NOTES FOR ALGEBRAIC GEOMETRY 1

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Contents

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0. Introduction: why schemes?

0.1. **Algebraic sets.** Before scheme theory, algebraic geometry focused on *algebraic sets.*

Definition 0.1.1. Let k be an algebraically closed field.

- The **Zariski topology** on the affine space $\mathbb{A}^n_k$ is the topology with a base consisting of all the subsets that are equal to the non-vanishing locus $U(f)$ of some polynomial $f \in k[x_1, \cdots, x_n]$.
- An **embedded affine algebraic set**[1] in $\mathbb{A}^n_k$ is a closed subspace for the Zariski topology.
- An **embedded quasi-affine algebraic set** is a Zariski open subset of an embedded affine algebraic set.

Example 0.1.2. Any finite subset of $\mathbb{A}^n_k$ is an embedded affine algebraic set.

Example 0.1.3. $\mathbb{Z}$ is not an embedded affine algebraic set in $\mathbb{A}^1_{\mathbb{C}}$.

Similarly one can define *embedded (quasi)-projective algebraic set* using *homogeneous* polynomials and the projective space $\mathbb{P}^n_k$.

There are tons of shortcomings in the above definition. An obvious one is that the notions of *embedded* algebraic sets are not *intrinsic*.

Example 0.1.4. The embedded affine algebraic sets $\mathbb{A}^1_k \subseteq \mathbb{A}^1_k$ and $\mathbb{A}^1_k \subseteq \mathbb{A}^2_k$ should be viewed as the same algebraic sets.

Notation 0.1.5. To remedy this, we need some notations.

- For an ideal $I \subseteq k[x_1, \cdots, x_n]$, let $Z(I) \subseteq \mathbb{A}^n_k$ be the locus of common zeros of polynomials in I.
- For a Zariski closed subset $X \subseteq \mathbb{A}^n_k$, let $I(X) \subseteq k[x_1, \cdots, x_n]$ be the ideal of all polynomials vanishing on X.

Recall an ideal I is called *radical* if $I = \sqrt{I}$.

Theorem 0.1.6 (Hilbert Nullstellensatz)**.** *We have a bijection:*

$$\begin{array}{rcl} \{\text{radical ideals of } k[x_1, \cdots, x_n]\} & \longleftrightarrow & \{\text{Zariski closed subsets of } \mathbb{A}^n_k\} \\ I & \longrightarrow & Z(I) \\ I(X) & \longleftarrow & X. \end{array}$$

Part of the theorem says the set of points of $\mathbb{A}^n_k$ is in bijection with the set of maximal ideals of $k[x_1, \cdots, x_n]$. As a corollary, $Z(I)$ is in bijection with the set of maximal ideals containing I. The latter can be further identified with maximal ideals of $R \coloneqq k[x_1, \cdots, x_n]/I$.

Note that I is radical iff R is *reduced*, i.e., contains no nilpotent elements. This justifies the following definition.

Definition 0.1.7. An **affine algebraic k-set** is a *maximal spectrum* $\mathsf{Spm}\, R$ (= sets of maximal ideals) of a *finitely generated* (commutative unital) *reduced* k-algebra R. We equip it with the **Zariski topology** with a base of open subsets given by

$$U(f) \coloneqq \{\mathfrak{m} \in \mathsf{Spm}\, R \mid f \notin \mathfrak{m}\},\ f \in R.$$

[1] Some people use the word *variety*, while some people reserve it for *irreducible* algebraic sets.

Example 0.1.8. $\mathsf{Spm}\, k[x] \simeq \mathbb{A}^1_k$.

We have the following *duality* between algebra and geometry.

Algbera	Geometry
finitely generated reduced k-algebra R	affine algebraic k-set X
maximal ideals $\mathfrak{m} \subseteq R$	points $x \in X$
elements $f \in R$	functions $\phi : X \to \mathbb{A}^1_k$
radical ideals $I \subseteq R$	Zariski closed subsets $Z \subseteq X$

Here an element $f \in R$ corresponds to the function

$$\phi : \mathsf{Spm}\, R \to k,\ \mathfrak{m} \mapsto \underline{f}$$

sending a maximal ideal $\mathfrak{m}$ to the image $\underline{f}$ of f in the *residue field* of $\mathfrak{m}$, which is canonically identified with the underlying set of $\mathbb{A}^1_k$ via the composition $k \to R \to R/\mathfrak{m}$.

The word *duality* means the correspondence $R \leftrightarrow X$ is *contravariant*. Indeed, given a homomorphism $f : R' \to R$, we obtain a *continuous* map

$$\mathsf{Spm}\, R \to \mathsf{Spm}\, R',\ \mathfrak{m} \mapsto f^{-1}(\mathfrak{m}).$$

Note however that not all continuous maps $\mathsf{Spm}\, R \to \mathsf{Spm}\, R'$ are obtained in this way, nor is R determined by the topological space $\mathsf{Spm}\, R$.

Exercise 0.1.9. Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology. Find those bijections coming from a homomorphism $k[x] \to k[x]$.

This motivates the following definition.

Definition 0.1.10. A **morphism** from $\mathsf{Spm}\, R$ to $\mathsf{Spm}\, R'$ is a continuous map coming from a homomorphism $R' \to R$.

Then one can define general algebraic k-sets by gluing affine algebraic k-sets using morphisms, just like how people define *structured* manifolds as glued from *structured* Euclidean spaces using maps preserving the addiontal structrues.

0.2. **Shortcomings.** The theory of algebraic k-sets provides a bridge between algebra and geometry. In particular, one can use topological/geometric methods, including various cohomology theories, to study *finitely generated reduced k-algebras*. However, these adjectives are non-necessary restrictions to this bridge.

First, number theory studies number fields and their rings of integers, such as $\mathbb{Q}$ and $\mathbb{Z}$. It is desirable to have geometric objects corresponding to them. Hence we would like to consider general commutative rings rather than k-algebras. Then one immediately realizes the maximal spectra Spm are not enough.

Example 0.2.1. The map $\mathbb{Z} \to \mathbb{Q}$ does not induce a map from $\mathsf{Spm}\, \mathbb{Q}$ to $\mathsf{Spm}\, \mathbb{Z}$. Namely, the inverse image of $(0) \subseteq \mathbb{Q}$ in $\mathbb{Z}$ is a non-maximal prime ideal.

This suggests for general algebra R, we should consider its *prime spectrum*, denoted by $\mathsf{Spec}\, R$, rather than just its maximal spectrum.

Second, even in the study of finitely generated algebras, one naturally encounters non-finitely generated ones.

Example 0.2.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal of a finitely generated algebra. The localization $R_\mathfrak{p}$ and its completion $\widehat{R}_\mathfrak{p}$ are in general not finitely generated.

Of course, one can restrict their attentions to *Noetherian* rings (and live a happy life). But let me object the false feeling that all natural rings one care are Noetherian.

Example 0.2.3. Noetherian rings are not stable under tensor products: $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not Noetherian.

Example 0.2.4. The ring of adeles of $\mathbb{Q}$ is not Noetherian.

Example 0.2.5. Natural moduli problems about Noetherian objects can fail to be Noetherian. My favorite ones includes: the space of formal connections, the space (stack) of formal group laws...

In short, there are important applications of non-Noetherian rings, and this course will deal with the latter.

Finally, we want to remove the restriction about reducedness.

Example 0.2.6. Reduced rings are not stable under tensor products: $k[x] \otimes_{k[x,y]} k[x,y]/(y-x^2)$ is not reduced. Geometrically, this means $Z(y)$ and $Z(y-x^2)$ do not intersect transversally inside $\mathbb{A}^2_k$.

One may notice that without reducedness, we should accordingly consider all ideals rather than just *radical* ideals, but then the construction $I \mapsto Z(I)$ would not be bijective. Indeed, ideals with the same nilpotent radical would give the same *topological subspace* of $\mathsf{Spec}\, R$.

But *this is a feature rather than a bug.* In Example 0.2.6, the ideal $(y, y-x^2) = (x^2, y)$ is not radical, and it carries geometric meanings that cannot be seen from its nilpotent radical (x,y). Namely, $f \in (x,y)$ iff $f(0,0) = 0$, while $f \in (x^2, y)$ iff $f(0,0) = \partial_x f(0,0) = 0$. Roughly speaking, this suggests that $(y, y-x^2)$ remembers that the curves $Z(y)$ and $Z(y-x^2)$ are tangent to each other at the point $(0,0) \in \mathbb{A}^2_k$, and the tangent vector is $\partial_x|_{(0,0)}$. Also note that the length of $k[x,y]/(y, y-x^2)$ is equal to 2, which is the number of intersection points predicted by the Bézout's theorem.

In summary, on the algebra side, we should consider *all* commutative rings. On the geometric side, the corresponding notion is called *affine schemes.* Our first task in this course is to develop the following duality:

Algebra	Geometry
commutative rings R	affine schemes X
prime ideals $\mathfrak{p} \subseteq R$	points $x \in X$
elements $f \in R$	functions $X \to \mathbb{A}^1_{\mathbb{Z}}$
ideals $I \subseteq R$	closed subschemes $Z \subseteq X$.

0.3. **Schemes as structured spaces.** In theory, one can *define* a morphism between affine schemes to be a continuous map coming from a ring homomorphism. Then one can define general *schemes* by gluing affine schemes using such morphisms. This mimics the definition of differentiable manifold in the sense that a scheme would be a topological space equipped with a *maximal* affine atlas.

In practice, the above approach is awkward to work with. We prefer a more efficient way to encode the additional structure on the topological space underlying a scheme. One extremely simple but powerful way to achieve this, maybe discovered by Serre and popularized by Grothendieck, is the notion of *sheaves* on topological spaces. Roughly speaking, a sheaf $\mathcal{F}$ on X is an *contravariant* assignment

$$U \mapsto \mathcal{F}(U)$$

sending open subsets $U \subseteq X$ to certain structures (e.g. sets, groups, rings) $\mathcal{F}(U)$, such that a certain gluing condition is satisfied. Here contravariancy means that for $U \subseteq V$, we should provide a map $\mathcal{F}(V) \to \mathcal{F}(U)$ preserving the prescribed structures.

Example 0.3.1. Let X be any topological space. The assignment

$$U \mapsto C(U, \mathbb{R})$$

sending $U \subseteq X$ to the ring of continuous functions on U would be a sheaf of commutative rings on X.

Similarly, for a smooth manifold X, $U \mapsto C^\infty(U, \mathbb{R})$ would be a sheaf of commutative rings on X. This motivates us to define:

Pre-Definition 0.3.2. A **scheme** is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X$ such that locally it is isomorphic to an affine scheme.

Here for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ should be the ring of *algebraic* functions on U, but we have not defined the latter notion yet. Nevertheless, for an *affine* scheme $X \simeq \mathsf{Spec}\, R$, the previous discussion suggests we should have $\mathcal{O}_X(X) \simeq R$. As we shall see in future lectures, we can bootstrap from this to get the definition of the entire sheaf $\mathcal{O}_X$.

The goal of this course is to define schemes and study their basic properties.