

## LECTURE 11

Last time, for a smooth  $k$ -scheme  $X$ , we introduced the sheaf of differential operators  $\mathcal{D}_X$  on  $X$  and defined quasi-coherent  $\mathcal{D}_X$ -modules, known as  *$\mathcal{D}$ -module on  $X$* . In this lecture, we introduce operations on  $\mathcal{D}$ -modules. We will mainly focus on the formal aspect of this theory, known as the *six functors formalism for  $\mathcal{D}$ -modules*. For more details, see [B] and [HTT].

### 1. CONVENTIONS ON DERIVED CATEGORIES

For the purpose of this course, we do not need the full power of the derived categories of  $\mathcal{D}$ -modules. However, these categories are useful in other topics of geometric representation theory, hence I choose to include the results about them in this lecture.

When talking about derived categories of  $\mathcal{D}$ -modules, we always assume  $X$  is quasi-projective. In this case, the abelian category  $\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r}$  has enough injective and locally projective objects, and any object admits a resolution by locally projective  $\mathcal{D}_X$ -modules with length  $\leq 2d_X$ , where  $d_X = \dim(X)$  is the dimension (function) of  $X$ . The latter implies  $\text{Ext}^i(-, -) \simeq 0$  for  $i > 2d_X$ .

We have the following triangulated categories equipped with natural t-structures:

- $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r})$ , the derived category of quasi-coherent  $\mathcal{D}_X$ -modules. This can be identified with the full subcategory of the derived category  $D(\mathcal{D}_X\text{-mod}^{l/r})$  of *all*  $\mathcal{D}_X$ -modules containing those complexes whose cohomologies are quasi-coherent.
- $D^b(\mathcal{D}_X\text{-mod}_c^{l/r})$ , the bounded derived category of coherent  $\mathcal{D}_X$ -modules. This can be identified with the full subcategory of the bounded derived category  $D^b(\mathcal{D}_X\text{-mod}^{l/r})$  of *all*  $\mathcal{D}_X$ -modules containing those complexes whose cohomologies are coherent.

When talking about functors between derived categories, even if such functors are left/right derived functors, we drop the decorations “L/R” from the notations. For example,  $- \otimes -$  in derived categories would mean  $- \otimes^L -$  in classical literatures. We choose to do so because we will encounter functors that are not derived functors.

*Remark 1.1.* The (essential) image of the fully faithful functor

$$D^b(\mathcal{D}_X\text{-mod}_c^{l/r}) \rightarrow D(\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r})$$

contains exactly the *compact* objects in the target, i.e., those objects  $\mathcal{M}$  such that  $\text{Hom}(\mathcal{M}, -)$  commutes with filtered (homotopy) colimits.

### 2. FORGET AND INDUCE

**Construction 2.1.** *We have adjoint functors*

$$\text{ind}^{l/r} : \mathcal{O}_X\text{-mod}_{\text{qc}} \rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} : \text{oblv}^{l/r}$$

*such that*

$$\text{ind}^l(\mathcal{F}) := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \text{ind}^r(\mathcal{D}) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

*Both functors are exact.*

## 3. TENSOR AND HOM

**Construction 3.1.** Let  $\mathcal{M}, \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  and  $\mathcal{M}', \mathcal{N}' \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$ . Then there are natural objects defined using the signed Leibniz rules:

- $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  defined by  $\partial \cdot (m \otimes n) = (\partial \cdot m) \otimes n + m \otimes (\partial \cdot n)$ ;
- $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$  defined by  $(m' \otimes n) \cdot \partial = (m' \cdot \partial) \otimes n - m' \otimes (\partial \cdot n)$ ;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  defined by  $(\partial \cdot \phi)(m) = \partial \cdot \phi(m) - \phi(\partial \cdot m)$ ;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  defined by  $(\partial \cdot \phi)(m') = -\phi(m') \cdot \partial + \phi(m \cdot \partial)$ ;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$  defined by  $(\phi \cdot \partial)(m) = \phi(m) \cdot \partial + \phi(m \cdot \partial)$ .

*Remark 3.2.* One way to memorize the above rules for left vs. right is using the known objects  $\mathcal{O}_X \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  and  $\omega_X := \Omega_X^n \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$ . For example,  $\text{Hom}_{\mathcal{O}_X}(\omega_X, \omega_X) \simeq \mathcal{O}_X$  has a left  $\mathcal{D}$ -module structure, while  $\text{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) \simeq \omega_X^{-1}$  in general has no  $\mathcal{D}$ -module structures.

*Remark 3.3.* One way to memorize the signed Leibniz rules: (i) put a minus sign when acting on a section of the source object in  $\text{Hom}_{\mathcal{O}_X}(-, -)$ ; (ii) put a minus sign when moving  $\partial$  from the one side of  $\cdot$  to the other side.

*Remark 3.4.* The tensor operations make  $\mathcal{D}_X\text{-mod}_{\text{qc}}^l$  a symmetric monoidal category such that the forgetful functor  $\mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{O}_X\text{-mod}_{\text{qc}}^l$  is naturally symmetric monoidal. The category  $\mathcal{D}_X\text{-mod}_{\text{qc}}^r$  is a module category of it.

The following result follows by unwinding the definitions:

**Lemma 3.5.** Let  $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  and  $\mathcal{M}' \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$ . We have adjoint functors

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} &\rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} : \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, -) \\ \mathcal{M}' \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^l &\rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^r : \text{Hom}_{\mathcal{O}_X}(\mathcal{M}', -) \end{aligned}$$

compatible with the similar adjunction between  $\mathcal{O}_X$ -modules.

*Remark 3.6.* The above compatibility means e.g. the isomorphism between  $\mathcal{D}_X$ -modules

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$$

provided by the adjunction in the lemma gives the corresponding isomorphism between  $\mathcal{O}_X$ -modules provided by the adjunction between  $\mathcal{O}_X$ -modules.

Recall  $\omega_X$  is a line bundle. It follows formally that:

**Corollary 3.7.** The following functors are inverse to each other:

$$\omega_X \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^r : \text{Hom}_{\mathcal{O}_X}(\omega_X, -).$$

*Remark 3.8.* In particular, for any right  $\mathcal{D}$ -module  $\mathcal{N}'$ , we obtain a left  $\mathcal{D}$ -module structure on  $\omega_X^{-1} \otimes \mathcal{N}'$ .

*Remark 3.9.* We also have similar results for the derived category of  $\mathcal{D}$ -modules and the corresponding derived functors. The left derived functor  $- \otimes_{\mathcal{O}_X} -$  has cohomological amplitude  $[-d_X, 0]$  while the right derived functor  $\text{Hom}_{\mathcal{O}_X}(-, -)$  has cohomological amplitude  $[0, d_X]$ .

*Remark 3.10.* When identifying the *derived* categories of left and right  $\mathcal{D}$ -modules, it is more convenient to use the complex  $\omega_X[d_X]$ , which is also known as the *dualizing complex*<sup>1</sup> on  $X$ .

<sup>1</sup>This notion makes sense even when  $X$  is singular.

In other words, we use the equivalences

$$\omega_X[d_X] \otimes_{\mathcal{O}_X} - : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \xrightarrow{\sim} D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \mathcal{H}om_{\mathcal{O}_X}(\omega_X[d_X], -).$$

**Construction 3.11.** We define a symmetric monoidal structure  $-\otimes^!-$  on  $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r)$  by translating the symmetric monoidal structure  $-\otimes_{\mathcal{O}_X}-$  of  $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l)$  via the above equivalences. In other words, as  $\mathcal{O}_X$ -modules, we have

$$\mathcal{M}' \otimes^! \mathcal{N}' \simeq \mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}' \otimes_{\mathcal{O}_X} \omega_X^{-1}[-d_X].$$

#### 4. EXTERNAL TENSOR

**Construction 4.1.** Let  $\mathcal{M}_i \in \mathcal{D}_{X_i}\text{-mod}_{\text{qc}}^{l/r}$ . Then there are natural objects

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 := \mathcal{M}_1 \otimes_k \mathcal{M}_2 \in \mathcal{D}_{X_1 \times X_2}\text{-mod}_{\text{qc}}^{l/r}$$

induced by the isomorphism  $\mathcal{D}_{X_1} \otimes_k \mathcal{D}_{X_2} \simeq \mathcal{D}_{X_1 \times X_2}$ . This is called the **external tensor product** of  $\mathcal{D}$ -modules.

#### 5. PULLBACKS

Let  $\phi : Y \rightarrow X$  be a map between smooth  $k$ -schemes.

**Construction 5.1.** We will construct a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_X\text{-mod}_{\text{qc}}^l & \xrightarrow{\phi^*} & \mathcal{D}_Y\text{-mod}_{\text{qc}}^l \\ \downarrow \text{oblv}^l & & \downarrow \text{oblv}^l \\ \mathcal{O}_X\text{-mod}_{\text{qc}} & \xrightarrow{\phi^*} & \mathcal{O}_Y\text{-mod}_{\text{qc}}. \end{array}$$

The functor

$$\phi^* : \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$$

is called the **(\*)pullback of left  $\mathcal{D}$ -modules**.

The construction is as follows. For  $\mathcal{M} \in \mathcal{O}_X\text{-mod}_{\text{qc}}$ , recall  $\phi^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{M}$ . Suppose  $\mathcal{M}$  is equipped with a left  $\mathcal{D}_X$ -module structure, then there is a left  $\mathcal{D}_Y$ -module structure on  $\phi^*(\mathcal{M})$  defined by the Leibniz rule:

$$\partial \cdot (f \otimes s) := \partial(f) \otimes s + f \bar{\partial} \cdot s,$$

where

- $\partial$  is a local section of  $\mathcal{T}_Y$  and  $\bar{\partial}$  is the image of it under the map  $\mathcal{T}_Y \rightarrow \phi^*\mathcal{T}_X = \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{T}_X$ ;
- $f$  is a local section of  $\mathcal{O}_Y$ ;
- $s$  is a local section of  $\phi^{-1}\mathcal{M}$ , and  $\bar{\partial} \cdot s$  is defined using the action of  $\phi^{-1}\mathcal{T}_X$  on  $\phi^{-1}\mathcal{M}$ .

**Remark 5.2.** One can show the pullback of left  $\mathcal{D}$ -modules are compatible with composition of maps between smooth  $k$ -schemes.

**Example 5.3.** The pullback of the object  $\mathcal{O}_X \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$  is  $\mathcal{O}_Y \in \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$ .

**Construction 5.4.** We write:

$$\mathcal{D}_{Y \rightarrow X} := \phi^* \mathcal{D}_X \simeq \mathcal{O}_Y \otimes_{\phi^{-1} \mathcal{O}_X} \phi^{-1} \mathcal{D}_X$$

and call it the **transfer module**.

The above construction gives a left  $\mathcal{D}_Y$ -module structure on  $\mathcal{D}_{Y \rightarrow X}$ . On the other hand, there is an obvious right  $\phi^{-1} \mathcal{D}_X$ -module structure on  $\mathcal{D}_{Y \rightarrow X}$ . One can show these two actions commute. In other words,  $\mathcal{D}_{Y \rightarrow X}$  is a  $(\mathcal{D}_Y, \phi^{-1} \mathcal{D}_X)$ -bimodule.

Note that for  $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ , we have

$$\phi^* \mathcal{M} \simeq \mathcal{D}_{Y \rightarrow X} \otimes_{\phi^{-1} \mathcal{D}_X} \phi^{-1} \mathcal{M}.$$

**Construction 5.5.** Note that the functor  $\phi^* : \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$  is right exact. We abuse notation and let

$$\phi^* : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \rightarrow D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l)$$

be the left derived functor of it. Note that it is compatible with the left derived functor  $\phi^* : D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \rightarrow D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$  and the forgetful functors.

*Remark 5.6.* We have:

- If  $\phi$  is flat, then  $\phi^*$  is t-exact, i.e., preserves the heart.
- If  $\phi$  is a closed embedding (which is automatically regular), then  $\phi^*$  has cohomological amplitude  $[-d_X + d_Y, 0]$ .

**Example 5.7.** Let  $\phi$  be a closed embedding. Since  $X$  and  $Y$  are smooth,  $\phi$  is a regular immersion. For any closed point  $p \in Y$ , we can find an étale coordinate system  $x_1, \dots, x_m$  of  $X$  near  $p$  such that  $Y$  is locally cut out by the ideal  $(x_{n+1}, \dots, x_m)$  ( $m = \dim(\mathcal{O}_{X,y})$  and  $n = \dim(\mathcal{O}_{Y,y})$ ). Let  $y_1, \dots, y_n$  be the restriction of  $x_1, \dots, x_n$  on  $Y$ . They form an étale coordinate system of  $Y$  near  $p$ . Then near the point  $p \in Y$ , we have

$$\mathcal{D}_{Y \rightarrow X} \simeq \mathcal{D}_Y \otimes_k k[\partial_{n+1}, \dots, \partial_m]$$

as left  $\mathcal{D}_Y$ -modules. In particular,  $\mathcal{D}_{Y \rightarrow X}$  is a locally free left  $\mathcal{D}_Y$ -module.

**Construction 5.8.** Let  $\phi^!$  be the unique functor that makes the following diagram commute

$$\begin{array}{ccc} D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) & \xrightarrow{\phi^*} & D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l) \\ \omega_X[d_X] \downarrow \simeq & & \simeq \downarrow \omega_Y[d_Y] \\ D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi^!} & D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r), \end{array}$$

The obtained functor

$$\phi^! : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \rightarrow D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r)$$

is called the **!-pullback of (complices of) right  $\mathcal{D}$ -modules**. It has cohomological amplitude  $[-d_Y, d_X - d_Y]$  and in general is not a derived functor.

*Remark 5.9.* We have:

- If  $\phi$  is flat, then  $\phi^!$  is t-exact up to a shift.
- If  $\phi$  is a closed embedding, then  $\phi^!$  has cohomological amplitude  $[0, d_X - d_Y]$ .

**Example 5.10.** By definition,  $\phi^!(\omega_X[d_X]) \simeq \omega_Y[d_Y]$ .

**Example 5.11.** If  $j : U \rightarrow X$  is an open embedding, then  $j^!$  is t-exact and the corresponding functor  $\mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_U\text{-mod}_{\text{qc}}^r$  is the restriction functor. Indeed, this follows from  $\omega_X|_U \simeq \omega_U$ .

**Fact 5.12.** If  $\phi : Y \rightarrow X$  is a closed embedding, then  $\phi^!$  is equivalent to the right derived functor of a functor

$$\phi^! : \mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^r.$$

**Construction 5.13.** Let  $\phi : Y \rightarrow X$  be a closed embedding. The functor  $\phi^! : \mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^r$  can be described as follows.

Recall we have adjoint functors

$$\phi_* : \mathcal{O}_Y\text{-mod}_{\text{qc}} \rightleftarrows \mathcal{O}_X\text{-mod}_{\text{qc}} : \phi^!$$

where for a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{M}$  and any open subset  $U \subset X$ , a section  $m$  of  $\phi^!(\mathcal{M})$  on  $U \cap Y$  corresponds to a section  $\tilde{m}$  of  $\mathcal{M}$  on  $U$  annihilated by the ideal  $\mathcal{I}_Y := \ker(\mathcal{O}_X \rightarrow \phi_* \mathcal{O}_Y)$ . Suppose  $\mathcal{M}$  is equipped with a right  $\mathcal{D}_X$ -module structure. For any local section  $\partial$  of  $\mathcal{T}_Y$ , we can extend it to a local section  $\tilde{\partial}$  of  $\mathcal{T}_X$ . Now for a local section  $m$  of  $\phi^!(\mathcal{M})$ , we define  $m \cdot \partial$  such that

$$\widetilde{m \cdot \partial} = \tilde{m} \cdot \tilde{\partial}.$$

One can show the local section  $m \cdot \partial$  is well-defined and does not depend on the choice of  $\tilde{\partial}$ . Moreover, this defines a right  $\mathcal{D}_Y$ -module structure on  $\phi^!(\mathcal{M})$ .

**Remark 5.14.** For any map  $\phi : Y \rightarrow X$  between finite type  $k$ -schemes, one can define a functor

$$\phi^! : D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \rightarrow D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$$

as follows.

If  $\phi$  is an open embedding, take  $\phi^! := \phi^*$ . If  $\phi$  is proper, take  $\phi^!$  to be the right adjoint of (the right derived functor)  $\phi_*$ . For the general case, choose a Nagata compactification  $Y \xrightarrow{j} \overline{Y} \xrightarrow{\overline{\phi}} X$  such that  $j$  is an open embedding and  $\overline{\phi}$  is proper, and take  $\phi^! := j^! \circ \overline{\phi}^!$ . One can show the functor  $\phi^!$  does not depend on the choice of the compactification, and these functors are compatible with compositions of maps. In fact, the construction  $\phi \mapsto \phi^!$  can be uniquely characterized by these properties (if stated properly).

When  $X$  and  $Y$  are smooth, the  $!$ -pullback functors of  $\mathcal{O}$ -modules and right  $\mathcal{D}$ -modules are compatible via the forgetful functors. In other words, we have a commutative diagram

$$(5.1) \quad \begin{array}{ccc} D(\mathcal{O}_Y\text{-mod}_{\text{qc}}) & \xleftarrow{\phi^!} & D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \\ \uparrow \text{oblv}^r & & \uparrow \text{oblv}^r \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xleftarrow{\phi^!} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \end{array}$$

**Fact 5.15.** In the (derived) setting of Construction 3.1, we have

$$\begin{aligned} \phi^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) &\simeq \phi^*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \phi^*(\mathcal{N}), \\ \phi^!(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}) &\simeq \phi^!(\mathcal{M}') \otimes_{\mathcal{O}_Y} \phi^*(\mathcal{N}), \\ \phi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^* \mathcal{M}, \phi^* \mathcal{N}), \\ \phi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^! \mathcal{M}', \phi^! \mathcal{N}'), \\ \phi^! \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^* \mathcal{M}, \phi^! \mathcal{N}'). \end{aligned}$$

**Fact 5.16.** For  $\text{pr}_i : X_1 \times X_2 \rightarrow X_i$ , we have

$$\begin{aligned} \mathcal{M}_1 \boxtimes \mathcal{M}_2 &\simeq \text{pr}_1^*(\mathcal{M}_1) \otimes_{\mathcal{O}_{X_1 \times X_2}} \text{pr}_2^*(\mathcal{M}_2) \\ \mathcal{M}'_1 \boxtimes \mathcal{M}'_2 &\simeq \text{pr}_1^!(\mathcal{M}_1) \overset{!}{\otimes} \text{pr}_2^!(\mathcal{M}_2). \end{aligned}$$

## 6. PUSHFORWARDS

**Construction 6.1.** Let  $\phi : Y \rightarrow X$  be a map between smooth  $k$ -schemes. Recall the transfer module

$$\mathcal{D}_{Y \rightarrow X} := \phi^* \mathcal{D}_X \simeq \mathcal{O}_Y \otimes_{\phi^{-1} \mathcal{O}_X} \phi^{-1} \mathcal{D}_X$$

is a bimodule for  $(\mathcal{D}_Y, \phi^{-1} \mathcal{D}_X)$ . We define a functor

$$\phi_{*, \text{dR}} : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \rightarrow D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r), \mathcal{N} \mapsto \phi_*(\mathcal{N} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}),$$

where

- The (left derived) tensor product functor  $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$  sends a complex of right  $\mathcal{D}_Y$ -modules to a complex of right  $\phi^{-1} \mathcal{D}_X$ -modules.
- The (right derived) functor  $\phi_*$  sends a complex of right  $\phi^{-1} \mathcal{D}_X$ -modules to a complex of right  $\mathcal{D}_X$ -modules via the homomorphism  $\mathcal{D}_X \rightarrow \phi_*(\phi^{-1} \mathcal{D}_X)$ .

We call  $\phi_{*, \text{dR}}$  the **direct image functor**, or **de Rham pushforward functor**, of (complices) of right  $\mathcal{D}$ -modules.

*Remark 6.2.* One can show the direct image functors of right  $\mathcal{D}$ -modules are compatible with composition of maps between smooth  $k$ -schemes.

*Remark 6.3.* The functor  $\phi_{*, \text{dR}}$  is called the *de Rham* pushforward functor because for  $\pi : X \rightarrow \text{pt}$ ,  $\pi_{*, \text{dR}}(\omega_X[-d_X])$  can be identified with the de Rham complex of  $X$ . For this reason, we also write

$$\Gamma_{\text{dR}}(X, -) := \pi_{*, \text{dR}}(-).$$

You are strongly encouraged to look at its proof in [G, Sect. 5.17].

*Remark 6.4.* Some authors use the notation  $\phi_*$  for  $\phi_{*, \text{dR}}$ .

*Remark 6.5.* The cohomological amplitude of  $\phi_{*, \text{dR}}$  is  $[-d_Y, d_Y]$ . Better estimation exist in the following cases:

- If  $\phi$  is affine, then the bounds can be  $[-d_Y, 0]$ .
- If  $\phi$  is smooth, then the bounds can be  $[-d_Y + d_X, d_Y]$ .
- If  $\phi$  is a closed embedding, then the functor is t-exact.

**Warning 6.6.** One can define a functor between the abelian categories using the same formula. However, that functor would not be  $\mathcal{H}^0(\phi_{*, \text{dR}})$  and is of less interests.

*Exercise 6.7.* This is **Homework 5, Problem 4**. Let  $x : \text{pt} \rightarrow X$  be a closed point of  $X$ . We write  $\delta_x := x_{*, \text{dR}}(k)$ . Prove:

- (1)  $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$  as a right  $\mathcal{D}_X$ -module.
- (2)  $\delta_x$  is set-theoretically support on at  $x$ , i.e., for the complement open  $U := X - x$ , we have  $\delta_x|_U = 0$ .
- (3) There exists a unique section  $\text{Dirac}_x$  of  $\delta_x$  such that  $\text{Dirac}_x \cdot f = f(x) \text{Dirac}_x$  for any local section  $f$  of  $\mathcal{O}_X$  defined near  $x$ .
- (4)  $\delta_x$  is generated by  $\text{Dirac}_x$  as a right  $\mathcal{D}_X$ -module.

*Remark 6.8.* The section  $\text{Dirac}_x$  should be viewed as the incarnation of the Dirac function in the theory of  $\mathcal{D}$ -modules.

**Lemma 6.9.** *The following diagram commutes:*

$$(6.1) \quad \begin{array}{ccc} D(\mathcal{O}_Y\text{-mod}_{\text{qc}}) & \xrightarrow{\phi_*} & D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \\ \text{ind}^r \downarrow & & \text{ind}^r \downarrow \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi_*, \text{dR}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \end{array}$$

*Sketch.* For  $\mathcal{F} \in D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$ , we have

$$\phi_{*, \text{dR}} \circ \text{ind}^r(\mathcal{F}) \simeq \phi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}) \simeq \phi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \phi^* \mathcal{D}_X) \simeq \phi_* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \text{ind}^r \circ \phi_*(\mathcal{F})$$

where the last isomorphism is the (derived) projection formula.  $\square$

We state the following results without proof.

**Proposition 6.10.** *If  $\phi : Y \rightarrow X$  is proper, then we have adjoint functors*

$$\phi_{*, \text{dR}} : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \rightleftarrows D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \phi^!.$$

**Proposition 6.11.** *If  $\phi : Y \rightarrow X$  is smooth, then we have adjoint functors*

$$\phi^![-2d_Y + 2d_X] : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \rightleftarrows D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \phi_{*, \text{dR}}.$$

*Remark 6.12.* If  $\phi : Y \rightarrow X$  is proper, then the square (5.1) can be obtained from (6.1) by passing to right adjoints.

**Construction 6.13.** *As in the case of pullback functors, we can define the **direct image functor** of left  $\mathcal{D}$ -modules:*

$$\begin{array}{ccc} D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l) & \xrightarrow{\phi_{*, \text{dR}}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \\ \omega_Y[d_Y] \downarrow \simeq & & \simeq \downarrow \omega_X[d_X] \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi_{*, \text{dR}}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r). \end{array}$$

## 7. KASHIWARA'S LEMMA

If  $\phi : Y \rightarrow X$  is a closed embedding, then the tensor product functor  $-\otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$  is t-exact because  $\mathcal{D}_{Y \rightarrow X}$  is locally free as a  $\mathcal{D}_Y$ -module. On the other hand, the functor  $\phi_*$  is also t-exact because  $\phi$  is affine. Therefore the functor  $\phi_{*, \text{dR}}$  is t-exact.

**Theorem 7.1** (Kashiwara's lemma). *Let  $\phi : Y \rightarrow X$  be a closed embedding between smooth  $k$ -schemes, then the exact functor*

$$\phi_{*, \text{dR}} : \mathcal{D}_Y\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_X\text{-mod}_{\text{qc}}^r$$

*is fully faithful and its essential image contains exactly right  $\mathcal{D}_X$ -modules that are set-theoretically supported on  $Y$ .*

*Remark 7.2.* Using Kashiwara's lemma, we can define  $\mathcal{D}_Y\text{-mod}_{\text{qc}}^r$  even for finite type singular  $k$ -scheme  $Y$ . Namely, if  $Y$  is affine, we can embed  $Y$  into a smooth ambient  $k$ -scheme  $X$  and define a right  $\mathcal{D}$ -module on  $Y$  to be a right  $\mathcal{D}$ -module on  $X$  that is set-theoretically supported on the image of  $Y$ . One can show the obtained abelian category does not depend on the choice of the embedding. When  $Y$  is not affine, we can define the category by gluing.

Moreover, all the previous constructions about right  $\mathcal{D}$ -modules can be generalized to the singular case.

A more canonical construction of  $\mathcal{D}_Y\text{-mod}_{\text{qc}}^r$  or even  $\mathcal{D}_Y\text{-mod}_{\text{qc}}^l$  for singular  $k$ -schemes is to use the theory of (Grothendieck's) crystals.

Another application of Kashiwara's lemma is the following result. See [G, Sect. 5.12] for a proof.

**Corollary 7.3.** *Let  $X$  be a smooth  $k$ -scheme, then any  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -module is locally free as an  $\mathcal{O}_X$ -module.*

## 8. BASE-CHANGE ISOMORPHISM AND PROJECTION FORMULA

**Fact 8.1.** *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\phi'} & X' \\ \downarrow f & & \downarrow g \\ Y & \xrightarrow{\phi} & X \end{array}$$

*be a Cartesian square of finite type  $k$ -schemes. Then we have equivalences*

$$g^! \circ \phi_{*,\text{dR}} \simeq \phi'_{*,\text{dR}} \circ f^!$$

*between functors  $D(\mathcal{D}_Y\text{-mod}_q c^r) \rightarrow D(\mathcal{D}_{X'}\text{-mod}_q c^r)$ .*

**Fact 8.2.** *Let  $\phi : Y \rightarrow X$  be any morphism between finite type  $k$ -schemes. Then we have*

$$\phi_{*,\text{dR}}(- \overset{!}{\otimes} \phi^!(\bullet)) \simeq \phi_{*,\text{dR}}(-) \overset{!}{\otimes} \bullet.$$

*Exercise 8.3.* This is **Homework 5, Problem 5**. Let  $x : \text{pt} \rightarrow X$  be a closed point of  $X$ . Prove<sup>2</sup>  $\delta_x \otimes^! \delta_x \simeq \delta_x$ .

## 9. DUALITY

The duality functor is only defined on *coherent*  $\mathcal{D}$ -modules.

**Fact 9.1.** *For any  $\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_c^r)$ , there exists a unique object  $\mathbb{D}\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_c^r)$  such that*

$$\Gamma_{\text{dR}}(X, \mathcal{M} \overset{!}{\otimes} -) \simeq \text{Hom}(\mathbb{D}\mathcal{M}, -)$$

*as functors  $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \rightarrow \text{Vect}$ . The obtained functor*

$$\mathbb{D} : D^b(\mathcal{D}_X\text{-mod}_c^r)^{\text{op}} \rightarrow D^b(\mathcal{D}_X\text{-mod}_c^r)$$

*is an anti-involution, i.e.,  $\mathbb{D} \circ \mathbb{D} \simeq \text{Id}$ .*

**Remark 9.2.** The construction of  $\mathbb{D}\mathcal{M}$  can be treated as a blackbox.

**Example 9.3.** We have  $\mathbb{D}(\omega_X) \simeq \omega_X$ .

**Construction 9.4.** *Let  $\phi : Y \rightarrow X$  be a map between finite type  $k$ -schemes. The standard functors  $\phi^!$  and  $\phi_{*,\text{dR}}$  in general do not preserve coherent complices. Hence we only have partially defined functors*

$$\phi_! := \mathbb{D} \circ \phi_{*,\text{dR}} \circ \mathbb{D}, \quad \phi_{\text{dR}}^* := \mathbb{D} \circ \phi^! \circ \mathbb{D}.$$

*They are called the **!-direct image functor** and the **de Rham pullback functor**.*

<sup>2</sup>A formal proof exists, but you are encouraged to do some direct calculations to see  $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = 0$  unless  $i = -d_X$  and  $\mathcal{H}^{-d}(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = \delta_x$ .



**Fact 9.5.** Let  $\phi : Y \rightarrow X$  be a map between finite type  $k$ -schemes. Then  $\phi_!$  is equivalent to the partially defined left adjoint of  $\phi^!$ . More precisely, we have

$$\mathrm{Hom}(\phi_! \mathcal{M}, -) \simeq \mathrm{Hom}(\mathcal{M}, \phi^!(-))$$

whenever  $\phi_! \mathcal{M}$  is well-defined. Similarly,  $\phi_{\mathrm{dR}}^*$  is equivalent to the partially defined left adjoint of  $\phi_{*, \mathrm{dR}}$ .

*Remark 9.6.* If  $\phi$  is proper, then  $\phi_! \simeq \phi_{*, \mathrm{dR}}$ . If  $\phi$  is smooth, then  $\phi_{\mathrm{dR}}^* \simeq \phi^![-2d_Y + 2d_X]$ .

## 10. HOLONOMIC D-MODULES

We do not give the standard definition of holonomic D-modules. Instead, we characterize them as follows:

**Fact 10.1.** Let  $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\mathbb{C}}^r$ , then  $\mathcal{M}$  is **holonomic** iff  $\mathbb{D}\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\mathbb{C}}^r$  (rather than just in the derived category).

**Fact 10.2.** Let  $\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_{\mathbb{C}}^r)$ , then  $\mathcal{M}$  has **holonomic cohomologies**, i.e.,  $\mathcal{H}^\bullet(\mathcal{M})$  are holonomic, iff for any closed point  $i : x \rightarrow X$ , the complex  $i^! \mathcal{M} \in D^b(\mathcal{D}_{\mathrm{pt}}\text{-mod}_{\mathbb{C}}^r) \simeq D^b(\mathrm{Vect})$  has finite dimensional cohomologies.

**Notation 10.3.** Let  $\mathcal{D}_X\text{-mod}_{\mathrm{hol}}^r$  be the abelian category of holonomic right  $\mathcal{D}_X$ -modules and  $D^b(\mathcal{D}_X\text{-mod}_{\mathrm{hol}}^r)$  be the bounded derived category.

**Fact 10.4.**  $D^b(\mathcal{D}_X\text{-mod}_{\mathrm{hol}}^r)$  is equivalent to the full subcategory of  $D^b(\mathcal{D}_X\text{-mod}_{\mathbb{C}}^r)$  containing complexes with holonomic cohomologies.

**Fact 10.5.** All the functors defined so far preserve bounded holonomic complexes.

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