# LECTURE 5

In this lecture, we define the quasi-category $\mathcal{QC}at$ of small quasi-categories, which models the $\infty$-category $\mathsf{Cat}_\infty$ of small $\infty$-categories.

## 1. Idea of the construction

1.1. Last time, for quasi-categories $\mathcal{C}$ and $\mathcal{D}$, we constructed the quasi-category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ of functors between them. Let $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$ be the core Kan complex of this quasi-category, which can be viewed as obtained from $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ by discarding all non-invertible natural transformations.

1.2. Using these Kan complexes, we can enrich the *ordinary* cateogry $\mathsf{QCat}$ over $\mathsf{Set}_\Delta$, and obtain a simplicial (enriched) category $\mathbb{QC}\mathbbm{at}$. The homotopy hypothesis suggests $\mathbb{QC}\mathbbm{at}$ is already a model for $\mathsf{Cat}_\infty$.

1.3. To translate this into rigorous mathematics, we will construct a Quillen equivalence between model categories

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

between the Joyal model category of simplicial sets and the standard model category of small simplicial categories. The simplicial category $\mathbb{QC}\mathbbm{at}$ is a fibrant object in[1] $\mathsf{Cat}_\Delta$ because it is enriched over fibrant objects in $\mathsf{Set}_\Delta$. The Joyal model structure is designed such that (bi)fibrant objects are exactly quasi-categories. Therefore

$$\mathcal{QC}at \overset{\text{def}}{=} \mathfrak{N}_\bullet(\mathbb{QC}\mathbbm{at})$$

is a fibrant object in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, i.e., a quasi-category.

## 2. Homotopy category of quasi-categories

2.1. The *homotopy* category of the desired quasi-category $\mathcal{QC}at$, in the sense of [Lecture 4, Definition 4.2], can be easily defined. It will also be the homotopy category of the desired model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, in the sense of [Lecture 2, Definition 2.19]. It models the homotopy category $\mathsf{hCat}_\infty$ of the desired $\infty$-category $\mathsf{Cat}_\infty$.

**Definition 2.2.** *Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be functors between quasi-categories. We call a morphism $\alpha : F_1 \to F_2$ in the quasi-category $\mathsf{Fun}(\mathcal{C},\mathcal{D})$ a* ***natural transformation*** *from $F_1$ to $F_2$.*

*We say $\alpha$ is* ***invertible****, or is an* ***equivalence****, if it is an isomorphism in $\mathsf{Fun}(\mathcal{C},\mathcal{D})$.*

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[1]Strictly speaking, $\mathbb{QC}\mathbbm{at}$ is not an object in $\mathsf{Cat}_\Delta$ because it is not small. Hence we need to replace the *large* model category $\mathsf{Cat}_\Delta$ of *small* simplicial categories by the *very large* model category $\widetilde{\mathsf{Cat}}_\Delta$ of *large* simplicial categories, and similarly replace $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ by the *very large* Joyal model category $\widetilde{\mathsf{Set}}_\Delta^{\mathsf{Joyal}}$ of large simplicial sets. Here one can use *Grothendieck universes* to give precise meanings to the above size conditions. We will ignore these set-theoretic issues until we encounter really problems about them.

**Theorem 2.3** (Ker.01DK). *Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be functors between quasi-categories. A natural transformation $\alpha : F_1 \to F_2$ is invertible iff its value at any object $x \in \mathcal{C}$ is an isomorphism, i.e. $\alpha(x) : F_1(x) \xrightarrow{\simeq} F_2(x)$.*

**Exercise 2.4.** *Show that the following construction defines an ordinary category $\mathsf{h}\mathcal{QC}at$:*

- *Objects of $\mathsf{h}\mathcal{QC}at$ are small quasi-categories;*
- *Morphisms $\mathsf{h}\mathcal{QC}at$ are equivalence classes of functors between quasi-categories:*
$$\mathsf{Hom}_{\mathsf{h}\mathcal{QC}at}(\mathcal{C}, \mathcal{D}) \overset{\text{def}}{=} \pi_0(\mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$$
- *Composition is given by $[G] \circ [F] \overset{\text{def}}{=} [G \circ F]$.*

## 3. Equivalences between quasi-categories

3.1. Note that we have a functor $\mathsf{QCat} \to \mathsf{h}\mathcal{QC}at$. Similar to [Lecture 4, Definition 5.1], we make the following definition.

**Definition 3.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories, i.e. a morphism in $\mathsf{QCat}$. We say $F$ is an **equivalence** if its image $[F]$ in $\mathsf{h}\mathcal{QC}at$ is an isomorphism.*

**Remark 3.3.** *In other words, $F : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there exists a functor $G : \mathcal{D} \to \mathcal{X}$ such that $G \circ F$ is equivalent to $\mathsf{Id}_{\mathcal{C}}$ and $F \circ G$ is equivalent to $\mathsf{Id}_{\mathcal{D}}$.*

**Theorem 3.4** (Ker.01JX). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories. Then $F$ is an equivalence iff*

(1) *$F$ is **fully faithful**: for any objects $x, y \in \mathcal{C}$, the functor $\mathsf{Maps}_{\mathcal{C}}(x, y) \to \mathsf{Maps}_{\mathcal{D}}(Fx, Fy)$ is an equivalence between $\infty$-groupoids.*
(2) *$F$ is **essentially surjective**: the map $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ is surjective. In other words, for any object $d \in \mathcal{D}$, there exist $c \in \mathcal{C}$ and an isomorphism $F(c) \xrightarrow{\simeq} d$.*

**Exercise 3.5.** *Being essentially surjective can be checked on the level of homotopy categories. Namely, $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective iff $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$ is so.*

**Exercise 3.6.** *Being fully faithful* cannot *be checked on the level of homotopy categories.*

**Remark 3.7.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories. So far we have at least four equivalence-like conditions on $F$:*

(a) *$F$ is an* isomorphism *in $\mathsf{Set}_{\Delta}$.*
(b) *$F$ is an* equivalence *between quasi-categories.*
(c) *$F$ is a* weak homotopy equivalence *in the Kan–Quillen model category $\mathsf{Set}_{\Delta}^{\mathsf{KQ}}$.*
(d) *$F$ induces an* equivalence between ordinary categorie*: $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$.*

*The relations between these notions are*

$$\begin{tikzcd} (a) \arrow[r, Rightarrow] & (b) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (c) \\ & (d). & \end{tikzcd}$$

**Exercise 3.8.** *The above implications are not inverible.*

1. $\mathsf{N}_\bullet(\{\bullet \rightleftarrows \bullet\}) \to \Delta^0$ *satisfies (b) but not (a).*
2. $\Delta^1 \to \Delta^0$ *satisfies (c) but not (b) or (d).*
3. $\mathsf{Sing}(S^2) \to \Delta^0$ *satisfies (d) but not (b) or (c).*

**Remark 3.9.** *Let* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *be the functor between* $\infty$*-categories modeled by* $F : \mathcal{C} \to \mathcal{D}$*. The above conditions tranlate into:*

(a) *This condition does not make sense because it is evil.*
(b) $\mathsf{F}$ *is an* equivalence *between* $\infty$*-categories.*
(c) $\mathsf{F}$ becomes an equivalence beween $\infty$-groupoids *after formally inverting all morphisms in* $\mathsf{C}$ *and* $\mathsf{D}$*.*
(d) $\mathsf{F}$ becomes an equivalence between ordinary categories *after formally forcing all non-invertible higher morphisms in* $\mathsf{C}$ *and* $\mathsf{D}$ *to be identities.*

**Remark 3.10.** *Let* $\mathsf{C}$ *be an* $\infty$*-category. We denote* $\mathsf{C}^\sharp$ *the* $\infty$*-groupoid obtained from* $\mathsf{C}$ *by formally inverting all morphisms. Note that this is* not *the core* $\infty$*-groupoid* $\mathsf{C}^\simeq$*, which is obtained by* discarding *all non-invertible morphisms (see [Lecture 4, Exercise 5.3]). We have*

$$\mathsf{C}^\simeq \to \mathsf{C} \to \mathsf{C}^\sharp.$$

## 4. Joyal model structure

4.1. Morphisms in the class (W) of $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ will be called *categorical equivalences*[2] between simplicial sets. When restricted to bifibrant objects, i.e. quasi-categories, they are supposed to model equivalences between the underlying $\infty$-categories. In other words:

> *Categorical equivalences between quasi-categories should be equivalences, in the sense of Definition 3.2.*

Hence the main task is to define categorical equivalences between *general* simplicial sets.

4.2. Recall when defining the classical model structure on $\mathsf{Set}_\Delta$ (see [Lecture 3, Theorem-Definition 6.2]), we declare weak homotopy equivalences between simplicial sets to be maps $f : X \to Y$ such that $|f| : |X| \to |Y|$ are weak homotopy equivalences in $\mathsf{Top}$.

It is possible to define categorical equivalences in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ in a similar manner. Namely, we can first define weak equivalences in $\mathsf{Cat}_\Delta$ and then transfer them via the functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$[3]. However, I find the construction of the functor $\mathfrak{C}$ not intuitive enough to provide actual feelings about categorical equivalences. Therefore we will follow Joyal's definition.

[2]Joyal called them *weak categorical equivalences*. We follow Lurie's terminology.

[3]This was the approach adopted in Lurie's HTT.2.2.5, which differs from that in Joyal's works (see [Joy08]). See HTT.2.2.5.9 and HTT.2.2.5.10 for more information.

4.3. Joyal's definition is motivated by the following observation:

**Exercise 4.4.** *Let $\mathcal{C}$ be a model category such that any object is cofibrant. Then a morphism $f : X \to Y$ belongs to (W) iff for any fibrant object $Z$, the map*

$$(\mathsf{Hom}_{\mathcal{C}}(Y,Z)/\sim) \to (\mathsf{Hom}_{\mathcal{C}}(X,Z)/\sim)$$

*is bijective. Here "$\sim$" stands for the homotopy relation between morphisms, see [Lecture 2, Proposition-Definition 2.18].*

*Hint: Yoneda lemma; the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ detects all weak equivalences.*

**Definition 4.5.** *Let $f : X \to Y$ be a morphism in $\mathsf{Set}_\Delta$. We say $f$ is a **categorical equivalence** if for any quasi-category $Z$, the map*

$$\pi_0(\mathsf{Fun}(Y,Z)^{\simeq}) \to \pi_0(\mathsf{Fun}(X,Z)^{\simeq})$$

*is bijective.*

**Proposition 4.6** ([Joy08, Corollary 2.29])**.** *Any inner horn inclusion $\Lambda^n_i \to \Delta^n$, $0 < i < n$ is a categorical equivalence.*

**Exercise 4.7.** *Convince yourself the above claim does* not *follow immediately from the definition of quasi-categories.*

**Exercise 4.8.** *Categorical equivalences are weak homotopy equivalences. Hint: [Lecture 3, §6.5].*

**Exercise 4.9** (Ker.01EG)**.** *Acyclic Kan fibrations are categorical equivalences.*

**Theorem-Definition 4.10** ([Joy08, Theorem 6.12])**.** *There exists a model structure on $\mathsf{Set}_\Delta$ given by*

- (W) *class of categorical equivalences;*
- (C) *class of monomorphisms;*
- (F) *class of morphisms satisfying the right lifting property against $(C \cap W)$.*

*We call it the **Joyal model structure** on $\mathsf{Set}_\Delta$. and denote this model category by $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.*

*Fibrant objects in this model category are exactly quasi-categories.*

4.11. The following result is tautological:

**Exercise 4.12.** *The functor $\mathsf{QCat} \to \mathsf{Set}_\Delta$ induces an equivalence*[4] $\mathsf{h}\mathcal{QC}at \to \mathsf{hSet}_\Delta^{\mathsf{Joyal}}$. *In particular, $\mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}]$ also models $\mathsf{hCat}_\infty$.*

**Definition 4.13.** *Fibrations in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ are called **categorical fibrations**.*

**Proposition 4.14.** *The identity functor gives a Quillen adjunction*

$$\mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}.$$

**Exercise 4.15.** *Let $\mathbb{L} : \mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}] \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}[W^{-1}] : \mathbb{R}$ be the derived adjunction of the above Quillen adjunction. Intepret it as an adjunction $\mathsf{hCat}_\infty \rightleftarrows \mathsf{hGrpd}_\infty$.*

[4]In fact, it is an *isomorphism*

**Remark 4.16.** *By definition, categorical fibrations $p : X \to Y$ in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ satisfy the right lifting properties against all inner horn inclusions. Morphisms satisfy these properties are called* ***inner fibrations****. Note that when $Y = \Delta^0$, inner fibrations over $Y$ are categorical fibrations. However, the inclusion*

$$\textit{inner fibrations} \subset \textit{categorical fibrations}$$

*is strict in general. For more information, see HTT.2.4.6.5.*

## 5. Simplicial categories

5.1. In this section, we define a model structure on the category $\mathsf{Cat}_\Delta$ of small simplicial categories.

5.2. Weak equivalences between simplicial categories are defined similarly as weak equivalences between topological categories. See [Lecture 2, §3].

**Definition 5.3.** *Let $\mathcal{C}$ be a simplicial category. Its* ***homotopy category*** *$\pi_0\mathcal{C}$ is defined by*

$$\mathsf{Ob}(\pi_0\mathcal{C}) \coloneqq \mathsf{Ob}(\mathcal{C}),\ \mathsf{Hom}_{\pi_0\mathcal{C}}(x,y) \coloneqq \pi_0\mathsf{Hom}_{\mathcal{C}}(x,y).$$

**Definition 5.4.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial categories. We say $F$ is a* ***weak equivalence*** *if:*

- *It induces an equivalence $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$.*
- *The morphism $\mathsf{Hom}_{\mathcal{C}}(x,y) \to \mathsf{Hom}_{\mathcal{D}}(Fx,Fy)$ is a weak equivalence in $\mathsf{Set}_\Delta^{\mathsf{KQ}}$ for any $x, y \in \mathsf{Ob}(\mathcal{C})$.*

**Theorem 5.5** (HTT.A.3.2)**.** *There exists a canonical model structure on $\mathsf{Cat}_\Delta$ such that:*

- *Weak equivalences are as in Defintion 5.4.*
- *Fibrant objects are exactly simplicial categories $\mathcal{C}$ such that $\mathsf{Hom}_{\mathcal{C}}(-,-)$ are Kan complexes.*

*We call it the* ***classical*** *or* ***Bergner model structure*** *on $\mathsf{Cat}_\Delta$.*

**Remark 5.6.** *It is not easy to describe fibrations and cofibrant objects in $\mathsf{Cat}_\Delta$, hence we will not do it. Note that [Lecture 2, Exercise 3.6] has a simplicial analogue, which suggests many naturally defined simplicial categories are not cofibrant.*

## 6. Quasi-categories and simplicial categories

6.1. In this section, we construct the desired Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet.$$

As in the construction of

$$|-| : \mathsf{Set}_\Delta^{\mathsf{KQ}} \rightleftarrows \mathsf{Top} : \mathsf{Sing},$$

we only need to find the correct definition for the restriction $\mathfrak{C}|_\Delta : \Delta \to \mathsf{Cat}_\Delta$ along the Yoneda embedding $\Delta \to \mathsf{Set}_\Delta \overset{\text{def}}{=} \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$. The entire functor $\mathfrak{C}$ will be the unique (up to unique equivalence) colimit-preserving functor that extends $\mathfrak{C}|_\Delta$, and the functor $\mathfrak{N}_\bullet$ will be its right adjoint.

6.2. In other words, we need to construct simplicial categories $\mathfrak{C}[\Delta^n] \in \mathsf{Cat}_\Delta$ equipped with face and degeneracy operators. To motivate this construction, we look at the following question:

> *Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. What is a functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$?*

Since we *expect* the model category $\mathsf{Cat}_\Delta$ to model $\mathsf{Cat}_\infty$, and since $\mathfrak{C}[\Delta^n]$ is cofibrant (because $\mathfrak{C}$ is expected to be a left Quillen functor), we obtain

> *A functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$ should model a functor $[n] \to \mathsf{C}$ between corresponding $\infty$-categories.*

6.3. For instance, for $n = 2$, it should be an invertible 2-morphism witnessing the following commutative diagram in $\mathsf{C}$:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z. \end{tikzcd}$$

Note that in the siplicial category $\mathcal{C}$, the composition is *concrete*: for 0-simplexes $f \in \mathsf{Hom}_{\mathcal{C}}(x,y)_0$ and $g \in \mathsf{Hom}_{\mathcal{C}}(y,z)_0$, we have a well-defined simplex $g \circ f \in \mathsf{Hom}_{\mathcal{C}}(x,z)_0$. Hence the above invertible 2-morpihsm corresponds to a 1-simplex $g \circ f \to h$[5] in $\mathsf{Hom}_{\mathcal{C}}(x,z)_1$.

Therefore we should define $\mathfrak{C}[\Delta^2]$ as

- It has three objects, labelled by $0, 1$ and $2$.
- The morphism simplicial sets are

$$\mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(i,j) \overset{\text{def}}{=\!=} \begin{cases} \varnothing & \text{if } i > j \\ \Delta^1 & \text{if } i = 0, j = 2 \\ \Delta^0 & \text{if otherwise} \end{cases}$$

- The only non-obvious composition is

$$\begin{aligned} \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,1) \times \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(1,2) &\to \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,2) \\ \Delta^0 \times \Delta^0 &\to \Delta^1, \end{aligned}$$

  which is given by the 0-simplex $0 \in \Delta^1$.

6.4. The above research can be conducted for general $n$. Namely, for any $\infty$-category, we *know* the correct meaning of a functor $[n] \to \mathsf{C}$, thanks to the fact that $[n]$ comes from a linearly ordered set[6]. For instance, 0-simplex in $\mathsf{Hom}_{\mathfrak{C}[\Delta^n]}(0,n)$ should be a subset $I \subset [n]$ containing $0$ and $n$; and each chain $I_0 \supset \cdots \supset I_m$ corresponds to a $m$-simplex.

In formula, we define:

**Definition 6.5.** *Let $J$ be any linearly ordered set. We define a simplicial category $\mathfrak{C}[\Delta^J]$ as follows:*

- *Objects are elements in $J$;*

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[5] One may also use $h \to g \circ f$. Thanks to the fibrant assumption about $\mathsf{Hom}_{\mathcal{C}}(x,z)$, the actual data are the same. But the obtained functor $\mathfrak{C}$ will differ by the involution $\mathsf{Set}_\Delta \to \mathsf{Set}_\Delta, X \mapsto X^{\mathsf{op}}$. Here we follow the conventions in Kerodon (see Ker.00KN), which is the opposite to HTT (see HTT.1.1.5.1.).

[6] On the other hand, homotopy coherent commutative diagrams discussed in [Lecture 1, Section 3] are beyond our reach, because they are not given by (partially) ordered set.

- $$\mathsf{Hom}_{\mathfrak{C}[\Delta^J]}(x,y) \overset{\text{def}}{=} \begin{cases} \varnothing & \textit{if } x > y \\ \mathsf{N}_\bullet(P_{x,y}) & \textit{if } x \le y \end{cases}$$
  *where* $P_{x,y} := \{I \subset [x,y] \mid x, y \in I\}$ *is equipped with the partial order* $\supset$, *and* $\mathsf{N}_\bullet(P_{x,y})$ *is the nerve of the corresponding category.*
- *Morphisms are induced by*
  $$(I \subset [x,y], I' \subset [y,z]) \mapsto (I \cup I' \subset [x,z])$$

6.6. The above construction is functorial in $J$ and therefore we have a functor

$$\Delta \to \mathsf{Cat}_\Delta,\ [n] \mapsto \mathfrak{C}[\Delta^n] \overset{\text{def}}{=} \mathfrak{C}[\Delta^{[n]}]$$

Note that $\mathfrak{C}[\Delta^n]$ is *not* fibrant. This can already be seen when $n = 2$.

Let

$$\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$$

be the (essentially) unique colimit-preserving functor that extends the above functor.

**Exercise 6.7.** *What is* $\mathfrak{C}[\Lambda^2_1]$*?*

**Theorem-Definition 6.8** (HTT.2.2.5.1, HTT.2.2.5.8)**.** *The functor* $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$ *has a right adjoint, called the* ***simplicial nerve functor***

$$\mathfrak{N}_\bullet : \mathsf{Cat}_\Delta \to \mathsf{Set}_\Delta.$$

*The adjoint pair*

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

*is a Quillen equivalence.*

6.9. Note that $\mathfrak{C}[\Delta^0] \simeq \{*\}$. Hence we may

- identify $X_0$ with $\mathsf{Ob}(\mathfrak{C}(X))$ for any simplicial set $X$;
- identify $\mathsf{Ob}(\mathcal{C})$ with $\mathfrak{N}_\bullet(\mathcal{C})_0$ for any simplicial category $\mathcal{C}$.

From now on, we will abuse notations by viewing them as the *same* sets.

6.10. To justify corresponding bifibrant objects in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ and $\mathsf{Cat}_\Delta$ indeed model the same $\infty$-category, we need the following results.

**Theorem 6.11** (Ker.01LE)**.** *Let* $\mathcal{C} \in \mathsf{Cat}_\Delta$ *be a fibrant simplicial category. Then there is a canonical weak homotopy equivalence between Kan complexes*

$$\mathsf{Hom}_{\mathcal{C}}(x,y) \xrightarrow{\simeq} \mathsf{Hom}^{\mathsf{L}}_{\mathfrak{N}_\bullet(\mathcal{C})}(x,y).$$

**Corollary 6.12.** *Let* $\mathcal{C} \in \mathsf{Cat}_\Delta$ *be a fibrant simplicial category. Then there is a canonical equivalence* $\mathsf{h}\mathcal{C} \xrightarrow{\simeq} \mathsf{h}\mathfrak{N}_\bullet(\mathcal{C})$[7].

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[7] In fact, this result is much more elementary. See Ker.00M4

## 7. The quasi-category of quasi-categories

7.1. This is a verbatim of §1.

**Definition 7.2.** *Let $\mathbb{QCat}$ be the simplicial category defined by:*

- *Objects are quasi-categories;*
- *For quasi-categories $\mathcal{C}$ and $\mathcal{D}$,*
$$\mathsf{Hom}_{\mathbb{QCat}}(\mathcal{C},\mathcal{D}) := \mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}.$$
- *The composition*
$$\mathsf{Fun}(\mathcal{C},\mathcal{D}) \times \mathsf{Fun}(\mathcal{D},\mathcal{E}) \to \mathsf{Fun}(\mathcal{C},\mathcal{E})$$
*is given by the unversal property of* $\mathsf{Fun}(-,-)$.

**Definition 7.3.** *The* ***quasi-category of small quasi-categories*** *is defined to be*
$$\mathcal{QC}at := \mathfrak{N}_\bullet(\mathbb{QCat}).$$
*Let* $\mathsf{Cat}_\infty$ *be the* $\infty$*-category modelled by it*[8]*. We call it the* $\infty$***-category of small*** $\infty$***-categories****.*

**Exercise 7.4.** *Let $\mathcal{C}$ and $\mathcal{D}$ be quasi-categories. There is a canonical weak homotopy equivalence between Kan complexes*
$$\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq} \xrightarrow{\simeq} \mathsf{Hom}^{\mathsf{L}}_{\mathcal{QC}at}(\mathcal{C},\mathcal{D}).$$
*In other words,* $\mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}$ *models the* $\infty$*-groupoids* $\mathsf{Maps}_{\mathsf{Cat}_\infty}(\mathsf{C},\mathsf{D})$ *as desired.*

**Variant 7.5.** *Similarly, we define the* ***quasi-category of small Kan complexes*** $\mathcal{K}an$*, which models the* $\infty$*-category* $\mathsf{Grpd}_\infty$ *of small* $\infty$*-groupoids.*

**Exercise 7.6.** *Show that the homotopy category of $\mathcal{QC}at$ is indeed equivalent to* $\mathsf{h}\mathcal{QC}at$ *in Exercise 2.4.*

## 8. Commutative diagrams

8.1. By previous discussion, in infinite category theory, for any simplicial set $S$, the *correct* notion of a commutative diagram of shape $S$ in a simplicial category $\mathcal{C}$ is a functor $\mathfrak{C}[S] \to \mathcal{C}$. We call such diagrams a **homotopy coherent commutative diagram** of shape $S$ in $\mathcal{C}$.

8.2. In most cases, the shape of the diagram is an ordinary category $\mathcal{D}$, we also abuse language and call a functor $\mathfrak{C}[\mathsf{N}_\bullet(\mathcal{D})] \to \mathcal{C}$ a **homotopy coherent commutative diagram** of shape $\mathcal{D}$ in $\mathcal{C}$. This is the notion that behaves well under standard infinite categorical constructions.

One can also ask for a diagram that **commutes on the nose**, which means a functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is viewed as enriched over discrete simplicial sets. In practice, it is hard to construct such diagrams, but most *non-formal* calculations eventually lead people to this realm[9].

One can also ask for a diagram that **commutes up to homotopy**, which means a functor $\mathcal{D} \to \mathsf{h}\mathcal{C}$. This is essentially a notion in ordinary category theory.

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[8] In rigorous mathematical words, we should *define* $\infty$-categories to be quasi-categories. However, I do want the readers to view quasi-categories as models for $\infty$-categories.

[9] My personal experience: *genuine* non-formal calculations are very few.

In summary:

$$\text{on the nose} \;\Rightarrow\; \text{homotopy-coherently} \;\Rightarrow\; \text{up to homotopy}\,.$$

**Exercise 8.3.** *Apply the above discussion to $\mathcal{C} := \mathbb{QCat}$ and obtain three notions of commutative diagram about quasi-categories.*

## Appendix A. $\mathsf{QCat}$ vs. $\mathcal{QC}at$

**Exercise A.1.** *Let $\mathcal{C}$ be an ordinary category. Construct an isomorphism $\mathsf{N}_\bullet(\mathcal{C}) \simeq \mathfrak{N}_\bullet(\mathcal{C})$, where in the RHS we view $\mathcal{C}$ as enriched over discrete simplicial sets.*

**Exercise A.2.** *Construct a functor $\mathsf{N}_\bullet(\mathsf{QCat}) \to \mathcal{QC}at$ that models $\mathsf{QCat} \to \mathsf{Cat}_\infty$.*

## References

[Joy08] André Joyal. The theory of quasi-categories and its applications. 2008.