# LECTURE 17

In this lecture, we introduce two examples of stable $\infty$-categories:

- the naive $\infty$-category $\mathsf{K}(\mathsf{A})$ of cochain complexes in an *additive category* $\mathsf{A}$
- the derived $\infty$-category $\mathsf{D}(\mathsf{A})$ of an *abelian* category $\mathsf{A}$

## 1. $\mathsf{K}(\mathsf{A})$ and $\mathsf{D}(\mathsf{A})$ via localization

**Definition 1.1.** *Let $\mathsf{A}$ be an additive category and $\mathsf{Ch}(\mathsf{A})$ be the* ordinary category *of cochain complexes in $\mathsf{A}$. Let $f, g : M^\bullet \to N^\bullet$ be morphisms in $\mathsf{Ch}(\mathsf{A})$. A* ***cochain homotopy*** *from $f$ to $g$ is a collection of maps $h^n : M^n \to N^{n-1}$ such that*

$$f^n - g^n = d \circ h^n + h^{n+1} \circ d$$

*for any $n$. We say $f$* ***and*** *$g$* ***are homotopic****, or $f \sim g$, if there exists a cochain homotopy between them.*

**Remark 1.2.** *Being homotopic is an equivalence relation on $\mathsf{Hom}_{\mathsf{Ch}(\mathsf{A})}(M^\bullet, N^\bullet)$.*

**Example 1.3.** *Let $F, G : X \to Y$ be continuous maps between topological spaces. Then any homotopy from $F$ to $G$ induces a cochain homotopy from $F^*$ to $G^*$, where $F^* : \mathsf{C}^\bullet(Y) \to \mathsf{C}^\bullet(X)$ is the cochain homomorphism between the singular cochain complexes.*

**Exercise 1.4.** *Let $\mathsf{A}$ be an abelian category (so that we can define cohomologies). If $f \sim g$, then $\mathsf{H}^n(f) = \mathsf{H}^n(g)$ as morphisms $\mathsf{H}^n(M^\bullet) \to \mathsf{H}^n(N^\bullet)$.*

**Definition 1.5.** *Let $\mathsf{A}$ be an additive category and $f : M^\bullet \to N^\bullet$ be a morphism in $\mathsf{Ch}(\mathsf{A})$. We say $f$ is a* ***cochain homotopy equivalence*** *if there exists $g : N^\bullet \to M^\bullet$ such that $f \circ g \sim \mathsf{id}_{\mathsf{N}^\bullet}$ and $g \circ f \simeq \mathsf{id}_{\mathsf{M}^\bullet}$. Let $S$ be the collection of cochain homotopy equivalences in $\mathsf{Ch}(\mathsf{A})$.*

**Definition 1.6.** *Let $\mathsf{A}$ be an abelian category and $f : M^\bullet \to N^\bullet$ be a morphism in $\mathsf{Ch}(\mathsf{A})$. We say $f$ is a* ***quasi-isomorphism*** *if it induces isomorphisms between the cohomologies. Let $W$ be the collection of quasi-isomorphisms in $\mathsf{Ch}(\mathsf{A})$.*

**Exercise 1.7.** *Let $\mathsf{A}$ be an abelian category. Show that a cochain homotopy equivalence in $\mathsf{Ch}(\mathsf{A})$ is a quasi-isomorphism.*

**Exercise 1.8.** *Let $\mathsf{A}$ be an abelian category and $0 \to L \to M \to N \to 0$ be a short exact sequence. Show that*

$$\begin{tikzcd}
{[\cdots} \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & {\cdots]} \\
{[\cdots} \arrow[r] & 0 \arrow[r] & N \arrow[r] & {\cdots]}
\end{tikzcd}$$

*is always a quasi-isomorphism, but is a cochain homotopy equivalence iff the given short exact sequence has a splitting.*

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**Definition 1.9.** *Let* $\mathsf{A}$ *be an additive category. The* ***naive*** $\infty$***-category of cochain complexes*** *in* $\mathsf{A}$ *is defined to be the* $\infty$*-categorical localization*[1]

$$\mathsf{K}(\mathsf{A}) \coloneqq \mathsf{Ch}(\mathsf{A})[S^{-1}].$$

**Definition 1.10.** *Let* $\mathsf{A}$ *be an abelian category. The* ***unbounded derived*** $\infty$***-category*** *of* $\mathsf{A}$ *is defined to be the* $\infty$*-categorical localization*

$$\mathsf{D}(\mathsf{A}) \coloneqq \mathsf{Ch}(\mathsf{A})[W^{-1}].$$

**Exercise 1.11.** *Write* $\mathsf{D}(A)$ *as a localization of* $\mathsf{K}(\mathsf{A})$*.*

**Warning 1.12.** *In the non-*$\infty$*-categorical literatures,* $\mathsf{K}(\mathsf{A})$ *and* $\mathsf{D}(\mathsf{A})$ *often means the 1-categorical localizations, which are the homotopy categories of our* $\infty$*-categories.*

1.13. It is well-known (see [Sta24, Tag 05QI]) that $\mathsf{hK}(\mathsf{A})$ has a structure of triangulated categories such that for any $f : M^\bullet \to N^\bullet$ in $\mathsf{Ch}(\mathsf{A})$, there is a distinguished triangle

$$M^\bullet \xrightarrow{f} N^\bullet \to \mathsf{cone}(f) \to M^\bullet[1]$$

for the corresponding objects in $\mathsf{hK}(A)$. Moreover, when $\mathsf{A}$ is abelian, this triangulated is compatible with the collection of quasi-isomorphisms and thereby induces a triangulated structure on the 1-categorical localization $\mathsf{hD}(\mathsf{A})$.

**Warning 1.14.** *The category* $\mathsf{Ch}(\mathsf{A})$ *is almost never triangulated because any abelian triangulated category is semisimple.*

1.15. The following result says the above triangulated categories come from stable $\infty$-categories.

**Theorem 1.16** (HA.1.3.2.10)**.** *Let* $\mathsf{A}$ *be an additive category. The* $\infty$*-category* $\mathsf{K}(\mathsf{A})$ *is stable and the triangulated structure on* $\mathsf{hK}(\mathsf{A})$ *coincides with the well-known one*[2]*.*

**Theorem 1.17** (HA.1.3.5.9, 1.3.5.21)**.** *Let* $\mathsf{A}$ *be a* Grothendieck[3] *abelian category. The* $\infty$*-category* $\mathsf{D}(\mathsf{A})$ *is presentable and stable and the triangulated structure on* $\mathsf{hD}(\mathsf{A})$ *coincides with the well-known one.*

**Warning 1.18.** *Let* $\mathsf{A}$ *be a Grothendieck abelian category. The* $\infty$*-category* $\mathsf{K}(\mathsf{A})$ *is almost never presentable. See this MathOverflow question.*

## 2. $\mathsf{K}(\mathsf{A})$ and $\mathsf{D}(\mathsf{A})$ via differential graded nerve

2.1. Definition 1.9 and Definition 1.10 are concise but inconvenient for calculations. In this section, we provide a more explicit construction of these $\infty$-categories.

[1] According to our convention, we use the same symbol to denote an ordinary category $\mathsf{C}$ and the corresponding $\infty$*-category*, realized as the *quasi-category* $\mathsf{N}_\bullet(\mathsf{C})$. Hence $\mathsf{K}(\mathsf{A})$ can be represented as the *quasi-categorical localization* $\mathsf{N}_\bullet(\mathsf{Ch}(\mathsf{A}))[S^{-1}]$, which is the notation used in Lurie's books.

[2] This claim follows from the proof of HA.1.3.2.10.

[3] This means $\mathsf{A}$ is presentable and taking filtered colimits is exact.

2.2. The alternative construction for $\mathsf{K}(\mathsf{A})$ will be a quasi-category such that

- an object is a cochain complex $M_0$
- a morphism is a cochain homomorphism $M_0 \to M_1$
- a 2-simplex is given by

$$\begin{tikzcd} & M_1 \arrow[dr, "f_{12}"] \arrow[d, Rightarrow, "f_{012}"] & \\ M_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & {} & M_2 \end{tikzcd}$$

  where $h_{012}$ stands for a homotopy from $f_{12} \circ f_{01}$ to $f_{02}$,
- $\cdots$

To give a description for higher simplices, it is convenient to introduce some terminologies.

**Construction 2.3.** *Let* $\mathsf{Ch}(\mathsf{Ab})$ *be the ordinary category of cochain complexes. For* $M, N \in \mathsf{Ch}(\mathsf{Ab})$*, the* ***graded tensor product*** $M \otimes N$ *is defined as follows.*

- *For each integer* $k$*,*
$$(M \otimes N)^k := \bigoplus_{i+j=k} M^i \otimes N^j.$$
- *For each integer* $k$*, the differential* $(M \otimes N)^k \to (M \otimes N)^{k+1}$ *is given by the* ***graded Lubniz rule****, which sends a pure tensor* $m \otimes n \in M^i \otimes N^j$ *to*
$$d(m \otimes n) := d(m) \otimes n + (-1)^j m \otimes d(n).$$

*There is a natural* monoidal structure *on* $\mathsf{Ch}(\mathsf{Ab})$ *with multiplication given by the graded tensor products.*

**Example 2.4.** *Let* $X$ *and* $Y$ *be topological spaces. Then we have* $\mathsf{C}^\bullet(X \times Y) \simeq \mathsf{C}^\bullet(X) \otimes \mathsf{C}^\bullet(Y)$.

**Exercise 2.5.** *Let* $M \in \mathsf{Ch}(\mathsf{Ab})$*. Show that a 0-cocycle in* $M$ *is the same as a morphism* $\mathbb{1} \to M$*, where* $\mathbb{1}$ *is the monoidal unit.*

**Definition 2.6.** *A* ***differential graded category****, or a* ***dg-category****, is a* $\mathsf{Ch}(\mathsf{Ab})$*-enriched category.*

2.7. By definition, a dg-category $\mathbb{C}$ consists of the following datum:

- A cochain complex $\mathsf{Hom}_{\mathbb{C}}(x, y) \in \mathsf{Ch}(\mathsf{Ab})$ for objects $x, y \in \mathbb{C}$ called the **mapping complex**
- A composition law given by cochain homomorphisms
$$- \circ - : \mathsf{Hom}_{\mathbb{C}}(x, y) \otimes \mathsf{Hom}_{\mathbb{C}}(y, z) \to \mathsf{Hom}_{\mathbb{C}}(x, z)$$
  which is associative in the obvious sense
- A *0-cocycle* $\mathsf{id}_x \in \mathsf{Hom}_{\mathbb{C}}(x, x)^0$ such that $f \circ \mathsf{id}_x = f$ and $\mathsf{id}_x \circ g = g$ for any $f \in \mathsf{Hom}_{\mathbb{C}}(x, y)^m$ and $g \in \mathsf{Hom}_{\mathbb{C}}(y, x)^n$.

**Exercise 2.8.** *Show that any* $\mathsf{id}_x \in \mathsf{Hom}_{\mathbb{C}}(x, x)^0$ *satisfying the above condition is automatically a cocycle.*

**Definition 2.9.** *Let* $\mathbb{C}$ *be a dg-category. The* ***underlying category*** $\mathsf{C}$ *of* $\mathbb{C}$ *is defined such that the* $\mathsf{Hom}$*-sets are given by*
$$\mathsf{Hom}_{\mathsf{C}}(x, y) := \mathsf{Hom}_{\mathsf{Ch}(\mathsf{A})}(\mathbb{1}, \mathsf{Hom}_{\mathbb{C}}(x, y)).$$

2.10. We will identify $\mathsf{Hom}_{\mathsf{C}}(x,y)$ with the abelian group of 0-cocycles in $\mathsf{Hom}_{\mathbb{C}}(x,y)$.

**Definition 2.11.** *Let $\mathbb{C}$ be a dg-category. The **homotopy category** $\mathsf{h}\mathbb{C}$ of $\mathbb{C}$ is defined such that the* Hom*-sets are given by*

$$\mathsf{Hom}_{\mathsf{h}\mathbb{C}}(x,y) := \mathsf{H}^0(\mathsf{Hom}_{\mathbb{C}}(x,y)).$$

**Example 2.12.** *For any additive category $\mathsf{A}$, the ordinary category $\mathsf{Ch}(\mathsf{A})$ has a natural enrichment $\mathbb{Ch}(\mathsf{A})$ over $\mathsf{Ch}(\mathsf{Ab})$ such that*

$$\mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^k := \prod_i \mathsf{Hom}_{\mathsf{A}}(M^i, N^{i+k})$$

*with the differential given by the graded Lubniz rule*

$$(df)(x) = d(f(x)) - (-1)^k f(dx)$$

*for $f \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^k$ and $x \in M^i$.*

**Exercise 2.13.** *Show that the underlying category of $\mathbb{Ch}(\mathsf{A})$ is indeed $\mathsf{Ch}(\mathsf{A})$. What is its homotopy category?*

**Exercise 2.14.** *Let $f,g \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^0$ be 0-cocycles, viewed as morphisms in the ordinary category $\mathsf{Ch}(\mathsf{A})$. Show that a homotopy from $f$ to $g$ is the same as an element $h \in \mathsf{Hom}_{\mathbb{Ch}(\mathsf{A})}(M,N)^1$ such that $dh = f - g$.*

**Proposition-Construction 2.15** (HA.1.3.1.10)**.** *Let $\mathbb{C}$ be a dg-category. We have a quasi-category $\mathsf{N}_{\mathsf{dg}}(\mathbb{C})$, called the **dg-nerve** of $\mathbb{C}$, defined as follows.*

- *A 0-simplex is an object in $\mathbb{C}$*
- *A 1-simplex consists of its boundary $\{X_0, X_1\}$ together with a 0-cocycle in $f_{01} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_1)^0$*
- *A 2-simplex consists of its boundary $\{f_{01}, f_{02}, f_{12}\}$ together with an element $g_{012} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_2)^{-1}$ such that*

$$df_{012} = -(f_{02} - f_{12} \circ f_{01}).$$

- *A 3-simplex consists of its boundary $\{f_{012}, f_{013}, f_{023}, f_{123}\}$ together with an element $f_{0123} \in \mathsf{Hom}_{\mathbb{C}}(X_0,X_3)^{-2}$ such that*

$$df_{0123} = -(f_{013} - f_{23} \circ f_{012}) + (f_{023} - f_{123} \circ f_{01}).$$

- *Higher simplices are given similarly (see HA.1.3.1.6).*

**Proposition 2.16.** *Let $\mathbb{C}$ be a dg-category and $X, Y$ be objects in $\mathbb{C}$. Then there are canonical isomorphisms*

$$\pi_i(\mathsf{Maps}_{\mathsf{N}_{\mathsf{dg}}(\mathbb{C})}(X,Y)) \simeq \mathsf{H}^{-i}(\mathsf{Hom}_{\mathbb{C}}(X,Y))$$

**Remark 2.17.** *We will prove the above proposition next time. In fact, the space $\mathsf{Maps}_{\mathsf{N}_{\mathsf{dg}}(\mathbb{C})}(X,Y)$ can be obtained from the truncation $\tau^{\leq 0}\mathsf{Hom}_{\mathbb{C}}(X,Y)$ in a canonical way, known as the* Dold–Kan correspondence.

**Exercise 2.18.** *Construct a monomorphism $\mathsf{N}(\mathsf{C}) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{C})$ that is bijective on $n$-simplices with $n \leq 1$, where $\mathsf{N}(\mathsf{C})$ is the usual nerve of the underlying ordinary category $\mathsf{C}$ of $\mathbb{C}$.*

**Exercise 2.19.** *Construct an equivalence $\mathsf{hN}_{\mathsf{dg}}(\mathbb{C}) \simeq \mathsf{h}\mathbb{C}$.*

**Proposition 2.20** (HA.1.3.4.5)**.** *Let $\mathsf{A}$ be an additive category. Then the functor $\mathsf{N}(\mathsf{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ exhibits $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ as the localization of $\mathsf{N}(\mathsf{Ch}(\mathsf{A}))$ for the collection of cochain homotopy equivalences. In other words, $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$ represents the $\infty$-category $\mathsf{K}(\mathsf{A})$.*

**Exercise 2.21.** *Let $\mathsf{A}$ be an abelian category and $0 \to L \to M \to N \to 0$ be a short exact sequence. Show that it represents a zero object in $\mathsf{K}(\mathsf{A})$ iff it admits a splitting.*

2.22. Our next goal is to realize $\mathsf{D}(\mathsf{A})$ as the dg-nerve of a dg-category when $\mathsf{A}$ is a Grothendieck abelian category. Recall such $\mathsf{A}$ has enough injective objects.

**Theorem 2.23.** *Let $\mathsf{A}$ be a Grothendieck abelian category. Then $\mathsf{Ch}(\mathsf{A})$ admits a left proper combinatorial model structure determined by the following.*

(W) *weak equivalences are quasi-isomorphisms*

(C) *cofibrations are degreewise monomoprhisms*

**Exercise 2.24.** *If $\mathsf{M} \in \mathsf{Ch}(\mathsf{A})$ is fibrant in the above model structure, then each $\mathsf{M}^n \in \mathsf{A}$ is injective. Conversely, if $\mathsf{M}$ is bounded below and each $\mathsf{M}^n$ is injective, then $\mathsf{M}$ is fibrant.*

**Proposition 2.25** (HA.1.3.5.13)**.** *Let $\mathsf{Ch}(\mathsf{A})^\circ \subseteq \mathsf{Ch}(\mathsf{A})$ be the full subcategory of bifibrant objects. Let $\mathbb{Ch}(\mathsf{A})^\circ \subseteq \mathbb{Ch}(\mathsf{A})$ be the corresponding dg-category. Then the embedding*

$$\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}))$$

*admits a left adjoint*

$$\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$$

*such that the composition*

$$\mathsf{N}(\mathsf{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})) \to \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$$

*exhibits $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$ as the localization of $\mathsf{N}(\mathsf{Ch}(\mathsf{A}))$ for the collection of quasi-isomorphisms. In other words, $\mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A})^\circ)$ represents the $\infty$-category $\mathsf{D}(\mathsf{A})$.*

2.26. The above proposition implies for any Grothendieck abelian category $\mathsf{A}$, we have an adjunction of exact functors

$$\mathsf{K}(\mathsf{A}) \rightleftarrows \mathsf{D}(\mathsf{A}) \tag{2.1}$$

such that the right adjoint functor is fully faithful, and the left adjoint is given by taking fibrant replacements.

**Exercise 2.27.** *Find the kernel of the left adjoint functor $\mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$.*

## 3. t-structures

**Proposition 3.1** (HA.1.3.5.19, 1.3.5.21)**.** *Let $\mathsf{A}$ be a Grothendieck abelian category.*

- *Let $\mathsf{D}(\mathsf{A})^{\le 0}$ be the full sub-$\infty$-category consisting of objects represented by cochain complexes $M$ with $H^n(M) \simeq 0$ for $n > 0$.*
- *Let $\mathsf{D}(\mathsf{A})^{\ge 0}$ be the full sub-$\infty$-category consisting of objects represented by cochain complexes $M$ with $H^n(M) \simeq 0$ for $n < 0$.*

*Then $(\mathsf{D}(\mathsf{A})^{\le 0}, \mathsf{D}(\mathsf{A})^{\ge 0})$ determines a t-structure on $\mathsf{D}(\mathsf{A})$. Moreover,*

- *The functor $\mathsf{H}^0 : \mathsf{Ch}(\mathsf{A}) \to \mathsf{A}$ induces an equivalence $\mathsf{D}(\mathsf{A})^\heartsuit \to \mathsf{A}$.*
- *This t-structure is accessible, left separated, right complete and compatible with filtered colimits.*

**Warning 3.2.** *The t-structure on* $\mathsf{D}(\mathsf{A})$ *is generally not left complete. See [Nee11] for a counterexample. The left completion is often denoted by* $\widehat{\mathsf{D}}(\mathsf{A})$.

**Example 3.3.** *Let* $\mathsf{A} = \mathsf{Mod}_R^{\heartsuit}$ *be the abelian category of* $R$*-modules for an associative ring* $R$*. In future lectures, we will construct*

- *a symmetric monoidal structure on* $\mathsf{Sptr}$ *given by* smash products*;*
- *an* $\mathbb{E}_1$-algebra *structure on the* Eilenberg–Maclane spectrum $\mathbb{H}R \in \mathsf{Sptr}$*;*
- *a stable* $\infty$*-category* $\mathsf{Mod}_{\mathbb{H}R}$ *of* $\mathbb{H}R$*-modules in* $\mathsf{Sptr}$*, equipped with a t-structure induced from that of* $\mathsf{Sptr}$*;*
- *a t-exact equivalence*

$$\mathsf{Mod}_{\mathbb{H}R} \simeq \mathsf{D}(\mathsf{Mod}_R^{\heartsuit}) =: \mathsf{D}(R).$$

*Moreover, when* $R$ *is* commutative*, we will equip* $\mathsf{D}(R)$ *with a symmetric monoidal structure given by* relative tensor product *over* $\mathbb{H}R$.

**Exercise 3.4.** *Let* $\mathsf{A}$ *be a Grothendieck abelian category. Show that the left bounded part of* $\mathsf{D}(\mathsf{A})$ *can be identified with*

$$\mathsf{D}^+(\mathsf{A}) \simeq \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}}))$$

*where* $\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}})$ *is the dg-category of left bounded complexes of injective objects in* $\mathsf{A}$*. Note that the proof should* not *work for* $\mathsf{D}^-(\mathsf{A})$.

3.5. Motivated by the above exercise, we make the following definition.

**Definition 3.6.** *Let* $\mathsf{A}$ *be an abelian category with enough injectives. The* ***left bounded derived-$\infty$-category*** *of* $\mathsf{A}$ *is defined to be*

$$\mathsf{D}^+(\mathsf{A}) := \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^+(\mathsf{A}_{\mathsf{inj}})).$$

*Dually, let* $\mathsf{A}$ *be an abelian category with enough projectives. The* ***right bounded derived-$\infty$-category*** *of* $\mathsf{A}$ *is defined to be*

$$\mathsf{D}^-(\mathsf{A}) := \mathsf{N}_{\mathsf{dg}}(\mathbb{Ch}^-(\mathsf{A}_{\mathsf{proj}})) \simeq \mathsf{D}^+(\mathsf{A}^{\mathsf{op}})^{\mathsf{op}}.$$

**Exercise 3.7.** *Show that* $\mathsf{D}^-(\mathsf{A})$ *is a stable sub-*$\infty$*-category of* $\mathsf{K}(\mathsf{A}_{\mathsf{proj}})$.

**Warning 3.8.** *The* $\infty$*-categories* $\mathsf{D}^+(\mathsf{A})$ *and* $\mathsf{D}^-(\mathsf{A})$ *are not presentable because countable coproducts may not exist.*

3.9. The following results characterize $\mathsf{D}^-(\mathsf{A})$ via universal properties.

**Proposition 3.10** (HA.1.3.2.19, 1.3.3.16)**.** *Let* $\mathsf{A}$ *be an abelian category with enough projectives. The obvious choice defines a left complete t-structure on* $\mathsf{D}^-(\mathsf{A})$ *whose heart is canonically equivalent to* $\mathsf{A}$.

**Theorem 3.11** (HA.1.3.3.2. 1.3.3.6)**.** *Let* $\mathsf{A}$ *be an abelian category with enough projectives. For any stable* $\infty$*-category* $\mathsf{C}$ *equipped with a left complete t-structure, the following data are equivalent:*

(i) *A right t-exact functor* $F : \mathsf{D}^-(\mathsf{A}) \to \mathsf{C}$ *that sends* $\mathsf{A}_{\mathsf{proj}}$ *into* $\mathsf{C}^{\heartsuit}$
(ii) *A right exact functor* $f : \mathsf{A} \to \mathsf{C}^{\heartsuit}$

*where* $f := \tau^{\geq 0} \circ F|_{\mathsf{A}}$*. Moreover, the following conditions are equivalent*

- $F$ *is t-exact.*
- $F$ *preserves the hearts.*
- $f$ *is exact.*

**Definition 3.12.** *Let* $\mathsf{A}$ *be an abelian category with enough projectives and* $\mathsf{C}$ *be a stable* $\infty$*-category equipped with a left complete t-structure. For a right exact functor* $f : \mathsf{A} \to \mathsf{C}^{\heartsuit}$*, the corresponding functor* $F$ *is called the* ***left derived functor*** *of* $f$ *and denote as*

$$\mathbb{L}f : \mathsf{D}^{-}(\mathsf{A}) \to \mathsf{C}.$$

**Exercise 3.13.** *Show that* $\mathbb{L}f(M)$ *can be calculated via a projective resolution of the complex* $M$*.*

**Exercise 3.14.** *Consider the t-exact functor* $\mathsf{D}^{-}(\mathsf{Ab}) \to \mathsf{Sptr}$ *corresponding to the equivalence* $\mathsf{Ab} \simeq \mathsf{Sptr}^{\heartsuit}$*,* $A \mapsto \mathbb{H}A$*. Taking* right completion*, we obtain a functor*

$$\pi_* : \mathsf{D}(\mathsf{Ab}) \to \mathsf{Sptr}.$$

*Show that:*

(1) *The functor can also be obtained via the universal property of* $\mathsf{D}^{+}(\mathsf{Ab})$ *described by the dual of Theorem 3.11.*
(2) *Make a guess for the composition* $\Omega^{\infty} \circ \pi_*$*.*

3.15. Let $\mathsf{A}$ be a Grothendieck abelian category with enough *projective* objects. The following result says the two definitions of $\mathsf{D}^{-}(\mathsf{A})$ coincide.

**Proposition 3.16** (HA.1.3.5.24)**.** *Let* $\mathsf{A}$ *be a Grothendieck abelian category with enough projective objects. Then the composition*

$$\mathsf{D}^{-}(\mathsf{A}) \xrightarrow{\subseteq} \mathsf{K}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$$

*is fully faithful with essential image given by the right bounded part of* $\mathsf{D}(\mathsf{A})$*. In particular,* $\mathsf{D}(\mathsf{A})$ *is left complete.*

3.17. In fact, there is also a canonical t-structure on $\mathsf{K}(\mathsf{A})$ when $\mathsf{A}$ is a Grothendieck abelian category. However, one needs to be careful about the coconnective part because there are nonzero objects in $\mathsf{K}(\mathsf{A})$ with zero cohomologies.

**Proposition 3.18** (HA.1.3.5.18)**.** *Let* $\mathsf{A}$ *be a Grothendieck abelian category.*

- *Let* $\mathsf{K}(\mathsf{A})^{\leq 0}$ *be the full sub-*$\infty$*-category consisting of objects represented by cochain complexes* $M$ *with* $H^n(M) \simeq 0$ *for* $n > 0$*.*
- *Let* $\mathsf{K}(\mathsf{A})^{\geq 0}$ *be the full sub-*$\infty$*-category consisting of objects represented by cochain complexes* $M$ *with* $M^n \simeq 0$ *for* $n < 0$ *such that* $M^n$ *is* injective *for any* $n$*.*

*Then* $(\mathsf{K}(\mathsf{A})^{\leq 0}, \mathsf{K}(\mathsf{A})^{\geq 0})$ *determines a t-structure on* $\mathsf{K}(\mathsf{A})$*.*

**Exercise 3.19.** *Identify* $\mathsf{K}(\mathsf{A})^{\geq 0}$ *with the essential image of the fully faithful functor*

$$\mathsf{D}(\mathsf{A})^{\geq 0} \to \mathsf{D}(\mathsf{A}) \to \mathsf{K}(\mathsf{A})$$

*where the first functor is the right adjoint in* (2.1)*.*

**Exercise 3.20.** *Show that both functors in* $\mathsf{K}(\mathsf{A}) \rightleftarrows \mathsf{D}(\mathsf{A})$ *are t-exact, and they induce equivalences between the hearts.*

**Exercise 3.21.** *Show that the t-structure on* $\mathsf{K}(\mathsf{A})$ *is right complete and compatible with filtered colimits, but is not left separated.*

## Appendix A. $\breve{\mathsf{D}}(\mathsf{A})$

**Construction A.1.** *Let* $\mathsf{A}$ *be a Grothendieck abelian category. Define the* ***unseparated derived*** $\boldsymbol{\infty}$***-category*** *of* $\mathsf{A}$ *to be*

$$\breve{\mathsf{D}}(\mathsf{A}) := \mathsf{N}^{\mathsf{dg}}(\mathbb{Ch}(\mathsf{A}_{\mathsf{inj}})) \simeq \mathsf{K}(\mathsf{A}_{\mathsf{inj}}).$$

*There is a t-structure on* $\breve{\mathsf{D}}(\mathsf{A})$ *defined similarly as that on* $\mathsf{K}(\mathsf{A})$ *such that*

- *The heart is canonically identified with* $\mathsf{A}$*;*
- *This t-structure is accessible, right complete and compatible with filtered colimits;*
- *This t-structure is (left)* ***anti-complete****.*

**Remark A.2.** *Roughly speaking, being anti-complete means the t-structure is "orthogonal" to complete ones. In fact, there is an essentially unique colimit-preserving t-exact functor*

$$\breve{\mathsf{D}}(\mathsf{A}) \to \mathsf{D}(\mathsf{A})$$

*that restricts to the identity functor on the hearts. Moreover, this functor exhibits* $\breve{\mathsf{D}}(\mathsf{A})$ *as the (left)* ***anti-completion*** *of* $\mathsf{D}(\mathsf{A})$*.*

**Example A.3.** *For Noetherian commutative ring* $R$*,*

$$\breve{\mathsf{D}}(\mathsf{Mod}_R^\heartsuit) \simeq \mathsf{Ind}(\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}_{R,\mathsf{fg}}^\heartsuit)),$$

*where* $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}_{R,\mathsf{fg}}^\heartsuit)$ *is the full sub-*$\infty$*-category of* $\mathsf{D}(\mathsf{Mod}_R^\heartsuit)$ *consisting of complexes with bounded finite generated cohomologies.*

A.4. **Suggested readings.** SAG.C.

## Appendix B. dg-category vs. $\mathbb{H}Z$-linear stable $\infty$-categories

B.1. The homotopy category $\mathsf{h}\mathbb{C}$ of any dg-category $\mathbb{C}$ has a canonical *candidate* for triangulated structures. If this candidate is indeed a triangulated structure, we say $\mathbb{C}$ is **pretriangulated**. The following notions are essentially equivalent:

- pretriangulated dg-categories over a commutative ring $R$;
- $\mathbb{H}R$-linear stable $\infty$-categories.

B.2. **Suggested readings.** [Coh13].

## References

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