NOTES FOR ALGEBRAIC GEOMETRY 1

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Contents

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0. Introduction: why schemes?

0.1. **Algebraic sets.** Before scheme theory, algebraic geometry focused on *algebraic sets*.

Definition 0.1.1. Let k be an algebraically closed field.

- The **Zariski topology** on the affine space $\mathbb{A}^n_k$ is the topology with a base consisting of all the subsets that are equal to the non-vanishing locus $U(f)$ of some polynomial $f \in k[x_1, \cdots, x_n]$.
- An **embedded affine algebraic set**[1] in $\mathbb{A}^n_k$ is a closed subspace for the Zariski topology.
- An **embedded quasi-affine algebraic set** is a Zariski open subset of an embedded affine algebraic set.

Example 0.1.2. Any finite subset of $\mathbb{A}^n_k$ is an embedded affine algebraic set.

Example 0.1.3. $\mathbb{Z}$ is not an embedded affine algebraic set in $\mathbb{A}^1_{\mathbb{C}}$.

Similarly one can define *embedded (quasi)-projective algebraic set* using *homogeneous* polynomials and the projective space $\mathbb{P}^n_k$.

There are tons of shortcomings in the above definition. An obvious one is that the notions of *embedded* algebraic sets are not *intrinsic*.

Example 0.1.4. The embedded affine algebraic sets $\mathbb{A}^1_k \subseteq \mathbb{A}^1_k$ and $\mathbb{A}^1_k \subseteq \mathbb{A}^2_k$ should be viewed as the same algebraic sets.

Notation 0.1.5. To remedy this, we need some notations.

- For an ideal $I \subseteq k[x_1, \cdots, x_n]$, let $Z(I) \subseteq \mathbb{A}^n_k$ be the locus of common zeros of polynomials in I.
- For a Zariski closed subset $X \subseteq \mathbb{A}^n_k$, let $I(X) \subseteq k[x_1, \cdots, x_n]$ be the ideal of all polynomials vanishing on X.

Recall an ideal I is called *radical* if $I = \sqrt{I}$.

Theorem 0.1.6 (Hilbert Nullstellensatz)**.** *We have a bijection:*

$$\begin{array}{rcl} \{\textit{radical ideals of } k[x_1, \cdots, x_n]\} & \longleftrightarrow & \{\textit{Zariski closed subsets of } \mathbb{A}^n_k\} \\ I & \longrightarrow & Z(I) \\ I(X) & \longleftarrow & X. \end{array}$$

Part of the theorem says the set of points of $\mathbb{A}^n_k$ is in bijection with the set of maximal ideals of $k[x_1, \cdots, x_n]$. As a corollary, $Z(I)$ is in bijection with the set of maximal ideals containing I. The latter can be further identified with maximal ideals of $R := k[x_1, \cdots, x_n]/I$.

Note that I is radical iff R is *reduced*, i.e., contains no nilpotent elements. This justifies the following definition.

Definition 0.1.7. An **affine algebraic k-set** is a *maximal spectrum* $\mathsf{Spm}\, R$ (= sets of maximal ideals) of a *finitely generated* (commutative unital) *reduced* k-algebra R. We equip it with the **Zariski topology** with a base of open subsets given by

$$U(f) := \{\mathfrak{m} \in \mathsf{Spm}\, R \mid f \notin \mathfrak{m}\}, \ f \in R.$$

[1] Some people use the word *variety*, while some people reserve it for *irreducible* algebraic sets.

Example 0.1.8. $\mathsf{Spm}\, k[x] \simeq \mathbb{A}^1_k$.

We have the following *duality* between algebra and geometry.

Algbera	Geometry
finitely generated reduced k-algebra R	affine algebraic k-set X
maximal ideals $\mathfrak{m} \subseteq R$	points $x \in X$
elements $f \in R$	functions $\phi : X \to \mathbb{A}^1_k$
radical ideals $I \subseteq R$	Zariski closed subsets $Z \subseteq X$

Here an element $f \in R$ corresponds to the function

$$\phi : \mathsf{Spm}\, R \to k,\ \mathfrak{m} \mapsto \underline{f}$$

sending a maximal ideal $\mathfrak{m}$ to the image $\underline{f}$ of f in the *residue field* of $\mathfrak{m}$, which is canonically identified with the underlying set of $\mathbb{A}^1_k$ via the composition $k \to R \to R/\mathfrak{m}$.

The word *duality* means the correspondence $R \leftrightarrow X$ is *contravariant*. Indeed, given a homomorphism $f : R' \to R$, we obtain a *continuous* map

$$\mathsf{Spm}\, R \to \mathsf{Spm}\, R',\ \mathfrak{m} \mapsto f^{-1}(\mathfrak{m}).$$

Note however that not all continuous maps $\mathsf{Spm}\, R \to \mathsf{Spm}\, R'$ are obtained in this way, nor is R determined by the topological space $\mathsf{Spm}\, R$.

Exercise 0.1.9. Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology. Find those bijections coming from a homomorphism $k[x] \to k[x]$.

This motivates the following definition.

Definition 0.1.10. A **morphism** from $\mathsf{Spm}\, R$ to $\mathsf{Spm}\, R'$ is a continuous map coming from a homomorphism $R' \to R$.

Then one can define general algebraic k-sets by gluing affine algebraic k-sets using morphisms, just like how people define *structured* manifolds as glued from *structured* Euclidean spaces using maps preserving the addiontal structrues.

0.2. **Shortcomings.** The theory of algebraic k-sets provides a bridge between algebra and geometry. In particular, one can use topological/geometric methods, including various cohomology theories, to study *finitely generated reduced k-algebras*. However, these adjectives are non-necessary restrictions to this bridge.

First, number theory studies number fields and their rings of integers, such as $\mathbb{Q}$ and $\mathbb{Z}$. It is desirable to have geometric objects corresponding to them. Hence we would like to consider general commutative rings rather than k-algebras. Then one immediately realizes the maximal spectra Spm are not enough.

Example 0.2.1. The map $\mathbb{Z} \to \mathbb{Q}$ does not induce a map from $\mathsf{Spm}\, \mathbb{Q}$ to $\mathsf{Spm}\, \mathbb{Z}$. Namely, the inverse image of $(0) \subseteq \mathbb{Q}$ in $\mathbb{Z}$ is a non-maximal prime ideal.

This suggests for general algebra R, we should consider its *prime spectrum*, denoted by $\mathsf{Spec}\, R$, rather than just its maximal spectrum.

Second, even in the study of finitely generated algebras, one naturally encounters non-finitely generated ones.

Example 0.2.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal of a finitely generated algebra. The localization $R_{\mathfrak{p}}$ and its completion $\widehat{R}_{\mathfrak{p}}$ are in general not finitely generated.

Of course, one can restrict their attentions to *Noetherian* rings (and live a happy life). But let me object the false feeling that all natural rings one care are Noetherian.

Example 0.2.3. Noetherian rings are not stable under tensor products: $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not Noetherian.

Example 0.2.4. The ring of adeles of $\mathbb{Q}$ is not Noetherian.

Example 0.2.5. Natural moduli problems about Noetherian objects can fail to be Noetherian. My favorite ones includes: the space of formal connections, the space (stack) of formal group laws...

In short, there are important applications of non-Noetherian rings, and this course will deal with the latter.

Finally, we want to remove the restriction about reducedness.

Example 0.2.6. Reduced rings are not stable under tensor products: $k[x] \otimes_{k[x,y]} k[x,y]/(y-x^2)$ is not reduced. Geometrically, this means $Z(y)$ and $Z(y-x^2)$ do not intersect transversally inside $\mathbb{A}^2_k$.

One may notice that without reducedness, we should accordingly consider all ideals rather than just *radical* ideals, but then the construction $I \mapsto Z(I)$ would not be bijective. Indeed, ideals with the same nilpotent radical would give the same *topological subspace* of $\mathsf{Spec}\, R$.

But *this is a feature rather than a bug*. In Example 0.2.6, the ideal $(y, y-x^2) = (x^2, y)$ is not radical, and it carries geometric meanings that cannot be seen from its nilpotent radical (x,y). Namely, $f \in (x,y)$ iff $f(0,0) = 0$, while $f \in (x^2, y)$ iff $f(0,0) = \partial_x f(0,0) = 0$. Roughly speaking, this suggests that $(y, y-x^2)$ remembers that the curves $Z(y)$ and $Z(y-x^2)$ are tangent to each other at the point $(0,0) \in \mathbb{A}^2_k$, and the tangent vector is $\partial_x|_{(0,0)}$. Also note that the length of $k[x,y]/(y,y-x^2)$ is equal to 2, which is the number of intersection points predicted by the Bézout's theorem.

In summary, on the algebra side, we should consider *all* commutative rings. On the geometric side, the corresponding notion is called *affine schemes*. Our first task in this course is to develop the following duality:

Algebra	Geometry
commutative rings R	affine schemes X
prime ideals $\mathfrak{p} \subseteq R$	points $x \in X$
elements $f \in R$	functions $X \to \mathbb{A}^1_{\mathbb{Z}}$
ideals $I \subseteq R$	closed subschemes $Z \subseteq X$.

0.3. **Schemes as structured spaces.** In theory, one can *define* a morphism between affine schemes to be a continuous map coming from a ring homomorphism. Then one can define general *schemes* by gluing affine schemes using such morphisms. This mimics the definition of differentiable manifold in the sense that a scheme would be a topological space equipped with a *maximal* affine atlas.

In practice, the above approach is awkward to work with. We prefer a more efficient way to encode the additional structure on the topological space underlying a scheme. One extremely simple but powerful way to achieve this, maybe discovered by Serre and popularized by Grothendieck, is the notion of *sheaves* on topological spaces. Roughly speaking, a sheaf $\mathcal{F}$ on X is an *contravariant* assignment

$$U \mapsto \mathcal{F}(U)$$

sending open subsets $U \subseteq X$ to certain structures (e.g. sets, groups, rings) $\mathcal{F}(U)$, such that a certain gluing condition is satisfied. Here contravariancy means that for $U \subseteq V$, we should provide a map $\mathcal{F}(V) \to \mathcal{F}(U)$ preserving the prescribed structures.

Example 0.3.1. Let X be a topological space. The assignment

$$U \mapsto C(U, \mathbb{R})$$

sending $U \subseteq X$ to the ring of continuous functions on U would be a sheaf of commutative rings on X.

Similarly, for a smooth manifold X, $U \mapsto C^\infty(U, \mathbb{R})$ would be a sheaf of commutative rings on X. This motivates us to define:

Pre-Definition 0.3.2. A **scheme** is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X$ such that locally it is isomorphic to an affine scheme.

Here for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ should be the ring of *algebraic* functions on U, but we have not defined the latter notion yet. Nevertheless, for an *affine* scheme $X \simeq \mathsf{Spec}\, R$, the previous discussion suggests we should have $\mathcal{O}_X(X) \simeq R$. As we shall see in future lectures, we can bootstrap from this to get the definition of the entire sheaf $\mathcal{O}_X$.

The goal of this course is to define schemes and study their basic properties.

Part I. (Pre)sheaves

1. Definition of (pre)sheaves

1.1. Presheaves.

Definition 1.1.1. Let X be a topological space and $(U(X), \subseteq)$ be the partially ordered set of open subsets of X. We define the **category $\mathfrak{U}(X)$ of open subsets** in X to be the category associated to the partially ordered set $(U(X), \subseteq)$.

The category $\mathfrak{U}(X)$ can be explicitly described as follows:

- An object in $\mathfrak{U}(X)$ is an open subset $U \subseteq X$.
- If $U \subseteq V$, then $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is a singleton; otherwise $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is empty.
- The identify morphisms and composition laws are defined in the unique way.

Definition 1.1.2. Let X be a topological space and $\mathcal{C}$ be a category.

- A **$\mathcal{C}$-valued presheaf on** X is a functor $\mathcal{F} : \mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C}$.
- A **morphism** $\mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued presheaves is a natural transformation between these functors.

Let Set be the category of sets. By definition, a **presheaf $\mathcal{F}$ of sets**, i.e., a Set-valued presheaf, on X consists of the following data:

- For any open subset $U \subseteq X$, we have a set $\mathcal{F}(U)$, which is called the **set of sections** of $\mathcal{F}$ on U.
- For $U \subseteq V$, we have a map
$$\mathcal{F}(V) \to \mathcal{F}(U),\ s \mapsto s|_U$$
which is called the **restriction map**.

These data should satisfy the following condition:

- For any open subset $U \subseteq X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
- For $U \subseteq V \subseteq W$, the restriction maps make the following diagram commute
$$\begin{tikzcd} & \mathcal{F}(V) \arrow[dr] & \\ \mathcal{F}(U) \arrow[ur] \arrow[rr] & & \mathcal{F}(W). \end{tikzcd}$$

Let $\mathcal{F}$ adn $\mathcal{F}'$ be presheaves of sets on X. By definition, a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ consists of the following data:

- For any open subset $U \subseteq X$, we have a map $\phi_U : \mathcal{F}(U) \to \mathcal{F}(U)'$.

These data should satisfy the following condition:

- For $U \subseteq V$, the following diagram commute
$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d] & \mathcal{F}'(V) \arrow[d] \\ \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{F}'(U), \end{tikzcd}$$
where the vertical maps are restriction maps.

Similarly one can explicitly describe the notion of presheaves of abelian groups (k-vector spaces, commutative algebras) and morphisms between them.

Example 1.1.3. Let X be a topological space and $\mathcal{C}$ be a category. For any object $A \in \mathcal{C}$, the constant functor

$$\mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C},\ U \mapsto A,\ f \mapsto \mathsf{id}_A$$

defines a $\mathcal{C}$-valued presheaf on X, which is called the **constant presheaf associated to** A. It is often denoted by $\underline{A}$.

Example 1.1.4. Let X be a topological space and $E \to X$ be a topological space over it. We define a presheaf Sect_E of sets as follows.

- For any $U \subseteq X$,
$$\mathsf{Sect}_E(U) := \mathsf{Hom}_X(U, E)$$
is the set of countinuous maps $U \to E$ defined over X, a.k.a. sections of E over U.
- For $U \subseteq V$, the restriction map $\mathsf{Sect}_E(V) \to \mathsf{Sect}_E(U)$ sends a section $s : V \to E$ to its restriction $s|_U : U \to E$.

We call it the **presheaf of sections for** $E \to X$.

Example 1.1.5. If $E \to X$ is a real vector bundle, we can naturally upgrade Sect_E to be a presheaf of real vector spaces on X.

Example 1.1.6. Consider the constant real line bundle $\mathbb{R} \times X$ on X. Note that $\mathsf{Sect}_{\mathbb{R}\times X}(U)$ can be identified with the set of continuous functions on U. It follows that we can upgrade $\mathsf{Sect}_{\mathbb{R}\times X}$ to be a presheaf of $\mathbb{R}$-algebra on X.

1.2. **Sheaves of sets.**

Roughly speaking, a sheaf is a presheaf whose sections on small open subsets can be uniquely glued to sections on larger ones.

Definition 1.2.1. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. We say $\mathcal{F}$ is a **sheaf** if it satisfies the following condition:

(*) For any open covering $U = \bigcup_{i\in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that
$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \text{ for any } i, j \in I,$$
there is a *unique* section $s \in \mathcal{F}(U)$ such that
$$s_i = s|_U \text{ for any } i \in I.$$

Remark 1.2.2. Using the language of category theory, the sheaf condition is equivalent to the following condition:

- For any open covering $U = \bigcup_{i\in I} U_i$, the diagram
$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$
is an *equalizer* diagram. Here the first map is
$$s \mapsto (s|_{U_i})_{i\in I}$$
the other two maps are
$$(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j})_{(i,j)\in I^2}$$
and
$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

In particular, the map $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ is an injection.

Remark 1.2.3. For $U = \varnothing$ and $I = \varnothing$, the sheaf condition says there is a unique section $s \in \mathcal{F}(\varnothing)$ subject to no property. In other words, the above definition forces $\mathcal{F}(\varnothing)$ to be a singleton.

Example 1.2.4. Let X be a topological space. The constant presheaf $\underline{A}$ associated to a set A is in general not a sheaf. Indeed, $\underline{A}(\varnothing)$ is A rather than a singleton.

We provide another reason for readers uncomfortable with the above. For a sheaf $\mathcal{F}$ and *disjoint* open subsets U_1 and U_2, the sheaf condition implies

$$\mathcal{F}(U_1 \sqcup U_2) \simeq \mathcal{F}(U_1) \times \mathcal{F}(U_2).$$

But in general A and $A \times A$ are not isomorphic.

Example 1.2.5. Let $E \to X$ be a continuous map between topological spaces. The presheaf Sect_E of sections on X is a sheaf. Indeed, this follows from the fact that continuous maps can be glued.

Example 1.2.6. Let $\{*\}$ be a 1-point space. Then a sheaf $\mathcal{F}$ of sets on $\{*\}$ is uniquely determined by the set $\mathcal{F}(\{*\})$ of global sectoins. We often abuse the notations and use a set A to denote the sheaf on $\{*\}$ whose set of global sections is A.

Exercise 1.2.7. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X.

(1) Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X and $\alpha : \mathcal{F}|_{\mathfrak{B}} \to \mathcal{F}'|_{\mathfrak{B}}$ be a natural transformation between their restrictions on the full subcategory $\mathfrak{B}^{\mathsf{op}} \subseteq \mathfrak{U}(X)^{\mathsf{op}}$. Show that α can be uniquely extended to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$.
(2) Show that the claims in (1) remain true if $\mathcal{F}$ is only assumed to be a presheaf.
(3) Show that the claims in (1) on existence and uniqueness can both fail if $\mathcal{F}'$ is only assumed to be a presheaf..

The above exercise says sheaves are determined by their restrictions on a topological base. A natural question is, given a functor $\mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$, under what conditions can we extend it to a sheaf $\mathfrak{U}(X) \to \mathsf{Set}$? This question is relevant to us because the Zariski topology of $\mathsf{Spec}\, R$ is defined using a base consisting of open subsets that can be easily described:

$$U(f) := \left\{\mathfrak{p} \in \mathsf{Spec}\, R \,|\, f \notin \mathfrak{p}\right\} \simeq \mathsf{Spec}\, R_f.$$

It would be convenient if we can recover a sheaf $\mathcal{F}$ on $\mathsf{Spec}\, R$ from its values on these open subsets. For instance, we wonder whether the contravariant functor

$$U(f) \mapsto R_f$$

can be extended to a sheaf of commutative rings. If yes, we would obtain the sheaf $\mathcal{O}_X$ of algebraic functions desired in the introduction. The following construction gives a positive answer to this question.

Construction 1.2.8. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X. For a functor $\mathcal{F} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$ and $U \in \mathfrak{U}(X)$, define

$$\mathcal{F}'(U) := \lim_{V \in \mathfrak{B}^{\mathsf{op}},\, V \subseteq U} \mathcal{F}(V).$$

In other words, an element in $s' \in \mathcal{F}'(U)$ is a collection of elements $s_V \in \mathcal{F}(V)$ for all open subsets $V \subseteq U$ contained in $\mathfrak{B}$ such that for $V_1 \subseteq V_2 \subseteq U$ with $V_1, V_2 \in \mathfrak{B}$, the map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$ sends s_{V_2} to s_{V_1}. This construction is clearly functorial in U, i.e., for $U_1 \subseteq U_2$, we have a natural map $\mathcal{F}'(U_2) \to \mathcal{F}'(U_1)$. One can check this defines a functor

$$\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$$

equipped with a canonical isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$. In other words, we have extended $\mathcal{F}$ to a *presheaf* $\mathcal{F}'$ of sets on X.

Remark 1.2.9. Using the language in category theory, the functor $\mathcal{F}'$ is the *right Kan extension* of $\mathcal{F}$ along the embedding $\mathfrak{B}^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Proposition 1.2.10. *In above, $\mathcal{F}'$ is a sheaf iff $\mathcal{F}$ satisfies the following condition:*

(**) *For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, and any collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that*

$$s_i|_V = s_j|_V \text{ for any } i, j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$

there is a unique *section $s \in \mathcal{F}(U)$ such that*

$$s_i = s|_{U_i} \text{ for any } i \in I.$$

Proof. The "only if" statement follows from the sheaf condition on $\mathcal{F}'$ and the isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$.

For the "if" statement, we verify the sheaf condition on $\mathcal{F}'$ directly. Let $U = \bigcup_{i\in I} U_i$ be an open covering, and $s'_i \in \mathcal{F}'(U_i)$ be a collection of sections such that

$$s'_i|_{U_i\cap U_j} = s'_j|_{U_i\cap U_j} \text{ for any } i, j \in I.$$

By Construction 1.2.8, each s'_i corresponds to a collection $s_{i,V} \in \mathcal{F}(V)$ for $V \subseteq U_i$, $V \in \mathfrak{B}$ that is compatible with restrictions.

We need to show there is a unique section $s' \in \mathcal{F}'(U)$ such that $s'|_{U_i} = s'_i$.

We first deal with the existence. For any $V \subseteq U$ with $V \in \mathfrak{B}$, since $\mathfrak{B}$ is a base, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. In other words, we can choose a map $f : J \to I$ such that $V_j \subseteq U_i$.

Consider the collection of sections

$$t_{j,V} \coloneqq s_{f(j),V_j} \in \mathcal{F}(V_j),\ j \in J. \tag{1.1}$$

One can check it does not depend on the choice of f and they satisfy the assumption in (**). Hence there is a unique section $s'_V \in \mathcal{F}(V)$ such that $s'_V|_{V_j} = s_{f(j),V_j}$.

One can check the obtained section s'_V does not depend on the open covering $V = \bigcup_{j\in J} V_j$ and the collections (s'_V), $V \subseteq U$, $V \in \mathfrak{B}$ is compatible with restrictions. Hence by Construction 1.2.8, it corresponds to an element $s' \in \mathcal{F}'(U)$. One can check that $s'|_{U_i} = s'_i$. This proves the claim about uniqueness.

It remains to prove the statement about uniqueness. Suppose there are two such sections s', s'' such that

$$s'|_{U_i} = s''|_{U_i} = s'_i \tag{1.2}$$

By Construction 1.2.8, they correspond to two collections $s'_V, s''_V \in \mathcal{F}(V)$ for $V \subseteq U$, $V \in \mathfrak{B}$. We only need to show $s'_V = s''_V$.

Note that if V is contained in some U_i, then (1.2) implies

$$s'_V = s''_V = s_{i,V}. \tag{1.3}$$

Now for general open subset $V \subseteq U$, $V \in \mathfrak{B}$, as before, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. Consider the collection of sections (1.1). By (1.3) (applied to each V_j), we have

$$s'_V|_{V_j} = s''_V|_{V_j} = t_{j,V}.$$

Hence by (**), we must have $s'_V = s''_V$ as desired.

□

1.3. $\mathcal{C}$-valued sheaves.

Definition 1.3.1. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on a topological space X. We say $\mathcal{F}$ is a **$\mathcal{C}$-valued sheaf** if for any test object $c \in \mathcal{C}$, the functor

$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$

is a sheaf of sets.

Remark 1.3.2. By Yoneda's lemma and Remark 1.2.2, $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf iff for any open covering $U = \bigcup_{i\in I} U_i$, the canonical diagram

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is an *equalizer* diagram in $\mathcal{C}$. Here the first morphism is given by restrictions along $U_i \subseteq U$, while the other two morphisms are given respectively by restrictions along $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$. In particular, the morphism

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

is a *monomorphism*[2].

As a corollary of the remark, we obtain:

Corollary 1.3.3. *Let $\mathcal{F}$ be a presheaf of abelian groups. Then $\mathcal{F}$ is a sheaf of abelian groups iff its underlying presheaf of sets $\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Ab} \to \mathsf{Set}$ is a sheaf of sets. Here the functor $\mathsf{Ab} \to \mathsf{Set}$ sends an abelian group to its underlying set.*

Exercise 1.3.4. Let $\mathcal{F}$ be a presheaf of abelian groups. Show that $\mathcal{F}$ is a sheaf of abelian groups iff for any open covering $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact. Here the second map is

$$s \mapsto (s|_{U_i})_{i\in I},$$

and the third map is

$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

Now suppose $\mathcal{F}$ is a sheaf, can you further extend this exact sequence to the right?

[2]This means for any test object $c \in \mathcal{C}$, the functor $\mathsf{Hom}_{\mathcal{C}}(c,-)$ sends this morphism to an injection between sets.

Remark 1.3.5. Let $\mathcal{C}$ be a category that admits small limits. Then Construction 1.2.8 and Proposition 1.2.10 can be generalized to $\mathcal{C}$-valued (pre)sheaves with condition (**) replaced by

- For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, any object $c \in \mathcal{C}$, and any collection of elements $s_i \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U_i))$, $i \in I$ such that
$$s_i|_V = s_j|_V \text{ for any } i,j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$
there is a *unique* element $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ such that
$$s_i = s|_{U_i} \text{ for any } i \in I.$$

In above $s|_V$ means the post-composition of $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ with the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.

Note however for $\mathcal{C} = \mathsf{Ab}$, we can keep condition (**) *as it is*, because the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ detects limits.

2. Stalks

2.1. Definition.

Definition 2.1.1. Let X be a topological space and $\mathcal{F}$ be a presheaf of sets on X. For a point $x \in X$, let $\mathfrak{U}(X,x) \subseteq \mathfrak{U}(X)$ be the full subcategory of open neighborhoods of x inside X. The **stalk of $\mathcal{F}$ at x** is

$$\mathcal{F}_x := \operatorname*{colim}_{U\in\mathfrak{U}(X,x)^{\mathsf{op}}} \mathcal{F}(U). \tag{2.1}$$

For a given section $s \in \mathcal{F}(U)$, the **germ of s at x**, denoted by s_x, is the image of s under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$.

Note that $\mathfrak{U}(X,x)^{\mathsf{op}}$ is the category associated to the *direct set*[3] $(U(X,x), \subseteq)$ of open neighborboods of x inside X. Hence the above colimit is a *direct colimit*[4]. It follows that $\mathcal{F}_x$ can be explicitly described as the quotient

$$\Big(\coprod_{U\in U(X,x)} \mathcal{F}(U)\Big)/\sim, \tag{2.2}$$

of the disjoint union of all $\mathcal{F}(U)$, $U \in U(X,x)$ by an equivalence relation $\sim$. Here two sections $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(U')$ are equivalent iff there exists $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Using this description, the germ s_x of a section $s \in \mathcal{F}(U)$ is just the equivalence class to which it belongs.

Remark 2.1.2. In general, let $\mathcal{C}$ be a category that admits direct colimits and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf. We can define the stalk of $\mathcal{F}$ at x using the same formula (2.1). Note that this construction is functorial in $\mathcal{F}$.

In particular, for a presheaf $\mathcal{F}$ of abelian groups, we can define its stalk $\mathcal{F}_x$, which is an abelian group. It is easy to see the underlying set $\mathcal{F}_x$ is given by (2.2) and the group structure is given by the formula

$$s_x + s'_x = (s|_V + s'|_V)_x,\ s \in \mathcal{F}(U), s' \in \mathcal{F}(U'), V \subseteq U \cap U'.$$

---

[3]A direct set is a partially ordered set $(I, \le)$ such that any finite subset of I admits an upper bound in I.

[4]Some people use the word *direct limit*. I strongly object this terminology.

Remark 2.1.3. Let $\mathcal{C}$ be a category such that taking direct colimits in $\mathcal{C}$ commutes with *finite* limits[5]. The the functor $\mathcal{F} \mapsto \mathcal{F}_x$ commutes with finite limits.

2.2. **Sheaves and stalks.** The following result says a section of a *sheaf* is determined by its germs.

Lemma 2.2.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X. Then for any open subset $U \subseteq X$, the map*

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x,\ s \mapsto (s_x)_{x \in U} \tag{2.3}$$

is injective. Moreover, a collection of elements $s(x) \in \mathcal{F}_x$, $x \in U$ is contained in the image of this map iff it satisfies the following condition

(***) *For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.*

Proof. We first show the map (2.3) is injective. Let $s, s' \in \mathcal{F}(U)$ such that all their germs are equal. By definition, for any $x \in U$, there exists $V \subseteq U$ such that $s|_V = s'|_V$. In particular, we can find an open covering $U = \bigcup_{i \in I} U_i$ such that $s|_{U_i} = s'|_{U_i}$. But this implies $s = s'$ because the sheaf condition implies

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

is injective.

It is obvious that any element in the image of (2.3) satisfies condition (***). To prove the converse, let $s(x) \in \mathcal{F}_x$, $x \in U$ be a collection of elements satisfying condition (***). By assumption, we can find an open covering $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that for any $x \in U_i$, we have

$$t(x) = (s_i)_x. \tag{2.4}$$

In particular, the germs of $s_i|_{U_i \cap U_j}$ and $s_j|_{U_i \cap U_j}$ are equal. Applying the injectivity of (2.3) to $U_i \cap U_j$, we obtain

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Hence by the sheaf condition, we can find a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. For any $x \in U$, pick $i \in I$ such that $x \in U_i$, we have

$$s_x = (s_i)_x = t(x),$$

where the first equality is due to the definition of stalks, while the second one is (2.4). In particular, $s(x) \in \mathcal{F}_x$, $x \in U$ is the image of s under the map (2.3). □

Remark 2.2.2. Similar claim for presheaves is false in general. Namely, for $U = X = \varnothing$, the empty product $\prod_{x \in \varnothing} \mathcal{F}_x$ is a singleton, while $\mathcal{F}(\varnothing)$ can be any set.

Corollary 2.2.3. *If $\alpha, \beta : \mathcal{F} \to \mathcal{F}'$ are morphisms between sheaves of sets such that $\alpha_x = \beta_x$ for any $x \in X$, then $\alpha = \beta$.*

Remark 2.2.4. Using the language in category theory, the above corollary says the functors $(-)_x : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Set}$, $x \in X$ are *jointly conservative.*

[5] This is true for Set, Ab and more generally any compactly generated category. An object c in a (locally small) category $\mathcal{C}$ is compact iff $\mathsf{Hom}_{\mathcal{C}}(c, -)$ preserves small filtered colimits. We say $\mathcal{C}$ is compactly generated if it admits small colimits and any object in $\mathcal{C}$ is isomorphic to a small filtered colimit of compact objects.

Proposition 2.2.5. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism between sheaves of sets on a topological space. Then α is an isomorphism iff for any $x \in X$, $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a bijection.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose α_x is a bijection for any $x \in X$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\alpha_U"] & \prod_{x\in U} \mathcal{F}_x \arrow[d, "(\alpha_x)_{x\in X}", "\simeq"'] \\ \mathcal{F}'(U) \arrow[r] & \prod_{x\in U} \mathcal{F}'_x. \end{tikzcd}$$

By Lemma 2.2.1, the horizontal maps are injective, hence so is α_U.

It remains to show α_U is surjective. Let $s' \in \mathcal{F}'(U)$ be a section, we will construct a section $s \in \mathcal{F}(U)$ mapping to it by α_U.

For any point $x \in U$, since α_x is bijective, we can find an open subset $V \subseteq X$ and a section $t \in \mathcal{F}(V)$ such that $\alpha_x(t_x) = s'_x$. By definition, $\alpha_x(t_x) = \alpha_V(t)_x$. Hence the germs of $\alpha_V(t)$ and s' at x are equal. By definition, there exists an open neighborhood W of x inside $U \cap V$ such that $\alpha_V(t)|_W = s'|_W$. Note that we also have $\alpha_V(t)|_W = \alpha_W(t|_W)$.

It follows that we can find an open covering $U = \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. In particular, we have

$$\alpha_{U_i\cap U_j}(s_i|_{U_i\cap U_j}) = \alpha_{U_i\cap U_j}(s_j|_{U_i\cap U_j}) = s|_{U_i\cap U_j}.$$

Since we have already shown $\alpha_{U_i\cap U_j}$ is injective, we obtain $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Hence by the sheaf condition for $\mathcal{F}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Using the sheaf condition for $\mathcal{F}'$, it is easy to see $\alpha_U(s) = s'$ as desired.

□

The above results imply that a *morphism* between sheaves are determined by the induced maps between the stalks. However, a sheaf itself is *not* determined by its stalks.

Exercise 2.2.6. Let X be a connected topological space and $E \to X$ and $E' \to X$ be two covering spaces of the same degree. Show that the sheaves Sect_E and $\mathsf{Sect}_{E'}$ on X have isomorphic stalks for any point $x \in X$, but they are not isomorphic unless there exists a homeomorphism $E \simeq E'$ defined over X.

Remark 2.2.7. Let $\mathcal{C}$ be a *compactly generated* category. Lemma 2.2.1 and Proposition 2.2.5 can be generalized to $\mathcal{C}$-valued sheaves. In other words:

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$, the morphism $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is a monomorphism.
- A morphism $\alpha : \mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued sheaves is an isomorphism iff $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for any $x \in X$.

These statements can be deduced from the special case for Set with the help of the following two observations:

- A morphism $d \to d'$ in $\mathcal{C}$ is a monomorphism (resp. isomorphism) iff for any *compact* object $c \in \mathcal{C}$, the map $\mathsf{Hom}_{\mathcal{C}}(c, d) \to \mathsf{Hom}_{\mathcal{C}}(c, d')$ is an injection (resp. bijection).

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$ and any *compact* object $c \in \mathcal{C}$, the stalk of the Set-valued sheaf
$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$
at $x \in X$ is canonically isomorphic to $\mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}_x)$.

The details are left to the curious readers.

2.3. **Skyscrapers.**

Definition 2.3.1. Let X be a topological space and $x \in X$ be a point. For any set A, we can define a presheaf $\delta_{x,A}$ of sets as follows.

- For an open subset $U \subseteq X$,
  - if $x \in U$, define $\delta_{x,A}(U) \coloneqq A$;
  - if $x \notin U$, define $\delta_{x,A}(U) \coloneqq \{*\}$.
- For open subsets $U \subseteq V$,
  - if $x \in U$ (and therefore $x \in V$), define the restriction map $\delta_{x,A}(U)$ to be id_A;
  - if $x \notin U$, define the restriction map to be the unique map $\delta_{x,A}(V) \to \delta_{x,A}(U) = \{*\}$.

One can check this indeed defines a presheaf $\delta_{x,A}$. We call the the **skyscraper sheaf** at x with value A.

Exercise 2.3.2. The presheaf $\delta_{x,A}$ is indeed a sheaf.

Lemma 2.3.3. *Let X be a topological space, $x \in X$ be a point and A be a set. The stalk of $\delta_{x,A}$ at a point $y \in X$ is canonically bijective to*

- *the set A if y is contained in $\overline{\{x\}}$, the closure of $\{x\}$ inside X;*
- *the singleton $\{*\}$ otherwise.*

Proof. If $y \in \overline{\{x\}}$, then any open neighborhood of y contains x. It follows that
$$(\delta_{x,A})_y \coloneqq \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} A$$
is a direct colimit of the constant diagram with values A. This implies $(\delta_{x,A})_y \simeq A$.

If $y \notin \overline{\{x\}}$, then there exists an open neighborhood V of y such that $x \notin V$. Note that $\mathfrak{U}(V,y)^{\mathsf{op}} \subseteq \mathfrak{U}(X,y)^{\mathsf{op}}$ is (co)final. If follows that
$$(\delta_{x,A})_y \coloneqq \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq (\delta_{x,A})_y \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \{*\}$$
is a direct colimit of the constant diagram with values $\{*\}$. This implies $(\delta_{x,A})_y \simeq \{*\}$.

□

Note that if A is equipped with the structure of an abelian group, the skyscraper $\delta_{x,A}$ can be upgraded to a sheaf of abelian groups. Then the abelian group $(\delta_{x,A})_y$ is either A or 0.

Proposition 2.3.4. *Let X be a topological space, $x \in X$ be a point and A be a set. For any presheaf $\mathcal{F}$ of sets on X, the composition*
$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \xrightarrow{(-)_x} \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, (\delta_{x,A})_x) \simeq \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \tag{2.5}$$
is an bijection.

Corollary 2.3.5. *The stalk functor*

$$\mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{Set},\ \mathcal{F} \mapsto \mathcal{F}_x$$

admits a right adjoint

$$\mathsf{Set} \to \mathsf{PShv}(X, \mathsf{Set}),\ A \mapsto \delta_{A,x}.$$

Proof of Proposition 2.3.4. We first construct a map

$$\mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \tag{2.6}$$

as follows. Given any map $f : \mathcal{F}_x \to A$, for any open subset $U \subseteq X$, we define a map $\alpha_U : \mathcal{F}(U) \to \delta_{x,A}(U)$ such that:

- If $x \in U$, α_U is the composition $\mathcal{F}(U) \to \mathcal{F}_x \xrightarrow{f} A$;
- If $x \notin U$, α_U is the unique map $\mathcal{F}(U) \to \{*\}$.

One can check these maps are compatible with restriction and therefore define a morphism $\alpha : \mathcal{F} \to \delta_{x,A}$. Now we define the map (2.6) to be $f \mapsto \alpha$.

One can check that (2.5) and (2.6) are inverse to each other. Hence both are bijections.

□

Remark 2.3.6. In general, for any category $\mathcal{C}$ admitting a final object[6] and any object $A \in \mathcal{C}$, one can define a $\mathcal{C}$-valued sheaf $\delta_{x,A}$. If $\mathcal{C}$ admits direct colimits, the stalks of $\delta_{x,A}$ are either A or the final object of $\mathcal{C}$, and the functor $A \mapsto \delta_{A,x}$ is right adjoint to $\mathcal{F} \mapsto \mathcal{F}_x$.

[6] An object $* \in \mathcal{C}$ is a final object iff for any $c \in \mathcal{C}$, there is a unique morphism $c \to *$.

3. Category of (pre)sheaves

Let X be a topological space and $\mathcal{C}$ be a category. Note that $\mathcal{C}$-valued presheaves on X form a category

$$\mathsf{PShv}(X,\mathcal{C}) \coloneqq \mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}),$$

and $\mathcal{C}$-valued sheaves form a full subcategory

$$\mathsf{Shv}(X,\mathcal{C}) \subseteq \mathsf{PShv}(X,\mathcal{C}).$$

In this section, we study the basic properties of these categories.

3.1. Sheafification.

Definition 3.1.1. Let $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$. The **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^\sharp \in \mathsf{Shv}(X,\mathsf{Set})$ equipped with a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ such that for any test sheaf $\mathcal{G}$, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha\circ\theta.$$

Proposition 3.1.2. *For any $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$, its sheafification $(\mathcal{F}^\sharp,\theta)$ exists, and is unique up to unique isomorphism. Moreover, the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ induces bijections $\mathcal{F}_x \to \mathcal{F}^\sharp_x$ between the stalks.*

Proof. The statement about uniqueness follows from Yoneda's lemma. To prove the existence, we construct a sheafification as follows.

We first construct the desired sheaf $\mathcal{F}^\sharp$. For any open subset $U \subseteq X$, let

$$\mathcal{F}^\sharp(U) \subseteq \prod_{x\in U}\mathcal{F}_x,$$

be the subset consisting of elements $(s(x))_{x\in U}$ satisfying the following condition:

- For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.

For $U \subseteq U'$, it is obvious that the projection map $\prod_{x\in U'}\mathcal{F}_x \to \prod_{x\in U}\mathcal{F}_x$ sends $\mathcal{F}^\sharp(U')$ into $\mathcal{F}^\sharp(U)$. Moreover, one can check the obtained maps $\mathcal{F}^\sharp(U') \to \mathcal{F}^\sharp(U)$ upgrade the assignment $U \mapsto \mathcal{F}^\sharp(U)$ to an object in $\mathsf{Shv}(X,\mathsf{Set})$.

Now we construct the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$. For any open subset $U \subseteq X$, consider the map

$$\mathcal{F}(U) \to \prod_{x\in U}\mathcal{F}_x,\ s \mapsto (s_x)_{x\in U}.$$

It is obvious that the image of this map is contained in $\mathcal{F}^\sharp(U)$. Moreover, the obtained maps $\mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ is functorial in U, therefore give a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$.

It remains to show $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ exhibits $\mathcal{F}^\sharp$ as a sheafification of $\mathcal{F}$. Let $\mathcal{G}$ be a test sheaf, we need to show

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha\circ\theta \tag{3.1}$$

is bijective. Let $\beta : \mathcal{F} \to \mathcal{G}$ be a morphism. For any open subset $U \subseteq X$, recall taking germs induces an injection

$$\mathcal{G}(U) \to \prod_{x\in U}\mathcal{G}_x$$

and its image is desribed in Lemma 2.2.1. Using that description, it is clear that there is a unique dotted map making the following diagram commute:

$$\begin{tikzcd} \mathcal{F}^\sharp(U) \arrow[r, "\subseteq"] \arrow[d, dotted] & \prod_{x\in U}\mathcal{F}_x \arrow[d, "(\beta_x)_{x\in U}"] \\ \mathcal{G}(U) \arrow[r] & \prod_{x\in U}\mathcal{G}_x. \end{tikzcd}$$

Moreover, the obtained map $\mathcal{F}^\sharp(U) \to \mathcal{G}(U)$ is functorial in U. Hence we obtain a morphism $\beta^\sharp : \mathcal{F}^\sharp \to \mathcal{G}$. Now one can check that the map

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}) \to \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}),\ \beta \mapsto \beta^\sharp$$

and (3.1) are inverse to each other. In particular, they are both bijective as desired. □

Corollary 3.1.3. *The fully faithful embedding* $\mathsf{Shv}(X,\mathsf{Set}) \to \mathsf{PShv}(X,\mathsf{Set})$ *admits a left adjoint which sends* $\mathcal{F}$ *to its sheafification* $\mathcal{F}^\sharp$.

Example 3.1.4. Let A be a set. The sheafification $\underline{A}^\sharp$ of the constant presheaf $\underline{A}$ is the sheaf

$$\mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set},\ U \mapsto C(U,A)$$

that sends U to the set of continuous maps from U to A (equipped with the discrete topology). We call it the **constant sheaf** associated to A.

Remark 3.1.5. The sheafification functor $\mathsf{PShv}(X,\mathsf{Set}) \to \mathsf{Shv}(X,\mathsf{Set})$ preserves finite limits. This follows by combining the following four facts:

- The sheafification functor commutes with taking stalks;
- Taking stalks commutes with finite limits (Remark 2.1.3).
- The functors of taking stalks are jointly conservative on the category of sheaves (Remark 2.2.4)

Namely, for any finite diagram of presheaves $\mathcal{F}_i$, $i \in I$, we have

$$(\lim_{i\in I}\mathcal{F}_i)^\sharp_x \simeq (\lim_{i\in I}\mathcal{F}_i)_x \simeq \lim_{i\in I}\mathcal{F}_{i,x} \simeq \lim_{i\in I}(\mathcal{F}_i^\sharp)_x \simeq (\lim_{i\in I}\mathcal{F}_i^\sharp)_x,\ \forall x \in X,$$

and this isomorphism is induced by the canonical morphism

$$(\lim_{i\in I}\mathcal{F}_i)^\sharp \to \lim_{i\in I}\mathcal{F}_i^\sharp.$$

This implies the above morphism is an isomorphism.

Remark 3.1.6. Suppose $\mathcal{F}$ is a presheaf of abelian groups. Let $\mathcal{F}^\sharp$ be the sheafification of the underlying Set-valued presheaf of $\mathcal{F}$ as constructed in the proof of the proposition. One can check that $\mathcal{F}^\sharp(U)$ is a subgroup of the abelian group $\prod_{x\in U}\mathcal{F}_x$. It follows that $\mathcal{F}^\sharp$ can be upgraded to a sheaf of abelian groups. Moreover, for any test sheaf $\mathcal{G}$ of abelian groups, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Ab})}(\mathcal{F}^\sharp,\mathcal{G}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Ab})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta.$$

In other words, $\mathsf{Shv}(X,\mathsf{Ab}) \to \mathsf{PShv}(X,\mathsf{Ab})$ admits a left adjoint which sends $\mathcal{F}$ to $\mathcal{F}^\sharp$.

Remark 3.1.7. In general, if $\mathcal{C}$ is a category admitting small limits and filtered colimits, then any $\mathcal{C}$-valued presheaf admits a sheafification that can be constructed as follows.

For $U \subseteq X$, we can define the *category* Cov_U *of open coverings of* U as follows:

- An object is an open covering $U = \bigcup_{i\in I} U_i$;
- A morphism from $(U_i)_{i\in I}$ to $(V_j)_{j\in J}$ is a map $J \to I$ such that $V_j \subseteq U_i$ for any $j \in J$.

One can show that Cov_U is filtered. Now for any $\mathcal{F} \in \mathsf{PShv}(X, \mathcal{C})$, we have a functor

$$\begin{array}{ccc} \mathsf{Cov}_U & \to & \mathcal{C} \\ (U_i)_{i\in I} & \mapsto & \lim[\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)]. \end{array}$$

sending a covering to the equalizer appeared in the sheaf condition. Note that the identity covering $\{U\}$ is sent to the object $\mathcal{F}(U)$. Now we define

$$\mathcal{F}^+(U) := \operatorname*{colim}_{[(U_i)_{i\in I}]\in \mathsf{Cov}_U} \lim[\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)].$$

By construction, there is a canonical morphism $\mathcal{F}(U) \to \mathcal{F}^+(U)$. Moreover, the above definition is contravariantly functorial in U, therefore we obtain an object $\mathcal{F}^+ \in \mathsf{PShv}(X, \mathcal{C})$ equipped with a canonical morphism $\mathcal{F} \to \mathcal{F}^+$.

In general, $\mathcal{F}^+$ is note a $\mathcal{C}$-valued sheaf. But one can check that for any open covering $U = \bigcup_{i\in I} U_i$, the morphism

$$\mathcal{F}^+(U) \to \prod_{i\in I} \mathcal{F}^+(U_i)$$

is a monomorphism. Using this property, one can show that $(\mathcal{F}^+)^+$ is a sheaf and the composition $\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$ exhibits $(\mathcal{F}^+)^+$ as a sheafification of $\mathcal{F}$.

3.2. Direct images.

Construction 3.2.1. Let $f : X \to X'$ be a continuous map between topological spaces. We have a functor

$$\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}},\ U' \mapsto f^{-1}(U').$$

For any category $\mathcal{C}$, it induces a functor

$$\mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}}, \mathcal{C}) \to \mathsf{Fun}(\mathfrak{U}(X')^{\mathsf{op}}, \mathcal{C}).$$

By definition, this gives a functor

$$f_* : \mathsf{PShv}(X, \mathcal{C}) \to \mathsf{PShv}(X', \mathcal{C}).$$

We call it the **direct image functor** (or **pushforward functor**) along f for $\mathcal{C}$-valued presheaves.

Note that for continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)_* \simeq g_* \circ f_*$.

Explicitly, given a $\mathcal{C}$-valued presheaf $\mathcal{F}$ on X, its **direct image** (or **pushforward**) along f is the presheaf $f_*\mathcal{F}$ defined by

$$f_*\mathcal{F}(U') := \mathcal{F}(f^{-1}(U')),$$

with restriction maps given by those maps for $\mathcal{F}$.

Proposition 3.2.2. *Let $f : X \to X'$ be a continuous map between topological spaces. If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is a sheaf.*

Proof. The sheaf condition for $f_*\mathcal{F}$ and an open covering $U' = \bigcup_{i\in I} U'_i$ is just the sheaf condition for $\mathcal{F}$ and the open covering $f^{-1}(U') = \bigcup_{i\in I} f^{-1}(U'_i)$. □

Example 3.2.3. Let $x \in X$ be a point and write $i : \{x\} \to X$ for the emebdding map. Let $\mathcal{C}$ be a category admitting a final object $*$. For any object $A \in \mathcal{C}$, we have

$$i_*(A) \simeq \delta_{x,A},$$

where we abuse notations and use A to denote the unique $\mathcal{C}$-valued sheaf on $\{x\}$ whose object of global sections is A.

Example 3.2.4. Let $p : X \to \{*\}$ be the obvious projection map. For any *sheaf* $\mathcal{F}$, the direct image $p_*\mathcal{F}$ is uniquely determined by $p_*\mathcal{F}(\{*\})$, which is $\mathcal{F}(X)$ by definition. Hence in this case, we also call p_* is **taking global sections functor**.

Warning 3.2.5. Direct image functors do *not* commute with sheafifications. In other words $f_*(\mathcal{F}^\sharp)$ and $(f_*\mathcal{F})^\sharp$ are in general not isomorphic. For a counterexample, take $\mathcal{F}$ to be a constant *presheaf*.

3.3. **Inverse images for presheaves.**

Construction 3.3.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We define a presheaf $f^{-1}_{\mathsf{PShv}}\mathcal{F}' \in \mathsf{PShv}(X, \mathsf{Set})$ by the following formula

$$f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U) := \operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V),$$

where $\mathfrak{U}(X', f(U)) \subseteq \mathfrak{U}(X')$ is the full subcategory of open neighborhoods of $f(U)$ inside X', and the restriction maps for $f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ are induced by those for $\mathcal{F}'$.

The construction $\mathcal{F}' \to f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for presheaves of sets.

Note that $\mathfrak{U}(X', f(U))^{\mathsf{op}}$ is the category associated to a direct set. Hence $f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U)$ can be calculated as a quotient of

$$\coprod_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V).$$

Example 3.3.2. Let X be a topological space and x be a point. Write $i : \{x\} \to X$ for the embedding. We have

$$\big(i^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(\{x\}) \simeq \mathcal{F}'_x.$$

Remark 3.3.3. The functor $f^{-1}_{\mathsf{PShv}}\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$ is the left Kan extension of $\mathcal{F}' : \mathfrak{U}(X')^{\mathsf{op}} \to \mathsf{Set}$ along the pullback functor $\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Remark 3.3.4. The functor f^{-1}_{PShv} commutes with finite limits because filtered colimits commute with finite limits in Set.

Lemma 3.3.5. *Let X be a topological space and $U \subseteq X$ be an open subset. Write $j : U \to X$ for the embedding map. Then j^{-1}_{PShv} sends sheaves to sheaves.*

Proof. For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V \subseteq U$, unwinding the definitions, we have

$$\big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V) \simeq \mathcal{F}(V).$$

Hence the sheaf condition for $j^{-1}_{\mathsf{PShv}}(\mathcal{F})$ follows from that for $\mathcal{F}$.

□

Warning 3.3.6. For general continuous map $f : X \to X'$, the functor f^{-1}_{PShv} does not send sheaves to sheaves. To see this, consider the projection map $p : X \to \{*\}$.

Construction 3.3.7. Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We construct a morphism

$$\mathcal{F}' \to f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}') \tag{3.2}$$

as follows. For any open subest $U' \subseteq X'$, by definition,

$$\big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U') \simeq \big(f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(f^{-1}(U')) \simeq \operatorname*{colim}_{V \in \mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}} \mathcal{F}'(V).$$

Note that U' is an object in $\mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}$. Hence we have a canonical map

$$\mathcal{F}'(U') \to \big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U').$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.2).

Moreover, we can upgrade these morphisms to a natural transformation

$$\mathsf{Id} \to f_* \circ f^{-1}_{\mathsf{PShv}}. \tag{3.3}$$

Construction 3.3.8. Dually, let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ be a presheaf. We construct a morphism

$$f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F}) \to \mathcal{F}. \tag{3.4}$$

as follows. For any open subest $U \subseteq X$, by definition,

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \simeq \operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \big(f_*(\mathcal{F})\big)(V) \simeq \operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}(f^{-1}(V)).$$

Note that for any $V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}$, we have $U \subseteq f^{-1}(V)$, which gives a restriction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$. One can check these maps are functorial in V and give a map

$$\operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U).$$

Hence we obtain a map

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \to \mathcal{F}(U).$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.4).

Moreover, we can upgrade these morphisms to a natural transformation

$$f^{-1}_{\mathsf{PShv}} \circ f_* \to \mathsf{Id}. \tag{3.5}$$

The following proposition follows from a boring diagram chasing. We omit the details.

Proposition 3.3.9. *Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. The following compositions are inverse to each other:*

$$\begin{aligned}\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F}) &\xrightarrow{f_*} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f_*\mathcal{F}) \\ &\xrightarrow{-\circ(3.2)} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F})\end{aligned}$$

and

$$\mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) \xrightarrow{f^{-1}_{\mathsf{PShv}}} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F}))$$
$$\xrightarrow{(3.4)\circ -} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F})$$

Corollary 3.3.10. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{PShv}(X', \mathsf{Set}).$$

Corollary 3.3.11. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1}_{\mathsf{PShv}} \simeq f^{-1}_{\mathsf{PShv}} \circ g^{-1}_{\mathsf{PShv}}$.*

Corollary 3.3.12. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any presheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}_{\mathsf{PShv}}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Remark 3.3.13. Let $\mathcal{C}$ be a category admitting direct colimits. One can define the functor f^{-1}_{PShv} for $\mathcal{C}$-valued presheaves using the same formula, and f^{-1}_{PShv} is canonically left adjoint to f_*.

3.4. **Inverse images for sheaves.**

Construction 3.4.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F} \in \mathsf{Shv}(X', \mathsf{Set})$ be a sheaf. We define

$$f^{-1}\mathcal{F} := (f^{-1}_{\mathsf{PShv}}\mathcal{F}')^{\sharp}$$

to be the sheafification of the presheaf-theoretic inverse image of $\mathcal{F}$.

The construction $\mathcal{F}' \to f^{-1}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for sheaves of sets.

Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. We have canonical bijections:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(f^{-1}(\mathcal{F}'), \mathcal{F}) \simeq \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F})$$
$$\simeq \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) \simeq \mathsf{Hom}_{\mathsf{Shv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}),$$

where

- the first bijection is due to the definition of sheafifications;
- the second bijection is that in Proposition 3.3.9;
- the last bijection is due to the fully faithful embedding $\mathsf{Shv}(X', \mathsf{Set}) \subseteq \mathsf{PShv}(X, \mathsf{Set})$.

Corollary 3.4.2. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Shv}(X', \mathsf{Set}).$$

Corollary 3.4.3. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1} \simeq f^{-1} \circ g^{-1}$.*

Corollary 3.4.4. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any sheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Exercise 3.4.5. The following diagram commutes:

$$\begin{tikzcd}
\mathsf{PShv}(X', \mathsf{Set}) \arrow[r, "f^{-1}_{\mathsf{PShv}}"] \arrow[d, "(-)^\sharp"] & \mathsf{PShv}(X, \mathsf{Set}) \arrow[d, "(-)^\sharp"] \\
\mathsf{Shv}(X', \mathsf{Set}) \arrow[r, "f^{-1}"] & \mathsf{Shv}(X, \mathsf{Set}).
\end{tikzcd}$$

Exercise 3.4.6. Show that f^{-1} sends a constant sheaf to the constant sheaf associated to the same set.

Example 3.4.7. Let X be a topological space and x be a point. Write $i : \{x\} \to X$ for the embedding. For $\mathcal{F} \in \mathsf{Shv}(X, \mathsf{Set})$, we have

$$i^{-1}(\mathcal{F}) \simeq \mathcal{F}_x,$$

where in the RHS we abuse notations by identifying a sheaf on $\{x\}$ with its set of global sections (see Example 1.2.6).

Remark 3.4.8. Let $\mathcal{C}$ be a category admitting small limits and filtered colimits. One can define the functor f^{-1} for $\mathcal{C}$-valued sheaves using the same formula, and f^{-1} is canonically left adjoint to f_*.

3.5. **Open base-change.**

Construction 3.5.1. Given a commutative square of topological spaces

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & X' \arrow[d, "v"] \\
Y \arrow[r, "g"] & Y',
\end{tikzcd} \tag{3.6}$$

consider the canonical natural isomorphism $v_* \circ f_* \simeq g_* \circ u_*$. Using the adjunctions $(g^{-1}_{\mathsf{PShv}}, g_*)$ and $(f^{-1}_{\mathsf{PShv}}, f_*)$, we obtain natural transformations

$$g^{-1}_{\mathsf{PShv}} \circ v_* \to g^{-1}_{\mathsf{PShv}} \circ v_* \circ f_* \circ f^{-1}_{\mathsf{PShv}} \simeq g^{-1}_{\mathsf{PShv}} \circ g_* \circ u_* \circ f^{-1}_{\mathsf{PShv}} \to u_* \circ f^{-1}_{\mathsf{PShv}},$$

where the first arrow is induced by $\mathsf{Id} \to f_* \circ f^{-1}_{\mathsf{PShv}}$ (see (3.3)), while the last arrow is induced by $g^{-1}_{\mathsf{PShv}} \circ g_* \to \mathsf{Id}$ (see (3.5)).

We call the above composition the **base-change natural transformation**[7] for presheaves associated to the square (3.6).

Similarly, we have the base-change natural transformation for sheaves

$$g^{-1} \circ v_* \to u_* \circ f^{-1}.$$

[7]Other name: Bech–Chevalley natural transformations.

Proposition 3.5.2. *Let $f : X \to X'$ be a continuous map between topological spaces and $U' \subseteq X'$ be an open subset. Write $U := f^{-1}(U')$ can consider the following diagram*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ U' \arrow[r, "j'"] & X'. \end{tikzcd}$$

Then both

$$(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$$

and

$$(j')^{-1} \circ f_* \to g_* \circ j^{-1}$$

are natural isomorphisms.

Proof. We will prove the claim for presheaves. That for sheaves follow from Lemma 3.3.5.

For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V' \subseteq U'$, unwinding the definitions, we have

$$\big((j')^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(V') \simeq \big(f_*(\mathcal{F})\big)(V') \simeq \mathcal{F}(f^{-1}(V'))$$

and

$$\big(g_* \circ j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V') \simeq \big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(g^{-1}(V')) \simeq \mathcal{F}(f^{-1}(V')).$$

One can check that via these identifications, the value of $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ at $\mathcal{F}$ and V' is given by the identity map on $\mathcal{F}(f^{-1}(V'))$. In particular, $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ is a natural isomorphism.

□

Remark 3.5.3. Informally, we say: *open* pullbacks commute with pushforwards.

Warning 3.5.4. In the setting of Proposition 3.5.2, one can also consider the natural transformations

$$f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$$

and

$$f^{-1} \circ j'_* \to j_* \circ g^{-1}.$$

However, they are *not* invertible in general.

Exercise 3.5.5. Let $X' = \{s, b\}$ be the topological space with two points whose open subsets are exactly given by $\varnothing, \{b\}$ and X'. Consider the following diagram

$$\begin{tikzcd} \varnothing \arrow[r, "j"] \arrow[d, "g"] & \{s\} \arrow[d, "f"] \\ \{b\} \arrow[r, "j'"] & X'. \end{tikzcd}$$

Show that $f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$ and $f^{-1} \circ j'_* \to j_* \circ g^{-1}$ are not invertible.

Part II. Definition of schemes

4. Spec(R)

4.1. Zariski topology.

Definition 4.1.1. Let R be a (unital) commutative ring. Write $\mathsf{Spec}(R)$ for the set of prime ideals of R. We equip it with the **Zariski topology** so that the subsets

$$U(f) := \{\mathfrak{p} \in \mathsf{Spec}(R) \,|\, f \notin \mathfrak{p}\},\ f \in R$$

form a topological base. The obtained topological space is called the **prime spectrum** of R. The open subsets of the form $U(f)$ are called the **standard open subsets**[8].

Note that $U(f) \cap U(g) = U(fg)$.

Construction 4.1.2. For any ideal $I \subseteq R$, consider $Z(I) = \{\mathfrak{p} \,|\, I \subseteq \mathfrak{p}\}$. By definition,

$$Z(I) \simeq \mathsf{Spec}(R) \smallsetminus \bigcup_{f \in I} U(f).$$

This implies the following result.

Lemma 4.1.3. *A subset Z of $\mathsf{Spec}(R)$ is closed iff it is of the form $Z(I)$ for some ideal $I \subseteq R$.*

Lemma 4.1.4. *Let $I, J \subseteq R$ be ideals. Then $Z(I) \subseteq Z(J)$ iff $J \subseteq \sqrt{I}$.*

Proof. Recall the radical $\sqrt{I}$ is equal to the intersection of prime ideals containing I, i.e.,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in Z(I)} \mathfrak{p}. \tag{4.1}$$

For the "if" statement, suppose $J \subseteq \sqrt{I}$. Then $J \subseteq \mathfrak{p}$ and therefore $\mathfrak{p} \in Z(J)$ for any $\mathfrak{p} \in Z(I)$. Hence we have $Z(I) \subset Z(J)$ as desired.

For the "only if" statement, suppose $Z(I) \subseteq Z(J)$. By (4.1), $\sqrt{J} \subseteq \sqrt{I}$. In particular, $J \subseteq \sqrt{I}$ as desired.

□

Corollary 4.1.5. *Let $I, J \subseteq R$ be ideals. Then $Z(I) = Z(J)$ iff $\sqrt{J} = \sqrt{I}$.*

Corollary 4.1.6. *A point $\mathfrak{p} \in \mathsf{Spec}(R)$ is closed iff $\mathfrak{p}$ is maximal.*

Corollary 4.1.7. *Let x and $y \in \mathsf{Spec}(R)$ be points given by prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. Then $x \in \overline{\{y\}}$ iff $\mathfrak{p} \supset \mathfrak{q}$.*

In above, we say x is a **specialization** of y, and y is a **generalization** of x.

Corollary 4.1.8. *The topological space $\mathsf{Spec}(R)$ is Kolmogorov, i.e., for any pair of distinct points, at least one of them has an open neighborhood not containing the other point.*

Remark 4.1.9. The space $\mathsf{Spec}(R)$ is in general not Haussdorff. Indeed, it is so iff the Krull dimension of R is zero.

[8] Other name: elementary open subsets.

Example 4.1.10. The points in $\mathsf{Spec}(\mathbb{Z})$ are listed as below:

(i) For each prime number p, there is a point $(p) \in \mathsf{Spec}(\mathbb{Z})$.
(ii) There is a point $(0) \in \mathsf{Spec}(\mathbb{Z})$.

A subset of $\mathsf{Spec}(\mathbb{Z})$ is closed iff it is finite collection of points in (i), or it is the entire space.

Note that points (i) are closed, while the point in (ii) is not closed. In fact, the closure of the latter is the entire space.

Example 4.1.11. For any field k, $\mathsf{Spec}(k)$ is a point.

Example 4.1.12. For any discrete valuation ring R, $\mathsf{Spec}(R)$ consists of two points: a closed point corresponding to its ideal of definition, and an open point corresponding to the zero ideal.

Exercise 4.1.13. Let k be an algebraically closed field. Describe the topological space $\mathsf{Spec}(k[x,y]/(xy))$.

Lemma 4.1.14. *The topological space* $\mathsf{Spec}(R)$ *is quasi-compact. In other words, any open covering of it admits a finite sub-covering.*

Proof. It is enough to show any open covering of the form $\mathsf{Spec}(R) = \bigcup_{f\in S} U(f)$ admits a finite sub-covering. Let $\langle S\rangle$ be the ideal generated by S. We obtain $Z(\langle S\rangle) = \varnothing$ and therefore $\langle S\rangle = R$. Hence there exists a finite subset $S' \subseteq S$ such that $1 \in \langle S'\rangle$ and therefore $R = \langle S'\rangle$. Hence we have

$$\varnothing = Z(\langle S'\rangle) = \mathsf{Spec}(R) \smallsetminus \bigcup_{f\in S'} U(f).$$

In other words, we have found a finite sub-covering given by $U(f)$, $f \in S'$. □

4.2. **Structure sheaf.** We are going to construct a canonical sheaf on $\mathsf{Spec}(R)$. For this purpose, we need to associate a set to any standard open subset. Note that a standard open subset $U(f)$ does *not* uniquely determine the element f. However, we have the following results.

Lemma 4.2.1. *For* $f, f' \in R$, $U(f) \subseteq U(f')$ *iff* $R \to R_f$ *(uniquely) factors through* $R \to R_{f'}$.

Proof. By definition, $U(f) \subseteq U(f')$ iff $Z(\langle f\rangle) \supset Z(\langle f'\rangle)$. By Lemma 4.1.4, this happens iff $f^n \in \langle f'\rangle$ for some $n \geq 0$. The latter condition is equivalent to f' being an unit under the map $R \to R_f$. By definition, this is equivalent to the condition that $R \to R_f$ factors through $R \to R_{f'}$. □

Corollary 4.2.2. *The open subsets* $U(f)$ *and* $U(f')$ *of* $\mathsf{Spec}(R)$ *are equal iff* R_f *and* $R_{f'}$ *are isomorphic as* R*-algebras.*

Proposition-Definition 4.2.3. *There exists an essentially unique*[9] *sheaf* $\mathcal{O}$ *of commutative rings on* $\mathsf{Spec}(R)$ *equipped with an isomorphism* $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ *such that for any* $f \in R$, *the* R*-algebra* $\mathcal{O}(U(f))$ *given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}(U(f))$$

[9] This means the pair $(\mathcal{O}, \phi)$ is unique up to a unique isomorphism.

is isomorphic to R_f.

The sheaf $\mathcal{O}_{\mathsf{Spec}(R)} := \mathcal{O}$ *is called the* ***structure sheaf*** *on* $\mathsf{Spec}(R)$. *When using this terminology, we treat the isomorphism* $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ *as implicit.*

Remark 4.2.4. Note that for an R-algebra A, being isomorphic to R_f is a *property rather than a structure*. Namely, there is at most one R-homomorphism from R_f to A.

Proof of Proposition-Definition 4.2.3. Let $\mathfrak{B}$ be the category of standard open subsets in $\mathsf{Spec}(R)$. Since a sheaf is uniquely determined by its restriction on a topological base, we only need to show there is a unique functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ equipped with an isomorphism $\varphi : R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ such that:

(a) The functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ satisfies the sheaf condition in Proposition 1.2.10.
(b) For any $f \in R$, $\mathcal{O}(U(f))$ is isomorphic to R_f as R-algebras.

By Lemma 4.2.1, there is a unique pair $(\mathcal{O}, \varphi)$ satisfying condition (b). Hence we only need to check condition (a). Unwinding the definitions, this amounts to the following fact in commutative algebra (applied to the case $M = A$). We leave the proof of it to the readers.

□

Lemma 4.2.5. *Let* A *be a commutative ring and* f, $(f_i)_{i\in I}$ *be elements in* R *such that* $U(f) = \bigcup_{i\in I} U(f_i)$. *For any* A*-module* M, *the following sequence is exact:*

$$0 \to M_f \to \prod_{i\in I} M_{f_i} \to \prod_{(i,j)\in I^2} M_{f_i f_j}$$

is exact. Here the second map is induced by the canonical maps $M \to M_{f_i}$, *and the third map is*

$$(s_i)_{i\in I} \mapsto (s_j - s_i)_{(i,j)\in I^2}.$$

Exercise 4.2.6. Let k be a field and $R = k[x,y]$. Consider the point $0 \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x,y). Let $U := \mathsf{Spec}(R) \smallsetminus 0$ be the complementary open subset. Find $\mathcal{O}(U)$.

Definition 4.2.7. An **affine scheme** is a topological space X equipped with a sheaf $\mathcal{O}$ of commutative rings on X such that $(X, \mathcal{O}) \simeq (\mathsf{Spec}R, \mathcal{O}_{\mathsf{Spec}(R)})$ for some commutative ring R.

Proposition 4.2.8. *Let* $x \in \mathsf{Spec}(R)$ *be the point corresponding to a prime ideal* $\mathfrak{p} \subseteq R$. *Then the* R*-algebra* $\mathcal{O}_x$ *given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}_x$$

is (uniquely) isomorphic to $R_{\mathfrak{p}}$ *as* R*-algebras. In particular,* $\mathcal{O}_x$ *is a local ring.*

Proof. By definition, we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{U}(\mathsf{Spec}(R),x)^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\mathfrak{B}_x \subseteq \mathfrak{U}(\mathsf{Spec}(R), x)$ be the full subcategory of standard open neighborhoods of x in $\mathsf{Spec}(R)$. By the definition of Zariski topology, $\mathfrak{B}_x^{\mathsf{op}} \to \mathfrak{U}(\mathsf{Spec}(R), x)^{\mathsf{op}}$ is (co)final. Hence we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{B}_x^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\phi : R \to A$ be any test R-algebra. We have

$$\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \lim_{U \in \mathfrak{B}_x} \mathsf{Hom}_R(\mathcal{O}(U), A).$$

Since $\mathcal{O}(U)$ is a localization of R for each $U \in \mathfrak{B}_x$, we have

- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \varnothing$ if $U = U(f)$ and $\phi(f)$ is not a unit;
- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \{*\}$ if $U = U(f)$ and $\phi(f)$ is a unit.

It follows that

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $U(f) \in \mathfrak{B}_x$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $U(f) \in \mathfrak{B}_x$.

Note that for an element $f \in R$, the standard open $U(f)$ is a neighborhood of x iff $f \notin \mathfrak{p}$. Hence we have

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $f \in R \smallsetminus \mathfrak{p}$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $f \in R \smallsetminus \mathfrak{p}$.

Note that $\mathsf{Hom}_R(R_\mathfrak{p}, A)$ has the same description. Hence by Yoneda lemma, there is a unique isomophism $\mathcal{O}_x \simeq R_\mathfrak{p}$ as R-algerbas.

□

4.3. **Functoriality.** Throughout this subsection, we fix the following notations:

- Let R and R' be commutative rings.
- Write $X := \mathsf{Spec}(R)$ and $X' := \mathsf{Spec}(R')$.
- Write $\mathcal{O}$ and $\mathcal{O}'$ respectively for the structure sheaves on X and X'.

Construction 4.3.1. Let $h : R \to R'$ be a homomorphism between commuative algebras. Consider the map

$$\phi : X' \to X,\ \mathfrak{p}' \mapsto h^{-1}(\mathfrak{p}').$$

By definition, for any $f \in R$,

$$\phi^{-1}(U(f)) = U(h(f)).$$

It follows that ϕ is a continuous map with respect to the Zariski topology.

Note that the assignment $h \mapsto \phi$ loses information: h *cannot* be reconstructed from ϕ.

Proposition 4.3.2. *Let $h : R \to R'$ be a homomorphism and $\phi : X' \to X$ be the corresponding continuous map. Then there exists a unique morphism in* $\mathsf{Shv}(X, \mathsf{CRing})$

$$\alpha : \mathcal{O} \to \phi_*(\mathcal{O}')$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_*\mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\
R \arrow[u, "\simeq"] \arrow[rr, "h"] & & R'. \arrow[u, "\simeq"']
\end{tikzcd}$$

Proof. Let $\mathfrak{B} \subseteq \mathfrak{U}(\mathsf{Spec}(R))$ be the full subcategory of standard open subsets. By Exercise 1.2.7, it is enough to show that there exists a unique natural transformation

$$\alpha : \mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}} \to \phi_*(\mathcal{O}')|_{\mathfrak{B}^{\mathrm{op}}}$$

that makes the diagram commute.

For any $U \in \mathfrak{B}^{\mathsf{op}}$, we claim there is a unique dotted homomorphism α_U making the following diagram commute

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, dotted, "\alpha_U"] & \phi_*\mathcal{O}'(U) \arrow[r, "\simeq"] & \mathcal{O}'(\phi^{-1}(U)) \\ R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u] \end{tikzcd}$$

Indeed, choose $f \in R$ such that $U = U(f)$. Then $\phi^{-1}(U) = U(h(f))$. Hence $\mathcal{O}(U) \simeq R_f$ and $\mathcal{O}'(\phi^{-1}(U)) \simeq R'_{h(f)}$. Via these identications, the claim becomes obvious.

It follows that α_U can be assembled into a natural transformation α satisfying the desired property. Moreover, such α is unique because each α_U is unique.

□

By adjunction, we obtain the following result.

Corollary 4.3.3. *Let $h : R \to R'$ be a homomorphism and $\phi : \mathsf{Spec}(R') \to \mathsf{Spec}(R)$ be the corresponding continuous map. Then there exists a unique morphism in $\mathsf{Shv}(X', \mathsf{CRing})$*

$$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$$

such that the following diagram commutes

$$\begin{tikzcd} \phi^{-1}\mathcal{O}(X') \arrow[r, "\beta_{X'}"] & \mathcal{O}'(X') \\ R \arrow[u] \arrow[r, "h"] & R'. \arrow[u, "\simeq"'] \end{tikzcd}$$

Moreover, for any point $x' \in X'$ and $x := \phi(x')$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

is a local homomorphism between local rings.

Proof. The first claim follows from Proposition 4.3.2 and the adjunction $\phi^{-1} \vdash \phi_*$. For the second claim, let $\mathfrak{p}' \subseteq R'$ be the prime ideal corresponding to x' and $\mathfrak{p} := \phi^{-1}(\mathfrak{p}')$. By Proposition 4.2.8, we can identify $\mathcal{O}_x \to \mathcal{O}'_{x'}$ with the unique R-homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$, which makes the desired claim manifest.

□

The following result says knowing h is equivalent to knowing a pair (ϕ, β).

Proposition-Construction 4.3.4. *There is a canonical bijection between the following sets:*

(i) *The set $\mathsf{Hom}_{\mathsf{CRing}}(R, R')$ of homomorphisms from R to R'.*
(ii) *The set of pairs (ϕ, β), where*
  - *$\phi : X' \to X$ is a continuous map,*
  - *$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ is a morphism in $\mathsf{Shv}(X', \mathsf{CRing})$*

  such that for any $x = \phi(x')$, $x' \in X'$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

  is a local homomorphism between local rings.

Proof. For any pair (ϕ, β) in (ii), let $\alpha : \mathcal{O} \to \beta_* \mathcal{O}'$ be the morphism corresponding to β via adjunction. There is a unique dotted homomorphism h that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_* \mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\ R \arrow[u, "\simeq"] \arrow[rr, dotted, "h"] & & R'. \arrow[u, "\simeq"'] \end{tikzcd}$$

This defines a map (ii)→(i). We have seen this map is surjective (Corollary 4.3.3). It remains to check it is injective.

Suppose (ϕ_1, β_1) and (ϕ_2, β_2) produce the same homomorphism $h : R \to R'$.

We first show $\phi_1 = \phi_2$. Let $x' \in X'$ be a point corresponding to a prime ideal $\mathfrak{p}' \subseteq R'$, consider $x_i := \phi_i(x_i)$. We will show $x_1 = x_2$. Let $\mathfrak{p}_i \subseteq R$ be the prime ideal corresponding to x_i. For $i = 1, 2$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{x_i} \arrow[r, "\simeq"] & (\phi_i^{-1}\mathcal{O})_{x'} \arrow[r, "(\beta_i)_{x'}"] & \mathcal{O}'_{x'} \\ R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u] \end{tikzcd}$$

By Proposition 4.2.8, $\mathcal{O}_{x_i} \simeq R_{\mathfrak{p}_i}$ and $\mathcal{O}'_{x'} \simeq R'_{\mathfrak{p}'}$. Hence the commutative diagram implies $h^{-1}(\mathfrak{p}') \subseteq \mathfrak{p}_i$. Moreover, since by assumption the top horizontal arrow is a local homomorphism, we must have $h^{-1}(\mathfrak{p}') = \mathfrak{p}_i$. In particular, $\mathfrak{p}_1 = \mathfrak{p}_2$ and therefore $x_1 = x_2$ as desired.

Now write $\phi = \phi_1 = \phi_2$. It remains to show $\beta_1 = \beta_2$. By the last paragraph, for any $x' \in X'$, we have $(\beta_1)_{x'} = (\beta_2)_{x'}$ because it can be identified with the *unique* homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ compatible with $h : R \to R'$. Now by Corollary 2.2.3, we obtain $\beta_1 = \beta_2$ as desired.

□

Exercise 4.3.5. Show that the conclusion of Proposition-Construction 4.3.4 would be false if we do not require $\beta_{x'}$ to be a local homomorphism. In other words, show that there exists a continuous map $\phi : X' \to X$ together with a morphism $\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ such that $\beta_{x'}$ is not a local homomorphism for some point $x' \in X'$.

5. Schemes as locally ringed spaces

5.1. **Locally ringed spaces.** Motivated by the construction $(\mathsf{Spec}(R), \mathcal{O})$, we make the following definition.

Definition 5.1.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between ringed spaces** is a pair $\phi = (\phi, \beta)$, where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves.

Let $\mathsf{Top}_{\mathsf{CRing}}$ be the category of ringed spaces and morphisms between them.

Remark 5.1.2. Equivalently, we can replace β by a morphism $\alpha : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$.

Construction 5.1.3. By definition, any morphism $(\phi, \beta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between ringed spaces factors as

$$(X, \mathcal{O}_X) \xrightarrow{(\mathrm{id}, \beta)} (X, \phi^{-1}(\mathcal{O}_Y)) \xrightarrow{(\phi, \mathrm{id})} (Y, \mathcal{O}_Y).$$

Definition 5.1.4. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that for any point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Let $(X, \mathcal{O}_X)$ be a locally ringed space and $x \in X$ be a point. The **residue field of** $(X, \mathcal{O}_X)$ **at** x is the field

$$\kappa_x := \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between locally ringed spaces** is a pair (ϕ, β), where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves such that for any point $x \in X$, the homomorphism

$$\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$$

is a local homomorphism.

Let $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the category of locally ringed spaces and morphisms between them.

Construction 5.1.5. Let $(\phi, \beta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homomorphism between locally ringed spaces. For any $x \in X$, the local homomorphism $\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ induces a homomorphism

$$\kappa_{Y,\phi(x)} \to \kappa_{X,x}.$$

Warning 5.1.6. The functor $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}} \to \mathsf{Top}_{\mathsf{CRing}}$ is faithful but not full.

Example 5.1.7. For any commutative ring R, we obtain a locally ringed space $(\mathsf{Spec}(R), \mathcal{O})$. We will abuse notation and denote this locally ringed space just by $\mathsf{Spec}(R)$, and treat its structure sheaf as implicit.

For any homomorphism $h : R \to R'$, we obtain a morphism $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ between locally ringed spaces as in Proposition-Construction 4.3.4. Moreover, the information of h is exactly encoded by this morphism.

Definition 5.1.8. An **affine scheme** is a locally ringed space that is isomorphic to $\mathsf{Spec}(R)$ for some R. A **morphism between affine schemes** is a morphism between locally ringed spaces. Let $\mathsf{Aff} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory of affine schemes.

Using these new terminilogies, we can reformulate Proposition-Construction 4.3.4 as follows.

Proposition-Construction 5.1.9. *The following functors are inverse to each other:*

$$\begin{array}{rcl} \mathsf{CRing}^{\mathsf{op}} & \simeq & \mathsf{Aff} \\ R & \mapsto & \mathsf{Spec}(R) \\ \mathcal{O}_X(X) & \leftarrow & X. \end{array}$$

Construction 5.1.10. Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. For any $f : Y \to X$, the pair $(Y, f^{-1}\mathcal{O}_X)$ defines a (locally) ringed space, and we have a canonical morphism

$$(Y, f^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$$

given by the pair $(f, \mathsf{id}_{f^{-1}\mathcal{O}_X})$.

When $f : Y \subseteq X$ is a subspace, we write $\mathcal{O}_X|_Y := f^{-1}\mathcal{O}_X$ and call the obtained (locally) ringed space $(Y, \mathcal{O}_X|_Y)$ the **restriction of** $(X, \mathcal{O}_X)$ **to** Y (or the **locally ringed subspace of** $(X, \mathcal{O}_X)$ **associated to** Y).

Example 5.1.11. Let R be a commutative ring and consider the locally ringed space $\mathsf{Spec}(R)$. For any element $f \in R$, by Construction 5.1.10, we obtain a locally ringed subspace of $\mathsf{Spec}(R)$ associated to $U(f)$. By construction, it can be identified with $\mathsf{Spec}(R_f)$. In particular, it is an affine scheme.

5.2. **Schemes.**

Definition 5.2.1. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open covering $X = \bigcup_{i \in I} U_i$ with each $(U_i, \mathcal{O}_X|_{U_i})$ being an affine scheme.

A **morphism** between schemes is a morphism between locally ringed spaces. Let $\mathsf{Sch} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory consisting of schemes.

Notation 5.2.2. We often abuse notation by writing X for a scheme $(X, \mathcal{O}_X)$ and treating its structure sheaf as implicit. Similarly, we often abuse notation by writing $\phi : X \to Y$ for a morphism between schemes and treating the morphism $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as implicit.

Warning 5.2.3. Nevertheless, one should keep in their minds that schemes are not determined by their underlying topological spaces; similarly morphisms between schemes are not determined by the underlying continuous maps.

Exercise 5.2.4. Let X be a scheme over $\mathsf{Spec}(\mathbb{F}_q)$.

(1) For any open subset $U \subseteq X$, show that the map $\beta_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, $f \mapsto f^q$ is a homomorphism, and these maps give an endomorphism $\beta : \mathcal{O}_X \to \mathcal{O}_X$ of the structure sheaf.
(2) Show that $\mathsf{Frob}_{X,q} := (\mathsf{id}_X, \beta)$ is an endomorphism of the scheme X defined over $\mathsf{Spec}(\mathbb{F}_q)$.

The morphism $\mathsf{Frob}_{X,q}$ is known as the **absolute q-Frobenius morphism of** X, and plays a central role in the study of schemes over finite fields.

The following results follow from their counterparts for affine schemes.

Lemma 5.2.5. *Let X be a scheme.*

- *The affine open subsets of X form a base for its topology.*
- *The space X is Kolmogorov.*

The following exercise provides examples of locally ringed spaces that are not schemes.

Exercise 5.2.6. Let X be a topological space. For any open subset $U \subseteq X$, let $\mathcal{C}_X(U)$ be the commutative ring of $\mathbb{R}$-valued[10] continuous functions on U. Note that $U \mapsto \mathcal{C}_X(U)$ defines a sheaf of commutative rings on X.

(1) Show that $(X, \mathcal{C}_X)$ is a locally ringed space.
(2) Show that a continuous map $X \to X'$ induces a morphism $(X, \mathcal{C}_X) \to (X', \mathcal{C}_{X'})$ between locally ringed spaces.
(3) Show that $(\mathbb{R}, \mathcal{C}_{\mathbb{R}})$ is not a scheme.

5.3. **Open immersions.**

Proposition-Definition 5.3.1. *Let X be a scheme. For any open subspace $U \subseteq X$, the corresponding locally ringed subspace is a scheme. We call it the* ***open subscheme of X associated to U****.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering such that each U_i is an affine scheme. We only need to show the locally ringed subspace associated to each $U_i \cap U$ can be covered by affine schemes. Without lose of generality, we can replace X with U_i and U with $U_i \cap U$, and therefore assume $X \simeq \mathsf{Spec}(R)$ is affine. Now by the definition of the Zariski topology, we can find elements $(f_j)_{j \in J}$ in R such that $U = \bigcup_{j \in J} U(f_j)$. By Example 5.1.11, each $U(f_j)$ is an affine scheme isomorphic to $\mathsf{Spec}(R_{f_j})$. Hence U is a scheme as desired. □

Definition 5.3.2. We say a morphism $f : Y \to X$ is an **open immersion** if there exists an (unique) open subscheme $U \subseteq X$ such that f factors as $Y \xrightarrow{\simeq} U \to X$.

Warning 5.3.3. An open subscheme U of X may fail to be affine even if X is affine. Also, an *affine* open subset of an affine scheme may fail to be a standard subset.

Exercise 5.3.4. Let k be a field and $R = k[x,y]$. Consider the point $(0,0) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x,y). Let $U := \mathsf{Spec}(R) \setminus \{(0,0)\}$ be the complementary open subset. Show that the scheme U is not affine.

Exercise 5.3.5. Let k be a field of characteristic 0 and $R := k[x,y]/(y^2 - x^3)$. Consider the point $(1,1) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal $(x-1, y-1)$. $U := \mathsf{Spec}(R) \setminus \{(1,1)\}$ be the complementary open subset. Show that the scheme U is affine but it is not a standard open subset of $\mathsf{Spec}(R)$.

Warning 5.3.6. Let X be a scheme and $Y \subseteq X$ be a subspace. The locally ringed subspace associated to Y is in general not a scheme.

Exercise 5.3.7. Let R be a local ring and $X := \mathsf{Spec}(R)$. Consider the unique closed point $x \in \mathsf{Spec}(R)$. Show that $(\{x\}, \mathcal{O}_X|_{\{x\}})$ is not a scheme unless R is a field.

[10] We equip $\mathbb{R}$ with the usual topology.

Exercise 5.3.8. An open immersion $f : Y \to X$ is a monomorphism in Sch. In other words, for any $Z \in \mathsf{Sch}$, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(Z, Y) \xrightarrow{f\circ -} \mathsf{Hom}_{\mathsf{Sch}}(Z, X)$$

is injective. Moreover, a morphism $h : Z \to X$ is contained in the image iff the underlying continuous map factors through the open subset $f(Y) \subset X$.

6. Gluing schemes

6.1. Statement.

Definition 6.1.1. A **gluing data of schemes** is a collection

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

where

- I is a set;
- For each $i \in I$, X_i is a scheme;
- For any pair $(i,j) \in I^2$, U_{ij} is an open subscheme of X_i;
- For any pair $(i,j) \in I^2$,
$$\phi_{ij} : U_{ij} \to U_{ji},$$
is an isomorphism between schemes.

The above data should satisfy the following conditions:

- For any $i \in I$, $U_{ii} = X_i$ and $\phi_{ii} = \mathsf{id}_{X_i}$.
- For any triple $(i,j,k) \in I^3$,
$$\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$
as open subsets of U_{ji}.
- For any triple $(i,j,k) \in I^3$, the following **cocycle condition** holds:
$$\phi_{jk}|_{U_{ji}\cap U_{jk}} \circ \phi_{ij}|_{U_{ij}\cap U_{ik}} = \phi_{ik}|_{U_{ij}\cap U_{ik}},$$
i.e., the following diagram commutes:
$$\begin{tikzcd} & U_{ij}\cap U_{ik} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\phi_{ik}", "\simeq"'] & \\ U_{ji}\cap U_{jk} \arrow[rr, "\phi_{jk}", "\simeq"'] & & U_{ki}\cap U_{kj}. \end{tikzcd}$$

Proposition-Definition 6.1.2. *Given a gluing data of schemes*

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

there exists an essentially unique collection

$$\big(X, (X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$$

where

- *X is a scheme;*
- *For each $i \in I$, X_i' is an open subscheme of X;*
- *For each $i \in I$,*
$$\varphi_i : X_i \xrightarrow{\simeq} X_i',$$
is an isomorphism;

such that

- *$X = \bigcup_{i\in I} X_i'$ as topological spaces;*
- *For any pair $(i,j) \in I^2$,*
$$\varphi_i(U_{ij}) = X_i' \cap X_j'$$
as open subsets of X_i';

- *For any pair $(i,j) \in I^2$, we have*
$$\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ji}} \circ \phi_{ij},$$
i.e., the following diagram commutes
$$\begin{tikzcd} & U_{ij} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\varphi_i", "\simeq"'] & \\ U_{ji} \arrow[rr, "\varphi_j", "\simeq"'] & & X'_i \cap X'_j. \end{tikzcd}$$

*We say the scheme X is **glued** from the given gluing data, and treat $\big((X'_i)_{i\in I}, (\varphi_i)_{i\in I}\big)$ as implicit.*

Proof. It is an exercise in point-set topology that the similar claim for topological spaces is correct. In other words, the gluing data gives an essentially unique topological space X equipped with open subspaces X'_i and homeomorphisms $\varphi_i : X_i \to X'_i$ satisfying the *topological* conditions listed in the statement.

Hence we only show there is an essentially unique $\mathcal{O}_X \in \mathsf{Shv}(X, \mathsf{CRing})$ equipped with isomorphisms $\varphi_i^{-1}(\mathcal{O}_X|_{X'_i}) \simeq \mathcal{O}_{X_i}$, that satisfies the remaining *sheaf-theoretic* conditions. Note that such a ringed space $(X, \mathcal{O}_X)$ will automatically be a scheme because its restriction to each X'_i is a scheme isomorphic to X_i.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X'_i$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. It is easy to see there exists an essentially unique functor
$$\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$$
equipped with isomorphisms
$$\beta_i : \varphi_i^{-1}((\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}})|_{X'_i}) \xrightarrow{\simeq} \mathcal{O}_i$$
satisying the desired conditions. Here $(\mathcal{O}_{\mathfrak{B}^{\mathsf{op}}})|_{X'_i}$ is the restriction of $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ along the fully faithful embedding $\mathfrak{U}(X'_i)^{\mathsf{op}} \to \mathfrak{B}^{\mathsf{op}}$. Moreover, one can check that the obtained $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ satisfies the sheaf condition in Proposition 1.2.10. Hence there is an essentially unique extension of $\mathcal{O}|_{\mathfrak{B}^{\mathsf{op}}}$ to a CRing-valued sheaf $\mathcal{O}_X$ on X, which fulfills our goal.

□

The following proposition describes how to construct morphisms out of a glued space.

Proposition 6.1.3. *Let*
$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big),$$
be a gluing data of schemes and
$$\big(X, (X'_i)_{i\in I}, (\varphi_i)_{i\in I}\big)$$
be its gluing output. For any scheme Y, the map
$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Sch}}(X, Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Sch}}(X_i, Y) \\ f & \mapsto & (f|_{X'_i} \circ \varphi_i)_{i\in I} \end{array}$$
is injective, and a collection of morphisms $(g_i : X_i \to Y)_{i\in I}$ is contained in the image iff $g_i|_{U_{ij}} = g_j|_{U_{ji}} \circ \phi_{ij}$ for any pair $(i,j) \in I^2$.

Proof. To simplify the notations, we identify X_i with X_i' and identify U_{ij} with the intersection $X_i \cap X_j$ inside X. Consequently, ϕ_{ij} and φ_i are identity morphisms.

We first prove the similar claim for topological spaces. Indeed, since $X = \bigcup_{i\in X} X_i$ as a topological space, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Top}}(X,Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Top}}(X_i,Y) \\ f & \mapsto & (f|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of continuous map $(f_i : X_i \to Y)_{i\in I}$ is contained in the image iff $f_i|_{X_i\cap X_j} = f_j|_{X_i\cap X_j}$ for any pair $(i,j) \in I^2$.

It follows that we only need to show that for a given continuous map $f : X \to Y$ and $f_i := f|_{X_i}$, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Shv}(X_i,\mathsf{CRing})}(f_i^{-1}\mathcal{O}_Y,\mathcal{O}_{X_i}) \\ \beta & \mapsto & (\beta|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of morphisms $(\beta_i : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{X_i})_{i\in I}$ is contained in the image iff $\beta_i|_{X_i\cap X_j} = \beta_j|_{X_i\cap X_j}$.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X_i$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. By Exercise 1.2.7, we have a bijection

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \xrightarrow{\simeq} & \mathsf{Hom}_{\mathsf{Fun}(\mathfrak{B}^{\mathsf{op}},\mathsf{CRing})}((f^{-1}\mathcal{O}_Y)|_{\mathfrak{B}^{\mathsf{op}}},(\mathcal{O}_X)|_{\mathfrak{B}^{\mathsf{op}}}) \\ \beta & \mapsto & \beta|_{\mathfrak{B}^{\mathsf{op}}}. \end{array}$$

Now the desired claim follows from the fact that the category $\mathfrak{B}$ can be covered by its full subcategories $\mathfrak{U}(X_i)$, and $\mathfrak{U}(X_i)\cap\mathfrak{U}(X_j) \simeq \mathfrak{U}(X_i\cap X_j)$. □

We can also describe morphisms *into* a glued space.

Proposition-Construction 6.1.4. *Let*

$$\big(I,(X_i)_{i\in I},(U_{ij})_{(i,j)\in I^2},(\phi_{ij})_{(i,j)\in I^2}\big),$$

be a gluing data of schemes and

$$\big(X,(X_i')_{i\in I},(\varphi_i)_{i\in I}\big)$$

be its gluing output. For any scheme Y, there is a canonical bijection between the following sets:

(i) *The set $\mathsf{Hom}_{\mathsf{Sch}}(Y,X)$ of morphisms $f : Y \to X$*
(ii) *The set of collections*

$$(f_i : Y_i \to X_i)_{i\in I},$$

where
- *Each Y_i is an open subscheme of Y and $Y = \bigcup_{i\in I} Y_i$ is an open covering;*
- *Each $f_i : Y_i \to X_i$ is a morphism;*

such that
- *For each pair $(i,j) \in I^2$, $f_i^{-1}(U_{ij}) = Y_i \cap Y_j$;*
- *For each pair $(i,j) \in I^2$, $f_j|_{Y_i\cap Y_j} = \phi_{ij} \circ f_i|_{Y_i\cap Y_j}$.*

Sketch. We first construct a map (i)→(ii). Give a morphism $f: Y \to X$, we declare Y_i to be the open subscheme associated to the open subset $f^{-1}(X'_i)$. In particular, $f|_{Y_i}$ gives a morphism $Y_i \to X'_i$. We declare f_i to be the composition

$$Y_i \to X'_i \xrightarrow{\varphi_i^{-1}} X_i.$$

One can verify the collection $(f_i : Y_i \to X_i)_{i\in I}$ satisfies the desired requirements. This gives a map (i)→(ii).

Now we construct a map (ii)→(i). Given a collection $(f_i : Y_i \to X_i)_{i\in I}$. Consider the compositions

$$g_i : Y_i \xrightarrow{f_i} X_i \xrightarrow{\varphi_i} X'_i \to X.$$

One can check $g_i|_{Y_i\cap Y_j} = g_j|_{Y_i\cap Y_j}$. It follows that there is a unique morphism $f: Y \to X$ such that $f|_{Y_i} = g_i$. This gives a map (ii)→(i).

Now one can check the above two maps are inverse to each other.

□

Remark 6.1.5. Results in this subsection also works for general (locally) ringed spaces.

6.2. **Examples.**

Example 6.2.1. Let $(X_i)_{i\in I}$ be a set of schemes, U_{ij} be the empty scheme for $i \neq j$, and ϕ_{ij} be the identity morphisms. This is obviously a gluing data. The scheme X glued from this gluing data is called the **disjoint union of** $(X_i)_{i\in I}$, and we denote it by $\coprod_{i\in I} X_i$. By Proposition 6.1.3, $\coprod_{i\in I} X_i$ is also the coproduct of $(X_i)_{i\in I}$ inside the category Sch.

Example 6.2.2. As one would expect, the n-dimensional projective space can be glued from $(n+1)$ affine spaces of dimension n. Below are the details.

Let R be any commutative ring. For $n \geq 0$, let $I := \{0, 1, \cdots, n\}$ and

$$X_i := \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]/(x_i^{(i)} - 1)).$$

Let

$$U_{ij} := U(x_j^{(i)}) \subseteq X_i.$$

Then we have

$$U_{ij} \simeq \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1)).$$

Note that we have an isomorphism

$$R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1) \simeq R[x_0^{(j)}, \cdots, x_n^{(j)}]_{x_i^{(j)}}/(x_j^{(j)} - 1)$$

that sends $x_k^{(i)}$ to $x_k^{(j)}/x_i^{(j)}$. This gives an isomorphism

$$U_{ij} \xrightarrow{\simeq} U_{ji}.$$

One can check the above gives a gluing data, hence we obtain an essentially unique scheme X glued from it.

We write $\mathbb{P}_R^n$ for the gluing result and call it the **n-dimensional projective space over R.**

Exercise 6.2.3. Let R be a commutative ring and k be an algebraically closed field.

(1) Find $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$. Deduce that $\mathbb{P}^n_R$ is not affine for $n \geq 1$.
(2) Show that the closed points of $\mathbb{P}^n_k$ can be canonically identified with elements in $(k^{n+1} \smallsetminus 0)/k^\times$, where $k^\times$ acts on the vector space k^{n+1} via scaler multiplication.

Exercise 6.2.4. Let R be any commutative ring and $I = \{1, 2\}$. Let

$$X_1 = X_2 := \mathbb{A}^1_R := \mathsf{Spec}(R[t])$$

and

$$U_{12} = U_{21} := U(t), U_{11} := X_1, U_{22} := X_2.$$

Let ϕ_{ij} be the identity morphisms. Consider the scheme X glued from the above gluing data. Show that X is not affine.

7. Morphisms to affine schemes

7.1. A criterion for being affine.

Construction 7.1.1. Let X and Y be schemes. For any morphism $f : X \to Y$, by definition we have a morphism $\alpha : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Taking global sections, we obtain a homomorphism

$$\alpha_Y : \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) \simeq \mathcal{O}_X(X).$$

One can check this defines a functor

$$\mathsf{Sch} \to \mathsf{CRing}^{\mathsf{op}},\ X \mapsto \mathcal{O}_X(X)$$

that sends a morphism $f : X \to Y$ to α_Y as above.

Theorem 7.1.2. *A scheme Y is affine iff for any scheme X, the natural map*

$$\mathsf{Hom}_{\mathsf{Sch}}(X, Y) \to \mathsf{Hom}_{\mathsf{CRing}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{7.1}$$

is a bijection.

Proof. We first prove the "only if" statement. Let $Y \simeq \mathsf{Spec}(R)$ be an affine scheme, we need to show the natural map

$$\mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \tag{7.2}$$

is a bijection.

We construct a map of the inverse direction as follows. Let $h : R \to \mathcal{O}_X(X)$ be a homomorphism. For any affine open subscheme $U \subseteq X$, by Proposition-Definition 5.1.9, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(U, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(U))$$

is a bijection. Let $f_U : U \to \mathsf{Spec}(R)$ be the morphism corresponding to the composition

$$R \xrightarrow{h} \mathcal{O}_X(X) \xrightarrow{(-)|_U} \mathcal{O}_X(U).$$

Unwinding the constructions, one can check for affine open subschemes $U \subseteq V \subseteq X$, we have $p_U = p_V|_U$. Recall affine open subsets form a base for the topology of X. Using Exercise 1.2.7, one can show there is a unique morphism $p : X \to \mathsf{Spec}(R)$ such that $p_U = p|_U$ for any affine open subscheme $U \subseteq X$. The construction $h \mapsto p$ as above gives a map

$$\mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \to \mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)). \tag{7.3}$$

One can check that (7.2) and (7.3) are inverse to each other. Namely, (7.3) $\circ$ (7.2) = id follows from the uniqueness property about the morphism p, while (7.2) $\circ$ (7.3) = id follows from the fact that an element in $\mathcal{O}_X(X)$ is determined by its restrictions in $\mathcal{O}_X(U)$'s.

Now we deduce the "if" statement from the "only if" one. Let Y be a scheme such that (7.1) is bijective for any scheme X. Write $R := \mathcal{O}_Y(Y)$. By the "if" statement, we have

$$\mathsf{Hom}_{\mathsf{Sch}}(Y, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, R).$$

In particular, there is a canonical morphism $q : Y \to \mathsf{Spec}(R)$ corresponding to id_R. Moreover, by construction, for any scheme X the following diagram commutes:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Sch}}(X,Y) \arrow[r, "\simeq"] \arrow[d, "q\circ-"] & \mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X)) \arrow[d, equal] \\
\mathsf{Hom}_{\mathsf{Sch}}(X,\mathsf{Spec}(R)) \arrow[r, "\simeq"] & \mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X)),
\end{tikzcd}$$

where the horizontal maps are (7.1) applied to Y and $\mathsf{Spec}(R)$ respectively, and they are bijective either by the "if" statement or by assumption. It follows that composing with q induces a bijection

$$\mathsf{Hom}_{\mathsf{Sch}}(X,Y) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{Sch}}(X,\mathsf{Spec}(R))$$

for any $X \in \mathsf{Sch}$. By Yoneda lemma, q is an isomorphism and therefore Y is affine as desired.

□

7.2. **Applications.**

Corollary 7.2.1. *For any scheme X, there is a unique morphism $X \to \mathsf{Spec}(\mathbb{Z})$. In other words, $\mathsf{Spec}(\mathbb{Z}) \in \mathsf{Sch}$ is a final object.*

Corollary 7.2.2. *For any scheme X, there is a unique morphism*

$$q : X \to \mathsf{Spec}(\mathcal{O}_X(X))$$

that induces the identity homomorphism between global sections.

Corollary 7.2.3. *The embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ admits a canonical left adjoint given by*

$$\mathsf{Sch} \to \mathsf{Aff},\ X \mapsto \mathsf{Spec}(\mathcal{O}_X(X)).$$

Exercise 7.2.4. Show that the embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ does not admit a right adjoint.

8. Functor of points

The main goal of this section is to construct a fully faithful functor $\mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ and describe its essential image.

8.1. R-points.

Definition 8.1.1. Let X be a scheme and R be a commutative ring. An **R-point of** X is a morphism $\mathsf{Spec}(R) \to X$. Let

$$X(R) \coloneqq \mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), X)$$

be the set of R-points of X.

A **field-valued point of** X is a k-point of X for some field k.

A **geometric point of** X is a k-point of X for some *separably closed* field k.

Proposition-Construction 8.1.2. *Let X be a scheme and k be a field. There is a bijection between*

- *the set $X(k)$ of k-points of X;*
- *the set of pairs (x, i), where $x \in X$ is a topological point*[11] *and $i : \kappa_x \to k$ is a homomorphism.*

Proof. By definition, a k-point of X is given by a pair (ϕ, β). Since $\mathsf{Spec}(k)$ has only one topological point, the continuous map ϕ is given by a topological point $x \in X$. Now the morphism β is given by a *local* homomorphism $\mathcal{O}_{X,x} \to k$, where we identify a sheaf $\mathcal{F}$ of commutative rings on the one-point space $\{*\}$ with $\mathcal{F}(\{*\})$. Now a local homomorphism $\mathcal{O}_{X,x} \to k$ uniquely factors as $\mathcal{O}_{X,x} \to \kappa_x \to k$. This gives the desired bijection.

□

Remark 8.1.3. For a field-valued point $s : \mathsf{Spec}(k) \to X$, we sometimes abuse notations and write it as $s \to X$, where s is understood as $\mathsf{Spec}(k)$. Similarly, for a topological point $x \in X$, we sometimes abuse notations and write the κ_x-point $\mathsf{Spec}(\kappa_s) \to X$ as $x \to X$.

Note however that a scheme is *not* determined by its field-valued points.

Exercise 8.1.4. Let k be a field, and consider $X \coloneqq \mathsf{Spec}(k)$ and $X' \coloneqq \mathsf{Spec}(k[\epsilon]/(\epsilon^2))$. Let $f : X \to X'$ be the morphism corresponding to the obvious homomorphism $k[\epsilon]/(\epsilon^2) \to k$. Show that f induces a bijection between the set of field-valued points of X and that of X'.

Construction 8.1.5. Let X be a scheme and $x \in X$ be a topological point. Let $U \subseteq X$ be an affine open subset containing x. Note that we have $U \simeq \mathsf{Spec}(\mathcal{O}_X(U))$. The canonical homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ induces a morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U))$. Now the composition

$$\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U)) \simeq U \to X$$

gives an $\mathcal{O}_{X,x}$-point of X.

Exercise 8.1.6. Show that:

(1) The above $\mathcal{O}_{X,x}$-point of X does not depend on the choice of U.
(2) The morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to X$ identifies the locally ringed space $\mathsf{Spec}(\mathcal{O}_{X,x})$ as the restriction of X to a certain subspace.

[11] A point in the underlying topological space.

8.2. Schemes as a functor.

Definition 8.2.1. Let X be a scheme. The functor

$$\mathsf{h}_X : \mathsf{CRing} \to \mathsf{Set},\ R \mapsto X(R)$$

is called the **functor of points of X**.

Note that the construction $X \mapsto \mathsf{h}_X$ is functorial. Hence we have a canonical functor

$$\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

Remark 8.2.2. By construction, the composition

$$\mathsf{CRing}^{\mathsf{op}} \simeq \mathsf{Aff} \to \mathsf{Sch} \xrightarrow{\mathsf{h}} \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

can be identified with the Yoneda embedding $R \mapsto \mathsf{Hom}_{\mathsf{CRing}}(R, -)$.

Theorem 8.2.3. *The functor* $\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ *is fully faithful.*

Remark 8.2.4. For a description of the essential image of this functor[12], see [EH00, Chapter VI].

Remark 8.2.5. The theorem suggests another way to develop scheme theory *without* using locally ringed spaces. Namely, one can *define* a scheme as a functor $\mathsf{CRing} \to \mathsf{Set}$ satisfying certain properties. In fact, in the 1970s, Grothendieck himself radically urged to abandon his earlier definition of schemes in favor of the functorial point of view. In my opinion, this approach at least has the following advantages.

- It makes a lot of constructions about schemes formal and therefore easier.
- It provides a more direct way to deal with moduli problems and deformation theory.
- It allows one to define more exotic geometric objects, such as algebraic spaces, stacks, indschemes...

No matter how much I love this functorial approach, however, I do *not* believe a learner should ignore the classical view of schemes as a structured topological space[13].

Definition 8.2.6. We say a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is **represented by a scheme** X if it is equipped with a natural isomorphism $F \simeq \mathsf{h}_X$.

Example 8.2.7. The functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to its underlying set is represented by the affine scheme $\mathbb{A}^1_{\mathbb{Z}} := \mathsf{Spec}(\mathbb{Z}[t])$.

Exercise 8.2.8. Show that the functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to the set $\mathsf{GL}_n(R)$ of $n \times n$ invertible matrices over R is represented by an affine scheme.

Exercise 8.2.9. Show that the constant functor $\mathsf{CRing} \to \mathsf{Set}$, $R \mapsto I$ is not represented by a scheme unless $I \simeq \{*\}$. What is the functor represented by the disjoint union $\bigsqcup_{i \in I} \mathsf{Spec}(\mathbb{Z})$?

---

[12] In standard terminology, a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is contained in the essential image iff F satisfies Zariski descents and admits an open covering by representable functors.

[13] Just imagine learning the projective space $\mathbb{P}^n_{\mathbb{Z}}$ for the first time using the following definition: it is the functor sending a commutative algebra R to the isomorphism classes of surjections from the free R-module $R^{\oplus(n+1)}$ to a rank 1 projective R-module P.

Proof of Theorem 8.2.3. For any pair of schemes (X,Y), we need to show

$$\mathsf{Hom}_{\mathsf{Sch}}(X,Y) \to \mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X,\mathsf{h}_Y) \tag{8.1}$$

is bijective.

We first construct a map of the inverse direction as follows. Let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation. For any affine open subscheme $U \subseteq X$, by definition we have identifications

$$\mathsf{Hom}_{\mathsf{Sch}}(U,Z) \simeq \mathsf{h}_Z(\mathcal{O}_X(U))$$

functorial in Z. Let $j_U \in \mathsf{h}_X(\mathcal{O}_X(U))$ be the element corresponding to the canonical immersion $U \to X$. Consider the morphism $f_U : U \to Y$ corresponding to the element $\theta(j_U) \in \mathsf{h}_Y(\mathcal{O}_X(U))$. One can check that for affine open subschemes $U \subseteq V \subseteq X$, we have $f_U = f_V|_U$. Using Exercise 1.2.7, one can show there is a unique morphism $f : X \to Y$ such that $f_U = f|_U$ for any affine open subscheme $U \subseteq X$. The construction $\theta \mapsto f$ as above gives a map

$$\mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X,\mathsf{h}_Y) \to \mathsf{Hom}_{\mathsf{Sch}}(X,Y). \tag{8.2}$$

It remains to check (8.1) and (8.2) are inverse to each other. Using the uniqueness property about the morphism f, it is easy to see $(8.2) \circ (8.1) = \mathsf{id}$.

It remains to show $(8.1) \circ (8.2) = \mathsf{id}$. For this, let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation and $f : X \to Y$ be the morphism constructed as above. Let $\mathsf{h}_f : \mathsf{h}_X \to \mathsf{h}_Y$ be the natural transformation induced by f via functoriality. We only need to show $\theta = \mathsf{h}_f$. In other words, for any R-point $x \in X(R)$, we need to show

$$\theta(x) = \mathsf{h}_f(x). \tag{8.3}$$

Note that by definition x is a morphism $x : \mathsf{Spec}(R) \to X$, and both sides in (8.3) are morphisms from $\mathsf{Spec}(R)$ to Y.

Unwinding the constructions, it is easy to see for any affine open subscheme $j_U : U \subseteq X$, we have

$$\theta \circ \mathsf{h}_{j_U} = \mathsf{h}_{\theta(j_U)} = \mathsf{h}_{f_U} = \mathsf{h}_f \circ \mathsf{h}_{j_U}.$$

In other words, we know (8.3) is true if x is contained in the image of $U(R) \to X(R)$ for some affine open subscheme U.

Now for general x, we can find a covering of $\mathsf{Spec}(R) = \bigcup_{i\in I} V_i$ by its affine open subschemes such that $x|_{V_i} : V_i \to X$ factors through some affine open subscheme of X. By the last paragraph, we see

$$\theta(x|_{V_i}) = \mathsf{h}_f(x|_{V_i}).$$

In other words, the restrictions of the morphisms $\theta(x)$ and $\mathsf{h}_f(x) : \mathsf{Spec}(R) \to Y$ to each V_i are equal. Now Exercise 1.2.7 implies these two morphisms are equal as desired.

□

Part III. Language of schemes

9. Fiber products

9.1. Definition of fiber products.

Recall we have the notion of fiber products in any category.

Definition 9.1.1. Let $\mathcal{C}$ be a category. We say a commutative square in $\mathcal{C}$

$$\begin{tikzcd} d \arrow[r, "g'"] \arrow[d, "f'"] & a \arrow[d, "f"] \\ b \arrow[r, "g"] & c \end{tikzcd} \tag{9.1}$$

is **Cartesian**, if for any object $x \in \mathcal{C}$, the commutative square

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}}(x,d) \arrow[r, "g'\circ -"] \arrow[d, "f'\circ -"] & \mathsf{Hom}_{\mathcal{C}}(x,a) \arrow[d, "f\circ -"] \\ \mathsf{Hom}_{\mathcal{C}}(x,b) \arrow[r, "g\circ -"] & \mathsf{Hom}_{\mathcal{C}}(x,c) \end{tikzcd}$$

is a Cartesian square in Set. In other words, if it induces a bijection

$$\mathsf{Hom}_{\mathcal{C}}(x,d) \to \mathsf{Hom}_{\mathcal{C}}(x,a) \underset{\mathsf{Hom}_{\mathcal{C}}(x,c)}{\times} \mathsf{Hom}_{\mathcal{C}}(x,b).$$

In this case, we also say (9.1) is a **pullback square** and say (9.1) **exhibits d as the pullback of the diagram** $a \xrightarrow{f} c \xleftarrow{g} b$. We also say d is the **fiber product** of $a \xrightarrow{f} c \xleftarrow{g} b$.

Remark 9.1.2. By Yoneda's lemma, the object d, equipped with the morphisms f' and g', is essentially unique. We often write

$$d \simeq a \underset{c}{\times} b$$

when the morphisms are clear from the context.

Remark 9.1.3. By Yoneda's lemma, the construction

$$[a \xrightarrow{f} c \xleftarrow{g} b] \mapsto a \underset{c}{\times} b$$

is functorial. In other words, for a commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "p"] & c \arrow[d, "r"] & b \arrow[l] \arrow[d, "q"] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd},$$

there is a unique dotted morphism $a \times_c b \to a' \times_c{}' b'$ that make the following diagram commute

$$\begin{tikzcd} a \arrow[d, "p"] & a \times_c b \arrow[l] \arrow[r] \arrow[d, dotted] & b \arrow[d, "q"] \\ a' & a' \times_{c'} b' \arrow[l] \arrow[r] & b' \end{tikzcd}.$$

We often abuse notation and denote this morphism

$$(p,q) : a \underset{c}{\times} b \to a' \underset{c}{\times}' b'$$

(but it also depends on other morphisms in the diagram).

Example 9.1.4. If c is a final object, then $a \times_c b \simeq a \times b$.

Example 9.1.5. Fiber products exist in Ab and the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves fiber products. Given a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in Ab, we have

$$A \underset{C}{\times} B \simeq \mathsf{ker}(A \bigoplus B \xrightarrow{(f,-g)} C).$$

Example 9.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose fiber products exist in $\mathcal{C}$. Then fiber products exist in $\mathsf{Fun}(\mathcal{D},\mathcal{C})$, and we have

$$(F_1 \underset{F_3}{\times} F_2)(d) \simeq F_1(d) \underset{F_3(d)}{\times} F_2(d)$$

for functors $F_i : \mathcal{D} \to \mathcal{C}$ and objects $d \in \mathcal{D}$.

Definition 9.1.7. Let $\mathcal{C}$ be a category. The **pushout of a diagram** $a \xleftarrow{f} c \xrightarrow{g} b$ is defined to be the pullback of the corresponding diagram in $\mathcal{C}^{\mathsf{op}}$. We also call it the **fiber coproduct of** $a \xleftarrow{f} c \xrightarrow{g} b$ and denote it by

$$a \coprod_c b.$$

9.2. **Fiber products of affine schemes.**

Exercise 9.2.1. Fiber coproducts exist in CRing and we have

$$A \coprod_C B \simeq A \underset{C}{\otimes} B.$$

Corollary 9.2.2. *Fiber products exist in* Aff. *Given a diagram* $A \leftarrow C \to B$*, we have*

$$\mathsf{Spec}(A \underset{C}{\otimes} B) \simeq \mathsf{Spec}(A) \underset{\mathsf{Spec}(C)}{\times} \mathsf{Spec}(B).$$

Warning 9.2.3. The underlying topological space of $\mathsf{Spec}(A \otimes_C B)$ is in general not the fiber product of the corresponding topological spaces. In other words, the forgetful functor $\mathsf{Aff} \to \mathsf{Top}$ does not preserve fiber products.

Exercise 9.2.4. Let $k \to k'$ be a finite *separable* extension of degree d and $\overline{k}$ be a algebraic closure of k. Show that

$$\mathsf{Spec}(k') \underset{\mathsf{Spec}(k)}{\times} \mathsf{Spec}(\overline{k}) \simeq \sqcup_d \mathsf{Spec}(\overline{k}).$$

Note that we also have the following formal corollary of Corollary 7.2.3:

Corollary 9.2.5. *The functor* $\mathsf{Aff} \to \mathsf{Sch}$ *preserves fiber products.*

9.3. Fiber product of schemes.

Theorem 9.3.1. *The category* Sch *admits fiber products.*

We will give a constructive proof of the theorem at the end of this section. For now, we prove a particular case of it.

Lemma 9.3.2. *Let* $f : X \to Y$ *be a morphism between schemes and* $U \subseteq Y$ *be an open subscheme. Then the fiber product* $X \times_Y U$ *exists, and the canonical morphism* $X \times_Y U \to X$ *is an open immersion onto the open subscheme* $f^{-1}(U)$*.*

Proof. We only need to show the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f'"] \arrow[d, "j'"] & U \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in Sch is Cartesian. Let $Z \in \mathsf{Sch}$ be a test object. We only need to show the commutative diagram

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Sch}}(Z, f^{-1}(U)) \arrow[r, "f'\circ-"] \arrow[d, "j'\circ-"] & \mathsf{Hom}_{\mathsf{Sch}}(Z, U) \arrow[d, "j\circ-"] \\ \mathsf{Hom}_{\mathsf{Sch}}(Z, X) \arrow[r, "f\circ-"] & \mathsf{Hom}_{\mathsf{Sch}}(Z, Y) \end{tikzcd}$$

in Set is Cartesian. Using the definition of open subschemes, the vertical maps are injective, and a morphism $g : Z \to X$ (resp. $h : Z \to Y$) is contained in the image iff the subspace $g(Z)$ (resp. $h(Z)$) is contained in $f^{-1}(U)$ (resp. U). It follows that $g : Z \to X$ is contained in the image of the left vertical map iff $f \circ g$ is contained in the image of the right vertical map. In other words, the above square is Cartesian as desired.

□

Corollary 9.3.3. *Let* $X \to Y \xleftarrow{j} U$ *be a diagram in* Sch *such that* j *is an open immersion. Then the forgetful functor* $\mathsf{Sch} \to \mathsf{Top}$ *preserves the fiber product of this diagram.*

Exercise 9.3.4. Let X be a scheme.

(1) Suppose X is affine. Show that the intersection of two affine open subsets in X is still affine.
(2) Show that (1) may fail for general X.

Proposition 9.3.5. *A diagram in* Sch

(9.2)
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

is Cartesian iff it induces a Cartesian diagram

(9.3)
$$\begin{tikzcd} \mathsf{h}_W \arrow[r] \arrow[d] & \mathsf{h}_X \arrow[d] \\ \mathsf{h}_Y \arrow[r] & \mathsf{h}_Z \end{tikzcd}$$

in $\mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$. *In particular, we have*

(9.4)
$$\mathsf{h}_{X\times_Z Y} \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y.$$

Proof. For the "only if" claim, let (9.2) be a Cartesian square. By definition, for any $R \in \mathsf{CRing}$, the functor $\mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), -)$ sends (9.2) to a Cartesian square in Set. In other words, the values of the functors in (9.3) at $R \in \mathsf{CRing}$ form a Cartesian square in Set. This formally implies (9.3) itself is a Cartesian square as desired (Example 9.1.6).

Note that the "only if" claim itself implies the isomorphism (9.4).

For the "if" claim, let (9.2) be a commutative square such that (9.3) is Cartesian. By definition, (9.2) corresponds to a morphism $f : W \to X \times_Z Y$ and (7.3) corresponds to an isomorphism

(9.5)
$$\mathsf{h}_W \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y.$$

Moreover, unwinding the definitions, we see the following diagram commute

$$\begin{tikzcd} \mathsf{h}_W \arrow[r, "(9.5)"] \arrow[d, "\mathsf{h}_f"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y \arrow[d, equal] \\ \mathsf{h}_{X\times_Z Y} \arrow[r, "(9.4)"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y. \end{tikzcd}$$

It follows that h_f is also an isomorphism. By Theorem 8.2.3, f is an isomorphism as desired.

□

9.4. **Existence of fiber products.** In this subsection, we prove Theorem 9.3.1. We first deduce the theorem from the following lemma.

Lemma 9.4.1. *Let $X \to S \leftarrow Y$ be a diagram in* Sch. *Let $X = \bigcup_{i\in I} X_i$ and $Y = \bigcup_{j\in J} Y_j$ be coverings by open subschemes. Suppose for each pair $(i, j) \in I \times J$, the fiber product $X_i \times_S Y_j$ exists in* Sch. *Then $X \times_S Y$ exists.*

Proof of Theorem 9.3.1. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram in Sch. We will show $X \times_S Y$ exists.

We first reduce to the case when S is affine. Let $S = \bigcup_{i\in I} S_i$ be an open covering by affine open subschemes. For each $i \in I$, let $X_i := f^{-1}(S_i) \subseteq X$ and $Y_i := g^{-1}(S_i)$ be the corresponding open subschemes. By Lemma 9.4.1, we only need to show $X_i \times_S Y_j$ exists for any pair $(i, j) \in I^2$. By Lemma 9.3.2, we have the following *Cartesian* square

$$\begin{tikzcd} & Y_i \cap Y_j \arrow[r] \arrow[d] & Y_j \arrow[d] \\ X_i \arrow[r] & S_i \arrow[r] & S. \end{tikzcd}$$

A diagram chasing shows that $X_i \times_S Y_j$ exists iff

$$X_i \underset{S_i}{\times} (Y_i \cap Y_j)$$

exists, and these two fiber products are canonically isomorphic. Note that S_i is affine by assumption. Hence we can reduce to the case when S is affine.

Apply Lemma 9.4.1 again, we can reduce to the case when X and Y are both affine. Now the claim follows from Corollary 9.2.5 and Corollary 9.2.2.

□[Theorem 9.3.1]

Sketch of Lemma 9.4.1. We will construct the desired fiber product using *gluing of schemes*. Write $P \coloneqq I \times J$ and $W_\alpha \coloneqq X_i \times_S Y_j$ for $\alpha = (i,j) \in P$. For $(i,k) \in I^2$, write $X_{ik} \coloneqq X_i \cap X_k$ and similarly $Y_{jl} \coloneqq Y_j \cap Y_l$.

For each pair $(\alpha, \beta) \in P^2$, we define an open subscheme $W_{\alpha\beta} \subseteq W_\alpha$ as follows. Write $\alpha = (i,j)$ and $\beta = (k,l)$. Since $X_{ik} \to X_i$ and $Y_{jl} \to Y_j$ are open subschemes, applying Lemma 9.3.2 twice, we see that

$$W_{\alpha\beta} \coloneqq X_{ik} \underset{X_i}{\times} W_\alpha \underset{Y_j}{\times} Y_{jl}$$

exists and can be identified with an open subscheme of W_α. Note that we have a canonical isomorphism

$$\phi_{\alpha\beta} : W_{\alpha\beta} \simeq X_{ik} \underset{X_i}{\times} (X_i \underset{S}{\times} Y_j) \underset{Y_j}{\times} Y_{jl} \simeq X_{ik} \underset{S}{\times} Y_{jl} \simeq X_{ki} \underset{S}{\times} Y_{lj} \simeq W_{\beta\alpha}.$$

One can check

$$(P, (W_\alpha)_{\alpha \in P}, (W_{\alpha\beta})_{(\alpha,\beta)\in P^2}, (\phi_{\alpha\beta})_{(\alpha,\beta)\in P^2})$$

is a gluing data of schemes (Definition 6.1.1). Let

$$(W, (W'_\alpha)_{\alpha\in P}, (\varphi_\alpha)_{\alpha \in P})$$

be the gluing output.

Now we construct a canonical morphism $p : W \to X$. By Proposition 6.1.3, we only need to construct morphisms $p_\alpha : W_\alpha \to X$ such that

$$p_\alpha|_{W_{\alpha\beta}} = p_\beta|_{W_{\beta\alpha}} \circ \phi_{\alpha\beta}.$$

We declare p_α to be the composition

$$W_\alpha = X_i \underset{S}{\times} Y_j \to X_i \to X.$$

One can check the collcetion $(p_\alpha)_{\alpha\in P}$ satisfies the above equations. Therefore we obtain a unique morphism $p : W \to X$ such that $p|_{W'_\alpha} \circ \varphi_\alpha = p_\alpha$.

Similarly, we use Proposition 6.1.3 to construct a canonical morphism $q : W \to Y$. By construction, the following diagram commutes:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S. \end{tikzcd}$$

It remains to show this diagram is Cartesian. Let $Z \in \mathsf{Sch}$ be a test object. We only need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z, -)$ sends the above diagram to a Cartesian square in Set. One can check this by applying Proposition 6.1.4.

□[Lemma 9.4.1]

10. Change of base

10.1. S-schemes.

Definition 10.1.1. Let S be a scheme. An **S-scheme is a scheme X equipped with a morphism** $X \to S$. When $S = \mathsf{Spec}(R)$, we also say X is a **R-scheme**.

Definition 10.1.2. Let X be an S-scheme. The **base-change of X along a morphism** $S' \to S$ is the S'-scheme

$$X_{S'} := X \underset{X}{\times} S'$$

equipped with its canonical projection to S'.

Remark 10.1.3. The reason for introducing the above terminology is to encourage the readers to view an S-scheme X as a *family of schemes over* S.

Exercise 10.1.4. Show that the base-change of an open immersion is still an open immersion. In other words, let $f : X \to Y$ be an open immersion between S-schemes and $S' \to S$ be any morphism. Then $f_{S'} : X_{S'} \to Y_{S'}$ is an open immersion, where $f_{S'}$ is the morphism

$$(f, \mathsf{id}_{S'}) : X \underset{S}{\times} S' \to Y \underset{S}{\times} S'.$$

10.2. Fibers.

Definition 10.2.1. Let X be an S-scheme and $s \in S$ be a topological point. The **fiber of X at s** is the κ_s-scheme

$$X_s := X \underset{S}{\times} s := X \underset{S}{\times} \mathsf{Spec}(\kappa_s),$$

where $\mathsf{Spec}(\kappa_s) \to S$ is the canonical κ_s-point lying over s.

Proposition 10.2.2. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. The* continuous map $X_s \to X$ *induces a homeomorphism*

$$X_s \xrightarrow{\simeq} p^{-1}(s)$$

between topological spaces.

Remark 10.2.3. One can reformulate the proposition as: the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ preserves the fiber product of $X \to S \leftarrow s$, where s is understood as $\mathsf{Spec}(\kappa_s)$.

Proof of Proposition 10.2.2. Note that the continuous map $q : X_s \to X$ indeed factors through the subspace $p^{-1}(s) \subseteq X$. This follows from applying the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ to the following commutative diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathsf{Spec}(\kappa_s) \arrow[r] & S \end{tikzcd}$$

We first show $X_s \to p^{-1}(s)$ is surjective. Let $x \in p^{-1}(S)$ be a topological point. We have a commutative diagram in Sch

$$\begin{tikzcd}
\mathsf{Spec}(\kappa_x) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\mathsf{Spec}(\kappa_s) \arrow[r] & S
\end{tikzcd}$$

which by definition corresponds to a morphism $\mathsf{Spec}(\kappa_x) \to X_s$. By construction, the composition $\mathsf{Spec}(\kappa_x) \to X_s \to X$ is the canonical κ_x-point at $x \in X$. This shows $x \in p^{-1}(S)$ is in the image of the map $X_s \to p^{-1}(S)$.

Now we reduce to the case when S is affine. In other words, we will show the claim is *local on* S. To do this, let $U \subseteq S$ be an affine open subscheme containing the point s. Let $X_U \coloneqq X \times_S U$ be the base-change of X to U. Consider the Cartesian squares

$$\begin{tikzcd}
(X_U)_s \arrow[r] \arrow[d] & X_U \arrow[r, "j_X"] \arrow[d, "p_U"] & X \arrow[d, "p"] \\
s \arrow[r] & U \arrow[r, "j"] & S.
\end{tikzcd}$$

It follows formally that the outer square is also Cartesian. In particular, we have

$$(X_U)_s \simeq X_s.$$

On the other hand, by Exercise 10.1.4, j_X is also an open immersion and its image is the open subset $p^{-1}(U) \subseteq X$. This implies j_X induces a homeomorphism

$$p_U^{-1}(s) \simeq p^{-1}(s).$$

Moreover, it is easy to see the diagram

$$\begin{tikzcd}
(X_U)_s \arrow[r] \arrow[d, "\simeq"] & p_U^{-1}(s) \arrow[d, "\simeq"] \\
X_s \arrow[r] & p^{-1}(s).
\end{tikzcd}$$

Hence to show the bottom horizontal map is a homeomorphism, we only need to show the top horizontal is a homeomorphism. This allows us to replace S with U (and therefore X with X_U) thereby assume S to be affine.

Now we reduce to the case when X is affine. In other words, we will show the claim is *local on* X. To do this, let $X = \bigcup_{i \in I} U_i$ be an open covering such that each U_i is affine. We only need to show the continuous map

$$X_s \cap q^{-1}(U_i) \to p^{-1}(s) \cap U_i$$

is a homeomorphism. Write p_i for the composition $U_i \to X \to S$. We have

$$p^{-1}(s) \cap U_i = p_i^{-1}(s)$$

as subspaces of U_i. On the other hand, by Lemma 9.3.2, $X_s \cap q^{-1}(U_i)$ is the underlying topological space of the fiber product

$$X_s \underset{X}{\times} U_i \simeq (s \underset{S}{\times} X) \underset{X}{\times} U_i \simeq s \underset{S}{\times} U_i \simeq (U_i)_s.$$

Hence we only need to show $(U_i)_s \to p_i^{-1}(s)$ is a homeomorphism. This allows us to replace X with U_i thereby assume X to be affine.

By the previous discussion, we can assume $S = \mathsf{Spec}(A)$ and $X = \mathsf{Spec}(B)$. Let $f : A \to B$ be the homomorphism corresponding to $p : X \to S$. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to the point $x \in S$. Recall $\kappa_{\mathfrak{p}} \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. By Corollary 9.2.2 and Corollary 9.2.5, we have

$$X_s \simeq \mathsf{Spec}(B \underset{A}{\otimes} \kappa_{\mathfrak{p}})$$

and the morphism $X_s \to X$ is induced by the homomorphism $B \simeq B \otimes_A A \to B \otimes_A \kappa_{\mathfrak{p}}$. We need to show $\mathsf{Spec}(B \otimes_A \kappa_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces a homeomorphism onto its image. This follows from the following exercise.

Exercise 10.2.4. Let $h : R \to R'$ be a homomorphism. Suppose any element $r' \in R'$ can be written as $h(r)u$ for some $r \in R$ and *invertible* element $u \in R'$. Then $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ induces a homeomorphism onto its image.

Namely, any element in $B \otimes_A \kappa_{\mathfrak{p}}$ can be written as

$$\sum_{i \in I} b_i \otimes \overline{a_i/c} = \big((\sum_{i \in I} b_i f(a_i)) \otimes 1 \big) \cdot (1 \otimes \overline{1/c})$$

where $a_i \in A$, $b_i \in B$, $c \in A \smallsetminus \mathfrak{p}$, and $\overline{b_i/c}$ is the image of b_i/c under the homomorphism $A_{\mathfrak{p}} \to \kappa_{\mathfrak{p}}$. Note that $(1 \otimes \overline{1/c})$ is invertible as desired.

$\square$[Proposition 10.2.2]

Warning 10.2.5. In Proposition 10.2.2, X_s is in general *not* isomorphic to the locally ringed subspace $(p^{-1}(s), \mathcal{O}_X|_{p^{-1}(s)})$ associated to $p^{-1}(s)$. See Warning 5.3.6.

In particular, for a topological points $x \in X_s$, the homomorphism $\mathcal{O}_{X,i(x)} \to \mathcal{O}_{X_s,x}$ is in general not an isomorphism, where $i : X_s \to X$ is the canonical morphism.

That said, we have the following result:

Proposition 10.2.6. *The morphism $i : X_s \to X$ induces an isomorphism $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$ for any topological point $x \in X_s$.*

Proof. We claim i induces a bijection between the following sets:

- The set of field-valued points of X_s;
- The set of field-valued points of X contained in $p^{-1}(s)$.

To prove the claim, let k be any field. By the definition of fiber,

$$X_s(k) \simeq X(k) \underset{S(k)}{\times} s(k).$$

Using Proposition-Construction 8.1.2, it is easy to see $s(k) \to S(k)$ is injective, and its image contains k-points of S lying over s. It follows that $X_s(k) \to X(k)$ is also injective, and its image contains k-points of X whose underlying topological point is sent to s by p. This proves the desired claim.

Since the underlying topological space of X_s is homeomorphic to $p^{-1}(s)$, the claim implies i induces a bijection between the following sets:

- The set of field-valued points of X_s lying over x;
- The set of field-valued points of X lying over $i(x)$.

Applying Proposition-Construction 8.1.2 again, we see $\kappa_{i(x)} \to \kappa_x$ induces a bijection between the sets of field extensions of $\kappa_{i(x)}$ and κ_x. This is possible only if $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$.

□

Exercise 10.2.7. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram of schemes. Show that the morphism

$$\bigsqcup_{(x,y,s)} \mathsf{Spec}(\kappa_x) \underset{\mathsf{Spec}(\kappa_s)}{\times} \mathsf{Spec}(\kappa_y) \to X \underset{S}{\times} Y$$

induces a bijection between topological points. Here $x \in X$, $y \in Y$ and $s \in S$ are topological points such that $f(x) = g(y) = s$.

10.3. **Base-change to local scheme.** We also have the following result about base-change along $\mathsf{Spec}(\mathcal{O}_{S,s}) \to S$.

Proposition 10.3.1. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. Write $S' := \mathsf{Spec}(\mathcal{O}_{S,s})$. The* morphism

$$X_{S'} := X \underset{S}{\times} S' \to X$$

induces an isomorphism between locally ringed spaces

$$X_{S'} \xrightarrow{\simeq} p^{-1}(S').$$

In particular, the locally ringed subspace $p^{-1}(S')$ of X is a scheme.

Proof. As in the proof of Proposition 10.2.2, we have a surjective morphism $X_{S'} \to p^{-1}(S')$ between locally ringed spaces, and we can assume X and S are affine. Let $f : A \to B$ and $\mathfrak{p} \subset A$ be as in the proof of Proposition 10.2.2. We need to show $\phi : \mathsf{Spec}(B \otimes_A A_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces an isomorphism

$$\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}}) \xrightarrow{\simeq} \phi(\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}})),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$. This follows from the following exercise:

Exercise 10.3.2. Let R be a commutative ring and $S \subseteq R$ be a multiplicative subset. Consider the canonical homomorphism $R \to R[S^{-1}]$. Show that $\phi : \mathsf{Spec}(R[S^{-1}]) \to \mathsf{Spec}(R)$ induces an isomorphism

$$\mathsf{Spec}(R[S^{-1}]) \xrightarrow{\simeq} \phi(\mathsf{Spec}(R)),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$.

□

11. Subschemes and immersions

We have studied open subschemes and open immersions in Sect. 5.3. In this section, we introduce general subschemes and immersions.

11.1. **Monomorphisms and epimorphisms.**

Recall in any category $\mathcal{C}$, we can define monomorphisms and epimorphisms.

Definition 11.1.1. A morphism $f : x \to y$ in $\mathcal{C}$ is a **monomorphism** if for any test object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(z,x) \xrightarrow{f\circ -} \mathsf{Hom}_{\mathcal{C}}(z,y)$$

is an injection.

A morphism $f : x \to y$ in $\mathcal{C}$ is a **epimorphism** if the corresponding morphism in $\mathcal{C}^{\mathsf{op}}$ is a monomorphism, i.e., for any test object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(y,z) \xrightarrow{-\circ f} \mathsf{Hom}_{\mathcal{C}}(x,z)$$

is an injection.

Example 11.1.2. In Set, monomorphisms are exactly injections, while epimorphisms are exactly surjections.

Exercise 11.1.3. Let $f : U \to X$ be an open immersion.

- Show that f is a monomorphism.
- Show that f is an epimorphism iff it is an isomorphism.

Warning 11.1.4. A morphism can simultaneously be a monomorphism and an epimorphism, but fail to be an isomorphism.

Exercise 11.1.5. Let $R \in \mathsf{CRing}$ and $f \in R$ be an element that is not a zero-divisor. Show that $R \to R_f$ is a monomorphism and an epimorphism in CRing.

Exercise 11.1.6. Show that the functor $\mathsf{Aff} \to \mathsf{Sch}$ sends monomorphisms to monomorphisms, but may fail to send epimorphisms to epimorphisms.

11.2. **Digression: epimorphisms between sheaves.**

Proposition 11.2.1. *Let X be a topological space and $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism in $\mathsf{Shv}(X, \mathsf{Set})$. The following conditions are equivalent:*

(i) *The morphism α is an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$.*
(ii) *For any point $x \in X$, the map $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a surjection.*

Proof. (i)⇒(ii): let α be an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$. Suppose α_x is not a surjection for some point $x \in X$. We can find a set A and maps $f, g : \mathcal{F}'_x \to A$ such that $f \neq g$ but $f \circ \alpha_x = g \circ \alpha_x$. By Proposition 2.3.4, f, g correspond to morphisms $\phi, \varphi : \mathcal{F}' \to \delta_{x,A}$ such that $\phi \neq \varphi$ but $\phi \circ \alpha = \varphi \circ \alpha$. But this contradicts the assumption that α is an epimorphism.

(ii)⇒(i): suppose α is a morphism that induces surjections between stalks. We will show α is an epimorphism. Let $\phi, \varphi : \mathcal{F}' \to \mathcal{F}''$ be morphisms in $\mathsf{Shv}(X, \mathsf{Set})$ such that $\phi \circ \alpha = \varphi \circ \alpha$. We only need to show $\phi_U = \varphi_U : \mathcal{F}'(U) \to \mathcal{F}''(U)$ for any open subset $U \subseteq X$. For any morphism $\psi : \mathcal{F}' \to \mathcal{F}''$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}'(U) \arrow[r] \arrow[d, "\psi_U"] & \prod_{x\in U} \mathcal{F}'_x \arrow[d, "(\psi_x)_{x\in U}"] \\
\mathcal{F}''(U) \arrow[r] & \prod_{x\in U} \mathcal{F}''_x
\end{tikzcd}$$

such that the horizontal arrows are injections (Lemma 2.2.1). Hence to show $\phi_U = \varphi_U$, we only need to show $\phi_x = \varphi_x$ for any $x \in U$. However, this follows from $\phi_x \circ \alpha_x = \varphi_x \circ \alpha_x$ and the assumption that α_x is a surjection.

□

Warning 11.2.2. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$. The map $\alpha_X : \mathcal{F}(X) \to \mathcal{F}'(X)$ is in general not a surjection.

11.3. **Closed immersions.**

Definition 11.3.1. Let $i : Y \to X$ be a morphism between schemes. We say i is a **closed immersion** if

(1) It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;
(2) The morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(Y, \mathsf{Set})$.

Let X be a scheme. A **closed subscheme** is an isomorphism class of closed immersions into X.

Remark 11.3.2. Condition (2) is equivalent to

(2Ab) The morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(Y, \mathsf{Ab})$.

Epimorphisms in $\mathsf{Shv}(Y, \mathsf{Ab})$ (or more generally in any abelian category) are often called *surjections*.

Proposition 11.3.3. *A morphism $i : Y \to X$ between schemes is a closed immersion iff*

(1) *It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;*
(2') *For any point $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection.*

Proof. We only need to show conditions (2) and (2') are equivalent if (1) holds. Under the latter assumption, it is easy to see:

- If $x \notin i(Y)$, $(i_*\mathcal{O}_Y)_x \simeq 0$;
- If $x = i(y)$, $(i_*\mathcal{O}_Y)_x \simeq \mathcal{O}_{Y,y}$.

Now the claim follows from Proposition 11.2.1.

□

Remark 11.3.4. By Proposition 11.2.1, we can also replace condition (2') by

(2") The morphism $i^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$.

Proposition 11.3.5. *Let $R \in \mathsf{CRing}$ and $I \subseteq R$ be an ideal. Let $i : \mathsf{Spec}(R/I) \to \mathsf{Spec}(R)$ be the morphism corresponding to the canonical surjection $\pi : R \to R/I$. Then i is a closed immersion.*

Proof. It is easy to see i induces a continuous bijective map from $\mathsf{Spec}(R/I)$ to the closed subspace $Z(I)$ of $\mathsf{Spec}(R)$. Moreover, for any standard open subset $U(\overline{f}) \in \mathsf{Spec}(R/I)$, $\overline{f} \in R/I$, its image is equal to the open subset $U(f) \cap Z(I) \subseteq Z(I)$, where $f \in R$ is any lifting of $\overline{f}$. This implies i induces a homeomorphism from $\mathsf{Spec}(R/I)$ to $Z(I)$.

Let $y \in \mathsf{Spec}(R/I)$ be a topological point and $\overline{\mathfrak{p}} \subseteq R/I$ be the corresponding prime ideal. By definition, $i(y)$ corresponds to the prime ideal $\mathfrak{p} := \pi^{-1}(\overline{\mathfrak{p}})$ and we have $\overline{\mathfrak{p}} = \mathfrak{p}/I$. The homomorphism $\mathcal{O}_{\mathsf{Spec}(R),i(y)} \to \mathcal{O}_{\mathsf{Spec}(R/I),y}$ can be identified with $R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}/I}$, which is obviously surjective.

□

Exercise 11.3.6. Let k be a field. Consider the closed immersion $i : \mathbb{A}^1_k \to \mathbb{A}^2_k$ corresponding to the surjection $k[x,y] \to k[x]$. Show that $\mathcal{O}_{\mathbb{A}^2_k}(U) \to (i_*\mathcal{O}_{\mathbb{A}^1_k})(U)$ is not surjective for general open subset $U \subseteq \mathbb{A}^2_k$.

Proposition 11.3.7. *A closed immersion $i : Y \to X$ is a monomorphism in* Sch.

Proof. Let Z be any test scheme. We need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z,Y) \to \mathsf{Hom}_{\mathsf{Sch}}(Z,X)$ is injective.

Suppose $f,g : Z \to Y$ are morphisms such that $i \circ f = i \circ g$. It is clear that the underlying continuous maps of f and g are equal. Write ϕ for this continuous map. Now the morphisms f and g are given by morphisms

$$\alpha, \beta : \mathcal{O}_Y \to \phi_*\mathcal{O}_Z.$$

We only need to show $\alpha = \beta$.

Let $\gamma : \mathcal{O}_X \to i_*\mathcal{O}_Y$ be the canonical morphism. The assumption $i \circ f = i \circ g$ implies

$$i_*(\alpha) \circ \gamma = i_*(\beta) \circ \gamma : \mathcal{O}_X \to i_* \circ \phi_*\mathcal{O}_X$$

Since γ is an epimorphism, we obtain

$$i_*(\alpha) = i_*(\beta)$$

In particular

$$i^{-1} \circ i_*(\alpha) = i^{-1} \circ i_*(\alpha).$$

Now the desired claim follows from the fact that $i^{-1} \circ i_* \simeq \mathrm{Id}$, which can be checked by unwinding the definitions.

□

11.4. **Ideal of definition.**

Construction 11.4.1. Let $i : Y \to X$ be a closed immersion. Consider the kernel

$$\mathcal{I}_Y := \mathsf{ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y),$$

which is defined to be the fiber product of $\mathcal{O}_X \to i_*\mathcal{O}_Y \leftarrow 0$ inside $\mathsf{Shv}(X,\mathsf{Ab})$. We call it the **ideal of definition** for the closed immersion $i : Y \to X$.

For any open subset $U \subseteq X$, it is easy to show that the functor

$$(-)(U) : \mathsf{Shv}(X,\mathsf{Ab}) \to \mathsf{Ab}$$

preserves fiber products (in fact arbitrary small limits). It follows that

$$\mathcal{I}_Y(U) \simeq \mathsf{ker}(\mathcal{O}_X(U) \to (i_*\mathcal{O}_Y)(U)) \simeq \mathsf{ker}(\mathcal{O}_X(U) \to \mathcal{O}_Y(i^{-1}U)).$$

In particular, $\mathcal{I}_Y(U) \subseteq \mathcal{O}_X(U)$ is an ideal, and therefore $\mathcal{I}_Y$ is a **sheaf of ideals for $\mathcal{O}_X$**[14].

[14] Other name: ideal sheaf of $\mathcal{O}_X$.

Note however that $\mathcal{O}_X(U) \to (i_*\mathcal{O}_Y)(U)$ may fail to be a surjection (see Exercise 11.3.6).

For any point $x \in X$, it is easy to show that the functor

$$(-)_x : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$$

preserves fiber products (in fact arbitrary *finite* limits). It follows that

$$\mathcal{I}_{Y,x} \simeq \mathsf{ker}(\mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x).$$

Hence

- If $x \notin i(Y)$, $\mathcal{I}_{Y,x} \simeq \mathcal{O}_{X,x}$;
- If $x = i(y)$, $\mathcal{I}_{Y,x} \simeq \mathsf{ker}(\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y})$.

Since $\mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x$ is surjective, we obtain a short exact sequence in Ab:

$$0 \to \mathcal{I}_{Y,x} \to \mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x \to 0$$

Lemma 11.4.2. *Let $i : Y \to X$ be a closed immersion and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. The canonical morphism*

$$\mathcal{O}_X/\mathcal{I}_Y := \mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X) \to i_*\mathcal{O}_Y$$

is an isomorphism, where the cokernel is taken in $\mathsf{Shv}(X, \mathsf{Ab})$.

Proof. Taking stalks commute with pushouts (in fact arbitrary small colimits), hence we have

$$\mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X)_x \simeq \mathsf{coker}(\mathcal{I}_{Y,x} \to \mathcal{O}_{X,x}) \simeq (i_*\mathcal{O}_Y)_x$$

for any point $x \in X$. This implies $\mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X) \to i_*\mathcal{O}_Y$.

□

Remark 11.4.3. In fact, for any surjection $\phi : \mathcal{F} \to \mathcal{F}'$ in $\mathsf{Shv}(X, \mathsf{Ab})$, we have $\mathsf{coker}(\mathsf{ker}(\phi) \to \mathcal{F}) \simeq \mathcal{F}'$. We will return to this fact in future lectures.

Exercise 11.4.4. Let $i : Y \to X$ be a closed immersion and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. Show that $\mathcal{O}_X/\mathcal{I}_Y$ is the sheafification of the presheaf

$$U \mapsto \mathcal{O}_X(U)/\mathcal{I}_Y(U).$$

Give an example where this sheafification step is necessary.

Lemma 11.4.5. *A closed immersion $i : Y \to X$ is determined by its ideal of definition up to isomorphism.*

Proof. Let $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be the ideal of definition. We can reconstruct the corresponding closed subscheme as follows. Consider $\mathcal{Q} := \mathcal{O}_X/\mathcal{I}_Y$. We have $\mathcal{Q} \simeq i_*\mathcal{O}_Y$, hence $\mathcal{Q}_x \simeq 0$ if $x \notin i(Y)$ while $\mathcal{Q}_x \simeq \mathcal{O}_{Y,y} \neq 0$ if $x = i(y)$. This implies $i(Y)$ is equal to the *support* of $\mathcal{Q}$.

$$\mathrm{supp}(\mathcal{Q}) := \{x \in X \mid \mathcal{Q}_x \neq 0\}.$$

In particular, the underlying topological space of Y can be identified with the subspace $\mathrm{supp}(\mathcal{Q}) \simeq \mathcal{O}$. Via this identification, the structure sheaf $\mathcal{O}_Y$ is given by $\mathcal{Q}|_{\mathrm{supp}(\mathcal{Q})}$, and the morphism is $Y \to X$ is isomorphic to the following composition

$$(\mathrm{supp}(\mathcal{Q}), \mathcal{Q}|_{\mathrm{supp}(\mathcal{Q})}) \to (X, \mathcal{Q}) \to (X, \mathcal{O})$$

of morphisms between locally ringed spaces.

□

Note however that not every sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ can be realized an ideal of definiton for a closed immersion.

Exercise 11.4.6. Let k be a field and $X \coloneqq \mathbb{A}^1_k$ and $0 \in X$ be the zero point. Show that

$$\mathcal{I}(U) \coloneqq \begin{cases} \mathcal{O}(U) & \text{for} \quad 0 \notin U \\ 0 & \text{for} \quad 0 \in U \end{cases}$$

defines an sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$, but it is not an ideal of definition for any closed immersion into X.

In future lectures, we will give a necessary and sufficient condition for a sheaf $\mathcal{I}$ of ideals to be an ideal of definition. Namely, we need $\mathcal{I}$ to be *quasi-coherent* as an $\mathcal{O}_X$-module.

11.5. **Locally closed immersions.**

Definition 11.5.1. Let $f : Y \to X$ be a morphism between schemes. We say f is a **locally closed immersion** if it can be written as a composition $Y \xrightarrow{i} Z \xrightarrow{j} X$ such that i is a closed immersion while j is an open immersion.

Proposition 11.5.2. *A morphism $f : Y \to X$ between schemes is a locally closed immersion iff*

(1) *It induces a homeomorphism $Y \xrightarrow{\simeq} f(Y)$ onto a locally closed subspace*[15] *of X;*
(2) *For any point $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection.*

Proof. The "only if" claim: let $f : Y \xrightarrow{i} Z \xrightarrow{j} X$ be a factorization of f such that i is a closed immersion and j is an open immersion. Then Y is homeomorphic to the closed subspace $i(Y) \subseteq Z$, and Z is homeomorphic to the open subspace $j(Z)$. Hence Y is homeomorphic to $j(i(Y)) = f(Y)$, which is a locally closed subspace of X. Moreover, for any point $y \in Y$, we have, the homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ factors as

$$\mathcal{O}_{X,f(y)} = \mathcal{O}_{X,j\circ i(y)} \to \mathcal{O}_{Z,i(y)} \to \mathcal{O}_{Y,y}.$$

Since j is an open immersion, the second map is an isomorphism; since i is a closed immersion, the last map is a surjection. Hence $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection as desired.

The "if" claim: by (1), we can find an open subset $Z \subseteq X$ such that $f(Y)$ is a closed subset of Z. Consider the open subscheme of X given by Z. Write $j : Z \to X$ for the corresponding open embedding. By Exercise 5.3.8, there is a unique morphism $i : Y \to Z$ such that $f = j \circ i$. It remains to show i is a closed immersion. Condition (1) implies i induces a homeomorphism $Y \simeq i(Y) = f(Y)$ onto the closed subspace $i(Y) \subseteq Z$. Condition (2) implies the composition

$$\mathcal{O}_{X,f(y)} = \mathcal{O}_{X,j\circ i(y)} \to \mathcal{O}_{Z,i(y)} \to \mathcal{O}_{Y,y}$$

is surjective. Since the second map is an isomorphism, the last map is a surjection as desired.

□

Warning 11.5.3. A locally closed immersion may fail to be written as an open immersion *followed* by a closed immersion. We will provide a counterexample in future lectures.

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[15] A subspace is locally closed iff it can be written as the intersection of an open subspace and a closed subspace.

Part IV. Quasi-coherent sheaves

12. $\mathcal{O}_X$-module sheaves

12.1. Definition.

A study of commutative rings cannot be complete without mentioning their *modules*.

Definition 12.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$**-module presheaf**[16] is an object $\mathcal{M} \in \mathsf{PShv}(X, \mathsf{Ab})$ equipped with the following structure:

- For any open subset $U \subset X$, the abelian group $\mathcal{M}(U)$ is equipped with an $\mathcal{O}_X(U)$-module structure,

such that

- For any open subsets $U \subseteq V \subseteq X$, the restriction map
$$\mathcal{M}(V) \to \mathcal{M}(U)$$
intertwines the action of the homomorphism
$$\mathcal{O}_X(V) \to \mathcal{O}_X(U).$$

An $\mathcal{O}_X$**-module (sheaf)** is an $\mathcal{O}_X$-module presheaf $\mathcal{M}$ such that the underlying object $\mathcal{M} \in \mathsf{PShv}(X, \mathsf{Ab})$ is a sheaf. When there is no danger of ambiguity, we just call it an $\mathcal{O}_X$**-module**.

Example 12.1.2. The structure sheaf $\mathcal{O}_X$ itself is an $\mathcal{O}_X$-module (sheaf).

Definition 12.1.3. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module presheaves. An $\mathcal{O}_X$**-linear morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a morphism $f : \mathcal{M} \to \mathcal{N}$ between sheaves such that:

- For any open subset $U \subset X$, the map $f_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{O}_X(U)$-linear.

We can define a category
$$\mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}},$$
where objects are $\mathcal{O}_X$-module presheaves, and morphisms are $\mathcal{O}_X$-linear morphisms. Let
$$\mathcal{O}_X\text{-}\mathsf{mod} \subseteq \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$$
be the full subcategory of $\mathcal{O}_X$-module (sheaves).

Example 12.1.4. A sheaf $\mathcal{I}$ of ideals of $\mathcal{O}_X$ has an obvious $\mathcal{O}_X$-module structure. The embedding $\mathcal{I} \to \mathcal{O}_X$ is an $\mathcal{O}_X$-linear morphism.

12.2. Sheafification.

Proposition 12.2.1. *The embedding functor $\mathcal{O}_X\text{-}\mathsf{mod} \to \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$ admits a left adjoint*
$$\mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{-}\mathsf{mod}$$
compatible with the sheafification functor
$$(-)^\sharp : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{Shv}(X, \mathsf{Ab})$$

Sketch. For $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$, consider its sheafification $\mathcal{M}^\sharp$, viewed as an object in $\mathsf{Shv}(X, \mathsf{Ab})$. We have an action of $\mathcal{O}_X$ on $\mathcal{M}^\sharp$ given by the formula
$$\mathcal{O}_X \times \mathcal{M}^\sharp \simeq \mathcal{O}_X^\sharp \times \mathcal{M}^\sharp \simeq (\mathcal{O}_X \times \mathcal{M})^\sharp \to \mathcal{M}^\sharp,$$
where

[16] Other name: presheaf of modules for $\mathcal{O}_X$.

- the first isomorphism is because $\mathcal{O}_X$ is a sheaf;
- the second isomorphism is because sheafification commutes with finite products (see Remark 3.1.5);
- the third morphism is provided with the action of $\mathcal{O}_X$ on $\mathcal{M}$.

One can check this defines an $\mathcal{O}_X$-module sheaf $\mathcal{M}^\sharp$, and the morphism $\phi : \mathcal{M} \to \mathcal{M}^\sharp$ is $\mathcal{O}_X$-linear. Moreover, for any test object $\mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$, the following composition

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\mathcal{M}^\sharp, \mathcal{N}) \xrightarrow{-\circ\phi} \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}}(\mathcal{M}, \mathcal{N})$$

is a bijection. This implies the desired statement.

□

12.3. **Pushforward and pullback.**

Construction 12.3.1. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Recall we have a morphism $\alpha : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ in $\mathsf{Shv}(Y, \mathsf{CRing})$.

For any $\mathcal{M} \in \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$, one can check the object

$$\phi_*(\mathcal{M}) \in \mathsf{PShv}(Y, \mathsf{Ab})$$

has a structure of $\phi_*(\mathcal{O}_X)$-module presheaf given by

$$\phi_*(\mathcal{O}_X) \times \phi_*(\mathcal{M}) \simeq \phi_*(\mathcal{O}_X \times \mathcal{M}) \to \phi_*(\mathcal{O}_X).$$

Restricting along α, we obtain a structure of $\mathcal{O}_Y$-module presheaf on $\phi_*(\mathcal{O}_X)$. We call the obtained object

$$\phi_*(\mathcal{O}_X) \in \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}}$$

the **pushforward (= direct image) of the $\mathcal{O}_X$-module presheaf $\mathcal{N}$ along ϕ.**

One can upgrade this construction to a functor

$$\phi_* : \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}}$$

compatible with the direct image functor for abelian sheaves. Moreover, it restricts to a functor

$$\phi_* : \mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_Y\text{–}\mathsf{mod}$$

Construction 12.3.2. For a chain $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y) \xrightarrow{\varphi} (Z, \mathcal{O}_Z)$, the composition

$$\mathcal{O}_X\text{–}\mathsf{mod}_{(\mathsf{PShv})} \xrightarrow{\phi_*} \mathcal{O}_Y\text{–}\mathsf{mod}_{(\mathsf{PShv})} \xrightarrow{\varphi_*} \mathcal{O}_Z\text{–}\mathsf{mod}_{(\mathsf{PShv})}$$

is canonically identified with $(\varphi \circ \phi)_*$.

Construction 12.3.3. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Recall we have a morphism $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in $\mathsf{Shv}(X, \mathsf{CRing})$. In particular, there is a morphism

$$\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y \to \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

in $\mathsf{PShv}(X, \mathsf{CRing})$.

For any $\mathcal{N} \in \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}}$, one can check the object

$$\phi^{-1}_{\mathsf{PShv}}(\mathcal{N}) \in \mathsf{PShv}(X, \mathsf{Ab})$$

has a structure of $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y)$-module given by[17]

$$\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y) \times \phi^{-1}_{\mathsf{PShv}}(\mathcal{N}) \simeq \phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y \times \mathcal{N}) \to \phi^{-1}_{\mathsf{PShv}}(\mathcal{N}).$$

---

[17]See Remark 3.3.4 for the first isomorphism.

Consider the $\mathcal{O}_X$-module presheaf on X

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U).$$

We denote it by

$$\phi^*_{\mathsf{PShv}}(\mathcal{N}) \in \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$$

and call it the **pullback (=inverse image) of the $\mathcal{O}_Y$-module presheaf $\mathcal{M}$ along ϕ.**

One can upgrade this construction to a functor

$$\phi^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}.$$

Proposition 12.3.4. *Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. The functor*

$$\phi^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$$

is canonically left adjoint to

$$\phi_* : \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}}.$$

Sketch. Let $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$ and $\mathcal{N} \in \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}}$. By definition, knowing a morphism $\phi^*_{\mathsf{PShv}}(\mathcal{N}) \to \mathcal{M}$ is equivalent to knowing $\mathcal{O}_X(U)$-linear maps

$$\mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U) \to \mathcal{M}(U)$$

compatible with the restriction maps. The latter is equivalent to knowing $(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)$-linear maps

$$(\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U) \to \mathcal{M}(U)$$

compatible with the restriction maps, i.e., a $\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y$-linear morphism

$$\phi^{-1}_{\mathsf{PShv}}\mathcal{N} \to \mathcal{M},$$

where $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y)$ acts on $\mathcal{M}$ via the homomorphism $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y) \to \mathcal{O}_X$. Using the adjunction $(\phi^{-1}_{\mathsf{PShv}}, \phi_*)$, the above data is equivalent to an $\mathcal{O}_Y$-linear morphism

$$\mathcal{N} \to \phi_*\mathcal{M}$$

as desired.

□

We now define pullback of $\mathcal{O}$-module *sheaves*.

Construction 12.3.5. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Define ϕ^* to be the following composition

$$\phi^* : \mathcal{O}_Y\text{-}\mathsf{mod} \xrightarrow{\subseteq} \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}} \xrightarrow{\phi^*_{\mathsf{PShv}}} \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}} \xrightarrow{\sharp} \mathcal{O}_X\text{-}\mathsf{mod}.$$

For $\mathcal{N} \in \mathcal{O}_Y\text{-}\mathsf{mod}$, we call $\phi^*\mathcal{N}$ the **pullback (=inverse image) of $\mathcal{N}$ along ϕ.**

In other words, $\phi^*\mathcal{N}$ is the sheafification of

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U).$$

Exercise 12.3.6. Show that $\phi^*\mathcal{N}$ can be identified with the *sheafification* of

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}\mathcal{N})(U). \tag{12.1}$$

As a corollary of the above exercise, we have:

Lemma 12.3.7. *Let $j : U \to X$ be an open immersion of schemes. For $\mathcal{M} \in \mathcal{O}_X$–$\mathsf{mod}$, we have $j^*\mathcal{M} \simeq j^{-1}\mathcal{M}$.*

Warning 12.3.8. The assignement (12.1) is in general *not* a sheaf. In other words, to define ϕ^*, we need to sheafify after tensoring up.

Exercise 12.3.9. Let k be a field, and consider the morphism $\phi : X \to Y$ between schemes given by

$$\bigsqcup_I \mathsf{Spec}(k[t]) \to \mathsf{Spec}(k),$$

where I is an infinite set. Let $\mathcal{N} \in \mathcal{O}_Y$–$\mathsf{mod} \simeq k$–$\mathsf{mod}$ be an infinite-dimensional object. Show that (12.1) is not a sheaf.

Exercise 12.3.10. Let k be a field. Let $\phi : \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the projection map given by $k[x] \to k[x,y]$ and $i : \mathsf{Spec}(k) \to \mathbb{A}^1_k$ be the closed immersion given by $k[x] \to k[x]/x \simeq k$. Consider $\mathcal{N} := i_*\mathcal{O}_{\mathsf{Spec}(k)}$. Show that (12.1) is not a sheaf.

The following result follows formally from Proposition 12.2.1 and Proposition 12.3.4.

Corollary 12.3.11. *Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. The functor*

$$\phi^* : \mathcal{O}_Y\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

is canonically left adjoint to

$$\phi_* : \mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_Y\text{–}\mathsf{mod}.$$

Construction 12.3.12. For a chain $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y) \xrightarrow{\varphi} (Z, \mathcal{O}_Z)$, the canonical natural isomorphism

$$\varphi_* \circ \phi_* \simeq (\varphi \circ \phi)_*$$

induces canonical natural isomorphisms

$$\phi^*_{\mathsf{PShv}} \circ \varphi^*_{\mathsf{PShv}} \simeq (\varphi \circ \phi)^*_{\mathsf{PShv}}, \;\; \phi^* \circ \varphi^* \simeq (\varphi \circ \phi)^*.$$

Construction 12.3.13. In general, one can define a symmetric monoidal structure on $\mathcal{O}_X$–mod such that the tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ of two objects $\mathcal{M}$ and $\mathcal{N}$ is the *sheafification* of the $\mathcal{O}_X$-module presheaf

$$U \mapsto \mathcal{M}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{N}(U).$$

13. Quasi-coherent modules

13.1. $\widetilde{M}$**.** In this subsection, let $A \in \mathsf{CRing}$ be a commutative ring and $X := \mathsf{Spec}(A)$ be the corresponding affine scheme. Consider the ringed space $X' := (*, A)$ whose underlying topological space is a singleton, and whose structure sheaf is given by $A \in \mathsf{CRing} \simeq \mathsf{Shv}(*, \mathsf{CRing})$. Let

$$\pi : X \to (*, A) = X'$$

be the morphism such that $\alpha : \mathcal{O}_{X'} \to \pi_* \mathcal{O}_X$ corresponds to the canonical isomorphism $A \simeq \mathcal{O}_X(X)$. Note that we can make the following identificaiton

$$A\text{–}\mathsf{mod} \simeq \mathcal{O}_{X'}\text{–}\mathsf{mod}.$$

Definition 13.1.1. Consider the functor

$$\pi^* : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}.$$

For $M \in A\text{–}\mathsf{mod}$, we write

$$\widetilde{M} := \pi^*(\mathcal{M}) \in \mathcal{O}_X\text{–}\mathsf{mod}$$

and call it the **$\mathcal{O}_X$-module associated to the A-module** M.

By Corollary 12.3.11, we have:

Corollary 13.1.2. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. We have a canonical adjunction*

$$\begin{array}{ccc} A\text{–}\mathsf{mod} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod} \\ M & \mapsto & \widetilde{M} \\ \mathcal{F}(X) & \leftarrow\!\shortmid & \mathcal{F}. \end{array}$$

Construction 13.1.3. Let $A \to B$ be a homomorphism in CRing and $f : Y \to X$ be the corresponding morphism between affine schemes. We have a canonical commutative diagram of ringed spaces:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & (*, B) \arrow[d] \\ X \arrow[r] & (*, A). \end{tikzcd}$$

By Construction 12.3.12, we have a canonical commutative diagram of functors

(13.1)
$$\begin{tikzcd} A\text{–}\mathsf{mod} & \mathcal{O}_X\text{–}\mathsf{mod} \arrow[l, "(-)(X)"'] \\ B\text{–}\mathsf{mod} \arrow[u, "\mathsf{res}"] & \mathcal{O}_Y\text{–}\mathsf{mod}, \arrow[l, "(-)(Y)"'] \arrow[u, "f_*"'] \end{tikzcd}$$

and a commutative diagram of the left adjoint functors

(13.2)
$$\begin{tikzcd} A\text{–}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] \arrow[d, "B \otimes_A -"'] & \mathcal{O}_X\text{–}\mathsf{mod} \arrow[d, "f^*"] \\ B\text{–}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_Y\text{–}\mathsf{mod}. \end{tikzcd}$$

On the other hand, using the method in the proof of Proposition-Definition 4.2.3, one can prove the following result:

Proposition 13.1.4. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. For any $M \in A\text{–}\mathsf{mod}$, there is an essentially unique $\mathcal{O}_X$-module $\mathcal{M}$ equipped with an A-linear isomorphism $M \xrightarrow{\simeq} \mathcal{M}(X)$ such that for any $f \in A$, the A-linear map $M \simeq \mathcal{M}(X) \to \mathcal{M}(U(f))$ induces an isomorphism*

$$M_f \xrightarrow{\simeq} \mathcal{M}(U(f)).$$

Let $\mathcal{M}$ and $\widetilde{M}$ be the $\mathcal{O}_X$-module sheaves as above. By definition, $\widetilde{M}$ is the *sheafification* of $\pi^*_{\mathsf{PShv}}(M)$. In particular, for any $f \in A$, there is a canonical $\mathcal{O}_X(U(f))$-linear map

$$\mathcal{O}_X(U(f)) \underset{A}{\otimes} M \simeq \big(\pi^*_{\mathsf{PShv}}(M)\big)(U(f)) \to \widetilde{M}(U(f)).$$

Recall that we have $A_f \simeq \mathcal{O}_X(U(f))$ and

$$\mathcal{M}(U(f)) \simeq M_f \simeq A_f \underset{A}{\otimes} M.$$

This gives a canonical A_f-linear map

$$\mathcal{M}(U(f)) \to \widetilde{M}(U(f)).$$

By Exercise 1.2.7, there is a unique morphism $\mathcal{M} \to \widetilde{M}$ in $\mathcal{O}_X\text{–}\mathsf{mod}$ whose evaluation on each $U(f)$ is given by the above map.

Proposition 13.1.5. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. For $M \in A\text{–}\mathsf{mod}$, let $\widetilde{M}$ and $\mathcal{M}$ be the $\mathcal{O}_X$-modules defined as above. Then the canonical morphism $\mathcal{M} \to \widetilde{M}$ is an isomorphism. In particular, for any $f \in A$, we have*

$$M_f \to \widetilde{M}(U(f)).$$

Proof. Let $\mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$ be a testing object, we only need to show the map

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\widetilde{M},\mathcal{N}) \to \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\mathcal{M},\mathcal{N}) \tag{13.3}$$

is a bijection. By the definition of sheafification, we have

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\widetilde{M},\mathcal{N}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}}(\pi^*_{\mathsf{PShv}}(M),\mathcal{N}). \tag{13.4}$$

Note that the values of $\pi^*_{\mathsf{PShv}}(M)$ and $\mathcal{M}$ on $U(f)$ are both canonically identified with M_f. By Exercise 1.2.7, we obtain an identification

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\mathcal{M},\mathcal{N}) \simeq \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}}(\pi^*_{\mathsf{PShv}}(M),\mathcal{N}). \tag{13.5}$$

One can check (13.4) is equal to the composition (13.5) $\circ$ (13.3). Hence (13.3) is also a bijection.

□

Corollary 13.1.6. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The functor*

$$\widetilde{(-)} : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

is fully faithful.

Proof. This functor admits a right adjoint $\mathcal{F} \mapsto \mathcal{F}(X)$, and the unit natural transformation $M \to \widetilde{M}(X)$ is invertible (by Proposition 13.1.5). This formally implies the left adjoint is fully faithful.

□

Warning 13.1.7. The functor $\widetilde{(-)} : A\text{-}\mathsf{mod} \to \mathcal{O}_X\text{-}\mathsf{mod}$ is not essentially surjective. For example, the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ in Exercise 11.4.6 is not isomorphic to $\widetilde{M}$ for any $M \in k[t]\text{-}\mathsf{mod}$, because $\mathcal{I}(X) \simeq 0$.

Corollary 13.1.8. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The functor*

$$\widetilde{(-)} : A\text{-}\mathsf{mod} \to \mathcal{O}_X\text{-}\mathsf{mod}$$

commutes with finite limits and small colimits.

Proof. Follows from Proposition 13.1.5 and the fact that $M \mapsto M_f$ commutes with finite limits and small colimits.

□

Corollary 13.1.9. *Let $\phi : A \to B$ be a homomorphism in CRing and $f : Y \to X$ be the corresponding morphism between affine schemes. The commutative diagram 13.1 induces a natural isomorphism $\widetilde{\mathsf{res}(-)} \xrightarrow{\simeq} f_*(\widetilde{-})$, i.e., a commutative diagram*

$$\begin{tikzcd} A\text{-}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_X\text{-}\mathsf{mod} \\ B\text{-}\mathsf{mod} \arrow[u, "\mathsf{res}"] \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_Y\text{-}\mathsf{mod}. \arrow[u, "f_*"'] \end{tikzcd} \tag{13.6}$$

Proof. For $M \in A\text{-}\mathsf{mod}$ and $f \in A$, Proposition 13.1.5 implies

$$\widetilde{\mathsf{res}(M)}(U(f)) \simeq \mathsf{res}(M)_f \simeq M_{\phi(f)} \simeq \widetilde{M}(U(\phi(f))) \simeq f_*\widetilde{M}(U(f)).$$

One can check theis composition is the value of the Bech–Chevalley natural transformation $\widetilde{\mathsf{res}(-)} \to f_*(\widetilde{-})$ at $U(f)$. This implies the desired claim.

□

13.2. **Definition of quasi-coherent sheaves.**

Definition 13.2.1. Let X be a scheme. We say an $\mathcal{O}_X$-module is **quasi-coherent** if for any open immersion $j : U = \mathsf{Spec}(A) \to X$, the $\mathcal{O}_U$-module $\mathcal{F}|_U := j^*\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some $M \in A\text{-}\mathsf{mod}$.

We write

$$\mathsf{QCoh}(X) = \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{qcoh}} \subseteq \mathcal{O}_X\text{-}\mathsf{mod}$$

for the full subcategory of quasi-coherent $\mathcal{O}_X$-modules.

Example 13.2.2. The structure sheaf $\mathcal{O}_X$ is quasi-coherent when viewed as a module of itself.

13.3. **Affine case.**

Proposition 13.3.1. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The following functors are well-defined and inverse to each other*

$$\begin{array}{ccc} A\text{-}\mathsf{mod} & \rightleftarrows & \mathsf{QCoh}(X) \\ M & \mapsto & \widetilde{M} \\ \mathcal{F}(X) & \leftarrow\!\shortmid & \mathcal{F}. \end{array}$$

Proof. For well-definedness, we only need to show $\widetilde{M}$ is quasi-coherent for $M \in A\text{-}\mathsf{mod}$. Let $U = \mathsf{Spec}(B) \to X$ be an open immersion given by a homomorphism $A \to B$. By Constuction 13.1.3, we have $\widetilde{M}|_U \simeq \widetilde{B \otimes_A M}$. By definition, this implies $\widetilde{M}$ is quasi-coherent.

By Corollary 13.1.2, we only need to show the functor $A\text{-}\mathsf{mod} \to \mathsf{QCoh}(X)$, $M \mapsto \widetilde{M}$ is an equivalence. By Corollary 13.1.6, we only need to show it is essentially surjective. But this follows from the definition.

□

Combining Proposition 13.3.1 with Corollary 13.1.8, we obtain:

Corollary 13.3.2. *Let X be an affine scheme. Finite limits and small colimits of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Combining Proposition 13.3.1 with (13.2), we obtain:

Corollary 13.3.3. *Let $f : X \to Y$ be a morphism between affine schemes. The functor f^* sends quasi-coherent $\mathcal{O}_Y$-modules to quasi-coherent $\mathcal{O}_X$-modules.*

Combining Proposition 13.3.1 with Corollary 13.1.9, we obtain:

Corollary 13.3.4. *Let $f : X \to Y$ be a morphism between affine schemes. The functor f_* sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules.*

13.4. **General case.**

Theorem 13.4.1. *Being quasi-coherent is a local condition. In other words, for a scheme X and an open covering $X = \bigcup_{i\in I} X_i$, an object $\mathcal{F} \in \mathcal{O}_X\text{-}\mathsf{mod}$ is quasi-coherent iff $\mathcal{F}|_{X_i}$ is quasi-coherent for each $i \in I$.*

Proof. The "only if" claim is obvious,

For the "if" claim, let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that each $\mathcal{F}|_{X_i}$ is quasi-coherent.

We only need to show $\mathcal{F}|_U$ is quasi-coherent for any affine open subscheme $U \subseteq X$. Let $U = \bigcup_{j\in J} U_j$ be a covering of U by affine open subschemes such that for any $j \in J$, $U_j \subseteq X_i$ for some $i \in I$. The assumption implies each $\mathcal{F}|_{U_j} \simeq (\mathcal{F}|_{X_i})|_{U_j}$ is quasi-coherent. Hence we can replace $(X, \{X_i\}, \mathcal{F})$ with $(U, \{U_j\}, \mathcal{F}|_U)$ and reduce to the case when X and X_i are affine.

In the case when X is affine, since its underlying topological space is quasi-compact, we can assume I is a finite set. Write $X_{ij} := X_i \cap X_j$, and let $f_i : X_i \to X$, $f_{ij} : X_{ij} \to X$ be the open immersions. Since $\mathcal{F}$ is a sheaf, by Exercise 1.3.4, we have an isomorphism

$$\mathcal{F} \simeq \mathsf{ker}(\prod_{i\in I} f_{i,*}(\mathcal{F}|_{X_i}) \to \prod_{(i,j)\in I^2} f_{ij,*}(\mathcal{F}|_{X_ij})).$$

By assumption, each $\mathcal{F}|_{X_i}$ is quasi-coherent and therefore so is $\mathcal{F}|_{X_ij}$. By Corollary 13.3.4, $f_{i,*}(\mathcal{F}|_{X_i})$ and $f_{ij,*}(\mathcal{F}|_{X_ij})$ are quasi-coherent. By Corollary 13.3.2, $\mathcal{F}$ is quasi-coherent as desired.

□

Corollary 13.4.2. *Let $f : X \to Y$ be a morphism between schemes. The functor f^* sends quasi-coherent $\mathcal{O}_Y$-modules to quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Using Theorem 13.4.1, we can assume Y is affine by replacing $X \to Y$ with its base-change $X \times_Y U \to U$, where U ranges over all affine open subschemes of Y. Using Theorem 13.4.1 again, we can assume X is affine by replacing X with V, where V ranges over all affine open subschemes of X. Now the desired claim follows from Corollary 13.3.3.

□

Corollary 13.4.3. *Let X be a scheme. Finite limits and small colimits of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. Follows from Corollary 13.3.2 and the fact that j^{-1} commutes with finite limits and small colimits for any open immersion j.

□

Lemma 13.4.4. *Let $f : X \to Y$ be a closed immersion. The functor f_* sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. The problem is local on Y, so we can assume Y is affine and therefore quasi-compact. This implies X is quasi-compact because f is a closed immersion. Let $X = \sqcup_{i \in I} X_i$ be a finite covering of X by affine open subschemes. We have

$$X_{ij} \coloneqq X_i \cap X_j \simeq X_i \underset{X}{\times} X_j \xrightarrow{\simeq} X_i \underset{Y}{\times} X_j,$$

where the last isomorphism is because $f : X \to Y$ is a monomorphism (Proposition 11.3.7). In particular X_{ij} is affine. Let $g_i : X_i \to X$ and $g_{ij} : X_{ij} \to X$ be the open immersions.

Let $\mathcal{F} \in \mathsf{QCoh}(X)$. We only need to show $f_*\mathcal{F}$ is quasi-coherent. As in the proof of Theorem 13.4.1, we have

$$\mathcal{F} \simeq \mathsf{ker}\Big(\prod_{i \in I} g_{i,*}(\mathcal{F}|_{X_i}) \to \prod_{(i,j) \in I^2} g_{ij,*}(\mathcal{F}|_{X_{ij}})\Big).$$

Since f_* is a right adjoint, it commutes with limits. This implies

$$f_*\mathcal{F} \simeq \mathsf{ker}\Big(\prod_{i \in I} (f \circ g_i)_*(\mathcal{F}|_{X_i}) \to \prod_{(i,j) \in I^2} (f \circ g_{ij})_*(\mathcal{F}|_{X_{ij}})\Big).$$

Now the desired claim follows from Corollary 13.4.3 and Corollary 13.3.4.

□

Remark 13.4.5. In the future lectures, we will show f_* preserves quasi-coherent modules if f is (relatively) *quasi-compact and quasi-separated.*

Warning 13.4.6. For general morphism $f : X \to Y$, f_* does not preserve quasi-coherent modules.

Exercise 13.4.7. Let k be a field and $X \coloneqq \mathbb{A}^1_k$.

(1) Show that infinite products of quasi-coherent $\mathcal{O}_X$-modules may fail to be quasi-coherent.
(2) Let I be an infinite set and consider the obvious morphism

$$(\mathsf{id})_{i \in I} : \bigsqcup_{i \in I} \mathbb{A}^1_k \to \mathbb{A}^1_k.$$

Show that pushforward along this morphism does not preserve quasi-coherent modules.

Exercise 13.4.8. Let X be a scheme and $\mathcal{M}, \mathcal{N} \in \mathsf{QCoh}(X)$. Show that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathsf{QCoh}(X)$.

14. Quasi-coherent algebras

14.1. Definition.

Definition 14.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-algebra (sheaf) is an object $\mathcal{A} \in \mathsf{Shv}(X, \mathsf{CRing})$ equipped with a morphism $\mathcal{O}_X \to \mathcal{A}$. Let

$$\mathcal{O}_X\text{-}\mathsf{alg} := \mathsf{Shv}(X, \mathsf{CRing})_{\mathcal{O}_X/}$$

be the category of $\mathcal{O}_X$-algebras.

Construction 14.1.2. Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf. Consider the $\mathcal{O}_X$-module

$$\mathcal{O}_X/\mathcal{I} := \mathsf{coker}(\mathcal{I} \to \mathcal{O}_X) \in \mathcal{O}_X\text{-}\mathsf{mod}.$$

Unwinding the definitions, it is isomorphic to the sheafification of the $\mathcal{O}_X$-module presheaf $(\mathcal{O}_X/\mathcal{I})_{\mathsf{PShv}}$, which sends U to $\mathcal{O}_X(U)/\mathcal{I}(U)$. Note that $\mathcal{O}_X(U)/\mathcal{I}(U)$ has an obvious $\mathcal{O}_X(U)$-algebra structure. It follows that $\mathcal{O}_X/\mathcal{I}$ inherits an $\mathcal{O}_X$-algebra (sheaf) structure.

Construction 14.1.3. There is a forgetful functor

$$\mathcal{O}_X\text{-}\mathsf{alg} \to \mathcal{O}_X\text{-}\mathsf{mod}$$

sending an $\mathcal{O}_X$-algebra $\mathcal{A}$ to the underlying abelian sheaf $\mathcal{A}$ equipped with the $\mathcal{O}_X$-module structure given by

$$\mathcal{O}_X(U) \times \mathcal{A}(U) \to \mathcal{A}(U) \times \mathcal{A}(U) \to \mathcal{A}(U).$$

Definition 14.1.4. Let X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is **quasi-coherent** if it is quasi-coherent when viewed as an $\mathcal{O}_X$-module. Let

$$\mathcal{O}_X\text{-}\mathsf{alg}_{\mathsf{qcoh}} \subseteq \mathcal{O}_X\text{-}\mathsf{alg}$$

be the full subcategory of quasi-coherent $\mathcal{O}_X$-algebras.

Definition 14.1.5. Let X be a scheme. A **quasi-coherent ideal of** $\mathcal{O}_X$ is a sheaf of ideals $\mathcal{I}$ that is quasi-coherent when viewed as an $\mathcal{O}_X$-module.

Example 14.1.6. Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal. Then $\mathcal{O}_X/\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-algebra.

Example 14.1.7. Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. For any $B \in A\text{-}\mathsf{alg}$, we can view it as an A-module and consider the quasi-coherent $\mathcal{O}_X$-module $\widetilde{B}$. Recall $\widetilde{B}$ is the sheafification of the $\mathcal{O}_X$-module presheaf

$$U \mapsto \mathcal{O}_X(U) \underset{A}{\otimes} B.$$

Note that $\mathcal{O}_X(U) \otimes_A B$ has an obvious $\mathcal{O}_X(U)$-algebra structure. It follows that $\widetilde{B}$ inherits an $\mathcal{O}_X$-algebra (sheaf) structure.

Exercise 14.1.8. Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. Show that the following functors are inverse to each other:

$$\begin{array}{ccc} A\text{-}\mathsf{alg} & \rightleftarrows & \mathcal{O}_X\text{-}\mathsf{alg}_{\mathsf{qcoh}} \\ B & \mapsto & \widetilde{B} \\ \mathcal{B}(X) & \leftarrow\!\shortmid & \mathcal{B}. \end{array}$$

14.2. The relative Spec construction.

Construction 14.2.1. Let S be a scheme and $p : X \to S$ be an S-scheme. The object $p_*\mathcal{O}_X \in \mathsf{Shv}(S, \mathsf{CRing})$ has an $\mathcal{O}_S$-algebra structure given by the canonical morphism $\mathcal{O}_X \to p_*\mathcal{O}_X$.

Example 14.2.2. Let $p : X \to S$ be a closed immersion of schemes and $\mathcal{I}_X \subseteq \mathcal{O}_S$ be its ideal of definition. Recall we have an equivalence between abelian sheaves:

$$\mathcal{O}_S/\mathcal{I}_X \simeq p_*\mathcal{O}_X.$$

One can check this is also an equivalence between $\mathcal{O}_X$-algebras. Note that they are quasi-coherent by Lemma 13.4.4. In particular,

$$\mathcal{I}_X \simeq \mathsf{ker}(\mathcal{O}_S \to p_*\mathcal{O}_X)$$

is a quasi-coherent ideal of $\mathcal{O}_S$.

Construction 14.2.3. Let S be a scheme, and $p : X \to S$, $q : Y \to S$ be S-schemes. Let $f : X \to Y$ be an S-morphism, i.e., a morphism such that $p = q \circ f$. We have a canonical $\mathcal{O}_S$-homomorphism

$$q_*\mathcal{O}_Y \to q_*(f_*\mathcal{O}_X) \simeq p_*\mathcal{O}_X.$$

One can check this defines a *contravariant* functor

$$\begin{array}{rcl} (\mathsf{Sch}_S)^{\mathsf{op}} & \to & \mathcal{O}_S\text{–}\mathsf{alg} \\ (X \xrightarrow{p} S) & \mapsto & p_*\mathcal{O}_X. \end{array}$$

The following theorem says the partially defined left adjoint to this functor is well defined on *quasi-coherent* $\mathcal{O}_S$-algebras.

Theorem-Definition 14.2.4. *Let S be a scheme. For any $\mathcal{A} \in \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}$, there is an essentially unique S-scheme $q : Y \to S$ equipped with an isomorphism $\mathcal{A} \xrightarrow{\simeq} q_*\mathcal{O}_Y$ such that for any testing S-scheme $p : X \to S$ with $p_*\mathcal{O}_X$ being quasi-coherent, the composition*

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, Y) \to \mathsf{Hom}_{\mathcal{O}_S\text{–}\mathsf{alg}}(q_*\mathcal{O}_Y, p_*\mathcal{O}_X) \xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{O}_S\text{–}\mathsf{alg}}(\mathcal{A}, p_*\mathcal{O}_X) \tag{14.1}$$

is a bijection. We denote this S-scheme by

$$\mathsf{Spec}_S(\mathcal{A}) \in \mathsf{Sch}_S.$$

Sketch. The (essential) uniqueness follows from Yoneda's lemma. We only need to prove the claim about existence.

We first treat the case when $S = \mathsf{Spec}(R)$ is affine.

Let $A := \mathcal{A}(S)$ be the R-algebra corresponding to $\mathcal{A}$. Write $Y := \mathsf{Spec}(A)$ and view it as an S-scheme via the morphism $q : \mathsf{Spec}(A) \to \mathsf{Spec}(R)$ that corresponds to $A \to R$. By Exercise 14.1.8, we have $\mathcal{A} \simeq q_*\mathcal{O}_Y$ because $\mathcal{A}(S) \simeq A \simeq (q_*\mathcal{O}_Y)(S)$.

Let $p : X \to S$ be a testing S-scheme such that $p_*\mathcal{O}_X$ is quasi-coherent, we need to verify (14.1) is a bijection. Let $\phi : R \simeq \mathcal{O}_S(S) \to \mathcal{O}_X(X)$ be the homomorphism induced by p. Unwinding the definitions, we have

$$\begin{aligned}
& \mathsf{Hom}_{\mathsf{Sch}_S}(X,Y) \\
\simeq\ & \mathsf{Hom}_{\mathsf{Sch}}(X,Y) \underset{\mathsf{Hom}_{\mathsf{Sch}}(X,S)}{\times} \{p\} \\
\simeq\ & \mathsf{Hom}_{\mathsf{CRing}}(A,\mathcal{O}_X(X)) \underset{\mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X))}{\times} \{\phi\} \\
\simeq\ & \mathsf{Hom}_{R\text{-}\mathsf{alg}}(A,\mathcal{O}_X(X)) \\
\simeq\ & \mathsf{Hom}_{\mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}}}(\mathcal{A},p_*\mathcal{O}_X)
\end{aligned}$$

where the second bijection is due to Theorem 7.1.2, and the fourth bijection is due to Exercise 14.1.8. One can check the above composition is equal to (14.1), hence the latter is also a bijection as desired.

Now we treat the case when S is a general scheme. Let $S = \bigcup_{i\in I} S_i$ be an covering of S_i by affine open subschemes and write $S_{ij} := S_i \cap S_j$. Note that $\mathcal{A}_i := \mathcal{A}|_{S_i}$ is a quasi-coherent $\mathcal{O}_{S_i}$-algebra. By the previous discussion, there exists an essentailly unique S_i-scheme $q_i : Y_i \to S_i$ equipped with an isomorphism $\mathcal{A}_i \simeq q_{i,*}\mathcal{O}_{Y_i}$ such that

$$\mathsf{Hom}_{\mathsf{Sch}_{S_i}}(X,Y_i) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p_*\mathcal{O}_X)$$

for any S_i-scheme $p : X \to S_i$ such that $p_*\mathcal{O}_{X_i}$ is quasi-coherent. For $(i,j) \in I^2$, consider the S_{ij}-scheme

$$q_{ij} : Y_{ij} := Y_i \underset{S_i}{\times} S_{ij} \to S_{ij}.$$

It is easy to check

$$q_{ij,*}\mathcal{O}_{Y_{ij}} \simeq (q_{i,*}\mathcal{O}_{Y_i})|_{S_{ij}} \simeq \mathcal{A}|_{S_{ij}}.$$

In particular, $q_{ij,*}\mathcal{O}_{Y_{ij}}$ is quasi-coherent.

For any testing S_{ij}-scheme $p : X \to S_{ij}$, we have

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathsf{Sch}_{S_i}}(X,Y_i) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p'_*\mathcal{O}_X),$$

where p' is the composition $X \xrightarrow{p} S_{ij} \xrightarrow{u_{ij}} S_i$. We have

$$\mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p'_*\mathcal{O}_X) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,s_{ij,*}\circ p_*\mathcal{O}_X) \simeq \mathsf{Hom}_{\mathcal{O}_{S_{ij}}\text{-}\mathsf{alg}}(\mathcal{A}|_{S_{ij}},p_*\mathcal{O}_X).$$

This implies

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathcal{O}_{S_{ij}}\text{-}\mathsf{alg}}(\mathcal{A}|_{S_{ij}},p_*\mathcal{O}_X).$$

Note that the RHS is invariant after switching i with j. Hence we obtain a canonical bijection

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathsf{Sch}_{S_{ji}}}(X,Y_{ji}).$$

One can check this bijection is functorial in X. Hence by Yoneda's lemma, we obtain an isomorphism

$$\phi_{ij} : Y_{ij} \to Y_{ji}$$

defined over $S_{ij} = S_{ji}$.

One can check

$$(I,(Y_i)_{i\in I},(Y_{ij})_{(i,j)\in I^2},(\phi_{ij})_{(i,j)\in I^2})$$

is a gluing data of schemes (see Definition 6.1.1). Let

$$(Y,(Y'_i)_{i\in I},(\varphi_i)_{i\in I})$$

be its gluing output (see Proposition-Definition 6.1.2). By Proposition 6.1.3, there is a canonical morphism

$$q : Y \to S$$

glued from the morphisms $Y_i \xrightarrow{q_i} S_i \to S$. By construction, we have an isomorphism

$$\alpha_i : \mathcal{A}|_{S_i} \simeq q_{i,*}\mathcal{O}_{Y_i} \simeq q_*\mathcal{O}_Y|_{S_i}.$$

Moreover, the restriction of α_i to S_{ij} is equal to the restriction of α_j to $S_{ji} = S_{ij}$. Hence by Exercise 1.2.7, there is a unique isomorphism

$$\alpha : \mathcal{A} \simeq q_*\mathcal{O}_Y$$

such that its restriction to each S_i is equal to α_i.

For $q : Y \to S$ and $\mathcal{A} \simeq q_*\mathcal{O}_Y$ constructed as above, using Proposition-Construction 6.1.4, one can check (14.1) is bijective whenever $p_*\mathcal{O}_X$ is quasi-coherent.

□

Corollary 14.2.5. *Let S be a scheme and $\mathsf{Sch}'_S \subseteq \mathsf{Sch}_S$ be the full subcategory consisting of S-schemes $p : X \to S$ such that $p_*\mathcal{O}_X$ is quasi-coherent. Then the functor*

$$(\mathsf{Sch}'_S)^{\mathsf{op}} \to \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}, \ \ (p : X \to S) \mapsto p_*\mathcal{O}_X$$

admits a fully faithful left adjoint

$$\mathsf{Spec}_S : \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}} \to (\mathsf{Sch}'_S)^{\mathsf{op}}.$$

Exercise 14.2.6. Let $f : S' \to S$ be a morphism between schemes. For $\mathcal{A} \in \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}$, consider

$$\mathcal{A}' := f^*\mathcal{A} \in \mathcal{O}_{S'}\text{–}\mathsf{alg}_{\mathsf{qcoh}}.$$

Show that

$$\mathsf{Spec}_S(\mathcal{A}) \underset{S}{\times} S' \simeq \mathsf{Spec}_{S'}(\mathcal{A}').$$

Exercise 14.2.7. Let S be a scheme and $\mathcal{A} \in \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}$. Show that for *any* S-scheme $p : X \to S$, the canonical map

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, \mathsf{Spec}_S(\mathcal{A})) \to \mathsf{Hom}_{\mathcal{O}_S\text{–}\mathsf{alg}}(\mathcal{A}, p_*\mathcal{O}_X)$$

is a bijection. Hint: base-change along $X \to S$.

15. Application: classification of closed immersions

15.1. The classification.

Theorem 15.1.1. *Let X be a scheme. The following functors are well-defined and inverse to each other*

$$\begin{array}{ccc} \{\textit{closed immersions into } X\}^{\mathsf{op}} & \rightleftarrows & \{\textit{quasi-coherent ideals of } \mathcal{O}_X\} \\ (i : Y \to X) & \mapsto & \mathsf{ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y) \\ \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) & \leftarrow\!\shortmid & \mathcal{I} \subseteq \mathcal{O}_X. \end{array}$$

Proof. For the claim of well-definedness, we only need to show for any quasi-coherent ideal $\mathcal{I}$, the structural morphism $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \to X$ is a closed immersion. By Exercise 14.2.6, this claim is local on X, hence we can assume $X = \mathsf{Spec}(A)$ is affine. Let $I \subseteq A$ be the ideal such that $\mathcal{I} \simeq \widetilde{I}$. By construction, $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \simeq \mathsf{Spec}(A/I)$ and the structural morphism $\mathsf{Spec}(A/I) \to \mathsf{Spec}(A)$ corresponds to the canonical homomorphism $A \to A/I$, which is indeed a closed immersion.

It remains to check these two functors are inverse to each other.

Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal of $\mathcal{O}_X$ and consider $i : Y = \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \to X$. By definition, we have a canonical isomorphism $\mathcal{O}_X/\mathcal{I} \simeq i_*\mathcal{O}_Y$ of quasi-coherent $\mathcal{O}_X$-algebras. This implies

$$\mathsf{ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y) \simeq \mathsf{ker}(\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}) \simeq \mathcal{I}.$$

In other words, the composition of the leftwards functor followed by the rightwards functor is equivalent to the identity functor.

On ther other hand, let $i : Y \to X$ be a closed immersion. By definition of Spec_X, the identity homomorphism $i_*\mathcal{O}_Y \to i_*\mathcal{O}_Y$ corresponds to an X-morphism $\phi : Y \to \mathsf{Spec}_X(i_*\mathcal{O}_Y)$. It remains to show this is an isomorphism. By construction, ϕ induces an isomorphism between the corresponding ideals of definition. Hence by Lemma 11.4.5, ϕ is an isomorphism as desired.

□

Corollary 15.1.2. *Let $X = \mathsf{Spec}(A)$ be an affine scheme. Any closed immersion into X is isomorphic to $\mathsf{Spec}(A/I) \to \mathsf{Spec}(A)$ for a unique ideal $I \subseteq X$.*

Corollary 15.1.3. *Closed immersions are stable under base-change.*

Proof. Let $i : Y \to X$ be a closed immersion between S-schemes and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. We identify Y with $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I})$. For any $S' \to S$, consider $X' := X \times_S S'$ and $Y' := Y \times_S S'$. We have

$$Y' \simeq \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \underset{S}{\times} S' \simeq \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \underset{X}{\times} X' \simeq \mathsf{Spec}_{X'}(f^*(\mathcal{O}_X/\mathcal{I})),$$

where $f : X' \to X$ is the canonical projection. Note that we have $f^*(\mathcal{O}_X/\mathcal{I}) \simeq \mathcal{O}_{X'}/f^*\mathcal{I}$ and $f^*\mathcal{I}$ is a quasi-coherent ideal of $\mathcal{O}_{X'}$. Hence $Y' \to X'$ is a closed immersion as desired.

□

Combining with Exercise 10.1.4, we obtain

Corollary 15.1.4. *Locally closed immersions are stable under base-change.*

Part V. Properties of schemes and morphisms

16. Quasi-compact morphisms and quasi-separated morphisms

16.1. Quasi-compact morphisms.

Definition 16.1.1. We say a scheme X is **quasi-compact** if its underlying topological space is quasi-compact, i.e., any open covering of X admits a finite subcovering.

Definition 16.1.2. Let $f : X \to Y$ be a morphism between schemes. We say f is **quasi-compact** if for any quasi-compact open subset $U \subseteq Y$, its inverse image $f^{-1}(U)$ is quasi-compact.

Lemma 16.1.3. *Let $f : X \to Y$ be a morphism such that Y is affine. Then f is quasi-compact iff X is quasi-compact.*

Proof. The "only if" claim is obvious because Y is quasi-compact. For the "if" claim, suppose X is quasi-compact. For any quasi-compact open subset U, we only need to show $X \times_Y U$ is quasi-compact. Choose a finite open covering $X = \bigcup_{i\in I} X_i$ by affine open subscheme X_i of X, and similarly choose $U = \bigcup_{j\in J} U_j$. Then

$$X \underset{Y}{\times} U \simeq \bigcup_{(i,j)\in I\times J} X_i \underset{Y}{\times} U_j.$$

Note that each $X_i \times_Y U_j$ is affine and therefore quasi-compact. This implies $X \times_Y U$ is quasi-compact. . □

Lemma 16.1.4. *Consider the class of quasi-compact morphisms.*

(i) *Quasi-compact morphisms are stable under compositions.*
(ii) *Quasi-compact morphisms are stable under base-changes.*
(iii) *Being a quasi-compact morphism is local on the target.*

Proof. (i) is obvious.

To prove (iii), let $f : X \to Y$ be a morphism and $Y = \sqcup_{i\in I} Y_i$ be an open covering such that each $f_i : X \times_Y Y_i \to Y_i$ is quasi-compact. We only need to show f is quasi-compact. Let $U \subseteq Y$ be a quasi-compact open subset. We can choose a finite covering $U = \bigcup_{j\in J} U_j$ of U by its affine open subschemes such that for any $j \in J$, $U_j \subseteq Y_i$ for some $i \in I$. Since f_i is quasi-compact and U_j is a quasi-compact open subset of Y_i, its inverse image

$$f_i^{-1}(U_j) \simeq (X \underset{Y}{\times} Y_i) \underset{Y_i}{\times} U_j \simeq X \underset{Y}{\times} U_j$$

is quasi-compact. It follows that

$$f^{-1}(U) \simeq X \underset{Y}{\times} U \simeq \bigcup_{j\in J} X \underset{Y}{\times} U_j$$

is also quasi-compact as desired.

To prove (ii), let $f : X \to Y$ be a quasi-compact morphism and $f' : X' \to Y'$ be its base-change. We only need to show f' is quasi-compact. By Lemma 16.1.3, for any affine open subscheme $U \subseteq Y$, the base-change $X \times_Y U \to U$ is quasi-compact. Hence by (iii), we can reduce to the case when Y and Y' are affine. Using Lemma 16.1.3 again, we see that X is quasi-compact. It follows that $X' \simeq X \times_Y Y'$ has a *finite* covering by its affine open subsets. This implies X' is quasi-compact. By Lemma 16.1.3 again, we see that f' is quasi-compact as desired.

□

Exercise 16.1.5. Show that a closed immersion is quasi-compact.

16.2. **Separated morphisms.**

Definition 16.2.1. Let $f : X \to Y$ be a morphism between schemes. The **(relative) diagonal morphism**

$$\Delta_f : X \to X \underset{Y}{\times} X$$

is defined to be the morphism corresponding to the commutative square

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "="'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let $p : X \to S$ be an S-scheme. We also write

$$\Delta_X := \Delta_p : X \to X \underset{S}{\times} X$$

and call it the **diagonal morphism of the S-scheme X.**

Lemma 16.2.2. *Let $f : X \to Y$ be a morphism between schemes. The moprhism $\Delta_f : X \to X \times_Y X$ is a locally closed immersion.*

Proof. Consider the projection map $p_1 : X \times_Y X \to X$. Note that $p_1 \circ \Delta_f = \mathsf{id}_X$. This implies Δ_f induces a homeomorphism from X to its image. Moreover, for any point $x \in X$, the identity homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ factors as

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X\times_Y X, \Delta_f(x)} \to \mathcal{O}_{X,x},$$

where the first homomorphism is induced by p_1 while the second one is induced by Δ_f. In particular, the second homomorphism is a surjection. By Proposition 11.5.2, Δ_f is a locally closed immersion.

□

Exercise 16.2.3. Show that $f : X \to Y$ is a monomorphism iff Δ_f is an isomorphism.

Definition 16.2.4. Let $f : X \to Y$ be a morphism between schemes. We say f is **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a *closed* immersion. We say a scheme X is **separted** if $X \to \mathsf{Spec}(\mathbb{Z})$ is separated.

By Lemma 16.2.2, we have:

Corollary 16.2.5. *A morphism $f : X \to Y$ is separated iff $\Delta_f(X)$ is a closed subset of $X \times_Y X$.*

By Exercise 16.2.3, we have:

Corollary 16.2.6. *Any monomorphism between schemes is separated. In particular, a locally closed immersion is separated.*

Lemma 16.2.7. *Consider the class of separated morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are separated. Then $g \circ f$ is separated.*

(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is separted. Then f is separated.*

(iii) *Separated morphisms are stable under base-changes.*
(iv) *Being separated is local on the targets.*

Proof. For (i) and (ii), we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[dr, "\Delta_{g\circ f}"'] & X \times_Y X \arrow[r] \arrow[d, "\Delta'_g"] & Y \arrow[d, "\Delta_g"] \\ & X \times_Z X \arrow[r] & Y \times_Z Y. \end{tikzcd}$$

Moreover, one can check the right square is Cartesian.

If f and g are separated, then Δ_f and Δ_g are closed immersions. By Corollary 15.1.3, Δ'_g is also a closed immersion. Hence $\Delta_{g\circ f} \simeq \Delta'_g \circ \Delta_f$ is also a closed immersion. This proves (i).

If $g \circ f$ is separated, then $\Delta_{g\circ f} \simeq \Delta'_g \circ \Delta_f$ is a closed immersion. In particular $\Delta'_g(\Delta_f(X))$ is a closed subset of $X \times_Z X$.

Note that Δ'_g is a locally closed immersion by Corollary 15.1.4 and Lemma 16.2.2. Hence Δ'_g induces a homeomorphism from $X \times_Y X$ to its image. Combining with the last paragraph, we obtain that $\Delta_f(X)$ is a closed subset of $X \times_Y X$. By Corollary 16.2.5, this implies (ii).

For (iii) and (iv), note that for a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Delta_{f'}"] \arrow[d] & X' \times_{Y'} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \arrow[r] & Y \end{tikzcd}$$

such that the right square and the outer square are both Cartesian. Hence the left square is also Cartesian. Now (iii) follows from Corollary 15.1.3, while (iv) follows from the fact that being a closed immersion is local on the target.

□

Exercise 16.2.8. A morphism out of an affine scheme is separated.

Exercise 16.2.9. Let X be a separated scheme. Show that the intersection of two affine open subsets of X is affine.

Definition 16.2.10. Let $f : X \to Y$ be a morphism between S-schemes. The *graph morphism (relative to S)*

$$\Gamma_f : X \to X \underset{S}{\times} Y$$

is defined to be the morphism corresponding to the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="] & Y \arrow[d] \\ X \arrow[r] & S. \end{tikzcd}$$

Exercise 16.2.11. Let $f : X \to Y$ be a morphism between S-schemes.

(1) Show that the graph morphism $\Gamma_f : X \to X \times_S Y$ is a locally closed immersion.
(2) If $Y \to S$ is separated, show that Γ_f is a closed immersion.

16.3. **Quasi-separated morphisms.**

Definition 16.3.1. Let $f : X \to Y$ be a morphism between schemes. We say f is **quasi-separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. We say a scheme X is **quasi-separated** if $X \to \mathsf{Spec}(\mathbb{Z})$ is quasi-separated.

By Exercise 16.1.5, we have

Corollary 16.3.2. *A separated morphism is quasi-separated.*

Lemma 16.3.3. *Let X be a scheme. The following conditions are equivalent:*

(i) *The scheme X is quasi-separated.*
(ii) *The intersection of any pair of quasi-compact open subsets of X is quasi-compact.*
(iii) *The intersection of any pair of affine open subsets of X is quasi-compact.*

Proof. For (i)⇒(ii), suppose X is quasi-separated. Let U_1 and U_2 be quasi-compact open subsets of X. By Lemma 16.1.4, $U_1 \times U_2$ is also quasi-compact. Note that by assumption $\Delta : X \to X \times X$ is quasi-compact, hence $U_1 \cap U_2 \simeq \Delta^{-1}(U_1 \times U_2)$ is quasi-compact as desired.

(ii)⇒(iii) is obvious.

For (iii)⇒(i), suppose the intersection of any pair of quasi-compact open subsets of X is quasi-compact. Let $X = \bigcup_{i \in I} U_i$ be a covering of X by affine open subschemes. Then $X \times X$ can be covered by its affine open subschemes $U_i \times U_j$, $(i,j) \in I^2$. By Lemma 16.1.4, we only need to show

$$U_i \cap U_j \simeq X \underset{X \times X}{\times} (U_i \times U_j) \to U_i \times U_j$$

is quasi-compact. But this follows from the assumption that $U_i \cap U_j$ is quasi-compact (Lemma 16.1.3).

□

The proof of the following results is similar to that of Lemma 16.2.7. We leave it to the readers.

Lemma 16.3.4. *Consider the class of quasi-separated morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are quasi-separated. Then $g \circ f$ is quasi-separated.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is quasi-separted. Then f is quasi-separated.*
(iii) *Quasi-separated morphisms are stable under base-changes.*

(iv) *Being quasi-separated is local on the targets.*

Exercise 16.3.5. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is quasi-separated and $g \circ f$ is quasi-compact. Show that f is quasi-compact.

16.4. Direct image along qcqs morphisms.

Theorem 16.4.1. *Let $f : X \to Y$ be a quasi-compact quasi-separated morphism. Then the functor $f_* : \mathcal{O}_X\text{-}\mathsf{mod} \to \mathcal{O}_Y\text{-}\mathsf{mod}$ sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules. Moreover, the obtained functor*

$$f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y).$$

commutes with (small) coproducts.

Proof. We first show the claims are local on Y. This follows from the following observations:

- The class of quasi-compact quasi-separated morphisms are closed under base-changes.
- For any $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}$ and open subscheme $U \in Y$, $f_*(\mathcal{M})|_U \simeq f'_*(\mathcal{M}|_V)$, where $f' : V \to U$ is the base-change of $f : X \to Y$.
- For an open covering $Z = \bigcup_{i\in I} Z_i$ of schemes, the functors $(-)|_{Z_i} : \mathcal{O}_Z\text{-}\mathsf{mod} \to \mathcal{O}_{Z_i}\text{-}\mathsf{mod}$ preserve and detect quasi-coherent modules.
- For an open covering $Z = \bigcup_{i\in I} Z_i$ of schemes, the functors $(-)|_{Z_i} : \mathsf{QCoh}(Z) \to \mathsf{QCoh}(Z_i)$ preserve and detect small coproducts (in fact, any small colimits).

As a consequence, we can assume $Y = \mathsf{Spec}(A)$ is affine, and therefore X is quasi-compact and quasi-separated.

We now reduce to the case when X is quasi-compact and *separated*. For this purpose, let $X = \bigcup_{i\in I} U_i$ be a finite covering of X by its affine open subschemes. Write $U_{ij} \coloneqq U_i \cap U_j$, which is quasi-compact by Lemma 16.3.3. Note that U_{ij} is also separated because U_i is separated (Exercise 16.2.8) and $U_{ij} \to U_i$ is separated (Corollary 16.2.6). Let $\mathcal{M} \in \mathsf{QCoh}(X)$ be a quasi-coherent $\mathcal{O}_X$-module. Recall we have

$$\mathcal{M} \simeq \mathsf{ker}(\prod_{i\in I} g_{i,*}(\mathcal{M}|_{U_i}) \to \prod_{(i,j)\in I^2} g_{ij,*}(\mathcal{M}|_{U_{ij}})),$$

where $g_i : U_i \to X$ and $g_{ij} : U_{ij} \to X$ are the open immersions. It follows that

$$f_*\mathcal{M} \simeq \mathsf{ker}(\prod_{i\in I}(f \circ g_i)_*(\mathcal{M}|_{U_i}) \to \prod_{(i,j)\in I^2} (f \circ g_{ij})_*(\mathcal{M}|_{U_{ij}})).$$

By Corollary 13.4.3, to show $f_*\mathcal{M}$ is quasi-coherent, we only need to show each $(f\circ g_i)_*(\mathcal{M}|_{U_i})$ and $(f\circ g_{ij})_*(\mathcal{M}|_{U_{ij}}))$ is quasi-coherent. Moreover, Corollary 13.4.3 also implies coproducts commute with finite limits in $\mathsf{QCoh}(Z)$. Hence to show the obtained functor $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ commutes with coproducts, we only need to show each $(f \circ g_i)_*$ and $(f \circ g_{ij})_*$ does. As a consequence, we can replace X with U_i or U_{ij}, and therefore assume X is quasi-compact and separated.

We now repeat the above paragraph. Note that under the new assumption that X is separated, $U_{ij} \simeq U_i \times_X U_j \to U_i \times U_j$ is a closed immersion because it is a base-change of $\Delta : X \to X \times X$. Since $U_i \times U_j$ is affine, so is U_{ij} (Corollary 15.1.2). As a consequence, after replacing X with U_i or U_{ij}, we can assume X is *affine*.

Now the desired claim follows from Corollary 13.3.4 and the fact that for any homomorphism $\phi : A \to B$, the restriction functor $B\text{–}\mathsf{mod} \to A\text{–}\mathsf{mod}$ commutes with small coproducts.

$\square$

Exercise 16.4.2. Consider $\mathbb{A}^\infty_{\mathbb{Z}} \coloneqq \mathsf{Spec}(\mathbb{Z}[t_1, t_2, \cdots])$ and its closed subscheme Z corresponding to the ideal $(t_1, t_2, \cdots)$. Let $U \coloneqq \mathbb{A}^\infty_{\mathbb{A}} \setminus Z$ be the complementary open subscheme. Define X to be the scheme glued from two pieces of $\mathbb{A}^\infty_{\mathbb{Z}}$ via the identity morphism on U.

(1) Show that X is quasi-compact, but is not quasi-separated.
(2) Consider the unique morphism $p : X \to \mathsf{Spec}(\mathbb{Z})$. Show that p_* does not preserve quasi-coherent modules.

17. Reducedness

17.1. **Reduced schemes.**

Definition 17.1.1. Let X be a scheme. We say X is reduced if for any open subset $U \subseteq X$, the commutative ring $\mathcal{O}_X(U)$ is reduced.

The following result follows from the sheaf condition for $\mathcal{O}_X$.

Lemma 17.1.2. *A scheme X is reduced iff for some topological base $\mathfrak{B}$ of X, the commutative ring $\mathcal{O}_X(U)$ is reduced for any $U \in \mathfrak{B}$.*

Corollary 17.1.3. *An affine scheme $\mathsf{Spec}(A)$ is reduced iff A is reduced.*

Lemma 17.1.4. *A scheme X is reduced iff the local ring $\mathcal{O}_{X,x}$ is reduced for any $x \in X$.*

Proof. The claim is local on X and the affine case is well-known. □

17.2. **Reduced subschemes.**

Let X be a scheme and $Z \subseteq X$ be a locally closed subset. Consider the category $\mathsf{SubSch}_{X,Z}$ of locally closed immersions $i : Y \to X$ such that $i(Y) = Z$.

Proposition-Definition 17.2.1. *There is an essentially unique object $Z_{\mathsf{red}} \to X$ in $\mathsf{SubSch}_{X,Z}$ such that Z_{red} is reduced. Moreover, it is an initial object in this category. We call Z_{red} the* ***reduced subscheme of** X **on** Z.*

Proof. Let $U \subseteq X$ be an open subscheme such that Z is a closed subset of U. Note that the obvious functor $\mathsf{SubSch}_{U,Z} \to \mathsf{SubSch}_{X,Z}$ is an equivalence. Hence we can replace X with U and therefore assume Z is closed in X.

For the case $X = \mathsf{Spec}(A)$ is affine, choose an ideal $J \subseteq A$ such that $Z = Z(J)$. By Theorem 15.1.1, the category $\mathsf{SubSch}_{X,Z}$ is equivalent to the opposite category of ideals $I \subseteq A$ such that $Z(I) = Z(J)$. The last condition is equivalent to $\sqrt{I} = \sqrt{J}$. Hence $\mathsf{Spec}(A/\sqrt{J}) \to \mathsf{Spec}(A)$ is the unique reduced object in this category, moreover it is an initial object because any I as above satisfies $I \subseteq \sqrt{J}$.

We now prove the general case. For any affine open subscheme $U \subseteq X$, consider the reduced closed subscheme $(U \cap Z)_{\mathsf{red}} \to U$. If $U \subseteq V$ are two affine open subschemes, one can check

$$(V \cap Z)_{\mathsf{red}} \underset{V}{\times} U \to U$$

is a reduced closed subscheme whose image is the topological space $U \cap Z$. Hence we have an isomorphism between closed subschemes of U:

$$(V \cap Z)_{\mathsf{red}} \underset{V}{\times} U \simeq (U \cap Z)_{\mathsf{red}}.$$

This implies there is a unique closed subscheme $Z_{\mathsf{red}} \to X$ such that

$$Z_{\mathsf{red}} \underset{X}{\times} U \simeq (U \cap Z)_{\mathsf{red}}$$

as closed subschemes of U. By Lemma 17.1.2, Z_{red} is indeed reduced.

It remains to show the object $Z_{\mathsf{red}} \to X$ constructed as above satisfies the desired properties. Let $Y \to X$ be any closed immersion with image Z. We only need to show that there is a unique X-morphism $Z_{\mathsf{red}} \to Y$, and this morphism is an isomorphism if Y is reduced. One can check that this claim is local on X, hence it follows from the affine case, which has been proved.

□

In particular, there is a unique reduced subscheme $X_{\mathsf{red}} \to X$ whose underlying topological space is X.

Lemma 17.2.2. *Let X be a scheme and $x \in X$ be a point. We have $\mathcal{O}_{X_{\mathsf{red}},x} \simeq (\mathcal{O}_{X,x})_{\mathsf{red}}$.*

Proof. The claim in local on X hence we can assume X is affine. Now the claim is a well-known fact in commutative algebra. □

Exercise 17.2.3. Let X be a scheme and $U \subseteq X$ be an open subset. Show that $\mathcal{O}_{X_{\mathsf{red}}}(U) \simeq \mathcal{O}_X(U)_{\mathsf{red}}$.

Lemma 17.2.4. *Let X and Y be S-schemes. The canonical morphism*

$$X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y$$

is a closed immersion and induces a homeomorphism between the topological spaces. In particular

$$(X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}})_{\mathsf{red}} \simeq (X \underset{S}{\times} Y)_{\mathsf{red}}.$$

Proof. The morphism is a closed immersion because it can be written as the composition

$$X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y,$$

and each of the two morphisms is a base-change of a closed immersion. To show this morphism induces a homeomorphism, note that this claim is local in X, Y and S. Hence we can assume $X = \mathsf{Spec}(A)$, $Y = \mathsf{Spec}(B)$ and $S = \mathsf{Spec}(R)$. Then the desired morphism is given by $A \otimes_R B \to A_{\mathsf{red}} \otimes_R B_{\mathsf{red}}$, which is a surjection with a nilpotent kernel. It follows that $\mathsf{Spec}(A_{\mathsf{red}} \otimes_R B_{\mathsf{red}}) \to \mathsf{Spec}(A \otimes_R B)$ is a homeomorphism.

□

Warning 17.2.5. Products of reduced schemes may fail to be reduced. For example, $\mathbb{Z}[t]/(t^2 - p) \otimes \mathbb{F}_p \simeq \mathbb{F}_p[t]/(t^2)$ is not reduced.

17.3. **Morphisms between reduced subschemes.**

Exercise 17.3.1. Let $f : X \to Y$ be a morphism between schemes. If X is a reduced scheme, then f uniquely factors through Y_{red}. Hint: reduce to the case when Y is affine.

Corollary 17.3.2. *Let $f : X \to Y$ be a morphism between schemes. There is a unique moprhism $f_{\mathsf{red}} : X_{\mathsf{red}} \to Y_{\mathsf{red}}$ such that the following diagram commutes:*

$$\begin{tikzcd} X_{\mathsf{red}} \arrow[r, dotted, "f_{\mathsf{red}}"] \arrow[d] & Y_{\mathsf{red}} \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Exercise 17.3.3. If $f : X \to Y$ be an open (resp. closed, locally closed) immersion, so is $f_{\mathsf{red}} : X_{\mathsf{red}} \to Y_{\mathsf{red}}$.

Exercise 17.3.4. Show that f is quasi-compact (resp. separated, quasi-separated) iff f_{red} is so.

18. Quasi-affine morphisms

18.1. **Affine morphisms.**

Definition 18.1.1. We say a morphism $f : X \to Y$ between schemes is **affine** if for any affine open subscheme $U \subseteq Y$, its inverse image $X \times_Y U$ is affine.

Example 18.1.2. Let S be an affine scheme. A morphism $f : X \to S$ is affine iff X is affine.

Lemma 18.1.3. *An affine morphism $f : X \to Y$ is quasi-compact and separated.*

Proof. By Lemma 16.1.4(iii) and Lemma 16.2.7(iv), we can assume Y is affine. Since f is affine, we see that X is affine. Now f is quasi-compact by Lemma 16.1.3 and is separated by Exercise 16.2.8. □

Proposition 18.1.4. *Let S be a scheme. The following functors are well-defined and inverse to each other:*

$$\begin{array}{ccc} \mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}} & \rightleftarrows & \{\textit{affine morphisms } p : X \to S\}^{\mathsf{op}} \\ \mathcal{A} & \mapsto & (\mathsf{Spec}_S(\mathcal{A}) \to S) \\ p_*\mathcal{O}_X & \leftarrow & (p : X \to S) \end{array} \tag{18.1}$$

Proof. The leftward functor is well-defined by Lemma 18.1.3 and Theorem 16.4.1.

To show the rightward functor is well-defined, we only need to show $\mathsf{Spec}_S(\mathcal{A}) \to S$ is affine. Let $U \subseteq S$ be an affine open subscheme, we have

$$\mathsf{Spec}_S(\mathcal{A}) \underset{S}{\times} U \simeq \mathsf{Spec}_U(\mathcal{A}|_U) \simeq \mathsf{Spec}(\mathcal{A}(U)),$$

which is indeed an affine scheme.

By Corollary 14.2.5, (18.1) is an adjoint pair, and the left adjoint is fully faihtful. Hence we only need to show that for any affine morphism $p : X \to S$, the S-morphism

$$X \to \mathsf{Spec}_S(p_*\mathcal{O}_X)$$

that corresponds to the identity morphism $p_*\mathcal{O}_X$ is an isomorphism. This claim is local on S, hence we can assume S is affine, and thereby X is affine. Now the above morphism becomes the canonical morphism

$$X \to \mathsf{Spec}(\mathcal{O}_X(X)),$$

which is indeed an isomorphism. □

Corollary 18.1.5. *Any closed immersion is affine.*

Lemma 18.1.6. *Consider the class of affine morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are affine. Then $g \circ f$ is affine.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is separated and $g \circ f$ is affine. Then f is affine.*
(iii) *Affine morphisms are stable under base-changes.*
(iv) *Being affine is local on the targets.*

Proof. (i) is obvious.

(iii) follows from Proposition 18.1.4 and Exercise 14.2.6.

To prove (ii), we factor f as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{\mathsf{pr}_2} Y$, where the first morphism is the graph of f relative to Z and pr_2 is the obvious projection. By Exercise 16.2.11(2), Γ_f is a closed immersion and therefore affine (Corollary 18.1.5). On the other hand, pr_2 is the base-change of $g \circ f : X \to Z$ along $Y \to Z$. By (iii), pr_2 is affine. It follows that f is affine because it is the composition of two affine morphisms.

To prove (iv), let $f : X \to Y$ be a moprhism and $Y = \bigcup_{i \in I} Y_i$ be an open covering such that each $f_i : X \times_Y Y_i \to Y_i$ is affine. By Lemma 18.1.3, each f_i is quasi-compact and separated. By Lemma 16.1.4(iii) and Lemma 16.2.7(iv), f is quasi-compact and separated. In particular, $f_* \mathcal{O}_X$ is quasi-coherent. By Corollary 14.2.5 and Proposition 18.1.4, we only need to show the Y-morphism

$$X \to \mathsf{Spec}_Y(f_* \mathcal{O}_X) \tag{18.2}$$

that corresponds to the identity morphism on $f_* \mathcal{O}_X$ is an isomorphism. The assumption implies the base-change of (18.2) to each Y_i is an isomorphism. It follows that (18.2) is indeed an isomorphism.

□

18.2. Quasi-coherent modules and affine morphisms.

Construction 18.2.1. Let S be a scheme and $f : X \to S$ be an affine morphism. Write $\mathcal{A} \coloneqq f_* \mathcal{O}_X$, which is a quasi-coherent $\mathcal{O}_S$-algebra. Consider the morphism $(X, \mathcal{O}_X) \to (S, \mathcal{A})$ between ringed spaces. Pullback and pushforward along this morphism provide an adjoint pair

$$\begin{array}{ccc} \mathcal{A}\text{–}\mathsf{mod} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod} \\ \mathcal{M} & \mapsto & \widetilde{\mathcal{M}} \\ f_* \mathcal{F} & \leftarrow\!\shortmid & \mathcal{F}, \end{array}$$

where recall

$$\widetilde{\mathcal{M}} \simeq \mathcal{O}_X \underset{f^{-1}\mathcal{A}}{\otimes} f^{-1} \mathcal{M}.$$

Let

$$\mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} \subseteq \mathcal{A}\text{–}\mathsf{mod}$$

be the full subcategory consisting of $\mathcal{F} \in \mathcal{A}\text{–}\mathsf{mod}$ such that $\mathcal{F}$ is quasi-coherent as an $\mathcal{O}_S$-module.

Proposition 18.2.2. *In the above setting, the following functors are well-defined and inverse to each other:*

$$\begin{array}{ccc} \mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qcoh}} \\ \mathcal{M} & \mapsto & \widetilde{\mathcal{M}} \\ f_* \mathcal{F} & \leftarrow\!\shortmid & \mathcal{F}, \end{array}$$

Proof. The claim is local on S and we can assume $S = \mathsf{Spec}(R)$. Recall that $\mathsf{QCoh}(S) \simeq R\text{–}\mathsf{mod}$. Write $A \coloneqq \mathcal{A}(S)$ and view it as a commutative algebra object in $R\text{–}\mathsf{mod}$. It follows that $\mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} \simeq A\text{–}\mathsf{mod}(R\text{–}\mathsf{mod}) \simeq A\text{–}\mathsf{mod}$. Now the desired equivalence becomes that in Corollary 13.1.2.

□

18.3. **Serre's criterion: statement.**

Theorem 18.3.1 (Serre)**.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism between schemes. The following conditions are equivalent:*

(1) *The morphism f is affine.*
(2) *The functor*
$$f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$$
is exact.

Proof. We claim both conditions are local on Y. For (1), this is Lemma 18.1.6(iv). For (2), let $Y = \sqcup_{i \in I} Y_i$ be an open covering and $f_i : X_i \to Y_i$ be the base-change of f along $Y_i \to Y$. For $\mathcal{F} \in \mathsf{QCoh}(X)$, it is easy to see that
$$f_*(\mathcal{F})|_{Y_i} \simeq f_{i,*}(\mathcal{F}|_{X_i}).$$
Note that the restriction functors $\mathsf{QCoh}(Y) \to \mathsf{QCoh}(Y_i)$ and $\mathsf{QCoh}(X) \to \mathsf{QCoh}(X_i)$ $(i \in I)$ preserve and detect exact sequences of quasi-coherent modules. It follows that f_* is exact iff each $f_{i,*}$ is exact. In other words, condition (2) is local on Y.

By the previous discussion, we can assume $Y = \mathsf{Spec}(R)$ is affine and therefore X is quasi-compact and quasi-separated. Recall we have $\mathsf{QCoh}(Y) \simeq R\text{–}\mathsf{mod}$ and the forgetful functor $R\text{–}\mathsf{mod} \to \mathsf{Ab}$ preserves and detect exact sequences. It follows that f_* is exact iff the functor
$$\mathsf{QCoh}(X) \xrightarrow{f_*} \mathsf{QCoh}(Y) \simeq R\text{–}\mathsf{mod} \to \mathsf{Ab}$$
is exact. By definition, this functor is just
$$\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}.$$
Now the claim follows from Theorem 18.3.2 below.

□

Theorem 18.3.2 (Serre)**.** *Let X be a quasi-compact scheme. The following conditions are equivalent:*

(1) *The scheme X is affine.*
(2) *The functor*
$$\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}$$
is exact.

Appendix A. Abelian categories

A.1. **Additive category.** In most textbooks, an additive category is defined as a category equipped with an *extra structure*[18] that admits finite products. However, it is a remarkable fact that such an extra structure is actually unique. In other words, being additive is *a property rather than a structure.*

Definition A.1.1. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ **admits** a zero object if it admits a final object $*$ and an initial object $\varnothing$, and the unique morphism $\varnothing \to *$ is an isomorphism.

If $\mathcal{A}$ admits a zero object, then any final object is also an initial object and vice versa. We often denote such an object by $0 \in \mathcal{A}$.

Example A.1.2. The category Ab of abelian groups admits a zero object, known as the zero abelian group.

Example A.1.3. The category Top of topological spaces admits both a final and an initial object, but these objects are not isomorphic. On the other hand, the category Top_* of *pointed* topological spaces admits a zero object.

Lemma A.1.4. *Let $\mathcal{A}$ be a category that admits a zero object. For any pair of objects $X, Y \in \mathcal{A}$, there exists a unique morphism $f : X \to Y$ that can factor through a zero object.*

Proof. Existence is obvious. For uniqueness, suppose f and f' are two such morphisms. We can write them as the following compositions:

$$f : X \xrightarrow{f_1} 0 \xrightarrow{f_2} Y,\ f' : X \xrightarrow{f'_1} 0' \xrightarrow{f'_2} Y,$$

where 0 and $0'$ are both zero objects. Let $h : 0 \to 0'$ be the unique morphism from 0 to $0'$. Since $0'$ is a final object, $f'_1 = h \circ f_1$. Since 0 is an initial object, $f_2 = f'_2 \circ h$. It follows that

$$f = f_2 \circ f_1 = f'_2 \circ h \circ f_1 = f'_2 \circ f'_1 = f'$$

as desired. □

Definition A.1.5. Let $\mathcal{A}$ be a category that admits a zero object, and X, Y be a pair of objects. The unique morphism $X \to Y$ that factors through a zero object is called the **zero morphism**. We often denote this morphism by $0 : X \to Y$.

Definition A.1.6. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ **admits finite biproducts** if it satisfies the following conditions.

(i) The category $\mathcal{A}$ admits finite products, finite coproducts and a zero object.
(ii) For any pair of objects X, Y, the map $X \coprod Y \to X \times Y$ given by the matrix $\begin{pmatrix} \mathsf{id}_X & 0 \\ 0 & \mathsf{id}_Y \end{pmatrix}$ is an isomorphism.

If $\mathcal{A}$ is a category that admits finite biproducts, then any finite product of $(X_i)_{i \in I}$ is also a finite coproduct and vice versa. We often denote such object by $\bigoplus_{i \in I} X_i$.

[18] Namely, an enrichment over the category Ab of abelian groups. In other words, all Hom-sets are upgraded to an abelian group, and the composition maps are bilinear maps between abelian groups.

Construction A.1.7. Let $\mathcal{A}$ be a category that admits finite biproducts. For any pair of objects X, Y, and any pair of morphisms $f, g : X \to Y$, let $f + g$ be the composition

$$X \xrightarrow{\begin{pmatrix} \mathsf{id}_X & \mathsf{id}_X \end{pmatrix}} X \bigoplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \bigoplus Y \xrightarrow{\begin{pmatrix} \mathsf{id}_Y \\ \mathsf{id}_Y \end{pmatrix}} Y.$$

One can show this endows $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a abelian semigroup[19] (i.e. the binary operation $+$ is unital, commutative and associative), whose identity element is the zero morphism $0 : X \to Y$. Moreover, one can check that the composition map

$$\mathsf{Hom}_{\mathcal{A}}(X, Y) \times \mathsf{Hom}_{\mathcal{A}}(Y, Z) \to \mathsf{Hom}_{\mathcal{A}}(X, Z)$$

is *bilinear*. In other words: $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ and $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$.

Definition A.1.8. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ is **additive** if it satisfies the following conditions:

(i) The category $\mathcal{A}$ admits finite biproducts.
(ii) For any pair of objects X, Y, the abelian semigroup $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ in Construction A.1.7 is an abelian group, i.e., any element $f \in \mathsf{Hom}_{\mathcal{A}}(X, Y)$ admits an inverse $-f$ such that $f + (-f) = 0$.

Example A.1.9. The category Ab of abelian groups is an additive category.

The following result implies our definition of additive category coincides with other approaches in the literature.

Exercise A.1.10. Let $\mathcal{A}$ be a category that admits finite products. Suppose for any pair of objects X, Y, there is a binary operation $+'$ on $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ such that:

(i) The pair $(\mathsf{Hom}_{\mathcal{A}}(X, Y), +')$ is an abelian group.
(ii) For objects X, Y, Z, the composition map

$$\mathsf{Hom}_{\mathcal{A}}(X, Y) \times \mathsf{Hom}_{\mathcal{A}}(Y, Z) \to \mathsf{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear with respect to the above abelian group structures.

Then $\mathcal{A}$ is additive and $+'$ is equal to the operation in Construction A.1.7.

A.2. **Kernel and cokernel.** Recall (co)kernels can be defined as fiber (co)products.

Definition A.2.1. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism.

- The **kernel** of f, often denoted by $\mathsf{ker}(f)$, is defined to be the fiber product of the span $X \xrightarrow{f} Y \xleftarrow{0} 0$.
- The **cokernel** of f, often denoted by $\mathsf{coker}(f)$, is defined to be the fiber coproduct of the cospan $Y \xleftarrow{f} X \xrightarrow{0} 0$.

Lemma A.2.2. *Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism that admits a kernel. Then the canonical morphism $i : \mathsf{ker}(f) \to X$ is a monomorphism.*

Proof. By definition, we only need to show for any test object $T \in \mathcal{A}$, post-composing with i induces an injection between sets:

$$\mathsf{Hom}_{\mathcal{A}}(T, \mathsf{ker}(f)) \to \mathsf{Hom}_{\mathcal{A}}(T, X).$$

[19] Other name: commutative monoid.

By definition of $\mathsf{ker}(-)$, we have a Cartesian square of sets

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{A}}(T, \mathsf{ker}(f)) \arrow[r, "i\circ-"] \arrow[d, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, X) \arrow[d, "f\circ-"] \\ \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[r, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, Y). \end{tikzcd}$$

By the definitions of zero objects and zero morphisms, $\mathsf{Hom}_{\mathcal{A}}(T, 0)$ is a singleton and the bottom horizontal arrow is an injection. It follows that the top horizontal arrow is an injection as desired.

□

Remark A.2.3. Dually, the canonical map $p : Y \to \mathsf{coker}(f)$ is an *epimorphism*, i.e., pre-composing with p induces an injection

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{coker}(f), T) \to \mathsf{Hom}_{\mathcal{A}}(Y, T)$$

for any test object $T \in \mathcal{A}$.

Example A.2.4. In Ab, any morphism admits a kernel and a cokernel.

Example A.2.5. The canonical morphism $\mathsf{ker}(X \xrightarrow{0} Y) \to X$ is an isomorphism, while $Y \to \mathsf{coker}(X \xrightarrow{0} Y)$ is an isomorphism.

Lemma A.2.6. *Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism. The following conditions are equivalent:*

(i) *The morphism f is a monomorphism.*
(ii) *The kernel $\mathsf{ker}(f)$ exists and is a zero object.*

Proof. By definition, condition (ii) is equivalent to:

(ii') For any test object $T \in \mathcal{A}$, the following diagram is a Cartesian square of sets:

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[r, "="] \arrow[d, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[d, "0\circ-"] \\ \mathsf{Hom}_{\mathcal{A}}(T, X) \arrow[r, "f\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, Y). \end{tikzcd}$$

By the definitions of zero objects and zero morphisms, ths condition is equivalent to

(ii") For any test object $T \in \mathcal{A}$, the map

$$\mathsf{Hom}_{\mathcal{A}}(T, X) \to \mathsf{Hom}_{\mathcal{A}}(T, Y)$$

sends nonzero morphisms to nonzero morphisms.

Since $\mathcal{A}$ is additive, the above map is a homomorphism of abelian groups. Recall zero morphisms are the identity elements in these groups. It follows that condition (ii") is equivalent to $\mathsf{Hom}_{\mathcal{A}}(T, X) \to \mathsf{Hom}_{\mathcal{A}}(T, Y)$ being injective, which is exactly condition (i).

□

Warning A.2.7. In general, being a monomorphism and an epimorphism does not imply being an isomorphism. Hence vanishing of kernel and cokernel in an *additive* category does not imply a morphism is an isomorphism. See the exercise below for a counterexample. However, we will see abelian categories do not have this caveat.

Exercise A.2.8. Let $\mathsf{Ab}_{\mathsf{tf}} \subseteq \mathsf{Ab}$ be the full subcategory of torsion free abelian groups, i.e., those abelian groups containing no finite order elements.

(1) Show that $\mathsf{Ab}_{\mathsf{tf}}$ is an additive category and any morphism in $\mathsf{Ab}_{\mathsf{tf}}$ admits a kernel and a cokernel.
(2) Find a morphism f in $\mathsf{Ab}_{\mathsf{tf}}$ such that $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects but f is not an isomorphism.

A.3. **Abelian category.**

Definition A.3.1. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism.

- If $\mathsf{coker}(f)$ exists, the **image** of f is defined to be
$$\mathsf{im}(f) := \mathsf{ker}(Y \to \mathsf{coker}(f)).$$
- If $\mathsf{ker}(f)$ exists, the **coimage** of f is defined to be
$$\mathsf{coim}(f) := \mathsf{coker}(\mathsf{ker}(f) \to X).$$

Remark A.3.2. Let $f' \in \mathsf{Hom}_{\mathcal{A}^{\mathsf{op}}}(Y, X)$ be the morphism corresponding to f. Note that $\mathcal{A}^{\mathsf{op}}$ is also an additive category and the monomorphism $\mathsf{ker}(f) \to X$ in $\mathcal{A}$ corresponds to the epimorphism $X \to \mathsf{coker}(f')$ in $\mathcal{A}^{\mathsf{op}}$. It follows that the monomorphism $\mathsf{im}(f) \to Y$ in $\mathcal{A}$ corresponds to the epimorphism $Y \to \mathsf{coim}(f')$ in $\mathcal{A}^{\mathsf{op}}$.

Exercise A.3.3. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism such that $\mathsf{ker}(f)$, $\mathsf{coker}(f)$, $\mathsf{im}(f)$ and $\mathsf{coim}(f)$ all exists. Let $p : X \to \mathsf{coim}(f)$ and $i : \mathsf{im}(f) \to Y$ be the canonical morphisms. Show that f admits a unique factorization as
$$X \xrightarrow{p} \mathsf{coim}(f) \to \mathsf{im}(f) \xrightarrow{i} Y.$$

In particular, we obtain a *canonical* morphism $\mathsf{coim}(f) \to \mathsf{im}(f)$.

Definition A.3.4. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ is **abelian** if it satisfies the following conditions:

(i) The category $\mathcal{A}$ is additive.
(ii) Any morphism in $\mathcal{A}$ admits a kernel and a cokernel.
(iii) For any morphism f in $\mathcal{A}$, the canonical morphism $\mathsf{coim}(f) \to \mathsf{im}(f)$ is an isomorphism.

Example A.3.5. The category Ab is an abelian category.

Remark A.3.6. By Remark A.3.2, $\mathcal{A}$ is abelian iff $\mathcal{A}^{\mathsf{op}}$ is so.

Lemma A.3.7. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ be a morphism. The following conditions are equivalent:*

(i) *The morphism f is a monomorphism.*
(ii) *The kernel $\mathsf{ker}(f)$ exists and is a zero object.*
(iii) *The morphism f is a kernel, can be written as $\mathsf{ker}(g) \to X$ for some morphism $g : Y \to Z$.*

Proof. We have seen (i)⇔(ii) in Lemma A.2.6, and (iii)⇒(i) in Lemma A.2.2. It remains to show (ii)⇒(iii).

Now suppose condition (ii) holds. By Example A.2.5, the canonical map from X to $\mathsf{coim}(f) = \mathsf{coker}(\mathsf{ker}(f) \to X)$ is an isomorphism. Since $\mathcal{A}$ is abelian, we obtain isomorphisms
$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f).$$

By definition, the canonical map $\mathsf{im}(f) \to Y$ is a kernel of the canonical map $Y \to \mathsf{coker}(f)$. It follwos that the composition

$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f) \to Y$$

is also a kernel of the canonical map $Y \to \mathsf{coker}(f)$. But by definition, this composition is just f. Hence f is a kernel as desired.

□

Proposition A.3.8. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ be a morphism. Then f is an isomorphism iff both $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose both $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects. By Example A.2.5, the canonical maps $\mathsf{ker}(Y \to \mathsf{coker}(f)) \to Y$ and $X \to \mathsf{coker}(\mathsf{ker}(f) \to X)$ are both isomorphisms. In other words, $\mathsf{im}(f) \to Y$ and $X \to \mathsf{coim}(f)$ are both isomorphisms. On the other hand, the canonical map $\mathsf{coim}(f) \to \mathsf{im}(f)$ is an isomorphism because $\mathcal{A}$ is abelian. It follows that the composition

$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f) \xrightarrow{\simeq} Y$$

is an isomorphism. But by definition, this composition is just f.

□

Construction A.3.9. Let $\mathcal{A}$ be an additive category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence such that $g \circ f = 0$. Since the canonical morphism $X \to \mathsf{coim}(f)$ is an epimorphism, the composition $\mathsf{coim}(f) \to Y \xrightarrow{g} Z$ is also zero. Hence the morphism $\mathsf{coim}(f) \to Y$ admits a unique factorization as $\mathsf{coim}(f) \to \mathsf{ker}(g) \to Y$ such that the second map is the canonical monomorphism. In particular, we obtain a canonical morphism

$$\mathsf{coim}(f) \to \mathsf{ker}(g).$$

Definition A.3.10. Let $\mathcal{A}$ be an abelian category. We say a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is **exact** if it satisfies the following conditions.

- The composition $g \circ f$ is zero.
- the canonical morphism $\mathsf{coim}(f) \to \mathsf{ker}(g)$ in Construction A.3.9 is an isomorphism.

We say a sequence $X_1 \to X_2 \to \cdots \to X_n$ is **exact** if any three consequential terms from an exact sequence.

A.4. **Additive functors.**

Proposition-Definition A.4.1. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a functor between additive categories. The following conditions are equivalent:*

(i) *The functor F preserves finite products.*
(ii) *The functor F preserves finite coproducts.*

*We say F is **additive** if it satisfies the above conditions.*

Proof. We will show (i)⇒(ii). The other implication follows by duality. Suppose F preserves finite products. To verify (ii), we only need to check F preserves initial objects and binary coproducts. The claim for initial objects follow from the axiom that additive categories admit zero objects. The claim for binary coproducts follow from the axoim that the canonical map $X \coprod Y \to X \times Y$ is an isomorphism.

□

The following result follows immediately from Construction A.1.7.

Lemma A.4.2. *Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between additive categories. Then for any pair of objects $X, Y \in \mathcal{A}$, the map*

$$\mathsf{Hom}_{\mathcal{A}}(X,Y) \to \mathsf{Hom}'_{\mathcal{A}}(F(X),F(Y))$$

is a homomorphism between abelian groups.

Note that addtive functors in general do not preserve (co)kernels.

Definition A.4.3. Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between abelian categories.

- We say F is **left exact** if F preserves kernels.
- We say F is **right exact** if F preserves cokernels.
- We say F is **exact** if F is both left exact and right exact.

Remark A.4.4. It is easy to see F is left exact iff it preserves exact sequences of the form $0 \to X_1 \to X_2 \to X_3$, while F is right exact iff it preserves exact sequences of the form $X_1 \to X_2 \to X_3 \to 0$. Also, F is exact iff it preserves all exact sequences.

Example A.4.5. Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$ be an object. Note that we have functors

$$\mathsf{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \to \mathsf{Ab},\ \mathsf{Hom}_{\mathcal{A}}(-,X) : \mathcal{A}^{\mathsf{op}} \to \mathsf{Ab}$$

One can check both functors are left exact.

Lemma A.4.6. *An additive functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories is left exact iff it preserves fiber products. Dually, F is right exact iff it preserves fiber coproducts.*

Proof. The "if" part is obvious because any kernel is a fiber product. For the "only if" part, let $X \xrightarrow{f} Y \xleftarrow{g} Z$ be any span diagram. Consider

$$W := \mathsf{ker}(X \bigoplus Z \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} Y).$$

One can check that the morphisms $W \to X \oplus Z \to X$ and $W \to X \oplus Z \to Z$ exhibits W as the fiber product of $X \xrightarrow{f} Y \xleftarrow{g} Z$. Since F preserves biprodcuts and kernels, it also preserves fiber products.

□

Remark A.4.7. Using the language of category theory, a functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories is left (resp. right) exact iff it preserves finite limits (resp. colimits).

A.5. **Abelian categories in theses notes.** The following claims are either proved in the notes, or follow immediately from the definitions.

Proposition A.5.1. *Let X be a topological space.*

- *The category $\mathsf{PShv}(X, \mathsf{Ab})$ is an abelian category.*
- *For any open subset $U \subseteq X$, the functor $(-)(U) : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is exact.*
- *For any point $x \in X$, the functor $(-)_x : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is exact.*

Let $f : X \to Y$ be a continuous map between topological spaces.

- *The functor $f_* : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{PShv}(Y, \mathsf{Ab})$ is exact.*
- *The functor $f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(Y, \mathsf{Ab}) \to \mathsf{PShv}(X, \mathsf{Ab})$ is exact.*

Proposition A.5.2. *Let X be a topological space.*

- *The category $\mathsf{Shv}(X, \mathsf{Ab})$ is an abelian category.*
- *For any open subset $U \subseteq X$, the functor $(-)(U) : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is left exact.*
- *For any point $x \in X$, the functor $(-)_x : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is exact.*
- *The fully faithful functor $\mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{PShv}(X, \mathsf{Ab})$ is left exact.*
- *The sheafification functor $\sharp : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{Shv}(X, \mathsf{Ab})$ is exact.*

Let $f : X \to Y$ be a continuous map between topological spaces.

- *The functor $f_* : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Shv}(Y, \mathsf{Ab})$ is left exact.*
- *The functor $f^{-1} : \mathsf{Shv}(Y, \mathsf{Ab}) \to \mathsf{Shv}(X, \mathsf{Ab})$ is exact.*

Proposition A.5.3. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

- *The categories $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ and $\mathcal{O}_X\text{–}\mathsf{mod}$ are abelian categories.*
- *The forgetful functors $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathsf{PShv}(X, \mathsf{Ab})$ and $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathsf{Shv}(X, \mathsf{Ab})$ are exact.*
- *The fully faithful functor $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ is left exact.*
- *The sheafification functor $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{–}\mathsf{mod}$ is exact.*

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces.

- *The functor $f_* : \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}}$ is exact.*
- *The functor $f_* : \mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_Y\text{–}\mathsf{mod}$ is left exact.*
- *The functor $f^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ is right exact.*
- *The functor $f^* : \mathcal{O}_Y\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$ is right exact.*

Proposition A.5.4. *Let X be a scheme.*

- *The category $\mathsf{QCoh}(X)$ is an abelian category.*
- *The fully faithful functor $\mathsf{QCoh}(X) \to \mathcal{O}_X\text{–}\mathsf{mod}$ is exact.*

Proposition A.5.5. *Let $f : X \to Y$ be a morphism between schemes. The functor $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is right exact. If f is an open immersion, it is exact.*

Remark A.5.6. In fact, f^* is exact iff f is *flat*.

Proposition A.5.7. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism between schemes. The functor $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ is left exact. It is exact iff f is affine.*

References

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*. Springer, 2000.