

## LECTURE 10

Last time, for a diagram  $u : K_1 \times K_2 \rightarrow C$ , we constructed a canonical morphism

$$\operatorname{colim}_{y \in K_2} \lim_{K_1} u(-, y) \rightarrow \lim_{x \in K_1} \operatorname{colim}_{K_2} u(x, -)$$

when assuming both sides exist. In this lecture, we discuss several cases when the above morphism is invertible.

### 1. FINITE DIAGRAMS

**Definition 1.1.** We say a simplicial set  $K$  is **finite** if it has finitely many non-degenerate simplices.

We say an  $\infty$ -category  $K$  is **essentially finite** if it can be represented by a quasi-category which is a finite simplicial set.

1.2. Recall we say an ordinary category  $K$  is essentially finite iff the set of equivalence classes of objects and the set of morphisms in  $K$  are finite.

**Exercise 1.3.** Let  $K$  be an essentially finite  $\infty$ -category. Show that  $hK$  is an essentially finite ordinary category.

**Exercise 1.4.** Let  $K$  be an ordinary category. If  $K$  is essentially finite as an  $\infty$ -category, then it is essentially finite as an ordinary category.

**Warning 1.5.** The converse is not true: even a finite ordinary category may fail to be essentially finite as an  $\infty$ -category.

**Exercise 1.6.** Let  $\operatorname{Idem}$  be the ordinary category defined by

- There is a unique object  $*$ ;
- $\operatorname{Hom}(*, *) := \{\operatorname{id}, e\}$ , with  $e \circ e = e$ .

Show that  $N_\bullet(\operatorname{Idem})$  is not categorically equivalent to any finite simplicial set.

**Exercise 1.7.** Let  $\mathbb{B}(\mathbb{Z}/2\mathbb{Z})$  be the ordinary category defined by

- There is a unique object  $*$ ;
- $\operatorname{Hom}(*, *) := \{\operatorname{id}, f\}$ , with  $f \circ f = \operatorname{id}$ .

Show that  $N_\bullet(\mathbb{B}(\mathbb{Z}/2\mathbb{Z}))$  is not categorically equivalent to any finite simplicial set.

1.8. Nevertheless, for finite partially ordered sets, the above abnormality does not appear.

**Exercise 1.9.** Let  $J$  be a finite partially ordered set, viewed as an ordinary category. Show that  $N_\bullet(J)$  is a finite simplicial set.

**Proposition 1.10** (Ker.02NB). For any simplicial set  $K$ , there exists a partially ordered set  $J$  and an initial morphism  $N_\bullet(J) \rightarrow K$ . If  $K$  is finite, we can take  $J$  to be finite.

**Corollary-Definition 1.11.** Let  $C$  be an  $\infty$ -category. The following are equivalent:

- $\mathcal{C}$  admits limits indexed by finite simplicial sets.
- $\mathcal{C}$  admits limits indexed by essentially finite  $\infty$ -categories.
- $\mathcal{C}$  admits limits indexed by finite partially ordered sets.

We say  $\mathcal{C}$  **admits finite limits** if it satisfies the above conditions.

Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\infty$ -categories such that  $\mathcal{C}$  admits finite limits. The following are equivalent:

- $F$  preserves limits indexed by finite simplicial sets.
- $F$  preserves limits indexed by essentially finite  $\infty$ -categories.
- $F$  preserves limits indexed by finite partially ordered sets.

We say  $F$  is **left exact** if it satisfies the above conditions.

Dually, for a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that  $\mathcal{C}$  admits finite colimits, we say  $F$  is **right exact** if it preserves finite colimits.

**Warning 1.12.** Let  $\mathcal{C}$  be an  $\infty$ -category that admits finite limits. It may not admit limits indexed by finite ordinary categories. For example,  $\mathcal{C}$  may fail to be idempotent complete.

**Remark 1.13.** In future lectures, we will define left/right exact functors between general  $\infty$ -categories.

**Proposition 1.14.** Let  $\mathcal{C}$  be an  $\infty$ -category. The following are equivalent:

- (i)  $\mathcal{C}$  admits finite limits.
- (ii)  $\mathcal{C}$  admits fiber products and a final object.
- (iii)  $\mathcal{C}$  admits finite products and equalizers.

We also have similar results for left exact functors.

*Sketch.* The implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) are standard. To show (ii) $\Rightarrow$ (i), let  $u : K \rightarrow \mathcal{C}$  be a finite diagram. We need to show  $\lim u$  exists. When  $K = \emptyset$ , this is the final object of  $\mathcal{C}$ . If  $K$  is nonempty, we can find a pushout square in  $\mathbf{Set}_\Delta$

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & K' \\ \downarrow \scriptstyle c & & \downarrow \scriptstyle c \\ \Delta^n & \xrightarrow{\quad c \quad} & K. \end{array}$$

Note that this is also a homotopy pushout square in  $\mathbf{Set}_\Delta^{\mathbf{Joyal}}$ . By the theorem on decomposition of diagrams (Ker.03DB), we have

$$\lim u \xrightarrow{\cong} \lim u|_{K'} \times_{\lim u|_{\partial\Delta^n}} \lim u|_{\Delta^n}$$

where the source exists if the target does. By induction, we reduce to prove the claim when  $K = \Delta^n$ , which follows from the fact that  $\Delta^0 \xrightarrow{0} \Delta^n$  is initial.  $\square$

## 2. FILTERED $\infty$ -CATEGORIES

**Definition 2.1.** We say an  $\infty$ -category  $\mathcal{C}$  is **filtered** if any finite diagram  $u : K \rightarrow \mathcal{C}$  can be extended to a diagram  $K^\triangleright \rightarrow \mathcal{C}$ .

Dually, we say  $\mathcal{C}$  is **cofiltered** if for any finite diagram  $u : K \rightarrow \mathcal{C}$  can be extended to a diagram  $K^\triangleleft \rightarrow \mathcal{C}$ .

2.2. Note that  $\mathcal{C}$  is filtered iff  $\mathcal{C}^{\text{op}}$  is cofiltered. Hence we will focus on filtered  $\infty$ -categories.

**Exercise 2.3.** *Being filtered is invariant under equivalences.*

2.4. An induction argument, which is similar to but more elaborate than that in the proof of Proposition 1.14, gives the following result:

**Lemma 2.5** (Ker.02Q0). *An  $\infty$ -category  $\mathcal{C}$  is filtered iff any diagram  $u : \partial\Delta^n \rightarrow \mathcal{C}$  with  $n \geq 0$ <sup>1</sup> can be extended to a diagram  $(\partial\Delta^n)^{\triangleright} \rightarrow \mathcal{C}$ .*

**Remark 2.6.** *Note that  $(\partial\Delta^n)^{\triangleright} \simeq \Lambda_{n+1}^{n+1}$ .*

2.7. Recall we say an ordinary category  $\mathcal{C}$  is filtered if

- (0) It is nonempty.
- (1) For any objects  $x_1, x_2$  in  $\mathcal{C}$ , there exists an object  $x$  equipped with morphisms  $x_i \rightarrow x$ .
- (2) For any diagram  $x \rightrightarrows y$  in  $\mathcal{C}$ , there exists  $y \rightarrow z$  such that the two compositions  $x \rightrightarrows y \rightarrow z$  are equal.

Note that condition (i) corresponds to the extension problem for  $K = \partial\Delta^i$ .

**Proposition 2.8** (Ker.02PS). *An  $\infty$ -category  $\mathcal{C}$  is filtered iff it satisfies the following conditions:*

- (0) *It is nonempty.*
- (1) *For any objects  $x_1, x_2$  in  $\mathcal{C}$ , there exists an object  $x$  equipped with morphisms  $x_i \rightarrow x$ .*
- (2) *For any objects  $x, y$  in  $\mathcal{C}$  and any diagram  $\partial\Delta^n \rightarrow \text{Maps}(x, y)$  with  $n \geq 1$ , there exists a morphism  $y \rightarrow z$  such that the composition*

$$\partial\Delta^n \rightarrow \text{Maps}(x, y) \rightarrow \text{Maps}(x, z)$$

*is null-homotopic, i.e., equivalent to a constant diagram<sup>2</sup>.*

**Remark 2.9.** *Roughly speaking, for  $n \geq 2$ , the extension problem*

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{u} & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

*corresponds to an extension problem*

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \text{Maps}(x, y) \\ \downarrow & \nearrow & \\ \Delta^{n-1} & & \end{array}$$

*such that  $x = u(0)$  and  $y = u(n)$ .*

**Exercise 2.10.** *Let  $\mathcal{C}$  be a filtered  $\infty$ -category. Show that  $\mathbf{h}\mathcal{C}$  is a filtered ordinary category.*

<sup>1</sup>By definition,  $\partial\Delta^0 = \emptyset$ .

<sup>2</sup>Exercise: for a Kan complex  $K$ , a morphism  $\partial\Delta^n \rightarrow K$  is null-homotopic iff it factors through  $\Delta^n$ .

**Exercise 2.11.** Let  $\mathcal{C}$  be an ordinary category. Show that  $\mathcal{C}$  is filtered as an ordinary category iff it is filtered as an  $\infty$ -category.

**Example 2.12.** An  $\infty$ -category is filtered if it admits a final object.

**Example 2.13.** An  $\infty$ -category is filtered if it admits finite colimits.

**Exercise 2.14.** The category  $\mathbf{Idem}$  is filtered.

### 3. FILTERED COLIMITS AND FINITE LIMITS

**Theorem 3.1** (Ker.05XS). Let  $\mathcal{C}$  be a small  $\infty$ -category. Then  $\mathcal{C}$  is filtered iff the functor

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathrm{Grpd}_{\infty}) \rightarrow \mathrm{Grpd}_{\infty}$$

preserves finite limits.

**Warning 3.2.** The claim may be false if  $\mathrm{Grpd}_{\infty}$  is replaced by general  $\infty$ -categories.

**Exercise 3.3.** Consider the discrete topological space  $\mathbb{Z}$  and its one-point-compactification  $\mathbb{Z} \cup \{\infty\}$ <sup>3</sup>. Consider the partially ordered set  $\mathcal{P}$  of closed subsets of  $\mathbb{Z} \cup \{\infty\}$ . Show that filtered colimits in  $\mathcal{P}$  may not commute with finite limits.

**Corollary 3.4.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a final functor between  $\infty$ -categories. If  $\mathcal{C}$  is filtered, so is  $\mathcal{D}$ .

**Challenge 3.5.** Can you find a proof of the above corollary without using Theorem 3.1?

3.6. Recall a partially ordered set  $J$  is *directed* if any finite subset of it admits an upper bound in  $J$ .

**Proposition 3.7** (Ker.02QA). Let  $\mathcal{C}$  be an  $\infty$ -category. The following are equivalent:

- $\mathcal{C}$  is filtered.
- There exists a directed partially ordered set  $J$  and a final functor  $J \rightarrow \mathcal{C}$ .

**Proposition 3.8.** Let  $\mathcal{C}$  be a filtered  $\infty$ -category and  $u : K \rightarrow \mathcal{C}$  be any finite diagram. Then the forgetful functor  $\mathcal{C}_{u/} \rightarrow \mathcal{C}$  is final.

*Proof.* By Quillen's Theorem A, we only need to show  $\mathcal{C}_{u/} \times_{\mathcal{C}} \mathcal{C}_{x/}$  is weakly contractible for any  $x \in \mathcal{C}$ . We have

$$\mathcal{C}_{u/} \times_{\mathcal{C}} \mathcal{C}_{x/} \simeq \mathcal{C}_{u \sqcup x/}$$

where  $u \sqcup x : K \sqcup \Delta^0 \rightarrow \mathcal{C}$  is the disjoint union of  $u$  and  $x$ . Now the claim follows from the following two exercises.  $\square$

**Exercise 3.9.** A filtered  $\infty$ -category is weakly contractible. Hint:  $K \times \Delta^1 \rightarrow K^{\triangleright}$ .

**Proposition 3.10.** Let  $\mathcal{C}$  be an  $\infty$ -category. The following are equivalent:

- (i)  $\mathcal{C}$  is filtered.
- (ii) For any finite diagram  $u : K \rightarrow \mathcal{C}$ , the  $\infty$ -category  $\mathcal{C}_{u/}$  is filtered.
- (iii) For any finite diagram  $u : K \rightarrow \mathcal{C}$ , the  $\infty$ -category  $\mathcal{C}_{u/}$  is weakly contractible.
- (iv) For any finite  $K \in \mathbf{Set}_{\Delta}$ , the diagonal functor  $\mathcal{C} \rightarrow \mathrm{Fun}(K, \mathcal{C})$ ,  $x \mapsto \underline{x}$  is final.

<sup>3</sup>An closed subset of  $\mathbb{Z} \cup \{\infty\}$  is either finite or contains  $\infty$ .

*Sketch.* The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) are left as exercises. We will prove (ii) $\Leftrightarrow$ (iii). By Quillen's Theorem A, (ii) is equivalent to:

$$\mathbf{C} \times_{\mathbf{Fun}(K, \mathbf{C})} \mathbf{Fun}(K, \mathbf{C})^{u/}$$

is weakly contractible for any  $u : K \rightarrow \mathbf{C}$ . It is easy to see the above fiber product is equivalent to  $\mathbf{C}^{u/}$ , which implies the claim.  $\square$

#### 4. SIFTED DIAGRAM

**Definition 4.1.** Let  $K$  be a simplicial set. We say  $K$  is **sifted** if for any finite set  $I$ , the diagonal morphism  $K \rightarrow K^I$  is final.

Dually, we say  $K$  is **cosifted** if for any finite set  $I$ , the diagonal morphism  $K \rightarrow K^I$  is initial.

4.2. Note that  $\mathbf{C}$  is sifted iff  $\mathbf{C}^{\text{op}}$  is cosifted. Hence we will focus on sifted  $\infty$ -categories.

**Exercise 4.3.** Being filtered is invariant under equivalences.

**Example 4.4.** Any filtered  $\infty$ -category is sifted.

**Exercise 4.5.** Show that a simplicial set  $K$  is sifted iff it is nonempty and  $K \rightarrow K \times K$  is final. Hint: If  $K \rightarrow K \times K$  is a weak homotopy equivalent and  $K \neq \emptyset$ , then  $K$  is weakly contractible.

**Corollary 4.6.** An  $\infty$ -category  $\mathbf{C}$  is sifted iff it is nonempty and  $\mathbf{C}_{x/} \times_{\mathbf{C}} \mathbf{C}_{y/}$  is weakly contractible for any pair of objects  $x, y \in \mathbf{C}$ .

**Warning 4.7.** In classical category theory, an ordinary category is called sifted if it is nonempty and  $\mathbf{C}_{x/} \times_{\mathbf{C}} \mathbf{C}_{y/}$  is connected for any pair of objects  $x, y \in \mathbf{C}$ . A sifted ordinary category may fail to be sifted as an  $\infty$ -category.

**Exercise 4.8.** Let  $\Delta_{\leq 1} \subset \Delta$  be the full subcategory consisting of  $[0]$  and  $[1]$ . Show that  $\Delta_{\leq 1}^{\text{op}}$  is sifted as an ordinary category but not as an  $\infty$ -category.

**Exercise 4.9.** Show that  $\Delta^{\text{op}}$  is sifted as an  $\infty$ -category.

**Proposition 4.10.** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a final functor between  $\infty$ -categories. If  $\mathbf{C}$  is sifted, so is  $\mathbf{D}$ .

*Proof.* For any finite set  $I$ , consider the commutative diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ \delta_1 \downarrow & \searrow & \downarrow \delta_2 \\ \mathbf{C}^I & \xrightarrow{F^I} & \mathbf{D}^I \end{array}$$

Since  $F$  is final, so is  $F^I$  ([Lecture 7, Proposition 3.16]). Since  $\mathbf{C}$  is sifted,  $\delta_1$  is final. It follows that  $\delta_2$  is also final as desired ([Lecture 7, Exercise 3.13]).  $\square$

**Exercise 4.11.** Let  $K$  be a sifted simplicial set. Suppose  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{E}$  are  $\infty$ -categories that admit  $K$ -indexed colimits, and  $F : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{E}$  is a functor that preserves  $K$ -indexed colimits in each variable<sup>4</sup>. Show that  $F$  preserves  $K$ -indexed colimits.

<sup>4</sup>This means  $F(c, -)$  and  $F(-, d)$  preserve  $K$ -indexed colimits for any  $c \in \mathbf{C}$  and  $d \in \mathbf{D}$ .

## 5. SIFTED COLIMITS AND FINITE PRODUCTS

**Theorem 5.1** (HTT.5.5.8.11, Ker.05XM). *Let  $\mathcal{C}$  be a small  $\infty$ -category. Then  $\mathcal{C}$  is sifted iff the functor*

$$\mathrm{colim} : \mathrm{Fun}(\mathcal{C}, \mathrm{Grpd}_\infty) \rightarrow \mathrm{Grpd}_\infty$$

*preserves finite products.*

## 6. UNIVERSALITY OF COLIMITS

**Construction 6.1.** *Let  $\mathcal{C}$  be an  $\infty$ -category that admits fiber products. Let  $X \rightarrow Y$  be a functor in  $\mathrm{Grpd}_\infty$ . As in classical category theory, we have an adjunction*

$$(6.1) \quad (\mathrm{Grpd}_\infty)_{/X} \rightleftarrows (\mathrm{Grpd}_\infty)_{/Y}$$

*where:*

- *The left adjoint is compatible with the forgetful functors;*
- *The right adjoint is given by  $- \times_Y X$ .*

**Remark 6.2.** *Alternatively, the above right adjoint can be constructed via the simplicial model category  $\mathrm{Set}_\Delta^{\mathrm{KQ}}$ .*

**Theorem 6.3** (Ker.05V5). *The functor*

$$- \times_Y X : (\mathrm{Grpd}_\infty)_{/Y} \rightarrow (\mathrm{Grpd}_\infty)_{/X}$$

*preserves small colimits.*

**Exercise 6.4.** *Let  $u : K \rightarrow \mathrm{Grpd}_\infty$ ,  $i \mapsto X_i$  be a small diagram and write  $X := \mathrm{colim} u$ . Show that there is a canonical isomorphism*

$$\mathrm{colim}_{i \in K} (X_i \times_X Y) \simeq Y.$$

APPENDIX A. FILTERED COLIMITS OF  $\infty$ -CATEGORIES

**Proposition A.1.** *Let  $\mathcal{C}$  be a small filtered ordinary category. Then any colimit diagram  $\bar{u} : \mathcal{C}^\triangleright \rightarrow \mathrm{Set}_\Delta$  is also a homotopy colimit diagram in  $\mathrm{Set}_\Delta^{\mathrm{Joyal}}$ .*

**Exercise A.2.** *The functor  $\mathrm{QCat} \rightarrow \mathrm{Cat}_\infty$  preserves small filtered  $\infty$ -colimits.*

**Exercise A.3.** *Let  $K$  be a small filtered  $\infty$ -category and  $K \rightarrow \mathrm{Cat}_\infty$ ,  $i \mapsto C_i$  be a diagram. Show that any object in  $\mathcal{C} := \mathrm{colim}_K C_i$  is equivalent to  $\mathrm{ins}_i(x)$  for some  $i \in K$  and  $x \in C_i$ .*

**Exercise A.4.** *Let  $K$  be a small filtered  $\infty$ -category and  $K \rightarrow \mathrm{Cat}_\infty$ ,  $i \mapsto C_i$  be a diagram. For  $i, j \in K$  and  $x \in C_i$ ,  $y \in C_j$ , show that*

$$\mathrm{Maps}_{\mathcal{C}}(\mathrm{ins}_i(x), \mathrm{ins}_j(y)) \xrightarrow{\simeq} \mathrm{colim}_{k \in K_{(i,j)}/} \mathrm{Maps}_{C_k}(x_k, y_k),$$

*where*

- $K_{(i,j)}/$  is the coslice  $\infty$ -category for  $\Delta^0 \sqcup \Delta^0 \xrightarrow{(i,j)} K$ ;
- For  $k \in K_{(i,j)}/$ ,  $x_k$  is the image of  $x$  under the functor  $C_i \rightarrow C_k$ , and similarly for  $y_k$ .

**A.5. Suggested readings.** Ker.03DD.

## APPENDIX B. STRONG UNIVERSALITY OF COLIMITS

**Theorem B.1.** *Let  $K \rightarrow \mathbf{Grpd}_\infty$ ,  $i \mapsto \mathbf{X}_i$  be a small diagram. Consider  $\mathbf{X} := \operatorname{colim}_{i \in K} \mathbf{X}_i$ . Then there is a canonical equivalence*

$$(B.1) \quad (\mathbf{Grpd}_\infty)_{/\mathbf{X}} \rightarrow \lim_{i \in K} (\mathbf{Grpd}_\infty)_{/\mathbf{X}_i},$$

where each evaluation functor is

$$- \times_{\mathbf{X}} \mathbf{X}_i : (\mathbf{Grpd}_\infty)_{/\mathbf{X}} \rightarrow (\mathbf{Grpd}_\infty)_{/\mathbf{X}_i}.$$

**Exercise B.2.** *Use Theorem 6.3 to show (B.1) is fully faithful.*

**Exercise B.3.** *Show that Theorem 6.3 remains true if  $\mathbf{Grpd}_\infty$  is replaced by  $\mathbf{Set}$ , but Theorem B.1 would fail.*

**B.4. Suggested readings.** Ker.05SB.