Diamonds 02/26/2022

Diamonds.

Goal: to construct a functor

$$\{\text{analytic pre-adic spaces}/\mathrm{Spa}\,\mathbb{Z}_p\} \to \{\text{diamonds}\}$$
$$X \longmapsto X^{\diamond}$$

Generalizing Tilting.

For X perfectoid space. $X \mapsto X^{\flat}$

Claim: X analytic $/\mathrm{Spa}\,\mathbb{Z}_p$.

$\exists$ $f: \tilde{X} \to X$. $\tilde{X}$ perfectoid space. f <u>pro-étale</u>

$$X = \mathrm{Coeq}\big(\underbrace{\tilde{X} \times_X \tilde{X}}_{\text{perfectoid}} \rightrightarrows \tilde{X}\big). \qquad R = \tilde{X} \times_X \tilde{X}$$

$$X^{\diamond} \text{"="} \mathrm{Coeq}\big((\tilde{X} \times_X \tilde{X})^{\flat} \rightrightarrows \tilde{X}^{\flat}\big).$$

$$\text{"="} \quad \tilde{X}^{\flat}/\tilde{R}^{\flat} \quad \leftarrow \text{exists in cat of pro-étale sheaves.}$$

Analogy:

Perfectoid spaces $\overset{\text{in char } p}{\rightsquigarrow}$ schemes.

Diamonds $\rightsquigarrow$ Algebraic spaces.

$\cap$ $\qquad\qquad\qquad\qquad$ $\cap$

pro-étale sheaves $\qquad$ fpqc sheaves.

<u>Pro-étale Morphism</u>

<u>Def</u>. A morphism $f: \mathrm{Spa}(B,B^+) \to \mathrm{Spa}(A,A^+)$ of affinoid perfectoid spaces is affinoid pro-étale.

If $(B,B^+) = \varinjlim (A_i, A_i^+)$.

$\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ étale.

Example. (1). $S = \varprojlim S_i$. X perfectoid.

$$X \times S = \varprojlim X \times S_i \to X.$$

$X \times S_i$ is disjoint union of $|S_i|$ copies of X.

Rmk. In cat of affinoid perfectoid spaces. all connected limits exist. S_i

(2). $\underline{S}_K = \mathrm{Spa}(K, O_K) \times S$. $x \in S$.

pick $\{U_i\}$. $x \in U_i$.

$$\mathrm{Spa}(K, O_K) \times \{x\} = \varprojlim \mathrm{Spa}(K, O_K) \times U_i \to \underline{S}_K.$$

is pro-étale but not open.

Rmk. Pro-étale cannot be checked pro-étale locally on the target. Means.

$$\begin{tikzcd} X' = X \times Y' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "\text{pro-étale}"'] & Y \end{tikzcd}$$

$X' \to Y'$ pro-étale $\not\Rightarrow$ $X \to Y$ pro-étale.

Example. $p \neq 2$. $Y = \mathrm{Spa}\, K\langle T^{1/p^\infty}\rangle$ $\qquad \tilde{D}_K = \varprojlim_{T \to T^p} D_K$.

$X = \mathrm{Spa}\, K\langle T^{1/2p^\infty}\rangle$.

$X \to Y$ is ramified at 0. but finite étale away from 0.

ϖ. p.u

$$Y'_n = \underbrace{\{x \mid |x| \le |\varpi|^n\}}_{Y'_{n+1}} \amalg \coprod_{i=1}^{n} \{x \mid |\varpi|^i \le t \le |\varpi|^{i-1}\}.$$

$$X'_n = \{x \mid |x| \le |\varpi|^n\} \amalg (\qquad) \times_Y X.$$

$X_n' \to Y_n'$ finite étale.

Take $Y' = \varprojlim Y_n' \to Y$.

$X' \to Y'$ is pro-étale.
but $X \to Y$ is not.

<u>properties</u>. (1). Composition of pro-étale is pro-étale.

(2). Fibre product of pro-étale is pro-étale.

(3).
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$
$h = gf$
h, g is pro-étale $\Rightarrow$ f is pro-étale

pro-étale site.

Perfd: Category of perfectoid spaces.

Perf : category of char p. perfectoid spaces.

$X_{\text{proét}}$: $Y \to X$. pro-étale. Y perfectoid spaces.

pro-étale cover: $\{f_i : Y_i \to Y\}_{i \in I}$ (1) each f_i pro-étale.

(2). $U \subset Y$. quasi-compact. open.

$\exists$. finite $I_U \subset I$. $U_i \subset Y_i$ q.c. open.

s.t. $U = \bigcup f_i(U_i)$.

<u>Prop</u>. (1). $\mathcal{O}_X, \mathcal{O}_X^+$. are pro-étale sheaves.

$\underline{\mathcal{O}_X}$: $X \mapsto \mathcal{O}_X(X)$. $\underline{\mathcal{O}_X^+}$: $X \mapsto \mathcal{O}_X^+(X)$.

(2). Representable presheaves are sheaves.

X: perfectoid space

$h_X(Y) = \mathrm{Hom}(Y, X)$. is a sheaf.

Proof. (1). known: $\mathcal{O}_X$, $\mathcal{O}_X^+$ are sheaves in analytic topology.

Assume. $X = \mathrm{Spa}(R, R^+)$ affinoid perfectoid.

$Y \to X$. pro-étale. Y affinoid.

$Y = \mathrm{Spa}(R_\infty, R_\infty^+) = \varprojlim \mathrm{Spa}(R_i, R_i^+)$.

Need to show $0 \to R \to R_\infty \to R_\infty \underset{R^+}{\otimes} R_\infty \to \cdots$ exact.

step 1. $0 \to R^+/\varpi \to R_i^+/\varpi \to \cdots$ almost exact.

$H^i(X_{\text{ét}}, \mathcal{O}_X^+/\varpi)$. almost R^+/ϖ

step 2. $0 \to R^+/\varpi \to R_\infty^+/\varpi \to \cdots$ almost exact.

$0 \to R^+ \to R_\infty^+ \to \cdots$ almost exact.

invert ϖ. $0 \to R \to R_\infty \to \cdots$ exact.

$\mathcal{O}_X$ is a sheaf. $\mathcal{O}_X^+$. $\{|f| \le 1\}$.

(2). $Y = \mathrm{Spa}(S, S^+)$. $\{f_i : Y_i \to Y\}$ pro-étale cover.

$X = \mathrm{Spa}(R, R^+)$. $Y_i \to X$. system of compatible morphisms $\rightsquigarrow Y \to X$.

$$\Updownarrow$$

$$(R, R^+) \to (\underset{\sim}{\mathcal{O}(Y_i)}, \underset{\sim}{\mathcal{O}^+(Y_i)}).$$

$$\Downarrow$$

$$(R, R^+) \to (\mathcal{O}(Y), \mathcal{O}^+(Y)).$$

Notation. S profinite set. $\underline{S}$ sheaf. $\underline{S} = \varprojlim \underline{S_i}$ ← constant sheaves.

(1). $\underline{S_i}$ is not representable by a perfectoid space.

but $X \times \underline{S_i}$. is representable.

(2). $\underline{S}$. is not the constant sheaf.

$$\underline{S}(T) = C^0(|T|, S) = C^0(\pi_0 |T|, S).$$

(profinite ↓ points to $\pi_0|T|$)

Quasi-Pro-étale morphisms.

prop (*) Let $f: X \to Y$ affi. perfectoid spaces. TFAE.

(1). $\exists$ $Y' \to Y$ surjective pro-étale,

s.t. $X \times_Y Y' \to Y'$ is pro-étale.

(2). for any geometric point of rank 1. $\mathrm{Spa}\, C \to Y$.

$$\underset{\substack{\| \\ \mathrm{Spa}\, C \times \underline{S}}}{\mathrm{Spa}\, C \times_Y X} \to \mathrm{Spa}\, C \text{ is pro-étale.} \qquad S \text{ pro-finite set.}$$

Call f quasi-pro-étale if locally (on source & target) it satisfies above conditions.

Rmk: pro-étale $\Rightarrow$ quasi-pro étale.

Def. A perfectoid space X is (strictly) totally disconnected if it is quasi-compact & quasi separated and all open cover (étale) splits.

Rmk. (1). X is affinoid.

(2). Each connected component of X is of the form $\mathrm{Spa}(k, k^+)$.

strict $\rightsquigarrow$ k algebraically closed.

Prop. For X affinoid perfectoid.

there exists $f: \tilde{X} \to X$ pro-étale & $\tilde{X}$ is strictly totally disconnected.

Lemma. If $f: Y \to X$ satisfies (2) in prop. (*)

& X is strictly tota. discon. $\Rightarrow$ f is pro-étale.

proof. $f: Y \to \underbrace{X \times_{\pi_0(X)} \pi_0(Y)}_{} \to X$.

assume $\pi_0(Y) \to \pi_0(X)$ homeomorphism.

$\Rightarrow f$ is an injection. $|f|: |Y| \to |X|$.

image is pro-constructible. g. $\{|g| \le 1\}$.

Equivalently. $f: X \to Y$ is quasi-pro-étale if for all

Y' strictly totally disconnected and $Y' \to Y$ pro-étale,

$X \times_Y Y' \to X$ is pro-étale.

Def. $f: \mathcal{F} \to \mathcal{G}$ is a morphism between pro-étale sheaves (on Perf).

f is quasi-pro-étale if $\forall\, Y'$ s.t.d. & $Y' \to \mathcal{G}$ any map.

$$\begin{array}{ccc} \mathcal{F} \times_{\mathcal{G}} Y' & \to & \mathcal{F} \\ \downarrow & & \downarrow \\ Y' & \to & \mathcal{G} \end{array}$$

(1) $\mathcal{F} \times_{\mathcal{G}} Y'$ is representable by X' perfectoid sp.

(2). $X' \to Y'$ is pro-étale.

Diamond

Def. A diamond Y is a pro-étale sheaf on Perf.

s.t. $Y = X/R$. quotient of sheaves. X, R perfectoid spaces

& $R \subset X \times X$. is a pro-étale equivalence relation.

Def. R pro-étale equivalence relation on X if

$R \subset X \times X$.

s.t. $R \to X$, are pro-étale.

Prop. $X \in \mathrm{Perf}$. R pro-étale eq. relation

(1). $Y = X/R$. is a diamond. $\leftarrow$ Def.

(2). $R \to X \times_Y X$ of sheaves is an isomorphism.

(3). $\tilde{X} \to X$ pro-étale. $\tilde{R} = R \times_{X \times X} (\tilde{X} \times \tilde{X})$ induced eq. rel.

then $\tilde{Y} = \tilde{X}/\tilde{R} \to X/R = Y$. is an isomorphism.

(4). $X \to Y$ is quasi-pro-étale, surjective.

proof. (4). for X' s.t.d.

$$\begin{tikzcd}
\tilde{X}' \times_{X'} W \arrow[rr] \arrow[d, "\text{pro-ét.}"] \arrow[ddr] & & R \arrow[dd, "\text{pro-étale}"] \\
\tilde{X}' \arrow[drr] \arrow[ddr, "\text{pro-étale}"'] & & \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& X' \arrow[r] & Y
\end{tikzcd}$$

$$\begin{aligned}
&\tilde{X}' \times_{X'} W \\
&= \tilde{X}' \times_{X'} (X' \times_Y X). \\
&= \tilde{X}' \times_X (X \times_Y X) \\
&= \tilde{X}' \times_X R.
\end{aligned}$$

Prop. If $f: h_X \to \mathcal{F}$. is quasi-pro-étale & surjective.

then $\mathcal{F}$ is a diamond.

pf.

$$\begin{tikzcd}
X \times_{\mathcal{F}} X \arrow[r] \arrow[d] & X \arrow[d] \\
X \arrow[r] & \mathcal{F}
\end{tikzcd}$$

assume X s.t.d.

$\Rightarrow$ $\underset{\substack{\mathcal{F} \\ \shortparallel \\ R.}}{X \times X}$ is representable. pro-étale over X.

Prop. If Y', Y pro-étale sheaves.

$f: Y' \to Y$. surjective quasi pro-étale
Y' diamond $\Leftrightarrow$ Y is a diamond

Example. $\mathrm{Spd}\ \mathbb{Q}_p$

$$\mathbb{Q}_p \to \mathbb{Q}_p^{cycl} = \mathbb{Q}_p(\mu_{p^\infty}) = \varinjlim \mathbb{Q}_p(\mu_{p^n}).$$

$$\mathrm{Spa}\ \mathbb{Q}_p^{cycl} \to \mathrm{Spa}\ \mathbb{Q}_p. \quad \mathbb{Z}_p^\times \text{ torsor.}$$

$$\mathbb{Z}_p^\times \circlearrowright \mathrm{Spa}(\mathbb{Q}_p^{cycl})^\flat = \mathrm{Spa}\ \mathbb{F}_p((t^{1/p^\infty})).$$

$$\mathrm{Spd}\ \mathbb{Q}_p := \mathrm{Spa}(\mathbb{Q}_p^{cycl})^\flat / \mathbb{Z}_p^\times.$$

Need to check. $\mathbb{Z}_p^\times \times \mathrm{Spa}(\mathbb{Q}_p^{cycl})^\flat \to \mathrm{Spa}(\mathbb{Q}_p^{cycl})^\flat \times \mathrm{Spa}(\mathbb{Q}_p^{cycl})^\flat$

is an injection.

on geometric points $\mathbb{Z}_p^\times$ acts freely on $\mathrm{Hom}(\mathbb{F}_p((t^{1/p^\infty})), C)$.

C algebraically closed

$$f_1, f_2 : \mathbb{F}_p((t^{1/p^\infty})) \hookrightarrow C.$$

$\gamma \circ f_1 = f_2$. $\qquad \gamma : t \mapsto (1+t)^\gamma - 1$.

$\Rightarrow \gamma = 1$.

Thm. $\{$Perfectoid spaces over $\mathrm{Spa}\ \mathbb{Q}_p\} \simeq$

$\{$perfectoid spaces with char p. with morphism to $\mathrm{Spd}\ \mathbb{Q}_p\}$.

pf. Both sides are fibred over Perf.

L.H.S. $\quad X \mapsto X^\flat$.

R.H.S. $\quad$ sheaf. $\mathrm{Spd}\ \mathbb{Q}_p$.

LHS presheaf on Perf.

$$\mathrm{Untilt}_{\mathbb{Q}_p} : \quad X \mapsto \{X^\#,\ \iota : X^{\#\flat} \xrightarrow{\sim} X,\ X^\# \to \mathrm{Spa}\ \mathbb{Q}_p\}.$$

Lemma. $\mathrm{Untilt}_{\mathbb{Q}_p}$ is actually a sheaf.

Over. $X = \mathrm{Spa}(R, R^+)$. affinoid perfectoid.

$\{ X^\#, \iota: X^{\#\flat} \xrightarrow{\sim} X.\ X^\# \to \mathrm{Spa}\, \mathbb{Q}_p \}$

$\simeq \{ \tilde{R} \mid R \to \tilde{R},\ \mathbb{Z}_p^\times\text{-torsor},\ (\mathbb{Q}_p^{cycl})^\flat \to \tilde{R} \text{ equivariant}_{\text{under action}}\ \mathbb{Z}_p^\times \}$.

"$\to$": $X^\# = \mathrm{Spa}(R^\#, R^{\#+})$.

define $\tilde{R}^\# = R^\# \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p^{cycl} = \widehat{\varinjlim R^\# \underset{\mathbb{Q}_p}{\otimes} \mathbb{Q}_p(\mu_{p^n})}$

$\mathbb{Z}_p^\times$ torsor over $R^\#$.

tilt $\rightsquigarrow$ $\tilde{R}$ $\mathbb{Z}_p^\times$ torsor over R.

"$\leftarrow$"

$\tilde{R}/R$ $\mathbb{Z}_p^\times$ torsor. $\tilde{R}$, $(\mathbb{Q}_p^{cyc})^\flat$ alg.

$\{ \text{perfectoid } (\mathbb{Q}_p^{cycl})^\flat\text{-alg.} \}$

$\simeq \{ \text{perfectoid } (\mathbb{Q}_p^{cyc})\text{-alg} \}$.

get$_{\text{canonical}}$ $\tilde{R}^\# \hookleftarrow \mathbb{Q}_p^{cyc}$. algebra. $\mathbb{Z}_p^\times$-equivariant.

$R^\#$. Untilt

can extend the construction to any Tate $\underline{\mathbb{Z}_p}$-algebra R.

Lemma. R. Tate-$\mathbb{Z}_p$-algebra (p topologically nilpotent).

Let $\varinjlim R_i$ be filtered direct limit of R_i finite étale over R. s.t. admits no nonsplit finite étale cover.

Endow $\varinjlim R_i$ topology s.t. $\varinjlim R_i^\circ$ are open & bounded.

Define $\tilde{R} = \widehat{\varinjlim R_i}$. Then $\tilde{R}$ is perfectoid.

Rmk. can choose R_i G_i-torsor over R. $G = \varprojlim G_i$

proof. ϖ any p.u. since p topologically nilpotent.

$\exists N$. $\varpi \mid p^N$ in R°.

consider $x^{p^N} - \varpi x = \varpi$. $\rightsquigarrow$ finite étale.

this has a solution. $x = \varpi_1$.

then. $\varpi_1^{p^N} / \varpi$. $\varpi_1^{p^N} / p^N \Rightarrow \varpi_1^p / p$.

To check: $\Phi: R^\circ/\varpi_1 \to R^\circ/\varpi_1^p$ is an isomorphism.

surjectivity: $f \in R^\circ$.

choose $x^p - \varpi_1^p x = f$.

$\Rightarrow$ solution x.

Lemma. $\mathrm{Spa}(\tilde{R}, \tilde{R}^+) \to \mathrm{Spa}(R, R^+)$ is a $\underline{G}$ torsor.

Need to check $\underline{G} \times \mathrm{Spa}(\tilde{R}^\flat, \tilde{R}^{\flat +}) \to \mathrm{Spa}(\tilde{R}^\flat, \tilde{R}^{\flat +}) \times \mathrm{Spa}(\tilde{R}^\flat, \tilde{R}^{\flat +})$ is an injection.

(then $\mathrm{Spd}(R, R^+)$

$= \mathrm{Spa}(\tilde{R}^\flat, \tilde{R}^{\flat +}) / \underline{G}$.)

$\underline{G}$ acts freely on $\mathrm{Hom}(\tilde{R}^\flat, C)$. C algebraically closed.

$f: \tilde{R}^\flat \to C \Rightarrow f^\#: \tilde{R} \to C^\#$

$$\mathrm{Hom}(\tilde{R}^\flat, C) \overset{?}{=} \underset{\sim\sim\sim}{\mathrm{Hom}(\tilde{R}, C^\#)}.$$

For $f : \tilde{R}^\flat \to C.$

$$\tilde{R}^\circ = W(\tilde{R}^{\flat\circ})/I.$$

$$W(f^\circ)/I : \quad \tilde{R}^\circ \to \mathcal{O}_{C^\#}.$$

$$\begin{array}{ccc} R_i & \xrightarrow{f} & C^\# \\ {\scriptstyle \gamma}\uparrow & \nearrow{\scriptstyle f} & \\ R_i & & \end{array}$$

$\gamma f = f \Rightarrow \gamma = 1.$

Now we have.

$\mathrm{Spd}(R, R^+)$. diamond for each Tate $\mathbb{Z}_p$-algebra R.

$\Diamond$: {Analytic-Pre-adic space/Spa $\mathbb{Z}_p$} $\to$ {diamonds}.

$$X \longmapsto X^\Diamond.$$

$X^\Diamond$: a pre-sheaf. for each T in Perf.

$$X^\Diamond(T) = \{ T^\#,\ \iota : T^{\#\flat} \xrightarrow{\sim} T,\ \text{with}\ T^\# \to X \}.$$

Prop. $X^\Diamond$ is a diamond.

Def. For Y diamond. a presentation $Y = X/R$.

Define $|Y| = |X|/|R|$.

Prop. (1) $|Y| = \{ \mathrm{Spa}(k, k^+) \to Y \}/\sim$.

$\sim$:

$$\begin{array}{ccccc} & & \mathrm{Spa}(k_1, k_1^+) & & \\ & \nearrow & & \searrow & \\ \mathrm{spa}(k_3, k_3) & & \longrightarrow & & Y \\ & \searrow & & \nearrow & \\ & & \mathrm{Spa}(k_2, k_2^+) & & \end{array}$$

(2) $|Y|$ is indep of $|X|/|R|$.

<u>Prop.</u> $|X^{\diamond}| = |X|$.

For $X = \mathrm{Spa}(R, R^+)$.

$$|\mathrm{Spa}(\tilde{R}, \tilde{R}^+)| = |\mathrm{Spa}(\tilde{R}^b, \tilde{R}^{b+})|$$

$$|X^{\diamond}| = |\mathrm{Spa}(\tilde{R}^b, \tilde{R}^{b+})|/G = |X|.$$