SEMINAR NOTES ON
GEOMETRIZATION OF THE LOCAL LANGLANDS COORRESPONDENCE

LSG

Contents

Notes taken by Lin.

1. Talk I by Lin: the big picture

... once we could merely *formulate* Fargues' conjecture, enough mathinery is available to apply Lafforgue's ideas to get the "automorphic-to-Galois" direction...

[FS21, Sect. I.11]

1.1. What is the local Langlands correspondence?

Notation 1.1.1. *We fix the following notations:*

- E *is a local field (e.g. ...* $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t))$*)*[1]*;*
- G *is a split reductive group over* E*;*
- $\hat{G}$ *is the Langlands dual group over* $\mathbb{Z}$*;*
- W_E *is the Weil group for* E*.*

Conjecture 1.1.2. *There is a canonical map between sets:*

$$\left\{\textit{irreducible objects } \operatorname{Rep}_{\mathbb{C}}(G(E))\right\} \to \left\{W_E \to \hat{G}(\mathbb{C})\right\}, \ \pi \mapsto \varphi_\pi$$

subject to certain compatibilities.

Remark 1.1.3. *Several remarks are in order.*

(0) *In general this is not a bijection.*
(1) *From easy to hard: archimedean, char p nonarchimedean, char 0 nonarchimedean. E.g., the conjecture is unknown for $E = \mathbb{Q}_p$ and general redcutive group.*
(2) *$G(E)$ is a topological group and we require π to be* smooth*, i.e., any vector is fixed by some compact open subgroup.*
(3) *W_E is a modification of $\operatorname{Gal}(\overline{E}/E)$. $W_{\mathbb{C}} = \mathbb{C}^\times$, $W_{\mathbb{R}}$ is the canonical extension of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times$. For nonarchimedean E with residue field $k = \mathbb{F}_q$,*

$$\begin{tikzcd}
1 \arrow[r] & I_E \arrow[r] \arrow[d, "\simeq"] & W_E \arrow[r] \arrow[d, "\subset"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z} \arrow[r] \arrow[d, "\subset"', "1\mapsto \mathrm{Frob}"] & 1 \\
1 \arrow[r] & I_E \arrow[r] & \operatorname{Gal}(\overline{E}/E) \arrow[r] & \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \arrow[r] & 1.
\end{tikzcd}$$

The necessity of W_E instead of $\operatorname{Gal}(\overline{E}/E)$ can be already seen in local class field theory.
(4) *W_E is a topological group and we require φ_π to be continuous. However, it is* not *equipped with the subspace topology from $\operatorname{Gal}(\overline{E}/E)$. Instead, I_E has the subspace topology from $\operatorname{Gal}(\overline{E}/E)$, which is pro-finite, and is forced to be open in W_E, which is therefore locally pro-finite.*
(5) *φ_π is the so-called L-parameter of π.*
(6) *Part of the compatible requirements are about L-functions and ϵ-factors. Lin knows nothing about them. Maybe someone can explain later?*

1.2. What is geometrization?

Answer 1.2.1. *Do global geommetric Langlands on the Fargues–Fontaine curve, which behaves like a genus 0 curve in nonarchimedean geometry.*

To explain what this mean, we need some basic notions in analytic geometry.

Notation 1.2.2. *From now on, we restrict to the case $E = \mathbf{Q}_p$. There is no essential difference for general E, and the char p case is even easier.*

Analogy 1.2.3. *Tutorials on analytic geometry will be provided by Lin and Yuchen in the next weeks. For now, let us be satisfied by the following:*

[1]The colors are chosen to be compatible with [SW20, Figure 12.1].

	algebraic geometry	*analytic geometry*
affine	$\operatorname{Spec} R, R \in \mathrm{CAlg}$	$\operatorname{Spa}(R, R^+)$, (R, R^+) *is a* Huber pair*:* $R^+ \subset R \in \mathrm{CAlg(Top)}$ *satisfying certain conditions*
globalization	scheme *as locally ringed spaces*	(pre-)adic space *as locally topologically ringed spaces*
point	$\operatorname{Spec} K, K$ *is a field*	$\operatorname{Spa}(K, K^+), (K, K^+)$ *is an* affinoid field analytic *if* K *is nondiscrete* nonanalytic *if* K *is discrete*

Notation 1.2.4. *In most cases, people make the canonical choice* $R^+ := R^\circ$ *being the subring of power-bounded elements and write* $\operatorname{Spa} R := \operatorname{Spa}(R, R^\circ)$.

Remark 1.2.5. *Among all the (pre)-adic spaces, there is a class of objects, called* perfectoid spaces, *that are well-adapted to connect char* 0 *and char* p. *Affine perfectoid spaces are given by* $\operatorname{Spa}(R, R^+)$ *such that* R *is a* perfectoid ring. *Basic examples of perfectoid rings include* $\mathbb{F}_p((t^{1/p^\infty}))$, *which is the completion of* $\bigcup_n \mathbb{F}_p((t))(t^{1/p^n})$, *and* $\mathbb{Q}_p^{\mathrm{cycl}}$, *which is the completion of* $\mathbb{Q}_p(\mu_{p^\infty})$. *Any perfectoid ring is defined over* $\mathbb{Z}_p$ *although the latter itself is not a perfectoid ring.*

Personally I think the following result (together with the *tilting equivalence* to be explained later) is the root of all the magic:

Theorem 1.2.6. *There is* no *final object in* Perfd*, however, products exist in the category* Perfd_p *of perfectoid spaces of characteristic* p.

Analogy 1.2.7. *Sanath will talk about the details about FF curves. For now, let us be satisfied by the following:*

	Global Geometric Langlands *for functional field*	*Geometrized Local Langlands*
geometry	*algebraic geometry over* $\mathbb{F}_p$	*"perfectoid geometry"*
test objects	*schemes* $S \in \mathrm{Sch}_{/\mathbb{F}_p}$	*char* p *perfectoid space* $S \in \mathrm{Perfd}_p$
final test object	$\operatorname{Spec} \mathbb{F}_p$	**not exist** $\operatorname{Spa} \mathbb{F}_p \notin \mathrm{Perfd}_p$
spaces	*prestacks* $\supset$ *fpqc-stacks* $\supset$ *algebraic spaces*	*prestacks* $\supset$ *v-stacks* $\supset$ *diamonds*
absolute *curve*	X *over* $\mathbb{F}_p$	**not exist/sci-fi**
relative *curve*	$X_S := S \times_{\mathbb{F}_p} X$	$\mathcal{X}_S := \mathcal{Y}_S/\mathrm{Frob}_S := (S \dot{\times} \operatorname{Spa} \mathbb{Q}_p)/\mathrm{Frob}_S$

Question 1.2.8. *Wait, how dare you multiply a char* 0 *object* $\operatorname{Spa} \mathbb{Q}_p$ *with a char* p *object* S*!*

Warning 1.2.9. *There is a dot over the product sign in the notation* $S \dot{\times} \operatorname{Spa} \mathbb{Q}_p$, *which means it is* not *a product, at least not naively. For example,* $\operatorname{Spa}(R, R^+) \dot{\times} \operatorname{Spa} \mathbb{Q}_p$ *is an open subspace of* $\operatorname{Spa} W(R^+)$, *where* $W(R^+)$ *is the ring of* p*-Witt vectors in* R^+. *In fact, it is the open subspace where the functions* $p, [\varpi] \in W(R^+)$ *are invetible, where* $\varpi \in R^+$ *is a* pseudo-uniformizer*. Whatever this means, we see* $S \dot{\times} \operatorname{Spa} \mathbb{Q}_p$ *is of char* 0.

Question 1.2.10. *Wait, if* $\mathcal{X}_S$ *is of char* 0, *is it okay to study it using char* p *test objects? For example, when talking about Hecke modifications, you need a notion of Cartier divisors of* $\mathcal{X}_S$ *relative to the base, but where is your base? It can't be* S *or* S/Frob_S *because they are of char* p.

Answer 1.2.11. *No, at least not in the naive way. The correct way to relate $\mathcal{X}_S$ to char p objects is via its associated diamond $(\mathcal{X}_S)^\diamond$, which we will explain now.*

Construction 1.2.12 ([FS21, Sect. 6.2])**.** *For any commutative ring R, the* (p-)tilt *of R is*

$$R^\flat := \lim(\dots \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} R).$$

A priori this is only a multiplicative monoid. If R is equipped with a good enough complete topology, such as a perfectoid ring, then one can define a ring structure where the addition law is

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) := (z^{(0)}, z^{(1)}, \dots),$$

where

$$z^{(i)} := \lim_{n\to\infty} (x^{(i+n)} + y^{(i+n)})^{p^n}.$$

In particular, when R is a perfectoid ring, we obtain a char p *perfectoid ring $R^\flat$. We say R is a* untilt of *$R^\flat$.*

We define

$$\mathrm{Spd}(R, R^+) := \mathrm{Spa}(R, R^+)^\flat.$$

Using gluing, we can define $X^\flat$ for any prefectoid space X.

Remark 1.2.13. *For perfectoid ring R, we have $(R^\flat)^\circ \simeq (R^\circ)^\flat$.*

Example 1.2.14. *Any char p prefectoid ring R is the tilt of itself, and is the only char p untilt of itself. But there are char* 0 *untilts.*

Example 1.2.15. *The tilt of $\mathbb{Q}_p^{\mathrm{cycl}}$ is $\mathbb{F}_p((t^{1/p^\infty}))$.*

Theorem 1.2.16 (Tilting Equivalence)**.** *For any perfectoid space X, the functor $Y \mapsto Y^\flat$ induces an equivalence between the categories of perfectoid spaces over X and $X^\flat$. This equivalence preserves (finite) étale covers.*

Definition 1.2.17. *For any pre-adic space X, define $X^\diamond$ to be the prestack*

$$X^\diamond : \mathrm{Perfd}_p^{\mathrm{op}} \to \mathrm{Set},\ S \mapsto \coprod_{S^\sharp \in \mathrm{Untilt}(S)} \mathrm{Maps}(S^\sharp, X).$$

Remark 1.2.18. *In fact, Lin thinks the following is correct. Consider*

$$\mathrm{PreStk} := \mathrm{Funct}(\mathrm{Perfd}^{\mathrm{op}}, \mathrm{Set}), \mathrm{PreStk}_p := \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set})$$

and define

$$\diamond : \mathrm{PreStk} \to \mathrm{PreStk}_p$$

to be the unique colimit-preserving functor extending $\flat$. Then when restricted to pre-adic spaces, one recovers the above definition.

Theorem 1.2.19. *For pre-adic space X, the underlying topological spaces of X and $X^\diamond$ are canonically homeomorphic.*

Remark 1.2.20. *In fact, we have the following slogen: "$\diamond$ only remembers topological information."*

Example 1.2.21. $\mathrm{Spd}\,\mathbb{Z}_p := (\mathrm{Spa}\,\mathbb{Z}_p)^\diamond$ *classifies all untilts;* $\mathrm{Spd}\,\mathbb{Q}_p := (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$ *classifies all char 0 untilts.*

Example 1.2.22. *If X is already a perfectoid space, then $X^\diamond \simeq X^\flat$ by the tilting equivalence.*

Example 1.2.23. *For char p pre-adic space X, the functor $X \mapsto X^\diamond$ is just*

$$\mathrm{PreAdic}_p \to \mathrm{Funct}(\mathrm{PreAdic}_p^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set}).$$

This is because only char p untilts $S^\sharp$ can map to X.

Example 1.2.24. *By the tilting equivalence, if X is the quotient of $R \rightrightarrows Y$ of perfectoid spaces connected by pro-étale maps, then $X^\diamond$ is the quotient of $R^\flat \rightrightarrows Y^\flat$. Essentially,* diamonds *are defined to be such quotients. In fact, any nalytic pre-adic space, which means all its residue fields are not discrete, over $\mathbb{Z}_p$ can be written as such a quotient.*

Remark 1.2.25. *Yifei will talk about the pro-étale topology and explain why it is powerful.*

Example 1.2.26. *Unfortunately/fortunately, $\mathrm{Spa}\,\mathbb{Q}_p$ is not a perfectoid space but it has a perfectoid pro-étale cover $\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cycl}} \to \mathrm{Spa}\,\mathbb{Q}_p$ whose Galois group is $\mathbb{Z}_p^\times$. Hence*

$$\mathrm{Spd}\,\mathbb{Q}_p \simeq \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}/\underline{\mathbb{Z}_p^\times} \simeq \mathrm{Spa}\,\mathbb{F}_p((t^{1/p^\infty}))/\underline{\mathbb{Z}_p^\times},$$

where $\underline{\mathbb{Z}_p^\times}$ is the discrete group diamond.

Theorem 1.2.27. *For any char p perfectoid space S, we have*

$$(S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)^\diamond \simeq S \times \mathrm{Spd}\,\mathbb{Q}_p.$$

Warning 1.2.28. *This is not formal. But Lin thinks $\dot{\times}$ was, or should have been, designed to make this correct.*

Remark 1.2.29. *For a char p perfectoid space S, a map $S \to \mathrm{Spd}\,\mathbb{Q}_p$ provides a char 0 untilt $S^\sharp$, which will provide a closed immersion $S^\sharp \to \mathcal{Y}_S := S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ once we know the precise definition of the target. This is a Cartier divisor and so is the composition $S^\sharp \to \mathcal{X}_S$. Also, the latter only depends on the composition $S \to \mathrm{Spd}\,\mathbb{Q}_p \to \mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$. This suggests $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$ should be the moduli prestack of Cartier divisors on FF curves.*

In fact, as we have seen (or will see) in geometric Langlands, we use $\overline{\mathbb{F}_p}$-points on the curve to define Hecke modifications. Hence we should restrict our attension to S defined over $\overline{\mathbb{F}_p}$ rather than $\mathbb{F}_p$. The effect is to change $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$ to $\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\mathrm{Frob}$, where $\mathbb{Q}_p^{\mathrm{ur}} = \mathrm{Frac}(W(\overline{\mathbb{F}_p}))$ is the maximal unramified extension of $\mathbb{Q}_p$.

We can finally define the moduli of divisors on FF curves:

Definition 1.2.30. *The* moduli diamond of degree 1 closed Cartier divisors on FF curves *is*

$$\mathrm{Div}^1 := \mathrm{Spd}\,\mathbb{Q}_p^{ur}/\mathrm{Frob}.$$

Warning 1.2.31. *Unlike in algebraic geometry, Div^1 is* not *the curve itself. They live in different characteristics. In fact, we do not have the absolute FF curve.*

Remark 1.2.32. *An amazing thing is this tilting/untilting game allows us to consider products of the "curve", or in fact, consider $\mathrm{Div}^1 \times \cdots \times \mathrm{Div}^1$. Unlike the self product of $\mathrm{Spa}\,\mathbb{Q}_p$, this product, which is taken in the category of diamonds, is not boring. Lin thinks this is essentially because Perfd_p lacks a final object but has products (which is false for* Perfd*).*

1.3. **Why Fargues–Fontaine curve?** Please read [FS21, Sect. I.11] (titled "The origin of the ideas"). For now, let me explain how the automorphic and Galois sides naturally appear in this geometrization picture.

For the Galois side:

Theorem 1.3.1. $\pi_1(\mathrm{Div}^1) \simeq W_E$.

Remark 1.3.2. *Heuristically this follows from the definition $\mathrm{Div}^1 := \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\mathrm{Frob}$. Indeed, W_E is the extension of $\mathbb{Z}\langle\mathrm{Frob}\rangle$ by $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{ur}})$.*

Question 1.3.3. *Wait, didn't you say FF curves behaved like genus 0 curves?*

Answer 1.3.4. *Yes. But $\mathcal{X}_C$ is not defined over C or any algebraically closed field. Instead, we have:*

Theorem 1.3.5. $\Gamma(\mathcal{X}_C, \mathcal{O}) \simeq \mathbb{Q}_p$.

For the automorphic side, we consider the v-stack Bun_G whose values $\mathrm{Bun}_G(S)$ classify G-torsors on $\mathcal{X}_S$. Taeuk will explain the precise meaning of the following:

Theorem 1.3.6. *Bun_G has a stratification labelled by the poset $B(G)$ such that each stratum is of the form $*/H$, where H is a group diamond which is an extension of a discrete group $\underline{M(\mathbb{Q}_p)}$ by a unipotent group, where M is an inner form of a Levi subgroup of G.*

Remark 1.3.7. *This is another example where FF curves behave like genus* 0.

Corollary 1.3.8. *For any* $\mathbb{Z}_l$*-algebra* Λ*, the category* $D(\mathrm{Bun}_G, \Lambda)$ *can be glued from categories* $\mathrm{Rep}_\Lambda(M(\mathbb{Q}_p))$ *for* M *being inner forms of Levi subgroups of* G.

Yifei will explain how to define $D(-, \Lambda)$ and play with them.

1.4. **What can be translated from geometric Langlands?**

Answer 1.4.1. *Essentially any pure geometric constructions in Geometric Landlands can be or at least should be translated. Things already done in [FS21]: geometric Satake, Lafforgue's automorphic-to-Galois construction via shitukas, formulation of categorical Langlands conjecture, the spectral action...*

1.5. **What else?** The story is not complete without talking about p-adic Hodge theory. After all, FF curves were born during the study of Fontaine's peroid rings. If people want, we can make a digression to it.

2. Talk 2 by Lin: adic spaces

> ... we should remember that valuations played a central role in Zariski's approach from the late 1930's, building up algebraic geometry by algebraic means. Later this role was reduced by Grothendieck and others in favour of prime ideals... Recently, in the Ragsquad[a] seminar and elsewhere, we experienced a revived interest in Zariski's work.
>
> [a]Recent advances in real algebraic geometry and quadratic forms.
>
> [HK94]

2.1. **Why adic spaces: a history of nonarchimedean geometry.** Huber introduces adic spaces as *a generalization of formal schemes and rigid analytic varieties*, which is also the title of [Hub94].

We are familiar with formal schemes:

Definition 2.1.1. *An **Noetherian adic ring** A is a Noetherian ring equipped with the I-adic topology for certain ideal $I \subset A$. For a Noetherian adic ring, define an affine **formal scheme** $\operatorname{Spf} A := \operatorname{colim}_n \operatorname{Spec} A/I^n$ which only depends on the completion of A. One can realize $\operatorname{Spf} A$ as a locally toplogically ringed space by taking the colimits in the category of such spaces.*

Example 2.1.2. $\operatorname{Spf} \mathbb{Z}_p$ *only has one point. There is an obvious map* $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spf} \mathbb{Z}_p$*, but there is no map from* $\operatorname{Spec} \mathbb{Q}_p$ *to it. Note that* $\operatorname{Spec} \mathbb{Q}_p$ *can not detect the topology of* $\mathbb{Q}_p$.

We are less familiar with Tate's rigid analytic geometry. Below is a briefing of it.

Definition 2.1.3. *A **nonarchimedean field** is a field K complete with respect to a nontrivial nonachimedean valuation[2] $K \to 0 \cup \mathbb{R}^+$.*

The corresponding valuation ring is denoted by $K^\circ := \{x \in K \,\big|\, |x| \le 1\}$.

*A **pseudo-uniformizer** ϖ in K is a topologically nilpotent unit in K, i.e., $0 < |\varpi| < 1$. Such element always exists because we assumed the valuation is nontrivial.*

Example 2.1.4. $\mathbb{Q}_p$ *is a nonarchimedean field and p is a (pseudo-)uniformizer.*

Example 2.1.5. *Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$. Then $\mathbb{C}_p$ is an algebraically closed nonarchimedean field and p is a pseudo-uniformizer.*

Roughly speaking, Tate's rigid geometry over a nonarchimedean field K is the geometry where the topology of K is remembered.

Remark 2.1.6. *Let us try to invent rigid geometry. For simplicity, we assume $K = \overline{K}$. Suppose we can define the affine line $(\mathbb{A}^1_K)^{\mathrm{rgd}}$ over K such that its points correspond to elements in K. What should be the open subvariety of $(\mathbb{A}^1_K)^{\mathrm{rgd}}$? Of course, we want the usual Zariski opens. But we also want the subset correponding to $\{x \in K \,|\, |x| \le c\} \subset K$ to be open for any $c \in \mathbb{R}^+$ because the latter is open in K. Naively, we want $K[T]$ to be the ring of functions on $(\mathbb{A}^1_K)^{\mathrm{rgd}}$, then what should be the ring of functions on $\{x \in K \,\big|\, |x| \le 1\} \subset K$? The answer is the* Tate algebra $K\langle T\rangle$.

Definition 2.1.7. *For any topological ring R, define the **Tate algebra***

$$R\langle T\rangle := \left\{ f = \sum a_i T^i \in R[[T]] \,\Big|\, \lim a_i = 0 \right\} \subset R[[T]].$$

More generally, we define $R\langle T_1, \dots, T_n\rangle$, and we have $R\langle T_1, \dots, T_m\rangle\langle T_{m+1}, \dots . T_n\rangle = R\langle T_1, \dots, T_n\rangle$.

*A **topologically finitely presented (or tfp) R-affinoid algebra** is a topological R-algebra A that is isomorphic to a quotient of some $R\langle T_1, \dots T_n\rangle$ by a finitely generated ideal[3]*

Remark 2.1.8. *If the topology on R is given by a norm $|-|$, then $R\langle T\rangle$ is the completion of $R[T]$ with respect to the* Gauss norm $|\sum a_i T^i| := \max |a_i|$.

[2]We always write valuation groups multiplicatively.

[3]For nonarchimedean field K, $K\langle T_1, \dots T_n\rangle$ is Noetherian. See [Con08, Theorem 1.1.5]. But there is no proof in *loc.cit.*. Conrad only mentioned using Weierstrass Preparation techniques. Also, when K° is Noetherian (or equivalently when the valuation is discrete), $K^\circ\langle T_1, \dots T_n\rangle$ and $K\langle T_1, \dots T_n\rangle$ are Noetherian by Example 2.1.9 below.

Example 2.1.9. *$K^\circ\langle T\rangle$ is the ϖ-adic completion of $K^\circ[T]$. $K\langle T\rangle = K^\circ\langle T\rangle[\varpi^{-1}]$.*

Remark 2.1.10. *For an algebraically closed nonarchimedean field K, the maximal spectrum $\operatorname{MaxSpec} K\langle T\rangle$ is $\{(T-x) \mid |x| \le 1\}$. Therefore I suggest to view anything inside $\langle - \rangle$ as being forced to have norm ≤ 1.*

Definition 2.1.11. *For a nonarchimedean field K, a **K-affinoid algebra** is a topological K-algebra A that is isomorphic to a quotient of some $K\langle T_1, \dots T_n\rangle$.*

Remark 2.1.12. *For a K-affinoid algebra, we want to define a topology on* MaxSpec A. *As explained before, there are two kinds of basic opens: the Zariski ones $\{x \in \operatorname{MaxSpec} A \mid g(x) \neq 0\}$ for $g \in A$, and the disk-like ones, or* Weierstrass *ones $\{x \in \operatorname{MaxSpec} A \mid |f(x)| \le 1\}$ for $f \in A$.*

However, we do not *want to use the topology generated by them. Because otherwise the topology on the* closed unit disk $\operatorname{MaxSpec} K\langle T\rangle$ *is not connected (in the case $K = \mathbb{Q}_p$, this topology would be the totally disconnected space $\mathbb{Z}_p$).*

At first glance, this seems like a dilemma: we want the topology on $\operatorname{MaxSpec} K\langle T\rangle$ *to resemble that of $\mathbb{Z}_p$ but also behave like a disk. How can this be possible?*

The key is to use Grothendieck topology instead of topology.

Definition 2.1.13. *For a K-affinoid algebra A, and elements $f_1, \dots, f_n, g \in A$ which generate the unit ideal, define*

$$U\left(\frac{f_1, \dots, f_n}{g}\right) := \operatorname{MaxSpec} A\langle f_1/g, \dots, f_n/g\rangle := \left\{x \in \operatorname{MaxSpec} A \,\middle|\, |f_i(x)| \le |g(x)| \neq 0\right\}$$

*and call it a **rational domain** of* MaxSpec A.

*The **Tate topology** on* MaxSpec A *is the "quasi-compact" Grothendieck topology: opens are given by rational domains and coverings are required to be finite.*

*There is an obvious definition of the structure sheaf on this site and we obtain a locally topologically ringed topoi, which is called the **affinoid rigid space*** $\operatorname{Sp}(A)$ *associated to A. Such notion can be globalized to define **rigid spaces**.*

Remark 2.1.14. *Several remarks are in order.*

- *Taking $f_i = 0$, we get the Zariski opens; taking $g = 1$, we get the Weierstrass opens.*
- *More precisely, $A\langle f_1/g, \dots, f_n/g\rangle := A\langle T_1, \dots T_n\rangle/(gT_1 - f_1, \dots, gT_n - f_n)$.*
- *Some authors use the condition $(f_1, \dots, f_n) = A$ instead of $(f_1, \dots, f_n, g) = A$. There is no difference because we can add $f_0 := g$.*
- *The condition $|g(x)| \neq 0$ is in fact redundant: if $g(x) = 0$, then $f_i = 0$ and $1 = 0$ because $(f_i, g) = A$. We keep it because later in the definition of rational domains of adic spaces, we no longer require $(f_1, \dots, f_n) = A$.*
- *But we certainly need the condition $|g(x)| \neq 0$* or *$(f_1, \dots, f_n) = A$. Otherwise taking $g = 0$ we will get Zariski* closed *subset.*
- *On the other hand, if we keep the condition $|g(x)| \neq 0$ and drop the condition $(f_1, \dots, f_n, g) = A$, we still get an open subset $\{x \in \operatorname{MaxSpec} A \mid |f_i(x)| \le |g(x)| \neq 0\}$ because it is the union of* $\operatorname{MaxSpec} A\langle f_1/g, \dots, f_n/g, \varpi^m/g\rangle$ *for large enough m. Note that this is indeed a cover in the Tate topology because when restricted to any rational domain inside the target, the above union stablizes. But $\{x \in \operatorname{MaxSpec} A \mid |f_i(x)| \le |g(x)| \neq 0\}$ itself is not quasi-compact. Hence if we want a well-behaved Grothendieck topology, we need* some *condition on f_i.*
- *The reason to use non-strict inequalities instead of strict inequalities is similar: strict inequalities still define open subset of* MaxSpec A, *but they are not quasi-compact.*
- *In the Tate topology,* $\operatorname{MaxSpec} K\langle T\rangle$ *is connected. Lin does not want to prove it now*[4]*, but we can see the naive way to disconnect it does not work: the loci given by $|T| \ge 1$ and $|T| < 1$ no longer cover* $\operatorname{MaxSpec} K\langle T\rangle$ *because it has no* finite *refinement by rational domains.*

[4] Lin thinks using sheaf theory to prove connectedness is better than point-set topology and believes we will learn such a proof in the futher. Nevertheless, see [Con08, Exercise 2.4.9].

Remark 2.1.15. *Several shortcomings of Tate's rigid geometry: all of these are correct but seem hopeless to prove using methods of rigid geometry:*

- *Flat morphisms are open under some finiteness assumptions.*
- *Flatness is preserved under base change along field extensions.*
- *Fpqc descents for rigid spaces, maps between them, coherent sheaves on them.*
- *There is a robust theory of étale sheaves on rigid spaces.*
- *...*

Roughly speaking, these shortcomings lead to the following evolution of theories: Raynaud's formal models*,* Berthelot's formal models*,* Berkovich spaces*,* Huber's adic spaces[5]*...*

Remark 2.1.16. *We will not spend more time reviewing the theories before Huber than the following sketches.*

- *Raynaud's thoery realize qcqs rigid spaces over K as "generic fibers"[6][7] of tfp formal schemes* flat *over K°. Such realizations are unique up to* (formal) blow-ups in the special fiber $\operatorname{Spec} \kappa(K^\circ)$.
- *Berthelot's theory allows taking generic fibers for formal schemes* of formally finite type *over K°. In the affine case, this means we allow quotients of $K^\circ[[T_1, \dots, T_m]]\langle T_{m+1}, \dots, T_n\rangle$. But the construction is very indirect: the generic fiber of* $\operatorname{Spf} K^\circ[[T]]$ *is not even affine.*
- *Berkovich's theory studies multiplicative seminorms (or equivalently, continuous valuations into $0 \cup \mathbb{R}^+$) of K-affinoid algebras. This is motivated by the fact that any point of* $\mathfrak{m} \in \operatorname{Sp} A$ *defines such a seminorm via $A \to A/\mathfrak{m} \xrightarrow{|-|} 0 \cup \mathbb{R}^+$.*
- *Huber's theory revives Zariski's work and studies all the continuous valuations of (nice) topological algebras A. (Some of) such valuations will be points of a locally topologically ringed space*[8] $\operatorname{Spa} A$*, the associated* adic space*. There are several amazing things about adic spaces:*
  - *It has a underlying topological space rather than a topoi.*
  - *It uniformizes formal schemes and rigid spaces in the same framework. Now the* taking generic fiber *construction can be done naively. For example* $\operatorname{Spa} \mathbb{Z}_p$ *now has two points.*
  - *For a K-affinoid algebra A, the underlying set* $|\operatorname{Spa} A|$ *is bijective to the set of all points of the topoi* $\operatorname{Sp} A$.
  - *For a K-affinoid algebra A,* $\operatorname{Spa} A$ *is the inverse limit of* all *the Raynaud's formal models (viewed as adic spaces).*
  - *It is well-behaved even for infinite type K°-algebras, which in particular allows perfectoid geometry.*
- *Clausen–Scholze's theory want to replace "topological space" by "sheaves on the site of pro-finite sets", or equivalently "pro-étale sheavess on fields" and rewrite everything...*

2.2. **Continuous valuations.** Affine adic spaces will be defined as a certain subspace of the space $\operatorname{Cont}(A)$ of continuous valuations of a Huber ring A. We study $\operatorname{Cont}(A)$ is this subsection.

Definition 2.2.1. *A topological ring A is* ***Huber*** *if it admits an open subring $A_0 \subset A$ which is I-adic with respect to a finitely generated ideal of definition $I \subset A_0$. Any such A_0 is called a* ***ring of definition*** *and I is called a* ***ideal of definition****.*

Remark 2.2.2. *Several remarks are in order*[9]*:*

- *A_0 and I only play an auxiliary role in the proofs and no constructions will depend on them;*
- *A subring $A_0 \subset A$ is a ring of definition iff A_0 is open and bounded*[10].

[5] Maybe *Clausen–Scholze's condensed mathematics* should be the next one.

[6] There is an obvious way to define such generic fibers for $\operatorname{Spf} K^\circ\langle T_1, \dots, T_n\rangle$. One extends this construction to any flat tfp affinoid K°-algebras R by showing it does not depend on presentations of R as a quotient of Tate algebras. One extends it to global case (i.e., to *admissible* formal schemes) by showing it is compatible with localizations.

[7] There is map $\operatorname{Sp} K \to \operatorname{Spf} K^\circ$ between locally topologically ringed topos. For a tfp formal scheme X, there is a canonical morphism from X to its Raynaud's generic fiber X_K. But Lin does not know if X_K is the fiber product $X \times_{\operatorname{Spf} K^\circ} \operatorname{Sp} K$.

[8] $\operatorname{Spa} A$ in fact has more structure: for any point x of it, we have a continuous valuation of the local ring $\mathcal{O}_{X,x}$.

[9] The proofs are elementary, and can be found in [Hub93, §1].

[10] A subset S of A is bounded iff for any open neighborhood U of 0, there exists an open neighbhorhodd V of 0 such that $VS \subset U$. For example, $K^\circ \subset K$ is bounded but $(K^\circ)^{-1} \subset K$ is not bounded.

- *Let $A^\circ \subset A$ be the subring of power bounded elements. Then any ring of definition A_0 is contained in A°. In fact, A° is the filtered union of all the rings of definitions. However, A° might not be a ring of definition (i.e. A° might be unbounded): $\mathbb{Q}_p[\epsilon]/(\epsilon^2)$ is a counter-example.*
- *All constructions will only depend on the completion $\widehat{A}$ of A, which is still a Huber ring. The closure of A_0 inside $\widehat{A}$ is a ring of definition, which is also the I-adic completion $\widehat{A_0}$ of A_0. Also $\widehat{A} = \widehat{A_0} \otimes_{A_0} A$ as plain rings.*

Definition 2.2.3. *A Huber ring A is* ***Tate*** *if it has a* ***pseudo-uniformizer****, i.e., a topologically nilpotent* unit *ϖ.*

A Huber ring A is ***analytic*** *if the ideal generated by topologically nilpotent* elements *is A.*

Warning 2.2.4. *Being Tate does not mean the topology on A is ϖ-adic: $\mathbb{Z}_p[[T]]$ equipped with (p,T)-adic topology is a counter-example.*

Example 2.2.5. *By Example 2.1.9, any tfp K-affinoid algebra is Tate.*

Definition 2.2.6. *A* ***continuous valuation*** *on a topological ring A is a map $|-| : A \to 0 \cup \Gamma$ where Γ is a totally ordered abelian group Γ, such that $|-|$ is multiplicative, nonarchimedean and continuous, i.e.,*

- $|0| = 0, |1| = 1, |ab| = |a||b|$;
- $|a+b| \le \max(|a|,|b|)$;
- *For any $\gamma \in \Gamma$, $\{a \in A \,\big|\, |a| < \gamma\}$ is open in A.*

Two continuous valuations $|-|, |-|'$ are equivalent if $|a| \le |b|$ iff $|a|' \le |b|'$.

Remark 2.2.7. *We can replace Γ by the subgroup generated by the image of $|-|$ and obtain an equivalent valuation. After such replacements, two valuations are equivalent iff there is an isomorphism $\Gamma \simeq \Gamma'$ sending $|-|$ to $|-|'$.*

Notation 2.2.8. *We want to view valuations as points. Hence for a continuous valuation $x : A \to 0 \cup \Gamma$ of A, we write $|f(x)| \coloneqq |f|_x \coloneqq x(f)$.*

Definition 2.2.9. *For a Huber ring A, define $\mathrm{Cont}(A)$ to be the set of all continuous valuations of A. We equip $\mathrm{Cont}(A)$ with the topology generated by open subsets $\{x \in \mathrm{Cont}(A) \,\big|\, |f(x)| \le |g(x)| \ne 0\}$ for $f, g \in A$.*

A ***rational domain*** *of $\mathrm{Cont}(A)$ is a subset of the form*

$$U\Big(\frac{f_1,\dots,f_n}{g}\Big) \coloneqq \{x \in \mathrm{Cont}(A) \,\big|\, |f_i(x)| \le |g(x)| \ne 0\}$$

such that $(f_1, \cdots, f_n)$ is an open *ideal.*

The topology of $\mathrm{Cont}(A)$ is well-behaved:

Theorem 2.2.10 ([Hub93, Corollary 3.2])**.** *For a Huber ring A, $\mathrm{Cont}(A)$ is a* spectral space*. Moreover, any rational domain of $\mathrm{Cont}(A)$ is quasi-compact.*

Definition 2.2.11. *A topological space X is a* ***spectral space*** *if it satisfies the following equivalent conditions:*

- *X is quasi-compact, has a basis of quasi-compact open subsets stable under finite intersections, and every irreducible closed subset has a unique generic point (i.e. is* ***sober****).*
- *X is a projective limit of finite T_0-spaces*[11]*.*
- *$X = \operatorname{Spec} B$ for some commuative algebra B.*

A map between spectral spaces is ***spectral*** *if the inverse image of any quasi-compact open is quasi-compact.*

Remark 2.2.12. *To guarantee rational domains are quasi-compact, the condition $(f_1, \cdots, f_n)$ being open is necessary*[12]*: $A = \mathbb{C}_p\langle T\rangle, f = 0, g = T$ is a conterexample.*

[11] A space is T_0 if for any two distinct points, one of them has a neighborhood not containing the other

[12] For curious reader, Huber's proof uses this condition in [Hub93, Page 464, Proof of (b)].

Definition 2.2.13. *There is an obvious continuous map* $\mathrm{Cont}(A) \to \mathrm{Spec}(A)$ *sending a valuation to its kernel. For* $x \in \mathrm{Cont}(A)$*, the* ***completed residue field*** $K(x)$ *is the completion of the* ***residue field*** $\mathrm{Frac}(A/\ker(x))$ *with respect to the induced valuation* $\mathrm{Frac}(A/\ker(x)) \to 0 \cup \Gamma_x$*.*

A point $x \in \mathrm{Cont}(A)$ *is* ***analytic*** *if* $K(x)$ *is non-discrete, i.e., if* $\ker(x)$ *is not an open ideal of* A*.*

Remark 2.2.14. *For an open prime ideal* $\mathfrak{p} \subset A$*, there is an obvious non-analytic point given by the valuation* $A \to A/\mathfrak{p} \to 0 \cup 1$*. Any non-analytic point is of this form.*

Remark 2.2.15. *A point* $x \in \mathrm{Cont}(A)$ *is non-analytic iff* $f(x) = 0$ *for any topologically nilpotent element* f*. It follows that* $\mathrm{Cont}(A)$ *is analytic, i.e., all points of it are analytic, iff* A *is analytic.*

Remark 2.2.16. *The subset of analytic points of* $\mathrm{Cont}(A)$ *is open. Namely, for an ideal of definition* $(f_1, \cdots, f_n)$*, it is the union of* $U(\frac{f_1,\ldots,f_n}{f_i})$*.*

To get a feeling about the topology of $\mathrm{Cont}(A)$, we describe specializations and generizations inside it[13].

Definition 2.2.17. *Define* $\mathrm{Spv}\, A$ *to be* $\mathrm{Cont}(A^{\mathrm{discrete}})$*. In other words, we drop the continuous requirement for valuations.*

Remark 2.2.18. *Note that* $\mathrm{Cont}(A)$ *is a subspace of* $\mathrm{Spv}\, A$*.*

Definition 2.2.19. *A point* y *is a* ***primary specialization*** *of* x *if there exists a convex subgroup* $H \subset \Gamma_x$ *such that* $y(f) = x(f)$ *if* $x(f) \in H$ *and* $y(f) = 0$ *otherwise. In this case, we write* $y = x|_H$*. Note that (non-trivial) primary specializations change unnderlying prime ideals*[14]*.*

A point y *is a* ***secondary specialization***[15] *of* x *if there exists a convex subgroup* $H \subset \Gamma_y$ *such that* $\Gamma_x = \Gamma_y/H$ *and* $x(f) = y(f)H$*. In this case, we write* $x = y/H$*. Note that secondary specializations do not change underlying prime ideals.*

Remark 2.2.20. *For any totally ordered abelian group* Γ*, the set of its convex subgroups is totally ordered by inclusion. Hence primary specializations of* x *form a chain; secondary generalization of* x *form a chain. The length of such chains is the* ***rank*** *of* x*.*

Remark 2.2.21. *For fixed prime ideal* $\mathfrak{p} \subset A$*, points of* $\mathrm{Spv}\, A$ *over* $\mathfrak{p} \in \mathrm{Spec}\, A$ *are in bijection with valuation rings of the residue field* $\mathrm{Frac}(A/\ker(x))$*. Also,* y *is a secondary specialization of* x *iff the valuation ring corresponding to* y *is contained in that of* x*. Therefore the rank of* x *is also the rank of the corresponding valuation ring.*

Lemma 2.2.22 ([HK94, Proposition 1.2.4])**.** *Inside* $\mathrm{Spv}\, A$*, any specialization of* x *is a primary specialization of a secondary specialization.*

Lemma 2.2.23 ([HK94, Lemma 1.2.5])**.** *For prime ideals* $\mathfrak{p} \subset \mathfrak{q}$ *and a valuation* x *over* $\mathfrak{p}$*, primary specialization* y *of* x *over* $\mathfrak{q}$ *is unique if exists. Such* y *exists for* x *iff similar* y' *exists for some/any secondary specializaiton/generalization* x' *of* x*.*

On the other hand, for a valuation y *over* $\mathfrak{q}$*, primary generalization* x *of* y *over* $\mathfrak{p}$ *exists iff similar* x' *exists for some/any secondary specializaiton/generalization* y' *of* y*.*

Lemma 2.2.22 implies the following result.

Corollary 2.2.24. *If* x *is continuous, then its specializations inside* $\mathrm{Spv}\, A$ *are continuous. In particular, inside* $\mathrm{Cont}(A)$*, any specialization of* x *is a primary specialization of a secondary specialization.*

Lemma 2.2.25. *Inside* $\mathrm{Cont}(A)$*, all generalizations of an analytic point* y *are secondary generalizations.*

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[13]Results below are not used in Scholze's papers that Lin has read, but they are important when building the theory. Readers who want to treat Huber's work as a black box can skip to the next subsection.

[14]A nice pun!

[15]Lin thinks this is a pun: "primary" means both "changing prime ideals" and "non-secondary".

Proof. We need to show y is not a non-trivial primary specialization of any continuous valuation x. Suppose there is such x and $y = x|_H$ for a convex subgroup $H \subset \Gamma_x$. Then $H \neq \Gamma_x$ because otherwise $y = x$. Hence there is some $\lambda \notin H$ with $\lambda < 1$. By convexity, any element $\leq \lambda$ is not in H. Hence $y(f) = 0$ whenever $x(f) < \lambda$. Since x is continuous, such f form an open subset of A, which implies y is not analytic. □

3. Talk 3 by Lin: adic spaces

3.1. **Affine adic spaces.** A caveat of $\mathrm{Cont}(A)$ is its rational domain might not be of the form $\mathrm{Cont}(B)$ for any B. Note that $B := A\langle f_1/g, \dots, f_n/g\rangle$ does not work[16] because a continuous valuation v of $A\langle T\rangle$ does *not* force $v(T) \leq 1$.

To remedy this, we need to introduce a subring $B^+ \subset B$ remembering which elements are required to have valuations ≤ 1.

Lemma 3.1.1 ([Hub93, Lemma 3.3])**.** *The following sets are in bijection:*

- *The set of subsets $F \subset \mathrm{Cont}(A)$ of the form $\cap_{f\in S}\{x \,|\, |f(x)| \leq 1\}$ as S runs over arbitrary subsets of A;*
- *The set of open and integrally closed subrings A^+ of A.*

The bijection sends F to $\{f \in A \,|\, |f(x)| \leq 1, x \in F\}$, and sends A^+ to $\{x \in \mathrm{Cont}(A) \,|\, |f(x)| \leq 1, f \in A^+\}$.

Also, if A^+ is contained in the subring A° of power bounded elements, then the corresponding subset is dense in $\mathrm{Cont}(A)$*.*

This motivates the following definition:

Definition 3.1.2. *Let A be a Huber ring. A rubring $A^+ \subset A$ is a **ring of integral elements** if A^+ is open and integrally closed in A and $A^+ \subset A^\circ$.*

*A **Huber pair** (A, A^+) is a Huber ring A and a ring of integral elements $A^+ \subset A$.*

For a Huber pair (A, A^+), define $\mathrm{Spa}(A, A^+)$ *to be the subspace of* $\mathrm{Cont}(A)$ *containing valuations x with $|f(x)| \leq 1$ for any $f \in A^+$.*

We write $\mathrm{Spa}\,A := \mathrm{Spa}(A, A^\circ)$.

***Rational domains** of* $\mathrm{Spa}(A, A^+)$ *are defined as restrictions of rational domains of* $\mathrm{Cont}(A)$*.*

Remark 3.1.3. *If we do not require $A^+ \subset A^\circ$, then* $\mathrm{Spa}(A, A^+)$ *will be too small and even empty. In particular, the functor* Spa *from the opposite category of complete Huber pairs to the category of adic spaces will not be fully faithful or even conservative.*

Theorem 3.1.4 (Huber, [Hub93, Theorem 3.5])**.** *For a Huber pair (A, A^+),* $\mathrm{Spa}(A, A^+)$ *is a spectral space and any rational domain of it is quasi-compact.*

Proposition 3.1.5 ([Hub93, Proposition 3.9])**.** $\mathrm{Spa}(A, A^+)$ *only depends on the completion of (A, A^+).*

Proposition 3.1.6 ([Hub94, Proposition 1.3])**.** *For a Huber pair (A, A^+) and a rational domain $U(f_i/g)$ of* $\mathrm{Spa}(A, A^+)$*, there exists a complete (B, B^+) over (A, A^+) such that the map* $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *factors through U and is universal for such maps. Moreover, this map is a hoomeomorphism onto U.*

Proof. (Sketch) $C := A[g^{-1}]$ is a Huber ring. Let C^+ be the integral closure of the image of $A^+[f_1/g, \dots, f_n/g]$ in C. Define (B, B^+) to be its completion. □

Notation 3.1.7. *For a rational domain $U(f_i/g)$ of* $\mathrm{Spa}(A, A^+)$*, write $(A\langle f_i/g\rangle, A\langle f_i/g\rangle^+)$ for the complete Huber pair in the above proposition.*

Now we equip $\mathrm{Spa}(A, A^+)$ with more structures so that we can define adic spaces.

[16] This is claimed in [SW20, §3.3], but Lin doesn't know a counter-example.

Definition 3.1.8. *For a Huber pair (A, A^+), define a presheaf $\mathcal{O}$ (resp. $\mathcal{O}^+$) of complete topological rings on* $\operatorname{Spa}(A, A^+)$ *such that its value on a rational domain $U(f_i/g)$ is $A\langle f_i/g\rangle$ (resp. $A\langle f_i/g\rangle^+$) and its values on general opens are given by RKE.*

*A Huber pair (A, A^+) is **sheafy** if $\mathcal{O}$ is a sheaf*[17].

Counter-Example 3.1.9. *See [Hub94, End of §1] for a non-sheaf Huber pair.*

Theorem 3.1.10. *Let (A, A^+) be a complete Huber pair. Then it is sheaf in the following case*[18]*:*

(1) *(Schemes) A is discrete.*
(2) *(Formal schemes) A is finitely generated over a Noetherian ring of definition A_0.*
(3) *(Rigid spaces) A is Tate and any $A\langle T_1, \dots, T_n\rangle$ is Noetherian.*
(4) *(Perfectoid spaces) A is perfectoid (which we have not defined yet).*

Theorem 3.1.11 ([KL13, Theorem 2.4.23])**.** *For complete* analytic[19] *Huber pair (A, A^+), $H^i(\operatorname{Spa}(A, A^+), \mathcal{O}) = 0$ for $i > 0$.*

3.2. **Adic spaces.**

Definition 3.2.1. *Consider the category of $(X, \mathcal{O}_X, |-|_{x\in X})$, where $(X, \mathcal{O}_X)$ is a locally topologically ringed space and for each $x \in X$, $|-|_x$ is a continuous valuation of $\mathcal{O}_{X,x}$. For such an object, we write $\mathcal{O}_X^+$ to be the subsheaf of $\mathcal{O}_X$ whose sections f satisfies $|f|_x \le 1$.*

An affinoid adic space *is an object* $\operatorname{Spa}(A, A^+)$ *in this category, where (A, A^+) is a* sheaf *Huber pair.*

*An **adic space** is an object in this category locally given by affinoid adic spaces.*

Theorem 3.2.2 ([Hub94, Theorem 2.1])**.** *The functor $(A, A^+) \mapsto \operatorname{Spa}(A, A^+)$ froom sheafy complete Huber pairs to adic spaces is fully faithful.*

Remark 3.2.3. *Non-sheaf Huber pairs can be studied via* pre-adic spaces*. We will define them when we need to use them.*

3.3. **Examples.**

Example 3.3.1. *The final object is* $\operatorname{Spa}\mathbb{Z}$.

Example 3.3.2 (The adic closed unit disk)**.** *The space* $\operatorname{Spa}\mathbb{Z}[T]$ *represents the functor $X \mapsto \mathcal{O}_X^+(X)$. For a nonarchimedean field $\operatorname{Spa}\mathbb{Z}[T] \times \operatorname{Spa} K = \operatorname{Spa} K\langle T\rangle$. Let us look into its topology.*

There are five types of points of $\operatorname{Spa} K = \operatorname{Spa} K\langle T\rangle$:

(1) *The classical points: for $x \in K^\circ$, the valuation $f \mapsto |f(x)|_K$ corresponds to a classical point denoted by x, i.e., an element of the set $\operatorname{Sp} K\langle T\rangle$. These are closed points.*

(2),(3) *The rays of the tree: for a disk $D(x, r) = \{y \in K^\circ \mid |y - x| \le r\}$ with $0 < r \le 1$ and $x \in K^\circ$, the valuation $f \mapsto \sup_{y\in D(x,r)} |f(y)|_K$ corresponds to a point denoted by x_r (but it only depends on the disk $D(x, r)$). If $r \in |K^\times|$ is a valuation, then the point is said to be of type (2), which is not a closed point, otherwise of type (3), which is a closed point. A branching occurs exactly at type (2) points: this means $D(x, r) = D(x', r)$ for some $x \ne x'$.*

(4) *Dead ends of the tree: for a sequence of disks $D_1 \supset D_2 \supset \dots$ in K° such that $\cap D_i = \varnothing$ (such families exist e.g. for $\mathbb{C}_p$), the valuation $f \mapsto \inf_i \sup_{x\in D_i} |f(x)|_K$ corresponds to a point D_∞. There are closed points.*

(5) *Rank 2 points: for any $x \in K^\circ$, $0 < r \le 1$, $r \in |K^\times|$, and a sign $\pm$, where positive sign $+$ is only allowed when $r \ne 1$, consider the abelian group $\mathbb{R}^+ \times \gamma^{\mathbb{Z}}$, where $\gamma = r \pm$infinitesimal*[20]*. Then the rank-2 valuation $f = \sum a_n (T - x)^n \mapsto \max |a_n|_K \gamma^n$ corresponds to a point denoted by $x_r^\pm$. The point x_r^+ depends only on $D(x, r)$ and x_r^- only on*[21] *$D(x, < r)$. The points $x_r^\pm$ are in the closure of the type (2) point x_r.*

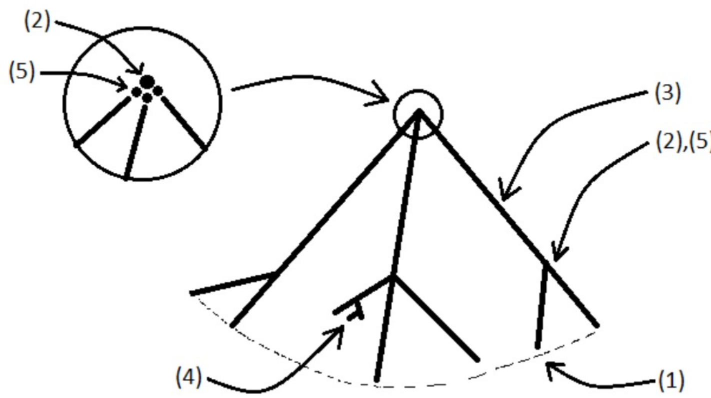
[17] This implies $\mathcal{O}^+$ is also a sheaf by Lemma 3.1.1

[18] See [Hub94, Theorem 2.2] for (1)-(3), [KL13, Theorem 3.6.14] for (4).

[19] A is analytic iff $\operatorname{Spa}(A, A^+)$ is analytic. Also, $x \in \operatorname{Spa}(A, A^+)$ is analytic iff a rational domain containing it is Tate. See [SW20, Proposition 4.3.1].

[20] This means e.g. if the sign is $+$, then $r < \gamma < r'$ for any $r' > r$. Lin suggests to view γ as a tie-breaker for the type (2) valuation x_r.

[21] Why?



Remark 3.3.3. *As mentioned in Remark 2.1.16,* $\operatorname{Spa} K\langle T\rangle$ *is the projective limit of all the formal models of* $\operatorname{Sp} K\langle T\rangle$*. We can start with the obvious formal model* $\operatorname{Spf} K^\circ\langle T\rangle$*, whose reduced scheme is the affine line* $\mathbb{A}^1_\kappa$ *over* $\kappa := \kappa(K^\circ)$*, then perform iterated blowups at closed points. Each blowup produces a* $\mathbb{P}^1_\kappa$*, and the strict transform of this* $\mathbb{P}^1_\kappa$ *survives in the projective limit and gives the closure of a point of type (2)*[22].

Example 3.3.4 (The adic affine line)**.** *The space* $\operatorname{Spa}(\mathbb{Z}[T],\mathbb{Z})$ *represents the functor* $X \mapsto \mathcal{O}_X(X)$*. For a nonarchimedean field* $\operatorname{Spa}(\mathbb{Z}[T],\mathbb{Z}) \times \operatorname{Spa} K = \bigcup_n \operatorname{Spa} K\langle \varpi^n T\rangle$*. Note that it is* not *quasi-compact.*

Example 3.3.5 (The closure of the adic closed unit disk in the adic affine line)**.** *The map* $\operatorname{Spa} K\langle T\rangle \to \operatorname{Spa}(\mathbb{Z}[T],\mathbb{Z}) \times \operatorname{Spa} K$ *is an* open immersion*. The closure is* $\operatorname{Cont}(K\langle T\rangle)$*: it contains an additional point* x_1^+ *defined similarly as type (5) points. Note that* x_1^+ *does not send* $K^\circ\langle T\rangle$ *to* ≤ 1*, hence is not a point of* $\operatorname{Spa} K\langle T\rangle$*.*

Example 3.3.6 (The open unit disk)**.** *The space* $\mathbb{D} := \operatorname{Spa}\mathbb{Z}[[T]]$*. For a nonarchimedean field* $\mathbb{D}_K := \operatorname{Spa}\mathbb{Z}[[T]] \times \operatorname{Spa} K = \bigcup_n \operatorname{Spa} K\langle T, T^n/\varpi\rangle$*.*

Example 3.3.7 (The punctured open unit disk)**.** *The space* $\mathbb{D}^\times := \operatorname{Spa}\mathbb{Z}((T))$*. For a nonarchimedean field* $\mathbb{D}^\times_K := \operatorname{Spa}\mathbb{Z}((T)) \times \operatorname{Spa} K = \mathbb{D}_K \smallsetminus \{T = 0\}$*.*

Example 3.3.8 (The constant adic space)**.** *For any profinite set* S*, the space* $\underline{S} := \operatorname{Spa} C^0(S,\mathbb{Z})$ *represents* $X \mapsto \operatorname{Maps}(|X|, S)$ *and* $|\underline{S}| = S \times |\operatorname{Spa}\mathbb{Z}|$*.*

Counter-Example 3.3.9 (Products do not exist in general)**.** $\operatorname{Spa}(\mathbb{Z}[T_1, T_2, \dots]) \times \operatorname{Spa} K$ *should be* $\operatorname{colim}_{n_i} \operatorname{Spa} K\langle \varpi^{n_1} T_1, \varpi^{n_2} T_2, \dots\rangle$*. But the connecting maps are not open immersions.*

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[22] How to relate the disk $D(x, r)$ to such a blowup?