

LECTURE 3

In this lecture, we give a quick review of the theory of algebraic groups. This theory is analogous to that of complex Lie groups, but the techniques are more algebraic and some proofs are subtler¹. Standard textbooks include: [B], [H] and [Sp]. See [M] for a modern treatment of this theory, which is also my favourite.

1. ALGEBRAIC GROUPS

Definition 1. An **algebraic group** over k is a finite type k -scheme G equipped with a group structure, i.e., a multiplication map $m : G \times G \rightarrow G$ subject to axioms similar to those for an abstract group.

Homomorphisms between algebraic groups are defined in the obvious way. Let \mathbf{Grp}_k be the category of algebraic groups.

Remark 2. As in the study of abstract groups, the unit and the inversion maps are determined by the multiplication map. We denote them respectively by:

$$e : \text{pt} \rightarrow G, \sigma : G \rightarrow G,$$

where $\text{pt} := \text{Spec}(k)$.

Construction 3. Let G be an algebraic group. For any commutative k -algebra A , write

$$G(A) := \text{Hom}(\text{Spec}(A), G)$$

be the set of maps $\text{Spec}(A) \rightarrow G$ between k -schemes. The group structure on G induces a group structure on $G(A)$.

Note that for $A \rightarrow B$, we have a homomorphism $G(A) \rightarrow G(B)$. Hence we obtain a functor

$$G(-) : \mathbf{CAlg}_k \rightarrow \mathbf{Grp}$$

from the category of commutative k -algebras to the category of (abstract) groups. By the Yoneda lemma, the algebraic group G is determined by this functor.

Example 4. The **additive group** \mathbb{G}_a is defined such that $\mathbb{G}_a(A) = A$, viewed as a commutative group under addition. The underlying k -scheme is the affine line \mathbb{A}^1 .

Example 5. The **multiplicative group** $\mathbb{G}_m = \mathbb{G}_m$ is defined such that $\mathbb{G}_m(A) = A^\times$, i.e. the subset of unit elements in A , viewed as a commutative group under multiplication. The underlying k -scheme is $\mathbb{A}^1 \setminus 0$, i.e., the affine line with the origin removed.

Example 6. One can define algebraic groups $G := \text{GL}_n, \text{SL}_n, \text{SO}_n$, etc. such that $G(A)$ is the group of matrices of the corresponding type with coefficients in A .

Example 7. One can define the algebraic group PGL_n such that $\mathcal{O}_{\text{PGL}_n}$ is a subring of $\mathcal{O}_{\text{GL}_n}$ ².

Date: Mar 11, 2024.

¹Especially if one allows positive-characteristic or non-algebraic-closed base field k .

²Warning: $\text{PGL}_n(A) \neq \text{GL}_n(A)/\text{GL}_1(A)$ for general A . In fact, viewed as functors in A , the LHS is the sheafification of the RHS in the fpqc topology.

Remark 8. Not every functor $\mathbf{CAlg}_k \rightarrow \mathbf{Grp}$ comes from an algebraic group. For example, any k -vector space V defines a functor

$$\mathrm{GL}_V(-) : \mathbf{CAlg}_k \rightarrow \mathbf{Grp}$$

that sends A to the group of A -linear automorphisms of $A \otimes V$. This functor is not represented by an algebraic group unless V is finite-dimensional.

2. HOPF ALGEBRAS

From now on, we assume G is affine³⁴.

Construction 9. An affine algebraic group G is determined by its ring of functions \mathcal{O}_G . The maps $m : G \times G \rightarrow G$ and $e : \mathrm{pt} \rightarrow G$ correspond to homomorphisms between algebras

$$\Delta : \mathcal{O}_G \rightarrow \mathcal{O}_G \otimes \mathcal{O}_G, \quad \epsilon : \mathcal{O}_G \rightarrow k,$$

which are called the **comultiplication** and **counit** maps of \mathcal{O}_G . Together with the usual multiplication and unit maps of \mathcal{O}_G , we obtain a **bialgebra** $(\mathcal{O}_G, \cdot, \Delta)$.

Note that this bialgebra is commutative but not cocommutative unless G is so.

The inverse map $\sigma : G \rightarrow G$ corresponds to a homomorphism $S : \mathcal{O}_G \rightarrow \mathcal{O}_G$, which is called the **antipode** of \mathcal{O}_G . This makes \mathcal{O}_G into a commutative **Hopf algebra**.

Remark 10. A Hopf algebra is a bialgebra A equipped with an antipode map $S : A \rightarrow A$ subject to a certain axiom. For our purposes, it is less useful to memorize this axiom than to imagine it amounts to say “ $\mathrm{Spec}(A)$ ”⁵ has an inversion map.

Example 11. For $G = \mathbb{G}_a$, we have $\mathcal{O}_G = k[t]$ and

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \quad \epsilon(f) = f(0), \quad S(f)(t) = f(-t).$$

Example 12. For $G = \mathbb{G}_m$, we have $\mathcal{O}_G = k[t, t^{-1}]$ and

$$\Delta(t) = t \otimes t, \quad \epsilon(f) = f(1), \quad S(f)(t) = f(t^{-1}).$$

Example 13. The universal enveloping algebra $U(\mathfrak{g})$ of any Lie algebra is a Hopf algebra. The comultiplication $\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathfrak{g} \subset U(\mathfrak{g})$ and its compatibility with the multiplication. Similarly, the antipode is determined by $S(x) = -x$, $x \in \mathfrak{g}$.

The Hopf algebra $U(\mathfrak{g})$ is cocommutative but not commutative unless \mathfrak{g} is abelian.

Remark 14. Using the Hopf algebra structure on $U(\mathfrak{g})$, the tensor product structure in $\mathfrak{g}\text{-mod}$ can be defined as follows. Let V_1 and V_2 be left $U(\mathfrak{g})$ -modules. Their tensor product $V_1 \otimes V_2$ is naturally a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ -module. Restricting along Δ , we can view $V_1 \otimes V_2$ as a $U(\mathfrak{g})$ -module.

Remark 15. There are also interesting Hopf algebras that are neither commutative nor cocommutative. For example, *quantum algebras* are such gadgets. See [L] for a standard textbook.

³Any affine algebraic group over field of characteristic 0 is smooth.

⁴Projective algebraic groups, a.k.a., abelian varieties, are also important and play a central role in modern mathematics.

⁵Note however that this does not make sense if A is not commutative.

3. TANGENT SPACES

As in the theory of Lie groups, the tangent space of an algebraic group at its unit is a Lie algebra. To describe this, let us review the definition of tangent spaces in algebraic geometry.

Definition 16. Let X be any k -scheme and $x \in X$ be a k -point, i.e., a map $x : \operatorname{Spec}(k) \rightarrow X$. The **tangent space** of X at x , denoted by $T_x X$, is the *set* of dotted arrows making the following diagram commute:

$$\begin{array}{ccc} \operatorname{Spec}(k) & \xrightarrow{x} & X \\ \downarrow & \nearrow & \\ \operatorname{Spec}(k[\epsilon]/\epsilon^2) & & \end{array}$$

Here the vertical map is given by the homomorphism $k[\epsilon]/\epsilon^2 \rightarrow k$, $\epsilon \mapsto 0$.

Elements in $T_x X$ are called **tangent vectors** of X at x .

Tangent vectors are related to *derivations*. Let us review its definition in the algebraic setting.

Definition 17. Let A be a k -algebra and M be an A -module. A **k -derivation** of A into M is a k -linear map $D : A \rightarrow M$ satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let $\operatorname{Der}_k(A, M)$ be the set of such k -derivations. This is naturally a k -vector space.

Construction 18. Let $X = \operatorname{Spec}(A)$ be an affine scheme and $x : \operatorname{Spec}(k) \rightarrow X$ be given by a homomorphism $\phi : A \rightarrow k$. View k as an A -module via this homomorphism and denote it by k_x . For any $D \in \operatorname{Der}_k(A, k_x)$, the map

$$A \rightarrow k[\epsilon]/\epsilon^2, f \mapsto \phi(f) + D(f)\epsilon$$

is a homomorphism and thereby gives a map $\operatorname{Spec}(k[\epsilon]/\epsilon^2) \rightarrow X$, which is an element in $T_x X$. It is easy to see this gives a bijection

$$\operatorname{Der}_k(A, k_x) \simeq T_x X.$$

In particular, we obtain a k -vector space structure on $T_x X$.

Construction 19. Let $f : X \rightarrow Y$ be a morphism between k -schemes. Let $x \in X$ and $y := f(x) \in Y$ be k -points. There is an obvious k -linear map

$$df : T_x X \rightarrow T_y Y$$

given by composing with f . We call it the **differential** of f .

We have the following obvious result:

Lemma 20. Let X and Y be k -schemes. Let $x \in X$ and $y \in Y$ be k -points. For $\partial_1 \in T_x X$ and $\partial_2 \in T_y Y$, write $\partial_1 \oplus \partial_2 \in T_{(x,y)}(X \times Y)$ for the map

$$\operatorname{Spec}(k[\epsilon]/\epsilon^2) \xrightarrow{(\partial_1, \partial_2)} X \times Y.$$

Then the map

$$(3.1) \quad T_x X \times T_y Y \rightarrow T_{(x,y)}(X \times Y), (\partial_1, \partial_2) \mapsto \partial_1 \oplus \partial_2$$

induces an isomorphism $T_x X \oplus T_y Y = T_{(x,y)}(X \times Y)$.

4. LIE ALGEBRAS AND ALGEBRAIC GROUPS

Notation 21. Let G be an algebraic group. Define

$$\mathrm{Lie}(G) := T_e G.$$

Example 22. For $G = \mathrm{GL}_n$, we have $\mathrm{Lie}(\mathrm{GL}_n) \simeq \mathfrak{gl}_n$. In particular, $\mathrm{Lie}(\mathrm{GL}_n)$ is naturally a Lie algebra. We have similar results for other classical subgroups of GL_n .

We state the following result without proof:

Theorem 23. There is a canonical functor $\mathrm{Grp}_k \rightarrow \mathrm{Lie}_k$ sending G to $\mathrm{Lie}(G)$ equipped with a natural Lie bracket, such that for $G = \mathrm{GL}_n$, the Lie bracket on $\mathrm{Lie}(\mathrm{GL}_n)$ is given by that on \mathfrak{gl}_n .

Remark 24. The above functor $\mathrm{Grp}_k \rightarrow \mathrm{Lie}_k$ is unique if stated properly. See [H, Theorem 10.23].

Warning 25. It is not true that every Lie algebra can be obtained from algebraic groups. See [Bou, I, §5, Exercise 6] for a counterexample.

Remark 26. Consider the multiplication map $m : G \times G \rightarrow G$ and its differential $dm : \mathrm{Lie}(G \times G) \rightarrow \mathrm{Lie}(G)$. One can show that the composition

$$\mathrm{Lie}(G) \oplus \mathrm{Lie}(G) \simeq \mathrm{Lie}(G \times G) \xrightarrow{dm} \mathrm{Lie}(G)$$

sends (∂_1, ∂_2) to $\partial_1 + \partial_2$. This is the algebraic analogue of the formula $\exp(tu) \cdot \exp(tv) = \exp(t(u+v)) + O(t^2)$, $u, v \in \mathrm{Lie}(G)$ that appears in the study of Lie groups.

Remark 27. Let G be an affine algebraic group and $\mathfrak{g} := \mathrm{Lie}(G)$ be its Lie algebra. The Hopf algebras \mathcal{O}_G and $U(\mathfrak{g})$ are related as follows.

Consider the dual $U(\mathfrak{g})^* := \mathrm{Hom}_k(U(\mathfrak{g}), k)$ as a topological vector space⁶. The cocommutative Hopf algebra structure on $U(\mathfrak{g})$ induces a commutative topological Hopf algebra structure on it⁷.

On the other hand, consider the maximal ideal $\mathfrak{m} \subset \mathcal{O}_G$ corresponding to the unit point $e : \mathrm{pt} \rightarrow G$. Let $\hat{\mathcal{O}}_G$ be the \mathfrak{m} -adic completion of \mathcal{O}_G . The commutative Hopf algebra structure on \mathcal{O}_G induces a commutative topological Hopf algebra structure on it.

We have

$$U(\mathfrak{g})^* \simeq \hat{\mathcal{O}}_G$$

as commutative topological Hopf algebras.

Example 28. Note that the Lie algebras of \mathbb{G}_a and \mathbb{G}_m are isomorphic, it follows that $\hat{\mathcal{O}}_{\mathbb{G}_a} \simeq k[[t]]$ and $\hat{\mathcal{O}}_{\mathbb{G}_m} = k[[t-1]]$ are isomorphic. Up to a scalar, this isomorphism is given by

$$k[[t-1]] \rightarrow k[[t]], f \mapsto f(\exp(t)).$$

Note that given a power series $a_0 + a_1(t-1) + a_2(t-1)^2 + \dots$, the series $a_0 + a_1(\exp(t)-1) + a_2(\exp(t)-1)^2 + \dots$ indeed converges in the t -adic topology.

⁶For a vector space V equipped with the discrete topology, the dual V^* is equipped with the weakest topology such that for any finite-dimensional subspace $V_0 \subset V$, the map $V^* \rightarrow V_0^*$ is continuous. Here V_0^* is equipped with the discrete topology. Equivalently, we can define V^* as an object in the pro-category $\mathrm{Pro}(\mathrm{Vect}_{k,\mathrm{fd}})$ of finite dimensional vector spaces.

⁷Here we must use the *complete tensor product* $U(\mathfrak{g})^* \hat{\otimes} U(\mathfrak{g})^*$ instead of the usual tensor product. By design, this is the dual of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

5. REPRESENTATIONS OF ALGEBRAIC GROUPS

In this section, G is an affine algebraic group.

Definition 29. A **representation of G** , or equivalently a **G -module** is a k -vector space V equipped with a natural transformation $G(-) \rightarrow \mathrm{GL}_V(-)$ as functors $\mathrm{CAlg}_k \rightarrow \mathrm{Grp}$.

Let V and W be G -modules, a **G -linear map** $\phi : V \rightarrow W$ is a k -linear map such that for any $A \in \mathrm{CAlg}_k$, the following diagram commutes

$$\begin{array}{ccc} G(A) & \longrightarrow & \mathrm{GL}_V(A) \\ \downarrow & & \downarrow \phi \circ - \\ \mathrm{GL}_W(A) & \xrightarrow{- \circ \phi} & \mathrm{Hom}_A(A \otimes V, A \otimes W). \end{array}$$

Let $\mathrm{Rep}(G)$ be the category of G -modules.

Example 30. Any V can be equipped with a trivial G -module structure such that the homomorphisms $G(A) \rightarrow \mathrm{GL}_V(A)$ are trivial.

Proposition 31. *The category $\mathrm{Rep}(G)$ is an abelian category and the forgetful functor $\mathrm{Rep}(G) \rightarrow \mathrm{Vect}_k$ is exact.*

Remark 32. If V is finite-dimensional, then GL_V is represented by an algebraic group. A G -module structure on V is just a homomorphism $G \rightarrow \mathrm{GL}_V$ between algebraic groups.

Warning 33. *Evaluate at $k \in \mathrm{CAlg}_k$, we obtain a group homomorphism $G(k) \rightarrow \mathrm{GL}_V(k) = \mathrm{GL}(V)$. But a G -module structure contains more information than such a homomorphism. This can be seen from the following exercise. (Note that for $k = \mathbb{C}$, the abstract groups $\mathbb{G}_a(\mathbb{C}) = \mathbb{C}$ and $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ are isomorphic via the exponential map.)*

Exercise 34. This is **Homework 1, Problem 4**.

- (1) Find all maps between k -schemes $\mathbb{A}^1 \rightarrow \mathbb{A}^1 \setminus 0$.
- (2) Find all 1-dimensional representations of the additive group \mathbb{G}_a .
- (3) Find all maps between k -schemes $\mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1 \setminus 0$.
- (4) Find all 1-dimensional representations of the multiplicative group \mathbb{G}_m .

Proposition 35. *Any G -action on a vector space V is locally finite, i.e., V is the union of its finite-dimensional subrepresentations.*

Proposition 36. *There is a canonical equivalence*

$$\mathrm{Rep}(G) \simeq \mathcal{O}_G\text{-comod}$$

from the category of G -modules to the category of \mathcal{O}_G -comodules. This equivalence is compatible with the forgetful functors to Vect_k .

Remark 37. The functor $\mathrm{Rep}(G) \simeq \mathcal{O}_G\text{-comod}$ is constructed as follows. Let $V \in \mathrm{Rep}(G)$. Consider $A := \mathcal{O}_G$ and the homomorphism $G(A) \rightarrow \mathrm{GL}_V(A)$. The identity map $G \rightarrow G$ can be written as $\mathrm{Spec}(A) \rightarrow G$ which corresponds to an element in $G(A)$ ⁸. Consider the image of this element in $\mathrm{GL}_V(A)$, which is a A -linear map $A \otimes V \rightarrow A \otimes V$. This is the same as a k -linear map $V \rightarrow A \otimes V$, i.e., a k -linear map

$$V \rightarrow \mathcal{O}_G \otimes V.$$

One can verify this defines a \mathcal{O}_G -comodule structure on V .

⁸Warning: this is *not* the unit element of this group.

Construction 38. *There is a forgetful functor*

$$\mathrm{Rep}(G) \rightarrow \mathrm{Lie}(G)\text{-mod}$$

that can be constructed in the following equivalent ways:

- *Let us first suppose $V \in \mathrm{Rep}(G)$ is finite-dimensional. Then we have a homomorphism between algebraic groups $G \rightarrow \mathrm{GL}_V$ which induces a homomorphism between their Lie algebras $\mathrm{Lie}(G) \rightarrow \mathrm{Lie}(\mathrm{GL}_V) = \mathfrak{gl}(V)$, i.e., a $\mathrm{Lie}(G)$ -module structure on V .
When V is infinite-dimensional, the functor $\mathrm{GL}_V(-)$ is no longer represented by an algebraic group, but there is a formal method to define its Lie algebra and make the above construction work.*
- *Any $V \in \mathrm{Rep}(G)$ has an \mathcal{O}_G -comodule structure. By co-restricting along the map $\mathcal{O}_G \rightarrow \hat{\mathcal{O}}_G$, we obtain a (continuous) comodule structure for $\hat{\mathcal{O}}_G \simeq U(\mathfrak{g})^*$. Passing to duality, we obtain a module structure for $U(\mathfrak{g})$.*

Lemma 39. *The adjoint representation $\mathrm{Lie}(G) \in \mathrm{Lie}(G)\text{-mod}$ has a canonical lift to an object in $\mathrm{Rep}(G)$. We call it the **adjoint action of G on $\mathrm{Lie}(G)$** .*

Remark 40. On the level of k -points, the conjugate action of $G(k)$ on G induces an (abstract) action of $G(k)$ on $\mathrm{Lie}(G)$. This can be generalized to an action of $G(A)$ on $A \otimes \mathrm{Lie}(G)$ for any $A \in \mathrm{CAlg}_k$ and thereby obtain the desired G -module structure on $\mathrm{Lie}(G)$.

Proposition 41. *If G is connected, then the functor $\mathrm{Rep}(G) \rightarrow \mathrm{Lie}(G)\text{-mod}$ is fully faithful. In particular, the G -invariance and $\mathrm{Lie}(G)$ -invariance for a G -module are the same.*

Definition 42. If G is connected, we say an object $V \in \mathrm{Lie}(G)\text{-mod}$ is **G -integrable** if it is contained in the essential image of the above functor.

Warning 43. *The similar claim for derived categories is false. In other words, extensions of G -integrable modules are not necessarily G -integrable. The multiplicative group \mathbb{G}_m is a counterexample.*

6. SEMISIMPLE ALGEBRAIC GROUPS

Theorem 44. *Any semisimple Lie algebra \mathfrak{g} can be realized as the Lie algebra of an algebraic group. In the category of connected algebraic groups G with $\mathrm{Lie}(G) \simeq \mathfrak{g}$, there is a final object G_{ad} and an initial object G_{sc} . Moreover, the homomorphisms*

$$G_{\mathrm{sc}} \rightarrow G \rightarrow G_{\mathrm{ad}}$$

are isogenies, i.e., are surjective and have finite kernels.

Example 45. For $\mathfrak{g} = \mathfrak{sl}_n$, we have $G_{\mathrm{sc}} = \mathrm{SL}_n$ and $G_{\mathrm{ad}} = \mathrm{PGL}_n$.

Definition 46. We say G is **semisimple**⁹ if it is connected and its Lie algebra is semisimple. For a semisimple algebraic group G , we say it is **of adjoint type** (resp. **simply connected**) if it is of the form G_{ad} (resp. G_{sc}).

Theorem 47. *If G_{sc} is a simply-connected semisimple algebraic group, then any finite-dimensional \mathfrak{g} -module is G_{sc} -integrable, i.e.,*

$$\mathrm{Rep}(G_{\mathrm{sc}})_{\mathrm{fd}} = \mathfrak{g}\text{-mod}_{\mathrm{fd}}.$$

Warning 48. *The similar claim for infinite-dimensional representation is false. This can be seen from the following exercise.*

⁹This is an ad hoc definition that only is only correct under our assumptions on k .

Exercise 49. This is **Homework 1, Problem 5**. Let G be any semisimple algebraic group with Lie algebra \mathfrak{g} . Prove: any Verma module of \mathfrak{g} is not G -integrable.

Theorem 50. *Let G be any semisimple algebraic group with Lie algebra \mathfrak{g} . Then the abelian categories $\text{Rep}(G)$ and $\mathfrak{g}\text{-mod}_{\text{fd}}$ are semisimple¹⁰. Simple objects in $\text{Rep}(G)$ are finite-dimensional, and an object $V \in \text{Rep}(G)$ is simple iff it is a simple object in $\mathfrak{g}\text{-mod}$.*

REFERENCES

- [B] Borel, Armand. Linear algebraic groups. Vol. 126. Springer Science & Business Media, 2012.
- [Bou] Bourbaki, Nicolas. Groupes et Algèbres de Lie: Elements de Mathématique. Hermann, 1968.
- [H] Humphreys, James E. Linear algebraic groups. Vol. 21. Springer Science & Business Media, 2012.
- [L] Lusztig, George. Introduction to quantum groups. Springer Science & Business Media, 2010.
- [M] Milne, James S. Algebraic groups: the theory of group schemes of finite type over a field. Vol. 170. Cambridge University Press, 2017.
- [Sp] Springer, Tony Albert. "Linear algebraic groups." In Algebraic Geometry IV: Linear Algebraic Groups Invariant Theory, pp. 1-121. Berlin, Heidelberg: Springer Berlin Heidelberg, 1998.

¹⁰I.e., any object can be written as a direct sum of simple objects.