SEMINAR NOTES ON
GEOMETRIZATION OF THE LOCAL LANGLANDS COORRESPONDENCE

LSG

Contents

1. Talk I by Lin: the big picture

... once we could merely *formulate* Fargues' conjecture, enough mathinery is available to apply Lafforgue's ideas to get the "automorphic-to-Galois" direction...

[FS21, Sect. I.11]

1.1. What is the local Langlands correspondence?

Notation 1.1.1. *We fix the following notations:*

- E *is a local field (e.g. ...* $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t))$*)*[1]*;*
- G *is a reductive group over* E*;*
- $\hat{G}$ *is the Langlands dual group over* $\mathbb{Z}$*;*
- W_E *is the Weil group for* E*.*

Conjecture 1.1.2. *There is a canonical map between sets:*

$$\Big\{\textit{irreducible objects } \mathrm{Rep}_{\mathbb{C}}(G(E))\Big\} \to \Big\{W_E \to \hat{G}(\mathbb{C})\Big\},\ \pi \mapsto \varphi_\pi$$

subject to certain compatibilities.

Remark 1.1.3. *Several remarks are in order.*

(0) *In general this is not a bijection.*
(1) *From easy to hard: archimedean, char p nonarchimedean, char 0 nonarchimedean. E.g., the conjecture is unknown for $E = \mathbb{Q}_p$ and general redcutive group.*
(2) $G(E)$ *is a topological group and we require π to be* smooth*, i.e., any vector is fixed by some compact open subgroup.*

Notes taken by Lin.

[1]The colors are chosen to be compatible with [SW20, Figure 12.1].

(3) *W_E is a modification of* $\mathrm{Gal}(\overline{E}/E)$. $W_{\mathbb{C}} = \mathbb{C}^\times$, *$W_{\mathbb{R}}$ is the canonical extension of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *by* $\mathbb{C}^\times$. *For nonarchimedean E with residue field $k = \mathbb{F}_q$,*

$$\begin{tikzcd} 1 \arrow[r] & I_E \arrow[r] \arrow[d, "\simeq"] & W_E \arrow[r] \arrow[d, hook'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[r] \arrow[d, hook', "1\mapsto \mathrm{Frob}"] & 1 \\ 1 \arrow[r] & I_E \arrow[r] & \mathrm{Gal}(\overline{E}/E) \arrow[r] & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \arrow[r] & 1. \end{tikzcd}$$

The necessity of W_E instead of $\mathrm{Gal}(\overline{E}/E)$ can be already seen in local class field theory.

(4) *W_E is a topological group and we require φ_π to be continuous. However, it is* not *equipped with the subspace topology from $\mathrm{Gal}(\overline{E}/E)$. Instead, I_E has the subspace topology from $\mathrm{Gal}(\overline{E}/E)$, which is pro-finite, and is forced to be open in W_E, which is therefore locally pro-finite.*

(5) *φ_π is the so-called L-parameter of π.*

(6) *Part of the compatible requirements are about L-functions and ϵ-factors. Lin knows nothing about them. Maybe someone can explain later?*

1.2. **What is geometrization?**

Answer 1.2.1. *Do global geommetric Langlands on the Fargues–Fontaine curve, which behaves like a genus 0 curve in nonarchimedean geometry.*

To explain what this mean, we need some basic notions in analytic geometry.

Notation 1.2.2. *From now on, we restrict to the case $E = \mathbf{Q}_p$. There is no essential difference for general E, and the char p case is even easier.*

Analogy 1.2.3. *Tutorials on analytic geometry will be provided by Lin and Yuchen in the next weeks. For now, let us be satisfied by the following:*

	algebraic geometry	*analytic geometry*
affine	$\mathrm{Spec}\, R, R \in \mathrm{CAlg}$	$\mathrm{Spa}(R, R^+)$, (R, R^+) *is a* Huber pair*:* $R^+ \subset R \in \mathrm{CAlg}(\mathrm{Top})$ *satisfying certain conditions*
globalization	scheme *as locally ringed spaces*	(pre-)adic space *as locally topologically ringed spaces*
point	$\mathrm{Spec}\, K, K$ *is a field*	$\mathrm{Spa}(K, K^+), (K, K^+)$ *is an* affinoid field analytic *if K is nondiscrete* nonanalytic *if K is discrete*

Notation 1.2.4. *In most cases, people make the canonical choice $R^+ := R^\circ$ being the subring of power-bounded elements and write* $\mathrm{Spa}\, R := \mathrm{Spa}(R, R^\circ)$.

Remark 1.2.5. *Among all the (pre)-adic spaces, there is a class of objects, called* perfectoid spaces, *that are well-adapted to connect char 0 and char p. Affine perfectoid spaces are given by* $\mathrm{Spa}(R, R^+)$ *such that R is a* perfectoid ring. *Basic examples of perfectoid rings include $\mathbb{F}_p((t^{1/p^\infty}))$, which is the completion of $\bigcup_n \mathbb{F}_p((t))(t^{1/p^n})$, and $\mathbb{Q}_p^{\mathrm{cycl}}$, which is the completion of $\mathbb{Q}_p(\mu_{p^\infty})$. Any perfectoid ring is defined over $\mathbb{Z}_p$ although the latter itself is not a perfectoid ring.*

Personally I think the following result (together with the *tilting equivalence* to be explained later) is the root of all the magic:

Theorem 1.2.6. *There is* no *final object in* Perfd*, however, products exist in the category* Perfd_p *of perfectoid spaces of characteristic p.*

Analogy 1.2.7. *Sanath will talk about the details about FF curves. For now, let us be satisfied by the following:*

	Global Geometric Langlands for functional field	*Geometrized Local Langlands*
geometry	*algebraic geometry over* $\mathbb{F}_p$	*"perfectoid geometry"*
test objects	*schemes* $S \in \mathrm{Sch}_{/\mathbb{F}_p}$	*char* p *perfectoid space* $S \in \mathrm{Perfd}_p$
final test object	$\mathrm{Spec}\,\mathbb{F}_p$	**not exist** $\mathrm{Spa}\,\mathbb{F}_p \notin \mathrm{Perfd}_p$
spaces	*prestacks* $\supset$ *fpqc-stacks* $\supset$ *algebraic spaces*	*prestacks* $\supset$ *v-stacks* $\supset$ *diamonds*
absolute *curve*	X *over* $\mathbb{F}_p$	**not exist/sci-fi**
relative *curve*	$X_S \coloneqq S \times_{\mathbb{F}_p} X$	$\mathcal{X}_S \coloneqq \mathcal{Y}_S/\mathrm{Frob}_S \coloneqq (S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)/\mathrm{Frob}_S$

Question 1.2.8. *Wait, how dare you multiply a char* 0 *object* $\mathrm{Spa}\,\mathbb{Q}_p$ *with a char* p *object* S*!*

Warning 1.2.9. *There is a dot over the product sign in the notation* $S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$*, which means it is* not *a product, at least not naively. For example,* $\mathrm{Spa}(R, R^+) \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ *is an open subspace of* $\mathrm{Spa}\,W(R^+)$*, where* $W(R^+)$ *is the ring of* p*-Witt vectors in* R^+*. In fact, it is the open subspace where the functions* $p, [\varpi] \in W(R^+)$ *are invetible, where* $\varpi \in R^+$ *is a* pseudo-uniformizer*. Whatever this means, we see* $S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ *is of char* 0*.*

Question 1.2.10. *Wait, if* $\mathcal{X}_S$ *is of char* 0*, is it okay to study it using char* p *test objects? For example, when talking about Hecke modifications, you need a notion of Cartier divisors of* $\mathcal{X}_S$ *relative to the base, but where is your base? It can't be* S *or* S/Frob_S *because they are of char* p*.*

Answer 1.2.11. *No, at least not in the naive way. The correct way to relate* $\mathcal{X}_S$ *to char* p *objects is via its associated diamond* $(\mathcal{X}_S)^\diamond$*, which we will explain now.*

Construction 1.2.12 ([FS21, Sect. 6.2])**.** *For any commutative ring* R*, the* (p-)tilt *of* R *is*

$$R^\flat \coloneqq \lim(\dots \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R).$$

A priori this is only a multiplicative monoid. If R *is equipped with a good enough complete topology, such as a perfectoid ring, then one can define a ring structure where the addition law is*

$$(x^{(0)}, x^{(1)}, \dots) + (y^{(0)}, y^{(1)}, \dots) \coloneqq (z^{(0)}, z^{(1)}, \dots),$$

where

$$z^{(i)} \coloneqq \lim_{n\to\infty} (x^{(i+n)} + y^{(i+n)})^{p^n}.$$

In particular, when R *is a perfectoid ring, we obtain a* char p *perfectoid ring* $R^\flat$*. We say* R *is a* untilt of $R^\flat$.

We define

$$\mathrm{Spd}(R, R^+) \coloneqq \mathrm{Spa}(R, R^+)^\flat.$$

Using gluing, we can define $X^\flat$ *for any prefectoid space* X*.*

Remark 1.2.13. *For perfectoid ring* R*, we have* $(R^\flat)^\circ \simeq (R^\circ)^\flat$*.*

Example 1.2.14. *Any char* p *prefectoid ring* R *is the tilt of itself, and is the only char* p *untilt of itself. But there are char* 0 *untilts.*

Example 1.2.15. *The tilt of* $\mathbb{Q}_p^{\mathrm{cycl}}$ *is* $\mathbb{F}_p((t^{1/p^\infty}))$*.*

Theorem 1.2.16 (Tilting Equivalence)**.** *For any perfectoid space* X*, the functor* $Y \mapsto Y^\flat$ *induces an equivalence between the categories of perfectoid spaces over* X *and* $X^\flat$*. This equivalence preserves (finite) étale covers.*

Definition 1.2.17. *For any pre-adic space X, define $X^\diamond$ to be the prestack*

$$X^\diamond : \mathrm{Perfd}_p^{\mathrm{op}} \to \mathrm{Set},\ S \mapsto \coprod_{S^\sharp \in \mathrm{Untilt}(S)} \mathrm{Maps}(S^\sharp, X).$$

Remark 1.2.18. *In fact, Lin thinks the following is correct. Consider*

$$\mathrm{PreStk} := \mathrm{Funct}(\mathrm{Perfd}^{\mathrm{op}}, \mathrm{Set}), \mathrm{PreStk}_p := \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set})$$

and define

$$\diamond : \mathrm{PreStk} \to \mathrm{PreStk}_p$$

to be the unique colimit-preserving functor extending $\flat$. Then when restricted to pre-adic spaces, one recovers the above definition.

Theorem 1.2.19. *For pre-adic space X, the underlying topological spaces of X and $X^\diamond$ are canonically homeomorphic.*

Remark 1.2.20. *In fact, we have the following slogen: "$\diamond$ only remembers topological information."*

Example 1.2.21. $\mathrm{Spd}\,\mathbb{Z}_p := (\mathrm{Spa}\,\mathbb{Z}_p)^\diamond$ *classifies all untilts;* $\mathrm{Spd}\,\mathbb{Q}_p := (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$ *classifies all char 0 untilts.*

Example 1.2.22. *If X is already a perfectoid space, then $X^\diamond \simeq X^\flat$ by the tilting equivalence.*

Example 1.2.23. *For char p pre-adic space X, the functor $X \mapsto X^\diamond$ is just*

$$\mathrm{PreAdic}_p \to \mathrm{Funct}(\mathrm{PreAdic}_p^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Funct}(\mathrm{Perfd}_p^{\mathrm{op}}, \mathrm{Set}).$$

This is because only char p untilts $S^\sharp$ can map to X.

Example 1.2.24. *By the tilting equivalence, if X is the quotient of $R \rightrightarrows Y$ of perfectoid spaces connected by pro-étale maps, then $X^\diamond$ is the quotient of $R^\flat \rightrightarrows Y^\flat$. Essentially,* diamonds are defined to be *such quotients. In fact, any nalytic pre-adic space, which means all its residue fields are not discrete, over $\mathbb{Z}_p$ can be written as such a quotient.*

Remark 1.2.25. *Yifei will talk about the pro-étale topology and explain why it is powerful.*

Example 1.2.26. *Unfortunately/fortunately, $\mathrm{Spa}\,\mathbb{Q}_p$ is not a perfectoid space but it has a perfectoid pro-étale cover $\mathrm{Spa}\,\mathbb{Q}_p^{\mathrm{cycl}} \to \mathrm{Spa}\,\mathbb{Q}_p$ whose Galois group is $\mathbb{Z}_p^\times$. Hence*

$$\mathrm{Spd}\,\mathbb{Q}_p \simeq \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{cycl}}/\underline{\mathbb{Z}_p^\times} \simeq \mathrm{Spa}\,\mathbb{F}_p((t^{1/p^\infty}))/\underline{\mathbb{Z}_p^\times},$$

where $\underline{\mathbb{Z}_p^\times}$ is the discrete group diamond.

Theorem 1.2.27. *For any char p perfectoid space S, we have*

$$(S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)^\diamond \simeq S \times \mathrm{Spd}\,\mathbb{Q}_p.$$

Warning 1.2.28. *This is not formal. But Lin thinks $\dot{\times}$ was, or should have been, designed to make this correct.*

Remark 1.2.29. *For a char p perfectoid space S, a map $S \to \mathrm{Spd}\,\mathbb{Q}_p$ provides a char 0 untilt $S^\sharp$, which will provide a closed immersion $S^\sharp \to \mathcal{Y}_S := S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ once we know the precise definition of the target. This is a Cartier divisor and so is the composition $S^\sharp \to \mathcal{X}_S$. Also, the latter only depends on the composition $S \to \mathrm{Spd}\,\mathbb{Q}_p \to \mathrm{Spd}\,\mathbb{Q}_p/\,\mathrm{Frob}$. This suggests $\mathrm{Spd}\,\mathbb{Q}_p/\,\mathrm{Frob}$ should be the moduli prestack of Cartier divisors on FF curves.*

In fact, as we have seen (or will see) in geometric Langlands, we use $\overline{\mathbb{F}_p}$-points on the curve to define Hecke modifications. Hence we should restrict our attension to S defined over $\overline{\mathbb{F}_p}$ rather than $\mathbb{F}_p$. The effect is to change $\mathrm{Spd}\,\mathbb{Q}_p/\,\mathrm{Frob}$ to $\mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}}/\,\mathrm{Frob}$, where $\mathbb{Q}_p^{\mathrm{ur}} = \mathrm{Frac}(W(\overline{\mathbb{F}_p}))$ is the maximal unramified extension of $\mathbb{Q}_p$.

We can finally define the moduli of divisors on FF curves:

Definition 1.2.30. *The* moduli diamond of degree 1 closed Cartier divisors on FF curves *is*

$$\mathrm{Div}^1 \coloneqq \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}} / \mathrm{Frob}.$$

Warning 1.2.31. *Unlike in algebraic geometry,* Div^1 *is* not *the curve itself. They live in different characteristics. In fact, we do not have the absolute FF curve.*

Remark 1.2.32. *An amazing thing is this tilting/untilting game allows us to consider products of the "curve", or in fact, consider* $\mathrm{Div}^1 \times \cdots \times \mathrm{Div}^1$*. Unlike the self product of* $\mathrm{Spa}\,\mathbb{Q}_p$*, this product, which is taken in the category of diamonds, is not boring. Lin thinks this is essentially because* Perfd_p *lacks a final object but has products (which is false for* Perfd*).*

1.3. **Why Fargues–Fontaine curve?** Please read [FS21, Sect. I.11] (titled "The origin of the ideas"). For now, let me explain how the automorphic and Galois sides naturally appear in this geometrization picture.

For the Galois side:

Theorem 1.3.1. $\pi_1(\mathrm{Div}^1) \simeq W_E$.

Remark 1.3.2. *Heuristically this follows from the definition* $\mathrm{Div}^1 \coloneqq \mathrm{Spd}\,\mathbb{Q}_p^{\mathrm{ur}} / \mathrm{Frob}$*. Indeed,* W_E *is the extension of* $\mathbb{Z}\langle\mathrm{Frob}\rangle$ *by* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{ur}})$.

Question 1.3.3. *Wait, didn't you say FF curves behaved like genus 0 curves?*

Answer 1.3.4. *Yes. But* $\mathcal{X}_C$ *is not defined over* C *or any algebraically closed field. Instead, we have:*

Theorem 1.3.5. $\Gamma(\mathcal{X}_C, \mathcal{O}) \simeq \mathbb{Q}_p$.

For the automorphic side, we consider the v-stack Bun_G whose values $\mathrm{Bun}_G(S)$ classify G-torsors on $\mathcal{X}_S$. Taeuk will explain the precise meaning of the following:

Theorem 1.3.6. Bun_G *has a stratification labelled by the poset* $B(G)$ *such that each stratum is of the form* $*/H$*, where* H *is a group diamond which is an extension of a discrete group* $\underline{M(\mathbb{Q}_p)}$ *by a unipotent group, where* M *is an inner form of a Levi subgroup of* G*.*

Remark 1.3.7. *This is another example where FF curves behave like genus* 0*.*

Corollary 1.3.8. *For any* $\mathbb{Z}_l$*-algebra* Λ*, the category* $D(\mathrm{Bun}_G, \Lambda)$ *can be glued from categories* $\mathrm{Rep}_\Lambda(M(\mathbb{Q}_p))$ *for* M *being inner forms of Levi subgroups of* G*.*

Yifei will explain how to define $D(-,\Lambda)$ and play with them.

1.4. **What can be translated from Geometric Langlands?**

Answer 1.4.1. *Essentially any pure geometric constructions in Geometric Landlands can be or at least should be translated. Things already done in [FS21]: geometric Satake, Lafforgue's automorphic-to-Galois construction via shitukas, formulation of categorical Langlands conjecture, the spectral action...*

1.5. **What else?** The story is not complete without talking about p-adic Hodge theory. After all, FF curves were born during the study of Fontaine's peroid rings. If people want, we can make a digression to it.

References

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.

[SW20] Peter Scholze and Jared Weinstein. *Berkeley lectures on p-adic geometry*. Princeton University Press, 2020.