LECTURE 19

In this lecture, we define Cartesian and coCartesian fibrations between ∞-categories.

1. locally Cartesian arrows

1.1. Informally speaking, a Cartesian fibration is a functor $p : \mathcal{C} \to \mathcal{D}$ such that the fibers $\{\mathcal{C}_d\}_{d\in\mathcal{D}}$ depend contravariantly functorially in a *homotopy coherent* way. For any object $c' \in \mathcal{C}_{d'}$ and a morphism $g : d \to d'$ in the base $\mathcal{D}$, the corresponding functor $g^\dagger : \mathcal{C}_{d'} \to \mathcal{C}_d$ will send c' to an object $c \in \mathcal{C}_d$ equipped with a morphism $f : c \to c'$ in $\mathcal{C}$ lying over g. We can depict the above discussion as the following diagram.

$$\begin{tikzcd} c \arrow[r, dotted, "f"] & c' & \in \mathcal{C} \arrow[d, "p"] \\ d \arrow[r, "g"] & d' & \in \mathcal{D} \end{tikzcd}$$

The morphism f can be characterized by the following universal property.

Definition 1.2. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories. We say a morphism $f : X \to Y$ in $\mathcal{C}$ is **locally p-Cartesian**, or **Cartesian over $\mathcal{D}$**, if for every object X' in the* naive[1] *fiber $\mathcal{C}_{p(X)} := \mathcal{C} \times_{\mathcal{D}} \{p(X)\}$, the following commutative square in Spc is Cartesian*

$$\begin{tikzcd} \mathsf{Maps}_{\mathcal{C}_{p(X)}}(X', X) \arrow[r, "f\circ -"] \arrow[d] & \mathsf{Maps}_{\mathcal{C}}(X', Y) \arrow[d] \\ \{*\} \arrow[r, "p(f)"] & \mathsf{Maps}_{\mathcal{D}}(p(X'), p(Y)). \end{tikzcd} \tag{1.1}$$

*We say p is a **locally Cartesian fibration** if for any $Y \in \mathcal{C}$ and morphism $g : x \to p(Y)$ in $\mathcal{D}$, there exists a locally p-Cartesian morphism $f : X \to Y$ lying over g.*

*Dually, we say f is **locally p-coCartesian** if the corresponding morphism in $\mathcal{C}^{\mathsf{op}}$ is locally p-Cartesian. We say p is a **locally coCartesian fibration** if p^{op} is a locally Cartesian one.*

Remark 1.3. *Informally speaking, $f : X \to Y$ is locally p-Cartesian iff the following data are equivalent:*

- *morphisms $X' \to X$ in the fiber $\mathcal{C}_{p(X)}$;*
- *morphisms $X' \to Y$ in $\mathcal{C}$ lying over $p(f)$.*

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[1] This means the fiber product in the ordinary category Set_Δ. Note that $\mathcal{C}_{p(X)}$ is a quasi-category because p is an inner fibration.

Remark 1.4. *Strictly speaking, the commutative square* (1.1) *in* Spc *is well-defined up to homotopy. Nevertheless, being Cartesian is invariant under homotopy. Alternatively, we can say the corresponding square in* hSpc*, which is well-defined, is a* homotopy pullback*. Note that the right vertical map can be realized as a Kan fibration between Kan complexes (HTT.2.4.4.1):*

$$\mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(X', Y) \to \mathsf{Hom}^{\mathsf{R}}_{\mathcal{D}}(p(X'), p(Y)).$$

Exercise 1.5. *If f is an isomorphism, then f is locally p-Cartesian.*

Exercise 1.6. *If f is homotopic to f', then f is locally p-Cartesian iff f' is.*

Warning 1.7. *The naive fiber product $\mathcal{C}_{p(X)} \coloneqq \mathcal{C} \times_{\mathcal{D}} \{p(X)\}$ may not be a homotopy or ∞-categorical fiber product.*

Exercise 1.8. *Show that $p : \mathsf{N}_\bullet([1]) \to \mathsf{N}_\bullet(\mathsf{Isom})$ is an inner fibration whose naive fibers are all equivalent to Δ^0, while whose homotopy fibers are all equivalent to Δ^1.*

Exercise 1.9. *Consider the inner fibrations $p : \mathsf{N}_\bullet([1]) \to \mathsf{N}_\bullet(\mathsf{Isom})$ and $p' : \mathsf{N}_\bullet([1]) \to \mathsf{N}_\bullet([0])$. Show that every edge in the source is locally p-Cartesian, while only degenerate edges are locally p'-Cartesian.*

Warning 1.10. *The above exercise implies Definition 1.2 is* not *invariant under categorical equivalences. Note however that this would not happen if $p : \mathcal{C} \to \mathcal{D}$ is a* categorical fibration*, because its naive fibers coincide with the homotopy fibers.*

2. Cartesian arrows

2.1. The following exercise says *locally* p-Cartesian arrows may not be closed under compositions. Hence we need a stronger condition to make the fibers functorial.

Example 2.2. *Consider the following map between posets*

$$[1] \times [1] \to [2],\ (0,0) \mapsto 0,\ (0,1) \mapsto 0,\ (1,0) \mapsto 1,\ (1,1) \mapsto 2.$$

Show that $p : \mathsf{N}_\bullet([1] \times [1]) \to \mathsf{N}_\bullet([2])$ is a locally Cartesian fibration, but locally p-Cartesian arrows are not closed under compositions.

Proposition 2.3 (HTT.2.4.2.7)**.** *Let $p : \mathcal{C} \to \mathcal{D}$ be a locally Cartesian fibration between quasi-categories and $f : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:*

(1) *For any $X' \in \mathcal{C}$, the following commutative square*

$$\begin{tikzcd} \mathsf{Maps}_{\mathcal{C}}(X', X) \arrow[r, "f\circ -"] \arrow[d] & \mathsf{Maps}_{\mathcal{C}}(X', Y) \arrow[d] \\ \mathsf{Maps}_{\mathcal{D}}(p(X'), p(X)) \arrow[r, "p(f)"] & \mathsf{Maps}_{\mathcal{D}}(p(X'), p(Y)). \end{tikzcd} \tag{2.1}$$

is Cartesian.

(2) *For any morphism $g : X' \to X$, g is locally p-Cartesian iff $f \circ g$ is locally p-Cartesian.*

(3) *For any morphism $g : X' \to X$, if g is locally p-Cartesian, then $f \circ g$ is locally p-Cartesian.*

Remark 2.4. *Informally speaking, (1) means the following data are equivalent:*

- *morphisms $X' \to X$ in $\mathcal{C}$;*

- *morphisms $X' \to Y$ in $\mathcal{C}$ and a factorization of $p(X') \to p(Y)$ through $p(X)$.*

Definition 2.5. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories and $f : X \to Y$ be a morphism in $\mathcal{C}$. We say f is* ***p-Cartesian****, or* ***Cartesian over $\mathcal{D}$*** *if it satisfies Condition (1) in Proposition 2.3. We say p is a* ***Cartesian fibration*** *if for any $Y \in C$ and edge $g : x \to p(Y)$ in D, there exists a p-Cartesian morphism in C lying over g.*

Dually, we say f is ***p-coCartesian*** *if the corresponding arrow in $\mathcal{C}^{\mathsf{op}}$ is p^{op}-Cartesian. We say p is a* ***coCartesian fibration*** *if p^{op} is a Cartesian one.*

Remark 2.6. *One can show f is p-Cartesian iff $\mathcal{C}_{/f} \to \mathcal{C}_{/Y} \times_{\mathcal{D}_{/p(Y)}} \mathcal{D}_{/p(f)}$ is a trivial Kan fibration in Set_{Δ} (HTT.2.4.4.3). In fact, one can use the latter condition to define Cartesian fibrations between general simplicial sets.*

Exercise 2.7. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories and $f : X \to Y$ be a p-Cartesian morphism. For a morphism $g : X' \to X$, show that g is p-Cartesian iff $f \circ g$ is so.*

2.8. The following exercises say Cartesian arrows are invariant under homotopies and categorical equivalences.

Exercise 2.9. *Let f be a morphism in $\mathcal{C}$ such that $p(f)$ is an isomorphism. Show that f is p-Cartesian iff it is an isomorphism.*

Exercise 2.10. *If f is homotopic to f', then f is p-Cartesian iff f' is.*

Exercise 2.11. *Suppose we have a commutative square of quasi-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[d, "p"] & \mathcal{C}' \arrow[d, "p'"] \\ \mathcal{D} \arrow[r, "s"] & \mathcal{D}' \end{tikzcd}$$

such that r and s are equivalences. Then f in $\mathcal{C}$ is p-Cartesian iff $r(f)$ is p'-Cartesian.

Definition 2.12. *Let $p : \mathsf{C} \to \mathsf{D}$ be a functor between ∞-categories. We say a morphism f in C is p-Cartesian if for any/all quasi-categorical realization $p : \mathcal{C} \to \mathcal{D}$, the corresponding morphism is p-Cartesian.*

2.13. In fact, the notion of Cartesian fibrations is also invariant under categorical equivalences, as long as we restrict to **categorical fibrations**. To explain this, we need the following result, which follows from HTT.2.4.6.5.

Proposition 2.14. *A Cartesian fibration between quasi-categories is a categorical fibration.*

Exercise 2.15. *Suppose we have a commutative square of quasi-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[d, "p"] & \mathcal{C}' \arrow[d, "p'"] \\ \mathcal{D} \arrow[r, "s"] & \mathcal{D}' \end{tikzcd}$$

such that r and s are equivalences and p and p' are categorical fibrations. Show that p is a Cartesian fibration iff p' is so.

Definition 2.16. *Let $p : \mathsf{C} \to \mathsf{D}$ be a functor between ∞-categories. We say p is* ***essentially a Cartesian fibration*** *if it can be realized as a Cartesian fibration between quasi-categories.*

Exercise 2.17. *Let $p : \mathsf{C} \to \mathsf{D}$ be a functor between ∞-categories. Show that p is essentially a Cartesian fibration iff for any quasi-categorical realization of p as a categorical fibrations $\mathcal{C} \to \mathcal{D}$, the corresponding functor is a Cartesian fibration between quasi-categories.*

2.18. The following exercises explain the relations between Cartesian arrows and locally Cartesian arrows.

Exercise 2.19. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories and $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that f is locally p-Cartesian iff the corresponding morphism in $\mathcal{C} \times_{\mathcal{D}} \Delta^1$ is Cartesian over Δ^1, where $\Delta^1 \to \mathcal{D}$ is given by $p(f)$.*

Exercise 2.20. *Show that any p-Cartesian arrow is locally p-Cartesian.*

Exercise 2.21. *Show that p is a Cartesian fibration iff it is a locally Cartesian fibration such that locally Cartesian arrows are closed under compositions.*

3. Uniqueness of Cartesian arrows

Proposition 3.1. *Let $p : \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories. Then a morphism $f : X \to Y$ in $\mathcal{C}$ is locally p-Cartesian iff the corresponding object in*

$$\mathcal{C}_{/Y} \underset{\mathcal{D}_{/p(Y)}}{\times} \{p(f)\}$$

is final.

Sketch. First, one can show $\mathcal{C}_{/Y} \times_{\mathcal{D}_{/p(Y)}} \{p(f)\}$ is quasi-category (so that it makes sense to talk about final objects). Using Exercise 2.19, one can reduce to the case when $\mathcal{D} = \Delta^1$, $p(X) = 0$ and $p(Y) = 1$. One can identify

$$\mathcal{C}_{/f} \to \mathcal{C}_{/Y} \underset{\mathcal{D}_{/p(Y)}}{\times} \mathcal{D}_{/p(f)}$$

with

$$\left(\mathcal{C}_{/Y} \underset{\mathcal{D}_{/p(Y)}}{\times} \{p(f)\}\right)_{/f} \to \mathcal{C}_{/Y} \underset{\mathcal{D}_{/p(Y)}}{\times} \{p(f)\}.$$

Then the claim follows from the fact that $z \in \mathcal{E}$ is final iff $\mathcal{E}_{/z} \to \mathcal{E}$ is a trivial Kan fibration (Ker.02HF). □

3.2. In particular, for fixed $Y \in \mathcal{C}$ and $g : x \to p(Y)$, (locally) p-Cartesian liftings $X \to Y$ of g is essentially unique if exists.

4. Right fibrations

Definition 4.1. *We say a functor $p : \mathcal{C} \to \mathcal{D}$ is a* ***right fibration*** *between quasi-categories if it is a Cartesian fibration such that any morphism in $\mathcal{C}$ is p-Cartesian. We say p is a* ***left fibration*** *if p^{op} is a right fibration.*

Exercise 4.2. *Let $p : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration between quasi-categories. Show that p is a right fibration iff each fiber $\mathcal{C}_y$, $y \in \mathcal{D}$ is a Kan complex.*

Exercise 4.3. *Let $p : C \to D$ be a morphism between simplicial sets such that D is a quasi-category. Show that p is a right fibration iff it satisfies the following right lifting properties for all $0 < i \le n$:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & D. \end{tikzcd}$$

Dually, show that p is a left fibration iff it satisfies the above right lifting properties for all $0 \le i < n$.

Remark 4.4. *Note that the above property makes sense when D is a general simplicial set. In fact, one can use it to define right fibrations between simplicial sets.*

Exercise 4.5. *Show that any Kan fibration is a right fibration.*

5. Examples

Exercise 5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be quasi-categories. Show that $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ is both a Cartesian and coCartesian fibration, and an arrow is (co)Cartesian iff its image in $\mathcal{C}$ is an isomorphism.*

Example 5.2 (HTT.2.1.2.2, 4.2.1.6)**.** *Let $u : K \to \mathcal{C}$ be any diagram in a quasi-category $\mathcal{C}$. Then the forgetful functors $\mathcal{C}_{/u} \to \mathcal{C}$ and $\mathcal{C}^{/u}$ are right fibrations.*

Example 5.3 (Ker.02VW)**.** *For any simplicial set C, let $\mathsf{Arr}(C) \coloneqq \mathsf{Fun}([1], C)$ be the simplicial set of arrows in C. If C is a quasi-category, so is $\mathsf{Arr}(\mathsf{C})$. The projection*

$$\mathsf{ev}_0 : \mathsf{Arr}(\mathsf{C}) \to \mathsf{C}$$

is a Cartesian fibration while the projection

$$\mathsf{ev}_1 : \mathsf{Arr}(\mathsf{C}) \to \mathsf{C}$$

is a coCartesian fibration. A morphism in $\mathsf{Arr}(C)$ is ev_0-Cartesian iff ev_1 sends it to an isomorphism.

Example 5.4 (Ker.03JF)**.** *For any simplicial set C, let $\mathsf{TwArr}(C)$ be the simplicial set*

$$\mathsf{TwArr}(C)_n \coloneqq \mathsf{Hom}_{\mathsf{Set}_\Delta}(\mathsf{N}_\bullet([n]^{\mathsf{op}} \star [n]), C).$$

*If C is a quasi-category, so is $\mathsf{TwArr}(\mathsf{C})$, which is called the quasi-category of **twisted arrows** in C. The projection*

$$\mathsf{TwArr}(C) \to C^{\mathsf{op}} \times C$$

induced by $[n]^{\mathsf{op}} \to [n]^{\mathsf{op}} \star [n] \leftarrow [n]$ is a left fibration. It follows that

$$\mathsf{TwArr}(C) \to C^{\mathsf{op}},\ \mathsf{TwArr}(C) \to C$$

are coCartesian fibrations.

Exercise 5.5. *What are the coCartesian arrows for $\mathsf{TwArr}(C) \to C^{\mathsf{op}}$ and $\mathsf{TwArr}(C) \to C$?*

Exercise 5.6. *What is $\mathsf{TwArr}(C^{\mathsf{op}})$?*

Example 5.7 (Ker.01VG)**.** *Let $p : \mathcal{C} \to \mathcal{D}$ be a Cartesian fibration between quasi-categories and K be any simplicial set. Then*

$$p^K : \mathsf{Fun}(K, \mathcal{C}) \to \mathsf{Fun}(K, \mathcal{D})$$

is a Cartesian fibration and an arrow is p^K-Cartesian iff its value at any vertex $x \in K$ is p-Cartesian.

Exercise 5.8. *Let* Ring *be the ordinary category of rings and* LMod *be the ordinary category of pairs (A, M) where A is a ring and M is a left A-module. A morphism $(A, M) \to (B, N)$ in* LMod *consists of a ring homomorphism $A \to B$ and a linear map $M \to N$ such that the following diagram commutes*

$$\begin{tikzcd} A \otimes M \arrow[r] \arrow[d] & M \arrow[d] \\ B \otimes N \arrow[r] & N. \end{tikzcd}$$

Show that $\mathsf{LMod} \to \mathsf{Ring}$ *is both a Cartesian and coCartesian fibration between ordinary categories.*

Appendix A. locally Cartesian fibraitons and lax functors

Let S be an ∞-category. There is a canonical equivalence between the following $(\infty, 2)$-categories:

- The $(\infty, 2)$-category of lax functors $\mathsf{S}^{\mathsf{op}} \to \mathbf{Cat}_\infty$, where $\mathbf{Cat}_\infty$ is the $(\infty, 2)$-category of $(\infty, 1)$-categories.
- The $(\infty, 2)$-category of locally Cartesian fibrations $\mathsf{C} \to \mathsf{S}$, where morphisms are functors defined over S that preserve Cartesian arrows over S.

A.1. **Suggested readings.** [Lur09].

References

[Lur09] Jacob Lurie. $(\infty, 2)$-categories and the Goodwillie calculus I. *arXiv preprint arXiv:0905.0462*, 2009.