Sheaf Theory.

Motivation To obtain a finer 'topology' on a scheme, and generalise sheaf theory in this new situation.

Idea Treat more morphisms $(U \to X)$ rather than Zariski open immersions as 'open subset'.

§ Presheaves and sheaves

A class E of morphisms of schemes usually satisfies

- Isoms are in E.
- Closed under composition.
- Closed under base change.

With $\underset{\text{class}}{E}/\underset{\text{scheme}}{X}$ we mention the full-subcat of Sch/X whose structure morphs are E-morphs.

e.g. E: (Zar) open immersions.

E = (ét) finite étale morphs.

E = (fl) flat and locally of finite type.

Fix X scheme, C/X full subcat of Sch/X. E class of morphism s.t. C/X closed under fibre product, and, $Y \to X$ in C/X. $U \to Y$ in E $\Rightarrow$ $U \to X$ in C/X.

Def An E-covering of $Y \in C/X$ is a family $(U_i \xrightarrow{g_i} Y)_i \subset E$, s.t. $Y = \bigcup g_i(U_i)$. An E-topology on C/X is the class of all such coverings. An E-site is the cat C/X together with an E-top.

Notation & Example • $(C/X)_E$. X_E when C is clear.

- small site $X_E := (E/X)_E$ (analogue of a space)
- big site $X_E = (LFT/X)_E$. (analogue of cat of spaces)
- $X_{Zar} = (Zar/X)_{Zar}$.
- $X_{ét} = (ét/X)_{ét}$
- $X_{fl} = (LFT/X)_{fl}$.

Def A presheaf on a site $(C/X)_E$ is a functor $(C/X)^{op} \longrightarrow Ab$. A morphism of presheaves is a natural trans.

$\Rightarrow$ Presheaves on $(C/X)_E$. $PSh((C/X)_E) = PSh(X_E)$ is an abelian category.

Example (1) constant presheaf $P_M(U) = M$.

(2) $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$

(3) $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$

(4) $\mathcal{F} \in \mathcal{O}_X\text{-mod}$ $W(\mathcal{F})(U) = \Gamma(U, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U)$.

e.g. • $W(\mathcal{O}_X) = \mathbb{G}_a$.

• X S-scheme. $W(\Omega^1_{X/S}) \to \Gamma(\cdot, \Omega^1_{\cdot/S})$.

is an isom for small étale site.

Def A presheaf P is a sheaf if

• (S_1) if $s \in P(U)$ and there is a covering $(U_i \to U)$ such that $res_{U_i, U}\, s = 0$ $\forall i$ then $s = 0$

• (S_2) if $(U_i \to U)$ is a covering, and (s_i) $s_i \in P(U_i)$ is that

$$res_{U_i \times_U U_j, U_i} s_i = res_{U_i \times_U U_j, U_j} s_j \quad \forall i, j.$$

then exists $s \in P(U)$, $res_{U_i, U}\, s = s_i$ $\forall i$.

Or equivalently

$$(S) \quad P(U) \longrightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)$$

is exact.

Example Consider $P \in Psh(X_{\acute{e}t})$ and $Y \to X$ a Galois covering with group G. $(Y \to X)$ is a covering, and $Y \times_X Y = \coprod_{\sigma \in G} Y_\sigma$. Projections $Y \times_X Y \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y$ is then

$$\coprod_{\sigma \in G} Y_\sigma \overset{(1,\dots,1)}{\underset{(\sigma_1,\dots,\sigma_n)}{\rightrightarrows}} Y,$$

and thus (S) becomes

$$P(X) \longrightarrow P(Y) \overset{(1,\dots,1)}{\underset{(\sigma_1,\dots,\sigma_n)}{\rightrightarrows}} P(Y)^n.$$

If P is a sheaf then $P(X) = P(Y)^G$.

Prop A P is a presheaf on étale or flat site on X. Then P is a sheaf iff

(a) for any U in C/X, $P|_U$ is a sheaf for U_{Zar}.

(b) for any covering $(U' \to U)$ with U and U' both affine, $P(U) \to P(U') \rightrightarrows P(U' \times_U U')$ is exact.

proof. Necessity is obvious.

To show sufficiency. (a) reduces (S) for $(U_i \to U)$ to $(U' \to U)$ where $U' = \coprod U_i$. and (b) proves exactness of (S) for $(U_i \to U)_{i \in I}$ where I is finite and U_i, U all affine.

Let $f: U' \to U$ an E-covering. f is open hence qc. Can write $U = \bigcup U_i$, U_i affine, and $f^{-1}(U_i) = \bigcup U'_{ik}$ finite union. Consider diagram

$$\begin{tikzcd} P(U) \arrow[r] \arrow[d] & P(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P(U' \times_U U') \arrow[d] \\ \prod_i P(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_k P(U'_{ik}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_{k,l} P(U'_{ik} \times_U U'_{il}) \\ \prod_{i,j} P(U_i \cap U_j) \arrow[r] & \prod_{i,j} \prod_{k,l} P(U'_{ik} \cap U_{jl}) & \end{tikzcd}$$

Diagram chase complete the proof $\square$

Cor For any qcoh $\mathcal{O}_X$-mod $\mathcal{F}$, $W(\mathcal{F})$ is a sheaf on X_{fl} and $X_{ét}$

e.g. $\mathbb{G}_a$, $\mathbb{G}_m$ are indeed sheaves on X_{fl}, $X_{ét}$.

Cor The presheaf defined by a commutative group scheme on X is a sheaf for flat, étale and Zariski sites on X.

Example. Let $\mathbb{G}_a = \mathrm{Spec}\, \mathbb{Z}[T]$. $\mathbb{G}_{a,X} = X \times_{\mathbb{Z}} \mathbb{G}_a$. Then $\mathrm{Mor}_X(U, \mathbb{G}_a) \cong \Gamma(U, \mathcal{O}_U) = \mathbb{G}_a(U)$. Thus $\mathbb{G}_a$ is a sheaf. So is $\mathbb{G}_m$ for $\mathbb{G}_m = \mathrm{Spec}\, \mathbb{Z}[T, T^{-1}]$.

Example Fix $X = \mathrm{Spec}\, K$, K a field. Fix geometric point $\bar{x} \longrightarrow X$ and $G = \pi_1(X, \bar{x}) = \mathrm{Gal}(K^{sep}/K)$.

$P \in PSh(X_{\acute{e}t})$. K'/K finite separable. Write $P(K') = P(\mathrm{Spec}\, K')$. $M_P = \varinjlim P(K')$. Then $G \curvearrowright M_P$ and $M_P = \bigcup M_P^H$, where $H \subset G$ runs through open subgrps of G $\Longrightarrow$ M_P is a discrete G-mod.

Conversely, give a discrete G-mod M, define a presheaf F_M by (a) $F_M(K') = M^H$, $H = \mathrm{Gal}(K^{sep}/K')$,
(b) $F_M(\prod K_i) = \prod K_i$.

F_M is a sheaf: (a) of Prop A is clear. It suffices to check (b) for $\mathrm{Spec}\, L' \to \mathrm{Spec}\, L$, both separable extension of K. Let L''/L Galois containing L'.

$$\begin{tikzcd}
F_M(L) \arrow[r] \arrow[d, equal] & F_M(L') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F_M(L' \otimes_L L') \arrow[d] \\
F_M(L) \arrow[r] & F_M(L'') \arrow[r, shift left] \arrow[r, shift right] & F_M(L'' \otimes_L L')
\end{tikzcd}$$

bottom row exact $\Rightarrow$ top row exact.

The correspondence $F \longleftrightarrow M_F$, $M \longleftrightarrow F_M$ induces an equivalence between $S(X_{\acute{e}t})$ and G-mod of discrete G-modules.

§ The cat of sheaves

Def A morphism of schemes $\pi: X' \to X$ is a **morphism of sites** $(C'/X')_{E'} \to (C/X)_E$ if

- $\forall Y \in C/X$. $Y_{(X')} \in C'/X'$
- $\forall$ E-morphism $U \to Y$ in C/X, $U_{(X')} \to Y_{(X')}$ is an E'-morphism.

Also **continuous morphism**.

Def Let $\pi: X'_{E'} \to X_E$ be continuous, P' a presheaf on $X'_{E'}$. The **direct image** $\pi_* P'$ on X_E is defined by $(\pi_* P')(U) = P'(U_{(X')})$. $\pi_*: PSh(X'_{E'}) \to PSh(X_E)$ is a functor. The **inverse image** functor π^{-1} is the left adjoint of π_*.

Construction $(\pi^{-1} P)(U') = \varinjlim P(U)$, where U runs over all squares

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ X' \arrow[r, "\pi"] & X \end{tikzcd}$$

Prop The functor π_* is exact and π^{-1} is right-exact. π^{-1} is left-exact if

- finite inverse limit exists in C/X, or
- π is in C/X and $C'/X' = (C/X)/X'$

Lemma The direct limit defining π^{-1} is cofiltered if finite inverse limit exists in C/X.

Prop If $\mathcal{F}$ is a sheaf, then $\pi_* \mathcal{F}$ is a sheaf.

Stalk of a (pre)sheaf in étale site. 'A point': given a sheaf is given an abelian group. Not true for one-point scheme unless it is a spectrum of a separably closed field. A point is a geometric point.

Let x be a set-theoretic point in X, and $u_x : \bar{x} \to x \to X$ a geometric point.

Def Let $P \in PSh(X_{ét})$. The **stalk** of P at $\bar{x}$ is $P_{\bar{x}} = (u_{\bar{x}}^{-1} P)(\bar{x})$. $P_{\bar{x}}$ is independent of choice of $k(\bar{x})$

Remark
- Taking stalk is exact
- $P_{\bar{x}}$ is a $Gal(k^{sep}/k)$-module, $k = k(x)$.
- Let U be an étale nbhd of $\bar{x}$. there is canonical $P(U) \to P_{\bar{x}}$, $s \mapsto s_{\bar{x}}$.

- For P the sheaf defined by a group scheme G that is LFT over X, we have [see. 3.3]

$$P_{\bar{x}} = \varinjlim G(U) = G(\varprojlim U) = G(\mathcal{O}_{X,\bar{x}}).$$

e.g. $(\mathbb{G}_a)_{\bar{x}} = \mathcal{O}_{X,\bar{x}}$ $(\mathbb{G}_m)_{\bar{x}} = \mathcal{O}_{X,\bar{x}}^*$.

Prop Let $\mathcal{F} \in Sh(X_{et})$. If $s \in \mathcal{F}(U)$ is nonzero, then there is an $x \in X$ and an $\bar{x}$-point of U s.t. $s_{\bar{x}}$ is nonzero.

Thm & Def For any presheaf P on X_E there is a sheaf P^{sh} on X_E and a morphism $\phi: P \to P^{sh}$ s.t. for any sheaf $\mathcal{F}$ there is natural isomorphism

$$Hom_{psh}(P, \mathcal{F}) \cong Hom_{sh}(P^{sh}, \mathcal{F}).$$

P^{sh} is the sheaf associate with P or the sheafification of P.

proof We prove for étale site.

For $\bar{x} = Spec\ \kappa(\bar{x})$, where $\kappa(\bar{x})$ is algebraically closed, P^{sh} is determined by $P(\bar{x})$ in an obvious way.

For general X. For each $x \in X$ choose $\bar{x} \to X$. write $P_{\bar{x}}^* = (u_{\bar{x}}^{-1} P)^{sh}$, i.e., the sheaf defined by $P_{\bar{x}}$.

Let $P^* = \prod_x u_{x,*} P_x^*$ which is a sheaf. There is natural $\phi : P \to P^*$ induced by

$$P \to u_{x,*} \circ u_x^{-1} P \to u_{x,*} (u_x^{-1} P)^{sh}.$$

Let P^{sh} be the intersection of all subsheaves containing $\phi(P)$ in P^*.

Consider

$$\begin{tikzcd}
P \arrow[r, "\phi"] \arrow[dr] & P^{sh} \arrow[r, hook] & P^* \arrow[d, "\psi"] \\
& F \arrow[r, hook] & F^*
\end{tikzcd}$$

where $P^* \xrightarrow{\psi} F^*$ is induced by ϕ and $F \hookrightarrow F^*$ since F is a sheaf. $\phi(P) \subset \psi^{-1}(F)$, thus $P^{sh} \subset \psi^{-1}(F)$, and $\psi_0 : P^{sh} \to \psi^{-1}(F) \to F$ is a morphism making diagram commute. If ψ_1 is also such a morphism, then $\ker(\psi_1 - \psi_0) \subset P^{sh}$ is a subsheaf containing $\phi(P)$, making $\ker(\psi_1 - \psi_0) = P^{sh}$ and $\psi_1 = \psi_0$. $\square$

Remark. • Thm & Def says that

$$(\cdot)^{sh} : PSh(X_E) \rightleftarrows Sh(X_E) : u$$

is an adjoint pair.

• P and P^{sh} have same stalks.

Prop • $(\cdot)^{sh} : PSh(X_E) \to Sh(X_E)$ is exact

• For étale topology.

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

being exact in $Sh(X_E)$ is equivalent to

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

being exact for any geometric point $\bar{x}$.

Example • Let μ_n be the subsheaf of $\mathbb{G}_m$ such that $\mu_n(U)$ = n-th roots of unit in $\Gamma(U, \mathcal{O}_U)$. It is represented by $\operatorname{Spec} \mathbb{Z}[T]/(T^n-1)$, thus a sheaf.

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$$

is exact in $PSh(X_E)$ and hence in $Sh(X_E)$. However, $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ is rarely epimorphic in $Sh(X_{Zar})$. On the other hand, for a strictly local ring $(A, \mathfrak{m})$, as long as $n \neq 0 \in K = A/\mathfrak{m}$, by Hensel lemma.

$$0 \to \mu_n(A) \to A^* \xrightarrow{n} A^* \to 0$$

It provides that $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$ is exact in $Sh(X_{Et})$ if $n \neq 0 \in \kappa(x)$ for any $x \in X$.

In $Sh(X_{fl})$ exactness holds even without restriction on residue field character. ($A \to A[T]/(T^n - a)$ is flat).

- Let X be an $\mathbb{F}_p$-scheme. Then

$$(\mathbb{Z}/p\mathbb{Z})_X = X \times_{\mathbb{F}_p} \mathbb{F}_p[T]/(T^p - T)$$

Thus the constant sheaf $(\mathbb{Z}/p\mathbb{Z})_X$ is given by

$$(\mathbb{Z}/p\mathbb{Z})_X(U) = \{a \in \Gamma(U, \mathcal{O}_U) : a^p - a = 0\}.$$

Consider

$$0 \to (\mathbb{Z}/p\mathbb{Z})_X \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0.$$

where $F(a) = a^p$. It is not exact in $Sh(X_{Zar})$ in general.

But is exact in $Sh(X_{ét})$ (and hence $Sh(X_{fl})$). ($A[T]/(T^p - T)$ is étale over A!)

- Again let X be an $\mathbb{F}_p$-scheme. Let α_p be defined by $\alpha_p(U) = \{a \in \Gamma(U, \mathcal{O}_U) : a^p = 0\}$. Then α_p is a sheaf represented by $\mathbb{F}_p[T]/(T^p)$.

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0$$

is always exact in $Sh(X_{fl})$, but not in $Sh(X_{ét})$ or $Sh(X_{Zar})$