- Recall. Grothendieck Spectral Sequence

$A \xrightarrow{f} B \xrightarrow{g} C$ s.t. f, g left exact.

f sends injectives to g-acyclic objects

$\rightsquigarrow E_2^{pq} = (R^p g)(R^q f)(A) \Rightarrow R^{p+q}(gf)(A)$.

Note that $d_r^{p0} = 0$ (since $d_r^{pq} : E_r^{pq} \longrightarrow E_r^{p+r, q-r+1}$).

$\Rightarrow Z_{r+1}^{p0}(E_r^{p0}) = E_r^{p0} \quad \forall p, r.$

This induces an epimorphism $R^p g(fA) \twoheadrightarrow E_\infty^{p0}$.

Filtration $R^p(gf)(A) = F^0R^1 \supseteq F^1R^p \supseteq \cdots \supseteq F^pR^p \supseteq F^{p+1}R^p = 0$

with $E_\infty^{i,p-i} \simeq F^iR^p/F^{i+1}R^p \rightsquigarrow$ inclusion $E_\infty^{p0} \simeq F^pR^p \hookrightarrow R^p(gf)(A)$.

Composition $\rightsquigarrow$

If $R^q f(A) = 0$ for $q > 0$ (e.g. f is exact), then $E_2^{pq} = 0$ for $q \neq 0$

$\Rightarrow B_{r+1}^{p0}(E_r^{p0}) = 0 \quad \forall p, r \quad \Rightarrow R^p g(fA) \xrightarrow{\sim} E_\infty^{p0}$.

For $0 \leq i < p$, $F^iR^p/F^{i+1}R^p \simeq E_\infty^{i,p-i} = 0 \Rightarrow E_\infty^{p0} \xrightarrow{\sim} R^p(gf)(A)$.

$\Rightarrow$ The morphism $R^p g(fA) \longrightarrow R^p(gf)(A)$ is an isomorphism.

- Ex. (Leray Spectral Sequence)

  $\pi : X'_{E'} \longrightarrow X_E \qquad F \in Sh(X'_{E'})$

  $\rightsquigarrow H^i(X, \pi_* F) \longrightarrow H^i(X', F)$.

  If π_* is exact, the above surjection is an isomorphism.

- Prop. (Finer Topology vs Coarser Topology)

  C/X a subcategory of $C'/X \qquad f : X \xrightarrow{id} X$

  $(C'/X)_E$ is a site. $E(C) := \{\{U_i \longrightarrow U\}_{i \in I} \in E \mid U \in (C/X)\}$.

  Assume that $(C/X)_{E(C)}$ is also a site.

That is, for any $U \in C/X$, $\{U_i \longrightarrow U\}_{i\in I} \in E(C)$

the composition $U_i \longrightarrow U \longrightarrow X$ should

also be in C/X. (C/X can't be arbitrary)

(e.g. big site v.s. small site)

$\Rightarrow$ (1) f_* is exact. $F \longrightarrow f_* f^* F$ is an isom of sheaves.

(2) $f^* : Sh((C/X)_E) \longrightarrow Sh((C'/X)_E)$

is fully faithful.

(3) $H^i_C(X, f_* F') \xrightarrow{\sim} H^i_{C'}(X, F')$

$H^i_C(X, F) \xrightarrow{\sim} H^i_{C'}(X, f^* F)$

$\forall\ F' \in Sh((C'/X)_E)$, $F \in Sh((C/X)_E)$.

• Prop. (More Coverings vs Fewer Coverings)

$C_2/X \subseteq C_1/X$ subcategory $E_2 \subseteq E_1$

$f: (C_1/X)_{E_1} \xrightarrow{id} (C_2/X)_{E_2}$ s.t. $\forall U \in C_2/X$ and covering

$\{U_i \longrightarrow U\}_{i\in I} \subseteq E_1$, $\exists$ covering $\{V_j \longrightarrow U\}_{j\in J} \subseteq E_2$

s.t.
$$\begin{tikzcd} V_j \arrow[r] \arrow[dr] & U_{i_j} \arrow[d] \\ & U \end{tikzcd}$$
"refinement"

$\Rightarrow$ f_* is exact $\Rightarrow$ $H^i(X_{E_2}, f_* F) \xrightarrow{\sim} H^i(X_{E_1}, F)$ $\forall F \in Sh(X_{E_1})$.

• Pf. It suffices to show f_* preserves surjection.

Take any $F \overset{\varphi}{\twoheadrightarrow} F''$ in $Sh(C_1/X)_{E_1}$, $U \in C_2/X$, $x \in \Gamma(U, F'')$.

$\Rightarrow \exists$ covering $\{U_i \longrightarrow U\}_{i\in I} \in E_1$, $x_i \in \Gamma(U_i, F)$ s.t.

$\varphi(U_i)(x_i) = x|_{U_i}$.

Choose a refinement $\{V_j\}_{j\in J}\in E_2 \longrightarrow \{U_i\}_{i\in I}$

$\Rightarrow \forall j, \exists i$ s.t. $\pi_*\varphi(V_j)(x_i|_{V_j}) = \varphi(U_i)(x_i)|_{V_j} = x|_{V_j}$

$\Rightarrow \pi_*\varphi$ is surj. in $Sh(C_2/X)_{E_2}$.

- Def. (Noetherian Site) A site is Noetherian iff $\forall$ covering $\{U_i \longrightarrow U\}_{i\in I}$, $\exists$ finite $J\subseteq I$ s.t. $\{U_j \longrightarrow U\}_{j\in J}$ is a subcovering.
- Prop. X_E is noetherian. Define $P(X_{E(f)})$ and $S(X_{E(f)})$ to be the category of presheaves and sheaves allowing only finite coverings.

  Then we have category equivalence

  $$P(X_{E(f)}) \simeq P(X_E) \quad S(X_{E(f)}) \simeq S(X_E)$$

- Prop. (Flat vs Zariski)

  $f: X_{fl} \xrightarrow{id.} X_{Zar}$ $\quad F\in QCoh(X,\mathcal{O}_X)$

  $\Rightarrow H^i(X_{Zar}, F) \xrightarrow{\sim} H^i(X_{fl}, W(F))$

  Pf. Note that $F = f_*W(F)$

  spectral seq $H^i(X_{Zar}, R^j\pi_*W(F)) \Rightarrow H^{i+j}(X_{fl}, W(F))$.

  It suffices to show $R^jf_*W(F)=0$ $\forall j>0$.

  $R^jf_*W(F)$ is the sheafification of $U \longmapsto H^j(U_{fl}, W(F))$

  $\Rightarrow$ It suffices to show $\forall U\simeq \operatorname{Spec}A\subseteq X$ open affine, $W(F)$ is flabby on U

  $\Leftrightarrow \forall U\simeq \operatorname{Spec}A\subseteq X$ open affine and finite affine covering $\mathcal{V}=\{V_i \longrightarrow U\}_{i\in I}$, the Čech cohomology $H^j(\mathcal{V}/U, W(F))=0$ $\forall j>0$.

We may replace $\mathcal{V}$ with $\coprod V_i \simeq \operatorname{Spec} B$, write $M := \Gamma(U, F)$

$\Rightarrow$ the Čech complex is

$$0 \longrightarrow M \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \longrightarrow \cdots$$

$$d^n : m \otimes b_1 \otimes \cdots \otimes b_n \longmapsto \sum_{i=0}^{n} (-1)^i m \otimes b_1 \otimes \cdots \otimes b_i \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_n$$

$A \longrightarrow B$ faithfully flat $\Rightarrow$ above seq is exact.

- Thm. (Flat vs Étale, group scheme)

$G \longrightarrow X$ smooth, quasi-projective, commutative group scheme

(by abuse of notation, G can be referred to as the sheaf it represents, i.e. $U \longmapsto \operatorname{Hom}_X(U, G)$)

$$\Rightarrow H^i(X_{\acute{e}t}, G) \xrightarrow{\sim} H^i(X_{fl}, G)$$

- Lemma. $X = \operatorname{Spec} A$ Henselian. $Y \longrightarrow X$ flat LFT.

$\Rightarrow \exists X' \longrightarrow X$ finite flat factors through Y:

$$\begin{tikzcd} & Y \arrow[dr] & \\ X' \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

- Lemma. A Henselian local ring. $X = \operatorname{Spec} A$.

X_{fl} big LFT flat site

X_E small E-site, $E = \{$finite flat morphisms$\}$

$f : X_{fl} \xrightarrow{\text{id}} X_E$

$\Rightarrow f_*$ is exact $\Rightarrow H^i(X_E, f_* F) \xrightarrow{\sim} H^i(X_{fl}, F)$ $\forall F \in \mathrm{Sh}(X_{E})$.

Pf. Recall. A is Henselian, B is a finite A-algebra

$\Rightarrow B = \prod B_i$, where B_i's are Henselian A-algebra.

Use "More Coverings vs Fewer Coverings".

That is, to prove $\forall\, \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ finite flat

and $\{U_i \longrightarrow \mathrm{Spec}\, B\}_{i\in I}$ LFT flat site

$\Rightarrow \exists$ refinement

$$\begin{tikzcd} \{\mathrm{Spec}\, R_j \arrow[rr] \arrow[dr, "\text{finite flat}"'] & & U_{i_j}\}_{j\in J} \arrow[dl, "\text{LFT flat}"] \\ & \mathrm{Spec}\, B & \end{tikzcd}$$

$\mathrm{Spec}\, B = \coprod \mathrm{Spec}\, B_i$, B_i Henselian $\Rightarrow$ WLOG, WMA B Henselian.

(Previous lemma) $\Rightarrow \square$

- **Prop.** $\tilde{X} \longrightarrow X$ finite flat.

  Base change functor $\times_X \tilde{X} : Sch/X \longrightarrow Sch/\tilde{X}$

  has a right adjoint, denoted as $R_{\tilde{X}/X} : Sch/\tilde{X} \longrightarrow Sch/X$

  (Weil restriction).

  Moreover, if $Z \longrightarrow \tilde{X}$ is of finite type,

  then $R_{\tilde{X}/X}(Z) \longrightarrow X$ is also of finite type.

- **Proof of (Flat vs Étale, group scheme)**

Goal: Prove $f : X_E \xrightarrow{id} X_{et}$ satisfies $R^i f(G) = 0\ \forall i>0$

(At stalks of X_{et}) It suffices to show for $X = \mathrm{Spec}\, A$,

where A is strictly Henselian, $H^i(X_E, G) = 0$ for $i>0$. (E defined before)

[Step 1] X affine scheme X_0 closed subscheme

Define functor $N : Sch/X \longrightarrow Ab$

$$Y \longmapsto \ker(G(Y) \longrightarrow G(Y_0))$$

where $Y_0 = X_0 \times_X Y$.

If X_0 is defined by an ideal $\mathcal{I}$ with $\mathcal{I}^2 = 0$, then

$N = W(M)$ for some coherent $\mathcal{O}_X$-mod M.

(Recall: $W(M)(Y) = \Gamma(Y, \pi^* M)$ for $(\pi : Y \longrightarrow X) \in Sch/X$).

Pf. Let $\omega_G = e^* \Omega^1_{G/X}$, where $e : X \longrightarrow G$ identity section.

Take $M = \underline{\mathcal{H}om}_{\mathcal{O}_X}(\omega_G, \mathcal{L})$.

[Step 2] X scheme. $X' \longrightarrow X$ faithfully flat & finite.

For every Y/X define $\underline{C}^\bullet(G)(Y)$ to be the Čech complex $C^\bullet(Y'/Y, G)$, where $Y' = Y \times_X X'$.

($\forall i$, $\underline{C}^i(G)$ is a functor $Sch/X \longrightarrow Ab$ w.r.t. Y)

$\underline{Z}^i(G)(Y) := \ker(d^i : \underline{C}^i(G)(Y) \longrightarrow \underline{C}^{i+1}(G)(Y))$

$\Rightarrow$ $\underline{C}^{i-1}(G) \longrightarrow \underline{Z}^i(G)$ is represented by a smooth morphism of X-group schemes.

Pf. $\underline{C}^i(G)(Y) = G(\underbrace{Y' \times_Y \cdots \times_Y Y'}_{i \text{ copies}}) = G(Y \times_X \underbrace{X' \times_X \cdots \times_X X'}_{i \text{ copies}})$

$:= Hom_X(Y \times_X X' \times_X \cdots \times_X X', G)$

$\simeq Hom_{X' \times \cdots \times X'}(Y \times_X X' \times_X \cdots \times_X X', G \times X' \times_X \cdots \times_X X')$

$\simeq Hom_X(Y, R_{X' \times \cdots \times X'/X}(G \times X' \times \cdots \times X'))$ (Weil restriction)

$\Rightarrow$ $\underline{C}^i(G)$ is representable. (Moreover, $\underline{C}^i(G) \longrightarrow \underline{C}^{i+1}(G)$ is f.p.)

$\Rightarrow$ $\underline{Z}^i(G) := \ker(\underline{C}^i(G) \longrightarrow \underline{C}^{i+1}(G))$ is representable.

Take any affine X-scheme T and closed subscheme T_0 defined by a closed subscheme.

Goal:
$$\begin{tikzcd} T_0 \arrow[r] \arrow[d] & T \arrow[d] \arrow[dl, dashed, "\exists"'] \\ \underline{C}^{i-1} \arrow[r] & \underline{Z}^i \end{tikzcd}$$

(surjectivity follow from smoothness of $G \to X$)

$$\begin{tikzcd}
0 \arrow[r] & C^{i-1}(T'/T,N) \arrow[r, "\varphi_c"] \arrow[d, "d_{T'}"] & \underline{C}^{i-1}(G)(T) \arrow[r, "\psi_c"] \arrow[d, "d_T"] & \underline{C}^{i-1}(G)(T_0) \arrow[r] \arrow[d, "d_{T_0}"] & 0 \\
0 \arrow[r] & Z^i(T'/T,N) \arrow[r, "\varphi_z"] & \underline{Z}^i(G)(T) \arrow[r, "\psi_z"] & \underline{Z}^i(G)(T_0) &
\end{tikzcd}$$

where N is the functor $Sch/T \longrightarrow Ab$

$$Y \longmapsto \ker(G(Y) \longrightarrow G(Y_0)).$$

Goal: Prove $\forall \alpha \in \underline{C}^{i-1}(G)(T_0), \beta \in \underline{Z}^i(G)(T)$ s.t. $d_{T_0}(\alpha) = \psi_z(\beta)$,

$\exists \gamma \in \underline{C}^{i-1}(G)(T)$ s.t. $\psi_c(\gamma) = \alpha, d_T(\gamma) = \beta$.

$$\Uparrow$$

$\forall \bar{\alpha} \in \underline{C}^{i-1}(G)(T), \beta \in \underline{Z}^i(G)(T)$ s.t. $d_{T_0}(\psi_c(\bar{\alpha})) = \psi_z(\beta)$,

$\exists \Delta \in \ker \psi_c = \operatorname{im} \varphi_c$ s.t. $d_T(\bar{\alpha} + \Delta) = \beta$

$$\Uparrow \qquad \boxed{\psi_z(\beta - d_T(\bar{\alpha})) = \psi_z(\beta) - d_{T_0}(\psi_c(\bar{\alpha})) = 0}$$

$\forall \bar{\alpha} \in \underline{C}^{i-1}(G)(T), \beta \in \underline{Z}^i(G)(T)$ s.t. $d_{T_0}(\psi_c(\bar{\alpha})) = \psi_z(\beta)$,

$\exists \delta \in C^{i-1}(T'/T,N)$ s.t. $\underset{\varphi_z(d_{T'}(\delta))}{\underset{\parallel}{d_T(\varphi_c(\delta))}} = \beta - d_T(\bar{\alpha})$ $(\in \ker \psi_z = \operatorname{im} \varphi_z)$

$$\Uparrow$$

$d_{T'} : C^{i-1}(T'/T,N) \longrightarrow Z^i(T'/T,N)$ is surjective.

(i.e. $C^\bullet(T'/T,N)$ is exact, i.e. $\check{H}^i(T'/T, W(M)) = 0, M \in Coh(\mathcal{O}_X)$).

Use (Flat vs Zariski).

[Step 3] $X = \operatorname{Spec} A$ Henselian. $X' \longrightarrow X$ faithfully flat.

$X_0 =$ closed point of X $\quad X_0' = X' \times_X X_0$

$\Rightarrow \check{H}^i(X'/X, G) \xrightarrow{\sim} \check{H}^i(X_0'/X, G) \quad \forall i > 0.$

Pf. (Recall) $X \longrightarrow \mathrm{Spec}\,A$ smooth, A Henselian res.field $= k$

$\Rightarrow X(A) \longrightarrow X(k)$ is surjective.

$\rightsquigarrow C^i(X'/X,G) \longrightarrow C^i(X'_0/X,G)$ is surj $\forall i$.

(Note that $X' \times \cdots \times X'$ is finite over $X = \mathrm{Spec}\,A$

$\Rightarrow X' \times \cdots \times X'$ is a disjoint union $\coprod_{\text{finite}} \mathrm{Spec}\,B_i$,

where B_i Henselian)

Goal: $\ker(C^\bullet(X'/X,G) \longrightarrow C^\bullet(X'_0/X,G))$ is exact. $(*)$

(if this holds, then

$$\begin{tikzcd}
\ker \arrow[r, "\mathrm{im}=Z"] \arrow[d, hook] & \ker \arrow[r, "\ker=Z"] \arrow[d, hook] & \ker \arrow[d, hook] \\
C^{i-1}(X'/X,G) \arrow[r, "d^{i-1}"] \arrow[d, two heads] & C^i(X'/X,G) \arrow[r, "d^i"] \arrow[d, two heads] & C^{i+1}(X'/X,G) \arrow[d, two heads] \\
C^{i-1}(X'_0/X_0,G) \arrow[r, "d_0^{i-1}"] & C^i(X'_0/X_0,G) \arrow[r, "d_0^i"] & C^{i+1}(X'_0/X_0,G)
\end{tikzcd}$$

$\ker d^i / \mathrm{im}\, d^{i-1} \simeq (\ker d^i / Z) / (\mathrm{im}\, d^{i-1}/Z) \simeq \ker d_0^i / \mathrm{im}\, d_0^{i-1}$)

Take any $z \in Z^i(X'/X,G) \subseteq C^i(X'/X,G)$

s.t. $z_0 = 0$ in $C^i(X'_0/X_0,G)$

$(*) \Leftrightarrow \exists\, c \in C^{i-1}(X'/X,G)$ s.t. $c_0 = 0$ in $C^i(X'_0/X_0,G)$

and $d^{i-1}(c) = z$.

[Step 2] $\Rightarrow \underline{C}^{i-1}(G) \longrightarrow \underline{Z}^i(G)$ represented by smooth morphism

$\Rightarrow (d^{i-1})^{-1}(z) \subseteq \underline{C}^{i-1}(G)$ smooth subscheme.

$\left.\begin{array}{l} (d^{i-1})^{-1}(z) \text{ has a section over } X_0 \text{ (zero section)} \\ X = \mathrm{Spec}\,A \text{ Henselian} \end{array}\right\} \Rightarrow$ lifts to a section over X.

[Step 4] The problem is reduced to $X = \operatorname{Spec} k$ separably closed.

Goal: $\check{H}^i(X'/X, G) = 0 \quad i > 0$.

$\underline{C}^{i-1} \longrightarrow \underline{Z}^i$ smooth $\Rightarrow$ it suffices to show

$\underline{C}^{i-1}(K) \longrightarrow \underline{Z}^i(K)$ surj. for K algebraically closed

$\Rightarrow$ The problem is reduced to $X = \operatorname{Spec} K$ algebraically closed.

Now
$$\begin{tikzcd} X' \arrow[r] & X = \operatorname{Spec} K \arrow[l, dashed, bend right] \end{tikzcd}$$
has a section

$$\Downarrow$$

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & C^{i-1}(X/X, G) \arrow[r, "id"] \arrow[d] & C^{i}(X/X, G) \arrow[r, "0"] \arrow[d] & C^{i+1}(X/X, G) \arrow[r, "id"] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{i-1}(X'/X, G) \arrow[r] \arrow[u, dashed, bend left] & C^{i}(X'/X, G) \arrow[r] \arrow[u, dashed, bend left] & C^{i+1}(X'/X, G) \arrow[r] \arrow[u, dashed, bend left] & \cdots
\end{tikzcd}$$

homotopy equivalence.

- Thm. (Analytic vs étale, over $\mathbb{C}$).

  $X/\mathbb{C}$ smooth. M <u>finite</u> abelian group important!

  $\Rightarrow H^i(X(\mathbb{C}), M) \simeq H^i(X_{et}, M)$.

- Ex. X smooth curve over $\mathbb{C}$ of genus g.

  $H^1(X(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z}^{2g}$.

  $H^1(X_{\text{ét}}, \mathbb{Z}) = \operatorname{Hom}_{\text{Conts}}(\pi_1^{\text{ét}}(X), \mathbb{Z}) = 0$

  $H^1(X_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \simeq H^1(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$.

- Idea. Analytic topology and étale topology are "not comparable".

  To solve this, introduce a site X_{cx}:

  ① base $= X^{an}$

  ② Objects & coverings are local isomorphisms.

Then X_{cx} "is finer than both analytic and étale topology".

$$\begin{tikzcd} & X_{cx} \arrow[dl, "①\simeq"'] \arrow[dr, "②\simeq"] & \\ X^{an} & & X_{ét} \end{tikzcd}$$

①$\simeq$ is obtained by Prop (More Coverings vs Fewer Coverings)

(* for locally contractible topological space X,

we have $H^i(X, \underline{M})$ (sheaf cohomology) $\simeq H^i(X, M)$ (singular cohomology))

Now it suffices to prove ②$\simeq$:

$\pi: X_{cx} \xrightarrow{id} X_{ét}$ induces $H^i(X_{ét}, M) \simeq H^i(X_{cx}, M)$.

- Def. (Elementary Fibration) A morphism of schemes $f: X \longrightarrow S$ is an elementary fibration iff f fits into the following diagram:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Y \arrow[l, "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

where:

① j is an open immersion s.t. $\forall s \in S$, X_s is dense in $\overline{X}_s$, and $Y = \overline{X} - X$.

② $\overline{f}$ smooth & projective, with geometrically irreducible fibers of $\dim = 1$

③ g is finite étale, and each fiber of g is nonempty.

- Lemma. k algebraically closed. X/k smooth.

$\Rightarrow \forall x \in X$, $\exists$ nbhd $U \ni x$ and a sequence of elementary fibrations

$$U = U_n \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow U_0 = \operatorname{Spec} k.$$

• Lemma. (Riemann existence theorem)

$X/\mathbb{C}$ LFT. $\qquad X^{an}$ analytization.

There's a category equivalence

$$\{\text{finite étale coverings } Y/X\} \longleftrightarrow \{\text{similar coverings of } X_{an}\}$$
$$Y \longmapsto Y^{an}$$

• Pf of $H^i(X_{cx}, M) \simeq H^i(X_{et}, M)$

Goal: $f: X_{cx} \xrightarrow{id} X_{ét} \Rightarrow R^j f_* M = 0 \ \forall j > 0.$

$$\Uparrow$$

• Lemma. $F \in Sh(X_{cx})$ locally constant torsion sheaf with finite fibers. $\quad \gamma \in H^i(X_{cx}, F), i > 0.$

$\Rightarrow \forall x \in X(\mathbb{C})$, $\exists$ étale $U \longrightarrow X$ s.t.

① image of U contains x

② $\gamma|_{U_{cx}} = 0.$

• Pf. Induction on $n = \dim X$.

The statement is local for $X \Rightarrow$

WMA F is the constant sheaf associated with finite abelian group M and $\exists$ elementary fibration

$f: X \longrightarrow S \quad (\dim S = n-1)$

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "\overline{f}"] & Y \arrow[l, "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

finiteness of M and $\operatorname{codim}_X Y=1$ are needed

X is dense in $\overline{X} \Rightarrow j_*F = \underline{M}$

$(\mathrm{VI}.5.1) \Rightarrow \boxed{\begin{array}{l} R^1 j_*F = i_*(\text{constant sheaf on } Y) \\ R^{>1} j_*F = 0 \end{array}}$

$\overline{f}$ proper & smooth $\Rightarrow \overline{f}$ preserves locally constant torsion sheaves.

Note we have spectral sequence $(R^p\overline{f})(R^q j)F \Rightarrow (R^{p+q}f)F$

$\Rightarrow R^1 f_*F$ is a locally constant torsion sheaf with finite fibers,

$R^{>1} f_*F = 0$.

Leray spectral sequence $H^p(S_{cx}, R^q f_*F) \Rightarrow H^{p+q}(X_{cx}, F)$

reduces to LES:

$$\cdots \longrightarrow H^i(S_{cx}, f_*F) \xrightarrow{\alpha} H^i(X_{cx}, F) \xrightarrow{\beta} H^{i-1}(S_{cx}, R^1 f_*F) \longrightarrow \cdots$$

By induction $\Rightarrow \forall \gamma \in H^{i-1}(S_{cx}, R^1 f_*F)$, $\gamma' \in H^i(S_{cx}, f_*F)$,

$\exists\, U' \longrightarrow S$ étale nbhd of $f(x)$ s.t. $\gamma|_{U'_{cx}} = \gamma'|_{U'_{cx}} = 0$.

① For $i>1$: Take any $\gamma'' \in H^i(X_{cx}, F)$ and let $\gamma' = \beta(\gamma'')$

$\Rightarrow \exists\, U' \longrightarrow S$ étale nbhd of $f(x)$ s.t. $\gamma'|_{U'_{cx}} = 0$

$\Rightarrow \beta_{U'}(\gamma''|_{U'_{cx}}) = 0 \Rightarrow \gamma''|_{U'_{cx}} \in \operatorname{im} \alpha_{U'}$. Write $\gamma''|_{U'_{cx}} = \alpha_U(\tilde{\gamma})$, $\tilde{\gamma} \in H^i(U'_{cx}, f_*F)$

$\Rightarrow \exists\, U'' \longrightarrow U'$ s.t. composition $U'' \longrightarrow S$ is an étale

nbhd of $f(x)$ and $\tilde{\gamma}|_{U''_{cx}} = 0 \Rightarrow \gamma''|_{U''_{cx}} = \alpha_{U''}(\tilde{\gamma}|_{U''_{cx}}) = 0$

(Take $U = U'' \times_S X$)

② For $i=1$: Use Riemann existence theorem

(Note: $H^1(X, M)$ classifies M-torsors over X)