LECTURE 1

The main goal of this course is to study representations of semisimple Lie algebras via geometric methods. We restrict ourselves to the case when the base field k is algebraically closed and of characteristic 0, such as the field $\mathbb{C}$ of complex numbers.

1. Semisimple Lie Algebras

This is just a quick review of the definitions about finite-dimensional semisimple Lie algebras. See [Hum, Chapter 0] for the abc's and [Ser] for a thorough textbook.

Definition 1. A **Lie algebra** (over k) is a vector space $\mathfrak{g}$ equipped with a binary operation $[-,-]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the **Lie bracket**, such that:

- The Lie bracket is **bilinear**, i.e., factors as $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$.
- The Lie bracket is **alternating**: $[x,x]=0$.
- The **Jacobi identity** holds: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. A **Lie algebra homomorphism** between them is a k-linear map $f:\mathfrak{g}_1\to\mathfrak{g}_2$ commuting with Lie brackets, i.e., $f([x,y])=[f(x),f(y)]$.

This defines a category Lie_k of Lie algebras.

Example 2. Any vector space V is equipped with a trivial Lie bracket: $[x,y]=0$. Such Lie algebras are called **abelian Lie algebras**.

Example 3. Let A be an associative algebra. Then the underlying vector space has a natural Lie algebra structure with Lie bracket given by $[x,y]:=xy-yx$. This defines a functor $\mathsf{oblv}:\mathsf{Alg}_k\to\mathsf{Lie}_k$ from the category of associative algebras to that of Lie algebras.

Example 4. Let V be a vector space and $\mathfrak{gl}(V)$ be the vector space of endomorphisms of V. By Example 3, $\mathfrak{gl}(V)$ is naturally a Lie algebra with Lie bracket given by $[f,g]=f\circ g-g\circ f$. This is the **general linear Lie algebra** of V.

If V is finite-dimensional, let $\mathfrak{sl}(V)\subset\mathfrak{gl}(V)$ be the subspace of endomorphisms f such that the trace $\mathsf{tr}(f)=0$.

When $V=k^{\oplus n}$, we write $\mathfrak{gl}_n:=\mathfrak{gl}(V)$, $\mathfrak{sl}_n:=\mathfrak{sl}(V)$. Note that $\mathfrak{gl}_n$ (resp. $\mathfrak{sl}_n$) can be identified with the space of $n\times n$ matrices (resp. whose traces are zero).

Fact 5. *We have* $[\mathfrak{gl}(V),\mathfrak{gl}(V)]=\mathfrak{sl}(V)$.

Definition 6. Let $\mathfrak{g}$ be a Lie algebra. A **representation** of $\mathfrak{g}$, or **$\mathfrak{g}$-module**, is a vector space V equipped with a Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{gl}(V)$. In other words, there is a bilinear map $(-\cdot-):\mathfrak{g}\times V\to V$ such that $[x,y]\cdot v=x\cdot(y\cdot v)-y\cdot(x\cdot v)$.

Let V_1 and V_2 be representations of $\mathfrak{g}$. A $\mathfrak{g}$-linear map between them is a k-linear map $f:V_1\to V_2$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g}\times V_1 \arrow[r] \arrow[d, "\mathsf{Id}\times f"'] & V_1 \arrow[d, "f"] \\
\mathfrak{g}\times V_2 \arrow[r] & V_2.
\end{tikzcd}$$

Date: Feb 26, 2024.

This defines a category $\mathfrak{g}-\mathsf{mod}$ of representations of $\mathfrak{g}$.

Fact 7. *The category $\mathfrak{g}-\mathsf{mod}$ is an abelian category. The forgetful functor $\mathfrak{g}-\mathsf{mod} \to \mathsf{Vect}_k$ is exact.*

Example 8. Let $\mathfrak{g}$ be a Lie algebra. The map $\mathsf{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $x \mapsto \mathsf{ad}_x := [x, -]$ defines a $\mathfrak{g}$-module structure on $\mathfrak{g}$ itself. This is called the **adjoint representation**.

Definition 9. Let $\mathfrak{g}$ be a Lie algebra. An **ideal** $\mathfrak{a} \subset \mathfrak{g}$ is a sub-representation of the adjoint representation. In other words, we require $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.

Remark 10. Note that an ideal $\mathfrak{a}$ is also a Lie subalgebra.

Example 11. Let $\mathfrak{g}$ be a Lie algebra, then $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ is an ideal. We call it the **derived Lie algebra** of $\mathfrak{g}$.

Definition 12. Let $\mathfrak{g}$ be a Lie algebra. We say $\mathfrak{g}$ is **simple** if:

- It is not abelian;
- The adjoint representation is simple (a.k.a. irreducible), i.e., $\mathfrak{g}$ has no ideal other than 0 and itself.

Example 13. The Lie algebra $\mathfrak{gl}_n$ is not simple because $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$ is a proper ideal of it. The Lie algebra $\mathfrak{sl}_n$ is simple for $n \geq 2$.

Remark 14. Finite-dimensional simple Lie algebras (over k) are fully classified. A similar classification for infinite-dimensional simple Lie algebras seems to be hopeless.

Definition 15. Let $\mathfrak{g}_i$, $i \in I$ be Lie algebras indexed by a set I. The direct sum $\oplus \mathfrak{g}_i$ of the underlying vector spaces has a natural Lie bracket given by $[(x_i)_{i \in I}, (y_i)_{i \in I}] := ([x_i, y_i])_{i \in I}$. The obtained Lie algebra is called the **direct sum** of the Lie algebras $\mathfrak{g}_i$.

Warning 16. *The direct sum $\oplus \mathfrak{g}_i$ is* not *the coproduct in the category Lie_k. Instead, if I is a finite set, then it is the* product *of $\mathfrak{g}_i$ in this category.*

Remark 17. Representation theory for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ can be obtained from those for $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in a non-trivial mechanism[1].

Definition 18. Let $\mathfrak{g}$ be a Lie algebra. We say $\mathfrak{g}$ is **semisimple** if it is a direct sum of simple Lie algebras.

Remark 19. The zero Lie algebra 0 is semisimple but not simple.

The main goal of this course is to study representations of finite-dimensional semisimple Lie algebras.

Convension 20. *From now on, unless otherwise stated, Lie algebras are assumed to be finite-dimensional.*

Exercise 21. This is not a homework!

(1) Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra. Show $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.
  - The opposite statement is generally *not* true. Below is a counterexample. Let $\mathfrak{h}$ be a simple Lie algebra and V be a nontrivial simple $\mathfrak{h}$-module. Define a bracket on the vector space by the formula $\mathfrak{h} \oplus V$ by $[(x, u), (y, v)] := ([x, y], x \cdot v - y \cdot u)$.

(2) Show this bracket defines a Lie algebra structure on $\mathfrak{h} \oplus V$. We denote this Lie algebra by $\mathfrak{h} \ltimes V$.

(3) Show $[\mathfrak{h} \ltimes V, \mathfrak{h} \ltimes V] = \mathfrak{h} \ltimes V$ but $\mathfrak{h} \ltimes V$ is not semisimple.

[1] The abelian category $(\mathfrak{g}_1 \oplus \mathfrak{g}_2)-\mathsf{mod}$ is the *tensor product* of $\mathfrak{g}_1-\mathsf{mod}$ and $\mathfrak{g}_2-\mathsf{mod}$.

2. Root Space Decomposition

Convension 22. *From now on, unless otherwise stated,* $\mathfrak{g}$ *means a finite-dimensional semisimple Lie algebra.*

Definition 23. A **Cartan subalgebra** $\mathfrak{t}$ of $\mathfrak{g}$ is a maximal abelian subalgebra of it.

Warning 24. *Cartan subalgebras for general finite-dimensional Lie algebras are defined in a different way and they are not abelian in general. That definition is equivalent to the above one if* $\mathfrak{g}$ *is semisimple.*

Theorem 25. *Cartan subalgebras of* $\mathfrak{g}$ *have a same dimension, which is called the* ***(semisimple) rank*** *of* $\mathfrak{g}$. *In fact, Cartan subalgebras are all conjugate to each other: for Cartan subalgebras* $\mathfrak{t}_1$ *and* $\mathfrak{t}_2$, *there exists* $x \in \mathfrak{g}$ *such that* $\mathsf{ad}_x(\mathfrak{t}_1) = \mathfrak{t}_2$.

Example 26. The rank of $\mathfrak{sl}_n$ is $n-1$. One Cartan subalgebra of it is the subspace of diagonal matrices.

Notation 27. *From now on, we fix a Cartan subalgebra* $\mathfrak{t}$ *of* $\mathfrak{g}$. *Let* $\mathfrak{t}^* := \mathsf{Hom}(\mathfrak{t}, k)$ *be the dual vector space of* $\mathfrak{t}$. *For any* $\alpha \in \mathfrak{t}^*$, *let* $\mathfrak{g}_\alpha \subset \mathfrak{g}$ *be the* α*-eigenspace for the adjoint* $\mathfrak{t}$*-action on* $\mathfrak{g}$, *i.e.,*

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for any } h \in \mathfrak{t}\}.$$

Remark 28. Note that $\mathfrak{g}_0 = \mathfrak{t}$ (because $\mathfrak{t}$ is maximal) and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (because of the Jacobi identity).

Proposition 29 (Root Space Decomposition[2])**.** *Let* $\mathfrak{g}$ *be a finite-dimensional semisimple Lie algebra with a fixed Cartan subalgebra* $\mathfrak{t}$. *Then we have a direct sum decomposition*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi \subset \mathfrak{h}^* \setminus 0$ *is the finite set containing those nonzero* α *such that* $\mathfrak{g}_\alpha$ *is nonempty. Moreover, if* $\alpha \in \Phi$, *then* $-\alpha \in \Phi$ *and* $\mathfrak{g}_\alpha$ *is 1-dimensional.*

Proposition 30. *There exists a (non-unique) subset* $\Phi^+ \subset \Phi$ *such that:*

- *We have a disjoint decomposition* $\Phi = \Phi^+ \sqcup -\Phi^+$*;*
- *If* $\alpha, \beta \in \Phi^+$ *and* $\alpha + \beta \in \Phi$, *then* $\alpha + \beta \in \Phi^+$.

Notation 31. *From now on, we fix such a subset* Φ^+. *Write* $\Phi^- = -\Phi^+$.

Definition 32. (For above choices), elements in Φ are called **roots** of $\mathfrak{g}$. Elements in Φ^+ (resp. Φ^-) are called **positive roots** (resp. **negative roots**). For $\alpha \in \Phi^+$, we say α is a **(positive) simple root** if it cannot be written as the sum of two positive roots. Let $\Delta \subset \Phi^+$ be the subset of simple roots.

Proposition 33. *The subset* $\Delta \subset \mathfrak{t}^*$ *is a basis. In particular, any positive root can be uniquely written as a linear combination of simple roots with non-negative coefficients.*

Definition 34. Define

$$\mathfrak{b} := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha, \ \mathfrak{n} := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

which are Lie subalgebras of $\mathfrak{g}$. We call $\mathfrak{b}$ the **Borel subalgebra** of $\mathfrak{g}$ (that corresponds to the choice of Φ^+) and $\mathfrak{n}$ the **nilpotent radical** of $\mathfrak{b}$.

---

[2] Some authors, including Humphreys, prefer the name *Cartan decomposition*. But there is a completely different Cartan decomposition in the study of real Lie algerbas.

Remark 35. In general, a Borel subalgebra $\mathfrak{b}$ of any Lie algebra $\mathfrak{g}$ is defined to be a maximal *solvable* subalgebra of it. Here solvable means the sequence $D^1(\mathfrak{b}) := \mathfrak{b}$, $D^{n+1}(\mathfrak{b}) := [D^n(\mathfrak{b}), D^n(\mathfrak{b})]$ satisfies $D^n(\mathfrak{b}) = 0$ for $n >> 0$. It is known that all Borel subalgebras are conjugate to each other.

The subalgebra $\mathfrak{n} \subset \mathfrak{b}$ is called the nilpotent radical because it contains exactly nilpotent elements in $\mathfrak{b}$, i.e., those elements x such that $(\mathsf{ad}_x)^{\circ n} = 0$ for $n >> 0$.

Note that we have $\mathfrak{t} \simeq \mathfrak{b}/\mathfrak{n}$.

Exercise 36. This is not a homework! For $\mathfrak{g} = \mathfrak{sl}_n$ and its standard Cartan subalgebra (Example 26).

(1) Find an explicit description of Φ and $\mathfrak{g}_\alpha$.
(2) Show there is a unique choice of Φ^+ such that the corresponding $\mathfrak{b}$ is the subspace of upper triangulated matrices.
(3) For the choice of Φ^+ in (2), find all the simple roots and write each root as a linear combination of these simple roots.

3. Root system

Definition 37. Let E be a finite-dimensional Euclidean space and $\Phi \subset E$ be a finite subset such that $0 \notin \Phi$. We say (E, Φ) is a **root system** if the following is satisfied:

- The subset Φ is a basis of E;
- For any $\alpha \in \Phi$, $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$;
- For $\alpha, \beta \in \Phi$, the number $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer;
- The subset Φ is closed under reflection along any $\alpha \in \Phi$, i.e., for $\alpha, \beta \in \Phi$, the element $\beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ is contained in Φ.

Definition 38. Let (E, Φ) be a root system. The **dual root system** is defined to be $(E^*, \check{\Phi})$, where E^* is the dual Euclidean space of E and $\check{\Phi}$ consists of those $\check{\alpha}$ for $\alpha \in \Phi$ defined by $\check{\alpha}(-) = 2\frac{(-,\alpha)}{(\alpha,\alpha)}$.

Exercise 39. This is not a homework! Show the double-dual of a root system is itself.

Let us return to the notations in the last section. Let $E := \mathbb{R}\Phi$ be the $\mathbb{R}$-vector space spaned by Φ (such that we have $E \otimes_{\mathbb{R}} k \simeq \mathfrak{t}^*$). We are going to show (E, Φ) is a root system. For this purpose, we need to define an inner product on E.

Definition 40. Let $\mathfrak{g}$ be any finite-dimensional Lie algebra. The **Killing form** on $\mathfrak{g}$ is the bilinear form $\mathsf{Kil} : \mathfrak{g} \times \mathfrak{g} \to k$, $\mathsf{Kil}(x, y) := \mathsf{tr}(\mathsf{ad}(x) \circ \mathsf{ad}(y))$

Proposition 41. *The Killing form is symmetric and (*ad*-)invariant, i.e.,*

- *For* $x, y \in \mathfrak{g}$, $\mathsf{Kil}(x, y) = \mathsf{Kil}(y, x)$*;*
- *For* $x, y, z \in \mathfrak{g}$, $\mathsf{Kil}(\mathsf{ad}_z(x), y) + \mathsf{Kil}(x, \mathsf{ad}_z(y)) = 0$.

Proposition 42. *If* $\mathfrak{g}$ *is simple, then any symmetric invariant bilinear form on* $\mathfrak{g}$ *is of the form* $c\mathsf{Kil}$ *for* $c \in k$.

Warning 43. *The similar claim is false if* k *is not algebraically closed.*

Theorem 44 (Cartan–Killing Criterion)**.** *The Lie algebra* $\mathfrak{g}$ *is semisimple iff its Killing form is non-degenerate. Moreover, in this case, the restriction of* Kil *on* $\mathfrak{t}$ *is also non-degenerate.*

Construction 45. *Since* $\mathsf{Kil}|_{\mathfrak{h}}$ *is non-degenerate, it induces an isomorphism* $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ *sending* x *to the unique element* x^* *such that* $x^*(-) = \mathsf{Kil}(x, -)$. *Consider the inverse* $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ *of this isomorphism, which also corresponds to a non-degenerate bilinear form on* $\mathfrak{h}^*$.

Lemma 46. *The restriction of the above bilinear form on $E \subset \mathfrak{h}^*$ is an inner product.*

Convension 47. *From now on, we always view E as an Eucilidean space via the above inner product.*

Theorem 48. *The pair (E, Φ) defined above is a root system.*

Note that $\mathfrak{h} \simeq (\mathfrak{h}^*)^* \simeq (E \otimes_{\mathbb{R}} k)^* \simeq E^* \otimes_{\mathbb{R}} k$. Hence E^* and thereby $\check{\Phi}$ can be viewed as a subset of E^*.

Definition 49. For any root $\alpha \in \Phi$, define the correponding **coroot** to be $\check{\alpha} \in \check{\Phi} \subset \mathfrak{h}$.

Remark 50. There is a (unique if stated properly) semisimple Lie algebra corresponding to the dual root system $(E^*, \check{\Phi})$, known as the *Langlands dual* Lie algebra $\check{\mathfrak{g}}$ of $\mathfrak{g}$.

References

[Hum] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[Ser] Serre, Jean-Pierre. Complex semisimple Lie algebras. Springer Science & Business Media, 2000.