

## LECTURE 3

In this lecture, we give a quick review of the theory of algebraic groups. This theory is analogous to that of complex Lie groups, but the techniques are more algebraic and some proofs are subtler<sup>1</sup>. Standard textbooks include: [B], [H] and [Sp]. See [M] for a modern treatment of this theory, which is also my favourite.

### 1. ALGEBRAIC GROUPS

**Definition 1.** An **algebraic group** over  $k$  is a finite type  $k$ -scheme  $G$  equipped with a group structure, i.e., a multiplication map  $m : G \times G \rightarrow G$  subject to axioms similar to those for an abstract group.

Homomorphisms between algebraic groups are defined in the obvious way. Let  $\mathbf{Grp}_k$  be the category of algebraic groups.

*Remark 2.* As in the study of abstract groups, the unit and the inversion maps are determined by the multiplication map. We denote them respectively by:

$$e : \text{pt} \rightarrow G, \sigma : G \rightarrow G,$$

where  $\text{pt} := \text{Spec}(k)$ .

**Construction 3.** Let  $G$  be an algebraic group. For any commutative  $k$ -algebra  $A$ , write

$$G(A) := \text{Hom}(\text{Spec}(A), G)$$

be the set of maps  $\text{Spec}(A) \rightarrow G$  between  $k$ -schemes. The group structure on  $G$  induces a group structure on  $G(A)$ .

Note that for  $A \rightarrow B$ , we have a homomorphism  $G(A) \rightarrow G(B)$ . Hence we obtain a functor

$$G(-) : \mathbf{CAlg}_k \rightarrow \mathbf{Grp}$$

from the category of commutative  $k$ -algebras to the category of (abstract) groups. By the Yoneda lemma, the algebraic group  $G$  is determined by this functor.

**Example 4.** The **additive group**  $\mathbb{G}_a$  is defined such that  $\mathbb{G}_a(A) = A$ , viewed as a commutative group under addition. The underlying  $k$ -scheme is the affine line  $\mathbb{A}^1$ .

**Example 5.** The **multiplicative group**  $\mathbb{G}_m = \mathbb{G}_m$  is defined such that  $\mathbb{G}_m(A) = A^\times$ , i.e. the subset of unit elements in  $A$ , viewed as a commutative group under multiplication. The underlying  $k$ -scheme is  $\mathbb{A}^1 \setminus 0$ , i.e., the affine line with the origin removed.

**Example 6.** One can define algebraic groups  $G := \text{GL}_n, \text{SL}_n, \text{SO}_n$ , etc. such that  $G(A)$  is the group of matrices of the corresponding type with coefficients in  $A$ .

**Example 7.** One can define the algebraic group  $\text{PGL}_n$  such that  $\mathcal{O}_{\text{PGL}_n}$  is a subring of  $\mathcal{O}_{\text{GL}_n}$ <sup>2</sup>.

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<sup>1</sup>Especially if one allows positive-characteristic or non-algebraic-closed base field  $k$ .

<sup>2</sup>Warning:  $\text{PGL}_n(A) \neq \text{GL}_n(A)/\text{GL}_1(A)$  for general  $A$ . In fact, viewed as functors in  $A$ , the LHS is the sheafification of the RHS in the fpqc topology.

*Remark 8.* Not every functor  $\mathbf{CAlg}_k \rightarrow \mathbf{Grp}$  comes from an algebraic group. For example, any  $k$ -vector space  $V$  defines a functor

$$\mathrm{GL}_V(-) : \mathbf{CAlg}_k \rightarrow \mathbf{Grp}$$

that sends  $A$  to the group of  $A$ -linear automorphisms of  $A \otimes V$ . This functor is not represented by an algebraic group unless  $V$  is finite-dimensional.

## 2. HOPF ALGEBRAS

From now on, we assume  $G$  is affine<sup>34</sup>.

**Construction 9.** An affine algebraic group  $G$  is determined by its ring of functions  $\mathcal{O}_G$ . The maps  $m : G \times G \rightarrow G$  and  $e : \mathrm{pt} \rightarrow G$  correspond to homomorphisms between algebras

$$\Delta : \mathcal{O}_G \rightarrow \mathcal{O}_G \otimes \mathcal{O}_G, \quad \epsilon : \mathcal{O}_G \rightarrow k,$$

which are called the **comultiplication** and **counit** maps of  $\mathcal{O}_G$ . Together with the usual multiplication and unit maps of  $\mathcal{O}_G$ , we obtain a **bialgebra**  $(\mathcal{O}_G, \cdot, \Delta)$ .

Note that this bialgebra is commutative but not cocommutative unless  $G$  is so.

The inverse map  $\sigma : G \rightarrow G$  corresponds to a homomorphism  $S : \mathcal{O}_G \rightarrow \mathcal{O}_G$ , which is called the **antipode** of  $\mathcal{O}_G$ . This makes  $\mathcal{O}_G$  into a commutative **Hopf algebra**.

*Remark 10.* A Hopf algebra is a bialgebra  $A$  equipped with an antipode map  $S : A \rightarrow A$  subject to a certain axiom. For our purposes, it is less useful to memorize this axiom than to imagine it amounts to say “ $\mathrm{Spec}(A)$ ”<sup>5</sup> has an inversion map.

**Example 11.** For  $G = \mathbb{G}_a$ , we have  $\mathcal{O}_G = k[t]$  and

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \quad \epsilon(f) = f(0), \quad S(f)(t) = f(-t).$$

**Example 12.** For  $G = \mathbb{G}_m$ , we have  $\mathcal{O}_G = k[t, t^{-1}]$  and

$$\Delta(t) = t \otimes t, \quad \epsilon(f) = f(1), \quad S(f)(t) = f(t^{-1}).$$

**Example 13.** The universal enveloping algebra  $U(\mathfrak{g})$  of any Lie algebra is a Hopf algebra. The comultiplication  $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$  is determined by  $\Delta(x) = x \otimes 1 + 1 \otimes x$ ,  $x \in \mathfrak{g} \subset U(\mathfrak{g})$  and its compatibility with the multiplication. Similarly, the antipode is determined by  $S(x) = -x$ ,  $x \in \mathfrak{g}$ .

The Hopf algebra  $U(\mathfrak{g})$  is cocommutative but not commutative unless  $\mathfrak{g}$  is abelian.

*Remark 14.* Using the Hopf algebra structure on  $U(\mathfrak{g})$ , the tensor product structure in  $\mathfrak{g}\text{-mod}$  can be defined as follows. Let  $V_1$  and  $V_2$  be left  $U(\mathfrak{g})$ -modules. Their tensor product  $V_1 \otimes V_2$  is naturally a  $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ -module. Restricting along  $\Delta$ , we can view  $V_1 \otimes V_2$  as a  $U(\mathfrak{g})$ -module.

*Remark 15.* There are also interesting Hopf algebras that are neither commutative nor cocommutative. For example, *quantum algebras* are such gadgets. See [L] for a standard textbook.

<sup>34</sup>Any affine algebraic group over field of characteristic 0 is smooth.

<sup>4</sup>Projective algebraic groups, a.k.a., abelian varieties, are also important and play a central role in modern mathematics.

<sup>5</sup>Note however that this does not make sense if  $A$  is not commutative.

## 3. TANGENT SPACES

As in the theory of Lie groups, the tangent space of an algebraic group at its unit is a Lie algebra. To describe this, let us review the definition of tangent spaces in algebraic geometry.

**Definition 16.** Let  $X$  be any  $k$ -scheme and  $x \in X$  be a  $k$ -point, i.e., a map  $x : \operatorname{Spec}(k) \rightarrow X$ . The **tangent space** of  $X$  at  $x$ , denoted by  $T_x X$ , is the *set* of dotted arrows making the following diagram commute:

$$\begin{array}{ccc} \operatorname{Spec}(k) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \\ \operatorname{Spec}(k[\epsilon]/\epsilon^2) & & \end{array}$$

Here the vertical map is given by the homomorphism  $k[\epsilon]/\epsilon^2 \rightarrow k$ ,  $\epsilon \mapsto 0$ .

Elements in  $T_x X$  are called **tangent vectors** of  $X$  at  $x$ .

Tangent vectors are related to *derivations*. Let us review its definition in the algebraic setting.

**Definition 17.** Let  $A$  be a  $k$ -algebra and  $M$  be an  $A$ -module. A  **$k$ -derivation** of  $A$  into  $M$  is a  $k$ -linear map  $D : A \rightarrow M$  satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let  $\operatorname{Der}_k(A, M)$  be the set of such  $k$ -derivations. This is naturally a  $k$ -vector space.

**Construction 18.** Let  $X = \operatorname{Spec}(A)$  be an affine scheme and  $x : \operatorname{Spec}(k) \rightarrow X$  be given by a homomorphism  $\phi : A \rightarrow k$ . View  $k$  as an  $A$ -module via this homomorphism and denote it by  $k_x$ . For any  $D \in \operatorname{Der}_k(A, k_x)$ , the map

$$A \rightarrow k[\epsilon]/\epsilon^2, f \mapsto \phi(f) + D(f)\epsilon$$

is a homomorphism and thereby gives a map  $\operatorname{Spec}(k[\epsilon]/\epsilon^2) \rightarrow X$ , which is an element in  $T_x X$ . It is easy to see this gives a bijection

$$\operatorname{Der}_k(A, k_x) \simeq T_x X.$$

In particular, we obtain a  $k$ -vector space structure on  $T_x X$ .

**Construction 19.** Let  $f : X \rightarrow Y$  be a morphism between  $k$ -schemes. Let  $x \in X$  and  $y := f(x) \in Y$  be  $k$ -points. There is an obvious  $k$ -linear map

$$df : T_x X \rightarrow T_y Y$$

given by composing with  $f$ . We call it the **differential** of  $f$ .

We have the following obvious result:

**Lemma 20.** Let  $X$  and  $Y$  be  $k$ -schemes. Let  $x \in X$  and  $y \in Y$  be  $k$ -points. For  $\partial_1 \in T_x X$  and  $\partial_2 \in T_y Y$ , write  $\partial_1 \oplus \partial_2 \in T_{(x,y)}(X \times Y)$  for the map

$$\operatorname{Spec}(k[\epsilon]/\epsilon^2) \xrightarrow{(\partial_1, \partial_2)} X \times Y.$$

Then the map

$$(3.1) \quad T_x X \times T_y Y \rightarrow T_{(x,y)}(X \times Y), (\partial_1, \partial_2) \mapsto \partial_1 \oplus \partial_2$$

induces an isomorphism  $T_x X \oplus T_y Y = T_{(x,y)}(X \times Y)$ .

## 4. LIE ALGEBRAS AND ALGEBRAIC GROUPS

**Notation 21.** Let  $G$  be an algebraic group. Define

$$\mathrm{Lie}(G) := T_e G.$$

**Example 22.** For  $G = \mathrm{GL}_n$ , we have  $\mathrm{Lie}(\mathrm{GL}_n) \simeq \mathfrak{gl}_n$ . In particular,  $\mathrm{Lie}(\mathrm{GL}_n)$  is naturally a Lie algebra. We have similar results for other classical subgroups of  $\mathrm{GL}_n$ .

We state the following result without proof:

**Theorem 23.** There is a canonical functor  $\mathrm{Grp}_k \rightarrow \mathrm{Lie}_k$  sending  $G$  to  $\mathrm{Lie}(G)$  equipped with a natural Lie bracket, such that for  $G = \mathrm{GL}_n$ , the Lie bracket on  $\mathrm{Lie}(\mathrm{GL}_n)$  is given by that on  $\mathfrak{gl}_n$ .

*Remark 24.* The above functor  $\mathrm{Grp}_k \rightarrow \mathrm{Lie}_k$  is unique if stated properly. See [M, Theorem 10.23].

**Warning 25.** It is not true that every Lie algebra can be obtained from algebraic groups. See [Bou, I, §5, Exercise 6] for a counterexample.

*Remark 26.* Consider the multiplication map  $m : G \times G \rightarrow G$  and its differential  $dm : \mathrm{Lie}(G \times G) \rightarrow \mathrm{Lie}(G)$ . One can show that the composition

$$\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) \simeq \mathrm{Lie}(G \times G) \xrightarrow{dm} \mathrm{Lie}(G)$$

sends  $(\partial_1, \partial_2)$  to  $\partial_1 + \partial_2$ . This is the algebraic analogue of the formula  $\exp(tu) \cdot \exp(tv) = \exp(t(u+v)) + O(t^2)$ ,  $u, v \in \mathrm{Lie}(G)$  that appears in the study of Lie groups.

*Remark 27.* Let  $G$  be an affine algebraic group and  $\mathfrak{g} := \mathrm{Lie}(G)$  be its Lie algebra. The Hopf algebras  $\mathcal{O}_G$  and  $U(\mathfrak{g})$  are related as follows.

Consider the dual  $U(\mathfrak{g})^* := \mathrm{Hom}_k(U(\mathfrak{g}), k)$  as a topological vector space<sup>6</sup>. The cocommutative Hopf algebra structure on  $U(\mathfrak{g})$  induces a commutative topological Hopf algebra structure on it<sup>7</sup>.

On the other hand, consider the maximal ideal  $\mathfrak{m} \subset \mathcal{O}_G$  corresponding to the unit point  $e : \mathrm{pt} \rightarrow G$ . Let  $\hat{\mathcal{O}}_G$  be the  $\mathfrak{m}$ -adic completion of  $\mathcal{O}_G$ . The commutative Hopf algebra structure on  $\mathcal{O}_G$  induces a commutative topological Hopf algebra structure on it.

We have

$$U(\mathfrak{g})^* \simeq \hat{\mathcal{O}}_G$$

as commutative topological Hopf algebras.

**Example 28.** Note that the Lie algebras of  $\mathbb{G}_a$  and  $\mathbb{G}_m$  are isomorphic, it follows that  $\hat{\mathcal{O}}_{\mathbb{G}_a} \simeq k[[t]]$  and  $\hat{\mathcal{O}}_{\mathbb{G}_m} = k[[t-1]]$  are isomorphic. Up to a scalar, this isomorphism is given by

$$k[[t-1]] \rightarrow k[[t]], f \mapsto f(\exp(t)).$$

Note that given a power series  $a_0 + a_1(t-1) + a_2(t-1)^2 + \dots$ , the series  $a_0 + a_1(\exp(t)-1) + a_2(\exp(t)-1)^2 + \dots$  indeed converges in the  $t$ -adic topology.

<sup>6</sup>For a vector space  $V$  equipped with the discrete topology, the dual  $V^*$  is equipped with the weakest topology such that for any finite-dimensional subspace  $V_0 \subset V$ , the map  $V^* \rightarrow V_0^*$  is continuous. Here  $V_0^*$  is equipped with the discrete topology. Equivalently, we can define  $V^*$  as an object in the pro-category  $\mathrm{Pro}(\mathrm{Vect}_{k,\mathrm{fd}})$  of finite dimensional vector spaces.

<sup>7</sup>Here we must use the *complete tensor product*  $U(\mathfrak{g})^* \hat{\otimes} U(\mathfrak{g})^*$  instead of the usual tensor product. By design, this is the dual of  $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ .

## 5. REPRESENTATIONS OF ALGEBRAIC GROUPS

In this section,  $G$  is an affine algebraic group.

**Definition 29.** A **representation** of  $G$ , or equivalently a  **$G$ -module** is a  $k$ -vector space  $V$  equipped with a natural transformation  $G(-) \rightarrow \mathrm{GL}_V(-)$  as functors  $\mathrm{CAlg}_k \rightarrow \mathrm{Grp}$ .

Let  $V$  and  $W$  be  $G$ -modules, a  **$G$ -linear map**  $\phi : V \rightarrow W$  is a  $k$ -linear map such that for any  $A \in \mathrm{CAlg}_k$ , the following diagram commutes

$$\begin{array}{ccc} G(A) & \longrightarrow & \mathrm{GL}_V(A) \\ \downarrow & & \downarrow \phi \circ - \\ \mathrm{GL}_W(A) & \xrightarrow{- \circ \phi} & \mathrm{Hom}_A(A \otimes V, A \otimes W). \end{array}$$

Let  $\mathrm{Rep}(G)$  be the category of  $G$ -modules.

**Example 30.** Any  $V$  can be equipped with a trivial  $G$ -module structure such that the homomorphisms  $G(A) \rightarrow \mathrm{GL}_V(A)$  are trivial.

**Proposition 31.** *The category  $\mathrm{Rep}(G)$  is an abelian category and the forgetful functor  $\mathrm{Rep}(G) \rightarrow \mathrm{Vect}_k$  is exact.*

*Remark 32.* If  $V$  is finite-dimensional, then  $\mathrm{GL}_V$  is represented by an algebraic group. A  $G$ -module structure on  $V$  is just a homomorphism  $G \rightarrow \mathrm{GL}_V$  between algebraic groups.

**Warning 33.** *Evaluate at  $k \in \mathrm{CAlg}_k$ , we obtain a group homomorphism  $G(k) \rightarrow \mathrm{GL}_V(k) = \mathrm{GL}(V)$ . But a  $G$ -module structure contains more information than such a homomorphism. This can be seen from the following exercise. (Note that for  $k = \mathbb{C}$ , the abstract groups  $\mathbb{G}_a(\mathbb{C}) = \mathbb{C}$  and  $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$  are isomorphic via the exponential map.)*

*Exercise 34.* This is **Homework 1, Problem 4**.

- (1) Find all maps between  $k$ -schemes  $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus 0$ .
- (2) Find all 1-dimensional representations of the additive group  $\mathbb{G}_a$ .
- (3) Find all maps between  $k$ -schemes  $\mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1 \setminus 0$ .
- (4) Find all 1-dimensional representations of the multiplicative group  $\mathbb{G}_m$ .

**Proposition 35.** *Any  $G$ -action on a vector space  $V$  is locally finite, i.e.,  $V$  is the union of its finite-dimensional subrepresentations.*

**Proposition 36.** *There is a canonical equivalence*

$$\mathrm{Rep}(G) \simeq \mathcal{O}_G\text{-comod}$$

*from the category of  $G$ -modules to the category of  $\mathcal{O}_G$ -comodules. This equivalence is compatible with the forgetful functors to  $\mathrm{Vect}_k$ .*

*Remark 37.* The functor  $\mathrm{Rep}(G) \simeq \mathcal{O}_G\text{-comod}$  is constructed as follows. Let  $V \in \mathrm{Rep}(G)$ . Consider  $A := \mathcal{O}_G$  and the homomorphism  $G(A) \rightarrow \mathrm{GL}_V(A)$ . The identity map  $G \rightarrow G$  can be written as  $\mathrm{Spec}(A) \rightarrow G$  which corresponds to an element in  $G(A)$ <sup>8</sup>. Consider the image of this element in  $\mathrm{GL}_V(A)$ , which is a  $A$ -linear map  $A \otimes V \rightarrow A \otimes V$ . This is the same as a  $k$ -linear map  $V \rightarrow A \otimes V$ , i.e., a  $k$ -linear map

$$V \rightarrow \mathcal{O}_G \otimes V.$$

One can verify this defines a  $\mathcal{O}_G$ -comodule structure on  $V$ .

<sup>8</sup>Warning: this is *not* the unit element of this group.

**Construction 38.** *There is a forgetful functor*

$$\mathrm{Rep}(G) \rightarrow \mathrm{Lie}(G)\text{-mod}$$

*that can be constructed in the following equivalent ways:*

- *Let us first suppose  $V \in \mathrm{Rep}(G)$  is finite-dimensional. Then we have a homomorphism between algebraic groups  $G \rightarrow \mathrm{GL}_V$  which induces a homomorphism between their Lie algebras  $\mathrm{Lie}(G) \rightarrow \mathrm{Lie}(\mathrm{GL}_V) = \mathfrak{gl}(V)$ , i.e., a  $\mathrm{Lie}(G)$ -module structure on  $V$ .  
When  $V$  is infinite-dimensional, the functor  $\mathrm{GL}_V(-)$  is no longer represented by an algebraic group, but there is a formal method to define its Lie algebra and make the above construction work.*
- *Any  $V \in \mathrm{Rep}(G)$  has an  $\mathcal{O}_G$ -comodule structure. By co-restricting along the map  $\mathcal{O}_G \rightarrow \hat{\mathcal{O}}_G$ , we obtain a (continuous) comodule structure for  $\hat{\mathcal{O}}_G \simeq U(\mathfrak{g})^*$ . Passing to duality, we obtain a module structure for  $U(\mathfrak{g})$ .*

**Lemma 39.** *The adjoint representation  $\mathrm{Lie}(G) \in \mathrm{Lie}(G)\text{-mod}$  has a canonical lift to an object in  $\mathrm{Rep}(G)$ . We call it the **adjoint action of  $G$  on  $\mathrm{Lie}(G)$** .*

*Remark 40.* On the level of  $k$ -points, the conjugate action of  $G(k)$  on  $G$  induces an (abstract) action of  $G(k)$  on  $\mathrm{Lie}(G)$ . This can be generalized to an action of  $G(A)$  on  $A \otimes \mathrm{Lie}(G)$  for any  $A \in \mathrm{CAlg}_k$  and thereby obtain the desired  $G$ -module structure on  $\mathrm{Lie}(G)$ .

**Proposition 41.** *If  $G$  is connected, then the functor  $\mathrm{Rep}(G) \rightarrow \mathrm{Lie}(G)\text{-mod}$  is fully faithful. In particular, the  $G$ -invariance and  $\mathrm{Lie}(G)$ -invariance for a  $G$ -module are the same.*

**Definition 42.** If  $G$  is connected, we say an object  $V \in \mathrm{Lie}(G)\text{-mod}$  is  **$G$ -integrable** if it is contained in the essential image of the above functor.

**Warning 43.** *The similar claim for derived categories is false. In other words, extensions of  $G$ -integrable modules are not necessarily  $G$ -integrable. The multiplicative group  $\mathbb{G}_m$  is a counterexample.*

## 6. SEMISIMPLE ALGEBRAIC GROUPS

**Theorem 44.** *Any semisimple Lie algebra  $\mathfrak{g}$  can be realized as the Lie algebra of an algebraic group. In the category of connected algebraic groups  $G$  with  $\mathrm{Lie}(G) \simeq \mathfrak{g}$ , there is a final object  $G_{\mathrm{ad}}$  and an initial object  $G_{\mathrm{sc}}$ . Moreover, the homomorphisms*

$$G_{\mathrm{sc}} \rightarrow G \rightarrow G_{\mathrm{ad}}$$

*are isogenies, i.e., are surjective and have finite kernels.*

**Example 45.** For  $\mathfrak{g} = \mathfrak{sl}_n$ , we have  $G_{\mathrm{sc}} = \mathrm{SL}_n$  and  $G_{\mathrm{ad}} = \mathrm{PGL}_n$ .

**Definition 46.** We say  $G$  is **semisimple**<sup>9</sup> if it is connected and its Lie algebra is semisimple. For a semisimple algebraic group  $G$ , we say it is **of adjoint type** (resp. **simply connected**) if it is of the form  $G_{\mathrm{ad}}$  (resp.  $G_{\mathrm{sc}}$ ).

**Theorem 47.** *If  $G_{\mathrm{sc}}$  is a simply-connected semisimple algebraic group, then any finite-dimensional  $\mathfrak{g}$ -module is  $G_{\mathrm{sc}}$ -integrable, i.e.,*

$$\mathrm{Rep}(G_{\mathrm{sc}})_{\mathrm{fd}} = \mathfrak{g}\text{-mod}_{\mathrm{fd}}.$$

**Warning 48.** *The similar claim for infinite-dimensional representation is false. This can be seen from the following exercise.*

<sup>9</sup>This is an ad hoc definition that only is only correct under our assumptions on  $k$ .

*Exercise 49.* This is **Homework 1, Problem 5**. Let  $G$  be any semisimple algebraic group with Lie algebra  $\mathfrak{g}$ . Prove: any Verma module of  $\mathfrak{g}$  is not  $G$ -integrable.

**Theorem 50.** *Let  $G$  be any semisimple algebraic group with Lie algebra  $\mathfrak{g}$ . Then the abelian categories  $\text{Rep}(G)$  and  $\mathfrak{g}\text{-mod}_{\text{fd}}$  are semisimple<sup>10</sup>. Simple objects in  $\text{Rep}(G)$  are finite-dimensional, and an object  $V \in \text{Rep}(G)$  is simple iff it is a simple object in  $\mathfrak{g}\text{-mod}$ .*

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<sup>10</sup>I.e., any object can be written as a direct sum of simple objects.