

LECTURE 21

In this lecture, we introduce ∞ -operads and monoidal ∞ -categories.

1. SYMMETRIC MONOIDAL ∞ -CATEGORY

1.1. Recall a **monoidal category** is an (ordinary) category \mathbf{C} equipped with the following structure:

- A binary functor $- \otimes - : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$, called the **monoidal product**;
- An object $\mathbb{1} \in \mathbf{C}$, called the **monoidal unit**;
- An invertible natural transformation of the form $X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$, called the **associator**;
- Invertible natural transformations of the form $\mathbb{1} \otimes X \simeq X$, $X \otimes \mathbb{1} \simeq X$, called the **left and right unitors**

such that certain diagrams, including the pentagon diagram, commute.

Informally, we can say a monoidal category is a category equipped with a multiplication which is unital and associative *up to coherent homotopy*. Here the coherence data are finite because ordinary categories form a 2-category. In fact, the definition of monoidal categories only invokes n -ary operators $\mathbf{C}^{\times n} \rightarrow \mathbf{C}$ for $0 \leq n \leq 4$.

1.2. Also recall a **symmetric monoidal category** is a monoidal category \mathbf{C} equipped with:

- An invertible natural transformation of the form $X \otimes Y \simeq Y \otimes X$, called the **swap natural transformation**

such that certain diagrams, including the inverse law, commute.

1.3. We would like to generalize the above to a notion of (*symmetric*) *monoidal ∞ -categories*. Now the coherence data should invoke n -ary operators for all $n \geq 0$.

Instead of writing down such coherence data, we would like to encode them as a functor $F : J \rightarrow \mathbf{Cat}_\infty$, a.k.a. a homotopy coherent diagram of ∞ -categories, where J is a clever-designed simplicial set such that

- For $n \geq 0$, there is a vertex $\langle n \rangle$ which is sent by F to the n -th power $\mathbf{C}^{\times n}$;
- There is an edge $\langle 2 \rangle \rightarrow \langle 1 \rangle$ encoding the monoidal product;
- There is an edge $\langle 0 \rangle \rightarrow \langle 1 \rangle$ encoding the monoidal unit;
- The coherence data are encoded by cells in J .

Note that the first point is actually a *structure rather than property*: we need to provide projections $p_k : F(\langle n \rangle) \rightarrow F(\langle 1 \rangle)$ for each $1 \leq k \leq n$ that exhibit $F(\langle n \rangle)$ as the n -th power of $F(\langle 1 \rangle)$. Hence we should have:

- There is an edge $\langle n \rangle \rightarrow \langle 1 \rangle$ encoding the projection functor $\mathrm{pr}_i : \mathbf{C}^{\times n} \rightarrow \mathbf{C}$ for each $i \in \{1, \dots, n\}$.

Also, in the *symmetric* monoidal setting, to encode the swap natural transformations, we need to be able to swap the factors in $\mathbf{C}^{\times n}$. Hence we should have:

- For $n \geq 0$, the symmetric group Σ_n should act on $\langle n \rangle$.

In fact, the above two requirements can be combined together as:

- For each *injective* map $\phi : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, there is an edge from $\langle n \rangle \rightarrow \langle m \rangle$ encoding the projection functor $\text{pr}_\phi : \mathbb{C}^{\times n} \rightarrow \mathbb{C}^{\times m}$.

These edges in J will be called the **inert morphisms**. In contrast, edges encoding data such as the product and the unit are called the **active morphisms**.

Our intuition says:

- In the symmetric monoidal setting, there is a unique **active** morphism $\langle n \rangle \rightarrow \langle 1 \rangle$.
- In the monoidal setting, active morphisms $\langle n \rangle \rightarrow \langle 1 \rangle$ should be acted freely and transitively by Σ_n .

Exercise 1.4. Show that injective maps $\phi : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ are in natural bijection with maps between marked sets

$$\alpha : \{\ast, 1, 2, \dots, n\} \rightarrow \{\ast, 1, 2, \dots, m\}$$

such that $\alpha^{-1}(\{j\})$ is a singleton for any $1 \leq j \leq m$.

1.5. The above discussion motivates the following definition by Segal.

Definition 1.6. Let Fin_\ast be the minimal model for the ordinary category of marked finite sets. For each $n \geq 0$, let $\langle n \rangle \in \text{Fin}_\ast$ be the object

$$\langle n \rangle := \{\ast\} \sqcup \langle n \rangle^\circ := \{\ast, 1, 2, \dots, n\}.$$

Let $\alpha : \langle n \rangle \rightarrow \langle m \rangle$ be a morphism in Fin_\ast , we say

- (1) The morphism α is **inert** if $\alpha^{-1}(\{j\})$ is a singleton for any non-marked element $j \in \langle m \rangle^\circ$.
- (2) The morphism α is **active** if $\alpha^{-1}(\{\ast\}) = \{\ast\}$.

We also write

$$\text{Comm}^\otimes := \text{Fin}_\ast.$$

Exercise 1.7. A morphism in Comm^\otimes is an isomorphism iff it is both inert and active.

Exercise 1.8. Any morphism $\langle n \rangle \rightarrow \langle m \rangle$ can be written as $\langle n \rangle \xrightarrow{\beta} \langle l \rangle \xrightarrow{\gamma} \langle m \rangle$ with β inert and γ active. Moreover, such expression is unique up to unique isomorphism.

Definition 1.9. A **symmetric monoidal ∞ -category** is a functor

$$F : \text{Comm}^\otimes \rightarrow \text{Cat}_\infty$$

that sends the inert morphisms $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$, $i \in \langle n \rangle^\circ$ to functors

$$F(\rho^i) : F(\langle n \rangle) \rightarrow F(\langle 1 \rangle)$$

that exhibit $F(\langle n \rangle)$ as $F(\langle 1 \rangle)^{\times n}$. We call $F(\langle 1 \rangle)$ its **underlying ∞ -category**.

Exercise 1.10. If $F(\langle 1 \rangle)$ is ordinary, then F determines a symmetric monoidal category.

1.11. The non-symmetric case can be encoded by the following variant of Comm^\otimes .

Definition 1.12. Let Assoc^\otimes be the ordinary category defined as follows:

- An object in Assoc^\otimes is a marked finite set $\langle n \rangle$ with $n \geq 0$.
- A morphism from $\langle n \rangle$ to $\langle m \rangle$ is a pair (α, \leq_j) where
 - α is a map $\langle n \rangle \rightarrow \langle m \rangle$ between marked sets
 - For each $j \in \langle m \rangle^\circ$, \leq_j is a linear ordering on $\alpha^{-1}(j)$.

For a morphism (α, \leq_j) , we say it is inert (resp. active) if α is so.

Definition 1.13. A **monoidal ∞ -category** is a functor

$$F : \text{Assoc}^\otimes \rightarrow \text{Cat}_\infty$$

that sends the inert morphisms $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$, $i \in \langle n \rangle^\circ$ to functors

$$F(\rho^i) : F(\langle n \rangle) \rightarrow F(\langle 1 \rangle)$$

that exhibit $F(\langle n \rangle)$ as $F(\langle 1 \rangle)^{\times n}$. We call $F(\langle 1 \rangle)$ its **underlying ∞ -category**.

Exercise 1.14. Show that Exercise 1.7-1.10 remain valid for Assoc^\otimes instead of Fin_* .

Exercise 1.15. Show that for both Comm^\otimes and Assoc^\otimes , the group of automorphisms on $\langle n \rangle$ can be identified with Σ_n .

1.16. There is a functor $\text{Assoc}^\otimes \rightarrow \text{Comm}^\otimes$ that sends a symmetric monoidal ∞ -category to a monoidal one via restriction.

Warning 1.17. Although $\text{Assoc}^\otimes \rightarrow \text{Comm}^\otimes$ behaves like a quotient functor (it is surjective on both objects and morphisms), one should not think a symmetric monoidal ∞ -category as a monoidal one satisfying certain properties. Rather, even for a functor $\text{Assoc}^\otimes \rightarrow \mathcal{D}$ into a $(2,1)$ -category \mathcal{D} , such as that of ordinary categories, a weak factorization through Comm^\otimes is a structure rather than property.

Remark 1.18. In future lectures, we will encode non-symmetric monoidal ∞ -categories in a more efficient way, where the role of Assoc^\otimes is replaced by Δ^{op} . This is known as the simplicial model for monoids. Roughly speaking, this amounts to cancel out the Σ_n -action $\text{Hom}(\langle n \rangle, \langle 1 \rangle)$ from the data encoded by a functor $\text{Assoc}^\otimes \rightarrow \text{Cat}_\infty$.

Comparing with Assoc^\otimes , this simplicial model has several advantages. For instance, one can define \mathbb{A}_n -algebras using the truncation $(\Delta_{\leq n})^{\text{op}}$. However, it is hard to directly compare monoidal structures encoded by the simplicial model with the symmetric monoidal ones because the Σ_n -action is hidden.

2. ∞ -OPERADS

Exercise 2.1. Show that for both Comm^\otimes and Assoc^\otimes , the following data are equivalent:

- An active morphism $\alpha : \langle n \rangle \rightarrow \langle m \rangle$;
- A decomposition $\langle n \rangle^\circ \simeq \bigsqcup_{j \in \langle m \rangle^\circ} \langle n_j \rangle^\circ$ and active morphisms $\alpha_j : \langle n_j \rangle \rightarrow \langle 1 \rangle$ labelled by $j \in \langle m \rangle^\circ$.

Hint: for each α , consider the factorization of $\rho_j \circ \alpha$ in Exercise 1.8.

2.2. In fact, the categories \mathbf{Comm}^\otimes and \mathbf{Assoc}^\otimes can be recovered from the following data:

- (i) For each $n \geq 0$, the set $\mathbf{O}(n)$ of active morphisms $\langle n \rangle \rightarrow \langle 1 \rangle$;
- (ii) The action of Σ_n on $\mathbf{O}(n)$;
- (iii) For $m \geq 0$ and n_j labelled by $j \in \langle m \rangle^\circ$, a map

$$\mathbf{O}(m) \times \left(\prod_j \mathbf{O}(n_j) \right) \rightarrow \mathbf{O}(n_1 + \cdots + n_m)$$

given by the above exercise.

In the classical literatures, a collection of such data satisfying certain compatibilities is called a (*symmetric*) *operad*. Here $\mathbf{O}(n)$ is the set of n -ary operators and (iii) are called *composition maps*.

2.3. One can also consider *operads with enriched structures*. For instance, we have the notions of *simplicial operads*, *topological operads*, *k-linear operads*, *dg-operads*, using enrichment over the *ordinary* symmetric monoidal categories \mathbf{Set}_Δ , \mathbf{Top} , \mathbf{Mod}_k^∇ , $\mathbf{Ch}(\mathbf{Ab})$...

We would like to have a notion of operads *weakly* enriched over the ∞ -category \mathbf{Spc} . In other words, each $\mathbf{O}(n)$ should be a space, and the composition maps are associative up to coherent homotopies.

Following the general philosophy in this course, it is better to *define* an ∞ -operad as an ∞ -category satisfying certain conditions, such as \mathbf{Comm}^\otimes and \mathbf{Assoc}^\otimes , rather than listing all these coherence data.

2.4. Before giving the definition of ∞ -operads, let us conduct one more generalization.

Let \mathbf{A} be a symmetric monoidal ∞ -category and \mathbf{M} be an ∞ -category. We want to define an \mathbf{A} -action on \mathbf{M} via a functor $F : \mathbf{CMod}^\otimes \rightarrow \mathbf{Cat}_\infty$. The index category \mathbf{CMod}^\otimes should contain \mathbf{Comm}^\otimes as a full subcategory, and the remaining part should encode the \mathbf{A} -module structure on \mathbf{M} . Hence we should at least have:

- objects $(\mathbf{a}, \dots, \mathbf{a})$ which comes from $\langle n \rangle \in \mathbf{Assoc}^\otimes$ and should be sent to $\mathbf{A}^{\times n}$ by F ;
- objects $(\mathbf{a}, \dots, \mathbf{a}, \mathbf{m})$ which should be sent to $\mathbf{A}^{\times n} \times \mathbf{M}$ by F ;
- a unique active morphism $(\mathbf{a}, \dots, \mathbf{a}, \mathbf{m}) \rightarrow \mathbf{m}$ encoding the action functor $\mathbf{A}^{\times n} \times \mathbf{M} \rightarrow \mathbf{M}$.

Note that we should also have objects of the form $(\mathbf{m}, \mathbf{a}, \dots, \mathbf{a})$ and more generally $(\mathbf{a}, \dots, \mathbf{a}, \mathbf{m}, \mathbf{a}, \dots, \mathbf{a})$. For technical reasons, it is more convenient to even allow sequences with multiple \mathbf{m} -terms. Of course, we will not allow any *active* morphism from (\mathbf{m}, \mathbf{m}) to \mathbf{m} .

Definition 2.5. Let \mathbf{CMod}^\otimes be the category defined as follows:

- An object is a pair $(\langle n \rangle, c)$, where $\langle n \rangle \in \mathbf{Fin}_*$ and $c : \langle n \rangle^\circ \rightarrow \{\mathbf{a}, \mathbf{m}\}$ is a map.
- A morphism from $(\langle n \rangle, c)$ to $(\langle m \rangle, d)$ is a morphism $\alpha : \langle n \rangle \rightarrow \langle m \rangle$ in \mathbf{Comm}^\otimes such that
 - If $c(i) = \mathbf{m}$, then either $\alpha(i) = *$ or $d(\alpha(i)) = \mathbf{m}$.
 - If $d(j) = \mathbf{m}$, then there is a **unique** $i \in \alpha^{-1}(\{j\})$ with $c(i) = \mathbf{m}$.

For such a morphism, we say it is *inert* (resp. *active*) if the underlying morphism in \mathbf{Comm}^\otimes is so.

2.6. Similarly, we can define a category \mathbf{LMod}^\otimes encoding monoids and their left modules.

Definition 2.7. Let \mathbf{LMod}^\otimes be the category defined as follows:

- An object is a pair $(\langle n \rangle, c)$, where $\langle n \rangle \in \mathbf{Fin}_*$ and $c : \langle n \rangle^\circ \rightarrow \{\mathbf{a}, \mathbf{m}\}$ is a map.
- A morphism from $(\langle n \rangle, c)$ to $(\langle m \rangle, d)$ is a morphism $(\alpha, \leq_j) : \langle n \rangle \rightarrow \langle m \rangle$ in \mathbf{Assoc}^\otimes such that
 - If $c(i) = \mathbf{m}$, then either $\alpha(i) = *$ or $d(\alpha(i)) = \mathbf{m}$.
 - If $d(j) = \mathbf{m}$, then there is a **unique** $i \in \alpha^{-1}(\{j\})$ with $c(i) = \mathbf{m}$, which is also the maximal element for the ordering \leq_j .

For such a morphism, we say it is *inert* (resp. *active*) if the underlying morphism in \mathbf{Assoc}^\otimes is so.

Dually, we define \mathbf{RMod}^\otimes by replacing maximal by minimal.

2.8. Categories such as \mathbf{CMod}^\otimes , \mathbf{LMod}^\otimes or \mathbf{RMod}^\otimes are *colored operads*, and the symbols \mathbf{a}, \mathbf{m} are the *colors*.

Exercise 2.9. Construct an ordinary colored operad \mathbf{BMod}^\otimes with three colors $(\mathbf{a}_l, \mathbf{m}, \mathbf{a}_r)$ encoding two monoids and a bimodule of them.

Exercise 2.10. Let \mathbf{O}^\otimes be either \mathbf{Comm}^\otimes , \mathbf{Assoc}^\otimes , \mathbf{CMod}^\otimes , \mathbf{LMod}^\otimes , \mathbf{RMod}^\otimes or \mathbf{BMod}^\otimes . Consider the natural functor $p : \mathbf{O}^\otimes \rightarrow \mathbf{Fin}_*$. Show that

- (1) For any inert morphism in \mathbf{Fin}_* , there are enough p -coCartesian liftings of it. Moreover, a morphism in \mathbf{O}^\otimes is inert iff it is a p -coCartesian over an inert morphism in \mathbf{Fin}_* .
- (2) Let $C \in \mathbf{O}_{\langle n \rangle}^\otimes$ be an object lying over $\langle n \rangle$, and $C \rightarrow C_i$ be p -coCartesian morphisms lying over $\rho^i : \langle n \rangle \rightarrow \langle 1 \rangle$, $i \in \langle n \rangle^\circ$. Then these morphisms exhibit C as a p -limit¹. In other words, for any testing object $D \in \mathbf{O}^\otimes$, the following square is Cartesian

$$\begin{array}{ccc} \mathbf{Maps}_{\mathbf{O}^\otimes}(D, C) & \longrightarrow & \prod_i \mathbf{Maps}_{\mathbf{O}^\otimes}(D, C_i) \\ \downarrow & & \downarrow \\ \mathbf{Maps}_{\mathbf{Fin}_*}(p(D), \langle n \rangle) & \longrightarrow & \prod_i \mathbf{Maps}_{\mathbf{Fin}_*}(p(D), \langle 1 \rangle). \end{array}$$

- (3) The covariant transport functors ρ_{\dagger}^i exhibit $\mathbf{O}_{\langle n \rangle}^\otimes$ as $(\mathbf{O}_{\langle 1 \rangle}^\otimes)^{\times n}$.

Definition 2.11. An ∞ -operad² is an ∞ -category \mathbf{O}^\otimes over \mathbf{Fin}_* satisfying the conclusions in the above exercise³. We call

$$\mathbf{O} := \mathbf{O}_{\langle 1 \rangle}^\otimes$$

the ∞ -category of colors in \mathbf{O}^\otimes .

Definition 2.12. Let $p : \mathbf{O}^\otimes \rightarrow \mathbf{Fin}_*$ be an ∞ -operad and $f : C \rightarrow C'$ be a morphism in \mathbf{O}^\otimes , we say

- The morphism f is **inert** if it is p -coCartesian over an inert morphism in \mathbf{Fin}_* .
- The morphism f is **active** if $p(f)$ is active.

¹See HTT.4.3.2 for a discussion about relative limits in general.

²A better terminology might be *colored* ∞ -operad.

³More precisely, we should say a functor $p : \mathbf{O}^\otimes \rightarrow \mathbf{Fin}_*$ exhibits \mathbf{O}^\otimes as an essential ∞ -operad, if any/all realization $\mathcal{O}^\otimes \rightarrow \mathbf{N}(\mathbf{Fin}_*)$ of p as an inner fibration satisfies the above conditions.

2.13. We often denote an object $C \in \mathcal{O}_{\langle n \rangle}^{\otimes}$ by $(C_1, \dots, C_n)^4$ with $C_i \in \mathcal{O}$ and treat the inert morphisms $C \rightarrow C_i$ as implicit.

Exercise 2.14. A morphism in \mathcal{O}^{\otimes} is an isomorphism iff it is both inert and active.

Exercise 2.15. Show that any morphism in \mathcal{O}^{\otimes} can be essentially uniquely written as a composition of an inert morphism followed by an active one.

2.16. Let \mathcal{O}^{\otimes} be an ∞ -operad. We can define an \mathcal{O} -monoidal ∞ -category as a functor $\mathcal{O}^{\otimes} \rightarrow \mathbf{Cat}_{\infty}$ satisfying certain conditions. In practice, it is better to describe such conditions via the corresponding coCartesian fibration over \mathcal{O}^{\otimes} .

Definition 2.17. Let \mathcal{O}^{\otimes} be an ∞ -operad. An **\mathcal{O} -monoidal ∞ -category** is an (essential) coCartesian fibration $\mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes}$ such that the composition $\mathcal{C}^{\otimes} \rightarrow \mathcal{O}^{\otimes} \rightarrow \mathbf{Fin}_*$ exhibits \mathcal{C}^{\otimes} as an ∞ -operad. We call $\mathcal{C} := \mathcal{C}_{\langle 1 \rangle}^{\otimes}$ the **underlying ∞ -category** of \mathcal{C}^{\otimes} .

Definition 2.18. Let \mathcal{C}^{\otimes} and \mathcal{C}'^{\otimes} be \mathcal{O} -monoidal ∞ -categories. An **\mathcal{O} -monoidal functor** is a functor $\mathcal{C}^{\otimes} \rightarrow \mathcal{C}'^{\otimes}$ defined over \mathcal{O}^{\otimes} that preserves all coCartesian arrows. Let

$$\mathbf{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{C}') \subseteq \mathbf{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{C}'^{\otimes})$$

be the full sub- ∞ -category of \mathcal{O} -monoidal functors.

Let

$$\mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty}) \subseteq (\mathbf{Cat}_{\infty})_{/\mathcal{O}^{\otimes}}$$

be the 1-full sub- ∞ -category of small \mathcal{O} -monoidal ∞ -categories and \mathcal{O} -monoidal functors between them.

Exercise 2.19. Show that for $\mathcal{O}^{\otimes} = \mathbf{Comm}^{\otimes}$ or \mathbf{Assoc}^{\otimes} , the above definition recovers the notions of (symmetric) monoidal ∞ -categories and monoidal functors via straightening.

Exercise 2.20. What are the initial and final objects in $\mathbf{Mon}_{\mathcal{O}}(\mathbf{Cat}_{\infty})$?

3. ALGEBRAS

3.1. One advantage of defining \mathcal{O} -monoidal ∞ -categories via coCartesian fibrations is that the definition of algebras inside them is extremely simple.

Definition 3.2. Let \mathcal{O}^{\otimes} and \mathcal{C}^{\otimes} be ∞ -operads. An **∞ -operad map** between them is a functor $p: \mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$ defined over \mathbf{Fin}_* that preserves inert morphisms. We also call such a functor **an \mathcal{O} -algebra in \mathcal{C}** . Let

$$\mathbf{Op}_{\infty} \subseteq (\mathbf{Cat}_{\infty})_{/\mathbf{Fin}_*}$$

be the 1-full sub- ∞ -category consisting of small ∞ -operads and ∞ -operad maps between them.

Let

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{C}) \subseteq \mathbf{Fun}_{\mathbf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{C}^{\otimes})$$

be the full sub- ∞ -category of ∞ -operad maps $\mathcal{O}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$.

Exercise 3.3. What are the initial and final objects in \mathbf{Op}_{∞} ?

⁴Lurie's books use $C_1 \oplus \dots \oplus C_n$, which might cause confusion because C is *neither* the (absolute) product nor coproduct of C_i 's.

3.4. To define associative algebra in a category, we only need to equip the latter with a monoidal structure rather than a symmetric monoidal one. More generally, we have the following relative version of Definition 3.2.

Definition 3.5. *Let*

$$\begin{array}{ccc} \mathcal{O}'^\otimes & & \mathcal{C}^\otimes \\ & \searrow q & \swarrow \\ & \mathcal{O}^\otimes & \end{array}$$

be maps between ∞ -operads. An \mathcal{O}' -algebra in \mathcal{C} (relative to \mathcal{O}) is an object in

$$(3.1) \quad \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})_{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})}^{\times} \{q\}.$$

When q is the identity map on \mathcal{O}^\otimes , we also write

$$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}/\mathcal{O}}(\mathcal{C}).$$

Warning 3.6. *The ∞ -categories $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ are different.*

Remark 3.7. *If $\mathcal{C}^\otimes \rightarrow \mathcal{O}^\otimes$ is realized as a categorical fibration, then the fiber product (3.1) can be calculated as the naive fiber product.*

Exercise 3.8. *Show that*

$$(\mathcal{C} \times_{\mathcal{O}} \mathcal{O}')^\otimes := \mathcal{C}^\otimes \times_{\mathcal{O}^\otimes} \mathcal{O}'^\otimes$$

is an ∞ -operad, and

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{C} \times_{\mathcal{O}} \mathcal{O}').$$

Exercise 3.9. *Show that restriction along $\langle 1 \rangle \in \mathrm{Fin}_*$ gives a conservative functor*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \rightarrow \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}', \mathcal{C}),$$

*which is called the **forgetful functor**.*

Definition 3.10. *Let \mathcal{C}^\otimes and \mathcal{C}'^\otimes be \mathcal{O} -monoidal ∞ -categories. An object in $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{C}')$ is called a **(right) lax \mathcal{O} -monoidal functor** from \mathcal{C}^\otimes to \mathcal{C}'^\otimes*

Remark 3.11. *Show that for $\mathcal{O}^\otimes = \mathrm{Comm}^\otimes$ or Assoc^\otimes , the above definition generalizes the notions of lax (symmetric) monoidal functors between ordinary categories.*

4. UNITALITY

4.1. Note that for any

We say an ∞ -operad \mathcal{O}^\otimes is unital if there is an essentially unique nullary operator into any color $X \in \mathcal{O}$.

APPENDIX A. MONOIDAL ENVELOPE

A.1. By definition, any symmetric monoidal ∞ -category is an ∞ -operad. The converse is false because an ∞ -operad $\mathcal{C}^\otimes \rightarrow \mathrm{Fin}_*$ might not have enough coCartesian arrows.

Exercise A.2. *Let $\mathcal{C}^\otimes \rightarrow \mathrm{Fin}_*$ be any symmetric monoidal ∞ -category and $\mathcal{C}' \subseteq \mathcal{C}$ be a full sub- ∞ -category. Let $\mathcal{C}'^\otimes \subset \mathcal{C}^\otimes$ be the full sub- ∞ -category consisting of objects (X_1, \dots, X_n) with $X_i \in \mathcal{C}'$. Show that \mathcal{C}'^\otimes is an ∞ -operad.*

Exercise A.3. *Conversely, show that any ∞ -operad \mathcal{C}^\otimes can be realized as a full sub- ∞ -category of a symmetric monoidal ∞ -category.*

A.4. In fact, there is a universal symmetric monoidal ∞ -category $\mathbf{Env}(\mathcal{C})^\otimes$ containing \mathcal{C}^\otimes such that for any test symmetric monoidal ∞ -category \mathcal{D}^\otimes , we have

$$\mathbf{Fun}^\otimes(\mathbf{Env}(\mathcal{C}), \mathcal{D}) \simeq \mathbf{Alg}_{\mathcal{C}}(\mathcal{D}).$$

This is called the *symmetric monoidal envelope* of \mathcal{C} .

A.5. **Suggested readings.** HA.2.2.4.