

## LECTURE 9

In this lecture, we introduce Kan extensions for  $\infty$ -categories, and use them to obtain several useful tools for computing (co)limits.

### 1. DEFINITION OF KAN EXTENSIONS

1.1. Let  $\delta : K_0 \rightarrow K$  be a morphism in  $\mathbf{Set}_\Delta$ . For an  $\infty$ -category  $\mathcal{D}$ , consider the functor

$$\mathbf{Fun}(K, \mathcal{D}) \xrightarrow{- \circ \delta} \mathbf{Fun}(K_0, \mathcal{D}).$$

We say  $F : K \rightarrow \mathcal{D}$  is a left (resp. right) Kan extension of  $F_0 : K_0 \rightarrow \mathcal{D}$  if it is the image of  $F_0$  under the *partially defined* left (resp. right) adjoint of the above functor. More precisely, we make the following definition.

**Definition 1.2.** Let  $\delta : K_0 \rightarrow K$  be a morphism in  $\mathbf{Set}_\Delta$  and  $F_0 : K_0 \rightarrow \mathcal{C}$  be a diagram in an  $\infty$ -category  $\mathcal{C}$ . For a diagram  $F : K \rightarrow \mathcal{C}$ , and a natural transformation  $\beta : F_0 \rightarrow F \circ \delta$ ,

$$\begin{array}{ccc} & K & \\ \delta \nearrow & \downarrow \beta & \searrow F \\ K_0 & \xrightarrow{F_0} & \mathcal{C} \end{array}$$

we say  $\beta$  **exhibits  $F$  as a left Kan extension of  $F_0$  along  $\delta$**  if for any diagram  $F' : K \rightarrow \mathcal{C}$

$$\mathbf{Maps}_{\mathbf{Fun}(K, \mathcal{C})}(F, F') \rightarrow \mathbf{Maps}_{\mathbf{Fun}(K_0, \mathcal{C})}(F_0 \circ \delta, F' \circ \delta) \rightarrow \mathbf{Maps}_{\mathbf{Fun}(K_0, \mathcal{C})}(F_0, F' \circ \delta)$$

is an equivalence between  $\infty$ -groupoids.

Dually, we say a natural transformation  $\beta : F \circ \delta \rightarrow F_0$ ,

$$\begin{array}{ccc} & K & \\ \delta \nearrow & \downarrow \beta & \searrow F \\ K_0 & \xrightarrow{F_0} & \mathcal{C} \end{array}$$

**exhibits  $F$  as a right Kan extension of  $F_0$  along  $\delta$**  if for any functor  $F' : K \rightarrow \mathcal{C}$ ,

$$\mathbf{Maps}_{\mathbf{Fun}(K, \mathcal{C})}(F', F) \rightarrow \mathbf{Maps}_{\mathbf{Fun}(K_0, \mathcal{C})}(F' \circ \delta, F_0 \circ \delta) \rightarrow \mathbf{Maps}_{\mathbf{Fun}(K_0, \mathcal{C})}(F' \circ \delta, F_0)$$

is an equivalence between  $\infty$ -groupoids.

1.3. Once we have established  $\infty$ -categorical Yoneda lemma, it is easy to show the pair  $(F, \beta)$  is essentially unique<sup>1</sup>. Hence we can talk about *the* right/left Kan extension of  $F_0$  along  $\delta$ , as long as we incorporate the natural transformation  $\beta$  as part of the data in its definition. We denote the corresponding diagrams to be

$$\mathrm{RKE}_\delta F_0, \mathrm{LKE}_\delta F_0.$$

When writing  $\mathrm{RKE}_\delta F_0$ , we always view it as an object in  $\mathrm{Fun}(K, \mathcal{C})$  equipped with a *canonical* lifting along

$$\mathrm{Fun}(K, \mathcal{C}) \xrightarrow[\mathrm{Fun}(K_0, \mathcal{C})]{\times} \mathrm{Fun}(K_0, \mathcal{C})_{/F_0} \rightarrow \mathrm{Fun}(K, \mathcal{C}).$$

1.4. Note that

$$(\mathrm{LKE}_\delta F_0)^{\mathrm{op}} \simeq \mathrm{RKE}_{\delta^{\mathrm{op}}} F_0^{\mathrm{op}}.$$

Hence in below, we focus on right Kan extensions.

1.5. As in the classical category theory, one can show taking adjoint functors is compatible with compositions. Therefore we have the following result:

**Proposition 1.6.** *Let  $K_0 \xrightarrow{\delta} K_1 \xrightarrow{\theta} K_2$  be morphisms in  $\mathrm{Set}_\Delta$  and  $F_0 : K_0 \rightarrow \mathcal{C}$  be a diagram in an  $\infty$ -category  $\mathcal{C}$ . Suppose  $\mathrm{RKE}_\delta F_0$  exists. Then we have a canonical equivalence*

$$\mathrm{RKE}_\theta(\mathrm{RKE}_\delta F_0) \xrightarrow{\simeq} \mathrm{RKE}_{\theta \circ \delta} F_0,$$

where the source exists iff the target does.

**Remark 1.7.** *The precise meaning of the above equivalence the following. Suppose  $\beta : F_1 \circ \delta \rightarrow F_0$  exhibits  $F_1$  as a right Kan extension of  $F_0$  along  $\delta$ , and  $\gamma : F_2 \circ \theta \rightarrow F_1$  exhibits  $F_2$  as a right Kan extension of  $F_1$  along  $\theta$ , then*

$$F_2 \circ \theta \circ \delta \xrightarrow{\gamma(\delta)} F_1 \circ \delta \xrightarrow{\beta} F_0$$

exhibits  $F_2$  as a right Kan extension of  $F_0$  along  $\theta \circ \delta$ .

**Theorem 1.8.** *Let  $\delta : K_0 \rightarrow \mathcal{K}$  be a morphism in  $\mathrm{Set}_\Delta$  and  $F_0 : K_0 \rightarrow \mathcal{C}$  be a diagram in an  $\infty$ -category  $\mathcal{C}$ . Suppose that*

- $\mathcal{K}$  is a quasi-category;
- For any object  $x \in \mathcal{K}$ , the limit of the diagram<sup>2</sup>

$$K_0 \times_{\mathcal{K}} \mathcal{K}_{x/} \xrightarrow{p_x} K_0 \xrightarrow{F_0} \mathcal{C}$$

exists.

Then  $\mathrm{RKE}_\delta F_0$  exists, and we have a canonical isomorphism

$$(1.1) \quad (\mathrm{RKE}_\delta F_0)(x) \simeq \lim_{K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}} (F_0 \circ p_x).$$

<sup>1</sup>This means the full sub- $\infty$ -category of

$$\mathrm{Fun}(K, \mathcal{C}) \xrightarrow[\mathrm{Fun}(K_0, \mathcal{C})]{\times} \mathrm{Fun}(K_0, \mathcal{C})_{/F_0}$$

consisting of pairs  $(F, \beta)$  that satisfy the definition of right Kan extensions is equivalent to [0].

<sup>2</sup>The fiber product  $K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$  is taken in the ordinary category  $\mathrm{Set}_\Delta$ . It also calculates the homotopy fiber product in  $\mathrm{Set}_\Delta^{\mathrm{Joyal}}$  because  $\mathcal{K}_{x/} \rightarrow \mathcal{K}$  is a categorical fibration. In particular, when  $K_0$  is also a quasi-category, we can view the above fiber product as taking in the quasi-category  $\mathcal{QCat}$ . As a consequence, we can state the proposition purely using the language of  $\infty$ -categories.

**Definition 1.9.** We say  $\mathrm{RKE}_\delta F_0$  is **pointwise**<sup>3</sup> if  $\delta$  and  $F_0$  satisfy the assumptions in Theorem 1.8.

**Remark 1.10.** Note that the construction  $x \mapsto K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$  is (contravariantly) functorial while the naive one  $x \mapsto K \times_{\mathcal{C}} \{x\}$  is not functorial.

**Remark 1.11.** The isomorphism (1.1) can be informally obtained as follows. Let  $F : \mathcal{K} \rightarrow \mathcal{C}$  be a fixed functor and  $\beta : F \circ \delta \rightarrow F_0$  be any natural transformation. For an object  $(y, f) \in K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$ , where  $y \in K_0$  and  $f : x \rightarrow \delta(y)$  is a morphism in  $\mathcal{C}$ , consider the composition

$$(1.2) \quad F(x) \xrightarrow{F(f)} F \circ \delta(y) \xrightarrow{\beta(y)} F_0(y).$$

This construction is functorial in  $(y, f)$  and therefore corresponds to a morphism

$$(1.3) \quad F(x) \rightarrow \lim_{(y, f) \in K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}} F_0(y)$$

as long as the target exists. Now the proposition claims (1.3) is invertible for any  $x$  iff  $\beta$  exhibits  $F$  as a right Kan extension of  $F_0$  along  $\delta$ . In other words, the natural transformation  $\beta$  is completely encoded in the morphisms (1.2) and their higher functorialities.

**Remark 1.12.** To translate the above construction into homotopy coherent language, we first notice that there is an obvious natural transformation from the constant functor  $\underline{x} : \mathcal{K}_{x/} \rightarrow \mathcal{K}$  to the forgetful functor  $\mathrm{oblv}$ . Precomposing with the projection functor  $K \times_{\mathcal{K}} \mathcal{K}_{x/} \rightarrow \mathcal{K}_{x/}$ , we obtain the left natural transformation in the following diagram:

$$\begin{array}{ccc} K_0 \times_{\mathcal{K}} \mathcal{K}_{x/} & \xrightarrow{\underline{x}} & \mathcal{K} \\ \searrow p_x & \Downarrow \delta & \downarrow \beta \\ & K_0 & \xrightarrow{F_0} \mathcal{C} \end{array}$$

Composing with  $\beta$ , we obtain a natural transformation  $\underline{F(x)} \rightarrow F_0 \circ p_x$ , which by definition corresponds to a lifting of  $\underline{F(x)}$  along

$$\mathcal{C}^{/F_0 \circ p_x} \rightarrow \mathcal{C},$$

where the source  $\mathcal{C}^{/F_0 \circ p_x}$  is the alternative slice category in [Lecture 6, §5]. Now the proposition claims that the above lifting is a final object (which is assumed to exist) for any  $x \in \mathcal{K}$  iff  $\beta$  exhibits  $F$  as the right Kan extension of  $F_0$  along  $\delta$ .

**Exercise 1.13.** For  $\pi : K \rightarrow \Delta^0$ , show that  $\mathrm{RKE}_\pi u$  exists iff  $\lim u$  exists. Moreover, we have

$$(\mathrm{RKE}_\pi u)(*) \simeq \lim u.$$

How to find the canonical lifting of  $(\mathrm{RKE}_\pi u)(*)$  in  $\mathcal{C}_{/u}$ ?

<sup>3</sup>Not all right Kan extensions are pointwise. Also, Lurie defined (see Ker.02Y9) a right Kan extension as

$$x \mapsto \lim_{K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}} (F_0 \circ p_x)$$

and proved (see Ker.0309) it satisfies the universal property in Definition 1.2. Therefore Lurie's right Kan extensions should be *pointwise* right Kan extensions in these notes.

**Warning 1.14.** Note that the fiber product  $K_0 \times_K K_{x/}$  also appears in the statement of Quillen's Theorem A, which claims the following two conditions are equivalent

- (i) The morphism  $K_0 \rightarrow K$  is final.
- (ii) The fiber product  $K_0 \times_K K_{x/}$  is weakly contractible for any  $x \in K$ .

Note however that (i) is related to colimits and therefore left Kan extensions, while Theorem 1.8 is related to right Kan extensions. I do not know how to relate these two results.

**Exercise 1.15.** Show that a right Kan extension  $\mathrm{RKE}_\delta F_0$  is pointwise iff it is preserved by any representable functor  $\mathrm{Maps}_C(c, -) : C \rightarrow \mathrm{Grpd}_\infty$ .

**Proposition 1.16.** Let  $\iota : K_0 \rightarrow K$  be a fully faithful functor between  $\infty$ -categories, and  $F_0 : K_0 \rightarrow C$  be any functor. Suppose the pointwise  $\mathrm{RKE}_\iota F_0$  exists<sup>4</sup>, then the canonical natural transformation

$$(\mathrm{RKE}_\iota F_0) \circ \iota \rightarrow F_0$$

is invertible.

*Sketch.* We only need to show for any  $x_0 \in K_0$  and  $x := \iota(x_0)$ , the evaluation morphism

$$\lim_{K_0 \times_K K_{x/}} (F_0 \circ p_x) \rightarrow F_0(x_0)$$

is invertible. We only need to show  $(x_0, x \xrightarrow{\iota} \iota(x_0)) \in K_0 \times_K K_{x/}$  is initial. But this follows from the obvious equivalence  $(K_0)_{x_0/} \rightarrow K_0 \times_K K_{x/}$ .  $\square$

**Exercise 1.17.** For an  $\infty$ -category  $K$ , consider the obvious embedding  $\iota : K \rightarrow K^\triangleleft$ . Show that the pointwise right Kan extension  $\mathrm{RKE}_\iota u$  exists iff the limit of  $u : K \rightarrow C$  exists. Moreover,  $\mathrm{RKE}_\iota u$  is a limit diagram extending  $u$ .

**Exercise 1.18.** Show that pointwise right Kan extension of any functor along  $K^\triangleleft \rightarrow \Delta^0$  always exists, and is given by evaluating the functor at the apex  $* \in K^\triangleleft$ .

## 2. LIMITS COMMUTE WITH LIMITS

2.1. From now on,  $u : K_1 \times K_2 \rightarrow C$  is a functor between  $\infty$ -categories<sup>5</sup>. Let  $\mathrm{pr}_i : K_1 \times K_2 \rightarrow K_i$  and  $\pi_i : K_i \rightarrow [0]$  be the projections.

**Exercise 2.2.** The pointwise right Kan extension  $\mathrm{RKE}_{\mathrm{pr}_1} u$  exists iff for any  $x \in K_1$ ,  $\lim_{K_2} u(x, -)$  exists. Moreover, for any  $x$ , there is a canonical isomorphism

$$(\mathrm{RKE}_{\mathrm{pr}_1} u)(x) \simeq \lim_{K_2} u(x, -).$$

2.3. As a consequence, we obtain a construction of

$$K_1 \rightarrow C, x \mapsto \lim_{K_2} u(x, -)$$

promised in the previous lectures. Combining with Proposition 1.6, we obtain the **distribution law of limits**.

<sup>4</sup>This means the right Kan extension exists and is pointwise

<sup>5</sup>In fact, these results can be generalized to the case when  $K_1$  and  $K_2$  are simplicial sets. However, one needs to slightly modify the constructions and statements. See HTT.5.5.2.3 for an example.

**Corollary 2.4.** *Suppose for any  $x \in K_1$ ,  $\lim_{K_2} u(x, -)$  exists. Then there is a canonical isomorphism*

$$\lim_{K_1 \times K_2} u \xrightarrow{\simeq} \lim_{x \in K_1} \lim_{K_2} u(x, -).$$

*Here the source exists iff the target does.*

**Remark 2.5.** *In future lectures, we will generalize the above result by replacing  $K_1 \times K_2 \rightarrow K_1$  by any Cartesian fibration between  $\infty$ -categories.*

**Corollary 2.6.** *Suppose:*

- *For any  $x \in K_1$ ,  $\lim_{K_2} u(x, -)$  exists.*
- *For any  $y \in K_2$ ,  $\lim_{K_1} u(-, y)$  exists.*

*Then there is a canonical isomorphism*

$$\lim_{x \in K_1} \lim_{K_2} u(x, -) \simeq \lim_{y \in K_2} \lim_{K_1} u(-, y).$$

*Here the LHS exists iff the RHS does.*

2.7. As an application, we obtain the following result about limits of  $\infty$ -categories.

**Theorem 2.8.** *Let  $u : K \rightarrow \mathbf{Cat}_\infty$ ,  $i \mapsto C_i$ , be a diagram of  $\infty$ -categories. Consider the evaluation functors*

$$\mathrm{ev}_i : \lim_{i \in K} C_i \rightarrow C_i.$$

*Then there is a canonical equivalence*

$$(2.1) \quad \mathrm{Maps}_{\lim_{i \in K} C_i}(x, y) \xrightarrow{\simeq} \lim_{i \in K} \mathrm{Maps}_{C_i}(\mathrm{ev}_i(x), \mathrm{ev}_i(y))$$

*between  $\infty$ -groupoids.*

*Sketch.* By definition,

$$\mathrm{Maps}_C(x, y) \simeq \{x\}_{\mathrm{Fun}(\{0\}, C)} \times_{\mathrm{Fun}(\{0\}, C)} \mathrm{Fun}(\Delta^1, C) \times_{\mathrm{Fun}(\{1\}, C)} \{y\}$$

can be written as a limit. Now the theorem follows from Corollary 2.6 and the fact that  $\mathrm{Fun}(J, -)$  preserves limits.  $\square$

**Remark 2.9.** *Note that we also have*

$$(2.2) \quad (\lim_{i \in K} C_i)^\simeq \xrightarrow{\simeq} \lim_{i \in K} C_i^\simeq.$$

*because taking cores is a right adjoint. In practice, most results, if not all, about  $\lim_{i \in K} C_i$  are proven using (2.1) and (2.2).*

### 3. HOW TO COMMUTE LIMITS WITH COLIMITS?

3.1. For simplicity, we assume  $C$  admits  $K_1$ -indexed limits and  $K_2$ -indexed colimits. In particular, Exercise 2.2 and its dual imply all the LKE and RKE in below exist.

**Construction 3.2.** *We have an obvious commutative diagram*

$$\begin{array}{ccc} \mathrm{Fun}(K_1 \times K_2, C) & \xleftarrow{-\circ \mathrm{pr}_1} & \mathrm{Fun}(K_1, C) \\ \mathrm{op}_2 \uparrow & & \uparrow -\circ \pi_1 \\ \mathrm{Fun}(K_2, C) & \xleftarrow{-\circ \pi_2} & \mathrm{Fun}([0], C). \end{array}$$

As in classical category theory, we can pass to left adjoints along the horizontal direction and obtain a natural transformation<sup>6</sup>

$$\begin{array}{ccc} \text{Fun}(K_1 \times K_2, C) & \xrightarrow{\text{LKE}_{\text{pr}_1}} & \text{Fun}(K_1, C) \\ \circ \text{pr}_2 \uparrow & \searrow & \uparrow - \circ \pi_1 \\ \text{Fun}(K_2, C) & \xrightarrow{\text{LKE}_{\pi_2}} & \text{Fun}([0], C). \end{array}$$

We can then pass to right adjoints along the vertical direction and obtain a natural transformation

$$(3.1) \quad \begin{array}{ccc} \text{Fun}(K_1 \times K_2, C) & \xrightarrow{\text{LKE}_{\text{pr}_1}} & \text{Fun}(K_1, C) \\ \text{RKE}_{\text{pr}_2} \downarrow & \nearrow \alpha & \downarrow \text{RKE}_{\pi_1} \\ \text{Fun}(K_2, C) & \xrightarrow{\text{LKE}_{\pi_2}} & \text{Fun}([0], C). \end{array}$$

In particular, for any  $u : K_1 \times K_2 \rightarrow C$ , we obtain a canonical morphism

$$\text{colim}_{y \in K_2} \lim_{K_1} u(-, y) \rightarrow \lim_{x \in K_1} \text{colim}_{K_2} u(x, -).$$

**Definition 3.3.** We say  $K_1$ -indexed limits commute with  $K_2$ -indexed colimits in  $C$  if (3.1) is invertible.

3.4. In the next lecture, we will explain the following results:

**Theorem 3.5.** In  $\text{Grpd}_\infty$ , filtered colimits commute with finite limits.

**Remark 3.6.** Once we have introduced compactly generated  $\infty$ -categories, it is easy to deduce that the above theorem holds in any such  $\infty$ -category.

**Warning 3.7.** The definitions of filtered or finite index  $\infty$ -categories are subtle. For example, filtered or finite ordinary category may fail to be filtered or finite as  $\infty$ -categories.

**Theorem 3.8.** In  $\text{Grpd}_\infty$ , sifted colimits commute with finite products.

**Theorem 3.9.** In  $\text{Grpd}_\infty$ , small colimits are preserved by base-changes.

#### APPENDIX A. DECOMPOSITION OF DIAGRAMS

**Exercise A.1.** Let  $K$  be a simplicial set and  $K_1, K_2 \subset K$  be simplicial subsets of  $K$  such that

$$K_1 \bigsqcup_{K_1 \cap K_2} K_2 \rightarrow K$$

is an isomorphism. Let  $u : K \rightarrow C$  be a diagram in an  $\infty$ -category. Show that there is a canonical isomorphism

$$\lim_K u \xrightarrow{\cong} \lim_{K_1} u \times_{\lim_{K_1 \cap K_2} u} \lim_{K_2} u.$$

Here the source exists if the target does.

**Exercise A.2.** Show that the equalizer of  $x \rightrightarrows y$  is isomorphic to  $x \times_{x \times y} x$ .

<sup>6</sup>We will explain this in details in future lectures.

**Exercise A.3.** *Let  $K$  be a simplicial set and  $\mathbf{Sub}(K)$  be the partially ordered set of simplicial subsets of  $K$ , viewed as an ordinary category. Let  $I \rightarrow \mathbf{Sub}(K)$ ,  $i \mapsto K_i$  be a functor between ordinary categories. Find a sufficient condition such that for any diagram  $u : K \rightarrow \mathbf{C}$ , there is a canonical isomorphism*

$$\lim_K u \xrightarrow{\cong} \lim_{i \in I} \lim_{K_i} u.$$

A.4. **Suggested readings.** HTT.4.2.3.