

Man-Thm:

$$\mathbb{L}_G : \mathrm{DMod}_{\mathrm{crit}}^+(\mathrm{Bun}_G) \longrightarrow \mathrm{Ind}\mathrm{Grp}(\mathrm{LS}_G^\circ)$$

compatible with derived Satake, Eisenstein series, duality etc.

Recall:

Generalized vanishing conjecture \Rightarrow

$$\mathrm{DMod}_{\mathrm{crit}}^+(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G^0} \mathrm{QGr}(\mathrm{LS}_G^\circ)$$

coeffn. ↓ ↓ P_G^{spec}

$$\mathrm{Wh}^+(\mathrm{Gr}_G)_{\mathrm{per}} \xleftarrow{\sim} \mathrm{Rep}(\tilde{G})_{\mathrm{per}}$$

(\mathbb{L}_G^0 will be $\Psi \circ \mathbb{L}_G$, $\Psi : \mathrm{Ind}\mathrm{Grp} \rightarrow \mathrm{QGr}$)

$$\mathrm{DMod}_{\mathrm{crit}}^+(\mathrm{Bun}_G)^c \xrightarrow{\mathbb{L}_G^0} \mathrm{QGr}(\mathrm{LS}_G^\circ)^{\geq 0} \quad (\text{by [FR]})$$

↓
IndGrp⁺(LS_G)^{≥ 0}
↓
IndGrp⁺(LS_G)_{Nilp}

\mathbb{L}_G is obtained by Ind-completion.

Shorthand:

$$\mathbb{D}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G} \mathrm{QC}^+(\mathrm{LS}_G^\circ)$$

By construction, compatible
with derived Satake

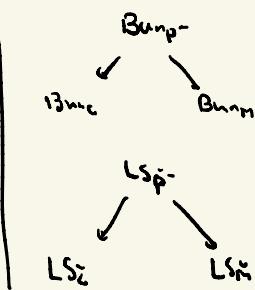
Thm:
(Eis, c7)

$$\mathbb{D}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G} \mathrm{QC}^+(\mathrm{LS}_G^\circ)$$

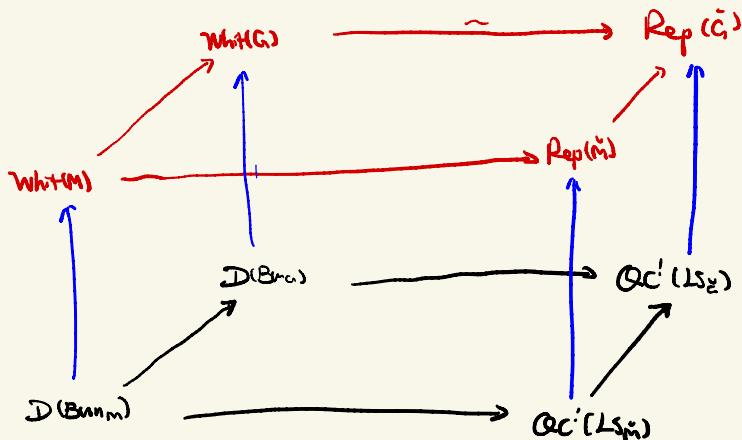
$$\mathrm{Eis} \uparrow \downarrow \mathrm{CT}_G$$

$$\mathbb{D}(\mathrm{Bun}_M)$$

$$\xrightarrow{\mathbb{L}_M} \mathrm{QC}^+(\mathrm{LS}_M^\circ)$$



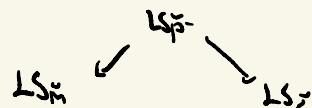
Proof of Eis:



Need top and two side squares commute.

④ Spectral side:

$$\begin{array}{ccc} \text{Rep}(M) & \xrightarrow{\quad ? \quad} & \text{Rep}(G) \\ \Gamma_{\text{spec}} \uparrow & & \uparrow \Gamma_{\text{spec}} \\ (\mathbb{Q}C^*(LS_M)) & \xrightarrow{E_{\text{spec}}} & (\mathbb{Q}C^*(LS_Z)) \end{array}$$



Paradigm: For $H_1 \rightarrow H_2$

$$\begin{array}{ccc} LS_{H_1} & \longrightarrow & (IBH_1)_+ \\ \downarrow & & \downarrow \\ LS_{H_2} & \longrightarrow & (IBH_2)_+ \end{array}$$

$$\begin{array}{ccc} (\mathbb{Q}C^*(LS_{H_2})) & \xrightarrow{\quad \overline{\rho} \quad} & \text{Rep}(H_2)_+ \\ \uparrow \text{push} & & \uparrow \text{inv} \\ (\mathbb{Q}C^*(LS_{H_1})) & \xrightarrow{\quad \overline{\rho} \quad} & \text{Rep}(H_1)_+ \end{array}$$

But no obvious diagram for pull. vs $\overline{\rho}$.

Dually pull vs loc

$$\begin{array}{ccc} \mathcal{QC}^!(LS_{H_2}) & \xleftarrow{\text{loc}} & \text{Rep}(H_2)_x \\ \downarrow \text{pull} & & \downarrow \text{res} \\ \mathcal{QC}^!(LS_{H_1}) & \xleftarrow{\text{loc}} & \text{Rep}(H_1)_x \end{array}$$

However, there is a diagram for
push vs loc:

Zero: If $H_1 \rightarrow H_2$ has unipotent kernel,

$$\begin{array}{ccc} \mathcal{QC}^!(LS_{H_2}) & \xleftarrow{\text{loc}} & \text{Rep}(H_2)_{\text{Ran}_x} \\ \uparrow \text{push} & & \uparrow \text{inv}^+ \\ \mathcal{QC}^!(LS_{H_1}) & \xleftarrow{\text{loc}} & \text{Rep}(H_1)_x \end{array}$$

Here $\text{Ran}_x = \{ I \in \text{Ran} \mid x \in I \}$

$$\begin{array}{ccc} \text{inv}^+ : \text{Rep}(H_1)_x & \xrightarrow{\text{ins}_x} & \text{Rep}(H_1)_{\text{Ran}_x} \\ & & \downarrow \text{inv} \\ & & \text{Rep}(H_2)_{\text{Ran}_x} \end{array}$$

$$\text{ins}(V_x)_{\{x, y_1, \dots, y_m\}} := V_x \otimes \mathbb{1}_{y_1} \otimes \dots \otimes \mathbb{1}_{y_m}$$

More generally : $y \xrightarrow{\pi} x$ D-scheme

If y is quasi-affine or coaffine, or more generally

$$QC(y) \simeq \mathbb{H}_{\pi_*}^0 \text{-mod}(QC(x)).$$

Note that $\pi_{*} \mathcal{O}_y$ is a complex

$$\underline{\text{Ex}}: A^2 \setminus 0 \longrightarrow *, \quad B \mathbb{G}_m \longrightarrow *$$

satisfy similar conditions

Prop :

$$QC(y(D_x)) \xrightarrow{\text{ins}} QC^!(y(D_x))_{\text{Param}}$$



$$QC(y(x)) \xrightarrow{P} \text{Vect}$$

where $y(D_x)_{\text{Param}}$ means at $(x, y_1, \dots, y_n) \in \mathbb{R}^n$

fiber is $y(D_x) \times y(D_{y_1}) \times \dots \times y(D_{y_n})$

$$\text{ins}(M) = M \otimes 0 \otimes \dots \otimes 0.$$

Prop \Rightarrow Lem : case when M is supported on

$$y(D_x) \hookrightarrow y(D_x^0).$$

& use a parametrized version

$(BH_1 \rightarrow BH_2 \text{ is coaffine}).$

Now Lem \Rightarrow

$$\begin{array}{ccc}
 QC^!(LS_{\tilde{\mu}}) & \xleftarrow{\text{loc}} & \text{Rep}(\tilde{\mu})_x \\
 \downarrow \text{push} & & \downarrow \text{inv} \\
 QC^!(LS_{\tilde{\mu}}) & \xleftarrow{\text{loc}} & \text{Rep}(\tilde{\mu})_{\text{Per}_{xc}}
 \end{array}$$

$$\Rightarrow QC^!(LS_{\tilde{\mu}}) \xleftarrow{\text{loc}} \text{Rep}(\tilde{\mu})_x$$

$$\begin{array}{ccc}
 \downarrow \text{CT} & & \downarrow \text{chev}^+ \\
 QC^!(LS_{\tilde{\mu}}) & \xleftarrow{\text{loc}} & \text{Rep}(\tilde{\mu})_{\text{Per}_{xc}}
 \end{array}$$

Explicitly,

$$chev^+(M_{(x_1, y_1, \dots, y_m)}) = M_x^{U^-} \otimes 1_{y_1}^{U^-} \otimes \dots \otimes 1_{y_m}^{U^-}$$

($1^{U^-} = C(U^-)$ chevalley complex).

$$\Rightarrow QC^!(LS_{\tilde{\mu}}) \xrightarrow{\tilde{P}} \text{Rep}(\tilde{\mu})_x$$

$$\begin{array}{ccc}
 \uparrow E^{\pm} & & \uparrow \text{dual functor} \\
 QC^!(LS_{\tilde{\mu}}) & \xrightarrow{P} & \text{Rep}(\tilde{\mu})_{\text{Per}_{xc}}
 \end{array}$$

• Geometric side

$$\begin{array}{ccc}
 D(B_{\mu_0}) & \xrightarrow{\text{coeff}} & \text{Whit}(G)_x \\
 \uparrow E_1^- & & \uparrow \text{dual functor} \\
 D(B_{\mu, m}) & \xrightarrow{\text{coeff}} & \text{Whit}(M)_{\text{Per}_{xc}}
 \end{array}$$

The dual functor

$$\text{Whit}(G) \longrightarrow \text{Whit}(M)_{\text{PM, c}}$$

is given by inserting unit for a functor

$$\text{Whit}(G) \xrightarrow{\mathcal{J}^-} \text{Whit}(M)$$

"

$$D(G_{\alpha})^{(\mathbb{L}N, \chi_{\alpha})} \qquad \qquad \qquad D(G_M)^{(\mathbb{L}N_M, \chi_M)}$$

Def 1:

$$\begin{aligned} S\mathcal{I}_{\bar{p}}^- &:= D(G_{\alpha})^{\mathbb{L}N^- \cdot \mathbb{L}^+ M} \quad (p = \beta \text{ or } N(K), T(O)) \\ &= (D(G_{\alpha}) \otimes D(G_M))^{\mathbb{L}p^-} \\ &= \text{Fun}_{\mathbb{L}p^-}(D(G_{\alpha}), D(G_M)) \end{aligned}$$

For any $F: D(G_{\alpha}) \longrightarrow D(G_M)$ $\mathbb{L}p^-$ -linear.

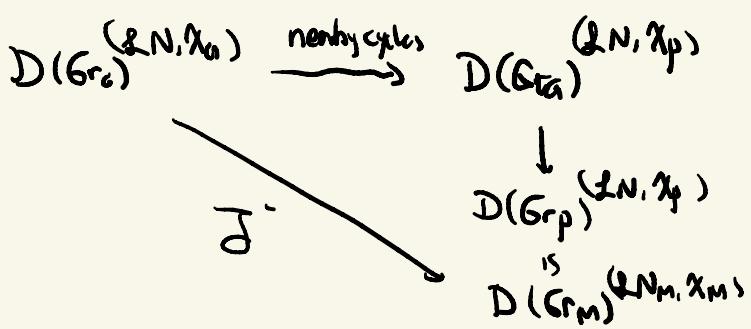
$$\rightsquigarrow D(G_{\alpha})^{(\mathbb{L}N, \chi_{\alpha})} \xrightarrow{\text{oblv}} D(G_M)^{(\mathbb{L}N_M, \chi_M)} \xrightarrow{F} D(G_M)^{(\mathbb{L}N_M, \chi_M)}$$

Now consider F corresponds to the unit of $S\mathcal{I}_{\bar{p}}$, i.e., the standard object on the orbit $\mathbb{L}N^- \cdot 1 \cdot \mathbb{L}^+ G / \mathbb{L}^+ G \hookrightarrow G_{\alpha}$.

Def 2: characters of $\mathbb{L}N$

$$\chi_{\alpha} \rightsquigarrow \chi_p$$

(non-deg) (trivial on $\mathbb{L}N$
gives χ_M on $\mathbb{L}N_M$)



Why Def 1 uses P^\vee , Def 2 uses P ?

2nd - adjointness. No time to explain.

The derived diagram

$$\begin{array}{ccc}
 D(\text{Bun}_G) & \longrightarrow & \text{Wh}(G)_{\text{perf.}} \\
 \uparrow E_5 & & \uparrow \text{dual functor} \\
 D(\text{Bun}_M) & \longrightarrow & \text{Wh}(M).
 \end{array}$$

can be proven pure geometrically.

- The top square, need

$$\begin{array}{ccc}
 \text{Wh}(G) & \xrightarrow{\sim} & \text{Rep}(k)^\vee \\
 \downarrow J^- & & \downarrow \text{chev.} \\
 \text{Wh}(M) & \xrightarrow{\sim} & \text{Rep}(k)^\vee
 \end{array}$$

Can be proven using

$$\begin{aligned}
 & \text{Thm} \\
 S_{\bar{\mathcal{G}}_p} &= D(\mathcal{G}_0) \xrightarrow{\text{functn}} S_{\bar{\mathcal{G}}_p}^{\text{spur}} \\
 &\simeq \text{Rep}^! \left(\frac{LS_{\bar{\mathcal{G}}_p}^{\text{ur}}(D) \times LS_p(\mathbb{F})}{LS_{\bar{\mathcal{G}}_p}^{\text{ur}}(D)} \right)
 \end{aligned}$$

(case $p = \mathbb{Q}_p$, this is derived Satake !)

Why expect this in local Langlands?

$$\begin{array}{ccc}
 \mathcal{L}_{\mathcal{G}}\text{-mod} & \longrightarrow & \text{ShuCat}(LS_{\bar{\mathcal{G}}}^{\text{ur}}(\mathbb{F})) \\
 (\downarrow \mathcal{L}_{\mathcal{G}}) & & \downarrow (\downarrow |_{LS_p(\mathbb{F})}) \\
 \mathcal{L}_{\mathcal{M}}\text{-mod} & \longrightarrow & \text{ShuCat}(LS_{\bar{\mathcal{M}}}^{\text{ur}}(\mathbb{F}))
 \end{array}$$

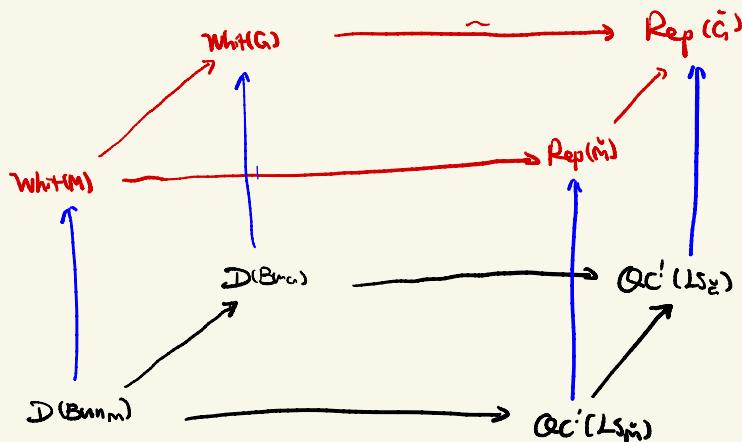
$$\begin{array}{ccc}
 \text{Start with } D(\mathcal{G}_0) & \rightsquigarrow & \text{Rep}^!(LS_{\bar{\mathcal{G}}}^{\text{ur}}(D)) \\
 & \downarrow & \downarrow \\
 D(\mathcal{G}_0)^{\mathcal{L}_{\mathcal{G}}} & \rightsquigarrow & \text{Rep}^!(LS_{\bar{\mathcal{G}}}(D) \times LS_p(\mathbb{F})) \\
 & & LS_p(\mathbb{F})
 \end{array}$$

$$D(\mathcal{G}_0) \rightsquigarrow \text{Rep}^!(LS_{\bar{\mathcal{M}}}^{\text{ur}}(\mathbb{F}))$$

Calculate $F_m(-, -)$ on both sides.

This gives theorem E.b.

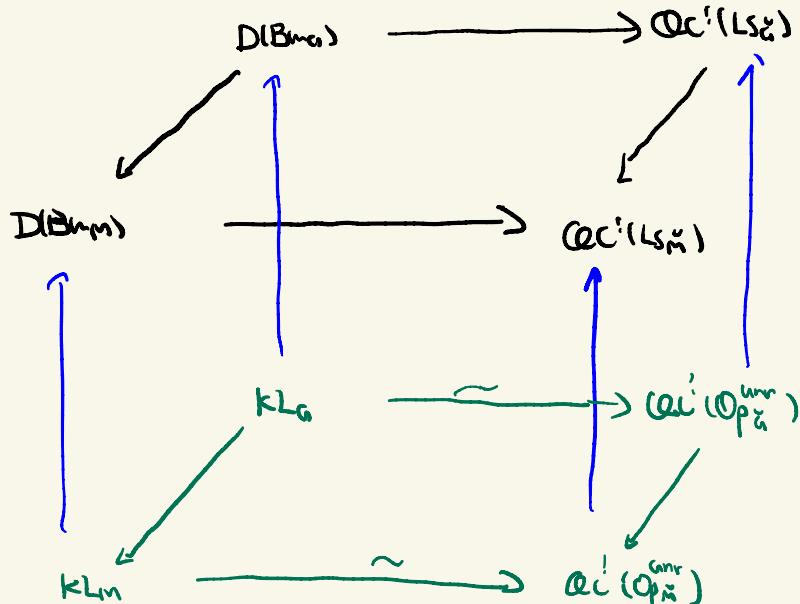
The same method does not give Thm CT;



No way to flip the arrows of direction



However : $\text{Wh}(G) \rightarrow KM$ duality suggests



$$\begin{array}{ccc}
 \text{Whit}_K(M) & \longrightarrow & \text{Whit}_K(G) \\
 \downarrow & & \uparrow \\
 D_K(B_{\text{dR}, M}) & \longrightarrow & D_K(B_{\text{dR}, G}) \\
 \rightsquigarrow K \rightarrow \infty & \text{QC}^!(\mathcal{O}_{B_{\text{dR}}^{unr}}) & \longrightarrow \text{QC}^!(\mathcal{O}_{B_{\text{dR}}^{unr}}^{\text{ur}}) \\
 \uparrow & & \uparrow \\
 \text{QC}^!(L_M) & \longrightarrow & \text{QC}^!(L_G)
 \end{array}$$

passing to duality gives the right face.

The functors

- $KL_G \longrightarrow KL_M$
- $M \longmapsto (\Sigma^\infty_+ U((t)), M)$

(proof is similar to spectral in Thm 56)

- $\text{QC}^!(\mathcal{O}_{B_{\text{dR}}^{unr}}^{\text{ur}}) \longrightarrow \text{QC}^!(\mathcal{O}_{B_{\text{dR}}^{unr}}^{\text{ur}})$
acted by unit in $S\mathbb{F}_{p^{\infty}, \text{spec}}$.

Prove the top & side faces commute.

However, we haven't shown the front or back
faces commute.

$$\begin{array}{ccc} D(B_{\text{tw}}) & \longrightarrow & \mathcal{O}\mathcal{C}^!(LS_L^{\vee}) \\ \uparrow & & \downarrow \\ K_{L_G} & \longrightarrow & \mathcal{O}\mathcal{C}^!(O_{P_L^{\text{unr}}}^{\text{unr}}) \end{array}$$

$$\begin{array}{ccc} \text{Nerv } W_{L_G} & \longrightarrow & R_{\mathcal{P}}(Y) \\ \uparrow & & \uparrow \\ D(B_{\text{tw}}) & & \mathcal{O}\mathcal{C}^!(LS_R^{\vee}) \\ \uparrow & & \uparrow \\ K_{L_R} & \longrightarrow & \mathcal{O}\mathcal{C}^!(O_{P_R^{\text{unr}}}^{\text{unr}}) \end{array}$$

chiral method.

□ Then CT.

Len: $\mathbb{H}_G^L D(B_{\text{tw}}) \longrightarrow \mathcal{O}\mathcal{C}^!(LS_R^{\vee})$
has a left adjoint. \mathbb{H}_G^L

Front: $\mathcal{O}\mathcal{C}^!(LS_L^{\vee})$ is generated by $E_{\text{spec}, p}^{\vee}(\mathcal{O}\mathcal{C}(LS_M^{\vee}))$

Only need existence of.

$\mathcal{QC}(LS_m^{\text{ur}})$

↓ Eigspac.

$$D(B_{\text{red}}) \xleftarrow{H^L} \mathcal{QC}'(LS_m^{\text{ur}})$$

i.e. The following has left adjoint.

$$D(B_{\text{red}}) \xrightarrow{H} \mathcal{QC}'(LS_m^{\text{ur}})$$

$$\begin{array}{ccc} D(B_{\text{red}}) & \xrightarrow{H} & \mathcal{QC}'(LS_m^{\text{ur}}) \\ \downarrow C_T & & \downarrow C_T \\ D(B_{\text{red}}) & \xrightarrow{H} & \mathcal{QC}'(LS_m^{\text{ur}}) \\ & \searrow H^{\text{coadj}} & \downarrow f \\ & & \mathcal{QC}(LS_m^{\text{ur}}) \end{array}$$

But we know H^{coadj} has a left adjoint

$$\begin{array}{ccc} \mathcal{Rep}(M)_{\text{perf}} & \xrightarrow{\text{act on } W_i} & \\ \downarrow & & \\ \mathcal{QC}(LS_m^{\text{ur}}) & \dashrightarrow & D(B_{\text{red}}) \end{array}$$

Thm: $H^L_{G_i} D(B_{\text{red}})_{E_i} \xrightarrow{\sim} \mathcal{QC}'(LS_m^{\text{ur}})_{\text{red}}$

if H_M is invertible for all proper M .

reduce to show
 \sim

$$H^L_{\text{cusp}} : D(B_{\text{red}})_{\text{cusp}} \xrightarrow{\sim} \mathcal{QC}(LS_i, \text{inv})$$

Next time