2024 Spring — Geometric Representation Theory (1)

HOMEWORK PROBLEMS

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Homework 2 (Due on April 1)

Problem 2.1 (Lecture 4, Exercise 9)**.** Let $\mathfrak{g} = \mathfrak{sl}_2$ and e, h, f be the standard basis. Consider

$$\Omega := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2).$$

Calculate its image in $\mathrm{Sym}(\mathfrak{t}) = k[h]$ under the k-linear surjection $U(\mathfrak{g}) \twoheadrightarrow \mathrm{Sym}(\mathfrak{t})$ (see equation (2.1) of Lecture 4).

Solution. Using PBW theorem, the prescribed map can be reinterpreted as

$$U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{n}^-) \otimes_k U(\mathfrak{t}) \otimes_k U(\mathfrak{n}) \longrightarrow U(\mathfrak{t}) \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{t}) = k[h].$$

Recall that the elements of the standard PBW basis of $U(\mathfrak{n}^-) \otimes_k U(\mathfrak{t}) \otimes_k U(\mathfrak{n})$ are of form $f^a \otimes h^b \otimes e^c$ with $a, b, c \in \mathbb{Z}_{\geqslant 0}$, which is the image of $f^a h^b e^c \in U(\mathfrak{g})$ along the first isomorphism above. Motivated by this, we write

$$\Omega = ef + fe + \frac{1}{2}h^2 = [e, f] + 2fe + \frac{1}{2}h^2 = 2fe + h + \frac{1}{2}h^2.$$

On the other hand, note that $f \otimes 1 \otimes e$ and $1 \otimes h \otimes 1$ are respectively sent to 0 and h in $\mathrm{Sym}(\mathfrak{t})$. Thus,

$$\begin{aligned} U(\mathfrak{sl}_2) &\longrightarrow \mathrm{Sym}(\mathfrak{t}) = k[h] \\ ef &\longmapsto 0 \\ h &\longmapsto h. \end{aligned}$$

It follows that the image of Ω is $h + \frac{1}{2}h^2$. $\square$

Problem 2.2 (Lecture 4, Exercise 14)**.** Let $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to k$ be any nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. For any basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ and its dual basis $x_1^*, \ldots, x_n^*$ with respect to the form κ[1], consider the Casimir element

$$\Omega_\kappa = \sum_{i=1}^{n} x_i \cdot x_i^* \in U(\mathfrak{g}).$$

(1) Prove: the Casimir element Ω_κ does not depend on the choice of the basis, and is contained in the center $Z(\mathfrak{g})$.
(2) For $\mathfrak{g} = \mathfrak{sl}_2$, $\kappa = \mathsf{Kil}$, and the canonical basis e, h, f, find Ω_{Kil} and prove it is not contained in $\mathrm{Sym}(\mathfrak{t}) \subset U(\mathfrak{g})$.

Solution. (1) Note that the nondegenerate symmetric pairing κ is specialized from the map $\mathfrak{g} \otimes \mathfrak{g} \to k$, and it uniquely corresponds to an isomorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}^*$, whose inverse is written as $\phi^{-1}\colon \mathfrak{g}^* \to \mathfrak{g}$. Since ϕ^{-1} is again an isomorphism, there is a unique nondegenerate symmetric pairing $\kappa^\vee\colon k \to \mathfrak{g} \otimes \mathfrak{g}$ determined by ϕ^{-1}. Consider its composite with the natural embedding map $\mathfrak{g} \otimes \mathfrak{g} \to U(\mathfrak{g})$; we claim that

[1]By definition, this means $\kappa(x_i, x_j^*)_{i,j}$ is the unit matrix.

$$\begin{array}{ccccc} k & \xrightarrow{\kappa^\vee} & \mathfrak{g}\otimes\mathfrak{g} & \longrightarrow & U(\mathfrak{g}) \\ 1 & \longmapsto & & \longrightarrow & \Omega_\kappa, \end{array}$$

i.e. the Casimir element is the image of $1 \in k$. Indeed, as $x_1, \dots, x_n$ is a basis of $\mathfrak{g}$, its dual basis of $\mathfrak{g}^*$ with respect to κ is given by $x_1^*, \dots, x_n^*$ with $x_i^* = \phi(x_i)$. So the image of $1 \in k$ in $U(\mathfrak{g})$ is given by that of $(\sum_i x_i) \otimes \left(\sum_i \phi^{-1}(x_i^*)\right)$. With the isomorphism ϕ^{-1} we are able to identify $\phi^{-1}(x_i^*)$ with x_i^* by abuse of notation, and hence the image is the same as $\sum_i x_i \otimes x_i^*$, which further maps to $\sum_i x_i \cdot x_i^* = \Omega_\kappa \in U(\mathfrak{g})$. This proves the claim. To conclude, we see Ω_κ is only determined by $1 \in k$ and the choice of $\kappa^\vee$, so it is independent of the choice of $x_1, \dots, x_n$.

We then check $\Omega_\kappa \in Z(\mathfrak{g})$. For any $V \in \mathfrak{g}\text{-}\mathsf{mod}$, there is a $\mathfrak{g}$-mod structure on V^* given by $(x \cdot f)(v) = f(-x \cdot v)$[2] for $f \in V^*$ and $x \in \mathfrak{g}$. With this $\mathfrak{g}$-action on $V = \mathfrak{g}$ and $V^* = \mathfrak{g}^*$, all prescribed maps $\phi, \phi^{-1}, \kappa, \kappa^\vee$ are $\mathfrak{g}$-linear. It follows that the preimage of Ω_κ in $\mathfrak{g}\otimes\mathfrak{g}$ is already $\mathfrak{g}$-invariant with respect to the diagonal $\mathfrak{g}$-action. By definition, this diagonal action is $x \cdot (u \otimes v) = [x, u] \otimes v + u \otimes [x, v]$, and its image in $U(\mathfrak{g})$ is $xuv - uxv + uxv - uvx = xuv - uvx$. Hence any $\mathfrak{g}$-invariant element in $\mathfrak{g}\otimes\mathfrak{g}$ is sent to $Z(\mathfrak{g})$, which implies $\Omega_\kappa \in Z(\mathfrak{g})$. (Formally, the argument above means the tensor $\mathfrak{g}$-module structure of $T(\mathfrak{g}) = \bigoplus_{n\geqslant 0} \mathfrak{g}^{\otimes n}$ is $\mathfrak{g}$-linearly compatible with the adjoint $\mathfrak{g}$-module stucture of $U(\mathfrak{g})$. So the $\mathfrak{g}$-invariant elements of $\mathfrak{g}\otimes\mathfrak{g}$ is mapped to the $\mathfrak{g}$-invariant elements of $U(\mathfrak{g})$, namely $Z(\mathfrak{g})$.)

(2) Using the relations $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$, we compute the matrices of ad_e, ad_f, and ad_h under the ordered basis $\mathtt{B} = \{e, f, h\}$ as

$$[\mathrm{ad}_e]_{\mathtt{B}} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad [\mathrm{ad}_f]_{\mathtt{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad [\mathrm{ad}_h]_{\mathtt{B}} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the definition,

$$\mathsf{Kil}(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = \mathrm{tr}([\mathrm{ad}_x]_{\mathtt{B}}[\mathrm{ad}_y]_{\mathtt{B}}).$$

So we deduce

$$\mathsf{Kil}(e, f) = \mathsf{Kil}(f, e) = 4, \quad \mathsf{Kil}(h, h) = 8.$$

Then, with respect to the Killing form, the ordered dual basis of $\mathtt{B}$ is given by

$$\mathtt{B}^* = \left\{ e^* = \frac{f}{4},\ f^* = \frac{e}{4},\ h^* = \frac{h}{8} \right\}.$$

Therefore,

$$\Omega_{\mathsf{Kil}} = \frac{1}{4}ef + \frac{1}{4}fe + \frac{1}{8}h^2 = \frac{1}{2}fe + \frac{1}{4}h + \frac{1}{8}h^2.$$

Here the right-hand side is written as a linear combination of elements in standard PBW basis. Note that $fe \notin k[h]$, and hence $\Omega_{\mathsf{Kil}} \notin \mathrm{Sym}(\mathfrak{t}) = k[h]$. $\square$

Alternative Solution. (1) Let $y_1, \dots, y_n$ be another basis of $\mathfrak{g}$ and $y_1^*, \dots, y_n^*$ be its dual basis. Then there are invertible matrices $A = (a_{ij}), B = (b_{ij}) \in \mathrm{M}_n(k)$ such that

$$x_i = \sum_{j=1}^n a_{ij} y_j, \quad x_i^* = \sum_{j=1}^n b_{ij} y_j^*.$$

Since κ is a bilinear form, $\kappa(x_i, x_j^*) = \kappa\left(\sum_k a_{ik} y_k, \sum_l b_{jl} y_l^*\right) = \sum_{k,l} a_{ik} b_{jl} \cdot \kappa(y_k, y_l^*)$. On the other hand, by definition we have $\kappa(x_i, x_j^*) = \kappa(y_i, y_j^*) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

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[2]Here the negative sign has various explanation: from the point of view of lie algebra, $\exp(-x) = \exp(x)^{-1}$; from the isomorphism $\mathfrak{gl}(V) = \mathfrak{gl}(V^*)^{\mathrm{op}}$; it is the unique action such that $k \to V \otimes V^*$ and $V \otimes V^* \to k$ are $\mathfrak{g}$-linear, where k is the trivial $\mathfrak{g}$-module.

It follows that $\delta_{ij} = \sum_{k,l} a_{ik}b_{jl} \cdot \delta_{kl} = \sum_k a_{ik}b_{jk}$. Changing the indices of the sum, this is equivalent to

$$\delta_{kl} = \sum_{i=1}^{n} a_{ik}b_{il}.$$

Therefore, to show the independence of the choice of basis, we have

$$\sum_{i=1}^{n} x_i \cdot x_i^* = \sum_{1\leqslant i,k,l\leqslant n} a_{ik}y_k \cdot b_{il}y_l^* = \sum_{1\leqslant k,l\leqslant n} \delta_{kl}(y_k \cdot y_l^*) = \sum_{k=1}^{n} y_k \cdot y_k^*.$$

Now it remains to check $\Omega_\kappa \in Z(\mathfrak{g})$, and it suffices to show Ω_κ commutes with any element in $U(\mathfrak{g})$. By definition of the universal enveloping algebra, it further suffices to show Ω_κ commutes with any $y \in \mathfrak{g}$. For this, as $x_1, \ldots, x_n$ is a chosen basis of $\mathfrak{g}$, we may assume $y = x_i$ and compute

$$\begin{aligned} x_i\Omega_\kappa - \Omega_\kappa x_i &= \sum_{j=1}^{n} x_i(x_j \cdot x_j^*) - (x_j \cdot x_j^*)x_i \\ &= \sum_{j=1}^{n} [x_i, x_j] \cdot x_j^* + x_j \cdot [x_i, x_j^*]. \end{aligned}$$

For this to be 0, from the assumption on κ we have that $\kappa([x_i, x_j], x_i^*) + \kappa(x_j, [x_i, x_j^*]) = 0$. So the desired result follows.

(2) The same as in the first solution. □

Problem 2.3 (Lecture 4, Exercise 31)**.** Let $\mathfrak{g} = \mathfrak{sl}_2$. Prove $Z(\mathfrak{sl}_2) \simeq k[\Omega_{\mathsf{Kil}}]$ where Ω_{Kil} is the Casimir element. (You can use [Lecture 4, Theorem 26] for this exercise.)

Solution. In Problem 2.2(2) we have proved that $\Omega_{\mathsf{Kil}} \notin \mathrm{Sym}(\mathfrak{t})$. However, in this problem Ω_{Kil} denotes (by abuse of notation) the image of $\Omega_{\mathsf{Kil}} \in Z(\mathfrak{sl}_2)$ in Problem 2.2(2) along the map $U(\mathfrak{sl}_2) \twoheadrightarrow \mathrm{Sym}(\mathfrak{t})$ in Problem 2.1 restricted to $Z(\mathfrak{sl}_2)$. Thus, combining the results before we deduce $k[\Omega_{\mathsf{Kil}}] = k[(h + h^2/2)/4] = k[2h + h^2]$. Also recall the Harish-Chandra isomorphism

$$Z(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{t})^{W_\bullet}.$$

It together with the fact that $\mathrm{Sym}(\mathfrak{t}) = k[h]$ for $\mathfrak{g} = \mathfrak{sl}_2$ reduces the proof to showing the set-theoretical equality

$$k[2h + h^2] = k[h]^{W_\bullet}.$$

We know the $W_\bullet$-action on $k[h]$ is given by $h \mapsto -h - 2$. It follows for $f(h) \in k[h]$ that if $f(h) \in k[h]^{W_\bullet}$ then $f(h) = f(-h-2)$, and in particular $f(0) = f(-2)$, implying that f is generated by a polynomial in h that simultaneously vanishes at 0 and -2, namely $2h + h^2$; therefore, $k[h]^{W_\bullet} \subset k[2h + h^2]$. Finally, the converse inclusion $k[h]^{W_\bullet} \supset k[2h + h^2]$ is clear because $2h + h^2 = 2(-h-2) + (-h-2)^2$. This completes the proof. □

Problem 2.4 (Lecture 5, Exercise 13)**.** Prove: the adjoint $\mathfrak{g}$-action on $U(\mathfrak{g})$, i.e.,

$$\mathfrak{g} \times U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}), \quad (x, u) \longmapsto \mathrm{ad}_x(u) = [x, u],$$

preserves each $\mathsf{F}^{\leqslant n}U(\mathfrak{g})$, and the induced $\mathfrak{g}$-action on $\mathrm{gr}^\bullet(U(\mathfrak{g})) \simeq \mathrm{Sym}(\mathfrak{g})$ is the adjoint action in [Lecture 5, Construction 11].

Solution. Since $U(\mathfrak{g})$ is the quotient of $\bigoplus_{n\geqslant 0} \mathfrak{g}^{\otimes n}$, each $\mathsf{F}^{\leqslant n}U(\mathfrak{g})$ contains elements generated by elements in $\mathfrak{g}$ of degree at most n; namely, each $u \in \mathsf{F}^{\leqslant n}U(\mathfrak{g})$ is of form $u = \sum_i a_i \cdot u_{i1} \cdots u_{in_i}$ for some $1 \leqslant n_i \leqslant n$. Without loss of generality we may assume $u = u_1 \cdots u_m \in \mathsf{F}^{\leqslant n}U(\mathfrak{g})$ for some $1 \leqslant m \leqslant n$ with $u_i \in \mathfrak{g}$. Then for $x \in \mathfrak{g}$,

$$\begin{aligned}\mathrm{ad}_x(u) &= x \cdot u - \sum_{i=1}^{m-1} u_1 \cdots u_i \cdot x \cdot u_{i+1} \cdots u_m + \sum_{i=1}^{m-1} u_1 \cdots u_i \cdot x \cdot u_{i+1} \cdots u_m - u \cdot x \\ &= \sum_{i=1}^{m} u_1 \cdots u_{i-1} \cdot [x, u_i] \cdot u_{i+1} \cdots u_m.\end{aligned}$$

Given this, note that for $u_i \in \mathfrak{g}$ we have $[x, u_i] \in \mathfrak{g}$; it then follows that $\mathrm{ad}_x(u) \in \mathsf{F}^{\leqslant n} U(\mathfrak{g})$. Thus the filtration of $U(\mathfrak{g})$ is preserved by the adjoint $\mathfrak{g}$-action.

By PBW theorem the image of $u_1 \cdots u_m \in U(\mathfrak{g})$ in $\mathrm{gr}^\bullet U(\mathfrak{g})$ is $u_1 \otimes \cdots \otimes u_m$. From the formula of $\mathrm{ad}_x(u)$ above, we see under the induced $\mathfrak{g}$-action by $x \in \mathfrak{g}$ on $\mathrm{Sym}(\mathfrak{g})$, we have

$$(x, u_1 \otimes \cdots \otimes u_m) \longmapsto \sum_{i=1}^{m} u_1 \otimes \cdots \otimes u_{i-1} \otimes (x \cdot u_i) \otimes u_{i+1} \otimes \cdots u_m.$$

This is the same as [Lecture 5, Construction 11]. $\square$

Problem 2.5 (Lecture 5, Exercise 23)**.** Let $\lambda \in P^+$ be a dominant integral weight and $n \geqslant 0$. Prove there exist scalars $c_{\lambda'} \in k$, $\lambda' \prec \lambda$, such that

$$\phi_{\mathrm{cl}}(a_{\lambda,n}) = a_{\lambda,n}|_{\mathfrak{t}} = \frac{1}{\#\mathrm{Stab}_W(\lambda)} b_{\lambda,n} + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n},$$

where $\mathrm{Stab}_W(\lambda) \subset W$ is the stabilizer of the W-action at λ.

Solution. Let L_λ be the unique finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Denote by $\mathrm{wt}(L_\lambda)$ the set of all weights of L_λ. For any $x \in \mathfrak{t}$, from the definition we have

$$a_{\lambda,n}(x) = \mathrm{tr}(x^n; L_\lambda) = \sum_{\mu \in \mathrm{wt}(L_\lambda)} \dim(L_\lambda)_\mu \cdot \mu^n(x).$$

Consider the group action of W on $\mathrm{wt}(L_\lambda)$[3] with finitely many orbits $W\lambda_0, W\lambda_1, \ldots, W\lambda_s$, where $\lambda_0 = \lambda$. Since λ is dominant, each $\lambda' \in \mathrm{wt}(L_\lambda)$ satisfies $\lambda' \prec \lambda$. Thus,

$$\begin{aligned}\alpha_{\lambda,n}(x) &= \sum_{\mu \in \mathrm{wt}(L_\lambda)} \dim(L_\lambda)_\mu \cdot \mu^n(x) \\ &= \sum_{i=0}^{s} \sum_{\mu \in W\lambda_i} \dim(L_\lambda)_{\lambda_i} \cdot \mu^n(x) \\ &= \sum_{i=0}^{s} \frac{1}{\#\mathrm{Stab}_W(\lambda_i)} \sum_{w \in W} \dim(L_\lambda)_{\lambda_i} \cdot (w\lambda_i)^n(x) \\ &= \frac{1}{\#\mathrm{Stab}_W(\lambda)} b_{\lambda,n}(x) + \sum_{\lambda' \prec \lambda} c_{\lambda'} b_{\lambda',n}(x)\end{aligned}$$

for some scalars $c_{\lambda'} \in k$. Here $b_{\lambda,n} \coloneqq \sum_{w \in W} w(\lambda^n)$, and the last equality is because $w\lambda_i \prec \lambda$ for $\lambda_i \neq \lambda$ and each $\lambda' \prec \lambda$ has the form $w\lambda_i$ for some w and some $i > 0$. $\square$

[3]The action of W on $\mathrm{wt}(L_\lambda)$ can be realized as follows. As a $\mathfrak{g}$-module, L is G_{sc}-integrable as it is finite-dimensional (in our case $G = G_{\mathrm{sc}}$); this induces the action of $N_T(G)$ on L. On the other hand, corresponding to the action of $\mathfrak{t}$, the action of T on L preserves all weight spaces, and in particular preserves the highest weight space L_λ. It follows that $W \coloneqq N_T(G)/T$ acts on $\mathrm{wt}(L)$, and thus on $\mathrm{wt}(L_\lambda)$.