E non-arch local field w/ residue field $\mathbb{F}_q$

π uniformizer of E

S perfectoid $/ \mathbb{F}_q$

will define X_S Fargues - Fontaine curve

analytic adic space $/ \mathcal{O}_E$

$$S = \mathrm{Spa}(R, R^+) \qquad \varpi \in R^+$$

$$\mathcal{Y}_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\}$$

$$\uparrow$$

$$Y_S = \text{''} \quad \text{''} \setminus \{\pi \cdot [\varpi] = 0\}$$

$$\downarrow$$

$$X_S = Y_S / \mathrm{Frob}$$

$$\mathcal{Y}_S^{\diamond} \qquad T = \mathrm{Spa}(A, A^+)$$

$$\mathrm{Maps}(T, \mathcal{Y}_S^{\diamond}) = \{ T^{\sharp},\ T^{\sharp} \to \mathcal{Y}_S \}$$

$$= \left\{ (A^{\sharp}, A^{\sharp,+}) \quad \begin{array}{l} W_{\mathcal{O}_E}(R^+) \to A^{\sharp,+} \\ [\varpi] \mapsto \text{invertible in } A^{\sharp} \end{array} \right\}$$

$$= \left\{ (A^{\sharp}, A^{\sharp,+}) \quad \begin{array}{l} \mathcal{O}_E \to A^{\sharp,+} \\ R^+ \to A^+ \\ \varpi \mapsto \text{invertible in } A \end{array} \right\}$$

$$= \mathrm{Maps}(T, S \times \mathrm{Spd}\, \mathcal{O}_E)$$

$$\begin{array}{ccl} \mathcal{Y}_S^{\diamond} & \simeq & S \times \operatorname{Spd} \mathcal{O}_E \\ \downarrow & & \\ Y_S^{\diamond} & \simeq & S \times \operatorname{Spd} E \\ \downarrow & & \\ X_S^{\diamond} & \simeq & (S \times \underbrace{\operatorname{Spd} E)}_{\text{Frob} \times 1.} \end{array}$$

$$\left\{ \begin{array}{c} \text{sections of } \mathcal{Y}_S^{\diamond} \\ \downarrow \\ S \end{array} \right\} \xleftarrow{\sim} \{ S \to \operatorname{Spd} \mathcal{O}_E \} \xleftarrow{\sim} \{ S^{\#} / \mathcal{O}_E \}$$

$$Y_S \qquad\qquad E \qquad\qquad S^{\#} / E$$

$$E_\infty = E(\pi^{1/p^\infty})$$

$$\begin{array}{c} \mathcal{Y}_S \times_{\operatorname{Spa} \mathcal{O}_E} \operatorname{Spa} \mathcal{O}_{E_\infty} \\ \downarrow \\ \mathcal{Y}_S \end{array} \qquad (\mathcal{Y}_S \times_{\operatorname{Spa}\mathcal{O}_E} \operatorname{Spa} \mathcal{O}_{E_\infty})^{\diamond} \simeq \mathbb{D}_{S,\text{perf}}$$

$$\mathbb{D}_{S,\text{perf}} \subset S \times \operatorname{Spa} \mathbb{F}_q [\![t^{1/p^\infty}]\!]$$

THM 1 (i) $\mathcal{Y}_S \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ is perfectoid.

(ii) $\mathcal{Y}_S$ analytic adic space.

THM 2 for any $S^{\#}/\mathcal{O}_E$, the map $S^{\#} \to \mathcal{Y}_S$ defines a closed Cartier divisor

$$\{ S^\# / E \} \longrightarrow \left\{ \begin{array}{l} \text{closed Cartier} \\ \text{divisors in } Y_S \end{array} \right\}$$

$$\|$$

$$\{ S \to \operatorname{Spd} E \}$$

$$\downarrow$$

$$\{ S \to \operatorname{Spd} E / \text{Frob} \} \longrightarrow \left\{ \begin{array}{l} \text{closed Cartier} \\ \text{divisor in } X_S \end{array} \right\}$$

$$\operatorname{Div}^1 = \operatorname{Spd} E / \text{Frob}$$

Y_S has "radius function"

$$r : |Y_S| \longrightarrow (0, \infty)$$

$$x \longmapsto \frac{\log |[\varpi](\tilde{x})|}{\log |\pi(\tilde{x})|}$$

$\tilde{x}$ rank-one generalization of x

Frobenius takes $[\varpi] \mapsto [\varpi]^q$

so increases r by factor of q

$r \longrightarrow \infty$ $\quad [\varpi] = 0$

$r \longrightarrow 0$ $\quad \pi = 0$

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proof of THM 1

$$Y_S = \bigcup_n \underbrace{\{ |\pi^n| \le |[\varpi]| \neq 0 \}}_{= \operatorname{Spa}(B, B^+)}$$

$$W_{\mathcal{O}_E}(R^+)\left\langle \frac{\pi^n}{[\varpi]} \right\rangle \hookrightarrow \text{int. cl.} \hookrightarrow W_{\mathcal{O}_E}(R^+)\left\langle \frac{\pi^n}{[\varpi]} \right\rangle \left[\tfrac{1}{\pi}\right]$$

$$\text{int. cl.} = B^+, \qquad W_{\mathcal{O}_E}(R^+)\left\langle \frac{\pi^n}{[\varpi]} \right\rangle \left[\tfrac{1}{\pi}\right] = B$$

(ii) $\quad \mathrm{Spa}(A, A^+) = \mathrm{Spa}(B, B^+) \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$ $\quad$ [suffices to consider $n=1$]

$$B^+ \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty} \hookrightarrow \text{int. cl.} \hookrightarrow B \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$$

$$\text{int. cl.} = A^+, \qquad B \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E_\infty} = A$$

$$A_0 = \left(W_{\mathcal{O}_E}(R^+) \widehat{\otimes}_{\mathcal{O}_E} \mathcal{O}_{E^\infty} \right) \left[\left(\frac{\pi}{[\varpi]} \right)^{1/p^\infty} \right]^\wedge_{[\varpi]}$$

$$A_0 \subset A^+$$

[$[\varpi]^{1/p^\infty} \in W$ $\quad \pi^{1/p^\infty} \in \mathcal{O}_{E_\infty}$]

$$A_0\left[\tfrac{1}{[\varpi]}\right] = A$$

enough to check that A_0 perfectoid.

PROP. R_0 π-adically complete, $\pi^p | p$, perfectoid $\Rightarrow$ $R_0\left[\tfrac{1}{\pi}\right]$ perfectoid.

$$A_0/[\varpi] \simeq \operatorname{colim} \left(R^+/\varpi \otimes_{\mathbb{F}_q} \mathcal{O}_{E(\pi^{1/p^i})}/\pi \right)[t^{1/p^i}] \Big/ \left(t^{1/p^i} \cdot [\varpi]^{1/p^i} = \pi^{1/p^i} \right)$$

$$\simeq \operatorname{colim} R^+/\varpi\, [t^{1/p^i}]$$

$$= R^+/\varpi\, [t^{1/p^\infty}] \qquad \square$$

(iii) R Tate $\mathbb{Z}_p$-alg.

R is semi perfectoid if $R \hookrightarrow R'$ for some perfectoid R'

and the inclusion splits as topological R-module map.

FACT $\operatorname{Spa}(R, R^+)$ sheafy if R is semi-perfectoid $\square$

X uniform analytic adic space

[uniform $\Rightarrow$ functions are determined by values on points]

Cartier divisor: locally free subsheaf $\mathcal{I} \hookrightarrow \mathcal{O}_X$

support: support of $\mathcal{O}_X / \mathcal{I}$

meromorphic function along divisor: $H^0(X, \operatorname{colim} \mathcal{I}^{-\otimes n})$

Z = support

divisor is closed if $(Z, \mathcal{O}_X/\mathcal{I}, |\cdot|_x \; x \in Z)$

is adic space.

PROP. (i) Z is nowhere dense

(ii) $\mathcal{O}_X \hookrightarrow \operatorname{colim} \mathcal{I}^{-\otimes n} \hookrightarrow j_* \mathcal{O}_U \qquad U = X \setminus Z$

PROP Cartier divisor is closed $\Leftrightarrow$ $\mathcal{I}(U) \hookrightarrow \mathcal{O}_X(U)$

has closed image for

any affinoid U.

proof of THM 2.

$$\begin{array}{ccc} S^\# & \longrightarrow & \mathcal{Y}_S \\ \shortparallel & & \\ \mathrm{Spa}(R^\#, R^{\#,+}) & & \end{array} \qquad \begin{array}{l} W_{\mathcal{O}_E}(R^+) \\ \downarrow \\ W_{\mathcal{O}_E}(R^+)/\xi = R^{\#,+} \end{array}$$

$$\xi = \pi - a\cdot[\varpi]$$

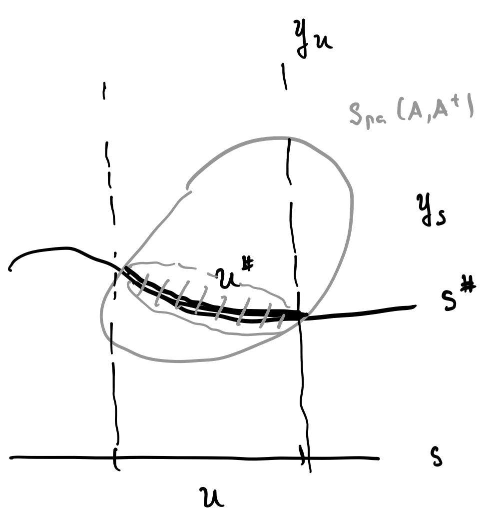
WTS: for any $\mathrm{Spa}(A, A^+) \hookrightarrow \mathcal{Y}_S$

$\xi : A \longrightarrow A$ injective w/ closed image.

CLAIM: this map is bounded below for the "sup norm"

$\|f\|_\infty = \sup\limits_{x \text{ rank one}} |f(x)|$ normalized so that $|[\varpi](x)| = \frac{1}{p}$.

first reduction: $S^\# \hookrightarrow \mathrm{Spa}(A, A^+)$.



$$\mathrm{Spa}(A, A^+) = \bigcup_n \{ |\xi| \le |[\varpi]|^n \neq 0 \} \cap \mathrm{Spa}(A, A^+)$$

some q.compactness argument

$\Rightarrow$ for some n

$$\{ |\xi| \leq |[\varpi]|^n \} \subseteq \mathcal{U}^\# \cap \mathrm{Spa}(A, A^+)$$

$\mathrm{Spa}(A, A^+)$ covered by this set and

$$\{ |\xi| \geq |[\varpi]|^n \}$$

can throw away this

second reduction: $\mathrm{Spa}(A, A^+) = \{ |\xi| \leq |[\varpi]|^n \} \underset{\mathcal{Y}_S}{\cap}$

CLAIM: $\| \xi \cdot f \|_\infty \geq \| [\varpi]^n \|_\infty \cdot \| f \|_\infty .$

will follow as $\| \cdot \|_\infty$ can be computed on the set of $x \in \{ |\xi| \leq |[\varpi]|^n \}$ admitting specialization outside of $\{ |\xi| \leq |[\varpi]|^n \}$ "Shilov boundary"

WLOG $S = \mathrm{Spa}(C, \mathcal{O}_C)$ C alg. closed

can consider $\mathcal{Y}_S \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}$

$|f(x^\flat)| = |f^\#(x)|$

tilting gets us to $\mathbb{D}_{C, \mathrm{perf}}$

"maximum modulus principle" $\square$

C alg closed non-arch field $/\mathbb{F}_q$ $S = \mathrm{Spa}(C, \mathcal{O}_C)$

$$\{ \text{untilts } C^\# / \mathcal{O}_E \} \longrightarrow |\mathcal{Y}_C|$$

image of this map is called classical points $|\mathcal{Y}_C|^{cl}$

$$\begin{array}{ccc}
|\mathbb{D}_C| \cong |(\mathcal{Y}_C \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty})^\diamond| & \xrightarrow{\sim} & |\mathcal{Y}_C \times_{\mathcal{O}_E} \mathcal{O}_{E_\infty}| \\
\cup & & \downarrow \\
|\mathbb{D}_C|^{cl} & & |\mathcal{Y}_C| \\
\| & & \cup \\
\{x \in C : |x| < 1\} & \dashrightarrow ? & |\mathcal{Y}_C|^{cl} \\
x & \longmapsto & \pi - [x]
\end{array}$$

[this map is actually open]

PROP (i) C'/C $x \in |\mathcal{Y}_C|$ classical iff its preimage in $|\mathcal{Y}_{C'}|$ is a classical point

(ii) $x \in |\mathcal{Y}_C|$ rank one, not classical

$\Rightarrow$ for some C'/C, preimage in $|\mathcal{Y}_{C'}|$ contains an open set.

<u>proof</u> (i) ($\Leftarrow$)

a family $X_i \to X$ of maps of perfectoid spaces is a v-cover if for every q. compact $\mathcal{U} \subseteq X$ there is a finite set of $\mathcal{U}_i \subseteq X_i$ which jointly cover $\mathcal{U}$

FACT : any perfectoid space is a v-sheaf.

classical $\overbrace{\ldots \times \mathrm{Spa}\, C'}^{\sim} \longrightarrow \mathrm{Spa}\, C'$

$$\begin{tikzcd}
\square \arrow[r] \arrow[d] & \operatorname{Spd} \mathcal{O}_E \times \operatorname{Spa} C' \arrow[d] \arrow[r] & \operatorname{Spa} C' \arrow[d, "v\text{-cover}"] \\
\operatorname{Spd} K_x \arrow[r] \arrow[rr, bend right, "x^\diamond", "\sim"'] & \operatorname{Spd} \mathcal{O}_E \times \operatorname{Spa} C \arrow[r] & \operatorname{Spa} C
\end{tikzcd}$$

$\Rightarrow$ x classical.

(ii) By openness of $|\mathbb{D}_C| \to |\mathcal{Y}_C|$ suffices to prove analogous statement for $\mathbb{D}_C$.

Consider WLOG $x = \rho$ - Gauss norm around 0.

$$\begin{tikzcd}
y \arrow[r] & \mathbb{D}_{C_x} \arrow[d] \\
x \arrow[r] \arrow[ur, dashed] & \mathbb{D}_C
\end{tikzcd}
\qquad
\begin{tikzcd}
C_{x,y} & C_x\langle T\rangle \arrow[l] \arrow[dl] \\
C_x & C\langle T\rangle \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
u & T \arrow[l, mapsto] \\
t & T \arrow[l, mapsto] \arrow[u, mapsto]
\end{tikzcd}$$

CLAIM: premage of x contains ball of radius ρ around $1 \otimes t$.

i.e. $|t-u| < \rho \Rightarrow$ for all $f = \sum_i a_n T^n \in C\langle T\rangle$,

$$|\textstyle\sum_i a_n t^n| = |\sum_i a_n u^n|.$$

$$\left|\textstyle\sum_i a_n (t^n - u^n)\right| < \sup |a_n| \rho^n = |\textstyle\sum_i a_n t^n| \qquad \square$$

$$|t^n - u^n| < \rho^n$$

THM. Let $\operatorname{Spa}(B, B^+) \subseteq \mathcal{Y}_C$
$\quad\;\; \| $
$\quad\;\; \mathcal{U}$

(i) $\pi_0(\mathcal{U})$ finite

(ii) for any $f \in B$ nonzero, $\{f = 0\} \subseteq |\mathcal{U}|^{cl}$

(iii) $|\mathcal{U}|^{cl} \to \operatorname{maxSpec}(B)$ is bijection

(iv) $\mathcal{U}$ connected $\Rightarrow$ B is P.I.D.

Remarks (i) $|\mathbb{D}_c| \to |\mathcal{Y}_c|$

(ii) $f(x) = 0$ x not classical, wlog x rank one

after C'/C find $\mathcal{U}' \subseteq \mathcal{U}$ w/ $f|_{\mathcal{U}'} = 0$

(iii) follows from (ii)

(iv) $f \in B$.

$\{f = 0\} \subseteq |\mathcal{U}|^{cl}$ spectral space w/ no generalizations

$\Rightarrow$ profinite

CLAIM: $f(x) = 0 \Rightarrow f = \xi_x^n \cdot g$ $(\xi_x) = m_x$ $g(x) \neq 0$

$\|\cdot\|_{\mathcal{U},\infty} = \sup_{\substack{x \in \text{Shilov} \\ \text{boundary}}} |\cdot|_x$ $\leftarrow$ finite set

scale ξ_x so that $|\xi_x| \geq 1$ on boundary.

if $f = \xi_x^n \cdot g_n \Rightarrow \|g_n\|_\infty \leq \|f\|_\infty$

$\|f\|_{\infty, \{|\xi_x| \leq |[\varpi]|\}} \leq \|[\varpi]^n\| \cdot \|f\|_\infty$

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if $n \to \infty$ get $\|f\|_{\infty, \{|\xi_x| \leq |[\varpi]|\}} = 0$

$\Rightarrow f = 0$.

$\mathcal{O}_x(\mathcal{U}) \hookrightarrow \mathcal{O}_x(\mathcal{U}')$