LECTURE 5

In this lecture, we define the quasi-category $\mathcal{Q}\mathcal{C}at$ of small quasi-categories, which models the ∞-category Cat_∞ of small ∞-categories.

1. Idea of the construction

1.1. Last time, for quasi-categories $\mathcal{C}$ and $\mathcal{D}$, we constructed the quasi-category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ of functors between them. Let $\mathsf{Fun}(\mathcal{C}, \mathcal{D})^\simeq$ be the core Kan complex of this quasi-category, which can be viewed as obtained from $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ by discarding all non-invertible natural transformations.

1.2. Using these Kan complexes, we can enrich the *ordinary* cateogry QCat over Set_Δ, and obtain a simplicial (enriched) category $\mathbb{Q}\mathbb{C}\mathfrak{at}$. The homotopy hypothesis suggests $\mathbb{Q}\mathbb{C}\mathfrak{at}$ is already a model for Cat_∞.

1.3. To translate this into rigorous mathematics, we will construct a Quillen equivalence between model categories
$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$
between the Joyal model category of simplicial sets and the standard model category of small simplicial categories. The simplicial category $\mathbb{Q}\mathbb{C}\mathfrak{at}$ is a fibrant object in[1] Cat_Δ because it is enriched over fibrant objects in Set_Δ. The Joyal model structure is designed such that (bi)fibrant objects are exactly quasi-categories. Therefore
$$\mathcal{Q}\mathcal{C}at \overset{\mathrm{def}}{=\!=} \mathfrak{N}_\bullet(\mathbb{Q}\mathbb{C}\mathfrak{at})$$
is a fibrant object in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, i.e., a quasi-category.

2. Homotopy category of quasi-categories

2.1. The *homotopy* category of the desired quasi-category $\mathcal{Q}\mathcal{C}at$, in the sense of [Lecture 4, Definition 4.2], can be easily defined. It will also be the homotopy category of the desired model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$, in the sense of [Lecture 2, Definition 2.19]. It models the homotopy category $\mathsf{h}\mathsf{Cat}_\infty$ of the desired ∞-category Cat_∞.

Definition 2.2. *Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be functors between quasi-categories. We call a morphism $\alpha : F_1 \to F_2$ in the quasi-category $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ a **natural transformation** from F_1 to F_2.*

*We say α is **invertible**, or is an **equivalence**, if it is an isomorphism in $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$.*

---

[1]Strictly speaking, $\mathbb{Q}\mathbb{C}\mathfrak{at}$ is not an object in Cat_Δ because it is not small. Hence we need to replace the *large* model category Cat_Δ of *small* simplicial categories by the *very large* model category $\widetilde{\mathsf{Cat}}_\Delta$ of *large* simplicial categories, and similarly replace $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ by the *very large* Joyal model category $\widetilde{\mathsf{Set}}_\Delta^{\mathsf{Joyal}}$ of large simplicial sets. Here one can use *Grothendieck universes* to give precise meanings to the above size conditions. We will ignore these set-theoretic issues until we encounter really problems about them.

Exercise 2.3. *Show that the following construction defines an ordinary category* h$\mathcal{QC}at$:

- *Objects of* h$\mathcal{QC}at$ *are small quasi-categories;*
- *Morphisms* h$\mathcal{QC}at$ *are equivalence classes of functors between quasi-categories:*
$$\mathsf{Hom}_{\mathsf{h}\mathcal{QC}at}(\mathcal{C}, \mathcal{D}) \stackrel{\text{def}}{=} \pi_0(\mathsf{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$$
- *Composition is given by* $[G] \circ [F] \stackrel{\text{def}}{=} [G \circ F]$.

3. Equivalences between quasi-categories

3.1.   Note that we have a functor $\mathsf{QCat} \to \mathsf{h}\mathcal{QC}at$. Similar to [Lecture 4, Definition 5.1], we make the following definition.

Definition 3.2. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between quasi-categories, i.e. a morphism in* QCat. *We say* F *is an* **equivalence** *if its image* $[F]$ *in* h$\mathcal{QC}at$ *is an isomorphism.*

3.3.   In other words, $F : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there exists a functor $G : \mathcal{D} \to \mathcal{X}$ such that $G \circ F$ is equivalent to $\mathsf{Id}_{\mathcal{C}}$ and $F \circ G$ is equivalent to $\mathsf{Id}_{\mathcal{D}}$.

Theorem 3.4 ([Lur, Tag 01JX])**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between quasi-categories. Then* F *is an equivalence iff*

(1) F *is* **fully faithful**: *for any objects* $x, y \in \mathcal{C}$, *the functor* $\mathsf{Maps}_{\mathcal{C}}(x, y) \to \mathsf{Maps}_{\mathcal{D}}(Fx, Fy)$ *is an equivalence between* ∞-*groupoids.*

(2) F *is* **essentially surjective**: *the map* $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ *is surjective. In other words, for any object* $d \in \mathcal{D}$, *there exist* $c \in \mathcal{C}$ *and an isomorphism* $F(c) \stackrel{\simeq}{\to} d$.

Exercise 3.5. *Being essentially surjective can be checked on the level of homotopy categories. Namely,* $F : \mathcal{C} \to \mathcal{D}$ *is essentially surjective iff* $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$ *is so.*

Exercise 3.6. *Being fully faithful* cannot *be checked on the level of homotopy categories.*

3.7.   Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between quasi-categories. So far we have at least four equivalence-like conditions on F:

(a) F is an *isomorphism* in Set_Δ.
(b) F is an *equivalence* between quasi-categories.
(c) F is a *weak homotopy equivalence* in the Kan–Quillen model category $\mathsf{Set}_\Delta^{\mathsf{KQ}}$.
(d) F induces an *equivalence between ordinary categorie*: $\mathsf{h}F : \mathsf{h}\mathcal{C} \to \mathsf{h}\mathcal{D}$.

The relations between these notions are

$$(a) \Longrightarrow (b) \Longrightarrow (c)$$
$$\Big\Downarrow$$
$$(d).$$

Exercise 3.8. *The above implications are not inverible.*

(1) $\mathsf{N}_\bullet(\{\bullet \rightrightarrows \bullet\}) \to \Delta^0$ *satisfies (b) but not (a).*
(2) $\Delta^1 \to \Delta^0$ *satisfies (c) but not (b) or (d).*
(3) $\mathsf{Sing}(S^1) \to \Delta^0$ *satisfies (d) but not (b) or (c).*

3.9. Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be the functor between ∞-categories modeled by $F : \mathcal{C} \to \mathcal{D}$. The above conditions tranlate into:

  (a) This condition does not make sense because it is evil.
  (b) F is an *equivalence* between ∞-categories.
  (c) F *becomes an equivalence beween ∞-groupoids* after formally inverting all morphisms in C and D.
  (d) F *becomes an equivalence between ordinary categories* after formally forcing all non-invertible higher morphisms in C and D to be identities.

Note that the construction in (c) is *not* taking the core ∞-groupoids. The latter are obtained by *discarding* all non-invertible morphisms. See [Lecture 4, Exercise 5.3].

4. Joyal model structure

4.1. Morphisms in the class (W) of $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ will be called *categorical equivalences*[2] between simplicial sets. When restricted to bifibrant objects, i.e. quasi-categories, they are supposed to model equivalences between the underlying ∞-categories. In other words:

> *Categorical equivalences between quasi-categories should be equivalences, in the sense of Definition 3.2.*

Hence the main task is to define categorical equivalences between *general* simplicial sets.

4.2. Recall when defining the classical model structure on Set_Δ (see [Lecture 3, Theorem-Definition 6.2]), we declare weak homotopy equivalences between simplicial sets to be maps $f : X \to Y$ such that $|f| : |X| \to |Y|$ are weak homotopy equivalences in Top.

It is possible to define categorical equivalences in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ in a similar manner. Namely, we can first define weak equivalences in Cat_Δ and then transfer them via the functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$[3]. However, I find the construction of the functor $\mathfrak{C}$ not intuitive enough to provide actual feelings about categorical equivalences. Therefore we will follow Joyal's definition.

4.3. Joyal's definition is motivated by the following observation:

Exercise 4.4. *Let $\mathcal{C}$ be a model category such that any object is cofibrant. Then a morphism $f : X \to Y$ belongs to (W) iff for any fibrant object Z, the map*

$$(\mathsf{Hom}_\mathcal{C}(Y, Z)/\sim) \to (\mathsf{Hom}_\mathcal{C}(X, Z)/\sim)$$

is bijective. Here "$\sim$" stands for the homotopy relation between morphisms, see [Lecture 2, Proposition-Definition 2.18].

Hint: Yoneda lemma; the functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ detects all weak equivalences.

---

[2]Joyal called them *weak categorical equivalences*. We follow Lurie's terminology.
[3]This was the approach adopted in Lurie's HTT.2.2.5, which differs from that in Joyal's works (see [Joy08]). See HTT.2.2.5.9 and HTT.2.2.5.10 for more information.

Definition 4.5. *Let $f : X \to Y$ be a morphism in Set_Δ. We say f is a **categorical equivalence** if for any quasi-category Z, the map*

$$\pi_0(\mathsf{Fun}(Y, Z)^\simeq) \to \pi_0(\mathsf{Fun}(X, Z)^\simeq)$$

is bijective.

Proposition 4.6 ([Joy08, Corollary 2.29]). *Any inner horn inclusion $\Lambda_i^n \to \Delta^n$, $0 < i < n$ is a categorical equivalence.*

Exercise 4.7. *Convince yourself the above claim does* not *follow immediately from the definition of quasi-categories.*

Exercise 4.8. *Categorical equivalences are weak homotopy equivalences. Hint: [Lecture 3, §6.5].*

Theorem-Definition 4.9 ([Joy08, Theorem 6.12]). *There exists a model structure on Set_Δ given by*

- (W) *class of categorical equivalences;*
- (C) *class of monomorphisms;*
- (F) *class of morphisms satisfying the right lifting property against $(C \cap W)$.*

*We call it the **Joyal model structure** on Set_Δ. and denote this model category by $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.*

Fibrant objects in this model category are exactly quasi-categories.

4.10. The following result is tautological:

Exercise 4.11. *The functor $\mathsf{QCat} \to \mathsf{Set}_\Delta$ induces an equivalence[4] $\mathsf{hQCat} \to \mathsf{hSet}_\Delta^{\mathsf{Joyal}}$. In particular, $\mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}]$ also models hCat_∞.*

Definition 4.12. *Fibrations in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ are called **categorical fibrations**.*

4.13.

Proposition 4.14. *The identity functor gives a Quillen adjunction*

$$\mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}.$$

Exercise 4.15. *Let $\mathbb{L} : \mathsf{Set}_\Delta^{\mathsf{Joyal}}[W^{-1}] \rightleftarrows \mathsf{Set}_\Delta^{\mathsf{KQ}}[W^{-1}] : \mathbb{R}$ be the derived adjunction of the above Quillen adjunction. Intepret it as an adjunction $\mathsf{hCat}_\infty \rightleftarrows \mathsf{hGrpd}_\infty$.*

4.16. By definition, categorical fibrations $p : X \to Y$ in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ satisfy the right lifting properties against all inner horn inclusions. Morphisms satisfy these properties are called **inner fibrations**. Note that when $Y = \Delta^0$, inner fibrations over Y are categorical fibrations. However, the inclusion

$$\text{inner fibrations} \subset \text{categorical fibrations}$$

is strict in general. For more information, see HTT.2.4.6.5.

5. Simplicial categories

5.1. In this section, we define a model structure on the category Cat_Δ of small simplicial categories.

---

[4]In fact, it is an *isomorphism*

5.2. Weak equivalences between simplicial categories are defined similarly as weak equivalences between topological categories. See [Lecture 2, §3].

Definition 5.3. *Let $\mathcal{C}$ be a simplicial category. Its **homotopy category** $\pi_0\mathcal{C}$ is defined by*

$$\mathsf{Ob}(\pi_0\mathcal{C}) \coloneqq \mathsf{Ob}(\mathcal{C}), \ \mathsf{Hom}_{\pi_0\mathcal{C}}(x,y) \coloneqq \pi_0\mathsf{Hom}_{\mathcal{C}}(x,y).$$

Definition 5.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between simplicial categories. We say F is a **weak equivalence** if:*

- *It induces an equivalence $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$.*
- *The morphism $\mathsf{Hom}_{\mathcal{C}}(x,y) \to \mathsf{Hom}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence in $\mathsf{Set}_{\Delta}^{\mathsf{KQ}}$ for any $x, y \in \mathsf{Ob}(\mathcal{C})$.*

Theorem 5.5 (HTT.A.3.2). *There exists a canonical model structure on Cat_{Δ} such that:*

- *Weak equivalences are as in Defintion 5.4.*
- *Fibrant objects are exactly simplicial categories $\mathcal{C}$ such that $\mathsf{Hom}_{\mathcal{C}}(-,-)$ are Kan complexes.*

*We call it the **classical** or **Bergner model structure** on Cat_{Δ}.*

5.6. It is not easy to describe fibrations and cofibrant objects in Cat_{Δ}, hence we will not do it. Note that [Lecture 2, Exercise 3.6] has a simplicial analogue, which suggests many naturally defined simplicial categories are not cofibrant.

6. Quasi-categories and simplicial categories

6.1. In this section, we construct the desired Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_{\Delta}^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_{\Delta} : \mathfrak{N}_{\bullet}.$$

As in the construction of

$$|-| : \mathsf{Set}_{\Delta}^{\mathsf{KQ}} \rightleftarrows \mathsf{Top} : \mathsf{Sing},$$

we only need to find the correct definition for the restriction $\mathfrak{C}|_{\Delta} : \Delta \to \mathsf{Cat}_{\Delta}$ along the Yoneda embedding $\Delta \to \mathsf{Set}_{\Delta} \overset{\mathrm{def}}{=} \mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{Set})$. The entire functor $\mathfrak{C}$ will be the unique (up to unique equivalence) colimit-preserving functor that extends $\mathfrak{C}|_{\Delta}$, and the functor $\mathfrak{N}_{\bullet}$ will be its right adjoint.
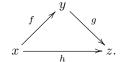
6.2. In other words, we need to construct simplicial categories $\mathfrak{C}[\Delta^n] \in \mathsf{Cat}_{\Delta}$ equipped with face and degeneracy operators. To motivate this construction, we look at the following question:

> *Let $\mathcal{C} \in \mathsf{Cat}_{\Delta}$ be a fibrant simplicial category. What is a functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$?*

Since we *expect* the model category Cat_{Δ} to model Cat_{∞}, and since $\mathfrak{C}[\Delta^n]$ is cofibrant (because $\mathfrak{C}$ is expected to be a left Quillen functor), we obtain

> *A functor $\mathfrak{C}[\Delta^n] \to \mathcal{C}$ should model a functor $[n] \to \mathsf{C}$ between corresponding ∞-categories.*

6.3.   For instance, for $n = 2$, it should be an invertible 2-morphism witnessing the following commutative diagram in C:

$$
\begin{array}{ccc}
 & y & \\
{}^{f}\nearrow & & \searrow{}^{g} \\
x & \xrightarrow[\;h\;]{} & z.
\end{array}
$$

Note that in the siplicial category $\mathcal{C}$, the composition is *concrete*: for 0-simplexes $f \in \mathsf{Hom}_{\mathcal{C}}(x,y)_0$ and $g \in \mathsf{Hom}_{\mathcal{C}}(y,z)_0$, we have a well-defined simplex $g \circ f \in \mathsf{Hom}_{\mathcal{C}}(x,z)_0$. Hence the above invertible 2-morpihsm corresponds to a 1-simplex $g \circ f \to h$[5] in $\mathsf{Hom}_{\mathcal{C}}(x,z)_1$.

Therefore we should define $\mathfrak{C}[\Delta^2]$ as

- It has three objects, labelled by $0, 1$ and 2.
- The morphism simplicial sets are

$$
\mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(i,j) \overset{\text{def}}{=\!=} \begin{cases} \varnothing & \text{if } i > j \\ \Delta^1 & \text{if } i = 0, j = 2 \\ \Delta^0 & \text{if otherwise} \end{cases}
$$

- The only non-obvious composition is

$$
\mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,1) \times \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(1,2) \to \mathsf{Hom}_{\mathfrak{C}[\Delta^2]}(0,2)
$$
$$
\Delta^0 \times \Delta^0 \to \Delta^1,
$$

which is given by the 0-simplex $1 \in \Delta^1$.

6.4.   The above research can be conductd for general n. Namely, for any ∞-category, we *know* the correct meaning of a functor $[n] \to \mathsf{C}$, thanks to the fact that $[n]$ comes from a linearly ordered set[6]. For instance, 0-simplex in $\mathsf{Hom}_{\mathfrak{C}[\Delta^n]}(0,n)$ should be a subset $I \subset [n]$ containing 0 and n; and each chain $I_0 \subset \cdots \subset I_m$ corresponds to a m-simplex.

In formula, we define:

Definition 6.5. *Let J be any linearly ordered set. We define a simplicial category* $\mathfrak{C}[\Delta^J]$ *as follows:*

- *Objects are elements in J;*
-
$$
\mathsf{Hom}_{\mathfrak{C}[\Delta^J]}(x,y) \overset{\text{def}}{=\!=} \begin{cases} \varnothing & \text{if } x > y \\ \mathsf{N}_{\bullet}(P_{x,y}) & \text{if } x \leq y \end{cases}
$$

  where $P_{x,y} := \{I \subset [x,y] \mid x, y \in I\}$ is a partially ordered set, and $\mathsf{N}_{\bullet}(P_{x,y})$ is the nerve of the corresponding category.
- *Morphisms are induced by*

$$
(I \subset [x,y], I' \subset [y,z]) \mapsto (I \cup I' \subset [x,z])
$$

---

[5]One may also use $h \to g \circ f$. Thanks to the fibrant assumption about $\mathsf{Hom}_{\mathcal{C}}(x,z)$, the actual data are the same. But the obtained functor $\mathfrak{C}$ will differ by the involution $\mathsf{Set}_{\Delta} \to \mathsf{Set}_{\Delta}, X \mapsto X^{\mathsf{op}}$. Here we follow the conventions in Kerodon (see [Lur, Tag 00KN]), which is the opposite to HTT (see HTT.1.1.5.1.).

[6]On the other hand, homotopy coherent commutative diagrams discussed in [Lecture 1, Section 3] are beyond our reach, because they are not given by (partially) ordered set.

6.6. The above construction is functorial in J and therefore we have a functor

$$\Delta \to \mathsf{Cat}_\Delta, \ [n] \mapsto \mathfrak{C}[\Delta^n] \overset{\text{def}}{=\joinrel=} \mathfrak{C}[\Delta^{[n]}]$$

Note that $\mathfrak{C}[\Delta^n]$ is *not* fibrant. This can already be seen when $n = 2$.

Let

$$\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$$

be the (essentially) unique colimit-preserving functor that extends the above functor.

Exercise 6.7. *What is $\mathfrak{C}[\Lambda_1^2]$?*

Theorem-Definition 6.8 (HTT.2.2.5.1, HTT.2.2.5.8). *The functor $\mathfrak{C} : \mathsf{Set}_\Delta \to \mathsf{Cat}_\Delta$ has a right adjoint, called the **simplicial nerve functor***

$$\mathfrak{N}_\bullet : \mathsf{Cat}_\Delta \to \mathsf{Set}_\Delta.$$

The adjoint pair

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}_\bullet$$

is a Quillen equivalence.

6.9. Note that $\mathfrak{C}[\Delta^0] \simeq \{*\}$. Hence we may
- identify X_0 with $\mathsf{Ob}(\mathfrak{C}(X))$ for any simplicial set X;
- identify $\mathsf{Ob}(\mathcal{C})$ with $\mathfrak{N}_\bullet(\mathcal{C})_0$ for any simplicial category $\mathcal{C}$.

From now on, we will abuse notations by viewing them as the *same* sets.

6.10. To justify corresponding bifibrant objects in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ and Cat_Δ indeed model the same ∞-category, we need the following results.

Theorem 6.11 ([Lur, Tag 01LE]). *Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical weak homotopy equivalence between Kan complexes*

$$\mathsf{Hom}_{\mathcal{C}}(x,y) \overset{\simeq}{\to} \mathsf{Hom}^{\mathsf{L}}_{\mathfrak{N}_\bullet(\mathcal{C})}(x,y).$$

Corollary 6.12. *Let $\mathcal{C} \in \mathsf{Cat}_\Delta$ be a fibrant simplicial category. Then there is a canonical equivalence $\mathsf{h}\mathcal{C} \overset{\simeq}{\to} \mathsf{h}\mathfrak{N}_\bullet(\mathcal{C})$[7].*

6.13. The construction of the above weak equivalence is too combinatorial to record in these notes. See [Lur, Tag 01LD].

7. THE QUASI-CATEGORY OF QUASI-CATEGORIES

7.1. This is a verbatim of §1.

Definition 7.2. *Let $\mathbb{QCat}$ be the simplicial category defined by:*
- *Objects are quasi-categories;*
- *For quasi-categories $\mathcal{C}$ and $\mathcal{D}$,*

$$\mathsf{Hom}_{\mathbb{QCat}}(\mathcal{C},\mathcal{D}) \coloneqq \mathsf{Fun}(\mathcal{C},\mathcal{D})^{\simeq}.$$

- *The composition*

$$\mathsf{Fun}(\mathcal{C},\mathcal{D}) \times \mathsf{Fun}(\mathcal{D},\mathcal{E}) \to \mathsf{Fun}(\mathcal{C},\mathcal{E})$$

is given by the unversal property of $\mathsf{Fun}(-,-)$.

---

[7]In fact, this result is much more elementary. See [Lur, Tag 00M4]

Definition 7.3. *The **quasi-category of small quasi-categories** is defined to be*

$$\mathcal{QC}at := \mathfrak{N}_\bullet(\mathbb{QCat}).$$

Let Cat_∞ *be the* ∞*-category modelled by it*[8]*. We call it the* ∞***-category of small*** ∞***-categories***.

Exercise 7.4. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be quasi-categories. There is a canonical weak homotopy equivalence between Kan complexes*

$$\mathsf{Fun}(\mathcal{C},\mathcal{D})^\simeq \xrightarrow{\simeq} \mathsf{Hom}^{\mathsf{L}}_{\mathcal{QC}at}(\mathcal{C},\mathcal{D}).$$

In other words, $\mathsf{Fun}(\mathcal{C},\mathcal{D})^\simeq$ *models the* ∞*-groupoids* $\mathsf{Maps}_{\mathsf{Cat}_\infty}(\mathsf{C},\mathsf{D})$ *as desired.*

Variant 7.5. *Similarly, we define the **quasi-category of small Kan complexes** $\mathcal{K}an$, which models the* ∞*-category* Grpd_∞ *of small* ∞*-groupoids.*

Exercise 7.6. *Show that the homotopy category of* $\mathcal{QC}at$ *is indeed equivalent to* $\mathsf{h}\mathcal{QC}at$ *in Exercise 2.3.*

8. Commutative diagrams

8.1.   By previous discussion, in infinite category theory, for any simplicial set S, the *correct* notion of a commutative diagram of shape S in a simplicial category $\mathcal{C}$ is a functor $\mathfrak{C}[S] \to \mathcal{C}$. We call such diagrams a **homotopy coherent commutative diagram** of shape S in $\mathcal{C}$.

8.2.   In most cases, the shape of the diagram is an ordinary category $\mathcal{D}$, we also abuse language and call a functor $\mathfrak{C}[\mathsf{N}_\bullet(\mathcal{D})] \to \mathcal{C}$ a **homotopy coherent commutative diagram** of shape $\mathcal{D}$ in $\mathcal{C}$. This is the notion that behaves well under standard infinite categorical constructions.

One can also ask for a diagram that **commutes on the nose**, which means a functor $\mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is viewed as enriched over discrete simplicial sets. In practice, it is hard to construct such diagrams, but most *non-formal* calculations eventually lead people to this realm[9].

One can also ask for a diagram that **commutes up to homotopy**, which means a functor $\mathcal{D} \to \mathsf{h}\mathcal{C}$. This is essentially a notion in ordinary category theory.

In summary:

$$\text{on the nose} \implies \text{homotopy-coherently} \implies \text{up to homotopy}.$$

Exercise 8.3. *Apply the above discussion to* $\mathcal{C} := \mathbb{QCat}$ *and obtain three notions of commutative diagram about quasi-categories.*

Appendix A. QCat vs. $\mathcal{QC}at$

Exercise A.1. *Let* $\mathcal{C}$ *be an ordinary category. Construct an isomorphism* $\mathsf{N}_\bullet(\mathcal{C}) \simeq \mathfrak{N}_\bullet(\mathcal{C})$*, where in the RHS we view* $\mathcal{C}$ *as enriched over discrete simplicial sets.*

Exercise A.2. *Construct a functor* $\mathsf{N}_\bullet(\mathsf{QCat}) \to \mathcal{QC}at$ *that models* $\mathsf{QCat} \to \mathsf{Cat}_\infty$.

---

[8]In rigorous mathematical words, we should *define* ∞-categories to be quasi-categories. However, I do want the readers to view quasi-categories as models for ∞-categories.

[9]My personal experience: *genuine* non-formal calculations are very few.

References

[Joy08] André Joyal. The theory of quasi-categories and its applications. 2008.

[Lur] Jacob Lurie. Kerodon. `https://kerodon.net`.