LECTURE 10

From this lecture on, we turn to the geometric side of the localization theory. The main player will be (algebraic) D-modules on the flag variety G/B. In this lecture, we define D-modules. There are many good references for this theory. For example, [HTT] is a thorough textbook, while [B] is a short notes.

1. Recollection: (co)tangent sheaves

Recall the following definitions in algebraic geometry.

Definition 1.1. Let A be a k-algebra and M be an A-module. A **k-derivation** of A into M is a k-linear map $D : A \to M$ satisfying the **Lebniz rule**

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

Let $\mathsf{Der}(A, M)$ be the set of such k-derivations. This is naturally an A-module.

Definition 1.2. Let $(X, \mathcal{O}_X)$ be any k-ringed space and $\mathcal{M}$ be an $\mathcal{O}_X$-module. A **k-derivation** of $\mathcal{O}_X$ into $\mathcal{M}$ is a k-linear morphism $D : \mathcal{O}_X \to \mathcal{M}$ such that for any open subscheme $U \subset X$, $D(U) : \mathcal{O}(U) \to \mathcal{M}(U)$ is a k-derivation. Let $\mathsf{Der}(\mathcal{O}_X, \mathcal{M})$ be the space of k-derivations.

Proposition-Definition 1.3. *Let X be a k-scheme. The functor $\mathcal{O}_X\text{-}\mathsf{mod} \to \mathsf{Vect}$, $\mathcal{M} \mapsto \mathsf{Der}(\mathcal{O}_X, \mathcal{M})$ is represented by a quasi-coherent $\mathcal{O}_X$-module Ω^1_X, i.e.*

$$\mathsf{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \simeq \mathsf{Der}(\mathcal{O}_X, \mathcal{M}).$$

*We call Ω^1_X the **sheaf of differentials**, or the **cotangent sheaf**, of X over k.*

The identity map on Ω^1_X corresponds to a k-derivation

$$d : \mathcal{O}_X \to \Omega^1_X,$$

*which is called the **universal k-derivation** of $\mathcal{O}_X$.*

*When $X = \mathsf{Spec}(A)$ is affine, let $\Omega^1_A \in A\text{-}\mathsf{mod}$ be such that $\Omega^1_X \simeq \widetilde{\Omega^1_A}$. We call Ω^1_A the **module of differentials** of A over k.*

Example 1.4. If $A = \mathsf{Sym}(V)$ for a k-vector space $V \in \mathsf{Vect}$, then $\Omega^1_A \simeq A \otimes_k V$ and $d : A \to \Omega^1_A$ sends $v \in V \subset \mathsf{Sym}(V)$ to $dv = 1 \otimes v \in A \otimes_k V$.

Construction 1.5. *Let $f : X \to Y$ be a morphism between k-schemes. For any $\mathcal{O}_X$-module $\mathcal{M}$, consider the composition*

$$\mathsf{Der}(\mathcal{O}_X, \mathcal{M}) \to \mathsf{Der}(f_*\mathcal{O}_X, f_*\mathcal{M}) \to \mathsf{Der}(\mathcal{O}_Y, f_*\mathcal{M})$$

that sends a k-derivation $D : \mathcal{O}_X \to \mathcal{M}$ to $\mathcal{O}_Y \to f_*\mathcal{O}_X \xrightarrow{f_*(D)} f_*\mathcal{M}$. By definition, we obtain maps

$$\mathsf{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{M}) \to \mathsf{Hom}_{\mathcal{O}_Y}(\Omega^1_Y, f_*\mathcal{M}) \simeq \mathsf{Hom}_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{M})$$

that are functorial in $\mathcal{M}$. This gives an $\mathcal{O}_X$-linear morphism

$$f^*\Omega^1_Y \to \Omega^1_X. \tag{1.1}$$

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Lemma 1.6. *If $f : X \to Y$ is an open embedding, or more generally an étale morphism, then $f^*\Omega^1_Y \to \Omega^1_X$ is an isomorphism.*

Lemma 1.7. *Let $f : X \to Y$ be a closed embedding corresponding to the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$. Then we have*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} f^*\Omega^1_Y \to \Omega^1_X \to 0,$$

where the map $\mathcal{I}/\mathcal{I}^2 \to f^*\Omega^1_Y \simeq \Omega^1_Y/\mathcal{I}\Omega^1_Y$ is induced by $\mathcal{I} \hookrightarrow \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y$.

Remark 1.8. Since any affine scheme is a closed subscheme of $\mathsf{Spec}(\mathsf{Sym}(V))$ for some $V \in \mathsf{Vect}$, the above lemmas, together with Example 1.4, allow us to calculate Ω^1_X for any k-scheme X.

Corollary 1.9. *If X is (locally) of finite type over k, then Ω^1_X is (locally) coherent.*

Corollary 1.10. *If X is a smooth k-scheme of dimension n, then Ω^1_X is locally free of rank n.*

From now on, we always assume X is a smooth k-scheme.

Construction 1.11. *Let X be a smooth k-scheme. Define $\Omega^n_X := \bigwedge^n_{\mathcal{O}_X}(\Omega^1_X)$, i.e., the anti-symmetric quotient of $\Omega^1_X \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \Omega^1_X$. As in the study of differential geometry, there is a unique complex*

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots$$

*such that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^m(\alpha \wedge d\beta)$, where $\alpha \in \Omega^m_X(U)$ is a m-form. This is the **de Rham complex** of X.*

Construction 1.12. *Let X be a smooth k-scheme. We define the **tangent sheaf** $\mathcal{T}_X$ of X over k to be the dual of Ω^1_X, i.e.,*

$$\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

Note that $\mathcal{T}_X$ is quasi-coherent, and for any open subset U, we have

$$\mathcal{T}(U) := \mathsf{Der}(\mathcal{O}_U, \mathcal{O}_U).$$

By definition, for $\partial \in \mathcal{T}(U)$ and $f \in \mathcal{O}(U)$, we have $\partial(f) = \langle \partial, df \rangle$.

Corollary 1.13. *We have equivalences between functors $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathsf{Vect}$:*

$$\mathsf{Der}_k(\mathcal{O}_X, -) \simeq \Gamma(X, \mathcal{T}_X \underset{\mathcal{O}_X}{\otimes} -).$$

Remark 1.14. The **tangent space** $T_{X,x}$ introduced in [Section 3, Lecture 3] can be identified with the stalk of $\mathcal{T}_X$ at x.

Corollary 1.15. *If X is a smooth k-scheme of dimension n, then $\mathcal{T}_X$ is locally free of rank n.*

Construction 1.16. *Let $f : X \to Y$ be a morphism between smooth k-schemes. The morphism* (1.1) *induces an $\mathcal{O}_X$-linear morphism*

$$df : \mathcal{T}_X \to f^*\mathcal{T}_Y.$$

Lemma 1.17. *Let $X = X_1 \times X_2$ be a smooth k-scheme. Then the $\mathcal{O}_X$-linear morphism*

$$(d\mathsf{pr}_1, d\mathsf{pr}_2) : \mathcal{T}_X \to \mathsf{pr}_1^*\mathcal{T}_{X_1} \oplus \mathsf{pr}_2^*\mathcal{T}_{X_2}$$

is an isomorphism.

Construction 1.18. *Let X be a smooth k-scheme. For any open subset U of X, $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ defines a Lie bracket on $\mathsf{Der}(\mathcal{O}_U, \mathcal{O}_U)$. By defition, $\mathcal{O}_U$ is a representation of the obtained Lie algebra.*

It follows that $\mathcal{T}_X$ is a sheaf of Lie algebras *on X, and $\mathcal{O}_X$ is a $\mathcal{T}_X$-module.*

Warning 1.19. *The category $\mathcal{O}_X$–mod has a natural symmetric monoidal structure but $\mathcal{T}_X$ is* not *a Lie algebra object in this symmetric monoidal category. In other words, the Lie bracket $[-,-]:\mathcal{T}_X\otimes_k\mathcal{T}_X\to\mathcal{T}_X$ does not factor through $\mathcal{T}_X\otimes_{\mathcal{O}_X}\mathcal{T}_X$.*

Construction 1.20. *As in the study of differential geometry, for any local section $\partial\in\mathcal{T}(U)$, we can define the* ***contraction*** *operator*

$$i_\partial:\Omega_X^{n+1}(U)\to\Omega_X^n(U)$$

and the ***Lie derivative***

$$\mathcal{L}_\partial:\Omega_X^n(U)\to\Omega_X^n(U).$$

They satisfy all the identities in differential geometry. In particular, we have the ***Cartan's magic formula****:*

$$\mathcal{L}_\partial(\omega)=i_\partial(d\omega)+d(i_\partial\omega).$$

We have the following useful result, which allows us to apply the techniques in differential geometry to the study of $\mathcal{T}_X$ and Ω_X. For a proof, see [HTT, Theorem A.5.1].

Proposition-Definition 1.21. *Let X be any n-dimensional smooth k-scheme and $p\in X$ be a closed point. Then there exists an affine open neighborhood U of p and functions $x_i\in\mathcal{O}(U)$, $i=1,\cdots,n$, such that*

(i) *$\{dx_i\}$ is a free basis of $\Omega^1(U)$ as an $\mathcal{O}(U)$-module;*
(ii) *Let $\{\partial_i\}$ be the dual basis of $\mathcal{T}(U)$, then $[\partial_i,\partial_j]=0$ and $\partial_i(x_j)=\delta_{ij}$.*
(iii) *The images of x_i in the local ring $\mathcal{O}_{X,p}$ generate the maximal ideal $\mathfrak{m}_{X,p}$.*

We call such a system $\{x_i\}$ an ***étale coordinate system*** *of X near p.*

Remark 1.22. (ii) actually follows from (i).

Remark 1.23. The functions x_i define an étale map $X\to\mathbb{A}^n$ sending $p\in X$ to the origin $0\in\mathbb{A}^n$.

2. Tangent sheaf vs. Lie algebra

Construction 2.1. *Let X be a finite type k-scheme equipped with a* right *action of an algebraic group G. Let $\mathfrak{g}:=\mathsf{Lie}(G)$ be the Lie algebra of G. We construct a Lie algebra homomorphism*

$$a:\mathfrak{g}\to\mathcal{T}(X)$$

as follows. Consider the action map $\mathsf{act}:X\times G\to X$ and the $\mathcal{O}_{X\times G}$-linear morphisms

$$\mathcal{O}_X\otimes\mathcal{T}_G\simeq\mathsf{pr}_2^*\mathcal{T}_G\hookrightarrow\mathcal{T}_{X\times G}\xrightarrow{d\mathsf{act}}\mathsf{act}^*\mathcal{T}_X.$$

By restricting along $X\xrightarrow{\mathsf{Id}\times e}X\times G$, we otain $\mathcal{O}_X$-linear morphisms

$$\mathcal{O}_X\otimes\mathfrak{g}\to\mathcal{T}_X,$$

which induces the desired k-linear map $a:\mathfrak{g}\to\mathcal{T}(X)$.

Remark 2.2. Using the language of differential geometry, for $x\in\mathfrak{g}$, $a(x)$ is the vector field of the flow $\mathsf{act}(-,\exp(tx)):X\to X$.

Lemma 2.3. *The above map $a:\mathfrak{g}\to\mathcal{T}(X)$ is a Lie algebra homomorphism.*

Proof. Unwinding the definitions, the corresponding map $\mathsf{Der}(\mathcal{O}_G,k_e)\to\mathsf{Der}(\mathcal{O}_X,\mathcal{O}_X)$ sends a k-derivation $\partial:\mathcal{O}_G\to k_e$ to the composition[1]

$$a(\partial_1):\mathcal{O}_X\to\mathcal{O}_X\otimes\mathcal{O}_G\xrightarrow{\mathsf{Id}\otimes\partial}\mathcal{O}_X\otimes k_e\simeq\mathcal{O}_X.$$

[1]For non-affine G, we use the morphism between $\mathcal{O}_X$-modules: $\mathcal{O}_X\to\mathsf{act}_*(\mathcal{O}_G\otimes\mathcal{O}_X)\xrightarrow{\partial\otimes\mathsf{Id}}\mathsf{act}_*(k_e\otimes\mathcal{O}_X)\simeq\mathcal{O}_X$.

Using the axioms of group actions, we can identify $a(\partial_1) \circ a(\partial_2)$ with

$$\mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_G \xrightarrow{\mathsf{Id} \otimes \Delta} \mathcal{O}_X \otimes \mathcal{O}_G \otimes \mathcal{O}_G \xrightarrow{\mathsf{Id} \otimes \partial_1 \otimes \partial_2} \mathcal{O}_X \otimes k_e \otimes k_e \simeq \mathcal{O}_X.$$

This implies

$$[a(\partial_1), a(\partial_2)] = a(\partial_1) \circ a(\partial_2) - a(\partial_2) \circ a(\partial_1) = a([\partial_1, \partial_2]),$$

where the last equation is due to [Remark 4.5, Lecture 3].

□

Remark 2.4. Let X be a finite type k-scheme equipped with a *left* action of an algebraic group G. We can obtain a right G-action by precomposing with $g \mapsto g^{-1}$. It follows that we also have a Lie algebra homomorphism $a : \mathfrak{g} \to \mathcal{T}(X)$. Using the language of differential geometry, for $x \in \mathfrak{g}$, $a(x)$ is the vector field of the flow $\mathsf{act}(\exp(-tx), -) : X \to X$.

Example 2.5. Consider the left and right multplication actions of G on itself. We obtain Lie algebra homomorphisms

$$\mathfrak{g} \xrightarrow{a_l} \mathcal{T}(G) \xleftarrow{a_r} \mathfrak{g}.$$

By construction, the image of a_r consists of *left* invariant vector fields on G, while the image of a_l consists of *right* invariant ones.

It is easy to see that the images of $a_l : \mathfrak{g} \to \mathcal{T}(G)$ and $a_r : \mathfrak{g} \to \mathcal{T}(G)$ commute with respect to the Lie bracket, i.e., $[a_l(x), a_r(y)] = 0$ for $x, y \in \mathfrak{g}$. Indeed, this follows by considering the $(G \times G)$-action on G given by $(g_1, g_2) \cdot x := g_1^{-1} x g_2$.

It is easy to show the obtained $\mathcal{O}_G$-linear maps, which we denote by the same symbols,

$$\mathcal{O}_G \otimes \mathfrak{g} \xrightarrow{a_l} \mathcal{T}_G \xleftarrow{a_r} \mathcal{O}_G \otimes \mathfrak{g}$$

are isomorphisms. Note that the stalks of the above maps at $e \in G$ are given by

$$\mathfrak{g} \xrightarrow{-\mathsf{Id}} \mathfrak{g} \xleftarrow{\mathsf{Id}} \mathfrak{g}.$$

Definition 2.6. Let X be a smooth k-scheme. A **quasi-coherent $\widetilde{\mathcal{T}}_X$-module** is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ equipped with a $\mathcal{T}_X$-module structure such that

(i) The map $\mathcal{T}_X \otimes_k \mathcal{M} \to \mathcal{M}$ is $\mathcal{O}_X$-linear, where $\mathcal{T}_X \otimes_k \mathcal{M}$ is viewed as an $\mathcal{O}_X$-module via the first factor.
(ii) The map $\mathcal{O}_X \otimes_k \mathcal{M} \to \mathcal{M}$ is $\mathcal{T}_X$-linear, where $\mathcal{O}_X \otimes_k \mathcal{M}$ is viewed as an $\mathcal{T}_X$-module via the diagonal action.

Let $\widetilde{\mathcal{T}}_X\text{-}\mathsf{mod}_{\mathsf{qc}}$ be the category of quasi-coherent $\widetilde{\mathcal{T}}_X$-module, where morphisms are defined in the obvious way.

Remark 2.7. Unwinding the definitions, (ii) means the action of $\mathcal{T}_X$ on $\mathcal{M}$ satisfies the **Lebniz rule**: for any local sections $\partial \in \mathcal{T}(U)$, $m \in \mathcal{M}(U)$ and $f \in \mathcal{O}(U)$, we have

$$\partial(f \cdot m) = f \cdot \partial(m) + \partial(f) \cdot m.$$

Remark 2.8. The notation stands $\widetilde{\mathcal{T}}_X$ stands for $\mathcal{O}_X \oplus \mathcal{T}_X$, viewed as a *Picard algebroid* on X in the sense of [BB][2].

Example 2.9. The structure sheaf $\mathcal{O}_X$, equipped with the standard $\mathcal{T}_X$-action, is a quasi-coherent $\widetilde{\mathcal{T}}_X$-module.

[2]A Picard algebroid $\mathcal{P}$ on X is a sheaf of Lie algebras equipped with a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{P} \to \mathcal{T}_X \to 0$, where $\mathcal{O}_X$ is viewed as a sheaf of Lie algebras, such that the action of $\mathcal{P}$ on its (Lie) ideal sheaf $\mathcal{O}_X$ factors through the canonical action of $\mathcal{T}_X$ on $\mathcal{O}_X$.

Remark 2.10. It is easy to show $\widetilde{\mathcal{T}}_X\text{−}\mathsf{mod}_{\mathsf{qc}}$ is an abelian category and the forgetful functor $\widetilde{\mathcal{T}}_X\text{−}\mathsf{mod}_{\mathsf{qc}} \to \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{qc}}$ is exact.

Construction 2.11. *Let X be a finite type k-scheme equipped with an action of an algebraic group G. We have a functor*

$$\Gamma : \widetilde{\mathcal{T}}_X\text{−}\mathsf{mod}_{\mathsf{qc}} \to \mathfrak{g}\text{−}\mathsf{mod},\ \mathcal{M} \mapsto \mathcal{M}(X),$$

where $\mathcal{M}(X)$ is viewed as a $\mathfrak{g}$-module via the Lie algebra homomorphism $a : \mathfrak{g} \to \mathcal{T}(X)$.

Remark 2.12. One version of the localizaiton theory says for $X = G/B$, the above functor induces an equivalence

$$\Gamma : \widetilde{\mathcal{T}}_X\text{−}\mathsf{mod}_{\mathsf{qc}} \xrightarrow{\simeq} \mathfrak{g}\text{−}\mathsf{mod}_{\chi_0},$$

where $\chi_0 = \varpi(0)$ is the central character of M_0.

3. Differential operators

Just like the associative algebra $U(\mathfrak{g})$ plays a significant role in the study of $\mathfrak{g}$-modules, there is a sheaf of associative algebras, known as the sheaf of differential operators $\mathcal{D}_X$, that plays a similar role in the study of $\mathcal{T}_X$-modules[3].

Definition 3.1. Let X be an affine smooth k-scheme. We define the notion of differential operator on X inductively.

A k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order $-n$ ($n > 0$) iff $D = 0$.

A k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ is a **differential operator of order** n ($n \geq 0$) if for any function $f \in \mathcal{O}(X)$, the k-linear map $[D, f] := D \circ f - f \circ D$, i.e.,

$$g \mapsto D(fg) - fD(g)$$

is a differential operator of order $n - 1$.

Let $\mathsf{F}^{\leq n}\mathcal{D}(X)$ be the space of differential operators of order n on X, and $\mathcal{D}(X) := \cup_n \mathsf{F}^{\leq n}\mathcal{D}(X)$ be the space of all differential operators on X.

Example 3.2. Multplication by any $f \in \mathcal{O}(X)$ is a differential operator of order 0 and the map $\mathcal{O}(X) \to \mathsf{F}^{\leq 0}\mathcal{D}(X) \simeq \mathsf{gr}^0\mathcal{D}(X)$ is an isomorphism.

Exercise 3.3. This is Homework 5, Problem 1. Let X be an affine smooth k-scheme. Prove: any k-derivation $\mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order 1, and the obtained map $\mathcal{O}(X) \oplus \mathcal{T}(X) \to \mathsf{F}^{\leq 1}\mathcal{D}(X)$ is an isomorphism.

Lemma 3.4. *Let X be an affine smooth k-scheme, and $D_1 \in \mathsf{F}^{\leq m}\mathcal{D}(X)$, $D_2 \in \mathsf{F}^{\leq n}\mathcal{D}(X)$ be differential operators. Prove:*

(1) *The composition $D_1 \circ D_2$ is a differential operator of order $m + n$.*
(2) *The commutator $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is a differential operator of order $m+n-1$.*

In other words, $\mathsf{F}^{\bullet}\mathcal{D}(X)$ is a filtered associative algebra such that $\mathsf{gr}^{\bullet}\mathcal{D}(X)$ is commutative.

Proof. Follow from the above exercise and induction. □

Remark 3.5. For *any* k-linear map $D : \mathcal{O}(X) \to \mathcal{O}(X)$ and functions $u, v \in \mathcal{O}(X)$, we have

$$[D, uv] = [D, u]v + u[D, v]. \tag{3.1}$$

[3]The sheaf $\mathcal{D}_X$ is so useful that $\mathcal{D}_X$-modules were introduced much earlier than $\mathcal{T}_X$-modules. The latter point of view was ignored until the study of general Lie algebroids.

It follows that in Definition 3.1, we can replace "for any function $f \in \mathcal{O}(X)$" by "for generators f of $\mathcal{O}(X)$".

Note that we also have

$$D(uv) = [D, u](v) + uD(v). \tag{3.2}$$

Remark 3.6. For a disjoint union $V_1 \sqcup V_2$ of affine smooth k-schemes, we have $\mathsf{F}^\bullet\mathcal{D}(V_1 \sqcup V_2) \simeq \mathsf{F}^\bullet\mathcal{D}(V_1) \oplus \mathsf{F}^\bullet\mathcal{D}(V_2)$.

Warning 3.7. *Definition 3.1 makes sense even when X is singular, but the obtained algebra $\mathcal{D}(X)$ is ill-behaved and is not the correct algebra to consider. In modern point of view, for singular X, the "correct" $\mathcal{D}(X)$ should be a DG algebra.*

4. Sheaf of differential operators

In this section, for any smooth k-scheme, we construct a sheaf $\mathcal{D}_X$ of filtered associative algebras such that for any affine open subscheme U, we have $\mathcal{D}_X(U) \simeq \mathcal{D}(U)$. The proofs in this section are technical and can be treated as blackboxes.

Throughout this section, let V be a *connected* affine smooth k-scheme and $f \in \mathcal{O}(V)$ be a nonzero function. Let $U \subset V$ be the affine open subscheme where $f \neq 0$, i.e. $\mathcal{O}(U) \simeq \mathcal{O}(V)_f$. Note that U is also connected. We identify $\mathcal{O}(V)$ as a subalgebra of $\mathcal{O}(U)$.

Lemma 4.1. *Any differential operator $D \in \mathcal{D}(U)$ is determined by its restriction $D|_{\mathcal{O}(V)} : \mathcal{O}(V) \to \mathcal{O}(U)$.*

Proof. Let $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$, for any $g \in \mathcal{O}(V)$ and $m \geq 0$, we have

$$D(g) = D(f^m \cdot \frac{g}{f^m}) = f^m \cdot D(\frac{g}{f^m}) + [D, f^m](\frac{g}{f^m}). \tag{4.1}$$

It follows that

$$D(\frac{g}{f^m}) = \frac{D(g)}{f^m} - \frac{[D, f^m](\frac{g}{f^m})}{f^m}. \tag{4.2}$$

Note that $[D, f^m] \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$ hence it is determined by $[D, f^m]|_{\mathcal{O}(V)} = [D|_{\mathcal{O}(V)}, f^m]$, which is determined by $D|_{\mathcal{O}(V)}$. Now (4.2) implies D is also determined by $D|_{\mathcal{O}(V)}$. □

Lemma 4.2. *For any differential operator $D_0 \in \mathsf{F}^{\leq n}\mathcal{D}(V)$, there is a unique differential operator $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{O}(V) \arrow[r, "D_0"] \arrow[d] & \mathcal{O}(V) \arrow[d] \\ \mathcal{O}(U) \arrow[r, dotted, "D"] & \mathcal{O}(U). \end{tikzcd}$$

Proof. The uniqueness follows from Lemma 4.1. It remains to show the existence. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$ and any $m \geq 0$, by induction hypothesis, there is a unique differential operator $[D_0, f^m]^\sharp \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$ that extends $[D_0, f^m] \in \mathsf{F}^{\leq n-1}\mathcal{D}(V)$. Motivated by (4.2), for any $g \in \mathcal{O}(V)$, we define

$$D(\frac{g}{f^m}) := \frac{D_0(g)}{f^m} - \frac{[D_0, f^m]^\sharp(\frac{g}{f^m})}{f^m}.$$

We need to show the map D is well-defined, i.e., $D(\frac{g}{f^m}) = D(\frac{f^l g}{f^{l+m}})$ for any $l \geq 0$. Note that we have

$$D_0(f^l g) = f^l D_0(g) + [D_0, f^l](g).$$

We also have $[D_0, f^{l+m}] = [D_0, f^l]f^m + f^m[D_0, f^l]$. This implies

$$[D_0, f^{l+m}]^\sharp = [D_0, f^l]^\sharp f^m + f^m[D_0, f^l]^\sharp.$$

Combining the above two equations, a direct calculation shows $D(\frac{g}{f^m}) = D(\frac{f^l g}{f^{l+m}})$ as desired.

It remains to show D is a differential operator of order n. By Remark 3.5, we only need to show $[D, f^{-1}]$ and $[D, h]$, $h \in \mathcal{O}(V)$ are differential operators of order $n-1$. By (3.1), we have $0 = [D, ff^{-1}] = f[D, f^{-1}] + [D, f]f^{-1}$. Hence $[D, f^{-1}] = -f^{-1}[D, f]f^{-1}$. Therefore we only need to show $D' := [D, h] \in \mathsf{F}^{\leq n-1}\mathcal{D}(U)$. Write $D_0' := [D_0, h] \in \mathsf{F}^{\leq n-1}\mathcal{D}(V)$. A direct calculations shows D' can be obtained from D_0' using the same formula that defines D from D_0, i.e.,

$$D'(\frac{g}{f^m}) := \frac{D_0'(g)}{f^m} - \frac{[D_0', f^m]^\sharp(\frac{g}{f^m})}{f^m}.$$

Hence we win by induction in n (again).

□

Lemma 4.3. *For any differential operator $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$, there exists an integer $N \geq 0$ such that $f^N D$ and Df^N send $\mathcal{O}(V)$ into $\mathcal{O}(V)$.*

Proof. We prove by induction in n. When $n < 0$, there is nothing to prove. For $n \geq 0$,let $g_i \in \mathcal{O}(V)$, $i \in I$ be a *finite* set of generators. By induction hypothesis, there exists an integer $N \geq 0$ such that $f^N[D, g_i]$ and $[D, g_i]f^N$ preserve $\mathcal{O}(V)$ for any $i \in I$. By (3.1), $f^N[D, g]$ and $[D, g]f^N$ preserve $\mathcal{O}(V)$ for any $g \in \mathcal{O}(V)$. By enlarging N, we can assume $f^N D(1) \in \mathcal{O}(V)$. Hence by (3.2), $f^N D(g) = f^N[D, g](1) + f^N g D(1) \in \mathcal{O}(V)$ and $D(f^{2N}g) = [D, f^N](f^N g) + f^N D(f^N g) \in \mathcal{O}(V)$.

□

Construction 4.4. *Let $\mathsf{F}^\bullet\mathcal{D}(V) \to \mathsf{F}^\bullet\mathcal{D}(U)$ be the map defined by Lemma 4.2. Recall $\mathcal{D}(U)$ is an $\mathcal{O}(U)$-bimodule and $\mathcal{D}(V)$ is an $\mathcal{O}(V)$-bimodule. It is easy to see the above map is compatible with the homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$. It follows that we have maps*

$$\mathcal{O}(U) \underset{\mathcal{O}(V)}{\otimes} \mathsf{F}^\bullet\mathcal{D}(V) \to \mathsf{F}^\bullet\mathcal{D}(U) \leftarrow \mathsf{F}^\bullet\mathcal{D}(V) \underset{\mathcal{O}(V)}{\otimes} \mathcal{O}(U), \tag{4.3}$$

such that the rightward map is left $\mathcal{O}(U)$-linear while the leftward map is right $\mathcal{O}(U)$-linear.

Lemma 4.5. *The maps* (4.3) *are isomorphisms.*

Proof. We first show the maps are injective. Note that any element in $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathsf{F}^\bullet\mathcal{D}(V)$ is a pure tensor $f^{-m} \otimes D_0$ for some $m \geq 0$ and $D_0 \in \mathsf{F}^n\mathcal{D}(V)$. The rightward map sends it to $f^{-m}D_0 \in \mathsf{F}^n\mathcal{D}(U)$. If $f^{-m}D_0 = 0$, then $D_0 = 0$ because $\mathcal{O}(V)$ is integral. This proves the rightward map is injective. A similar argument shows the leftward map is injective.

To prove the maps are surjective, let $D \in \mathsf{F}^{\leq n}\mathcal{D}(U)$. By Lemma 4.3, there exists $N \geq 0$ such that $D_0 := f^N D$ and $D_0' := Df^N$ preserve $\mathcal{O}(V)$. We view D_0 and D_0' as elements in $\mathsf{F}^{\leq n}\mathcal{D}(V)$. By Lemma 4.1, the rightward (resp. leftward) map sends $f^{-N} \otimes D_0$ (resp. $D_0' \otimes f^{-N}$) to D.

□

Construction 4.6. *Let X be a smooth k-scheme. By Lemma 4.5 and Remark 3.6, for any n, there exists a unique sheaf $\mathsf{F}^{\leq n}\mathcal{D}_X$ such that:*

- *For any affine open subscheme $U \subset X$,*

$$\mathsf{F}^{\leq n}\mathcal{D}_X(U) := \mathsf{F}^{\leq n}\mathcal{D}(U);$$

- *For $U \subset V \subset X$ as before, the connecting map $\mathsf{F}^{\le n}\mathcal{D}_X(V) \to \mathsf{F}^{\le n}\mathcal{D}_X(U)$ is given by the map $\mathsf{F}^{\le n}\mathcal{D}(U) \to \mathsf{F}^{\le n}\mathcal{D}(V)$ in Construction 4.4.*

*We call $\mathsf{F}^{\le n}\mathcal{D}_X$ the **sheaf of differential operators of order** n on X.*

*Define the **sheaf of differential operators** on X to be*

$$\mathcal{D}_X := \cup_n \mathsf{F}^{\le n}\mathcal{D}_X.$$

Note that $\mathsf{F}^\bullet\mathcal{D}_X$ is naturally a sheaf of filtered associtive k-algebras such that $\mathsf{gr}^\bullet\mathcal{D}_X$ is commutative.

Remark 4.7. Unwinding the definitions, for any open subscheme $V \subset X$, $\mathcal{D}_X(V)$ can be identified with the algebra of k-linear morphisms $\mathcal{O}_V \to \mathcal{O}_V$ such that for any affine open subscheme $U \subset V$, the obtained map $\mathcal{O}(U) \to \mathcal{O}(U)$ is a differential operator.

Corollary 4.8. *Let X be a smooth k-scheme. We have natural isomorphisms $\mathcal{O}_X \simeq \mathsf{F}^{\le 0}\mathcal{D}_X$ and $\mathcal{O}_X \oplus \mathcal{T}_X \simeq \mathsf{F}^{\le 1}\mathcal{D}_X$.*

Corollary 4.9. *Let X be a smooth k-scheme. Then $\mathcal{D}_X$ is naturally an $\mathcal{O}_X$-bimodule, and it is quasi-coherent for both the left and right $\mathcal{O}_X$-module structures.*

5. The PBW theorem for $\mathcal{D}_X$

Construction 5.1. *Let X be a smooth k-scheme. By Corollary 4.8, we have a homomorphism $\mathcal{O}_X \to \mathsf{gr}^0\mathcal{D}_X$ and a morphism $\mathcal{T}_X \to \mathsf{gr}^1\mathcal{D}_X$ between their modules. By the universal property of symmetric algebras, we obtain a graded homomorphism*

$$\mathsf{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathsf{gr}^\bullet\mathcal{D}_X. \tag{5.1}$$

The proof of the following theorem can be treated as a blackbox[4].

Theorem 5.2 (PBW for $\mathcal{D}_X$)**.** *The above graded homomorphism is an isomorphism.*

Proof. We can assume X is affine and connected. Hence we only need to show

$$\mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{gr}^n\mathcal{D}(X)$$

is an isomorphism. We prove by induction in n. For $n < 0$, there is nothing to prove. For $n \ge 0$ and $D \in \mathsf{F}^{\le n}\mathcal{D}(X)$, consider the composition

$$\mathcal{O}(X) \xrightarrow{[D,-]} \mathsf{F}^{\le n-1}\mathcal{D}(X) \twoheadrightarrow \mathsf{gr}^{n-1}\mathcal{D}(X).$$

By (3.1), this is a k-derivation of $\mathcal{O}(X)$ into $\mathsf{gr}^{n-1}\mathcal{D}(X)$, where the latter is viewed as an $\mathcal{O}(X)$-module via the homomorphism $\mathcal{O}(X) \to \mathsf{gr}^0\mathcal{D}(X)$. Moreover, this k-derivation only depends on the image of D in $\mathsf{gr}^n\mathcal{D}(X)$. In other words, we have a k-linear map

$$\mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Der}_k(\mathcal{O}(X), \mathsf{gr}^{n-1}\mathcal{D}(X)),\ D \mapsto [D,-].$$

Note that $[gD, f] = g[D, f]$ for any $f, g \in \mathcal{O}(X)$. It follows that the above map is $\mathcal{O}(X)$-linear. By induction hypothesis and Corollary 1.13, we have

$$\mathsf{Der}_k(\mathcal{O}(X), \mathsf{gr}^{n-1}\mathcal{D}(X)) \simeq \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{gr}^{n-1}\mathcal{D}(X) \simeq \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)).$$

Composing with the multplication map

$$\mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)),$$

[4]In fact, for the purpose of *this* course, we can define $\mathcal{D}_X$ to be the subsheaf of $\mathcal{H}om_k(\mathcal{O}_X, \mathcal{O}_X)$ generated by the images $\mathcal{O}_X$ and $\mathcal{T}_X$ under compositions. Then the PBW theorem becomes obvious. However, it is not obvious to show this definition coincides with ours.

we obtain an $\mathcal{O}(X)$-linear map

$$\mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)).$$

By considering affine open subschemes, we obtain an $\mathcal{O}_X$-linear morphism

$$\mathsf{gr}^n\mathcal{D}_X \to \mathsf{Sym}^m_{\mathcal{O}_X}(\mathcal{T}_X).$$

We claim it is inverse to (5.1) up to multplication by n.

A direct calculation shows

$$\mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X)) \to \mathsf{gr}^n\mathcal{D}(X) \to \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X))$$

is given by multplication by n. Indeed, for $\partial_1, \cdots, \partial_n \in \mathcal{T}(X)$, we have

$$[\partial_1\cdots\partial_n, f] \equiv \sum_{i=1}^{n} \partial_i(f)\partial_1\cdots\widehat{\partial_i}\cdots\partial_n \quad \in \quad \mathsf{gr}^{\leq n-1}\mathcal{D}(X).$$

Hence $\partial_1\cdots\partial_n$ is sent to the element

$$\sum_{i=1}^{n} \partial_i \otimes \partial_1\cdots\widehat{\partial_i}\cdots\partial_n \in \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{Sym}^{n-1}_{\mathcal{O}(X)}(\mathcal{T}(X)),$$

which is sent to $n\partial_1\cdots\partial_n \in \mathsf{Sym}^n_{\mathcal{O}(X)}(\mathcal{T}(X))$ as desired.

It remains to show

$$\mathsf{gr}^n\mathcal{D}_X \to \mathsf{Sym}^m_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathsf{gr}^n\mathcal{D}_X \tag{5.2}$$

is given by multplication by n. Note that the problem is Zariski local in X, hence we can assume X admits an étale coordinate system $x_1, \cdots, x_d$. Now let $D \in \mathsf{F}^{\leq n}\mathcal{D}_X$ and

$$\sum_{i=1}^{d} \partial_i \otimes D_i \in \mathcal{T}(X) \underset{\mathcal{O}(X)}{\otimes} \mathsf{gr}^{n-1}\mathcal{D}(X)$$

be its image. By definition, for any function $f \in \mathcal{O}(X)$, we have

$$[D, f] \equiv \sum_{i=1}^{d} \partial_i(f) D_i \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X)$$

Taking $f = x_i$, we obtain $D_i \equiv [D, x_i]$. Therefore for any $f \in \mathcal{O}(X)$,

$$[D, f] \equiv \sum_{i=1}^{d} \partial_i(f)[D, x_i] \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X) \tag{5.3}$$

and the composition (5.2) sends D to $\sum_{i=1}^{d} \partial_i[D, x_i]$. It remains to show

$$nD \equiv \sum_{i=1}^{d} \partial_i[D, x_i] \quad \in \quad \mathsf{gr}^n\mathcal{D}(X)$$

To prove this, we use induction in n (again). For $n \leq 0$, the claim is obvious. For $n > 0$, we only need to show

$$n[D, f] \equiv \sum_{i=1}^{d} [\partial_i[D, x_i], f] \quad \in \quad \mathsf{gr}^{n-1}\mathcal{D}(X) \tag{5.4}$$

for any $f \in \mathcal{O}(X)$.

A direct calculation shows

$$[\partial_i[D, x_i], f] = \partial_i[[D, f], x_i] + \partial_i(f)[D, x_i]. \tag{5.5}$$

By induction hypothesis, we have

$$(n-1)[D,f] \equiv \sum_{i=1}^{d} \partial_i[[D,f],x_i] \ \in\ \mathrm{gr}^{n-1}\mathcal{D}(X). \tag{5.6}$$

Now (5.3)+(5.6) implies (5.4) by (5.5).

□

Corollary 5.3. *Let X be a affine smooth k-scheme, then the associative algebra $\mathcal{D}(X)$ is generated by the images of $\mathcal{O}(X)$ and $\mathcal{T}(X)$ subject to the following relations:*

$$f_1 \star f_2 = f_1 f_2,\ \partial_1 \star \partial_2 - \partial_2 \star \partial_1 = [\partial_1,\partial_2],\ f \star \partial = f\partial,\ \partial \star f - f \star \partial = \partial(f)$$

for $f, f_1, f_2 \in \mathcal{O}(X)$ and $\partial, \partial_1, \partial_2 \in \mathcal{T}(X)$. Here $\star$ (temporarily) denotes the multplication in $\mathcal{D}(X)$.

Remark 5.4. Equivalently, we can say $\mathcal{D}_X$ is the universal enveloping (associative) algebra of the Picard algebroid $\widetilde{\mathcal{T}}_X \coloneqq \mathcal{O}_X \oplus \mathcal{T}_X$.

Example 5.5. For $X = \mathbb{A}^d$, we have $\mathcal{D}(X) = k[x_1,\cdots,x_d,\partial_1,\cdots,\partial_d]$ such that $[x_i,x_j] = [\partial_i,\partial_j] = 0$ and $[\partial_i, x_j] = \delta_{i,j}$. This is known as the **Weyl algebra**.

6. Definition of D-modules

Definition 6.1. Let X be a smooth k-scheme. A **left (resp. right) $\mathcal{D}_X$-module** is a sheaf $\mathcal{M}$ of k-vector spaces equipped with a left (resp. right) action by $\mathcal{D}_X$.

Let $\mathcal{D}_X\text{-}\mathsf{mod}^l$ (resp. $\mathcal{D}_X\text{-}\mathsf{mod}^r$) be the category of left (resp. right) $\mathcal{D}_X$-modules, where morphisms are defined in the obvious way.

Construction 6.2. *Let X be a smooth k-scheme. Restricting along the homomorphism $\mathcal{O}_X \to \mathcal{D}_X$, we obtain forgetful functors*

$$\mathsf{oblv}^l : \mathcal{D}_X\text{-}\mathsf{mod}^l \to \mathcal{O}_X\text{-}\mathsf{mod},\ \mathsf{oblv}^r : \mathcal{D}_X\text{-}\mathsf{mod}^r \to \mathcal{O}_X\text{-}\mathsf{mod}.$$

Definition 6.3. We say a left (resp. right) $\mathcal{D}_X$-module $\mathcal{M}$ is **quasi-coherent** if the underlying $\mathcal{O}_X$-module is quasi-coherent.

When X is quasi-compact, we say a quasi-coherent left (resp. right) $\mathcal{D}_X$-module $\mathcal{M}$ is **coherent** if it is locally finitely generated.

Let $\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{(q)c}}$ (resp. $\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{(q)c}}$) be the category of left (resp. right) (quasi-)coherent $\mathcal{D}_X$-modules.

Remark 6.4. It is easy to see the above categories are abelian categories and the forgetful functors are exact.

Warning 6.5. *In general, a coherent $\mathcal{D}_X$-module is not coherent as an $\mathcal{O}_X$-module. When there is danger of ambiguity, we use the terminologies "$\mathcal{D}_X$-coherent" vs "$\mathcal{O}_X$-coherent".*

The following result follows from the PBW theorem:

Proposition 6.6. *Restricting along $\mathcal{O}_X \oplus \mathcal{T}_X \to \mathcal{D}_X$ defines an equivalence*

$$\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \simeq \widetilde{\mathcal{T}}_X\text{-}\mathsf{mod}_{\mathsf{qc}}.$$

Remark 6.7. In fact, we could have defined a notion of *quasi-coherent right $\widetilde{\mathcal{T}}_X$-modules* such that they form an abelian category equivalent to $\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$.

7. Examples of D-modules

Example 7.1. The sheaf $\mathcal{D}_X$ itself is a left and a right $\mathcal{D}_X$-module.

Example 7.2. The sheaf $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module with the action given by $D \cdot f := D(f)$.

Exercise 7.3. This is Homework 5, Problem 2. Let X be a smooth k-scheme of dimension n. Prove: there is a unique right $\mathcal{D}_X$-module structure on Ω_X^n such that for local sections $f \in \mathcal{O}(U)$, $\partial \in \mathcal{T}(U)$ and $\omega \in \Omega^n(U)$, the right action is given by

$$\omega \cdot f = f\omega,\ \omega \cdot \partial = -\mathcal{L}_\partial(\omega)$$

Example 7.4. For $X = \mathbb{A}^1 = \mathsf{Spec}(k[x])$, we define a left D-module $\mathcal{M}_{e^x}$ whose underlying $\mathcal{O}_X$-module is isomorphic to $\mathcal{O}_X$, with a generator denoted by "e^x". The $\mathcal{T}_X$-action is determined by the formular $\partial_x \cdot e^x = e^x$. It is easy to see $\mathcal{M}_{e^x} \simeq \mathcal{D}_X/\mathcal{D}_X \cdot (\partial_x - 1)$, where $\mathcal{D}_X \cdot (\partial_x - 1)$ is the left ideal of $\mathcal{D}_X$ generated by the section $\partial_x - 1$.

Example 7.5. For $X = \mathbb{A}^1 - 0 = \mathsf{Spec}(k[x^\pm])$ and $\lambda \in k$, we define a left D-module $\mathcal{M}_{x^\lambda}$ whose underlying $\mathcal{O}_X$-module is isomorphic to $\mathcal{O}_X$, with a generator denoted by "x^λ". The $\mathcal{T}_X$-action is determined by the formular $\partial_x \cdot x^\lambda = \lambda x^{-1} \cdot x^\lambda$. It is easy to see $\mathcal{M}_{x^\lambda} \simeq \mathcal{D}_X/\mathcal{D}_X \cdot (\partial_x - \lambda x^{-1})$, where $\mathcal{D}_X \cdot (\partial_x - \lambda x^{-1})$ is the left ideal of $\mathcal{D}_X$ generated by the section $\partial_x - \lambda x^{-1}$.

Exercise 7.6. This is Homework 5, Problem 3. In Example 7.5, prove $\mathcal{M}_{x^\lambda}$ is isomorphic to $\mathcal{O}_X$ as left $\mathcal{D}_X$-modules iff $\lambda \in \mathbb{Z}$.

8. $\mathcal{D}_X$ vs. $U(\mathfrak{g})$

Construction 8.1. *Let X be a smooth k-scheme equipped with an action by an algebraic group G. Consider the Lie algebra homomorphisms*

$$\mathfrak{g} \xrightarrow{a} \mathcal{T}(X) \to \mathcal{D}(X)$$

where $\mathcal{D}(X)$ is viewed as a Lie algebra via the forgetful functor $\mathsf{Alg} \to \mathsf{Lie}$. By the universal property of $U(\mathfrak{g})$, we obtain a homomorphism

$$U(\mathfrak{g}) \xrightarrow{a} \mathcal{D}(X).$$

By construction, this homomorphism is compatible with the PBW filtrations on both sides.

This induces a functor

$$\Gamma : \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \to U(\mathfrak{g})\text{-}\mathsf{mod},\ \mathcal{M} \mapsto \mathcal{M}(X).$$

By construction, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \arrow[r, "\simeq"] \arrow[d, "\Gamma"] & \widetilde{\mathcal{T}}_X\text{-}\mathsf{mod}_{\mathsf{qc}} \arrow[d, "\Gamma"] \\
U(\mathfrak{g})\text{-}\mathsf{mod} \arrow[r, "\simeq"] & \mathfrak{g}\text{-}\mathsf{mod}.
\end{tikzcd}$$

Remark 8.2. The localizaiton theory says for $X = G/B$, the homomorphism $U(\mathfrak{g}) \xrightarrow{a} \mathcal{D}(X)$ induces an isomorphism

$$a : U(\mathfrak{g})_{\chi_0} := U(\mathfrak{g}) \underset{Z(\mathfrak{g})}{\otimes} k_{\chi_0} \xrightarrow{\simeq} \mathcal{D}(X).$$

and the functor Γ induces an equivalence

$$\Gamma : \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \xrightarrow{\simeq} U(\mathfrak{g})_{\chi_0}\text{-}\mathsf{mod}.$$

References

[B] Bernstein, Joseph. Algebraic theory of D-modules, 1984, abailable at `https://gauss.math.yale.edu/~il282/Bernstein_D_mod.pdf`.

[BB] Beilinson, Alexander, and Joseph Bernstein. "A proof of Jantzen conjectures." ADVSOV (1993): 1-50.

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/catO.pdf`.

[H] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[HTT] Hotta, Ryoshi, and Toshiyuki Tanisaki. D-modules, perverse sheaves, and representation theory. Vol. 236. Springer Science & Business Media, 2007.