Given a variety $X$ over $\mathbb{C}$. $X(\mathbb{C})$ acquires a topology from that on $\mathbb{C}$. One can define $\beta^r(X) = \dim H^r(X(\mathbb{C}), \mathbb{Q})$. And :

Thm. (Lefschetz fixed point). Let $X$ be a compact triangulatebale space. For a continuous map $f: X \to X$. then define the Lefschetz number $\Lambda_f$ of $f$ to be $\sum_{k\geq 0} (-1)^k \operatorname{tr}(H_k(f, \mathbb{Q}))$. If $\Lambda_f \neq 0$, then $f$ has a fixed point.

(Lefschetz-Hopf theorem). If $f$ has only finite fixed points, then $\sum_{f(x)=x} \operatorname{ind}(f, x) = \Lambda_f$.

Thus number of fixed points can be described by cohomology.

However, the zariski topology is too coarse. Given an irreducible topological space $X$. then any constant presheaf $A$ is a sheaf. thus flasque. Hence $H^n(X, A) = 0$ for all $n \geq 1$.

However. we know smooth projective $\mathbb{C}$-algebraic variety coorresponds to some complex manifold. with that we have a better cohomology.

Though later. we would know that there does not exists a cohomology theory on schemes compatible with the ones induces by complex topology. Still, we have $H^i_{et}(X, A) \cong H^i(X(\mathbb{C}), A)$ given that $A$ is a finite abelian group. And every étale coverings can be refined by a covering of the complex topology $\Rightarrow$ $\forall$ Abelian group $\Lambda$. there are canonical maps from $H^i_{et}(X, \Lambda)$ to $H^i(X(\mathbb{C}), \Lambda)$.

By taking $\Lambda = \mathbb{Z}/l^n\mathbb{Z}$. and passing to the inverse limit over $n$, we have an isomorphism $H^r(X_{et}, \mathbb{Z}_l) \to H^r(X(\mathbb{C}), \mathbb{Z}_l)$.

When tensored with $\mathbb{Q}_l$. this becomes an iso. $H^r(X_{et}, \mathbb{Q}_l) \to H^r(X(\mathbb{C}), \mathbb{Q}_l)$.

There's an example where étale cohomology and the usual cohomology differs.

Take $X$ to be a projective nonsingular curve of genus $g$, then

$H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{2g}$, while $H^1(X_{et}, \mathbb{Z}) = 0$.

They have similar properties. $H^1(X(\mathbb{C}), \Lambda) = \mathrm{Hom}(\pi_1(X, x), \Lambda)$

$H^1(X_{et}, \Lambda) = (\pi_1^{alg}(X, x), \Lambda)$.

A review of concepts:

For $f: X \to Y$.

$f$ is quasi-compact if $\forall$ affine open subset $U$ of $Y$, $f^{-1}(U)$ is quasi-compact.

$f$ is locally of finite type if for each affine open subset $V = \operatorname{Spec} B$ of $Y$, and any point $x$ in $f^{-1}(Y)$, $\exists\, x \in U = \operatorname{Spec} A \subseteq f^{-1}(V)$. s.t. $A$ is a finitely generated algebra.

$f$ finite type if quasi-compact + locally of finite type

$f$ is affine if $\forall$ affine open subset $\overset{= \operatorname{Spec} A}{V}$ of $Y$. $f^{-1}(V)$ is affine,

and finite if $f^{-1}(V) = \operatorname{Spec} B$ is a finitely generated $A$-mod.

Closed immersion: $\overset{Y \to X}{f}$ is a homeomorphism of $sp(Y)$ onto a closed subset of $sp(X)$, and $f^{\#}: \mathcal{O}_X \to f_* \mathcal{O}_Y$ is surjective.

(locally is $\operatorname{spec} A/\mathfrak{a} \to \operatorname{spec} A$).

Convention: All rings are noetherian! All schemes are locally noetherian.

Central concept: flat morphism: analogue in AG of a map whose fibers form a continuous varing family.

Eg. A surjective morphism of smooth varieties is flat iff all fiber have the same dimension.

Recall $f: X \to Y$, the fiber of $y \in Y$ is defined as $X_y = X \times_Y \operatorname{Spec} k(y)$, which is homeomorphic to $f^{-1}(y)$.

finite type and finite fibers

And an étale morphism is a flat $\underset{\uparrow}{\text{quasi-finite}}$ morphism $Y \to X$ with no ramification (that is branch) points. Locally $Y$ looks like an integral extension. (flat and unramified).

An étale morphism induces an isomorphism on the ~~tagent~~ tangent space, and induces an isomorphism on the completions of the local rings at a point.

Let $f: Y \to X$ be a morphism locally of finite type,

we say $f$ is unramified if $\mathcal{O}_{Y,y}/m_x\mathcal{O}_{Y,y}$ over $k(x)$ is finite separable, for all $x \in X$. equivalently $Y_x$ is a

sum $\coprod \operatorname{Spec} k_i$, where $k_i$ are finite separable field extensions

of $k(x)$.

For example, consider a map $f$ from smooth curve $Y$ to smooth curve $X$, then

each local ring is a discrete valuation ring, (except the generic point)

$X = \operatorname{Spec} A$. $Y = \operatorname{Spec} B$. Let $p \in A$. $p = q_1^{e_1} \cdots q_n^{e_n}$, then

$f$ maps $q_i$ to $p$. ($q_i$ distinct)

Unramified at the generic point $\Rightarrow$ $\operatorname{Frac} B / \operatorname{Frac} A$ is a

finite separable extension.

Unramified at $q_i \Rightarrow$ $B_{q_i} / pB_{q_i}$ over $A/p$ is finite

separable, this implies $pB_{q_i} = B_{q_i} \Rightarrow e_i = 1$. and

$(B/q_i) \,/\, (A/p)$ is finite separable

This coincides with unramified extensions in ANT.

Normalization:

An integral scheme $X$ is normal if all its local rings are integrally closed. The normalization of $X$ in field $L \supseteq R(X)$ can be defined as follow: For each $U = \operatorname{Spec} A$, let $\hat{A}$ be the integral closure of $A$ in $L$, one can glue all these $\operatorname{Spec} \hat{A}$ to a scheme, which is universal among all these scheme $Y$ with $R(Y) \subseteq L$, and an affine morphism $g: Y \to X$, s.t. $\Gamma(g^{-1}(U), \mathcal{O}_Y)$ is integral over $\Gamma(U, \mathcal{O}_X)$ for $U$ affine.

Prop. When $X$ is normal and $f: X' \to X$ is the normalization of $X$ in some finite separable extension of $R(X)$, then $f$ is finite.

Proof: Pure algebraic number theory result:

If $L/K$ is separable, $A$ integrally closed in $K$, then the integral closure $B$ of $A$ is contained in a free $A$-module

Prop. 1.3. (a). A closed immersion is finite.

(b). The composite of two finite morphisms is finite.

(c). Any base change of a finite morphism is finite.

Prop 1.4. Any finite morphism $f: Y \to X$ is proper, i.e. separated, finite-type, and universaly closed.

Proof. Use separaness is locally on target.

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A morphism $X \to \operatorname{Spec} k$, with $k$ a field. there is a topological characterization of finiteness:

Prop 1.5: $f: X \to \operatorname{Spec} k$, finite-type, $k$ a field, then TFAE

(a) $X$ is affine and $\Gamma(X, \mathcal{O}_X)$ is an artin ring.

(b). $X$ is finite and discrete.

(c). $X$ is discrete.

(d). $f$ is finite.

Note that a ring $R$ is artinian, iff $\dim R = 0$ and $R$ has finite maximal ideal, also $\Leftrightarrow$ $\dim R = 0$ and $R$ Noetherian.

And $f$ finite $\Rightarrow$ $X$ affine, $\Gamma(X, \mathcal{O}_X)$ is finite over $k$.

$\Rightarrow$ $\dim X = 0$.

Defn. $f: Y \to X$ is quasi-finite if it is of finite type and has finite fibers, i.e. $f^{-1}(x)$ is discrete for all $x \in X$.

Then similarly an $A$-algebra $B$ is quasi-finite if it is of finite type and $B \otimes_A \mathrm{Frac}(A/p)$ is a finite $k(p)$-module, $\forall$ prime $p \subset A$.

Eg. Let $A$ be a discrete valuation ring with uniformizer $\pi$, then $A[T]/(p(T))$ is quasi-finite over $A$ $\Leftrightarrow$ $(\mathrm{Frac}\, A)[T]/(p(T))$ is finite over $\mathrm{Frac}\, A$ (which always holds), and $A/(\pi)[T]/(p(T))$ is finite $\Rightarrow$ some coefficient of $p$

is a unit, $\Leftrightarrow$ $P$ does not vanishes in $A/(\pi)$.

Prop 1.7. (a). Any immersion is quasi-finite.

(b). Composition of two quasi-finite morphisms is quasi-finite.

(c). Any base change of a quasi-finite morphism is quasi-finite.

Proof: (c). $f: Y \to X$ quasi-finite. $X' \to X$, $x' \mapsto x$. then

$$= f^{-1}(x') \otimes k(x') = f^{-1}(x) \otimes k(x) \otimes k(x') = f^{-1}(x) \otimes_{k(x)} k(x').$$

$$f'^{-1}_{(x')}(x') = f^{-1}(x) \otimes_{k(x)} k(x')$$

is finite over $k(x')$.