LECTURE 6

In this lecture, we define ∞-(co)limits, i.e., (co)limits in an ∞-category.

1. Idea of the definition

1.1. In classical category, we have the following characterization of limits:

A diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ is a limit diagram iff it corresponds to a final object in the slice category $\mathcal{C}_{/u}$, where $u : K \to \mathcal{C}$.

We will define ∞-limits in exactly the same way, once the following are accomplished:

(1) Define final and initial objects in an ∞-category;
(2) Define slice and coslice ∞-categories.

2. Final and initial objects

2.1. Recall an ∞-groupoid is **contractible** if it is equivalent to $[0]$.

Definition 2.2. *Let C be an ∞-category and $x \in \mathsf{C}$ be an object.*

- *We say $x \in \mathsf{C}$ is **final** if for any object $y \in \mathsf{C}$, the ∞-groupoid $\mathsf{Maps}_{\mathsf{C}}(y, x)$ is contractible.*
- *We say $x \in \mathsf{C}$ is **initial** if for any object $y \in \mathsf{C}$, the ∞-groupoid $\mathsf{Maps}_{\mathsf{C}}(x, y)$ is contractible.*

2.3. Note that $x \in \mathsf{C}$ is initial iff the corresponding object in C^{op} is final. Hence in below, we focus on final objects.

Exercise 2.4. *Being final is invariant under equivalences:*

(1) *If $x \to y$ is an isomorphism in C, then x is final iff y is so.*
(2) *If $F : \mathsf{C} \to \mathsf{C}'$ is an equivalence between ∞-categories, then $x \in \mathsf{C}$ is final iff $F(x)$ in C' is so.*

Exercise 2.5. *The apex of $\mathsf{C}^{\triangleright}$ is final.*

Example 2.6. *The empty category is initial in Cat_{∞}. The singleton $[0]$ is final in Cat_{∞}.*

Proposition 2.7. *The full sub-∞-category consisting of final objects in an ∞-category C is either empty or a contractible ∞-groupoid. In other words, final objects in an ∞-category C are essentially unique if exist.*

Proof. We can replace C by the full sub-∞-category consisting of final objects, and assume any object in C is final. We need to show C is either empty or a contractible ∞-groupoid. Indeed, if C is not empty, then the functor $\mathsf{C} \to [0]$ is fully faithful and essentially surjective. $\square$

Exercise 2.8. *Let C be an ∞-category and x be an object. If x is final, then its image under $\mathsf{C} \to \mathsf{hC}$ is final. The converse is ture if C has a final object.*

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Warning 2.9. *In general, the functor $\mathsf{C} \to \mathsf{hC}$ cannot* detect *final objects. For example, the following lemma implies a non-contractible ∞-groupoid C has no final objects, but $\mathsf{hC} \simeq [0]$ when C is simply-connected.*

Proposition 2.10. *An ∞-groupoid C contains a final object x iff C is contractible.*

Sketch 1. For pointed space X, $\pi_n(\Omega X) \simeq \pi_{n+1}(X)$.

Sketch 2. The functor $[0] \xrightarrow{x} \mathsf{C}$ is fully faithful and essentially surjective. □

2.11. In fact, we have the following result, which will be proved in §4.

Proposition 2.12. *If an ∞-category C contains a final object x, then it is weakly contractible*[1].

3. Slice and coslice: general case

3.1. In [Lecture 4, §8], for a quasi-category $\mathcal{C}$ and an object $x \in \mathcal{C}$, we defined the slice and coslice quasi-categories $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$. In this section, we generalize this construction to a general diagram $u : K \to \mathsf{C}$ in an ∞-category C to define ∞-categories $\mathsf{C}_{/u}$ and $\mathsf{C}_{u/}$.

3.2. We first define them via quasi-categories.

Definition 3.3. *Let $u : K \to X$ be a morphism in Set_Δ. The* ***slice simplicial set of** X **over** u* *is the simplicial set $X_{/u}$ defined by*[2]

$$\mathsf{Hom}(Y, X_{/u}) \simeq \mathsf{Hom}(Y \star K, X) \underset{\mathsf{Hom}(K,X)}{\times} \{u\}.$$

Dually, the ***coslice simplicial set of** X **under** f* *is the simplicial set $X_{u/}$ defined by*

$$\mathsf{Hom}(Y, X_{u/}) \simeq \mathsf{Hom}(K \star Y, X) \underset{\mathsf{Hom}(K,X)}{\times} \{u\}.$$

3.4. Note that $(X_{u/})^{\mathsf{op}} \simeq (X^{\mathsf{op}})_{/u^{\mathsf{op}}}$. Hence in below, we focus on the slice construction.

Proposition 3.5 (Ker.018F)**.** *Let $u : K \to \mathcal{C}$ be a morphism in Set_Δ such that $\mathcal{C}$ is a quasi-category. Then $\mathcal{C}_{/u}$ is a quasi-category.*

Proposition 3.6 (Ker.02GL, 02NC, 02NR)**.** *The slice construction is invariant under equivalences:*

(1) *If $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence between quasi-categories, then for any diagram $u : K \to \mathcal{C}$, the induced functor $\mathcal{C}_{/u} \to \mathcal{C}'_{/F\circ u}$ is an equivalence.*
(2) *If $v : K' \to K$ is a categorical equivalence between simplicial sets, then for any diagram $u : K \to \mathcal{C}$ in a quasi-category $\mathcal{C}$, the restriction functor $\mathcal{C}_{/u} \to \mathcal{C}_{/u\circ v}$ is an equivalence.*

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[1] In other words, C becomes a contractible ∞-groupoid after formally inverting all morphisms. See [Lecture 5, §3.7-3.10].

[2] See [Lecture 4, Definition 8.5] for the definition of the join $Y \star K$.

3.7. By the philosophy in [Lecture 5, §9], for a diagram $u : K \to \mathsf{C}$ in an ∞-category C, there are well-defined ∞-categories $\mathsf{C}_{/u}$ and $\mathsf{C}_{u/}$. Moreover, the same is true if K is replaced by an ∞-category K.

We call them the **slice/coslice ∞-category of C over F**.

Exercise 3.8. *Let $u : K \to \mathsf{C}$ be a diagram in an ∞-category C, and $v : J \to \mathsf{C}_{/u}$ be a diagram in $\mathsf{C}_{/u}$. Show that $(\mathsf{C}_{/u})_{/v} \simeq \mathsf{C}_{/w}$, where $w : J \star K \to \mathsf{C}$ is the diagram corresponding to v.*

Construction 3.9. *Let $u : K \to X$ be a morphism in Set_Δ. By the Yoneda lemma, the morphisms $Y \to Y \star K$ induce a morphism $X_{/u} \to X$, which is called the **forgetful morphism** from $X_{/u}$ to X.*

*Let $u : K \to \mathsf{C}$ be a diagram in an ∞-category C. The above construction gives the **forgetful functor** $\mathsf{C}_{/u} \to \mathsf{C}$.*

3.10. We will prove the following result in §5.

Proposition 3.11. *For any diagram $u : K \to \mathsf{C}$ in an ∞-category C, the forgetful functor $\mathsf{C}_{/u} \to \mathsf{C}$ is **conservative**[3].*

Proposition 3.12 (Ker.018F, Ker.00TE)**.** *Let $u : K \to \mathsf{C}$ be a diagram in an ∞-category C. For a morphism $f : x \to y$ in C and a lifting $\widetilde{y} \in \mathsf{C}_{/u}$ of y, there is an essentially unique morphism $\widetilde{x} \to \widetilde{y}$ that lifts f.*

Remark 3.13. *The precise meaning of the proposition is:*

(1) *The functor*

$$\mathsf{Fun}([1], \mathsf{C}_{/u}) \to \mathsf{Fun}(\{1\}, \mathsf{C}_{/u}) \underset{\mathsf{Fun}(\{1\}, \mathsf{C})}{\times} \mathsf{Fun}([1], \mathsf{C})$$

is an equivalence between ∞-categories.

Here the fiber product is calculated inside the ∞-category Cat_∞, which will be defined in the next section.

Alternatively, we can realize C as a quasi-category $\mathcal{C}$, then the proposition says[4]*:*

(2) *The functor*

$$\mathsf{Fun}(\Delta^1, \mathcal{C}_{/u}) \to \mathsf{Fun}(\{1\}, \mathcal{C}_{/u}) \underset{\mathsf{Fun}(\{1\}, \mathcal{C})}{\times} \mathsf{Fun}(\Delta^1, \mathcal{C})$$

is an acyclic Kan fibration.

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[3] This means a morphism in $\mathsf{C}_{/u}$ is an isomorphism iff its image in C is so.

[4] For the purpose of these notes, we translate results in Lurie's books into model-independent language as often as we can. Therefore, the statements stated in these notes are not exactly the same as those in the cited references. To compare them, one often needs additional knowledge about the model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.

For instance, to deduce (1) from (2), one needs to know the fiber product in (1) can be realized as that in (2). By HTT.4.2.4.1 (which will be discussed in future lectures), the former can be calculated as the corresponding *homotopy* fiber product in the simplicial model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$. By (the dual of) HTT.A.2.4.4, this homotopy fiber product coincides with the naive fiber product because $\mathsf{Fun}(\Delta^1, \mathcal{C}) \to \mathsf{Fun}(\{1\}, \mathcal{C})$ is a categorical fibration (HTT.2.4.6.5), and all the three terms are fibrant in $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.

Here the fiber product is calculated inside the ordinary category QCat.

Functors $\mathsf{D} \to \mathsf{C}$ *satisfying the lifting property in Proposition 3.12 (such as* $\mathsf{C}_{/u} \to \mathsf{C}$*) are called* ***right fibrations***[5]. *Dually,* $\mathsf{C}_{u/} \to \mathsf{C}$ *is a* ***left fibration***. *We will systematically study these fibrations in future lectures.*

3.14. Now we discuss functoriality of the slice construction in u.

Proposition-Construction 3.15. *Let* K *be a simplicial set and* C *be an* ∞*-category. Let* $u_0 \to u_1$ *be a morphism in* $\mathsf{Fun}(K, \mathsf{C})$*, and* $u : K \times \Delta^1 \to \mathsf{C}$ *be the corresponding diagram. Consider the functors*

$$\mathsf{C}_{/u_0} \leftarrow \mathsf{C}_{/u} \to \mathsf{C}_{/u_1}.$$

obtained by restricting along $K \rightrightarrows K \times \Delta^1$.

(1) *The functor* $\mathsf{C}_{/u_0} \leftarrow \mathsf{C}_{/u}$ *is an equivalence.*
(2) *The functor* $\mathsf{C}_{/u} \to \mathsf{C}_{/u_1}$ *is an equivalence if* $u_0 \to u_1$ *is an isomorphism in* $\mathsf{Fun}(K, \mathsf{C})$.

By inverting the equivalence $\mathsf{C}_{/u_0} \leftarrow \mathsf{C}_{/u}$*, we obtain a canonical functor* $\mathsf{C}_{/u_0} \to \mathsf{C}_{/u_1}$ *compatible with the forgetful functors to* C.

Proof. (1) is left as an exercise for the *next* lecture. To prove (2), consider the quasi-category $\mathcal{I} := \mathsf{N}_\bullet(\mathsf{I})$, where I is the ordinary category such that:

- There are two objects in I;
- For $x, y \in \mathsf{Ob}(\mathsf{I})$, there is a unique morphism $x \to y$.

Since $u_0 \to u_1$ is an isomorphism, we can find morphism v making the following diagram commute:

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "u"] \arrow[d, hook, "\subset"'] & \mathsf{C} \\ K \times \mathcal{I}. \arrow[ur, dotted, "v"'] & \end{tikzcd}$$

Hence we have the following commutative diagram

$$\begin{tikzcd} & \mathsf{C}_{/v} \arrow[dl, "p'"'] \arrow[d, "r"] \arrow[dr, "q'"] & \\ \mathsf{C}_{/u_0} & \mathsf{C}_{/u} \arrow[l, "p"] \arrow[r, "q"'] & \mathsf{C}_{/u_1}. \end{tikzcd}$$

By (1), p is an equivalence. By Proposition 3.6(2), both p' and q' are equivalences. It follows that r, and therefore q, is an equivalence. □

Exercise 3.16. *Let* K *be a simplicial set and* $\mathcal{C}$ *be a quasi-category. Can you construct a functor* $\mathsf{Fun}(K, \mathcal{C}) \to \mathsf{QCat}$, $u \mapsto \mathcal{C}_{/u}$*? Here* QCat *is the ordinary category of quasi-categories.*

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[5]More precisely, such functors are *modelled* by right fibrations between quasi-categories.

4. Final objects and slice constructions

Proposition 4.1 (Ker.02J2)**.** *Let* C *be an* ∞*-category and* $f : x \to y$ *be a morphism. Then the following are equivalent:*

- f *is an isomorphism;*
- *The object in* $\mathsf{C}_{x/}$ *given by* f *is initial;*
- *The object in* $\mathsf{C}_{/y}$ *given by* f *is final.*

Proposition 4.2. *Let* C *be an* ∞*-category. Then an object* x *is final iff the functor* $\mathsf{C}_{/x} \to \mathsf{C}$ *is an equivalence.*

Sketch. The $\Leftarrow$ direction of Proposition 4.2 follows immediately from Proposition 4.1. For the $\Rightarrow$ direction, let $\mathcal{C}$ be a quasi-category with a final object x. Note that the fiber of $\mathcal{C}_{/x} \to \mathcal{C}$ at any object $y \in \mathcal{C}$ is $\mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(y, x)$, which by assumption is categorical equivalent to Δ^0. In future lectures, we will deduce the claim in the proposition from this and the fact that $\mathcal{C}_{/x} \to \mathcal{C}$ is a *right fibration.* □

Proof of Proposition 2.12. By definition, we only need to show for any ∞-groupoid D, the functor

$$- \circ \pi : \mathsf{Fun}([0], \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{D})$$

is an equivalence, where $\pi : \mathsf{C} \to [0]$ is the projection functor. Note that this functor has an obvious left inverse

$$- \circ x : \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}([0], \mathsf{D})$$

induced by the functor $x : [0] \to \mathsf{C}$. Hence it remains to construct an invertible natural transformation from

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{\mathsf{Id}} \mathsf{Fun}(\mathsf{C}, \mathsf{D}), \tag{4.1}$$

to

$$\mathsf{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{-\circ x} \mathsf{Fun}([0], \mathsf{D}) \xrightarrow{-\circ \pi} \mathsf{Fun}(\mathsf{C}, \mathsf{D}). \tag{4.2}$$

We claim any such natural tranformation is automatically invertible. To prove the claim, note that $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ is an ∞-groupoid because D is so. Therefore

$$\mathsf{Fun}(\mathsf{Fun}(\mathsf{C}, \mathsf{D}), \mathsf{Fun}(\mathsf{C}, \mathsf{D}))$$

is also an ∞-groupoid, which implies the claim.

It remains to construct *a* natural transformation from (4.1) to (4.2). For this purpose, it is enough to construct a natural transformation from Id_{C} to the composition $\mathsf{C} \xrightarrow{\pi} [0] \xrightarrow{x} \mathsf{C}$. Consider the obvious functors

$$\mathsf{C} \leftarrow \mathsf{C}_{/x} \to \mathsf{Fun}([1], \mathsf{C}).$$

By Proposition 4.2, the leftwards functor is an equivalence. Hence we can invert it and obtain a functor $\mathsf{C} \to \mathsf{Fun}([1], \mathsf{C})$, which corresponds to a functor $[1] \to \mathsf{Fun}(\mathsf{C}, \mathsf{C})$. Unwinding the definitions, this gives a natural transformation from Id_{C} to $\mathsf{C} \xrightarrow{\pi} [0] \xrightarrow{x} \mathsf{C}$. □[Proposition 2.12]

5. Blunt joins

5.1. To prove Proposition 3.11, we need the following equivalent construction of slice ∞-categories.

Let $u : K \to \mathcal{C}$ be a diagram in a quasi-category $\mathcal{C}$. As in [Lecture 4, §9.1], we have a canonical functor

$$\mathcal{C}_{/u} \to \mathcal{C} \underset{\mathsf{Fun}(K\times\{0\},\mathcal{C})}{\times} \mathsf{Fun}(K \times \Delta^1, \mathcal{C}) \underset{\mathsf{Fun}(K\times\{1\},\mathcal{C})}{\times} \{u\}. \tag{5.1}$$

On the level of objects, this functor sends a diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ to the composition $K \times \Delta^1 \to K^{\triangleleft} \to \mathcal{C}$, which corresponds to a natural transformation $\underline{x} \to u$, where $\underline{x} : K \to \mathcal{C}$ is the constant functor with value $x := \overline{u}(*)$.

To simplify the notations, we make the following definition.

Definition 5.2. *Let J and K be simplicial sets. Define the **blunt join** of J and K as*

$$J \diamond K := J \coprod_{J\times K\times\{0\}} (J \times K \times \Delta^1) \coprod_{J\times K\times\{1\}} K.$$

Warning 5.3. *The blunt join of quasi-categories might not be a quasi-category.*

Exercise 5.4. *Show that $\Delta^0 \diamond \Delta^1$ is* not *a quasi-category.*

5.5. Using the above notation, the functor (5.1) can be rewritten as

$$\mathcal{C}_{/u} \to \mathsf{Fun}(\Delta^0 \diamond K, \mathcal{C}) \underset{\mathsf{Fun}(K,\mathcal{C})}{\times} \{u\} =: \mathcal{C} \underset{\mathsf{Fun}(K,\mathcal{C})}{\overset{\to}{\times}} \{u\} =: \mathcal{C}^{/u}. \tag{5.2}$$

Dually, we have a canonical functor

$$\mathcal{C}_{u/} \to \{u\} \underset{\mathsf{Fun}(K,\mathcal{C})}{\times} \mathsf{Fun}(K \diamond \Delta^0, \mathcal{C}) =: \{u\} \underset{\mathsf{Fun}(K,\mathcal{C})}{\overset{\to}{\times}} \mathcal{C} =: \mathcal{C}^{u/}. \tag{5.3}$$

Exercise 5.6. *There is a unique morphism $J \diamond K \to J \star K$ making the following diagram commute:*

$$\begin{tikzcd} & J \diamond K \arrow[dr, dotted] & \\ J \coprod K \arrow[ur] \arrow[rr] & & J \star K. \end{tikzcd}$$

Proposition 5.7 (Ker.01HU, 01HV, 01HW)**.** *The functors $- \star -$ and $- \diamond -$ preserve categorical equivalences in both factors, and the morphism $J \diamond K \to J \star K$ is a categorical equivalence.*

Remark 5.8. *As a result, we have a well-defined binary operator on ∞-categories that can be modelled by $- \star -$.*

Exercise 5.9. *Let C and D be ∞-categories. Characterize $\mathsf{C} \star \mathsf{D}$ by universal properties.*

Proposition 5.10 (Ker.01KU)**.** *The functors* (5.2) *and* (5.3) *are invertible.*

Remark 5.11. *One can show the fiber product in* (5.2)*, which is taken in* QCat*, also calculates the homotopy fiber product in* $\mathsf{Set}_{\Delta}^{\mathsf{Joyal}}$ *and therefore the fiber product in QCat (see Footnote 4). Therefore, for a diagram $u : K \to \mathsf{C}$ in an ∞-category, we have canonical equivalences*

$$\mathsf{C}_{/u} \xrightarrow{\simeq} \mathsf{Fun}(\Delta^0 \diamond K, \mathsf{C}) \underset{\mathsf{Fun}(K,\mathsf{C})}{\times} \{u\} \xleftarrow{\simeq} \mathsf{Fun}(\Delta^0 \star K, \mathsf{C}) \underset{\mathsf{Fun}(K,\mathsf{C})}{\times} \{u\}. \tag{5.4}$$

Proof of Proposition 3.11. We only need to prove the similar claim for $\mathcal{C}^{/u}$. By definition, a morphism in this quasi-category is a natural transformation $\alpha : F_1 \to F_2$ between two functors $F_1, F_2 : K \times \Delta^1 \to \mathcal{C}$ such that

(0) $\alpha|_{K\times\{0\}}$ is a constant natural transformation between two constant functors $\underline{x_1}, \underline{x_2} : K \to \mathcal{C}$;
(1) $\alpha|_{K\times\{1\}}$ is the identity natural transformation at the functor $u : K \to \mathcal{C}$.

Note that the forgetful functor $\mathcal{C}^{/u} \to \mathcal{C}$ sends α to the morphism $x_1 \to x_2$ in (0). By [Lecture 5, Theorem 2.3], α is invertible iff $x_1 \to x_2$ is so.

□[Proposition 3.11]

Exercise 5.12. *Let $\widetilde{u} : J \times K \to \mathcal{C}$ be a diagram in a quasi-category $\mathcal{C}$. Write $u : J \to \mathsf{Fun}(K, \mathcal{C})$ for the corresponding diagram in $\mathsf{Fun}(K, \mathcal{C})$. Construct a canonical isomorphism*

$$\mathcal{C}^{/\widetilde{u}} \xrightarrow{\simeq} \mathsf{Fun}(K, \mathcal{C})^{/u} \underset{\mathsf{Fun}(K,\mathcal{C})}{\times} \mathcal{C}. \tag{5.5}$$

6. Definition of limits and colimits

Definition 6.1. *Let K be a simplicial set and C be an ∞-category. We say a diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}$ is a **limit diagram** if the corresponding object in $\mathsf{C}_{/u}$ is final, where $u := \overline{u}|_K$ is the restriction of $\overline{u}$ on K. We also say $\overline{u}$ **exhibits** $\overline{u}(*)$ **as a limit of the diagram** $u : K \to \mathsf{C}$.*

*Dually, we say a diagram $\overline{u} : K^{\triangleright} \to \mathsf{C}$ is a **colimit diagram** if the corresponding object in $\mathsf{C}_{u/}$ is initial. We also say $\overline{u}$ **exhibits** $\overline{u}(*)$ **as a colimit of the diagram** $u : K \to \mathsf{C}$.*

6.2. Note that a diagram $\overline{u} : K^{\triangleright} \to \mathsf{C}$ is a colimit diagram iff $\overline{u}^{\mathsf{op}} : (K^{\mathsf{op}})^{\triangleleft} \to \mathsf{C}^{\mathsf{op}}$ is a limit diagram. Hence in below, we focus on limit diagrams.

Remark 6.3. *For a diagram $\overline{u} : K^{\triangleleft} \to \mathsf{C}$, being a limit diagram is a property. However, for an object x, being a limit of a given diagram $u : K \to \mathsf{C}$ is a structure.*

Exercise 6.4. *The limit of an empty diagram is the final object.*

6.5. Proposition 2.7 allows us to talk about *the* limit object of a diagram $u : K \to \mathsf{C}$, as long as we incorporate the extended diagram $\overline{u}$ as part of the data in its definition. There are various standard notations for the limit object:

$$\mathsf{lim}\, u,\ \underset{K}{\mathsf{lim}}\, u,\ \underset{y\in K}{\mathsf{lim}}\, u(y), \cdots.$$

When using these notations, we often assume such a limit exsits, and view it as an object in C equipped with a *canonical* lifting along the forgetful functor $\mathsf{C}_{/u} \to \mathsf{C}$.

In particular, there are canonical morphisms,

$$\mathsf{ev}_y : \mathsf{lim}\, u \to u(y),\ y \in K,$$

called the **evaluating morphisms**. Dually, we have the **inserting morphisms**

$$\mathsf{ins}_y : u(y) \to \mathsf{colim}\, u,\ y \in K$$

Appendix A. Twisted arrows

Exercise A.1. *Let* C *be an ordinary category. Construct a category* $\mathsf{Tw}(\mathsf{C})$ *such that*

- *Objects in* $\mathsf{Tw}(\mathsf{C})$ *are morphisms* $x \to y$ *in* C*.*
- *The assignment* $(x \to y) \mapsto x$ *is contravariant, i.e., given by a functor* $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}^{\mathsf{op}}$*.*
- *The assignment* $(x \to y) \mapsto y$ *is covariant, i.e., given by a functor* $\mathsf{Tw}(\mathsf{C}) \to \mathsf{C}$*.*

We call it the ***category of twisted arrows*** *in* C*.*

Exercise A.2. *Generalize the above construction to* ∞*-categories.*

Exercise A.3. *Construct a functor* $\mathsf{Tw}(\mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ *sending a twisted arrow* $\mathsf{K} \xrightarrow{u} \mathsf{C}$ *to* $\mathsf{C}_{/u}$*.*

A.4. **Suggested readings.** Ker.03GB.