NOTES FOR ALGEBRAIC GEOMETRY 1

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Contents

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0. Introduction: why schemes?

0.1. **Algebraic sets.** Before scheme theory, algebraic geometry focused on *algebraic sets*.

Definition 0.1.1. Let k be an algebraically closed field.

- The **Zariski topology** on the affine space $\mathbb{A}^n_k$ is the topology with a base consisting of all the subsets that are equal to the non-vanishing locus $U(f)$ of some polynomial $f \in k[x_1, \cdots, x_n]$.
- An **embedded affine algebraic set**[1] in $\mathbb{A}^n_k$ is a closed subspace for the Zariski topology.
- An **embedded quasi-affine algebraic set** is a Zariski open subset of an embedded affine algebraic set.

Example 0.1.2. Any finite subset of $\mathbb{A}^n_k$ is an embedded affine algebraic set.

Example 0.1.3. $\mathbb{Z}$ is not an embedded affine algebraic set in $\mathbb{A}^1_{\mathbb{C}}$.

Similarly one can define *embedded (quasi)-projective algebraic set* using *homogeneous* polynomials and the projective space $\mathbb{P}^n_k$.

There are tons of shortcomings in the above definition. An obvious one is that the notions of *embedded* algebraic sets are not *intrinsic*.

Example 0.1.4. The embedded affine algebraic sets $\mathbb{A}^1_k \subseteq \mathbb{A}^1_k$ and $\mathbb{A}^1_k \subseteq \mathbb{A}^2_k$ should be viewed as the same algebraic sets.

Notation 0.1.5. To remedy this, we need some notations.

- For an ideal $I \subseteq k[x_1, \cdots, x_n]$, let $Z(I) \subseteq \mathbb{A}^n_k$ be the locus of common zeros of polynomials in I.
- For a Zariski closed subset $X \subseteq \mathbb{A}^n_k$, let $I(X) \subseteq k[x_1, \cdots, x_n]$ be the ideal of all polynomials vanishing on X.

Recall an ideal I is called *radical* if $I = \sqrt{I}$.

Theorem 0.1.6 (Hilbert Nullstellensatz)**.** *We have a bijection:*

$$\begin{array}{rcl} \{\textit{radical ideals of } k[x_1, \cdots, x_n]\} & \longleftrightarrow & \{\textit{Zariski closed subsets of } \mathbb{A}^n_k\} \\ I & \longrightarrow & Z(I) \\ I(X) & \longleftarrow & X. \end{array}$$

Part of the theorem says the set of points of $\mathbb{A}^n_k$ is in bijection with the set of maximal ideals of $k[x_1, \cdots, x_n]$. As a corollary, $Z(I)$ is in bijection with the set of maximal ideals containing I. The latter can be further identified with maximal ideals of $R \coloneqq k[x_1, \cdots, x_n]/I$.

Note that I is radical iff R is *reduced*, i.e., contains no nilpotent elements. This justifies the following definition.

Definition 0.1.7. An **affine algebraic k-set** is a *maximal spectrum* $\mathsf{Spm}\, R$ (= sets of maximal ideals) of a *finitely generated* (commutative unital) *reduced* k-algebra R. We equip it with the **Zariski topology** with a base of open subsets given by

$$U(f) \coloneqq \{\mathfrak{m} \in \mathsf{Spm}\, R \mid f \notin \mathfrak{m}\},\ f \in R.$$

[1]Some people use the word *variety*, while some people reserve it for *irreducible* algebraic sets.

Example 0.1.8. $\mathsf{Spm}\, k[x] \simeq \mathbb{A}^1_k$.

We have the following *duality* between algebra and geometry.

Algbera	Geometry
finitely generated reduced k-algebra R	affine algebraic k-set X
maximal ideals $\mathfrak{m} \subseteq R$	points $x \in X$
elements $f \in R$	functions $\phi : X \to \mathbb{A}^1_k$
radical ideals $I \subseteq R$	Zariski closed subsets $Z \subseteq X$

Here an element $f \in R$ corresponds to the function

$$\phi : \mathsf{Spm}\, R \to k,\ \mathfrak{m} \mapsto \underline{f}$$

sending a maximal ideal $\mathfrak{m}$ to the image $\underline{f}$ of f in the *residue field* of $\mathfrak{m}$, which is canonically identified with the underlying set of $\mathbb{A}^1_k$ via the composition $k \to R \to R/\mathfrak{m}$.

The word *duality* means the correspondence $R \leftrightarrow X$ is *contravariant.* Indeed, given a homomorphism $f : R' \to R$, we obtain a *continuous* map

$$\mathsf{Spm}\, R \to \mathsf{Spm}\, R',\ \mathfrak{m} \mapsto f^{-1}(\mathfrak{m}).$$

Note however that not all continuous maps $\mathsf{Spm}\, R \to \mathsf{Spm}\, R'$ are obtained in this way, nor is R determined by the topological space $\mathsf{Spm}\, R$.

Exercise 0.1.9. Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology. Find those bijections coming from a homomorphism $k[x] \to k[x]$.

This motivates the following definition.

Definition 0.1.10. A **morphism** from $\mathsf{Spm}\, R$ to $\mathsf{Spm}\, R'$ is a continuous map coming from a homomorphism $R' \to R$.

Then one can define general algebraic k-sets by gluing affine algebraic k-sets using morphisms, just like how people define *structured* manifolds as glued from *structured* Euclidean spaces using maps preserving the addiontal structrues.

0.2. **Shortcomings.** The theory of algebraic k-sets provides a bridge between algebra and geometry. In particular, one can use topological/geometric methods, including various cohomology theories, to study *finitely generated reduced k-algebras.* However, these adjectives are non-necessary restrictions to this bridge.

First, number theory studies number fields and their rings of integers, such as $\mathbb{Q}$ and $\mathbb{Z}$. It is desirable to have geometric objects corresponding to them. Hence we would like to consider general commutative rings rather than k-algebras. Then one immediately realizes the maximal spectra Spm are not enough.

Example 0.2.1. The map $\mathbb{Z} \to \mathbb{Q}$ does not induce a map from $\mathsf{Spm}\, \mathbb{Q}$ to $\mathsf{Spm}\, \mathbb{Z}$. Namely, the inverse image of $(0) \subseteq \mathbb{Q}$ in $\mathbb{Z}$ is a non-maximal prime ideal.

This suggests for general algebra R, we should consider its *prime spectrum*, denoted by $\mathsf{Spec}\, R$, rather than just its maximal spectrum.

Second, even in the study of finitely generated algebras, one naturally encounters non-finitely generated ones.

Example 0.2.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal of a finitely generated algebra. The localization $R_{\mathfrak{p}}$ and its completion $\widehat{R}_{\mathfrak{p}}$ are in general not finitely generated.

Of course, one can restrict their attentions to *Noetherian* rings (and live a happy life). But let me object the false feeling that all natural rings one care are Noetherian.

Example 0.2.3. Noetherian rings are not stable under tensor products: $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not Noetherian.

Example 0.2.4. The ring of adeles of $\mathbb{Q}$ is not Noetherian.

Example 0.2.5. Natural moduli problems about Noetherian objects can fail to be Noetherian. My favorite ones includes: the space of formal connections, the space (stack) of formal group laws...

In short, there are important applications of non-Noetherian rings, and this course will deal with the latter.

Finally, we want to remove the restriction about reducedness.

Example 0.2.6. Reduced rings are not stable under tensor products: $k[x] \otimes_{k[x,y]} k[x,y]/(y-x^2)$ is not reduced. Geometrically, this means $Z(y)$ and $Z(y-x^2)$ do not intersect transversally inside $\mathbb{A}^2_k$.

One may notice that without reducedness, we should accordingly consider all ideals rather than just *radical* ideals, but then the construction $I \mapsto Z(I)$ would not be bijective. Indeed, ideals with the same nilpotent radical would give the same *topological subspace* of $\mathsf{Spec}\, R$.

But *this is a feature rather than a bug.* In Example 0.2.6, the ideal $(y, y-x^2) = (x^2, y)$ is not radical, and it carries geometric meanings that cannot be seen from its nilpotent radical (x,y). Namely, $f \in (x,y)$ iff $f(0,0) = 0$, while $f \in (x^2, y)$ iff $f(0,0) = \partial_x f(0,0) = 0$. Roughly speaking, this suggests that $(y, y-x^2)$ remembers that the curves $Z(y)$ and $Z(y-x^2)$ are tangent to each other at the point $(0,0) \in \mathbb{A}^2_k$, and the tangent vector is $\partial_x|_{(0,0)}$. Also note that the length of $k[x,y]/(y,y-x^2)$ is equal to 2, which is the number of intersection points predicted by the Bézout's theorem.

In summary, on the algebra side, we should consider *all* commutative rings. On the geometric side, the corresponding notion is called *affine schemes.* Our first task in this course is to develop the following duality:

Algebra	Geometry
commutative rings R	affine schemes X
prime ideals $\mathfrak{p} \subseteq R$	points $x \in X$
elements $f \in R$	functions $X \to \mathbb{A}^1_{\mathbb{Z}}$
ideals $I \subseteq R$	closed subschemes $Z \subseteq X$.

0.3. **Schemes as structured spaces.** In theory, one can *define* a morphism between affine schemes to be a continuous map coming from a ring homomorphism. Then one can define general *schemes* by gluing affine schemes using such morphisms. This mimics the definition of differentiable manifold in the sense that a scheme would be a topological space equipped with a *maximal* affine atlas.

In practice, the above approach is awkward to work with. We prefer a more efficient way to encode the additional structure on the topological space underlying a scheme. One extremely simple but powerful way to achieve this, maybe discovered by Serre and popularized by Grothendieck, is the notion of *sheaves* on topological spaces. Roughly speaking, a sheaf $\mathcal{F}$ on X is an *contravariant* assignment

$$U \mapsto \mathcal{F}(U)$$

sending open subsets $U \subseteq X$ to certain structures (e.g. sets, groups, rings) $\mathcal{F}(U)$, such that a certain gluing condition is satisfied. Here contravariancy means that for $U \subseteq V$, we should provide a map $\mathcal{F}(V) \to \mathcal{F}(U)$ preserving the prescribed structures.

Example 0.3.1. Let X be any topological space. The assignment

$$U \mapsto C(U, \mathbb{R})$$

sending $U \subseteq X$ to the ring of continuous functions on U would be a sheaf of commutative rings on X.

Similarly, for a smooth manifold X, $U \mapsto C^\infty(U, \mathbb{R})$ would be a sheaf of commutative rings on X. This motivates us to define:

Pre-Definition 0.3.2. A **scheme** is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X$ such that locally it is isomorphic to an affine scheme.

Here for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ should be the ring of *algebraic* functions on U, but we have not defined the latter notion yet. Nevertheless, for an *affine* scheme $X \simeq \mathsf{Spec}\, R$, the previous discussion suggests we should have $\mathcal{O}_X(X) \simeq R$. As we shall see in future lectures, we can bootstrap from this to get the definition of the entire sheaf $\mathcal{O}_X$.

The goal of this course is to define schemes and study their basic properties.

Part I. (Pre)sheaves

1. Definition of (pre)sheaves

1.1. Presheaves.

Definition 1.1.1. Let X be a topological space and $(U(X), \subseteq)$ be the partially ordered set of open subsets of X. We define the **category $\mathfrak{U}(X)$ of open subsets** in X to be the category associated to the partially ordered set $(U(X), \subseteq)$.

The category $\mathfrak{U}(X)$ can be explicitly described as follows:

- An object in $\mathfrak{U}(X)$ is an open subset $U \subseteq X$.
- If $U \subseteq V$, then $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is a singleton; otherwise $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is empty.
- The identify morphisms and composition laws are defined in the unique way.

Definition 1.1.2. Let X be a topological space and $\mathcal{C}$ be a category.

- A **$\mathcal{C}$-valued presheaf on** X is a functor $\mathcal{F} : \mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C}$.
- A **morphism** $\mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued presheaves is a natural transformation between these functors.

Let Set be the category of sets. By definition, a **presheaf $\mathcal{F}$ of sets**, i.e., a Set-valued presheaf, on X consists of the following data:

- For any open subset $U \subseteq X$, we have a set $\mathcal{F}(U)$, which is called the **set of sections** of $\mathcal{F}$ on U.
- For $U \subseteq V$, we have a map
$$\mathcal{F}(V) \to \mathcal{F}(U),\ s \mapsto s|_U$$
which is called the **restriction map**.

These data should satisfy the following condition:

- For any open subset $U \subseteq X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
- For $U \subseteq V \subseteq W$, the restriction maps make the following diagram commute
$$\begin{tikzcd} & \mathcal{F}(V) \arrow[dr] & \\ \mathcal{F}(U) \arrow[ur] \arrow[rr] & & \mathcal{F}(W). \end{tikzcd}$$

Let $\mathcal{F}$ adn $\mathcal{F}'$ be presheaves of sets on X. By definition, a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ consists of the following data:

- For any open subset $U \subseteq X$, we have a map $\phi_U : \mathcal{F}(U) \to \mathcal{F}(U)'$.

These data should satisfy the following condition:

- For $U \subseteq V$, the following diagram commute
$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d] & \mathcal{F}'(V) \arrow[d] \\ \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{F}'(U), \end{tikzcd}$$
where the vertical maps are restriction maps.

Similarly one can explicitly describe the notion of presheaves of abelian groups (k-vector spaces, commutative algebras) and morphisms between them.

Example 1.1.3. Let X be a topological space and $\mathcal{C}$ be a category. For any object $A \in \mathcal{C}$, the constant functor

$$\mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C},\ U \mapsto A,\ f \mapsto \mathsf{id}_A$$

defines a $\mathcal{C}$-valued presheaf on X, which is called the **constant presheaf associated to** A. It is often denoted by $\underline{A}$.

Example 1.1.4. Let X be a topological space and $E \to X$ be a topological space over it. We define a presheaf Sect_E of sets as follows.

- For any $U \subseteq X$,
$$\mathsf{Sect}_E(U) := \mathsf{Hom}_X(U, E)$$
is the set of countinuous maps $U \to E$ defined over X, a.k.a. sections of E over U.
- For $U \subseteq V$, the restriction map $\mathsf{Sect}_E(V) \to \mathsf{Sect}_E(U)$ sends a section $s : V \to E$ to its restriction $s|_U : U \to E$.

We call it the **presheaf of sections for** $E \to X$.

Example 1.1.5. If $E \to X$ is a real vector bundle, we can naturally upgrade Sect_E to be a presheaf of real vector spaces on X.

Example 1.1.6. Consider the constant real line bundle $\mathbb{R} \times X$ on X. Note that $\mathsf{Sect}_{\mathbb{R}\times X}(U)$ can be identified with the set of continuous functions on U. It follows that we can upgrade $\mathsf{Sect}_{\mathbb{R}\times X}$ to be a presheaf of $\mathbb{R}$-algebra on X.

1.2. **Sheaves of sets.** Roughly speaking, a sheaf is a presheaf whose sections on small open subsets can be uniquely glued to sections on larger ones.

Definition 1.2.1. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. We say $\mathcal{F}$ is a **sheaf** if it satisfies the following condition:

(*) For any open covering $U = \bigcup_{i\in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for any } i, j \in I,$$
there is a *unique* section $s \in \mathcal{F}(U)$ such that
$$s_i = s|_U \text{ for any } i \in I.$$

Remark 1.2.2. Using the language of category theory, the sheaf condition is equivalent to the following condition:

- For any open covering $U = \bigcup_{i\in I} U_i$, the diagram
$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$
is an *equalizer* diagram. Here the first map is
$$s \mapsto (s|_{U_i})_{i\in I}$$
the other two maps are
$$(s_i)_{i\in I} \mapsto (s_i|_{U_i\cap U_j})_{(i,j)\in I^2}$$
and
$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

In particular, the map $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is an injection.

Remark 1.2.3. For $U = \varnothing$ and $I = \varnothing$, the sheaf condition says there is a unique section $s \in \mathcal{F}(\varnothing)$ subject to no property. In other words, the above definition forces $\mathcal{F}(\varnothing)$ to be a singleton.

Example 1.2.4. Let X be a topological space. The constant presheaf $\underline{A}$ associated to a set A is in general not a sheaf. Indeed, $\underline{A}(\varnothing)$ is A rather than a singleton.

We provide another reason for readers uncomfortable with the above. For a sheaf $\mathcal{F}$ and *disjoint* open subsets U_1 and U_2, the sheaf condition implies

$$\mathcal{F}(U_1 \sqcup U_2) \simeq \mathcal{F}(U_1) \times \mathcal{F}(U_2).$$

But in general A and $A \times A$ are not isomorphic.

Example 1.2.5. Let $E \to X$ be a continuous map between topological spaces. The presheaf Sect_E of sections on X is a sheaf. Indeed, this follows from the fact that continuous maps can be glued.

Example 1.2.6. Let $\{*\}$ be a 1-point space. Then a sheaf $\mathcal{F}$ of sets on $\{*\}$ is uniquely determined by the set $\mathcal{F}(\{*\})$ of global sectoins. We often abuse the notations and use a set A to denote the sheaf on $\{*\}$ whose set of global sections is A.

Exercise 1.2.7. Let X be a topological space and $\mathfrak{B}$ be a base of open subsets of X.

(1) Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X and $\alpha : \mathcal{F}|_{\mathfrak{B}} \to \mathcal{F}'|_{\mathfrak{B}}$ be a natural transformation between their restrictions on the full subcategory $\mathfrak{B}^{\mathsf{op}} \subseteq \mathfrak{U}(X)^{\mathsf{op}}$. Show that α can be uniquely extended to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$.
(2) Show that for presheaves, similar claims about existence and uniqueness are both false in general.

The above exercise says sheaves are determined by their restrictions on a topological base. A natural question is, given a functor $\mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$, under what conditions can we extend it to a sheaf $\mathfrak{U}(X) \to \mathsf{Set}$? This question is relevant to us because the Zariski topology of $\mathsf{Spec}\, R$ is defined using a base consisting of open subsets that can be easily described:

$$U(f) := \left\{ \mathfrak{p} \in \mathsf{Spec}\, R \,|\, f \notin \mathfrak{p} \right\} \simeq \mathsf{Spec}\, R_f.$$

It would be convenient if we can recover a sheaf $\mathcal{F}$ on $\mathsf{Spec}\, R$ from its values on these open subsets. For instance, we wonder whether the contravariant functor

$$U(f) \mapsto R_f$$

can be extended to a sheaf of commutative rings. If yes, we would obtain the sheaf $\mathcal{O}_X$ of algebraic functions desired in the introduction. The following construction gives a positive answer to this question.

Construction 1.2.8. Let X be a topological space and $\mathfrak{B}$ be a base of open subsets of X. For a functor $\mathcal{F} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$ and $U \in \mathfrak{U}(X)$, define

$$\mathcal{F}'(U) := \lim_{V \in \mathfrak{B}^{\mathsf{op}},\, V \subseteq U} \mathcal{F}(V).$$

In other words, an element in $s' \in \mathcal{F}'(U)$ is a collection of elements $s_V \in \mathcal{F}(V)$ for all open subsets $V \subseteq U$ contained in $\mathfrak{B}$ such that for $V_1 \subseteq V_2 \subseteq U$ with $V_1, V_2 \in \mathfrak{B}$, the map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$ sends s_{V_2} to s_{V_1}. This construction is clearly functorial in U, i.e., for $U_1 \subseteq U_2$, we have a natural map $\mathcal{F}'(U_2) \to \mathcal{F}'(U_1)$. One can check this defines a functor

$$\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$$

equipped with a canonical isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$. In other words, we have extended $\mathcal{F}$ to a *presheaf* $\mathcal{F}'$ of sets on X.

Remark 1.2.9. Using the language in category theory, the functor $\mathcal{F}'$ is the *right Kan extension* of $\mathcal{F}$ along the embedding $\mathfrak{B}^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Proposition 1.2.10. *In above, $\mathcal{F}'$ is a sheaf iff $\mathcal{F}$ satisfies the following condition:*

(**) *For any open covering $U = \bigcup_{i \in I} U_i$ in $\mathfrak{B}$, and any collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that*

$$s_i|_V = s_j|_V \text{ for any } i, j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$

there is a unique *section $s \in \mathcal{F}(U)$ such that*

$$s_i = s|_{U_i} \text{ for any } i \in I.$$

Proof. The "only if" statement follows from the sheaf condition on $\mathcal{F}'$ and the isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$.

For the "if" statement, we verify the sheaf condition on $\mathcal{F}'$ directly. Let $U = \bigcup_{i \in I} U_i$ be an open covering, and $s'_i \in \mathcal{F}'(U_i)$ be a collection of sections such that

$$s'_i|_{U_i \cap U_j} = s'_j|_{U_i \cap U_j} \text{ for any } i, j \in I.$$

By Construction 1.2.8, each s'_i corresponds to a collection $s_{i,V} \in \mathcal{F}(V)$ for $V \subseteq U_i$, $V \in \mathfrak{B}$ that is compatible with restrictions.

We need to show there is a unique section $s' \in \mathcal{F}'(U)$ such that $s'|_{U_i} = s'_i$.

We first deal with the existence. For any $V \subseteq U$ with $V \in \mathfrak{B}$, since $\mathfrak{B}$ is a base, we can choose an open covering $V = \bigcup_{j \in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. In other words, we can choose a map $f : J \to I$ such that $V_j \subseteq U_i$.

Consider the collection of sections

$$t_{j,V} \coloneqq s_{f(j),V_j} \in \mathcal{F}(V_j),\ j \in J. \tag{1.1}$$

One can check it does not depend on the choice of f and they satisfy the assumption in (**). Hence there is a unique section $s'_V \in \mathcal{F}(V)$ such that $s'_V|_{V_j} = s_{f(j),V_j}$.

One can check the obtained section s'_V does not depend on the open covering $V = \bigcup_{j \in J} V_j$ and the collections (s'_V), $V \subseteq U$, $V \in \mathfrak{B}$ is compatible with restrictions. Hence by Construction 1.2.8, it corresponds to an element $s' \in \mathcal{F}'(U)$. One can check that $s'|_{U_i} = s'_i$. This proves the claim about uniqueness.

It remains to prove the statement about uniqueness. Suppose there are two such sections s', s'' such that

$$s'|_{U_i} = s''|_{U_i} = s'_i \tag{1.2}$$

By Construction 1.2.8, they correspond to two collections $s'_V, s''_V \in \mathcal{F}(V)$ for $V \subseteq U$, $V \in \mathfrak{B}$. We only need to show $s'_V = s''_V$.

Note that if V is contained in some U_i, then (1.2) implies

$$s'_V = s''_V = s_{i,V}. \tag{1.3}$$

Now for general open subset $V \subseteq U$, $V \in \mathfrak{B}$, as before, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. Consider the collection of sections (1.1). By (1.3) (applied to each V_j), we have

$$s'_V|_{V_j} = s''_V|_{V_j} = t_{j,V}.$$

Hence by (**), we must have $s'_V = s''_V$ as desired.

□

1.3. $\mathcal{C}$-valued sheaves.

Definition 1.3.1. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on a topological space X. We say $\mathcal{F}$ is a **$\mathcal{C}$-valued sheaf** if for any testing object $c \in \mathcal{C}$, the functor

$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$

is a sheaf of sets.

Remark 1.3.2. By Yoneda's lemma and Remark 1.2.2, $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf iff for any open covering $U = \bigcup_{i\in I} U_i$, the canonical diagram

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is an *equalizer* diagram in $\mathcal{C}$. Here the first morphism is given by restrictions along $U_i \subseteq U$, while the other two morphisms are given respectively by restrictions along $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$. In particular, the morphism

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

is a *monomorphism*[2].

As a corollary of the remark, we obtain:

Corollary 1.3.3. *Let $\mathcal{F}$ be a presheaf of abelian groups. Then $\mathcal{F}$ is a sheaf of abelian groups iff its underlying presheaf of sets $\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Ab} \to \mathsf{Set}$ is a sheaf of sets. Here the functor $\mathsf{Ab} \to \mathsf{Set}$ sends an abelian group to its underlying set.*

Exercise 1.3.4. Let $\mathcal{F}$ be a presheaf of abelian groups. Show that $\mathcal{F}$ is a sheaf of abelian groups iff for any open covering $U = \bigcup_{i\in I} U_i$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \to \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact. Here the second map is

$$s \mapsto (s|_{U_i})_{i\in I},$$

and the third map is

$$(s_i)_{i\in I} \mapsto (s_j|_{U_i\cap U_j} - s_i|_{U_i\cap U_j})_{(i,j)\in I^2}.$$

Now suppose $\mathcal{F}$ is a sheaf, can you further extend this exact sequence to the right?

Remark 1.3.5. Let $\mathcal{C}$ be a category that admits small limits. Then Construction 1.2.8 and Proposition 1.2.10 can be generalized to $\mathcal{C}$-valued (pre)sheaves with condition (**) replaced by

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[2]This means for any testing object $c \in \mathcal{C}$, the functor $\mathsf{Hom}_{\mathcal{C}}(c,-)$ sends this morphism to an injection between sets.

- For any open covering $U = \bigcup_{i \in I} U_i$ in $\mathfrak{B}$, any object $c \in \mathcal{C}$, and any collection of elements $s_i \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U_i))$, $i \in I$ such that
$$s_i|_V = s_j|_V \text{ for any } i, j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$
there is a *unique* element $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ such that
$$s_i = s|_{U_i} \text{ for any } i \in I.$$

In above $s|_V$ means the post-composition of $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ with the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.

Note however for $\mathcal{C} = \mathsf{Ab}$, we can keep condition (**) *as it is*, because the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ detects limits.

2. Stalks

2.1. Definition.

Definition 2.1.1. Let X be a topological space and $\mathcal{F}$ be a presheaf of sets on X. For a point $x \in X$, let $\mathfrak{U}(X, x) \subseteq \mathfrak{U}(X)$ be the full subcategory of open neighborhoods of x inside X. The **stalk of $\mathcal{F}$ at x** is

$$\mathcal{F}_x := \underset{U \in \mathfrak{U}(X,x)^{\mathsf{op}}}{\mathsf{colim}} \mathcal{F}(U). \tag{2.1}$$

For a given section $s \in \mathcal{F}(U)$, the **germ of s at x**, denoted by s_x, is the image of s under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$.

Note that $\mathfrak{U}(X, x)^{\mathsf{op}}$ is the category associated to the *direct set*[3] $(U(X, x), \subseteq)$ of open neighborboods of x inside X. Hence the above colimit is a *direct colimit*[4]. It follows that $\mathcal{F}_x$ can be explicitly described as the quotient

$$\Big(\coprod_{U \in U(X,x)} \mathcal{F}(U) \Big) / \sim, \tag{2.2}$$

of the disjoint union of all $\mathcal{F}(U)$, $U \in U(X, x)$ by an equivalence relation $\sim$. Here two sections $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(U')$ are equivalent iff there exists $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Using this description, the germ s_x of a section $s \in \mathcal{F}(U)$ is just the equivalence class to which it belongs.

Remark 2.1.2. In general, let $\mathcal{C}$ be a category that admits direct colimits and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf. We can define the stalk of $\mathcal{F}$ at x using the same formula (2.1). Note that this construction is functorial in $\mathcal{F}$.

In particular, for a presheaf $\mathcal{F}$ of abelian groups, we can define its stalk $\mathcal{F}_x$, which is an abelian group. It is easy to see the underlying set $\mathcal{F}_x$ is given by (2.2) and the group structure is given by the formula

$$s_x + s'_x = (s|_V + s'|_V)_x,\ s \in \mathcal{F}(U), s' \in \mathcal{F}(U'), V \subseteq U \cap U'.$$

[3]A direct set is a partially ordered set $(I, \leq)$ such that any finite subset of I admits an upper bound in I.

[4]Some people use the word *direct limit*. I strongly object this terminology.

2.2. **Sheaves and stalks.** The following result says a section of a *sheaf* is determined by its germs.

Lemma 2.2.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X. Then for any open subset $U \subseteq X$, the map*

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x,\ s \mapsto (s_x)_{x \in U} \tag{2.3}$$

is injective. Moreover, a collection of elements $s(x) \in \mathcal{F}_x$, $x \in U$ is contained in the image of this map iff it satisfies the following condition

(***) *For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.*

Proof. We first show the map (2.3) is injective. Let $s, s' \in \mathcal{F}(U)$ such that all their germs are equal. By definition, for any $x \in U$, there exists $V \subseteq U$ such that $s|_V = s'|_V$. In particular, we can find an open covering $U = \bigcup_{i \in I} U_i$ such that $s|_{U_i} = s'|_{U_i}$. But this implies $s = s'$ because the sheaf condition implies

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

is injective.

It is obvious that any element in the image of (2.3) satisfies condition (***). To prove the converse, let $s(x) \in \mathcal{F}_x$, $x \in U$ be a collection of elements satisfying condition (***). By assumption, we can find an open covering $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that for any $x \in U_i$, we have

$$t(x) = (s_i)_x. \tag{2.4}$$

In particular, the germs of $s_i|_{U_i \cap U_j}$ and $s_j|_{U_i \cap U_j}$ are equal. Applying the injectivity of (2.3) to $U_i \cap U_j$, we obtain

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Hence by the sheaf condition, we can find a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. For any $x \in U$, pick $i \in I$ such that $x \in U_i$, we have

$$s_x = (s_i)_x = t(x),$$

where the first equality is due to the definition of stalks, while the second one is (2.4). In particular, $s(x) \in \mathcal{F}_x$, $x \in U$ is the image of s under the map (2.3). □

Remark 2.2.2. Similar claim for presheaves is false in general. Namely, for $U = X = \varnothing$, the empty product $\prod_{x \in \varnothing} \mathcal{F}_x$ is a singleton, while $\mathcal{F}(\varnothing)$ can be any set.

Corollary 2.2.3. *If $\alpha, \beta : \mathcal{F} \to \mathcal{F}'$ are morphisms between sheaves of sets such that $\alpha_x = \beta_x$ for any $x \in X$, then $\alpha = \beta$.*

Proposition 2.2.4. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism between sheaves of sets on a topological space. Then α is an isomorphism iff for any $x \in X$, $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a bijection.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose α_x is a bijection for any $x \in X$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\alpha_U"] & \prod_{x\in U} \mathcal{F}_x \arrow[d, "(\alpha_x)_{x\in X}", "\simeq"'] \\ \mathcal{F}'(U) \arrow[r] & \prod_{x\in U} \mathcal{F}'_x. \end{tikzcd}$$

By Lemma 2.2.1, the horizontal maps are injective, hence so is α_U.

It remains to show α_U is surjective. Let $s' \in \mathcal{F}'(U)$ be a section, we will construct a section $s \in \mathcal{F}(U)$ mapping to it by α_U.

For any point $x \in U$, since α_x is bijective, we can find an open subset $V \subseteq X$ and a section $t \in \mathcal{F}(V)$ such that $\alpha_x(t_x) = s'_x$. By definition, $\alpha_x(t_x) = \alpha_V(t)_x$. Hence the germs of $\alpha_V(t)$ and s' at x are equal. By definition, there exists an open neightborhood W of x inside $U \cap V$ such that $\alpha_V(t)|_W = s'|_W$. Note that we also have $\alpha_V(t)|_W = \alpha_W(t|_W)$.

It follows that we can find an open covering $U = \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. In particular, we have

$$\alpha_{U_i\cap U_j}(s_i|_{U_i\cap U_j}) = \alpha_{U_i\cap U_j}(s_j|_{U_i\cap U_j}) = s|_{U_i\cap U_j}.$$

Since we have already shown $\alpha_{U_i\cap U_j}$ is injective, we obtain $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Hence by the sheaf condition for $\mathcal{F}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Using the sheaf condition for $\mathcal{F}'$, it is easy to see $\alpha_U(s) = s'$ as desired.

□

The above results imply that a *morphism* between sheaves are determined by the induced maps between the stalks. However, a sheaf itself is *not* determined by its stalks.

Exercise 2.2.5. Let X be a connected topological space and $E \to X$ and $E' \to X$ be two covering spaces of the same degree. Show that the sheaves Sect_E and $\mathsf{Sect}_{E'}$ on X have isomorphic stalks for any point $x \in X$, but they are not isomorphic unless there exists a homeomorphism $E \simeq E'$ defined over X.

Remark 2.2.6. Let $\mathcal{C}$ be a *compactly generated* category[5]. Lemma 2.2.1 and Proposition 2.2.4 can be generalized to $\mathcal{C}$-valued sheaves. In other words:

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$, the morphism $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is a monomorphism.
- A morphism $\alpha : \mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued sheaves is an isomorphism iff $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for any $x \in X$.

These statements can be deduced from the special case for Set with the help of the following two observations:

- A morphism $d \to d'$ in $\mathcal{C}$ is a monomorphism (resp. isomorphism) iff for any *compact* object $c \in \mathcal{C}$, the map $\mathsf{Hom}_{\mathcal{C}}(c,d) \to \mathsf{Hom}_{\mathcal{C}}(c,d')$ is an injection (resp. bijection).

[5]An object c in a (locally small) category $\mathcal{C}$ is compact iff $\mathsf{Hom}_{\mathcal{C}}(c,-)$ preserves small filtered colimits. We say $\mathcal{C}$ is compactly generated if it admits small colimits and any object in $\mathcal{C}$ is isomorphic to a small filtered colimit of compact objects. It is known that compactly generated categories also admit small limits.

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$ and any *compact* object $c \in \mathcal{C}$, the stalk of the Set-valued sheaf
$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$
at $x \in X$ is canonically isomorphic to $\mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}_x)$.

The details are left to the curious readers.

2.3. **Skyscapters.**

Definition 2.3.1. Let X be a topological space and $x \in X$ be a point. For any set A, we can define a presheaf $\delta_{x,A}$ of sets as follows.

- For an open subset $U \subseteq X$,
  - if $x \in U$, define $\delta_{x,A}(U) \coloneqq A$;
  - if $x \notin U$, define $\delta_{x,A}(U) \coloneqq \{*\}$.
- For open subsets $U \subseteq V$,
  - if $x \in U$ (and therefore $x \in V$), define the restriction map $\delta_{x,A}(U)$ to be id_A;
  - if $x \notin U$, define the restriction map to be the unique map $\delta_{x,A}(V) \to \delta_{x,A}(U) = \{*\}$.

One can check this indeed defines a presheaf $\delta_{x,A}$. We call the the **skyscapter sheaf** at x with value A.

Exercise 2.3.2. The presheaf $\delta_{x,A}$ is indeed a sheaf.

Lemma 2.3.3. *Let X be a topological space, $x \in X$ be a point and A be a set. The stalk of $\delta_{x,A}$ at a point $y \in X$ is canonically bijective to*

- *the set A if y is contained in $\overline{\{x\}}$, the closure of $\{x\}$ inside X;*
- *the singleton $\{*\}$ otherwise.*

Proof. If $y \in \overline{\{x\}}$, then any open neighborhood of y contains x. It follows that
$$(\delta_{x,A})_y \coloneqq \operatorname*{colim}_{U \in \mathfrak{U}(X,y)^{\mathsf{op}}} \delta_{x,A}(U) \simeq \operatorname*{colim}_{U \in \mathfrak{U}(X,y)^{\mathsf{op}}} A$$
is a direct colimit of the constant diagram with values A. This implies $(\delta_{x,A})_y \simeq A$.

If $y \notin \overline{\{x\}}$, then there exists an open neighborhood V of y such that $x \notin V$. Note that $\mathfrak{U}(V,y)^{\mathsf{op}} \subseteq \mathfrak{U}(X,y)^{\mathsf{op}}$ is (co)final. If follows that
$$(\delta_{x,A})_y \coloneqq \operatorname*{colim}_{U \in \mathfrak{U}(X,y)^{\mathsf{op}}} \delta_{x,A}(U) \simeq (\delta_{x,A})_y \simeq \operatorname*{colim}_{U \in \mathfrak{U}(V,y)^{\mathsf{op}}} \delta_{x,A}(U) \simeq \operatorname*{colim}_{U \in \mathfrak{U}(V,y)^{\mathsf{op}}} \{*\}$$
is a direct colimit of the constant diagram with values $\{*\}$. This implies $(\delta_{x,A})_y \simeq \{*\}$.

□

Note that if A is equipped with the structure of an abelian group, the skyscapter $\delta_{x,A}$ can be upgraded to a sheaf of abelian groups. Then the abelian group $(\delta_{x,A})_y$ is either A or 0.

Proposition 2.3.4. *Let X be a topological space, $x \in X$ be a point and A be a set. For any presheaf $\mathcal{F}$ of sets on X, the composition*

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \xrightarrow{(-)_x} \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, (\delta_{x,A})_x) \simeq \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \tag{2.5}$$

is an bijection.

Corollary 2.3.5. *The stalk functor*

$$\mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{Set},\ \mathcal{F} \mapsto \mathcal{F}_x$$

admits a right adjoint

$$\mathsf{Set} \to \mathsf{PShv}(X, \mathsf{Set}),\ A \mapsto \delta_{A,x}.$$

Proof of Proposition 2.3.4. We first construct a map

$$\mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \tag{2.6}$$

as follows. Given any map $f : \mathcal{F}_x \to A$, for any open subset $U \subseteq X$, we define a map $\alpha_U : \mathcal{F}(U) \to \delta_{x,A}(U)$ such that:

- If $x \in U$, α_U is the composition $\mathcal{F}(U) \to \mathcal{F}_x \xrightarrow{f} A$;
- If $x \notin U$, α_U is the unique map $\mathcal{F}(U) \to \{*\}$.

One can check these maps are compatible with restriction and therefore define a morphism $\alpha : \mathcal{F} \to \delta_{x,A}$. Now we define the map (2.6) to be $f \mapsto \alpha$.

One can check that (2.5) and (2.6) are inverse to each other. Hence both are bijections.

□

Remark 2.3.6. In general, for any category $\mathcal{C}$ admitting a final object[6] and any object $A \in \mathcal{C}$, one can define a $\mathcal{C}$-valued sheaf $\delta_{x,A}$. If $\mathcal{C}$ admits direct colimits, the stalks of $\delta_{x,A}$ are either A or the final object of $\mathcal{C}$, and the functor $A \mapsto \delta_{A,x}$ is right adjoint to $\mathcal{F} \mapsto \mathcal{F}_x$.

[6]An object $* \in \mathcal{C}$ is a final object iff for any $c \in \mathcal{C}$, there is a unique morphism $c \to *$.

3. Category of (pre)sheaves

Let X be a topological space and $\mathcal{C}$ be a category. Note that $\mathcal{C}$-valued presheaves on X form a category

$$\mathsf{PShv}(X,\mathcal{C}) \coloneqq \mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}),$$

and $\mathcal{C}$-valued sheaves form a full subcategory

$$\mathsf{Shv}(X,\mathcal{C}) \subseteq \mathsf{PShv}(X,\mathcal{C}).$$

In this section, we study the basic properties of these categories.

3.1. **Sheafification.**

Definition 3.1.1. Let $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$. The **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^\sharp \in \mathsf{Shv}(X,\mathsf{Set})$ equipped with a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ such that for any testing sheaf $\mathcal{G}$, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta.$$

Proposition 3.1.2. *For any $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$, its sheafification $(\mathcal{F}^\sharp,\theta)$ exists, and is unique up to unique isomorphism. Moreover, the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ induces bijections $\mathcal{F}_x \to \mathcal{F}^\sharp_x$ between the stalks.*