Fix

C an alg. closed, perfectoid field of char p

$X_C := \mathrm{Proj}\left(\bigoplus_{n\geq 0} B^{\varphi=p^n}\right)$ Fargues-Fontaine Curve.

Graded module $\bigoplus M_i$ over $\bigoplus_{n\geq 0} B^{\varphi=p^n}$

$\rightsquigarrow \widetilde{M} \in QCoh\left(\mathrm{Proj}\left(\bigoplus_{n\geq 0} B^{\varphi=p^n}\right)\right)$

$$\left[\begin{array}{c} x \in X_C \text{ closed pt } \{x\} = V(f \in B^{\varphi=p}) \\ \widetilde{M}(X_C \setminus x) = M[\tfrac{1}{f}]^0 \end{array}\right]$$

$\exists$ canonical map $M^0 \longrightarrow H^0(X_C, \widetilde{M})$

Take $M(m) = \bigoplus_{n\in\mathbb{Z}} B^{\varphi=p^{n+m}}$.

$\mathcal{O}(m) := \widetilde{M(m)}$ is a line bundle.

$\exists$ canonical map $B^{\varphi=p^m} \longrightarrow H^0(X_C, \mathcal{O}(m))$

Thm $\rho: \mathbb{Z} \longrightarrow \operatorname{Pic} X_C$

$$m \longmapsto \mathcal{O}(m)$$

is an isomorphism of abelian groups.

Pf (sketch)

Surjectivity: Let $x = V(f + B^{\varphi = p})$

Under $B^{\varphi=p} \longrightarrow H^0(X_C, \mathcal{O}(1))$

f goes to a global section of $\mathcal{O}(1)$

that has zero of order 1 at x

$\Rightarrow$ we obtain isomorphism

$$\mathcal{O}(x) \simeq \mathcal{O}(1).$$

thus

$$\begin{tikzcd} & \operatorname{Div} X_C \arrow[dl, "\deg"'] \arrow[d] & x \arrow[d, mapsto] \\ \mathbb{Z} \arrow[r, "\rho"] & \operatorname{Pic} X_C & \mathcal{O}(x) \end{tikzcd}$$

commutes.

X_C is a dedekind scheme, so

$\mathrm{Div}\, X_C \longrightarrow \mathrm{Pic}\, X_C$ is surjective.

$\therefore \quad \mathbb{Z} \xrightarrow{e} \mathrm{Pic}\, X_C$ is surjective.

Injectivity:

$$B^{\varphi = p^m} \xrightarrow{\sim} H^0(X_C, \mathcal{O}(m))$$

is an isomorphism, and

$\mathbb{Q}_p = B^{\varphi=1} \not\cong B^{\varphi = p^m}$ for any $m > 0$,

so $\mathcal{O}_{X_C} \not\cong \mathcal{O}(m)$ for any $m > 0$.

$\square$

Definitions

$$\deg \mathcal{L} := \rho^{-1}(\mathcal{L})$$

For vector bundle $\mathcal{E}$,

$$\deg \mathcal{E} := \deg \bigwedge^{\operatorname{rk} \mathcal{E}} \mathcal{E}$$

$$\operatorname{slope} \mathcal{E} := \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}.$$

Given $0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$, SES of vb

$$\operatorname{rk} \mathcal{E} = \operatorname{rk} \mathcal{E}' + \operatorname{rk} \mathcal{E}''$$

$$\bigwedge^{\operatorname{rk} \mathcal{E}} \mathcal{E} \simeq \bigwedge^{\operatorname{rk} \mathcal{E}'} \mathcal{E}' \otimes \bigwedge^{\operatorname{rk} \mathcal{E}''} \mathcal{E}''$$

$$\therefore \deg \mathcal{E} = \deg \mathcal{E}' + \deg \mathcal{E}''.$$

$$\Longrightarrow \quad \operatorname{slope} \mathcal{E}' = \operatorname{slope} \mathcal{E} = \operatorname{slope} \mathcal{E}''$$

OR

$$\operatorname{slope} \mathcal{E}' < \operatorname{slope} \mathcal{E} < \operatorname{slope} \mathcal{E}''$$

OR

$$\operatorname{slope} \mathcal{E}' > \operatorname{slope} \mathcal{E} > \operatorname{slope} \mathcal{E}''.$$

Def'n

$\mathcal{E}$ is semistable if $\mathcal{E}' \subseteq \mathcal{E} \Rightarrow \text{slope}\, \mathcal{E}' \leq \text{slope}\, \mathcal{E}$

$\mathcal{E}$ is stable if $\mathcal{E}' \subseteq \mathcal{E} \Rightarrow \text{slope}\, \mathcal{E}' < \text{slope}\, \mathcal{E}$

Facts

- $\overset{\text{semistable}}{\mathcal{E}} \twoheadrightarrow \mathcal{E}'' \Rightarrow \text{slope}\, \mathcal{E} \leq \text{slope}\, \mathcal{E}''$
- $\overset{\text{semistable}}{\mathcal{E}} \xrightarrow{f} \overset{\text{semistable}}{\mathcal{F}}$, $\text{slope}\, \mathcal{E} > \text{slope}\, \mathcal{F} \Rightarrow f = 0$.
- Semistable vector bundles of fixed slope λ form abelian category $\text{Vect}_\lambda(X_e)$.

Def'n A Harder-Narasimhan filtration is

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = \mathcal{E}$$

such that

① The $\mathcal{E}_i / \mathcal{E}_{i-1}$ are semistable

② $\text{slope}\, \mathcal{E}_1/\mathcal{E}_0 > \text{slope}\, \mathcal{E}_2/\mathcal{E}_1 > \cdots > \text{slope}\, \mathcal{E}_m/\mathcal{E}_{m-1}$.

Thm Every vector bundle $\mathcal{E}$ on X_C has a unique Harder–Narasimhan filtration.

Pf

Uniqueness:

Induction on $\mathrm{rk}\, \mathcal{E}$ – let

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = \mathcal{E}$$

$$0 = \mathcal{E}_0' \subsetneq \mathcal{E}_1' \subsetneq \cdots \subsetneq \mathcal{E}_n' = \mathcal{E}$$

be two Harder-Narasimhan filtrations.

Want to show $\mathcal{E}_1 = \mathcal{E}_1'$.

First, we will show

$$\mathrm{slope}\, \mathcal{E}_1 = \mathrm{slope}\, \mathcal{E}_1'.$$

If not, WLOG $\mathrm{slope}\, \mathcal{E}_1 > \mathrm{slope}\, \mathcal{E}_1'$.

Then slope $\mathcal{E}_1 >$ slope $\mathcal{E}_i'/\mathcal{E}_{i-1}'$ $\quad \forall\, 1 \leq i \leq n,$

so $\quad \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i'/\mathcal{E}_{i-1}') = 0 \quad \forall\, 1 \leq i \leq n,$

$\Longrightarrow \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}) = 0$ ↯.

$\therefore$ slope $\mathcal{E}_1 =$ slope $\mathcal{E}_1'$

But slope $\mathcal{E}_1 >$ slope $\mathcal{E}_i'/\mathcal{E}_{i-1}'$ $\quad \forall\, 2 \leq i \leq n,$

so $\quad \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i'/\mathcal{E}_{i-1}') = 0 \quad \forall\, 2 \leq i \leq n,$

$\Longrightarrow \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}/\mathcal{E}_1') = 0.$

$\Longrightarrow \mathcal{E}_1 \hookrightarrow \mathcal{E} \twoheadrightarrow \mathcal{E}/\mathcal{E}_1'$ is 0

$\Longrightarrow \mathcal{E}_1 \subseteq \mathcal{E}_1'.$

By symmetry, $\mathcal{E}_1' \subseteq \mathcal{E}_1$ so $\mathcal{E}_1 = \mathcal{E}_1'$.

Then apply inductive hypothesis on

$$0 = \mathcal{E}_1/\mathcal{E}_1 \subsetneq \mathcal{E}_2/\mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_n/\mathcal{E}_1$$

$$0 = \mathcal{E}_1'/\mathcal{E}_1' \subsetneq \mathcal{E}_2'/\mathcal{E}_1' \subsetneq \cdots \subsetneq \mathcal{E}_n'/\mathcal{E}_1'$$

Existence:

Induction on $\operatorname{rk}\mathcal{E}$ -

$$S := \{ \operatorname{slope}\mathcal{E}' \mid \mathcal{E}' \subseteq \mathcal{E} \}$$

$\lambda := \max S$.

Take $\mathcal{E}' \subseteq \mathcal{E}$ with $\operatorname{slope}\mathcal{E}' = \lambda$

with largest possible rank.

$\mathcal{E}'$ is semistable. $\mathcal{E}'' := \mathcal{E}/\mathcal{E}'$ has H-N filt.

$$0 = \mathcal{E}_0'' \subsetneq \mathcal{E}_1'' \subsetneq \cdots \subsetneq \mathcal{E}_m'' = \mathcal{E}''$$

$$0 = \mathcal{E}_{-1} \subsetneq \mathcal{E}' = \overline{\mathcal{E}}_0'' \subsetneq \overline{\mathcal{E}}_1'' \subsetneq \cdots \subsetneq \overline{\mathcal{E}}_m'' = \mathcal{E}$$

If $\operatorname{slope}\mathcal{E}' \leq \operatorname{slope}\overline{\mathcal{E}}_1''/\mathcal{E}' = \operatorname{slope}\mathcal{E}_1''$,

$$0 \to \mathcal{E}' \to \overline{\mathcal{E}}_1'' \to \mathcal{E}_1'' \to 0$$

so $\operatorname{slope}\overline{\mathcal{E}}_1'' \geq \operatorname{slope}\mathcal{E}'$ $\lightning$ $\square$.

<u>Prop</u>

Let $d \in \mathbb{Z}$, $n \in \mathbb{N}$.

$\exists$ a semistable vector bundle $\mathcal{E}$ on X_C

$$\deg \mathcal{E} = d, \quad \operatorname{rk} \mathcal{E} = n.$$

<u>Idea</u>: $[E : \mathbb{Q}_p] = n$.

$$\begin{array}{l} \text{Let } X_{C,E} := X_C \times_{\operatorname{Spec} \mathbb{Q}_p} \operatorname{Spec} E. \\ \quad \pi \big\downarrow \text{ degree } n \text{ cover} \\ \quad X_C \end{array}$$

For $\mathcal{F} \in \mathrm{Vect}_{X_{C,E}}$,

$$\deg \pi_* \mathcal{F} = \deg \mathcal{F}$$

$$\operatorname{rk} \pi_* \mathcal{F} = n \operatorname{rk} \mathcal{F}$$

$\pi_* \mathcal{F}$ semistable iff $\mathcal{F}$ semistable

Take $\mathcal{L}$ a deg d line bundle on $X_{C,E}$,
$\mathcal{E} = \pi_* \mathcal{L}$.

This is independent of $\mathcal{L}$ and E!

Like for $E = \oplus_{P}$,

$$\operatorname{Pic} X_{C,E} \simeq \mathbb{Z}$$

$\rightsquigarrow$ independence of $\mathcal{L}$.

For independence of E, we introduce

Isocrystals Let k be alg. closed field of char p
$K = W(k)[\frac{1}{p}]$

An isocrystal over k is

a K-vector space V

an abelian group isomorphism $\varphi_V : V \to V$

such that $\varphi_V(\lambda v) = \varphi_K(\lambda)\varphi_V(v)$.

Example $\gcd(m,n) = 1$, $n \in \mathbb{N}$.

$V_{\frac{m}{n}} := K^n$,

$\varphi_{V_{\frac{m}{n}}}(x_1, x_2, \ldots, x_n)$

$$= (\varphi_K(x_2), \varphi_K(x_3), \ldots, p^m \varphi_K(x_1)).$$

Dieudonné - Manin Classification

- Isoc_k is semisimple.
- The simple objects are precisely $V_{\frac{m}{n}}$.

Now let $k = \overline{\mathbb{F}_p}$ = residue field of C

$\rightsquigarrow \quad W(\overline{\mathbb{F}_p}) \longrightarrow A_{inf}$

$\rightsquigarrow \quad K = W(\overline{\mathbb{F}_p})[\frac{1}{p}] \longrightarrow B.$

Given $V \in \mathrm{Isoc}_K$, let

$$\mathcal{E}_V := \bigoplus_{n \geq 0} \mathrm{Hom}_K(V, B)^{\varphi = p^n}.$$

Fact: this quasicoherent sheaf on X_C is a vector bundle.

This gives

$$\mathrm{Isoc}_k^{op} \longrightarrow \mathrm{Vect}(X)$$

<u>Warning</u>: This is <u>not</u> an equivalence of categories.

$\mathcal{O}(\frac{m}{n}) := \mathcal{E}_{V_{\frac{m}{n}}}$.

If $\gcd(m,n)=1$, consider $[E:\mathbb{Q}_p]=n$,

$$\begin{array}{ccc} X_{C,E} & & \mathcal{O}_E(m) \\ \pi \big\downarrow & & \\ X_C & & \pi_* \mathcal{O}_E(m). \end{array}$$

<u>Claim</u>: $\pi_* \mathcal{O}_E(m) = \mathcal{O}(\frac{m}{n})$

<u>Pf</u> (sketch) Let $\begin{array}{cl} E & \\ | & e \quad \text{tot. ramified} \\ E_0 & \\ | & d \quad \text{unramified.} \\ \mathbb{Q}_p & \end{array}$

① $\pi_* \mathcal{O}_E(m) = \mathcal{E}_V$ for $V = E^\vee \otimes_{\mathbb{Q}_p} K$

w/ appropriate φ_V

For $d=1$,

$$\varphi_V(x \otimes y) = \pi^m x \otimes \varphi_K(y)$$

② $V \simeq V_{\frac{m}{n}}$

□

Thm

1) If $\mathcal{E} \in \mathrm{Vect}(X)$, then Harder-Narasimhan filtration splits,

i.e. $0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_m = \mathcal{E}$

w/ $\lambda_i = \text{slope } \mathcal{E}_i/\mathcal{E}_{i-1}$

then $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{O}(\lambda_i)^{n_i}$

and $\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{O}(\lambda_i)^{n_i}$.

2) For $\mu \in \mathbb{Q}$,

$$\left\{ \begin{array}{c} \text{isoclinic isocrystals of} \\ \text{slope } \mu \end{array} \right\}^{op} \longrightarrow \left\{ \begin{array}{c} \text{semistable vector bundles of} \\ \text{slope } \mu \end{array} \right\}$$

$$V \longmapsto \mathcal{E}_V$$

is an equivalence of categories.