1 Direct and Inverse Images of Sheaves

Suppose $\pi : X' \to X$ is continuous, the **direct image** of a sheaf $\mathscr{F}'$ on $X'_{E'}$ is defined to be $\pi_*\mathscr{F}'$, and the **inverse image** of a sheaf $\mathscr{F}$ on X_E is defined to be $\pi^*\mathscr{F} := (\pi^{-1}\mathscr{F})^{\text{sh}}$. There are canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{Sh}(X_E)}(\mathscr{F}, \pi_*\mathscr{F}') \simeq \mathrm{Hom}_{\mathbf{PSh}(X'_{E'})}(\pi^{-1}\mathscr{F}, \mathscr{F}') \simeq \mathrm{Hom}_{\mathbf{Sh}(X'_{E'})}(\pi^*\mathscr{F}, \mathscr{F})$$

which shows that π^* and π_* are adjoint functors, thus π_* is left exact and commutes with limits, and π^* is right exact and commutes with colimits.

Theorem 1.1. Let $\pi : X' \to X$ be a continuous homomorphism.

1. For any sheaf $\mathscr{F}$ on $X_{\text{ét}}$ and any $x' \in X'$, $(\pi^*\mathscr{F})_{\bar{x}'} \simeq \mathscr{F}(\overline{\pi(x')})$. In particular, if π is the canonical morphism $\pi : \operatorname{Spec} \mathscr{O}_{X,\overline{x}} \to X$, then
$$\mathscr{F}_{\bar{x}} = (\pi^*\mathscr{F})_{\bar{x}} = \Gamma(\operatorname{Spec} \mathscr{O}_{X,\overline{x}}, \mathscr{F})$$
2. Assume that π is quasi-compact. Let $x \in X$, $\overline{x} = \operatorname{Spec} \kappa(x)^{\text{sep}}$, f be the canonical morphism $\widetilde{X} = \operatorname{Spec} \mathscr{O}_{X,\overline{x}} \to X$, and let $\widetilde{X}' = X' \times_X \widetilde{X}$
$$\begin{tikzcd} \widetilde{X}' \arrow[r, "f'"] \arrow[d, "\tilde{\pi}"'] & X' \arrow[d, "\pi"] \\ \widetilde{X} \arrow[r, "f"'] & X \end{tikzcd}$$
then $(\pi_*\mathscr{F})_{\bar{x}} = \Gamma(\widetilde{X}', f'^*\mathscr{F})$ for any sheaf $\mathscr{F}$ on X.

Proof. *1.* Write $x = \pi(x')$, we may take $\overline{x} = \overline{x}'$ and so have a commutative diagram

$$\begin{tikzcd} & \overline{x}' = \overline{x} \arrow[dl, "u_{x'}"'] \arrow[dr, "u_x"] & \\ X' \arrow[rr, "\pi"'] & & X \end{tikzcd}$$

Then $(\pi^*\mathscr{F})_{\bar{x}'} = (u_{x'}^*\pi^*\mathscr{F})(\overline{x}') = (u_x^*\mathscr{F})(\overline{x}) = \mathscr{F}_{\bar{x}}$. Meanwhile, for the second statement, $(u_x^*\mathscr{F})(\overline{x}) = \Gamma(\operatorname{Spec} \mathscr{O}_{X,\overline{x}}, \pi^*\mathscr{F})$ since every étale neighborhood $\overline{x} \to U$ of $\overline{x}$ factors through $\overline{x} \to \operatorname{Spec} \mathscr{O}_{X,\overline{x}}$ according to the definition of strict Henselization.

We may need a lemma to finish our proof of *2*:

Lemma 1.2. Let X be a scheme and let $Y = \varprojlim Y_i$, where (Y_i) is a filtered inverse system of X-schemes such that the transition morphisms $Y_i \leftarrow Y_j$ are affine. Assume that Y_i are quasi-compact, let Z be an X-scheme of finite type, then any X-morphism $Y \to Z$ factors through $Y \to Y_i$ for some i. In other words, $\mathrm{Hom}_X(Y, Z) = \varinjlim \mathrm{Hom}_X(Y_i, Z)$.

Proof. The proof is clear when all X, Y_i, Z are affine as for every A-algebra C of finite type, every homomorphism $C \to \varinjlim B_i$ factors through some B_i. □

Proof of *2*. By definition $(f'^*\mathscr{F})(\widetilde{X}') = \varinjlim \mathscr{F}(U')$, where the limit is taken over all commutative diagrams with $U' \to X'$ étale:

$$\begin{tikzcd} \widetilde{X}' \arrow[r] \arrow[dr] & U' \arrow[d] \\ & X' \end{tikzcd}$$

On the other hand, from the definitions of π_* and stalks, we see that $(\pi_*\mathscr{F})_{\bar{x}} = \varinjlim \mathscr{F}(U_{X'})$ where the limit is taken over all such diagrams that come by base extension from a commutative diagram with $U' \to X'$ étale:

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & U \arrow[d] \\ & X \end{tikzcd}$$

As $\widetilde{X}' = \varprojlim U_{X'}$, where U takes over all affine étale neighborhood of x, every morphism $\widetilde{X}' \to U'$ factors through some $U_{X'}$ for some étale neighborhood U of $x \in X$ according to lemma 1.2. Thus the two colimits are isomorphic. $\square$

Remark 1.1. If $\mathscr{F}$ is a sheaf on $X'_{\text{ét}}$ defined by a group scheme G that is locally of finite-type over X', then $(f'^*\mathscr{F})(\widetilde{X}') = G(\widetilde{X}')$.

Corollary 1.3. *1.* Let $i : Z \to X$ be a closed immersion, and let $\mathscr{F}$ be a sheaf on $Z_{\text{ét}}$. Let $x \in X$, then

$$(i_*\mathscr{F})_{\bar{x}} = \begin{cases} 0 & x \notin i(Z) \\ \mathscr{F}_{\bar{z}} & x = i(z),\ z \in Z \end{cases}$$

2. Let $j : U \to X$ be an open immersion, $x_0 \in U$, $x = j(x_0)$, then $(j_*\mathscr{F})_{\bar{x}} = \mathscr{F}_{\bar{x}_0}$.

3. Let $\pi : X' \to X$ be a finite morphism and $\mathscr{F}$ a sheaf on $X'_{\text{ét}}$. For any $x \in X$, $(\pi_*\mathscr{F})_{\bar{x}} = \prod \mathscr{F}_{\bar{x}'}^{d(x')}$, where the product is taken over all x' whose image under π is x, and $d(x')$ is the separable degree of $\kappa(x')$ over $\kappa(x)$.

Proof. *1.* For $x \notin i(Z)$, $\widetilde{Z}$ is empty. For $x \in i(Z)$, let $\mathscr{I}$ be the sheaf of ideals defining Z, we have $\widetilde{Z} = \operatorname{Spec}(\mathscr{O}_{X,\bar{x}}/\mathscr{I}) = \operatorname{Spec}(\mathscr{O}_{Z,\bar{z}})$.

2. Clear.

3. Notice that $\widetilde{X} = \operatorname{Spec} \mathscr{O}_{X,\bar{x}} \times_X X'$ splits into a disjoint union of $\operatorname{Spec}(\mathscr{O}_{X',\bar{x}'}^{d(x')})$, the conclusion follows at once. $\square$

Corollary 1.4. If π is a finite morphism, then π_* is exact.

Proof. This is direct from the proposition. $\square$

We now consider the situation: X is a scheme, U is an open subscheme of X, and Z is a subscheme of X whose underlying set is the complementary closed subset $Z = X - U$. We denote the inclusion maps by i and j:

$$Z \xrightarrow{i} X \xleftarrow{j} U$$

If $\mathscr{F}$ is a sheaf on $X_{\text{ét}}$, we get sheaves $\mathscr{F}_1 = i^*\mathscr{F}$, $\mathscr{F}_2 = j^*\mathscr{F}$ on Z and U, and a canonical map $\phi_{\mathscr{F}} : \mathscr{F}_1 = i^*\mathscr{F} \to i^*j_*\mathscr{F}_2 = i^*j_*j^*\mathscr{F}$. Define $\mathbf{T}$ to be the category consists of tuples $(\mathscr{F}_1, \mathscr{F}_2, \phi)$, where $\mathscr{F}_1$ is a sheaf on $Z_{\text{ét}}$, $\mathscr{F}_2$ is a sheaf on $U_{\text{ét}}$, $\phi : \mathscr{F}_1 \to i^*j_*\mathscr{F}_2$ is a $Z_{\text{ét}}$-sheaf homomorphism.

Theorem 1.5. There is an equivalence between the categories $\mathbf{Sh}(X_{\text{ét}})$ and $\mathbf{T}(X_{\text{ét}})$ under which $\mathscr{F} \in \mathbf{Sh}(X_{\text{ét}})$ corresponds to the tuple $(i^*\mathscr{F}, j^*\mathscr{F}, \phi_{\mathscr{F}})$.

Proof. Let T denotes the functor $\mathscr{F} \to (i^*\mathscr{F}, j^*\mathscr{F}, \phi_{\mathscr{F}})$. The quasi-inverse S of T can be defined by the following Cartesian diagram:

$$\begin{tikzcd} S(\mathscr{F}_1, \mathscr{F}_2, \phi) \arrow[r] \arrow[d] & j_*\mathscr{F}_2 \arrow[d] \\ i_*\mathscr{F}_1 \arrow[r, "i_*\phi"] & i_*i^*j_*\mathscr{F}_2 \end{tikzcd}$$

We claim that S is the quasi-inverse of T.

$ST \simeq \mathrm{id}$: It suffices to show that

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & j_*j^*\mathscr{F} \arrow[d] \\ i_*i^*\mathscr{F} \arrow[r] & i_*i^*j_*j^*\mathscr{F} \end{tikzcd}$$

is Cartesian. Since taking stalk is conservative and commutes with fiber products, we only need to check this diagram on stalk. For $x \in U$, the diagram becomes

$$\begin{tikzcd} \mathscr{F}_{\bar{x}} \arrow[r] \arrow[d] & \mathscr{F}_{\bar{x}} \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

after taking stalks at $\overline{x}$, which is clearly Cartesian. Meanwhile, the diagram becomes

$$\begin{tikzcd} \mathscr{F}_{\bar{x}} \arrow[r] \arrow[d] & (j_*j^*\mathscr{F})_{\bar{x}} \arrow[d] \\ \mathscr{F}_{\bar{x}} \arrow[r] & (j_*j^*\mathscr{F})_{\bar{x}} \end{tikzcd}$$

after taking stalks at $\overline{x}$.

$TS \simeq \mathrm{id}$: Notice that i^*, j^* commutes with Cartesian diagram, we have the following Cartesian diagrams:

$$\begin{tikzcd} i^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \arrow[r] \arrow[d] & i^*j_*\mathscr{F}_2 \arrow[d] \\ i^*i_*\mathscr{F}_1 \arrow[r] & i^*i_*i^*j_*\mathscr{F}_2 \end{tikzcd} \qquad \begin{tikzcd} j^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \arrow[r] \arrow[d] & j^*j_*\mathscr{F}_2 \arrow[d] \\ j^*i_*\mathscr{F}_1 \arrow[r] & j^*i_*i^*j_*\mathscr{F}_2 \end{tikzcd}$$

As $i^*i_*(-) = j^*j_*(-) = \mathrm{id}$, $j^*i_*(-) = 0$, we may simplify the diagrams into

$$\begin{tikzcd} i^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \arrow[r] \arrow[d] & i^*j_*\mathscr{F}_2 \arrow[d] \\ \mathscr{F}_1 \arrow[r] & i^*j_*\mathscr{F}_2 \end{tikzcd} \qquad \begin{tikzcd} j^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \arrow[r] \arrow[d] & \mathscr{F}_2 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

hence $i^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \simeq \mathscr{F}_1$, $j^*S(\mathscr{F}_1, \mathscr{F}_2, \phi) \simeq \mathscr{F}_2$, and the morphism ϕ is canonically the same. □

Remark 1.2. We may now reformulate i^*, i_*, j^*, j_* with this identification as

$$\begin{array}{rccc} i^*: & (\mathscr{F}_1, \mathscr{F}_2, \phi) & \mapsto & \mathscr{F}_1 \\ i_*: & \mathscr{F}_1 & \mapsto & (\mathscr{F}_1, 0, 0) \\ j^*: & (\mathscr{F}_1, \mathscr{F}_2, \phi) & \mapsto & \mathscr{F}_2 \\ j_*: & \mathscr{F}_2 & \mapsto & (i^*j_*\mathscr{F}_2, \mathscr{F}_2, \mathrm{id}) \end{array}$$

A sheaf $\mathscr{F}$ is said to be **supported** on a subscheme Y of X if $\mathscr{F}_x = 0$ for all $x \notin Y$.

Corollary 1.6. If $i : Z \to X$ is a closed immersion, then the functor $i_* : \mathbf{Sh}(Z_{\text{ét}}) \to \mathbf{Sh}(X_{\text{ét}})$ induces an equivalence between $\mathbf{Sh}(Z_{\text{ét}})$ and the full subcategory of $\mathbf{Sh}(X_{\text{ét}})$ comprising those sheaves with support on $i(Z)$.

We now define the **proper direct image** $j_!$ along j and the **twisted inverse image** $i^!$ along i as:

$$i^!(\mathscr{F}_1, \mathscr{F}_2, \phi) := \ker \phi, \ \ j_!(\mathscr{F}_2) := (0, \mathscr{F}_2, 0)$$

By definition we have $\mathrm{Hom}(j_!\mathscr{F}_2, \mathscr{G}) = \mathrm{Hom}(\mathscr{F}_2, j^*\mathscr{G})$, $\mathrm{Hom}(i_*\mathscr{F}_1, \mathscr{G}) = \mathrm{Hom}(\mathscr{F}_1, i^!\mathscr{G})$.

Remark 1.3. Generally, for an object $j : U \to X$ an object in $\mathscr{C}/X$ for some site $(\mathscr{C}/X)_E$ and a presheaf $\mathscr{F}$, we may define its proper direct image $j_!\mathscr{F}$ to be

$$j_!\mathscr{F}(V) = \bigoplus_{\phi \in \mathrm{Hom}_X(V,U)} \mathscr{F}(V_\phi)$$

which clearly defines the left adjoint of j^{-1}. For sheaves $\mathscr{F}$ we define $j_!\mathscr{F}$ to be the sheafification of the proper direct image of $\mathscr{F}$ as a presheaf, in this case $j_!$ becomes the left adjoint of j^*.

Let $\mathscr{A}$ be a sheaf of rings on a site $(\mathscr{C}/X)_E$, write $\mathbf{Sh}(X_E, \mathscr{A})$ for the category of sheaves of $\mathscr{A}$-modules. If $\mathscr{A}$ is the constant sheaf $\underline{\mathbb{Z}}$, then $\mathbf{Sh}(X_E, \mathscr{A}) = \mathbf{Sh}(X_E)$.

For any pair $\mathscr{F}_1$ and $\mathscr{F}_2$ of sheaves of $\mathscr{A}$-modules, $\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ is defined to be the sheaf associated with the presheaf $U \mapsto \mathscr{F}_1(U) \otimes_{\mathscr{A}(U)} \mathscr{F}_2(U)$, $\underline{\mathrm{Hom}}(\mathscr{F}_1, \mathscr{F}_2)$ is defined to be the sheaf $U \mapsto \mathrm{Hom}_{\mathbf{Sh}(U, \mathscr{A}|_U)}(\mathscr{F}_1|_U, \mathscr{F}_2|_U)$.

Proposition 1.7. We write $\mathrm{Bilin}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2; \mathscr{F}_3)$ for the sets of $\mathscr{A}$-bilinear maps $\mathscr{F}_1 \times \mathscr{F}_2 \to \mathscr{F}_3$, then we have canonical isomorphisms

$$\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2, \mathscr{F}_3) \simeq \mathrm{Hom}_{\mathscr{A}}(\mathscr{F}_1, \underline{\mathrm{Hom}}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2)) \simeq \mathrm{Bilin}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2; \mathscr{F}_3)$$

Proof. As we may treat $\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ as the quotient sheaf of the free sheaf generated by $\mathscr{F}_1 \times \mathscr{F}_2$ regarded as a sheaf of sets by the subsheaf generated from bilinearity, the isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2, \mathscr{F}_3) \simeq \mathrm{Bilin}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2; \mathscr{F}_3)$$

is clear. The proof that $\mathrm{Bilin}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2; \mathscr{F}_3) \simeq \mathrm{Hom}_{\mathscr{A}}(\mathscr{F}_1, \underline{\mathrm{Hom}}_{\mathscr{A}}(\mathscr{F}_2, \mathscr{F}_3))$. $\square$

A sheaf $\mathscr{F}$ of $\mathscr{A}$-modules on $X_{\text{ét}}$ is **pseudocoherent** at a geometric point $\overline{x}$ if there exists an étale neighborhood $U \to X$ of $\overline{x}$ and an exact sequence $(\mathscr{A}|_U)^m \to (\mathscr{A}|_U)^n \to \mathscr{F}|_U \to 0$ of sheaves on $U_{\text{ét}}$ with m, n finite.

Proposition 1.8. Let $\overline{x}$ be a geometric point of $\overline{x}$.

1. For any sheaf of sets $\mathscr{I}$ on $X_{\text{ét}}$, the stalk of the free sheaf of $\mathscr{A}$-modules $F\mathscr{I}$ associated to $\mathscr{I}$ at $\overline{x}$ is equal to $F\mathscr{I}_{\overline{x}}$, the free $\mathscr{A}_{\overline{x}}$-module associated with $\mathscr{I}_{\overline{x}}$.
2. If $\mathscr{F}_1$ is pseudocoherent at $\overline{x}$, then
$$\underline{\mathrm{Hom}}_{\mathscr{A}}(\mathscr{F}_1, \mathscr{F}_2)_{\overline{x}} = \mathrm{Hom}_{\mathscr{A}_{\overline{x}}}(\mathscr{F}_{1\overline{x}}, \mathscr{F}_{2\overline{x}})$$
3. For any pair of sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$, $(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2)_{\overline{x}} = \mathscr{F}_{1\overline{x}} \otimes_{\mathscr{A}_{\overline{x}}} \mathscr{F}_{2\overline{x}}$.

Proof. *1.* The stalk of the free sheaf of $\mathscr{A}$-modules is equal to the stalk of the free presheaf of $\mathscr{A}$-modules, hence

$$(F\mathscr{I})_{\overline{x}} = \varinjlim \mathscr{A}(U)^{\mathscr{I}(U)} = \mathscr{A}_{\overline{x}}^{\mathscr{I}_{\overline{x}}}$$

2. The conclusion holds for $\mathscr{F}_1 = \mathscr{A}$, hence holds for pseudocoherent $\mathscr{F}_1$ according to five lemma.

3. Tensor products commute with filtered colimits. $\square$