LECTURE 9

In this lecture, we introduce Kan extensions for ∞-categories, and use them to obtain several useful tools for computing (co)limits.

1. Definition of Kan extensions

1.1. Let $\delta : K_0 \to K$ be a morphism in Set_Δ. For an ∞-category D, consider the functor

$$\mathsf{Fun}(K, \mathsf{C}) \xrightarrow{-\circ\delta} \mathsf{Fun}(K_0, \mathsf{C}).$$

We say $F : K \to \mathsf{D}$ is a left (resp. right) Kan extension of $F_0 : K_0 \to \mathsf{D}$ if it is the image of F_0 under the *partially defined* left (resp. right) adjoint of the above functor. More precisely, we make the following definition.

Definition 1.2. *Let $\delta : K_0 \to K$ be a morphism in Set_Δ and $F_0 : K_0 \to \mathsf{C}$ be a diagram in an ∞-category C. For a diagram $F : K \to \mathsf{C}$, and a natural transformation $\beta : F_0 \to F \circ \delta$,*

$$\begin{tikzcd} & K \arrow[dr, dotted, "F"] & \\ K_0 \arrow[ur, "\delta"] \arrow[rr, "F_0"'] \arrow[rr, phantom, shift left=4, "\Uparrow\beta"] & & \mathsf{C} \end{tikzcd}$$

*we say β **exhibits F as a left Kan extension** of F_0 along δ if for any diagram $F' : K \to \mathsf{C}$*

$$\mathsf{Maps}_{\mathsf{Fun}(K,\mathsf{C})}(F, F') \to \mathsf{Maps}_{\mathsf{Fun}(K_0,\mathsf{C})}(F \circ \delta, F' \circ \delta) \to \mathsf{Maps}_{\mathsf{Fun}(K_0,\mathsf{C})}(F_0, F' \circ \delta)$$

is an equivalence between ∞-groupoids.

Dually, we say a natural transformation $\beta : F \circ \delta \to F_0$,

$$\begin{tikzcd} & K \arrow[dr, dotted, "F"] & \\ K_0 \arrow[ur, "\delta"] \arrow[rr, "F_0"'] \arrow[rr, phantom, shift left=4, "\Downarrow\beta"] & & \mathsf{C} \end{tikzcd}$$

***exhibits F as a right Kan extension** of F_0 along δ if for any functor $F' : K \to \mathsf{C}$,*

$$\mathsf{Maps}_{\mathsf{Fun}(K,\mathsf{C})}(F', F) \to \mathsf{Maps}_{\mathsf{Fun}(K_0,\mathsf{C})}(F' \circ \delta, F \circ \delta) \to \mathsf{Maps}_{\mathsf{Fun}(K_0,\mathsf{C})}(F' \circ \delta, F_0)$$

is an equivalence between ∞-groupoids.

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1.3. Once we have established ∞-categorical Yoneda lemma, it is easy to show the pair (F,β) is essentially unique[1]. Hence we can talk about *the* right/left Kan extension of F_0 along δ, as long as we incorporate the natural transformation β as part of the data in its definition. We denote the corresponding diagrams to be

$$\mathsf{RKE}_\delta F_0,\ \mathsf{LKE}_\delta F_0.$$

When writing $\mathsf{RKE}_\delta F_0$, we always view it as an object in $\mathsf{Fun}(K,\mathsf{C})$ equipped with a *canonical* lifting along

$$\mathsf{Fun}(K,\mathsf{C}) \underset{\mathsf{Fun}(K_0,\mathsf{C})}{\times} \mathsf{Fun}(K_0,\mathsf{C})_{/F_0} \to \mathsf{Fun}(K,\mathsf{C}).$$

1.4. Note that

$$(\mathsf{LKE}_\delta F_0)^{\mathsf{op}} \simeq \mathsf{RKE}_{\delta^{\mathsf{op}}} F_0^{\mathsf{op}}.$$

Hence in below, we focus on right Kan extensions.

1.5. As in the classical category theory, one can show taking adjoint functors is compatible with compositions. Therefore we have the following result:

Proposition 1.6. *Let $K_0 \xrightarrow{\delta} K_1 \xrightarrow{\theta} K_2$ be morphisms in Set_Δ and $F_0 : K_0 \to \mathsf{C}$ be a diagram in an ∞-category C. Suppose $\mathsf{RKE}_\delta F_0$ exsits. Then we have a canonical equivalence*

$$\mathsf{RKE}_\theta(\mathsf{RKE}_\delta F_0) \xrightarrow{\simeq} \mathsf{RKE}_{\theta\circ\delta} F_0,$$

where the source exists iff the target does.

Remark 1.7. *The precise meaning of the above equivalence the following. Suppose $\beta : F_1\circ\delta \to F_0$ exhibits F_1 as a right Kan extension of F_0 along δ, and $\gamma : F_2\circ\theta \to F_1$ exhibits F_2 as a right Kan extension of F_1 along θ, then*

$$F_2\circ\theta\circ\delta \xrightarrow{\gamma(\delta)} F_1\circ\delta \xrightarrow{\beta} F_0$$

exhibits F_2 as a right Kan extension of F_0 along $\theta\circ\delta$.

Theorem 1.8. *Let $\delta : K_0 \to \mathcal{K}$ be a morphism in Set_Δ and $F_0 : K_0 \to \mathsf{C}$ be a diagram in an ∞-category C. Suppose that*

- *$\mathcal{K}$ is a quasi-category;*
- *For any object $x \in \mathcal{K}$, the limit of the diagram*[2]

$$K_0 \underset{\mathcal{K}}{\times} \mathcal{K}_{x/} \xrightarrow{p_x} K_0 \xrightarrow{F_0} \mathsf{C}$$

  exists.

Then $\mathsf{RKE}_\delta F_0$ exists, and we have a canonical isomorphism

$$(\mathsf{RKE}_\delta F_0)(x) \simeq \lim_{K_0\times_{\mathcal{K}}\mathcal{K}_{x/}} (F_0\circ p_x). \tag{1.1}$$

[1] This means the full sub-∞-category of

$$\mathsf{Fun}(K,\mathsf{C}) \underset{\mathsf{Fun}(K_0,\mathsf{C})}{\times} \mathsf{Fun}(K_0,\mathsf{C})_{/F_0}$$

consisting of pairs (F,β) that satisfy the definition of right Kan extensions is equivalent to $[0]$.

[2] The fiber product $K_0\times_{\mathcal{K}}\mathcal{K}_{x/}$ is taken in the ordinary category Set_Δ. It also calculates the homotopy fiber product in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ because $\mathcal{K}_{x/}\to\mathcal{K}$ is a categorical fibration. In particular, when K_0 is also a quasi-category, we can view the above fiber product as taking in the quasi-category $\mathcal{QC}at$. As a consequence, we can state the proposition purely using the language of ∞-categories.

Definition 1.9. *We say* $\mathsf{RKE}_\delta F_0$ *is* ***pointwise***[3] *if* δ *and* F_0 *satisfy the assumptions in Theorem 1.8.*

Remark 1.10. *Note that the construction* $x \mapsto K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$ *is (contravariantly) functorial while the naive one* $x \mapsto K \times_{\mathsf{C}} \{x\}$ *is not functorial.*

Remark 1.11. *The isomorphism* (1.1) *can be informally obtained as follows. Let* $F : \mathcal{K} \to \mathsf{C}$ *be a fixed functor and* $\beta : F \circ \delta \to F_0$ *be any natural transformation. For an object* $(y, f) \in K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$, *where* $y \in K_0$ *and* $f : x \to \delta(y)$ *is a morphism in* C, *consider the composition*

$$F(x) \xrightarrow{F(f)} F \circ \delta(y) \xrightarrow{\beta(y)} F_0(y). \tag{1.2}$$

This construction is functorial in (y, f) *and therefore corresponds to a morphism*

$$F(x) \to \lim_{(y,f) \in K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}} F_0(y) \tag{1.3}$$

as long as the target exists. Now the proposition claims (1.3) is invertible for any x *iff* β *exhibits* F *as a right Kan extension of* F_0 *along* δ. *In other words, the natural transformation* β *is completely encoded in the morphisms (1.2) and their higher functorialities.*

Remark 1.12. *To translate the above construction into homotopy coherent language, we first notice that there is an obvious natural transformation from the constant functor* $\underline{x} : \mathcal{K}_{x/} \to \mathcal{K}$ *to the forgetful functor* oblv. *Precomposing with the projection functor* $K \times_{\mathcal{K}} \mathcal{K}_{x/} \to \mathcal{K}_{x/}$, *we obtain the left natural transformation in the following diagram:*

$$\begin{tikzcd}
K_0 \times_{\mathcal{K}} \mathcal{K}_{x/} \arrow[rr, "\underline{x}"] \arrow[dr, "p_x"'] & \arrow[d, phantom, "\Downarrow"] & \mathcal{K} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \beta"] & \\
& K_0 \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & & \mathsf{C}.
\end{tikzcd}$$

Composing with β, *we obtain a natural transformation* $\underline{F(x)} \to F_0 \circ p_x$, *which by definition corresponds to a lifting of* $F(x)$ *along*

$$\mathsf{C}^{/F_0 \circ p_x} \to \mathsf{C},$$

where the source $\mathsf{C}^{/F_0 \circ p_x}$ *is the alternative slice category in [Lecture 6, §5]. Now the proposition claims that the above lifting is a final object (which is assumed to exist) for any* $x \in \mathcal{K}$ *iff* β *exhibits* F *as the right Kan extension of* F_0 *along* δ.

Exercise 1.13. *For* $\pi : K \to \Delta^0$, *show that* $\mathsf{RKE}_\pi u$ *exists iff* $\lim u$ *exists. Moreover, we have*

$$(\mathsf{RKE}_\pi u)(*) \simeq \lim u.$$

How to find the canonical lifting of $(\mathsf{RKE}_\pi u)(*)$ *in* $\mathsf{C}_{/u}$*?*

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[3] Not all right Kan extensions are pointwise. Also, Lurie *defined* (see Ker.02Y9) a right Kan extension as

$$x \mapsto \lim_{K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}} (F_0 \circ p_x)$$

and proved (see Ker.0309) it satisfies the universal property in Definition 1.2. Therefore Lurie's right Kan extensions should be *pointwise* right Kan extensions in these notes.

Warning 1.14. *Note that the fiber product $K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$ also appears in the statement of Quillen's Theorem A, which claims the following two conditions are equivalent*

(i) *The morphism $K_0 \to \mathcal{K}$ is final.*
(ii) *The fiber product $K_0 \times_{\mathcal{K}} \mathcal{K}_{x/}$ is weakly contractible for any $x \in \mathcal{K}$.*

Note however that (i) is related to colimits *and therefore* left *Kan extensions, while Theorem 1.8 is related to* right *Kan extensions. I do not know how to relate these two results.*

Exercise 1.15. *Show that a right Kan extension $\mathsf{RKE}_\delta F_0$ is pointwise iff it is preserved by any represetable functor $\mathsf{Maps}_{\mathsf{C}}(c,-) : \mathsf{C} \to \mathsf{Grpd}_\infty$.*

Proposition 1.16. *Let $\iota : \mathsf{K}_0 \to \mathsf{K}$ be a fully faithful functor between ∞-categories, and $F_0 : \mathsf{K}_0 \to \mathsf{C}$ be any functor. Suppose the pointwise $\mathsf{RKE}_\iota F_0$ exists*[4]*, then the canonical natural transformation*

$$(\mathsf{RKE}_\iota F_0) \circ \iota \to F_0$$

is invertible.

Sketch. We only need to show for any $x_0 \in K_0$ and $x := \iota(x_0)$, the evaluation morphism

$$\lim_{\mathsf{K}_0 \times_{\mathsf{K}} \mathsf{K}_{x/}} (F_0 \circ p_x) \to F_0(x_0)$$

is invertible. We only need to show $(x_0, x \xrightarrow{=} \iota(x_0)) \in \mathsf{K}_0 \times_{\mathsf{K}} \mathsf{K}_{x/}$ is initial. But this follows from the obvious equivalence $(\mathsf{K}_0)_{x_0/} \to \mathsf{K}_0 \times_{\mathsf{K}} \mathsf{K}_{x/}$. □

Exercise 1.17. *For an ∞-category K, consider the obvious embedding $\iota : \mathsf{K} \to \mathsf{K}^{\triangleleft}$. Show that the pointwise right Kan extension $\mathsf{RKE}_\iota u$ exists iff the limit of $u : \mathsf{K} \to \mathsf{C}$ exists. Moreover, $\mathsf{RKE}_\iota u$ is a limit diagram extending u.*

Exercise 1.18. *Show that pointwise right Kan extension of any functor along $K^{\triangleleft} \to \Delta^0$ always exists, and is given by evaluating the functor at the apex $* \in K^{\triangleleft}$.*

2. Limits commute with limits

2.1. From now on, $u : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{C}$ is a functor between ∞-categories[5]. Let $\mathsf{pr}_i : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{K}_i$ and $\pi_i : \mathsf{K}_i \to [0]$ be the projections.

Exercise 2.2. *The pointwise right Kan extension $\mathsf{RKE}_{\mathsf{pr}_1} u$ exists iff for any $x \in \mathsf{K}_1$, $\lim_{\mathsf{K}_2} u(x,-)$ exists. Moreover, for any x, there is a canonical isomorphism*

$$(\mathsf{RKE}_{\mathsf{pr}_1} u)(x) \simeq \lim_{\mathsf{K}_2} u(x,-).$$

2.3. As a consequence, we obtain *a* construction of

$$\mathsf{K}_1 \to \mathsf{C},\ x \mapsto \lim_{\mathsf{K}_2} u(x,-)$$

promised in the previous lectures. Combining with Proposition 1.6, we obtain the **distribution law of limits**.

[4]This means the right Kan extension exists and is pointwise

[5]In fact, these results can be generalized to the case when K_1 and K_2 are simplicial sets. However, one needs to slightly modify the constructions and statements. See HTT.5.5.2.3 for an example.

Corollary 2.4. *Suppose for any $x \in \mathsf{K}_1$, $\lim_{\mathsf{K}_2} u(x,-)$ exists. Then there is a canonical isomorphism*

$$\lim_{\mathsf{K}_1 \times \mathsf{K}_2} u \xrightarrow{\simeq} \lim_{x \in \mathsf{K}_1} \lim_{\mathsf{K}_2} u(x,-).$$

Here the source exists iff the target does.

Remark 2.5. *In future lectures, we will generalize the above result by replacing* $\mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{K}_1$ *by any* Cartesian fibration *between ∞-categories.*

Corollary 2.6. *Suppose:*

- *For any $x \in \mathsf{K}_1$, $\lim_{\mathsf{K}_2} u(x,-)$ exists.*
- *For any $y \in \mathsf{K}_2$, $\lim_{\mathsf{K}_1} u(-,y)$ exists.*

Then there is a canonical isomorphism

$$\lim_{x \in \mathsf{K}_1} \lim_{\mathsf{K}_2} u(x,-) \simeq \lim_{y \in \mathsf{K}_2} \lim_{\mathsf{K}_1} u(-,y).$$

Here the LHS exists iff the RHS does.

2.7. As an application, we obtain the following result about limits of ∞-categories.

Theorem 2.8. *Let $u : K \to \mathsf{Cat}_\infty$, $i \mapsto \mathsf{C}_i$, be a diagram of ∞-categories. Consider the evaluation functors*

$$\mathsf{ev}_i : \lim_{i \in K} \mathsf{C}_i \to \mathsf{C}_i.$$

Then there is a canonical equivalence

$$\mathsf{Maps}_{\lim_{i \in K} \mathsf{C}_i}(x,y) \xrightarrow{\simeq} \lim_{i \in K} \mathsf{Maps}_{\mathsf{C}_i}(\mathsf{ev}_i(x), \mathsf{ev}_i(y)) \tag{2.1}$$

between ∞-groupoids.

Sketch. By definition,

$$\mathsf{Maps}_{\mathsf{C}}(x,y) \simeq \{x\} \underset{\mathsf{Fun}(\{0\},\mathsf{C})}{\times} \mathsf{Fun}(\Delta^1, \mathsf{C}) \underset{\mathsf{Fun}(\{1\},\mathsf{C})}{\times} \{y\}$$

can be written as a limit. Now the theorem follows from Corollary 2.6 and the fact that $\mathsf{Fun}(J,-)$ preserves limits. □

Remark 2.9. *Note that we also have*

$$(\lim_{i \in K} \mathsf{C}_i)^{\simeq} \xrightarrow{\simeq} \lim_{i \in K} \mathsf{C}_i^{\simeq}. \tag{2.2}$$

because taking cores is a right adjoint. In practice, most results, if not all, about $\lim_{i \in K} \mathsf{C}_i$ *are proven using* (2.1) *and* (2.2).

3. How to commute limits with colimits?

3.1. For simplicity, we assume C admits K_1-indexed limits and K_2-indexed colimits. In particular, Exercise 2.2 and its dual imply all the LKE and RKE in below exist.

Construction 3.2. *We have an obvious commutative diagram*

$$\begin{tikzcd}
\mathsf{Fun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{C}) & \mathsf{Fun}(\mathsf{K}_1, \mathsf{C}) \arrow[l, "-\circ \mathsf{pr}_1"'] \\
\mathsf{Fun}(\mathsf{K}_2, \mathsf{C}) \arrow[u, "-\circ \mathsf{pr}_2"] & \mathsf{Fun}([0], \mathsf{C}). \arrow[l, "-\circ \pi_2"] \arrow[u, "-\circ \pi_1"']
\end{tikzcd}$$

As in classical category theory, we can pass to left adjoints along the horizontal direction and obtain a natural transformation[6]

$$
\begin{tikzcd}
\mathsf{Fun}(\mathsf{K}_1\times\mathsf{K}_2,\mathsf{C}) \arrow[r, "\mathsf{LKE}_{\mathsf{pr}_1}"] \arrow[dr, Rightarrow] & \mathsf{Fun}(\mathsf{K}_1,\mathsf{C}) \\
\mathsf{Fun}(\mathsf{K}_2,\mathsf{C}) \arrow[u, "-\circ\mathsf{pr}_2"] \arrow[r, "\mathsf{LKE}_{\pi_2}"'] & \mathsf{Fun}([0],\mathsf{C}). \arrow[u, "-\circ\pi_1"']
\end{tikzcd}
\tag{3.1}
$$

Lemma 3.3. *The natural transformation* (3.1) *is invertible.*

Sketch. Let $x\in\mathsf{K}_1$ be an object and $v\in\mathsf{Fun}(\mathsf{K}_2,\mathsf{C})$ be a diagram. We need to show

$$\mathsf{LKE}_{\mathsf{pr}_1}(v\circ\mathsf{pr}_2)(x)\to\mathsf{LKE}_{\pi_2}v(\pi_1(x))$$

is invertible. This morphism can be identified with

$$\operatorname*{colim}_{(\mathsf{K}_1)_{/x}\times_{\mathsf{K}_1}(\mathsf{K}_1\times\mathsf{K}_2)} v'\to\operatorname*{colim}_{\mathsf{K}_2} v,$$

where v' is the composition $(\mathsf{K}_1)_{/x}\times_{\mathsf{K}_1}(\mathsf{K}_1\times\mathsf{K}_2)\to\mathsf{K}_2\xrightarrow{v}\mathsf{C}$. Hence we only need to show the morphism $(\mathsf{K}_1)_{/x}\times_{\mathsf{K}_1}(\mathsf{K}_1\times\mathsf{K}_2)\to\mathsf{K}_2$ is final, which follows from [Lecture 7, Proposition 3.15]. □

Construction 3.4. *By the above lemma, we have a commutative diagram*

$$
\begin{tikzcd}
\mathsf{Fun}(\mathsf{K}_1\times\mathsf{K}_2,\mathsf{C}) \arrow[r, "\mathsf{LKE}_{\mathsf{pr}_1}"] & \mathsf{Fun}(\mathsf{K}_1,\mathsf{C}) \\
\mathsf{Fun}(\mathsf{K}_2,\mathsf{C}) \arrow[u, "-\circ\mathsf{pr}_2"] \arrow[r, "\mathsf{LKE}_{\pi_2}"'] & \mathsf{Fun}([0],\mathsf{C}). \arrow[u, "-\circ\pi_1"']
\end{tikzcd}
$$

We can then pass to right adjoints along the vertical direction and obtain a natural transformation

$$
\begin{tikzcd}
\mathsf{Fun}(\mathsf{K}_1\times\mathsf{K}_2,\mathsf{C}) \arrow[r, "\mathsf{LKE}_{\mathsf{pr}_1}"] \arrow[d, "\mathsf{RKE}_{\mathsf{pr}_2}"'] & \mathsf{Fun}(\mathsf{K}_1,\mathsf{C}) \arrow[d, "\mathsf{RKE}_{\pi_1}"] \\
\mathsf{Fun}(\mathsf{K}_2,\mathsf{C}) \arrow[ur, Rightarrow, "\alpha"] \arrow[r, "\mathsf{LKE}_{\pi_2}"'] & \mathsf{Fun}([0],\mathsf{C}).
\end{tikzcd}
\tag{3.2}
$$

In particular, for any $u:\mathsf{K}_1\times\mathsf{K}_2\to\mathsf{C}$, we obtain a canonical morphism

$$\operatorname*{colim}_{y\in\mathsf{K}_2}\lim_{\mathsf{K}_1}u(-,y)\to\lim_{x\in\mathsf{K}_1}\operatorname*{colim}_{\mathsf{K}_2}u(x,-).$$

Definition 3.5. *We say K_1-indexed limits commute with K_2-indexed colimits in C if* (3.2) *is invertible.*

3.6. In the next lecture, we will explain the following results:

Theorem 3.7. *In Grpd_∞, filtered colimits commute with finite limits.*

Remark 3.8. *Once we have introduced* compactly generated ∞-categories*, it is easy to deduce that the above theorem holds in any such ∞-category.*

[6]We will explain this in details in future lectures.

Warning 3.9. *The definitions of filtered or finite index* ∞-categories *are subtle. For example, finite ordinary category may fail to be finite as* ∞*-categories.*

Theorem 3.10. *In* Grpd_∞*, sifted colimits commute with finite products.*

Theorem 3.11. *In* Grpd_∞*, small colimits are preserved by base-changes.*

Appendix A. Decomposition of diagrams

Exercise A.1. *Let* K *be a simplicial set and* $K_1, K_2 \subset K$ *be simplicial subsets of* K *such that*

$$K_1 \coprod_{K_1 \cap K_2} K_2 \to K$$

is an isomorphism. Let $u : K \to \mathsf{C}$ *be a diagram in an* ∞*-category. Show that there is a canonical isomorphism*

$$\lim_K u \xrightarrow{\simeq} \lim_{K_1} u \underset{\lim_{K_1 \cap K_2} u}{\times} \lim_{K_2} u.$$

Here the source exists if the target does.

Exercise A.2. *Show that the equalizer of* $x \rightrightarrows y$ *is isomorphic to* $x \times_{x \times y} x$*.*

Exercise A.3. *Let* K *be a simplicial set and* $\mathsf{Sub}(K)$ *be the partially ordered set of simplicial subsets of* K*, viewed as an ordinary category. Let* $I \to \mathsf{Sub}(K)$*,* $i \mapsto K_i$ *be a functor between ordinary categories. Find a sufficient condition such that for any diagram* $u : K \to \mathsf{C}$*, there is a canonical isomorphism*

$$\lim_K u \xrightarrow{\simeq} \lim_{i \in I} \lim_{K_i} u.$$

A.4. **Suggested readings.** HTT.4.2.3.