

# Lecture 1

Sep. 10, 2024

# A slogan

$\infty$ -category theory = category theory + homotopy theory.

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- ▶ Examples: **Set**, **Grp**, **Ring**, **Top**...

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We will abandon all of them.

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Axioms: all given by **equalities**. (Doctrine (2))

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- ▶ **Strict monoidal category**: supply **associators**:

$$X \otimes (Y \otimes Z) \xrightarrow{\simeq} (X \otimes Y) \otimes Z$$

subject to certain coherent conditions.

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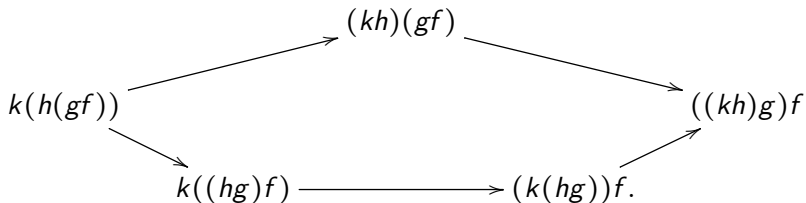
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  - ▶ *Principle of isomorphism.*
- ▶ For  $n \geq 2$ , we should never require two morphisms in a weak  $n$ -category to be **equal**.
- ▶ For  $n > k$ , we should never require two  $k$ -morphisms in a weak  $n$ -category to be **equal**.

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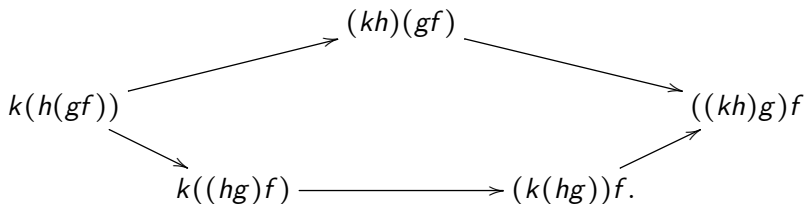
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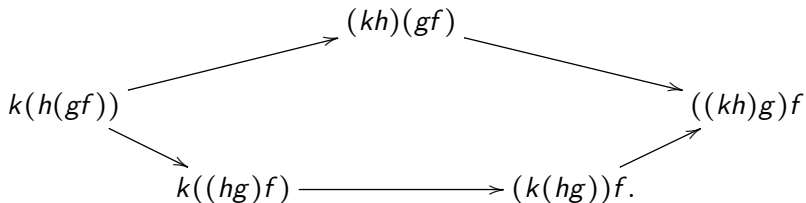
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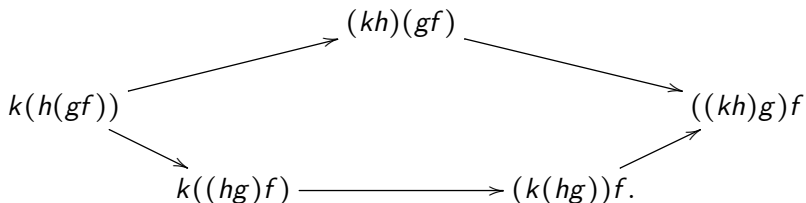
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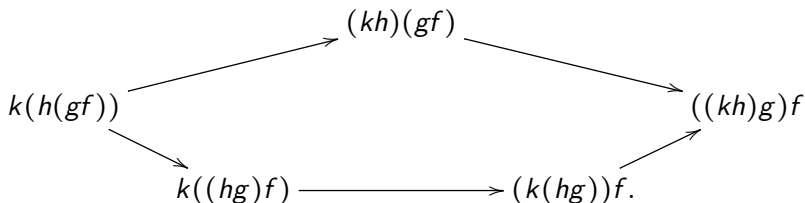
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- ▶ Coherence data for associativity can be **hidden** in the homotopy theory of spaces.
- ▶  **$\infty$ -groupoids**: morphisms and higher morphisms are all invertible.

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- ▶ Induction on  $k$ .
- ▶  $(\infty, 0)$ -categories :=  $\infty$ -groupoids := homotopy types.
- ▶ Induction step: coherence data for associativity is already incorporated into the theory of  $\infty$ -groupoids!

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- ▶  $\mathbf{hTop}$ : inverting homotopy equivalences in  $\mathbf{Top}$ .
- ▶  $\mathbf{hTop}_{\leq n}$ : full subcategory of **homotopy  $n$ -types**:  $\pi_k \simeq 0, k > n$ .

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*Show  $\pi_{\leq 1} : \mathbf{hTop}_{\leq 1} \rightarrow \mathbf{Grpd}$  is an equivalence.*

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## Exercise 5

*Can we define morphisms as **homotopy classes** of pathes?*

$\pi_{\leq 2}$  is not strict

- ▶ Pathes:  $f$ ,  $g$ ,  $h$ .

## $\pi_{\leq 2}$ is not strict

- Pathes:  $f, g, h$ .

$$(h \circ (g \circ f))(t) = \begin{cases} f(4t) & \text{for } t \in [0, 1/4] \\ g(4t - 1) & \text{for } t \in [1/4, 1/2] \\ h(2t - 1) & \text{for } t \in [1/2, 1] \end{cases}$$

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## Exercise 6

Challenge: show  $\pi_{\leq 2} : \mathbf{hTop}_{\leq 2} \rightarrow \mathbf{2-Grpd}$  is an equivalence.

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Thank you!