

LECTURE 10

Last time, for a diagram $u : K_1 \times K_2 \rightarrow C$, we constructed a canonical morphism

$$\operatorname{colim}_{y \in K_2} \lim_{K_1} u(-, y) \rightarrow \lim_{x \in K_1} \operatorname{colim}_{K_2} u(x, -)$$

when assuming both sides exist. In this lecture, we discuss several cases when the above morphism is invertible.

1. FINITE DIAGRAMS

Definition 1.1. We say a simplicial set K is **finite** if it has finitely many non-degenerate simplices.

We say an ∞ -category K is **essentially finite** if it can be represented by a quasi-category which is a finite simplicial set.

1.2. Recall we say an ordinary category K is essentially finite iff the set of equivalence classes of objects and the set of morphisms in K are finite.

Exercise 1.3. Let K be an essentially finite ∞ -category. Show that hK is an essentially finite ordinary category.

Exercise 1.4. Let K be an ordinary category. If K is essentially finite as an ∞ -category, then it is essentially finite as an ordinary category.

Warning 1.5. The converse is not true: even a finite ordinary category may fail to be essentially finite as an ∞ -category.

Exercise 1.6. Let Idem be the ordinary category defined by

- There is a unique object $*$;
- $\operatorname{Hom}(*, *) := \{\operatorname{id}, e\}$, with $e \circ e = e$.

Show that $N_\bullet(\operatorname{Idem})$ is not equivalent to a finite quasi-category.

Exercise 1.7. Let $\mathbb{B}(\mathbb{Z}/2\mathbb{Z})$ be the ordinary category defined by

- There is a unique object $*$;
- $\operatorname{Hom}(*, *) := \{\operatorname{id}, f\}$, with $f \circ f = \operatorname{id}$.

Show that $N_\bullet(\mathbb{B}(\mathbb{Z}/2\mathbb{Z}))$ is not equivalent to a finite quasi-category.

1.8. Nevertheless, for finite partially ordered set, the above abnormality does not appear.

Exercise 1.9. Let J be a finite partially ordered set, viewed as an ordinary category. Show that $N_\bullet(J)$ is a finite simplicial set.

Proposition 1.10 (Ker.02NB). For any simplicial set K , there exists a partially ordered set J and an initial morphism $N_\bullet(J) \rightarrow K$. If K is finite, we can take J to be finite.

Corollary-Definition 1.11. Let C be an ∞ -category. The following are equivalent:

- \mathcal{C} admits limits indexed by finite simplicial sets.
- \mathcal{C} admits limits indexed by essentially finite ∞ -categories.
- \mathcal{C} admits limits indexed by finite partially ordered sets.

We say \mathcal{C} **admits finite limits** if it satisfies the above conditions.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞ -categories such that \mathcal{C} admits finite limits. The following are equivalent:

- F preserves limits indexed by finite simplicial sets.
- F preserves limits indexed by essentially finite ∞ -categories.
- F preserves limits indexed by finite partially ordered sets.

We say F is **left exact** if it satisfies the above conditions.

Dually, for a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that \mathcal{C} admits finite colimits, we say F is **right exact** if it preserves finite colimits.

Warning 1.12. Let \mathcal{C} be an ∞ -category that admits finite limits. It may not admit limits indexed by finite ordinary categories. For example, \mathcal{C} may fail to be idempotent complete.

Remark 1.13. In future lectures, we will define left/right exact functors between general ∞ -categories.

Proposition 1.14. Let \mathcal{C} be an ∞ -category. The following are equivalent:

- (i) \mathcal{C} admits finite limits.
- (ii) \mathcal{C} admits fiber products and a final object.
- (iii) \mathcal{C} admits finite products and equalizers.

We also have similar results for left exact functors.

Sketch. The implications (i) \Rightarrow (ii) \Leftrightarrow (iii) are standard. To show (ii) \Rightarrow (i), let $u : K \rightarrow \mathcal{C}$ be a finite diagram. We need to show $\lim u$ exists. When $K = \emptyset$, this is the final object of \mathcal{C} . If K is nonempty, we can find a pushout square in \mathbf{Set}_Δ

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & K' \\ \downarrow \scriptstyle c & & \downarrow \scriptstyle c \\ \Delta^n & \xrightarrow{\quad c \quad} & K. \end{array}$$

Note that this is also a homotopy pushout square in $\mathbf{Set}_\Delta^{\mathbf{Joyal}}$. By the theorem on decomposition of diagrams (Ker.03DB), we have

$$\lim u \xrightarrow{\cong} \lim u|_{K'} \times_{\lim u|_{\partial\Delta^n}} \lim u|_{\Delta^n}$$

where the source exists if the target does. By induction, we reduce to prove the claim when $K = \Delta^n$, which follows from the fact that $\Delta^0 \xrightarrow{0} \Delta^n$ is initial. \square

2. FILTERED ∞ -CATEGORIES

Definition 2.1. We say an ∞ -category \mathcal{C} is **filtered** if any finite diagram $u : K \rightarrow \mathcal{C}$ can be extended to a diagram $K^\triangleright \rightarrow \mathcal{C}$.

Dually, we say \mathcal{C} is **cofiltered** if for any finite diagram $u : K \rightarrow \mathcal{C}$ can be extended to a diagram $K^\triangleleft \rightarrow \mathcal{C}$.

2.2. Note that \mathcal{C} is filtered iff \mathcal{C}^{op} is cofiltered. Hence we will focus on filtered ∞ -categories.

Exercise 2.3. *Being filtered is invariant under equivalences.*

2.4. An induction argument, which is similar to but more elaborate than that in the proof of Proposition 1.14, gives the following result:

Lemma 2.5 (Ker.02Q0). *An ∞ -category \mathcal{C} is filtered iff any diagram $u : \partial\Delta^n \rightarrow \mathcal{C}$ with $n \geq 0$ ¹ can be extended to a diagram $(\partial\Delta^n)^{\triangleright} \rightarrow \mathcal{C}$.*

Remark 2.6. *Note that $(\partial\Delta^n)^{\triangleright} \simeq \Lambda_{n+1}^{n+1}$.*

2.7. Recall we say an ordinary category \mathcal{C} is filtered if

- (0) It is nonempty.
- (1) For any objects x_1, x_2 in \mathcal{C} , there exists an object x equipped with morphisms $x_i \rightarrow x$.
- (2) For any diagram $x \rightrightarrows y$ in \mathcal{C} , there exists $y \rightarrow z$ such that the two compositions $x \rightrightarrows y \rightarrow z$ are equal.

Note that condition (i) corresponds to the extension problem for $K = \partial\Delta^i$.

Proposition 2.8 (Ker.02PS). *An ∞ -category \mathcal{C} is filtered iff it satisfies the following conditions:*

- (0) *It is nonempty.*
- (1) *For any objects x_1, x_2 in \mathcal{C} , there exists an object x equipped with morphisms $x_i \rightarrow x$.*
- (2) *For any objects x, y in \mathcal{C} and any diagram $\partial\Delta^n \rightarrow \text{Maps}(x, y)$ with $n \geq 1$, there exists a morphism $y \rightarrow z$ such that the composition*

$$\partial\Delta^n \rightarrow \text{Maps}(x, y) \rightarrow \text{Maps}(x, z)$$

is null-homotopic, i.e., equivalent to a constant diagram².

Remark 2.9. *Roughly speaking, for $n \geq 2$, the extension problem*

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

corresponds to an extension problem

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \text{Maps}(x, y) \\ \downarrow & \nearrow & \\ \Delta^{n-1} & & \end{array}$$

such that $x = u(0)$ and $y = u(n)$.

Exercise 2.10. *Let \mathcal{C} be a filtered ∞ -category. Show that $\mathbf{h}\mathcal{C}$ is a filtered ordinary category.*

¹By definition, $\partial\Delta^0 = \emptyset$.

²Exercise: for a Kan complex K , a morphism $\partial\Delta^n \rightarrow K$ is null-homotopic iff it factors through Δ^n .

Exercise 2.11. Let \mathcal{C} be an ordinary category. Show that \mathcal{C} is filtered as an ordinary category iff it is filtered as an ∞ -category.

Example 2.12. An ∞ -category is filtered if it admits a final object.

Example 2.13. An ∞ -category is filtered if it admits finite colimits.

Exercise 2.14. The category \mathbf{Idem} is filtered.

3. FILTERED COLIMITS AND FINITE LIMITS

Theorem 3.1 (Ker.05XS). Let \mathcal{C} be a small ∞ -category. Then \mathcal{C} is filtered iff the functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathrm{Grpd}_{\infty}) \rightarrow \mathrm{Grpd}_{\infty}$$

preserves finite limits.

Warning 3.2. The claim may be false if Grpd_{∞} is replaced by general ∞ -categories.

Exercise 3.3. Consider the discrete topological space \mathbb{Z} and its one-point-compactification $\mathbb{Z} \cup \{\infty\}$ ³. Consider the partially ordered set \mathcal{P} of closed subsets of $\mathbb{Z} \cup \{\infty\}$. Show that filtered colimits in \mathcal{P} may not commute with finite limits.

Corollary 3.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a final functor between ∞ -categories. If \mathcal{C} is filtered, so is \mathcal{D} .

Challenge 3.5. Can you find a proof of the above corollary without using Theorem 3.1?

3.6. Recall a partially ordered set \mathcal{J} is *directed* if any finite subset of it admits an upper bound in \mathcal{J} .

Proposition 3.7 (Ker.02QA). Let \mathcal{C} be an ∞ -category. The following are equivalent:

- \mathcal{C} is filtered.
- There exists a directed partially ordered set \mathcal{J} and a final functor $\mathcal{J} \rightarrow \mathcal{C}$.

Proposition 3.8. Let \mathcal{C} be a filtered ∞ -category and $u : K \rightarrow \mathcal{C}$ be any finite diagram. Then the forgetful functor $\mathcal{C}_{u/} \rightarrow \mathcal{C}$ is final.

Proof. By Quillen's Theorem A, we only need to show $\mathcal{C}_{u/} \times_{\mathcal{C}} \mathcal{C}_{x/}$ is weakly contractible for any $x \in \mathcal{C}$. We have

$$\mathcal{C}_{u/} \times_{\mathcal{C}} \mathcal{C}_{x/} \simeq \mathcal{C}_{u \sqcup x/}$$

where $u \sqcup x : K \sqcup \Delta^0 \rightarrow \mathcal{C}$ is the disjoint union of u and x . Now the claim follows from the following two exercises. \square

Exercise 3.9. A filtered ∞ -category is weakly contractible. Hint: $K \times \Delta^1 \rightarrow K^{\triangleright}$.

Proposition 3.10. Let \mathcal{C} be an ∞ -category. The following are equivalent:

- (i) \mathcal{C} is filtered.
- (ii) For any finite diagram $u : K \rightarrow \mathcal{C}$, the ∞ -category $\mathcal{C}_{u/}$ is filtered.
- (iii) For any finite diagram $u : K \rightarrow \mathcal{C}$, the ∞ -category $\mathcal{C}_{u/}$ is weakly contractible.
- (iv) For any finite $K \in \mathbf{Set}_{\Delta}$, the diagonal functor $\mathcal{C} \rightarrow \mathrm{Fun}(K, \mathcal{C})$, $x \mapsto \underline{x}$ is final.

³An closed subset of $\mathbb{Z} \cup \{\infty\}$ is either finite or contains ∞ .

Sketch. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) are left as exercises. We will prove (ii) \Leftrightarrow (iii). By Quillen's Theorem A, (ii) is equivalent to:

$$\mathbf{C} \times_{\mathbf{Fun}(K, \mathbf{C})} \mathbf{Fun}(K, \mathbf{C})^{u/}$$

is weakly contractible for any $u : K \rightarrow \mathbf{C}$. It is easy to see the above fiber product is equivalent to $\mathbf{C}^{u/}$, which implies the claim. \square

4. SIFTED DIAGRAM

Definition 4.1. Let K be a simplicial set. We say K is **sifted** if for any finite set I , the diagonal morphism $K \rightarrow K^I$ is final.

Dually, we say K is **cosifted** if for any finite set I , the diagonal morphism $K \rightarrow K^I$ is initial.

4.2. Note that \mathbf{C} is sifted iff \mathbf{C}^{op} is cosifted. Hence we will focus on sifted ∞ -categories.

Exercise 4.3. Being filtered is invariant under equivalences.

Example 4.4. Any filtered ∞ -category is sifted.

Exercise 4.5. Show that a simplicial set K is sifted iff it is nonempty and $K \rightarrow K \times K$ is final. Hint: If $K \rightarrow K \times K$ is a weak homotopy equivalent and $K \neq \emptyset$, then K is weakly contractible.

Corollary 4.6. An ∞ -category \mathbf{C} is sifted iff it is nonempty and $\mathbf{C}_{x/} \times_{\mathbf{C}} \mathbf{C}_{y/}$ is weakly contractible for any pair of objects $x, y \in \mathbf{C}$.

Warning 4.7. In classical category theory, an ordinary category is called sifted if it is nonempty and $\mathbf{C}_{x/} \times_{\mathbf{C}} \mathbf{C}_{y/}$ is connected for any pair of objects $x, y \in \mathbf{C}$. A sifted ordinary category may fail to be sifted as an ∞ -category.

Exercise 4.8. Let $\Delta_{\leq 1} \subset \Delta$ be the full subcategory consisting of $[0]$ and $[1]$. Show that $\Delta_{\leq 1}^{\text{op}}$ is sifted as an ordinary category but not as an ∞ -category.

Exercise 4.9. Show that Δ^{op} is sifted as an ∞ -category.

Definition 4.10. The colimit of a Δ^{op} -indexed diagram is called the **geometric realization** of such diagram. The limit of a Δ -indexed diagram is called the **totalization** of such diagram.

4.11. It is known that geometric realizations and filtered colimits *almost* generate all sifted diagrams. See §A.10.

Proposition 4.12. Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a final functor between ∞ -categories. If \mathbf{C} is sifted, so is \mathbf{D} .

Proof. For any finite set I , consider the commutative diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ \delta_1 \downarrow & \searrow & \downarrow \delta_2 \\ \mathbf{C}^I & \xrightarrow{F^I} & \mathbf{D}^I \end{array}$$

Since F is final, so is F^I ([Lecture 7, Proposition 3.16]). Since \mathbf{C} is sifted, δ_1 is final. It follows that δ_2 is also final as desired ([Lecture 7, Exercise 3.13]). \square

Exercise 4.13. Let K be a sifted simplicial set. Suppose \mathcal{C} , \mathcal{D} and \mathcal{E} are ∞ -categories that admit K -indexed colimits, and $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ is a functor that preserves K -indexed colimits in each variable⁴. Show that F preserves K -indexed colimits.

5. SIFTED COLIMITS AND FINITE PRODUCTS

Theorem 5.1 (HTT.5.5.8.11, Ker.05XM). Let \mathcal{C} be a small ∞ -category. Then \mathcal{C} is sifted iff the functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathrm{Grpd}_\infty) \rightarrow \mathrm{Grpd}_\infty$$

preserves finite products.

6. UNIVERSALITY OF COLIMITS

Construction 6.1. Let \mathcal{C} be an ∞ -category that admits fiber products. Let $X \rightarrow Y$ be a functor in Grpd_∞ . As in classical category theory, we have an adjunction

$$(6.1) \quad (\mathrm{Grpd}_\infty)_{/X} \rightleftarrows (\mathrm{Grpd}_\infty)_{/Y}$$

where:

- The left adjoint is compatible with the forgetful functors;
- The right adjoint is given by $- \times_Y X$.

Remark 6.2. Alternatively, the above right adjoint can be constructed via the simplicial model category $\mathrm{Set}_\Delta^{\mathrm{KQ}}$.

Theorem 6.3 (Ker.05V5). The functor

$$- \times_Y X : (\mathrm{Grpd}_\infty)_{/Y} \rightarrow (\mathrm{Grpd}_\infty)_{/X}$$

preserves small colimits.

Exercise 6.4. Let $u : K \rightarrow \mathrm{Grpd}_\infty$, $i \mapsto X_i$ be a small diagram and write $X := \mathrm{colim} u$. Show that there is a canonical isomorphism

$$\mathrm{colim}_{i \in K} (X_i \times_X Y) \simeq Y.$$

APPENDIX A. FILTERED COLIMITS OF ∞ -CATEGORIES

Proposition A.1. Let \mathcal{C} be a small filtered ordinary category. Then any colimit diagram $\bar{u} : \mathcal{C}^\triangleright \rightarrow \mathrm{Set}_\Delta$ is also a homotopy colimit diagram in $\mathrm{Set}_\Delta^{\mathrm{Joyal}}$.

A.2. We introduce two applications of Proposition A.1. The first application is a description of filtered colimits of ∞ -categories.

Exercise A.3. The functor $\mathrm{QCat} \rightarrow \mathrm{Cat}_\infty$ preserves small filtered ∞ -colimits.

Exercise A.4. Let K be a small filtered ∞ -category and $K \rightarrow \mathrm{Cat}_\infty$, $i \mapsto \mathcal{C}_i$ be a diagram. Show that any object in $\mathcal{C} := \mathrm{colim}_K \mathcal{C}_i$ is equivalent to $\mathrm{ins}_i(x)$ for some $i \in K$ and $x \in \mathcal{C}_i$.

⁴This means $F(c, -)$ and $F(-, d)$ preserve K -indexed colimits for any $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Exercise A.5. Let K be a small filtered ∞ -category and $K \rightarrow \mathbf{Cat}_\infty$, $i \mapsto C_i$ be a diagram. For $i, j \in K$ and $x \in C_i$, $y \in C_j$, show that

$$\mathrm{Maps}_C(\mathrm{ins}_i(x), \mathrm{ins}_j(y)) \xleftarrow{\simeq} \mathrm{colim}_{k \in K_{(i,j)}/} \mathrm{Maps}_{C_k}(x_k, y_k),$$

where

- $K_{(i,j)}/$ is the coslice ∞ -category for $\Delta^0 \sqcup \Delta^0 \xrightarrow{(i,j)} K$;
- For $k \in K_{(i,j)}/$, x_k is the image of x under the functor $C_i \rightarrow C_k$, and similarly for y_k .

A.6. The second application is to decompose general colimits into filtered colimits of finite colimits.

Exercise A.7. Let K be any simplicial set. Show that the partially ordered set $\mathrm{Fin}(K)$ of finite simplicial subsets of K is filtered, and

$$K \simeq \mathrm{hocolim}_{J \in \mathrm{Fin}(K)} J.$$

Exercise A.8. For any diagram $u : K \rightarrow C$, there is a canonical equivalence

$$\mathrm{colim}_{J \in \mathrm{Fin}(K)} \mathrm{colim}_J u|_J \xrightarrow{\simeq} \mathrm{colim}_K u,$$

where the RHS exists if the LHS does.

Corollary A.9. An ∞ -category C admits small colimits iff it admits small filtered colimits and finite colimits. For such C , a functor $F : C \rightarrow D$ preserves small colimits iff it preserves filtered colimits and finite colimits.

A.10. The above result says

$$\text{small colimits} = \text{small filtered colimits} + \text{finite colimits}.$$

In fact, we also have:

$$\text{small colimits} = \text{small sifted colimits} + \text{finite coproducts}.$$

$$\text{small colimits} = \text{small filtered colimits} + \text{geometric realization} + \text{finite coproducts}.$$

See HA.1.3.3.10.

A.11. **Suggested readings.** Ker.03DD.

APPENDIX B. STRONG UNIVERSALITY OF COLIMITS

Theorem B.1. Let $K \rightarrow \mathbf{Grpd}_\infty$, $i \mapsto X_i$ be a small diagram. Consider $X := \mathrm{colim}_{i \in K} X_i$. Then there is a canonical equivalence

$$(B.1) \quad (\mathbf{Grpd}_\infty)_{/X} \rightarrow \lim_{i \in K} (\mathbf{Grpd}_\infty)_{/X_i},$$

where each evaluation functor is

$$- \times_X X_i : (\mathbf{Grpd}_\infty)_{/X} \rightarrow (\mathbf{Grpd}_\infty)_{/X_i}.$$

Exercise B.2. Use Theorem 6.3 to show (B.1) is fully faithful.

Exercise B.3. Show that Theorem 6.3 remains true if \mathbf{Grpd}_∞ is replaced by \mathbf{Set} , but Theorem B.1 would fail.

B.4. **Suggested readings.** Ker.05SB.