

LECTURE 5

Last time we constructed a homomorphism $\phi : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})$ fitting into the following diagram

$$(0.1) \quad \begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\phi} & \text{Sym}(\mathfrak{t}) \\ & \searrow \chi_\lambda = \xi_{-, \lambda} & \downarrow \text{ev}_\lambda \\ & & k_\lambda, \end{array}$$

where for any $\lambda \in \mathfrak{t}^*$, $\chi_\lambda = \xi_{-, \lambda}$ is the central character of the Verma module M_λ . We stated the following result, which will be proved today.

Theorem 1 (Harish-Chandra). *The homomorphism ϕ induces an isomorphism*

$$\phi_{\text{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \text{Sym}(\mathfrak{t})^{W_\bullet}$$

from $Z(\mathfrak{g})$ to the invariance of $\text{Sym}(\mathfrak{t})$ with respect to the dotted W -action.

1. STEP 1: IMAGE IS W_\bullet -INVARIANT

Let us first prove the image of ϕ is indeed contained in $\text{Sym}(\mathfrak{t})^{W_\bullet}$. Since W is generated by simple reflections s_α , $\alpha \in \Delta$, we only need to show $\phi(z) = s_\alpha \cdot \phi(z)$ for any $z \in Z(\mathfrak{g})$. In other words, we need to show

$$(1.1) \quad \text{ev}_\lambda(\phi(z)) = \text{ev}_\lambda(s_\alpha \cdot \phi(z))$$

for any $\lambda \in \mathfrak{t}^*$. In fact, we only need to prove this for a Zariski dense subset of \mathfrak{t}^* . Let us first give this dense subset. The following result is obvious after drawing a picture.

Lemma 2. *Let $\alpha \in \Delta$ be a simple root. The subset $\{\lambda \in \mathfrak{t}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}\}^1$ is a Zariski dense subset of \mathfrak{t}^* .*

Example 3. For $\mathfrak{g} = \mathfrak{sl}_2$, this subset is $\mathbb{Z}^{\geq -1} \subset \mathbb{A}^1$. In fact, in [Exam. 15, Lect. 4], we have used this subset to show the image of $\phi : Z(\mathfrak{sl}_2) \rightarrow k[h]$ is invariant under $h \mapsto -h - 2$. The argument below is an immediate generalization.

Lemma 4. *Let $\alpha \in \Delta$ be a simple root and $\lambda \in \mathfrak{t}^*$. Suppose $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}$. Then M_λ contains $M_{s_\alpha \cdot \lambda}$ as a submodule.*

Corollary 5. *The image of $\phi : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})$ is indeed contained in $\text{Sym}(\mathfrak{t})^{W_\bullet}$.*

Proof. By previous discussion, we only need to prove (1.1) for α and λ satisfying the assumption of Lemma 4. By (0.1), the LHS and RHS of (1.1) are exactly the central character of M_λ and $M_{s_\alpha \cdot \lambda}$. Now Lemma 4 implies they are equal because the central character of a module is equal to the central character of any nonzero submodule of it. □

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¹In other words, $\lambda + \rho$ is **dominant and integral**.

Proof of Lemma 4. Recall

$$s_\alpha(\lambda) = \lambda - 2 \frac{(\lambda, \alpha)}{(\alpha, \alpha)} \alpha = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha.$$

It follows that

$$s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \check{\alpha} \rangle \alpha = \lambda - m\alpha$$

for $m \in \mathbb{Z}^{\geq 0}$. The lemma is obvious for $m = 0$. We assume $m > 0$. Then we have $\langle \lambda, \check{\alpha} \rangle = m - 1$ because $\langle \rho, \check{\alpha} \rangle = 1$ (see [H1, Cor. to Lem. 10.2(B)]).

Let $f_\alpha \in \mathfrak{n}^-$ be a nonzero vector of weight $-\alpha$. Consider the vector $f_\alpha^m \cdot v_\lambda$, which is of weight $\lambda - m\alpha = s_\alpha \cdot \lambda$. We only need to show

$$\mathfrak{n} \cdot (f_\alpha^m \cdot v_\lambda) = 0.$$

Indeed, if this is true, the map $k_{s_\alpha \cdot \lambda} \rightarrow M_\lambda$, $c \mapsto c(f_\alpha^m \cdot v_\lambda)$ is \mathfrak{b} -linear, and thereby induces a \mathfrak{g} -linear map $M_{s_\alpha \cdot \lambda} \rightarrow M_\lambda$. This map is injective because as a morphism between $U(\mathfrak{n}^-)$ -modules, it is given by $\cdot f_\alpha^m : U(\mathfrak{n}^-) \rightarrow U(\mathfrak{n}^-)$, which is injective by the PBW theorem.

It remains to show \mathfrak{n} annihilates $f_\alpha^m \cdot v_\lambda$. For each simple root $\beta \in \Delta$, let $e_\beta \in \mathfrak{n}$ be a nonzero vector of weight β . Note that the vectors $(e_\beta)_{\beta \in \Delta}$ generate \mathfrak{n} under Lie brackets². Hence we only need to show $e_\beta \cdot f_\alpha^m \cdot v_\lambda = 0$. There are two cases:

- If $\alpha \neq \beta$, then $[e_\beta, f_\alpha] = 0$ ³ and

$$e_\beta \cdot f_\alpha^m \cdot v_\lambda = f_\alpha^m \cdot e_\beta \cdot v_\lambda = 0$$

because $\mathfrak{n} \cdot v_\lambda = 0$.

- If $\alpha = \beta$, $[e_\alpha, f_\alpha] \in \mathfrak{t}$ is proportionate to $\check{\alpha}$ (see [H1, Prop. 8.3(d)]). Rescale e_α , we may assume $[e_\alpha, f_\alpha] = \check{\alpha}$. Then $[\check{\alpha}, f_\alpha] = \langle -\alpha, \check{\alpha} \rangle f_\alpha = -2f_\alpha$. Now the following calculation is essentially that in [Exe. 24, Lect. 2]. We have

$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1 \leq i \leq m} f_\alpha^{m-i} \cdot [e_\alpha, f_\alpha] \cdot f_\alpha^{i-1} \cdot v_\lambda + f_\alpha^m \cdot e_\alpha \cdot v_\lambda.$$

Recall we have $e_\alpha \cdot v_\lambda = 0$. Also,

$$\check{\alpha} \cdot f_\alpha^j = \sum_{1 \leq i \leq j} f_\alpha^{j-i} \cdot [\check{\alpha}, f_\alpha] \cdot f_\alpha^{i-1} + f_\alpha^j \cdot \check{\alpha} = -2j f_\alpha^j + f_\alpha^j \cdot \check{\alpha}.$$

Hence

$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1 \leq i \leq m} (-2(i-1)f_\alpha^{m-1} + f_\alpha^{m-1} \cdot \check{\alpha}) v_\lambda = (-m(m-1) + m\langle \lambda, \check{\alpha} \rangle) v_\lambda = 0$$

as desired.

□[Lemma 4]

2. STEP 2: FILTRATIONS

By Step 1, we have a homomorphism

$$\phi_{\text{HC}} : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})^{W^\bullet}.$$

In this step, we equip both sides with filtrations, and show ϕ_{HC} is compatible with them. The punchline is the following easy fact:

Fact 6. *Let V_1 and V_2 be two vector spaces equipped with $\mathbb{Z}^{\geq 0}$ -indexed (exhausted) filtrations. Suppose $\varphi : V_1 \rightarrow V_2$ is a k -linear map compatible with the filtrations. Then φ is an isomorphism iff $\text{gr}^\bullet \varphi : \text{gr}^\bullet V_1 \rightarrow \text{gr}^\bullet V_2$ is so.*

²Because $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{[\alpha, \beta]}$ whenever $\alpha, \beta, \alpha + \beta \in \Phi^+$.

³Otherwise it is a nonzero vector with weight $\beta - \alpha$ but the latter is not a root because $\Phi = \Phi^+ \sqcup \Phi^-$.

Construction 7. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on $Z(\mathfrak{g})$ with $F^{\leq i} Z(\mathfrak{g}) := Z(\mathfrak{g}) \cap F^{\leq i} U(\mathfrak{g})$. Note that $\text{gr}^\bullet Z(\mathfrak{g})$ is a subalgebra of $\text{gr}^\bullet U(\mathfrak{g}) \simeq \text{Sym}(\mathfrak{g})$.

Construction 8. The PBW filtration on $U(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t})$ is preserved by the dotted W -action. Hence it induces a filtration on $U(\mathfrak{t})^{W^\bullet}$.

Warning 9. The dotted W -action on $\text{Sym}(\mathfrak{t})$ does not preserve the grading. But the usual (linear) W -action does.

Lemma 10. The homomorphism $\phi_{\text{HC}} : Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})^{W^\bullet}$ is compatible with the above filtrations.

Proof. This is obvious from the description of ϕ as

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t}).$$

□

3. STEP 3: CALCULATING THE GRADED PIECES

Construction 11. Recall $\text{Sym}(\mathfrak{g})$ has a natural \mathfrak{g} -module structure constructed as follows. For any $V \in \mathfrak{g}$, there is a natural \mathfrak{g} -module structure on $V^{\otimes n}$ given by

$$\mathfrak{g} \times V^{\otimes n} \rightarrow V^{\otimes n}, (x, \otimes_i v_i) \mapsto \sum_i (v_1 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n).$$

This action is compatible with the symmetric group Σ_n -action on $V^{\otimes n}$ and thereby induces a \mathfrak{g} -module structure on $\text{Sym}^n(V)$. Taking direct sum, we obtain a \mathfrak{g} -module structure on $\text{Sym}(V)$.

In the case $V = \mathfrak{g}$, to distinguish with the multiplication structure on $\text{Sym}(\mathfrak{g})$, we denote this action by

$$\mathfrak{g} \times \text{Sym}(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{g}), (x, u) \mapsto \text{ad}_x(u),$$

and call it the **adjoint action**.

Note that by definition, for $x \in \mathfrak{g}$ and $u, v \in \text{Sym}(\mathfrak{g})$, we have

$$\text{ad}_x(u \cdot v) = \text{ad}_x(u) \cdot v + u \cdot \text{ad}_x(v).$$

In particular, the \mathfrak{g} -invariance

$$\text{Sym}(\mathfrak{g})^{\mathfrak{g}} := \{u \in \text{Sym}(\mathfrak{g}) \mid \text{ad}_x(u) = 0 \text{ for any } x \in \mathfrak{g}\}$$

is a subalgebra of $\text{Sym}(\mathfrak{g})$.

Lemma 12. There is a unique dotted graded isomorphism making the following diagram commute

$$\begin{array}{ccc} \text{gr}^\bullet Z(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \text{Sym}(\mathfrak{g})^{\mathfrak{g}} \\ \downarrow \subset & & \downarrow \subset \\ \text{gr}^\bullet U(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \text{Sym}(\mathfrak{g}), \end{array}$$

where the bottom isomorphism is given by the PBW theorem.

To prove this lemma, we use the following exercise:

Exercise 13. This is **Homework 2, Problem 4**. Prove: the adjoint \mathfrak{g} -action on $U(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \rightarrow U(\mathfrak{g}), (x, u) \mapsto \text{ad}_x(u) = [x, u],$$

preserves each $F^{\leq n} U(\mathfrak{g})$, and the induced \mathfrak{g} -action on $\text{gr}^\bullet(U(\mathfrak{g})) \simeq \text{Sym}(\mathfrak{g})$ is the adjoint action in Construction 11.

⁴Note that $\text{gr}^\bullet U(\mathfrak{t})$ is also isomorphic to $\text{Sym}(\mathfrak{t})$. To distinguish them, we always use $U(\mathfrak{t})$ to denote the filtered commutative ring while use $\text{Sym}(\mathfrak{t})$ to denote the graded commutative ring.

Proof of Lemma 12. By the above exercise, we have a short exact sequence of *finite-dimensional* \mathfrak{g} -modules:

$$0 \rightarrow F^{\leq n-1}U(\mathfrak{g}) \rightarrow F^{\leq n}U(\mathfrak{g}) \rightarrow \mathrm{Sym}^n(\mathfrak{g}) \rightarrow 0.$$

Since $\mathfrak{g}\text{-mod}_{\mathrm{fd}}$ is semisimple, this short exact sequence splits. Hence taking \mathfrak{g} -invariance, we obtain⁵

$$0 \rightarrow F^{\leq n-1}Z(\mathfrak{g}) \rightarrow F^{\leq n}Z(\mathfrak{g}) \rightarrow \mathrm{Sym}^n(\mathfrak{g})^{\mathfrak{g}} \rightarrow 0$$

This gives the desired isomorphism $\mathrm{gr}^n Z(\mathfrak{g}) \simeq \mathrm{Sym}^n(\mathfrak{g})^{\mathfrak{g}}$.

□[Lemma 12]

A similar proof⁶ gives:

Lemma 14. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathrm{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t})^W \\ \downarrow \subset & & \downarrow \subset \\ \mathrm{gr}^\bullet U(\mathfrak{t}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t}), \end{array}$$

where the right-top corner is the invariance for the linear W -action on $\mathrm{Sym}(\mathfrak{t})$.

Combining the above two lemmas, we obtain:

Corollary 15. *There is a unique dotted graded homomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathrm{gr}^\bullet Z(\mathfrak{g}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \\ \downarrow \mathrm{gr}^\bullet \phi_{\mathrm{HC}} & & \downarrow \phi_{\mathrm{cl}} \\ \mathrm{gr}^\bullet(U(\mathfrak{t})^{W\bullet}) & \xrightarrow{\quad \simeq \quad} & \mathrm{Sym}(\mathfrak{t})^W. \end{array}$$

4. STEP 4: CHEVALLEY ISOMORPHISM

It remains to show

$$\phi_{\mathrm{cl}} : \mathrm{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \mathrm{Sym}(\mathfrak{t})^W$$

is an isomorphism. Let us first give an explicit construction of this homomorphism. We need the following characterization of ϕ .

Construction 16. *Consider the composition*

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k.$$

With respect to the adjoint \mathfrak{t} -action, the source has weight 0. Hence the composition factors through the 0-weight subspace of the target, which is exactly $U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} k \simeq U(\mathfrak{t})$. By definition, the obtained map

$$Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})$$

is just ϕ .

⁵ Warning: in general, taking invariance is only *left exact*. (Memory method: it is given by $\mathrm{Hom}_{\mathfrak{g}}(k, -)$.) Hence we need the existence of a splitting.

⁶Note that $\mathrm{Rep}(W)_{\mathrm{fd}}$ is also semisimple

Construction 17. It follows ϕ_{cl} can be constructed as follows. Consider the composition

$$(4.1) \quad \text{Sym}(\mathfrak{g})^{\mathfrak{g}} \hookrightarrow \text{Sym}(\mathfrak{g}) \twoheadrightarrow \text{Sym}(\mathfrak{g}/\mathfrak{n}).$$

It factors through $\text{Sym}(\mathfrak{b}/\mathfrak{n}) \simeq \text{Sym}(\mathfrak{t})$. The obtained map

$$\text{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \text{Sym}(\mathfrak{t})$$

can be identified with $\text{gr}^{\bullet}\phi$. Since ϕ factors through $U(\mathfrak{t})^{W^{\bullet}}$, the map $\text{gr}^{\bullet}\phi$ factors through $\text{gr}^{\bullet}(U(\mathfrak{t})^{W^{\bullet}}) \simeq \text{Sym}(\mathfrak{t})^W$. The obtained map

$$\text{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \text{Sym}(\mathfrak{t})^W$$

is just ϕ_{cl} .

Remark 18. The geometric meaning of the above construction is as follows.

Note that $\text{Sym}(\mathfrak{g})^{\mathfrak{g}} = \text{Sym}(\mathfrak{g})^G$ because G -invariance is equal to \mathfrak{g} -invariance⁷. Hence (4.1) corresponds to the morphisms

$$(\mathfrak{g}/\mathfrak{n})^* \rightarrow \mathfrak{g}^* \rightarrow \mathfrak{g}^*//G.$$

Since $\text{gr}^{\bullet}\phi$ factors through $\text{gr}^{\bullet}(U(\mathfrak{t})^{W^{\bullet}}) \simeq \text{Sym}(\mathfrak{t})^W$. The above composition factors through $(\mathfrak{g}/\mathfrak{n})^* \rightarrow (\mathfrak{b}/\mathfrak{n})^* \simeq \mathfrak{t}^* \rightarrow \mathfrak{t}^*//W$. In other words, we have

$$\begin{array}{ccc} (\mathfrak{g}/\mathfrak{n})^* & \longrightarrow & \mathfrak{g}^*//G \\ \downarrow & & \uparrow \text{dotted} \\ \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^*//W \end{array}$$

such that the dotted arrow is given by $\text{Spec}(\phi_{\text{cl}})$.

It is convenient to get rid of the dual spaces using the Killing form. Namely, Kil induces an isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ compatible with the G -actions, while $\text{Kil}|_{\mathfrak{t}}$ induces an isomorphism $\mathfrak{t} \simeq \mathfrak{t}^*$ compatible with the W -actions⁸. Via the first isomorphism, the subspaces $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ corresponds to $(\mathfrak{g}/\mathfrak{b})^* \subset (\mathfrak{g}/\mathfrak{n})^* \subset \mathfrak{g}$. Then the above commutative diagram is identified with

$$(4.2) \quad \begin{array}{ccc} \mathfrak{b} & \longrightarrow & \mathfrak{g}//G \\ \downarrow & & \uparrow \text{dotted} \\ \mathfrak{t} & \longrightarrow & \mathfrak{t}//W. \end{array}$$

We abuse notations and also view the dotted arrow as $\text{Spec}(\phi_{\text{cl}})$.

Remark 19. Since the projection $\mathfrak{b} \rightarrow \mathfrak{t}$ has a splitting $\mathfrak{t} \rightarrow \mathfrak{b}$. The above claim implies $\mathfrak{t} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}//G$ factors through $\mathfrak{t}//W$. We can also prove this using group-theoretic method. Namely, consider the normalizer $N_G(T)$ of T inside G . Recall we have $W \simeq N_G(T)/T$ such that the linear W -action on \mathfrak{t} can be identified with the adjoint action of $N_G(T)/T$. Then the morphism $\mathfrak{t} \rightarrow \mathfrak{g}//G$ factors through $\mathfrak{t}//W$ because $N_G(T)$ is a subgroup of G .

Warning 20. I do not know any group-theoretic proof of (4.2). This is because not $\mathfrak{b} \rightarrow \mathfrak{g}/G$ (the quotient stack) does not factor through \mathfrak{t} : two elements in \mathfrak{b} that have the same image in \mathfrak{t} are not necessarily conjugate to each other.

⁷Because $\text{Rep}(G) \rightarrow \mathfrak{g}\text{-mod}$ is fully faithful when G is connected.

⁸Both claims follow from the fact that Kil is invariant with respect to the adjoint \mathfrak{g} -action and thereby to the adjoint G -action. Here we use $W \simeq N_G(T)/T$.

Theorem 21 (Chevalley). *The homomorphism*

$$\phi_{\text{cl}} : \text{Sym}(\mathfrak{g})^{\mathfrak{g}} \rightarrow \text{Sym}(\mathfrak{t})^W$$

is an isomorphism. In other words, the natural morphism $\mathfrak{t} // W \rightarrow \mathfrak{g} // G$ is an isomorphism.

Proof. As in Remark 18, we can identify ϕ_{cl} with the restriction map $\text{Fun}(\mathfrak{g})^G \rightarrow \text{Fun}(\mathfrak{t})^W$, which is also $\text{Sym}(\mathfrak{g}^*)^G \rightarrow \text{Sym}(\mathfrak{t}^*)^W$.

This map is injective because if an adjoint-invariant function on \mathfrak{g} vanishes on \mathfrak{t} , then it vanishes on each semisimple elements. But the latter are Zariski dense in \mathfrak{g} .

To prove the surjectivity, we first find generators of $\text{Sym}(\mathfrak{t}^*)^W$ as follows. Recall the subset P^+ of **dominant integral weights**, i.e.,

$$P^+ := \{\lambda \in \mathfrak{t}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0} \text{ for all } \alpha \in \Delta\}.$$

Note that P^+ spans \mathfrak{t}^* . In fact, for each $\alpha \in \Delta$, we can find **fundamental dominant weights** ω_α such that

$$\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha\beta}, \quad \alpha, \beta \in \Delta.$$

Then $\{\omega_\alpha\}$ is a basis of \mathfrak{t}^* and $P^+ = \mathbb{Z}^{\geq 0}\{\omega_\alpha\}$. A direct calculation shows that $\{\lambda^n \mid \lambda \in P^+\}$ span $\text{Sym}^n(\mathfrak{t}^*)$. Hence the sums

$$b_{\lambda,n} := \sum_{w \in W} w(\lambda^n), \quad \lambda \in P^+, n \geq 0$$

span $\text{Sym}(\mathfrak{t}^*)^W$.

It remains to show each $b_{\lambda,n}$ is contained in the image of ϕ_{cl} . We need the following well-known fact.

Theorem 22 (Weyl). *For any $\lambda \in P^+$, there is a unique finite-dimensional irreducible \mathfrak{g} -module L_λ with highest weight λ .*

For $\lambda \in P^+, n \geq 0$, consider the function $a_{\lambda,n} \in \text{Fun}(\mathfrak{g})$ defined by

$$a_{\lambda,n}(x) := \text{tr}(x^n; L_\lambda),$$

i.e., its value at any $x \in \mathfrak{g}$ is the trace of the action of x^n on L_λ . It is easy to see $a_{\lambda,n}$ is \mathfrak{g} -invariant⁹ and thereby G -invariant.

Now the following exercise implies each $b_{\lambda,n}$ is contained in the image of ϕ_{cl} . Indeed, this follows from induction on λ with respect to the partial ordering $<$ ¹⁰.

Exercise 23. This is **Homework 2, Problem 5**. Let $\lambda \in P^+$ be a dominant integral weight and $n \geq 0$. Prove there exists scalars $c_{\lambda'} \in k$, $\lambda' < \lambda$ such that

$$\phi_{\text{cl}}(a_{\lambda,n}) = a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\text{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' < \lambda} c_{\lambda'} b_{\lambda',n},$$

where $\text{Stab}_W(\lambda) \subset W$ is the stabilizer of the W -action at λ .

□

Combining all the previous discussion, we finish the proof of Theorem 1.

⁹Recall L_λ is G_{sc} -integrable ([Thm. 47, Lect. 3]). Hence $a_{\lambda,n}$ is G_{sc} -invariant, and thereby \mathfrak{g} -invariant.

¹⁰Recall $\lambda' \leq \lambda$ iff $\lambda - \lambda' \in \mathbb{Z}^{\geq 0}\Phi^+$. See [Defn. 22, Lect. 2].

5. MORE ON THE QUOTIENT $\mathfrak{t} // W$

We state some results on $\mathfrak{t} // W \simeq \mathfrak{t} // W_\bullet$ without proof.

Proposition 24 ([H2, Sect. 1.10]). *The morphism $\varpi : \mathfrak{t} \rightarrow \mathfrak{t} // W_\bullet$ is surjective, and W_\bullet acts transitively on the fiber at each closed point.*

Remark 25. In other words, any character $\chi : Z(\mathfrak{g}) \rightarrow k$ is the central character χ_λ for some Verma module V_λ , and $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$ for some $w \in W$.

Corollary 26 (Linkage principle). *Verma modules M_λ and M_μ belong to the same block iff $\mu = w \cdot \lambda$ for some $w \in W$. In particular, there are only finitely many Verma modules and irreducible objects¹¹ in each block.*

Proposition 27 ([G, Sect. 7]). *We have*

- (1) *The schemes $\mathfrak{t} // W \simeq \mathfrak{g} // G$ are smooth.*
- (2) *The map $\mathfrak{t} \rightarrow \mathfrak{t} // W$ is flat.*
- (3) *The map $\mathfrak{g} \rightarrow \mathfrak{g} // G$ is flat.*

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¹¹Recall M_λ has a unique irreducible quotient L_λ , and all irreducible objects of \mathcal{O} are of this form. See [Thm. 25, Prop. 31, Lec. 2].