LECTURE 9

1. Standard filtrations

Last time we proved $\mathcal{O}$ has enough projectives and injectives. Hence for $M, N \in \mathcal{O}$, the vector space $\mathsf{Ext}^i_{\mathcal{O}}(M, N), i \geq 0$ is well-defined. It can be calculated in either of the following methods:

- Choose a projective resolution $P^\bullet \to M$ and consider the chain complex $\mathsf{Hom}_{\mathcal{O}}(P^{-\bullet}, N)$. Then $\mathsf{Ext}^i_{\mathcal{O}}(M, N)$ is the i-th cohomology of this complex.
- Choose an injective resolution $N \to I^\bullet$ and consider the chain complex $\mathsf{Hom}_{\mathcal{O}}(M, I^\bullet)$. Then $\mathsf{Ext}^i_{\mathcal{O}}(M, N)$ is the i-th cohomology of this complex.

We will prove the following result:

Theorem-Definition 1.1. *For an object $M \in \mathcal{O}$, the following conditions are equivalent:*

(a) *The object M admits a finite filtration such that the subquotients are isomorphic to Verma modules.*
(b) *For any weight μ and $i > 0$, $\mathsf{Ext}^i_{\mathcal{O}}(M, M^\vee_\mu) = 0$.*
(c) *For any weight μ, $\mathsf{Ext}^1_{\mathcal{O}}(M, M^\vee_\mu) = 0$.*

A filtration as in (a) is called a ***standard filtration*** *of M. Let $\mathcal{O}^\Delta \subset \mathcal{O}$ be the full subcategory of objects admitting standard filtrations.*

Via contragradient duality, the above result is equivalent to:

Theorem-Definition 1.2. *For an object $M \in \mathcal{O}$, the following conditions are equivalent:*

(a) *The object M admits a finite filtration such that the subquotients are isomorphic to dual Verma modules.*
(b) *For any weight λ and $i > 0$, $\mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, M) = 0$.*
(c) *For any weight λ, $\mathsf{Ext}^1_{\mathcal{O}}(M_\lambda, M) = 0$.*

A filtration as in (a) is called a ***costandard filtration*** *of M. Let $\mathcal{O}^\nabla \subset \mathcal{O}$ be the full subcategory of objects admitting costandard filtrations.*

As a particular case, we obtain

Corollary 1.3. *For weights λ, μ and $i > 0$, we have $\mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, M^\vee_\mu) = 0$.*

We will also prove the following result.

Theorem 1.4. *Every projective object of $\mathcal{O}$ admits a standard filtration.*

To prove these results, we need some preparations.

Lemma 1.5. *The* generalized Verma modules[1] $M_{\lambda,n}$ *admit standard filtrations.*

Proof. Recall $M_{\lambda,n} \simeq \mathsf{ind}^{\mathfrak{g}}_{\mathfrak{b}}(U(\mathfrak{b})/I'_{\lambda,n})$, where $I'_{\lambda,n}$ is the left ideal of $U(\mathfrak{b})$ generated by the following elements:

- The element $t - \lambda(t)$ for any $t \in \mathfrak{t}$;

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[1] See [Proof of Lemma 4.10, Lecture 8].

- The element $x_1 x_2 \cdots x_n$ for $x_i \in \mathfrak{n}^+$.

By considering the weights, the finite-dimensional $\mathfrak{b}$-module $U(\mathfrak{b})/I'_{\lambda,n}$ has a filtration such that the subquotients are 1-dimensional. Since the functor $\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}$ is exact, we obtain a standard filtration of $M_{\lambda,n}$.

□

Lemma 1.6. *Let $M \in \mathcal{O}^{\Delta}$ be an object admitting a standard filtration $\mathsf{F}^{\leq \bullet} M$ with length m. Let $v \in M$ be a nonzero highest weight vector with weight λ, and $M_\lambda \to M$ be the unique $\mathfrak{g}$-linear map sending v_λ to v. Let i be the smallest index such that the image $\mathsf{Im}(f)$ is contained in $\mathsf{F}^{\leq i} M$. Then:*

1. *The composition $M_\lambda \to M \to \mathsf{gr}^i M$ is an isomorphism. In particular, $M_\lambda \to M$ is injective.*
2. *The quotient M/M_λ admits a standard filtration with length $m-1$.*

Proof. By assumption $\mathsf{gr}^i M$ is a Verma module M_μ and the composition $M_\lambda \to M \to \mathsf{gr}^i M$ is nonzero. This implies $\lambda \leq \mu$. Sinnce λ is a highest weight of M. We get $\lambda = \mu$. This implies $M_\lambda \to M \to \mathsf{gr}^i M$ is an isomorphism because it preserves the nonzero highest weight vectors. This proves (1).

For (2), we have a short exact seqeunce $0 \to \mathsf{F}^{\leq i-1} M \to M/M_\lambda \to M/\mathsf{F}^{\leq i} M \to 0$. By assumption, $\mathsf{F}^{\leq i-1} M$ has a standard filtration of length $i-1$, and $M/\mathsf{F}^{\leq i} M$ has one of length $m-i$. It follows that M/M_λ admits a standard filtration with length $(i-1)+(m-i) = m-1$.

□

Lemma 1.7. *For direct sum decomposition $M = M_1 \oplus M_2$ in $\mathcal{O}$, if M admits a standard filtration, so do M_1 and M_2.*

Proof. We use induction on the length of the standard filtration of M. When the length is 0, the claim is trivial. If the length is $m > 0$, then $M \neq 0$. Without lose of generality, we can assume M_1 contains a nonzero highest weight vector v of M with weight λ. By Lemma 1.6, we have injections $M_\lambda \hookrightarrow M_1 \hookrightarrow M$ such that M/M_λ admits a standard filtration of length $m-1$. Since $M/M_\lambda \simeq M_1/M_\lambda \oplus M_2$, by induction hypothesis, we have $M_1/M_\lambda, M_2 \in \mathcal{O}^{\Delta}$. It follows that we also have $M_1 \in \mathcal{O}^{\Delta}$.

□

Proof of Theorem 1.4. In the proof of [Theorem 4.3, Lecture 8], we proved any object in $\mathcal{O}$ is a quotient of a direct sum of some $M_{\lambda,n}$. In general, if a projective object is a quotient of another object, then it is a direct summand of the latter. It follows that any projective object is a direct summand of a direct sum of some $M_{\lambda,n}$. Then the claim follows from Lemma 1.5 and Lemma 1.7.

□[Theorem 1.4]

Lemma 1.8. *Let $0 \to K \to M \to N \to 0$ be a short exact sequence such that $N \in \mathcal{O}^{\Delta}$. Then $M \in \mathcal{O}^{\Delta}$ iff $K \in \mathcal{O}^{\Delta}$.*

Warning 1.9. *$K, M \in \mathcal{O}^{\Delta}$ does not imply $N \in \mathcal{O}^{\Delta}$. This can be seen from the $\mathfrak{sl}_2$-case and the short exact sequence $0 \to M_{-l-2} \to M_l \to L_l \to 0$.*

Proof. The "if" part is obvious. For the "only if" part, let $M \in \mathcal{O}^{\Delta}$. Using induction, it is easy to reduce to the case when N is a Verma module M_μ. Let $M_\lambda \to M$ be as in Lemma 1.6. Then $M/M_\lambda \in \mathcal{O}^{\Delta}$. Note that the composition $M_\lambda \to M \to M_\mu$ is either 0 or an isomorphism by considering the weights. If this is the zero map, then $M/M_\lambda \to M_\mu$ is still a surjection and we can finish the proof by using induction. Otherwise $M \simeq K \oplus M_\mu$ and the claim follows from Lemma 1.7.

□

Proof of Theorem-Definition 1.1. We first prove $(a) \Rightarrow (b)$. We prove by induction on i. For any fixed i, using the long exact sequences

$$\cdots \to \mathsf{Ext}^i_{\mathcal{O}}(\mathsf{F}^{\le k-1}M, M^\vee_\mu) \to \mathsf{Ext}^i_{\mathcal{O}}(\mathsf{F}^{\le k}M, M^\vee_\mu) \to \mathsf{Ext}^i_{\mathcal{O}}(\mathsf{gr}^k M, M^\vee_\mu) \to \cdots,$$

we only need to show $\mathsf{Ext}^i_{\mathcal{O}}(\mathsf{gr}^k M, M^\vee_\mu) = 0$ for any k. By assumption, $\mathsf{gr}^k M \simeq M_\lambda$ for some weight λ. The case $i = 1$ is just [Lemma 3.16, Lecture 8]. For $i > 1$, let $0 \to N \to P \to M_\lambda \to 0$ be a short exact sequence such that P is projective. We have a long exact sequence

$$\cdots \to \mathsf{Ext}^{i-1}_{\mathcal{O}}(P, M^\vee_\mu) \to \mathsf{Ext}^{i-1}_{\mathcal{O}}(N, M^\vee_\mu) \to \mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, M^\vee_\mu) \to \mathsf{Ext}^i_{\mathcal{O}}(P, M^\vee_\mu) \to \cdots.$$

Since P is projective, we have $\mathsf{Ext}^{i-1}_{\mathcal{O}}(N, M^\vee_\mu) \simeq \mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, M^\vee_\mu)$. By Theorem 1.4 and Lemma 1.8, $N \in \mathcal{O}^\Delta$. Hence by induction hypothesis $\mathsf{Ext}^{i-1}_{\mathcal{O}}(N, M^\vee_\mu) \simeq 0$ as desired.

$(b) \Rightarrow (c)$ is obvious.

It remains to show $(c) \Rightarrow (a)$.

We prove by induction on $\mathsf{length}(M)$. The case $\mathsf{length}(M) = 0$ is obvious. If $\mathsf{length}(M) > 0$, let λ be a highest weight vector of M and $n := \dim(M^{\mathsf{wt}=\lambda})$. Then there is a nonzero $\mathfrak{g}$-linear map $M_\lambda^{\oplus n} \to M$. Let[2] N_1 and N_2 be the kernel and cokernel of this map, and $M' \neq 0$ be the image of it. We have short exact sequences

$$0 \to N_1 \to M_\lambda^{\oplus n} \to M' \to 0,$$
$$0 \to M' \to M \to N_2 \to 0.$$

They induce long exact sequences

$$0 \to \mathsf{Hom}_{\mathcal{O}}(M', M^\vee_\mu) \to \mathsf{Hom}_{\mathcal{O}}(M_\lambda^{\oplus n}, M^\vee_\mu) \to \mathsf{Hom}_{\mathcal{O}}(N_1, M^\vee_\mu) \to \mathsf{Ext}^1_{\mathcal{O}}(M', M^\vee_\mu) \to \cdots$$

$$\cdots \to \mathsf{Hom}_{\mathcal{O}}(M, M^\vee_\mu) \to \mathsf{Hom}_{\mathcal{O}}(M', M^\vee_\mu) \to \mathsf{Ext}^1_{\mathcal{O}}(N_2, M^\vee_\mu) \to \mathsf{Ext}^1_{\mathcal{O}}(M, M^\vee_\mu) \to$$
$$\to \mathsf{Ext}^1_{\mathcal{O}}(M', M^\vee_\mu) \to \mathsf{Ext}^2_{\mathcal{O}}(N_2, M^\vee_\mu) \to \cdots$$

By assumption, $\mathsf{Ext}^1_{\mathcal{O}}(M, M^\vee_\mu) = 0$. We claim $\mathsf{Ext}^1_{\mathcal{O}}(N_2, M^\vee_\mu) = 0$. Indeed:

- If $\lambda \neq \mu$, then $\mathsf{Hom}_{\mathcal{O}}(M_\lambda^{\oplus n}, M^\vee_\mu) = 0$ ([Lemma 3.14, Lecture 8]). By the first sequence, $\mathsf{Hom}_{\mathcal{O}}(M', M^\vee_\mu) = 0$. By the second sequence, $\mathsf{Ext}^1_{\mathcal{O}}(N_2, M^\vee_\mu) = 0$.
- If $\lambda = \mu$, note that μ is a highest weight of both M' and M and by construction, $(M')^{\mathsf{wt}=\mu} \simeq M^{\mathsf{wt}=\mu}$. Hence we have

  $$\mathsf{Hom}_{\mathcal{O}}(M', M^\vee_\mu) \simeq ((M')^{\mathsf{wt}=\mu})^* \simeq (M^{\mathsf{wt}=\mu})^* \simeq \mathsf{Hom}_{\mathcal{O}}(M, M^\vee_\mu).$$

  By the second sequence, $\mathsf{Ext}^1_{\mathcal{O}}(N_2, M^\vee_\mu) = 0$.

Note that $\mathsf{length}(N_2) < \mathsf{length}(M)$. By the induction hypothesis, $N_2 \in \mathcal{O}^\Delta$. Using $(a) \Rightarrow (b)$, we get $\mathsf{Ext}^2_{\mathcal{O}}(N_2, M^\vee_\mu) = 0$. By the second seqeunce, $\mathsf{Ext}^1_{\mathcal{O}}(M', M^\vee_\mu) = 0$. Now we have two cases:

- If $N_2 \neq 0$, then $\mathsf{length}(M') < \mathsf{length}(M)$. By the induction hypothesis, $M' \in \mathcal{O}^\Delta$. Then $M \in \mathcal{O}^{\mathsf{length}}$ because it is an extension of N_2 by M'.
- If $N_2 = 0$, then $M \simeq M'$. An argument similar to that in the last paragraph implies $\mathsf{Hom}_{\mathcal{O}}(N_1, M^\vee_\mu) = 0$. Note that μ can be any weight. This forces $N_1 = 0$ and therefore $M \simeq M_\lambda^{\oplus n}$. Then it is clear $M \in \mathcal{O}^\Delta$.

□[Theorem-Definition 1.1]

Lemma 1.10. *Let $M \in \mathcal{O}_\chi$ and $M' \in \mathcal{O}_{\chi'}$ such that $\chi \neq \chi'$. Then $\mathsf{Ext}^i(M, N) = 0$ for any $i \geq 0$.*

[2]The following language can be rewritten in the language of spectral sequences.

Proof. Let $P^\bullet \to M$ be a projective resolution of M. We can replace each P^{-n} by its image under the functor $\mathcal{O} \to \mathcal{O}_\chi$ and obtain a projective resolution of M contained in $\mathcal{O}_\chi$. Now the claim follows from the case $i = 0$.

□

Exercise 1.11. This is Homework 4, Problem 4. Let λ be a weight. Prove:

(1) If $M \in \mathcal{O}$ such that $\mathsf{wt}(M) \cap \{\mu \,|\, \mu \geq \lambda\} = \varnothing$, then $\mathsf{Ext}^i_{\mathcal{O}}(M, M_\lambda^\vee) = 0$ for $i \geq 0$[3].
(2) $\mathsf{Ext}^i_{\mathcal{O}}(L_\lambda, M_\lambda^\vee) = 0$ for $i > 0$.
(3) Combining (1) and (2), deduce $\mathsf{Ext}^i_{\mathcal{O}}(M_\lambda, L_\mu) = 0$ and $\mathsf{Ext}^i_{\mathcal{O}}(L_\mu, M_\lambda^\vee) = 0$ for $i > 0$ and $\mu \not> \lambda$.

2. BGG reciprocity

The following result follows from Theorem-Definition 1.1 by dévissage.

Proposition-Definition 2.1. *Let $M \in \mathcal{O}^\Delta$, then for any standard filtration of M, the multiplicity of M_λ in the subquotients does not depend on the filtration. We denote this number by $(M : M_\lambda)$. Moreover, we have*

$$(M : M_\lambda) \simeq \dim(\mathsf{Hom}_{\mathcal{O}}(M, M_\lambda^\vee)).$$

Theorem 2.2 (BGG reciprocity). *For weights $\lambda, \mu \in \mathfrak{t}^*$, we have*

$$(P_\mu : M_\lambda) = [M_\lambda^\vee : L_\mu] = [M_\lambda : L_\mu].$$

Proof. The last identity follows from $L_\mu^\vee \simeq L_\mu$. The first one follows from Proposition-Definition 2.1 and [Corollary 4.9, Lecture 8].

□

Remark 2.3. The previous discussions on standard filtrations and BGG reciprocity can be axiomized using the language of **highest weight categories**. See [CPS].

Remark 2.4. In $\mathcal{O}$, or any highest weight category, if an object admits both a standard filtration and a costandard filtration, then it is called a **tilting object**. One can show the set of indecomposable tilting objects is bijective to the set of irreducible objects. Tilting objects play important roles in representation theory. For a geometry-oriented[4] introduction, see [BBM].

3. Translation functors

Construction 3.1. *Let V be a finite-dimensional $\mathfrak{g}$-module. Consider the functor*

$$\mathfrak{g}\text{−}\mathsf{mod} \to \mathfrak{g}\text{−}\mathsf{mod},\ M \mapsto V \otimes M,$$

where (recall) the $\mathfrak{g}$-module structure on $V \otimes M$ is given by the Lebniz rule. It is easy to see this functor preserves $\mathcal{O}$. We denote the obtained functor by $T_V : \mathcal{O} \to \mathcal{O}$. Note that T_V is exact.

Lemma 3.2. *The functor $T_V : \mathcal{O} \to \mathcal{O}$ commutes with contragradient duality.*

Proof. For any finite dimensional $\mathfrak{g}$-module, we have $V^\vee \simeq V$ because $L_\lambda^\vee \simeq L_\lambda$.

□

Lemma 3.3. *The functor T_{V^*} is both left and right adjoint to T_V. In particular, T_V preserves both projectives and injectives.*

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[3]Hint: Step 1: reduce to the case $M = L_\mu$ with $\varpi(\mu) = \varpi(\lambda)$ and $\mu \not\geq \lambda$. Step 2: consider $0 \to K \to M_\mu \to L_\mu \to 0$ and note that $\mathsf{wt}(K) < \mu$.

[4]You should ask experts in representation theory for other good references.

Proof. We have the unit and pairing maps $k \to V \otimes V^*$ and $V^* \otimes V \to k$, which induce natural transformations $T_k \to T_{V\otimes V^*}$ and $T_{V^*\otimes V} \to T_k$. Note that $T_k \simeq \mathsf{Id}$ and $T_{V\otimes V'} \simeq T_V \circ T_{V'}$. Hence we obtain natural transformations $\mathsf{Id} \to T_V \circ T_{V^*}$ and $T_{V^*} \circ T_V \to \mathsf{Id}$. Now the duality data between V and V^* are translated exactly to the djunction data between T_V and T_{V^*}.

□

Remark 3.4. It follows formally that T_{V^*} and T_V are adjoint in the derived sense, i.e. $\mathsf{Ext}^i(T_V(M), N) \simeq \mathsf{Ext}^i(M, T_{V^*}(N))$.

Lemma 3.5. *For any weight λ, the module $T_V(M_\lambda)$ admits a standard filtration $\mathsf{F}^{\le k}(T_V(M_\lambda))$ such that the highest weights of $\mathsf{gr}^k(T_V(M_\lambda))$ is (non-strictly) decreasing in k. Moreover,*

$$(T_V(M_\lambda) : M_\mu) = \dim V^{\mathsf{wt}=\mu-\lambda}.$$

Proof. Follows from the *projection formula*

$$\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(V_1) \otimes V_2 \simeq \mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(V_1 \otimes V_2),\ V_1 \in \mathfrak{b}\text{−}\mathsf{mod}, V_2 \in \mathfrak{g}\text{−}\mathsf{mod}.$$

Indeed, any finite-dimensional weight $\mathfrak{b}$-module, such as $V \otimes k_\lambda$, admits a finite filtration whose subquotients are 1-dimensional modules with decreasing weights.

□

Construction 3.6. *Let χ_1, χ_2 be two central characters. For any finite-dimensional $\mathfrak{g}$-module V, consider the composition*

$$T_{\chi_1,V,\chi_2} : \mathcal{O}_{\chi_1} \to \mathcal{O} \xrightarrow{T_V} \mathcal{O} \to \mathcal{O}_{\chi_2}.$$

We call such functors the ***translation functors****. It follows that T_{χ_1,V,χ_2} is exact, and is both left and right adjoint to T_{χ_2,V^*,χ_1}.*

Construction 3.7. *Let λ and μ be weights such that $\mu - \lambda$ is integral. Then the W-orbit (for the linear action) $W(\mu - \lambda)$ contains a unique dominant integral weigth ν. Consider the finite-dimensional $\mathfrak{g}$-module L_ν. We write*

$$T_{\lambda,\mu} := T_{\varpi(\lambda),L_\nu,\varpi(\mu)} : \mathcal{O}_{\varpi(\lambda)} \to \mathcal{O}_{\varpi(\mu)}.$$

Note that $w_0(-\nu)$ is also dominant and integral. By definition, we have

$$T_{\mu,\lambda} = T_{\varpi(\mu),L_{-w_0(\nu)},\varpi(\lambda)},$$

which is both left and right adjoint to $T_{\lambda,\mu}$ because $L_{-w_0(\nu)} \simeq L_\nu^*$[5]

Recall the following definition:

Definition 3.8. Let (E, Φ) be the root system of $\mathfrak{g}$. For $\lambda, \mu \in E$, we say they belong to the same **dot-Weyl facet** if the signs of $\langle \lambda + \rho, \check{\alpha} \rangle$ and $\langle \mu + \rho, \check{\alpha} \rangle$ are the same for any $\alpha \in \Phi$.

For $\lambda \in E$, let F_λ be the dot-Weyl facet containing it.

Definition 3.9. For a dot-Weyl facet F_λ, its **upper closure** F_λ^+ is the subset of $\mu \in E$ such that

$$\langle \mu + \rho, \check{\alpha} \rangle \begin{cases} > 0 & \text{if } \langle \lambda + \rho, \check{\alpha} \rangle > 0, \\ = 0 & \text{if } \langle \lambda + \rho, \check{\alpha} \rangle = 0, \\ \le 0 & \text{if } \langle \lambda + \rho, \check{\alpha} \rangle < 0 \end{cases}$$

[5]Here L_ν^* is the usual linear dual of L_μ. It is not the contragradient dual. Note that $w_0(\nu)$ is indeed the lowest weight of L_μ.

Remark 3.10. When equipped with the standard topology, F_λ is a locally closed subset of E, and both the closure $\overline{F}_\lambda$ and the upper closure F_λ^+ are unions of dot-Weyl facets.

Note that each dot-Weyl facet is contained in the upper closure of a unique **dot-Weyl chamber**, i.e., open dot-Weyl facet. Also, λ is dot-regular iff F_λ is a dot-Weyl chamber.

Example 3.11. For $\mathfrak{sl}_2$ and the coordinate $l \coloneqq \langle \lambda, \check{\alpha} \rangle$, there are three facets: $(-\infty, -1)$, $\{-1\}$ and $(-1, \infty)$. Note that $\{-1\}$ is contained in the upper closure of $(-\infty, -1)$.

We have the following theorem. For complete proofs and its generalization to non-integral weights, see [H, Sect. 7][6].

Theorem 3.12. *Let λ and μ be dot-antidominant integral weights such that $F_\mu \subset \overline{F_\lambda}$. Then for any $w \in W$*

$$T_\lambda^\mu(M_{w\cdot\lambda}) \simeq M_{w\cdot\mu},\ T_\lambda^\mu(M_{w\cdot\lambda}^\vee) \simeq M_{w\cdot\mu}^\vee$$

and

$$T_\lambda^\mu(L_{w\cdot\lambda}) \simeq \begin{cases} L_{w\cdot\mu} & \text{if } F_{w\cdot\mu} \subset F_{w\cdot\lambda}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch. Let $\nu \in W(\mu - \lambda)$ be the unique dominant integral weight in this orbit. Write $V \coloneqq L_\nu$. By Lemma 3.5, $T_\lambda^\mu(M_{w\cdot\lambda})$ admits a standard filtration and the multiplicity

$$(T_\lambda^\mu(M_{w\cdot\lambda}) : M_{w'\cdot\mu}) = \dim V^{\mathsf{wt}=w'\cdot\mu - w\cdot\lambda}.$$

The RHS is nonzero unless $w'\cdot\mu - w\cdot\lambda \le \nu$. Now a combinatorial argument (see [H, Lemma 7.5]) shows the latter can happen only if $w'\cdot\mu = w\cdot\mu$, and in this case the multplicity is 1 because $w\cdot\mu - w\cdot\lambda = w(\mu-\lambda) \in W(\nu)$. This implies $T_\lambda^\mu(M_{w\cdot\lambda}) \simeq M_{w\cdot\mu}$.

The statement for dual Verma modules follows from the contragradient duality.

Now consider the chain $M_{w\cdot\lambda} \twoheadrightarrow L_{w\cdot\lambda} \hookrightarrow M_{w\cdot\lambda}^\vee$. Since T_λ^μ is exact, we obtain a chain $M_{w\cdot\mu} \twoheadrightarrow T_\lambda^\mu(L_{w\cdot\lambda}) \hookrightarrow M_{w\cdot\mu}^\vee$. This forces $T_\lambda^\mu(L_{w\cdot\lambda}) \simeq L_{w\cdot\mu}$ or 0 ([Lemma 3.14, Lecture 8]).

It remains to pinpoint these two cases. Since the exact functor T_λ^μ sends $M_{w\cdot\lambda}$ to $M_{w\cdot\mu}$, there exists a unique composition factor $L_{w'\cdot\lambda}$ of $M_{w\cdot\lambda}$ such that $T_\lambda^\mu(L_{w'\cdot\lambda}) \simeq L_{w\cdot\mu}$. By the last paragraph, we must have $w'\cdot\mu = w\cdot\mu$. Now we have two cases:

- If $F_{w\cdot\mu}$ is contained in the upper closure $F_{w\cdot\lambda}^+$, then a combinatorial argument shows $w'\cdot\lambda = w\cdot\lambda$ and therefore $T_\lambda^\mu(L_{w\cdot\lambda}) \simeq L_{w\cdot\mu}$ as desired.
- If $F_{w\cdot\mu}$ is not contained in the upper closure $F_{w\cdot\lambda}^+$, then there exists a reflection s_α such that $s_\alpha w\cdot\lambda < w\cdot\lambda$ while $s_\alpha w\cdot\mu = w\cdot\mu$. By Verma's theorem, there exists a proper embedding $M_{s_\alpha w\cdot\lambda} \subset M_{w\cdot\lambda}$ which is sent to an isomorphism by T_λ^μ. Hence $L_{w\cdot\lambda}$, which is a quotient of $M_{w\cdot\lambda}/M_{s_\alpha w\cdot\lambda}$, is sent to 0 as desired.

□

The following result is a formal consequence of the above theorem:

Theorem 3.13. *Let λ and μ be dot-antidominant integral weights such that $F_\lambda = F_\mu$. Then $T_\lambda^\mu : \mathcal{O}_{\varpi(\lambda)} \to \mathcal{O}_{\varpi(\mu)}$ is an equivalence.*

Proof. By the previous theorem, the exact functor $F \coloneqq T_\lambda^\mu$ is left adjoint to the exact functor $G \coloneqq T_\mu^\lambda$, and they induce a bijection between the sets of irreducible objects. In general, such adjoint functors are inverse to each other.

Proof: we only to show the adjunctions $\mathsf{Id} \to G \circ F$ and $F \circ G \to \mathsf{Id}$ are equivalences. We will prove the first equivalence and the second follows similarly. Note that for any object M, M and $G \circ F(M)$ have the same composition factors and multiplicities. Hence we only need to show $M \to G \circ F(M)$ is injective. Since F sends nonzero objects to nonzero objects, we only need to

[6]See [G, Sect. 4.23] for a simplified proof in a special case.

show $F(M) \to F \circ G \circ F(M)$ is injective. By the axiom of adjoint functors, this morphism has a left inverse, hence is indeed injective.

□

Remark 3.14. The above theorems essentially reduce the study about any integral block $\mathcal{O}_\chi$ to the principle block $\mathcal{O}_{\varpi(0)}$.

Example 3.15. Any dot-regular integral block $\mathcal{O}_\chi$ is equivalent to the principle block $\mathcal{O}_{\varpi(0)}$. Indeed, this is the special case of the above theorem when $\lambda = 0$ and $\mu \in \varpi^{-1}(\chi)$ is dot-antidominant. Note that the equivalence T_0^μ preserves (dual) Verma modules and irreducible modules.

Example 3.16. For $\mathfrak{g} = \mathfrak{sl}_2$ and the coordinate $l := \langle \lambda, \check{\alpha} \rangle$, then $\mathcal{O}_{\varpi(0)} \simeq \mathcal{O}_{\varpi(l)}$ for $l \in \mathbb{Z}^{\geq 0}$ such that the short exact sequence $0 \to M_{-2} \to M_0 \to L_0 \to 0$ is sent to $0 \to M_{-l-2} \to M_l \to L_l \to 0$.

On the other hand, the translation functor $\mathcal{O}_{\varpi(0)} \to \mathcal{O}_{\varpi(-1)}$ sends this sequence to $0 \to M_{-1} \to M_{-1} \to 0 \to 0$. Note that L_{-2} is sent to L_{-1} and $-1 \in F_{-2}^+$, while L_0 is sent to 0 and $-1 \notin F_0^+$.

Construction 3.17. *Let μ be any dot-antidominant integral weight. The functor $T_{-\rho}^\mu : \mathcal{O}_{\varpi(-\rho)} \to \mathcal{O}_{\varpi(\mu)}$ is* not *covered by the above theorems (although its adjoint $T_\mu^{-\rho}$ is). Recall the most singular block $\mathcal{O}_{\varpi(-\rho)}$ is semi-simple ([Example 4.16, Lecture 7]) and contains a unique irreducible object $L_{-\rho}$, which is both projective and injective. It follows that*

$$\Xi_\mu := T_{-\rho}^\mu (L_{-\rho})$$

is both injective and projective.

Exercise 3.18. This is Homework 4, Problem 5. Let μ be any dot-antidominant integral weight. Prove[7]:

(1) For any $w \in W$, $(\Xi_\mu : M_{w \cdot \mu}) = 1$ and there is a surjection $\Xi_\mu \twoheadrightarrow M_\mu$.
(2) For any $w \in W$, $(P_\mu : M_{w \cdot \mu}) \geq 1$ and there is a surjection $P_\mu \twoheadrightarrow M_\mu$.
(3) There exists an isomorphism $\Xi_\mu \simeq P_\mu$ compatible with the surjections to M_μ.
(4) For any $w \in W$, $[M_{w \cdot \mu} : L_\mu] = 1$[8].

Remark 3.19. For dot-antidominant weight μ, the projective P_μ is called the **big projective module**, which plays an important role in representation theory.

References

[BBM] Beilinson, A., R. Bezrukavnikov, and I. Mirkovic. Tilting Exercises. Moscow Mathematical Journal 4, no. 3 (2004): 547-557.

[CPS] Scott, L., Brian Parshall, and E. Cline. Finite dimensional algebras and highest weight categories. (1988): 85-99.

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/catO.pdf`.

[H] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[7] Hint: Lemma 3.5 for (1); Theorem 2.2 and [Corollary 4.15, Lecture 7] for (2). For (3), first find a surjection $\Xi_\mu \twoheadrightarrow P_\mu$ then use (1) and (2).

[8] See [G, Proposition 4.20] for a different proof of this fact.