LECTURE 10

From this lecture on, we turn to the geometric side of the localization theory. The main player will be (algebraic) D-modules on the flag variety G/B. In this lecture, we introduce the basics of D-modules. There are many good references for this theory. For example, [HTT] is a thorough textbook, while [B] is a short notes.

1. Recollection: sheaf of differential operators

Recall the following definitions in algebraic geometry.

Definition 1.1. Let A be a k-algebra and M be an A-module. A **k-derivation** of A into M is a k-linear map $D : A \to M$ satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let $\mathsf{Der}_k(A, M)$ be the set of such k-derivations. This is naturally an A-module.

Proposition-Definition 1.2. *The functor $A{-}\mathsf{mod} \to A{-}\mathsf{mod}$, $M \mapsto \mathsf{Der}_k(A, M)$ is represented by an A-module $\Omega^1_{A/k}$, i.e.,*

$$\mathsf{Hom}_A(\Omega^1_{A/k}, M) \simeq \mathsf{Der}_k(A, M).$$

*We call $\Omega^1_{A/k}$ the **module of (Kähler) differentials** of A over k. In particular, the identity map on $\Omega^1_{A/k}$ corresponds to a k-derivative $d : A \to \Omega^1_{A/k}$, which we call the **universal k-derivative** of A.*

Construction 1.3. *For any homomorphism $f : A \to B$ and $M \in B{-}\mathsf{mod}$, there is an obvious A-linear map*

$$\mathsf{Der}_k(B, M) \to \mathsf{Der}_k(A, M),$$

where in the RHS we view M as an A-module by restricting along f. The two sides are represented by $\Omega^1_{B/k}$ and $B \otimes_A \Omega^1_{A/k}$ respectively. It follows that there is a B-linear map

$$B \underset{A}{\otimes} \Omega^1_{A/k} \to \Omega^1_{B/k}.$$

Lemma 1.4. *If $f : A \to B$ is a localization map, then $B \otimes_A \Omega^1_{A/k} \to \Omega^1_{B/k}$ is an isomorphism.*

Construction 1.5. *Let X be a k-scheme. The above lemma implies*

$$\Omega^1_{X/k}(U) \coloneqq \Omega^1_{\mathcal{O}_X(U)/k}$$

*defines a quasi-coherent $\mathcal{O}_X$-module $\Omega^1_{X/k}$. We call it the **sheaf of (Kähler) differentials**, or the **cotangent sheaf**, of X over k.*

Lemma 1.6. *If X is a smooth k-scheme of dimension n, then $\Omega^1_{X/k}$ is locally free of rank n.*

Construction 1.7. *Let X be a smooth k-scheme. The above lemme implies $\Omega^1_{X/k}$ is a vector bundle and in particular dualizable. We define the **tangent sheaf** $\mathcal{T}_X$ of X over k to be the dual vector bundle. By definition, we have*

$$\mathcal{T}_X(U) \simeq \mathsf{Der}_k(\mathcal{O}_X(U), \mathcal{O}_X(U)).$$

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Remark 1.8. The **tangent space** $T_{X,x}$ introduced in [Section 3, Lecture 3] can be identified with the stalk of $\mathcal{T}_X$ at x.

Definition 1.9. Let X be a k-scheme. The **tangent sheaf** $\mathcal{T}_X$ is the quasi-coherent sheaf on X

References

[B] Bernstein, Joseph. Algebraic theory of D-modules, 1984, abailable at `https://gauss.math.yale.edu/~il282/Bernstein_D_mod.pdf`.

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/catO.pdf`.

[H] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[HTT] Hotta, Ryoshi, and Toshiyuki Tanisaki. D-modules, perverse sheaves, and representation theory. Vol. 236. Springer Science & Business Media, 2007.