

LECTURE 11

Last time, for a smooth k -scheme X , we introduced the sheaf of differential operators \mathcal{D}_X on X and defined quasi-coherent \mathcal{D}_X -modules, known as *\mathcal{D} -module on X* . In this lecture, we introduce operations on \mathcal{D} -modules. We will mainly focus on the formal aspect of this theory, known as the *six functors formalism for \mathcal{D} -modules*. For more details, see [B] and [HTT].

1. CONVENTIONS ON DERIVED CATEGORIES

For the purpose of this course, we do not need the full power of the derived categories of \mathcal{D} -modules. However, these categories are useful in other topics of geometric representation theory, hence I choose to include the results about them in this lecture.

When talking about derived categories of \mathcal{D} -modules, we always assume X is quasi-projective. In this case, the abelian category $\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r}$ has enough injective and locally projective objects, and any object admits a resolution by locally projective \mathcal{D}_X -modules with length $\leq 2d_X$, where $d_X = \dim(X)$ is the dimension (function) of X . The latter implies $\text{Ext}^i(-, -) \simeq 0$ for $i > 2d_X$.

We have the following triangulated categories equipped with natural t-structures:

- $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r})$, the derived category of quasi-coherent \mathcal{D}_X -modules. This can be identified with the full subcategory of the derived category $D(\mathcal{D}_X\text{-mod}^{l/r})$ of *all* \mathcal{D}_X -modules containing those complexes whose cohomologies are quasi-coherent.
- $D^b(\mathcal{D}_X\text{-mod}_c^{l/r})$, the bounded derived category of coherent \mathcal{D}_X -modules. This can be identified with the full subcategory of the bounded derived category $D^b(\mathcal{D}_X\text{-mod}^{l/r})$ of *all* \mathcal{D}_X -modules containing those complexes whose cohomologies are coherent.

When talking about functors between derived categories, even if such functors are left/right derived functors, we drop the decorations “L/R” from the notations. For example, $- \otimes -$ in derived categories would mean $- \otimes^L -$ in classical literatures. We choose to do so because we will encounter functors that are not derived functors.

Remark 1.1. The (essential) image of the fully faithful functor

$$D^b(\mathcal{D}_X\text{-mod}_c^{l/r}) \rightarrow D(\mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r})$$

contains exactly the *compact* objects in the target, i.e., those objects \mathcal{M} such that $\text{Hom}(\mathcal{M}, -)$ commutes with filtered (homotopy) colimits.

2. FORGET AND INDUCE

Construction 2.1. *We have adjoint functors*

$$\text{ind}^{l/r} : \mathcal{O}_X\text{-mod}_{\text{qc}} \rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} : \text{oblv}^{l/r}$$

such that

$$\text{ind}^l(\mathcal{F}) := \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}, \text{ind}^r(\mathcal{D}) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

Both functors are exact.

3. TENSOR AND HOM

Construction 3.1. Let $\mathcal{M}, \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ and $\mathcal{M}', \mathcal{N}' \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$. Then there are natural objects defined using the signed Leibniz rules:

- $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ defined by $\partial \cdot (m \otimes n) = (\partial \cdot m) \otimes n + m \otimes (\partial \cdot n)$;
- $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$ defined by $(m' \otimes n) \cdot \partial = (m' \cdot \partial) \otimes n - m' \otimes (\partial \cdot n)$;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ defined by $(\partial \cdot \phi)(m) = \partial \cdot \phi(m) - \phi(\partial \cdot m)$;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ defined by $(\partial \cdot \phi)(m') = -\phi(m') \cdot \partial + \phi(m' \cdot \partial)$;
- $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$ defined by $(\phi \cdot \partial)(m) = \phi(m) \cdot \partial + \phi(m \cdot \partial)$.

Remark 3.2. One way to memorize the above rules for left vs. right is using the known objects $\mathcal{O}_X \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ and $\omega_X := \Omega_X^n \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$. For example, $\text{Hom}_{\mathcal{O}_X}(\omega_X, \omega_X) \simeq \mathcal{O}_X$ has a left \mathcal{D} -module structure, while $\text{Hom}_{\mathcal{O}_X}(\omega_X, \mathcal{O}_X) \simeq \omega_X^{-1}$ in general has no \mathcal{D} -module structures.

Remark 3.3. One way to memorize the signed Leibniz rules: (i) put a minus sign when acting on a section of the source object in $\text{Hom}_{\mathcal{O}_X}(-, -)$; (ii) put a minus sign when moving ∂ from the one side of \cdot to the other side.

Remark 3.4. The tensor operations make $\mathcal{D}_X\text{-mod}_{\text{qc}}^l$ a symmetric monoidal category such that the forgetful functor $\mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{O}_X\text{-mod}_{\text{qc}}^l$ is naturally symmetric monoidal. The category $\mathcal{D}_X\text{-mod}_{\text{qc}}^r$ is a module category of it.

The following result follows by unwinding the definitions:

Lemma 3.5. Let $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ and $\mathcal{M}' \in \mathcal{D}_X\text{-mod}_{\text{qc}}^r$. We have adjoint functors

$$\begin{aligned} \mathcal{M} \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} &\rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^{l/r} : \text{Hom}_{\mathcal{O}_X}(\mathcal{M}, -) \\ \mathcal{M}' \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^l &\rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^r : \text{Hom}_{\mathcal{O}_X}(\mathcal{M}', -) \end{aligned}$$

compatible with the similar adjunction between \mathcal{O}_X -modules.

Recall ω_X is a line bundle. It follows formally that:

Corollary 3.6. The following functors are inverse to each other:

$$\omega_X \otimes_{\mathcal{O}_X} - : \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightleftarrows \mathcal{D}_X\text{-mod}_{\text{qc}}^r : \text{Hom}_{\mathcal{O}_X}(\omega_X, -).$$

Remark 3.7. In particular, for any right \mathcal{D} -module \mathcal{N}' , we obtain a left \mathcal{D} -module structure on $\omega_X^{-1} \otimes \mathcal{N}'$.

Remark 3.8. We also have similar results for the derived category of \mathcal{D} -modules and the corresponding derived functors. The left derived functor $- \otimes_{\mathcal{O}_X} -$ has cohomological amplitude $[-d_X, 0]$ while the right derived functor $\text{Hom}_{\mathcal{O}_X}(-, -)$ has cohomological amplitude $[0, d_X]$.

Remark 3.9. When identifying the *derived* categories of left and right \mathcal{D} -modules, it is more convenient to use the complex $\omega_X[d_X]$, which is also known as the *dualizing complex*¹ on X . In other words, we use the equivalences

$$\omega_X[d_X] \otimes_{\mathcal{O}_X} - : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \rightleftarrows D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \text{Hom}_{\mathcal{O}_X}(\omega_X[d_X], -).$$

¹This notion makes sense even when X is singular.

Construction 3.10. We define a symmetric monoidal structure $-\otimes^!-$ on $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r)$ by translating the symmetric monoidal structure $-\otimes_{\mathcal{O}_X}-$ of $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l)$ via the above equivalences. In other words, as \mathcal{O}_X -modules, we have

$$\mathcal{M}' \otimes^! \mathcal{N}' \simeq \mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}' \otimes_{\mathcal{O}_X} \omega_X^{-1}[-d_X].$$

4. EXTERNAL TENSOR

Construction 4.1. Let $\mathcal{M}_i \in \mathcal{D}_{X_i}\text{-mod}_{\text{qc}}^{l/r}$. Then there are natural objects

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 := \mathcal{M}_1 \otimes_k \mathcal{M}_2 \in \mathcal{D}_{X_1 \times X_2}\text{-mod}_{\text{qc}}^{l/r}$$

induced by the isomorphism $\mathcal{D}_{X_1} \otimes_k \mathcal{D}_{X_2} \simeq \mathcal{D}_{X_1 \times X_2}$. This is called the **external tensor product** of \mathcal{D} -modules.

5. PULLBACKS

Let $\phi: Y \rightarrow X$ be a map between smooth k -schemes.

Construction 5.1. We will construct a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_X\text{-mod}_{\text{qc}}^l & \xrightarrow{\phi^*} & \mathcal{D}_Y\text{-mod}_{\text{qc}}^l \\ \downarrow \text{oblv}^l & & \downarrow \text{oblv}^l \\ \mathcal{O}_X\text{-mod}_{\text{qc}} & \xrightarrow{\phi^*} & \mathcal{O}_Y\text{-mod}_{\text{qc}}. \end{array}$$

The functor

$$\phi^*: \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$$

is called the **(*)pullback of left \mathcal{D} -modules**.

The construction is as follows. For $\mathcal{M} \in \mathcal{O}_X\text{-mod}_{\text{qc}}$, recall $\phi^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{M}$. Suppose \mathcal{M} is equipped with a left \mathcal{D}_X -module structure, then there is a left \mathcal{D}_Y -module structure on $\phi^*(\mathcal{M})$ defined by the Leibniz rule:

$$\partial \cdot (f \otimes s) := \partial(f) \otimes s + f \bar{\partial} \cdot s,$$

where

- ∂ is a local section of \mathcal{T}_Y and $\bar{\partial}$ is the image of it under the map $\mathcal{T}_Y \rightarrow \phi^*\mathcal{T}_X = \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{T}_X$;
- f is a local section of \mathcal{O}_Y ;
- s is a local section of $\phi^{-1}\mathcal{M}$, and $\bar{\partial} \cdot s$ is defined using the action of $\phi^{-1}\mathcal{T}_X$ on $\phi^{-1}\mathcal{M}$.

Remark 5.2. One can show the pullback of left \mathcal{D} -modules are compatible with composition of maps between smooth k -schemes.

Example 5.3. The pullback of the object $\mathcal{O}_X \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$ is $\mathcal{O}_Y \in \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$.

Construction 5.4. We write:

$$\mathcal{D}_{Y \rightarrow X} := \phi^*\mathcal{D}_X \simeq \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{D}_X$$

and call it the **transfer module**.

The above construction gives a left \mathcal{D}_Y -module structure on $\mathcal{D}_{Y \rightarrow X}$. On the other hand, there is an obvious right $\phi^{-1}\mathcal{D}_X$ -module structure on $\mathcal{D}_{Y \rightarrow X}$. One can show these two actions commute. In other words, $\mathcal{D}_{Y \rightarrow X}$ is a $(\mathcal{D}_Y, \phi^{-1}\mathcal{D}_X)$ -bimodule.

Note that for $\mathcal{M} \in \mathcal{D}_X\text{-mod}_{\text{qc}}^l$, we have

$$\phi^* \mathcal{M} \simeq \mathcal{D}_{Y \rightarrow X} \otimes_{\phi^{-1} \mathcal{D}_X} \phi^{-1} \mathcal{M}.$$

Construction 5.5. Note that the functor $\phi^* : \mathcal{D}_X\text{-mod}_{\text{qc}}^l \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^l$ is right exact. We abuse notation and let

$$\phi^* : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \rightarrow D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l)$$

be the left derived functor of it. Note that it is compatible with the left derived functor $\phi^* : D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \rightarrow D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$ and the forgetful functors.

Remark 5.6. We have:

- If ϕ is flat, then ϕ^* is t-exact, i.e., preserves the heart.
- If ϕ is a closed embedding (which is automatically regular), then ϕ^* has cohomological amplitude $[-d_X + d_Y, 0]$.

Example 5.7. Let ϕ be a closed embedding. Since X and Y are smooth, ϕ is a regular immersion. For any closed point $p \in Y$, we can find an étale coordinate system x_1, \dots, x_m of X near p such that Y is locally cut out by the ideal (x_{n+1}, \dots, x_m) ($m = \dim(\mathcal{O}_{X,y})$ and $n = \dim(\mathcal{O}_{Y,y})$). Let y_1, \dots, y_n be the restriction of x_1, \dots, x_n on Y . They form an étale coordinate system of Y near p . Then near the point $p \in Y$, we have

$$\mathcal{D}_{Y \rightarrow X} \simeq \mathcal{D}_Y \otimes_k [\partial_{n+1}, \dots, \partial_m]$$

as left \mathcal{D}_Y -modules. In particular, $\mathcal{D}_{Y \rightarrow X}$ is a locally free left \mathcal{D}_Y -module.

Construction 5.8. Let $\phi^!$ be the unique functor that makes the following diagram commute

$$\begin{array}{ccc} D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) & \xrightarrow{\phi^*} & D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l) \\ \omega_X[d_X] \downarrow \simeq & & \simeq \downarrow \omega_Y[d_Y] \\ D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi^!} & D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r), \end{array}$$

The obtained functor

$$\phi^! : D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \rightarrow D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r)$$

is called the **!-pullback of (complices of) right \mathcal{D} -modules**. It has cohomological amplitude $[-d_Y, d_X - d_Y]$ and in general is not a derived functor.

Remark 5.9. We have:

- If ϕ is flat, then $\phi^!$ is t-exact up to a shift.
- If ϕ is a closed embedding, then $\phi^!$ has cohomological amplitude $[0, d_X - d_Y]$.

Example 5.10. By definition, $\phi^!(\omega_X[d_X]) \simeq \omega_Y[d_Y]$.

Example 5.11. If $j : U \rightarrow X$ is an open embedding, then $j^!$ is t-exact and the corresponding functor $\mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_U\text{-mod}_{\text{qc}}^r$ is the restriction functor. Indeed, this follows from $\omega_X|_U \simeq \omega_U$. In this case, we write $j^! = j^*$.

Warning 5.12. In general, for a right \mathcal{D}_X -module \mathcal{M} , there is no \mathcal{D} -module structure on its \mathcal{O} -module pullback $\phi^* \mathcal{M}$.

Fact 5.13. If $\phi : Y \rightarrow X$ is a closed embedding, then $\phi^!$ is equivalent to the right derived functor of a functor

$$\phi^! : \mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^r.$$

Construction 5.14. Let $\phi : Y \rightarrow X$ be a closed embedding. The functor $\phi^! : \mathcal{D}_X\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_Y\text{-mod}_{\text{qc}}^r$ can be described as follows.

Recall we have adjoint functors

$$\phi_* : \mathcal{O}_Y\text{-mod}_{\text{qc}} \rightleftarrows \mathcal{O}_X\text{-mod}_{\text{qc}} : \phi^!$$

where for a quasi-coherent \mathcal{O}_X -module \mathcal{M} and any open subset $U \subset X$, a section m of $\phi^!(\mathcal{M})$ on $U \cap Y$ corresponds to a section \tilde{m} of \mathcal{M} on U annihilated by the ideal $\mathcal{I}_Y := \ker(\mathcal{O}_X \rightarrow \phi_*\mathcal{O}_Y)$. Suppose \mathcal{M} is equipped with a right \mathcal{D}_X -module structure. For any local section ∂ of \mathcal{T}_Y , we can extend it to a local section $\tilde{\partial}$ of \mathcal{T}_X . Now for a local section m of $\phi^!(\mathcal{M})$, we define $m \cdot \partial$ such that

$$\widetilde{m \cdot \partial} = \tilde{m} \cdot \tilde{\partial}.$$

One can show the local section $m \cdot \partial$ is well-defined and does not depend on the choice of $\tilde{\partial}$. Moreover, this defines a right \mathcal{D}_Y -module structure on $\phi^!(\mathcal{M})$.

Remark 5.15. For any map $\phi : Y \rightarrow X$ between finite type k -schemes, one can define a functor

$$\phi^! : D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \rightarrow D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$$

as follows.

If ϕ is an open embedding, take $\phi^! := \phi^*$. If ϕ is proper, take $\phi^!$ to be the right adjoint of (the right derived functor) ϕ_* . For the general case, choose a Nagata compactification $Y \xrightarrow{j} \bar{Y} \xrightarrow{\bar{\phi}} X$ such that j is an open embedding and $\bar{\phi}$ is proper, and take $\phi^! := j^! \circ \bar{\phi}^!$. One can show the functor $\phi^!$ does not depend on the choice of the compactification, and these functors are compatible with compositions of maps. In fact, the construction $\phi \mapsto \phi^!$ can be uniquely characterized by these properties (if stated properly).

When X and Y are smooth, the $!$ -pullback functors of \mathcal{O} -modules and right \mathcal{D} -modules are compatible via the forgetful functors. In other words, we have a commutative diagram

$$(5.1) \quad \begin{array}{ccc} D(\mathcal{O}_Y\text{-mod}_{\text{qc}}) & \xleftarrow[\phi^!]{\quad} & D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \\ \uparrow \text{oblv}^r & & \uparrow \text{oblv}^r \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xleftarrow[\phi^!]{\quad} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \end{array}$$

Fact 5.16. In the (derived) setting of Construction 3.1, we have

$$\begin{aligned} \phi^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) &\simeq \phi^*(\mathcal{M}) \otimes_{\mathcal{O}_Y} \phi^*(\mathcal{N}), \\ \phi^!(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}) &\simeq \phi^!(\mathcal{M}') \otimes_{\mathcal{O}_Y} \phi^*(\mathcal{N}), \\ \phi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^* \mathcal{M}, \phi^* \mathcal{N}), \\ \phi^* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^* \mathcal{M}', \phi^* \mathcal{N}'), \\ \phi^! \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^! \mathcal{M}, \phi^! \mathcal{N}'). \end{aligned}$$

Fact 5.17. For $\text{pr}_i : X_1 \times X_2 \rightarrow X_i$, we have

$$\begin{aligned} \mathcal{M}_1 \boxtimes \mathcal{M}_2 &\simeq \text{pr}_1^*(\mathcal{M}_1) \otimes_{\mathcal{O}_{X_1 \times X_2}} \text{pr}_2^*(\mathcal{M}_2) \\ \mathcal{M}'_1 \boxtimes \mathcal{M}'_2 &\simeq \text{pr}_1^!(\mathcal{M}_1) \overset{!}{\otimes} \text{pr}_2^!(\mathcal{M}_2). \end{aligned}$$

6. PUSHFORWARDS

Construction 6.1. Let $\phi : Y \rightarrow X$ be a map between smooth k -schemes. Recall the transfer module

$$\mathcal{D}_{Y \rightarrow X} := \phi^* \mathcal{D}_X \simeq \mathcal{O}_Y \otimes_{\phi^{-1} \mathcal{O}_X} \phi^{-1} \mathcal{D}_X$$

is a bimodule for $(\mathcal{D}_Y, \phi^{-1} \mathcal{D}_X)$. We define a functor

$$\phi_{*, \text{dR}} : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \rightarrow D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r), \mathcal{N} \mapsto \phi_*(\mathcal{N} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}),$$

where

- The (left derived) tensor product functor $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$ sends a complex of right \mathcal{D}_Y -modules to a complex of right $\phi^{-1} \mathcal{D}_X$ -modules.
- The (right derived) functor ϕ_* sends a complex of right $\phi^{-1} \mathcal{D}_X$ -modules to a complex of right \mathcal{D}_X -modules via the homomorphism $\mathcal{D}_X \rightarrow \phi_*(\phi^{-1} \mathcal{D}_X)$.

We call $\phi_{*, \text{dR}}$ the **direct image functor**, or **de Rham pushforward functor**, of (complices) of right \mathcal{D} -modules.

Remark 6.2. One can show the direct image functors of right \mathcal{D} -modules are compatible with composition of maps between smooth k -schemes.

Remark 6.3. The functor $\phi_{*, \text{dR}}$ is called the *de Rham pushforward functor* because for $\pi : X \rightarrow \text{pt}$, $\pi_{*, \text{dR}}(\omega_X[-d_X])$ can be identified with the de Rham complex of X . For this reason, we also write

$$\Gamma_{\text{dR}}(X, -) := \pi_{*, \text{dR}}(-).$$

You are strongly encouraged to look at its proof in [G, Sect. 5.17].

Remark 6.4. Some authors use the notation ϕ_* for $\phi_{*, \text{dR}}$.

Remark 6.5. The cohomological amplitude of $\phi_{*, \text{dR}}$ is $[-d_Y, d_Y]$. Better estimation exist in the following cases:

- If ϕ is affine, then the bounds can be $[-d_Y, 0]$.
- If ϕ is smooth, then the bounds can be $[-d_Y + d_X, d_Y]$.
- If ϕ is a closed embedding, then the functor is t-exact.

Warning 6.6. One can define a functor between the abelian categories using the same formula. However, that functor would not be $\mathcal{H}^0(\phi_{*, \text{dR}})$ and is of less interests.

Example 6.7. If $j : U \rightarrow X$ is an open embedding, then $\mathcal{D}_{U \rightarrow X} \simeq j^* \mathcal{D}_X \simeq \mathcal{D}_U$. It follows that $j_{*, \text{dR}} \mathcal{M} \simeq j_* \mathcal{M}$. In other words, there is a right \mathcal{D}_X -module structure on the \mathcal{O} -module direct image of \mathcal{M} . In this case, we write $j_{*, \text{dR}} = j_*$.

Exercise 6.8. This is **Homework 5, Problem 4**. Let $x : \text{pt} \rightarrow X$ be a closed point of X . We write $\delta_x := x_{*, \text{dR}}(k)$. Prove:

- (1) $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ as a right \mathcal{D}_X -module.
- (2) δ_x is set-theoretically supported on at x , i.e., for the complement open $U := X - x$, we have $\delta_x|_U = 0$.
- (3) There exists a unique section Dirac_x of δ_x such that $\text{Dirac}_x \cdot f = f(x) \text{Dirac}_x$ for any local section f of \mathcal{O}_X defined near x .
- (4) δ_x is generated by Dirac_x as a right \mathcal{D}_X -module.

Remark 6.9. The section Dirac_x should be viewed as the incarnation of the Dirac function in the theory of \mathcal{D} -modules.

Lemma 6.10. *The following diagram commutes:*

$$(6.1) \quad \begin{array}{ccc} D(\mathcal{O}_Y\text{-mod}_{\text{qc}}) & \xrightarrow{\phi_*} & D(\mathcal{O}_X\text{-mod}_{\text{qc}}) \\ \text{ind}^r \downarrow & & \text{ind}^r \downarrow \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi_{*,\text{dR}}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \end{array}$$

Sketch. For $\mathcal{F} \in D(\mathcal{O}_Y\text{-mod}_{\text{qc}})$, we have

$$\phi_{*,\text{dR}} \circ \text{ind}^r(\mathcal{F}) \simeq \phi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}) \simeq \phi_*(\mathcal{F} \otimes_{\mathcal{O}_Y} \phi^* \mathcal{D}_X) \simeq \phi_* \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X \simeq \text{ind}^r \circ \phi_*(\mathcal{F})$$

where the second last isomorphism is the (derived) projection formula. \square

We state the following results without proof.

Proposition 6.11. *If $\phi : Y \rightarrow X$ is proper, then we have adjoint functors*

$$\phi_{*,\text{dR}} : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \xleftarrow{\sim} D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \phi^!.$$

Remark 6.12. If $\phi : Y \rightarrow X$ is proper, then the square (5.1) can be obtained from (6.1) by passing to right adjoints.

Proposition 6.13. *If $\phi : Y \rightarrow X$ is smooth, then we have adjoint functors*

$$\phi^![-2d_Y + 2d_X] : D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) \xleftarrow{\sim} D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) : \phi_{*,\text{dR}}.$$

Example 6.14. If $j : U \rightarrow X$ is a closed embedding, then $j^! \simeq j^*$ is left adjoint to $j_{*,\text{dR}} = j_*$. Note that the right adjoint functor is fully faithful.

Construction 6.15. *As in the case of pullback functors, we can define the **direct image functor** of left \mathcal{D} -modules:*

$$\begin{array}{ccc} D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^l) & \xrightarrow{\phi_{*,\text{dR}}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^l) \\ \omega_Y[d_Y] \downarrow \simeq & & \simeq \downarrow \omega_X[d_X] \\ D(\mathcal{D}_Y\text{-mod}_{\text{qc}}^r) & \xrightarrow{\phi_{*,\text{dR}}} & D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r). \end{array}$$

7. KASHIWARA'S LEMMA

If $\phi : Y \rightarrow X$ is a closed embedding, then the tensor product functor $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$ is t-exact because $\mathcal{D}_{Y \rightarrow X}$ is locally free as a \mathcal{D}_Y -module. On the other hand, the functor ϕ_* is also t-exact because ϕ is affine. Therefore the functor $\phi_{*,\text{dR}}$ is t-exact.

Theorem 7.1 (Kashiwara's lemma). *Let $\phi : Y \rightarrow X$ be a closed embedding between smooth k -schemes, then the exact functor*

$$\phi_{*,\text{dR}} : \mathcal{D}_Y\text{-mod}_{\text{qc}}^r \rightarrow \mathcal{D}_X\text{-mod}_{\text{qc}}^r$$

is fully faithful and its essential image contains exactly right \mathcal{D}_X -modules that are set-theoretically supported on Y .

Remark 7.2. Using Kashiwara's lemma, we can define $\mathcal{D}_Y\text{-mod}_{\text{qc}}^r$ even for finite type singular k -scheme Y . Namely, if Y is affine, we can embed Y into a smooth ambient k -scheme X and define a right \mathcal{D} -module on Y to be a right \mathcal{D} -module on X that is set-theoretically supported on the image of Y . One can show the obtained abelian category does not depend on the choice of the embedding. When Y is not affine, we can define the category by gluing.

Moreover, all the previous constructions about right \mathcal{D} -modules can be generalized to the singular case.

A more canonical construction of $\mathcal{D}_Y\text{-mod}_{\text{qc}}^r$ or even $\mathcal{D}_Y\text{-mod}_{\text{qc}}^l$ for singular k -schemes is to use the theory of (Grothendieck's) crystals.

Another application of Kashiwara's lemma is the following result. See [G, Sect. 5.12] for a proof.

Corollary 7.3. *Let X be a smooth k -scheme, then any \mathcal{O}_X -coherent \mathcal{D}_X -module is locally free as an \mathcal{O}_X -module.*

8. BASE-CHANGE ISOMORPHISM AND PROJECTION FORMULA

Fact 8.1. *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\phi'} & X' \\ \downarrow f & & \downarrow g \\ Y & \xrightarrow{\phi} & X \end{array}$$

be a Cartesian square of finite type k -schemes. Then we have equivalences

$$g^! \circ \phi_{*,\text{dR}} \simeq \phi'_{*,\text{dR}} \circ f^!$$

between functors $D(\mathcal{D}_Y\text{-mod}_q c^r) \rightarrow D(\mathcal{D}_{X'}\text{-mod}_q c^r)$.

Fact 8.2. *Let $\phi : Y \rightarrow X$ be any morphism between finite type k -schemes. Then we have*

$$\phi_{*,\text{dR}}(- \overset{!}{\otimes} \phi^!(\bullet)) \simeq \phi_{*,\text{dR}}(-) \overset{!}{\otimes} \bullet.$$

Exercise 8.3. This is **Homework 5, Problem 5**. Let $x : \text{pt} \rightarrow X$ be a closed point of X . Prove² $\delta_x \otimes^! \delta_x \simeq \delta_x$.

Corollary 8.4. *Let $U \xrightarrow{j} X \xleftarrow{i} Y$ be finite type k -schemes such that i is a closed embedding and j is its complementary open embedding. Then for any $\mathcal{M} \in D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r)$, we have a distinguished triangle*

$$i_{*,\text{dR}} \circ i^! \mathcal{M} \rightarrow \mathcal{M} \rightarrow j_{*,\text{dR}} \circ j^! \mathcal{M} \xrightarrow{+1}.$$

Sketch. Consider the cone \mathcal{N} of the morphism $\mathcal{M} \rightarrow j_{*,\text{dR}} \circ j^* \mathcal{M}$ provided by the adjoint pair $(j^!, j_{*,\text{dR}})$. Note that $j^! \mathcal{N}$ is isomorphic to the cone of $j^! \mathcal{M} \rightarrow j^! \circ j_{*,\text{dR}} \circ j^* \mathcal{M}$. And the latter morphism is an isomorphism because $j_{*,\text{dR}} = j_*$ is fully faithful (see Example 6.14). Hence \mathcal{N} is a complex of right \mathcal{D}_X -modules that are set-theoretically supported on Y . By Kashiwara's lemma, we have $\mathcal{N} \simeq i_{*,\text{dR}} \circ i^!(\mathcal{N})$. Note that $i_{*,\text{dR}} \circ i^! \mathcal{N}$ is isomorphic to the cone of $i_{*,\text{dR}} \circ i^! \mathcal{M} \rightarrow i_{*,\text{dR}} \circ i^! \circ j_{*,\text{dR}} \circ j^! \mathcal{M}$, and the target is isomorphic to 0 by the base-change isomorphism. It follows that $i_{*,\text{dR}} \circ i^! \mathcal{N} \simeq i_{*,\text{dR}} \circ i^! \mathcal{M}[1]$ as desired. \square

²A formal proof exists, but you are encouraged to do some direct calculations to see $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = 0$ unless $i = -d_X$ and $\mathcal{H}^{-d}(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = \delta_x$.

9. DUALITY

The duality functor is only defined on *coherent* \mathcal{D} -modules.

Fact 9.1. *For any $\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_c^r)$, there exists a unique object $\mathbb{D}\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_c^r)$ such that*

$$\Gamma_{\text{dR}}(X, \mathcal{M} \overset{!}{\otimes} -) \simeq \text{Hom}(\mathbb{D}\mathcal{M}, -)$$

as functors $D(\mathcal{D}_X\text{-mod}_{\text{qc}}^r) \rightarrow \text{Vect}$. The obtained functor

$$\mathbb{D} : D^b(\mathcal{D}_X\text{-mod}_c^r)^{\text{op}} \rightarrow D^b(\mathcal{D}_X\text{-mod}_c^r)$$

is an anti-involution, i.e., $\mathbb{D} \circ \mathbb{D} \simeq \text{Id}$.

Remark 9.2. The construction of $\mathbb{D}\mathcal{M}$ can be treated as a blackbox. For completeness,

$$\mathbb{D}\mathcal{M} \simeq \text{Hom}_{\mathcal{D}_X^r}(\mathcal{M}, \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X[d]),$$

where

- $\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ has two right \mathcal{D}_X -module structures;
- The first one comes from the right multiplication of \mathcal{D}_X on itself. We use this right \mathcal{D}_X^r -structure to define the inner $\text{Hom}_{\mathcal{D}_X^r}$ object.
- The second one comes from the right \mathcal{D}_X -module structure on the tensor product of a right \mathcal{D}_X -module (i.e. ω_X) and a left \mathcal{D}_X -module (i.e. \mathcal{D}_X).
- The second right \mathcal{D}_X -module structure survives after taking the inner $\text{Hom}_{\mathcal{D}_X^r}$, and the RHS is viewed as a right \mathcal{D}_X -module using this structure.

Example 9.3. We have $\mathbb{D}(\omega_X) \simeq \omega_X$.

Construction 9.4. *Let $\phi : Y \rightarrow X$ be a map between finite type k -schemes. The standard functors $\phi^!$ and $\phi_{*,\text{dR}}$ in general do not preserve coherent complices. Hence we only have partially defined functors*

$$\phi_! := \mathbb{D} \circ \phi_{*,\text{dR}} \circ \mathbb{D}, \quad \phi_{\text{dR}}^* := \mathbb{D} \circ \phi^! \circ \mathbb{D}.$$

*They are called the **!-direct image functor** and the **de Rham pullback functor**.*

Fact 9.5. *Let $\phi : Y \rightarrow X$ be a map between finite type k -schemes. Then $\phi_!$ is equivalent to the partially defined left adjoint of $\phi^!$. More precisely, we have*

$$\text{Hom}(\phi_!\mathcal{M}, -) \simeq \text{Hom}(\mathcal{M}, \phi^!(-))$$

whenever $\phi_!\mathcal{M}$ is well-defined. Similarly, ϕ_{dR}^ is equivalent to the partially defined left adjoint of $\phi_{*,\text{dR}}$.*

Remark 9.6. If ϕ is proper, then $\phi_! \simeq \phi_{*,\text{dR}}$. If ϕ is smooth, then $\phi_{\text{dR}}^* \simeq \phi^![-2d_Y + 2d_X]$.

10. HOLONOMIC D-MODULES

We do not give the standard definition of holonomic D-modules. Instead, we characterize them as follows:

Fact 10.1. *Let $\mathcal{M} \in \mathcal{D}_X\text{-mod}_c^r$, then \mathcal{M} is **holonomic** iff $\mathbb{D}\mathcal{M} \in \mathcal{D}_X\text{-mod}_c^r$ (rather than just in the derived category).*

Fact 10.2. *Let $\mathcal{M} \in D^b(\mathcal{D}_X\text{-mod}_c^r)$, then \mathcal{M} has **holonomic cohomologies**, i.e., $\mathcal{H}^\bullet(\mathcal{M})$ are holonomic, iff for any closed point $i : x \rightarrow X$, the complex $i^!\mathcal{M} \in D^b(\mathcal{D}_{\text{pt}}\text{-mod}_c^r) \simeq D^b(\text{Vect})$ has finite dimensional cohomologies.*

Notation 10.3. Let $\mathcal{D}_X\text{-mod}_{\text{hol}}^r$ be the abelian category of holonomic right \mathcal{D}_X -modules and $D^b(\mathcal{D}_X\text{-mod}_{\text{hol}}^r)$ be the bounded derived category.

Fact 10.4. $D^b(\mathcal{D}_X\text{-mod}_{\text{hol}}^r)$ is equivalent to the full subcategory of $D^b(\mathcal{D}_X\text{-mod}_{\text{c}}^r)$ containing complexes with holonomic cohomologies.

Fact 10.5. All the functors defined so far preserve bounded holonomic complexes.

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