

LECTURE 24

In this lecture, we define and study several symmetric monoidal ∞ -categories.

1. CARTESIAN SYMMETRIC MONOIDAL STRUCTURE

1.1. Let \mathcal{C} be an ∞ -category that admits finite products. In this section, we endow \mathcal{C} with a symmetric monoidal structure $\mathcal{C}^\times \rightarrow \mathbf{Comm}^\otimes$ such that the multiplication functor is given by

$$\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}, (X, Y) \mapsto X \times Y.$$

Definition 1.2. We say a symmetric monoidal structure $\mathcal{C}^\otimes \rightarrow \mathbf{Comm}^\otimes$ on \mathcal{C} is **Cartesian** if it satisfies the following properties:

- (1) The unit object $1_{\mathcal{C}} \in \mathcal{C}$ is final.
- (2) For objects $X, Y \in \mathcal{C}$, the morphisms

$$X \xleftarrow{\simeq} X \otimes 1_{\mathcal{C}} \leftarrow X \otimes Y \rightarrow 1_{\mathcal{C}} \otimes Y \xrightarrow{\simeq} Y$$

exhibits $X \otimes Y$ as a product of X and Y in \mathcal{C} .

Theorem 1.3 (HA.2.4.1.19). The functor $\mathcal{C}^\otimes \mapsto \mathcal{C}$ defines an equivalence between

- (i) The ∞ -category of (small) Cartesian symmetric monoidal ∞ -categories, and symmetric monoidal functors between them;
- (ii) The ∞ -category of (small) ∞ -categories that admit finite products, and finite-product-preserving functors between them.

1.4. In particular, for an ∞ -category \mathcal{C} that admits finite products, there is an essentially unique Cartesian symmetric monoidal structure on it, which we denote by $\mathcal{C}^\times \rightarrow \mathbf{Fin}_*$. Note that when using this notation, we need to equip it with an identification $\mathcal{C}_{\langle 1 \rangle}^\times \simeq \mathcal{C}$.

1.5. We can equip \mathbf{Cat}_∞ with the Cartesian symmetric monoidal structure $\mathbf{Cat}_\infty^\times \rightarrow \mathbf{Fin}_*$ and consider \mathcal{O} -algebra object in it. Note that such objects are *a priori* different from \mathcal{O} -monoidal ∞ -categories: the former is a functor $\mathcal{O}^\otimes \rightarrow \mathbf{Cat}_\infty^\times$ while the latter is a functor $\mathcal{O}^\otimes \rightarrow \mathbf{Cat}_\infty$ (via straightening). To compare these notions, we need a functor

$$\mathbf{Cat}_\infty^\times \rightarrow \mathbf{Cat}_\infty$$

that sends an object $(D_1, \dots, D_n) \in (\mathbf{Cat}_\infty^\times)_{\langle n \rangle}$ to $\prod_{i=1}^n D_i \in \mathbf{Cat}_\infty$. As a sanity check, note that the inert morphism $(D_1, \dots, D_n) \rightarrow D_k$ can be sent to the projection functor $D_1 \times \dots \times D_n \rightarrow D_k$. It turns out this can be generalized to any Cartesian symmetric monoidal structure \mathcal{C}^\times , including $\mathbf{Cat}_\infty^\times$.

Exercise 1.6. Convince yourself that there is no similar functor $\mathcal{C}^\otimes \rightarrow \mathcal{C}$ for general symmetric monoidal ∞ -category \mathcal{C} .

1.7. To give the precise construction of the desired functor $C^\times \rightarrow C$, we need some terminologies.

Definition 1.8. Let $p : O^\otimes \rightarrow \text{Fin}_*$ be an ∞ -operad and D be an ∞ -category that admits finite products. Let

$$\text{Mon}_O(D) \subseteq \text{Fun}(O^\otimes, D)$$

be the full sub- ∞ -category consisting of functors $M : O^\otimes \rightarrow D$ such that:

- (i) For any multi-color $X = (X_1, \dots, X_n)$ in $O_{\langle n \rangle}^\otimes$, the canonical morphisms $M(X) \rightarrow M(X_i)$ induce an equivalence

$$M(X) \xrightarrow{\cong} \prod_{i=1}^n M(X_i).$$

Objects in $\text{Mon}_O(C)$ are called **O-monoids** in C .

Example 1.9. When $C = \text{Cat}_\infty$, an O-monoid in Cat_∞ is just an O-monoidal ∞ -category via straightening.

Definition 1.10. Let $p : O^\otimes \rightarrow \text{Fin}_*$ be a symmetric monoidal ∞ -category and D be an ∞ -category that admits finite products. Let

$$\text{Fun}^\times(O^\otimes, D) \subseteq \text{Mon}_O(D)$$

be the full sub- ∞ -category consisting of O-monoids $M : O^\otimes \rightarrow D$ such that:

- (ii) For any p -coCartesian morphism $X = (X_1, \dots, X_n) \rightarrow Y$ over the unique active morphism $\langle n \rangle \rightarrow \langle 1 \rangle$, the image under M is an equivalence

$$M(X) \xrightarrow{\cong} M(Y).$$

Remark 1.11. Informally, (i) and (ii) say that M sends $\otimes_{i=1}^n X_i$ to $\prod_{i=1}^n M(X_i)$.

Proposition 1.12 (HA.2.4.1.6). Let C and D be ∞ -categories that admit finite products. Then restriction along $C \rightarrow C^\times$ defines an equivalence

$$\text{Fun}^\times(C^\times, D) \xrightarrow{\cong} \text{Fun}^\times(C, D),$$

where

$$\text{Fun}^\times(C, D) \subset \text{Fun}(C, D)$$

is the full sub- ∞ -category of functors $C \rightarrow D$ that preserves finite products.

1.13. It follows that there is an essentially unique functor

$$\pi : C^\times \rightarrow C$$

in $\text{Fun}^\times(C^\times, C)$ such that the composition $C \rightarrow C^\times \xrightarrow{\pi} C$ is the identity functor.

Proposition 1.14 (HA.2.4.1.7). Let C be an ∞ -category that admits finite products. We equip C with the Cartesian symmetric monoidal structure. Then:

- (1) For any ∞ -operad $O^\otimes \rightarrow \text{Fin}_*$, composing with $\pi : C^\times \rightarrow C$ induces an equivalence

$$\text{Alg}_O(C) \xrightarrow{\cong} \text{Mon}_O(C).$$

- (2) For any symmetric monoidal ∞ -category $O^\otimes \rightarrow \text{Fin}_*$, composing with $\pi : C^\times \rightarrow C$ induces an equivalence

$$\text{Fun}^\otimes(O, C) \xrightarrow{\cong} \text{Fun}^\times(O^\otimes, C).$$

Remark 1.15. Note that (2) implies the functor in Theorem 1.3 is fully faithful. Indeed, in HA, Lurie first gave an explicit construction of a Cartesian symmetric monoidal structure \mathcal{C}^\times , then verified it satisfies the claims in Proposition 1.12 and 1.14, and finally deduced Theorem 1.3 from them. For details, see HA.2.4.1.

Remark 1.16. In HA, Lurie used the following terminologies:

- Objects in $\text{Mon}_O(\mathcal{C})$ are also called lax Cartesian structures on O^\otimes ;
- Objects in $\text{Fun}^\times(O^\otimes, \mathcal{D})$ are called weak Cartesian structures on O^\otimes .
- A Cartesian structure on O^\otimes is a weak one such that the composition $O \rightarrow O^\otimes \rightarrow \mathcal{D}$ is an equivalence.

2. COCARTESIAN SYMMETRIC MONOIDAL STRUCTURE

2.1. Using the automorphism $\text{Cat}_\infty \rightarrow \text{Cat}_\infty$, $\mathcal{C} \mapsto \mathcal{C}^{\text{op}}$, we obtain the dual version of Definition 1.2 and Theorem 1.3:

Definition 2.2. We say a symmetric monoidal structure $\mathcal{C}^\otimes \rightarrow \text{Fin}_*$ on \mathcal{C} is **co-Cartesian** if it satisfies the following properties:

- (1) The unit object $1_{\mathcal{C}} \in \mathcal{C}$ is initial.
- (2) For objects $X, Y \in \mathcal{C}$, the morphisms

$$X \xrightarrow{\sim} X \otimes 1_{\mathcal{C}} \rightarrow X \otimes Y \leftarrow 1_{\mathcal{C}} \otimes Y \xleftarrow{\sim} Y$$

exhibits $X \otimes Y$ as a coproduct of X and Y in \mathcal{C} .

Corollary 2.3. The functor $\mathcal{C}^\otimes \mapsto \mathcal{C}$ defines an equivalence between

- (i) The ∞ -category of (small) coCartesian symmetric monoidal ∞ -categories, and symmetric monoidal functors between them;
- (ii) The ∞ -category of (small) ∞ -categories that admit finite coproducts, and finite-coproduct-preserving functors between them.

2.4. For an ∞ -category \mathcal{C} that admits finite coproducts, we denote the coCartesian symmetric monoidal structure on it by $\mathcal{C}^\sqcup \rightarrow \text{Fin}_*$.

2.5. Note however that by Exercise 1.6, Proposition 1.12 and 1.14 cannot survive in the coCartesian setting. As a compensate, we have the following result:

Proposition 2.6 (HA.2.4.3.9). Let \mathcal{C} be an ∞ -category that admits finite coproducts. We equip \mathcal{C} with the coCartesian symmetric monoidal structure. Then for any unital ∞ -operad O^\otimes , the forgetful functor

$$\text{Alg}_O(\mathcal{C}) \rightarrow \text{Fun}(O, \mathcal{C})$$

is an equivalence.

Remark 2.7. Let $A : O^\otimes \rightarrow \mathcal{C}^\sqcup$ be an O -algebra. The above proposition says the functor A is determined by its restriction on O . Informally speaking, for any active morphism $X := (X_1, \dots, X_n) \rightarrow Y$ in O^\otimes , its image $A(X) \rightarrow A(Y)$ can be described as follows. First, this morphism essentially uniquely factors as

$$A(X) \xrightarrow{f} \bigsqcup_{i=1}^n A(X_i) \xrightarrow{g} A(Y)$$

such that f is coCartesian over Fin_* while g is contained in \mathcal{C} . The morphism g is determined by morphisms $A(X_i) \rightarrow A(Y)$, which by definition, must be the image of $X_i \rightarrow (X_1, \dots, X_n) \rightarrow Y$ under A . Here the morphism $X_i \rightarrow (X_1, \dots, X_n)$ is guaranteed by the assumption that O^\otimes is unit.

Exercise 2.8. Describe the essential image of $\text{Fun}^\otimes(\mathcal{O}, \mathcal{C}) \subseteq \text{Alg}_\mathcal{O}(\mathcal{C})$ in $\text{Fun}(\mathcal{O}, \mathcal{C})$.

3. TENSOR PRODUCTS OF ∞ -CATEGORIES

Theorem-Definition 3.1 (HA.4.8.1.15). Let $\widehat{\text{Cat}}_\infty$ be equipped with the Cartesian symmetric monoidal structure. Consider the 1-full sub- ∞ -category

$$(\text{Pr}^\text{L})^\otimes \subseteq (\widehat{\text{Cat}}_\infty)^\times$$

defined as follows:

- An object (C_1, \dots, C_n) is contained in $(\text{Pr}^\text{L})^\otimes$ iff each C_k is presentable.
- A morphism $(C_1, \dots, C_m) \rightarrow (D_1, \dots, D_n)$ defined over $\alpha : \langle m \rangle \rightarrow \langle n \rangle$ is contained in $(\text{Pr}^\text{L})^\otimes$ iff for each $j \in \langle n \rangle^\circ$, the corresponding functor

$$\prod_{i \in \alpha^{-1}(j)} C_i \rightarrow D_j$$

preserves colimits in each factor.

Then $(\text{Pr}^\text{L})^\otimes \rightarrow \text{Fin}_*$ defines a symmetric monoidal structure on Pr^L . We call the multiplication functor

$$- \otimes - : \text{Pr}^\text{L} \times \text{Pr}^\text{L} \rightarrow \text{Pr}^\text{L}$$

the *Lurie tensor product*.

3.2. By definition, for $C_i \in \text{Pr}^\text{L}$, there is a coCartesian morphism $(C_1, \dots, C_n) \rightarrow \otimes_{i=1}^n C_i$ in $(\text{Pr}^\text{L})^\otimes \rightarrow \text{Fin}_*$. This morphism is almost *never* coCartesian in the bigger fibration $(\widehat{\text{Cat}}_\infty)^\times \rightarrow \text{Fin}_*$. It follows that we have an essentially unique factorization in $(\widehat{\text{Cat}}_\infty)^\times$

$$(C_1, \dots, C_n) \rightarrow \prod_{i=1}^n C_i \rightarrow \bigotimes_{i=1}^n C_i.$$

such that the first morphism is coCartesian and the second is contained in $\widehat{\text{Cat}}_\infty$.

Exercise 3.3. For any $D \in \text{Pr}^\text{L}$, precomposing with $\prod_{i=1}^n C_i \rightarrow \otimes_{i=1}^n C_i$ induces an equivalence between the spaces of

- colimit-preserving functors $\otimes_{i=1}^n C_i \rightarrow D$;
- functors $\prod_{i=1}^n C_i \rightarrow D$ that preserve colimits in each factor.

The same is true if we consider ∞ -categories of these functors.

Exercise 3.4. Show that $\text{Spc} \in \text{Pr}^\text{L}$ is the unit object. What is the functor

$$\text{Spc} \times \text{Spc} \rightarrow \text{Spc} \otimes \text{Spc} \simeq \text{Spc}?$$

Exercise 3.5. Show that for any $C, D, E \in \text{Pr}^\text{L}$

$$\text{LFun}(C \otimes D, E) \simeq \text{LFun}(C, \text{LFun}(D, E)).$$

In particular, $\text{LFun}(-, -)$ calculates the *internal mapping objects* with respect to the Lurie tensor product.

Exercise 3.6. For small ∞ -categories C_0 and D_0 , show that

$$\text{PShv}(C_0) \otimes \text{PShv}(D_0) \simeq \text{PShv}(C_0 \times D_0).$$

Theorem 3.7 (HA.4.8.1.17). For $C, D \in \text{Pr}^\text{L}$, there is a canonical equivalence

$$C \otimes D \simeq \text{RFun}(C^{\text{op}}, D),$$

where $\text{RFun}(-, -)$ consists of functors that preserve small limits.

Remark 3.8. *Roughly speaking, the above equivalence is obtained as follows. For any testing object $E \in \mathbf{Pr}^L$, we have functors functorial in E :*

$$\begin{aligned}
 & \mathbf{LFun}(C \otimes D, E) \\
 \simeq & \mathbf{LFun}(C, \mathbf{LFun}(D, E)) \\
 \rightarrow & \mathbf{LFun}(C, \mathbf{RFun}(E, D)^{\mathrm{op}}) \\
 \simeq & \mathbf{LFun}(C, \mathbf{LFun}(E^{\mathrm{op}}, D^{\mathrm{op}})) \\
 \simeq & \mathbf{LFun}(E^{\mathrm{op}}, \mathbf{LFun}(C, D^{\mathrm{op}})) \\
 \simeq & \mathbf{RFun}(E, \mathbf{RFun}(C^{\mathrm{op}}, D))^{\mathrm{op}} \\
 \leftarrow & \mathbf{LFun}(\mathbf{RFun}(C^{\mathrm{op}}, D), E),
 \end{aligned}$$

where the non-invertible functors are obtained by passing to right adjoints:

$$\mathbf{LFun}(X, Y) \rightarrow \mathbf{RFun}(Y, X)^{\mathrm{op}}, F \mapsto F^R.$$

Such functors is always fully faithful and its essential image contains of accessible right functor $Y \rightarrow X$. A careful study shows that the essential images of $\mathbf{LFun}(C \otimes D, E)$ and $\mathbf{LFun}(\mathbf{RFun}(C^{\mathrm{op}}, D), E)$ under these embeddings coincide.

Warning 3.9. *The equivalence $C \otimes D \simeq \mathbf{RFun}(C^{\mathrm{op}}, D)$ is not functorial in C and D in the naive way.*

Exercise 3.10. *Let $F : C_1 \rightarrow C_2$ be a morphism in \mathbf{Pr}^L . Show that the following diagram commutes*

$$\begin{array}{ccc}
 C_2 \otimes D & \xrightarrow{\simeq} & \mathbf{RFun}(C_2^{\mathrm{op}}, D) \\
 \downarrow (F \otimes \mathrm{Id}_D)^R & & \downarrow - \circ F^{\mathrm{op}} \\
 C_1 \otimes D & \xrightarrow{\simeq} & \mathbf{RFun}(C_1^{\mathrm{op}}, D).
 \end{array}$$

Exercise 3.11. *Let $G : D_1 \rightarrow D_2$ be a morphism in \mathbf{Pr}^L . Show that the following diagram commutes*

$$\begin{array}{ccc}
 C \otimes D_2 & \xrightarrow{\simeq} & \mathbf{RFun}(C^{\mathrm{op}}, D_2) \\
 \downarrow (\mathrm{Id}_C \otimes G)^R & & \downarrow G^R \circ - \\
 C \otimes D_1 & \xrightarrow{\simeq} & \mathbf{RFun}(C^{\mathrm{op}}, D_1).
 \end{array}$$

Exercise 3.12. *Use either Exercise 3.10 or 3.11 to deduce that the Lurie tensor product $- \otimes - : \mathbf{Pr}^L \times \mathbf{Pr}^L \rightarrow \mathbf{Pr}^L$ preserves colimits in each factor.*

Exercise 3.13. *Let $C, D \in \mathbf{Pr}^L$. Recall we have adjunctions $D \rightleftarrows D_*$, where the left adjoint sends X to $X_+ := X \sqcup \{*\}$.*

- (1) *Show that there is a canonical equivalence $\mathbf{RFun}(C^{\mathrm{op}}, D_*) \rightarrow \mathbf{RFun}(C^{\mathrm{op}}, D)_*$ compatible with forgetful functors to $\mathbf{RFun}(C^{\mathrm{op}}, D)$.*
- (2) *Deduce that there is a canonical equivalence $C \otimes D_* \simeq (C \otimes D)_*$ compatible with the functors out of $C \otimes D$. In particular, $\mathbf{Spc}_* \otimes \mathbf{Spc}_* \simeq \mathbf{Spc}_*$.*

Exercise 3.14. *Use the above exercise to show that for any $C \in \mathbf{Pr}^L$*

$$C \otimes \mathbf{Sptr} \simeq \mathbf{Sptr}(C_*).$$

In particular, $\mathbf{Sptr} \otimes \mathbf{Sptr} \simeq \mathbf{Sptr}$.

4. SMASH PRODUCTS

Definition 4.1. Let $\mathcal{C}^\otimes \rightarrow \mathbf{Fin}_*$ be a symmetric monoidal ∞ -category. We say a commutative algebra $A \in \mathbf{Alg}_{\mathbf{Comm}}(\mathcal{C})$ is **idempotent** if $A \otimes A \rightarrow A$ is an equivalence.

Let

$$\mathbf{Alg}_{\mathbf{Comm}}^{\text{idem}}(\mathcal{C}) \subseteq \mathbf{Alg}_{\mathbf{Comm}}(\mathcal{C})$$

be the full sub- ∞ -category of idempotent commutative algebras in \mathcal{C} .

Proposition 4.2 (HA.4.8.2.9). Let $\mathcal{C}^\otimes \rightarrow \mathbf{Fin}_*$ be a symmetric monoidal ∞ -category. Then the forgetful functor

$$\mathbf{Alg}_{\mathbf{Comm}}^{\text{idem}}(\mathcal{C}) \rightarrow \mathbf{Alg}_{\mathbf{Comm}}(\mathcal{C}) \simeq \mathbf{Alg}_{\mathbf{Comm}}(\mathcal{C})_{\mathbb{1}_{\mathcal{C}}} \rightarrow \mathcal{C}_{\mathbb{1}_{\mathcal{C}}}$$

is fully faithful, and its essential image contains those objects $\mathbb{1}_{\mathcal{C}} \rightarrow X$ such that

$$X \simeq \mathbb{1}_{\mathcal{C}} \otimes X \rightarrow X \otimes X$$

is an isomorphism.

Exercise 4.3. Let $\mathcal{C} := \mathbf{Pr}^{\mathbf{L}}$ be equipped with the Lurie symmetric monoidal structure. Show that the morphisms $\mathbb{1}_{\mathcal{C}} \simeq \mathbf{Spc} \rightarrow \mathbf{Spc}_*$ and $\mathbb{1}_{\mathcal{C}} \simeq \mathbf{Spc} \rightarrow \mathbf{Sptr}$ satisfy the conditions in the proposition. Deduce that we have essentially unique upgrades of \mathbf{Sptr} and \mathbf{Spc}_* to objects in

$$\mathbf{Alg}_{\mathbf{Comm}}(\widehat{\mathbf{Cat}_{\infty}}) \simeq \mathbf{Mon}_{\mathbf{Comm}}(\widehat{\mathbf{Cat}_{\infty}})$$

such that

- These symmetric monoidal structures are compatible with small colimits;
- The unit objects are given respectively by $\mathbb{S} \in \mathbf{Sptr}$ and $\mathbb{S}^0 \in \mathbf{Spc}_*$.

Definition 4.4. We denote the multiplication functors of the above symmetric monoidal structures by

$$- \wedge - : \mathbf{Spc}_* \times \mathbf{Spc}_* \rightarrow \mathbf{Spc}_*$$

$$- \otimes - : \mathbf{Sptr} \times \mathbf{Sptr} \rightarrow \mathbf{Sptr},$$

and call them the **smash products**.

Exercise 4.5. Show that $\mathbf{Spc} \rightarrow \mathbf{Spc}_* \rightarrow \mathbf{Sptr}$ are naturally symmetric monoidal functors.

Proposition 4.6 (HA.4.8.2.10). Let $\mathcal{C}^\otimes \rightarrow \mathbf{Fin}_*$ be a symmetric monoidal ∞ -category and $A \in \mathbf{Alg}_{\mathbf{Comm}}^{\text{idem}}(\mathcal{C})$. Then $\mathbf{Mod}_A^{\mathbf{Comm}}(\mathcal{C})^\otimes$ is a symmetric monoidal ∞ -category and the forgetful functor

$$\mathbf{Mod}_A^{\mathbf{Comm}}(\mathcal{C})^\otimes \rightarrow \mathcal{C}^\otimes$$

is fully faithful. Moreover, an object $M \in \mathcal{C}$ is contained in the essential image iff

$$M \simeq \mathbb{1}_{\mathcal{C}} \otimes M \rightarrow A \otimes M$$

is an isomorphism.

Corollary 4.7. The functor

$$\mathbf{Mod}_{\mathbf{Spc}_*}^{\mathbf{Comm}}(\mathbf{Pr}^{\mathbf{L}}) \rightarrow \mathbf{Pr}^{\mathbf{L}}$$

is fully faithful with its essential image consists exactly of pointed presentable ∞ -categories. Moreover, these ∞ -categories are stable under the Lurie tensor product.

Corollary 4.8. *The functor*

$$\mathrm{Mod}_{\mathrm{Sptr}}^{\mathrm{Comm}}(\mathrm{Pr}^{\mathrm{L}}) \rightarrow \mathrm{Pr}^{\mathrm{L}}$$

is fully faithful with its essential image consists exactly of stable presentable ∞ -categories. Moreover, these ∞ -categories are stable under the Lurie tensor product.

Remark 4.9. *Note that the above corollary implies Sptr is the initial object in $\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{PrSt}^{\mathrm{L}})$, which gives an intrinsic characterization of the smash products of spectra.*

APPENDIX A. MORE ON SMASH PRODUCTS

A.1. The smash product on the ∞ -category Sptr can be rectified to a symmetric monoidal structure on a simplicial model category. The most successful one is the model category of *symmetric spectra*.

A.2. **Suggested readings.** [HSS00].

REFERENCES

- [HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *Journal of the American Mathematical Society*, 13(1):149–208, 2000.