NOTES FOR ALGEBRAIC GEOMETRY 1

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Contents

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0. Introduction: why schemes?

0.1. **Algebraic sets.** Before scheme theory, algebraic geometry focused on *algebraic sets*.

Definition 0.1.1. Let k be an algebraically closed field.

- The **Zariski topology** on the affine space $\mathbb{A}_k^n$ is the topology with a base consisting of all the subsets that are equal to the non-vanishing locus $U(f)$ of some polynomial $f \in k[x_1, \cdots, x_n]$.
- An **embedded affine algebraic set**[1] in $\mathbb{A}_k^n$ is a closed subspace for the Zariski topology.
- An **embedded quasi-affine algebraic set** is a Zariski open subset of an embedded affine algebraic set.

Example 0.1.2. Any finite subset of $\mathbb{A}_k^n$ is an embedded affine algebraic set.

Example 0.1.3. $\mathbb{Z}$ is not an embedded affine algebraic set in $\mathbb{A}_{\mathbb{C}}^1$.

Similarly one can define *embedded (quasi)-projective algebraic set* using *homogeneous* polynomials and the projective space $\mathbb{P}_k^n$.

There are tons of shortcomings in the above definition. An obvious one is that the notions of *embedded* algebraic sets are not *intrinsic*.

Example 0.1.4. The embedded affine algebraic sets $\mathbb{A}_k^1 \subseteq \mathbb{A}_k^1$ and $\mathbb{A}_k^1 \subseteq \mathbb{A}_k^2$ should be viewed as the same algebraic sets.

Notation 0.1.5. To remedy this, we need some notations.

- For an ideal $I \subseteq k[x_1, \cdots, x_n]$, let $Z(I) \subseteq \mathbb{A}_k^n$ be the locus of common zeros of polynomials in I.
- For a Zariski closed subset $X \subseteq \mathbb{A}_k^n$, let $I(X) \subseteq k[x_1, \cdots, x_n]$ be the ideal of all polynomials vanishing on X.

Recall an ideal I is called *radical* if $I = \sqrt{I}$.

Theorem 0.1.6 (Hilbert Nullstellensatz)**.** *We have a bijection:*

$$\begin{array}{rcl} \left\{\text{radical ideals of } k[x_1, \cdots, x_n]\right\} & \longleftrightarrow & \left\{\text{Zariski closed subsets of } \mathbb{A}_k^n\right\} \\ I & \longrightarrow & Z(I) \\ I(X) & \longleftarrow & X. \end{array}$$

Part of the theorem says the set of points of $\mathbb{A}_k^n$ is in bijection with the set of maximal ideals of $k[x_1, \cdots, x_n]$. As a corollary, $Z(I)$ is in bijection with the set of maximal ideals containing I. The latter can be further identified with maximal ideals of $R := k[x_1, \cdots, x_n]/I$.

Note that I is radical iff R is *reduced*, i.e., contains no nilpotent elements. This justifies the following definition.

Definition 0.1.7. An **affine algebraic k-set** is a *maximal spectrum* $\mathsf{Spm}\, R$ (= sets of maximal ideals) of a *finitely generated* (commutative unital) *reduced* k-algebra R. We equip it with the **Zariski topology** with a base of open subsets given by

$$U(f) := \left\{\mathfrak{m} \in \mathsf{Spm}\, R \mid f \notin \mathfrak{m}\right\}, \ f \in R.$$

[1]Some people use the word *variety*, while some people reserve it for *irreducible* algebraic sets.

Example 0.1.8. $\mathsf{Spm}\, k[x] \simeq \mathbb{A}^1_k$.

We have the following *duality* between algebra and geometry.

Algbera	Geometry
finitely generated reduced k-algebra R	affine algebraic k-set X
maximal ideals $\mathfrak{m} \subseteq R$	points $x \in X$
elements $f \in R$	functions $\phi : X \to \mathbb{A}^1_k$
radical ideals $I \subseteq R$	Zariski closed subsets $Z \subseteq X$

Here an element $f \in R$ corresponds to the function

$$\phi : \mathsf{Spm}\, R \to k,\ \mathfrak{m} \mapsto \underline{f}$$

sending a maximal ideal $\mathfrak{m}$ to the image $\underline{f}$ of f in the *residue field* of $\mathfrak{m}$, which is canonically identified with the underlying set of $\mathbb{A}^1_k$ via the composition $k \to R \to R/\mathfrak{m}$.

The word *duality* means the correspondence $R \leftrightarrow X$ is *contravariant.* Indeed, given a homomorphism $f : R' \to R$, we obtain a *continuous* map

$$\mathsf{Spm}\, R \to \mathsf{Spm}\, R',\ \mathfrak{m} \mapsto f^{-1}(\mathfrak{m}).$$

Note however that not all continuous maps $\mathsf{Spm}\, R \to \mathsf{Spm}\, R'$ are obtained in this way, nor is R determined by the topological space $\mathsf{Spm}\, R$.

Exercise 0.1.9. Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology. Find those bijections coming from a homomorphism $k[x] \to k[x]$.

This motivates the following definition.

Definition 0.1.10. A **morphism** from $\mathsf{Spm}\, R$ to $\mathsf{Spm}\, R'$ is a continuous map coming from a homomorphism $R' \to R$.

Then one can define general algebraic k-sets by gluing affine algebraic k-sets using morphisms, just like how people define *structured* manifolds as glued from *structured* Euclidean spaces using maps preserving the addiontal structrues.

0.2. **Shortcomings.** The theory of algebraic k-sets provides a bridge between algebra and geometry. In particular, one can use topological/geometric methods, including various cohomology theories, to study *finitely generated reduced k-algebras.* However, these adjectives are non-necessary restrictions to this bridge.

First, number theory studies number fields and their rings of integers, such as $\mathbb{Q}$ and $\mathbb{Z}$. It is desirable to have geometric objects corresponding to them. Hence we would like to consider general commutative rings rather than k-algebras. Then one immediately realizes the maximal spectra Spm are not enough.

Example 0.2.1. The map $\mathbb{Z} \to \mathbb{Q}$ does not induce a map from $\mathsf{Spm}\, \mathbb{Q}$ to $\mathsf{Spm}\, \mathbb{Z}$. Namely, the inverse image of $(0) \subseteq \mathbb{Q}$ in $\mathbb{Z}$ is a non-maximal prime ideal.

This suggests for general algebra R, we should consider its *prime spectrum*, denoted by $\mathsf{Spec}\, R$, rather than just its maximal spectrum.

Second, even in the study of finitely generated algebras, one naturally encounters non-finitely generated ones.

Example 0.2.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal of a finitely generated algebra. The localization $R_{\mathfrak{p}}$ and its completion $\widehat{R}_{\mathfrak{p}}$ are in general not finitely generated.

Of course, one can restrict their attentions to *Noetherian* rings (and live a happy life). But let me object the false feeling that all natural rings one care are Noetherian.

Example 0.2.3. Noetherian rings are not stable under tensor products: $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not Noetherian.

Example 0.2.4. The ring of adeles of $\mathbb{Q}$ is not Noetherian.

Example 0.2.5. Natural moduli problems about Noetherian objects can fail to be Noetherian. My favorite ones includes: the space of formal connections, the space (stack) of formal group laws...

In short, there are important applications of non-Noetherian rings, and this course will deal with the latter.

Finally, we want to remove the restriction about reducedness.

Example 0.2.6. Reduced rings are not stable under tensor products: $k[x] \otimes_{k[x,y]} k[x,y]/(y-x^2)$ is not reduced. Geometrically, this means $Z(y)$ and $Z(y-x^2)$ do not intersect transversally inside $\mathbb{A}^2_k$.

One may notice that without reducedness, we should accordingly consider all ideals rather than just *radical* ideals, but then the construction $I \mapsto Z(I)$ would not be bijective. Indeed, ideals with the same nilpotent radical would give the same *topological subspace* of $\mathsf{Spec}\, R$.

But *this is a feature rather than a bug.* In Example 0.2.6, the ideal $(y, y-x^2) = (x^2, y)$ is not radical, and it carries geometric meanings that cannot be seen from its nilpotent radical (x,y). Namely, $f \in (x,y)$ iff $f(0,0) = 0$, while $f \in (x^2,y)$ iff $f(0,0) = \partial_x f(0,0) = 0$. Roughly speaking, this suggests that $(y, y-x^2)$ remembers that the curves $Z(y)$ and $Z(y-x^2)$ are tangent to each other at the point $(0,0) \in \mathbb{A}^2_k$, and the tangent vector is $\partial_x|_{(0,0)}$. Also note that the length of $k[x,y]/(y,y-x^2)$ is equal to 2, which is the number of intersection points predicted by the Bézout's theorem.

In summary, on the algebra side, we should consider *all* commutative rings. On the geometric side, the corresponding notion is called *affine schemes*. Our first task in this course is to develop the following duality:

Algebra	Geometry
commutative rings R	affine schemes X
prime ideals $\mathfrak{p} \subseteq R$	points $x \in X$
elements $f \in R$	functions $X \to \mathbb{A}^1_{\mathbb{Z}}$
ideals $I \subseteq R$	closed subschemes $Z \subseteq X$.

0.3. **Schemes as structured spaces.** In theory, one can *define* a morphism between affine schemes to be a continuous map coming from a ring homomorphism. Then one can define general *schemes* by gluing affine schemes using such morphisms. This mimics the definition of differentiable manifold in the sense that a scheme would be a topological space equipped with a *maximal* affine atlas.

In practice, the above approach is awkward to work with. We prefer a more efficient way to encode the additional structure on the topological space underlying a scheme. One extremely simple but powerful way to achieve this, maybe discovered by Serre and popularized by Grothendieck, is the notion of *sheaves* on topological spaces. Roughly speaking, a sheaf $\mathcal{F}$ on X is an *contravariant* assignment

$$U \mapsto \mathcal{F}(U)$$

sending open subsets $U \subseteq X$ to certain structures (e.g. sets, groups, rings) $\mathcal{F}(U)$, such that a certain gluing condition is satisfied. Here contravariancy means that for $U \subseteq V$, we should provide a map $\mathcal{F}(V) \to \mathcal{F}(U)$ preserving the prescribed structures.

Example 0.3.1. Let X be a topological space. The assignment

$$U \mapsto C(U, \mathbb{R})$$

sending $U \subseteq X$ to the ring of continuous functions on U would be a sheaf of commutative rings on X.

Similarly, for a smooth manifold X, $U \mapsto C^{\infty}(U, \mathbb{R})$ would be a sheaf of commutative rings on X. This motivates us to define:

Pre-Definition 0.3.2. A **scheme** is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X$ such that locally it is isomorphic to an affine scheme.

Here for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ should be the ring of *algebraic* functions on U, but we have not defined the latter notion yet. Nevertheless, for an *affine* scheme $X \simeq \mathsf{Spec}\, R$, the previous discussion suggests we should have $\mathcal{O}_X(X) \simeq R$. As we shall see in future lectures, we can bootstrap from this to get the definition of the entire sheaf $\mathcal{O}_X$.

The goal of this course is to define schemes and study their basic properties.

Part I. (Pre)sheaves

1. Definition of (pre)sheaves

1.1. Presheaves.

Definition 1.1.1. Let X be a topological space and $(U(X), \subseteq)$ be the partially ordered set of open subsets of X. We define the **category $\mathfrak{U}(X)$ of open subsets** in X to be the category associated to the partially ordered set $(U(X), \subseteq)$.

The category $\mathfrak{U}(X)$ can be explicitly described as follows:

- An object in $\mathfrak{U}(X)$ is an open subset $U \subseteq X$.
- If $U \subseteq V$, then $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is a singleton; otherwise $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is empty.
- The identify morphisms and composition laws are defined in the unique way.

Definition 1.1.2. Let X be a topological space and $\mathcal{C}$ be a category.

- A **$\mathcal{C}$-valued presheaf on** X is a functor $\mathcal{F} : \mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C}$.
- A **morphism** $\mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued presheaves is a natural transformation between these functors.

Let Set be the category of sets. By definition, a **presheaf $\mathcal{F}$ of sets**, i.e., a Set-valued presheaf, on X consists of the following data:

- For any open subset $U \subseteq X$, we have a set $\mathcal{F}(U)$, which is called the **set of sections** of $\mathcal{F}$ on U.
- For $U \subseteq V$, we have a map
$$\mathcal{F}(V) \to \mathcal{F}(U),\ s \mapsto s|_U$$
which is called the **restriction map**.

These data should satisfy the following condition:

- For any open subset $U \subseteq X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
- For $U \subseteq V \subseteq W$, the restriction maps make the following diagram commute
$$\begin{tikzcd} & \mathcal{F}(V) \arrow[dr] & \\ \mathcal{F}(U) \arrow[ur] \arrow[rr] & & \mathcal{F}(W). \end{tikzcd}$$

Let $\mathcal{F}$ adn $\mathcal{F}'$ be presheaves of sets on X. By definition, a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ consists of the following data:

- For any open subset $U \subseteq X$, we have a map $\phi_U : \mathcal{F}(U) \to \mathcal{F}(U)'$.

These data should satisfy the following condition:

- For $U \subseteq V$, the following diagram commute
$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d] & \mathcal{F}'(V) \arrow[d] \\ \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{F}'(U), \end{tikzcd}$$
where the vertical maps are restriction maps.

Similarly one can explicitly describe the notion of presheaves of abelian groups (k-vector spaces, commutative algebras) and morphisms between them.

Example 1.1.3. Let X be a topological space and $\mathcal{C}$ be a category. For any object $A \in \mathcal{C}$, the constant functor
$$\mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C},\ U \mapsto A,\ f \mapsto \mathsf{id}_A$$
defines a $\mathcal{C}$-valued presheaf on X, which is called the **constant presheaf associated to** A. It is often denoted by $\underline{A}$.

Example 1.1.4. Let X be a topological space and $E \to X$ be a topological space over it. We define a presheaf Sect_E of sets as follows.

- For any $U \subseteq X$,
$$\mathsf{Sect}_E(U) := \mathsf{Hom}_X(U, E)$$
is the set of countinuous maps $U \to E$ defined over X, a.k.a. sections of E over U.
- For $U \subseteq V$, the restriction map $\mathsf{Sect}_E(V) \to \mathsf{Sect}_E(U)$ sends a section $s : V \to E$ to its restriction $s|_U : U \to E$.

We call it the **presheaf of sections for** $E \to X$.

Example 1.1.5. If $E \to X$ is a real vector bundle, we can naturally upgrade Sect_E to be a presheaf of real vector spaces on X.

Example 1.1.6. Consider the constant real line bundle $\mathbb{R} \times X$ on X. Note that $\mathsf{Sect}_{\mathbb{R}\times X}(U)$ can be identified with the set of continuous functions on U. It follows that we can upgrade $\mathsf{Sect}_{\mathbb{R}\times X}$ to be a presheaf of $\mathbb{R}$-algebra on X.

1.2. **Sheaves of sets.** Roughly speaking, a sheaf is a presheaf whose sections on small open subsets can be uniquely glued to sections on larger ones.

Definition 1.2.1. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. We say $\mathcal{F}$ is a **sheaf** if it satisfies the following condition:

(*) For any open covering $U = \bigcup_{i\in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that
$$s_i|_{U_i \cap U_j} = s_j|_{U_i\cap U_j} \text{ for any } i, j \in I,$$
there is a *unique* section $s \in \mathcal{F}(U)$ such that
$$s_i = s|_U \text{ for any } i \in I.$$

Remark 1.2.2. Using the language of category theory, the sheaf condition is equivalent to the following condition:

- For any open covering $U = \bigcup_{i\in I} U_i$, the diagram
$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$
is an *equalizer* diagram. Here the first map is
$$s \mapsto (s|_{U_i})_{i\in I}$$
the other two maps are
$$(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j})_{(i,j)\in I^2}$$
and
$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

In particular, the map $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ is an injection.

Remark 1.2.3. For $U = \varnothing$ and $I = \varnothing$, the sheaf condition says there is a unique section $s \in \mathcal{F}(\varnothing)$ subject to no property. In other words, the above definition forces $\mathcal{F}(\varnothing)$ to be a singleton.

Example 1.2.4. Let X be a topological space. The constant presheaf $\underline{A}$ associated to a set A is in general not a sheaf. Indeed, $\underline{A}(\varnothing)$ is A rather than a singleton.

We provide another reason for readers uncomfortable with the above. For a sheaf $\mathcal{F}$ and *disjoint* open subsets U_1 and U_2, the sheaf condition implies

$$\mathcal{F}(U_1 \sqcup U_2) \simeq \mathcal{F}(U_1) \times \mathcal{F}(U_2).$$

But in general A and $A \times A$ are not isomorphic.

Example 1.2.5. Let $E \to X$ be a continuous map between topological spaces. The presheaf Sect_E of sections on X is a sheaf. Indeed, this follows from the fact that continuous maps can be glued.

Example 1.2.6. Let $\{*\}$ be a 1-point space. Then a sheaf $\mathcal{F}$ of sets on $\{*\}$ is uniquely determined by the set $\mathcal{F}(\{*\})$ of global sectoins. We often abuse the notations and use a set A to denote the sheaf on $\{*\}$ whose set of global sections is A.

Exercise 1.2.7. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X.

(1) Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X and $\alpha : \mathcal{F}|_{\mathfrak{B}} \to \mathcal{F}'|_{\mathfrak{B}}$ be a natural transformation between their restrictions on the full subcategory $\mathfrak{B}^{\mathsf{op}} \subseteq \mathfrak{U}(X)^{\mathsf{op}}$. Show that α can be uniquely extended to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$.
(2) Show that for presheaves, similar claims about existence and uniqueness are both false in general.

The above exercise says sheaves are determined by their restrictions on a topological base. A natural question is, given a functor $\mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$, under what conditions can we extend it to a sheaf $\mathfrak{U}(X) \to \mathsf{Set}$? This question is relevant to us because the Zariski topology of $\mathsf{Spec}\, R$ is defined using a base consisting of open subsets that can be easily described:

$$U(f) \coloneqq \left\{\mathfrak{p} \in \mathsf{Spec}\, R \,|\, f \notin \mathfrak{p}\right\} \simeq \mathsf{Spec}\, R_f.$$

It would be convenient if we can recover a sheaf $\mathcal{F}$ on $\mathsf{Spec}\, R$ from its values on these open subsets. For instance, we wonder whether the contravariant functor

$$U(f) \mapsto R_f$$

can be extended to a sheaf of commutative rings. If yes, we would obtain the sheaf $\mathcal{O}_X$ of algebraic functions desired in the introduction. The following construction gives a positive answer to this question.

Construction 1.2.8. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X. For a functor $\mathcal{F} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$ and $U \in \mathfrak{U}(X)$, define

$$\mathcal{F}'(U) \coloneqq \lim_{V \in \mathfrak{B}^{\mathsf{op}},\, V \subseteq U} \mathcal{F}(V).$$

In other words, an element in $s' \in \mathcal{F}'(U)$ is a collection of elements $s_V \in \mathcal{F}(V)$ for all open subsets $V \subseteq U$ contained in $\mathfrak{B}$ such that for $V_1 \subseteq V_2 \subseteq U$ with $V_1, V_2 \in \mathfrak{B}$, the map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$ sends s_{V_2} to s_{V_1}. This construction is clearly functorial in U, i.e., for $U_1 \subseteq U_2$, we have a natural map $\mathcal{F}'(U_2) \to \mathcal{F}'(U_1)$. One can check this defines a functor

$$\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$$

equipped with a canonical isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$. In other words, we have extended $\mathcal{F}$ to a *presheaf* $\mathcal{F}'$ of sets on X.

Remark 1.2.9. Using the language in category theory, the functor $\mathcal{F}'$ is the *right Kan extension* of $\mathcal{F}$ along the embedding $\mathfrak{B}^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Proposition 1.2.10. *In above, $\mathcal{F}'$ is a sheaf iff $\mathcal{F}$ satisfies the following condition:*

(**) *For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, and any collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that*

$$s_i|_V = s_j|_V \text{ for any } i, j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$

there is a unique *section $s \in \mathcal{F}(U)$ such that*

$$s_i = s|_{U_i} \text{ for any } i \in I.$$

Proof. The "only if" statement follows from the sheaf condition on $\mathcal{F}'$ and the isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$.

For the "if" statement, we verify the sheaf condition on $\mathcal{F}'$ directly. Let $U = \bigcup_{i\in I} U_i$ be an open covering, and $s'_i \in \mathcal{F}'(U_i)$ be a collection of sections such that

$$s'_i|_{U_i\cap U_j} = s'_j|_{U_i\cap U_j} \text{ for any } i, j \in I.$$

By Construction 1.2.8, each s'_i corresponds to a collection $s_{i,V} \in \mathcal{F}(V)$ for $V \subseteq U_i$, $V \in \mathfrak{B}$ that is compatible with restrictions.

We need to show there is a unique section $s' \in \mathcal{F}'(U)$ such that $s'|_{U_i} = s'_i$.

We first deal with the existence. For any $V \subseteq U$ with $V \in \mathfrak{B}$, since $\mathfrak{B}$ is a base, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. In other words, we can choose a map $f : J \to I$ such that $V_j \subseteq U_i$.

Consider the collection of sections

$$t_{j,V} \coloneqq s_{f(j),V_j} \in \mathcal{F}(V_j),\ j \in J. \tag{1.1}$$

One can check it does not depend on the choice of f and they satisfy the assumption in (**). Hence there is a unique section $s'_V \in \mathcal{F}(V)$ such that $s'_V|_{V_j} = s_{f(j),V_j}$.

One can check the obtained section s'_V does not depend on the open covering $V = \bigcup_{j\in J} V_j$ and the collections (s'_V), $V \subseteq U$, $V \in \mathfrak{B}$ is compatible with restrictions. Hence by Construction 1.2.8, it corresponds to an element $s' \in \mathcal{F}'(U)$. One can check that $s'|_{U_i} = s'_i$. This proves the claim about uniqueness.

It remains to prove the statement about uniqueness. Suppose there are two such sections s', s'' such that

$$s'|_{U_i} = s''|_{U_i} = s'_i \tag{1.2}$$

By Construction 1.2.8, they correspond to two collections $s'_V, s''_V \in \mathcal{F}(V)$ for $V \subseteq U$, $V \in \mathfrak{B}$. We only need to show $s'_V = s''_V$.

Note that if V is contained in some U_i, then (1.2) implies

$$s'_V = s''_V = s_{i,V}. \tag{1.3}$$

Now for general open subset $V \subseteq U$, $V \in \mathfrak{B}$, as before, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. Consider the collection of sections (1.1). By (1.3) (applied to each V_j), we have

$$s'_V|_{V_j} = s''_V|_{V_j} = t_{j,V}.$$

Hence by (**), we must have $s'_V = s''_V$ as desired.

□

1.3. $\mathcal{C}$-valued sheaves.

Definition 1.3.1. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on a topological space X. We say $\mathcal{F}$ is a **$\mathcal{C}$-valued sheaf** if for any testing object $c \in \mathcal{C}$, the functor

$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$

is a sheaf of sets.

Remark 1.3.2. By Yoneda's lemma and Remark 1.2.2, $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf iff for any open covering $U = \bigcup_{i\in I} U_i$, the canonical diagram

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is an *equalizer* diagram in $\mathcal{C}$. Here the first morphism is given by restrictions along $U_i \subseteq U$, while the other two morphisms are given respectively by restrictions along $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$. In particular, the morphism

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

is a *monomorphism*[2].

As a corollary of the remark, we obtain:

Corollary 1.3.3. *Let $\mathcal{F}$ be a presheaf of abelian groups. Then $\mathcal{F}$ is a sheaf of abelian groups iff its underlying presheaf of sets $\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Ab} \to \mathsf{Set}$ is a sheaf of sets. Here the functor $\mathsf{Ab} \to \mathsf{Set}$ sends an abelian group to its underlying set.*

Exercise 1.3.4. Let $\mathcal{F}$ be a presheaf of abelian groups. Show that $\mathcal{F}$ is a sheaf of abelian groups iff for any open covering $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact. Here the second map is

$$s \mapsto (s|_{U_i})_{i\in I},$$

and the third map is

$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

Now suppose $\mathcal{F}$ is a sheaf, can you further extend this exact sequence to the right?

Remark 1.3.5. Let $\mathcal{C}$ be a category that admits small limits. Then Construction 1.2.8 and Proposition 1.2.10 can be generalized to $\mathcal{C}$-valued (pre)sheaves with condition (**) replaced by

[2]This means for any testing object $c \in \mathcal{C}$, the functor $\mathsf{Hom}_{\mathcal{C}}(c,-)$ sends this morphism to an injection between sets.

- For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, any object $c \in \mathcal{C}$, and any collection of elements $s_i \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U_i))$, $i \in I$ such that
$$s_i|_V = s_j|_V \text{ for any } i,j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$
there is a *unique* element $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ such that
$$s_i = s|_{U_i} \text{ for any } i \in I.$$

In above $s|_V$ means the post-composition of $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ with the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.

Note however for $\mathcal{C} = \mathsf{Ab}$, we can keep condition (**) *as it is*, because the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ detects limits.

2. Stalks

2.1. Definition.

Definition 2.1.1. Let X be a topological space and $\mathcal{F}$ be a presheaf of sets on X. For a point $x \in X$, let $\mathfrak{U}(X,x) \subseteq \mathfrak{U}(X)$ be the full subcategory of open neighborhoods of x inside X. The **stalk of $\mathcal{F}$ at x** is

$$\mathcal{F}_x := \operatorname*{colim}_{U \in \mathfrak{U}(X,x)^{\mathsf{op}}} \mathcal{F}(U). \tag{2.1}$$

For a given section $s \in \mathcal{F}(U)$, the **germ of s at x**, denoted by s_x, is the image of s under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$.

Note that $\mathfrak{U}(X,x)^{\mathsf{op}}$ is the category associated to the *direct set*[3] $(U(X,x), \subseteq)$ of open neighborboods of x inside X. Hence the above colimit is a *direct colimit*[4]. It follows that $\mathcal{F}_x$ can be explicitly described as the quotient

$$\Big(\coprod_{U \in U(X,x)} \mathcal{F}(U) \Big) / \sim, \tag{2.2}$$

of the disjoint union of all $\mathcal{F}(U)$, $U \in U(X,x)$ by an equivalence relation $\sim$. Here two sections $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(U')$ are equivalent iff there exists $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Using this description, the germ s_x of a section $s \in \mathcal{F}(U)$ is just the equivalence class to which it belongs.

Remark 2.1.2. In general, let $\mathcal{C}$ be a category that admits direct colimits and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf. We can define the stalk of $\mathcal{F}$ at x using the same formula (2.1). Note that this construction is functorial in $\mathcal{F}$.

In particular, for a presheaf $\mathcal{F}$ of abelian groups, we can define its stalk $\mathcal{F}_x$, which is an abelian group. It is easy to see the underlying set $\mathcal{F}_x$ is given by (2.2) and the group structure is given by the formula

$$s_x + s'_x = (s|_V + s'|_V)_x,\ s \in \mathcal{F}(U), s' \in \mathcal{F}(U'), V \subseteq U \cap U'.$$

[3]A direct set is a partially ordered set $(I, \leq)$ such that any finite subset of I admits an upper bound in I.

[4]Some people use the word *direct limit*. I strongly object this terminology.

2.2. **Sheaves and stalks.** The following result says a section of a *sheaf* is determined by its germs.

Lemma 2.2.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X. Then for any open subset $U \subseteq X$, the map*

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x,\ s \mapsto (s_x)_{x \in U} \tag{2.3}$$

is injective. Moreover, a collection of elements $s(x) \in \mathcal{F}_x$, $x \in U$ is contained in the image of this map iff it satisfies the following condition

(***) *For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.*

Proof. We first show the map (2.3) is injective. Let $s, s' \in \mathcal{F}(U)$ such that all their germs are equal. By definition, for any $x \in U$, there exists $V \subseteq U$ such that $s|_V = s'|_V$. In particular, we can find an open covering $U = \bigcup_{i \in I} U_i$ such that $s|_{U_i} = s'|_{U_i}$. But this implies $s = s'$ because the sheaf condition implies

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

is injective.

It is obvious that any element in the image of (2.3) satisfies condition (***). To prove the converse, let $s(x) \in \mathcal{F}_x$, $x \in U$ be a collection of elements satisfying condition (***). By assumption, we can find an open covering $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that for any $x \in U_i$, we have

$$t(x) = (s_i)_x. \tag{2.4}$$

In particular, the germs of $s_i|_{U_i \cap U_j}$ and $s_j|_{U_i \cap U_j}$ are equal. Applying the injectivity of (2.3) to $U_i \cap U_j$, we obtain

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Hence by the sheaf condition, we can find a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. For any $x \in U$, pick $i \in I$ such that $x \in U_i$, we have

$$s_x = (s_i)_x = t(x),$$

where the first equality is due to the definition of stalks, while the second one is (2.4). In particular, $s(x) \in \mathcal{F}_x$, $x \in U$ is the image of s under the map (2.3).

□

Remark 2.2.2. Similar claim for presheaves is false in general. Namely, for $U = X = \varnothing$, the empty product $\prod_{x \in \varnothing} \mathcal{F}_x$ is a singleton, while $\mathcal{F}(\varnothing)$ can be any set.

Corollary 2.2.3. *If $\alpha, \beta : \mathcal{F} \to \mathcal{F}'$ are morphisms between sheaves of sets such that $\alpha_x = \beta_x$ for any $x \in X$, then $\alpha = \beta$.*

Proposition 2.2.4. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism between sheaves of sets on a topological space. Then α is an isomorphism iff for any $x \in X$, $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a bijection.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose α_x is a bijection for any $x \in X$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\alpha_U"] & \prod_{x\in U} \mathcal{F}_x \arrow[d, "(\alpha_x)_{x\in X}", "\simeq"'] \\ \mathcal{F}'(U) \arrow[r] & \prod_{x\in U} \mathcal{F}'_x. \end{tikzcd}$$

By Lemma 2.2.1, the horizontal maps are injective, hence so is α_U.

It remains to show α_U is surjective. Let $s' \in \mathcal{F}'(U)$ be a section, we will construct a section $s \in \mathcal{F}(U)$ mapping to it by α_U.

For any point $x \in U$, since α_x is bijective, we can find an open subset $V \subseteq X$ and a section $t \in \mathcal{F}(V)$ such that $\alpha_x(t_x) = s'_x$. By definition, $\alpha_x(t_x) = \alpha_V(t)_x$. Hence the germs of $\alpha_V(t)$ and s' at x are equal. By definition, there exists an open neightborhood W of x inside $U \cap V$ such that $\alpha_V(t)|_W = s'|_W$. Note that we also have $\alpha_V(t)|_W = \alpha_W(t|_W)$.

It follows that we can find an open covering $U = \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. In particular, we have

$$\alpha_{U_i\cap U_j}(s_i|_{U_i\cap U_j}) = \alpha_{U_i\cap U_j}(s_j|_{U_i\cap U_j}) = s|_{U_i\cap U_j}.$$

Since we have already shown $\alpha_{U_i\cap U_j}$ is injective, we obtain $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Hence by the sheaf condition for $\mathcal{F}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Using the sheaf condition for $\mathcal{F}'$, it is easy to see $\alpha_U(s) = s'$ as desired.

□

The above results imply that a *morphism* between sheaves are determined by the induced maps between the stalks. However, a sheaf itself is *not* determined by its stalks.

Exercise 2.2.5. Let X be a connected topological space and $E \to X$ and $E' \to X$ be two covering spaces of the same degree. Show that the sheaves Sect_E and $\mathsf{Sect}_{E'}$ on X have isomorphic stalks for any point $x \in X$, but they are not isomorphic unless there exists a homeomorphism $E \simeq E'$ defined over X.

Remark 2.2.6. Let $\mathcal{C}$ be a *compactly generated* category[5]. Lemma 2.2.1 and Proposition 2.2.4 can be generalized to $\mathcal{C}$-valued sheaves. In other words:

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$, the morphism $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is a monomorphism.
- A morphism $\alpha : \mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued sheaves is an isomorphism iff $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for any $x \in X$.

These statements can be deduced from the special case for Set with the help of the following two observations:

- A morphism $d \to d'$ in $\mathcal{C}$ is a monomorphism (resp. isomorphism) iff for any *compact* object $c \in \mathcal{C}$, the map $\mathsf{Hom}_{\mathcal{C}}(c, d) \to \mathsf{Hom}_{\mathcal{C}}(c, d')$ is an injection (resp. bijection).

[5]An object c in a (locally small) category $\mathcal{C}$ is compact iff $\mathsf{Hom}_{\mathcal{C}}(c, -)$ preserves small filtered colimits. We say $\mathcal{C}$ is compactly generated if it admits small colimits and any object in $\mathcal{C}$ is isomorphic to a small filtered colimit of compact objects. It is known that compactly generated categories also admit small limits.

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$ and any *compact* object $c \in \mathcal{C}$, the stalk of the Set-valued sheaf
$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$
at $x \in X$ is canonically isomorphic to $\mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}_x)$.

The details are left to the curious readers.

2.3. Skyscrapers.

Definition 2.3.1. Let X be a topological space and $x \in X$ be a point. For any set A, we can define a presheaf $\delta_{x,A}$ of sets as follows.

- For an open subset $U \subseteq X$,
  - if $x \in U$, define $\delta_{x,A}(U) := A$;
  - if $x \notin U$, define $\delta_{x,A}(U) := \{*\}$.
- For open subsets $U \subseteq V$,
  - if $x \in U$ (and therefore $x \in V$), define the restriction map $\delta_{x,A}(U)$ to be id_A;
  - if $x \notin U$, define the restriction map to be the unique map $\delta_{x,A}(V) \to \delta_{x,A}(U) = \{*\}$.

One can check this indeed defines a presheaf $\delta_{x,A}$. We call the the **skyscraper sheaf** at x with value A.

Exercise 2.3.2. The presheaf $\delta_{x,A}$ is indeed a sheaf.

Lemma 2.3.3. *Let X be a topological space, $x \in X$ be a point and A be a set. The stalk of $\delta_{x,A}$ at a point $y \in X$ is canonically bijective to*

- *the set A if y is contained in $\overline{\{x\}}$, the closure of $\{x\}$ inside X;*
- *the singleton $\{*\}$ otherwise.*

Proof. If $y \in \overline{\{x\}}$, then any open neighborhood of y contains x. It follows that
$$(\delta_{x,A})_y := \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} A$$
is a direct colimit of the constant diagram with values A. This implies $(\delta_{x,A})_y \simeq A$.

If $y \notin \overline{\{x\}}$, then there exists an open neighborhood V of y such that $x \notin V$. Note that $\mathfrak{U}(V,y)^{\mathsf{op}} \subseteq \mathfrak{U}(X,y)^{\mathsf{op}}$ is (co)final. If follows that
$$(\delta_{x,A})_y := \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq (\delta_{x,A})_y \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \{*\}$$
is a direct colimit of the constant diagram with values $\{*\}$. This implies $(\delta_{x,A})_y \simeq \{*\}$.

□

Note that if A is equipped with the structure of an abelian group, the skyscraper $\delta_{x,A}$ can be upgraded to a sheaf of abelian groups. Then the abelian group $(\delta_{x,A})_y$ is either A or 0.

Proposition 2.3.4. *Let X be a topological space, $x \in X$ be a point and A be a set. For any presheaf $\mathcal{F}$ of sets on X, the composition*

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \xrightarrow{(-)_x} \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, (\delta_{x,A})_x) \simeq \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \tag{2.5}$$

is an bijection.

Corollary 2.3.5. *The stalk functor*

$$\mathsf{PShv}(X,\mathsf{Set}) \to \mathsf{Set},\ \mathcal{F} \mapsto \mathcal{F}_x$$

admits a right adjoint

$$\mathsf{Set} \to \mathsf{PShv}(X,\mathsf{Set}),\ A \mapsto \delta_{A,x}.$$

Proof of Proposition 2.3.4. We first construct a map

$$\mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \tag{2.6}$$

as follows. Given any map $f : \mathcal{F}_x \to A$, for any open subset $U \subseteq X$, we define a map $\alpha_U : \mathcal{F}(U) \to \delta_{x,A}(U)$ such that:

- If $x \in U$, α_U is the composition $\mathcal{F}(U) \to \mathcal{F}_x \xrightarrow{f} A$;
- If $x \notin U$, α_U is the unique map $\mathcal{F}(U) \to \{*\}$.

One can check these maps are compatible with restriction and therefore define a morphism $\alpha : \mathcal{F} \to \delta_{x,A}$. Now we define the map (2.6) to be $f \mapsto \alpha$.

One can check that (2.5) and (2.6) are inverse to each other. Hence both are bijections.

□

Remark 2.3.6. In general, for any category $\mathcal{C}$ admitting a final object[6] and any object $A \in \mathcal{C}$, one can define a $\mathcal{C}$-valued sheaf $\delta_{x,A}$. If $\mathcal{C}$ admits direct colimits, the stalks of $\delta_{x,A}$ are either A or the final object of $\mathcal{C}$, and the functor $A \mapsto \delta_{A,x}$ is right adjoint to $\mathcal{F} \mapsto \mathcal{F}_x$.

[6]An object $* \in \mathcal{C}$ is a final object iff for any $c \in \mathcal{C}$, there is a unique morphism $c \to *$.

3. Category of (pre)sheaves

Let X be a topological space and $\mathcal{C}$ be a category. Note that $\mathcal{C}$-valued presheaves on X form a category

$$\mathsf{PShv}(X,\mathcal{C}) \coloneqq \mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}),$$

and $\mathcal{C}$-valued sheaves form a full subcategory

$$\mathsf{Shv}(X,\mathcal{C}) \subseteq \mathsf{PShv}(X,\mathcal{C}).$$

In this section, we study the basic properties of these categories.

3.1. Sheafification.

Definition 3.1.1. Let $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$. The **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^\sharp \in \mathsf{Shv}(X,\mathsf{Set})$ equipped with a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ such that for any testing sheaf $\mathcal{G}$, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta.$$

Proposition 3.1.2. *For any $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$, its sheafification $(\mathcal{F}^\sharp,\theta)$ exists, and is unique up to unique isomorphism. Moreover, the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ induces bijections $\mathcal{F}_x \to \mathcal{F}^\sharp_x$ between the stalks.*

Proof. The statement about uniqueness follows from Yoneda's lemma. To prove the existence, we construct a sheafification as follows.

We first construct the desired sheaf $\mathcal{F}^\sharp$. For any open subset $U \subseteq X$, let

$$\mathcal{F}^\sharp(U) \subseteq \prod_{x\in U} \mathcal{F}_x,$$

be the subset consisting of elements $(s(x))_{x\in U}$ satisfying the following condition:

- For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.

For $U \subseteq U'$, it is obvious that the projection map $\prod_{x\in U'} \mathcal{F}_x \to \prod_{x\in U} \mathcal{F}_x$ sends $\mathcal{F}^\sharp(U')$ into $\mathcal{F}^\sharp(U)$. Moreover, one can check the obtained maps $\mathcal{F}^\sharp(U') \to \mathcal{F}^\sharp(U)$ upgrade the assignment $U \mapsto \mathcal{F}^\sharp(U)$ to an object in $\mathsf{Shv}(X,\mathsf{Set})$.

Now we construct the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$. For any open subset $U \subseteq X$, consider the map

$$\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x,\ s \mapsto (s_x)_{x\in U}.$$

It is obvious that the image of this map is contained in $\mathcal{F}^\sharp(U)$. Moreover, the obtained maps $\mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ is functorial in U, therefore give a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$.

It remains to show $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ exhibits $\mathcal{F}^\sharp$ as a sheafification of $\mathcal{F}$. Let $\mathcal{G}$ be a testing sheaf, we need to show

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta \tag{3.1}$$

is bijective. Let $\beta : \mathcal{F} \to \mathcal{G}$ be a morphism. For any open subset $U \subseteq X$, recall taking germs induces an injection

$$\mathcal{G}(U) \to \prod_{x\in U} \mathcal{G}_x$$

and its image is desribed in Lemma 2.2.1. Using that description, it is clear that there is a unique dotted map making the following diagram commute:

$$\begin{tikzcd} \mathcal{F}^\sharp(U) \arrow[r, "\subseteq"] \arrow[d, dotted] & \prod_{x\in U}\mathcal{F}_x \arrow[d, "(\beta_x)_{x\in U}"] \\ \mathcal{G}(U) \arrow[r] & \prod_{x\in U}\mathcal{G}_x. \end{tikzcd}$$

Moreover, the obtained map $\mathcal{F}^\sharp(U)\to\mathcal{G}(U)$ is functorial in U. Hence we obtain a morphism $\beta^\sharp:\mathcal{F}^\sharp\to\mathcal{G}$. Now one can check that the map

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G})\to\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}),\ \beta\mapsto\beta^\sharp$$

and (3.1) are inverse to each other. In particular, they are both bijective as desired. □

Corollary 3.1.3. *The fully faithful embedding* $\mathsf{Shv}(X,\mathsf{Set})\to\mathsf{PShv}(X,\mathsf{Set})$ *admits a left adjoint which sends* $\mathcal{F}$ *to its sheafification* $\mathcal{F}^\sharp$*.*

Example 3.1.4. Let A be a set. The sheafification $\underline{A}^\sharp$ of the constant presheaf $\underline{A}$ is the sheaf

$$\mathfrak{U}(X)^{\mathsf{op}}\to\mathsf{Set},\ U\mapsto C(U,A)$$

that sends U to the set of continuous maps from U to A (equipped with the discrete topology). We call it the **constant sheaf** associated to A.

Remark 3.1.5. Suppose $\mathcal{F}$ is a presheaf of abelian groups. Let $\mathcal{F}^\sharp$ be the sheafification of the underlying Set-valued presheaf of $\mathcal{F}$ as constructed in the proof of the proposition. One can check that $\mathcal{F}^\sharp(U)$ is a subgroup of the abelian group $\prod_{x\in U}\mathcal{F}_x$. It follows that $\mathcal{F}^\sharp$ can be upgraded to a sheaf of abelian groups. Moreover, for any testing sheaf $\mathcal{G}$ of abelian groups, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Ab})}(\mathcal{F}^\sharp,\mathcal{G})\xrightarrow{\simeq}\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Ab})}(\mathcal{F},\mathcal{G}),\ \alpha\mapsto\alpha\circ\theta.$$

In other words, $\mathsf{Shv}(X,\mathsf{Ab})\to\mathsf{PShv}(X,\mathsf{Ab})$ admits a left adjoint which sends $\mathcal{F}$ to $\mathcal{F}^\sharp$.

Remark 3.1.6. In general, if $\mathcal{C}$ is a category admitting small limits and filtered colimits, then any $\mathcal{C}$-valued presheaf admits a sheafification that can be constructed as follows.

For $U\subseteq X$, we can define the *category* Cov_U *of open coverings of* U as follows:

- An object is an open covering $U=\bigcup_{i\in I}U_i$;
- A morphism from $(U_i)_{i\in I}$ to $(V_j)_{j\in J}$ is a map $J\to I$ such that $V_j\subseteq U_i$ for any $j\in J$.

One can show that Cov_U is filtered. Now for any $\mathcal{F}\in\mathsf{PShv}(X,\mathcal{C})$, we have a functor

$$\begin{array}{ccc} \mathsf{Cov}_U & \to & \mathcal{C} \\ (U_i)_{i\in I} & \mapsto & \lim[\prod_{i\in I}\mathcal{F}(U_i)\rightrightarrows\prod_{(i,j)\in I^2}\mathcal{F}(U_i\cap U_j)]. \end{array}$$

sending a covering to the equalizer appeared in the sheaf condition. Note that the identity covering $\{U\}$ is sent to the object $\mathcal{F}(U)$. Now we define

$$\mathcal{F}^+(U):=\operatorname*{colim}_{[(U_i)_{i\in I}]\in\mathsf{Cov}_U}\lim[\prod_{i\in I}\mathcal{F}(U_i)\rightrightarrows\prod_{(i,j)\in I^2}\mathcal{F}(U_i\cap U_j)].$$

By construction, there is a canonical morphism $\mathcal{F}(U) \to \mathcal{F}^+(U)$. Moreover, the above definition is contravariantly functorial in U, therefore we obtain an object $\mathcal{F}^+ \in \mathsf{PShv}(X,\mathcal{C})$ equipped with a canonical morphism $\mathcal{F} \to \mathcal{F}^+$.

In general, $\mathcal{F}^+$ is note a $\mathcal{C}$-valued sheaf. But one can check that for any open covering $U = \bigcup_{i\in I} U_i$, the morphism

$$\mathcal{F}^+(U) \to \prod_{i\in I} \mathcal{F}^+(U_i)$$

is a monomorphism. Using this property, one can show that $(\mathcal{F}^+)^+$ is a sheaf and the composition $\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$ exhibits $(\mathcal{F}^+)^+$ as a sheafification of $\mathcal{F}$.

3.2. **Direct images.**

Construction 3.2.1. Let $f : X \to X'$ be a continuous map between topological spaces. We have a functor

$$\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}},\ U' \mapsto f^{-1}(U').$$

For any category $\mathcal{C}$, it induces a functor

$$\mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}) \to \mathsf{Fun}(\mathfrak{U}(X')^{\mathsf{op}},\mathcal{C}).$$

By definition, this gives a functor

$$f_* : \mathsf{PShv}(X,\mathcal{C}) \to \mathsf{PShv}(X',\mathcal{C}).$$

We call it the **direct image functor** (or **pushforward functor**) along f for $\mathcal{C}$-valued presheaves.

Note that for continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g\circ f)_* \simeq g_* \circ f_*$.

Explicitly, given a $\mathcal{C}$-valued presheaf $\mathcal{F}$ on X, its **direct image** (or **pushforward**) along f is the presheaf $f_*\mathcal{F}$ defined by

$$f_*\mathcal{F}(U') \coloneqq \mathcal{F}(f^{-1}(U')),$$

with restriction maps given by those maps for $\mathcal{F}$.

Proposition 3.2.2. *Let $f : X \to X'$ be a continuous map between topological spaces. If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is a sheaf.*

Proof. The sheaf condition for $f_*\mathcal{F}$ and an open covering $U' = \bigcup_{i\in I} U'_i$ is just the sheaf condition for $\mathcal{F}$ and the open covering $f^{-1}(U') = \bigcup_{i\in I} f^{-1}(U'_i)$.

□

Example 3.2.3. Let $x \in X$ be a point and write $i : \{x\} \to X$ for the emebdding map. Let $\mathcal{C}$ be a category admitting a final object $*$. For any object $A \in \mathcal{C}$, we have

$$i_*(A) \simeq \delta_{x,A},$$

where we abuse notations and use A to denote the unique $\mathcal{C}$-valued sheaf on $\{x\}$ whose object of global sections is A.

Example 3.2.4. Let $p : X \to \{*\}$ be the obvious projection map. For any *sheaf* $\mathcal{F}$, the direct image $p_*\mathcal{F}$ is uniquely determined by $p_*\mathcal{F}(\{*\})$, which is $\mathcal{F}(X)$ by definition. Hence in this case, we also call p_* is **taking global sections functor**.

Warning 3.2.5. Direct image functors do *not* commute with sheafifications. In other words $f_*(\mathcal{F}^\sharp)$ and $(f_*\mathcal{F})^\sharp$ are in general not isomorphic. For a counterexample, take $\mathcal{F}$ to be a constant *presheaf*.

3.3. Inverse images for presheaves.

Construction 3.3.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We define a presheaf $f^{-1}_{\mathsf{PShv}}\mathcal{F}' \in \mathsf{PShv}(X, \mathsf{Set})$ by the following formula

$$f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U) := \operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V),$$

where $\mathfrak{U}(X', f(U)) \subseteq \mathfrak{U}(X')$ is the full subcategory of open neighborhoods of $f(U)$ inside X', and the restriction maps for $f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ are induced by those for $\mathcal{F}'$.

The construction $\mathcal{F}' \to f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for presheaves of sets.

Note that $\mathfrak{U}(X', f(U))^{\mathsf{op}}$ is the category associated to a direct set. Hence $f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U)$ can be calculated as a quotient of

$$\bigsqcup_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V).$$

Example 3.3.2. Let X be a topological space and x be a point. Write $i : \{x\} \to X$ for the embedding. We have

$$\big(i^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(\{x\}) \simeq \mathcal{F}'_x.$$

Lemma 3.3.3. *Let X be a topological space and $U \subseteq X$ be an open subset. Write $j : U \to X$ for the embedding map. Then j^{-1}_{PShv} sends sheaves to sheaves.*

Proof. For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V \subseteq U$, unwinding the definitions, we have

$$\big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V) \simeq \mathcal{F}(V).$$

Hence the sheaf condition for $j^{-1}_{\mathsf{PShv}}(\mathcal{F})$ follows from that for $\mathcal{F}$.

□

Warning 3.3.4. For general continuous map $f : X \to X'$, the functor f^{-1}_{PShv} does not send sheaves to sheaves. To see this, consider the projection map $p : X \to \{*\}$.

Remark 3.3.5. The functor $f^{-1}_{\mathsf{PShv}}\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$ is the left Kan extension of $\mathcal{F}' : \mathfrak{U}(X')^{\mathsf{op}} \to \mathsf{Set}$ along the pullback functor $\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Construction 3.3.6. Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We construct a morphism

$$\mathcal{F}' \to f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}') \tag{3.2}$$

as follows. For any open subest $U' \subseteq X'$, by definition,

$$\big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U') \simeq \big(f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(f^{-1}(U')) \simeq \operatorname*{colim}_{V \in \mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}} \mathcal{F}'(V).$$

Note that U' is an object in $\mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}$. Hence we have a canonical map

$$\mathcal{F}'(U') \to \big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U').$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.2).

Moreover, we can upgrade these morphisms to a natural transformation

$$\mathsf{Id} \to f_* \circ f^{-1}_{\mathsf{PShv}}. \tag{3.3}$$

Construction 3.3.7. Dually, let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ be a presheaf. We construct a morphism

$$f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F}) \to \mathcal{F}. \tag{3.4}$$

as follows. For any open subest $U \subseteq X$, by definition,

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \simeq \operatorname*{\mathsf{colim}}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \big(f_*(\mathcal{F})\big)(V) \simeq \operatorname*{\mathsf{colim}}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}(f^{-1}(V)).$$

Note that for any $V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}$, we have $U \subseteq f^{-1}(V)$, which gives a restriction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$. One can check these maps are functorial in V and give a map

$$\operatorname*{\mathsf{colim}}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U).$$

Hence we obtain a map

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \to \mathcal{F}(U).$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.4).

Moreover, we can upgrade these morphisms to a natural transformation

$$f^{-1}_{\mathsf{PShv}} \circ f_* \to \mathsf{Id}. \tag{3.5}$$

The following proposition follows from a boring diagram chasing. We omit the details.

Proposition 3.3.8. *Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. The following compositions are inverse to each other:*

$$\begin{aligned}\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F}) &\xrightarrow{f_*} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f_*\mathcal{F}) \\ &\xrightarrow{-\circ(3.2)} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F})\end{aligned}$$

and

$$\begin{aligned}\mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) &\xrightarrow{f^{-1}_{\mathsf{PShv}}} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})) \\ &\xrightarrow{(3.4)\circ-} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F})\end{aligned}$$

Corollary 3.3.9. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{PShv}(X', \mathsf{Set}).$$

Corollary 3.3.10. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1}_{\mathsf{PShv}} \simeq f^{-1}_{\mathsf{PShv}} \circ g^{-1}_{\mathsf{PShv}}$.*

Corollary 3.3.11. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any presheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}_{\mathsf{PShv}}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Remark 3.3.12. Let $\mathcal{C}$ be a category admitting direct colimits. One can define the functor f^{-1}_{PShv} for $\mathcal{C}$-valued presheaves using the same formula, and f^{-1}_{PShv} is canonically left adjoint to f_*.

3.4. **Inverse images for sheaves.**

Construction 3.4.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F} \in \mathsf{Shv}(X', \mathsf{Set})$ be a sheaf. We define

$$f^{-1}\mathcal{F} := (f^{-1}_{\mathsf{PShv}}\mathcal{F}')^\sharp$$

to be the sheafification of the presheaf-theoretic inverse image of $\mathcal{F}$.

The construction $\mathcal{F}' \to f^{-1}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for sheaves of sets.

Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. We have canonical bijections:

$$\begin{aligned} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(f^{-1}(\mathcal{F}'), \mathcal{F}) &\simeq \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F}) \\ &\simeq \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) \simeq \mathsf{Hom}_{\mathsf{Shv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}), \end{aligned}$$

where

- the first bijection is due to the definition of sheafifications;
- the second bijection is that in Proposition 3.3.8;
- the last bijection is due to the fully faithful embedding $\mathsf{Shv}(X', \mathsf{Set}) \subseteq \mathsf{PShv}(X, \mathsf{Set})$.

Corollary 3.4.2. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Shv}(X', \mathsf{Set}).$$

Corollary 3.4.3. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1} \simeq f^{-1} \circ g^{-1}$.*

Corollary 3.4.4. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any sheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Exercise 3.4.5. The following diagram commutes:

$$\begin{tikzcd} \mathsf{PShv}(X', \mathsf{Set}) \arrow[r, "f^{-1}_{\mathsf{PShv}}"] \arrow[d, "(-)^\sharp"] & \mathsf{PShv}(X, \mathsf{Set}) \arrow[d, "(-)^\sharp"] \\ \mathsf{Shv}(X', \mathsf{Set}) \arrow[r, "f^{-1}"] & \mathsf{Shv}(X, \mathsf{Set}). \end{tikzcd}$$

Exercise 3.4.6. Show that f^{-1} sends a constant sheaf to the constant sheaf associated to the same set.

Example 3.4.7. Let X be a topological space and x be a point. Write $i:\{x\}\to X$ for the embedding. For $\mathcal{F}\in\mathsf{Shv}(X,\mathsf{Set})$, we have

$$i^{-1}(\mathcal{F})\simeq\mathcal{F}_x,$$

where in the RHS we abuse notations by identifying a sheaf on $\{x\}$ with its set of global sections (see Example 1.2.6).

Remark 3.4.8. Let $\mathcal{C}$ be a category admitting small limits and filtered colimits. One can define the functor f^{-1} for $\mathcal{C}$-valued sheaves using the same formula, and f^{-1} is canonically left adjoint to f_*.

3.5. **Open base-change.**

Construction 3.5.1. Given a commutative square of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & X' \arrow[d, "v"] \\ Y \arrow[r, "g"] & Y', \end{tikzcd} \tag{3.6}$$

consider the canonical natural isomorphism $v_*\circ f_*\simeq g_*\circ u_*$. Using the adjunctions $(g^{-1}_{\mathsf{PShv}},g_*)$ and $(f^{-1}_{\mathsf{PShv}},f_*)$, we obtain natural transformations

$$g^{-1}_{\mathsf{PShv}}\circ v_*\to g^{-1}_{\mathsf{PShv}}\circ v_*\circ f_*\circ f^{-1}_{\mathsf{PShv}}\simeq g^{-1}_{\mathsf{PShv}}\circ g_*\circ u_*\circ f^{-1}_{\mathsf{PShv}}\to u_*\circ f^{-1}_{\mathsf{PShv}},$$

where the first arrow is induced by $\mathsf{Id}\to f_*\circ f^{-1}_{\mathsf{PShv}}$ (see (3.3)), while the last arrow is induced by $g^{-1}_{\mathsf{PShv}}\circ g_*\to\mathsf{Id}$ (see (3.5)).

We call the above composition the **base-change natural transformation**[7] for presheaves associated to the square (3.6).

Similarly, we have the base-change natural transformation for sheaves

$$g^{-1}\circ v_*\to u_*\circ f^{-1}.$$

Proposition 3.5.2. *Let $f:X\to X'$ be a continuous map between topological spaces and $U'\subseteq X'$ be an open subset. Write $U:=f^{-1}(U')$ can consider the following diagram*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ U' \arrow[r, "j'"] & X'. \end{tikzcd}$$

Then both

$$(j')^{-1}_{\mathsf{PShv}}\circ f_*\to g_*\circ j^{-1}_{\mathsf{PShv}}$$

and

$$(j')^{-1}\circ f_*\to g_*\circ j^{-1}$$

are natural isomorphisms.

[7]Other name: Bech–Chevalley natural transformations.

Proof. We will prove the claim for presheaves. That for sheaves follow from Lemma 3.3.3.

For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V' \subseteq U'$, unwinding the definitions, we have

$$\big((j')^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(V') \simeq \big(f_*(\mathcal{F})\big)(V') \simeq \mathcal{F}(f^{-1}(V'))$$

and

$$\big(g_* \circ j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V') \simeq \big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(g^{-1}(V')) \simeq \mathcal{F}(f^{-1}(V')).$$

One can check that via these identifications, the value of $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ at $\mathcal{F}$ and V' is given by the identity map on $\mathcal{F}(f^{-1}(V'))$. In particular, $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ is a natural isomorphism.

□

Remark 3.5.3. Informally, we say: *open* pullbacks commute with pushforwards.

Warning 3.5.4. In the setting of Proposition 3.5.2, one can also consider the natural transformations

$$f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$$

and

$$f^{-1} \circ j'_* \to j_* \circ g^{-1}.$$

However, they are *not* invertible in general.

Exercise 3.5.5. Let $X' = \{s, b\}$ be the topological space with two points whose open subsets are exactly given by $\varnothing, \{b\}$ and X'. Consider the following diagram

$$\begin{tikzcd} \varnothing \arrow[r, "j"] \arrow[d, "g"] & \{s\} \arrow[d, "f"] \\ \{b\} \arrow[r, "j'"] & X'. \end{tikzcd}$$

Show that $f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$ and $f^{-1} \circ j'_* \to j_* \circ g^{-1}$ are not invertible.

Part II. Definition of schemes

4. $\mathsf{Spec}(R)$

4.1. Zariski topology.

Definition 4.1.1. Let R be a (unital) commutative ring. Write $\mathsf{Spec}(R)$ for the set of prime ideals of R. We equip it with the **Zariski topology** so that the subsets

$$U(f) := \{\mathfrak{p} \in \mathsf{Spec}(R) \,|\, f \notin \mathfrak{p}\},\ f \in R$$

form a topological base. The obtained topological space is called the **prime spectrum** of R. The open subsets of the form $U(f)$ are called the **standard open subsets**[8].

Note that $U(f) \cap U(g) = U(fg)$.

Construction 4.1.2. For any ideal $I \subseteq R$, consider $Z(I) = \{\mathfrak{p} \,|\, I \subseteq \mathfrak{p}\}$. By definition,

$$Z(I) \simeq \mathsf{Spec}(R) \smallsetminus \bigcup_{f\in I} U(f).$$

This implies the following result.

Lemma 4.1.3. *A subset Z of $\mathsf{Spec}(R)$ is closed iff it is of the form $Z(I)$ for some ideal $I \subseteq R$.*

Lemma 4.1.4. *Let $I, J \subseteq R$ be ideals. Then $Z(I) \subseteq Z(J)$ iff $J \subseteq \sqrt{I}$.*

Proof. Recall the radical $\sqrt{I}$ is equal to the intersection of prime ideals containing I, i.e.,

$$\sqrt{I} = \bigcap_{\mathfrak{p}\in Z(I)} \mathfrak{p}. \tag{4.1}$$

For the "if" statement, suppose $J \subseteq \sqrt{I}$. Then $J \subseteq \mathfrak{p}$ and therefore $\mathfrak{p} \in Z(J)$ for any $\mathfrak{p} \in Z(I)$. Hence we have $Z(I) \subset Z(J)$ as desired.

For the "only if" statement, suppose $Z(I) \subseteq Z(J)$. By (4.1), $\sqrt{J} \subseteq \sqrt{I}$. In particular, $J \subseteq \sqrt{I}$ as desired.

□

Corollary 4.1.5. *Let $I, J \subseteq R$ be ideals. Then $Z(I) = Z(J)$ iff $\sqrt{J} = \sqrt{I}$.*

Corollary 4.1.6. *A point $\mathfrak{p} \in \mathsf{Spec}(R)$ is closed iff $\mathfrak{p}$ is maximal.*

Corollary 4.1.7. *Let x and $y \in \mathsf{Spec}(R)$ be points given by prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. Then $x \in \overline{\{y\}}$ iff $\mathfrak{p} \supset \mathfrak{q}$.*

In above, we say x is a **specialization** of y, and y is a **generalization** of x.

Corollary 4.1.8. *The topological space $\mathsf{Spec}(R)$ is Kolmogorov, i.e., for any pair of distinct points, at least one of them has an open neighborhood not containing the other point.*

Remark 4.1.9. The space $\mathsf{Spec}(R)$ is in general not Haussdorff. Indeed, it is so iff the Krull dimension of R is zero.

[8]Other name: elementary open subsets.

Example 4.1.10. The points in $\mathsf{Spec}(\mathbb{Z})$ are listed as below:

(i) For each prime number p, there is a point $(p) \in \mathsf{Spec}(\mathbb{Z})$.
(ii) There is a point $(0) \in \mathsf{Spec}(\mathbb{Z})$.

A subset of $\mathsf{Spec}(\mathbb{Z})$ is closed iff it is finite collection of points in (i), or it is the entire space.

Note that points (i) are closed, while the point in (ii) is not closed. In fact, the closure of the latter is the entire space.

Example 4.1.11. For any field k, $\mathsf{Spec}(k)$ is a point.

Example 4.1.12. For any discrete valuation ring R, $\mathsf{Spec}(R)$ consists of two points: a closed point corresponding to its ideal of definition, and an open point corresponding to the zero ideal.

Exercise 4.1.13. Let k be an algebraically closed field. Describe the topological space $\mathsf{Spec}(k[x,y]/(xy))$.

Lemma 4.1.14. *The topological space* $\mathsf{Spec}(R)$ *is quasi-compact. In other words, any open covering of it admits a finite sub-covering.*

Proof. It is enough to show any open covering of the form $\mathsf{Spec}(R) = \bigcup_{f\in S} U(f)$ admits a finite sub-covering. Let $\langle S\rangle$ be the ideal generated by S. We obtain $Z(\langle S\rangle) = \varnothing$ and therefore $\langle S\rangle = R$. Hence there exists a finite subset $S' \subseteq S$ such that $1 \in \langle S'\rangle$ and therefore $R = \langle S'\rangle$. Hence we have

$$\varnothing = Z(\langle S'\rangle) = \mathsf{Spec}(R) \smallsetminus \bigcup_{f\in S'} U(f).$$

In other words, we have found a finite sub-covering given by $U(f)$, $f \in S'$.

□

4.2. **Structure sheaf.** We are going to construct a canonical sheaf on $\mathsf{Spec}(R)$. For this purpose, we need to associate a set to any standard open subset. Note that a standard open subset $U(f)$ does *not* uniquely determine the element f. However, we have the following results.

Lemma 4.2.1. *For* $f, f' \in R$, $U(f) \subseteq U(f')$ *iff* $R \to R_f$ *(uniquely) factors through* $R \to R_{f'}$.

Proof. By definition, $U(f) \subseteq U(f')$ iff $Z(\langle f\rangle) \supset Z(\langle f'\rangle)$. By Lemma 4.1.4, this happens iff $f^n \in \langle f'\rangle$ for some $n \geq 0$. The latter condition is equivalent to f' being an unit under the map $R \to R_f$. By definition, this is equivalent to the condition that $R \to R_f$ factors through $R \to R_{f'}$.

□

Corollary 4.2.2. *The open subsets* $U(f)$ *and* $U(f')$ *of* $\mathsf{Spec}(R)$ *are equal iff* R_f *and* $R_{f'}$ *are isomorphic as* R*-algebras.*

Proposition-Definition 4.2.3. *There exists an essentially unique*[9] *sheaf* $\mathcal{O}$ *of commutative rings on* $\mathsf{Spec}(R)$ *equipped with an isomorphism* $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ *such that for any* $f \in R$, *the* R*-algebra* $\mathcal{O}(U(f))$ *given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}(U(f))$$

[9]This means the pair $(\mathcal{O}, \phi)$ is unique up to a unique isomorphism.

is isomorphic to R_f.

The sheaf $\mathcal{O}_{\mathsf{Spec}(R)} := \mathcal{O}$ *is called the* ***structure sheaf*** *on* $\mathsf{Spec}(R)$. *When using this terminology, we treat the isomorphism* $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ *as implicit.*

Remark 4.2.4. Note that for an R-algebra A, being isomorphic to R_f is a *property rather than a structure*. Namely, there is at most one R-homomorphism from R_f to A.

Proof of Proposition-Definition 4.2.3. Let $\mathfrak{B}$ be the category of standard open subsets in $\mathsf{Spec}(R)$. Since a sheaf is uniquely determined by its restriction on a topological base, we only need to show there is a unique functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ equipped with an isomorphism $\varphi : R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ such that:

(a) The functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ satisfies the sheaf condition in Proposition 1.2.10.
(b) For any $f \in R$, $\mathcal{O}(U(f))$ is isomorphic to R_f as R-algebras.

By Lemma 4.2.1, there is a unique pair $(\mathcal{O}, \varphi)$ satisfying condition (b). Hence we only need to check condition (a). Unwinding the definitions, this amounts to the following easy fact in commutative algebra. We leave the proof of it to the readers. □

Lemma 4.2.5. *Let* R *be a commutative ring and* f, $(f_i)_{i\in I}$ *be elements in* R *such that* $U(f) = \bigcup_{i\in I} U(f_i)$. *Then the following sequence is exact:*

$$0 \to A_f \to \prod_{i\in I} A_{f_i} \to \prod_{(i,j)\in I^2} A_{f_i f_j}$$

is exact. Here the second map is induced by the canonical maps $A \to A_{f_i}$, *and the third map is*

$$(s_i)_{i\in I} \mapsto (s_j - s_i)_{(i,j)\in I^2}.$$

Exercise 4.2.6. Let k be a field and $R = k[x,y]$. Consider the point $0 \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x,y). Let $U := \mathsf{Spec}(R) \smallsetminus 0$ be the complementary open subset. Find $\mathcal{O}(U)$.

Definition 4.2.7. An **affine scheme** is a topological space X equipped with a sheaf $\mathcal{O}$ of commutative rings on X such that $(X, \mathcal{O}) \simeq (\mathsf{Spec}R, \mathcal{O}_{\mathsf{Spec}(R)})$ for some commutative ring R.

Proposition 4.2.8. *Let* $x \in \mathsf{Spec}(R)$ *be the point corresponding to a prime ideal* $\mathfrak{p} \subseteq R$. *Then the* R*-algebra* $\mathcal{O}_x$ *given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}_x$$

is (uniquely) isomorphic to $R_{\mathfrak{p}}$ *as* R*-algebras. In particular,* $\mathcal{O}_x$ *is a local ring.*

Proof. By definition, we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{U}(\mathsf{Spec}(R),x)^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\mathfrak{B}_x \subseteq \mathfrak{U}(\mathsf{Spec}(R), x)$ be the full subcategory of standard open neighborhoods of x in $\mathsf{Spec}(R)$. By the definition of Zariski topology, $\mathfrak{B}_x^{\mathsf{op}} \to \mathfrak{U}(\mathsf{Spec}(R), x)^{\mathsf{op}}$ is (co)final. Hence we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{B}_x^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\phi : R \to A$ be any testing R-algebra. We have

$$\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \lim_{U \in \mathfrak{B}_x} \mathsf{Hom}_R(\mathcal{O}(U), A).$$

Since $\mathcal{O}(U)$ is a localization of R for each $U \in \mathfrak{B}_x$, we have

- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \varnothing$ if $U = U(f)$ and $\phi(f)$ is not a unit;
- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \{*\}$ if $U = U(f)$ and $\phi(f)$ is a unit.

It follows that

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $U(f) \in \mathfrak{B}_x$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $U(f) \in \mathfrak{B}_x$.

Note that for an element $f \in R$, the standard open $U(f)$ is a neighborhood of x iff $f \notin \mathfrak{p}$. Hence we have

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $f \in R \smallsetminus \mathfrak{p}$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $f \in R \smallsetminus \mathfrak{p}$.

Note that $\mathsf{Hom}_R(R_\mathfrak{p}, A)$ has the same description. Hence by Yoneda lemma, there is a unique isomophism $\mathcal{O}_x \simeq R_\mathfrak{p}$ as R-algerbas.

□

4.3. **Functoriality.** Throughout this subsection, we fix the following notations:

- Let R and R' be commutative rings.
- Write $X := \mathsf{Spec}(R)$ and $X' := \mathsf{Spec}(R')$.
- Write $\mathcal{O}$ and $\mathcal{O}'$ respectively for the structure sheaves on X and X'.

Construction 4.3.1. Let $h : R \to R'$ be a homomorphism between commuative algebras. Consider the map

$$\phi : X' \to X,\ \mathfrak{p}' \mapsto h^{-1}(\mathfrak{p}').$$

By definition, for any $f \in R$,

$$\phi^{-1}(U(f)) = U(h(f)).$$

It follows that ϕ is a continuous map with respect to the Zariski topology.

Note that the assignment $h \mapsto \phi$ loses information: h *cannot* be reconstructed from ϕ.

Proposition 4.3.2. *Let $h : R \to R'$ be a homomorphism and $\phi : X' \to X$ be the corresponding continuous map. Then there exists a unique morphism in* $\mathsf{Shv}(X, \mathsf{CRing})$

$$\alpha : \mathcal{O} \to \phi_*(\mathcal{O}')$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_*\mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\
R \arrow[u, "\simeq"] \arrow[rr, "h"] & & R'. \arrow[u, "\simeq"']
\end{tikzcd}$$

Proof. Let $\mathfrak{B} \subseteq \mathfrak{U}(\mathsf{Spec}(R))$ be the full subcategory of standard open subsets. By Exercise 1.2.7, it is enough to show that there exists a unique natural transformation

$$\alpha : \mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}} \to \phi_*(\mathcal{O}')|_{\mathfrak{B}^{\mathrm{op}}}$$

that makes the diagram commute.

For any $U \in \mathfrak{B}^{\mathsf{op}}$, we claim there is a unique dotted homomorphism α_U making the following diagram commute

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, dotted, "\alpha_U"] & \phi_*\mathcal{O}'(U) \arrow[r, "\simeq"] & \mathcal{O}'(\phi^{-1}(U)) \\ R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u] \end{tikzcd}$$

Indeed, choose $f \in R$ such that $U = U(f)$. Then $\phi^{-1}(U) = U(h(f))$. Hence $\mathcal{O}(U) \simeq R_f$ and $\mathcal{O}'(\phi^{-1}(U)) \simeq R'_{h(f)}$. Via these identications, the claim becomes obvious.

It follows that α_U can be assembled into a natural transformation α satisfying the desired property. Moreover, such α is unique because each α_U is unique.

□

By adjunction, we obtain the following result.

Corollary 4.3.3. *Let $h : R \to R'$ be a homomorphism and $\phi : \mathsf{Spec}(R') \to \mathsf{Spec}(R)$ be the corresponding continuous map. Then there exists a unique morphism in* $\mathsf{Shv}(X', \mathsf{CRing})$

$$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$$

such that the following diagram commutes

$$\begin{tikzcd} \phi^{-1}\mathcal{O}(X') \arrow[r, "\beta_{X'}"] & \mathcal{O}'(X') \\ R \arrow[u] \arrow[r, "h"] & R'. \arrow[u, "\simeq"] \end{tikzcd}$$

Moreover, for any point $x' \in X'$ and $x := \phi(x')$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

is a local homomorphism between local rings.

Proof. The first claim follows from Proposition 4.3.2 and the adjunction $\phi^{-1} \vdash \phi_*$. For the second claim, let $\mathfrak{p}' \subseteq R'$ be the prime ideal corresponding to x' and $\mathfrak{p} := \phi^{-1}(\mathfrak{p}')$. By Proposition 4.2.8, we can identify $\mathcal{O}_x \to \mathcal{O}'_{x'}$ with the unique R-homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$, which makes the desired claim manifest.

□

The following result says knowing h is equivalent to knowing a pair (ϕ, β).

Proposition-Construction 4.3.4. *There is a canonical bijection between the following sets:*

(i) *The set $\mathsf{Hom}_{\mathsf{CRing}}(R, R')$ of homomorphisms from R to R'.*
(ii) *The set of pairs (ϕ, β), where*
   – *$\phi : X' \to X$ is a continuous map,*
   – *$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ is a morphism in $\mathsf{Shv}(X', \mathsf{CRing})$*

   such that for any $x = \phi(x')$, $x' \in X'$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

   is a local homomorphism between local rings.

Proof. For any pair (ϕ, β) in (ii), let $\alpha : \mathcal{O} \to \beta_* \mathcal{O}'$ be the morphism corresponding to β via adjunction. There is a unique dotted homomorphism h that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_* \mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\ R \arrow[u, "\simeq"] \arrow[rr, dotted, "h"] & & R'. \arrow[u, "\simeq"'] \end{tikzcd}$$

This defines a map (ii)→(i). We have seen this map is surjective (Corollary 4.3.3). It remains to check it is injective.

Suppose (ϕ_1, β_1) and (ϕ_2, β_2) produce the same homomorphism $h : R \to R'$.

We first show $\phi_1 = \phi_2$. Let $x' \in X'$ be a point corresponding to a prime ideal $\mathfrak{p}' \subseteq R'$, consider $x_i \coloneqq \phi_i(x_i)$. We will show $x_1 = x_2$. Let $\mathfrak{p}_i \subseteq R$ be the prime ideal corresponding to x_i. For $i = 1, 2$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{x_i} \arrow[r, "\simeq"] & (\phi_i^{-1}\mathcal{O})_{x'} \arrow[r, "(\beta_i)_{x'}"] & \mathcal{O}'_{x'} \\ R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u] \end{tikzcd}$$

By Proposition 4.2.8, $\mathcal{O}_{x_i} \simeq R_{\mathfrak{p}_i}$ and $\mathcal{O}'_{x'} \simeq R'_{\mathfrak{p}'}$. Hence the commutative diagram implies $h^{-1}(\mathfrak{p}') \subseteq \mathfrak{p}_i$. Moreover, since by assumption the top horizontal arrow is a local homomorphism, we must have $h^{-1}(\mathfrak{p}') = \mathfrak{p}_i$. In particular, $\mathfrak{p}_1 = \mathfrak{p}_2$ and therefore $x_1 = x_2$ as desired.

Now write $\phi = \phi_1 = \phi_2$. It remains to show $\beta_1 = \beta_2$. By the last paragraph, for any $x' \in X'$, we have $(\beta_1)_{x'} = (\beta_2)_{x'}$ because it can be identified with the *unique* homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ compatible with $h : R \to R'$. Now by Corollary 2.2.3, we obtain $\beta_1 = \beta_2$ as desired.

□

Exercise 4.3.5. Show that the conclusion of Proposition-Construction 4.3.4 would be false if we do not require $\beta_{x'}$ to be a local homomorphism. In other words, show that there exists a continuous map $\phi : X' \to X$ together with a morphism $\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ such that $\beta_{x'}$ is not a local homomorphism for some point $x' \in X'$.

5. Schemes as locally ringed spaces

5.1. **Locally ringed spaces.** Motivated by the construction $(\mathsf{Spec}(R), \mathcal{O})$, we make the following definition.

Definition 5.1.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between ringed spaces** is a pair $\phi = (\phi, \beta)$, where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves.

Let $\mathsf{Top}_{\mathsf{CRing}}$ be the category of ringed spaces and morphisms between them.

Remark 5.1.2. Equivalently, we can replace β by a morphism $\alpha : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$.

Definition 5.1.3. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that for any point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Let $(X, \mathcal{O}_X)$ be a locally ringed space and $x \in X$ be a point. The **residue field of** $(X, \mathcal{O}_X)$ **at** x is the field

$$\kappa_x := \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between locally ringed spaces** is a pair (ϕ, β), where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves such that for any point $x \in X$, the homomorphism

$$\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$$

is a local homomorphism.

Let $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the category of locally ringed spaces and morphisms between them.

Construction 5.1.4. Let $(\phi, \beta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homomorphism between locally ringed spaces. For any $x \in X$, the local homomorphism $\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ induces a homomorphism

$$\kappa_{Y,\phi(x)} \to \kappa_{X,x}.$$

Warning 5.1.5. The functor $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}} \to \mathsf{Top}_{\mathsf{CRing}}$ is faithful but not full.

Example 5.1.6. For any commutative ring R, we obtain a locally ringed space $(\mathsf{Spec}(R), \mathcal{O})$. We will abuse notation and denote this locally ringed space just by $\mathsf{Spec}(R)$, and treat its structure sheaf as implicit.

For any homomorphism $h : R \to R'$, we obtain a morphism $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ between locally ringed spaces as in Proposition-Construction 4.3.4. Moreover, the information of h is exactly encoded by this morphism.

Definition 5.1.7. An **affine scheme** is a locally ringed space that is isomorphic to $\mathsf{Spec}(R)$ for some R. A **morphism between affine schemes** is a morphism between locally ringed spaces. Let $\mathsf{Aff} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory of affine schemes.

Using these new terminilogies, we can reformulate Proposition-Construction 4.3.4 as follows.

Proposition-Construction 5.1.8. *The following functors are inverse to each other:*

$$\begin{array}{rcl} \mathsf{CRing}^{\mathsf{op}} & \simeq & \mathsf{Aff} \\ R & \mapsto & \mathsf{Spec}(R) \\ \mathcal{O}_X(X) & \leftarrow & X. \end{array}$$

Construction 5.1.9. Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. For any $f : Y \to X$, the pair $(Y, f^{-1}\mathcal{O}_X)$ defines a (locally) ringed space, and we have a canonical morphism

$$(Y, f^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$$

given by the pair $(f, \mathsf{id}_{f^{-1}\mathcal{O}_X})$.

When $f : Y \subseteq X$ is a subspace, we write $\mathcal{O}_X|_Y := f^{-1}\mathcal{O}_X$ and call the obtained (locally) ringed space $(Y, \mathcal{O}_X|_Y)$ the **restriction of** $(X, \mathcal{O}_X)$ **to** Y (or the **locally ringed subspace of** $(X, \mathcal{O}_X)$ **associated to** Y).

Example 5.1.10. Let R be a commutative ring and consider the locally ringed space $\mathsf{Spec}(R)$. For any element $f \in R$, by Construction 5.1.9, we obtain a locally ringed subspace of $\mathsf{Spec}(R)$ associated to $U(f)$. By construction, it can be identified with $\mathsf{Spec}(R_f)$. In particular, it is an affine scheme.

5.2. **Schemes.**

Definition 5.2.1. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open covering $X = \bigcup_{i \in I} U_i$ with each $(U_i, \mathcal{O}_X|_{U_i})$ being an affine scheme.

A **morphism** between schemes is a morphism between locally ringed spaces. Let $\mathsf{Sch} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory consisting of schemes.

Notation 5.2.2. We often abuse notation by writing X for a scheme $(X, \mathcal{O}_X)$ and treating its structure sheaf as implicit. Similarly, we often abuse notation by writing $\phi : X \to Y$ for a morphism between schemes and treating the morphism $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as implicit.

Warning 5.2.3. Nevertheless, one should keep in their minds that schemes are not determined by their underlying topological spaces; similarly morphisms between schemes are not determined by the underlying continuous maps.

Exercise 5.2.4. Let X be a scheme over $\mathsf{Spec}(\mathbb{F}_q)$.

(1) For any open subset $U \subseteq X$, show that the map $\beta_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, $f \mapsto f^q$ is a homomorphism, and these maps give an endomorphism $\beta : \mathcal{O}_X \to \mathcal{O}_X$ of the structure sheaf.
(2) Show that $\mathsf{Frob}_{X,q} := (\mathsf{id}_X, \beta)$ is an endomorphism of the scheme X defined over $\mathsf{Spec}(\mathbb{F}_q)$.

The morphism $\mathsf{Frob}_{X,q}$ is known as the **absolute** q**-Frobenius morphism of** X, and plays a central role in the study of schemes over finite fields.

The following results follow from their counterparts for affine schemes.

Lemma 5.2.5. *Let X be a scheme.*

- *The affine open subsets of X form a base for its topology.*
- *The space X is Kolmogorov.*

The following exercise provides examples of locally ringed spaces that are not schemes.

Exercise 5.2.6. Let X be a topological space. For any open subset $U \subseteq X$, let $\mathcal{C}_X(U)$ be the commutative ring of $\mathbb{R}$-valued[10] continuous functions on U. Note that $U \mapsto \mathcal{C}_X(U)$ defines a sheaf of commutative rings on X.

(1) Show that $(X, \mathcal{C}_X)$ is a locally ringed space.
(2) Show that a continuous map $X \to X'$ induces a morphism $(X, \mathcal{C}_X) \to (X', \mathcal{C}_{X'})$ between locally ringed spaces.
(3) Show that $(\mathbb{R}, \mathcal{C}_{\mathbb{R}})$ is not a scheme.

5.3. **Open immersions.**

Proposition-Definition 5.3.1. *Let X be a scheme. For any open subspace $U \subseteq X$, the corresponding locally ringed subspace is a scheme. We call it the **open subscheme of** X **associated to** U.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering such that each U_i is an affine scheme. We only need to show the locally ringed subspace associated to each $U_i \cap U$ can be covered by affine schemes. Without lose of generality, we can replace X with U_i and U with $U_i \cap U$, and therefore assume $X \simeq \mathsf{Spec}(R)$ is affine. Now by the definition of the Zariski topology, we can find elements $(f_j)_{j \in J}$ in R such that $U = \bigcup_{j \in J} U(f_j)$. By Example 5.1.10, each $U(f_j)$ is an affine scheme isomorphic to $\mathsf{Spec}(R_{f_j})$. Hence U is a scheme as desired.

□

Definition 5.3.2. We say a morphism $f : Y \to X$ is an **open immersion** if there exists an (unique) open subscheme $U \subseteq X$ such that f factors as $Y \xrightarrow{\simeq} U \to X$.

Warning 5.3.3. An open subscheme U of X may fail to be affine even if X is affine. Also, an *affine* open subset of an affine scheme may fail to be a standard subset.

Exercise 5.3.4. Let k be a field and $R = k[x, y]$. Consider the point $(0, 0) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x, y). Let $U := \mathsf{Spec}(R) \setminus \{(0, 0)\}$ be the complementary open subset. Show that the scheme U is not affine.

Exercise 5.3.5. Let k be a field of characteristic 0 and $R := k[x, y]/(y^2 - x^3)$. Consider the point $(1, 1) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal $(x-1, y-1)$. $U := \mathsf{Spec}(R) \setminus \{(1, 1)\}$ be the complementary open subset. Show that the scheme U is affine but it is not a standard open subset of $\mathsf{Spec}(R)$.

Warning 5.3.6. Let X be a scheme and $Y \subseteq X$ be a subspace. The locally ringed subspace associated to Y is in general not a scheme.

Exercise 5.3.7. Let R be a local ring and $X := \mathsf{Spec}(R)$. Consider the unique closed point $x \in \mathsf{Spec}(R)$. Show that $(\{x\}, \mathcal{O}_X|_{\{x\}})$ is not a scheme unless R is a field.

Exercise 5.3.8. An open immersion $f : Y \to X$ is a monomorphism in Sch. In other words, for any $Z \in \mathsf{Sch}$, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(Z, Y) \xrightarrow{f \circ -} \mathsf{Hom}_{\mathsf{Sch}}(Z, X)$$

is injective.

[10] We equip $\mathbb{R}$ with the usual topology.

6. Gluing schemes

6.1. Statement.

Definition 6.1.1. A **gluing data of schemes** is a collection

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

where

- I is a set;
- For each $i \in I$, X_i is a scheme;
- For any pair $(i,j) \in I^2$, U_{ij} is an open subscheme of X_i;
- For any pair $(i,j) \in I^2$,
$$\phi_{ij} : U_{ij} \to U_{ji},$$
is an isomorphism between schemes.

The above data should satisfy the following conditions:

- For any $i \in I$, $U_{ii} = X_i$ and $\phi_{ii} = \mathsf{id}_{X_i}$.
- For any triple $(i,j,k) \in I^3$,
$$\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$
as open subsets of U_{ji}.
- For any triple $(i,j,k) \in I^3$, the following **cocycle condition** holds:
$$\phi_{jk}|_{U_{ji}\cap U_{jk}} \circ \phi_{ij}|_{U_{ij}\cap U_{ik}} = \phi_{ik}|_{U_{ij}\cap U_{ik}},$$
i.e., the following diagram commutes:
$$\begin{tikzcd} & U_{ij}\cap U_{ik} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\phi_{ik}", "\simeq"'] & \\ U_{ji}\cap U_{jk} \arrow[rr, "\phi_{jk}", "\simeq"'] & & U_{ki}\cap U_{kj}. \end{tikzcd}$$

Proposition-Definition 6.1.2. *Given a gluing data of schemes*

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

there exists an essentially unique collection

$$\big(X, (X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$$

where

- *X is a scheme;*
- *For each $i \in I$, X_i' is an open subscheme of X;*
- *For each $i \in I$,*
$$\varphi_i : X_i \xrightarrow{\simeq} X_i',$$
is an isomorphism;

such that

- *$X = \bigcup_{i\in I} X_i'$ as topological spaces;*
- *For any pair $(i,j) \in I^2$,*
$$\varphi_i(U_{ij}) = X_i' \cap X_j'$$
as open subsets of X_i';

- *For any pair* $(i,j) \in I^2$, *we have*
$$\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ji}} \circ \phi_{ij},$$
i.e., the following diagram commutes
$$\begin{tikzcd} & U_{ij} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\varphi_i", "\simeq"'] & \\ U_{ji} \arrow[rr, "\varphi_j", "\simeq"'] & & X_i' \cap X_j'. \end{tikzcd}$$

We say the scheme X *is* ***glued*** *from the given gluing data, and treat* $\big((X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$ *as implicit.*

Proof. It is an exercise in point-set topology that the similar claim for topological spaces is correct. In other words, the gluing data gives an essentially unique topological space X equipped with open subspaces X_i' and homeomorphisms $\varphi_i : X_i \to X_i'$ satisfying the *topological* conditions listed in the statement.

Hence we only show there is an essentially unique $\mathcal{O}_X \in \mathsf{Shv}(X, \mathsf{CRing})$ equipped with isomorphisms $\varphi_i^{-1}(\mathcal{O}_X|_{X_i'}) \simeq \mathcal{O}_{X_i}$, that satisfies the remaining *sheaf-theoretic* conditions. Note that such a ringed space $(X, \mathcal{O}_X)$ will automatically be a scheme because its restriction to each X_i' is a scheme isomorphic to X_i.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X_i'$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. It is easy to see there exists an essentially unique functor
$$\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$$
equipped with isomorphisms
$$\beta_i : \varphi_i^{-1}((\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}})|_{X_i'}) \xrightarrow{\simeq} \mathcal{O}_i$$
satisying the desired conditions. Here $(\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}})|_{X_i'}$ is the restriction of $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ along the fully faithful embedding $\mathfrak{U}(X_i')^{\mathsf{op}} \to \mathfrak{B}^{\mathsf{op}}$. Moreover, one can check that the obtained $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ satisfies the sheaf condition in Proposition 1.2.10. Hence there is an essentially unique extension of $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ to a CRing-valued sheaf $\mathcal{O}_X$ on X, which fulfills our goal.

□

The following proposition describes how to construct morphisms out of a glued space.

Proposition 6.1.3. *Let*
$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big),$$
be a gluing data of schemes and
$$\big(X, (X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$$
be its gluing output. For any scheme Y, *the map*
$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Sch}}(X,Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Sch}}(X_i, Y) \\ f & \mapsto & (f|_{X_i'} \circ \varphi_i)_{i\in I} \end{array}$$
is injective, and a collection of morphisms $(g_i : X_i \to Y)_{i\in I}$ *is contained in the image iff* $g_i|_{U_{ij}} = g_j|_{U_{ji}} \circ \phi_{ij}$ *for any pair* $(i,j) \in I^2$.

Proof. To simplify the notations, we identify X_i with X_i' and identify U_{ij} with the intersection $X_i \cap X_j$ inside X. Consequently, ϕ_{ij} and φ_i are identity morphisms.

We first prove the similar claim for topological spaces. Indeed, since $X = \bigcup_{i \in X} X_i$ as a topological space, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Top}}(X,Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Top}}(X_i,Y) \\ f & \mapsto & (f|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of continuous map $(f_i : X_i \to Y)_{i\in I}$ is contained in the image iff $f_i|_{X_i\cap X_j} = f_j|_{X_i \cap X_j}$ for any pair $(i,j) \in I^2$.

It follows that we only need to show that for a given continuous map $f : X \to Y$ and $f_i \coloneqq f|_{X_i}$, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Shv}(X_i,\mathsf{CRing})}(f_i^{-1}\mathcal{O}_Y,\mathcal{O}_{X_i}) \\ \beta & \mapsto & (\beta|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of morphisms $(\beta_i : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{X_i})_{i\in I}$ is contained in the image iff $\beta_i|_{X_i\cap X_j} = \beta_j|_{X_i\cap X_j}$.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X_i$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. By Exercise 1.2.7, we have a bijection

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \xrightarrow{\simeq} & \mathsf{Hom}_{\mathsf{Fun}(\mathfrak{B}^{\mathsf{op}},\mathsf{CRing})}((f^{-1}\mathcal{O}_Y)|_{\mathfrak{B}^{\mathsf{op}}},(\mathcal{O}_X)|_{\mathfrak{B}^{\mathsf{op}}}) \\ \beta & \mapsto & \beta|_{\mathfrak{B}^{\mathsf{op}}}. \end{array}$$

Now the desired claim follows from the fact that the category $\mathfrak{B}$ can be covered by its full subcategories $\mathfrak{U}(X_i)$, and $\mathfrak{U}(X_i) \cap \mathfrak{U}(X_j) \simeq \mathfrak{U}(X_i \cap X_j)$. □

We can also describe morphisms *into* a glued space.

Proposition-Construction 6.1.4. *Let*

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big),$$

be a gluing data of schemes and

$$\big(X, (X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$$

be its gluing output. For any scheme Y, there is a canonical bijection between the following sets:

(i) *The set* $\mathsf{Hom}_{\mathsf{Sch}}(Y,X)$ *of morphisms* $f : Y \to X$
(ii) *The set of collections*

$$(f_i : Y_i \to X_i)_{i\in I},$$

where
- *Each Y_i is an open subscheme of Y and $Y = \bigcup_{i\in I} Y_i$ is an open covering;*
- *Each $f_i : Y_i \to X_i$ is a morphism;*

such that
- *For each pair $(i,j) \in I^2$, $f_i^{-1}(U_{ij}) = Y_i \cap Y_j$;*
- *For each pair $(i,j) \in I^2$, $f_j|_{Y_i\cap Y_j} = \phi_{ij} \circ f_i|_{Y_i\cap Y_j}$.*

Sketch. We first construct a map (i)→(ii). Give a morphism $f: Y \to X$, we declare Y_i to be the open subscheme associated to the open subset $f^{-1}(X_i')$. In particular, $f|_{Y_i}$ gives a morphism $Y_i \to X_i'$. We declare f_i to be the composition

$$Y_i \to X_i' \xrightarrow{\varphi_i^{-1}} X_i.$$

One can verify the collection $(f_i : Y_i \to X_i)_{i\in I}$ satisfies the desired requirements. This gives a map (i)→(ii).

Now we construct a map (ii)→(i). Given a collection $(f_i : Y_i \to X_i)_{i\in I}$. Consider the compositions

$$g_i : Y_i \xrightarrow{f_i} X_i \xrightarrow{\varphi_i} X_i' \to X.$$

One can check $g_i|_{Y_i\cap Y_j} = g_j|_{Y_i\cap Y_j}$. It follows that there is a unique morphism $f : Y \to X$ such that $f|_{Y_i} = g_i$. This gives a map (ii)→(i).

Now one can check the above two maps are inverse to each other.

□

Remark 6.1.5. Results in this subsection also works for general (locally) ringed spaces.

6.2. **Examples.**

Example 6.2.1. Let $(X_i)_{i\in I}$ be a set of schemes, U_{ij} be the empty scheme for $i \neq j$, and ϕ_{ij} be the identity morphisms. This is obviously a gluing data. The scheme X glued from this gluing data is called the **disjoint union of** $(X_i)_{i\in I}$, and we denote it by $\bigsqcup_{i\in I} X_i$. By Proposition 6.1.3, $\bigsqcup_{i\in I} X_i$ is also the coproduct of $(X_i)_{i\in I}$ inside the category Sch.

Example 6.2.2. As one would expect, the n-dimensional projective space can be glued from $(n+1)$ affine spaces of dimension n. Below are the details.

Let R be any commutative ring. For $n \geq 0$, let $I := \{0, 1, \cdots, n\}$ and

$$X_i := \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]/(x_i^{(i)} - 1)).$$

Let

$$U_{ij} := U(x_j^{(i)}) \subseteq X_i.$$

Then we have

$$U_{ij} \simeq \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1)).$$

Note that we have an isomorphism

$$R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1) \simeq R[x_0^{(j)}, \cdots, x_n^{(j)}]_{x_i^{(j)}}/(x_j^{(j)} - 1)$$

that sends $x_k^{(i)}$ to $x_k^{(j)}/x_i^{(j)}$. This gives an isomorphism

$$U_{ij} \xrightarrow{\simeq} U_{ji}.$$

One can check the above gives a gluing data, hence we obtain an essentially unique scheme X glued from it.

We write $\mathbb{P}_R^n$ for the gluing result and call it the **n-dimensional projective space over** R.

Exercise 6.2.3. Let R be a commutative ring and k be an algebraically closed field.

(1) Find $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$. Deduce that $\mathbb{P}^n_R$ is not affine for $n \geq 1$.
(2) Show that the closed points of $\mathbb{P}^n_k$ can be canonically identified with elements in $(k^{n+1} \smallsetminus 0)/k^\times$, where $k^\times$ acts on the vector space k^{n+1} via scaler multiplication.

Exercise 6.2.4. Let R be any commutative ring and $I = \{1, 2\}$. Let

$$X_1 = X_2 := \mathbb{A}^1_R := \mathsf{Spec}(R[t])$$

and

$$U_{12} = U_{21} := U(t), U_{11} := X_1, U_{22} := X_2.$$

Let ϕ_{ij} be the identity morphisms. Consider the scheme X glued from the above gluing data. Show that X is not affine.

7. Morphisms to affine schemes

7.1. A criterion for being affine.

Construction 7.1.1. Let X and Y be schemes. For any morphism $f: X \to Y$, by definition we have a morphism $\alpha: \mathcal{O}_Y \to f_*\mathcal{O}_X$. Taking global sections, we obtain a homomorphism

$$\alpha_Y: \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) \simeq \mathcal{O}_X(X).$$

One can check this defines a functor

$$\mathsf{Sch} \to \mathsf{CRing}^{\mathsf{op}},\ X \mapsto \mathcal{O}_X(X)$$

that sends a morphism $f: X \to Y$ to α_Y as above.

Theorem 7.1.2. *A scheme Y is affine iff for any scheme X, the natural map*

$$\mathsf{Hom}_{\mathsf{Sch}}(X, Y) \to \mathsf{Hom}_{\mathsf{CRing}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{7.1}$$

is a bijection.

Proof. We first prove the "only if" statement. Let $Y \simeq \mathsf{Spec}(R)$ be an affine scheme, we need to show the natural map

$$\mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \tag{7.2}$$

is a bijection.

We construct a map of the inverse direction as follows. Let $h: R \to \mathcal{O}_X(X)$ be a homomorphism. For any affine open subscheme $U \subseteq X$, by Proposition-Definition 5.1.8, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(U, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(U))$$

is a bijection. Let $f_U: U \to \mathsf{Spec}(R)$ be the morphism corresponding to the composition

$$R \xrightarrow{h} \mathcal{O}_X(X) \xrightarrow{(-)|_U} \mathcal{O}_X(U).$$

Unwinding the constructions, one can check for affine open subschemes $U \subseteq V \subseteq X$, we have $p_U = p_V|_U$. Recall affine open subsets form a base for the topology of X. Using Exercise 1.2.7, one can show there is a unique morphism $p: X \to \mathsf{Spec}(R)$ such that $p_U = p|_U$ for any affine open subscheme $U \subseteq X$. The construction $h \mapsto p$ as above gives a map

$$\mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \to \mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)). \tag{7.3}$$

One can check that (7.2) and (7.3) are inverse to each other. Namely, (7.3) $\circ$ (7.2) $=$ id follows from the uniqueness property about the morphism p, while (7.2) $\circ$ (7.3) $=$ id follows from the fact that an element in $\mathcal{O}_X(X)$ is determined by its restrictions in $\mathcal{O}_X(U)$'s.

Now we deduce the "if" statement from the "only if" one. Let Y be a scheme such that (7.1) is bijective for any scheme X. Write $R := \mathcal{O}_Y(Y)$. By the "if" statement, we have

$$\mathsf{Hom}_{\mathsf{Sch}}(Y, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, R).$$

In particular, there is a canonical morphism $q : Y \to \mathsf{Spec}(R)$ corresponding to id_R. Moreover, by construction, for any scheme X the following diagram commutes:

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Sch}}(X, Y) \arrow[r, "\simeq"] \arrow[d, "q\circ -"] & \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \arrow[d, equal] \\ \mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)) \arrow[r, "\simeq"] & \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)), \end{tikzcd}$$

where the horizontal maps are (7.1) applied to Y and $\mathsf{Spec}(R)$ respectively, and they are bijective either by the "if" statement or by assumption. It follows that composing with q induces a bijection

$$\mathsf{Hom}_{\mathsf{Sch}}(X, Y) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R))$$

for any $X \in \mathsf{Sch}$. By Yoneda lemma, q is an isomorphism and therefore Y is affine as desired.

□

7.2. **Applications.**

Corollary 7.2.1. *For any scheme X, there is a unique morphism $X \to \mathsf{Spec}(\mathbb{Z})$. In other words, $\mathsf{Spec}(\mathbb{Z}) \in \mathsf{Sch}$ is a final object.*

Corollary 7.2.2. *For any scheme X, there is a unique morphism*

$$q : X \to \mathsf{Spec}(\mathcal{O}_X(X))$$

that induces the identity homomorphism between global sections.

Corollary 7.2.3. *The embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ admits a canonical left adjoint given by*

$$\mathsf{Sch} \to \mathsf{Aff},\ X \mapsto \mathsf{Spec}(\mathcal{O}_X(X)).$$

Exercise 7.2.4. Show that the embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ does not admit a right adjoint.

8. Functor of points

The main goal of this section is to construct a fully faithful functor $\mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ and describe its essential image.

8.1. R-points.

Definition 8.1.1. Let X be a scheme and R be a commutative ring. An **R-point of X** is a morphism $\mathsf{Spec}(R) \to X$. Let

$$X(R) := \mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), X)$$

be the set of R-points of X.

A **field-valued point of** X is a k-point of X for some field k.

A **geometric point of** X is a k-point of X for some *separably closed* field k.

Proposition-Construction 8.1.2. *Let X be a scheme and k be a field. There is a bijection between*

- *the set $X(k)$ of k-points of X;*
- *the set of pairs (x, i), where $x \in X$ is a topological point*[11] *and $i : \kappa_x \to k$ is a homomorphism.*

Proof. By definition, a k-point of X is given by a pair (ϕ, β). Since $\mathsf{Spec}(k)$ has only one topological point, the continuous map ϕ is given by a topological point $x \in X$. Now the morphism β is given by a *local* homomorphism $\mathcal{O}_{X,x} \to k$, where we identify a sheaf $\mathcal{F}$ of commutative rings on the one-point space $\{*\}$ with $\mathcal{F}(\{*\})$. Now a local homomorphism $\mathcal{O}_{X,x} \to k$ uniquely factors as $\mathcal{O}_{X,x} \to \kappa_x \to k$. This gives the desired bijection.

□

Remark 8.1.3. For a field-valued point $s : \mathsf{Spec}(k) \to X$, we sometimes abuse notations and write it as $s \to X$, where s is understood as $\mathsf{Spec}(k)$. Similarly, for a topological point $x \in X$, we sometimes abuse notations and write the κ_x-point $\mathsf{Spec}(\kappa_s) \to X$ as $x \to X$.

Note however that a scheme is *not* determined by its field-valued points.

Exercise 8.1.4. Let k be a field, and consider $X := \mathsf{Spec}(k)$ and $X' := \mathsf{Spec}(k[\epsilon]/(\epsilon^2))$. Let $f : X \to X'$ be the morphism corresponding to the obvious homomorphism $k[\epsilon]/(\epsilon^2) \to k$. Show that f induces a bijection between the set of field-valued points of X and that of X'.

Construction 8.1.5. Let X be a scheme and $x \in X$ be a topological point. Let $U \subseteq X$ be an affine open subset containing x. Note that we have $U \simeq \mathsf{Spec}(\mathcal{O}_X(U))$. The canonical homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ induces a morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U))$. Now the composition

$$\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U)) \simeq U \to X$$

gives an $\mathcal{O}_{X,x}$-point of X.

Exercise 8.1.6. Show that:

(1) The above $\mathcal{O}_{X,x}$-point of X does not depend on the choice of U.
(2) The morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to X$ identifies the locally ringed space $\mathsf{Spec}(\mathcal{O}_{X,x})$ as the restriction of X to a certain subspace.

---

[11] A point in the underlying topological space.

8.2. **Schemes as a functor.**

Definition 8.2.1. Let X be a scheme. The functor

$$\mathsf{h}_X : \mathsf{CRing} \to \mathsf{Set},\ R \mapsto X(R)$$

is called the **functor of points of** X.

Note that the construction $X \mapsto \mathsf{h}_X$ is functorial. Hence we have a canonical functor

$$\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

Remark 8.2.2. By construction, the composition

$$\mathsf{CRing}^{\mathsf{op}} \simeq \mathsf{Aff} \to \mathsf{Sch} \xrightarrow{\mathsf{h}} \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

can be identified with the Yoneda embedding $R \mapsto \mathsf{Hom}_{\mathsf{CRing}}(R, -)$.

Theorem 8.2.3. *The functor* $\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ *is fully faithful.*

Remark 8.2.4. For a description of the essential image of this functor[12], see [EH00, Chapter VI].

Remark 8.2.5. The theorem suggests another way to develop scheme theory *without* using locally ringed spaces. Namely, one can *define* a scheme as a functor $\mathsf{CRing} \to \mathsf{Set}$ satisfying certain properties. In fact, in the 1970s, Grothendieck himself radically urged to abandon his earlier definition of schemes in favor of the functorial point of view. In my opinion, this approach at least has the following advantages.

- It makes a lot of constructions about schemes formal and therefore easier.
- It provides a more direct way to deal with moduli problems and deformation theory.
- It allows one to define more exotic geometric objects, such as algebraic spaces, stacks, indschemes...

No matter how much I love this functorial approach, however, I do *not* believe a learner should ignore the classical view of schemes as a structured topological space[13].

Definition 8.2.6. We say a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is **represented by a scheme** X if it is equipped with a natural isomorphism $F \simeq \mathsf{h}_X$.

Example 8.2.7. The functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to its underlying set is represented by the affine scheme $\mathbb{A}^1_{\mathbb{Z}} := \mathsf{Spec}(\mathbb{Z}[t])$.

Exercise 8.2.8. Show that the functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to the set $\mathsf{GL}_n(R)$ of $n \times n$ invertible matrices over R is represented by an affine scheme.

Exercise 8.2.9. Show that the constant functor $\mathsf{CRing} \to \mathsf{Set}$, $R \mapsto I$ is not represented by a scheme unless $I \simeq \{*\}$. What is the functor represented by the disjoint union $\bigsqcup_{i \in I} \mathsf{Spec}(\mathbb{Z})$?

[12] In standard terminology, a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is contained in the essential image iff F satisfies Zariski descents and admits an open covering by representable functors.

[13] Just imagine learning the projective space $\mathbb{P}^n_{\mathbb{Z}}$ for the first time using the following definition: it is the functor sending a commutative algebra R to the isomorphism classes of surjections from the free R-module $R^{\oplus(n+1)}$ to a rank 1 projective R-module P.

Proof of Theorem 8.2.3. For any pair of schemes (X, Y), we need to show

$$\mathsf{Hom}_{\mathsf{Sch}}(X, Y) \to \mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X, \mathsf{h}_Y) \tag{8.1}$$

is bijective.

We first construct a map of the inverse direction as follows. Let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation. For any affine open subscheme $U \subseteq X$, by definition we have identifications

$$\mathsf{Hom}_{\mathsf{Sch}}(U, Z) \simeq \mathsf{h}_Z(\mathcal{O}_X(U))$$

functorial in Z. Let $j_U \in \mathsf{h}_X(\mathcal{O}_X(U))$ be the element corresponding to the canonical immersion $U \to X$. Consider the morphism $f_U : U \to Y$ corresponding to the element $\theta(j_U) \in \mathsf{h}_Y(\mathcal{O}_X(U))$. One can check that for affine open subschemes $U \subseteq V \subseteq X$, we have $f_U = f_V|_U$. Using Exercise 1.2.7, one can show there is a unique morphism $f : X \to Y$ such that $f_U = f|_U$ for any affine open subscheme $U \subseteq X$. The construction $\theta \mapsto f$ as above gives a map

$$\mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X, \mathsf{h}_Y) \to \mathsf{Hom}_{\mathsf{Sch}}(X, Y). \tag{8.2}$$

It remains to check (8.1) and (8.2) are inverse to each other. Using the uniqueness property about the morphism f, it is easy to see $(8.2) \circ (8.1) = \mathsf{id}$.

It remains to show $(8.1) \circ (8.2) = \mathsf{id}$. For this, let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation and $f : X \to Y$ be the morphism constructed as above. Let $\mathsf{h}_f : \mathsf{h}_X \to \mathsf{h}_Y$ be the natural transformation induced by f via functoriality. We only need to show $\theta = \mathsf{h}_f$. In other words, for any R-point $x \in X(R)$, we need to show

$$\theta(x) = \mathsf{h}_f(x). \tag{8.3}$$

Note that by definition x is a morphism $x : \mathsf{Spec}(R) \to X$, and both sides in (8.3) are morphisms from $\mathsf{Spec}(R)$ to Y.

Unwinding the constructions, it is easy to see for any affine open subscheme $j_U : U \subseteq X$, we have

$$\theta \circ \mathsf{h}_{j_U} = \mathsf{h}_{\theta(j_U)} = \mathsf{h}_{f_U} = \mathsf{h}_f \circ \mathsf{h}_{j_U}.$$

In other words, we know (8.3) is true if x is contained in the image of $U(R) \to X(R)$ for some affine open subscheme U.

Now for general x, we can find a covering of $\mathsf{Spec}(R) = \bigcup_{i \in I} V_i$ by its affine open subschemes such that $x|_{V_i} : V_i \to X$ factors through some affine open subscheme of X. By the last paragraph, we see

$$\theta(x|_{V_i}) = \mathsf{h}_f(x|_{V_i}).$$

In other words, the restrictions of the morphisms $\theta(x)$ and $\mathsf{h}_f(x) : \mathsf{Spec}(R) \to Y$ to each V_i are equal. Now Exercise 1.2.7 implies these two morphisms are equal as desired.

□

Part III. Language of schemes

9. Fiber products

9.1. Definition of fiber products. Recall we have the notion of fiber products in any category.

Definition 9.1.1. Let $\mathcal{C}$ be a category. We say a commutative square in $\mathcal{C}$

$$\begin{tikzcd} d \arrow[r, "g'"] \arrow[d, "f'"] & a \arrow[d, "f"] \\ b \arrow[r, "g"] & c \end{tikzcd} \tag{9.1}$$

is **Cartesian**, if for any object $x \in \mathcal{C}$, the commutative square

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}}(x,d) \arrow[r, "g'\circ-"] \arrow[d, "f'\circ-"] & \mathsf{Hom}_{\mathcal{C}}(x,a) \arrow[d, "f\circ-"] \\ \mathsf{Hom}_{\mathcal{C}}(x,b) \arrow[r, "g\circ-"] & \mathsf{Hom}_{\mathcal{C}}(x,c) \end{tikzcd}$$

is a Cartesian square in Set. In other words, if it induces a bijection

$$\mathsf{Hom}_{\mathcal{C}}(x,d) \to \mathsf{Hom}_{\mathcal{C}}(x,a) \underset{\mathsf{Hom}_{\mathcal{C}}(x,c)}{\times} \mathsf{Hom}_{\mathcal{C}}(x,b).$$

In this case, we also say (9.1) is a **pullback square** and say (9.1) **exhibits** d **as the pullback of the diagram** $a \xrightarrow{f} c \xleftarrow{g} b$. We also say d is the **fiber product** of $a \xrightarrow{f} c \xleftarrow{g} b$.

Remark 9.1.2. By Yoneda's lemma, the object d, equipped with the morphisms f' and g', is essentially unique. We often write

$$d \simeq a \underset{c}{\times} b$$

when the morphisms are clear from the context.

Remark 9.1.3. By Yoneda's lemma, the construction

$$[a \xrightarrow{f} c \xleftarrow{g} b] \mapsto a \underset{c}{\times} b$$

is functorial. In other words, for a commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "p"] & c \arrow[d, "r"] & b \arrow[l] \arrow[d, "q"] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd},$$

there is a unique dotted morphism $a \times_c b \to a' \times_c{}' b'$ that make the following diagram commute

$$\begin{tikzcd} a \arrow[d, "p"] & a \times_c b \arrow[l] \arrow[r] \arrow[d, dotted] & b \arrow[d, "q"] \\ a' & a' \times_{c'} b' \arrow[l] \arrow[r] & b'. \end{tikzcd}$$

We often abuse notation and denote this morphism

$$(p,q) : a \underset{c}{\times} b \to a' \underset{c}{\times}' b'$$

(but it also depends on other morphisms in the diagram).

Example 9.1.4. If c is a final object, then $a \times_c b \simeq a \times b$.

Example 9.1.5. Fiber products exist in Ab and the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves fiber products. Given a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in Ab, we have

$$A \underset{C}{\times} B \simeq \mathsf{ker}(A \bigoplus B \xrightarrow{(f,-g)} C).$$

Example 9.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose fiber products exist in $\mathcal{C}$. Then fiber products exist in $\mathsf{Fun}(\mathcal{D}, \mathcal{C})$, and we have

$$(F_1 \underset{F_3}{\times} F_2)(d) \simeq F_1(d) \underset{F_3(d)}{\times} F_2(d)$$

for functors $F_i : \mathcal{D} \to \mathcal{C}$ and objects $d \in \mathcal{D}$.

Definition 9.1.7. Let $\mathcal{C}$ be a category. The **pushout of a diagram** $a \xleftarrow{f} c \xrightarrow{g} b$ is defined to be the pullback of the corresponding diagram in $\mathcal{C}^{\mathsf{op}}$. We also call it the **fiber coproduct of** $a \xleftarrow{f} c \xrightarrow{g} b$ and denote it by

$$a \coprod_c b.$$

9.2. **Fiber products of affine schemes.**

Exercise 9.2.1. Fiber coproducts exist in CRing and we have

$$A \coprod_C B \simeq A \underset{C}{\otimes} B.$$

Corollary 9.2.2. *Fiber products exist in* Aff. *Given a diagram* $A \leftarrow C \to B$, *we have*

$$\mathsf{Spec}(A \underset{C}{\otimes} B) \simeq \mathsf{Spec}(A) \underset{\mathsf{Spec}(C)}{\times} \mathsf{Spec}(B).$$

Warning 9.2.3. The underlying topological space of $\mathsf{Spec}(A \otimes_C B)$ is in general not the fiber product of the corresponding topological spaces. In other words, the forgetful functor $\mathsf{Aff} \to \mathsf{Top}$ does not preserve fiber products.

Exercise 9.2.4. Let $k \to k'$ be a finite *separable* extension of degree d and $\overline{k}$ be a algebraic closure of k. Show that

$$\mathsf{Spec}(k') \underset{\mathsf{Spec}(k)}{\times} \mathsf{Spec}(\overline{k}) \simeq \sqcup_d \mathsf{Spec}(\overline{k}).$$

Note that we also have the following formal corollary of Corollary 7.2.3:

Corollary 9.2.5. *The functor* $\mathsf{Aff} \to \mathsf{Sch}$ *preserves fiber products.*

Exercise 9.2.6. Let X be an affine scheme. Show that the intersection of two affine open subschemes of X is still an affine open subscheme.

9.3. **Fiber product of schemes.**

Theorem 9.3.1. *The category* Sch *admits fiber products.*

We will give a constructive proof of the theorem at the end of this section. For now, we prove a particular case of it.

Lemma 9.3.2. *Let* $f : X \to Y$ *be a morphism between schemes and* $U \subseteq Y$ *be an open subscheme. Then the fiber product* $X \times_Y U$ *exists, and the canonical morphism* $X \times_Y U \to X$ *is an open immersion onto the open subscheme* $f^{-1}(U)$*.*

Proof. We only need to show the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f'"] \arrow[d, "j'"] & U \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in Sch is Cartesian. Let $Z \in \mathsf{Sch}$ be a testing object. We only need to show the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{Sch}}(Z, f^{-1}(U)) \arrow[r, "f'\circ-"] \arrow[d, "j'\circ-"] & \mathrm{Hom}_{\mathsf{Sch}}(Z, U) \arrow[d, "j\circ-"] \\ \mathrm{Hom}_{\mathsf{Sch}}(Z, X) \arrow[r, "f\circ-"] & \mathrm{Hom}_{\mathsf{Sch}}(Z, Y) \end{tikzcd}$$

in Set is Cartesian. Using the definition of open subschemes, the vertical maps are injective, and a morphism $g : Z \to X$ (resp. $h : Z \to Y$) is contained in the image iff the subspace $g(Z)$ (resp. $h(Z)$) is contained in $f^{-1}(U)$ (resp. U). It follows that $g : Z \to X$ is contained in the image of the left vertical map iff $f \circ g$ is contained in the image of the right vertical map. In other words, the above square is Cartesian as desired.

□

Corollary 9.3.3. *Let* $X \to Y \xleftarrow{j} U$ *be a diagram in* Sch *such that* j *is an open immersion. Then the forgetful functor* $\mathsf{Sch} \to \mathsf{Top}$ *preserves the fiber product of this diagram.*

Exercise 9.3.4. Let X be a scheme.

(1) Show that the intersection of affine open subsets inside an affine open scheme X is still affine.
(2) Show that (1) may fail for general X.

Proposition 9.3.5. *A diagram in* Sch

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd} \tag{9.2}$$

is Cartesian iff it induces a Cartesian diagram

$$\begin{tikzcd} \mathsf{h}_W \arrow[r] \arrow[d] & \mathsf{h}_X \arrow[d] \\ \mathsf{h}_Y \arrow[r] & \mathsf{h}_Z \end{tikzcd} \tag{9.3}$$

in $\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})$. *In particular, we have*

$$\mathsf{h}_{X\times_Z Y} \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y. \tag{9.4}$$

Proof. For the "only if" claim, let (9.2) be a Cartesian square. By definition, for any $R \in \mathsf{CRing}$, the functor $\mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), -)$ sends (9.2) to a Cartesian square in Set. In other words, the values of the functors in (9.3) at $R \in \mathsf{CRing}$ form a Cartesian square in Set. This formally implies (9.3) itself is a Cartesian square as desired (Example 9.1.6).

Note that the "only if" claim itself implies the isomorphism (9.4).

For the "if" claim, let (9.2) be a commutative square such that (9.3) is Cartesian. By definition, (9.2) corresponds to a morphism $f : W \to X \times_Z Y$ and (7.3) corresponds to an isomorphism

$$\mathsf{h}_W \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y. \tag{9.5}$$

Moreover, unwinding the definitions, we see the following diagram commute

$$\begin{tikzcd} \mathsf{h}_W \arrow[r, "(9.5)"] \arrow[d, "\mathsf{h}_f"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y \arrow[d, equal] \\ \mathsf{h}_{X\times_Z Y} \arrow[r, "(9.4)"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y. \end{tikzcd}$$

It follows that h_f is also an isomorphism. By Theorem 8.2.3, f is an isomorphism as desired. □

9.4. **Base-change and fibers.**

Definition 9.4.1. Let S be a scheme. An **S-scheme is a scheme X equipped with a morphism** $X \to S$. When $S = \mathsf{Spec}(R)$, we also say X is a **R-scheme**.

Definition 9.4.2. Let X be an S-scheme. The **base-change of X along a morphism** $S' \to S$ is the S'-scheme

$$X_{S'} := X \underset{X}{\times} S'$$

equipped with its canonical projection to S'.

Let X be an S-scheme and $s \in S$ be a topological point. The **fiber of X at s** is the κ_s-scheme

$$X_s := X \underset{S}{\times} s := X \underset{S}{\times} \mathsf{Spec}(\kappa_s),$$

where $\mathsf{Spec}(\kappa_s) \to S$ is the canonical κ_s-point lying over s.

Remark 9.4.3. The reason for introducing the above terminology is to encourage the readers to view an S-scheme X as a *family of schemes over S*.

Exercise 9.4.4. Show that the base-change of an open immersion is still an open immersion. In other words, let $f : X \to Y$ be an open immersion between S-schemes and $S' \to S$ be any morphism. Then $f_{S'} : X_{S'} \to Y_{S'}$ is an open immersion, where f_U is the morphism

$$(f, \mathsf{id}_{S'}) : X \underset{S}{\times} S' \to Y \underset{S}{\times} S'.$$

Proposition 9.4.5. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. The* continuous map $X_s \to X$ *induces a homeomorphism $X_s \xrightarrow{\simeq} p^{-1}(s)$ between topological spaces.*

Remark 9.4.6. One can reform the proposition as: the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ preserves the fiber product of $X \to S \leftarrow s$, where s is understood as $\mathsf{Spec}(\kappa_s)$.

Proof of Proposition 9.4.5. Note that the continuous map $q : X_s \to X$ indeed factors through the subspace $p^{-1}(s) \subseteq X$. This follows from applying the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ to the following commutative diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathsf{Spec}(\kappa_s) \arrow[r] & S \end{tikzcd}$$

We first show $X_s \to p^{-1}(s)$ is surjective. Let $x \in p^{-1}(S)$ be a topological point. We have a commutative diagram in Sch

$$\begin{tikzcd} \mathsf{Spec}(\kappa_x) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathsf{Spec}(\kappa_s) \arrow[r] & S \end{tikzcd}$$

which by definition corresponds to a morphism $\mathsf{Spec}(\kappa_x) \to X_s$. By construction, the composition $\mathsf{Spec}(\kappa_x) \to X_s \to X$ is the canonical κ_x-point at $x \in X$. This shows $x \in p^{-1}(S)$ is in the image of the map $X_s \to p^{-1}(S)$.

Now we reduce to the case when S is affine. In other words, we will show the claim is *local in* S. To do this, let $U \subseteq S$ be an affine open subscheme containing the point s. Let $X_U := X \times_S U$ be the base-change of X to U. Consider the Cartesian squares

$$\begin{tikzcd} (X_U)_s \arrow[r] \arrow[d] & X_U \arrow[r, "j_X"] \arrow[d, "p_U"] & X \arrow[d, "p"] \\ s \arrow[r] & U \arrow[r, "j"] & S. \end{tikzcd}$$

It follows formally that the outer square is also Cartesian. In particular, we have

$$(X_U)_s \simeq X_s.$$

On the other hand, by Exercise 9.4.4, j_X is also an open immersion and its image is the open subset $p^{-1}(U) \subseteq X$. This implies j_X induces a homeomorphism

$$p_U^{-1}(s) \simeq p^{-1}(s).$$

Moreover, it is easy to see the diagram

$$\begin{tikzcd}
(X_U)_s \arrow[r] \arrow[d, "\simeq"] & p_U^{-1}(s) \arrow[d, "\simeq"] \\
X_s \arrow[r] & p^{-1}(s).
\end{tikzcd}$$

Hence to show the bottom horizontal map is a homeomorphism, we only need to show the top horizontal is a homeomorphism. This allows us to replace S with U (and therefore X with X_U) thereby assume S to be affine.

Now we reduce to the case when X is affine. In other words, we will show the claim is *local in* X. To do this, let $X = \bigcup_{i\in I} U_i$ be an open covering such that each U_i is affine. We only need to show the continuous map

$$X_s \cap q^{-1}(U_i) \to p^{-1}(s) \cap U_i$$

is a homeomorphism. Write p_i for the composition $U_i \to X \to S$. We have

$$p^{-1}(s) \cap U_i = p_i^{-1}(s)$$

as subspaces of U_i. On the other hand, by Lemma 9.3.2, $X_s \cap q^{-1}(U_i)$ is the underlying topological space of the fiber product

$$X_s \underset{X}{\times} U_i \simeq (s \underset{S}{\times} X) \underset{X}{\times} U_i \simeq s \underset{S}{\times} U_i \simeq (U_i)_s.$$

Hence we only need to show $(U_i)_s \to p_i^{-1}(s)$ is a homeomorphism. This allows us to replace X with U_i thereby assume X to be affine.

By the previous discussion, we can assume $S = \mathsf{Spec}(A)$ and $X = \mathsf{Spec}(B)$. Let $f : A \to B$ be the homomorphism corresponding to $p : X \to S$. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to the point $x \in S$. Recall $\kappa_{\mathfrak{p}} \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. By Corollary 9.2.2 and Corollary 9.2.5, we have

$$X_s \simeq \mathsf{Spec}(B \underset{A}{\otimes} \kappa_{\mathfrak{p}})$$

and the morphism $X_s \to X$ is induced by the homomorphism $B \simeq B \otimes_A A \to B \otimes_A \kappa_{\mathfrak{p}}$. We need to show $\mathsf{Spec}(B \otimes_A \kappa_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces a homeomorphism onto its image. This follows from the following exercise.

Exercise 9.4.7. Let $h : R \to R'$ be a homomorphism. Suppose any element $r' \in R'$ can be written as $h(r)u$ for some $r \in R$ and *invertible* element $u \in R'$. Then $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ induces a homeomorphism onto its image.

Namely, any element in $B \otimes_A \kappa_{\mathfrak{p}}$ can be written as

$$\sum_{i\in I} b_i \otimes \overline{a_i/c} = \Big((\sum_{i\in I} b_i f(a_i)) \otimes 1 \Big) \cdot (1 \otimes \overline{1/c})$$

where $a_i \in A$, $b_i \in B$, $c \in A \smallsetminus \mathfrak{p}$, and $\overline{b_i/c}$ is the image of b_i/c under the homomorphism $A_{\mathfrak{p}} \to \kappa_{\mathfrak{p}}$. Note that $(1 \otimes \overline{1/c})$ is invertible as desired.

□[Proposition 9.4.5]

Warning 9.4.8. In Proposition 9.4.5, X_s is in general *not* isomorphic to the locally ringed subspace $(p^{-1}(s), \mathcal{O}_X|_{p^{-1}(s)})$ associated to $p^{-1}(s)$. See Warning 5.3.6.

In particular, for a topological points $x \in X_s$, the homomorphism $\mathcal{O}_{X,i(x)} \to \mathcal{O}_{X_s,x}$ is in general not an isomorphism, where $i : X_s \to X$ is the canonical morphism.

That said, we have the following result:

Proposition 9.4.9. *The morphism $i : X_s \to X$ induces an isomorphism $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$ for any topological point $x \in X_s$.*

Proof. We claim i induces a bijection between the following sets:

- The set of field-valued points of X_s;
- The set of field-valued points of X contained in $p^{-1}(s)$.

To prove the claim, let k be any field. By the definition of fiber,

$$X_s(k) \simeq X(k) \underset{S(k)}{\times} s(k).$$

Using Proposition-Construction 8.1.2, it is easy to see $s(k) \to S(k)$ is injective, and its image contains k-points of S lying over s. It follows that $X_s(k) \to X(k)$ is also injective, and its image contains k-points of X whose underlying topological point is sent to s by p. This proves the desired claim.

Since the underlying topological space of X_s is homeomorphic to $p^{-1}(s)$, the claim implies i induces a bijection between the following sets:

- The set of field-valued points of X_s lying over x;
- The set of field-valued points of X lying over $i(x)$.

Applying Proposition-Construction 8.1.2 again, we see $\kappa_{i(x)} \to \kappa_x$ induces a bijection between the sets of field extensions of $\kappa_{i(x)}$ and κ_x. This is possible only if $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$.

□

Exercise 9.4.10. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram of schemes. Show that the morphism

$$\bigsqcup_{(x,y,s)} \mathsf{Spec}(\kappa_x) \underset{\mathsf{Spec}(\kappa_s)}{\times} \mathsf{Spec}(\kappa_y) \to X \underset{S}{\times} Y$$

induces a bijection between topological points. Here $x \in X$, $y \in Y$ and $s \in S$ are topological points such that $f(x) = g(y) = s$.

We also have the following result about base-change along $\mathsf{Spec}(\mathcal{O}_{S,s}) \to S$.

Proposition 9.4.11. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. Write $S' \coloneqq \mathsf{Spec}(\mathcal{O}_{S,s})$. The* morphism

$$X_{S'} \coloneqq X \underset{S}{\times} S' \to X$$

induces an isomorphism between locally ringed spaces

$$X_{S'} \xrightarrow{\simeq} p^{-1}(S').$$

In particular, the locally ringed subspace $p^{-1}(S')$ of X is a scheme.

Proof. As in the proof of Proposition 9.4.5, we have a surjective morphism $X_{S'} \to p^{-1}(S')$ between locally ringed spaces, and we can assume X and S are affine. Let $f : A \to B$ and $\mathfrak{p} \subset A$ be as in the proof of Proposition 9.4.5. We need to show $\phi : \mathsf{Spec}(B \otimes_A A_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces an isomorphism

$$\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}}) \xrightarrow{\simeq} \phi(\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}})),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$. This follows from the following exercise:

Exercise 9.4.12. Let R be a commutative ring and $S \subseteq R$ be a multiplicative subset. Consider the canonical homomorphism $R \to R[S^{-1}]$. Show that $\phi : \mathsf{Spec}(R[S^{-1}]) \to \mathsf{Spec}(R)$ induces an isomorphism

$$\mathsf{Spec}(R[S^{-1}]) \xrightarrow{\simeq} \phi(\mathsf{Spec}(R)),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$.

□

9.5. **Existence of fiber products.** In this subsection, we prove Theorem 9.3.1. We first deduce the theorem from the following lemma.

Lemma 9.5.1. *Let $X \to S \leftarrow Y$ be a diagram in* Sch*. Let $X = \bigcup_{i \in I} X_i$ and $Y = \bigcup_{j \in J} Y_j$ be coverings by open subschemes. Suppose for each pair $(i,j) \in I \times J$, the fiber product $X_i \times_S Y_j$ exists in* Sch*. Then $X \times_S Y$ exists.*

Proof of Theorem 9.3.1. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram in Sch. We will show $X \times_S Y$ exists.

We first reduce to the case when S is affine. Let $S = \bigcup_{i \in I} S_i$ be an open covering by affine open subschemes. For each $i \in I$, let $X_i := f^{-1}(S_i) \subseteq X$ and $Y_i := g^{-1}(S_i)$ be the corresponding open subschemes. By Lemma 9.5.1, we only need to show $X_i \times_S Y_j$ exists for any pair $(i,j) \in I^2$. By Lemma 9.3.2, we have the following *Cartesian* square

$$\begin{tikzcd} & Y_i \cap Y_j \arrow[r] \arrow[d] & Y_j \arrow[d] \\ X_i \arrow[r] & S_i \arrow[r] & S. \end{tikzcd}$$

A diagram chasing shows that $X_i \times_S Y_j$ exists iff

$$X_i \underset{S_i}{\times} (Y_i \cap Y_j)$$

exists, and these two fiber products are canonically isomorphic. Note that S_i is affine by assumption. Hence we can reduce to the case when S is affine.

Apply Lemma 9.5.1 again, we can reduce to the case when X and Y are both affine. Now the claim follows from Corollary 9.2.5 and Corollary 9.2.2.

□[Theorem 9.3.1]

Sketch of Lemma 9.5.1. We will construct the desired fiber product using *gluing of schemes*. Write $P := I \times J$ and $W_\alpha := X_i \times_S Y_j$ for $\alpha = (i,j) \in P$. For $(i,k) \in I^2$, write $X_{ik} := X_i \cap X_k$ and similarly $Y_{jl} := Y_j \cap Y_l$.

For each pair $(\alpha, \beta) \in P^2$, we define an open subscheme $W_{\alpha\beta} \subseteq W_\alpha$ as follows. Write $\alpha = (i,j)$ and $\beta = (k,l)$. Since $X_{ik} \to X_i$ and $Y_{jl} \to Y_j$ are open subschemes, applying Lemma 9.3.2 twice, we see that

$$W_{\alpha\beta} := X_{ik} \underset{X_i}{\times} W_\alpha \underset{Y_j}{\times} Y_{jl}$$

exists and can be identified with an open subscheme of W_α. Note that we have a canonical isomorphism

$$\phi_{\alpha\beta} : W_{\alpha\beta} \simeq X_{ik} \underset{X_i}{\times} (X_i \underset{S}{\times} Y_j) \underset{Y_j}{\times} Y_{jl} \simeq X_{ik} \underset{S}{\times} Y_{jl} \simeq X_{ki} \underset{S}{\times} Y_{lj} \simeq W_{\beta\alpha}.$$

One can check

$$(P, (W_\alpha)_{\alpha\in P}, (W_{\alpha\beta})_{(\alpha,\beta)\in P^2}, (\phi_{\alpha\beta})_{(\alpha,\beta)\in P^2})$$

is a gluing data of schemes (Definition 6.1.1). Let

$$(W, (W'_\alpha)_{\alpha\in P}, (\varphi_\alpha)_{\alpha\in P})$$

be the gluing output.

Now we construct a canonical morphism $p : W \to X$. By Proposition 6.1.3, we only need to construct morphisms $p_\alpha : W_\alpha \to X$ such that

$$p_\alpha|_{W_{\alpha\beta}} = p_\beta|_{W_{\beta\alpha}} \circ \phi_{\alpha\beta}.$$

We declare p_α to be the composition

$$W_\alpha = X_i \underset{S}{\times} Y_j \to X_i \to X.$$

One can check the collcetion $(p_\alpha)_{\alpha\in P}$ satisfies the above equations. Therefore we obtain a unique morphism $p : W \to X$ such that $p|_{W'_\alpha} \circ \varphi_\alpha = p_\alpha$.

Similarly, we use Proposition 6.1.3 to construct a canonical morphism $q : W \to Y$. By construction, the following diagram commutes:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S. \end{tikzcd}$$

It remains to show this diagram is Cartesian. Let $Z \in \mathsf{Sch}$ be a testing object. We only need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z, -)$ sends the above diagram to a Cartesian square in Set. One can check this by applying Proposition 6.1.4.

□[Lemma 9.5.1]

10. Subschemes and immersions

We have studied open subschemes and open immersions in Sect. 5.3. In this section, we introduce general subschemes and immersions.

10.1. Monomorphisms and epimorphisms.

Recall in any category $\mathcal{C}$, we can define monomorphisms and epimorphisms.

Definition 10.1.1. A morphism $f : x \to y$ in $\mathcal{C}$ is a **monomorphism** if for any testing object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(z,x) \xrightarrow{f\circ -} \mathsf{Hom}_{\mathcal{C}}(z,y)$$

is an injection.

A morphism $f : x \to y$ in $\mathcal{C}$ is a **epimorphism** if the corresponding morphism in $\mathcal{C}^{\mathsf{op}}$ is a monomorphism, i.e., for any testing object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(y,z) \xrightarrow{-\circ f} \mathsf{Hom}_{\mathcal{C}}(x,z)$$

is an injection.

Example 10.1.2. In Set, monomorphisms are exactly injections, while epimorphisms are exactly surjections.

Exercise 10.1.3. Let $f : U \to X$ be an open immersion.

- Show that f is a monomorphism.
- Show that f is an epimorphism iff it is an isomorphism.

Warning 10.1.4. A morphism can simultaneously be a monomorphism and an epimorphism, but fail to be an isomorphism.

Exercise 10.1.5. Let $R \in \mathsf{CRing}$ and $f \in R$ be an element that is not a zero-divisor. Show that $R \to R_f$ is a monomorphism and an epimorphism in CRing.

Exercise 10.1.6. Show that the functor $\mathsf{Aff} \to \mathsf{Sch}$ sends monomorphisms to monomorphisms, but may fail to send epimorphisms to epimorphisms.

10.2. Digression: epimorphisms between sheaves.

Proposition 10.2.1. *Let X be a topological space and $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism in $\mathsf{Shv}(X,\mathsf{Set})$. The following conditions are equivalent:*

(i) *The morphism α is an epimorphism in $\mathsf{Shv}(X,\mathsf{Set})$.*
(ii) *For any point $x \in X$, the map $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a surjection.*

Proof. (i)⇒(ii): let α be an epimorphism in $\mathsf{Shv}(X,\mathsf{Set})$. Suppose α_x is not a surjection for some point $x \in X$. We can find a set A and maps $f,g : \mathcal{F}'_x \to A$ such that $f \neq g$ but $f \circ \alpha_x = g \circ \alpha_x$. By Proposition 2.3.4, f,g correspond to morphisms $\phi,\varphi : \mathcal{F}' \to \delta_{x,A}$ such that $\phi \neq \varphi$ but $\phi \circ \alpha = \varphi \circ \alpha$. But this contradicts the assumption that α is an epimorphism.

(ii)⇒(i): suppose α is a morphism that induces surjections between stalks. We will show α is an epimorphism. Let $\phi,\varphi : \mathcal{F}' \to \mathcal{F}''$ be morphisms in $\mathsf{Shv}(X,\mathsf{Set})$ such that $\phi \circ \alpha = \varphi \circ \alpha$. We only need to show $\phi_U = \varphi_U : \mathcal{F}'(U) \to \mathcal{F}''(U)$ for any open subset $U \subseteq X$. For any morphism $\psi : \mathcal{F}' \to \mathcal{F}''$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}'(U) \arrow[r] \arrow[d, "\psi_U"] & \prod_{x\in U} \mathcal{F}'_x \arrow[d, "(\psi_x)_{x\in U}"] \\ \mathcal{F}''(U) \arrow[r] & \prod_{x\in U} \mathcal{F}''_x \end{tikzcd}$$

such that the horizontal arrows are injections (Lemma 2.2.1). Hence to show $\phi_U = \varphi_U$, we only need to show $\phi_x = \varphi_x$ for any $x \in U$. However, this follows from $\phi_x \circ \alpha_x = \varphi_x \circ \alpha_x$ and the assumption that α_x is a surjection.

□

Warning 10.2.2. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$. The map $\alpha_X : \mathcal{F}(X) \to \mathcal{F}'(X)$ is in general not a surjection.

10.3. Closed immersions.

Definition 10.3.1. Let $i : Y \to X$ be a morphism between schemes. We say i is a **closed immersion** if

(1) It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;
(2) The morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(Y, \mathsf{Set})$.

Let X be a scheme. A **closed subscheme** is an isomorphism class of closed immersions into X.

Proposition 10.3.2. *a morphism $i : Y \to X$ between schemes is a closed immersion iff*

(1) *It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;*
(2') *For any point $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection.*

Proof. We only need to show conditions (2) and (2') are equivalent if (1) holds. Under the latter assumption, it is easy to see:

- If $x \notin i(Y)$, $(i_*\mathcal{O}_Y)_x \simeq 0$;
- If $x = i(y)$, $(i_*\mathcal{O}_Y)_x \simeq \mathcal{O}_{Y,y}$.

Now the claim follows from Proposition 10.2.1.

□

Remark 10.3.3. By Proposition 10.2.1, we can also replace condition (2') by

(2") The morphism $i^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$.

Proposition 10.3.4. *Let $R \in \mathsf{CRing}$ and $I \subseteq R$ be an ideal. Let $i : \mathsf{Spec}(R/I) \to \mathsf{Spec}(R)$ be the morphism corresponding to the canonical surjection $\pi : R \to R/I$. Then i is a closed immersion.*

Proof. It is easy to see i induces a continuous bijective map from $\mathsf{Spec}(R/I)$ to the closed subspace $Z(I)$ of $\mathsf{Spec}(R)$. Moreover, for any standard open subset $U(\overline{f}) \in \mathsf{Spec}(R/I)$, $\overline{f} \in R/I$, its image is equal to the open subset $U(f) \cap Z(I) \subseteq Z(I)$, where $f \in R$ is any lifting of $\overline{f}$. This implies i induces a homeomorphism from $\mathsf{Spec}(R/I)$ to $Z(I)$.

Let $y \in \mathsf{Spec}(R/I)$ be a topological point and $\overline{\mathfrak{p}} \subseteq R/I$ be the corresponding prime ideal. By definition, $i(y)$ corresponds to the prime ideal $\mathfrak{p} := \pi^{-1}(\overline{\mathfrak{p}})$ and we have

$\overline{\mathfrak{p}} = \mathfrak{p}/I$. The homomorphism $\mathcal{O}_{\mathsf{Spec}(R),i(y)} \to \mathcal{O}_{\mathsf{Spec}(R/I),y}$ can be identified with $R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}/I}$, which is obviously surjective.

□

Example 10.3.5. Let k be a field. Consider the closed immersion $i : \mathbb{A}^1_k \to \mathbb{A}^2_k$ corresponding to the surjection $k[x,y] \to k[x]$. Show that $\mathcal{O}_{\mathbb{A}^2_k}(U) \to (i_*\mathcal{O}_{\mathbb{A}^1_k})(U)$ is not surjective for general open subset $U \subseteq \mathbb{A}^2_k$.

Proposition 10.3.6. *A closed immersion $i : Y \to X$ is a monomorphism in* Sch.

Proof. Let Z be any testing scheme. We need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z,Y) \to \mathsf{Hom}_{\mathsf{Sch}}(Z,X)$ is injective.

Suppose $f,g : Z \to Y$ are morphisms such that $i \circ f = i \circ g$. It is clear that the underlying continuous maps of f and g are equal. Write ϕ for this continuous map. Now the morphisms f and g are given by morphisms

$$\alpha, \beta : \mathcal{O}_Y \to \phi_*\mathcal{O}_Z.$$

We only need to show $\alpha = \beta$.

Let $\gamma : \mathcal{O}_X \to i_*\mathcal{O}_Y$ be the canonical morphism. The assumption $i \circ f = i \circ g$ implies

$$i_*(\alpha) \circ \gamma = i_*(\beta) \circ \gamma : \mathcal{O}_X \to i_* \circ \phi_*\mathcal{O}_X$$

Since γ is an epimorphism, we obtain

$$i_*(\alpha) = i_*(\beta)$$

In particular

$$i^{-1} \circ i_*(\alpha) = i^{-1} \circ i_*(\alpha).$$

Now the desired claim follows from the fact that $i^{-1} \circ i_* \simeq \mathrm{Id}$, which can be checked by unwinding the definitions.

□

References

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*. Springer, 2000.