Geometrization of the Local Langlands

§1 : What's local Langlands.

- E is a local field $\left(\text{e.g. } \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t)) \right)$
- $G_{/E}$ is a split reductive group over E.
- $\hat{G}_{/\mathbb{Z}}$ is Langlands dual of G
- W_E : Weil group of E

<u>Conjecture</u> : There's a map between sets:

$$\left\{ \begin{array}{l} \text{irreducible objects} \\ \text{in } \mathrm{Rep}_{\mathbb{C}}(G(E)) \end{array} \right\} \longrightarrow \left\{ W_E \to \hat{G}(\mathbb{C}) \right\}$$

$$\pi \longmapsto \varphi_\pi$$

"L-parameter of π".

subjects to some compatibilies.

Rmk. (0) This is not a bijection

(1). From easy to hard for E:

archimedian , char p non-arch , char 0 non-arch.
($\mathbb{R}$, $\mathbb{C}$)   (e.g. $\mathbb{F}_q((t))$)   ($\mathbb{Q}_p$)

(2). $G(E)$ is a topological group

we require π to be "smooth"

(any vector is fixed by some quasi-cpt open of $G(E)$)

(3). W_E is a modification of $Gal(\bar{E}/E)$

- $W_{\mathbb{C}} = \mathbb{C}^\times$
- $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to Gal(\mathbb{C}/\mathbb{R}) \to 1$
- For non-arch E, residue field $k = \mathbb{F}_q$.

$$\begin{tikzcd}
1 \arrow[r] & I_E \arrow[r] \arrow[d, equal] & W_E \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, hook, "1 \mapsto \mathrm{Frob}"] & 1 \\
1 \arrow[r] & I_E \arrow[r] & Gal(\bar{E}/E) \arrow[r] & Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1
\end{tikzcd}$$

I_E: inertia grp

<u>Warning</u>: The topology on W_E is <u>not</u> the subspace topology induced from $\mathrm{Gal}(\bar{E}|E)$.

The correct topology: equip I_E with the subspace topology, and force it to be open inside W_E.

<u>Explain W_E</u>: It can already be seen in local class field theory.

["class formation"]

(4). We require φ_π to be continuous.

(5). Part of the compatibilities for $\pi \mapsto \varphi_\pi$ is about L-factors and ε-factors.

§2: What's geometrization?

<u>Answer</u>: Do global geometric Langlands on the <u>Fargues-Fontaine curve</u>, which behaves like a genus 0 curve in nonarchimedean geometry.

$E = \mathbb{Q}_p$. (For $\mathbb{F}_q((t))$, the story is easier)

<u>Analogy</u>

	alg. geo.	analytic geometry
affine	Spec R, $R \in \mathrm{CAlg}$	$\mathrm{Spa}(R, R^+)$ (R, R^+) is a <u>Huber</u> pair $R^+ \subset R \in \mathrm{CAlg}(\mathrm{Top})$ satisfies .....
globalization	Schemes as locally ringed spaces	(pre-)adic-space as locally <u>topologically</u> ringed space

point	$\mathrm{Spec}\, K$, K field	$\mathrm{Spa}(K, K^+)$.
		(K, K^+): <u>affinoid field</u>
		analytic $=$ non-discrete | non-analytic $=$ discrete

<u>Def</u>: <u>analytic affinoid field</u>.

K has a complete topology given by a non-arch valuation $K \to \mathbb{R}^+ \cup \{0\}$. ...

<u>Not</u>: In most cases people consider

$$\mathrm{Spa}\, R := \mathrm{Spa}(R, R^\circ)$$

$R^\circ \subset R$ subring of power-bounded objects.

(<u>Ex</u>: $R = \mathbb{Q}_p$, $R^\circ = \mathbb{Z}_p$).

<u>Rmk</u>: Among all the pre-adic spaces,

$$\{\text{perfectoid spaces}\} \subset \{\text{pre-adic spaces}\}$$

behave very well when you want to connect char 0 and char p geometry.

$\{\text{affine perfectoid spaces}\}^{op}$

$\simeq \{\text{perfectoid rings}\}$

<u>Ex</u>: 1) $\widehat{\mathbb{F}_p((t^{1/p^\infty}))}$

$:= \widehat{\bigcup_n \mathbb{F}_p((t^{1/p^n}))}$

2) $\mathbb{Q}_p^{cycl}$

$:= \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$

perfectoid rings are "super" ramified.

<u>Thm</u>: 1) There is no final object in $\mathrm{Perfd} = \{\text{perfectoid spaces}\}$.

2) Inside $\mathrm{Perfd}_p = \{\text{char } p \text{ perfectoid spaces}\}$, we have products.

Analogy:

	Global GL for function field	geometrization
geometry	alg. geo. over $\mathbb{F}_p$	"perfectoid geometry"
test objects.	schemes / $\mathbb{F}_p$	char p perfectoid spaces $S \in \mathrm{Perf}_{\mathbb{F}_p}$.
spaces	prestacks $\supset$ fpqc-stacks $\supset$ alg. spaces.	prestacks $\supset$ v-stack $\supset$ diamonds
absolute curve	X curve over $\mathbb{F}_p$.	Not exist/sci-fi
relative curve	$\forall$ S scheme $X_S = X \times_{\mathbb{F}_p} S$	$X_S = \mathcal{Y}_S / \mathrm{Frob}_S$ $\mathcal{Y}_S = S \dot{\times} \mathrm{Spa}\, \mathbb{Q}_p$

Question: S is of char p

$\mathbb{Q}_p$ is of char 0.

What's $S \dot{\times} \mathrm{Spa}\, \mathbb{Q}_p$

Answer: It's not a product. $\dot{\times}$

Ex: If $S = \mathrm{Spa}(R, R^+)$

$$S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p \underset{\text{open}}{\subset} \boxed{\mathrm{Spa}\, W(R^+)}$$

$W(R^+)$: ring of p-Witt-vectors in R^+

is the open locus where p and $[\varpi]$ are invertible
$$[\varpi] \in W(R^+)$$

In particular, $S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ is of char 0.

Question: If $X_S := (S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)/\mathrm{Frob}_S$

is of char 0, then how to study it using char p testing objects?

Ex: How to define the notion of Cartier divisors on X_S?

What's the base?

It can't be $\underbrace{S \text{ or } S/\text{Frob}S}_{\text{char } p}$.

<u>Assum.</u> In fact, we do not use X_S.

Instead $(X_S)^{\diamond}$: the associated diamond of it.

(of char p).

<u>Construction</u> (Tilting):

$\forall\ R \in \mathrm{CAlg}$. (p)-tilt:

$$R^{\flat} := \lim\left(\cdots R \xrightarrow{\text{Frob}} R \xrightarrow{\text{Frob}} R \right).$$

A priori, $R^{\flat}$ is only a multiplicative monoid.

<u>Homework</u>: if R is nice enough

(e.g. R is perfectoid),

you can define $(R^\flat, +)$

$(x^{(0)}, x^{(1)}, \ldots) \in R^\flat$

$\mathrm{Frob}(x^{(n+1)}) = x^{(n)}$.

$(x^{(0)}, x^{(1)}, \ldots) + (y^{(0)}, y^{(1)}, \ldots)$

$= (z^{(0)}, \ldots\ldots)$

$$z^{(i)} := \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n}$$

$\Rightarrow$ For R perfectoid.

$R^\flat$ is a perfectd ring of char p.

Spd