SEMINAR NOTES ON
GEOMETRIZATION OF THE LOCAL LANGLANDS COORRESPONDENCE

LSG

CONTENTS

1. TALK I BY LIN: THE BIG PICTURE

> ... once we could merely *formulate* Fargues' conjecture, enough mathinery is available to apply Lafforgue's ideas to get the "automorphic-to-Galois" direction...
>
> [FS21, Sect. I.11]

1.1. **What is the local Langlands correspondence?**

Notation 1.1.1. *We fix the following notations:*
- *E is a local field (e.g. ... $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t)))$[1];*
- *G is a reductive group over E;*
- *$\hat{G}$ is the Langlands dual group over $\mathbb{Z}$;*
- *W_E is the Weil group for E.*

Conjecture 1.1.2. *There is a canonical map between sets:*

$$\left\{ irreducible\ objects\ \mathrm{Rep}_{\mathbb{C}}(G(E)) \right\} \to \left\{ W_E \to \hat{G}(\mathbb{C}) \right\}, \pi \mapsto \varphi_\pi$$

subject to certain compatibilities.

Remark 1.1.3. *Several remarks are in order.*
- (0) *In general this is not a bijection.*
- (1) *From easy to hard: archimedean, char p nonarchimedean, char 0 nonarchimedean. E.g., the conjecture is unknown for $E = \mathbb{Q}_p$ and general redcutive group.*
- (2) *$G(E)$ is a topological group and we require π to be* smooth, *i.e., any vector is fixed by some compact open subgroup.*

Notes taken by Lin.
[1]The colors are chosen to be compatible with [SW20, Figure 12.1].

(3) W_E *is a modification of* $\mathrm{Gal}(\overline{E}/E)$. $W_{\mathbb{C}} = \mathbb{C}^{\times}$, $W_{\mathbb{R}}$ *is the canonical extension of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *by* $\mathbb{C}^{\times}$. *For nonarchimedean* E *with residue field* $k = \mathbb{F}_q$,

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & I_E & \longrightarrow & W_E & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow{\simeq} & & \subset\downarrow & \lrcorner & & \subset\downarrow{1\mapsto\mathrm{Frob}} & \\
1 & \longrightarrow & I_E & \longrightarrow & \mathrm{Gal}(\overline{E}/E) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) & \longrightarrow & 1.
\end{array}
$$

The necessity of W_E *instead of* $\mathrm{Gal}(\overline{E}/E)$ *can be already seen in local class field theory.*

(4) W_E *is a topological group and we require* φ_{π} *to be continuous. However, it is* not *equipped with the subspace topology from* $\mathrm{Gal}(\overline{E}/E)$. *Instead,* I_E *has the subspace topology from* $\mathrm{Gal}(\overline{E}/E)$, *which is pro-finite, and is forced to be open in* W_E, *which is therefore locally pro-finite.*

(5) φ_{π} *is the so-called L-parameter of* π.

(6) *Part of the compatible requirements are about L-functions and* ϵ*-factors. Lin knows nothing about them. Maybe someone can explain later?*

1.2. **What is geometrization?**

Answer 1.2.1. *Do global geommetric Langlands on the Fargues–Fontaine curve, which behaves like a genus* 0 *curve in nonarchimedean geometry.*

To explain what this mean, we need some basic notions in analytic geometry.

Notation 1.2.2. *From now on, we restrict to the case* $E = \mathbf{Q}_p$. *There is no essential difference for general* E, *and the char p case is even easier.*

Analogy 1.2.3. *Tutorials on analytic geometry will be provided by Lin and Yuchen in the next weeks. For now, let us be satisfied by the following:*

	algebraic geometry	*analytic geometry*
affine	$\mathrm{Spec}\,R, R \in \mathrm{CAlg}$	$\mathrm{Spa}(R, R^+),$
		(R, R^+) *is a* Huber pair*:* $R^+ \subset R \in \mathrm{CAlg}(\mathrm{Top})$
		satisfying certain conditions
globalization	scheme	(pre-)adic space
	as locally ringed spaces	*as locally topologically ringed spaces*
point	$\mathrm{Spec}\,K, K$ *is a field*	$\mathrm{Spa}(K, K^+), (K, K^+)$ *is an* affinoid field
		analytic *if* K *is nondiscrete*
		nonanalytic *if* K *is discrete*

Notation 1.2.4. *In most cases, people make the canonical choice* $R^+ := R^{\circ}$ *being the subring of power-bounded elements and write* $\mathrm{Spa}\,R := \mathrm{Spa}(R, R^{\circ})$.

Remark 1.2.5. *Among all the (pre)-adic spaces, there is a class of objects, called* perfectoid spaces, *that are well-adapted to connect char* 0 *and char p. Affine perfectoid spaces are given by* $\mathrm{Spa}(R, R^+)$ *such that R is a* perfectoid ring. *Basic examples of perfectoid rings include* $\mathbb{F}_p((t^{1/p^{\infty}}))$, *which is the completion of* $\bigcup_n \mathbb{F}_p((t))(t^{1/p^n})$, *and* $\mathbb{Q}_p^{\mathrm{cycl}}$, *which is the completion of* $\mathbb{Q}_p(\mu_{p^{\infty}})$. *Any perfectoid ring is defined over* $\mathbb{Z}_p$ *although the latter itself is not a perfectoid ring.*

Personally I think the following result (together with the *tilting equivalence* to be explained later) is the root of all the magic:

Theorem 1.2.6. *There is* no *final object in* Perfd, *however, products exist in the category* Perfd_p *of perfectoid spaces of characteristic p.*

Analogy 1.2.7. *Sanath will talk about the details about FF curves. For now, let us be satisfied by the following:*

	Global Geometric Langlands for functional field	Geometrized Local Langlands
geometry	*algebraic geometry over* $\mathbb{F}_p$	*"perfectoid geometry"*
test objects	*schemes* $S \in \mathrm{Sch}_{/\mathbb{F}_p}$	*char p perfectoid space* $S \in \mathrm{Perfd}_p$
final test object	$\mathrm{Spec}\,\mathbb{F}_p$	**not exist** $\mathrm{Spa}\,\mathbb{F}_p \notin \mathrm{Perfd}_p$
spaces	*prestacks* $\supset$ *fpqc-stacks* $\supset$ *algebraic spaces*	*prestacks* $\supset$ *v-stacks* $\supset$ *diamonds*
absolute *curve*	X *over* $\mathbb{F}_p$	**not exist/sci-fi**
relative *curve*	$X_S := S \times_{\mathbb{F}_p} X$	$\mathcal{X}_S := \mathcal{Y}_S/\mathrm{Frob}_S := (S \dot\times \mathrm{Spa}\,\mathbb{Q}_p)/\mathrm{Frob}_S$

Question 1.2.8. *Wait, how dare you multiply a char 0 object $\mathrm{Spa}\,\mathbb{Q}_p$ with a char p object S!*

Warning 1.2.9. *There is a dot over the product sign in the notation $S \dot\times \mathrm{Spa}\,\mathbb{Q}_p$, which means it is not a product, at least not naively. For example, $\mathrm{Spa}(R, R^+) \dot\times \mathrm{Spa}\,\mathbb{Q}_p$ is an open subspace of $\mathrm{Spa}\,W(R^+)$, where $W(R^+)$ is the ring of p-Witt vectors in R^+. In fact, it is the open subspace where the functions $p, [\varpi] \in W(R^+)$ are invetible, where $\varpi \in R^+$ is a* pseudo-uniformizer*. Whatever this means, we see $S \dot\times \mathrm{Spa}\,\mathbb{Q}_p$ is of char 0.*

Question 1.2.10. *Wait, if $\mathcal{X}_S$ is of char 0, is it okay to study it using char p test objects? For example, when talking about Hecke modifications, you need a notion of Cartier divisors of $\mathcal{X}_S$ relative to the base, but where is your base? It can't be S or S/Frob_S because they are of char p.*

Answer 1.2.11. *No, at least not in the naive way. The correct way to relate $\mathcal{X}_S$ to char p objects is via its associated diamond $(\mathcal{X}_S)^\diamond$, which we will explain now.*

Construction 1.2.12 ([FS21, Sect. 6.2])**.** *For any commutative ring R, the (p-)tilt of R is*

$$R^\flat := \lim(\ldots \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R).$$

A priori this is only a multiplicative monoid. If R is equipped with a good enough complete topology, such as a perfectoid ring, then one can define a ring structure where the addition law is

$$(x^{(0)}, x^{(1)}, \ldots) + (y^{(0)}, y^{(1)}, \ldots) := (z^{(0)}, z^{(1)}, \ldots),$$

where

$$z^{(i)} := \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n}.$$

In particular, when R is a perfectoid ring, we obtain a char p perfectoid ring $R^\flat$*. We say R is a* untilt *of $R^\flat$.*

We define

$$\mathrm{Spd}(R, R^+) := \mathrm{Spa}(R, R^+)^\flat.$$

Using gluing, we can define $X^\flat$ for any prefectoid space X.

Remark 1.2.13. *For perfectoid ring R, we have $(R^\flat)^\circ \simeq (R^\circ)^\flat$.*

Example 1.2.14. *Any char p prefectoid ring R is the tilt of itself, and is the only char p untilt of itself. But there are char 0 untilts.*

Example 1.2.15. *The tilt of $\mathbb{Q}_p^{\mathrm{cycl}}$ is $\mathbb{F}_p((t^{1/p^\infty}))$.*

Theorem 1.2.16 (Tilting Equivalence)**.** *For any perfectoid space X, the functor $Y \mapsto Y^\flat$ induces an equivalence between the categories of perfectoid spaces over X and $X^\flat$. This equivalence preserves (finite) étale covers.*

Definition 1.2.17. *For any pre-adic space X, define $X^{\diamond}$ to be the prestack*

$$X^{\diamond} : \mathrm{Perfd}_p^{\mathrm{op}} \to \mathrm{Set}, \ S \mapsto \bigsqcup_{S^{\sharp} \in \mathrm{Untilt}(S)} \mathrm{Maps}(S^{\sharp}, X).$$

Remark 1.2.18. *In fact, Lin thinks the following is correct. Consider*

$$\mathrm{PreStk} \coloneqq \mathrm{Funct}(\mathrm{Perfd}^{\mathrm{op}}, \mathrm{Set}), \mathrm{PreStk}_p \coloneqq \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set})$$

and define

$$\diamond : \mathrm{PreStk} \to \mathrm{PreStk}_p$$

to be the unique colimit-preserving functor extending $\flat$. Then when restricted to pre-adic spaces, one recovers the above definition.

Theorem 1.2.19. *For pre-adic space X, the underlying topological spaces of X and $X^{\diamond}$ are canonically homeomorphic.*

Remark 1.2.20. *In fact, we have the following slogen: "$\diamond$ only remembers topological information."*

Example 1.2.21. $\mathrm{Spd}\,\mathbb{Z}_p \coloneqq (\mathrm{Spa}\,\mathbb{Z}_p)^{\diamond}$ *classifies all untilts;* $\mathrm{Spd}\,\mathbb{Q}_p \coloneqq (\mathrm{Spa}\,\mathbb{Q}_p)^{\diamond}$ *classifies all char 0 untilts.*

Example 1.2.22. *If X is already a perfectoid space, then $X^{\diamond} \simeq X^{\flat}$ by the tilting equivalence.*

Example 1.2.23. *For char p pre-adic space X, the functor $X \mapsto X^{\diamond}$ is just*

$$\mathrm{PreAdic}_p \to \mathrm{Funct}(\mathrm{PreAdic}_p^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set}).$$

This is because only char p untilts $S^{\sharp}$ can map to X.

Example 1.2.24. *By the tilting equivalence, if X is the quotient of $R \rightrightarrows Y$ of perfectoid spaces connected by pro-étale maps, then $X^{\diamond}$ is the quotient of $R^{\flat} \rightrightarrows Y^{\flat}$. Essentially, diamonds are defined to be such quotients. In fact, any nalytic pre-adic space, which means all its residue fields are not discrete, over $\mathbb{Z}_p$ can be written as such a quotient.*

Remark 1.2.25. *Yifei will talk about the pro-étale topology and explain why it is powerful.*

Example 1.2.26. *Unfortunately/fortunately, $\mathrm{Spa}\,\mathbb{Q}_p$ is not a perfectoid space but it has a perfectoid pro-étale cover $\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cycl}} \to \mathrm{Spa}\,\mathbb{Q}_p$ whose Galois group is $\mathbb{Z}_p^{\times}$. Hence*

$$\mathrm{Spd}\,\mathbb{Q}_p \simeq \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}/\underline{\mathbb{Z}_p^{\times}} \simeq \mathrm{Spa}\,\mathbb{F}_p((t^{1/p^{\infty}}))/\underline{\mathbb{Z}_p^{\times}},$$

where $\underline{\mathbb{Z}_p^{\times}}$ is the discrete group diamond.

Theorem 1.2.27. *For any char p perfectoid space S, we have*

$$(S \mathbin{\dot{\times}} \mathrm{Spa}\,\mathbb{Q}_p)^{\diamond} \simeq S \times \mathrm{Spd}\,\mathbb{Q}_p.$$

Warning 1.2.28. *This is not formal. But Lin thinks $\dot{\times}$ was, or should have been, designed to make this correct.*

Remark 1.2.29. *For a char p perfectoid space S, a map $S \to \mathrm{Spd}\,\mathbb{Q}_p$ provides a char 0 untilt $S^{\sharp}$, which will provide a closed immersion $S^{\sharp} \to \mathcal{Y}_S \coloneqq S \mathbin{\dot{\times}} \mathrm{Spa}\,\mathbb{Q}_p$ once we know the precise definition of the target. This is a Cartier divisor and so is the composition $S^{\sharp} \to \mathcal{X}_S$. Also, the latter only depends on the composition $S \to \mathrm{Spd}\,\mathbb{Q}_p \to \mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$. This suggests $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$ should be the moduli prestack of Cartier divisors on FF curves.*

In fact, as we have seen (or will see) in geometric Langlands, we use $\overline{\mathbb{F}_p}$-points on the curve to define Hecke modifications. Hence we should restrict our attension to S defined over $\overline{\mathbb{F}_p}$ rather than $\mathbb{F}_p$. The effect is to change $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$ to $\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\mathrm{Frob}$, where $\mathbb{Q}_p^{\mathrm{ur}} = \mathrm{Frac}(W(\overline{\mathbb{F}_p}))$ is the maximal unramified extension of $\mathbb{Q}_p$.

We can finally define the moduli of divisors on FF curves:

Definition 1.2.30. *The* moduli diamond of degree 1 closed Cartier divisors on FF curves *is*

$$\mathrm{Div}^1 := \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\mathrm{Frob}.$$

Warning 1.2.31. *Unlike in algebraic geometry,* Div^1 *is* not *the curve itself. They live in different characteristics. In fact, we do not have the absolute FF curve.*

Remark 1.2.32. *An amazing thing is this tilting/untilting game allows us to consider products of the "curve", or in fact, consider* $\mathrm{Div}^1 \times \cdots \times \mathrm{Div}^1$. *Unlike the self product of* $\mathrm{Spa}\,\mathbb{Q}_p$, *this product, which is taken in the category of diamonds, is not boring. Lin thinks this is essentially because* Perfd_p *lacks a final object but has products (which is false for* Perfd*).*

1.3. **Why Fargues–Fontaine curve?** Please read [FS21, Sect. I.11] (titled "The origin of the ideas"). For now, let me explain how the automorphic and Galois sides naturally appear in this geometrization picture.

For the Galois side:

Theorem 1.3.1. $\pi_1(\mathrm{Div}^1) \simeq W_E$.

Remark 1.3.2. *Heuristically this follows from the definition* $\mathrm{Div}^1 := \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\mathrm{Frob}$. *Indeed,* W_E *is the extension of* $\mathbb{Z}\langle\mathrm{Frob}\rangle$ *by* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{ur}})$.

Question 1.3.3. *Wait, didn't you say FF curves behaved like genus 0 curves?*

Answer 1.3.4. *Yes. But* $\mathcal{X}_C$ *is not defined over* C *or any algebraically closed field. Instead, we have:*

Theorem 1.3.5. $\Gamma(\mathcal{X}_C, \mathcal{O}) \simeq \mathbb{Q}_p$.

For the automorphic side, we consider the v-stack Bun_G whose values $\mathrm{Bun}_G(S)$ classify G-torsors on $\mathcal{X}_S$. Taeuk will explain the precise meaning of the following:

Theorem 1.3.6. Bun_G *has a stratification labelled by the poset* $B(G)$ *such that each stratum is of the form* $*/H$, *where* H *is a group diamond which is an extension of a discrete group* $M(\mathbb{Q}_p)$ *by a unipotent group, where* M *is an inner form of a Levi subgroup of* G.

Remark 1.3.7. *This is another example where FF curves behave like genus* 0.

Corollary 1.3.8. *For any* $\mathbb{Z}_l$-*algebra* Λ, *the category* $D(\mathrm{Bun}_G, \Lambda)$ *can be glued from categories* $\mathrm{Rep}_\Lambda(M(\mathbb{Q}_p))$ *for* M *being inner forms of Levi subgroups of* G.

Yifei will explain how to define $D(-, \Lambda)$ and play with them.

1.4. **What can be translated from Geometric Langlands?**

Answer 1.4.1. *Essentially any pure geometric constructions in Geometric Landlands can be or at least should be translated. Things already done in [FS21]: geometric Satake, Lafforgue's automorphic-to-Galois construction via shitukas, formulation of categorical Langlands conjecture, the spectral action...*

1.5. **What else?** The story is not complete without talking about p-adic Hodge theory. After all, FF curves were born during the study of Fontaine's peroid rings. If people want, we can make a digression to it.

REFERENCES

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*. Princeton University Press, 2020.