Talk 3: Adic Spaces

Last time:

Huber rings: A

$$\left(\begin{array}{ll} \text{ring of definition} & A_0 \overset{\text{open}}{\subset} A \\ & \cup \\ \text{ideal of definition.} & I_0 \\ \text{s.t. } A_0 \text{ equipped w/ } I_0\text{-adic topology} & \end{array}\right)$$

$$\mathrm{Cont}(A) = \left\{ \begin{array}{l} \text{Continuous multiplicative} \\ \text{valuations: (non-Arch).} \\ v : A \to \cup\{0\} \;\; \ldots \end{array} \right\}$$

equipped it with a topology such that

$$U(f_i/g) = \left\{ x \in \mathrm{Cont}(A) \;\middle|\; \begin{array}{l} |f_i(x)| \\ \leq |g(x)| \neq 0 \end{array} \right\}$$

($(f_1, \cdots, f_n)$ is an open ideal of A).

form quasi-compact open basis of this topology.

<u>Thm</u>: (Huber): $\mathrm{Cont}(A)$ is a spectral space (behaves like affine Spectrum)

A caveat of $\mathrm{Cont}(A)$ is: rational domains might not of the form $\mathrm{Cont}(B)$ for some $A \to B$.

(Counter-example:
the closure of the closed unit disk inside the affine line.)

$U(f_i/g)$ is a rational domain

want to define

$$B = \underline{A\langle f_i/g\rangle} = \underline{A\langle T_i\rangle}/(gT_i - f_i)$$

does not work because

$$v \in \mathrm{Cont}(A) \not\Rightarrow v\left(\frac{f_i}{g}\right) \leq 1$$

In other words

$$\mathrm{Cont}(B) \supsetneq U(f_i/g).$$

Therefore, we need $B^+ \subset B$ to remember which element is required to be $|\ |_v \leq 1$ for any $v \in U(f_i/g)$

Lemma (Huber):

The following are in bijection.

- $\left\{ F \subset \mathrm{Cont}(A) \;\middle|\; F = \bigcap_{f \in S} \{x \mid |f(x)| \le 1\},\ S \subset A \right\}$

$F \mapsto \{f \in A \mid |f(x)| \le 1\ \forall x \in F\}$ $\Updownarrow$ $\{x \in \mathrm{Cont}(A) \mid |f(x)| \le 1\ \forall f \in A^+\} \mapsfrom A^+$

- $\left\{ A^+ \subset A \;\middle|\; A^+ \text{ open and integrally closed} \right\}$

Def: A Huber ring.

then a <u>ring of integral elements of A</u> is $A^+ \subset A$ open and integrally closed subring

$A^+ \subset A^\circ$ (power-bounded elements)

- A <u>Huber pair</u> is (A, A^+)

- $\mathrm{Spa}(A, A^+)$

$$= \left\{ x \in \mathrm{Cont}(A) \;\middle|\; \begin{array}{l} |f(x)| \le 1 \\ \forall f \in A^+ \end{array} \right\}$$

$\mathrm{Spa}\, A = \mathrm{Spa}(A, A^\circ)$

equip $\mathrm{Spa}(A, A^+)$ with the topo.

s.t. $U(f_i / g) := \{ x \in \mathrm{Spa}(A, A^+) \mid$

$|f_i(x)| \le |g(x)| \ne 0 \}$

form open basis.

[$\mathrm{Spa}(A, A^+) \subset \mathrm{Cont}(A)$

with subset topology]

$\llcorner$ It's only pro-constructible in Cont(A)
not open

Thm (Huber):

Spa (A, A^+) is spectral space
& $U(f_i/g)$ are quasi-cpt.

Rmk: Why $A^+ \subset A^\circ$.

We need A^+ to be not so big
such that Spa(A, A^+) is not
small.

If we do not require $A^+ \subset A^\circ$,
then Spa(A, A^+) can be empty
even for complete A.

Thm (Huber)

$\mathrm{Spa}(A, A^+)$ only depends on the completion of (A, A^+).

Prop (Huber)

For any rational domain

$\underline{U(f_i/g)} \subset \mathrm{Spa}(A, A^+)$.

$\exists!$ $(B, B^+) \leftarrow (A, A^+)$

s.t. $U(f_i/g)$

$\cong$

$\mathrm{Spa}(B, B^+)$.

Proof (Sketch)

$C := \boxed{A[g^{-1}]}$

$C^+ :=$ integral closure of

$\underline{A^+}[\underline{f_i/g}^{-1}]$
inside $A[g^{-1}]$

$(B, B^+) := (C, C^+)^\wedge$

$\square$.

Notn: $(B, B^+) =: \left(\boxed{A\langle f_i/g \rangle}, \right.$

$\left. A\langle f_i/g \rangle^+ \right)$.

Rmk 1: Next time I'll tell you whether $A\langle f_i/g \rangle$ is the Tate algebra

Def: For Huber (A, A^+) define *presheaves* $\mathcal{O}, \mathcal{O}^+$ of complete topological rings on $\mathrm{Spa}(A, A^+)$

$$\Gamma(U(f_i/g), \mathcal{O}) = A\langle f_i/g\rangle$$

$$\Gamma(U(f_i/g), \mathcal{O}^+) = A\langle f_i/g\rangle^+$$

$$\Gamma(\text{open}, \mathcal{O}) = \varprojlim_{U(f_i/g) \subset \text{open}} A\langle f_i/g\rangle$$

Warning: $\mathcal{O}$ is not a sheaf in general.

Def: (A, A^+) is sheafy if $\mathcal{O}$ is a sheaf.

($\Rightarrow$ $\mathcal{O}^+$ is a sheaf),

Thm: For complete (A, A^+), the following cases are sheafy.

(1) (scheme) A is discrete.

(2) (formal scheme)

A is finitely generated over a Noetherian ring of definition A_0.

(3) (~~rigid spaces~~).

A is Tate and strongly Noetherian

($A\langle T_1, \ldots, T_n\rangle$ is Noetherian for any n)

[Tate: has a top. nilp. unit]
ϖ

(4). Perfectoid spaces.

Rmk: To deal with non-sheafy Huber pairs, need pre-adic space.

Thm: (Kedlaya-Liu)
If (A, A^+) is a complete sheafy pair and is analytic

then
$H^i(\mathrm{Spa}(A, A^+), \mathcal{O}) = 0$
for $i > 0$.

Recall: Last time, we say a point $x \in \mathrm{Cont}(A)$ is analytic if $k(x)$ is nondiscrete.

$\mathrm{Spa}(A, A^+)$ is analytic if all its points are so.

Question: $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}^+) \neq 0$ in general.

It's "almost zero" in good cases.

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§ adic spaces.

Using gluing to define

adic spaces.

Def: $\{ (X, \mathcal{O}_X, |-|_{x \in X})$

| $(X, \mathcal{O}_X)$ is a locally topologically ringed space
| $\forall x \in X$, $|-|_x$ is a continuous valuation of $\mathcal{O}_{X,x}$ $\}$

$\cup$

$\mathrm{Spa}(A, A^+)$ if (A, A^+) is sheafy

Define the notion of open embeddings.

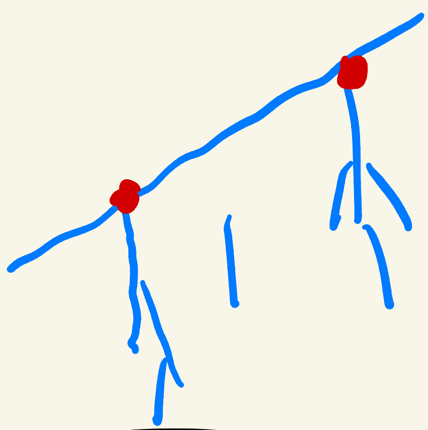
Define adic spaces to be objects

In this category locally given by $\mathrm{Spa}(A, A^+)$ for sheafy (A, A^+).

Trivial valuation:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & 0 \cup \{1\} \\ & \underline{A}/\mathfrak{p} \arrow[ur] & \end{tikzcd}$$

$\mathfrak{p} \subset A$ open



$$\mathcal{O}^+(U) = \left\{ f \in \mathcal{O}(U) \,\middle|\, |f|_x \le 1 \ \forall x \in U \right\}$$

Examples !

Ex: $\mathrm{Spa}\,\mathbb{Z} = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ is the final object.

Ex: $\mathrm{Spa}\,\mathbb{Z}[T] = \mathrm{Spa}(\mathbb{Z}[T], \underset{\sim}{\mathbb{Z}[T]})$

adic closed unit disk

$$\mathrm{Hom}(X, \underline{\mathrm{Spa}\,\mathbb{Z}[T]})$$

$$\shortparallel$$

$$\text{“}(\mathbb{Z}[T], \mathbb{Z}[T]) \to (\mathcal{O}_X, \mathcal{O}_X^+)\text{”}$$

$$\shortparallel$$

$$\Gamma(X, \mathcal{O}_X^+)$$

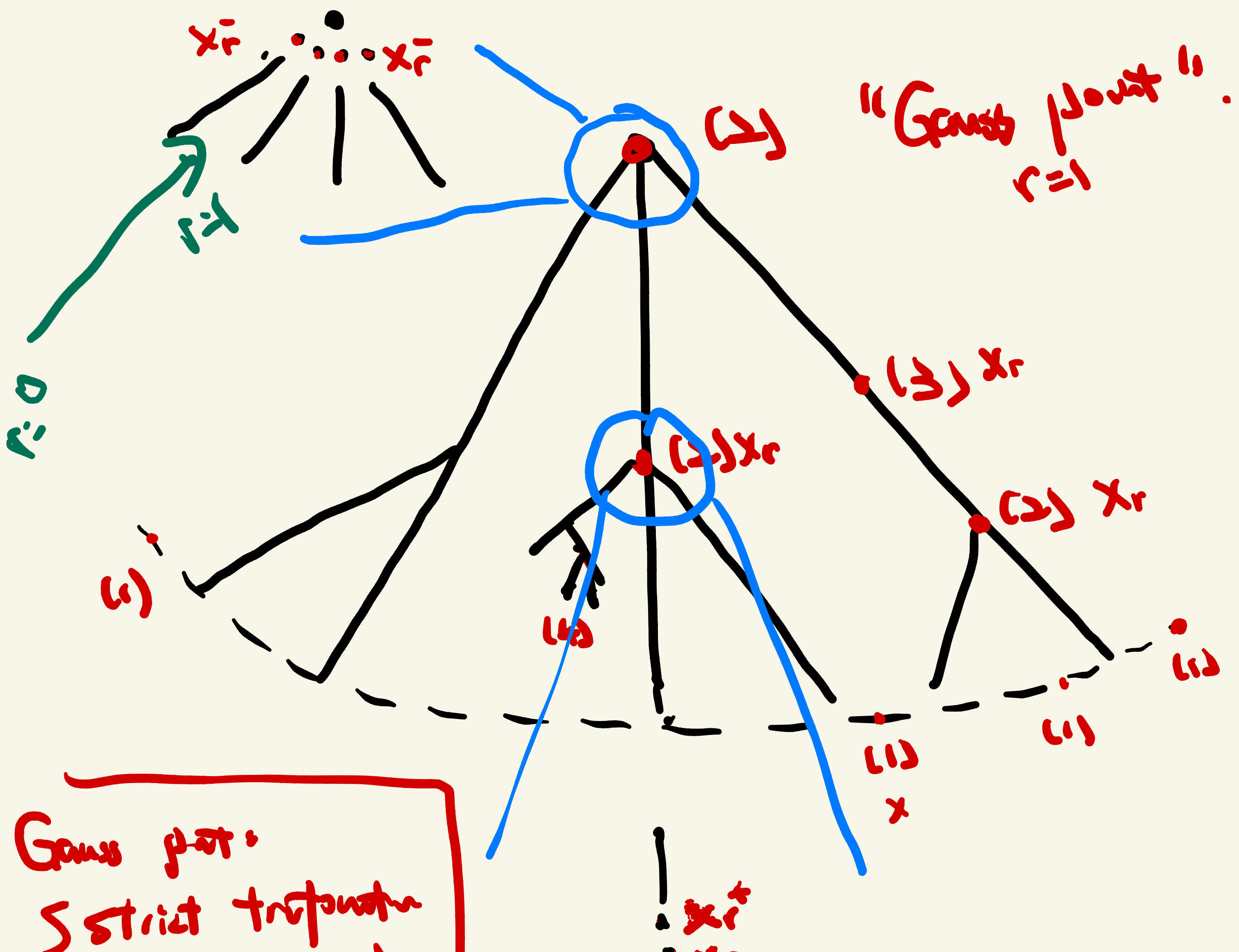
(K non-Arch)

$$\mathrm{Spa}\,\mathbb{Z}[T] \times \mathrm{Spa}\,K$$

II

$$\mathrm{Spa}\, k\langle T\rangle$$
$$\shortparallel$$
$$\mathrm{Spa}\left(k\langle T\rangle,\ k^{\circ}\langle T\rangle\right)$$

Look at topology of $\mathrm{Spa}\, k\langle T\rangle$



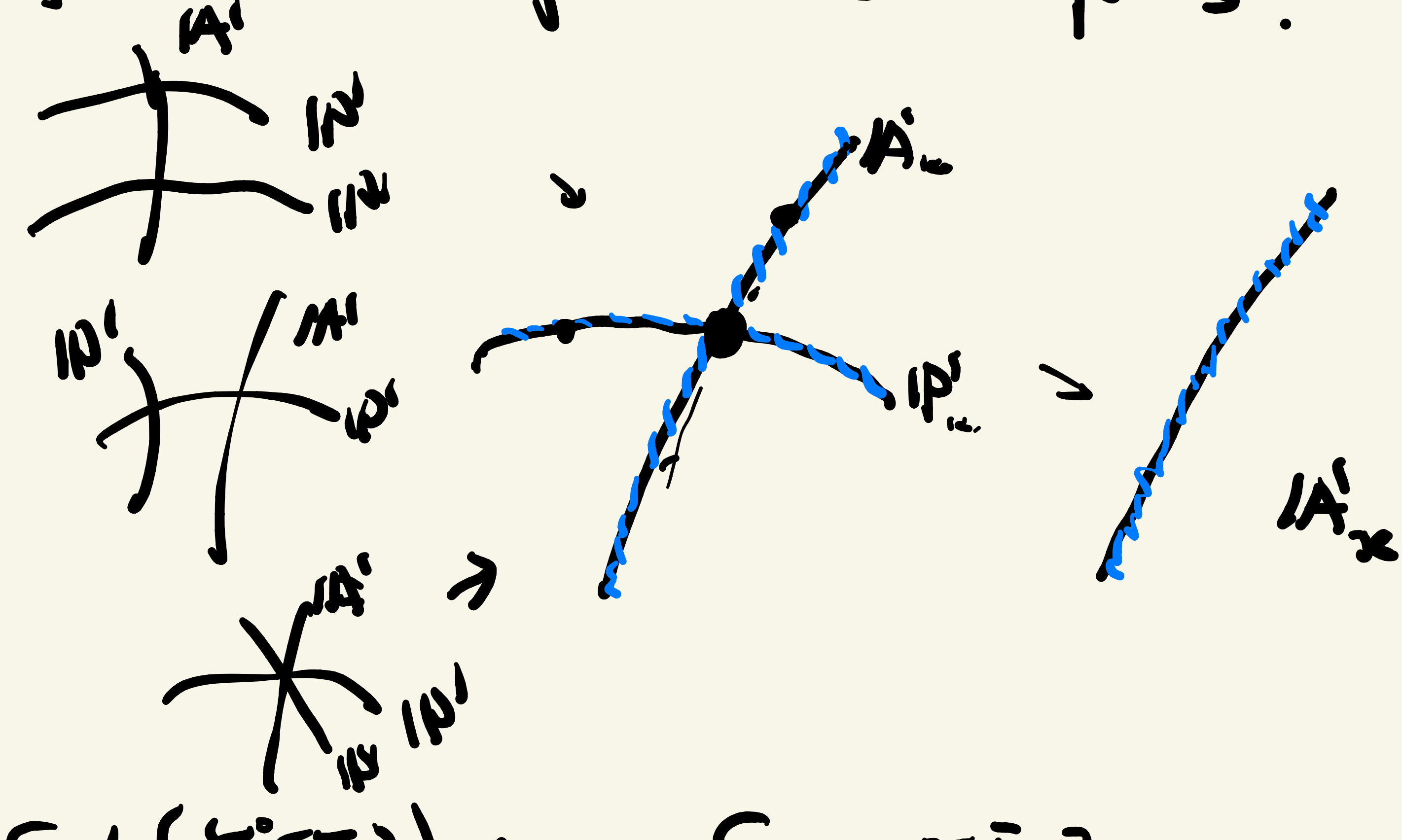
Gauss pts:
{strict transformation of /A¹ }

Rmk: As said before:

$$\mathrm{Spa}\ K\langle T\rangle = \lim \{\text{formal models of } \mathrm{Spa}\ K\langle T\rangle\}.$$

What are formal models of $\mathrm{Spa}\,K\langle T\rangle$?

Start from $\mathrm{Spf}\ \underline{K^\circ}\langle T\rangle$, do (formal) blow-up at closed points.



$$\mathrm{Spf}\,(K^\circ\langle T\rangle)_{red} = \mathrm{Spec}\ \kappa[T]$$

$A = \mathbb{Z}_p\langle T\rangle.$

$\mathfrak{J} = (p, T).$

exceptional divisor?

$\mathrm{Proj}(A \oplus J \oplus J^2 \oplus \cdots)$

$\mathrm{Proj}(A/\mathfrak{J} \oplus \mathfrak{J}/\mathfrak{J}^2 \oplus \cdots),$

$A/\mathfrak{J} = \mathbb{F}_p$

$\dim(\mathfrak{J}/\mathfrak{J}^2) = 2 \qquad \mathfrak{J}/\mathfrak{J}^2 = \mathbb{F}_p\{p\} \oplus \mathbb{F}_p\{T\}.$

$\dim(J^2/J^3) = 3 \qquad \{p^2, pT, T^2\}.$

$\mathrm{Proj}(A/\mathfrak{J} \oplus J/J^2 \oplus \cdots)$

$= \mathbb{F}_p[u, v] \qquad \deg u = 1$

$\deg v = 1$

Five types of points, i.e. valuations

of $K\langle T\rangle$ (s.t. $|K^\circ\langle T\rangle| \leq 1$).

1). $\boxed{\text{Classical points}}$ (those points can be seen in Tate's theory $\mathrm{Sp}\, K\langle T\rangle$).

$$x \in K^\circ, \quad \begin{aligned} K\langle T\rangle &\longrightarrow 0 \cup \mathbb{R}^+ \\ f &\longmapsto |f(x)|_K \end{aligned}$$

closed pts.

(Recall K is equipped with a valuation

$|\_|_K : K \to 0 \cup \mathbb{R}^+$)

2) $\boxed{\text{"The rays of the tree"}}$

3) $\forall$ disk $D(x,r) = \{ y \in K^\circ \mid |y-x| \leq r \}$

$x \in K^\circ$, $0 < r \leq 1$.

define

$$K\langle T\rangle \longrightarrow 0 \cup \mathbb{R}^{+}$$

$$f \longmapsto \sup_{y\in D(x,r)} |f(y)|_K$$

This val. only depends on $D(x,r)$.

we call x_r

If $r \in |K^*|$ ($\exists\, z,\ |z|_K = r$)

then called type 2). "branching pts"

non-closed pts

If $r \notin |K^*|$ ($\nexists\, z,\ |z|_K = r$)

called type 3).

closed pts

"non-branching pts".

X_r only depends on $D(x,r)$.

But , $\exists\ x \neq x'$

s.t. $D(x,r) = D(x',r)$

[actually x' can be anything in $D(x,r)$.]

If $r \in |K^\times|$. when r decrease,

$D(x, r-\varepsilon) \neq D(x', r-\varepsilon)$
for some x'

[$x' = x + z$, $|z|_K = r$]

4). Dead ends:

(If K is $\mathbb{Q}_p$ no such thing
If K is $\mathbb{C}_p$, then are).

$\forall$ sequence of disks

$D_1 \supset D_2 \supset \cdots\cdots$ in K^0

such that $\bigcap D_n = \phi$.

$$K\langle T\rangle \longrightarrow O \cup R^+$$

$$f \longmapsto \inf \sup_{y \in D_n} |f(y)|$$

D_∞. closed point.

3). Rank 2 points.

$\forall\ x \in K^0 \qquad 0 < r \leq 1,\ r \in |K^\times|$

and a sign $\pm$ (only allow + if $r = 1$).

there is a valuation:

val. grp is $\mathbb{R}^+ \times \gamma^{\mathbb{Z}}$

$$\gamma = r \pm \text{infinitesimal.}$$

$$\left(\begin{array}{l} \text{If you choose } +\text{, then} \\ \qquad r < \gamma < r+\varepsilon \quad \forall \varepsilon. \\ \text{If you choose } -\text{, then} \\ \qquad r-\varepsilon < \gamma < r \quad \forall \varepsilon \end{array}\right)$$

$$x_r^{\pm} : K\langle T\rangle \longrightarrow \mathcal{O}(\mathbb{R}^+ \times \gamma^{\mathbb{Z}})$$
$$\cup$$
$$f = \sum a_n (T-x)^n \longmapsto \max |a_n|_k \gamma^n$$

$$\left\{\begin{array}{l} |a_n|_k \gamma^n = |a_n r^n|_k \pm \text{infinitesimal} \\ \underline{|f|_{x_r^{\pm}} = |f|_{x_r} \pm \text{infinitesimal}} \\ \text{b.c} \quad |f|_{x_r} = \sup\limits_{|y-x| \le r} |f(y)|_k \end{array}\right.$$

$$= \sup_{|y-x|\le r} \sum |a_n (y-x)^n|_K$$

$$= \max |a_n| r^n$$

We use r as tie-breakers for valuations of type (2).

If $\|f\|_{X_r^{\pm}} \le |g|_{X_r^{\pm}}$

$\Rightarrow |f|_{X_r} \le |g|_{X_r}$

Therefore $X_r^{\pm}$ is in the closure of X_r

If $r = 1$, you define X_r^{+} similarly.

it will not extend

$K^\circ\langle T\rangle$ to ≤ 1.

X_r^+ only depends on $\boxed{D(x,r)}$

- r only depends on $D(x,r)$
- $D(x,r) = D(y,r)$

  $|y-x| \leq r$

  use this to prove

  $X_r^+ = Y_r^+$

X_r^- only depends on $\underline{D(x,<r)}$ $\overline{D(x,<r)}$

- r only depends on $D(x,<r)$

  If $|K^\times|$ is nondiscrete r is a diameter

  otherwise it is the smallest number $>$ diameter

- $D(x,<r) = D(y,<r)$

iff $|x-y|<r$.

$\Rightarrow \bar{x}_r = \bar{y}_r$

For fixed $D(x,r)$.

$\exists$ multiple choices

$D(y, \cancel{\epsilon} r) \subset D(x,r)$

~~each~~ each choice gives a point

$\bar{y}_r \in \overline{X_r}$

$\kappa(K) := \mathring{K}/m$.

many choices.

Ex: (Affine line)

$$\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})$$

$$\mathrm{Hom}(X, \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z})) = \mathcal{O}_X$$

$$\mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \mathrm{Spa}\, K = \bigcup_n \boxed{\mathrm{Spa}\, K\langle \varpi^n T\rangle}$$

(ϖ is pseudo-uniformizer of K).

(K is covered by

$D(0, |\varpi^{-n}|)$).

This product is not quasi-cpt.

Ex: (closed of closed disk inside affine line)

$$\boxed{\boxed{\mathrm{Spa}\, K\langle T\rangle} \to \mathrm{Spa}(\mathbb{Z}[T], \mathbb{Z}) \times \mathrm{Spa}\, K}$$

is an open embedding.

But its not a closed one.

It's closure is $\mathrm{Cont}(K\langle T\rangle)$.

$$\cup$$

$$X_1^+$$

X_1^+ is the only point not in $\mathrm{Spa}\, K\langle T\rangle$.

<u>Ex</u> (The open disk).

$\mathbb{D} := \mathrm{Spa}\ \mathbb{Z}\langle T \rangle$.

$\mathbb{D}_K := \mathrm{Spa}\ \mathbb{Z}\langle T\rangle \times \mathrm{Spa}\,K$.

$= \bigcup \mathrm{Spa}\ K\langle \underline{T}, \underline{T^n/\varpi} \rangle$

(cover it by $D(0, \underline{\leq \omega^{1/n}})$

It's not quasi-cpt.

<u>Ex</u>: products do not exist in general.

$\underline{\mathrm{Spa}\ \mathbb{Z}[T_1, T_2, \dots]} \times \mathrm{Spa}\,K$

does not exist!

$$= \operatorname{colim} \underline{\operatorname{Spa} k\langle \varpi_1^{n_1} T_1, \varpi_2^{n_2} T_2, \ldots \rangle}$$

(It's an ind-adic space rather than adic-space).

(The connecting maps are not open embeddings!)