LECTURE 2

Last time, we explained the following principle:

Slogan 0.1.

$$\textit{theory of } \infty\textit{-groupoids} = \textit{homotopy theory of topological spaces.}$$

$$\textit{theory of } (\infty,1)\textit{-categories} = \textit{homotopy theory of topological categories.}$$

In this lecture, we give precise meanings to the RHS's.

1. What is a homotopy theory?

1.1. Recall the homotopy theory of topological spaces is encoded by the *homotopy category* hTop, which can be defined in the following two equivalent ways:

- As the category obtained from Top, the category of *all* spaces, by inverting *weak* homotopy equivalence.
- As the category of nice topological spaces and homotopy classes of continuous maps between such spaces.

1.2. This suggests a homotopy theory should be understand as follows:

- In a given category $\mathcal{C}$, such as Top, we single out a collection W of morphisms, called **weak equivalences**, and study the category $\mathsf{h}\mathcal{C}$ obtained from $\mathcal{C}$ by inverting the morphisms in W.
- We find a full subcategory $\mathcal{C}^\circ \subset \mathcal{C}$ of *nice* objects in $\mathcal{C}$, such that for X, $Y \in \mathcal{C}^\circ$, the set $\mathsf{Hom}_{\mathsf{h}\mathcal{C}}(X,Y)$ can be calculated as the quotient of $\mathsf{Hom}_{\mathcal{C}}(X,Y)/\sim$. Here '$\sim$' stands for the equivalence relation defined by

$$\mathsf{Hom}_{\mathcal{C}}(CX,Y) \rightrightarrows \mathsf{Hom}_{\mathcal{C}}(X,Y), \tag{1.1}$$

  where CX is a **cylinder** of X, behaving like $X \times [0,1]$ when $X \in \mathsf{Top}$.

1.3. However, the passage from Top to hTop, or more generally from $\mathcal{C}$ to $\mathsf{h}\mathcal{C}$, loses a significant amount of *homotopy-invariant* information. For example, the notions of homotopy limits/colimits cannot be extracted from the category hTop.

Definition 1.4. *The* ***homotopy pushout*** *of a diagram $X \leftarrow Y \to Z$ in* Top *is defined to be*

$$X \overset{\mathsf{h}}{\underset{Y}{\sqcup}} Z := X \underset{Y\times\{0\}}{\sqcup} (Y \times [0,1]) \underset{Y\times\{1\}}{\sqcup} Z.$$

The ***homotopy pullback*** *of a diagram $X \to Y \leftarrow Z$ in* Top *is defined to be*

$$X \overset{\mathsf{h}}{\underset{Y}{\times}} Z := X \underset{Y^{\{0\}}}{\times} (Y^{[0,1]}) \underset{Y^{\{1\}}}{\times} Z.$$

Exercise 1.5. *Check the above definitions are homotopy invariant, i.e., the homotopy types of the results only depend on the images of the diagrams under* $\mathsf{h} : \mathsf{Top} \to \mathsf{hTop}$.

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Exercise 1.6. *Consider the following diagrams in* hTop*. Prove:*

(1) *The pushout of* $* \leftarrow S^1 \to D^2$ *in* hTop *is equivalent to* $*$*, while the homotopy pushout is* S^2*.*
(2) *The pushout of* $* \leftarrow S^1 \xrightarrow{2} S^1$ *in* hTop *does not exist, while the homotopy pushout is* $\mathbb{RP}^2$*.*
(3) *The pullback of* $* \to S^1 \leftarrow \mathbb{R}^1$ *in* hTop *is equivalent to* $\mathbb{R}^1$*, while the homotopy pullback is* $\mathbb{Z}$*.*
(4) *The pullback of* $* \to \mathbb{CP}^\infty \leftarrow \mathbb{RP}^\infty$ *in* hTop *does not exist, while the homotopy pullback is* S^1*.*

1.7. Note that in (1)/(3), the homotopy pushout/pullback can also be calculated as the homotopy types of the *usual* pushout/pullback inside Top. This is related to the facts that $S^1 \to D^2$ is a nice inclusion while $\mathbb{R}^1 \to S^1$ is a nice surjection.

2. Model categories

In the 1960s, Quillen (see [Qui67]) realized that classical homotopy theory can be carried out in any category equipped with three classes of morphisms: weak equivalences, cofibrations and fibrations, as long as they satisfy a list of axioms motivated by the example of Top. This motivated the definition of model categories.

Definition 2.1. *A* ***weak factorization system*** *on a category* $\mathcal{C}$ *is a pair* (L, R) *of classes of morphisms such that*

- *Every morphism can factor as* $p \circ i$ *such that* $i \in L$ *and* $p \in R$*.*
- L *is precisely the class of morphisms having the* ***left lifting property*** *against every morphism in* R*. In other words,* $i \in L$ *iff for any* $p \in R$ *and a commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y, \end{tikzcd}$$

  there exists a morphism $B \to X$ *making the two triangles commute.*
- R *is precisely the class of morphisms having the* ***right lifting property*** *against every morphism in* L*.*

Example 2.2. *The category* Set *has a weak factorization system* $(\mathsf{inj}, \mathsf{surj})$ *given by classes of injections and surjections.*

2.3. The following is a typical way to construct weak factorization systems. Let $\mathcal{C}$ be a category admits all small colimits, and I be a *set* of morphisms such that the sources of these morphisms are *small* in a suitable sense. Then there is a weak factorization system (L, R) such that

- R is the class of morphisms with right lifting property with respect to I;
- L is the *weakly saturated class*[1] generated by I.

This is known as Quillen's *small object argument.* See HTT.A.1.2.

Definition 2.4. *A* ***model structure*** *on a category* $\mathcal{C}$ *is a choice of three classes of morphisms* W*,* C *and* F*, respectively called* ***weak equivalences***, ***cofibrations*** *and* ***fibrations****, such that*

[1]Being weakly saturated means closed under pushouts, transfinite compositions and retracts. See HTT.A.1.2.2.

- *W contains all isomorphisms and is closed under 2-out-of-3*[2].
- *$(C, F\cap W)$ and $(C\cap W, F)$ are weak factorization systems on $\mathcal{C}$.*

We say $\mathcal{C}$ is a model category if it is locally small[3] *and has small limits and colimits*[4], *and is equipped with a model structure.*

Definition 2.5. *In a model category $\mathcal{C}$, morphisms in $C\cap W$ (resp. $F\cap W$) are called* ***acyclic cofibrations*** *(resp.* ***acyclic fibrations****).*

Example 2.6. *Quillen's classical model structure on* Top *is given by:*

(W) *A weak equivalence is a weak homotopy equivalence.*
(C) *A cofibration is a* retract of a relative cell complex.
(F) *A fibration is a* Serre fibration.

Here the weak factorization system $(C, F\cap W)$ is obtained by applying the small object argument to $\{S^{n-1}\to D^n\}$, while $(C\cap W, F)$ is obtained by applying to $\{D^n\times\{0\}\to D^n\times[0,1]\}$.

Example 2.7. *Let $\mathcal{A}$ be a nice*[5] *abelian category, such as Mod_R for a ring R. Let $\mathsf{Ch}^{\le 0}(\mathcal{A})$ be the category of (cochain) complexes in non-positive degrees*[6]. *Quillen's* ***projective model structure*** *on $\mathsf{Ch}^{\le 0}(\mathcal{A})$ is given by*

(W) *A weak equivalence is a a quasi-isomorphism, i.e., a map that induces isomorphisms between cohomologies.*
(C) *A cofibration is a degreewise monomorphism with degreewise* projective *cokernel.*
(F) *A fibration is a degreewise epimorphism.*

Dually, there is an ***injective model structure*** *on $\mathsf{Ch}^{\ge 0}(\mathcal{A})$.*

Example 2.8. *Let k be a field of characteristic zero. Let $\mathsf{dgcAlg}_k^{\le 0}$ be the category of differential graded-commutative algebras over k in non-positive degrees. The* ***projective model structure*** *on $\mathsf{dgcAlg}_k^{\le 0}$ is given by*

(W) *A weak equivalence is a a quasi-isomorphism.*
(F) *A fibration is a degreewise epimorphism.*
(C) *The class of cofibrations is determined by $W\cap F$.*

2.9. In the above two examples, the story becomes more complicated but interesting when we consider all *unbounded* complexes.

Definition 2.10. *Let $\mathcal{C}$ be a model category, and X, Y be objects in $\mathcal{C}$.*

(1) *A* ***cylinder object*** *for X is an object CX together with a factorization of the co-diagonal map $X\sqcup X\to X$ as*
$$X\sqcup X\xrightarrow{i} CX\xrightarrow{p} X$$
such that i is a cofibration and p is a weak equivalence.

[2]For composable morphisms f, g, if two out of the three morphisms f, g, gf are in W, so is the third.
[3]This means the $\mathsf{Hom}_{\mathcal{C}}(-,-)$ are sets rather than proper classes.
[4]Quillen's original definition only requries *finite* limits/colimits.
[5]More precisely, $\mathcal{A}$ admits small limits and colimits, and has enough projectives.
[6]We use cohomological convention.

(2) *A **path object** for Y is an object PY together with a factorization of the diagonal map $Y \to Y \times Y$ as*

$$Y \xrightarrow{i} PY \xrightarrow{p} Y \times Y$$

such that i is a weak equivalence and p is a fibration.

2.11. Note that the existence of cylinder objects and path objects follows from the factorization axiom in model structures.

Exercise 2.12. *Consider the model categories in Example 2.7.*

- *Find path objects in the projective model category $\mathsf{Ch}^{\leq 0}(\mathcal{A})$.*
- *Find cylinder objects in the injective model category $\mathsf{Ch}^{\geq 0}(\mathcal{A})$.*

Definition 2.13. *Let $\mathcal{C}$ be a model category. An object X is **fibrant** (resp. **cofibrant**) if $X \to *$ (resp. $\varnothing \to X$) is a fibration (resp. cofibration). Here $*$ and $\varnothing$ stand for the final and initial objects in $\mathcal{C}$.*

*An object is **bifibrant** if it is both fibrant and cofibrant. Let*

$$\begin{tikzcd} & \mathcal{C}^{\mathsf{c}} \arrow[dr] & \\ \mathcal{C}^{\circ} \arrow[ur] \arrow[dr] & & \mathcal{C} \\ & \mathcal{C}^{\mathsf{f}} \arrow[ur] & \end{tikzcd}$$

be the full subcategories of fibrant, cofibrant and bifibrant objects.

2.14. As we will soon see, bifibrant objects are the "nice" objects in the sense of Sect. 1.2.

Exercise 2.15. *For any object x, there exist weak equivalences $x' \to x$ and $x \to x''$ such that x' is cofibrant and x'' if fibrant. Such object x' (resp. x'') is called a **cofibrant replacement** (resp. **fibrant replacement**) of x*

Exercise 2.16. *What are cofibrant replacements in the projective model category $\mathsf{Ch}^{\leq 0}(\mathcal{A})$? How about fibrant replacements in the injective model category $\mathsf{Ch}^{\geq 0}(\mathcal{A})$?*

2.17. In order to get a feeling about the axioms in the definition of model categories, the readers are encouraged to prove the following result on their own.

Proposition-Definition 2.18. *Let $\mathcal{C}$ be a model category, X be a cofibrant object and Y be a fibrant object. For morphisms $f,\ g : X \to Y$, the following conditions are equivalent:*

(1) *For every/some cylinder object CX for X, there exists a commutative diagram*

$$\begin{tikzcd} X \sqcup X \arrow[rr, "{(f,g)}"] \arrow[dr] & & Y \\ & CX \arrow[ur, dotted] & \end{tikzcd}$$

(2) *For every/some path object PY for Y, there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "{(f,g)}"] \arrow[dr, dotted] & & Y \times Y \\ & PY \arrow[ur] & \end{tikzcd}$$

*We say f and g are **homotopic**, or $f \sim g$, if they satisfy the above equivalent conditions. The relation $\sim$ is an equivalence relation on the set $\mathsf{Hom}_{\mathcal{C}}(X,Y)$.*

Definition 2.19. *Let $\mathcal{C}$ be a model category. The **homotopy category** $\mathsf{h}\mathcal{C}$ of $\mathcal{C}$ is defined as follows:*

- *Objects are the bifibrant objects of $\mathcal{C}$;*
- *Morphisms are homotopy classes of morphisms in $\mathcal{C}$.*

2.20. Quillen proved the homotopy category is canonically equivalent to the localization of $\mathcal{C}$ by W, i.e., the category $\mathcal{C}[W^{-1}]$ obtained from $\mathcal{C}$ by inverting weak equivalences[7]. More precisely:

Theorem 2.21 (Quillen)**.** *The following functors are equivalences*

$$\begin{tikzcd} & \mathcal{C}^{\mathsf{f}}[W^{-1}] \arrow[dl] & & \\ \mathcal{C}[W^{-1}] & & \mathcal{C}^{\circ}[W^{-1}] \arrow[ul] \arrow[dl] \arrow[r] & \mathsf{h}\mathcal{C} \\ & \mathcal{C}^{\mathsf{c}}[W^{-1}] \arrow[ul] & & \end{tikzcd}$$

Exercise 2.22. *What does the theorem say when $\mathcal{C}$ is the projective model category $\mathsf{Ch}^{\leq 0}(\mathcal{A})$ or the injective model category $\mathsf{Ch}^{\geq 0}(\mathcal{A})$?*

2.23. Note that $\mathcal{C}[W^{-1}]$ depends only on $(\mathcal{C}, W)$, while $\mathsf{h}\mathcal{C}$ depends on the model structure of $\mathcal{C}$, i.e., on the choices of fibrations and cofibrations. One may compare these auxillary choices with local coordinates on a manifold: both provide powerful tools to do calculations, but more or less obstruct births of an intrinsic theory.

As will be explained in future lectures, this "intrinsic" theory of model categories is exactly the theory of $(\infty, 1)$-categories.

3. Topological categories

3.1. Now let us try to develop the homotopy theory of topological categories via the formalism of model categories.

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[7]This is the category characterized by the following universal property: knowing a functor $\mathcal{C}[W^{-1}] \to \mathcal{D}$ is equivalent to knowing a functor $\mathcal{C} \to \mathcal{D}$ that sends morphisms in W to isomorphisms.

3.2. There is only one reasonable choice for weak equivalences, defined as follows.

Definition 3.3. *Let $\mathcal{C}$ be a topological category. Its* ***homotopy category*** $\pi_0\mathcal{C}$ *is defined by*

$$\mathsf{Ob}(\pi_0\mathcal{C}) := \mathsf{Ob}(\mathcal{C}),\ \mathsf{Hom}_{\pi_0\mathcal{C}}(x,y) := \pi_0\mathsf{Hom}_{\mathcal{C}}(x,y).$$

Definition 3.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between topological categories. We say F is a* ***weak equivalence*** *if:*

- *It induces an equivalence $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$.*
- *It induces weak equivalences between the* Hom*-spaces, i.e. the continuous map* $\mathsf{Hom}_{\mathcal{C}}(x,y) \to \mathsf{Hom}_{\mathcal{D}}(Fx,Fy)$ *is a weak equivalence for any x, $y \in$* $\mathsf{Ob}(\mathcal{C})$.

3.5. However, it is not an easy task to define fibrations and cofibrations between topological categories. Even if one finds such a definition, the obtained model structure will not be practical for calculation. Namely, ulike the case of topological spaces, many seemingly innocent topological spaces can fail to bifibrant.

Exercise 3.6. *Consider the following commutative diagram of spaces:*

$$\begin{tikzcd} S^1 \times S^1 \arrow[rr, "h'"] \arrow[dr, "h"'] & & S^1 \\ & S^1 \times D^2, \arrow[ur, "f"'] & \end{tikzcd}$$

where $g(t_1,t_2) := (t_1t_2,t_2)$ and $h(t_1,t_2) := t_1$. Let $\mathcal{C}$ be the topological category with three objects x, y, z and

$$\mathsf{Hom}_{\mathcal{C}}(x,y) := S^1,\ \mathsf{Hom}_{\mathcal{C}}(y,z) := S^1,\ \mathsf{Hom}_{\mathcal{C}}(x,z) := S^1 \times D^2$$

such that the composition map

$$\mathsf{Hom}_{\mathcal{C}}(x,y) \times \mathsf{Hom}_{\mathcal{C}}(y,z) \to \mathsf{Hom}_{\mathcal{C}}(x,z)$$

is given by h. Similarly, let $\mathcal{D}$ be the topological category with three objects x', y', z' and

$$\mathsf{Hom}_{\mathcal{D}}(x',y') := S^1,\ \mathsf{Hom}_{\mathcal{C}}(y',z') := S^1,\ \mathsf{Hom}_{\mathcal{C}}(x',z') := S^1$$

such that the composition law is given by h'. Let

$$F : \mathcal{C} \to \mathcal{D},\ x \mapsto x',\ y \mapsto y',\ z \mapsto z'$$

be the obvious functor induced by f. Prove:

1. *The functor H is a weak equivalence.*
2. *There exists no functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ is homotopic to* $\mathsf{id}_{\mathcal{D}}$ *in any reasonable sense.*
3. *Deduce that $\mathcal{C}$ is not cofibrant or $\mathcal{D}$ is not fibrant (for any model structure on* $\mathsf{Cat}_{\mathsf{Top}}$ *such that weak equivalences are given by Definition 3.4).*

3.7. Another difficulty in developing the theory of $(\infty,1)$-categories via topological categories is to define the $(\infty,1)$-category Cat_∞ of small $(\infty,1)$-categories. Namely, if $(\infty,1)$-categories are understood *only* via topological categories, we need to define a topological category $\mathsf{Cat}^{+}_{\mathsf{Top}}$ of small topological categories such that

$$\pi_0\mathsf{Hom}_{\mathsf{Cat}^{+}_{\mathsf{Top}}}(\mathcal{C},\mathcal{D}) \simeq \mathsf{Hom}_{\mathsf{hCat}_{\mathsf{Top}}}(\mathcal{C},\mathcal{D}).$$

The following exercises suggest this is not an easy task.

Exercise 3.8. *For any abstract group G, let $\mathbb{B}G$ be the topological category with a single object $*$ such that $\mathsf{Hom}_{\mathbb{B}G}(*,*) := G$. Use homotopy hypothesis to show*

$$\mathsf{Hom}_{\mathsf{Cat}^+_{\mathsf{Top}}}(\mathbb{B}*, \mathbb{B}G)$$

is weakly homotopic to the Eilenburg–MacLane space $K(G,1)$.

Challenge 3.9. *Define a functor $\mathsf{Grp} \to \mathsf{Top}_*$ such that $G \in \mathsf{Grp}$ is sent to a representative of $K(G,1)$.*

3.10. In future lectures, we will study and compare two more models of $(\infty,1)$-categories:

(1) Joyal's model structure on the category Set_Δ of **simplicial sets**, whose bifibrant objects are **quasi-categories**.
(2) Bergner's model structure on the category Cat_Δ of **simplicial categories**, whose bifibrant objects cannot be easily described.

We will show that the *category of small quasi-categories* $\mathsf{QCat} \subset \mathsf{Set}_\Delta$ has a natural simplicial enrichment, i.e, we have a *simplicial category of small quasi-categories*, which serves as a model of Cat_∞, the $(\infty,1)$-category of $(\infty,1)$-categories.

3.11. In fact, QCat is *Cartesian closed* and therefore naturally enriched by itself. This reflects the idea that functors between $(\infty,1)$-categories form an $(\infty,1)$-category.

Appendix A. More homotopy (co)limits

Exercise A.1. *Let $\mathcal{C}$ be a model category. Define the notion of homotopy pushouts in $\mathcal{C}$. Prove that the homotopy pushout of $X \leftarrow Y \to Z$ can be calculated as pushout in $\mathcal{C}$ in either of the following cases:*

- *The morphism $Y \to Z$ is a cofibration, and X, Y are cofibrant objects.*
- *The morphism $Y \to Z$ is a cofibration, and $\mathcal{C}$ is **left proper**.*

Exercise A.2. *Show that homotopy pushouts/pullbacks are not functorial in* hTop. *In other words, we do not have desired functors* $\mathsf{Span}(\mathsf{hTop}) \to \mathsf{hTop}$ *or* $\mathsf{coSpan}(\mathsf{hTop}) \to \mathsf{hTop}$.

Exercise A.3. *Let $X \xrightarrow{p} Y \xleftarrow{q} Z$ be a diagram in Top and S be a testing object. Compare the data encoded in morphisms*

$$S \to X \underset{Y}{\times} Z,\ S \to X \overset{\mathsf{h}}{\underset{Y}{\times}} Z,\ \mathsf{h}S \to \mathsf{h}X \underset{\mathsf{h}Y}{\times} \mathsf{h}Z.$$

A.4. Convince yourself that:

> *To capture all homotopy-invariant information, it is not enough to know two maps are homotopic, rather, we need to know* how *they are homotopic.*

Exercise A.5. *Let*

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ Y_1 \arrow[r] & Y & Y_2 \arrow[l] \\ & Z \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

be a commutative diagram in Top. *Give a definition of the homotopy limit of this diagram, and study it as in Exercise A.3.*

A.6. Convince yourself that:

> *To capture all homotopy-invariant information, it is not enough to record homotopies between maps, rather, we need to record all the higher homotopies.*

References

[Qui67] Daniel G Quillen. Homotopical algebra. *Lecture Notes in Mathematics*, 1967.