NEARBY CYCLES ON DRINFELD-GAITSGORY-VINBERG INTERPOLATION GRASSMANNIAN AND LONG INTERTWINING FUNCTOR

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Dedicated to the memory of Ernest Borisovich Vinberg

Abstract. Let G be a reductive group and U, U^- be the unipotent radicals of a pair of opposite parabolic subgroups P, P^-. We prove that the DG categories of $U((t))$-equivariant and $U^-((t))$-equivariant D-modules on the affine Grassmannian Gr_G are canonically dual to each other. We show that the unit object witnessing this duality is given by nearby cycles on the Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian defined in [FKM20]. We study various properties of the mentioned nearby cycles, in particular compare them with the nearby cycles studied in [Sch18], [Sch16]. We also generalize our results to the Beilinson-Drinfeld Grassmannian Gr_{G,X^I} and to the affine flag variety Fl_G.

This version of the paper contains fewer appendices than the version submitted to arXiv.

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0. Introduction

0.1. **Motivation: nearby cycles and the long intertwining functor.** Let G be a reductive group over an algebraically closed field k of characteristic 0. For simplicity, we assume $[G,G]$ to be simply connected. Fix a pair (B,B^-) of opposite Borel subgroups of G. Let Fl_f be the flag variety of G, and N,N^- be the unipotent radicals of B,B^- respectively. Recall the following well-known fact (see e.g. [BB83] and [CGD19, Proposition 1.4.2]):

Fact 1. *The long-intertwining functor*

$$\Upsilon : \mathrm{DMod}(\mathrm{Fl}_f)^N \xrightarrow{\mathbf{oblv}^N} \mathrm{DMod}(\mathrm{Fl}_f) \xrightarrow{\mathbf{Av}_*^{N^-}} \mathrm{DMod}(\mathrm{Fl}_f)^{N^-} \tag{0.1}$$

is an equivalence.

In the above formula,

- $\mathrm{DMod}(\mathrm{Fl}_f)^N$ is the DG category of D-modules on Fl_f that are constant along the N-orbits.
- $\mathbf{oblv}^N$ is the forgetful functor.
- $\mathbf{Av}_*^{N^-}$ is the right adjoint of $\mathbf{oblv}^{N^-}$.

The DG category $\mathrm{DMod}(\mathrm{Fl}_f)^N$ is equivalent to $\mathrm{DMod}(\mathrm{Fl}_f/N)$ (see [DG13] for the definition). Verdier duality on the algebraic stack Fl_f/N

provides an equivalence

$$\mathrm{DMod}(\mathrm{Fl}_f/N) \simeq \mathrm{DMod}(\mathrm{Fl}_f/N)^\vee.$$

Here $\mathcal{C}^\vee$ is the dual DG category of $\mathcal{C}$, whose definition will be reviewed below. Let us first reinterpret Fact 1 as:

Fact 2. *The DG categories* $\mathrm{DMod}(\mathrm{Fl}_f)^N$ *and* $\mathrm{DMod}(\mathrm{Fl}_f)^{N^-}$ *are canonically dual to each other.*

Recall that a duality datum between two DG categories $\mathcal{C}, \mathcal{D}$ consists of a *unit (a.k.a. co-evaluation) functor* $c : \mathrm{Vect}_k \to \mathcal{C} \otimes_k \mathcal{D}$ and a *counit (a.k.a. evaluation) functor* $e : \mathcal{D} \otimes_k \mathcal{C} \to \mathrm{Vect}_k$, where $\otimes_k$ is the Lurie tensor product for DG categories, and Vect_k, the DG category of k-vector spaces, is the monoidal unit for $\otimes_k$. The pair (c, e) are required to make the following compositions isomorphic to the identity functors:

$$\begin{aligned}
&\mathcal{C} \simeq \mathrm{Vect}_k \underset{k}{\otimes} \mathcal{C} \xrightarrow{c \otimes \mathbf{Id}_{\mathcal{C}}} \mathcal{C} \underset{k}{\otimes} \mathcal{D} \underset{k}{\otimes} \mathcal{C} \xrightarrow{\mathbf{Id}_{\mathcal{C}} \otimes e} \mathcal{C} \underset{k}{\otimes} \mathrm{Vect}_k \simeq \mathcal{C} \\
&\mathcal{D} \simeq \mathcal{D} \underset{k}{\otimes} \mathrm{Vect}_k \xrightarrow{\mathbf{Id}_{\mathcal{D}} \otimes c} \mathcal{D} \underset{k}{\otimes} \mathcal{C} \underset{k}{\otimes} \mathcal{D} \xrightarrow{e \otimes \mathbf{Id}_{\mathcal{D}}} \mathrm{Vect}_k \underset{k}{\otimes} \mathcal{D} \simeq \mathcal{D}.
\end{aligned} \tag{0.2}$$

It follows formally that the counit for the duality in Fact 2 is the following composition:
(0.3)

$$\mathrm{DMod}(\mathrm{Fl}_f)^{N^-} \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Fl}_f)^N \xrightarrow{\mathbf{oblv}^{N^-} \otimes \mathbf{oblv}^N} \mathrm{DMod}(\mathrm{Fl}_f) \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Fl}_f) \to$$
$$\xrightarrow{- \overset{!}{\otimes} -} \mathrm{DMod}(\mathrm{Fl}_f) \xrightarrow{C_{\mathrm{dR}}} \mathrm{Vect}_k,$$

where $\otimes^!$ is the !-tensor product, and C_{dR} is taking the de-Rham cohomology complex.

Here is a natural question:

Question 1. *What is the unit functor for the duality in Fact 2?*

Of course, the question is uninteresting if we only want *one* formula for the unit. For example, it is the composition

$$\mathrm{Vect}_k \xrightarrow{\mathbf{unit}} \mathrm{DMod}(\mathrm{Fl}_f)^N \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Fl}_f)^N \xrightarrow{\mathbf{Id} \otimes \Upsilon^{-1}} \mathrm{DMod}(\mathrm{Fl}_f)^N \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Fl}_f)^{N^-}.$$

However, it becomes interesting when we want a more *symmetric* formula. So we restate Question 1 as

Question 2. *Can one find a symmetric formula for the unit of the duality in Fact 2?*

Let us look into the nature of the desired unit object. Tautologically we have

$$\mathrm{DMod}(\mathrm{Fl}_f)^N \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Fl}_f)^{N^-} \simeq \mathrm{DMod}(\mathrm{Fl}_f \times \mathrm{Fl}_f)^{N \times N^-}.$$

Also, knowing a continuous k-linear functor $\mathrm{Vect}_k \to \mathcal{C}$ is equivalent to knowing an object in $\mathcal{C}$. Hence the unit is essentially given by an $(N \times N^-)$-equivariant complex $\mathcal{K}$ of D-modules on $\mathrm{Fl}_f \times \mathrm{Fl}_f$. We start by asking the following question:

Question 3. *What is the support of the object $\mathcal{K}$?*

It turns out that this seemingly boring question has an interesting answer. Recall that both the N and N^- orbits on Fl_f are labelled by the Weyl group W. For $w \in W$, let Δ^w and $\Delta^{w,-}$ respectively be the !-extensions of the IC D-modules on the orbits NwB/B and N^-wB/B. It follows formally that we have

$$\mathrm{Hom}(\Delta^{w_1} \boxtimes \Delta^{w_2,-}, \mathcal{K}) \simeq \mathrm{Hom}(\Delta^{w_2,-}, \mathbb{D}^{\mathrm{Ver}} \circ \Upsilon(\Delta^{w_1})), \tag{0.4}$$

where

$$\mathbb{D}^{\mathrm{Ver}} : \mathrm{DMod}_{\mathrm{coh}}(\mathrm{Fl}_f) \simeq \mathrm{DMod}_{\mathrm{coh}}(\mathrm{Fl}_f)^{\mathrm{op}}$$

is the contravariant Verdier duality functor. It's well-known that $\mathbb{D}^{\mathrm{Ver}} \circ \Upsilon(\Delta^w) \simeq \Delta^{w,-}$. Hence (0.4) is nonzero only if N^-w_2B/B is contained in the closure of N^-w_1B/B, i.e. only if $w_1 \le w_2$, where "$\le$" is the Bruhat order. Therefore $\mathcal{K}$ is supported on the closures of

$$\coprod_{w \in W} (N \times N^-)(w \times w)(B \times B)/(B \times B). \tag{0.5}$$

The disjoint union (0.5) has a more geometric incarnation. To describe it, let us choose a regular dominant co-character $\mathbb{G}_m \to T$, the adjoint action of T on G induces a $\mathbb{G}_m$-action on Fl_f. The attractor, repeller, fixed loci (see [DG14] or Definition 1.2.11 for definitions) of this action are

$$\coprod_{w \in W} NwB/B,\ \coprod_{w \in W} N^-wB/B,\ \coprod_{w \in W} wB/B.$$

Hence (0.5) is identified with the 0-fiber of the *Drinfeld-Gaitsgory interpolation* $\widetilde{\mathrm{Fl}_f} \to \mathbb{A}^1$ for this action (see [DG14] or § 1.2.15 for its definition).

An important property of this interpolation is that there is a locally closed embedding

$$\widetilde{\mathrm{Fl}_f} \hookrightarrow \mathrm{Fl}_f \times \mathrm{Fl}_f \times \mathbb{A}^1, \tag{0.6}$$

defined over $\mathbb{A}^1$, such that its 1-fiber is the diagonal embedding $\mathrm{Fl}_f \hookrightarrow \mathrm{Fl}_f \times \mathrm{Fl}_f$, while its 0-fiber is the obvious embedding of (0.5) into $\mathrm{Fl}_f \times \mathrm{Fl}_f$. This motivates the following guess, which is a baby-version (=finite type version) of the main theorem of this paper:

Guess 1. *Consider the trivial family $\mathrm{Fl}_f \times \mathrm{Fl}_f \times \mathbb{A}^1 \to \mathbb{A}^1$. Up to a cohomological shift, $\mathcal{K}$ is isomorphic to the nearby cycles sheaf of the constant D-module supported on $\widetilde{\mathrm{Fl}_f} \times_{\mathbb{A}^1} \mathbb{G}_m$.*

The guess is in fact correct. For example, it can be proved using [BFO12, Theorem 6.1] and the localization theory[1]. On the other hand, in the main

[1] We are grateful to Yuchen Fu for pointing out this to us.

text of this paper, we will prove an affine version of this claim; our method can be applied to the finite type case as well (see § 3.6).

0.2. **Main theorems.**

0.2.1. *Inv-inv duality.* Consider the loop group $G((t))$ of G. Let Gr_G be the affine Grassmannian. Let P be a standard parabolic subgroup and P^- be its opposite parabolic subgroup. Let U, U^- respectively be the unipotent radical of P, P^-. Consider the DG category $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$ defined as in [Gai18b]. We will prove the following theorem (see Corollary 1.3.9(1)):

Theorem 1. *The DG categories* $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$ *and* $\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}$ *are dual to each other, with the counit functor given by*

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))} \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \xrightarrow{\mathbf{oblv}^{U^-((t))} \otimes \mathbf{oblv}^{U((t))}}$$

$$\to \mathrm{DMod}(\mathrm{Gr}_G) \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{-\overset{!}{\otimes}-} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{C_{\mathrm{dR}}} \mathrm{Vect}_k .$$

0.2.2. *The unit of the duality.* As one would expect, we will prove

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))} \simeq \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)^{U((t)) \times U^-((t))}.$$

Hence the unit functor is given by an $(U((t)) \times U^-((t)))$-equivariant object $\mathcal{K}$ in $\mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$.

Choose a dominant co-character $\gamma : \mathbb{G}_m \to T$ that is regular with repect to P. The adjoint action of T on G induces a $\mathbb{G}_m$-action on Gr_G. Consider the corresponding Drinfeld-Gaitsgory interpolation $\widetilde{\mathrm{Gr}}_G^{\gamma}$ and the embedding

$$\widetilde{\mathrm{Gr}}_G^{\gamma} \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1.$$

We will prove the following theorem (see Corollary 1.3.9(2)):

Theorem 2. *Consider the trivial family* $\mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1 \to \mathbb{A}^1$. *Up to a cohomological shift,* $\mathcal{K}$ *is canonically isomorphic to the nearby cycles sheaf of the dualizing D-module supported on* $\widetilde{\mathrm{Gr}}_G^{\gamma} \times_{\mathbb{A}^1} \mathbb{G}_m$.

0.2.3. *The long intertwining functor.* It is easy to see that the naive long-intertwining functor[2]

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \xrightarrow{\mathbf{oblv}^{U((t))}} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{\mathbf{Av}_*^{U^-((t))}} \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))}$$

is the zero functor unless $P = G$. This is essentially due to the fact that $U((t))$ is ind-infinite dimensional. Instead, we will deduce from Theorem 1 the following theorem (see § 1.1.7 and Corollary 1.3.13):

Theorem 3. *The functor*

(0.7)
$$\Upsilon : \mathrm{DMod}(\mathrm{Gr}_G)^{U((t))} \xrightarrow{\mathbf{oblv}^{U((t))}} \mathrm{DMod}(\mathrm{Gr}_G) \xrightarrow{\mathbf{pr}_{U^-((t))}} \mathrm{DMod}(\mathrm{Gr}_G)_{U^-((t))}$$

is an equivalence.

---

[2]The functor $\mathbf{Av}_*^{U^-((t))}$ below is *non-continuous.*

In the above formula, $\mathrm{DMod}(\mathrm{Gr}_G)_{U^-((t))}$ is the category of coinvariants for the $U^-((t))$-action on Gr_G. It can be defined as the localization of $\mathrm{DMod}(\mathrm{Gr}_G)$ that kills the kernels of $\mathbf{Av}_*^{\mathcal{N}}$ for all subgroup scheme $\mathcal{N}$ of $U^-((t))$.

In the special case when $P = B$, Theorem 3 can be deduced from a result of S. Raskin, which says (0.7) becomes an equivalence if we further take $T[[t]]$-invariants. See § 1.1.7 for a sketch of this reduction. However, our proof of Theorem 3 is independent to Raskin's result. Moreover, for general parabolics, to the best of our knowledge, Theorem 3 is *not* a direct consequence of any known results.

0.3. **Nearby cycles on** VinGr**.** Theorem 2 motivates us to study the nearby cycles mentioned in its statement. We denote this nearby cycles by $\Psi_\gamma \in \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$. Note that by Theorem 2, it only depends on P (and not on γ). We summarize known results about Ψ_γ as follows.

0.3.1. *Support.* Let r be the semi-simple rank of G. In [FKM20], the authors defined the *Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian* VinGr_G. There is a closed embedding

$$\mathrm{VinGr}_G \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^r,$$

which is a multi-variable degeneration of the diagonal embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G$. The co-character γ chosen before extends to a map $\mathbb{A}^1 \to \mathbb{A}^r$. Let

$$\mathrm{VinGr}_G^\gamma \hookrightarrow \mathrm{Gr}_G \times \mathrm{Gr}_G \times \mathbb{A}^1$$

be the sub-degeneration obtained by pullback along this map.

We will see that $\mathrm{VinGr}_G^\gamma \times_{\mathbb{A}^1} \mathbb{G}_m$ is isomorphic to $\widetilde{\mathrm{Gr}}_G^\gamma \times_{\mathbb{A}^1} \mathbb{G}_m$ as closed sub-indscheme of $\mathrm{Gr}_G \times \mathrm{Gr}_G$. Hence the support of Ψ_γ is contained in the 0-fiber of VinGr_G^γ, and it can also be calculated as the nearby cycles sheaf of the dualizing D-module on VinGr_G^γ.

0.3.2. *Equivariant structure.* (See Proposition 2.4.1(2))

We will prove Ψ_γ is constant along any $(U((t)) \times U^-((t)))$-orbit of $\mathrm{Gr}_G \times \mathrm{Gr}_G$.

We will prove Ψ_γ has a canonical equivariant structure for the diagonal $M[[t]]$-action on $\mathrm{Gr}_G \times \mathrm{Gr}_G$.

0.3.3. *Monodromy.* (See Proposition 2.4.1(1))

As a nearby cycles sheaf, Ψ_γ carries a monodromy endomorphism. We will prove that this endomorphism is locally unipotent.

0.3.4. *Factorization.* (See Corollary 3.4.4)

For any non-empty finite set I, consider the *Beillinson-Drinfeld Grassmannian* $\mathrm{Gr}_{G,I}$ and the similarly defined nearby cycles sheaf $\Psi_{\gamma,I} \in \mathrm{DMod}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})$. By [FKM20], we also have a relative version $\mathrm{VinGr}_{G,I}$ of VinGr_G. As before $\Psi_{\gamma,I}$ can also be calculated as the nearby cycles sheaf of the dualizing D-module on $\mathrm{VinGr}_{G,I}^\gamma$.

We will prove that the assignment $I \rightsquigarrow \Psi_{\gamma,I}$ factorizes. In other words, Ψ_γ can be upgraded to a *factorization algebra* in the factorization category $\mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ in the sense of [Ras15a].

0.3.5. *Local-global compatibility.* (see Theorem 1.5.1)

Let X be a connected projective smooth curve over k. In [Sch18] and [Sch16], S. Schieder defined the *Drinfeld-Lafforgue-Vinberg multi-variable degeneration*

$$\mathrm{VinBun}_G(X) \to \mathbb{A}^r,$$

which is a degeneration of $\mathrm{Bun}_G(X)$, the moduli stack of G-torsors on X. In [FKM20], the authors showed that the relationship between $\mathrm{VinGr}_{G,I}$ and $\mathrm{VinBun}_G(X)$ is similar to the relationship between $\mathrm{Gr}_{G,I}$ and $\mathrm{Bun}_G(X)$. In particular, there is a local-to-global map

$$\pi_I : \mathrm{VinGr}_{G,I} \to \mathrm{VinBun}_G(X)$$

defined over $\mathbb{A}^r$, which is a multi-variable degeneration of the map $\mathrm{Gr}_{G,I} \to \mathrm{Bun}_G(X)$.

In [Sch18] and [Sch16], S. Schieder calculated the nearby cycles sheaf $\Psi_{\gamma,\mathrm{glob}}$ of the dualizing D-module[3] for the sub-degeneration $\mathrm{VinBun}_G(X)^\gamma \to \mathbb{A}^1$. By construction, the map $\mathrm{VinGr}_{G,I} \to \mathrm{VinBun}_G(X)$ induces a map

$$\Psi_{\gamma,I} \to (\pi_I|_{C_P})^!(\Psi_{\gamma,\mathrm{glob}}),$$

where $\pi_I|_{C_P}$ is the 0-fiber of the map $\mathrm{VinGr}^\gamma_{G,I} \to \mathrm{VinBun}_G(X)^\gamma$. We will show that this is an isomorphism. Let us mention that in the proof of this isomorphism, we will *not* use Schieder's calculation.

0.4. **Variants, generalizations and upcoming work.**

0.4.1. $M[[t]]$*-equivariant versions.* Theorem 1 formally implies (see Corollary 1.4.5(1))

$$\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))M[[t]]} \text{ and } \mathrm{DMod}(\mathrm{Gr}_G)^{U^-((t))M[[t]]}$$

are dual to each other. As before, the unit of this duality is given by an object

$$\mathbb{D}^{\frac{\infty}{2}} \in \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)^{(M\times M)[[t]]}.$$

On the other hand, we have an object (see § 0.3.2)

$$\Psi_\gamma \in \mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)^{M[[t]],\mathrm{diag}}$$

We will prove the following theorem (see Corollary 1.4.5(2)):

Theorem 4. *Up to a cohomological shift, $\mathbb{D}^{\frac{\infty}{2}}$ is canonically isomorphic to $\mathbf{Av}_*^{M[[t]]\to(M\times M)[[t]]}(\Psi_\gamma)$.*

[3] S. Schieder actually worked with algebraic geometry on $\mathbb{F}_p$ and mixed l-adic sheaves. Let us ignore this difference for a moment.

0.4.2. *Tamely-ramified case.* Let Fl_G be the affine flag variety. As before, the choice of γ induces a $\mathbb{G}_m$-action on Fl_G. Our main theorems remain valid if we replace Gr_G by Fl_G. See Subsection 3.6.

0.4.3. *Other sheaf-theoretic contexts.* Although we work with D-modules, our main theorems are also valid (after minor modifications) in other sheaf-theoretic contexts listed in [Gai18a, § 1.2], which we refer as the *constructible contexts*. However, in order to prove them in the constructible contexts, we need a theory of group actions on categories in these sheaf-theoretic contexts. When developing this theory, one encounters some technical issues on homotopy-coherence, which are orthogonal to the main topic of this paper. Hence we will treat these issues in another article and use remarks in this paper to explain the required modifications. Once the aforementioned issues are settled down, these remarks become real theorems.

0.4.4. *t-structure.* As explained in [Gai18b] and [Gai17a], any objects in

$$\mathrm{DMod}(\mathrm{Gr}_G \times \mathrm{Gr}_G)^{(N\times N^-)((t))}$$

have no cohomologies in the standard t-strcuture. Nevertheless, D. Gaitsgory defined reasonable t-structures on this category and its factorization version. Calculations by the author show that, up to a cohomological shift, $\Psi_{2\rho}$ and its factorization version are contained in the heart of Gaitsgory's t-structures. The proof would appear eleswhere.

0.4.5. *Extended strange functional equation.* Let X be a connected projective smooth curve over k and Ran_X be its Ran space. Let $\mathrm{SI}_{\mathrm{Ran}}$ be the Ran version of the factorization category $\mathrm{DMod}(\mathrm{Gr}_G)^{N((t))T[[t]]}$, and $\mathrm{SI}^-_{\mathrm{Ran}}$ be the similar category defined using N^-.

In a future paper, following the suggestion of D. Gaitsgory, we will write down his definition of an extended (=parameterized) geometric Eisenstein series functor

$$\mathrm{Eis}_{\mathrm{ext}} : \mathrm{SI}_{\mathrm{Ran}} \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Bun}_T(X)) \to \mathrm{DMod}(\mathrm{Bun}_G(X)),$$

whose evaluations on $\Delta^0_{\mathrm{Ran}}, \mathrm{IC}^{\frac{\infty}{2}}_{\mathrm{Ran}}, \nabla^0_{\mathrm{Ran}} \in \mathrm{SI}_{\mathrm{Ran}}$ (see [Gai17a] for their definitions) are respectively, up to cohomological shifts, the functors $\mathrm{Eis}_!, \mathrm{Eis}_{!*}, \mathrm{Eis}_*$ defined in [BG02], [DG16] and [Gai17b]. Using the opposite Borel subgroup, we obtain another functor

$$\mathrm{Eis}^-_{\mathrm{ext}} : \mathrm{SI}^-_{\mathrm{Ran}} \underset{k}{\otimes} \mathrm{DMod}(\mathrm{Bun}_T(X)) \to \mathrm{DMod}(\mathrm{Bun}_G(X)).$$

By the miraculous duality in [Gai17b], $\mathrm{DMod}(\mathrm{Bun}_G(X))$ is self-dual, so is $\mathrm{DMod}(\mathrm{Bun}_T(X))$. By our main theorems, $\mathrm{SI}_{\mathrm{Ran}}$ and $\mathrm{SI}^-_{\mathrm{Ran}}$ are dual to each other. We will then use our main theorems to prove the following claim.

Claim 1. *Via the above dualities,* $\mathrm{Eis}_{\mathrm{ext}}$ *and* $\mathrm{Eis}^-_{\mathrm{ext}}$ *are conjugate to each other.*

This claim generalizes the main results in [DG16] and [Gai17b].

0.5. **Organization of this paper.** We give more precise statements of our main theorems in § 1. We do some preparations in § 2. We prove the main theorems in § 3 except for the local-global compatibility. We prove the local-global compatiblity in § 4.

As mentioned in the abstract. This version of the paper contains fewer appendices than the version submitted to arXiv, which is also available on the author's website. The results in these omitted appendices belong to the following types:

- special cases or variants of them are proved in the literature but those proofs cannot be generalized immediately, or
- they are folklores but no proofs exist in the literature.

We will cite the contents in these appendices as [Fulltext, labels]. For example, [Fulltext, Appendix B] is an appendix discussing group actions on DG categories. We will make sure that common contents have the same labels in both versions.

There are two appendices remained in this version. In Appendix C, we collect some geometric miscellanea. In Appendix D, we prove $\mathrm{DMod}(\mathrm{Gr}_G)^{U((t))}$, $\mathrm{DMod}(\mathrm{Gr}_G)_{U((t))}$ (and their factorization versions) are compactly generated.

0.6. **Notations and conventions.** Our conventions follow closely to those in [Gai18a] and [Gai18b]. We summarize them as below.

Convension 0.6.1. *(Categories) Unless otherwise stated, a* category *means an $(\infty,1)$-category in the sense of [Lur09]. Consequently, a $(1,1)$-category is refered to an* ordinary category*. We use same symbols to denote an ordinary category and its simplicial nerve. The reader can distinguish them according to the context.*

For two objects $c_1, c_2 \in C$ in a category C, we write $\mathrm{Maps}_C(c_1, c_2)$ *for the* mapping space *between them, which is in fact an object in the homotopy category of spaces. We omit the subscript C if there is no ambiguity.*

When saying there exists an unique *object satisfying certain properties in a category, we always mean* unique up to a contractible space of choices*.*

Following [GR17, Chapter 1, Subsection 1.2], a functor $F : C \to D$ is fully faithful *(resp.* 1-fully faithful*) if it induces isomorphisms (resp. monomorphisms) on mapping spaces.*

To avoid awkward language, we ignore all set-theoretical difficulties in category theory. Nevertheless, we do not do anything illegal like applying the adjoint functor theorem to non-accessible categories.

Notation 0.6.2. *(Compositions) Let C be a $(\infty,2)$-category. Let $f, f', f'' : c_1 \to c_2$ and $g, g' : c_2 \to c_3$ be morphisms in C. Let $\alpha : f \to f'$, $\alpha' : f' \to f''$ and $\beta : g \to g'$ be 2-morphisms in C. We follow the stardard conventions in the category theory:*

- *The composition of f and g is denoted by $g \circ f : c_1 \to c_3$;*
- *The vertical composition of α and α' is denoted by $\alpha' \circ \alpha : f \to f''$;*

- *The horizontal composition of α and β is denoted by $\beta\bigstar\alpha : g\circ f \to g'\circ f'$.*

Note that these compositions are actually well-defined up to a contractible space of choices.

We use similar symbols to denote the compositions of functors, vertical composition of natural transformations and horizontal composition of natural transformations.

Convension 0.6.3. *(Algebraic geometry) Unless otherwise stated, all algebro-geometric objects are defined over a fixed algebraically closed ground field k of characteristic 0, and are classical (i.e. non-derived).*

A prestack *is a contravariant functor*

$$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Groupoids}$$

from the ordinary category of affine schemes to the category of groupoids[4].

A prestack $\mathcal{Y}$ is reduced *if it is the left Kan extension of its restriction along $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{red}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, where $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{red}}$ is the category of reduced affine schemes. A map $\mathcal{Y}_1 \to \mathcal{Y}_2$ between prestacks is called a* nil-isomorphism *if its value on any* reduced *affine test scheme is an isomorphism.*

A prestack $\mathcal{Y}$ is called locally of finite type *or* lft *if it is the left Kan extension of its restriction along $(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \subset (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}$, where $\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}}$ is the category of finite type affine schemes. For the reader's convenience, we usually denote general prestacks by mathcal fonts (e.g. $\mathcal{Y}$), and leave usual fonts (e.g. Y) for lft prestacks.*

An algebraic stack *is a lft 1-Artin stack in the sense of [GR17, Chapter 2, § 4.1]. All algebraic stacks in this paper (are assumed to or can be shown to) have affine diagonals. In particular, as prestacks, they satisfy fpqc descent.*

An ind-algebraic stack *is a prestack isomorphic to a filtered colimit of algebraic stacks connected by schematic closed embeddings.*

An indscheme *is a prestack isomorphic to a filtered colimit of schemes connected by closed embeddings. All indschemes in this paper are (assumed to or can be shown to be) isomorphic to a filtered colimit of* quasi-compact quasi-separated *schemes connected by closed embeddings. In particular, they are indschemes in the sense of [GR14].*

Notation 0.6.4. *(Affine line) For a prestack $\mathcal{Y}$ over $\mathbb{A}^1$, we write $\overset{\circ}{\mathcal{Y}}$ (resp. $\mathcal{Y}_0$) for the base-change $\mathcal{Y}\times_{\mathbb{A}^1}\mathbb{G}_m$ (resp. $\mathcal{Y}\times_{\mathbb{A}^1} 0$), and $j : \overset{\circ}{\mathcal{Y}} \hookrightarrow \mathcal{Y}$ (resp. $i : \mathcal{Y}_0 \hookrightarrow \mathcal{Y}$) for the corresponding schematic open (resp. closed) embedding.*

Notation 0.6.5. *(Curves and disks) We fix a connected smooth projective curve X. For a positive integer n, we write $X^{(n)}$ for its n-th symmetric product.*

We write $\mathcal{D} := \mathrm{Spf}\, k[[t]]$ for the formal disk, *$\mathcal{D}' := \mathrm{Spec}\, k[[t]]$ for the* adic disk, *and $\mathcal{D}^\times := \mathrm{Spec}\, k((t))$ for the* punctured disk. *For a closed point x*

[4]All the prestacks in this paper would actually have *ordinary* groupoids as values.

on X, we have similarly defined prestacks $\mathcal{D}_x$, $\mathcal{D}'_x$ and $\mathcal{D}^\times_x$, which are non-canonically isomorphic to $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}^\times$.

Generally, for an affine test scheme S and an affine *closed subscheme $\Gamma \hookrightarrow X \times S$, we write $\mathcal{D}_\Gamma$ for the formal completion of Γ inside $X \times S$. We write $\mathcal{D}'_\Gamma$ for the schematic approximation*[5] *of $\mathcal{D}_\Gamma$. We write $\mathcal{D}^\times_\Gamma$ for the open subscheme $\mathcal{D}'_\Gamma - \Gamma$. We have maps*

$$\begin{tikzcd} \mathcal{D}^\times_\Gamma \arrow[r] & \mathcal{D}'_\Gamma \arrow[d] & \mathcal{D}_\Gamma \arrow[l] \\ & X \times S. & \end{tikzcd}$$

Notation 0.6.6. *(Loops and arcs) For a prestack $\mathcal{Y}$, we write $\mathcal{L}\mathcal{Y}$ (resp. $\mathcal{L}^+\mathcal{Y}$) for its* loop prestack (resp. arc prestack) *defined as follows. For an affine test scheme $S := \operatorname{Spec} R$, the groupoid $\mathcal{L}\mathcal{Y}(S)$ (resp. $\mathcal{L}^+\mathcal{Y}(S)$) classifies maps $\operatorname{Spec} R((t)) \to \mathcal{Y}$ (resp. $\operatorname{Spf} R[[t]] \to \mathcal{Y}$).*

Similarly, for a non-empty finite set I, we write $\mathcal{L}\mathcal{Y}_I$ (resp. $\mathcal{L}^+\mathcal{Y}_I$) for the loop prestack (resp. arc prestack) relative to X^I. *For an affine test scheme S, the groupoid $\mathcal{L}\mathcal{Y}_I(S)$ (resp. $\mathcal{L}^+\mathcal{Y}_I(S)$) classifies*

(i) *maps $x_i : S \to X$ labelled by I, and*
(ii) *a map $\mathcal{D}^\times_\Gamma \to \mathcal{Y}$ (resp. $\mathcal{D}_\Gamma \to \mathcal{Y}$), where $\Gamma \hookrightarrow X \times S$ is the schema-theoretic sum of the graphs of x_i.*

Notation 0.6.7. *(Reductive groups) We fix a connected reductive group G. For simplicity, we assume $[G, G]$ to be simply connected*[6].

We fix a pair of opposite Borel subgroups (B, B^-) *of it, therefore a* Cartan subgroup T. *We write Z_G for the center of G and $T_{\mathrm{ad}} := T/Z_G$ for the* adjoint torus.

We write $r := r_G$ for the semi-simple rank *of G, $\mathcal{I}$ for the* Dynkin diagram, *Λ_G (resp. $\check{\Lambda}_G$) for the* coweight (resp. weight) lattice, *and $\Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ fot the sub-monoid spanned by all positive simple co-roots $(\alpha_i)_{i \in \mathcal{I}}$.*

For any subset $\mathcal{J} \subset \mathcal{I}$, consider the corresponding standard parabolic subgroup P, *the* standard opposite parabolic subgroup P^- *and the* standard Levi subgroup M *(such that the Dynkin diagram of M is $\mathcal{J}$). We write U_P (resp. U_P^-) for the* unipotent radical *of P (resp. P^-). We omit the subscripts if it is clear from contexts. We write N (resp. N^-) for U_B (resp. U_B^-).*

We write $\Lambda_{G,P}$ for the quotient of Λ by the $\mathbb{Z}$-span of $(\alpha_i)_{i \in \mathcal{J}}$, and $\Lambda_{G,P}^{\mathrm{pos}}$ for the image of Λ_G^{pos} in $\Lambda_{G,P}$. The monoid $\Lambda_{G,P}^{\mathrm{pos}}$ defines a partial order $\le_P$ on $\Lambda_{G,P}$. We omit the subscript "P" if it is clear from the contexts.

Notation 0.6.8. *(Colored divisors) Each $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$ can be uniquely written as the image of $\sum_{i \in \mathcal{I} - \mathcal{I}_M} n_i \alpha_i$ for $n_i \in \mathbb{Z}^{\ge 0}$. We define the* configuration space

[5] $\mathcal{D}_\Gamma$ is an ind-affine indscheme. Its schematic approximation is $\operatorname{Spec} A$, where A is the topological ring of functions on $\mathcal{D}_\Gamma$.

[6] For general reductive groups, we have confidence that our results are correct after conducting the modifications in [Wan18, Appendix C.6]. However, we have not checked all the details.

$X^\theta := \prod_{i\in\mathcal{I}} X^{(n_i)}$, *whose* S*-points are* $\Lambda_{G,P}^{\mathrm{pos}}$-valued (relative Cartier) divisors *on* $X \times SS$*. We write* $X_{G,P}^{\mathrm{pos}}$ *for the disjoint union of all* $X^\theta, \theta \in \Lambda_{G,P}^{\mathrm{pos}}$*, and omit the subscript if it is clear from the context.*

For $\theta_i \in \Lambda_{G,P}^{\mathrm{pos}}, 1 \le i \le n$*, we write* $(\prod_{i=1}^n X^{\theta_i})_{\mathrm{disj}}$ *for the open subscheme of* $\prod_{i=1}^n X^{\theta_i}$ *classifying those* n*-tuples of divisors* $(D_1, \cdots, D_n)$ *with disjoint supports. For a prestack* $\mathcal{Y}$ *over* $\prod_{i=1}^n X^{\theta_i}$*, we write* $\mathcal{Y}_{\mathrm{disj}}$ *for its base-change to this open subscheme.*

Convension 0.6.9. *(DG categories) We study* DG categories *over* k*. Unless otherwise stated, DG categories are assumed to be* cocomplete *(i.e., containing colimits), and functors between them are assumed to be* continuous *(i.e. preserving colimits). The category forming by them is denoted by* DGCat*.*

DGCat *carries a closed symmetric monoidal structure, known as the* Lurie tensor product $\otimes$ *(which was denoted by* $\otimes_k$ *in the introduction). The unit object for it is* Vect *(which was denoted by* Vect_k *in the introduction). For* $\mathcal{C}, \mathcal{D} \in \mathrm{DGCat}$*, we write* $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$ *for the object in* DGCat *characterized by the universal property*

$$\mathrm{Maps}(\mathcal{E}, \mathrm{Funct}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Maps}(\mathcal{E} \otimes \mathcal{C}, \mathcal{D}).$$

Let $\mathcal{M}$ *be a DG category, we write* $\mathcal{M}^c$ *for its full subcategory consisting of compact objects, which is a* non-cocomplete DG category*.*

Notation 0.6.10. *(D-modules) Let* Y *be a finite type scheme. We write* $\mathrm{D}(Y)$ *for the DG category of D-modules on* Y*, which was denoted by* $\mathrm{DMod}(Y)$ *in the introduction. We write* ω_Y *for the dualizing D-module on* Y*.*

0.7. **Gaitsgory's exposition of this work.** D. Gaitsgory wrote a short exposition [Gai20] of this work (module Section 4), which might be helpful for the reader. However, there are some inaccuracies in this exposition. Most importantly, one cannot use [Gai20, Proposition 3.3.3] to deduce our main theorem because assumption (i) of *loc.cit.* is not satisfied (see Footnote 32).

0.8. **Acknowledgements.** This paper owes its existence to my teacher Dennis Gaitsgory. Among other things, he suggested the problem in § 0.4.5 and brought [FKM20] into my attention, which lead to the discovery of the main theorems.

I want to thank David Yang. Among other things, he resolved a pseudo contradiction which almost made me give up believing in the main theorems.

I'm grateful to Yuchen Fu, Kevin Lin, James Tao, Jonathan Wang, Ziquan Yang and Yifei Zhao for their discussions and help.

I want to thank Yuchen Fu and Dennis Gaitsgory for comments on the first draft of the paper.

I am very grateful to the anonymous referees for their numerous comments and suggestions.

1. Statements of the results

1.1. **The inv-inv duality and the second adjointness.** Let us first introduce the categorical main players of this paper. We use the theory of group actions on categories, which is reviewed in [Fulltext, Appendix B].

Definition 1.1.1. *Consider the action* $\mathcal{L}G_I \curvearrowright \mathrm{Gr}_{G,I}$*. It provides*[7] *an object* $\mathrm{D}(\mathrm{Gr}_{G,I}) \in \mathcal{L}G_I\text{-mod}$*. Consider the categories of invariants and coinvariants*

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \text{ and } \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$$

for the $\mathcal{L}U_I$*-action obtained by restriction. We write*

$$\mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I}) \text{ and } \mathbf{pr}_{\mathcal{L}U_I} : \mathrm{D}(\mathrm{Gr}_{G,I}) \to \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$$

for the corresponding forgetful and projection functors.

Remark 1.1.2. Similar to [Ras16, Remark 2.19.1], $\mathcal{L}U_I$ is an ind-pro-unipotent group scheme. It follows formally that (see [Fulltext, § B.3.1]), $\mathbf{oblv}^{\mathcal{L}U_I}$ is fully faithful, and $\mathbf{pr}_{\mathcal{L}U_I}$ is a localization functor, i.e., has a fully faithful (non-continuous) right adjoint.

Remark 1.1.3. Using [Fulltext, Formula (B.16)], it is easy to show that when P is the Borel subgroup B, our definition of $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}N_I}$ coincides with that in [Gai17a].

The following proposition is proved in § 2.3.

Proposition 1.1.4. *Both* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ *are compactly generated, and they are canonically dual to each other in* DGCat*.*

The following theorem is our first main result. A more complete version is proved in § 1.3.

Theorem 1.1.5. *(The inv-inv-duality)*

The categories $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$ *are dual to each other in* DGCat*, with the counit given by*

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-} \otimes \mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \to \mathrm{Vect},$$

where the last functor is the counit of the Verdier self-duality.

Remark 1.1.6. Explicitly, the pairing $\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \to \mathrm{Vect}$ sends $\mathcal{F} \boxtimes \mathcal{G}$ to $C_{\mathrm{dR},*}(\mathcal{F} \otimes^! \mathcal{G})$.

[7] By [Ras16, Corollary 2.13.4], $\mathcal{L}G_I$ is placid. Hence we can apply [Fulltext, § B.4] to this action.

1.1.7. *Motivation: the categorical second adjointness.* It was conjectured (in unpublished notes) by S. Raskin that for any $\mathcal{C} \in \mathcal{L}G$-mod, the functor

$$\mathbf{pr}_{\mathcal{L}N^-} \circ \mathbf{oblv}^{\mathcal{L}N} : \mathcal{C}^{\mathcal{L}N} \to \mathcal{C}_{\mathcal{L}N^-} \tag{1.1}$$

is an equivalence, where N is the unipotent radical for B. He explained that this conjecture can be viewed as a categorification of Bernstein's second adjointness [8].

For $\mathcal{C} = \mathrm{D}(\mathrm{Gr}_G)$, the conjecture is an easy consequence of [Ras16, Theorem 6.2.1, Corollary 6.2.3]. For reader's convenience, we sketch this proof, which we learned from D. Gaitsgory. By construction, the functor (1.1) is $\mathcal{L}T$-linear. Using Raskin's results, one can show (1.1) induces an equivalence:

$$(\mathrm{D}(\mathrm{Gr}_G)^{\mathcal{L}N})^{\mathcal{L}^+T} \simeq (\mathrm{D}(\mathrm{Gr}_G)_{\mathcal{L}N^-})^{\mathcal{L}^+T} \tag{1.2}$$

Using the fact that every $\mathcal{L}N$-orbit of Gr_G is stabilized by $\mathcal{L}^+T$, one can prove that the adjoint pairs

$$\mathbf{oblv}^{\mathcal{L}^+T} : (\mathrm{D}(\mathrm{Gr}_G)^{\mathcal{L}N})^{\mathcal{L}^+T} \rightleftharpoons \mathrm{D}(\mathrm{Gr}_G)^{\mathcal{L}N} : \mathbf{Av}_*^{\mathcal{L}^+T},$$
$$\mathbf{oblv}^{\mathcal{L}^+T} : (\mathrm{D}(\mathrm{Gr}_G)_{\mathcal{L}N^-})^{\mathcal{L}^+T} \rightleftharpoons \mathrm{D}(\mathrm{Gr}_G)_{\mathcal{L}N^-} : \mathbf{Av}_*^{\mathcal{L}^+T}$$

are both monadic. Then the Barr-Beck-Lurie theorem gives the desired result.

We also learned form Gaitsgory that the above equivalence can be generalized to the factorization case. I.e., the functor

$$\mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}N_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}N_I^-}, \tag{1.3}$$

is an equivalence. We sketch his proof as follows. Using the étale descent, we obtain the desired equivalence when I is a singleton. Also, one can show (e.g. using § 3.4.1) the functor (1.3) preserves compact objects, hence have *continuous* right adjoints. With some additional work, one can show these right adjoints $(\mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I})^R$ are strictly $\mathrm{D}(X^I)$-linear. It follows that the assignments $I \rightsquigarrow \mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I}$ and $I \rightsquigarrow (\mathbf{pr}_{\mathcal{L}N_I^-} \circ \mathbf{oblv}^{\mathcal{L}N_I})^R$ factorize, hence so do the adjunction natural transformations. We only need to show the adjunction natural transformations are invertible. Using factorization properties and the five lemma, one can reduce to the known case when I is a singleton.

1.1.8. *A new proof.* Combining Theorem 1.1.5 and Proposition 1.1.4, we obtain[9]:

Corollary 1.1.9. *The functor*

$$\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-}$$

is an equivalence.

[8] However, D. Yang told us he found a counter-example for this conjecture recently.

[9] A priori we only obtain *an* equivalence $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \simeq \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-}$. However, by the construction of the duality in Proposition 1.1.4, it is easy to see that this equivalence is given by the functor $\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I}$.

Remark 1.1.10. Consequently, we obtain a new proof of the equivalence (1.3) that does not rely on Raskin's results.

This new proof has three advantages:

- it works for general parabolics P rather than the Borel B (the monadicity in § 1.1.7 fails for general P);
- it works for the factorization version;
- it allows us to describe an quasi-inverse of the equivalence via a geometric construction (see Corollary 1.3.13), which we believe is of independent interest.

1.2. **Geometric players.** In order to state our other theorems, we introduce the geometric players of this paper, which are all certain versions of mapping stacks. The basic properties of mapping stacks are reviewed in Appendix C.1.

These geometric objects are well-studied in the literature. See for example [Wan18], [Sch16], [FKM20] and [DG16].

Notation 1.2.1. *The collection of simple positive roots of* G *provides an identification* $T_{\mathrm{ad}} \simeq \mathbb{G}_m^r$. *Define* $T_{\mathrm{ad}}^+ := \mathbb{A}^r \supset \mathbb{G}_m^r \simeq T_{\mathrm{ad}}$, *which is a semi-group completion of the adjoint torus* T_{ad}.

T_{ad}^+ *is stratified by the set of standard parabolic subgroups. Namely, for a standard parabolic subgroup* P *of* G *corresponding to a subset* $\mathcal{I}_M \subset \mathcal{I}$, *the stratum* $T_{\mathrm{ad},P}^+$ *is defined as the locus consisting of points* $(x_i)_{i\in\mathcal{I}}$ *such that* $x_i = 0$ *for* $i \notin \mathcal{I}_M$ *and* $x_i \neq 0$ *otherwise. A stratum* $T_{\mathrm{ad},P}^+$ *is contained in the closure of another stratum* $T_{\mathrm{ad},Q}^+$ *if and only if* $P \subset Q$.

Write C_P *for the unique point in* $T_{\mathrm{ad},P}^+$ *whose every coordinate is equal to either* 0 *or* 1. *In particular* C_B *is the zero element in* T_{ad}^+ *and* C_G *is the unit element.*

1.2.2. *The semi-group* Vin_G. The *Vinberg semi-group* Vin_G is an affine normal semi-group equipped with a flat semi-group homomorphism to T_{ad}^+. Its open subgroup of invertible elements is isomorphic to $G_{\mathrm{enh}} := (G\times T)/Z_G$, where Z_G acts on $G\times T$ anti-diagonally. Its fiber at C_P is canonoically isomorphic to

$$\mathrm{Vin}_G|_{C_P} \simeq \overline{(G/U \times G/U^-)/M},$$

where the RHS is the affine closure of $(G/U\times G/U^-)/M$[10], where M acts diagonally on $G/U^- \times G/U$ by *right* multiplication.

The $(G_{\mathrm{enh}}, G_{\mathrm{enh}})$-action on Vin_G induces a (G,G)-action on Vin_G, which preserves the projection $\mathrm{Vin}_G \to T_{\mathrm{ad}}^+$. On the fiber $\mathrm{Vin}_G|_{C_P}$, this action extends the left multiplication action of $G\times G$ on $(G/U\times G/U^-)/M$.

There is a canonical section $\mathfrak{s} : T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$, which is also a semi-group homomorphism. Its restriction on $T_{\mathrm{ad}} := T/Z_G$ is given by

$$T/Z_G \to (G\times T)/Z_G,\ t \mapsto (t^{-1}, t).$$

[10] This scheme is strongly quasi-affine in the sense of [BG02, Subsection 1.1].

The $(G\times G)$-orbit of the section $\mathfrak{s}$ is an open subscheme of Vin_G, known as the *defect-free locus* ${}_0\mathrm{Vin}_G$.

$$(G\times T)/Z_G \simeq \mathrm{Vin}_G \underset{T_{\mathrm{ad}}^+}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{Vin}_G \subset \mathrm{Vin}_G\,. \tag{1.4}$$

The fiber ${}_0\mathrm{Vin}_G|_{C_P}$ is given by $(G/U\times G/U^-)/M$, and the canonical section intersects it at the point $(1,1)$.

Example 1.2.3. When $G=\mathrm{SL}_2$, the base T_{ad}^+ is isomorphic to $\mathbb{A}^1$. The semi-group Vin_G is isomorphic to the monoid $\mathrm{M}_{2,2}$ of 2×2 matrices. The projection $\mathrm{Vin}_G\to\mathbb{A}^1$ is given by the determinant function. The canonical section is $\mathbb{A}^1\to\mathrm{M}_{2,2}$, $t\mapsto \mathrm{diag}(1,t)$. The action of $\mathrm{SL}_2\times\mathrm{SL}_2$ on $\mathrm{M}_{2,2}$ is given by $(g_1,g_2)\cdot A = g_1Ag_2^{-1}$.

Warning 1.2.4. *There is no consensus convention for the order of the two G-actions on Vin_G in the literature. Even worse, this order is* not *self-consistent in either [Sch16]*[11] *or [FKM20]*[12].

In this paper, we usethe order in [Wan17] and [Wan18]. We ask the reader to keep an eye on this issue when we cite other references.

Definition 1.2.5. *Let* $\mathrm{Bun}_G := \mathbf{Maps}(X,\mathrm{pt}/G)$ *be the moduli stack of G-torsors on X. Following [Sch16], the* Drinfeld-Lafforgue-Vinberg degeneration *of* Bun_G *is defined as (see Definition C.1.1 for the notation* $\mathbf{Maps}_{\mathrm{gen}}$*):*

$$\mathrm{VinBun}_G := \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash\mathrm{Vin}_G/G \supset G\backslash{}_0\mathrm{Vin}_G/G). \tag{1.5}$$

Definition 1.2.6. *The* defect-free locus *of* VinBun_G *is defined as*

$${}_0\mathrm{VinBun}_G := \mathbf{Maps}(X, G\backslash{}_0\mathrm{Vin}_G/G).$$

Remark 1.2.7. The maps $G\backslash\mathrm{Vin}_G/G\to T_{\mathrm{ad}}^+$ and $G\backslash\mathrm{Vin}_G/G\to G\backslash\mathrm{pt}/G$ induce a map (see Example C.1.2):

$$\mathrm{VinBun}_G\to\mathrm{Bun}_{G\times G}\times T_{\mathrm{ad}}^+.$$

The chain (1.4) induces open embeddings:

$$\mathrm{VinBun}_G \underset{T_{\mathrm{ad}}^+}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{VinBun}_G \subset \mathrm{VinBun}_G\,. \tag{1.6}$$

Remark 1.2.8. The parabolic stratification on the base T_{ad}^+ (see Notation 1.2.1) induces a *parabolic stratification* on VinBun_G. By [Wan18, (C.2)], each stratum $\mathrm{VinBun}_{G,P}$ is constant along $T_{\mathrm{ad},P}^+$.

Example 1.2.9. When $G=\mathrm{SL}_2$, for an affine test scheme S, the groupoid $\mathrm{VinBun}_G(S)$ classifies triples (E_1,E_2,ϕ), where E_1,E_2 are rank 2 vector bundles on $X\times SS$ whose determinant line bundles are trivialized, and $\phi: E_1\to E_2$ is a map such that its restriction at any geometric point s of S is an injection between *quasi-coherent sheaves* on $X\times s$. Since the determinant line bundles of E_1 and E_2 are trivialized, we can define the determinant

[11] [Sch16, Lemma 2.1.11] and [Sch16, § 6.1.2] are not consistent.

[12] [FKM20, Remark 3.14] and [FKM20, § 3.2.7] are not consistent.

$\det(\phi)$, which is a function on S because X is proper. Therefore we obtain a map $\mathrm{VinBun}_G \to \mathbb{A}^1 \simeq T_{\mathrm{ad}}^+$, which is the canonical projection.

In this paper, we are mostly interested in the following $\mathbb{A}^1$-degeneration of Bun_G obtained from VinBun_G.

Construction 1.2.10. *Let P be a standard parabolic subgroup of G and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. There exists a unique morphism of monoids $\overline{\gamma} : \mathbb{A}^1 \to T_{\mathrm{ad}}^+$ extending the obvious map $\mathbb{G}_m \to Z_M \hookrightarrow T \twoheadrightarrow T_{\mathrm{ad}}$. Define*

$$\mathrm{Vin}_G^\gamma := \mathrm{Vin}_G \underset{(T_{\mathrm{ad}}^+,\overline{\gamma})}{\times} \mathbb{A}^1$$

and similarly VinBun_G^γ.

We also define

$$_0\mathrm{VinBun}_G^\gamma := \mathrm{VinBun}_G^\gamma \underset{\mathrm{VinBun}_G}{\times} {}_0\mathrm{VinBun}_G\,.$$

The above $\mathbb{A}^1$-family is closed related to the *Drinfeld-Gaitsgory interpolation* constructed in [Dri13] and [DG14]. To describe it, we need some definitions.

Definition 1.2.11. *Let Z be any lft prestack equipped with a $\mathbb{G}_m$-action. Consider the $\mathbb{G}_m$-actions on $\mathbb{A}^1$ and $\mathbb{A}^1_- := \mathbb{P}^1 - \{\infty\}$. We define the* attractor, repeller, *and* fixed loci *for Z respectively by:*

$$Z^{\mathrm{att}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z),\ Z^{\mathrm{rep}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1_-, Z),\ Z^{\mathrm{fix}} := \mathbf{Maps}^{\mathbb{G}_m}(\mathrm{pt}, Z),$$

where $\mathbf{Maps}^{\mathbb{G}_m}(W,Z)$ is the lft prestack that classifies $\mathbb{G}_m$-equivariant maps $W \to Z$.

Construction 1.2.12. *By construction, we have maps*

$$p^+ : Z^{\mathrm{att}} \to Z,\ i^+ : Z^{\mathrm{fix}} \to Z^{\mathrm{att}},\ q^+ : Z^{\mathrm{att}} \to Z^{\mathrm{fix}}$$

induced respectively by the $\mathbb{G}_m$-equivariant maps $\mathbb{G}_m \to \mathbb{A}^1$, $\mathbb{A}^1 \to \mathrm{pt}$, $\mathrm{pt} \overset{0}{\to} \mathbb{A}^1$. We also have similar maps p^-, i^-, q^- for the repeller locus. Note that i^+ (resp. i^-) is a right inverse for q^+ (resp. q^-). We also have $p^+ \circ i^+ \simeq p^- \circ i^-$.

Example 1.2.13. Let P be a standard parabolic subgroup of G and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. The adjoint action of G on itself induces a $\mathbb{G}_m$-action on G. We have $G^{\gamma,\mathrm{att}} \simeq P$, $G^{\gamma,\mathrm{rep}} \simeq P^-$ and $G^{\gamma,\mathrm{fix}} \simeq M$.

Example 1.2.14. In the above example, the adjoint action of G on itself induces a G-action on $\mathrm{Gr}_{G,I}$. Hence we obtain a $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$. There are isomorphisms[13]

$$\mathrm{Gr}_{P,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}},\ \mathrm{Gr}_{P^-,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}},\ \mathrm{Gr}_{M,I} \simeq \mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}}$$

[13] When X is the affine line $\mathbb{A}^1$, the claim is proved in [HR18, Theorem A]. As explained in [HR18, Remark 3.18i), Footnote 3], one can deduce the general case from this special case. For completeness, we provide this argument in [Fulltext, § C.2].

defined over $\mathrm{Gr}_{G,I}$. Moreover, these isomorphisms are compatible with the maps $\mathrm{Gr}_{P^{\pm},I} \to \mathrm{Gr}_{M,I}$ and $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{att\ or\ rep}} \to \mathrm{Gr}_{G,I}^{\mathrm{fix}}$.

1.2.15. *Drinfeld-Gaitgory interpolation.* Let Z be any finite type scheme acted on by $\mathbb{G}_m$. [DG14, § 2.2.1] constructed the *Drinfeld-Gaitsgory interpolation*

$$\widetilde{Z} \to Z \times Z \times \mathbb{A}^1,$$

where $\widetilde{Z}$ is a finite type scheme. The $\mathbb{G}_m$-locus $\widetilde{Z} \times_{\mathbb{A}^1} \mathbb{G}_m$ is isomorphic to the graph of the $\mathbb{G}_m$-action, i.e., the image of the map

$$\mathbb{G}_m \times Z \to Z \times Z \times \mathbb{G}_m,\ (s,z) \mapsto (z, s\cdot z, s).$$

The 0-fiber $\widetilde{Z} \times_{\mathbb{A}^1} 0$ is isomorphic to $Z^{\mathrm{att}} \times_{Z^{\mathrm{fix}}} Z^{\mathrm{rep}}$.

Moreover, by [DG14, § 2.5.11], the map $\widetilde{Z} \to Z \times Z \times \mathbb{A}^1$ is a locally closed embedding if we assume:

- (♣) Z admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space $\mathbb{P}(V)$, where $\mathbb{G}_m$-acts linearly on V.

Remark 1.2.16. The construction $Z \rightsquigarrow \widetilde{Z}$ is functorial in Z and is compatible with Cartesian products.

Example 1.2.17. The $\mathbb{G}_m$-action on G in Example 1.2.13 satisfies condition (♣). Indeed, using a faithful representation $G \to \mathrm{GL}_n$, we reduce the claim to the case $G = \mathrm{GL}_n$, which is obvious.

Notation 1.2.18. *We denote the Drinfeld-Gaitsgory interpolation for the action in Example 1.2.13 by* $\widetilde{G}^\gamma$.

Remark 1.2.19. The above action $\mathbb{G}_m \curvearrowright G$ is compatible with the group structure on G. Hence by Remark 1.2.16, $\widetilde{G}^\gamma$ is a group scheme over $\mathbb{A}^1$. Note that its 1-fiber (resp. 0-fiber) is isomorphic to G (resp. $P \times_M P^-$).

Fact 1.2.20. *The following facts are proved in [DG16]:*

- *There is a* $(G\times G)$*-equivariant isomorphism*

$$ {}_0\mathrm{Vin}_G^\gamma \simeq (G\times G\times \mathbb{A}^1)/\widetilde{G}^\gamma \tag{1.7}$$

  that sends the canonical section $\mathfrak{s} : \mathbb{A}^1 \to {}_0\mathrm{Vin}_G^\gamma$ *to the unit section of the RHS. In particular,*

$$G\backslash {}_0\mathrm{Vin}_G^\gamma / G \simeq \mathbb{B}\widetilde{G}^\gamma,$$

  where $\mathbb{B}\widetilde{G}^\gamma := \mathbb{A}^1/\widetilde{G}^\gamma$ *is the classifying stack.*
- *There is an isomorphism*

$${}_0\mathrm{VinBun}_G^\gamma \simeq \mathrm{Bun}_{\widetilde{G}^\gamma} := \mathbf{Maps}(X, \mathbb{B}\widetilde{G}^\gamma).$$

  In particular, there are isomorphisms

$${}_0\mathrm{VinBun}_G|_{C_P} \simeq \mathrm{Bun}_{P\times_M P^-} \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathrm{Bun}_{P^-}$$

  defined over $\mathrm{Bun}_{G\times G} \simeq \mathrm{Bun}_G \times \mathrm{Bun}_G$.

Warning 1.2.21. *The isomorphism* $\mathrm{Bun}_{P\times_M P^-} \simeq \mathrm{Bun}_P \times_{\mathrm{Bun}_M} \mathrm{Bun}_{P^-}$ *is due to*

$$\mathbb{B}(P \underset{M}{\times} P^-) \simeq P\backslash M/P^- \simeq \mathbb{B}P \underset{\mathbb{B}M}{\times} \mathbb{B}P^-. \tag{1.8}$$

However, the map $\mathbb{B}(G_2 \times_{G_1} G_3) \to \mathbb{B}G_2 \times_{\mathbb{B}G_1} \mathbb{B}G_3$ *is not an isomorphism in general (for example when* $G_2 = P$*,* $G_3 = P^-$ *and* $G_1 = G$*).*

We also need the following local analogue of VinBun_G.

Definition 1.2.22. *Let* I *be a non-empty finite set. Following [FKM20], we define the* Drinfeld-Gaitsgory-Vinberg interpolation Grassmannian *as (see Definition C.1.3 for the notation below):*

$$\mathrm{VinGr}_{G,I} \coloneqq \mathbf{Maps}_{I,/T_{\mathrm{ad}}^+}(X, G\backslash \mathrm{Vin}_G/G \leftarrow T_{\mathrm{ad}}^+),$$

where the map $T_{\mathrm{ad}}^+ \to G\backslash \mathrm{Vin}_G/G$ *is induced by the canonical section* $\mathfrak{s}: T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$*.*

The defect-free locus of $\mathrm{VinGr}_{G,I}$ *is defined as:*

$${}_0\mathrm{VinGr}_{G,I} \coloneqq \mathbf{Maps}_{I,/T_{\mathrm{ad}}^+}(X, G\backslash {}_0\mathrm{Vin}_G/G \leftarrow T_{\mathrm{ad}}^+).$$

Remark 1.2.23. As before, the map $G\backslash \mathrm{Vin}_G/G \to (G\backslash \mathrm{pt}/G) \times T_{\mathrm{ad}}^+$ induces a map

$$\mathrm{VinGr}_{G,I} \to \mathrm{Gr}_{G\times G,I} \times T_{\mathrm{ad}}^+.$$

By [FKM20, Lemma 3.7], this map is a schematic closed embedding. Hence $\mathrm{VinGr}_{G,I}$ is an ind-projective indscheme.

As before, we have open embeddings

$$\mathrm{VinGr}_{G,I} \underset{T_{\mathrm{ad}}^+}{\times} T_{\mathrm{ad}} \subset {}_0\mathrm{VinGr}_{G,I} \subset \mathrm{VinGr}_{G,I}. \tag{1.9}$$

Construction 1.2.24. *By Construction C.1.7, there is a* local-to-global map

$$\pi_I : \mathrm{VinGr}_{G,I} \to \mathrm{VinBun}_G \tag{1.10}$$

fitting into the following commutative diagram

$$\begin{tikzcd} \mathrm{VinGr}_{G,I} \arrow[r] \arrow[d] & \mathrm{VinBun}_G \arrow[d] \\ \mathrm{Gr}_{G\times G,I} \times T_{\mathrm{ad}}^+ \arrow[r] & \mathrm{Bun}_{G\times G} \times T_{\mathrm{ad}}^+. \end{tikzcd}$$

It follows from the construction that ${}_0\mathrm{VinGr}_{G,I}$ *is the pre-image of* ${}_0\mathrm{VinBun}_G$ *under* π_I*.*

Remark 1.2.25. Recall that the assignment $I \rightsquigarrow \mathrm{Gr}_{G,I}$ *factorizes* in the sense of Beilinson-Drinfeld. It is known that the assignment $I \rightsquigarrow \mathrm{VinGr}_{G,I}$ *factorizes in families* over T_{ad}^+. Recall that this means we have isomorphisms

$$\begin{aligned} \mathrm{VinGr}_{G,I} \underset{X^I}{\times} X^J &\simeq \mathrm{VinGr}_{G,J}, \text{ for } I \twoheadrightarrow J, \\ \mathrm{VinGr}_{G,I_1\sqcup I_2} \underset{X^{I_1\sqcup I_2}}{\times} (X^{I_1}\times X^{I_2})_{\mathrm{disj}} &\simeq (\mathrm{VinGr}_{G,I_1} \underset{T_{\mathrm{ad}}^+}{\times} \mathrm{VinGr}_{G,I_1})_{\mathrm{disj}}, \end{aligned}$$

satisfying certain compatibilities.

Construction 1.2.26. *Let γ be as in Construction 1.2.10, we have the following degenerations of* $\mathrm{Gr}_{G,I}$*:*

(a) *The* $\mathbb{A}^1$*-degeneration*

$$\mathrm{VinGr}^{\gamma}_{G,I} := \mathrm{VinGr}_{G,I} \underset{(T^+_{\mathrm{ad}},\bar{\gamma})}{\times} \mathbb{A}^1,$$

which is a closed sub-indscheme of $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$.

(b) *The* $\mathbb{A}^1$*-degeneration*

$$\mathrm{Gr}_{\widetilde{G}^\gamma,I} := \mathbf{Maps}_{I,/\mathbb{A}^1}(X, \mathbb{B}\widetilde{G}^\gamma \leftarrow \mathbb{A}^1),$$

which is equipped with a map

$$\mathrm{Gr}_{\widetilde{G}^\gamma,I} \to \mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \simeq \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,$$

Lemma 1.2.27. *(1) There is an isomorphism*

$$_0\mathrm{VinGr}^{\gamma}_{G,I} \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I}$$

defined over $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$.

(2) Consider the $\mathbb{G}_m$*-action on* $\mathrm{Gr}_{G,I}$ *induced by* γ *and the graph of this action:*

$$\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{G}_m,\ (x,t) \mapsto (x, t\cdot x, t). \tag{1.11}$$

Then there are isomorphisms

$$\mathrm{VinGr}^{\gamma}_{G,I} \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I} \underset{\mathbb{A}^1}{\times} \mathbb{G}_m \simeq \mathrm{Gr}_{G,I} \times \mathbb{G}_m$$

defined over $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{G}_m$.

Proof. (1) follows from the $(G\times G)$-equivariant isomorphism (1.7). The first isomorphism in (2) follows from (1) and the chain (1.9). The second isomorphism in (2) follows from the isomorphism $\widetilde{G} \times_{\mathbb{A}^1} \mathbb{G}_m \simeq G \times \mathbb{G}_m$ between group schemes over $\mathbb{G}_m$.

□[Lemma 1.2.27]

Remark 1.2.28. Note that

$$_0\mathrm{VinGr}_{G,I}|_{C_P} \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I}|_{C_P} \simeq \mathrm{Gr}_{P,I} \underset{\mathrm{Gr}_{M,I}}{\times} \mathrm{Gr}_{P^-,I}$$

is preserved by the $\mathcal{L}(U\times U^-)_I$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$.

Remark 1.2.29. In fact, one can show $\mathrm{VinGr}_{G,I}|_{C_P}$ is preserved by the above action. This is a formal consequence of the fact that the $(U\times U^-)$-action on $\mathrm{Vin}_G|_{C_P}$ fixes the canonical section $\mathfrak{s}|_{C_P} : \mathrm{pt} \to \mathrm{Vin}_G|_{C_P}$. We do not need this fact in this paper hence we do not provide the details of its proof.

1.3. Nearby cycles and the unit of the inv-inv duality.

Construction 1.3.1. *Let I be a non-empty finite set, P be a standard parabolic subgroup and $\gamma : \mathbb{G}_m \to Z_M$ be a co-character dominant and regular with respect to P. Consider the indscheme*

$$Z := \mathrm{VinGr}^{\gamma}_{G,I} \to \mathbb{A}^1$$

defined in Construction 1.2.26.

By Lemma 1.2.27(2), we have $\overset{\circ}{Z} \simeq \mathrm{Gr}_{G,I} \times \mathbb{G}_m$. Consider the corresponding nearby cycles functor

$$\Psi_{\mathrm{VinGr}^{\gamma}_{G,I}} : \mathrm{D}_{\mathrm{rh}}(\mathrm{Gr}_{G,I} \times \mathbb{G}_m) \to \mathrm{D}(\mathrm{VinGr}_{G,I}|_{C_P}),$$

where the subscript "rh" means the full subcategory of regular ind-holonomic D-modules (see [Fulltext, § A.4.6] for what this means). The dualizing D-module $\omega_{\overset{\circ}{Z}}$ is regular ind-holonomic. Hence we obtain an object

$$\Psi_{\gamma,I,\mathrm{Vin}} := \Psi_{\mathrm{VinGr}^{\gamma}_{G,I}}(\omega_{\overset{\circ}{Z}}) \in \mathrm{D}(\mathrm{VinGr}_{G,I}|_{C_P}).$$

Construction 1.3.2. *Let*

$$\Psi_{\gamma,I} \in \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})$$

be the direct image of $\Psi_{\gamma,I,\mathrm{Vin}}$ for the closed embedding $\mathrm{VinGr}_{G,I}|_{C_P} \hookrightarrow \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$.

Consider the constant family

$$\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \to \mathbb{A}^1.$$

Since taking the nearby cycles commutes with proper push-forward functors, $\Psi_{\gamma,I}$ can also be calculated as the nearby cycles sheaf of $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m})$ along this constant family, where Γ_I was defined in (1.11).

Variant 1.3.3. *We can replace the above full nearby cycles by the unipotent ones and obtain similarly defined objects $\Psi^{\mathrm{un}}_{\gamma,I,\mathrm{Vin}}$ and $\Psi^{\mathrm{un}}_{\gamma,I}$.*

We have (see Proposition 2.4.1(1)):

Proposition 1.3.4. *The maps*

$$\Psi^{\mathrm{un}}_{\gamma,I,\mathrm{Vin}} \to \Psi_{\gamma,I,\mathrm{Vin}},\ \Psi^{\mathrm{un}}_{\gamma,I} \to \Psi_{\gamma,I}$$

are isomorphisms, i.e., the monodromy endomorphisms on $\Psi_{\gamma,I,\mathrm{Vin}}$ and $\Psi_{\gamma,I}$ are locally unipotent.

Construction 1.3.5. *It follows formally from the Verdier duality that we have an equivalence*

$$F : \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}) \simeq \mathrm{Funct}(\mathrm{D}(\mathrm{Gr}_{G,I}), \mathrm{D}(\mathrm{Gr}_{G,I}))$$

that sends an object $\mathcal{K}$ to

$$F_{\mathcal{M}}(-) := \mathrm{pr}_{2,*}(\mathrm{pr}_1^!(-) \overset{!}{\otimes} \mathcal{M}).$$

The functor $F_{\mathcal{M}}$ is the functor given by the kernel $\mathcal{M}$ *in the sense of [Gai16].*

Write $\iota : \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \hookrightarrow \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}$ *for the obvious closed embedding. Consider the object*

$$\mathcal{K} := \iota_*(\Psi_{\gamma,I}[-1]) \in \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}).$$

Also consider $\mathcal{K}^\sigma := \sigma_* \mathcal{K}$*, where* σ *is the involution on* $\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}$ *given by switching the two factors. Using these objects as kernels, we obtain functors*

$$F_{\mathcal{K}}, F_{\mathcal{K}^\sigma} : \mathrm{D}(\mathrm{Gr}_{G,I}) \to \mathrm{D}(\mathrm{Gr}_{G,I}).$$

Remark 1.3.6. The above functor $F_{\mathcal{K}}$ is the *functor given by the kernel* $\mathcal{K}$ in the sense of [Gai16].

The following theorem is proved in § 3.5:

Theorem 1.3.7. *(1) We have a canonical isomorphism in* $\mathrm{Funct}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}, \mathrm{D}(\mathrm{Gr}_{G,I}))$*:*

$$F_{\mathcal{K}}|_{\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}} \simeq \mathbf{oblv}^{\mathcal{L}U_I^-}.$$

(2) We have a canonical isomorphism in $\mathrm{Funct}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{G,I}))$*:*

$$F_{\mathcal{K}^\sigma}|_{\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}} \simeq \mathbf{oblv}^{\mathcal{L}U_I}.$$

1.3.8. *Unit of the inv-inv duality.* In § 2.4, we prove that the object $\Psi_{\gamma,I}$ is contained in the full subcategory

$$\mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times_{X^I} \mathcal{L}U_I^-} \subset \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I}).$$

Moreover, this full subcategory can be identified with (see Corollary 2.3.6(2))

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{D}(X^I)}{\otimes} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}.$$

It follows formally (see [Fulltext, Lemma B.1.8(3)]) that the kernel $\mathcal{K}$ is contained in the full subcategory[14]

$$\mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times \mathcal{L}U_I^-} \subset \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}).$$

Again, this full subcategory can be identified with (see Corollary 2.3.6(1))

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$$

The following result says that $\mathcal{K}$ is the unit of the inv-inv duality.

Corollary 1.3.9. *(1) The functor*

$$\mathrm{Vect} \xrightarrow{\mathcal{K}\otimes -} \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times \mathcal{L}U_I^-} \simeq \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$$

is the unit of a duality datum, and the corresponding counit is the functor in Theorem 1.1.5.

---

[14]The reader might have noticed that this claim is a formal consequence of Theorem 1.3.7. However, we need to prove this fact before we prove the theorem.

(2) The categories $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *and* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$ *are dual to each other in*[15] $\mathrm{D}(X^I)$-mod, *with the unit given by*

$$\mathrm{Vect} \xrightarrow{\Psi_{\gamma,I}[-1]\otimes -} \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times_{X^I} \mathcal{L}U_I^-} \simeq$$
$$\simeq \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \underset{\mathrm{D}(X^I)}{\otimes} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-},$$

and the counit given by

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}\otimes \mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \to \mathrm{D}(X^I),$$

where the last functor is the counit[16] *of the Verdier self-duality for* $\mathrm{D}(\mathrm{Gr}_{G,I})$ *as a* $\mathrm{D}(X^I)$*-module category.*

Proof. To prove (1), we check the axioms for the dualities. By symmetry, we only need to show the composition

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \xrightarrow{-\boxtimes\mathcal{K}} \mathrm{D}(\mathrm{Gr}_{G,I}\times \mathrm{Gr}_{G,I}\times \mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-\times \mathcal{L}U_I\times\mathcal{L}U_I^-} \xrightarrow{\langle -,-\rangle\otimes\mathbf{Id}} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$$

is isomorphic to the identity functor. We only need to show its composition with the fully faithful functor $\mathbf{oblv}^{\mathcal{L}U_I^-}: \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to \mathrm{D}(\mathrm{Gr}_{G,I})$ is isomorphic to $\mathbf{oblv}^{\mathcal{L}U_I^-}$. By definition, this composition is just the functor given by the kernel $\mathcal{K}$, i.e., the functor $F_{\mathcal{K}}|_{\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}}$. Hence we are done by Theorem 1.3.7.

Using [Fulltext, Lemma B.7.2], one can similarly prove (2).

□[Corollary 1.3.9]

Warning 1.3.10. *Our proof of Theorem 1.3.7, and therefore of Corollary 1.3.9, logically depends on the dualizability results in Proposition 1.1.4. Hence we* cannot *avoid Appendix D.*

Remark 1.3.11. In the constructible contexts, Theorem 1.3.7 remains correct, and can be proved similarly. We also have a version of Corollary 1.3.9(1). See Remark 2.3.8 for more details.

However, we do *not* have a version of Corollary 1.3.9(2) in the constructible contexts. For example, we do *not* even know if $\mathrm{Shv}_c(\mathrm{Gr}_{G,I})$ is self-dual as a $\mathrm{Shv}_c(S)$-module category, where Shv_c is the DG category of complexes of constructible sheaves.

Remark 1.3.12. As a by-product, the object $\Psi_{\gamma,I}$ does not depend on the choice of γ.

[15] $\mathrm{D}(X^I)$ is equipped with the symmetric monoidal structure given by the !-tensor products.

[16] It is given by

$$\mathrm{D}(\mathrm{Gr}_{G,I})\otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\otimes^!} \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{*\text{-pushforward}} \mathrm{D}(X^I).$$

We can now give the following description of the inverse of the equivalence in Corollary 1.1.9:

Corollary 1.3.13. *(1) The functor $F_{\mathcal{K}}$ factors uniquely as*

$$F_{\mathcal{K}} : \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}} \mathrm{D}(\mathrm{Gr}_{G,I}),$$

and the functor in the middle is inverse to

$$\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}.$$

(2) The functor $F_{\mathcal{K}^\sigma}$ factors uniquely as

$$F_{\mathcal{K}^\sigma} : \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I^-}} \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-} \to \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I}),$$

and the functor in the middle is inverse to

$$\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I^-}.$$

Proof. We prove (1) and obtain (2) by symmetry. By Proposition 1.1.4, $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ are dual to each other. Moreover, it is formal (see [Fulltext, Lemma B.1.11]) that the counit functor of this duality fits into a commutative diagram

$$\begin{tikzcd} \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \arrow[r, "\mathbf{Id} \otimes \mathbf{oblv}^{\mathcal{L}U_I}"] \arrow[d, "\mathbf{pr}_{\mathcal{L}U_I} \otimes \mathbf{Id}"] & \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \arrow[d] \\ \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \arrow[r, "\mathbf{counit}"] & \mathrm{Vect}, \end{tikzcd} \tag{1.12}$$

where the right vertical functor is the counit for the Verdier self-duality.

On the other hand, by Corollary 1.3.9(1) and (1.12), the composition

$$\mathbf{counit} \circ ((\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-}) \otimes \mathbf{Id}) : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{Vect}$$

is also the counit of a duality. Hence by uniqueness of the dual category, the functor $\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{oblv}^{\mathcal{L}U_I^-}$ is an equivalence. Denote the inverse of this equivalence by θ.

Note that the desired factorization of $F_{\mathcal{K}}$ is unique if it exists because $\mathbf{pr}_{\mathcal{L}U_I}$ is a localization and $\mathbf{oblv}^{\mathcal{L}U_I^-}$ is a full embedding. Hence it remains to show that $\mathbf{oblv}^{\mathcal{L}U_I^-} \circ \theta \circ \mathbf{pr}_{\mathcal{L}U_I}$ is isomorphic to $F_{\mathcal{K}}$. By uniqueness of the dual category, the functor θ is given by the composition

$$\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \xrightarrow{\mathbf{Id} \otimes \mathbf{unit}^{\text{inv-inv}}} \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-} \to$$
$$\xrightarrow{\mathbf{counit} \otimes \mathbf{Id}} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-},$$

where $\mathbf{unit}^{\text{inv-inv}}$ is the unit of the duality between $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}$. Now the desired claim can be checked directly using Corollary (1.3.9)(1).

□[Corollary 1.3.13]

Remark 1.3.14. In a future paper (mentioned in § 0.4.5), we will prove the following description of the values of $\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I}$ on the compact generators of $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. Write $\mathbf{s}_I : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{G,I}$ for the closed embedding. Let $\mathcal{F}$ be a compact object in $\mathrm{D}(\mathrm{Gr}_{M,I})$. Then $\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{oblv}^{\mathcal{L}U_I}$ sends the compact object (see Lemma 2.3.4(2))

$$\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{F}) \in \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$$

to $\mathbf{pr}_{\mathcal{L}U_I^-} \circ \mathbf{s}_{I,*}(\mathcal{F})$. This formally implies under the inv-inv duality, the dual object of $\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{F})$ is $\mathbf{Av}_!^{\mathcal{L}U_I^-} \circ \mathbf{s}_{I,*}(\mathbb{D}\mathcal{F})$.

1.4. **Variant: $\mathcal{L}^+M$-equivariant version.** In this subsection, we describe an $\mathcal{L}^+M$-equivariant version of the main theorems.

Construction 1.4.1. *Consider the following short exact sequence of group indschemes:*

$$\mathcal{L}U_I \to \mathcal{L}P_I \to \mathcal{L}M_I.$$

It admits a splitting $\mathcal{L}M_I \to \mathcal{L}P_I$. It follows formally (see [Fulltext, Lemma B.5.2]) that $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ can be upgraded to objects in $\mathcal{L}M_I\text{-mod}$. *Also, the functors $\mathbf{oblv}^{\mathcal{L}U_I}$ and $\mathbf{pr}_{\mathcal{L}U_I}$ have $\mathcal{L}M_I$-linear structures.*

We define

$$(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^{\mathcal{L}^+M_I} \text{ and } (\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-})^{\mathcal{L}^+M_I}.$$

As one would expect (see [Fulltext, Corollary B.6.3]), they are isomorphic to

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \text{ and } \mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I},$$

where $(\mathcal{L}U\mathcal{L}^+M)_I$ is the subgroup indscheme of $\mathcal{L}G_I$ generated by $\mathcal{L}U_I$ and $\mathcal{L}^+M_I$.

Construction 1.4.2. *We prove in Proposition 2.4.1 that $\Psi_{\gamma,I}$ can be upgraded to an object*

$$\Psi_{\gamma,I} \in \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I,\mathrm{diag}}.$$

It follows formally (see [Fulltext, Lemma B.7.9(1)]) that the functors $F_{\mathcal{K}}$ and $F_{\mathcal{K}^\sigma}$ defined in § 1.3 can be upgraded to $\mathcal{L}^+M_I$-linear functors.

The following result is deduced from Theorem 1.3.7 in § 3.5.7:

Corollary 1.4.3. *(1) We have a canonical isomorphism in* $\mathrm{Funct}_{\mathcal{L}^+M_I}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}, \mathrm{D}(\mathrm{Gr}_{G,I}))$

$$F_{\mathcal{K}}|_{\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-}} \simeq \mathbf{oblv}^{\mathcal{L}U_I^-}.$$

(2) We have a canonical isomorphism in $\mathrm{Funct}_{\mathcal{L}^+M_I}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{G,I}))$

$$F_{\mathcal{K}^\sigma}|_{\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}} \simeq \mathbf{oblv}^{\mathcal{L}U_I}.$$

1.4.4. *The inv-inv duality: equivariant version.* Since $\mathcal{L}^+M_I$ is a group scheme (rather than indscheme), as one would expect (see [Fulltext, Corollary B.6.1 and Lemma B.2.5]), we have an equivalence[17]

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I} \simeq \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}^+M_I}.$$

Moreover, $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}$ is self-dual.

We define

$$\mathbb{D}^{\frac{\infty}{2}} := \mathbf{Av}_*^{(\mathcal{L}^+M_I,\mathrm{diag})\to(\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I)}(\Psi_{\gamma,I}[-1]),$$

where the functor

$$\mathbf{Av}_* : \mathrm{D}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I,\mathrm{diag}} \to (\mathrm{D}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}$$

is the right adjoint of the obvious forgetful functor.

The equivariant structures on $\Psi_{\gamma,I}[-1]$ formally imply (see [Fulltext, Lemma B.5.2]) that $\mathbb{D}^{\frac{\infty}{2}}$ can be upgraded to an object in

$$(\mathrm{D}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U_I^-})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}.$$

Moreover, as one would expect (see [Fulltext, Lemma B.1.12 and Corollary B.6.3]), this category is isomorphic to

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}\underset{\mathrm{D}(X^I)}{\otimes}\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}.$$

The following result follows formally (see [Fulltext, Lemma B.7.9(2)]) from Corollary 1.4.3:

Corollary 1.4.5. *(1)* $\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}$ *and* $\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}$ *are dual to each other in* DGCat, *with the counit given by*

$$\begin{aligned}&\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}\otimes\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}\to\\ \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I^-}\otimes\mathbf{oblv}^{\mathcal{L}U_I}}\ &\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}\otimes\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}\to\mathrm{Vect}\end{aligned}$$

where the last functor is the counit of the self-duality of $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}^+M_I}$ *in* DGCat.

(2) The unit of the duality in (1) is

$$\begin{aligned}\mathrm{Vect}\ &\xrightarrow{\mathbb{D}^{\frac{\infty}{2}}\otimes-}\ (\mathrm{D}(\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I\times_{X^I}\mathcal{L}U_I^-})^{\mathcal{L}^+M_I\times_{X^I}\mathcal{L}^+M_I}\\ &\simeq\ \mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}\underset{\mathrm{D}(X^I)}{\otimes}\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}\\ &\to\ \mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I}\otimes\mathrm{D}(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I}.\end{aligned}$$

[17]Via this equivalence, $\mathbf{pr}_{\mathcal{L}^+M_I}$ corresponds to $\mathbf{Av}_*^{\mathcal{L}^+M_I}$

Remark 1.4.6. The last functor in the above composition is induced by $\Delta_* : \mathrm{D}(X^I) \to \mathrm{D}(X^I \times X^I)$. Namely, for any $\mathcal{M}, \mathcal{N} \in \mathrm{D}(X^I)$-mod, we have a functor

$$\mathcal{M} \underset{\mathrm{D}(X^I)}{\otimes} \mathcal{N} \simeq (\mathcal{M} \otimes \mathcal{N}) \underset{\mathrm{D}(X^I \times X^I)}{\otimes} \mathrm{D}(X^I) \overset{\mathbf{Id} \otimes \Delta_*}{\to} \mathcal{M} \otimes \mathcal{N}.$$

Remark 1.4.7. We also have a version of the above corollary for the corresponding duality as $\mathrm{D}(X^I)$-module categories. We omit it because the notation is too heavy.

Remark 1.4.8. In the constructible contexts, (1) remains correct. However, the functor

$$\mathrm{Shv}_c(\mathrm{Gr}_{G,I})^{(\mathcal{L}U\mathcal{L}^+M)_I} \underset{\mathrm{Shv}_c(X^I)}{\otimes} \mathrm{Shv}_c(\mathrm{Gr}_{G,I})^{(\mathcal{L}U^-\mathcal{L}^+M)_I} \to$$
$$\to (\mathrm{Shv}_c(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I \times_{X^I} \mathcal{L}U_I^-})^{\mathcal{L}^+M_I \times_{X^I} \mathcal{L}^+M_I}$$

is *not* an equivalence. To make (2) correct, one needs to replace the equivalence in (2) by the right adjoint of the above functor.

As before, Corollary 1.4.3 and 1.4.5 formally imply

Corollary 1.4.9. *The inverse functors in Corollary 1.3.13 are compatible with the $\mathcal{L}^+M_I$-linear structures on those functors.*

1.5. **Local-global compatibility.** Consider the algebraic stack $Y :=\mathrm{VinBun}_G^\gamma$ over $\mathbb{A}^1$. In [Sch16], Schieder studied the corresponding unipotent nearby cycles sheaf of the dualizing sheaf, which we denote by $\Psi^{\mathrm{un}}_{\gamma,\mathrm{glob}}$.

Consider the local-to-global map $\pi_I : \mathrm{VinGr}^\gamma_{G,I} \to \mathrm{VinBun}^\gamma_G$. It induces a morphism

$$\Psi^{\mathrm{un}}_{\gamma,I,\mathrm{Vin}} \to (\pi_I)^!_0(\Psi^{\mathrm{un}}_{\gamma,\mathrm{glob}}), \tag{1.13}$$

where $(\pi_I)_0$ is the 0-fiber of π_I. The following theorem is proved in § 4.3.

Theorem 1.5.1. *The morphism (1.13) is an isomorphism.*

2. Preparations

We need some preparations before proving Theorem 1.3.7 and Theorem 1.5.1.

In § 2.1, we review the definition of nearby cycles.

In § 2.2, we review a theorem of T. Braden, which is our main tool in the proof of the main theorems.

In § 2.3, we study the structure of the categorical players $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$.

In § 2.4, we show $\Psi_{\gamma,I}$ has the desired equivariant structures.

In § 2.5, we define a certain $\mathbb{G}_m$-action on $\mathrm{VinGr}^\gamma_{G,I}$ and study its attractor, repeller and fixed loci.

Convension 2.0.1. *We need a theory of D-modules on general prestacks. As explained in [Ras15b], there are two different theories* $\mathrm{D}^!$ *and* D^*, *where the natural functorialities are given respectively by* $!$*-pullback and* $*$*-pushforward functors. A quick review of [Ras15b] is provided in [Fulltext, § A.4]. In the main body of this paper, unless otherwise stated, we only use the theory* $\mathrm{D}^!$. *Hence we omit the superscript "!" from the notation* $\mathrm{D}^!$.

Also, in the main body of this paper, when discusssing $*$*-pushforward of D-modules, we always restrict to one of the following two cases:*

- *we work with lft prestacks and only use the* $*$*-pushforward functors for ind-finite type ind-schematic maps;*
- *we work with all prestacks and only use the* $*$*-pushforward functors for schematic and finitely presented maps.*

We have base-change isomorphisms between $!$*-pullback and* $*$*-pushforward functors in both cases. The reader can easily distinguish these two cases by looking at the fonts we are using (see Convension 0.6.3).*

Remark 2.0.2. It is well-known that the category of D-modules on finite type schemes are insensitive to non-reduced structures, i.e., for a nil-isomorphism $f : Y_1 \to Y_2$ both $f^!$ and f_* are equivalences. More or less by construction, the theories $\mathrm{D}^!$ and D^* are also insensitive to nil-isomorphisms between prestacks. We will use this fact repeatedly in this paper without mentioning it.

2.1. **Unipotent nearby cycles functor.** Let $f : \mathcal{Z} \to \mathbb{A}^1$ be an $\mathbb{A}^1$-family of prestacks. In this subsection, we review *a* definition of the unipotent nearby cycles functor for the family f. This definition is equivalent to Beilinson's well-known construction (see [Bei87]) when $\mathcal{Z}$ is a finite type scheme.

Construction 2.1.1. *Let* $p : S \to \mathrm{pt}$ *be any finite type scheme. Recall the cohomology complex of* S

$$C^\bullet(S) := p_* \circ p^*(k).$$

The adjoint pair (p^*, p_*) *defines a monad structure on* $p_* \circ p^*$. *Hence* $C^\bullet(S)$ *can be upgraded to an associative algebra in* Vect.

The algebra $C^\bullet(S)$ *acts naturally on the constant D-module* $k_S := p^*(k)$. *The action morphism is given by*

$$C^\bullet(S) \otimes k_S \simeq p^* \circ p_* \circ p^*(k) \to p^*(k) \simeq k_S,$$

where the second morphism is given by the adjoint pair (p^*, p_*).

Construction 2.1.2. *Consider the case* $S = \mathbb{G}_m$. *The map* $1 : \mathrm{pt} \hookrightarrow \mathbb{G}_m$ *defines an augmentation of* $C^\bullet(\mathbb{G}_m)$:

$$p_* \circ p^*(k) \to p_* \circ 1_* \circ 1^* \circ p^*(k) \simeq (p \circ 1)_* \circ (p \circ 1)^*(k) \simeq k.$$

Construction 2.1.3. *Let* $f : \mathcal{Z} \to \mathbb{G}_m$ *be a prestack over* $\mathbb{G}_m$. *For any* $\mathcal{F} \in \mathrm{D}(\mathcal{Z})$, *we have*

$$\mathcal{F} \simeq f^!(k_{\mathbb{G}_m}) \otimes^! \mathcal{F}[2].$$

Hence Construction 2.1.1 provides a natural $C^\bullet(\mathbb{G}_m)$-action on $\mathcal{F}$

The above action is compatible with !-pullback functors along maps defined over $\mathbb{G}_m$. By the base-change isomorphisms, it is also compatbile with $*$-pushforward functors whenever the latter are defined.

Notation 2.1.4. *Let $\mathcal{Z}$ be any prestack over $\mathbb{A}^1$. We write $\mathrm{D}(\overset{\circ}{\mathcal{Z}})^{\mathrm{good}}$ for the full subcategory of $\mathrm{D}(\overset{\circ}{\mathcal{Z}})$ consisting of objects $\mathcal{F}$ such that the partially defined left adjoint $j_!$ of $j^!$ is defined on $\mathcal{F}$. This condition is equivalent to $i^* \circ j_*(\mathcal{F})$ being defined on $\mathcal{F}$.*

Definition 2.1.5. *Let $f : \mathcal{Z} \to \mathbb{G}_m$ be a prestack over $\mathbb{G}_m$. We define the* unipotent nearby cycles sheaf *of $\mathcal{F} \in \mathrm{D}(\overset{\circ}{\mathcal{Z}})^{\mathrm{good}}$ to be*

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) := k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} i^! \circ j_!(\mathcal{F}), \tag{2.1}$$

where $C^\bullet(\mathbb{G}_m)$ acts on the RHS vis $\mathcal{F}$, and the augmentation $C^\bullet(\mathbb{G}_m)$-module is defined in Construction 2.1.2.

Fact 2.1.6. *By the base-change isomorphisms, Ψ_f^{un} commutes with $*$-pushforward functors along schematic proper maps (resp. !-pullback functors along schematic smooth maps).*

Remark 2.1.7. By the excision triangle, we also have:

$$\Psi_f^{\mathrm{un}}(\mathcal{F}) \simeq k \underset{C^\bullet(\mathbb{G}_m)}{\otimes} i^* \circ j_*(\mathcal{F})[-1]. \tag{2.2}$$

Remark 2.1.8. When $\mathcal{Z}$ is a finite type scheme and $\mathcal{F}$ is regular ind-holonomic, by [Cam18, Proposition 3.1.2(1)][18], the above definition coincides with the well-known definition in [Bei87]

Construction 2.1.9. *A direct calculation provides an isomorphism between augmented DG-algebras*

$$\mathrm{Maps}_{C^\bullet(\mathbb{G}_m)\text{-}\mathrm{mod}^r}(k,k) \simeq k[[t]],$$

where the RHS is contained in $\mathrm{Vect}^\heartsuit$. Hence $\Psi_f^{\mathrm{un}}(\mathcal{F})$ is equipped with an action of $k[[t]]$. The action of $t \in k[[t]]$ on $\Psi_f^{\mathrm{un}}(\mathcal{F})$ is the monodromy endomorphism *in the literature.*

By the Koszul duality, we have

$$i^* \circ j_*(\mathcal{F})[-1] \simeq i^! \circ j_!(\mathcal{F}) \simeq k \underset{k[[t]]}{\otimes} \Psi_f^{\mathrm{un}}(\mathcal{F}). \tag{2.3}$$

[18]Although [Cam18] stated the result below with the assumption that there is a $\mathbb{G}_m$-action on $\mathcal{Z}$, it was only used in the proof of [Cam18, Proposition 3.1.2(2)].

2.1.10. *Full nearby cycles functor.* Suppose Z is an indscheme of ind-finite type. Consider the category $\mathrm{D}_{\mathrm{rh}}(\overset{\circ}{Z})$ of regular ind-holonomic D-modules on $\overset{\circ}{Z}$. It is well-known that

$$\mathrm{D}_{\mathrm{rh}}(\overset{\circ}{Z}) \subset \mathrm{D}(\overset{\circ}{Z})^{\mathrm{good}}.$$

Hence the unipotent nearby cycles functor is always defined for regular ind-holonomic D-modules on $\overset{\circ}{Z}$.

On the other hand, there is also a *full nearby cycles functor*

$$\Psi_f : \mathrm{D}_{\mathrm{rh}}(\overset{\circ}{Z}) \to \mathrm{D}(Z_0).$$

Ψ_f satisfies the same standard properties as the unipotent one. Moreover, there is a Künneth formula for the *full* nearby cycles functors (e.g. see [BB93, Lemma 5.1.1] and the remark below it), which is not shared by the unipotent ones.

We have a canonical map $\Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f(\mathcal{F})$ for any regular ind-holonomic $\mathcal{F}$.

The following lemma is a folklore result (e.g. see [AB09, Claim 2])[19]:

Lemma 2.1.11. *Suppose that Z is equipped with a $\mathbb{G}_m$-action such that it can be written as a filtered colimit of closed subschemes stabilized by $\mathbb{G}_m$, and suppose the map $f : Z \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant. Let $\mathcal{F}$ be an regular ind-holonomic regular D-module on $\overset{\circ}{Z}$ such that both $\mathcal{F}$ and $\Psi_f(\mathcal{F})$ are unipotently $\mathbb{G}_m$-monodromic[20]. Then the obvious map $\Psi_f^{\mathrm{un}}(\mathcal{F}) \to \Psi_f(\mathcal{F})$ is an isomorphism.*

2.2. **Braden's theorem and the contraction principle.** In this subsection, we review Braden's theorem and the contraction principle. We first make the following observation

Remark 2.2.1. Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action. Then Z can be written as a filtered colimit $Z \simeq \mathrm{colim}_\alpha Z_\alpha$ with each Z_α being a finite type closed subscheme stabilized by $\mathbb{G}_m$. Indeed, for any presentation $Z \simeq \mathrm{colim}_\alpha Z'_\alpha$ of Z, we can define Z_α as the closure of the image of the map $\mathbb{G}_m \times Z'_\alpha \to Z$.

[19] An erroneous version of the lemma, which did not require $\Psi_f(\mathcal{F})$ to be unipotently $\mathbb{G}_m$-monodromic, appeared in an earlier version of [Gai01]. (A counterexample: for a non-trivial Kummer local system χ on $\mathbb{G}_m$, the sheaf $\chi^{-1} \boxtimes \chi$ on $\mathbb{G}_m \times \mathbb{G}_m$ is unipotently monodromic for the diagonal action, however, for the projection $\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$, the full nearby cycles and unipotent nearby cycles functors are different for $\chi^{-1} \boxtimes \chi$.) This wrong claim was cited by [Sch16, Lemma 8.0.4], which was then used in the proof of the factorization property of the global nearby cycles. We will *not* use this result from [Sch16]. Instead, our Corollary 3.4.4 and Theorem 1.5.1 implies it.

[20] See Definition 2.2.8 below.

Remark 2.2.2. Let $\mathbb{G}_m \curvearrowright Z$ be an action as above. Using [DG14, Lemma 1.4.9(ii), Corollary 1.5.3(ii)][21], we have $Z^{\mathrm{att}} \simeq \mathrm{colim}_\alpha Z^{\mathrm{att}}_\alpha$, and it exhibits Z^{att} as an ind-finite type indscheme. Using [DG14, Proposition 1.3.4], we also have similar result for Z^{fix}.

Definition 2.2.3. *A* retraction *consists of two lft prestacks* (Y, Y^0) *together with morphisms* $i : Y^0 \to Y$*,* $q : Y \to Y^0$ *and an isomorphism* $q \circ i \simeq \mathrm{Id}_{Y^0}$*. We abuse notation by calling* (Y, Y^0) *a retraction and treat the other data as implicit.*

Construction 2.2.4. *Let* Z *be an ind-finite type indscheme equipped with a* $\mathbb{G}_m$*-action. There are retractions* $(Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ *and* $(Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$*.*

Construction 2.2.5. *Let* (Y, Y^0) *be a retraction. We have natural transformations*

$$q_* \to q_* \circ i_* \circ i^* = (q \circ i)_* \circ i^* = i^*, \tag{2.4}$$

$$i^! \to i^! \circ q^! \circ q_! = (q \circ i)^! \circ q_! = q_!. \tag{2.5}$$

between functors $\mathrm{D}(Y) \to \mathrm{Pro}(\mathrm{D}(Y^0))$ *(see e.g. [DG14, Appendix A] for the definition of pro-categories). We refer them as the* contraction natural transformations*.*

Remark 2.2.6. In order to construct (2.4), we need to assume the $*$-pushforward functors are well-defined. See Convension 2.0.1.

Definition 2.2.7. *We say a retraction* (Y, Y^0) *is* $*$-nice *(resp.* $!$*-nice) for an object* $\mathcal{F} \in \mathrm{D}(Z)$ *if the values of (2.4) (resp. (2.5)) on* $\mathcal{F}$ *are isomorphisms.*

Definition 2.2.8. *Let* Z *first be a finite type scheme acted on by* $\mathbb{G}_m$*. The category*

$$\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \subset \mathrm{D}(Z)$$

of unipotently $\mathbb{G}_m$-monodromic D-modules[22] *on* Z *is defined as the full DG-subcategory of* $\mathrm{D}(Z)$ *generated under colimits by the image of the* $!$*-pullback functor* $\mathrm{D}(Z/\mathbb{G}_m) \to \mathrm{D}(Z)$*.*

Let Z *be an ind-finite type indscheme equipped with a* $\mathbb{G}_m$*-action. We define*

$$\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} := \lim_{!\text{-pullback}} \mathrm{D}(Z_\alpha)^{\mathbb{G}_m\text{-um}}.$$

Remark 2.2.9. It is clear that the $!$-pullback functor $\mathrm{D}(Z_\beta) \to \mathrm{D}(Z_\alpha)$ sends unipotently $\mathbb{G}_m$-monodromic objects to unipotently $\mathbb{G}_m$-monodromic ones. Hence the above limit is well-defined. Also, a standard argument shows that it does not depend on the choice of writing Z as $\mathrm{colim}_\alpha Z_\alpha$.

[21] There is a typo in the statement of [DG14, Lemma 1.4.9]: it should be "$Y \subset Z$ be a $\mathbb{G}_m$-stable *subspace*" rather than "... open subspace".

[22] [DG14] referred to them as just $\mathbb{G}_m$-monodromic D-modules. We keep the adverb *unipotently* because we need to consider other monodromies when discussing nearby cycles.

By passing to left adjoints, we also have

$$\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \simeq \underset{*\text{-pushforward}}{\operatorname{colim}} \mathrm{D}(Z_\alpha)^{\mathbb{G}_m\text{-um}}. \tag{2.6}$$

Here we use the general paradigm that a limit diagram connected by right adjoints induces a colimit diagram connected by left adjoints (see e.g. [GR17, Chapter 1, § 2.5]).

Theorem 2.2.10. *(Contraction principle) Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action. The retractions $(Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ and $(Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$ are both !-nice and $*$-nice for any object in $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}}$.*

Remark 2.2.11. When Z is a finite type scheme, the contraction principle is proved in [DG15, Theorem C.5.3]. The case of ind-finite type indschemes can be formally deduced because of (2.6).

In order to state Braden's theorem, we need more definitions.

Definition 2.2.12. *A commutative square of lft prestacks*

$$\begin{tikzcd} V' \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ V \arrow[r, "g"] & W \end{tikzcd} \tag{2.7}$$

is quasi-Cartesian *if the map $j : V' \to W' \times_W V$ induces an open embedding on reduced prestacks.*

Construction 2.2.13. *For a quasi-Cartesian square as in Definition 2.2.12, we extend it to a commutative diagram*

$$\begin{tikzcd} V' \arrow[dr, "j"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "q"'] & & \\ & W' \times_W V \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] & W' \arrow[d, "r"] \\ & V \arrow[r, "g"] & W. \end{tikzcd}$$

Consider the category of D-modules on these prestacks. We have the following base-change transformation

$$g^! \circ r_* \simeq \mathrm{pr}_{2,*} \circ \mathrm{pr}_1^! \to \mathrm{pr}_{2,*} \circ j_* \circ j^! \circ \mathrm{pr}_1^! \simeq q_* \circ (g')^!. \tag{2.8}$$

Using the adjoint pairs

$$q^* : \mathrm{Pro}(\mathrm{D}(V)) \rightleftharpoons \mathrm{Pro}(\mathrm{D}(V')) : q_*,$$
$$r^* : \mathrm{Pro}(\mathrm{D}(W)) \rightleftharpoons \mathrm{Pro}(\mathrm{D}(W')) : r_*,$$

we obtain a natural transformation

$$q^* \circ g^! \to (g')^! \circ r^*. \tag{2.9}$$

Definition 2.2.14. *A quasi-Cartesian square (2.7) is* nice *for an object $\mathcal{F} \in \mathrm{D}(W)$ if the value of (2.9) on $\mathcal{F}$ is an isomorphism in $\mathrm{D}(V')$.*

Warning 2.2.15. *One can obtain another quasi-Cartesian square from (2.7) by exchanging the positions of V and W'. However, the above definition is not preserved by this symmetry.*

Construction 2.2.16. *Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action. By [DG14, Proposition 1.9.4], there are quasi-Cartesian diagrams*

$$\begin{tikzcd} Z^{\mathrm{fix}} \arrow[r, "i^+"] \arrow[d, "i^-"] & Z^{\mathrm{att}} \arrow[d, "p^+"] \\ Z^{\mathrm{rep}} \arrow[r, "p^-"] & Z, \end{tikzcd} \qquad \begin{tikzcd} Z^{\mathrm{fix}} \arrow[r, "i^-"] \arrow[d, "i^+"] & Z^{\mathrm{rep}} \arrow[d, "p^-"] \\ Z^{\mathrm{att}} \arrow[r, "p^+"] & Z \end{tikzcd}$$

Theorem 2.2.17. *(Braden) Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action. The above two quasi-Cartesian diagrams are nice for any object in $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}}$.*

Remark 2.2.18. When Z is a finite type scheme, Braden's theorem was proved in [Bra03] for perverse sheaves and in [DG14] for all D-modules. The case of ind-finite type indschemes can be formally deduced because of (2.6).

Remark 2.2.19. Using the contraction principle, Braden's theorem can be reformulated as the existence of a canonical adjoint pair[23]

$$q^{\pm}_* \circ p^{\pm,!} : \mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \rightleftharpoons \mathrm{D}(Z^{\mathrm{fix}}) : p^{\mp}_* \circ q^{\mp,!}.$$

In fact, this is how [DG14] proved Braden's theorem.

For the purpose of this paper, we also introduce the following definition:

Definition 2.2.20. *A* Braden 4-tuple *consists of four prestacks (Z, Z^+, Z^-, Z^0) together with*

- *a quasi-Cartesian square (see Definition 2.2.12):*

$$\begin{tikzcd} Z^0 \arrow[r, "i^+"] \arrow[d, "i^-"] & Z^+ \arrow[d, "p^+"] \\ Z^- \arrow[r, "p^-"] & Z. \end{tikzcd}$$

- *morphisms $q^+ : Z^+ \to Z^0$ and $q^- : Z^- \to Z^0$ and isomorphisms $q^+ \circ i^+ \simeq \mathrm{Id}_{Z^0} \simeq q^- \circ i^-$.*

We abuse notation by calling (Z, Z^+, Z^-, Z^0) a Braden 4-tuple and treat the other data as implicit.

Given a Braden 4-tuple (Z, Z^+, Z^-, Z^0), we define its opposite Braden 4-tuple *to be (Z, Z^-, Z^+, Z^0).*

Construction 2.2.21. *Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action. We have a Braden 4-tuple $(Z, Z^{\mathrm{att}}, Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$.*

[23] Note that the image of the functor $p^-_* \circ q^{-,!} : \mathrm{D}(Z^{\mathrm{fix}}) \to \mathrm{D}(Z)$ is contained in $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}}$.

Example 2.2.22. The *inverse* of the dilation $\mathbb{G}_m$-action on $\mathbb{A}^1$ induces the Braden 4-tuple

$$\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0).$$

Example 2.2.23. By Example 1.2.14, we obtain a Braden 4-tuple $(\mathrm{Gr}_{G,I}, \mathrm{Gr}_{P,I}, \mathrm{Gr}_{P^-,I}, \mathrm{Gr}_{M,I})$.

Remark 2.2.24. See § 4.1 for a Braden 4-tuple that is not obtained from Construction 2.2.21.

Definition 2.2.25. *For a Braden 4-tuple as in Definition 2.2.20, we say it is* $*$-nice *for an object* $\mathcal{F} \in \mathrm{D}(Z)$ *if*

(i) *The corresponding quasi-Cartesian square is nice for* $\mathcal{F}$*;*
(ii) *The retraction* (Z^-, Z^0) *is* $*$*-nice for* $p^{-,!} \circ \mathcal{F}$*.*

Remark 2.2.26. We do not need the notion of !-niceness in this paper.

Then Braden's theorem and the contraction principle imply

Theorem 2.2.27. *Let* Z *be an ind-finite type indscheme equipped with a* $\mathbb{G}_m$*-action. Then* $(Z, Z^{\mathrm{att}}, Z^{\mathrm{rep}}, Z^{\mathrm{fix}})$ *and* $(Z, Z^{\mathrm{rep}}, Z^{\mathrm{att}}, Z^{\mathrm{fix}})$ *are* $*$*-nice for any objects in* $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}}$.

2.3. **Categorical players.** The goal of this subsection is to descibe the compact generators of $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$. The proofs are provided in Appendix D.

2.3.1. *Strata.* It is well-known (see [Fulltext, § C.3]) that the map $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is bijective on field-valued points, and the connected components of $\mathrm{Gr}_{P,I}$ induce a stratification on $\mathrm{Gr}_{G,I}$ labelled by $\Lambda_{G,P}$. For $\lambda \in \Lambda_{G,P}$, the corresponding stratum is denoted by

$$_\lambda\,\mathrm{Gr}_{G,I} := (\mathrm{Gr}_{P,I}^\lambda)_{\mathrm{red}},$$

where $\mathrm{Gr}_{P,I}^\lambda$ is the connected component of $\mathrm{Gr}_{P,I}$ corresponding to λ (see *loc.cit.* for our convention for it). By *loc.cit.*, the map $_\lambda\,\mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I}$ is a schematic locally closed embedding.

Consider the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{P,I}$. Note that $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is $\mathcal{L}P_I$-equivariant. Therefore the functors $\mathbf{p}_I^{+,!}$ and $\mathbf{p}_{I,*}^+$ can be upgraded to morphisms in $\mathcal{L}P_I$-mod. Therefore they induce $\mathcal{L}M_I$-linear functors:

$$\mathbf{p}_{I,*}^{+,\mathrm{inv}} : \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \tag{2.10}$$

$$\mathbf{p}_I^{+,!,\mathrm{inv}} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}. \tag{2.11}$$

On the other hand, consider the $\mathcal{L}M_I$-equivariant map $\mathbf{q}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{M,I}$. Note that the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{P,I}$ preserves the fibers of $\mathbf{q}_I^+$. Hence there are $\mathcal{L}M_I$-functors

$$\mathbf{q}_I^{+,!,\mathrm{inv}} : \mathrm{D}(\mathrm{Gr}_{M,I}) \to \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}, \tag{2.12}$$

$$\mathbf{q}_{I,*,\mathrm{co}}^+ : \mathrm{D}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{M,I}) \tag{2.13}$$

(see [Fulltext, Formula (B.11)]). Sometimes we omit the superscripts "inv" from these notations if there is no danger of ambiguity.

Lemma 2.3.2. *Let* $\mathbf{i}_I^+ : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{P,I}$ *be the map induced by* $M \hookrightarrow P$. *We have*

(1) (c.f. [Gai17a, Proposition 1.4.2]) The functor (2.12) is an equivalence, with an inverse given by

$$\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{P,I}) \xrightarrow{\mathbf{i}_I^{+,!}} \mathrm{D}(\mathrm{Gr}_{M,I}).$$

(2) The functor (2.13) is an equivalence, with an inverse given by

$$\mathrm{D}(\mathrm{Gr}_{M,I}) \xrightarrow{\mathbf{i}_{I,*}^+} \mathrm{D}(\mathrm{Gr}_{P,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I}.$$

Proof. Follows formally (see [Fulltext, Lemma B.4.1]) from the fact that $\mathcal{L}U_I$ acts transitively along the fibers of $\mathbf{q}_I^+$.

□[Lemma 2.3.2]

Lemma 2.3.3. *Let* $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_{G,I})$. *Suppose* $\mathbf{p}_I^{+,!}(\mathcal{F}) \in \mathrm{D}(\mathrm{Gr}_{P,I})$ *is contained in* $\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}$, *then* $\mathcal{F}$ *is contained in* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$.

Proof. It follows formally that (see [Fulltext, Formula B.9]), we can replace $\mathcal{L}U_I$ by one of its pro-smooth group subscheme U_α. It remains to prove that $\mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}_*^{U_\alpha}(\mathcal{F}) \to \mathcal{F}$ is an isomorphism. Since $\mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is bijective on field-valued points, $\mathbf{p}_I^{+,!}$ is conservative. Hence it remains to prove

$$\mathbf{p}_I^{+,!} \circ \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}_*^{U_\alpha}(\mathcal{F}) \to \mathbf{p}_I^!(\mathcal{F})$$

is an isomorphism. By [Ras16, Corollary 2.17.10], we have

$$\mathbf{p}_I^{+,!} \circ \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}_*^{U_\alpha} \simeq \mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}_*^{U_\alpha} \circ \mathbf{p}_I^{+,!}.$$

On the other hand, the assumption on $\mathbf{p}_I^{+,!}(\mathcal{F})$ implies

$$\mathbf{oblv}^{U_\alpha} \circ \mathbf{Av}_*^{U_\alpha} \circ \mathbf{p}_I^{+,!}(\mathcal{F}) \simeq \mathbf{p}_I^{+,!}(\mathcal{F}).$$

This proves the desired isomorphism.

□[Lemma 2.3.3]

The following two lemmas are proved in Appendix D.

Lemma 2.3.4. *(c.f. [Gai17a, Proposition 1.5.3, Corollary 1.5.6])*

(1) Consider the $\mathbb{G}_m$*-action on* $\mathrm{Gr}_{G,I}$ *in Example 1.2.14. We have*

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \subset \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}} \subset \mathrm{D}(\mathrm{Gr}_{G,I}).$$

(2) Let $\mathbf{s}_I : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{G,I}$ *be the map induced by* $M \hookrightarrow G$. *Then the composition*

$$\mathrm{D}(\mathrm{Gr}_{M,I}) \xrightarrow{\mathbf{s}_{I,*}} \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{Av}_!^{\mathcal{L}U_I}} \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})$$

factors through $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, *where* $\mathbf{Av}_!^{\mathcal{L}U_I}$ *is the left adjoint of the forgetful functor. Moreover, the image of this functor generates* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *under colimits and shifts. Consequently,* $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ *is compactly generated.*

(3) The functor (2.10) has a left adjoint[24]

$$\mathbf{p}_I^{+,*,\mathrm{inv}} : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I},$$

which can be canonically identified with

$$\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{p}_I^{-,!}} \mathrm{D}(\mathrm{Gr}_{P^-,I}) \xrightarrow{\mathbf{q}_{I,*}^-} \mathrm{D}(\mathrm{Gr}_{M,I}) \simeq \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}.$$

In particular, $\mathbf{p}_I^{+,*,\mathrm{inv}}$ is $\mathcal{L}M_I$-linear.

(4) The functor (2.11) has a $\mathrm{D}(X^I)$-linear[25] *left adjoint*

$$\mathbf{p}_{I,!}^{+,\mathrm{inv}} : \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}.$$

Lemma 2.3.5. *(1) The functor*

$$\mathrm{D}(\mathrm{Gr}_{M,I}) \xrightarrow{\mathbf{s}_{I,*}} \mathrm{D}(\mathrm{Gr}_{G,I}) \xrightarrow{\mathbf{pr}_{\mathcal{L}U_I}} \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$$

sends compact objects to compact objects. Moreover, its image generates $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$. Consequently, $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is compactly generated.

(2) $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is dualizable in DGCat, and its dual is canonically identified with $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. Moreover, this identification is compatible with the $\mathcal{L}M_I$-actions on them.

The following technical result follows formally from Lemma 2.3.5(2) (see [Fulltext, Lemma B.1.12 and Lemma A.3.4]).

Corollary 2.3.6. *Let $\mathcal{H}_1, \mathcal{H}_2 \in \{X^I, \mathcal{L}U_I, \mathcal{L}U_I^-\}$ be group indschemes over X^I.*

(1) We have a commutative diagram

$$\begin{tikzcd} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{H}_1} \otimes \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{H}_2} \arrow[r] \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1} \otimes \mathbf{oblv}^{\mathcal{H}_2}"] & \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I})^{\mathcal{H}_1 \times \mathcal{H}_2} \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1 \times \mathcal{H}_2}"] \\ \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathrm{D}(\mathrm{Gr}_{G,I}) \arrow[r, "\boxtimes"] & \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I}), \end{tikzcd}$$

where all the four functors are fully faithful, and the horizontal functors are equivalences.

(2) We have a commutative diagram

$$\begin{tikzcd} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{H}_1} \otimes_{\mathrm{D}(X^I)} \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{H}_2} \arrow[r] \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1} \otimes \mathbf{oblv}^{\mathcal{H}_2}"] & \mathrm{D}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})^{\mathcal{H}_1 \times_{X^I} \mathcal{H}_2} \arrow[d, "\mathbf{oblv}^{\mathcal{H}_1 \times_{X^I} \mathcal{H}_2}"] \\ \mathrm{D}(\mathrm{Gr}_{G,I}) \otimes_{\mathrm{D}(X^I)} \mathrm{D}(\mathrm{Gr}_{G,I}) \arrow[r, "\boxtimes_{X^I}"] & \mathrm{D}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}). \end{tikzcd}$$

where all the four functors are fully faithful, and the horizontal functors are equivalences.

Remark 2.3.7. Corollary 2.3.6 is also (obviously) correct if we replace

[24] We do not know whether the following stronger claim is true: the functor $\mathbf{p}_I^{+,*}$ is well-defined on $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \subset \mathrm{D}(\mathrm{Gr}_{G,I})$.

[25] One can actually prove it is $\mathcal{L}M_I$-linear. Also, one can prove any (right or left) lax $\mathrm{D}(X^I)$-linear functor is strict.

- the invariants categories by the coinvariants categories;
- the forgetful functors **oblv** by the localization functors **pr**.

Remark 2.3.8. In the constructible contexts, we still have the commutative diagram in (1). However, the horizontal functors are no longer equivalences. Nevertheless, one can prove that the commutative diagram is right adjointable along the horizontal direction.

2.4. **Equivariant structure.** In this subsection, we prove that $\Psi_{\gamma,I}$ has our desired equivariant structures and deduce Proposition 1.3.4 from it.

Consider the $\mathcal{L}(G\times G)_I$-action on $\mathrm{Gr}_{G\times G,I}$. Recall we have an object

$$\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}\in\mathcal{L}(M\times M)_I\text{-mod}.$$

By restriction along the diagonal embedding $\mathcal{L}M_I\hookrightarrow\mathcal{L}(M\times M)_I$, we view $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$ as an object in $\mathcal{L}M_I$-mod. We have:

Proposition 2.4.1. *(1) The map $\Psi^{\mathrm{un}}_{\gamma,I}\to\Psi_{\gamma,I}$ is an isomorphism.*

(2) The object $\Psi^{\mathrm{un}}_{\gamma,I}\simeq\Psi_{\gamma,I}$ is contained in the full subcategory $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$. Moreover, it can be canonically upgraded to an object in $(\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I})^{\mathcal{L}^+M_I,\mathrm{diag}}$.

Remark 2.4.2. Note that (1) implies Proposition 1.3.4 because taking (unipotent) nearby cycles commutes with proper push-forward functors.

Remark 2.4.3. It is quite possible that one can actually upgrade $\Psi_{\gamma,I}$ to an object in $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(P\times_M P^-)}$. However, because $\mathcal{L}M_I$ is not an ind-group scheme, our current techniques cannot prove it.

Proof. The rest of this subsection is devoted to the proof of the proposition. As one would expect, we have Cartesian squares (see [Fulltext, Lemma B.5.2 and Lemma B.5.1]):

$$\begin{tikzcd}
(\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I})^{\mathcal{L}^+M_I,\mathrm{diag}} \arrow[r] \arrow[d] & \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}^+M_I,\mathrm{diag}} \arrow[d] \\
\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \arrow[r] & \mathrm{D}(\mathrm{Gr}_{G\times G,I}),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \arrow[r] \arrow[d] & \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1} \arrow[d] \\
\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I^-,2} \arrow[r] & \mathrm{D}(\mathrm{Gr}_{G\times G,I}),
\end{tikzcd}$$

where the superscripts 1 (resp. 2) indicate that $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$) acts on $\mathrm{Gr}_{G\times G,I}\simeq\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}$ via the first (resp. second) factor.

Hence we can prove the proposition in three steps:

(i) The objects $\Psi_{\gamma,I}$ and $\Psi^{\mathrm{un}}_{\gamma,I}$ are contained in $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$ and $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I^-,2}$.

(ii) The morphism $\Psi^{\mathrm{un}}_{\gamma,I}\to\Psi_{\gamma,I}$ is an isomorphism.

(iii) The object $\Psi_{\gamma,I}$ can be canonically upgraded to an object in $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}^+M_I,\mathrm{diag}}$.

2.4.4. *Proof of (i).* Recall the co-character γ provides a $\mathbb{G}_m$-action on G (see Example 1.2.13). Note that $U \hookrightarrow G$ is stabilized by this action. By construction, this action is compatible with the group structure on U. In particular, the corresponding Drinfeld-Gaitsgory interpolation $\widetilde{U}^\gamma$ is a group scheme over $\mathbb{A}^1$ and the map $\widetilde{U}^\gamma \to U \times U \times \mathbb{A}^1$ is a group homomorphism (relative to $\mathbb{A}^1$).

Note that the above $\mathbb{G}_m$-action on U is contractive, i.e., its attractor locus is isomorphic to itself. Hence by [DG14, Proposition 1.4.5], the $\mathbb{G}_m$-action on U can be extended to an $\mathbb{A}^1$-action on U, where $\mathbb{A}^1$ is equipped with the multiplication monoid structure. Note that the fixed locus of the $\mathbb{G}_m$-action on U is $1 \hookrightarrow U$. Hence by [DG14, Proposition 2.4.4], the map $\widetilde{U}^\gamma \to U \times U \times \mathbb{A}^1$ can be identified with

$$U \times \mathbb{A}^1 \to U \times U \times \mathbb{A}^1,\ (g,t) \mapsto (g, t\cdot g, t). \tag{2.14}$$

In particular, its 1-fiber is the diagonal embedding, while its 0-fiber is the closed embedding onto the *first* U-factor.

By taking loops, we obtain from (2.14) a homomorphism between group indschemes over $X^I \times \mathbb{A}^1$

$$a : \mathcal{L}U_I \times \mathbb{A}^1 \to \mathcal{L}U_I \underset{X^I}{\times} \mathcal{L}U_I \times \mathbb{A}^1$$

such that its 1-fiber is the diagonal embedding, while its 0-fiber is the closed embedding onto the first $\mathcal{L}U_I$-factor. Similarly, we have a morphism between group indschemes over $X^I \times \mathbb{A}^1$:

$$r : \mathcal{L}U_I^- \times \mathbb{A}^1 \to \mathcal{L}U_I^- \underset{X^I}{\times} \mathcal{L}U_I^- \times \mathbb{A}^1$$

whose 1-fiber is the diagonal embedding and 0-fiber is the closed embedding onto the *second* $\mathcal{L}U_I^-$-factor. In fact, the map a (resp. r) is the Drinfeld-Gaitsgory interpolation for the $\mathbb{G}_m$-action on $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$), if we generalize the definitions in [DG14] to arbitrary prestacks.

Via the group homomorphism a and r, we have an action of $\mathcal{L}U_I \times \mathbb{A}^1$ (resp. $\mathcal{L}U_I^- \times \mathbb{A}^1$) on $\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1$ relative to $X^I \times \mathbb{A}^1$. Equivalently, we have an action of $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$) on $\mathrm{Gr}_{G\times G,I}$ relative to X^I. We use symbols "a" (resp. "r") to distinguish these actions from other ones.

Now consider the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I}$ (relative to X^I). By construction, this action is compatible with the $\mathbb{G}_m$-actions on $\mathcal{L}U_I$ (as a group indscheme) and on $\mathrm{Gr}_{G,I}$ (as a plain indscheme). This implies we have the following compatibility

$$(\mathcal{L}U_I \times \mathbb{A}^1 \xrightarrow{a} \mathcal{L}U_I \underset{X^I}{\times} \mathcal{L}U_I \times \mathbb{A}^1) \curvearrowright (\widetilde{\mathrm{Gr}}^\gamma_{G,I} \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1).$$

Hence by Lemma 1.2.27(2), the $(\mathcal{L}U_I, a)$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$ stabilizes the schematic closed embedding

$$\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \hookrightarrow \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{G}_m,\ (x,t) \mapsto (x, t\cdot x, t). \tag{2.15}$$

Note that the restricted $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I} \times \mathbb{G}_m$ is the usual one.

We also have similar results on the $(\mathcal{L}U_I^-, r)$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$. Now (i) is implied by the following stronger result (and its mirror version).

Lemma 2.4.5. *(1) Both the unipotent nearby cycles functor $\Psi^{\mathrm{un}}_{\gamma,I}$ and $i^* \circ j_*$ send the category*

$$\mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)^{\mathcal{L}U_I,a} \bigcap \mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)^{\mathrm{good}}$$

into $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$.

(2) The full nearby cycles functor $\Psi_{\gamma,I}$ sends the category

$$\mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)^{\mathcal{L}U_I,a} \bigcap \mathrm{D}_{\mathrm{rh}}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$$

into $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}U_I,1}$.

Proof. Write $\mathcal{L}U_I$ as a filtered colimit $\mathcal{L}U_I \simeq \mathrm{colim}_\alpha \mathcal{N}_\alpha$ of its closed pro-unipotent group subschemes. We only need to prove the lemma after replacing $\mathcal{L}U_I$ by $\mathcal{N}_\alpha$ for any α. Then (1) follows from [Fulltext, Proposition B.8.1].

To prove (2), we claim we can choose the above presentation $\mathcal{L}U_I \simeq \mathrm{colim}_\alpha \mathcal{N}_\alpha$ such that for each α, we can find a presentation $(\mathrm{Gr}_{G,I})_{\mathrm{red}} \simeq \mathrm{colim}\, Y_\beta$ such that each Y_β is a finite type closed subscheme of $(\mathrm{Gr}_{G,I})_{\mathrm{red}}$ stabilized by $\mathcal{N}_\alpha$. Indeed, similar to [Ras16, Remark 2.19.1], we can make each $\mathcal{N}_\alpha$ conjugate to $\mathcal{L}^+U_I$. Hence we only need to find a presentation $(\mathrm{Gr}_{G,I})_{\mathrm{red}} \simeq \mathrm{colim}\, Y_\beta$ such that each Y_β is stabilized by $\mathcal{L}^+U_I$. Then we can choose Y_β to be the Schubert cells of $(\mathrm{Gr}_{G,I})_{\mathrm{red}}$ (which are even stabilized by $\mathcal{L}^+G_I$). This proves the claim.

For any $\mathcal{N}_\alpha$ as above, since full nearby cycles functors commute with proper pushforward functors, it suffices to prove the claim after replacing $\mathrm{Gr}_{G,I}$ by Y_β (for any β). Then the $\mathcal{N}_\alpha$-action on Y_β factors through a smooth quotient group H. We can replace $\mathcal{N}_\alpha$ by H. Then we are done by using [Fulltext, Formula (B.16)]) and the fact that taking full nearby cycles commutes with smooth pullback functors.

□[Lemma 2.4.5]

2.4.6. *Proof of (ii).* Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1$ given by $s\cdot(x,y,t) = (x, s\cdot y, st)$. Note that the projection $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant. Also note that the schematic closed embedding (2.15) is stabilized by this action. Hence by Lemma 2.1.11, it suffices to prove that the object $\Psi_{\gamma,I} \in \mathrm{D}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})$ is unipotently $\mathbb{G}_m$-monodromic, where $\mathbb{G}_m$ acts on the second factor.

By (i), we have $\Psi_{\gamma,I} \in \mathrm{D}(\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-,2}$. Then we are done because

$$\mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I^-,2} \subset \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um},2}$$

by Lemma 2.3.4(1) (and Corollary 2.3.6(2)). This proves (ii).

2.4.7. *Proof of (iii).* Note that the Drinfeld-Gaitsgory interpolation $\widetilde{M}^\gamma \times \mathbb{A}^1 \to M \times M \times \mathbb{A}^1$ is isomorphic to the diagonal embedding $M \times \mathbb{A}^1 \to M \times M \times \mathbb{A}^1$. By an argument similar to that in § 2.4.4, we see the diagonal action of $\mathcal{L}^+ M_I$ on $\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m$ stabilizes the schematic closed embedding (2.15) and the restricted $\mathcal{L}^+ M_I$-action on $\mathrm{Gr}_{G,I} \times \mathbb{G}_m$ is the usual one.

Now let $\mathcal{C}$ be the full sub-category of $\mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$ generated by $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$ under colimits and shifts. By the previous discussion, $\mathcal{C}$ is a sub-$\mathcal{L}^+ M_I$-module of $\mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$. It follows formally that (see [Fulltext, Proposition B.8.1]), we obtain an $\mathcal{L}^+ M_I$-linear structure on the functor $\Psi^{\mathrm{un}}_{\gamma,I} : \mathcal{C} \to \mathrm{D}(\mathrm{Gr}_{G\times G,I})$. Therefore $\Psi^{\mathrm{un}}_{\gamma,I}$ induces a functor between the $\mathcal{L}^+ M_I$-invariants categories. Then we are done because $\Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$ can be naturally upgraded to an object in $\mathrm{D}(\mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)^{\mathcal{L}^+ M_I,\mathrm{diag}}$.

□[Proposition 2.4.1]

Remark 2.4.8. By Proposition 2.4.1(2), we also have

$$i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \in \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}(U\times U^-)_I}.$$

2.5. **Geometric players - II.** In this subsection, we study a certain $\mathbb{G}_m$-action on $\mathrm{VinGr}^\gamma_{G,I}$, which is used repeatedly in this paper.

Consider the action $T_{\mathrm{ad}} \curvearrowright \mathrm{Gr}_{G,I}$ induced by the adjoint action $T_{\mathrm{ad}} \curvearrowright G$. We have

Proposition 2.5.1. *The action*

$$(T_{\mathrm{ad}} \times T_{\mathrm{ad}}) \times (\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times T^+_{\mathrm{ad}}) \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times T^+_{\mathrm{ad}},$$
$$(s_1, s_2) \cdot (x, y, t) \coloneqq (s_1^{-1} \cdot x, s_2^{-1} \cdot y, s_1 t s_2^{-1}).$$

preserves both $\mathrm{VinGr}_{G,I}$ *and* ${}_0\mathrm{VinGr}_{G,I}$.

Remark 2.5.2. The claim is obvious when restricted to $T_{\mathrm{ad}} \subset T^+_{\mathrm{ad}}$.

2.5.3. *A general paradigm.* Proposition 2.5.1 can be proved using the Tannakian description of VinGr_G in [FKM20, § 3.1.2]. However, we prefer to prove it in an abstract way. The construction below is a refinement of that in [Wan18, Appendix C.3].

Consider the following paradigm. Let $1 \to K \to H \to Q \to 1$ be an exact sequence of affine algebraic groups. Let $Z \to B$ be a map between finite type affine schemes. Suppose we have an H-action on Z and a Q-action on B compatible in the obvious sense. Then we have a Q-equivariant map $p : K\backslash Z \to B$.

Suppose we are further given a section $B \hookrightarrow Z$ to the map $Z \to B$. Then we obtain a map $f : B \to Z \to K\backslash Z$ such that $p \circ f = \mathrm{Id}_B$.

Suppose we are further given a splitting $s: Q \hookrightarrow H$ compatible with the actions $Q \curvearrowright B$, $H \curvearrowright Z$ and the section $B \to Z$. Consider the restricted Q-action on Z. By assumption, the map $B \to Z$ is Q-equivariant. On the other hand, there is a Q-equivariant structure on $Z \to K\backslash Z$ because of the splitting $s: Q \hookrightarrow H$. Hence we obtain a Q-equivariant structure on $f: B \to K\backslash Z$.

Combining the above paragraphs, we obtain a Q-action on the retraction $(K\backslash Z, B, p, f)$. This construction is functorial in $B \hookrightarrow Z \to B$ in the obvious sense.

In the special case when $Z = B$ and K acts trivially on B, we obtain a Q-action on the chain $B \to K\backslash \mathrm{pt} \times B \to B$. More or less by definition, this action is also induced by the given Q-action on B and the adjoint action $Q \curvearrowright K$ provided by the section s.

Applying Construction C.1.3 to these retractions, (using Lemma C.1.5) we obtain Q-actions on $\mathbf{Maps}_{I,/B}(X, K\backslash Z \leftarrow B)$ and $\mathbf{Maps}_{I,/B}(X, K\backslash \mathrm{pt} \times B \leftarrow B)$. Moreover, the map $(B \hookrightarrow Z \to B) \to (B \simeq B \simeq B)$ induces a Q-equivariant map

$$\mathbf{Maps}_{I,/B}(X, K\backslash Z \leftarrow B) \to \mathbf{Maps}_{I,/B}(X, K\backslash \mathrm{pt} \times B \leftarrow B).$$

2.5.4. *Proof of Proposition 2.5.1.* Let us come back to the problem. Recall we have the following exact sequence of algebraic groups $1 \to G \to G_{\mathrm{enh}} \to T_{\mathrm{ad}} \to 1$, where $G_{\mathrm{enh}} := (G \times T)/Z_G$ is the group of invertible elements in Vin_G. Also recall we have a canonical section $\mathfrak{s}: T_{\mathrm{ad}}^+ \to \mathrm{Vin}_G$ whose restriction to T_{ad} is $T/Z_G \to (G \times T)/Z_G$, $t \mapsto (t^{-1}, t)$. Note that the corresponding T_{ad}-action on G provided by $\mathfrak{s}$ is the *inverse* of the usual adjoint action. Now applying the above paradigm to

$$(1 \to K \to H \to Q \to 1) := (1 \to G \times G \to G_{\mathrm{enh}} \times G_{\mathrm{enh}} \to T_{\mathrm{ad}} \times T_{\mathrm{ad}} \to 1)$$

$$(B \to Z \to B) := (T_{\mathrm{ad}}^+ \xrightarrow{\mathfrak{s}} \mathrm{Vin}_G \to T_{\mathrm{ad}}^+)$$

we obtain a $(T_{\mathrm{ad}} \times T_{\mathrm{ad}})$-equivariant structure on the map $\mathrm{VinGr}_{G,I} \to \mathrm{Gr}_{G\times G,I} \times T_{\mathrm{ad}}^+$, where $Q = (T_{\mathrm{ad}} \times T_{\mathrm{ad}})$ acts on the RHS via the usual action on $B = T_{\mathrm{ad}}^+$ and the *inverse* of the usual action on $\mathrm{Gr}_{K,I} = \mathrm{Gr}_{G\times G,I}$. This is exactly the action described in the problem. This proves the claim for $\mathrm{VinGr}_{G,I}$.

Replacing Z by ${}_0\mathrm{Vin}_G$, we obtain the claim for ${}_0\mathrm{VinGr}_{G,I}$.

□[Proposition 2.5.1]

Corollary 2.5.5. *Let $\mathbb{G}_m \curvearrowright \mathrm{Gr}_{G,I}$ be the action in Example 1.2.14. Then the action*

(2.16)

$$\mathbb{G}_m \times (\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1) \to \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,\ s\cdot(x,y,t) := (s\cdot x, s^{-1}\cdot y, s^{-2}t)$$

preserves both $\mathrm{VinGr}_{G,I}^\gamma$ *and* ${}_0\mathrm{VinGr}_{G,I}^\gamma$.

Construction 2.5.6. *Consider the above action $\mathbb{G}_m \curvearrowright (\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I} \times \mathbb{A}^1)$. The Braden 4-tuple for the action (2.16) is*

$$\mathrm{Br}_I^\gamma := (\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1, \mathrm{Gr}_{P\times P^-,I} \times 0, \mathrm{Gr}_{P^-\times P,I} \times \mathbb{A}^1, \mathrm{Gr}_{M\times M,I} \times 0).$$

Hence by [DG14, Lemma 1.4.9(ii)], the attractor (resp. repeller, fixed) locus for the action on $\mathrm{VinGr}^{\gamma}_{G,I}$ *is given by*

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1)}{\times} (\mathrm{Gr}_{P\times P^-,I}\times 0), \tag{2.17}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1)}{\times} (\mathrm{Gr}_{P^-\times P,I}\times \mathbb{A}^1), \tag{2.18}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \simeq \mathrm{VinGr}^{\gamma}_{G,I} \underset{(\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1)}{\times} (\mathrm{Gr}_{M\times M,I}\times 0). \tag{2.19}$$

We denote the corresponding Braden 4*-tuple by*

$$\mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \coloneqq (\mathrm{VinGr}^{\gamma}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I}, \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}).$$

2.5.7. *An alternate description.* The reader is advised to skip the rest of this subsection and return when necessary.

The formulae in Construction 2.5.6 are not satisfactory because for example they do not describe[26] the map $\mathbf{q}^{+}_{\mathrm{Vin},I} : \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \to \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I}$. In this sub-subsection, we use mapping stacks to give an alternative description of the Braden 4-tuple $\mathrm{Br}^{\gamma}_{\mathrm{Vin},I}$. Once we have this alternative description, we exhibit how to use them to study the geometry of $\mathrm{VinGr}_{G,I}$ in the rest of this subsection.

We assume the reader is familiar with the constructions in § C.4.2-C.4.3 and § C.4.6.

By Lemma C.1.13, we can rewrite (2.17)-(2.19) as

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, P\backslash \mathrm{Vin}_G|_{C_P}/P^- \leftarrow \mathrm{pt}), \tag{2.20}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \simeq \mathbf{Maps}_{I,/\mathbb{A}^1}(X, P^-\backslash \mathrm{Vin}^{\gamma}_G/P \leftarrow \mathbb{A}^1), \tag{2.21}$$

$$\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, M\backslash \mathrm{Vin}_G|_{C_P}/M \leftarrow \mathrm{pt}), \tag{2.22}$$

where the sections are all induced by the canonical section $\mathfrak{s}: T^+_{\mathrm{ad}} \to \mathrm{Vin}_G$.

Recall we have a $(P\times P^-)$-equivariant closed embedding $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$ (see § C.4.2). By definition, the canonical section $\mathfrak{s}|_{C_P} : \mathrm{pt} \to \mathrm{Vin}_G|_{C_P}$ factors through this embedding. Hence the map $\mathrm{pt} \to P\backslash \mathrm{Vin}_G|_{C_P}/P^-$ factors as $\mathrm{pt} \to P\backslash\overline{M}/P^- \hookrightarrow P\backslash \mathrm{Vin}_G|_{C_P}/P^-$, where the last map is a schematic closed embedding. By Lemma C.1.8 and (2.20), we obtain an isomorphism:

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, P\backslash\overline{M}/P^- \leftarrow \mathrm{pt}). \tag{2.23}$$

Similarly we have an isomorphism

$$\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, M\backslash\overline{M}/M \leftarrow \mathrm{pt}). \tag{2.24}$$

[26]Of course, the map $\mathbf{q}^{+}_{\mathrm{Vin},I}$ is the unique one that is compatible with the map $\mathrm{Gr}_{P\times P^-,I} \to \mathrm{Gr}_{M\times M,I}$. But this description is not convenient in practice.

Under these descriptions, we claim the commutative diagram

$$
\begin{tikzcd}
 & & \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \\
 & \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{Vin},I}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{Vin},I}"'] & \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \arrow[u, "\mathbf{q}^+_{\mathrm{Vin},I}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{Vin},I}"] \\
\mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} & \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \arrow[l, "\mathbf{q}^-_{\mathrm{Vin},I}"] \arrow[r, "\mathbf{p}^-_{\mathrm{Vin},I}"'] & \mathrm{VinGr}^{\gamma}_{G,I}
\end{tikzcd}
\tag{2.25}
$$

is induced by a commutative diagram

$$
\begin{tikzcd}
 & & (M\backslash\overline{M}/M \leftarrow \mathrm{pt}) \\
 & (M\backslash\overline{M}/M \leftarrow \mathrm{pt}) \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{sect}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{sect}}"'] & (P\backslash\overline{M}/P^- \leftarrow \mathrm{pt}) \arrow[u, "\mathbf{q}^+_{\mathrm{sect}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{sect}}"] \\
(M\backslash\overline{M}/M \leftarrow \mathrm{pt}) & (P^-\backslash\mathrm{Vin}^{\gamma}_G/P \leftarrow \mathbb{A}^1) \arrow[l, "\mathbf{q}^-_{\mathrm{sect}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{sect}}"'] & (G\backslash\mathrm{Vin}^{\gamma}_G/G \leftarrow \mathbb{A}^1),
\end{tikzcd}
\tag{2.26}
$$

where the only non-obvious morphism is $\mathbf{q}^-_{\mathrm{sect}}$, which is induced by the commutative diagram (C.17). Indeed, (2.25) is induced by (2.26) because the maps in (2.25) are uniquely determined by their compatibilities with the maps in the Braden 4-tuple

$$\mathrm{Br}^{\gamma}_I := (\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1, \mathrm{Gr}_{P\times P^-,I}\times 0, \mathrm{Gr}_{P^-\times P,I}\times\mathbb{A}^1, \mathrm{Gr}_{M\times M,I}\times 0).$$

2.5.8. *Stratification on* $\mathrm{VinGr}_{G,I}|_{C_P}$. As before, the map

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \simeq \mathrm{VinGr}_{G,I}|_{C_P} \underset{\mathrm{Gr}_{G\times G,I}}{\times} \mathrm{Gr}_{P\times P^-,I} \to \mathrm{VinGr}_{G,I}|_{C_P}$$

is bijective on field valued points. Hence the connected components of $\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I}$ provide a stratification on $\mathrm{VinGr}_{G,I}|_{C_P}$. On the other hand, [Sch16] defined a *defect stratification* on $\mathrm{VinBun}_G|_{C_P}$ (see § C.4.5 for a quick review). Let ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ be the disjoint union of all the defect strata. The following result says these two stratifications are compatible via the local-to-global-map.

Proposition 2.5.9. *There is a commutative diagram*

$$
\begin{tikzcd}
\mathrm{Gr}_{P\times P^-,I} \arrow[d] & \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \arrow[l] \arrow[r] \arrow[d] & \mathrm{VinGr}_{G,I}|_{C_P} \arrow[d] \\
\mathrm{Bun}_{P\times P^-} & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[l] \arrow[r] & \mathrm{VinBun}_G|_{C_P}
\end{tikzcd}
$$

such that its right square is Cartesian.

Proof. We have the following commutative diagram

$$\begin{tikzcd}
(P\backslash \mathrm{pt}/P^- \leftarrow \mathrm{pt}) \arrow[d] & (P\backslash \overline{M}/P^- \leftarrow \mathrm{pt}) \arrow[l] \arrow[r] \arrow[d] & (G\backslash \mathrm{Vin}_G|_{C_P}/G \leftarrow \mathrm{pt}) \arrow[d] \\
(P\backslash \mathrm{pt}/P^- \supset P\backslash \mathrm{pt}/P^-) & (P\backslash \overline{M}/P^- \supset P\backslash M/P^-) \arrow[l] \arrow[r] & (G\backslash \mathrm{Vin}_G|_{C_P}/G \supset G\backslash {}_0\mathrm{Vin}_G|_{C_P}/G).
\end{tikzcd}$$

By Construction C.1.7, we obtain the desired commutative diagram in the problem. It remains to show its right square is Cartesian. By Lemma C.1.14, it suffices to show the map

$$\mathrm{pt} \to \mathrm{pt} \underset{(G\backslash \mathrm{Vin}_G|_{C_P}/G)}{\times} (P\backslash \overline{M}/P^-)$$

is an isomorphism. Using the Cartesian diagram (C.8), the RHS is isomorphic to

$$\mathrm{pt} \underset{(G\backslash {}_0\mathrm{Vin}_G|_{C_P}/G)}{\times} (P\backslash M/P^-).$$

Then we are done because ${}_0\mathrm{Vin}_G|_{C_P} \simeq (G\times G)/(P\times_M P^-)$.

□[Proposition 2.5.9]

Corollary 2.5.10. *Let $\lambda, \mu \in \Lambda_{G,P}$ be two elements. Then the fiber product*

$$\mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \underset{\mathrm{Gr}_{P\times P^-,I}}{\times} (\mathrm{Gr}^{\lambda}_{P,I} \times \mathrm{Gr}^{\mu}_{P,I})$$

is empty unless $\lambda \le \mu$, where $\mathrm{Gr}^{\lambda}_{P,I}$ is the connected component of $\mathrm{Gr}_{P,I}$ corresponding to λ.

Proof. Using Proposition 2.5.9, it suffices to show the fiber product

$${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \underset{\mathrm{Bun}_{P\times P^-}}{\times} (\mathrm{Bun}^{-\lambda}_{P} \times \mathrm{Bun}^{-\mu}_{P^-})$$

is empty unless $\lambda \le \mu$. Then we are done by (C.14) and (C.12).

□[Corollary 2.5.10]

For any $\delta \in \Lambda_{G,P}$, there is a closed sub-indscheme ${}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I}$ of $\mathrm{Gr}_{G\times G,I}$ whose field-valued points are the union of the field-valued points contained in strata $\mathrm{Gr}^{\lambda,\mu}_{P\times P^-}$ such that $\lambda - \mu \le \delta$ (See [Fulltext, Corollary C.3.11] for its definition). We have:

Corollary 2.5.11. *(c.f. [FKM20, Lemma 3.13])* $(\mathrm{VinGr}_{G,I}|_{C_P})_{\mathrm{red}}$ *is contained in* ${}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I}$.

Proof. Note that $(\mathrm{VinGr}_{G,I}|_{C_P})_{\mathrm{red}}$ is also a closed sub-indscheme of $\mathrm{Gr}_{G\times G,I}$. Hence it suffices to show the set of field valued points of $\mathrm{VinGr}_{G,I}|_{C_P}$ is a subset of that of ${}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I}$. Then we are done by Corollary 2.5.10.

□[Corollary 2.5.11]

Proposition 2.5.12. *The following commutative square is Cartesian:*

$$\begin{tikzcd} \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{att}} \arrow[r] \arrow[d] & \mathrm{Gr}_{P\times P^-,I} \arrow[d] \\ \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}} \arrow[r] & \mathrm{Gr}_{M\times M,I}. \end{tikzcd}$$

Proof. Follows from Lemma C.1.13.

□[Proposition 2.5.12]

Remark 2.5.13. One can use Proposition 2.5.12 to prove the claim in Remark 1.2.29.

2.5.14. *Defect-free version.* By Proposition 2.5.1, the $\mathbb{G}_m$-action (2.16) also stabilizes ${}_0\mathrm{VinGr}_{G,I}^{\gamma} \simeq \mathrm{Gr}_{\widetilde{G}^\gamma,I}$. Let $\mathrm{Br}^{\gamma}_{{}_0\mathrm{Vin},I}$ be the Braden 4-tuple for this restricted action.

On the other hand, there is a Braden 4-tuple

$$(\mathrm{Gr}_{\widetilde{G}^\gamma,I}, \mathrm{Gr}_{P\times_M P^-,I}\times 0, \mathrm{Gr}_{M,I}\times\mathbb{A}^1, \mathrm{Gr}_{M,I}\times 0),$$

where the only non-obvious map $p^-: \mathrm{Gr}_{M,I}\times\mathbb{A}^1 \to \mathrm{Gr}_{\widetilde{G}^\gamma,I}$ is given by the composition

$$\mathrm{Gr}_{M,I}\times\mathbb{A}^1 \simeq \mathrm{Gr}_{\widetilde{M}^\gamma,I} \to \mathrm{Gr}_{\widetilde{G}^\gamma,I}.$$

We have

Proposition 2.5.15. *There is a canonical isomorphism between Braden 4-tuples*

$$\mathrm{Br}^{\gamma}_{{}_0\mathrm{Vin},I} \simeq (\mathrm{Gr}_{\widetilde{G}^\gamma,I}, \mathrm{Gr}_{P\times_M P^-,I}\times 0, \mathrm{Gr}_{M,I}\times\mathbb{A}^1, \mathrm{Gr}_{M,I}\times 0).$$

Proof. The statements concerning the attractor and fixed loci follow directly from Proposition 2.5.1 because the $\mathbb{G}_m$-action on ${}_0\mathrm{VinGr}_{G,I}|_{C_P} \simeq \mathrm{Gr}_{P\times_M P^-,I}$ is contractive.

Let us calculate the repeller locus. By [DG14, Lemma 1.4.9(i)], the map

$${}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \to \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \underset{\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}}{\times} {}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}$$

is an isomorphism. On the other hand, we have a Cartesian square (see (C.17))

$$\begin{tikzcd} (P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P \leftarrow \mathbb{A}^1) \arrow[r] \arrow[d] & (P^-\backslash \mathrm{Vin}_G^{\gamma}/P \leftarrow \mathbb{A}^1) \arrow[d, "\mathbf{q}^-_{\mathrm{sect}}"] \\ (M\backslash M/M \leftarrow \mathrm{pt}) \arrow[r] & (M\backslash \overline{M}/M \leftarrow \mathrm{pt}). \end{tikzcd}$$

Note that $P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P \simeq M\backslash M/M\times\mathbb{A}^1$ by (C.16). Hence by Lemma C.1.13, we have an isomorphism

$$\mathrm{Gr}_{M,I}\times\mathbb{A}^1 \simeq \mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \underset{\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}}{\times} {}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{fix}}.$$

This provides the desired isomorphism ${}_0\mathrm{VinGr}_{G,I}^{\gamma,\mathrm{rep}} \simeq \mathrm{Gr}_{M,I} \times \mathbb{A}^1$. It follows from construction that this isomorphism is compatible with the natural maps in the Braden 4-tuples.

□[Proposition 2.5.15]

3. Proofs - I

3.0.1. *Organization of this section.* Our proofs of Theorem 1.3.7 and Theorem 1.5.1 use a same strategy, which we axiomize in § 3.1.

In § 3.2, we prove a technical conservativity result.

In § 3.3 and 3.4, as warm-up exercises, we use the framework in § 3.1 to prove two results about $\Psi_{\gamma,I}$: (i) its restriction to the defect-free locus is constant; (ii) the assignment $I \rightsquigarrow \Psi_{\gamma,I}[-1]$ factorizes.

In § 3.5, we use the above framework to prove Theorem 1.3.7.

In § 3.6, we sketch how to generalize our main theorems to (affine) flag varieties.

The proof of Theorem 1.5.1 is postponed to § 4 because we need more sheaf-theoretic input.

3.1. **An axiomatic framework.** The essence of our proofs of Theorem 1.3.7 and Theorem 1.5.1 is to use Braden's theorem and the contraction principle to show taking unipotent nearby cycles commutes with certain pull-push functors. In this subsection, we give an axiomatic framework for these arguments.

3.1.1. *The main result.* Suppose we are given the following data:

- A $\mathbb{G}_m$-action on $\mathbb{A}^1$ given by $s \cdot t := s^n t$, where n is a negative integer;
- Three ind-finite type indschemes U, V and W acted on by $\mathbb{G}_m$;
- A correspondence $\alpha := (U \xleftarrow{f} V \xrightarrow{g} W)$ over $\mathbb{A}^1$ compatible with the $\mathbb{G}_m$-actions;
- An object $\mathring{\mathcal{F}} \in \mathrm{D}(\mathring{W})^{\mathbb{G}_m\text{-um}}$;
- A full subcategory $\mathcal{C} \subset \mathrm{D}(U_0)$.

By construction, we can extend α to a correspondence between Braden 4-tuples:

$$\alpha_{\mathrm{ext}} := (\alpha, \alpha^+, \alpha^-, \alpha^0) : (U, U^+, U^-, U^0) \leftarrow (V, V^+, V^-, V^0) \to (W, W^+, W^-, W^0),$$

defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$ (see Example 2.2.22), where the superscripts "$+, -, 0$" stands for attractor, repeller and fixed loci. As usual, we use the following notations:

$$\mathring{\alpha} := (\mathring{U} \xleftarrow{\mathring{f}} \mathring{V} \xrightarrow{\mathring{g}} \mathring{W}),\ \alpha_0 := (U_0 \xleftarrow{f_0} V_0 \xrightarrow{g_0} W_0)$$

Note that when restricted to 0-fibers, we obtain a correspondence between Braden 4-tuples:

$$(U_0, U_0^+, U_0^-, U_0^0) \leftarrow (V_0, V_0^+, V_0^-, V_0^0) \to (W_0, W_0^+, W_0^-, W_0^0).$$

The following result is a special case of our main result (see Thoerem 3.1.11 below):

Corollary 3.1.2. *Suppose the above data satisfy the following conditions (up to non-reduced structures)*[27]*:*

(P1) *The map* $V^0 \to U^0 \times_{U^+} V^+$ *is an isomorphism.*
(P2) *The map* $V^- \to U^- \times_U V$ *is an isomorphism.*
(P3) *The map* $V^- \to W^- \times_{W^0} V^0$ *is an isomorphism.*
(Q) *The map* $V^+ \to W^+ \times_W V$ *is an open embedding.*
(G1) *The object* $\mathring{\mathcal{F}}$ *is contained in* $\mathrm{D}(\mathring{W})^{\mathrm{good}}$ *(see Notation 2.1.4).*
(G2) *The object* $(\mathring{f})_* \circ (\mathring{g})^! (\mathring{\mathcal{F}})$ *is contained in* $\mathrm{D}(\mathring{U})^{\mathrm{good}}$.
(C) *The following composition is conservative*[28]*:*

$$\mathcal{C} \hookrightarrow \mathrm{D}(U_0) \xrightarrow{p^{+,*}_{U_0}} \mathrm{Pro}(\mathrm{D}(U_0^+)) \xrightarrow{i^{+,!}_{U_0}} \mathrm{Pro}(\mathrm{D}(U_0^0)).$$

(M) *The objects* $i^* \circ f_* \circ g^! \circ j_*(\mathring{\mathcal{F}})$ *and* $f_{0,*} \circ g_0^! \circ i^* \circ j_*(\mathring{\mathcal{F}})$ *are contained in* $\mathcal{C} \subset \mathrm{Pro}(\mathrm{D}(U_0))$,

then there are canonical isomorphisms

$$i^* \circ f_* \circ g^! \circ j_*(\mathring{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ i^* \circ j_*(\mathring{\mathcal{F}}),$$
$$\Psi^{\mathrm{un}} \circ (\mathring{f})_* \circ (\mathring{g})^! (\mathring{\mathcal{F}}) \simeq f_{0,*} \circ g_0^! \circ \Psi^{\mathrm{un}}(\mathring{\mathcal{F}}).$$

To state and prove the generalization of this result, we need some definitions that generalize those in § 2.2.

Definition 3.1.3. *Let* $\alpha' := (U' \leftarrow V' \to W')$ *and* $\alpha := (U \leftarrow V \to W)$ *be two correspondences of lft prestacks. A* 2-morphism $\mathfrak{s} : \alpha' \to \alpha$ *between them is a commutative diagram*

$$\begin{tikzcd} \alpha' \arrow[d, Rightarrow, "\mathfrak{s}"] & U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ \alpha & U & V \arrow[l, "f"] \arrow[r, "g"'] & W. \end{tikzcd}$$

A 2-morphism $\mathfrak{s} : \alpha' \to \alpha$ *is* right quasi-Cartesian *if the right square in the above diagram is quasi-Cartesian.*

Construction 3.1.4. *For a right quasi-Cartesian 2-morphism as in Definition 3.1.3, (2.8) induces a natural transformation*

$$f_* \circ g^! \circ r_* \to f_* \circ q_* \circ (g')^! \simeq p_* \circ f'_* \circ (g')^!.$$

Passing to left adjoints, we obtain a natural transformation

$$\mathfrak{s}^* : p^* \circ f_* \circ g^! \to f'_* \circ (g')^! \circ r^*, \tag{3.1}$$

[27](P) for *pullback*; (Q) for *quasi-Cartesian*; (C) for *conservative*; (G) for *good*; (M) for *morphism*.

[28]For instance, this condition is satisfied if $U_0^+ \to U_0$ is a *finite* stratification and $\mathcal{C}$ is the full subcategory of D-modules that are constant along each stratum.

between functors $\mathrm{Pro}(\mathrm{D}(W)) \to \mathrm{Pro}(\mathrm{D}(U'))$, *which we refer as the* $*$-transformation associated to $\mathfrak{s}$.

Example 3.1.5. Let $(\mathcal{Y}, \mathcal{Y}^0, q, i)$ be a retraction (see Definition 2.2.3). The natural transformation $q_* \to i^*$ in Construction 2.2.5 is the $*$-transformation associated to the following 2-morphism between correspondences:

$$\begin{tikzcd} \mathcal{Y}^0 \arrow[d, "="] & \mathcal{Y}^0 \arrow[l, "="'] \arrow[r, "="] \arrow[d, "i"] & \mathcal{Y}^0 \arrow[d, "i"] \\ \mathcal{Y}^0 & \mathcal{Y} \arrow[l, "q"'] \arrow[r, "="] & \mathcal{Y}. \end{tikzcd} \tag{3.2}$$

Definition 3.1.6. *(1) A right quasi-Cartesian 2-morphism* $\mathfrak{s}$ *as above is* pro-nice *for an object* $\mathcal{F} \in \mathrm{Pro}(\mathrm{D}(W))$ *if* $\mathfrak{s}^*(\mathcal{F}) : p^* \circ f_* \circ g^!(\mathcal{F}) \to f'_* \circ (g')^! \circ r^*(\mathcal{F})$ *is an isomorphism.*

(2) Let $T : \mathrm{Pro}(\mathrm{D}(U')) \to \mathcal{C}$ *be any functor. We say* $\mathfrak{s}$ *is* T-pro-nice *for* $\mathcal{F}$ *if* $\mathbf{Id}_T \bigstar \mathfrak{s}^*(\mathcal{F}) : T \circ p^* \circ f_* \circ g^!(\mathcal{F}) \to T \circ f'_* \circ (g')^! \circ r^*(\mathcal{F})$ *is an isomorphism (see Notation 0.6.2).*

(3) We say $\mathfrak{s}$ *is* nice *for* $\mathcal{F}$ *if it is pro-nice for* $\mathcal{F}$ *and* $\mathfrak{s}^*(\mathcal{F})$ *is a morphism in* $\mathrm{D}(U')$.

Definition 3.1.7. *Let* $\alpha := (U \leftarrow V \to W)$ *and* $\beta := (W \leftarrow \mathcal{Y} \to \mathcal{Z})$ *be two correspondences of prestacks. Their* composition *is defined to be* $\alpha \circ \beta := (U \leftarrow V \times_W \mathcal{Y} \to \mathcal{Z})$.

The horizontal *and* vertical compositions of 2-morphisms *between correspondences are defined in the obvious way.*

The following two lemmas can be proved by diagram chasing. We leave the details to the reader.

Lemma 3.1.8. *Let* α, α' *and* α'' *be three correspondences of prestacks. Let* $\mathfrak{t} : \alpha'' \to \alpha'$ *and* $\mathfrak{s} : \alpha' \to \alpha$ *be two 2-morphisms. We depict them as*

$$\begin{tikzcd} \alpha'' \arrow[d, Rightarrow, "\mathfrak{t}"] & U'' \arrow[d, "l"] & V'' \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[d, "m"] & W'' \arrow[d, "n"] \\ \alpha' \arrow[d, Rightarrow, "\mathfrak{s}"] & U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] \\ \alpha & U & V \arrow[l, "f"'] \arrow[r, "g"] & W. \end{tikzcd}$$

Suppose $\mathfrak{s}$ *is right quasi-Cartesian. We have:*

(1) $\mathfrak{s} \circ \mathfrak{t}$ *is right quasi-Cartesian iff* $\mathfrak{t}$ *is right quasi-Cartesian.*

(2) Suppose the conditions in (1) are satisfied, then there is a canonical equivalence

$$(\mathfrak{s} \circ \mathfrak{t})^* \simeq (\mathfrak{t}^* \bigstar \mathbf{Id}_{r^*}) \circ (\mathbf{Id}_{l^*} \bigstar \mathfrak{s}^*).$$

Lemma 3.1.9. *Let* α, α', β *and* β' *be four correspondences of prestacks such that* $\alpha \circ \beta$ *and* $\alpha' \circ \beta'$ *can be defined. Let* $\mathfrak{s} : \alpha' \to \alpha$ *and* $\mathfrak{t} : \beta' \to \beta$ *be* *two 2-morphisms. We depict them as*

$$
\begin{tikzcd}
\cdot & \cdot \arrow[l, bend right, "\alpha'"'] \arrow[l, bend left, "\alpha"] \arrow[l, phantom, "\Downarrow\mathfrak{s}"] & \cdot \arrow[l, bend right, "\beta'"'] \arrow[l, bend left, "\beta"] \arrow[l, phantom, "\Downarrow\mathfrak{t}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U' \arrow[d, "p"] & V' \arrow[l, "f'"'] \arrow[r, "g'"] \arrow[d, "q"] & W' \arrow[d, "r"] & \mathcal{Y}' \arrow[l, "d'"'] \arrow[r, "e'"] \arrow[d, "m"] & \mathcal{Z}' \arrow[d, "n"] \\
U & V \arrow[l, "f"'] \arrow[r, "g"] & W & \mathcal{Y} \arrow[l, "d"'] \arrow[r, "e"] & \mathcal{Z}.
\end{tikzcd}
$$

Suppose $\mathfrak{s}$ and $\mathfrak{t}$ are both right quasi-Cartesian. We have

(1) $\mathfrak{s}\bigstar\mathfrak{t}$ is right quasi-Cartesian.

(2) There is a canonical equivalence

$$(\mathfrak{s}\bigstar\mathfrak{t})^* \simeq (\mathbf{Id}_{f'_*\circ(g')^!}\bigstar\mathfrak{t}^*)\circ(\mathfrak{s}^*\bigstar\mathbf{Id}_{d_*\circ e^!}).$$

3.1.10. *Axioms.* We are ready to state the generalization of Corollary 3.1.2. Suppose we are given the following data:

- A correspondence of prestacks $\alpha := (U \xleftarrow{f} V \xrightarrow{g} W)$ over $\mathbb{A}^1$.
- Objects $\mathring{\mathcal{F}} \in \mathrm{D}(\mathring{W})$ and $\mathcal{F} := j_*(\mathring{\mathcal{F}}) \in \mathrm{D}(W)$.
- An extension of α to a correspondence between Braden 4-tuples

$$\alpha_{\mathrm{ext}} := (\alpha,\alpha^+,\alpha^-,\alpha^0) : (U,U^+,U^-,U^0) \leftarrow (V,V^+,V^-,V^0) \to (W,W^+,W^-,W^0),$$

  defined over the base Braden 4-tuple $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1,0,\mathbb{A}^1,0)$ (see Example 2.2.22).
- A full subcategory $\mathcal{C} \subset \mathrm{D}(U_0)$, where as usual $U_0 := U\times_{\mathbb{A}^1} 0$.

Suppose the above data satisfy the conditions in Corollary 3.1.2 and the following additional axioms:

(N1) The Braden 4-tuple (W,W^+,W^-,W^0) is $*$-nice for $\mathcal{F}$.
(N2) The Braden 4-tuple (U,U^+,U^-,U^0) is $*$-nice for $f_*\circ g^!(\mathcal{F})$.
(N3) The Braden 4-tuple (W_0,W_0^+,W_0^-,W_0^0) is $*$-nice for $i^*(\mathcal{F})$.
(N4) The Braden 4-tuple (U_0,U_0^+,U_0^-,U_0^0) is $*$-nice for $f_{0,*}\circ g_0^!\circ i^*(\mathcal{F})$.

Then taking the unipotent nearby cycles for $\mathring{\mathcal{F}}$ commutes with !-pull-$*$-push along the correspondence α. More precisely, we have

Theorem 3.1.11. *In the above setting, there are canonical isomorphisms*

$$i^*\circ f_*\circ g^!\circ j_*(\mathring{\mathcal{F}}) \simeq f_{0,*}\circ g_0^!\circ i^*\circ j_*(\mathring{\mathcal{F}}), \tag{3.3}$$

$$\Psi^{\mathrm{un}}\circ(\mathring{f})_*\circ(\mathring{g})^!(\mathring{\mathcal{F}}) \simeq f_{0,*}\circ g_0^!\circ\Psi^{\mathrm{un}}(\mathring{\mathcal{F}}). \tag{3.4}$$

Proof. The essence of this proof is diagram chasing on a 4-cube, which we cannot draw on a paper.

By Axioms (G1) and (G2), both sides of (3.3) and (3.4) are well-defined. By (2.2), it suffices to prove the equivalence (3.3). Hence it suffices to show the morphism $\mathfrak{z}^*(\mathcal{F})$ is an isomorphism, i.e., the 2-morphism $\mathfrak{z} : \alpha_0 \to \alpha$ is nice for $\mathcal{F}$.

By Axioms (C) and (M), it suffices to prove that $\mathfrak{z}$ is $(i^{+,!}_{U_0}\circ p^{+,*}_{U_0})$-pro-nice for $\mathcal{F}$.

By Axiom (Q), the 2-morphism $\mathfrak{p}^+ : \alpha^+ \to \alpha$ is right quasi-Cartesian. Hence so is its 0-fiber $\mathfrak{p}_0^+ : \alpha_0^+ \to \alpha_0$. Consider the commutative diagram

$$\begin{tikzcd} \alpha_0^+ \arrow[r, "\mathfrak{p}_0^+"] \arrow[d, "\mathfrak{z}^+"] & \alpha_0 \arrow[d, "\mathfrak{z}"] \\ \alpha^+ \arrow[r, "\mathfrak{p}^+"] & \alpha. \end{tikzcd}$$

By Lemma 3.1.8, it suffices to prove

(1) $\mathfrak{p}_0^+$ is pro-nice for $i^*(\mathcal{F})$;
(2) $\mathfrak{z} \circ \mathfrak{p}_0^+$ is pro-nice for $\mathcal{F}$.

Note that we have $\mathfrak{z} \circ \mathfrak{p}_0^+ \simeq \mathfrak{p}^+ \circ \mathfrak{z}^+$. Also note that $\mathfrak{z}^+ : \alpha_0^+ \to \alpha^+$ is an isomorphism (because our Braden 4-tuples are defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$). Using Lemma 3.1.8 again, we see that (2) can be replaced by

(2') $\mathfrak{p}^+$ is pro-nice for $\mathcal{F}$.

It remains to prove (1) and (2'). We will use Axioms (P1)-(P3) and (N1)-(N2) to prove (2'). One can obtain (1) similarly[29] from Axioms (P1)-(P3) and (N3)-(N4).

Consider 2-morphisms $\mathfrak{u}$, $\mathfrak{p}^+$ and $\mathfrak{u}\bigstar\mathfrak{p}^+$ depicted as

$$\begin{tikzcd} U^0 \arrow[d, "i_U^-"] & U^0 \arrow[l, "="'] \arrow[r, "i_U^+"] \arrow[d, "i_U^-"] & U^+ \arrow[d, "p_U^+"] & V^+ \arrow[l, "f^+"'] \arrow[r, "g^+"] \arrow[d, "p_V^+"] & W^+ \arrow[d, "p_W^+"] \\ U^- & U^- \arrow[l, "="'] \arrow[r, "p_U^-"] & U & V \arrow[l, "f"'] \arrow[r, "g"] & W \end{tikzcd}
\qquad
\begin{tikzcd} U^0 \arrow[d, "i_U^-"] & U^0 \times_{U^+} V^+ \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "g^+\circ \mathrm{pr}_2"] \arrow[d, "(i_U^-, p_V^+)"] & W^+ \arrow[d, "p_W^+"] \\ U^- & U^- \times_U V \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "g\circ \mathrm{pr}_2"] & W. \end{tikzcd}$$

$\mathfrak{u}$ $\qquad$ $\mathfrak{p}^+$ $\qquad$ $\mathfrak{u}\bigstar\mathfrak{p}^+$

By Lemma 3.1.9, it suffices to prove

(i) $\mathfrak{u}$ is pro-nice for $f_* \circ g^!(\mathcal{F})$;
(ii) $\mathfrak{u}\bigstar\mathfrak{p}^+$ is pro-nice for $\mathcal{F}$.

Note that (i) is implied by (the quasi-Cartesian part of) Axiom (N2). It remains to prove (ii). Consider 2-morphisms $\mathfrak{i}^-$, $\mathfrak{w}$ and $\mathfrak{i}^-\bigstar\mathfrak{w}$ depicted as

$$\begin{tikzcd} U^0 \arrow[d, "i_U^-"] & V^0 \arrow[l, "f^0"'] \arrow[r, "g^0"] \arrow[d, "i_V^-"] & W^0 \arrow[d, "i_W^-"] & W^0 \arrow[l, "="'] \arrow[r, "i_W^+"] \arrow[d, "i_W^-"] & W^+ \arrow[d, "p_W^+"] \\ U^- & V^- \arrow[l, "f^-"'] \arrow[r, "g^-"] & W^- & W^- \arrow[l, "="'] \arrow[r, "p_W^-"] & W \end{tikzcd}
\qquad
\begin{tikzcd} U^0 \arrow[d, "i_U^-"] & V^0 \arrow[l, "f^0"'] \arrow[r, "i_W^+\circ g^0"] \arrow[d, "i_V^-"] & W^+ \arrow[d, "p_W^+"] \\ U^- & V^- \arrow[l, "f^-"'] \arrow[r, "g^-"] & W. \end{tikzcd}$$

$\mathfrak{i}^-$ $\qquad$ $\mathfrak{w}$ $\qquad$ $\mathfrak{i}^-\bigstar\mathfrak{w}$

By Axioms (P1) and (P2), $\mathfrak{i}^-\bigstar\mathfrak{w}$ is nil-isomorphic to $\mathfrak{u}\bigstar\mathfrak{p}^+$. By Lemma 3.1.9 again, it suffices to prove

(a) $\mathfrak{w}$ is pro-nice for $\mathcal{F}$;
(b) $\mathfrak{i}^-$ is pro-nice for $p_W^{-,!}(\mathcal{F})$.

[29] Note that the 0-fiber versions of Axioms (P1)-(P3) are implied by themselves.

Note that (a) is implied by (the quasi-Cartesian part of) (N1). It remains to prove (b). Consider the 2-morphism (3.2) associated to the retraction (U^-, U^0). We denote it by $\mathfrak{c}_U$. Similarly we define $\mathfrak{c}_W$. By Axiom (P3), $\mathfrak{c}_U \bigstar \mathfrak{i}^-$ is nil-isomorphic[30] to $\mathbf{Id}_{\alpha^0} \bigstar \mathfrak{c}_W$. Using Lemma 3.1.9 again, we reduce (b) to (the retraction part of) Axioms (N1) and (N2) (because of Example 3.1.5).

□[Theorem 3.1.11]

3.2. **Two auxiliary results.** In this subsection, we prove two results which play key technical roles in our proofs of the main theorems. Namely, they serve respectively as Axioms (C) and (M) in § 3.1.10.

For $\lambda \in \Lambda_{G,P}$, there is a closed sub-indscheme ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ of $\mathrm{Gr}_{G,I}$ whose field-valued points are the union of the field-valued points contained in strata $\mathrm{Gr}^{\mu}_{P,I}$ such that $\mu \le \lambda$ (see [Fulltext, Proposition C.3.2] for its definition). As explained in § D.2.4, the $\mathcal{L}U_I$-action on $\mathrm{Gr}_{G,I}$ preserves ${}_{\le\lambda}\mathrm{Gr}_{G,I}$. Hence we have a fully faithful functor

$$\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}.$$

Similarly, for $\delta \in \Lambda_{G,P}$, the closed subscheme ${}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I}$ of $\mathrm{Gr}_{G\times G,I}$ (see Corollary 2.5.11) is preserved by the $\mathcal{L}(U\times U^-)_I$-action on $\mathrm{Gr}_{G\times G,I}$. Hence we have a fully faithful functor

$$\mathrm{D}({}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}.$$

We have:

Lemma 3.2.1. *(1) For $\lambda \in \Lambda_{G,P}$, the following composition is conservative*

$$\begin{aligned}&\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I}) \to \\ &\xrightarrow{\mathbf{p}_I^{+,*}} \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{P,I})) \xrightarrow{\mathbf{i}_I^{+,!}} \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{M,I})).\end{aligned} \tag{3.5}$$

(2) For $\delta \in \Lambda_{G,P}$, the following composition is conservative

$$\begin{aligned}&\mathrm{D}({}_{\mathrm{diff}\le\delta}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G\times G,I}) \to \\ &\xrightarrow{*\text{-pullback}} \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{P\times P^-,I})) \xrightarrow{!\text{-pullback}} \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{M\times M,I})).\end{aligned}$$

Warning 3.2.2. *We warn that (1) would be* false *if one replaces ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ by the entire $\mathrm{Gr}_{G,I}$. For example, using Braden's theorem, it is easy to see the dualizing D-module $\omega_{\mathrm{Gr}_{G,I}}$ is sent to zero by that composition because the fibers of $\mathrm{Gr}_{P,I} \to \mathrm{Gr}_{M,I}$ are infinitely dimensional.*

Proof. We will prove (1). The proof for (2) is similar.

Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$ in Example 1.2.14. By Lemma 2.3.4(1), Braden's theorem and the contraction principle, the composition (3.5) is

[30] We ask the reader to pardon us for not drawing these compositions.

isomorphic to

$$\mathrm{D}({}_{\leq\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I}) \to$$
$$\xrightarrow{\mathbf{p}_I^{-,!}} \mathrm{D}(\mathrm{Gr}_{P^-,I}) \xrightarrow{\mathbf{q}_{I,*}^-} \mathrm{D}(\mathrm{Gr}_{M,I}) \hookrightarrow \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{M,I})).$$

Hence by Lemma 2.3.4(3), it is also isomorphic to

$$\mathrm{D}({}_{\leq\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \hookrightarrow \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \xrightarrow{\mathbf{p}_I^{+,*,\mathrm{inv}}} \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \simeq$$
$$\simeq \mathrm{D}(\mathrm{Gr}_{M,I}) \hookrightarrow \mathrm{Pro}(\mathrm{D}(\mathrm{Gr}_{M,I})).$$

Then we are done by Lemma D.3.2.

□[Lemma 3.2.1]

Lemma 3.2.3. *The object* $i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \in \mathrm{D}(\mathrm{Gr}_{G\times G,I})$ *is contained in*

$$\mathrm{D}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{D}(\mathrm{Gr}_{G\times G,I}).$$

Proof. By Remark 2.4.8, $i^* \circ j_* \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$ is contained in $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$. It remains to show it is also contained in $\mathrm{D}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,I}) \subset \mathrm{D}(\mathrm{Gr}_{G\times G,I})$. By Lemma 1.2.27, the support of this object is contained in $\mathrm{VinGr}_{G,I}|_{C_P} \hookrightarrow \mathrm{Gr}_{G\times G,I}$. Hence we are done by Corollary 2.5.11.

□[Lemma 3.2.3]

3.3. **Warm-up: restriction to the defect-free locus.**

Recall (see Lemma 1.2.27) that we have an identification

$${}_0\mathrm{VinGr}_{G,I}^{\gamma} \simeq \mathrm{Gr}_{\widetilde{G}^{\gamma},I}$$

as locally closed sub-indscheme of

$$\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \simeq \mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1.$$

Note that the 0-fiber of $\mathrm{Gr}_{\widetilde{G}^{\gamma},I}$ is $\mathrm{Gr}_{P\times_M P^-,I}$, which is an open sub-indscheme of $\mathrm{VinGr}_{G,I}|_{C_P}$.

Consider the map ${}_0\mathrm{VinGr}_{G,I}^{\gamma} \to \mathbb{A}^1$. Let ${}_0\Psi_{\gamma,I,\mathrm{Vin}}$ (resp. ${}_0\Psi_{\gamma,I,\mathrm{Vin}}^{\mathrm{un}}$) be the full (resp. unipotent) nearby cycles sheaf of the dualizing D-module for this family.

Also consider the map $\mathbb{A}^1 \to \mathbb{A}^1$. Let Ψ_{triv} (resp. $\Psi_{\mathrm{triv}}^{\mathrm{un}}$) be the full (resp. unipotent) nearby cycles sheaf of the dualizing D-module for this family. It is well-known that $\Psi_{\mathrm{triv}}^{\mathrm{un}} \simeq \Psi_{\mathrm{triv}} \simeq k[1]$. We have

Proposition 3.3.1. *The maps*

$${}_0\Psi_{\gamma,I,\mathrm{Vin}} \to \omega \otimes \Psi_{\mathrm{triv}} \simeq \omega[1] \text{ and } {}_0\Psi_{\gamma,I,\mathrm{Vin}}^{\mathrm{un}} \to \omega \otimes \Psi_{\mathrm{triv}}^{\mathrm{un}} \simeq \omega[1]$$

are isomorphisms, where ω *is the dualizing D-module on* ${}_0\mathrm{VinGr}_{G,I}|_{C_P}$.

Proof. By Proposition 1.3.4 (which we have already proved in § 2.4) and the fact that taking (unipotent) nearby cycles commutes with open restrictions, we have ${}_0\Psi^{\mathrm{un}}_{\gamma,I,\mathrm{Vin}} \simeq {}_0\Psi_{\gamma,I,\mathrm{Vin}}$. Hence it is enough to prove the claim for the unipotent nearby cycles.

We equip ${}_0\mathrm{VinGr}^{\gamma}_{G,I}$ with the $\mathbb{G}_m$-action in § 2.5.14. We also equip $\mathbb{A}^1$ with the $\mathbb{G}_m$-action given by $s \cdot t := s^{-2}t$. Then we are done by applying Corollary 3.1.2 to

- the integer $n = -2$;
- the correspondence $({}_0\mathrm{VinGr}^{\gamma}_{G,I} \xleftarrow{=} {}_0\mathrm{VinGr}^{\gamma}_{G,I} \to \mathbb{A}^1)$;
- the object $\mathring{\mathcal{F}} := \omega_{\mathrm{Gr}_{G,I} \times \mathbb{G}_m}$;
- the subcategory $\mathrm{D}({}_0\mathrm{VinGr}_{G,I}|_{C_P})^{\mathcal{L}(U \times U^-)_I} \subset \mathrm{D}({}_0\mathrm{VinGr}_{G,I}|_{C_P})$ (see Remark 1.2.28).

Indeed, Axioms (P1-P3) and (Q) follows from Proposition 2.5.15. Axioms (G1) and (G2) are obvious because $\mathring{\mathcal{F}}$ is regular ind-holonomic. Axiom (C) follows from Lemma 3.2.1(2) and Lemma 2.5.11. Axiom (M) follows from Lemma 3.2.3.

□[Proposition 3.3.1]

3.4. **Warm-up: factorization.**

3.4.1. *Factorization of the algebraic players.* We first review the factorization structures on the algebraic players $\mathrm{D}(\mathrm{Gr}_G)^{\mathcal{L}U}$ and $\mathrm{D}(\mathrm{Gr}_G)_{\mathcal{L}U}$.

As one would expect (using [Fulltext, Lemma B.1.8(2)], Corollary 2.3.6 and Remark 2.3.7), the factorization structures on $I \rightsquigarrow \mathrm{D}(\mathrm{Gr}_{G,I}), \mathrm{D}(\mathrm{Gr}_{P,I})$ induces factorization structures on

$$I \rightsquigarrow \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{P,I})_{\mathcal{L}U_I},$$

such that the assignments of functors $I \rightsquigarrow \mathbf{oblv}^{\mathcal{L}U_I}, \mathbf{pr}_{\mathcal{L}U_I}$ are factorizable functors. Moreover, by the base-change isomorphisms, the functors in § 2.3.1 factorizes.

By its proof, the equivalences in Lemma 2.3.2 factorizes.

3.4.2. *Factorization of the nearby cycles.* Let $I \twoheadrightarrow J$ be a surjection between non-empty finite sets. Consider the corresponding diagonal embedding $\Delta_{J\to I} : X^J \to X^I$. For any prestack $\mathcal{Z}$ over X^I, we abuse notation by denoting the closed embedding $\mathcal{Z} \times_{X^I} X^J \to \mathcal{Z}$ by the same symbol $\Delta_{J\to I}$.

By Remark 1.2.25, the assignment $I \rightsquigarrow (\Gamma_I : \mathrm{Gr}_{G,I} \times \mathbb{G}_m \hookrightarrow \mathrm{Gr}_{G\times G,I} \times \mathbb{G}_m)$ factorizes in family (relative to $\mathbb{G}_m$). Hence we have the base-change isomorphism:

$$\Gamma_{J,*}(\omega_{\mathrm{Gr}_{G,J}\times\mathbb{G}_m}) \simeq \Delta^!_{J\to I} \circ \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}),$$

which induces a morphism

$$\Psi_{\gamma,J} \to \Delta^!_{J\to I}(\Psi_{\gamma,I}).$$

Proposition 3.4.3. *The above morphism $\Psi_{\gamma,J} \to \Delta^!(\Psi_{\gamma,I})$ is an isomorphism.*

Proof. Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1$ and $\mathrm{Gr}_{G\times G,J} \times \mathbb{A}^1$ defined in Corollary 2.5.5. We apply Corollary 3.1.2 to

- the integer $n = -2$;
- the correspondence $(\mathrm{Gr}_{G\times G,J} \times \mathbb{A}^1 \xleftarrow{=} \mathrm{Gr}_{G\times G,J} \times \mathbb{A}^1 \to \mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1)$;
- the object $\mathring{\mathcal{F}} \coloneqq \Gamma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$;
- the subcategory $\mathrm{D}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,J})^{\mathcal{L}(U\times U^-)_J} \subset \mathrm{D}(\mathrm{Gr}_{G\times G,J})$.

Axioms (P1-P3) and (Q) follows from Construction 2.5.6. Axioms (G1) and (G2) are obvious because $\mathring{\mathcal{F}}$ is regular ind-holonomic. Axiom (C) is just Lemma 3.2.1(2). Axiom (M) is just Lemma 3.2.3.

□[Proposition 3.4.3]

Corollary 3.4.4. *The assignment*

$$I \rightsquigarrow \Psi_{\gamma,I}[-1] \in \mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$$

gives a factorization algebra $\Psi[-1]_{\gamma,\mathrm{fact}}$ in the factorization category $\mathrm{D}(\mathrm{Gr}_{G\times G})^{\mathcal{L}(U\times U^-)}_{\mathrm{fact}}$.

Proof. By Proposition 3.4.3, the assignment $I \rightsquigarrow \Psi_{\gamma,I}[-1]$ is compatible with diagonal restrictions. It has the factorization property because of the Künneth formula for the nearby cycles.

□[Corollary 3.4.4]

Remark 3.4.5. It follows from the proof of Proposition 2.4.1(2) that $\Psi[-1]_{\gamma,\mathrm{fact}}$ can be upgraded to a factorization algebra in the factorization category $(\mathrm{D}(\mathrm{Gr}_{G\times G})^{\mathcal{L}(U\times U^-)})^{\mathcal{L}^+M}_{\mathrm{fact}}$. Moreover, one can show that $\Psi[-1]_{\gamma,\mathrm{fact}}$ is a *unital* factorization algebra. We do not need these facts in this paper, hence we do not provide proofs.

3.5. **Proof of Theorem 1.3.7.** We prove Theorem 1.3.7 (and Corollary 1.4.3) in this subsection. To simplify the notations, we denote all unipotent nearby cycles functors by Ψ^{un}. By symmetry, it is enough to prove (2).

3.5.1. *Preparation.* Consider the diagonal embedding

$$\Delta : \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \hookrightarrow \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1,\ (x,y,t) \mapsto (x,x,y,t).$$

We have the following diagram

$$\begin{tikzcd}
 & \mathrm{Gr}_{G,I}\times\mathbb{G}_m \arrow[r, "\Gamma^\sigma"] \arrow[d, "\Gamma^\sigma_I"] & \mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\times\mathbb{G}_m \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathbf{Id}\times\Gamma^\sigma_I"] & \mathrm{Gr}_{G,I} \\
\mathrm{Gr}_{G,I}\times\mathbb{G}_m & \mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{G}_m \arrow[l, "\mathrm{pr}_{23}"'] \arrow[r, "\mathring{\Delta}"] & \mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{G}_m, &
\end{tikzcd}$$

where Γ^σ and Γ^σ_I are given by the formula[31]:$(x,t)\mapsto(t\cdot x,x,t)$, the maps pr_1 and pr_{23} are the projections onto the factors indicated by the subscripts. Note that the square in this diagram is Cartesian.

We also have the following correspondence:

$$\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I}\overset{\mathrm{pr}_2}{\leftarrow}\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I}\overset{\Delta_0}{\to}\mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\underset{X^I}{\times}\mathrm{Gr}_{G,I}.$$

We claim:

(i) the functor $\Psi^{\mathrm{un}}[-1]\circ\mathrm{pr}_{23,*}\circ(\overset{\circ}{\Delta})^!\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!$ is well-defined on $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, and is isomorphic to $\mathbf{oblv}^{\mathcal{L}U_I}$.

(ii) the functor $\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!$ is well-defined, and we have

$$\mathrm{pr}_{2,*}\circ\Delta_0^!\circ\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!\simeq F_{\mathcal{K}^\sigma}.$$

Note that these two claims translate the theorem into a statement that taking certain unipotent nearby cycles commutes with certain pull-push functors (see (3.8) below).

3.5.2. *Proof of (ii).* By Lemma 3.5.3 below, for any $\mathcal{G}\in\mathrm{D}(\mathrm{Gr}_{G,I})$, the object

$$(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!(\mathcal{G})\simeq\mathcal{G}\boxtimes\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})$$

is contained in $\mathrm{D}(\mathrm{Gr}_{G,I}\times\mathrm{Gr}_{G,I}\times_{X^I}\mathrm{Gr}_{G,I}\times\mathbb{G}_m)^{\mathrm{good}}$, and we have

$$\begin{aligned}\Psi^{\mathrm{un}}[-1]\circ(\mathbf{Id}\times\Gamma^\sigma_I)_*\circ\mathrm{pr}_1^!(\mathcal{G})&\simeq\Psi^{\mathrm{un}}[-1](\mathcal{G}\boxtimes\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}))\simeq\\&\simeq\mathcal{G}\boxtimes\Psi^{\mathrm{un}}[-1]\circ\Gamma^\sigma_{I,*}(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m})\simeq\mathcal{G}\boxtimes\mathcal{K}^\sigma.\end{aligned}$$

Then (ii) follows from the definition of $F_{\mathcal{K}^\sigma}$.

Lemma 3.5.3. *Let Z be an ind-finite type indscheme over $\mathbb{A}^1$, and Y be any ind-finite type indscheme. Let $\mathcal{F}\in\mathrm{D}(\overset{\circ}{Z})$ and $\mathcal{G}\in\mathrm{D}(Y)$. Suppose the !-restriction of $\mathcal{F}$ on any finite type closed subscheme of $\overset{\circ}{Z}$ is holonomic, then the object $\mathcal{G}\boxtimes\mathcal{F}$ is contained in $\mathrm{D}(Y\times\overset{\circ}{Z})^{\mathrm{good}}$ and we have $j_!(\mathcal{G}\boxtimes\mathcal{F})\simeq\mathcal{G}\boxtimes j_!(\mathcal{F})$.*

Proof. (Sketch) Let we first assume Y and Z to be finite type schemes. When $\mathcal{G}$ is compact (i.e. coherent), the claim follows from the Verdier duality. The general case can be obtained from this by a standard devissage argument.

□[Lemma 3.5.3]

3.5.4. *Proof of (i).* Consider the automorphism α on $\mathrm{Gr}_{G,I}\times\mathbb{G}_m$ given by $(x,t)\mapsto(t\cdot x,t)$. By the base-change isomorphisms, the functor in (i) is isomorphic to

$$\Psi^{\mathrm{un}}\circ\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})[-1]\simeq k\underset{C^\bullet(\mathbb{G}_m)}{\otimes}(i^*\circ j_*\circ\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m}))[-2].$$

[31]Note that the order is different from that for Γ_I.

Suppose $\mathcal{G}$ is contained in $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. By Lemma 2.3.4(1), $\mathcal{G}$ is unipotently $\mathbb{G}_m$-monodromic. Therefore $\mathcal{G}\boxtimes\omega_{\mathbb{G}_m}\in\mathrm{D}(\mathrm{Gr}_{G,I}\times\mathbb{G}_m)$ is unipotently $\mathbb{G}_m$-monodromic for the diagonal action, which implies $\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})\in\mathrm{D}(\mathrm{Gr}_{G,I}\times\mathbb{G}_m)$ is unipotently $\mathbb{G}_m$-monodromic for the $\mathbb{G}_m$-action on the second factor. Hence we can apply the contraction principle to $j_*\circ\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})$ and obtain

$$i^*\circ j_*\circ\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})[-2]\simeq\mathrm{pr}_{1,*}\circ j_*\circ\alpha^!(\mathcal{G}\boxtimes\omega_{\mathbb{G}_m})[-2],\tag{3.6}$$

where $\mathrm{pr}_1:\mathrm{Gr}_{G,I}\times\mathbb{A}^1\to\mathrm{Gr}_{G,I}$ is the projection. In particular, the LHS of (3.6) is well-defined. Hence the functor in (i) is well-defined on $\mathcal{G}$.

By the base-change isomorphisms, the RHS of (3.6) is isomorphic to $\mathrm{act}_*(\mathcal{G}\boxtimes k_{\mathbb{G}_m})$, where $\mathrm{act}:\mathrm{Gr}_{G,I}\times\mathbb{G}_m\to\mathrm{Gr}_{G,I}$ is the action map. It remains to prove

$$k\underset{C^\bullet(\mathbb{G}_m)}{\otimes}\mathrm{act}_*(\mathcal{G}\boxtimes k_{\mathbb{G}_m})\simeq\mathcal{G}.$$

This formula is well-known for any $\mathcal{G}\in\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}}$. For completeness, we provide a formal proof.

Consider the adjoint pair

$$\mathbf{oblv}:\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m}\rightleftharpoons\mathrm{D}(\mathrm{Gr}_{G,I}):\mathbf{Av}_*.$$

We have $\mathrm{act}_*(\mathcal{G}\boxtimes k_{\mathbb{G}_m})\simeq\mathbf{oblv}\circ\mathbf{Av}_*(\mathcal{G})$. Write T for the co-monad $\mathbf{oblv}\circ\mathbf{Av}_*$ and $\epsilon:T\to\mathbf{Id}$ for its counit. Using the base-change isomorphism, we have $T\circ T\simeq C^\bullet(\mathbb{G}_m)\otimes T$. Now consider the simplicial object that defines $e\otimes_{C^\bullet(\mathbb{G}_m)}\mathrm{act}_*(\mathcal{G}\boxtimes k_{\mathbb{G}_m})$. It follows from definition that it is isomorphic to the simplicial object

$$T(\mathcal{G})\rightleftarrows T\circ T(\mathcal{G})\quad T\circ T\circ T(\mathcal{G})\qquad\cdots,$$

where all the rightward maps are induced by the co-multiplication on T and all the leftward maps are induced by the counit of T. This simplicial object has an augmentation

$$\mathcal{G}\longleftarrow T(\mathcal{G})\rightleftarrows T\circ T(\mathcal{G})\quad T\circ T\circ T(\mathcal{G})\qquad\cdots.\tag{3.7}$$

It suffices to prove that this augmentation exhibits $\mathcal{G}$ as the geometric realization of the simplicial diagram. Since $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathbb{G}_m\text{-um}}\subset\mathrm{D}(\mathrm{Gr}_{G,I})$ is generated under colimits and shifts by the image of $\mathbf{oblv}$. It suffices to prove (3.7) is a colimit diagram for any $\mathcal{G}$ contained in the essential image of $\mathbf{oblv}$. However, in this case, this augmented simplicial diagram splits. This proves (i).

3.5.5. *Proof of Theorem 1.3.7.* By (i) and (ii), it remains to prove that for any $\mathcal{G}$ contained in $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, the natural map

$$\Psi^{\mathrm{un}}\circ\mathrm{pr}_{23,*}\circ(\mathring{\Delta})^!\circ(\mathbf{Id}\times\Gamma_I^\sigma)_*\circ\mathrm{pr}_1^!(\mathcal{G})\to\mathrm{pr}_{2,*}\circ\Delta_0^!\circ\Psi^{\mathrm{un}}\circ(\mathbf{Id}\times\Gamma_I^\sigma)_*\circ\mathrm{pr}_1^!(\mathcal{G})\tag{3.8}$$

is an isomorphism[32].

Note that it is enough to prove this for a set of compact generators $\mathcal{G}$ of $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$. Hence by Lemma 2.3.4(2) and (4), we can assume that $\mathcal{G}$ is supported on ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$ for some $\lambda \in \Lambda_{G,P}$.

We apply Corollary 3.1.2 to

- the integer $n = -1$;
- the correspondence $(U \leftarrow V \rightarrow W) :=$

$$(\mathrm{Gr}_{G,I} \times \mathbb{A}^1 \overset{\mathrm{pr}_{23}}{\leftarrow} \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1 \overset{\Delta}{\rightarrow} \mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I} \times \mathbb{A}^1),$$

  where $\mathbb{G}_m$ acts on W by $s \cdot (x,t,z,t) := (x,y,s\cdot z, s^{-1}t)$, on V by restriction, and on U by $s \cdot (z,t) := (s \cdot z, s^{-1}t)$.
- the object $\overset{\circ}{\mathcal{F}} := (\mathbf{Id} \times \Gamma_I^{\sigma})_* \circ \mathrm{pr}_1^!(\mathcal{G})$;
- the subcategory $\mathrm{D}({}_{\leq\lambda}\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \subset \mathrm{D}(\mathrm{Gr}_{G,I})$.

Axioms (P1-P3) and (Q) can be checked directly using Example 1.2.14. Axioms (G1) and (G2) follow from (i) and (ii). Axiom (C) is just Lemma 3.2.1(1). It remains to check Axiom (M).

Write $\mathcal{F} := j_*(\overset{\circ}{\mathcal{F}})$. Unwinding the definition, we only need to prove that both sides of

$$i^* \circ \mathrm{pr}_{23,*} \circ \Delta^!(\mathcal{F}) \rightarrow \mathrm{pr}_{2,*} \circ \Delta_0^! \circ i^*(\mathcal{F}) \tag{3.9}$$

are contained in the full subcategory $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, and are supported on ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$.

For the LHS of (3.9), in § 3.5.4, we proved that it is isomorphic to $\mathrm{act}_*(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})$. Since each stratum ${}_{\mu}\mathrm{Gr}_{G,I} \simeq (\mathrm{Gr}^{\mu}_{P,I})_{\mathrm{red}}$ is preserved by the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$, so is ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$. Hence $\mathrm{act}_*(\mathcal{G} \boxtimes \omega_{\mathbb{G}_m})$ is supported on ${}_{\leq\lambda}\mathrm{Gr}_{G,I}$ because $\mathcal{G}$ is so. To prove it is contained in $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, by Lemma 2.3.3, it suffices to prove that its !-pullback to $\mathrm{Gr}_{P,I}$ is contained in $\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}$. Hence it suffices to show !-pull-$*$-push along the correspondence

$$\mathrm{Gr}_{P,I} \overset{\mathrm{act}}{\leftarrow} \mathrm{Gr}_{P,I} \times \mathbb{G}_m \overset{\mathrm{pr}_1}{\rightarrow} \mathrm{Gr}_{P,I}$$

preserves the subcategory $\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \subset \mathrm{D}(\mathrm{Gr}_{P,I})$. However, this follows from Lemma 2.3.2(1) and the fact that the $\mathbb{G}_m$-action on $\mathrm{Gr}_{P,I}$ contracts it onto $\mathrm{Gr}_{M,I}$.

To prove that the RHS of (3.9) is contained in $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$, it suffices to show that

$$i^*(\mathcal{F}) \in \mathrm{D}(\mathrm{Gr}_{G,I} \times \mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I,3},$$

---

[32] Although $\Psi^{\mathrm{un}} \circ \mathrm{pr}_{23,*} \simeq \mathrm{pr}_{2,*} \circ \Psi^{\mathrm{un}}$ because pr_{23} is ind-proper, we do *not* know if the stronger claim

$$\Psi^{\mathrm{un}} \circ (\overset{\circ}{\Delta})^! \circ (\mathbf{Id} \times \Gamma_I^{\sigma})_* \circ \mathrm{pr}_1^!(\mathcal{G}) \simeq \Delta_0^! \circ \Psi^{\mathrm{un}} \circ (\mathbf{Id} \times \Gamma_I^{\sigma})_* \circ \mathrm{pr}_1^!(\mathcal{G})$$

is correct. The reason is that the support of the LHS might be the entire $\mathrm{Gr}_{G\times G,I}$ hence Axiom (M) is not satisfied (see Warning 3.2.2). We warn that this was falsely claimed in Gaitsgory's exposition [Gai20] of this work.

where 3 indicates that we are considering the $\mathcal{L}U_I$-action on the third factor. We have

$$i^*(\mathcal{F}) \simeq \mathcal{G} \boxtimes i^* \circ j_* \circ \Gamma_I^\sigma(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}).$$

Hence it suffices to prove that

$$i^* \circ j_* \circ \Gamma_I^\sigma(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \in \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I,2},$$

or equivalently

$$i^* \circ j_* \circ \Gamma_I(\omega_{\mathrm{Gr}_{G,I}\times\mathbb{G}_m}) \in \mathrm{D}(\mathrm{Gr}_{G,I} \underset{X^I}{\times} \mathrm{Gr}_{G,I})^{\mathcal{L}U_I,1}.$$

However, this is just Remark 2.4.8.

For the claim about the support of the RHS, by the base-change isomorphisms, it suffices to prove the following statement. If a stratum $\mathrm{Gr}^{\mu_1}_{P^-,I} \times_{X^I} \mathrm{Gr}^{\mu_2}_{P,I}$ has non-empty intersection with both $\sigma(\mathrm{VinGr}_{G,I}|_{C_P})$ and ${}_{\leq\lambda}\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$, then $\mu_2 \leq \lambda$. By Corollary 2.5.10, the first non-empty intersection implies $\mu_2 \leq \mu_1$. On the other hand, the second non-empty intersection implies $\mu_1 \leq \lambda$ by definition. Hence we have $\mu_2 \leq \lambda$ as desired. This finishes the proof of the theorem.

□[Theorem 1.3.7]

Remark 3.5.6. One can similarly prove the main theorem in the constructible contexts.

3.5.7. *Proof of Corollary 1.4.3.* By (3.8), we have the following natural transformation

$$\Psi^{\mathrm{un}} \circ \mathrm{pr}_{23,*} \circ (\overset{\circ}{\Delta})^! \circ (\mathbf{Id}\times\Gamma_I^\sigma)_* \circ \mathrm{pr}_1^! \to \mathrm{pr}_{2,*} \circ \Delta_0^! \circ \Psi^{\mathrm{un}} \circ (\mathbf{Id}\times\Gamma_I^\sigma)_* \circ \mathrm{pr}_1^!$$

between two functors $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \to \mathrm{D}(\mathrm{Gr}_{G,I})$. By [Fulltext, Proposition B.8.1], both sides can be upgraded to $\mathcal{L}^+M_I$-linear functors. It follows from construction that the above natural transformation is compatible with these $\mathcal{L}^+M_I$-linear structures.

It remains to prove that the isomorphisms in § 3.5.1(i) and (ii) are compatible with the $\mathcal{L}^+M_I$-linear structures. This is tautological for (ii) because both $\mathcal{L}^+M_I$-linear structures come from [Fulltext, Proposition B.8.1] (see § 2.4.7). For the isomorphism in (i), unwinding the proof in § 3.5.4, it suffices to show that (3.7) induces a diagram in $\mathrm{Funct}_{\mathcal{L}^+M_I}(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}, \mathrm{D}(\mathrm{Gr}_{G,I}))$:

$$\mathbf{oblv}^{\mathcal{L}U_I} \longleftarrow T\circ\mathbf{oblv}^{\mathcal{L}U_I} \leftleftarrows\!\!\!\rightarrows T\circ T\circ\mathbf{oblv}^{\mathcal{L}U_I} \;\substack{\leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow \\ \leftarrow}\; T\circ T\circ T\circ\mathbf{oblv}^{\mathcal{L}U_I}.$$

But this is obvious.

□[Corollary 1.4.3]

3.6. **Generalization to the (affine) flag variety.** Our main theorems (except for the local-to-global compatibility) remain valid if we replace $\mathrm{Gr}_{G,I}$ by the affine flag variety Fl_G (resp. the finite flag variety Fl_f), and correspondingly replace $\mathrm{VinGr}^\gamma_{G,I}$ by the closure of the Drinfeld-Gaitsgory interpolations. This is because in the proof of the main theorems we only use the following properties of $\mathrm{Gr}_{G,I} \to X^I$, which are all shared by $\mathrm{Fl}_G \to \mathrm{pt}$ (resp. $\mathrm{Fl}_f \to \mathrm{pt}$):

- $\mathrm{Gr}_{G,I} \to X^I$ is ind-proper;
- The attractor locus $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}}$ (resp. repeller locus $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}}$) is stabilized by $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$), and the fixed locus $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{fix}}$ is fixed by both $\mathcal{L}U_I$ and $\mathcal{L}U_I^-$;
- The fibers of the projection map $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}} \to \mathrm{Gr}_{G,I}^{\gamma,\mathrm{fix}}$ (resp. $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}} \to \mathrm{Gr}_{G,I}^{\gamma,\mathrm{fix}}$) are acted transitively by $\mathcal{L}U_I$ (resp. $\mathcal{L}U_I^-$);
- The map $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}} \times_{X^I} \mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}} \to \mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ is surjective on k-points, and its restriction to each connected component of the source is a locally closed embedding. In particular, there is a stratification on $\mathrm{Gr}_{G,I} \times_{X^I} \mathrm{Gr}_{G,I}$ labelled by the set L of the connected components of $\mathrm{Gr}_{G,I}^{\gamma,\mathrm{att}} \times_{X^I} \mathrm{Gr}_{G,I}^{\gamma,\mathrm{rep}}$.
- There exists a partial order on L such that for $\lambda, \mu \in L$, the reduced closure of the stratum labelled by λ has empty intersection with the stratum labelled by μ unless $\mu \le \lambda$.
- For any $\lambda, \mu \in L$, there are only finitely many elements between them.
- Let $L_0 \subset L$ be the subset of those strata that have non-empty intersections with $\mathrm{VinGr}_{G,I}|_{C_P}$. Then L_0 is bounded from above.

We leave the details to the curious reader.

4. Proofs - II

In this section, we prove Theorem 1.5.1. We want to apply Theorem 3.1.11 to the correspondence

$$\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \leftarrow \mathrm{VinGr}_{G,I}^{\gamma} \xrightarrow{\pi_I} \mathrm{VinBun}_G^{\gamma}. \tag{4.1}$$

The Braden 4-tuples for $\mathrm{Gr}_{G\times G,I}$ and $\mathrm{VinGr}_{G,I}$ are provided by Construction 2.5.6. The only missing ingredient is a suitable Braden 4-tuple $\mathrm{Br}_{\mathrm{glob}}^{\gamma}$ for $\mathrm{VinBun}_G^{\gamma}$, which we propose to be

$$(\mathrm{VinBun}_G^{\gamma}, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y_{\mathrm{rel}}^{P,\gamma}, H_{M,G\text{-}\mathrm{pos}}),$$

where

- ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ is the disjoint union of the *defect strata* of $\mathrm{VinBun}_G|_{C_P}$ constructed in [Sch16] (see § C.4.5);
- $Y_{\mathrm{rel}}^{P,\gamma}$ is (the relative) *Schieder's local model* for $\mathrm{VinBun}_G^{\gamma}$ constructed in [Sch16] (see § C.4.7);
- $H_{M,G\text{-}\mathrm{pos}}$ is the *G-position Hecke stack* for Bun_M studied in [BFGM02], [BG06], [Sch16] (see § C.4.4).

In § 4.1, we construct the Braden 4-tuple $\mathrm{Br}_{\mathrm{glob}}^{\gamma}$ and the morphism $\mathrm{Br}_{\mathrm{Vin},I}^{\gamma} \to \mathrm{Br}_{\mathrm{glob}}^{\gamma}$.

To prove Theorem 1.5.1, we only need to check the axioms in § 3.1.10. The first four axioms, which are geometric, are checked in § 4.1. The other axioms, which are sheaf-theoretic, are actually known results. Namely, those relevant to VinGr_G^γ and $\mathrm{Gr}_{G\times G,I}$ have been verified in § 3, while those relevant to VinBun_G^γ were either proved or sketched in [Sch16]. We review these results in § 4.2.

In § 4.3, we finish the proof of Theorem 1.5.1.

4.1. **Geometric players - III.** As usual, we fix a standard parabolic P and a co-character $\gamma:\mathbb{G}_m\to Z_M$ that is dominant and regular with respect to P. We assume the reader is familiar with the constructions in Appendix C.4.

Recall we have

$$\begin{aligned}
\mathrm{VinBun}_G^\gamma &:= \mathbf{Maps}_{\mathrm{gen}}(X, G\backslash \mathrm{Vin}_G^\gamma/G \supset G\backslash_0\mathrm{Vin}_G^\gamma/G)\\
{}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} &:= \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/P^- \supset P\backslash M/P^-)\\
Y_{\mathrm{rel}}^{P,\gamma} &:= \mathbf{Maps}_{\mathrm{gen}}(X, P^-\backslash \mathrm{Vin}_G^\gamma/P \supset P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P)\\
H_{M,G\text{-}\mathrm{pos}} &:= \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M).
\end{aligned}$$

By (C.17), we have the following commutative diagram (c.f. (2.26))

(4.2)
$$\begin{tikzcd}
 & & (M\backslash\overline{M}/M \supset M\backslash M/M) \\
 & (M\backslash\overline{M}/M \supset M\backslash M/M) \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{pair}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{pair}}"'] & (P\backslash\overline{M}/P^- \supset P\backslash M/P^-) \arrow[u, "\mathbf{q}^+_{\mathrm{pair}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{pair}}"] \\
(M\backslash\overline{M}/M \supset M\backslash M/M) & (P^-\backslash \mathrm{Vin}_G^\gamma/P \supset P^-\backslash \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P) \arrow[l, "\mathbf{q}^-_{\mathrm{pair}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{pair}}"'] & (G\backslash \mathrm{Vin}_G^\gamma/G \supset G\backslash_0\mathrm{Vin}_G^\gamma/G).
\end{tikzcd}$$

It induces a commutative diagram

(4.3)
$$\begin{tikzcd}
 & & H_{M,G\text{-}\mathrm{pos}} \\
 & H_{M,G\text{-}\mathrm{pos}} \arrow[ur, "="] \arrow[r, "\mathbf{i}^+_{\mathrm{glob}}"'] \arrow[dl, "="'] \arrow[d, "\mathbf{i}^-_{\mathrm{glob}}"'] & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[u, "\mathbf{q}^+_{\mathrm{glob}}"'] \arrow[d, "\mathbf{p}^+_{\mathrm{glob}}"] \\
H_{M,G\text{-}\mathrm{pos}} & Y_{\mathrm{rel}}^{P,\gamma} \arrow[l, "\mathbf{q}^-_{\mathrm{glob}}"] \arrow[r, "\mathbf{p}^-_{\mathrm{glob}}"'] & \mathrm{VinBun}_G^\gamma.
\end{tikzcd}$$

Proposition-Definition 4.1.1. *The above commutative square defines a Braden* 4*-tuple (see Definition 2.2.20):*

$$(\mathrm{VinBun}_G^\gamma, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y_{\mathrm{rel}}^{P,\gamma}, H_{M,G\text{-}\mathrm{pos}}),$$

such that $\mathbf{i}^-_{\mathrm{glob}}$*,* $\mathbf{p}^+_{\mathrm{glob}}$ *and* $\mathbf{q}^-_{\mathrm{glob}}$ *are ind-finite type ind-schematic.*

We call it the global Braden 4-tuple $\mathrm{Br}^\gamma_{\mathrm{glob}}$.

Proof. To show $(\mathrm{VinBun}_G^\gamma, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y_{\mathrm{rel}}^{P,\gamma}, H_{M,G\text{-}\mathrm{pos}})$ defines a Braden 4-tuple, we only need to show that the square in (4.3) is quasi-Cartesian. This follows from Lemma C.1.12(1) and the open embedding

$$\mathrm{pt}/M \to (\mathrm{pt}/P)\underset{(\mathrm{pt}/G)}{\times}(\mathrm{pt}/P).$$

The map $\mathbf{p}^+_{\mathrm{glob}}$ is ind-finite type ind-schematic because its restriction to each connected component is a schematic locally closed embedding (see [Sch16, Proposition 3.3.2(a)]). Hence $\mathbf{i}^-_{\mathrm{glob}}$ is also ind-finite type ind-schematic because the square in (4.3) is quasi-Cartesian.

It remains to show $\mathbf{q}^-_{\mathrm{glob}}$ is ind-finite type ind-schematic. We claim it is affine and of finite type. We only need to prove the similar claim for $Y^{P,\gamma} \to \mathrm{Gr}_{M,G\text{-pos}}$ (because these two retractions are equivalent in the smooth topology, see [Fulltext, Lemma C.5.5]). However, this follows from [Sch16, Lemma 6.5.6] and [DG14, Theorem 1.5.2(2)].

□[Proposition-Definition 4.1.1]

Proposition-Construction 4.1.2. *The correspondence*

$$\mathrm{Gr}_{G\times G,I}\times\mathbb{A}^1 \leftarrow \mathrm{VinGr}^{\gamma}_{G,I} \xrightarrow{\pi_I} \mathrm{VinBun}^{\gamma}_G$$

can be extended to a correspondence between Braden 4*-tuples*

$$\mathrm{Br}^{\gamma}_I \leftarrow \mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \to \mathrm{Br}^{\gamma}_{\mathrm{glob}}$$

defined over $\mathrm{Br}_{\mathrm{base}} := (\mathbb{A}^1, 0, \mathbb{A}^1, 0)$. *Moreover, this extension satisfies Axioms (P1)-(P3) and (Q) in § 3.1.10.*

Proof. The morphism $\mathrm{Br}^{\gamma}_I \leftarrow \mathrm{Br}^{\gamma}_{\mathrm{Vin},I}$ was constructed in Construction 2.5.6. The morphism $\mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \to \mathrm{Br}^{\gamma}_{\mathrm{glob}}$ is induced by the obvious morphism from the diagram (2.26) to (4.2) (see Construction C.1.7).

Axioms (P1)-(P2) follow from the calculation in Construction 2.5.6. Axiom (Q) follows from Proposition 2.5.9. It remains to verify Axiom (P3). In other words, we only need to show the commutative diagram

$$\begin{tikzcd} \mathrm{VinGr}^{\gamma,\mathrm{rep}}_{G,I} \arrow[r] \arrow[d] & \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \arrow[d] \\ Y^{P,\gamma}_{\mathrm{rel}} \arrow[r] & H_{M,G\text{-pos}} \end{tikzcd}$$

is Cartesian. Recall it is obtained by applying Construction C.1.7 to the following commutative diagram

$$\begin{tikzcd} (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \leftarrow \mathbb{A}^1) \arrow[r, "\mathbf{q}^-_{\mathrm{sect}}"] \arrow[d] & (M\backslash \overline{M}/M \leftarrow \mathrm{pt}) \arrow[d] \\ (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \supset P^-\backslash \mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G/P) \arrow[r, "\mathbf{q}^-_{\mathrm{pair}}"] & (M\backslash \overline{M}/M \supset M\backslash M/M). \end{tikzcd}$$

By Lemma C.1.14, it suffices to show the map

$$\mathbb{A}^1 \to \mathrm{pt} \underset{(M\backslash \overline{M}/M)}{\times} (P^-\backslash \mathrm{Vin}^{\gamma}_G/P)$$

is an isomorphism. Using the Cartesian diagram (C.17), the RHS is isomorphic to

$$\mathrm{pt} \underset{(M\backslash M/M)}{\times} (P^-\backslash \mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G/P).$$

Then we are done by the $(M \times M)$-equivariant isomorphism (C.16).

$\square$Proposition-Construction 4.1.2

4.2. **Input from [Sch16].** We need some sheaf-theoretic results on VinBun_G and its relative local models. They were implicit (but without proofs) in [Sch16]. For completeness, we provide proofs for them.

Recall the $\mathbb{G}_m$-locus of VinBun_G^γ is given by $\mathrm{Bun}_G \times \mathbb{G}_m$. In this subsection, we write ω for $\omega_{\mathrm{Bun}_G \times \mathbb{G}_m}$.

Lemma 4.2.1. *The object* $\mathbf{p}^{+,!}_{\mathrm{glob}} \circ i^* \circ j_*(\omega)$ *is contained in the essential image of* $\mathbf{q}^{+,!}_{\mathrm{glob}}$.

Remark 4.2.2. This lemma is a corollary of (the Verdier dual of) [Sch16, Theorem 4.3.1]. However, the proof of [Sch16, Theorem 4.3.1] implicitly used (the Verdier dual of) this lemma. Namely, what S. Schieder called the *interplay principle* only proved his theorem up to a possible twist by local systems pulled back from $\mathrm{Bun}_{P\times P^-}$, and one needs the above lemma to rule out such twists[33].

For the mixed sheaf context as in [Sch16], thanks to the sheaf-function-correspondence, the lemma can be easily proved by showing that the stalks are constant along $\mathbf{q}^+_{\mathrm{glob}}$ (a similar argument can be found in [BG02, Subsection 6.3]). However, in the D-module context, one needs more work. We prove it in [Fulltext, Appendix E].

Corollary 4.2.3. *Consider the correspondence*

$$\mathrm{Gr}_{G\times G,I} \overset{(\iota_I)_0}{\leftarrow} \mathrm{VinGr}_{G,I}|_{C_P} \overset{(\pi_I)_0}{\rightarrow} \mathrm{VinBun}_G|_{C_P}.$$

We have

$$(\iota_I)_{0,*} \circ (\pi_I)_0^! \circ i^* \circ j_*(\omega) \in \mathrm{D}({}_{\mathrm{diff}\le 0}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}.$$

Proof. By Corollary 2.5.11, this object is indeed supported on ${}_{\mathrm{diff}\le 0}\,\mathrm{Gr}_{G\times G,I}$. It remains to show it is contained in $\mathrm{D}(\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I}$.

By Lemma 2.3.3, it suffices to show the !-pullback of the desired object along $\mathrm{Gr}_{P\times P^-,I} \to \mathrm{Gr}_{G\times G,I}$ is contained in $\mathrm{D}(\mathrm{Gr}_{P\times P^-,I})^{\mathcal{L}(U\times U^-)_I}$. Let $\mathcal{G}$ be this !-pullback. By Proposition 4.1.2, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_{M\times M,I} & \mathrm{VinGr}^{\gamma,\mathrm{fix}}_{G,I} \arrow[l] \arrow[r] & H_{M,G\text{-pos}} \\
\mathrm{Gr}_{P\times P^-,I} \arrow[u] \arrow[d] & \mathrm{VinGr}^{\gamma,\mathrm{att}}_{G,I} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[u] \arrow[d] \\
\mathrm{Gr}_{G\times G,I} & \mathrm{VinGr}_{G,I}|_{C_P} \arrow[l] \arrow[r] & \mathrm{VinBun}_G|_{C_P}.
\end{tikzcd}$$

[33] See [BG06, proof of Proposition 4.4] for an analog of this logic for the interplay principle between the Zastava spaces and $\overline{\mathrm{Bun}}_B$.

The bottom left square is Cartesian by the calculations in Construction 2.5.6, the bottom right square is Cartesian by Proposition 2.5.9, and the top left square is Cartesian by Proposition 2.5.12. By the base-change isomorphisms and Lemma 4.2.1, $\mathcal{G}$ is contained in the essential image of the !-pullback functor $\mathrm{D}(\mathrm{Gr}_{M\times M,I}) \to \mathrm{D}(\mathrm{Gr}_{P\times P^-,I})$. Then we are done by Lemma 2.3.2(1).

$\square$[Corollary 4.2.3]

Lemma 4.2.4. *(1) The global Braden 4-tuple* $\mathrm{Br}^{\gamma}_{\mathrm{glob}}$ *is* $*$*-nice for* $j_*(\omega)$ *(see Definition 2.2.25).*

(2) The 0*-fiber of* $\mathrm{Br}^{\gamma}_{\mathrm{glob}}$*:*

$$(\mathrm{Br}^{\gamma}_{\mathrm{glob}})_0 := (\mathrm{VinBun}_G|_{C_P}, {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, Y^{P,\gamma}_{\mathrm{rel}}|_{C_P}, H_{M,G\text{-}\mathrm{pos}})$$

is $*$*-nice for* $i^* \circ j_*(\omega)$*.*

Proof. We only prove (1). The proof of (2) is similar.

We first show that the retraction $(Y^{P,\gamma}_{\mathrm{rel}}, H_{M,G\text{-}\mathrm{pos}})$ is both $*$-nice and !-nice for $\mathfrak{p}^{-,!}_{\mathrm{glob}} \circ j_*(\omega)$. We only need to prove the similar claim for $(Y^{P,\gamma}, \mathrm{Gr}_{M,G\text{-}\mathrm{pos}})$ (because these two retractions are equivalent in the smooth topology, see [Fulltext, Lemma C.5.5]). However, this follows from [Sch16, Lemma 6.5.6] and the contraction principle.

Note that the retraction $({}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}, H_{M,G\text{-}\mathrm{pos}})$ is both $*$-nice and !-nice for $\mathfrak{p}^{+,*}_{\mathrm{glob}} \circ j_*(\omega)$ by the *stacky* contraction principle in [DG15]. Indeed, there is an $\mathbb{A}^1$-action on $\mathrm{Bun}_P \times \mathrm{Bun}_{P^-}$ that contracts it onto $\mathrm{Bun}_M \times \mathrm{Bun}_M$ in the sense of [*loc.cit.*, § C.5]. Hence by change of the base, there is an $\mathbb{A}^1$-action on ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$ that contracts it onto $H_{M,G\text{-}\mathrm{pos}}$.

It remains to show the quasi-Cartesian square in $\mathrm{Br}^{\gamma}_{\mathrm{glob}}$ is nice for $j_*(\omega)$. This can be proved by using the framework in [Dri13, Appendix C]. See [Che, Theorem 6.1.3] for a similar result for the quasi-Cartesian square

$$\begin{tikzcd} H_{M,G\text{-}\mathrm{pos}} \arrow[r] \arrow[d] & {}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P} \arrow[d] \\ Y^P_{\mathrm{rel}} \arrow[r] & \mathrm{VinBun}_{G,\geq C_P}. \end{tikzcd}$$

(The proof there also works for the γ-version.)

$\square$[Lemma 4.2.4]

4.3. **Proof of Theorem 1.5.1.** We apply Theorem 3.1.11 to

- the correspondence $\mathrm{Gr}_{G\times G,I} \times \mathbb{A}^1 \leftarrow \mathrm{VinGr}^{\gamma}_{G,I} \xrightarrow{\pi_I} \mathrm{VinBun}^{\gamma}_G$;
- the object $\overset{\circ}{\mathcal{F}} := \omega_{\mathrm{Bun}_G \times \mathbb{G}_m}$;
- the correspondence between Braden 4-tuples $\mathrm{Br}^{\gamma}_I \leftarrow \mathrm{Br}^{\gamma}_{\mathrm{Vin},I} \to \mathrm{Br}^{\gamma}_{\mathrm{glob}}$ defined in Proposition-Construction 4.1.2;
- the subcategory $\mathrm{D}({}_{\mathrm{diff}\leq 0}\mathrm{Gr}_{G\times G,I})^{\mathcal{L}(U\times U^-)_I} \subset \mathrm{D}(\mathrm{Gr}_{G\times G,I})$.

The Axioms (P1)-(P3) and (Q) are verified in Proposition-Construction 4.1.2. Axioms (G1) and (G2) are obvious because $\overset{\circ}{\mathcal{F}}$ is regular ind-holonomic. Axiom (C) is just Lemma 3.2.1(2). Axiom (M) is just Corollary 4.2.3 and Lemma 3.2.3. Axioms (N1) and (N3) are just Lemma 4.2.4. Axioms (N2), (N4) follow from Braden's theorem and the contraction principle. □[Theorem 1.5.1]

Appendix C. Geometric miscellanea

C.1. **Mapping stacks.** In this appendix, we recall the notion of mapping stacks (and its variants) and prove some results about them.

Definition C.1.1. *Let Y be an algebraic stack (see Convension 0.6.3). We write $\mathbf{Maps}(X,Y)$ for the prestack classifying maps $X \to Y$.*

Let $V \subset Y$ be an open embedding. We write $\mathbf{Maps}_{\mathrm{gen}}(X, Y \supset V)$ for the prestack whose value on an affine test scheme S is the groupoid of maps $\alpha : X \times S \to Y$ such that the open subscheme $\alpha^{-1}(V)$ has non-empty intersections with any geometric fiber of $X \times S \to S$. Note that there is an open embedding

$$\mathbf{Maps}_{\mathrm{gen}}(X, Y \supset V) \to \mathbf{Maps}(X,Y)$$

because X is projective.

Example C.1.2. If Y is a finite type affine scheme, then $\mathbf{Maps}(X,Y) \simeq Y$.

Definition C.1.3. *Let B be a finite type affine scheme and $Y \xrightarrow{p} B$ be an algebraic stack over it. Let $f : B \to Y$ be a section of p. Let I be a non-empty finite set. We write $\mathbf{Maps}_{I,/B}(X, Y \xleftarrow{f} B)$ for the prestack whose value on an affine test scheme S is the groupoid classifying:*

1. *maps $x_i : S \to X$ labelled by I,*
2. *a commutative diagram*

$$\begin{tikzcd} (X \times S) - \cup \Gamma_{x_i} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\subset"] & S \arrow[r, "\beta"] & B \arrow[d, "f"] \\ X \times S \arrow[rr, "\alpha"] & & Y. \end{tikzcd}$$

Note that $\mathbf{Maps}_{I,/B}(X, Y \xleftarrow{f} B)$ is defined over $X^I \times B$. Using Noetherian reduction, it is easy to see it is a lft prestack.

Example C.1.4. We have $\mathrm{Gr}_{G,I} \simeq \mathbf{Maps}_{I,/\mathrm{pt}}(X, \mathrm{pt}/G \leftarrow \mathrm{pt})$.

Lemma C.1.5. *Let (B, Y, p, f) be as in Definition C.1.3. Let A be any finite type affine scheme. We have a canonical isomorphism*

$$\mathbf{Maps}_{I,/A \times B}(X, A \times Y \xleftarrow{\mathrm{Id} \times f} A \times B) \simeq A \times \mathbf{Maps}_{I,/B}(X, Y \xleftarrow{f} B).$$

Proof. Follows from Example C.1.2. □[Lemma C.1.5]

Remark C.1.6. In Definition C.1.3, for fixed $\alpha : X \times S \to Y$, the desired map $\beta : S \to B$ is unique if it exists. Indeed, the map $p\circ\alpha : X \times S \to B$ must factor through a map $\beta' : S \to B$ because of Example C.1.2. Then the commutative diagram (2) forces $\beta = \beta'$.

Construction C.1.7. *Let $(B, Y \supset V, p, f)$ be a 4-tuple such that $Y \supset V$ is as in Definition C.1.1 and (B, Y, p, f) is as in Definition C.1.3. Suppose the section $f : B \to Y$ factors through U, then there is a natural map*

$$\mathbf{Maps}_{I,/B}(X, Y \overset{f}{\leftarrow} B) \to \mathbf{Maps}_{\mathrm{gen}}(X, Y \supset V).$$

Lemma C.1.8. *Let B be a finite type affine scheme and $g : Y_1 \hookrightarrow Y_2$ be a schematic closed embedding between algebraic stacks over B. Let $f_1 : B \to Y_1$ be a section of $Y_1 \to B$. Let $f_2 : B \to Y_2$ be the section of $Y_2 \to B$ induced by f_1. Then we have an isomorphism:*

$$\mathbf{Maps}_{I,/B}(X, Y_1 \overset{f_1}{\leftarrow} B) \simeq \mathbf{Maps}_{I,/B}(X, Y_2 \overset{f_2}{\leftarrow} B).$$

Proof. Let S be any finite type affine scheme. Let $x_i : S \to X$, $\alpha : X \times S \to Y_2$ and $\beta : S \to B$ be as in Definition C.1.3. By Lemma C.1.9 below, the schema-theoretic closure of $(X \times S) - \cup\Gamma_{x_i}$ inside $X \times S$ is $X \times S$. Therefore the commutative diagram in Definition C.1.3(2) forces α to factor through $Y_1 \hookrightarrow Y_2$. Then we are done because such a factorization is unique. □[Lemma C.1.8]

Lemma C.1.9. *Let S be a finite type affine scheme and $x_i : S \to X$ be maps labelled by a finite set I. Let $\Gamma_{x_i} \hookrightarrow X \times S$ be the graph of x_i. Then the schema-theoretic closure of $(X \times S) - \cup\Gamma_{x_i}$ inside $X \times S$ is $X \times S$.*

Proof. This lemma is well-known. For the reader's convenience, we provide a proof here[45]. Let Γ be the schema-theoretic sum of the graphs of the maps x_i. Then $\Gamma \hookrightarrow X \times S$ is a relative effective Cartier divisor for $X \times S \to S$. Write $U_x : (X \times S) - \Gamma$. Let $\iota : U_x \to X \times S$ be the open embedding. We only need to show $\mathcal{O}_{X\times S} \to \iota_*(\mathcal{O}_U)$ is an injection. Note that the set-theoretic support of the kernel of this map is contained in Γ. Hence we are done by Lemma C.1.10 below. □[Lemma C.1.9]

Lemma C.1.10. *Let Y be any Noetherian scheme and $D \hookrightarrow Y$ be an effective Cartier divisor. Let $\mathcal{M}$ be a flat coherent $\mathcal{O}_Y$-module and $\mathcal{N}$ be a sub-module of it. Suppose the set-theoretic support of $\mathcal{N}$ is contained in D, then $\mathcal{N} = 0$.*

Proof. Let $\mathcal{I}$ be the sheaf of ideals for D. By assumption, it is invertible. Since Y is Noetherian, $\mathcal{N}$ is also a coherent $\mathcal{O}_Y$-module. Hence by assumption, there exists a positive integer n such that the map $\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} \to \mathcal{N}$ is

[45]We learn the proof below from Ziquan Yang.

zero. Consider the commutative square

$$\begin{tikzcd} \mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\ \mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{M} \arrow[r] & \mathcal{M}. \end{tikzcd}$$

The right vertical map is injective by assumption. Hence the left vertical map is injective because $\mathcal{I}^n$ is $\mathcal{O}_Y$-flat. The bottom map is injective because $\mathcal{M}$ is $\mathcal{O}_Y$-flat. Hence we see the top map is also injective. This forces $\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{N} = 0$. Then we are done because $\mathcal{I}^n$ is invertible.

□[Lemma C.1.10]

C.1.11. *Cartesian squares.* The following three lemmas can be proved by unwinding the definitions. We leave the details to the reader.

Lemma C.1.12. *Suppose we are given the following commutative diagram of open embeddings between algebraic stacks:*

(C.1)
$$\begin{tikzcd} (Y_1 \supset V_1) \arrow[r] \arrow[d] & (Y_2 \supset V_2) \arrow[d] \\ (Y_3 \supset V_3) \arrow[r] & (Y_4 \supset V_4). \end{tikzcd}$$

(1) If the commutative square formed by Y_i is strictly *quasi-Cartesian (see Definition 2.2.12), then $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ sends (C.1) to a strictly quasi-Cartesian square.*

(2) If the two commutative squares formed respectively by Y_i and V_i are both Cartesian, then $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ sends (C.1) to a Cartesian square.

Lemma C.1.13. *Let* **Sect** *be the category of 4-tuples (B,Y,p,f) as in Definition C.1.3. Then the functor*

$$\mathbf{Sect} \to \mathrm{PreStk}_{\mathrm{lft}},\ (B,Y,p,f) \mapsto \mathbf{Maps}_{I,/B}(X, Y \xleftarrow{f} B)$$

commutes with fiber products.

Lemma C.1.14. *Let*

$$(B_1, Y_1 \supset V_1, p_1, f_1) \to (B_2, Y_2 \supset V_2, p_2, f_2)$$

be a morphism between two 4-tuples satisfy the conditions in Construction C.1.7. Suppose the natural map $B_1 \to B_2 \times_{Y_2} Y_1$ is an isomorphism. Then the natural commutative square

$$\begin{tikzcd} \mathbf{Maps}_{I,/B_1}(X, Y_1 \xleftarrow{f_1} B_1) \arrow[r] \arrow[d] & \mathbf{Maps}_{\mathrm{gen}}(X, Y_1 \supset V_1) \arrow[d] \\ \mathbf{Maps}_{I,/B_2}(X, Y_2 \xleftarrow{f_2} B_2) \arrow[r] & \mathbf{Maps}_{\mathrm{gen}}(X, Y_2 \supset V_2), \end{tikzcd}$$

is Cartesian.

C.4. **The geometric objects in [Sch16]: Constructions.** In this appendix, we review some geometric constructions in [Sch16]. We personally think some proofs in [Sch16] are too concise. Hence we provide details to them in [Fulltext, Appendix C.5].

C.4.1. *The degeneration* Vin_G^γ. Throughout this appendix, we fix a standard parabolic subgroup P and a co-character $\gamma : \mathbb{G}_m \to Z_M$ as in Construction 1.2.10. Recall the homomorphism $\overline{\gamma} : \mathbb{A}^1 \to T_{\mathrm{ad}}^+$ between semi-groups. Consider the fiber product $\mathrm{Vin}_G^\gamma := \mathrm{Vin}_G \times_{T_{\mathrm{ad}}^+} \mathbb{A}^1$. By construction Vin_G^γ is an algebraic monoid, and we have monoid homomophisms

$$\mathbb{A}^1 \xrightarrow{\mathfrak{s}^\gamma} \mathrm{Vin}_G^\gamma \to \mathbb{A}^1.$$

C.4.2. *The monoid* $\overline{M}$. The unproven claims in this sub-subsection can be found in [Sch16, § 3.1] and [Wan17].

Consider the closed embedding $M \simeq P/U \hookrightarrow G/U$. It is well-known that G/U is strongly quasi-affine (see e.g. [BG02, Theorem 1.1.2]). Let $\overline{M}$ be the closure of M inside $\overline{G/U}$. [Wan17, § 3] shows that $\overline{M}$ is normal and the group structure on M extends uniquely to a monoid structure on $\overline{M}$ such that its open subgroup of invertible elements is isomorphic to M.

On the other hand, by [Wan17, Theorem 4.1.4], the closed embedding

$$G/U \simeq (G/U \times P/U^-)/M \hookrightarrow (G/U \times G/U^-)/M \simeq {}_0\mathrm{Vin}_G|_{C_P}$$

extends uniquely to a closed embedding $\overline{G/U} \hookrightarrow \mathrm{Vin}_G|_{C_P}$. Hence the closed embedding[54]

$$M \to (G/U \times G/U^-)/M \simeq {}_0\mathrm{Vin}_G|_{C_P}\ m \mapsto (m,1)$$

extends uniquely to a closed embedding $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$. Moreover, $\overline{M}$ is also isomorphic to the closure of M inside $\mathrm{Vin}_G|_{C_P}$. By construction, $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$ is stabilized by the $(P \times P^-)$-action and fixed by the $(U \times U^-)$-action. Hence we have a commutative square of schemes acted on by $(P \times P^-)$:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\ {}_0\mathrm{Vin}_G|_{C_P} \arrow[r] & \mathrm{Vin}_G|_{C_P}. \end{tikzcd} \tag{C.8}$$

Note that this square is Cartesian because $M \hookrightarrow {}_0\mathrm{Vin}_G|_{C_P}$ is already a closed embedding.

54 Note that the image of $(m,1)$ and $(1,m^{-1})$ in $(G/U \times G/U^-)/M$ are equal.

C.4.3. *The monoid $\overline{A_M}$.* The unproven claims in this sub-subsection can be found in [Sch16, § 3.1.7].

Consider the abelianization[55] $A_M := M/[M,M] \simeq P/[P,P]$. It can be embedded into $G/[P,P]$ (which is strongly quasi-affine). Its closure $\overline{A_M}$ inside the affine closure $\overline{G/[P,P]}$ is known to be normal. The commutative group structure on A_M extends to a commutative monoid structure on $\overline{A_M}$ whose open subgroup of invertible elements is A_M.

The projection $M \twoheadrightarrow M/[M,M]$ induces a map $\overline{M} \to \overline{A_M}$, which is $(M\times M)$-equivariant by construction. Hence we have the following commutative diagram of schemes acted on by $(M\times M)$:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\ A_M \arrow[r] & \overline{A_M}, \end{tikzcd} \tag{C.9}$$

which is *Cartesian* by [Fulltext, Lemma C.5.1].

C.4.4. *The stack $H_{M,G\text{-pos}}$.* The unproven claims in this sub-subsection can be found in [Sch16, § 3.1.5] and [Wan18, Appendix A].

Recall that X^{pos} is defined as the disjoint union of X^θ for $\theta \in \Lambda_{G,P}^{\mathrm{pos}}$. By [Sch16, § 3.1.7], we have

$$X^{\mathrm{pos}} \simeq \mathbf{Maps}_{\mathrm{gen}}(X, A_M\backslash\overline{A_M} \supset A_M\backslash A_M),$$

where A_M acts on $\overline{A_M}$ via multiplication. Under this isomorphism, the addition map $X^{\mathrm{pos}}\times X^{\mathrm{pos}} \to X^{\mathrm{pos}}$ is induced by the *commutative* monoid structure on $\overline{A_M}$.

The *G-positive affine Grassmannian* is defined as (see § C.4.2 for the definition of $\overline{M}$)

$$\mathrm{Gr}_{M,G\text{-pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, \overline{M}/M \supset M/M),$$

where M acts on $\overline{M}$ by right multiplication. The map $\overline{M}/M \to \mathrm{pt}/M$ induces a map $\mathrm{Gr}_{M,G\text{-pos}} \to \mathrm{Bun}_M$.

By (C.9), the composition

$$\overline{M}/M \to \overline{A_M}/A_M \simeq A_M\backslash\overline{A_M} \tag{C.10}$$

sends M/M into $A_M\backslash A_M$. Hence we have a projection $\mathrm{Gr}_{M,G\text{-pos}} \to X^{\mathrm{pos}}$. We define[56]

$$\mathrm{Gr}^\theta_{M,G\text{-pos}} := \mathrm{Gr}_{M,G\text{-pos}} \underset{X^{\mathrm{pos}}}{\times} X^\theta.$$

[55] [Sch16] denoted it by T_M. We use the notation A_M to avoid confusions with the Cartan subgroup of M.

[56] Note that the last map in the composition (C.10) is induced by the group homomorphism $A_M \to A_M$, $t \mapsto t^{-1}$. Hence $\mathrm{Gr}^\theta_{M,G\text{-pos}}$ lives over $\mathrm{Bun}_M^{-\theta}$, which is compatible with the conventions in the literature.

By [Wan18, § 5.7], the definition above coincides with the definition in [BFGM02, Sub-section 1.8]. In particular, $\mathrm{Gr}^{\theta}_{M,G\text{-}\mathrm{pos}}$ is represented by a scheme of finite type.

The *G-positive Hecke stack* is defined as

$$H_{M,G\text{-}\mathrm{pos}} := \mathbf{Maps}_{\mathrm{gen}}(X, M\backslash\overline{M}/M \supset M\backslash M/M). \tag{C.11}$$

As before, we have a projection $H_{M,G\text{-}\mathrm{pos}} \to X^{\mathrm{pos}}$ induced by the composition

$$M\backslash\overline{M}/M \to A_M\backslash\overline{A_M}/A_M \to A_M\backslash\overline{A_M},$$

where the last map is induced by the group morphism

$$A_M \times A_M \to A_M,\ (s,t) \mapsto st^{-1}.$$

The base-change of this map to X^{θ} is denoted by $H^{\theta}_{M,G\text{-}\mathrm{pos}}$.

The map $M\backslash\overline{M}/M \to M\backslash\mathrm{pt}/M$ induces a map

$$\overleftarrow{\mathfrak{h}} \times \overrightarrow{\mathfrak{h}} : H_{M,G\text{-}\mathrm{pos}} \to \mathrm{Bun}_M \times \mathrm{Bun}_M\,.$$

Hence we obtain a disjoint union decomposition[57]

$$H_{M,G\text{-}\mathrm{pos}} = \coprod_{\theta\in\Lambda^{\mathrm{pos}}_{G,P}} H^{\theta}_{M,G\text{-}\mathrm{pos}} = \coprod_{\theta\in\Lambda^{\mathrm{pos}}_{G,P}} \coprod_{\lambda_1-\lambda_2=\theta} H^{\lambda_1,\lambda_2}_{M,G\text{-}\mathrm{pos}} \tag{C.12}$$

where for $\lambda_1,\lambda_2 \in \Lambda_{G,P}$, $H^{\lambda_1,\lambda_2}_{M,G\text{-}\mathrm{pos}}$ lives over the connected component $\mathrm{Bun}^{\lambda_1}_M \times \mathrm{Bun}^{\lambda_2}_M$.

Note that the fiber of $\overleftarrow{\mathfrak{h}}$ at the point $\mathcal{F}^{\mathrm{triv}}_M$ of Bun_M is $\mathrm{Gr}_{M,G\text{-}\mathrm{pos}}$.

C.4.5. *The stack* ${}_{\mathrm{str}}\mathrm{VinBun}_G|_{C_P}$. The unproven claims in this sub-subsection can be found in [Sch16, § 3.2].

The *defect stratification* on $\mathrm{VinBun}_G|_{C_P}$ is a stratification labelled by $\Lambda^{\mathrm{pos}}_{G,P}$. For $\theta \in \Lambda^{\mathrm{pos}}_{G,P}$, the corresponding stratum is

$${}_{\theta}\,\mathrm{VinBun}_G|_{C_P} \simeq (\mathrm{Bun}_{P\times P^-}) \underset{\mathrm{Bun}_{M\times M}}{\times} H^{\theta}_{M,G\text{-}\mathrm{pos}}. \tag{C.13}$$

We write ${}_{\mathrm{str}}\,\mathrm{VinBun}_G|_{C_P}$ for the disjoint union of all the defect strata. By Lemma C.1.12(2), we have

$$\begin{aligned} {}_{\mathrm{str}}\,\mathrm{VinBun}_G|_{C_P} &\simeq \mathrm{Bun}_{P\times P^-} \underset{\mathrm{Bun}_{M\times M}}{\times} H_{M,G\text{-}\mathrm{pos}} \simeq \\ &\simeq \mathbf{Maps}_{\mathrm{gen}}(X, P\backslash\overline{M}/P^- \supset P\backslash M/P^-). \end{aligned} \tag{C.14}$$

Recall we have a $(P\times P^-)$-equivariant closed embedding (see C.4.2) $\overline{M} \hookrightarrow \mathrm{Vin}_G|_{C_P}$, which sends M into ${}_0\mathrm{Vin}_G|_{C_P}$. Hence we obtain a map

$$(P\backslash\overline{M}/P^- \supset P\backslash M/P^-) \to (G\backslash\,\mathrm{Vin}_G|_{C_P}/G \supset G\backslash{}_0\mathrm{Vin}_G|_{C_P}/G).$$

Applying $\mathbf{Maps}_{\mathrm{gen}}(X,-)$ to it, we obtain a map

$${}_{\mathrm{str}}\,\mathrm{VinBun}_G|_{C_P} \to \mathrm{VinBun}_G|_{C_P}$$

[57]Our labels λ_1,λ_2 below are in the opposite order against that in [Sch16] because of Warning 1.2.4. Our order is compatible with [Wan18, § 5.3].

By [Sch16, Proposition 3.2.2], the connected components of the source provide a stratification for $\mathrm{VinBun}_G|_{C_P}$.

C.4.6. *The open Bruhat cell* $\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}$. Consider the $(P^-\times P)$-action on Vin_G^γ induced from the $(G\times G)$-action on Vin_G. Also consider the canonical section (see § 1.2.2) $\mathfrak{s}^\gamma:\mathbb{A}^1\to\mathrm{Vin}_{G,\gamma}$. By [Fulltext, Lemma C.5.2], the stabilizer subgroup of this section is given by

$$M\times\mathbb{A}^1\hookrightarrow P^-\times P\times\mathbb{A}^1,\ (m,t)\mapsto(m,m,t). \tag{C.15}$$

Hence we obtain a locally closed embedding $(P^-\times P)/M\times\mathbb{A}^1\hookrightarrow\mathrm{Vin}_G^\gamma$. By the dimension reason, this is an open embedding. We define the corresponding open subscheme of Vin_G^γ to be the *open Bruhat cell* $\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}$. It is contained in the defect-free locus of Vin_G^γ by § 1.2.2.

Consider the composition $(P^-\times P)/M\twoheadrightarrow(M\times M)/M\simeq M$, where the last map is given by $(a,b)\mapsto ab^{-1}$. It induces an $(M\times M)$-equivariant isomorphism

$$U^-\backslash\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/U\simeq M\times\mathbb{A}^1. \tag{C.16}$$

In particular, there is a $(P^-\times P)$-equivariant map $\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}\to M$. By [Fulltext, Lemma C.5.3], it can be extended to a map $\mathrm{Vin}_G^\gamma\to\overline{M}$ fitting into the following *Cartesian* square of schemes acted on by $(P^-\times P)$:

$$\begin{tikzcd} \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}} \arrow[r] \arrow[d] & \mathrm{Vin}_G^\gamma \arrow[d] \\ M \arrow[r] & \overline{M}. \end{tikzcd} \tag{C.17}$$

Moreover, the composition $\overline{M}\hookrightarrow\mathrm{Vin}_G|_{C_P}\hookrightarrow\mathrm{Vin}_G^\gamma\to\overline{M}$ is the identity map since its restriction on M is so.

Combining the Cartesian squares (C.18) and (C.17), we obtain a *Cartesian* square of schemes acted on by $(P^-\times P)$:

$$\begin{tikzcd} \mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}} \arrow[r] \arrow[d] & \mathrm{Vin}_G^\gamma \arrow[d] \\ A_M \arrow[r] & \overline{A_M}. \end{tikzcd} \tag{C.18}$$

C.4.7. *Schieder's local models.* (c.f. [Sch16, § 6.1.6])

[Sch16] constructed what known as *Schieder's local models* for VinBun_G, which model the singularities of VinBun_G in the same sense as how the parabolic Zastava spaces model the Drinfeld compactifications $\widetilde{\mathrm{Bun}}_P$ in [BFGM02].

The *absolute local model* is defined as

$$Y^{P,\gamma}:=\mathbf{Maps}_{\mathrm{gen}}(X,U^-\backslash\mathrm{Vin}_G^\gamma/P\supset U^-\backslash\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P).$$

The *relative local model* is defined as

$$Y^{P,\gamma}_{\mathrm{rel}}:=\mathbf{Maps}_{\mathrm{gen}}(X,P^-\backslash\mathrm{Vin}_G^\gamma/P\supset P^-\backslash\mathrm{Vin}_G^{\gamma,\mathrm{Bruhat}}/P). \tag{C.19}$$

We similarly define the defect-free locus ${}_0Y^{P,\gamma}$ and ${}_0Y^{P,\gamma}_{\mathrm{rel}}$. It is known that each connected component of ${}_0Y^{P,\gamma}$ is a finite type scheme.

Consider the isomorphism.

$$P^-\backslash \mathrm{Vin}^{\gamma}_G/P \simeq (P^-\backslash \mathrm{pt}/P) \underset{(G\backslash \mathrm{pt}/G)}{\times} (G\backslash \mathrm{Vin}^{\gamma}_G/G).$$

Since $\mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G$ is an open subscheme of ${}_0\mathrm{Vin}^{\gamma}_G$, by Lemma C.1.12(1), we obtain an open embedding

$$Y^{P,\gamma}_{\mathrm{rel}} \to \mathrm{VinBun}^{\gamma}_G \underset{\mathrm{Bun}_{G\times G}}{\times} \mathrm{Bun}_{P^-\times P}. \tag{C.20}$$

In particular, there is a *local-model-to-global* map

$$\mathbf{p}^-_{\mathrm{glob}} : Y^{P,\gamma}_{\mathrm{rel}} \to \mathrm{VinBun}^{\gamma}_G,$$

induced by the morphism

$$\mathbf{p}^-_{\mathrm{pair}} : (P^-\backslash \mathrm{Vin}^{\gamma}_G/P \supset P^-\backslash \mathrm{Vin}^{\gamma,\mathrm{Bruhat}}_G/P) \to (G\backslash \mathrm{Vin}^{\gamma}_G/G \supset G\backslash {}_0\mathrm{Vin}^{\gamma}_G/G).$$

Appendix D. Compact generation of $\mathrm{D}(\mathrm{Gr}_G)^{\mathcal{L}U}$ and $\mathrm{D}(\mathrm{Gr}_G)_{\mathcal{L}U}$

The goal of this appendix is to prove Lemma 2.3.4 and Lemma 2.3.5. The proofs below are suggested by D. Gaitsgory.

D.1. Parameterized Braden's theorem.

We need a parameterized version of Braden's theorem. We start with an auxiliary lemma

Lemma-Definition D.1.1. *Let Z be an ind-finite type indscheme equipped with a $\mathbb{G}_m$-action, and $\mathcal{D}$ be any DG category. Then the obvious functor*

$$\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \otimes \mathcal{D} \to \mathrm{D}(Z) \otimes \mathcal{D}$$

is fully faithful.

We define $(\mathrm{D}(Z) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}$ to be the essential image of the above functor.

Proof. It suffices to show that the fully faithful functor $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \to \mathrm{D}(Z)$ has a continuous right adjoint. Recall that both $\mathrm{D}(Z)^{\mathbb{G}_m} \simeq \mathrm{D}(Z/\mathbb{G}_m)$ and $\mathrm{D}(Z)$ are compactly generated, and the functor $\mathbf{oblv}^{\mathbb{G}_m}$ between them sends compact objects to compact objects. This formally implies that $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}}$ is compactly generated and the functor $\mathrm{D}(Z)^{\mathbb{G}_m\text{-um}} \to \mathrm{D}(Z)$ sends compact objects to compact objects. In particular, this functor has a continuous right adjoint.

□[Lemma-Construction D.1.1]

D.1.2. *Parameterized Braden's theorem.* Let Z and $\mathcal{D}$ be as in Lemma-Definition D.1.1. Consider the functor

$$\mathrm{D}(Z^{\mathrm{fix}})\otimes\mathcal{D} \xrightarrow{q^{-,!}\otimes\mathbf{Id}} \mathrm{D}(Z^{\mathrm{rep}})\otimes\mathcal{D} \xrightarrow{p^-_*\otimes\mathbf{Id}} \mathrm{D}(Z)\otimes\mathcal{D}.$$

By definition, its image is contained in the full subcategory $(\mathrm{D}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}$. Therefore we obtain a functor

$$(p^-_*\circ q^{-,!})\otimes\mathbf{Id}:\mathrm{D}(Z^{\mathrm{fix}})\otimes\mathcal{D}\to(\mathrm{D}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}.$$

Remark 2.2.19 implies

Theorem D.1.3. *(Parameterized Braden's theorem) There is a canonical adjoint pair*

$$(q^+_*\circ p^{+,!})\otimes\mathbf{Id}:(\mathrm{D}(Z)\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}\rightleftharpoons\mathrm{D}(Z^{\mathrm{fix}})\otimes\mathcal{D}:(p^-_*\circ q^{-,!})\otimes\mathbf{Id}.$$

Remark D.1.4. There is also a parameterized version of the contraction principle. We do not use it in this paper.

D.2. **Parameterized version of Lemma 2.3.4.** In this subsection. We prove a parameterized version of Lemma 2.3.4. We need the addition parameter to help us to deal with the coinvariants category latter.

Lemma D.2.1. *Let $\mathcal{D}$ be any DG category.*

(0) We have a canonical equivalence

$$\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\simeq(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

(1) We have[61]

$$(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\subset(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}\subset\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D}.$$

(2) The composition

$$\mathrm{D}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}\xrightarrow{\mathbf{s}_{I,*}\otimes\mathbf{Id}}\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D}\xrightarrow{\mathbf{Av}_!^{\mathcal{L}U_I}}(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\tag{D.1}$$

is well-defined, and the image of it generates $(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$ under colimits and shifts. Moreover, the left-lax $\mathrm{D}(X^I)$-linear structure on this functor is strict.

(3) The functor

$$(\mathbf{p}^+_{I,*}\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\to(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

has a left adjoint canonically isomorphic to

$$(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\xrightarrow{\mathbf{oblv}^{\mathcal{L}U_I}}\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D}\xrightarrow{(\mathbf{q}^-_{I,*}\circ\mathbf{p}_I^{-,!})\otimes\mathbf{Id}}$$
$$\to\mathrm{D}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}\simeq\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\simeq(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

(4) The functor

$$(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\to(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

[61] The category $(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathbb{G}_m\text{-um}}$ is defined in Lemma-Definition D.1.1.

has a left adjoint canonically isomorphic to

$$(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\simeq\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\simeq$$
$$\simeq\mathrm{D}(\mathrm{Gr}_{M,I})\otimes\mathcal{D}\xrightarrow{(D.1)}(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

D.2.2. *Proof of Lemma D.2.1.* The rest of this subsection is devoted to the proof of the lemma. We first note that (0) follows formally (see [Fulltext, Lemma B.1.12(4)]) from Lemma 2.3.2(2). Also, (4) is tautological once we know (D.1) is well-defined.

We first recall the following well-known result:

Lemma D.2.3. *Let Y be any ind-finite type indscheme and $\mathcal{D}\in\mathrm{DGCat}$.*

(1) Suppose Y is written as $\mathrm{colim}_{\alpha\in I}Y_\alpha$, where Y_α are closed sub-indschemes of Y. Then the natural functor

$$\mathrm{D}(Y)\otimes\mathcal{D}\to\lim_{!\text{-pullback}}\mathrm{D}(Y_\alpha)\otimes\mathcal{D}$$

is an equivalence.

(2) Suppose Y is written as $\mathrm{colim}_{\beta\in J}U_\beta$, where U_β are open sub-indschemes of Y and J is filtered. *Then the natural functor*

$$\mathrm{D}(Y)\otimes\mathcal{D}\to\lim_{!\text{-pullback}}\mathrm{D}(U_\beta)\otimes\mathcal{D}$$

is an equivalence.

Proof. We first prove (1). By definition, we have

$$\mathrm{D}(Y)\otimes\mathcal{D}\simeq\underset{*\text{-pushforward}}{\mathrm{colim}}\mathrm{D}(Y_\alpha)\otimes\mathcal{D}.$$

Then we are done by passing to left adjoints.

Now let us prove (2). Write Y as the filtered colimit of its closed subschemes $Y\simeq\mathrm{colim}_{\alpha\in I}Y_\alpha$. For $\alpha\in I$ and $\beta\in J$, let Y_α^β be the intersection of Y_α with U_β (inside Y). By (1), we have

$$\mathrm{D}(Y)\otimes\mathcal{D}\simeq\lim_{!\text{-pullback}}\mathrm{D}(Y_\alpha)\otimes\mathcal{D},$$
$$\mathrm{D}(U_\beta)\otimes\mathcal{D}\simeq\lim_{!\text{-pullback}}\mathrm{D}(Y_\alpha^\beta)\otimes\mathcal{D}.$$

Hence it remains to prove that for a fixed $\alpha\in I$, the natural functor

$$\mathrm{D}(Y_\alpha)\otimes\mathcal{D}\to\lim_{!\text{-pullback}}\mathrm{D}(Y_\alpha^\beta)\otimes\mathcal{D}$$

is an isomorphism. However, this is obvious because for large enough β, the subscheme Y_α is contained inside U_β and hence $Y_\alpha^\beta\simeq Y_\alpha$.

□[Lemma D.2.3]

D.2.4. *Proof of (1).* Recall the stratification on $\mathrm{Gr}_{G,I}$ defined by $\mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ (see § 2.3.1). Since the map $\mathbf{p}_I^+ : \mathrm{Gr}_{P,I} \to \mathrm{Gr}_{G,I}$ is $\mathcal{L}U_I$-equivariant and $\mathcal{L}U_I$ is ind-reduced, the sub-indschemes ${}_\lambda\mathrm{Gr}_{G,I}$, ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ and ${}_{\ge\lambda}\mathrm{Gr}_{G,I}$ of $\mathrm{Gr}_{G,I}$ are all preserved by the $\mathcal{L}U_I$-action.

By [Fulltext, Proposition C.3.2(3)] and Lemma D.2.3(1), we have

$$\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D} \simeq \lim_{!\text{-pullback}} \mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D}. \tag{D.2}$$

Hence

$$(\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \simeq \lim_{!\text{-pullback}} (\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$$

because taking invariants is a right adjoint.

On the other hand, we also have

$$\begin{aligned}(\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} &\simeq \operatorname*{colim}_{*\text{-pushforward}} (\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \simeq \\ &\simeq \lim_{!\text{-pullback}} (\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}.\end{aligned}$$

Hence to prove (1), it suffices to replace $\mathrm{Gr}_{G,I}$ by ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ (for all $\lambda \in \Lambda_{G,P}$).

Note that ${}_{\le\lambda}\mathrm{Gr}_{G,I}$ is the union of its open sub-indschemes ${}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$. Moreover, it is easy to see that the relation "$\ge$" defines a *filtered* partial ordering on $\{\mu \in \Lambda_{G,P} | \mu \le \lambda\}$. Hence by Lemma D.2.3(2), we have

$$\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D} \simeq \lim_{!\text{-pullback}} \mathrm{D}({}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D}. \tag{D.3}$$

Therefore

$$(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I} \simeq \lim_{!\text{-pullback}} (\mathrm{D}({}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}. \tag{D.4}$$

On the other hand, a similar argument as in the proof of Lemma D.2.3(2) shows

$$(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \simeq \lim_{!\text{-pullback}} (\mathrm{D}({}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}.$$

Hence to prove (1), it suffices to replace $\mathrm{Gr}_{G,I}$ by ${}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ (for all $\lambda, \mu \in \Lambda_{G,P}$ with $\mu \le \lambda$). Note that ${}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Using induction and the excision triangle, we can further replace $\mathrm{Gr}_{G,I}$ by a single stratum ${}_\theta\mathrm{Gr}_{G,I} \simeq (\mathrm{Gr}^\theta_{P,I})_{\mathrm{red}}$. Then we are done by (0) and Lemma 2.3.2(1). This proves (1).

D.2.5. *Proof of (3).* Consider the $\mathbb{G}_m$-action on $\mathrm{Gr}_{G,I}$. The attractor (resp. repeller, fixed) locus is $\mathrm{Gr}_{P,I}$ (resp. $\mathrm{Gr}_{P^-,I}$, $\mathrm{Gr}_{M,I}$). Applying Theorem D.1.3 to the inverse of this action, we obtain an adjoint pair

$$(\mathbf{q}^-_{I,*} \circ \mathbf{p}_I^{-,!}) \otimes \mathbf{Id} : (\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}} \rightleftharpoons \mathrm{D}(\mathrm{Gr}_{M,I}) \otimes \mathcal{D} : (\mathbf{p}^+_* \circ \mathbf{q}^{+,!}) \otimes \mathbf{Id}.$$

By (0) and Lemma 2.3.2(1), the image of the above right adjoint is contained in $(\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$, which itself is contained in $(\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathbb{G}_m\text{-um}}$ by (1). Hence we can formally obtain the adjoint pair in (3) from the above adjoint pair. This proves (3).

D.2.6. *Proof of (2).* We first prove that (D.1) is well-defined and strictly $\mathrm{D}(X^I)$-linear. It suffices to prove $(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}$ in (4) has a strictly $\mathrm{D}(X^I)$-linear left adjoint. To do this, we can replace $\mathrm{Gr}_{P,I}$ by $\mathrm{Gr}_{P,I}^{\lambda}$. Consider the following maps

$$_{\lambda}\mathrm{Gr}_{G,I}\xrightarrow{\lambda j}{}_{\le\lambda}\mathrm{Gr}_{G,I}\xrightarrow{\le\lambda\mathbf{p}_I^+}\mathrm{Gr}_{G,I}.$$

Since $_{\le\lambda}\mathbf{p}_I^+$ is a schematic closed embedding, we have an adjoint pair

$$({}_{\le\lambda}\mathbf{p}_{I,*}^+\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\rightleftharpoons(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}:({}_{\le\lambda}\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}.$$

Hence it suffices to prove that

$$({}_{\lambda}j^!\otimes\mathbf{Id})^{\mathrm{inv}}:(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\to(\mathrm{D}({}_{\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}$$

has a strictly $\mathrm{D}(X^I)$-linear left adjoint. For any $\mu_1\le\mu_2\le\lambda$, consider the following commutative square induced by !-pullback functors:

$$\begin{tikzcd}
(\mathrm{D}({}_{\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \arrow[r, "="] & (\mathrm{D}({}_{\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \\
(\mathrm{D}({}_{\le\lambda,\ge\mu_1}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \arrow[u] \arrow[r] & (\mathrm{D}({}_{\le\lambda,\ge\mu_2}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}. \arrow[u]
\end{tikzcd}$$

Using (D.4), the existence of the desired left adjoint follows formally (see [Fulltext, Lemma A.1.3]) from the following claim: the above square is left-adjointable along the vertical direction and the relevant left adjoints are strictly $\mathrm{D}(X^I)$-linear. By the base-change isomorphism, the above square is right adjointable along the horizontal direction. Hence it suffices to prove that the vertical functors have strictly $\mathrm{D}(X^I)$-linear left adjionts. Note that $_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Hence we are done by using (3) and the excision triangle. This proves (D.1) is well-defined and strictly $\mathrm{D}(X^I)$-linear.

It remains to prove the image of (D.1) generates the target category under colimits and shifts. It suffices to prove $(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}$ is conservative. We only need to prove $\mathbf{p}_I^{+,!}\otimes\mathbf{Id}$ is conservative. Suppose $y\in\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D}$ and $\mathbf{p}_I^{+,!}\otimes\mathbf{Id}(y)\simeq0$. We need to show $y\simeq0$. By (D.2) and (D.3), it suffices to show the !-restriction of y to $\mathrm{D}({}_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I})\otimes\mathcal{D}$ is zero for any $\lambda,\mu\in\Lambda_{G,P}$. Note that $_{\le\lambda,\ge\mu}\mathrm{Gr}_{G,I}$ contains only finite many strata. Hence we are done by using the excision triangle.

□[Lemma D.2.1]

D.3. **Proof of Lemma 2.3.4, 2.3.5.** Note that Lemma 2.3.4 can be obtained[62] from Lemma D.2.1 by letting $\mathcal{D}\coloneqq\mathrm{Vect}$.

The rest of this subsection is devoted to the proof of Lemma 2.3.5. Let $\mathcal{D}\in\mathrm{DGCat}$ be a test DG category. Consider the tautological functor

$$\alpha:\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\to(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

[62]Of course, in order to get the *compact* generation of $\mathrm{D}(\mathrm{Gr}_{G,I})$, we need to use the compact generation of $\mathrm{D}(\mathrm{Gr}_{M,I})$.

We have

Lemma D.3.1. *The following two commutative squares are left adjointable along horizontal diresctions.*

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D} \arrow[r, "\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id}"] \arrow[d, "\alpha"] & \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D} \arrow[d, "\beta"', "\simeq"] \\
(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \arrow[r, "(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}"] & (\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I},
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D} \arrow[r, "\mathbf{p}_{I,*}^{+,\mathrm{inv}}\otimes\mathbf{Id}"] \arrow[d, "\beta"', "\simeq"] & \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}\otimes\mathcal{D} \arrow[d, "\alpha"] \\
(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I} \arrow[r, "(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv}}"] & (\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.
\end{tikzcd}$$

Proof. First note that β is indeed an equivalence by Lemma D.2.1(0).

The claim for the second commutative square is a corollary of Lemma D.1(3). It remains to prove the claim for the first commutative square. By Lemma D.1(4), the relevant left adjoints are well-defined.

Let x be any object in $\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}$. It suffices to prove the morphism

$$(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv},L}\circ\beta(x)\to\alpha\circ(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L(x) \tag{D.5}$$

is an isomorphism. Note that we have

$$\mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I}\otimes\mathcal{D}\simeq\coprod_{\lambda\in\Lambda_{G,P}}(\mathrm{D}(\mathrm{Gr}_{P,I}^{\lambda})^{\mathcal{L}U_I}\otimes\mathcal{D}).$$

Without loss of generality, we can assume x is contained in the direct summand labelled by λ.

Consider the closed embedding ${}_{\le\lambda}\mathrm{Gr}_{G,I}\to\mathrm{Gr}_{G,I}$. It induces a fully faithful functor

$$(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\hookrightarrow(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

It is easy to see that both sides of (D.5) are contained in this full subcategory. Hence by Lemma D.3.2 below, it suffices to prove that the map

$$(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv},L}\circ(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv},L}\circ\beta\to(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv},L}\circ\alpha\circ(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L$$

is an isomorphism. By the left adjointability of the second square, the RHS is isomorphic to $\beta\circ(\mathbf{p}_{I,*}^{+,\mathrm{inv}}\otimes\mathbf{Id})^L\circ(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})^L$. Then we are done because of the obvious isomorphism

$$(\mathbf{p}_I^{+,!,\mathrm{inv}}\otimes\mathbf{Id})\circ(\mathbf{p}_{I,*}^{+,\mathrm{inv}}\otimes\mathbf{Id})\simeq(\mathbf{p}_I^{+,!}\otimes\mathbf{Id})^{\mathrm{inv}}\circ(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv}}.$$

□[Lemma D.3.1]

Lemma D.3.2. *Let $\lambda\in\Lambda_{G,P}$. The following composition is conservative*

$$(\mathrm{D}({}_{\le\lambda}\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\hookrightarrow(\mathrm{D}(\mathrm{Gr}_{G,I})\otimes\mathcal{D})^{\mathcal{L}U_I}\xrightarrow{(\mathbf{p}_{I,*}^{+}\otimes\mathbf{Id})^{\mathrm{inv},L}}(\mathrm{D}(\mathrm{Gr}_{P,I})\otimes\mathcal{D})^{\mathcal{L}U_I}.$$

Proof. Suppose that $y \in (\mathrm{D}({}_{\leq\lambda}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$ is sent to zero by the above composition. We need to show that $y \simeq 0$. By (D.4), if suffices to prove that the !-restrictions of y to $(\mathrm{D}({}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$ is zero for any $\mu \leq \lambda$. Note that these !-restrictions are equal to $*$-restrictions. Also note that ${}_{\leq\lambda,\geq\mu}\mathrm{Gr}_{G,I}$ contains only finitely many strata. Hence we are done by using induction and the excision triangle.

□[Lemma D.3.2]

Lemma D.3.3. *Let $\mathcal{D}$ be any DG category. The tautological functor*

$$\alpha : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D} \to (\mathrm{D}(\mathrm{Gr}_{G,I}) \otimes \mathcal{D})^{\mathcal{L}U_I}$$

is an isomorphism.

Proof. By Lemma D.2.1(2)(4) and Lemma D.3.1, the image of α generates the target under colimits and shifts. It remains to prove that α is fully faithful, which can be proved by diagram chasing with help of Lemma D.3.1. We exhibit it as follows.

Let $y \in \mathrm{D}(\mathrm{Gr}_{P,I})^{\mathcal{L}U_I} \otimes \mathcal{D}$ and $z \in \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D}$. We have

$$\begin{aligned}
& \mathrm{Maps}((\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L(y), z) \\
\simeq\ & \mathrm{Maps}(y, (\mathbf{p}_I^{+!,\mathrm{inv}} \otimes \mathbf{Id})(z)) \\
\simeq\ & \mathrm{Maps}(\beta(y), \beta \circ (\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})(z)) \\
\simeq\ & \mathrm{Maps}(\beta(y), (\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv}} \circ \alpha(z)) \\
\simeq\ & \mathrm{Maps}((\mathbf{p}_I^{+,!} \otimes \mathbf{Id})^{\mathrm{inv},L} \circ \beta(y), \alpha(z)) \\
\simeq\ & \mathrm{Maps}(\alpha \circ (\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L(y), \alpha(z)).
\end{aligned}$$

Then we are done because the category $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \otimes \mathcal{D}$ is generated under colimits and shifts by $(\mathbf{p}_I^{+,!,\mathrm{inv}} \otimes \mathbf{Id})^L(y)$.

□[Lemma D.3.3]

D.3.4. *Proof of Lemma 2.3.5.* Lemma D.3.3 formally implies (see [Fulltext, Lemma B.1.12(4)]) that the category $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is dualizable in DGCat. It follows formally (see [Fulltext, Lemma B.1.11]) that $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ and $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ are dual to each other. Since $\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ is compactly generated (by Lemma 2.3.4, which we have already proved), its dual category $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ is also compactly generated. Moreover, we have an equivalence

$$(\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^c \simeq (\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I})^{c,\mathrm{op}}. \tag{D.6}$$

Consider the pairing functor for the above duality:

$$\langle -,- \rangle : \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I} \times \mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I} \to \mathrm{Vect}\,.$$

For any $\mathcal{F} \in \mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I}$ and any *compact* object $\mathcal{G}$ in $\mathrm{D}(\mathrm{Gr}_{M,I})$, we have

$$\langle \mathcal{F}, \mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{G}) \rangle \simeq \langle \mathbf{s}_I^! \circ \mathbf{oblv}^{\mathcal{L}U_I} \circ \mathcal{F}, \mathcal{G} \rangle_{\mathrm{Verdier}} \simeq$$

$$\simeq \mathrm{Maps}(\mathbb{D}(\mathcal{G}), \mathbf{s}_I^! \circ \mathbf{oblv}^{\mathcal{L}U_I} \circ \mathcal{F}) \simeq \mathrm{Maps}(\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*} \circ \mathbb{D}(\mathcal{G}), \mathcal{F}).$$

Hence the object (which is well-defined by Lemma 2.3.4(2))

$$\mathbf{Av}_!^{\mathcal{L}U_I} \circ \mathbf{s}_{I,*} \circ \mathbb{D}(\mathcal{G}) \in (\mathrm{D}(\mathrm{Gr}_{G,I})^{\mathcal{L}U_I})^c$$

is sent by (D.6) to the object $\mathbf{pr}_{\mathcal{L}U_I} \circ \mathbf{s}_{I,*}(\mathcal{G})$. Consequently, the latter object is compact. All such objects generate the category $\mathrm{D}(\mathrm{Gr}_{G,I})_{\mathcal{L}U_I}$ under colimits and shifts because of Lemma 2.3.4(2).

□[Lemma 2.3.5]

References

[AB09] Sergey Arkhipov and Roman Bezrukavnikov. Perverse sheaves on affine flags and langlands dual group. *Israel Journal of Mathematics*, 170(1):135, 2009.

[BB83] Alexander Beilinson and Joseph Bernstein. A generalization of casselman's submodule theorem. In *Representation theory of reductive groups*, pages 35–52. Springer, 1983.

[BB93] Alexander Beilinson and Joseph Bernstein. A proof of jantzen conjectures. *Advances in Soviet mathematics*, 16(Part 1):1–50, 1993.

[Bei87] AA Beilinson. How to glue perverse sheaves. In *K-Theory, Arithmetic and Geometry*, pages 42–51. Springer, 1987.

[BFGM02] Alexander Braverman, Michael Finkelberg, Dennis Gaitsgory, and Ivan Mirković. Intersection cohomology of drinfeld's compactifications. *Selecta Mathematica*, 8(3):381–418, 2002.

[BFO12] Roman Bezrukavnikov, Michael Finkelberg, and Victor Ostrik. Character d-modules via drinfeld center of harish-chandra bimodules. *Inventiones mathematicae*, 188(3):589–620, 2012.

[BG02] A Braverman and D Gaitsgory. Geometric eisenstein series. *Inventiones mathematicae*, 2(150):287–384, 2002.

[BG06] Alexander Braverman and Dennis Gaitsgory. Deformations of local systems and eisenstein series. *arXiv preprint math/0605139*, 2006.

[Bra03] Tom Braden. Hyperbolic localization of intersection cohomology. *Transformation groups*, 8(3):209–216, 2003.

[Cam18] Justin Campbell. Nearby cycles of whittaker sheaves on drinfeld's compactification. *Compositio Mathematica*, 154(8):1775–1800, 2018.

[CGD19] Tsao-Hsien Chen, Dennis Gaitsgory, and Alexander Yom Din. On the casselman-jacquet functor. *Representations of Reductive Groups*, 2019.

[Che] Lin Chen. Deligne-lusztig duality on the moduli stack of bundles. *Available at the author's website.*

[DG13] Vladimir Drinfeld and Dennis Gaitsgory. On some finiteness questions for algebraic stacks. *Geometric and Functional Analysis*, 23(1):149–294, 2013.

[DG14] Vladimir Drinfeld and Dennis Gaitsgory. On a theorem of braden. *Transformation groups*, 19(2):313–358, 2014.

[DG15] V Drinfeld and D Gaitsgory. Compact generation of the category of d-modules on the stack of g-bundles on a curve. *Cambridge Journal of Mathematics*, 3(1):19–125, 2015.

[DG16] Vladimir Drinfeld and Dennis Gaitsgory. Geometric constant term functor (s). *Selecta Mathematica*, 22(4):1881–1951, 2016.

[Dri13] Vladimir Drinfeld. On algebraic spaces with an action of g_m. *arXiv preprint arXiv:1308.2604*, 2013.

[FKM20] Michael Finkelberg, Vasily Krylov, and Ivan Mirković. Drinfeld–gaitsgory–vinberg interpolation grassmannian and geometric satake equivalence. *Journal of Topology*, 13(2):683–729, 2020.

[Gai01] D Gaitsgory. Construction of central elements in the affine hecke algebra via nearby cycles. *Inventiones Mathematicae*, 144(2):253, 2001.

[Gai16] Dennis Gaitsgory. Functors given by kernels, adjunctions and duality. *Journal of Algebraic Geometry*, 25(3):461–548, 2016.

[Gai17a] Dennis Gaitsgory. The semi-infinite intersection cohomology sheaf-ii: the ran space version. *arXiv preprint arXiv:1708.07205*, 2017.

[Gai17b] Dennis Gaitsgory. A "strange" functional equation for eisenstein series and miraculous duality on the moduli stack of bundles. In *Annales Scientifiques de l'École Normale Supérieure*. Societe Mathematique de France, 2017.

[Gai18a] Dennis Gaitsgory. The local and global versions of the whittaker category. *arXiv preprint arXiv:1811.02468*, 2018.

[Gai18b] Dennis Gaitsgory. The semi-infinite intersection cohomology sheaf. *Advances in Mathematics*, 327:789–868, 2018.

[Gai20] Dennis Gaitsgory. Second adjointness via nearby cycles: A report on the work of lin chen. 2020.

[GR14] Dennis Gaitsgory and Nick Rozenblyum. Dg indschemes. *Perspectives in representation theory*, 610:139–251, 2014.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry*, volume 1. American Mathematical Soc., 2017.

[HR18] Thomas J Haines and Timo Richarz. The test function conjecture for local models of weil-restricted groups. *arXiv preprint arXiv:1805.07081*, 2018.

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009.

[Ras15a] Sam Raskin. Chiral categories. 2015.

[Ras15b] Sam Raskin. D-modules on infinite dimensional varieties. 2015.

[Ras16] Sam Raskin. Chiral principal series categories ii: the factorizable whittaker category. 2016.

[Sch16] Simon Schieder. Geometric bernstein asymptotics and the drinfeld-lafforgue-vinberg degeneration for arbitrary reductive groups. *arXiv preprint arXiv:1607.00586*, 2016.

[Sch18] Simon Schieder. Picard–lefschetz oscillators for the drinfeld–lafforgue–vinberg degeneration for sl2. *Duke mathematical journal*, 167(5):835–921, 2018.

[Wan17] Jonathan Wang. On the reductive monoid associated to a parabolic subgroup. *Journal of Lie Theory*, 27(3):637–655, 2017.

[Wan18] Jonathan Wang. On an invariant bilinear form on the space of automorphic forms via asymptotics. *Duke Mathematical Journal*, 167(16):2965–3057, 2018.

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