

F.2.4 (Proof of Proposition 2.1.8)**.** By Lemma D.1.11(4) and Lemma F.2.3, the category $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is dualizable in DGCat. Hence by Lemma D.1.10, $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ and $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ are canonically dual to each other. Since $\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ is compactly generated (by Lemma 2.1.7, which we have already proved), its dual category $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is also compactly generated. Moreover, we have a canonical equivalence

$$(\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^c \simeq (\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})^{c,\mathrm{op}}. \tag{F.7}$$

Consider the pairing functor for the above duality:

$$\langle -,- \rangle : \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \times \mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \to \mathrm{Vect}\,.$$

For any $\mathcal{F} \in \mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and any *compact* object $\mathcal{G}$ in $\mathrm{DMod}(\mathrm{Gr}_{M,I})$, we have

$$\langle \mathcal{F}, \mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{G}) \rangle \simeq \langle \mathbf{s}_I^! \circ \mathbf{oblv}^{\mathcal{L}U_I} \circ \mathcal{F}, \mathcal{G} \rangle_{\mathrm{Verdier}} \simeq$$
$$\simeq \mathrm{Maps}(\mathbb{D}(\mathcal{G}), \mathbf{s}_I^! \circ \mathbf{oblv}^{\mathcal{L}U_I} \circ \mathcal{F}) \simeq \mathrm{Maps}(\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*} \circ \mathbb{D}(\mathcal{G}), \mathcal{F}).$$

Hence the object (which is well-defined by Lemma 2.1.7(2))

$$\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*} \circ \mathbb{D}(\mathcal{G}) \in (\mathrm{DMod}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^c$$

is sent by (F.7) to the object $\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{G})$. Consequently, the latter object is compact. All such objects generate the category $\mathrm{DMod}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ under colimits and extensions because of Lemma 2.1.7(2).

□[Proposition 2.1.8]

Appendix G

Well-definedness results about $\mathrm{I}(G, P)$

The goal of this appendix is to prove Proposition 2.3.9 and Proposition 2.3.17, which were claimed in [Gai15a].

G.1 Proof of Proposition 2.3.9

Let $\widetilde{\mathrm{Bun}}_P$ be the Drinfeld's compactification constructed in [BG02]. Recall it is defined as

$$\widetilde{\mathrm{Bun}}_P \coloneqq \mathbf{Maps}_{\mathrm{gen}}(G\backslash\overline{G/U}/M \supset G\backslash(G/U)/M),$$

where $\overline{G/U}$ is the affine closure of G/U. By [Bar14, Remark 4.1.9], the map $\mathrm{Bun}_P \to \mathrm{Bun}_G^{P\text{-gen}}$ factors as

$$\mathrm{Bun}_P \xrightarrow{j} \widetilde{\mathrm{Bun}}_P \xrightarrow{\widetilde{\iota}_P} \mathrm{Bun}_G^{P\text{-gen}},$$

and the restriction of the map $\widetilde{\iota}_P$ on each connected component of $\widetilde{\mathrm{Bun}}_P$ is proper. Also, the map $\widetilde{\iota}_P$ is obtained by applying $\mathbf{Maps}_{\mathrm{gen}}(X, -)$ to the canonical morphism

$$(G\backslash\overline{G/U}/M \supset G\backslash(G/U)/M) \to (\mathbb{B}G \leftarrow \mathbb{B}P).$$

The above properness implies $\widetilde{\iota}_{P,!}$ is well-defined. On the other hand, it was proved in [DG16, § 1.1.6] that the composition

$$\mathrm{DMod}(\widetilde{\mathrm{Bun}}_P) \xrightarrow{j^!} \mathrm{DMod}(\mathrm{Bun}_P) \xrightarrow{\mathfrak{q}_{P,*}} \mathrm{DMod}(\mathrm{Bun}_M)$$

has a left adjoint isomorphic to

$$j_! \circ \mathfrak{q}_P^*(-) \simeq j_!(k_{\mathrm{Bun}_P}) \overset{!}{\otimes} \mathfrak{q}_P^!(-)[\mathrm{shift}], \tag{G.1}$$

where [shift] is a cohomological shift locally constant on Bun_M. Combining the above two results, we obtain the well-definedness of $\iota_{P,!} \circ \mathfrak{q}_P^*$.

To prove the second claim, we need to calculate $\iota_P^! \circ \iota_{P,!} \circ \mathfrak{q}_P^*$. Consider the diagram

$$\begin{tikzcd} \mathrm{Bun}_P \times_{\mathrm{Bun}_G^{P\text{-gen}}} \widetilde{\mathrm{Bun}}_P \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & \mathrm{Bun}_P \arrow[d, "\iota_P"] \\ \widetilde{\mathrm{Bun}}_P \arrow[r, "\widetilde{\iota}_P"] & \mathrm{Bun}_G^{P\text{-gen}} . \end{tikzcd}$$

By the base-change isomorphism, we have

$$\iota_P^! \circ \widetilde{\iota}_{P,!} \simeq \mathrm{pr}_{1,!} \circ \mathrm{pr}_2^! .$$

A direct calculation shows

$$\mathrm{Bun}_P \underset{\mathrm{Bun}_G^{P\text{-gen}}}{\times} \widetilde{\mathrm{Bun}}_P \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{G/U}/M \leftarrow P\backslash(P/U)/M).$$

Let $\overline{M}$ be the closure of P/U in $\overline{G/U}$, then we have

$$\mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{G/U}/M \leftarrow P\backslash(P/U)/M) \simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/M \leftarrow P\backslash(P/U)/M).$$

Now the RHS is isomorphic to $\mathrm{Bun}_P \times_{\mathrm{Bun}_M} H_{M,G\text{-}\mathrm{pos}}$, where

$$H_{M,G\text{-}\mathrm{pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M)$$

is the G-positive Hecke stack for M-tosors (see [Sch16, § 3.1.5]). Recall that the canonical map

$$i : \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}} \to \widetilde{\mathrm{Bun}}_P$$

is bijective on geometric points, and the connected components of the source provide a stratification on $\widetilde{\mathrm{Bun}}_P$ (known as the *defect stratification*).

We obtain

$$\iota_P^! \circ \iota_{P,!} \circ \mathfrak{q}_P^* \simeq \mathrm{pr}_{1,!} \circ i^! \circ j_! \circ \mathfrak{q}_P^*.$$

Hence it remains to show the functor $i^! \circ j_! \circ \mathfrak{q}_P^*$ factors through

$$\mathrm{DMod}(H_{M,G\text{-}\mathrm{pos}}) \xrightarrow{*\text{-pull}} \mathrm{DMod}(\mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} H_{M,G\text{-}\mathrm{pos}}). \tag{G.2}$$

By (G.1), we only need to show $i^! \circ j_!(k_{\mathrm{Bun}_P})$ is contained in the image of (G.2). However, this is well-known and can be proved using the Hecke actions in [BG02, § 6.2].

□[Proposition 2.3.9]

G.2 Proof of Proposition 2.3.17

Let M (resp. L) be the Levi quotient group of P (resp. Q). Let P_L be the image of P in L, which is a parabolic subgroup of L. Consider the correspondence

$$\mathrm{Bun}_L \leftarrow \mathrm{Bun}_{P_L} \rightarrow \mathrm{Bun}_M$$

and the corresponding geometric Eisenstein series functor

$$\mathrm{Eis}_{P_L,!} : \mathrm{DMod}(\mathrm{Bun}_M) \rightarrow \mathrm{DMod}(\mathrm{Bun}_L)$$

defined in [BG02]. Recall that it is defined as the $*$-pull-!-push along the above correspondence.

It is easy to check the composition of the correspondences

$$\begin{array}{ccccc} \mathrm{Bun}_L & \leftarrow & \mathrm{Bun}_{P_L} & \rightarrow & \mathrm{Bun}_M, \\ \mathrm{Bun}_G^{Q\text{-gen}} & \xleftarrow{\iota_Q} & \mathrm{Bun}_Q & \xrightarrow{\mathfrak{q}_Q} & \mathrm{Bun}_L \end{array}$$

is isomorphic to the composition of the correpondences

$$\begin{array}{ccccc} \mathrm{Bun}_G^{P\text{-gen}} & \xleftarrow{\iota_P} & \mathrm{Bun}_P & \xrightarrow{\mathfrak{q}_P} & \mathrm{Bun}_M, \\ \mathrm{Bun}_G^{Q\text{-gen}} & \xleftarrow{\mathfrak{p}^{\mathrm{enh}}_{P\to Q}} & \mathrm{Bun}_G^{P\text{-gen}} & \xrightarrow{=} & \mathrm{Bun}_G^{P\text{-gen}} \end{array}$$

Hence by the base-change isomorphisms, we have

$$\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!} \circ \iota_{P,!} \circ \mathfrak{q}_P^* \simeq \iota_{Q,!} \circ \mathfrak{q}_Q^* \circ \mathrm{Eis}_{P_L,!}\,. \tag{G.3}$$

In particular, the LHS is well-defined. Hence by Remark 2.3.11, $\mathfrak{p}^{\mathrm{enh}}_{P\to Q,!}$ is well-defined. This proves (1).

To prove (2), since $\mathrm{I}(G,P)$ is compactly generated (see Remark 2.3.11), we only need to prove $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q}$ preserves compact objects. By Remark 2.3.11 again, it suffices to prove $\mathrm{Eis}^{\mathrm{enh}}_{P\to Q} \circ \iota_{M,!}$ preserves compact objects. By (G.3), we have

$$\mathrm{Eis}^{\mathrm{enh}}_{P\to Q} \circ \iota_{M,!} \simeq \iota_{L,!} \circ \mathrm{Eis}_{P_L,!}\,.$$

Then we are done because both $\iota_{L,!}$ and $\mathrm{Eis}_{P_L,!}$ preserve compact objects.

□[Proposition 2.3.17]

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