LECTURE 27

In this lecture, we explain the following analogy:

Classical Algebra	∞-categorical Algebra
Sets	Spaces
Abelian groups	Spectra
Tensor products	Smash products
Associative rings	$\mathbb{E}_1$-rings
Commutative rings	$\mathbb{E}_\infty$-rings

1. Spectra vs abelian groups

1.1. In this section, we explain the analogy between spactra and abelian groups.

1.2. An abelian group is a commutative monoid G in Set such that each element has an inverse. The latter condition means the maps

$$(p_1, m) : G \times G \to G \times G,\ (m, p_2) : G \times G \to G \times G$$

are invertible. This notion makes sense in any Cartesian symmetric monoidal ∞-category.

Definition 1.3. *Let C be an ∞-category that admits finite products. We say an associative monoid $G \in \mathsf{Mon}_{\mathbb{E}_1}(\mathsf{C})$ is a **group in** C if the morphisms (p_1, m) and (m, p_2) are invertible.*

*For $1 \leq k \leq \infty$, we say an $\mathbb{E}_k$-monoid $G \in \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{C})$ is an $\mathbb{E}_k$**-group**[1] **in** C if its image under*

$$\mathsf{Mon}_{\mathbb{E}_k}(\mathsf{C}) \to \mathsf{Mon}_{\mathbb{E}_1}(\mathsf{C})$$

is a group.

Let

$$\mathsf{Grp}_{\mathbb{E}_k}(\mathsf{C}) \subseteq \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{C})$$

be the full sub-∞-category of $\mathbb{E}_k$-groups in C.

Example 1.4. *Let $X \in \mathsf{Spc}_*$ be any pointed space. For $0 < k < \infty$, the* k-fold loop space $\Omega^k X$ *has a canonical $\mathbb{E}_k$-group structure desribed as follows.*

Choose a topological realization of X, then $\Omega^k X$ can be realized as the topological space of maps $f : [-1, 1]^k \to X$ such that f sends the boundary to the base point of X. For any m-ary operator $\gamma : \square^k \times \langle m \rangle^\circ \to \square^k$ in ${}^{\mathsf{t}}\mathbb{E}_k$, we have a homomorphism

$$\mathsf{mult}_\gamma : (\Omega^k X)^{\times m} \to \Omega^k X$$

that sends $(f_1, \cdots, f_m) \in (\Omega^k X)^{\times m}$ to the map

$$f(s) := \begin{cases} f_i(t) & \text{if } s = \gamma(t, i) \\ * & \text{otherwise} \end{cases}$$

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[1] Alternative terminology: *grouplike $\mathbb{E}_k$-monoids.*

One can check this defines an ${}^{\mathsf{t}}\mathbb{E}_k$*-group* $\Omega^k X$ *in* Top *(where everything is enriched over topological spaces). Passing to* ∞*-categories, we obtain an* $\mathbb{E}_k$*-group in* Spc*.*

The above construction in functorial in X *and gives a functor*

$$\Omega^k : \mathsf{Spc}_* \to \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}). \tag{1.1}$$

Exercise 1.5. *For any* ∞*-category* C *that admits finite products, construct a functor* $\Omega^k : \mathsf{C}_* \to \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{C})$*. Hint: Yoneda lemma.*

1.6. The functor (1.1) is not an equivalence. In fact, it is not even conservative. Indeed, ΩX can only see the neutral connected component of X. It turns out this is the only reason that (1.1) is not an equivalence.

Theorem 1.7 (Boardman–Vogt, May)**.** *For* $0 < k < \infty$*, let* $(\mathsf{Spc}_*)_{\geq k} \subseteq \mathsf{Spc}_*$ *be the full sub-*∞*-category of* k*-connective spaces*[2]*. Then the functor*

$$\Omega^k : (\mathsf{Spc}_*)_{\geq k} \to \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}) \tag{1.2}$$

is an equivalence.

Remark 1.8. *The degenerate case* $k = 0$ *is:* $\mathsf{Spc}_* \simeq \mathsf{Mon}_{\mathbb{E}_0}(\mathsf{Spc})$*.*

Exercise 1.9. *For* $0 < k < \infty$*, construct a canonical commutative diagram*

$$\begin{tikzcd}
(\mathsf{Spc}_*)_{\geq k+1} \arrow[r, "\Omega"] \arrow[d, "\Omega^{k+1}"] & (\mathsf{Spc}_*)_{\geq k} \arrow[d, "\Omega^k"] \\
\mathsf{Grp}_{\mathbb{E}_{k+1}}(\mathsf{Spc}) \arrow[r] & \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}).
\end{tikzcd}$$

1.10. By the above exercise, we can pass to inverse limits in k and obtain the following characterization of connective spectra[3].

Corollary 1.11. *There is a canonical equivalence*

$$\Omega^\infty : \mathsf{Sptr}_{\geq 0} \xrightarrow{\simeq} \mathsf{Grp}_{\mathsf{E}_\infty}(\mathsf{Spc}),$$

compatible with the forgetful functors to Spc*.*

Remark 1.12. *In other words, connective spectra are just symmetric monoidal* ∞*-groupoids.*

Remark 1.13. *We denote the inverse of* (1.2) *by*

$$\mathbb{B}^k : \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}) \to (\mathsf{Spc}_*)_{\geq k},$$

and call it the k***-fold delooping functor****. Informally, this functor can be described as follows.*

For $k = 1$*, given any associative monoid* G *in a Cartesian monoidal* ∞*-category* C*, we can form the* Bar construction

$$* \times * \leftleftarrows * \times G \times * \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; * \times G \times G \times * \cdots$$

which is a simplicial object in C*. The colimit* $\mathbb{B}G$ *of this diagram, when exists, calculates the relative tensor product of* $*$ *with itself as a bimodule for* G*. One can check the image of the obtained functor*

$$\mathbb{B} : \mathsf{Grp}_{\mathbb{E}_1}(\mathsf{Spc}) \to \mathsf{Spc}_*$$

[2] This means $\pi_i \simeq 0$ for $i < k$.

[3] This means $\pi_i \simeq 0$ for $i < 0$.

is contained in $(\mathsf{Spc}_*)_{\geq 0}$ *and indeed gives the desired delooping functor.*

For general $0 < k < \infty$*, we note that the composition*

$$\mathsf{Grp}_{\mathbb{E}_1}(\mathsf{Spc}) \xrightarrow{\mathbb{B}} (\mathsf{Spc}_*)_{\geq 1} \to \mathsf{Spc},$$

commutes with finite prodcuts and therefore induces a functor

$$\mathbb{B} : \mathsf{Grp}_{\mathbb{E}_{k+1}}(\mathsf{Spc}) \simeq \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Grp}_{\mathbb{E}_1}(\mathsf{Spc})) \to \mathsf{Mon}_{\mathbb{E}_k}(\mathsf{Spc}),$$

where the first equivalence follows from the Dunn Additivity Theorem. One can check this functor factors as

$$\mathbb{B} : \mathsf{Grp}_{\mathbb{E}_{k+1}}(\mathsf{Spc}) \to \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}).$$

It follows that we have a functor

$$\mathbb{B}^k : \mathsf{Grp}_{\mathbb{E}_k}(\mathsf{Spc}) \to \mathsf{Spc}_*.$$

One can check its image is contained in $(\mathsf{Spc}_*)_{\geq k}$ *and indeed gives the desired* k*-fold delooping functor.*

2. Smash products vs. tensor products

2.1. In this section, we explain the analogy between smash products of spectra and tensor products of abelian groups.

2.2. Recall we have a functor

$$\pi_\bullet : \mathsf{Sptr} \to \mathsf{grAb}$$

sending a spectrum X to its homotopy groups

$$\pi_m X \simeq \pi_0 \mathsf{Maps}_{\mathsf{Sptr}}(\mathbb{S}[m], X).$$

Construction 2.3. *Consider the isomorphism* $\mathbb{S} \otimes \mathbb{S} \xrightarrow{\simeq} \mathbb{S}$ *in* Sptr *given by the unital structure. Using the fact that* $- \otimes -$ *commutes with colimits in each factor, for integers* m *and* n*, we obtain an isomorphism*

$$\mathbb{S}[m] \otimes \mathsf{S}[n] \xrightarrow{\simeq} \mathbb{S}[m+n].$$

Hence for spectra X *and* Y*, we obtain a morphism between spaces:*

$$\begin{aligned}\mathsf{Maps}_{\mathsf{Sptr}}(\mathsf{S}[m], X) \times \mathsf{Maps}_{\mathsf{Sptr}}(\mathsf{S}[n], Y) &\to \mathsf{Maps}_{\mathsf{Sptr}}(\mathbb{S}[m] \otimes \mathsf{S}[n], X \otimes Y) \\ &\xrightarrow{\simeq} \mathsf{Maps}_{\mathsf{Sptr}}(\mathbb{S}[m+n], X \otimes Y).\end{aligned}$$

Taking π_0*, we obtain a map*

$$\pi_m X \times \pi_n Y \to \pi_{m+n}(X \otimes Y).$$

One can check this map is bilinear, and therefore gives a morphism in Ab*:*

$$\pi_m X \otimes \pi_n Y \to \pi_{m+n}(X \otimes Y),$$

which can be assembled to a morphism

$$\pi_\bullet X \otimes \pi_\bullet Y \to \pi_\bullet(X \otimes Y),$$

where the source is the graded *tensor product. One can check this gives a right lax symmetric monoidal structure*[4][5] *on the functor* $\pi_\bullet$*. In other words, we have an* ∞*-operad map*

$$\pi_\bullet : \mathsf{Sptr}^{\otimes} \to \mathsf{grAb}^{\otimes}.$$

[4]There is no higher datum in this structure because grAb is ordinary.

[5]Warning: the functor $\mathsf{grAb} \to Ab$, $(M_i) \mapsto \bigoplus M_i$ is not symmetric monoidal.

Warning 2.4. *The functor $\pi_\bullet$ is not symmetric monoidal. Indeed, $\pi_\bullet(\mathbb{S})$ differs dramatically from $\mathbb{Z}$.*

2.5. The following exercises say π_0 is symmetric monoidal when restricted to $\mathsf{Sptr}_{\geq 0}$.

Exercise 2.6. *The full sub-∞-category $\mathsf{Sptr}_{\geq 0} \subseteq \mathsf{Sptr}$ is closed under smash products. Hint: $\mathsf{Sptr}_{\geq 0}$ is generated by $\mathbb{S}$ under colimits and extensions.*

Exercise 2.7. *The lax symmetric monoidal structure on $\pi_0 : \mathsf{Sptr}_{\geq 0} \to \mathsf{Ab}$ is strict.*

Exercise 2.8. *Describe the smash products of spectra via the corresponding $\mathbb{E}_\infty$-monoidal ∞-groupoids. Hint: you may first look at the $\mathbb{E}_0$-case.*

3. $\mathbb{E}_k$-RINGS

Definition 3.1. *We equip Sptr with the smash product symmetric monoidal structure. For $0 \leq k \leq \infty$, an* ***$\mathbb{E}_k$-ring*** *is an $\mathbb{E}_k$-algebra in Sptr.*

Construction 3.2. *Since $\mathsf{Sptr} \to \mathsf{grAb}$ is naturally right lax symmetric monoidal, we obtain a functor*

$$\mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Sptr}) \to \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{grAb}).$$

Therefore:

- *For an $\mathbb{E}_1$-ring A, we obtain an associative ring $\pi_\bullet A$.*
- *For an $\mathbb{E}_k$-ring A with $k \geq 2$, we obtain a* graded *commutative ring $\pi_\bullet A$.*

Construction 3.3. *Note that $\pi_0 : \mathsf{Sptr}_{\geq 0} \to \mathsf{Ab}$ can be identified with the functor $\tau_{\leq 0}$ via $\mathsf{Ab} \simeq \mathsf{Sptr}^\heartsuit$, and it admits a right adjoint*

$$\mathsf{Ab} \to \mathsf{Sptr}_{\geq 0},\ R \mapsto \mathbb{H}R$$

sending R to the Eilenberg–Maclane spectrum $\mathbb{H}R$. Since the left adjoint functor is naturally symmetric monoidal, the right adjoint admits a right lax symmetric monoidal structure. In particular, we obtain a functor

$$\mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Ab}) \to \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Sptr}_{\geq 0}) \to \mathsf{Alg}_{\mathbb{E}_k}(\mathsf{Sptr}).$$

In other words:

- *For an associative ring R, we obtain an $\mathbb{E}_1$-ring structure on $\mathbb{H}R$.*
- *For a commutative ring R, we obtain an $\mathbb{E}_\infty$-ring structure on $\mathbb{H}R$.*

3.4. Our final task is to relate modules of $\mathbb{H}R$ and R. For simplicity, we work in the $\mathbb{E}_\infty$-case.

From now on, let R be a commutative ring, Mod_R be the abelian category of R-modules and $\mathsf{Mod}_{\mathbb{H}R}(\mathsf{Sptr})$ be the ∞-category of $\mathbb{H}R$-modules in Sptr.

Construction 3.5. *Consider the relative tensor product symmetric monoidal structure on Mod_R. Passing to derived ∞-categories, we obtain a* presentable *symmetric monoidal structure on $\mathsf{D}(R) := \mathsf{D}(\mathsf{Mod}_R)$ (see HA.7.1.2.12 for details). Since $\mathsf{D}(R)$ is stable, we obtain a commutative algebra object $\mathsf{D}(R) \in \mathsf{Alg}_{\mathsf{Comm}}(\mathsf{PrSt}^\mathsf{L})$. Recall Sptr is the unit of PrSt^L. We obtain a morphism $\mathsf{Sptr} \to \mathsf{D}(R)$ in $\mathsf{Alg}_{\mathsf{Comm}}(\mathsf{PrSt}^\mathsf{L})$. In other words, a symmetric monoidal functor*

$$F : \mathsf{Sptr} \to \mathsf{D}(R).$$

3.6. Note that the underlying functor $\mathsf{Sptr} \to \mathsf{D}(R)$ is the unique colimit-preserving functor sending $\mathbb{S} \in \mathsf{Sptr}$ to $R \in \mathsf{D}(R)$. On the other hand, the left adjoint of the composition

$$G : \mathsf{D}(R) \to \mathsf{D}(\mathbb{Z}) \xrightarrow{\mathsf{DK}} \mathsf{Sptr}$$

also sends $\mathbb{S}$ to R. It follows that we have an adjunction

$$F : \mathsf{Sptr} \rightleftarrows \mathsf{D}(R) : G$$

such that F is symmetric monoidal while G is right lax symmetric monoidal[6]. Note that G also preserves colimits.

Exercise 3.7. *Identify the composition*

$$G \circ F : \mathsf{Sptr} \to \mathsf{D}(R) \to \mathsf{Sptr}$$

with $\mathbb{H}R \otimes -$.

Theorem 3.8 (HA.7.1.2.13)**.** *There is a canonical equivalence of symmetric monoidal* ∞*-categories*

$$\mathsf{Mod}_{\mathbb{H}R}(\mathsf{Sptr}) \simeq \mathsf{D}(R)$$

such that the adjunction $F : \mathsf{Sptr} \rightleftarrows \mathsf{D}(R) : G$ *can be identified with*

$$\mathsf{ind}_u : \mathsf{Mod}_{\mathbb{S}}(\mathsf{Sptr}) \rightleftarrows \mathsf{Mod}_{\mathbb{H}R}(\mathsf{Sptr}) : \mathsf{res}_u$$

along the unit morphism $u : \mathbb{S} \to \mathbb{H}R$*.*

Appendix A. The Barr–Beck–Lurie Theorem

A.1. The key input in the proof of Theorem 3.8 is the follwing theorem.

Theorem A.2 (Barr–Beck–Lurie)**.** *Let* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *be an adjunction between* ∞*-categories such that*

(i) *The functor* G *is conservative;*
(ii) *The functor* G *preserves* G*-split simplicial colimits*[7]*.*

Then there is a canonical equivalence $\mathsf{D} \simeq \mathsf{LMod}_T(\mathsf{C})$*, where* T *is a* monad *acting on* C *such that the underlying endomorphism is* $G \circ F$*. Moreover, the adjunction* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *can be identified with*

$$\mathsf{ind}_T : \mathsf{C} \to \mathsf{LMod}_T(\mathsf{C}) : \mathsf{oblv}_T.$$

A.3. **Suggested readings.** HA.4.7.

[6]More precisely, we have an adjunction $\mathsf{Sptr}^{\otimes} \rightleftarrows \mathsf{D}(R)^{\otimes}$ defined over Fin_*

[7]This means: Let $V \in \mathsf{D}_{\Delta} := \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{D})$ be a simplicial object such that $G(V)$ admits a splitting. Then V admits a colimit in D, and this colimit is preserved by G.