## LECTURE 11

In this lecture, we discuss the $\infty$-categorical Yoneda lemma.

### 1. Representable functors

**Definition 1.1.** *Let $\mathsf{C}$ be an $\infty$-category and $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ be a functor. We say an object $\eta \in F(x)$ **exhibits** $F$ **as represented by an object** $x \in \mathsf{C}$ if, for any object $y$, the composition*

$$\mathsf{Maps}_{\mathsf{C}^{\mathsf{op}}}(x, y) \to \mathsf{Maps}_{\mathsf{Grpd}_\infty}(F(x), F(y)) \xrightarrow{\mathsf{ev}_\eta} F(y) \tag{1.1}$$

*is an equivalence between $\infty$-groupoids. We say $F$ is **representable** if such $x$ and $\eta$ exist.*

*Dually, for a functor $G : \mathsf{C} \to \mathsf{Grpd}_\infty$, we say an object $\eta \in G(x)$ **exhibits** $G$ **as corepresented by an object** $x \in \mathsf{C}^{\mathsf{op}}$ if, for any object $y$, the composition*

$$\mathsf{Maps}_{\mathsf{C}}(x, y) \to \mathsf{Maps}_{\mathsf{Grpd}_\infty}(G(x), G(y)) \xrightarrow{\mathsf{ev}_\eta} G(y)$$

*is an equivalence between $\infty$-groupoids. We say $G$ is **corepresentable** if such $x$ and $\eta$ exist.*

1.2. Note that $\eta \in F(x)$ exhibits $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ as represented by $x \in \mathsf{C}$ iff the same object exhibits $F$ as corepresented by the corresponding object $x \in \mathsf{C}^{\mathsf{op}}$. Hence we will focus on representable functors.

**Remark 1.3.** *The composition* (1.1) *is defined up to homotopy. Note however that equivalences are invariant under homotopy.*

**Remark 1.4.** *The composition* (1.1) *sends $\mathsf{id}_x$ to $\eta \in F(x)$.*

1.5. It is easy to see Definition 1.1 is invariant under equivalences. In particular:

(1) Suppose $\eta_1, \eta_2 \in F(x)$ are isomorphic, then $\eta_1$ exhibits $F$ as represented by $x$ iff $\eta_2$ does so.
(2) Let $f : x \to x'$ be an isomorphism, then $\eta \in F(x)$ exhibits $F$ as represented by $x$ iff $F(f)(\eta) \in F(x')$ exhibits $F$ as represented by $x'$.

**Warning 1.6.** *It may happen that both $\eta_1$ and $\eta_2$ exhibit $F$ as represented by $x$, but they are not isomorphic. Namely, if $\eta_1$ exhibits $F$ as represented by $x$, then any automorphism $f : x \to x$ produces an object $F(f)(\eta_1)$ satisfying the same property. One can show $\eta_1$ and $F(f)(\eta_1)$ are isomorphic iff $f$ is homotopic to $\mathsf{id}_x$.*

**Exercise 1.7.** *Let $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ be any functor. Show that there is a functor $\pi_0 F : \mathsf{hC}^{\mathsf{op}} \to \mathsf{Set}$, unique up to unique equivalence, that fits into the following*

---

*Date*: Oct. 25, 2024.

*commutative diagram*

$$\begin{tikzcd} \mathsf{C}^{\mathsf{op}} \arrow[r, "F"] \arrow[d] & \mathsf{Grpd}_\infty \arrow[d, "\pi_0"] \\ \mathsf{hC}^{\mathsf{op}} \arrow[r, "\pi_0 F"'] & \mathsf{Set}. \end{tikzcd}$$

*Show that if $F$ is representable, then $\pi_0 F$ is representable as a functor between ordinary categories.*

**Warning 1.8.** *The converse of Exercise 1.7 is false. In other words, in Definition 1.1, we cannot replace* (1.1) *by the map*

$$\pi_0 \mathsf{Maps}_{\mathsf{C}^{\mathsf{op}}}(x, y) \to \pi_0 \mathsf{Maps}_{\mathsf{Grpd}_\infty}(F(x), F(y)) \xrightarrow{\mathsf{ev}_\eta} \pi_0 F(y).$$

**Exercise 1.9.** *Consider the constant functor $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ with value $\{*\}$. The unique object in $F(x)$ exhibits $F$ as represented by $x$ iff $x$ is final in $\mathsf{C}$. Note however that $\pi_0 F$ is representable iff $\mathsf{hC}$ has a final object.*

1.10. However, we also have the following result:

**Proposition 1.11.** *Let $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ be a representable functor. Then $\eta \in F(x)$ exhibits $F$ as represented by $x \in \mathsf{C}$ iff for any object $y$, $\eta$ induces a bijection*

$$\pi_0 \mathsf{Maps}_{\mathsf{C}^{\mathsf{op}}}(x, y) \to \pi_0 F(y)$$

*between sets.*

*Proof.* By assumption, there exists $\eta' \in F(x')$ that exhibits $F$ as represented by $x'$. Hence by definition, the object $\eta \in F(x)$ corresponds to a morphism $x' \xleftarrow{\alpha} x$ in $\mathsf{C}$ such that $F(x') \to F(x)$ sends $\eta'$ to an object isomorphic to $\eta$. It follows that we have a commutative diagram

$$\begin{tikzcd} \pi_0 \mathsf{Maps}_{\mathsf{C}^{\mathsf{op}}}(x, y) \arrow[r] \arrow[d, "- \circ \alpha"] & \pi_0 F(y) \arrow[d, equal] \\ \pi_0 \mathsf{Maps}_{\mathsf{C}^{\mathsf{op}}}(x', y) \arrow[r, "\simeq"'] & \pi_0 F(y). \end{tikzcd}$$

Unwinding the definitions, we have:

$\eta$ exhibits $F$ as represented by $x$
$\Leftrightarrow$ $\alpha$ is an isomorphism in $\mathsf{C}^{\mathsf{op}}$
$\Leftrightarrow$ $\alpha$ is an isomorphism in $\mathsf{hC}^{\mathsf{op}}$
$\Leftrightarrow$ For any $y$, the left vertical map is bijective
$\Leftrightarrow$ For any $y$, the top horizontal map is bijective.

□

**Remark 1.12.** *The ordinary category $\mathsf{hC}$ has a natural (strict) enrichment over the ordinary symmetric monoidal category $\mathsf{hGrpd}_\infty$ (see [Lecture 4, §10]), at least when $\mathsf{C}$ is locally small. Denote this $\mathsf{hGrpd}_\infty$-enriched category[1] by $\mathbb{h}\mathsf{C}$. One can similarly define the notion of representable $\mathsf{hGrpd}_\infty$-enriched functors $\mathbb{h}\mathsf{C}^{\mathsf{op}} \to \mathbb{h}\mathsf{Grpd}_\infty$. Then tautologically, a functor $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ is representable iff the enriched functor $\mathbb{h}F$ is so.*

[1]Warning: in HTT, Lurie denoted it just by $\mathsf{hC}$, the same symbol as the ordinary homotopy category.

## 2. Representable functors as left Kan extensions

2.1. Note that a pair $(F,\eta)$ can be viewed as a diagram

$$\begin{tikzcd} & \mathsf{C}^{\mathrm{op}} \arrow[dr, dotted, "F"] & \\ \Delta^0 \arrow[ur, "x"] \arrow[rr, "\{*\}"'] & \arrow[u, phantom, "\Uparrow\eta"] & \mathsf{Grpd}_\infty \end{tikzcd} \tag{2.1}$$

**Proposition 2.2.** *Let $\mathsf{C}$ be an $\infty$-category and $(F,\eta)$ as above. Then $\eta$ exhibits $F$ as represented by $x$ iff*

(i) *For each $y \in \mathsf{C}^{\mathrm{op}}$, the $\infty$-groupoid $\mathsf{Maps}_{\mathsf{C}^{\mathrm{op}}}(x,y)$ is essentially small;*
(ii) *In (2.1), $\eta$ exhibits $F$ as a left Kan extension of the constant functor $\{*\} : \Delta^0 \to \mathsf{Grpd}_\infty$ along $x : \Delta^0 \to \mathsf{C}^{\mathrm{op}}$.*

*Sketch.* First note that if $F$ is represented by $x$, then each $\mathsf{Maps}_{\mathsf{C}^{\mathrm{op}}}(x,y)$ is equivalent to $F(y) \in \mathsf{Grpd}_\infty$ and therefore essentially small. Hence we restrict to the case when (i) is true. By definition,

$$(\mathsf{C}^{\mathrm{op}})_{/y} \underset{\mathsf{C}^{\mathrm{op}}}{\times} \Delta^0 \simeq \mathsf{Maps}_{\mathsf{C}^{\mathrm{op}}}(x,y).$$

It follows that

$$(\mathsf{LKE}_x\{*\})(y) \simeq \underset{\mathsf{Maps}_{\mathsf{C}^{\mathrm{op}}}(x,y)}{\mathsf{colim}} \{*\} \simeq \mathsf{Maps}_{\mathsf{C}^{\mathrm{op}}}(x,y). \tag{2.2}$$

Now (ii) is equivalent to:

- The natural transformation $\mathsf{LKE}_x\{*\} \to F$ induced by $\eta$ is invertible.

This is equivalent to

- For any $y \in \mathsf{C}^{\mathrm{op}}$, the morphism $(\mathsf{LKE}_x\{*\})(y) \to F(y)$ induced by $\eta$ is invertible.

Unwinding the definitions, the above morphism is homotopic to the composition of (1.1) and (2.2), which implies the desired claim. □

**Corollary 2.3.** *Let $\mathsf{C}$ be an $\infty$-category and $x \in \mathsf{C}$ be a fixed object satisfying the size condition (i). Then the pairs $(F, \eta \in F(x))$ such that $\eta$ exhibits $F$ as represented by $x$ are essentially unique.*

**Remark 2.4.** *The precise meaning of the above corollary is as follows. The pairs $(F, \eta \in F(x))$ are classified by the fiber product of the following $\infty$-categories:*

$$\begin{tikzcd} & (\mathsf{Grpd}_\infty)_{\{*\}/} \arrow[d] \\ \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Grpd}_\infty) \arrow[r, "\mathsf{ev}_x"] & \mathsf{Grpd}_\infty. \end{tikzcd}$$

*Now the proposition claims the full sub-$\infty$-category of this fiber product consisting of those pairs satisfying the representability condition, is equivalent to $[0]$.*

2.5. By the above corollary, after enlarging the size bound of objects in $\mathsf{Grpd}_\infty$, we can talk about *the* functor $\mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ represented by an object $x \in \mathsf{C}$, as long as we incorporate the object $\eta$ as part of the data in its definition. We *denote* this functor by $\mathsf{h}_x$. When using these notation, we always view it as an object in $\mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$ equipped with a *canonical* object in $\mathsf{h}_x(x)$. We often denote this object by $\mathsf{id}_x$ because of Remark 1.4. Now Proposition 2.2 can be informally write as

$$\mathsf{h}_x \xleftarrow{\simeq} \mathsf{LKE}_x\{*\}$$

Dually, the functor corepresented by $x$ is denoted by $\mathsf{h}^x$.

**Exercise 2.6.** *Describe the corepresentable functor* $\mathsf{h}^x$ *as a* left *Kan extension.*

## 3. Yoneda lemma

**Theorem 3.1.** *Let* $\mathsf{C}$ *be an* $\infty$*-category and* $\mathsf{h}_x : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ *be the functor represented by* $x$*. Then for any functor* $G : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$*, the composition*

$$\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(\mathsf{h}_x, G) \to \mathsf{Maps}_{\mathsf{Grpd}_\infty}(\mathsf{h}_x(x), G(x)) \xrightarrow{\mathsf{ev}_{\mathsf{id}_x}} G(x)$$

*is an equivalence between* $\infty$*-categories.*

*Proof.* Via the equivalence $\mathsf{h}_x \xleftarrow{\simeq} \mathsf{LKE}_x\{*\}$, this composition is

$$\begin{aligned}\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(\mathsf{LKE}_x\{*\}, G) \to \mathsf{Maps}_{\mathsf{Fun}(\Delta^0,\mathsf{Grpd}_\infty)}((\mathsf{LKE}_x\{*\})|_x, G|_x) \to \\ \to \mathsf{Maps}_{\mathsf{Fun}(\Delta^0,\mathsf{Grpd}_\infty)}(\{*\}, G|_x) \simeq G(x),\end{aligned}$$

which is an equivalence by the definition of $\mathsf{LKE}$. □

**Exercise 3.2.** *Prove the last claim in Warning 1.6.*

3.3. By Proposition 2.2, the functor $\mathsf{h}_x$ is characterized by the conclusion of Theorem 2.2. In fact, the following stronger result is true.

**Proposition 3.4.** *Let* $\mathsf{C}$ *be an* $\infty$*-category and* $x \in \mathsf{C}$ *be an object satisfying the size condition (i). For a functor* $F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$ *and* $\eta \in F(x)$*, the following are equivalent:*

1. $\eta$ *exhibits* $F$ *as a functor represented by* $x$*;*
2. *For any functor* $G : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$*,* $\eta$ *induces an equivalence*
$$\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(F, G) \to G(x)$$
*between* $\infty$*-categories.*
3. *For any functor* $G : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$*,* $\eta$ *induces an equivalence*
$$\pi_0\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(F, G) \to \pi_0 G(x)$$
*between sets.*

*Proof.* Only (3)⇒(1) requires proof. The size assumption implies $\mathsf{h}_x$ exists. By Theorem 3.1, there is a natural transformation $\alpha : \mathsf{h}_x \to F$ sending $\mathsf{id}_x \in \mathsf{h}_x(x)$ to an object isomorphic to $\eta \in F(x)$. It follows that we have a commutative diagram

$$\begin{tikzcd}
\pi_0\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(F, G) \arrow[r, "\simeq"] \arrow[d, "-\circ\alpha"] & \pi_0 G(x) \arrow[d, equal] \\
\pi_0\mathsf{Maps}_{\mathsf{Fun}(\mathsf{C}^{\mathsf{op}},\mathsf{Grpd}_\infty)}(\mathsf{h}_x, G) \arrow[r, "\simeq"'] & \pi_0 G(x).
\end{tikzcd}$$

This implies $\alpha$ is an isomorphism in $\mathsf{hFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$ and therefore an isomorphism in $\mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$, which implies (1). □

**Remark 3.5.** *In this remark, we compare Proposition 1.11 and Proposition 3.4.*

*Once we have developed the theory of* partially defined *adjoint functors. We can rephrase the conditions in Definition 1.1 and Proposition 1.11 respectively as*

(a) *$\eta$ exhibits $x$ as the* potential value *of $\{*\} \in \mathsf{Grpd}_\infty$ under the left adjoint functor of*
$$F : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty.$$
(a′) *$\eta$ exhibits $x$ as the* potential value *of $\{*\} \in \mathsf{hGrpd}_\infty$ under the left adjoint functor of*
$$\mathsf{h}F : \mathsf{hC}^{\mathsf{op}} \to \mathsf{hGrpd}_\infty.$$

*Proposition 1.11 says if the partially defined left adjoint of $F$ is defined at $\{*\}$, then its value can be detected by the homotopy categories.*

*Similarly, we can rephrase (2) and (3) in Proposition 3.4 respectively as:*

(b) *$\eta$ exhibits $F$ as the* potential value *of $\{*\} \in \mathsf{Grpd}_\infty$ under the left adjoint functor of*
$$\mathsf{ev}_x : \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty) \to \mathsf{Grpd}_\infty,$$
(b′) *$\eta$ exhibits $F$ as the* potential value *of $\{*\} \in \mathsf{hGrpd}_\infty$ under the left adjoint functor of*
$$\mathsf{h}(\mathsf{ev}_x) : \mathsf{hFun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty) \to \mathsf{hGrpd}_\infty,$$

*Proposition 3.4 says if the partially defined left adjoint of $\mathsf{ev}_x$ is defined at $\{*\}$, then its value can be detected by the homotopy categories.*

*In fact, similar claim is true for the partially defined adjoint functor of any functor.*

**Exercise 3.6.** *Apply the above paradigm to final objects in an $\infty$-category. Compare with [Lecture 6, Exercise 2.8].*

**Exercise 3.7.** *Find what is wrong with the following argument.*

- *False claim: If the limit of a diagram $u : K \to \mathsf{C}$ exists, then it can be calculated in the homotopy category $\mathsf{hC}$.*
- *Fake proof: the limit can be written as a $\mathsf{LKE}$, which is the value of a partially defined left adjoint.*

## 4. Yoneda embedding

4.1. In this section, we provide *a* construction of the functor
$$\mathsf{Maps}_{\mathcal{C}}(-,-) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Grpd}_\infty$$
via simplicial categories. In future lectures, we will provide a pure quasi-categorical construction using the theory of left fibrations.

**Construction 4.2.** *Let $\mathcal{C}$ be a locally small[2] quasi-category and consider the simplicial category $\mathbb{C} := \mathfrak{C}(\mathcal{C})$. Let $\mathbb{C} \to \mathbb{C}'$ be a fibrant replacement. Consider the composition of simplicial functors:*
$$\mathfrak{C}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}) \to \mathfrak{C}(\mathcal{C}^{\mathsf{op}}) \times \mathfrak{C}(\mathcal{C}) \simeq \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to (\mathbb{C}')^{\mathsf{op}} \times \mathbb{C}' \to \mathbb{Kan},$$

[2] This means $\mathsf{Maps}_{\mathcal{C}}(x, y)$ is small for any objects $x$ and $y$.

*where the last functor is given by* $\mathsf{Hom}_{\mathsf{C}'}(-,-)$*. By adjunction, we obtain a functor*

$$\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathfrak{N}_\bullet(\mathbb{K}\mathrm{an}) =: \mathcal{K}an.$$

**Exercise 4.3.** *The equivalence class of the obtained functor* $\mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{K}an$ *does not depend on the choice of* $\mathbb{C}'$*. In other words, we obtain a well-defined* element *in*

$$\pi_0(\mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{K}an)^{\simeq}).$$

**Exercise 4.4.** *The above equivalence class is invariant under equivalences in* $\mathcal{C}$*. In other words, for an equivalence* $\mathcal{C} \to \mathcal{D}$*, the map*

$$\pi_0(\mathsf{Fun}(\mathcal{D}^{\mathsf{op}} \times \mathcal{D}, \mathcal{K}an)^{\simeq}) \to \pi_0(\mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{K}an)^{\simeq}).$$

*preserves the corresponding elements.*

4.5. As a consequence, for a locally small $\infty$-category $\mathsf{C}$, we obtain a functor

$$\mathsf{Maps}_{\mathsf{C}}(-,-) : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{Grpd}_\infty$$

whose equivalence class does not depend on any choice.

**Exercise 4.6.** *Up to equivalence, the above functor lifts the functor* $\mathsf{C}^{\mathsf{op}} \times \mathsf{C} \to \mathsf{hGrpd}_\infty$ *constructed in [Lecture 4, §10].*

**Exercise 4.7.** *For any object* $x \in \mathsf{C}$*, a morphism* $f \in \mathsf{Maps}_{\mathsf{C}}(y,x)$ *exhibits the functor*

$$\mathsf{Maps}_{\mathsf{C}}(-,x) : \mathsf{C}^{\mathsf{op}} \to \mathsf{Grpd}_\infty$$

*as represented by* $y$ *iff* $f$ *is invertible. In particular, the morphism* $\mathsf{id}_x$ *provides a canonical equivalence*

$$\mathsf{Maps}_{\mathsf{C}}(-,x) \simeq \mathsf{h}_x.$$

*Dually, there is a canonical equivalence*

$$\mathsf{Maps}_{\mathsf{C}}(x,-) \simeq \mathsf{h}^x.$$

**Corollary 4.8.** *Let* $\mathsf{C}$ *be a locally small* $\infty$*-category. The functor*

$$\mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty),\ x \mapsto \mathsf{Maps}_{\mathsf{C}}(-,x)$$

*is fully faithful and its essential image consists of representable functors.*

**Definition 4.9.** *Let* $\mathsf{C}$ *be a locally small* $\infty$*-category.*

- ***A (covariant) Yoneda embedding*** *is a functor* $\mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$ *equivalent to* $x \mapsto \mathsf{Maps}_{\mathsf{C}}(-,x)$*.*
- ***A (cotravariant) Yoneda embedding*** *is a functor* $\mathsf{C}^{\mathsf{op}} \to \mathsf{Fun}(\mathsf{C}, \mathsf{Grpd}_\infty)$ *equivalent to* $x \mapsto \mathsf{Maps}_{\mathsf{C}}(x,-)$*.*

**Definition 4.10.** *Let* $\mathsf{C}$ *be an* essentially small $\infty$*-category. We write*

$$\mathsf{PShv}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$$

*and call it the* $\infty$***-category of presheaves*** *on* $\mathsf{C}$*.*

**Remark 4.11.** *In the above definition, we require* $\mathsf{C}$ *to be essentially small rather than locally small to guarantee* $\mathsf{PShv}(\mathsf{C})$ *is locally small. Note however that* $\mathsf{PShv}(\mathsf{C})$ *is almost never small.*

4.12. Let $\mathsf{C}$ be an essentially small $\infty$-category. Recall $\mathsf{PShv}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\mathsf{op}}, \mathsf{Grpd}_\infty)$ admits all small limits and colimits, which can be calculated pointwisely ([Lecture 7, Theorem 2.14]).

The following result is equivalent to [Lecture 7, Theorem 2.11]. See HTT.5.1.3.2 for a proof using simplicial categories.

**Theorem 4.13.** *Let $\mathsf{C}$ be an essentially small $\infty$-category. Then a Yoneda embedding $\mathsf{C} \to \mathsf{PShv}(\mathsf{C})$ preserves and detects all limits.*

**Warning 4.14.** *The limit of a small diagram $u : K \to \mathsf{C}$ might not exist. This happens iff the limit of $K \to \mathsf{C} \to \mathsf{PShv}(\mathsf{C})$ is not a representable presheaf.*

## 5. Cocompletion

**Definition 5.1.** *We say a functor $\mathsf{C} \to \widehat{\mathsf{C}}$* ***exhibits $\widehat{\mathsf{C}}$ as a cocompletion of*** *$\mathsf{C}$ if the following conditions are satisified:*

- *The $\infty$-category $\widehat{\mathsf{C}}$ admits small colimits.*
- *For any $\infty$-category $\mathsf{D}$ that admits small colimits, the restriction functor*

$$\mathsf{LFun}(\widehat{\mathsf{C}}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

  *is an equivalence, where $\mathsf{LFun}(\widehat{\mathsf{C}}, \mathsf{D}) \subset \mathsf{Fun}(\widehat{\mathsf{C}}, \mathsf{D})$ is the full sub-$\infty$-category consisting of functors that preserve small colimits.*

**Theorem 5.2** (Ker.04BE)**.** *Let $\mathsf{C}$ be an essentially small $\infty$-category. A Yoneda embedding $\iota : \mathsf{C} \to \mathsf{PShv}(\mathsf{C})$ exhibits $\mathsf{PShv}(\mathsf{C})$ as a cocompletion of $\mathsf{C}$.*

*Idea of the proof.* We only need to show the restriction functor

$$- \circ \iota : \mathsf{LFun}(\mathsf{PShv}(\mathsf{C}), \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \tag{5.1}$$

is an equivalence. We claim $\mathsf{LKE}_\iota$ exists and induces a functor

$$\mathsf{LKE}_\iota : \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{LFun}(\mathsf{PShv}(\mathsf{C}), \mathsf{D}) \tag{5.2}$$

that is inverse to the above restriction functor.

In fact, for any $F : \mathsf{C} \to \mathsf{D}$, the *pointwise* $\mathsf{LKE}_\iota F : \mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$ exists because

$$\mathsf{C}_{/\mathcal{M}} := \mathsf{PShv}(\mathsf{C})_{/\mathcal{M}} \underset{\mathsf{PShv}(\mathsf{C})}{\times} \mathsf{C} \tag{5.3}$$

is essentially small[3] and therefore

$$(\mathsf{LKE}_\iota F)(\mathcal{M}) \simeq \underset{(\mathsf{h}_x \to \mathcal{M}) \in \mathsf{C}_{/\mathcal{M}}}{\mathsf{colim}} F(x).$$

One can show $\mathsf{C}_{/\mathsf{colim}\,\mathcal{M}_i} \simeq \mathsf{colim}\, \mathsf{C}_{/\mathcal{M}_i}$ and use decomposition of diagrams to show the functor $\mathsf{LKE}_\iota F$ preserves small colimits.

It remains to show the obtained functor (5.2) is an inverse to (5.1). It is a right inverse because $\iota$ is fully faithful. To show it is a left inverse, we need to show any colimit-preserving $G : \mathsf{PShv}(\mathsf{C}) \to \mathsf{D}$ satisfies

$$\mathsf{LKE}_\iota(G \circ \iota) \xrightarrow{\simeq} G. \tag{5.4}$$

[3]It is in this step that we use $\mathsf{C}$ is essentially small rather than just locally small (Ker.03WG). The theorem would fail under the latter weaker assumption (Ker.03WF).

Since $G$ is colimit-preserving and the above $\mathsf{LKE}$ is pointwise, we only need to treat the universal case when $G = \mathsf{Id}$. Since small colimits in $\mathsf{PShv}(\mathsf{C})$ are preserved and detected by $\mathsf{h}^{\mathrm{op}}_{\mathcal{M}} : \mathsf{PShv}(\mathsf{C}) \to \mathsf{Grpd}^{\mathrm{op}}_\infty$, we only need to treat the cases when $G = \mathsf{h}^{\mathrm{op}}_{\mathcal{M}}$.

In other words, we need to show the commutative diagram

$$\begin{tikzcd} & \mathsf{PShv}(\mathsf{C})^{\mathrm{op}} \arrow[dr, dotted, "\mathsf{h}_{\mathcal{M}}"] & \\ \mathsf{C}^{\mathrm{op}} \arrow[ur, "\iota^{\mathrm{op}}"] \arrow[rr, "\mathcal{M}"'] & & \mathsf{Grpd}_\infty \end{tikzcd}$$

induces

$$\mathsf{h}_{\mathcal{M}} \xrightarrow{\simeq} \mathsf{RKE}_{\iota^{\mathrm{op}}}\mathcal{M}.$$

Here the equivalence $\mathcal{M} \simeq \mathsf{h}_{\mathcal{M}} \circ \iota^{\mathrm{op}}$ follows from the Yoneda lemma for $\mathsf{C}$. We only need to show for any $\mathcal{N} \in \mathsf{PShv}(\mathsf{C})^{\mathrm{op}}$,

$$\mathsf{h}_{\mathcal{M}}(\mathcal{N}) \xrightarrow{\simeq} (\mathsf{RKE}_{\iota^{\mathrm{op}}}\mathcal{M})(\mathcal{N}).$$

Applying the Yoneda lemma to $\mathsf{PShv}(\mathsf{C})$, we only need to show[4]

$$\mathsf{h}_{\mathcal{M}}(\mathcal{N}) \xrightarrow{\simeq} \mathsf{Maps}(\mathsf{h}_{\mathcal{N}}, \mathsf{RKE}_{\iota^{\mathrm{op}}}\mathcal{M})$$

Unwinding the definitions, this functor can be identified with

$$\mathsf{h}_{\mathcal{M}}(\mathcal{N}) \simeq \mathsf{Maps}(\mathcal{N}, \mathcal{M}) \simeq \mathsf{Maps}(\mathsf{h}_{\mathcal{N}} \circ \iota^{\mathrm{op}}, \mathcal{M}) \simeq \mathsf{Maps}(\mathsf{h}_{\mathcal{N}}, \mathsf{RKE}_{\iota^{\mathrm{op}}}\mathcal{M}),$$

where the last equivalence uses the universal property of the $\mathsf{RKE}$. □

5.3. In the above proof, we actually proved the following:

**Corollary 5.4.** *Let $\mathsf{C}$ be an essentially small $\infty$-category. Then $\mathsf{PShv}(\mathsf{C})$ is generated by the image of $\iota : \mathsf{C} \to \mathsf{PShv}(\mathsf{C})$ under small colimits.*

**Remark 5.5.** *The precise meaning of the above corollary is: any object in $\mathcal{M} \in \mathsf{PShv}(\mathsf{C})$ is isomorphic to a small colimit of objects contained in the image of $\iota$. Indeed, we have*

$$\mathcal{M} \simeq \underset{(\mathsf{h}_x \to \mathcal{M}) \in \mathsf{C}_{/\mathcal{M}}}{\mathsf{colim}} \iota(x).$$

*by applying* (5.4) *to $G := \mathsf{Id}$.*

## Appendix A. Completely compact objects

**Definition A.1.** *Let $\mathsf{D}$ be a locally small $\infty$-category which admits small colimits. We say an object $d \in \mathsf{D}$ is* ***completely compact*** *if the functor*

$$\mathsf{Maps}_{\mathsf{D}}(d, -) : \mathsf{D} \to \mathsf{Grpd}_\infty$$

*preserves small colimits.*

**Exercise A.2.** *Suppose $d$ is completely compact in $\mathsf{D}$. Show that any retract of $d$ is also completely compact. Here recall we say $c$ is a retract of $d$ if $\mathsf{id}_c$ factors as $c \to d \to c$.*

---

[4]This is where the magic happens. In the general setting, to check a functor $T : \mathsf{U}^{\mathrm{op}} \to \mathsf{V}$ is a $\mathsf{RKE}$ of $\mathsf{U}_0^{\mathrm{op}} \to \mathsf{V}$, one needs to use *all* functors $S : \mathsf{U}^{\mathrm{op}} \to \mathsf{V}$ as testing functors and calculate $\mathsf{Maps}(S, T)$. However, when $\mathsf{V} = \mathsf{Grpd}_\infty$, the above argument says it is enough to use representable functors as testing functors.

**Exercise A.3.** *Let* $\mathsf{C}$ *be an essentially small* $\infty$*-category. Show that representable presheaves* $\mathsf{h}_x \in \mathsf{PShv}(\mathsf{C})$ *are completely compact.*

**Theorem A.4** (HTT.5.1.6.8)**.** *Let* $\mathsf{C}$ *be an essentially small* $\infty$*-category. Then a presheaf* $\mathcal{M} \in \mathsf{PShv}(\mathsf{C})$ *is completely compact iff it is a retract of some representable presheaf.*

A.5. **Suggested readings.** : HTT.5.1.6.