

LECTURE 16

In this lecture, we introduce t-structures on stable ∞ -categories.

1. DEFINITIONS

1.1. Let \mathcal{C} be a stable ∞ -category. A t-structure on \mathcal{C} is just a t-structure on the triangulated category $\mathrm{h}\mathcal{C}$.

Definition 1.2. Let \mathcal{C} be a stable ∞ -category. A **t-structure** on \mathcal{C} is defined to be a pair of full sub- ∞ -categories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of \mathcal{C} such that when we write

$$\mathcal{C}^{\leq n} := \mathcal{C}^{\leq 0}[-n], \quad \mathcal{C}^{\geq n} := \mathcal{C}^{\geq 0}[-n]$$

we have:

- (0) Both $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are stable under isomorphisms in \mathcal{C} .
- (1) For $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 1}$, we have $\mathrm{Maps}_{\mathcal{C}}(X, Y) \simeq \{0\}$.
- (2) We have inclusions $\mathcal{C}^{\leq -1} \subseteq \mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$.
- (3) For any $X \in \mathcal{C}$, there exists a fiber-cofiber sequence $X' \rightarrow X \rightarrow X''$ such that $X' \in \mathcal{C}^{\leq 0}$ and $X'' \in \mathcal{C}^{\geq 1}$.

Warning 1.3. We use the **cohomological convention**. To compare with notations in the homological convention, let

$$\mathcal{C}_{\leq n} := \mathcal{C}^{\geq -n}, \quad \mathcal{C}_{\geq n} := \mathcal{C}^{\leq -n}$$

Exercise 1.4. The assignment

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \mapsto (\mathrm{h}\mathcal{C}^{\leq 0}, \mathrm{h}\mathcal{C}^{\geq 0})$$

gives a bijection between t-structures on \mathcal{C} with t-structures on the triangulated category $\mathrm{h}\mathcal{C}$.

Exercise 1.5. The assignment

$$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}) \mapsto ((\mathcal{C}^{\geq 0})^{\mathrm{op}}, (\mathcal{C}^{\leq 0})^{\mathrm{op}})$$

gives a bijection between t-structures on \mathcal{C} with t-structures on $\mathcal{C}^{\mathrm{op}}$.

Exercise 1.6. Show that for $m < n$, $\mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n} \simeq \{0\}$.

Lemma 1.7. Let $n \geq 0$. For any $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq -n}$, the mapping space $\mathrm{Maps}_{\mathcal{C}}(X, Y)$ is a homotopy n -type.

Sketch. First note that the connected components of $\mathrm{Maps}_{\mathcal{C}}(X, Y)$ are weakly homotopy equivalent to each other because it is the loop space of $\mathrm{Maps}_{\mathcal{C}}(X, \Sigma Y)$. Hence we only need to show $\pi_m \mathrm{Maps}_{\mathcal{C}}(X, Y) \simeq 0$ for $m > n$ and the base point $0 \in \mathrm{Maps}_{\mathcal{C}}(X, Y)$. This follows from observation that $\Omega^{n+1} \mathrm{Maps}_{\mathcal{C}}(X, Y) \simeq \mathrm{Maps}_{\mathcal{C}}(X, \Omega^{n+1} Y)$ is weakly contractible because $\Omega^{n+1} Y \in \mathcal{C}^{\geq 1}$. \square

Definition 1.8. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. The **heart** of this t -structure is defined to be

$$\mathcal{C}^\heartsuit := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}.$$

Theorem 1.9. The ∞ -category \mathcal{C}^\heartsuit is an ordinary abelian category. Moreover, $0 \rightarrow X'' \rightarrow X \rightarrow X' \rightarrow 0$ is a short exact sequence in \mathcal{C}^\heartsuit iff $X'' \rightarrow X \rightarrow X'$ is a fiber-cofiber sequence of \mathcal{C} contained in \mathcal{C}^\heartsuit .

Sketch. By Lemma 1.7, \mathcal{C}^\heartsuit is an ordinary category. It follows it can be identified with $\mathrm{h}\mathcal{C}^{\leq 0} \cap \mathrm{h}\mathcal{C}^{\geq 0}$, which is the heart of the triangulated category $\mathrm{h}\mathcal{C}$. Now the claims follow from the corresponding well-known claims for triangulated categories. \square

Warning 1.10. A t -structure is not determined by its heart. Most information in a stable ∞ -category cannot be recovered from the heart. For instance, for $X, Y \in \mathcal{C}^\heartsuit$, the groups $\mathrm{Ext}_{\mathcal{C}}^\bullet(X, Y)$ and $\mathrm{Ext}_{\mathcal{C}^\heartsuit}^\bullet(X, Y)$ are not isomorphic in general.

Definition 1.11. Let \mathcal{C} and \mathcal{D} be stable ∞ -categories equipped with t -structures. For an exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$, we say

- the functor F is **left t -exact** if it sends $\mathcal{C}^{\geq 0}$ into $\mathcal{D}^{\geq 0}$
- the functor F is **right t -exact** if it sends $\mathcal{C}^{\leq 0}$ into $\mathcal{D}^{\leq 0}$
- the functor F is **t -exact** if it is both left t -exact and right t -exact.

Exercise 1.12. Show that the left adjoint of a left t -exact functor is right t -exact.

Exercise 1.13. Construct an exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ that preserves the hearts but is not t -exact.

2. COHOMOLOGIES

Proposition-Definition 2.1. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. For any n , the inclusion functor $\mathcal{C}^{\leq n} \rightarrow \mathcal{C}$ admits a right adjoint $\tau^{\leq n}$, and the inclusion functor $\mathcal{C}^{\geq n} \rightarrow \mathcal{C}$ admits a left adjoint $\tau^{\geq n}$. These functors are called the **truncation functors** for the t -structure.

Remark 2.2. One can memorize the above handedness as $\mathcal{C}^{\leq 0} \rightleftarrows \mathcal{C} \rightleftarrows \mathcal{C}^{\geq 1}$.

Exercise 2.3. Prove the above proposition by verifying the following claim. Let $X \in \mathcal{C}$ and $X' \rightarrow X \rightarrow X''$ be any fiber-cofiber sequence with $X' \in \mathcal{C}^{\leq 0}$ and $X'' \in \mathcal{C}^{\geq 1}$, then

- (1) For any $Y \in \mathcal{C}^{\leq 0}$, the morphism $X' \rightarrow X$ induces equivalences

$$\mathrm{Maps}_{\mathcal{C}}(Y, X') \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{C}}(Y, X).$$

- (2) For any $Z \in \mathcal{C}^{\geq 1}$, the morphism $X \rightarrow X''$ induces equivalences

$$\mathrm{Maps}_{\mathcal{C}}(X, Z) \xrightarrow{\sim} \mathrm{Maps}_{\mathcal{C}}(X'', Z).$$

Deduce that $X' \simeq \tau^{\leq 0} X$ and $X'' \simeq \tau^{\geq 1} X$.

Exercise 2.4. We have Cartesian squares

$$\begin{array}{ccc} \mathcal{C}^{\leq 0} & \xrightarrow{\subseteq} & \mathcal{C} \\ \downarrow & & \downarrow \tau^{\geq 1} \\ \{0\} & \xrightarrow{\subseteq} & \mathcal{C}^{\geq 1} \end{array} \quad \begin{array}{ccc} \mathcal{C}^{\geq 0} & \xrightarrow{\subseteq} & \mathcal{C} \\ \downarrow & & \downarrow \tau^{\leq -1} \\ \{0\} & \xrightarrow{\subseteq} & \mathcal{C}^{\leq -1} \end{array}$$

In particular, $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ determine each other.

Exercise 2.5. The full sub- ∞ -category $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}$ is stable under colimits, while $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}$ is stable under limits.

Exercise 2.6. A limit in $\mathcal{C}^{\leq 0}$ is isomorphic to the $\tau^{\leq 0}$ -truncation of the corresponding limit in \mathcal{C} . Dually, a colimit in $\mathcal{C}^{\geq 0}$ is isomorphic to the $\tau^{\geq 0}$ -truncation of the corresponding colimit in \mathcal{C} .

2.7. The following exercises say that for a fiber-cofiber sequence $X' \rightarrow X \rightarrow X''$, amplitude estimations of two terms give an estimation for the third one.

Exercise 2.8. Let $X' \rightarrow X \rightarrow X''$ be a fiber-cofiber sequence.

- (1) If $X', X'' \in \mathcal{C}^{\leq 0}$, then $X \in \mathcal{C}^{\leq 0}$.
- (2) If $X \in \mathcal{C}^{\leq 0}$ and $X'' \in \mathcal{C}^{\leq -1}$, then $X' \in \mathcal{C}^{\leq 0}$.
- (3) If $X \in \mathcal{C}^{\leq 0}$ and $X' \in \mathcal{C}^{\leq 1}$, then $X'' \in \mathcal{C}^{\leq 0}$.

In particular, the full sub- ∞ -categories $\mathcal{C}^{\leq 0} \subseteq \mathcal{C} \supseteq \mathcal{C}^{\geq 0}$ are stable under extensions.

Exercise 2.9. Give examples to show the above estimations are optimal.

Exercise 2.10. The truncation functors $\tau_{\leq \bullet}$ and $\tau_{\geq \bullet}$ commute with each other.

Remark 2.11. The precise meaning of the above exercise consists of the following. For m and n , we have

- $\tau_{\geq m} \circ \tau_{\geq n} \simeq \tau_{\geq \max\{m, n\}}$, $\tau_{\leq m} \circ \tau_{\leq n} \simeq \tau_{\leq \max\{m, n\}}$.
- The commutative square

$$\begin{array}{ccc} \mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n} & \xrightarrow{\quad \subseteq \quad} & \mathcal{C}^{\leq m} \\ \downarrow \subseteq & & \downarrow \subseteq \\ \mathcal{C}^{\geq n} & \xrightarrow{\quad \subseteq \quad} & \mathcal{C} \end{array}$$

is left adjointable along the horizontal direction, and the induced commutative square

$$\begin{array}{ccc} \mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathcal{C}^{\leq m} \\ \downarrow \subseteq & & \downarrow \subseteq \\ \mathcal{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathcal{C} \end{array}$$

is right adjointable along the vertical direction, i.e., induces a commutative square

$$\begin{array}{ccc} \mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathcal{C}^{\leq m} \\ \tau^{\leq m} \uparrow & & \tau^{\leq m} \uparrow \\ \mathcal{C}^{\geq n} & \xleftarrow{\tau^{\geq n}} & \mathcal{C} \end{array}$$

Definition 2.12. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. Consider the functor

$$H^n : \mathcal{C} \xrightarrow{\tau^{\leq n} \circ \tau^{\geq n}} \mathcal{C}^{\leq n} \cap \mathcal{C}^{\geq n} \xrightarrow{[-n]} \mathcal{C}^\heartsuit.$$

For $X \in \mathcal{C}$, we call $H^n(X) \in \mathcal{C}^\heartsuit$ the n -th cohomology of X .

Warning 2.13. It may happen that $H^n(X) \simeq 0$ while X is not isomorphic to 0. For instance, $\mathcal{C}^{\leq 0} := \mathcal{C}$ and $\mathcal{C}^{\geq 0} := \{0\}$ give a t -structure with $\mathcal{C}^\heartsuit \simeq \{0\}$. There are lots of interesting examples in

- non-regular *algebraic geometry*
- infinite type *algebraic geometry*
- infinite dimensional *representation theory*
- ...

2.14. The following result follows from the corresponding well-known result for triangulated categories.

Proposition 2.15. *Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. For any fiber-cofiber sequence $X' \rightarrow X \rightarrow X''$, we have a long exact sequence in \mathcal{C}^\heartsuit*

$$\cdots \rightarrow H^n(X') \rightarrow H^n(X) \rightarrow H^n(X'') \xrightarrow{\delta} H^{n+1}(X') \rightarrow \cdots,$$

where the connecting morphism δ is induced by the morphism $X'' \rightarrow X'[1]$.

2.16. The following result relates t -exact functors with exact functors between the hearts.

Exercise 2.17. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a left t -exact functor. Show that the composition*

$$H^0 F : \mathcal{C}^\heartsuit \rightarrow \mathcal{C}^{\geq 0} \xrightarrow{F} \mathcal{D}^{\geq 0} \xrightarrow{\tau^{\leq 0}} \mathcal{D}^\heartsuit$$

is left exact.

Warning 2.18. *As in Warning 1.10, even a t -exact functor F cannot be recovered from $H^0 F$.*

Remark 2.19. *Next time, we will define various versions of derived ∞ -categories of an abelian categories \mathcal{A} , which are equipped with t -structures whose hearts can be identified with \mathcal{A} . By definition, all information about these derived ∞ -categories can be recovered from \mathcal{A} .*

Under certain assumptions, a left/right t -exact functor F out of these derived ∞ -categories can be identified with the right/left derived functor of $H^0 F$.

3. BOUNDED, SEPARATED AND COMPLETE

Definition 3.1. *Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. For an object $X \in \mathcal{C}$,*

- we say X is **connective** if $X \in \mathcal{C}^{\leq 0}$;
- we say X is **coconnective** if $X \in \mathcal{C}^{\geq 0}$;
- we say X is **n -connective** if $X \in \mathcal{C}^{\leq -n}$;
- we say X is **n -coconnective** if $X \in \mathcal{C}^{\geq -n}$;
- we say X is **eventually connective**, or **right bounded** if

$$X \in \mathcal{C}^- := \bigcup \mathcal{C}^{\leq n}$$

- we say X is **eventually coconnective**, or **left bounded** if

$$X \in \mathcal{C}^+ := \bigcup \mathcal{C}^{\geq n}$$

- we say X is **bounded** if X is both left bounded and right bounded:

$$X \in \mathcal{C}^b := \mathcal{C}^+ \cap \mathcal{C}^-.$$

Warning 3.2. *As in Warning 2.13, the above properties cannot be tested via the cohomologies.*

Exercise 3.3. *Show that if X is left bounded, then $X \in \mathcal{C}^{\geq 0}$ iff $H^i(X) \simeq 0$ for $i < 0$.*

Definition 3.4. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure.

- We say \mathcal{C} is **right bounded** if $\mathcal{C} \simeq \mathcal{C}^-$.
- We say \mathcal{C} is **left bounded** if $\mathcal{C} \simeq \mathcal{C}^+$.
- We say \mathcal{C} is **bounded** if $\mathcal{C} \simeq \mathcal{C}^b$.
- We say \mathcal{C} is **right separated** if

$$\mathcal{C}^{\geq \infty} := \bigcap_n \mathcal{C}^{\geq n} \simeq \{0\}.$$

- We say \mathcal{C} is **left separated** if

$$\mathcal{C}^{\leq -\infty} := \bigcap_n \mathcal{C}^{\leq -n} \simeq \{0\}.$$

Definition 3.5. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. The **left completion** $\widehat{\mathcal{C}}$ of \mathcal{C} is defined to be the limit of the following diagram

$$\dots \xrightarrow{\tau} \mathcal{C}^{\geq -2} \xrightarrow{\tau} \mathcal{C}^{\geq -1} \xrightarrow{\tau} \mathcal{C}^{\geq 0} \xrightarrow{\tau} \dots.$$

We say \mathcal{C} is **left complete** if the functor $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is an equivalence.

Remark 3.6. Dually, the **right completion** is defined to be the limit of

$$\dots \xrightarrow{\tau} \mathcal{C}^{\leq 2} \xrightarrow{\tau} \mathcal{C}^{\leq 1} \xrightarrow{\tau} \mathcal{C}^{\leq 0} \xrightarrow{\tau} \dots.$$

Most unbounded t -structures appearing naturally are right complete, hence people do not give a notation to the right completion. Note however that the right completion can be identified with $(\widehat{\mathcal{C}^{\text{op}}})^{\text{op}}$.

Exercise 3.7. Show that the left completion $\widehat{\mathcal{C}}$ is stable and the functor $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is exact.

Exercise 3.8. Show that the left completion $\widehat{\mathcal{C}}$ admits an essential unique t -structure such that $\mathcal{C}^{\geq 0} \xrightarrow{\simeq} (\widehat{\mathcal{C}})^{\geq 0}$.

Exercise 3.9. Show that \mathcal{C} is left separated iff the functor $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ detects zero objects. In particular, \mathcal{C} is left separated if it is left complete.

Warning 3.10. A left separated t -structure may not be left complete. For example, any left bounded t -structure is left separated, but is almost never left complete.

Exercise 3.11. Suppose \mathcal{C} admits countable products and $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}$ is stable under countable products¹. Then \mathcal{C} is left separated iff it is left complete.

Exercise 3.12. Show that \mathcal{C} is left complete iff it is **Postnikov complete**. The latter means

- Every object $X \in \mathcal{C}$ is the limit of its **Postnikov tower**:

$$X \simeq \lim [\dots \rightarrow \tau^{\geq n-1} X \rightarrow \tau^{\geq n} X \rightarrow \tau^{\geq n+1} X \rightarrow \dots]$$

- Any **Postnikov tower** in \mathcal{C} converges. In other words, any collection $X_n \in \mathcal{C}^{\geq n}$ equipped with isomorphisms $\tau^{\geq n} X_{n-1} \xrightarrow{\simeq} X_n$ is the Postnikov tower of

$$X := \lim [\dots \rightarrow X_{n-1} \rightarrow X_n \rightarrow X_{n+1} \rightarrow \dots] \in \mathcal{C}$$

¹These conditions are often referred as: taking countable products is right t -exact.

4. t-STRUCTURES ON PRESENTABLE STABLE ∞ -CATEGORIES

Proposition-Definition 4.1 (HA.1.4.4.13). *Let \mathcal{C} be a presentable stable ∞ -category equipped with a t -structure. The following conditions are equivalent:*

- *The ∞ -category $\mathcal{C}^{\leq 0}$ is presentable.*
- *The ∞ -category $\mathcal{C}^{\leq 0}$ is accessible.*
- *The ∞ -category $\mathcal{C}^{\geq 0}$ is presentable.*
- *The ∞ -category $\mathcal{C}^{\geq 0}$ is accessible.*
- *The composition $\mathcal{C} \xrightarrow{\tau_{\leq 0}} \mathcal{C}^{\leq 0} \rightarrow \mathcal{C}$ is accessible.*
- *The composition $\mathcal{C} \xrightarrow{\tau_{\geq 0}} \mathcal{C}^{\geq 0} \rightarrow \mathcal{C}$ is accessible.*

*We say a t -structure on \mathcal{C} is **accessible** if it satisfies the above conditions.*

Proposition 4.2 (HA.1.4.4.11). *Let \mathcal{C} be a presentable stable ∞ -category. A full sub- ∞ -category $\mathcal{C}' \subseteq \mathcal{C}$ determines an accessible t -structure with $\mathcal{C}^{\leq 0} \simeq \mathcal{C}'$ iff*

- *\mathcal{C}' is accessible*
- *$\mathcal{C}' \subseteq \mathcal{C}$ is closed under colimits and extensions.*

Remark 4.3. *In practice, one can apply the above proposition to the smallest full sub- ∞ -category \mathcal{C}' generated by a small collection of objects under small colimits and extensions. Such \mathcal{C}' is always accessible.*

Definition 4.4. *Let \mathcal{C} be a presentable stable ∞ -category. We say a t -structure on \mathcal{C} is **compatible with filtered colimits** if taking filtered colimits is t -exact.*

Exercise 4.5. *A t -structure on \mathcal{C} is compatible with filtered colimits iff taking small coproducts is t -exact.*

Definition 4.6. *Let \mathcal{C} be a presentable stable ∞ -category. We say a t -structure on \mathcal{C} is **compactly generated** if $\mathcal{C}^{\leq 0}$ is generated by a small collection of compact objects in \mathcal{C} under small colimits and extensions.*

Exercise 4.7. *Let \mathcal{C} be a presentable stable ∞ -category equipped with a compactly generated t -structure.*

- (1) *Show that the t -structure is right complete and compatible with filtered colimits.*
- (2) *Show that \mathcal{C} is compactly generated, and compact objects in \mathcal{C} are exactly given by $X[n]$ such that X is compact object in $\mathcal{C}^{\leq 0}$.*

Exercise 4.8. *Let \mathcal{C}_0 be a small stable ∞ -category equipped with a bounded t -structure. Show that $\mathcal{C} := \mathrm{Ind}(\mathcal{C}_0)$ has a compactly generated t -structure with $\mathcal{C}^{\leq 0} := \mathrm{Ind}(\mathcal{C}_0^{\leq 0})$.*

Warning 4.9. *Compact objects in \mathcal{C} may not be stable under truncations. Hence not all compactly generated t -structures come from ind-completion. There are lots of interesting examples in the settings listed in Warning 2.13.*

Remark 4.10. *In practice, most t -structures on presentable stable ∞ -categories are right complete and compatible with filtered colimits, or even compactly generated.*

5. t-STRUCTURE ON Sptr

Proposition-Construction 5.1 (HA.1.4.3.4). *Let \mathcal{C} be a pointed presentable ∞ -category. We have an accessible t -structure on $\mathrm{Sptr}(\mathcal{C})$ given by the following.*

- Let $\mathrm{Sptr}(\mathcal{C})^{\geq 1}$ be the full sub- ∞ -category of objects X such that $\Omega^\infty(X) \simeq 0$.
- Let $\mathrm{Sptr}(\mathcal{C})^{\leq 0}$ be the full sub- ∞ -category generated by the essential image of the functor Σ^∞ under extensions and small colimits.

Exercise 5.2. Show that $\mathrm{Sptr}(\mathrm{Sptr}(\mathcal{C})^{\leq 0}) \rightarrow \mathrm{Sptr}(\mathcal{C})$ is a t -exact equivalence.

Proposition 5.3 (HA.1.4.3.6). Let Sptr be equipped with the above t -structure. Then

- (1) This t -structure is compactly generated² and left complete.
- (2) The Eilenberg–MacLane spectrum functor $A \mapsto \mathbb{H}A$ gives an equivalence $\mathrm{Ab} \xrightarrow{\sim} \mathrm{Sptr}^\heartsuit$.

Exercise 5.4. Show that the cohomology functor $H^i : \mathrm{Sptr} \rightarrow \mathrm{Sptr}^\heartsuit$ can be identified with π_{-i} via the equivalence $\mathrm{Ab} \xrightarrow{\sim} \mathrm{Sptr}^\heartsuit$.

Warning 5.5. Let A_1 and A_2 be abelian groups. In general, the extension groups $\mathrm{Ext}_{\mathrm{Sptr}}^i(\mathbb{H}A_1, \mathbb{H}A_2)$ and $\mathrm{Ext}^i(A_1, A_2)$ are not isomorphic. For instance, the graded ring $\mathrm{Ext}_{\mathrm{Sptr}}^\bullet(\mathbb{H}\mathbb{F}_q, \mathbb{H}\mathbb{F}_q)$ is the **mod p Steenrod algebra**.

5.6. Next time, we will construct a natural homomorphism

$$\mathrm{Ext}^i(A_1, A_2) \rightarrow \mathrm{Ext}_{\mathrm{Sptr}}^i(\mathbb{H}A_1, \mathbb{H}A_2)$$

which should be viewed as derived direct images along $\mathrm{Spec} \mathbb{Z} \rightarrow \mathrm{Spec} S$.

APPENDIX A. PRESTABLE ∞ -CATEGORIES

A.1. Let \mathcal{C} be a stable ∞ -category equipped with a t -structure. Sometimes it is more convenient to study $\mathcal{C}^{\leq 0}$ rather than \mathcal{C} . It is possible to find several axioms that characterize ∞ -categories of the form $\mathcal{C}^{\leq 0}$. Such ∞ -categories are called **prestable ∞ -categories**.

A.2. **Suggested readings.** SAG.C.

²Compact generation is not proved in HA, but it follows from the fact that Spc_* is compactly generated.