NOTES FOR ALGEBRAIC GEOMETRY 1

LIN CHEN

Contents

Date: Dec. 19, 2025.

0. Introduction: why schemes?

0.1. **Algebraic sets.** Before scheme theory, algebraic geometry focused on *algebraic sets*.

Definition 0.1.1. Let k be an algebraically closed field.

- The **Zariski topology** on the affine space $\mathbb{A}_k^n$ is the topology with a base consisting of all the subsets that are equal to the non-vanishing locus $U(f)$ of some polynomial $f \in k[x_1, \cdots, x_n]$.
- An **embedded affine algebraic set**[1] in $\mathbb{A}_k^n$ is a closed subspace for the Zariski topology.
- An **embedded quasi-affine algebraic set** is a Zariski open subset of an embedded affine algebraic set.

Example 0.1.2. Any finite subset of $\mathbb{A}_k^n$ is an embedded affine algebraic set.

Example 0.1.3. $\mathbb{Z}$ is not an embedded affine algebraic set in $\mathbb{A}_{\mathbb{C}}^1$.

Similarly one can define *embedded (quasi)-projective algebraic set* using *homogeneous* polynomials and the projective space $\mathbb{P}_k^n$.

There are tons of shortcomings in the above definition. An obvious one is that the notions of *embedded* algebraic sets are not *intrinsic*.

Example 0.1.4. The embedded affine algebraic sets $\mathbb{A}_k^1 \subseteq \mathbb{A}_k^1$ and $\mathbb{A}_k^1 \subseteq \mathbb{A}_k^2$ should be viewed as the same algebraic sets.

Notation 0.1.5. To remedy this, we need some notations.

- For an ideal $I \subseteq k[x_1, \cdots, x_n]$, let $Z(I) \subseteq \mathbb{A}_k^n$ be the locus of common zeros of polynomials in I.
- For a Zariski closed subset $X \subseteq \mathbb{A}_k^n$, let $I(X) \subseteq k[x_1, \cdots, x_n]$ be the ideal of all polynomials vanishing on X.

Recall an ideal I is called *radical* if $I = \sqrt{I}$.

Theorem 0.1.6 (Hilbert Nullstellensatz)**.** *We have a bijection:*

$$\begin{aligned} \left\{\textit{radical ideals of } k[x_1, \cdots, x_n]\right\} &\longleftrightarrow \left\{\textit{Zariski closed subsets of } \mathbb{A}_k^n\right\} \\ I &\longrightarrow Z(I) \\ I(X) &\longleftarrow X. \end{aligned}$$

Part of the theorem says the set of points of $\mathbb{A}_k^n$ is in bijection with the set of maximal ideals of $k[x_1, \cdots, x_n]$. As a corollary, $Z(I)$ is in bijection with the set of maximal ideals containing I. The latter can be further identified with maximal ideals of $R := k[x_1, \cdots, x_n]/I$.

Note that I is radical iff R is *reduced*, i.e., contains no nilpotent elements. This justifies the following definition.

Definition 0.1.7. An **affine algebraic k-set** is a *maximal spectrum* $\mathsf{Spm}\, R$ (= sets of maximal ideals) of a *finitely generated* (commutative unital) *reduced* k-algebra R. We equip it with the **Zariski topology** with a base of open subsets given by

$$U(f) := \left\{\mathfrak{m} \in \mathsf{Spm}\, R \mid f \notin \mathfrak{m}\right\},\ f \in R.$$

[1]Some people use the word *variety*, while some people reserve it for *irreducible* algebraic sets.

Example 0.1.8. $\mathsf{Spm}\, k[x] \simeq \mathbb{A}^1_k$.

We have the following *duality* between algebra and geometry.

Algbera	Geometry
finitely generated reduced k-algebra R	affine algebraic k-set X
maximal ideals $\mathfrak{m} \subseteq R$	points $x \in X$
elements $f \in R$	functions $\phi : X \to \mathbb{A}^1_k$
radical ideals $I \subseteq R$	Zariski closed subsets $Z \subseteq X$

Here an element $f \in R$ corresponds to the function

$$\phi : \mathsf{Spm}\, R \to k,\ \mathfrak{m} \mapsto \underline{f}$$

sending a maximal ideal $\mathfrak{m}$ to the image $\underline{f}$ of f in the *residue field* of $\mathfrak{m}$, which is canonically identified with the underlying set of $\mathbb{A}^1_k$ via the composition $k \to R \to R/\mathfrak{m}$.

The word *duality* means the correspondence $R \leftrightarrow X$ is *contravariant*. Indeed, given a homomorphism $f : R' \to R$, we obtain a *continuous* map

$$\mathsf{Spm}\, R \to \mathsf{Spm}\, R',\ \mathfrak{m} \mapsto f^{-1}(\mathfrak{m}).$$

Note however that not all continuous maps $\mathsf{Spm}\, R \to \mathsf{Spm}\, R'$ are obtained in this way, nor is R determined by the topological space $\mathsf{Spm}\, R$.

Exercise 0.1.9. Show that any bijection $\mathbb{A}^1_k \to \mathbb{A}^1_k$ is continuous for the Zariski topology. Find those bijections coming from a homomorphism $k[x] \to k[x]$.

This motivates the following definition.

Definition 0.1.10. A **morphism** from $\mathsf{Spm}\, R$ to $\mathsf{Spm}\, R'$ is a continuous map coming from a homomorphism $R' \to R$.

Then one can define general algebraic k-sets by gluing affine algebraic k-sets using morphisms, just like how people define *structured* manifolds as glued from *structured* Euclidean spaces using maps preserving the addiontal structrues.

0.2. **Shortcomings.** The theory of algebraic k-sets provides a bridge between algebra and geometry. In particular, one can use topological/geometric methods, including various cohomology theories, to study *finitely generated reduced k-algebras*. However, these adjectives are non-necessary restrictions to this bridge.

First, number theory studies number fields and their rings of integers, such as $\mathbb{Q}$ and $\mathbb{Z}$. It is desirable to have geometric objects corresponding to them. Hence we would like to consider general commutative rings rather than k-algebras. Then one immediately realizes the maximal spectra Spm are not enough.

Example 0.2.1. The map $\mathbb{Z} \to \mathbb{Q}$ does not induce a map from $\mathsf{Spm}\, \mathbb{Q}$ to $\mathsf{Spm}\, \mathbb{Z}$. Namely, the inverse image of $(0) \subseteq \mathbb{Q}$ in $\mathbb{Z}$ is a non-maximal prime ideal.

This suggests for general algebra R, we should consider its *prime spectrum*, denoted by $\mathsf{Spec}\, R$, rather than just its maximal spectrum.

Second, even in the study of finitely generated algebras, one naturally encounters non-finitely generated ones.

Example 0.2.2. Let $\mathfrak{p} \subseteq R$ be a prime ideal of a finitely generated algebra. The localization $R_\mathfrak{p}$ and its completion $\widehat{R}_\mathfrak{p}$ are in general not finitely generated.

Of course, one can restrict their attentions to *Noetherian* rings (and live a happy life). But let me object the false feeling that all natural rings one care are Noetherian.

Example 0.2.3. Noetherian rings are not stable under tensor products: $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not Noetherian.

Example 0.2.4. The ring of adeles of $\mathbb{Q}$ is not Noetherian.

Example 0.2.5. Natural moduli problems about Noetherian objects can fail to be Noetherian. My favorite ones includes: the space of formal connections, the space (stack) of formal group laws...

In short, there are important applications of non-Noetherian rings, and this course will deal with the latter.

Finally, we want to remove the restriction about reducedness.

Example 0.2.6. Reduced rings are not stable under tensor products: $k[x] \otimes_{k[x,y]} k[x,y]/(y-x^2)$ is not reduced. Geometrically, this means $Z(y)$ and $Z(y-x^2)$ do not intersect transversally inside $\mathbb{A}^2_k$.

One may notice that without reducedness, we should accordingly consider all ideals rather than just *radical* ideals, but then the construction $I \mapsto Z(I)$ would not be bijective. Indeed, ideals with the same nilpotent radical would give the same *topological subspace* of $\mathsf{Spec}\, R$.

But *this is a feature rather than a bug.* In Example 0.2.6, the ideal $(y, y-x^2) = (x^2, y)$ is not radical, and it carries geometric meanings that cannot be seen from its nilpotent radical (x,y). Namely, $f \in (x,y)$ iff $f(0,0) = 0$, while $f \in (x^2, y)$ iff $f(0,0) = \partial_x f(0,0) = 0$. Roughly speaking, this suggests that $(y, y-x^2)$ remembers that the curves $Z(y)$ and $Z(y-x^2)$ are tangent to each other at the point $(0,0) \in \mathbb{A}^2_k$, and the tangent vector is $\partial_x|_{(0,0)}$. Also note that the length of $k[x,y]/(y, y-x^2)$ is equal to 2, which is the number of intersection points predicted by the Bézout's theorem.

In summary, on the algebra side, we should consider *all* commutative rings. On the geometric side, the corresponding notion is called *affine schemes.* Our first task in this course is to develop the following duality:

Algebra	Geometry
commutative rings R	affine schemes X
prime ideals $\mathfrak{p} \subseteq R$	points $x \in X$
elements $f \in R$	functions $X \to \mathbb{A}^1_{\mathbb{Z}}$
ideals $I \subseteq R$	closed subschemes $Z \subseteq X$.

0.3. **Schemes as structured spaces.** In theory, one can *define* a morphism between affine schemes to be a continuous map coming from a ring homomorphism. Then one can define general *schemes* by gluing affine schemes using such morphisms. This mimics the definition of differentiable manifold in the sense that a scheme would be a topological space equipped with a *maximal* affine atlas.

In practice, the above approach is awkward to work with. We prefer a more efficient way to encode the additional structure on the topological space underlying a scheme. One extremely simple but powerful way to achieve this, maybe discovered by Serre and popularized by Grothendieck, is the notion of *sheaves* on topological spaces. Roughly speaking, a sheaf $\mathcal{F}$ on X is an *contravariant* assignment

$$U \mapsto \mathcal{F}(U)$$

sending open subsets $U \subseteq X$ to certain structures (e.g. sets, groups, rings) $\mathcal{F}(U)$, such that a certain gluing condition is satisfied. Here contravariancy means that for $U \subseteq V$, we should provide a map $\mathcal{F}(V) \to \mathcal{F}(U)$ preserving the prescribed structures.

Example 0.3.1. Let X be a topological space. The assignment

$$U \mapsto C(U, \mathbb{R})$$

sending $U \subseteq X$ to the ring of continuous functions on U would be a sheaf of commutative rings on X.

Similarly, for a smooth manifold X, $U \mapsto C^\infty(U, \mathbb{R})$ would be a sheaf of commutative rings on X. This motivates us to define:

Pre-Definition 0.3.2. A **scheme** is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X$ such that locally it is isomorphic to an affine scheme.

Here for an open subset $U \subseteq X$, $\mathcal{O}_X(U)$ should be the ring of *algebraic* functions on U, but we have not defined the latter notion yet. Nevertheless, for an *affine* scheme $X \simeq \mathsf{Spec}\, R$, the previous discussion suggests we should have $\mathcal{O}_X(X) \simeq R$. As we shall see in future lectures, we can bootstrap from this to get the definition of the entire sheaf $\mathcal{O}_X$.

The goal of this course is to define schemes and study their basic properties.

Part I. (Pre)sheaves

1. Definition of (pre)sheaves

1.1. Presheaves.

Definition 1.1.1. Let X be a topological space and $(U(X), \subseteq)$ be the partially ordered set of open subsets of X. We define the **category $\mathfrak{U}(X)$ of open subsets** in X to be the category associated to the partially ordered set $(U(X), \subseteq)$.

The category $\mathfrak{U}(X)$ can be explicitly described as follows:

- An object in $\mathfrak{U}(X)$ is an open subset $U \subseteq X$.
- If $U \subseteq V$, then $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is a singleton; otherwise $\mathsf{Hom}_{\mathfrak{U}(X)}(U, V)$ is empty.
- The identify morphisms and composition laws are defined in the unique way.

Definition 1.1.2. Let X be a topological space and $\mathcal{C}$ be a category.

- A **$\mathcal{C}$-valued presheaf on** X is a functor $\mathcal{F} : \mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C}$.
- A **morphism** $\mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued presheaves is a natural transformation between these functors.

Let Set be the category of sets. By definition, a **presheaf $\mathcal{F}$ of sets**, i.e., a Set-valued presheaf, on X consists of the following data:

- For any open subset $U \subseteq X$, we have a set $\mathcal{F}(U)$, which is called the **set of sections** of $\mathcal{F}$ on U.
- For $U \subseteq V$, we have a map
$$\mathcal{F}(V) \to \mathcal{F}(U),\ s \mapsto s|_U$$
which is called the **restriction map**.

These data should satisfy the following condition:

- For any open subset $U \subseteq X$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
- For $U \subseteq V \subseteq W$, the restriction maps make the following diagram commute
$$\begin{tikzcd} & \mathcal{F}(V) \arrow[dr] & \\ \mathcal{F}(U) \arrow[ur] \arrow[rr] & & \mathcal{F}(W). \end{tikzcd}$$

Let $\mathcal{F}$ adn $\mathcal{F}'$ be presheaves of sets on X. By definition, a morphism $\phi : \mathcal{F} \to \mathcal{F}'$ consists of the following data:

- For any open subset $U \subseteq X$, we have a map $\phi_U : \mathcal{F}(U) \to \mathcal{F}(U)'$.

These data should satisfy the following condition:

- For $U \subseteq V$, the following diagram commute
$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d] & \mathcal{F}'(V) \arrow[d] \\ \mathcal{F}(U) \arrow[r, "\phi_U"] & \mathcal{F}'(U), \end{tikzcd}$$
where the vertical maps are restriction maps.

Similarly one can explicitly describe the notion of presheaves of abelian groups (k-vector spaces, commutative algebras) and morphisms between them.

Example 1.1.3. Let X be a topological space and $\mathcal{C}$ be a category. For any object $A \in \mathcal{C}$, the constant functor

$$\mathfrak{U}(X)^{\mathsf{op}} \to \mathcal{C},\ U \mapsto A,\ f \mapsto \mathsf{id}_A$$

defines a $\mathcal{C}$-valued presheaf on X, which is called the **constant presheaf associated to** A. It is often denoted by $\underline{A}$.

Example 1.1.4. Let X be a topological space and $E \to X$ be a topological space over it. We define a presheaf Sect_E of sets as follows.

- For any $U \subseteq X$,
$$\mathsf{Sect}_E(U) := \mathsf{Hom}_X(U, E)$$
is the set of countinuous maps $U \to E$ defined over X, a.k.a. sections of E over U.
- For $U \subseteq V$, the restriction map $\mathsf{Sect}_E(V) \to \mathsf{Sect}_E(U)$ sends a section $s : V \to E$ to its restriction $s|_U : U \to E$.

We call it the **presheaf of sections for** $E \to X$.

Example 1.1.5. If $E \to X$ is a real vector bundle, we can naturally upgrade Sect_E to be a presheaf of real vector spaces on X.

Example 1.1.6. Consider the constant real line bundle $\mathbb{R} \times X$ on X. Note that $\mathsf{Sect}_{\mathbb{R}\times X}(U)$ can be identified with the set of continuous functions on U. It follows that we can upgrade $\mathsf{Sect}_{\mathbb{R}\times X}$ to be a presheaf of $\mathbb{R}$-algebra on X.

1.2. **Sheaves of sets.** Roughly speaking, a sheaf is a presheaf whose sections on small open subsets can be uniquely glued to sections on larger ones.

Definition 1.2.1. Let $\mathcal{F}$ be a presheaf of sets on a topological space X. We say $\mathcal{F}$ is a **sheaf** if it satisfies the following condition:

(*) For any open covering $U = \bigcup_{i \in I} U_i$ and any collection of sections $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \text{ for any } i, j \in I,$$
there is a *unique* section $s \in \mathcal{F}(U)$ such that
$$s_i = s|_U \text{ for any } i \in I.$$

Remark 1.2.2. Using the language of category theory, the sheaf condition is equivalent to the following condition:

- For any open covering $U = \bigcup_{i \in I} U_i$, the diagram
$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$
is an *equalizer* diagram. Here the first map is
$$s \mapsto (s|_{U_i})_{i \in I}$$
the other two maps are
$$(s_i)_{i \in I} \mapsto (s_i|_{U_i \cap U_j})_{(i,j) \in I^2}$$
and
$$(s_i)_{i \in I} \mapsto (s_j|_{U_i \cap U_j})_{(i,j) \in I^2}.$$

In particular, the map $\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$ is an injection.

Remark 1.2.3. For $U = \varnothing$ and $I = \varnothing$, the sheaf condition says there is a unique section $s \in \mathcal{F}(\varnothing)$ subject to no property. In other words, the above definition forces $\mathcal{F}(\varnothing)$ to be a singleton.

Example 1.2.4. Let X be a topological space. The constant presheaf $\underline{A}$ associated to a set A is in general not a sheaf. Indeed, $\underline{A}(\varnothing)$ is A rather than a singleton.

We provide another reason for readers uncomfortable with the above. For a sheaf $\mathcal{F}$ and *disjoint* open subsets U_1 and U_2, the sheaf condition implies

$$\mathcal{F}(U_1 \sqcup U_2) \simeq \mathcal{F}(U_1) \times \mathcal{F}(U_2).$$

But in general A and $A \times A$ are not isomorphic.

Example 1.2.5. Let $E \to X$ be a continuous map between topological spaces. The presheaf Sect_E of sections on X is a sheaf. Indeed, this follows from the fact that continuous maps can be glued.

Example 1.2.6. Let $\{*\}$ be a 1-point space. Then a sheaf $\mathcal{F}$ of sets on $\{*\}$ is uniquely determined by the set $\mathcal{F}(\{*\})$ of global sectoins. We often abuse the notations and use a set A to denote the sheaf on $\{*\}$ whose set of global sections is A.

Exercise 1.2.7. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X.

(1) Let $\mathcal{F}$ and $\mathcal{F}'$ be sheaves on X and $\alpha : \mathcal{F}|_{\mathfrak{B}} \to \mathcal{F}'|_{\mathfrak{B}}$ be a natural transformation between their restrictions on the full subcategory $\mathfrak{B}^{\mathsf{op}} \subseteq \mathfrak{U}(X)^{\mathsf{op}}$. Show that α can be uniquely extended to a morphism $\phi : \mathcal{F} \to \mathcal{F}'$.
(2) Show that the claims in (1) remain true if $\mathcal{F}$ is only assumed to be a presheaf.
(3) Show that the claims in (1) on existence and uniqueness can both fail if $\mathcal{F}'$ is only assumed to be a presheaf..

The above exercise says sheaves are determined by their restrictions on a topological base. A natural question is, given a functor $\mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$, under what conditions can we extend it to a sheaf $\mathfrak{U}(X) \to \mathsf{Set}$? This question is relevant to us because the Zariski topology of $\mathsf{Spec}\, R$ is defined using a base consisting of open subsets that can be easily described:

$$U(f) \coloneqq \left\{ \mathfrak{p} \in \mathsf{Spec}\, R \,|\, f \notin \mathfrak{p} \right\} \simeq \mathsf{Spec}\, R_f.$$

It would be convenient if we can recover a sheaf $\mathcal{F}$ on $\mathsf{Spec}\, R$ from its values on these open subsets. For instance, we wonder whether the contravariant functor

$$U(f) \mapsto R_f$$

can be extended to a sheaf of commutative rings. If yes, we would obtain the sheaf $\mathcal{O}_X$ of algebraic functions desired in the introduction. The following construction gives a positive answer to this question.

Construction 1.2.8. Let X be a topological space and $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be a base of open subsets of X. For a functor $\mathcal{F} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{Set}$ and $U \in \mathfrak{U}(X)$, define

$$\mathcal{F}'(U) \coloneqq \lim_{V \in \mathfrak{B}^{\mathsf{op}},\, V \subseteq U} \mathcal{F}(V).$$

In other words, an element in $s' \in \mathcal{F}'(U)$ is a collection of elements $s_V \in \mathcal{F}(V)$ for all open subsets $V \subseteq U$ contained in $\mathfrak{B}$ such that for $V_1 \subseteq V_2 \subseteq U$ with $V_1, V_2 \in \mathfrak{B}$, the map $\mathcal{F}(V_2) \to \mathcal{F}(V_1)$ sends s_{V_2} to s_{V_1}. This construction is clearly functorial in U, i.e., for $U_1 \subseteq U_2$, we have a natural map $\mathcal{F}'(U_2) \to \mathcal{F}'(U_1)$. One can check this defines a functor

$$\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$$

equipped with a canonical isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$. In other words, we have extended $\mathcal{F}$ to a *presheaf* $\mathcal{F}'$ of sets on X.

Remark 1.2.9. Using the language in category theory, the functor $\mathcal{F}'$ is the *right Kan extension* of $\mathcal{F}$ along the embedding $\mathfrak{B}^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Proposition 1.2.10. *In above, $\mathcal{F}'$ is a sheaf iff $\mathcal{F}$ satisfies the following condition:*

(**) *For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, and any collection of elements $s_i \in \mathcal{F}(U_i)$, $i \in I$ such that*

$$s_i|_V = s_j|_V \textit{ for any } i, j \in I \textit{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$

there is a unique *section $s \in \mathcal{F}(U)$ such that*

$$s_i = s|_{U_i} \textit{ for any } i \in I.$$

Proof. The "only if" statement follows from the sheaf condition on $\mathcal{F}'$ and the isomorphism $\mathcal{F}'|_{\mathfrak{B}^{\mathsf{op}}} \simeq \mathcal{F}$.

For the "if" statement, we verify the sheaf condition on $\mathcal{F}'$ directly. Let $U = \bigcup_{i\in I} U_i$ be an open covering, and $s'_i \in \mathcal{F}'(U_i)$ be a collection of sections such that

$$s'_i|_{U_i\cap U_j} = s'_j|_{U_i\cap U_j} \text{ for any } i, j \in I.$$

By Construction 1.2.8, each s'_i corresponds to a collection $s_{i,V} \in \mathcal{F}(V)$ for $V \subseteq U_i$, $V \in \mathfrak{B}$ that is compatible with restrictions.

We need to show there is a unique section $s' \in \mathcal{F}'(U)$ such that $s'|_{U_i} = s'_i$.

We first deal with the existence. For any $V \subseteq U$ with $V \in \mathfrak{B}$, since $\mathfrak{B}$ is a base, we can choose an open covering $V = \bigcup_{j\in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. In other words, we can choose a map $f : J \to I$ such that $V_j \subseteq U_i$.

Consider the collection of sections

$$t_{j,V} \coloneqq s_{f(j),V_j} \in \mathcal{F}(V_j),\ j \in J. \tag{1.1}$$

One can check it does not depend on the choice of f and they satisfy the assumption in (**). Hence there is a unique section $s'_V \in \mathcal{F}(V)$ such that $s'_V|_{V_j} = s_{f(j),V_j}$.

One can check the obtained section s'_V does not depend on the open covering $V = \bigcup_{j\in J} V_j$ and the collections (s'_V), $V \subseteq U$, $V \in \mathfrak{B}$ is compatible with restrictions. Hence by Construction 1.2.8, it corresponds to an element $s' \in \mathcal{F}'(U)$. One can check that $s'|_{U_i} = s'_i$. This proves the claim about uniqueness.

It remains to prove the statement about uniqueness. Suppose there are two such sections s', s'' such that

$$s'|_{U_i} = s''|_{U_i} = s'_i \tag{1.2}$$

By Construction 1.2.8, they correspond to two collections $s'_V, s''_V \in \mathcal{F}(V)$ for $V \subseteq U$, $V \in \mathfrak{B}$. We only need to show $s'_V = s''_V$.

Note that if V is contained in some U_i, then (1.2) implies

$$s'_V = s''_V = s_{i,V}. \tag{1.3}$$

Now for general open subset $V \subseteq U$, $V \in \mathfrak{B}$, as before, we can choose an open covering $V = \bigcup_{j \in J} V_j$ in $\mathfrak{B}$ such that each V_j is contained in some U_i. Consider the collection of sections (1.1). By (1.3) (applied to each V_j), we have

$$s'_V|_{V_j} = s''_V|_{V_j} = t_{j,V}.$$

Hence by (**), we must have $s'_V = s''_V$ as desired.

□

1.3. $\mathcal{C}$-valued sheaves.

Definition 1.3.1. Let $\mathcal{C}$ be a category and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on a topological space X. We say $\mathcal{F}$ is a **$\mathcal{C}$-valued sheaf** if for any test object $c \in \mathcal{C}$, the functor

$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$

is a sheaf of sets.

Remark 1.3.2. By Yoneda's lemma and Remark 1.2.2, $\mathcal{F}$ is a $\mathcal{C}$-valued sheaf iff for any open covering $U = \bigcup_{i \in I} U_i$, the canonical diagram

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

is an *equalizer* diagram in $\mathcal{C}$. Here the first morphism is given by restrictions along $U_i \subseteq U$, while the other two morphisms are given respectively by restrictions along $U_i \cap U_j \subseteq U_i$ and $U_i \cap U_j \subseteq U_j$. In particular, the morphism

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

is a *monomorphism*[2].

As a corollary of the remark, we obtain:

Corollary 1.3.3. *Let $\mathcal{F}$ be a presheaf of abelian groups. Then $\mathcal{F}$ is a sheaf of abelian groups iff its underlying presheaf of sets $\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathsf{Ab} \to \mathsf{Set}$ is a sheaf of sets. Here the functor $\mathsf{Ab} \to \mathsf{Set}$ sends an abelian group to its underlying set.*

Exercise 1.3.4. Let $\mathcal{F}$ be a presheaf of abelian groups. Show that $\mathcal{F}$ is a sheaf of abelian groups iff for any open covering $U = \bigcup_{i \in I} U_i$, the sequence

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

is exact. Here the second map is

$$s \mapsto (s|_{U_i})_{i \in I},$$

and the third map is

$$(s_i)_{i \in I} \mapsto (s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j})_{(i,j) \in I^2}.$$

Now suppose $\mathcal{F}$ is a sheaf, can you further extend this exact sequence to the right?

[2] This means for any test object $c \in \mathcal{C}$, the functor $\mathsf{Hom}_{\mathcal{C}}(c,-)$ sends this morphism to an injection between sets.

Remark 1.3.5. Let $\mathcal{C}$ be a category that admits small limits. Then Construction 1.2.8 and Proposition 1.2.10 can be generalized to $\mathcal{C}$-valued (pre)sheaves with condition (**) replaced by

- For any open covering $U = \bigcup_{i\in I} U_i$ in $\mathfrak{B}$, any object $c \in \mathcal{C}$, and any collection of elements $s_i \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U_i))$, $i \in I$ such that
$$s_i|_V = s_j|_V \text{ for any } i,j \in I \text{ and } V \subseteq U_i \cap U_j, V \in \mathfrak{B},$$
there is a *unique* element $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ such that
$$s_i = s|_{U_i} \text{ for any } i \in I.$$

In above $s|_V$ means the post-composition of $s \in \mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}(U))$ with the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.

Note however for $\mathcal{C} = \mathsf{Ab}$, we can keep condition (**) *as it is*, because the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ detects limits.

2. Stalks

2.1. Definition.

Definition 2.1.1. Let X be a topological space and $\mathcal{F}$ be a presheaf of sets on X. For a point $x \in X$, let $\mathfrak{U}(X,x) \subseteq \mathfrak{U}(X)$ be the full subcategory of open neighborhoods of x inside X. The **stalk of $\mathcal{F}$ at x** is

$$\mathcal{F}_x \coloneqq \underset{U\in\mathfrak{U}(X,x)^{\mathsf{op}}}{\mathsf{colim}} \mathcal{F}(U). \tag{2.1}$$

For a given section $s \in \mathcal{F}(U)$, the **germ of s at x**, denoted by s_x, is the image of s under the canonical map $\mathcal{F}(U) \to \mathcal{F}_x$.

Note that $\mathfrak{U}(X,x)^{\mathsf{op}}$ is the category associated to the *direct set*[3] $(U(X,x), \subseteq)$ of open neighborboods of x inside X. Hence the above colimit is a *direct colimit*[4]. It follows that $\mathcal{F}_x$ can be explicitly described as the quotient

$$\Big(\coprod_{U\in U(X,x)} \mathcal{F}(U)\Big)/\sim, \tag{2.2}$$

of the disjoint union of all $\mathcal{F}(U)$, $U \in U(X,x)$ by an equivalence relation $\sim$. Here two sections $s \in \mathcal{F}(U)$ and $s' \in \mathcal{F}(U')$ are equivalent iff there exists $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. Using this description, the germ s_x of a section $s \in \mathcal{F}(U)$ is just the equivalence class to which it belongs.

Remark 2.1.2. In general, let $\mathcal{C}$ be a category that admits direct colimits and $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf. We can define the stalk of $\mathcal{F}$ at x using the same formula (2.1). Note that this construction is functorial in $\mathcal{F}$.

In particular, for a presheaf $\mathcal{F}$ of abelian groups, we can define its stalk $\mathcal{F}_x$, which is an abelian group. It is easy to see the underlying set $\mathcal{F}_x$ is given by (2.2) and the group structure is given by the formula

$$s_x + s'_x = (s|_V + s'|_V)_x,\ s \in \mathcal{F}(U), s' \in \mathcal{F}(U'), V \subseteq U \cap U'.$$

---

[3]A direct set is a partially ordered set $(I, \le)$ such that any finite subset of I admits an upper bound in I.

[4]Some people use the word *direct limit*. I strongly object this terminology.

Remark 2.1.3. Let $\mathcal{C}$ be a category such that taking direct colimits in $\mathcal{C}$ commutes with *finite* limits[5]. The the functor $\mathcal{F} \mapsto \mathcal{F}_x$ commutes with finite limits.

2.2. **Sheaves and stalks.** The following result says a section of a *sheaf* is determined by its germs.

Lemma 2.2.1. *Let $\mathcal{F}$ be a sheaf of sets on a topological space X. Then for any open subset $U \subseteq X$, the map*

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x, \ s \mapsto (s_x)_{x \in U} \tag{2.3}$$

is injective. Moreover, a collection of elements $s(x) \in \mathcal{F}_x$, $x \in U$ is contained in the image of this map iff it satisfies the following condition

(***) *For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.*

Proof. We first show the map (2.3) is injective. Let $s, s' \in \mathcal{F}(U)$ such that all their germs are equal. By definition, for any $x \in U$, there exists $V \subseteq U$ such that $s|_V = s'|_V$. In particular, we can find an open covering $U = \bigcup_{i \in I} U_i$ such that $s|_{U_i} = s'|_{U_i}$. But this implies $s = s'$ because the sheaf condition implies

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i)$$

is injective.

It is obvious that any element in the image of (2.3) satisfies condition (***). To prove the converse, let $s(x) \in \mathcal{F}_x$, $x \in U$ be a collection of elements satisfying condition (***). By assumption, we can find an open covering $U = \bigcup_{i \in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that for any $x \in U_i$, we have

$$t(x) = (s_i)_x. \tag{2.4}$$

In particular, the germs of $s_i|_{U_i \cap U_j}$ and $s_j|_{U_i \cap U_j}$ are equal. Applying the injectivity of (2.3) to $U_i \cap U_j$, we obtain

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Hence by the sheaf condition, we can find a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. For any $x \in U$, pick $i \in I$ such that $x \in U_i$, we have

$$s_x = (s_i)_x = t(x),$$

where the first equality is due to the definition of stalks, while the second one is (2.4). In particular, $s(x) \in \mathcal{F}_x$, $x \in U$ is the image of s under the map (2.3). □

Remark 2.2.2. Similar claim for presheaves is false in general. Namely, for $U = X = \varnothing$, the empty product $\prod_{x \in \varnothing} \mathcal{F}_x$ is a singleton, while $\mathcal{F}(\varnothing)$ can be any set.

Corollary 2.2.3. *If $\alpha, \beta : \mathcal{F} \to \mathcal{F}'$ are morphisms between sheaves of sets such that $\alpha_x = \beta_x$ for any $x \in X$, then $\alpha = \beta$.*

Remark 2.2.4. Using the language in category theory, the above corollary says the functors $(-)_x : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Set}$, $x \in X$ are *jointly conservative.*

[5]This is true for Set, Ab and more generally any compactly generated category. An object c in a (locally small) category $\mathcal{C}$ is compact iff $\mathsf{Hom}_{\mathcal{C}}(c, -)$ preserves small filtered colimits. We say $\mathcal{C}$ is compactly generated if it admits small colimits and any object in $\mathcal{C}$ is isomorphic to a small filtered colimit of compact objects.

Proposition 2.2.5. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism between sheaves of sets on a topological space. Then α is an isomorphism iff for any $x \in X$, $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a bijection.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose α_x is a bijection for any $x \in X$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\alpha_U"] & \prod_{x\in U} \mathcal{F}_x \arrow[d, "(\alpha_x)_{x\in X}", "\simeq"'] \\ \mathcal{F}'(U) \arrow[r] & \prod_{x\in U} \mathcal{F}'_x. \end{tikzcd}$$

By Lemma 2.2.1, the horizontal maps are injective, hence so is α_U.

It remains to show α_U is surjective. Let $s' \in \mathcal{F}'(U)$ be a section, we will construct a section $s \in \mathcal{F}(U)$ mapping to it by α_U.

For any point $x \in U$, since α_x is bijective, we can find an open subset $V \subseteq X$ and a section $t \in \mathcal{F}(V)$ such that $\alpha_x(t_x) = s'_x$. By definition, $\alpha_x(t_x) = \alpha_V(t)_x$. Hence the germs of $\alpha_V(t)$ and s' at x are equal. By definition, there exists an open neighborhood W of x inside $U \cap V$ such that $\alpha_V(t)|_W = s'|_W$. Note that we also have $\alpha_V(t)|_W = \alpha_W(t|_W)$.

It follows that we can find an open covering $U = \bigcup_{i\in I} U_i$ and sections $s_i \in \mathcal{F}(U_i)$ such that $\alpha_{U_i}(s_i) = s|_{U_i}$. In particular, we have

$$\alpha_{U_i\cap U_j}(s_i|_{U_i\cap U_j}) = \alpha_{U_i\cap U_j}(s_j|_{U_i\cap U_j}) = s|_{U_i\cap U_j}.$$

Since we have already shown $\alpha_{U_i\cap U_j}$ is injective, we obtain $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Hence by the sheaf condition for $\mathcal{F}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$. Using the sheaf condition for $\mathcal{F}'$, it is easy to see $\alpha_U(s) = s'$ as desired.

□

The above results imply that a *morphism* between sheaves are determined by the induced maps between the stalks. However, a sheaf itself is *not* determined by its stalks.

Exercise 2.2.6. Let X be a connected topological space and $E \to X$ and $E' \to X$ be two covering spaces of the same degree. Show that the sheaves Sect_E and $\mathsf{Sect}_{E'}$ on X have isomorphic stalks for any point $x \in X$, but they are not isomorphic unless there exists a homeomorphism $E \simeq E'$ defined over X.

Remark 2.2.7. Let $\mathcal{C}$ be a *compactly generated* category. Lemma 2.2.1 and Proposition 2.2.5 can be generalized to $\mathcal{C}$-valued sheaves. In other words:

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$, the morphism $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is a monomorphism.
- A morphism $\alpha : \mathcal{F} \to \mathcal{F}'$ between $\mathcal{C}$-valued sheaves is an isomorphism iff $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is an isomorphism for any $x \in X$.

These statements can be deduced from the special case for Set with the help of the following two observations:

- A morphism $d \to d'$ in $\mathcal{C}$ is a monomorphism (resp. isomorphism) iff for any *compact* object $c \in \mathcal{C}$, the map $\mathsf{Hom}_{\mathcal{C}}(c, d) \to \mathsf{Hom}_{\mathcal{C}}(c, d')$ is an injection (resp. bijection).

- For any $\mathcal{C}$-valued sheaf $\mathcal{F}$ and any *compact* object $c \in \mathcal{C}$, the stalk of the Set-valued sheaf
$$\mathfrak{U}(X)^{\mathsf{op}} \xrightarrow{\mathcal{F}} \mathcal{C} \xrightarrow{\mathsf{Hom}_{\mathcal{C}}(c,-)} \mathsf{Set}$$
at $x \in X$ is canonically isomorphic to $\mathsf{Hom}_{\mathcal{C}}(c, \mathcal{F}_x)$.

The details are left to the curious readers.

2.3. **Skyscrapers.**

Definition 2.3.1. Let X be a topological space and $x \in X$ be a point. For any set A, we can define a presheaf $\delta_{x,A}$ of sets as follows.

- For an open subset $U \subseteq X$,
  - if $x \in U$, define $\delta_{x,A}(U) := A$;
  - if $x \notin U$, define $\delta_{x,A}(U) := \{*\}$.
- For open subsets $U \subseteq V$,
  - if $x \in U$ (and therefore $x \in V$), define the restriction map $\delta_{x,A}(U)$ to be id_A;
  - if $x \notin U$, define the restriction map to be the unique map $\delta_{x,A}(V) \to \delta_{x,A}(U) = \{*\}$.

One can check this indeed defines a presheaf $\delta_{x,A}$. We call the the **skyscraper sheaf** at x with value A.

Exercise 2.3.2. The presheaf $\delta_{x,A}$ is indeed a sheaf.

Lemma 2.3.3. *Let X be a topological space, $x \in X$ be a point and A be a set. The stalk of $\delta_{x,A}$ at a point $y \in X$ is canonically bijective to*

- *the set A if y is contained in $\overline{\{x\}}$, the closure of $\{x\}$ inside X;*
- *the singleton $\{*\}$ otherwise.*

Proof. If $y \in \overline{\{x\}}$, then any open neighborhood of y contains x. It follows that
$$(\delta_{x,A})_y := \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} A$$
is a direct colimit of the constant diagram with values A. This implies $(\delta_{x,A})_y \simeq A$.

If $y \notin \overline{\{x\}}$, then there exists an open neighborhood V of y such that $x \notin V$. Note that $\mathfrak{U}(V,y)^{\mathsf{op}} \subseteq \mathfrak{U}(X,y)^{\mathsf{op}}$ is (co)final. If follows that
$$(\delta_{x,A})_y := \underset{U \in \mathfrak{U}(X,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq (\delta_{x,A})_y \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \delta_{x,A}(U) \simeq \underset{U \in \mathfrak{U}(V,y)^{\mathsf{op}}}{\mathsf{colim}} \{*\}$$
is a direct colimit of the constant diagram with values $\{*\}$. This implies $(\delta_{x,A})_y \simeq \{*\}$.

□

Note that if A is equipped with the structure of an abelian group, the skyscraper $\delta_{x,A}$ can be upgraded to a sheaf of abelian groups. Then the abelian group $(\delta_{x,A})_y$ is either A or 0.

Proposition 2.3.4. *Let X be a topological space, $x \in X$ be a point and A be a set. For any presheaf $\mathcal{F}$ of sets on X, the composition*

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \xrightarrow{(-)_x} \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, (\delta_{x,A})_x) \simeq \mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \tag{2.5}$$

is an bijection.

Corollary 2.3.5. *The stalk functor*

$$\mathsf{PShv}(X,\mathsf{Set}) \to \mathsf{Set},\ \mathcal{F} \mapsto \mathcal{F}_x$$

admits a right adjoint

$$\mathsf{Set} \to \mathsf{PShv}(X,\mathsf{Set}),\ A \mapsto \delta_{A,x}.$$

Proof of Proposition 2.3.4. We first construct a map

$$\mathsf{Hom}_{\mathsf{Set}}(\mathcal{F}_x, A) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F}, \delta_{x,A}) \tag{2.6}$$

as follows. Given any map $f : \mathcal{F}_x \to A$, for any open subset $U \subseteq X$, we define a map $\alpha_U : \mathcal{F}(U) \to \delta_{x,A}(U)$ such that:

- If $x \in U$, α_U is the composition $\mathcal{F}(U) \to \mathcal{F}_x \xrightarrow{f} A$;
- If $x \notin U$, α_U is the unique map $\mathcal{F}(U) \to \{*\}$.

One can check these maps are compatible with restriction and therefore define a morphism $\alpha : \mathcal{F} \to \delta_{x,A}$. Now we define the map (2.6) to be $f \mapsto \alpha$.

One can check that (2.5) and (2.6) are inverse to each other. Hence both are bijections.

□

Remark 2.3.6. In general, for any category $\mathcal{C}$ admitting a final object[6] and any object $A \in \mathcal{C}$, one can define a $\mathcal{C}$-valued sheaf $\delta_{x,A}$. If $\mathcal{C}$ admits direct colimits, the stalks of $\delta_{x,A}$ are either A or the final object of $\mathcal{C}$, and the functor $A \mapsto \delta_{A,x}$ is right adjoint to $\mathcal{F} \mapsto \mathcal{F}_x$.

[6]An object $* \in \mathcal{C}$ is a final object iff for any $c \in \mathcal{C}$, there is a unique morphism $c \to *$.

3. Category of (pre)sheaves

Let X be a topological space and $\mathcal{C}$ be a category. Note that $\mathcal{C}$-valued presheaves on X form a category

$$\mathsf{PShv}(X,\mathcal{C}) \coloneqq \mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}),$$

and $\mathcal{C}$-valued sheaves form a full subcategory

$$\mathsf{Shv}(X,\mathcal{C}) \subseteq \mathsf{PShv}(X,\mathcal{C}).$$

In this section, we study the basic properties of these categories.

3.1. **Sheafification.**

Definition 3.1.1. Let $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$. The **sheafification** of $\mathcal{F}$ is a sheaf $\mathcal{F}^\sharp \in \mathsf{Shv}(X,\mathsf{Set})$ equipped with a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ such that for any test sheaf $\mathcal{G}$, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta.$$

Proposition 3.1.2. *For any $\mathcal{F} \in \mathsf{PShv}(X,\mathsf{Set})$, its sheafification $(\mathcal{F}^\sharp,\theta)$ exists, and is unique up to unique isomorphism. Moreover, the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ induces bijections $\mathcal{F}_x \to \mathcal{F}^\sharp_x$ between the stalks.*

Proof. The statement about uniqueness follows from Yoneda's lemma. To prove the existence, we construct a sheafification as follows.

We first construct the desired sheaf $\mathcal{F}^\sharp$. For any open subset $U \subseteq X$, let

$$\mathcal{F}^\sharp(U) \subseteq \prod_{x\in U} \mathcal{F}_x,$$

be the subset consisting of elements $(s(x))_{x\in U}$ satisfying the following condition:

- For any $x \in U$, there exists a neighborhood V of x inside U and a section $s_V \in \mathcal{F}(V)$ such that for any $y \in V$, we have $s(y) = (s_V)_y$.

For $U \subseteq U'$, it is obvious that the projection map $\prod_{x\in U'} \mathcal{F}_x \to \prod_{x\in U} \mathcal{F}_x$ sends $\mathcal{F}^\sharp(U')$ into $\mathcal{F}^\sharp(U)$. Moreover, one can check the obtained maps $\mathcal{F}^\sharp(U') \to \mathcal{F}^\sharp(U)$ upgrade the assignment $U \mapsto \mathcal{F}^\sharp(U)$ to an object in $\mathsf{Shv}(X,\mathsf{Set})$.

Now we construct the morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$. For any open subset $U \subseteq X$, consider the map

$$\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x,\ s \mapsto (s_x)_{x\in U}.$$

It is obvious that the image of this map is contained in $\mathcal{F}^\sharp(U)$. Moreover, the obtained maps $\mathcal{F}(U) \to \mathcal{F}^\sharp(U)$ is functorial in U, therefore give a morphism $\theta : \mathcal{F} \to \mathcal{F}^\sharp$.

It remains to show $\theta : \mathcal{F} \to \mathcal{F}^\sharp$ exhibits $\mathcal{F}^\sharp$ as a sheafification of $\mathcal{F}$. Let $\mathcal{G}$ be a test sheaf, we need to show

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}) \to \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G}),\ \alpha \mapsto \alpha \circ \theta \tag{3.1}$$

is bijective. Let $\beta : \mathcal{F} \to \mathcal{G}$ be a morphism. For any open subset $U \subseteq X$, recall taking germs induces an injection

$$\mathcal{G}(U) \to \prod_{x\in U} \mathcal{G}_x$$

and its image is desribed in Lemma 2.2.1. Using that description, it is clear that there is a unique dotted map making the following diagram commute:

$$\begin{tikzcd} \mathcal{F}^\sharp(U) \arrow[r, "\subseteq"] \arrow[d, dotted] & \prod_{x\in U}\mathcal{F}_x \arrow[d, "(\beta_x)_{x\in U}"] \\ \mathcal{G}(U) \arrow[r] & \prod_{x\in U}\mathcal{G}_x. \end{tikzcd}$$

Moreover, the obtained map $\mathcal{F}^\sharp(U)\to\mathcal{G}(U)$ is functorial in U. Hence we obtain a morphism $\beta^\sharp:\mathcal{F}^\sharp\to\mathcal{G}$. Now one can check that the map

$$\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(\mathcal{F},\mathcal{G})\to\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(\mathcal{F}^\sharp,\mathcal{G}),\ \beta\mapsto\beta^\sharp$$

and (3.1) are inverse to each other. In particular, they are both bijective as desired. □

Corollary 3.1.3. *The fully faithful embedding* $\mathsf{Shv}(X,\mathsf{Set})\to\mathsf{PShv}(X,\mathsf{Set})$ *admits a left adjoint which sends* $\mathcal{F}$ *to its sheafification* $\mathcal{F}^\sharp$*.*

Example 3.1.4. Let A be a set. The sheafification $\underline{A}^\sharp$ of the constant presheaf $\underline{A}$ is the sheaf

$$\mathfrak{U}(X)^{\mathsf{op}}\to\mathsf{Set},\ U\mapsto C(U,A)$$

that sends U to the set of continuous maps from U to A (equipped with the discrete topology). We call it the **constant sheaf** associated to A.

Remark 3.1.5. The sheafification functor $\mathsf{PShv}(X,\mathsf{Set})\to\mathsf{Shv}(X,\mathsf{Set})$ preserves finite limits. This follows by combining the following four facts:

- The sheafification functor commutes with taking stalks;
- Taking stalks commutes with finite limits (Remark 2.1.3).
- The functors of taking stalks are jointly conservative on the category of sheaves (Remark 2.2.4)

Namely, for any finite diagram of presheaves $\mathcal{F}_i$, $i\in I$, we have

$$(\lim_{i\in I}\mathcal{F}_i)^\sharp_x\simeq(\lim_{i\in I}\mathcal{F}_i)_x\simeq\lim_{i\in I}\mathcal{F}_{i,x}\simeq\lim_{i\in I}(\mathcal{F}_i^\sharp)_x\simeq(\lim_{i\in I}\mathcal{F}_i^\sharp)_x,\ \forall x\in X,$$

and this isomorphism is induced by the canonical morphism

$$(\lim_{i\in I}\mathcal{F}_i)^\sharp\to\lim_{i\in I}\mathcal{F}_i^\sharp.$$

This implies the above morphism is an isomorphism.

Remark 3.1.6. Suppose $\mathcal{F}$ is a presheaf of abelian groups. Let $\mathcal{F}^\sharp$ be the sheafification of the underlying Set-valued presheaf of $\mathcal{F}$ as constructed in the proof of the proposition. One can check that $\mathcal{F}^\sharp(U)$ is a subgroup of the abelian group $\prod_{x\in U}\mathcal{F}_x$. It follows that $\mathcal{F}^\sharp$ can be upgraded to a sheaf of abelian groups. Moreover, for any test sheaf $\mathcal{G}$ of abelian groups, pre-composing with θ induces an bijection:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Ab})}(\mathcal{F}^\sharp,\mathcal{G})\xrightarrow{\simeq}\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Ab})}(\mathcal{F},\mathcal{G}),\ \alpha\mapsto\alpha\circ\theta.$$

In other words, $\mathsf{Shv}(X,\mathsf{Ab})\to\mathsf{PShv}(X,\mathsf{Ab})$ admits a left adjoint which sends $\mathcal{F}$ to $\mathcal{F}^\sharp$.

Remark 3.1.7. In general, if $\mathcal{C}$ is a category admitting small limits and filtered colimits, then any $\mathcal{C}$-valued presheaf admits a sheafification that can be constructed as follows.

For $U\subseteq X$, we can define the *category* Cov_U *of open coverings of* U as follows:

- An object is an open covering $U = \bigcup_{i\in I} U_i$;
- A morphism from $(U_i)_{i\in I}$ to $(V_j)_{j\in J}$ is a map $J \to I$ such that $V_j \subseteq U_i$ for any $j \in J$.

One can show that Cov_U is filtered. Now for any $\mathcal{F} \in \mathsf{PShv}(X,\mathcal{C})$, we have a functor

$$\begin{array}{ccc} \mathsf{Cov}_U & \to & \mathcal{C} \\ (U_i)_{i\in I} & \mapsto & \lim[\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)]. \end{array}$$

sending a covering to the equalizer appeared in the sheaf condition. Note that the identity covering $\{U\}$ is sent to the object $\mathcal{F}(U)$. Now we define

$$\mathcal{F}^+(U) := \operatorname*{colim}_{[(U_i)_{i\in I}]\in\mathsf{Cov}_U} \lim[\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j)].$$

By construction, there is a canonical morphism $\mathcal{F}(U) \to \mathcal{F}^+(U)$. Moreover, the above definition is contravariantly functorial in U, therefore we obtain an object $\mathcal{F}^+ \in \mathsf{PShv}(X,\mathcal{C})$ equipped with a canonical morphism $\mathcal{F} \to \mathcal{F}^+$.

In general, $\mathcal{F}^+$ is note a $\mathcal{C}$-valued sheaf. But one can check that for any open covering $U = \bigcup_{i\in I} U_i$, the morphism

$$\mathcal{F}^+(U) \to \prod_{i\in I} \mathcal{F}^+(U_i)$$

is a monomorphism. Using this property, one can show that $(\mathcal{F}^+)^+$ is a sheaf and the composition $\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$ exhibits $(\mathcal{F}^+)^+$ as a sheafification of $\mathcal{F}$.

3.2. **Direct images.**

Construction 3.2.1. Let $f : X \to X'$ be a continuous map between topological spaces. We have a functor

$$\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}},\ U' \mapsto f^{-1}(U').$$

For any category $\mathcal{C}$, it induces a functor

$$\mathsf{Fun}(\mathfrak{U}(X)^{\mathsf{op}},\mathcal{C}) \to \mathsf{Fun}(\mathfrak{U}(X')^{\mathsf{op}},\mathcal{C}).$$

By definition, this gives a functor

$$f_* : \mathsf{PShv}(X,\mathcal{C}) \to \mathsf{PShv}(X',\mathcal{C}).$$

We call it the **direct image functor** (or **pushforward functor**) along f for $\mathcal{C}$-valued presheaves.

Note that for continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)_* \simeq g_* \circ f_*$.

Explicitly, given a $\mathcal{C}$-valued presheaf $\mathcal{F}$ on X, its **direct image** (or **pushforward**) along f is the presheaf $f_*\mathcal{F}$ defined by

$$f_*\mathcal{F}(U') := \mathcal{F}(f^{-1}(U')),$$

with restriction maps given by those maps for $\mathcal{F}$.

Proposition 3.2.2. *Let $f : X \to X'$ be a continuous map between topological spaces. If $\mathcal{F}$ is a sheaf, then $f_*\mathcal{F}$ is a sheaf.*

Proof. The sheaf condition for $f_*\mathcal{F}$ and an open covering $U' = \bigcup_{i\in I} U'_i$ is just the sheaf condition for $\mathcal{F}$ and the open covering $f^{-1}(U') = \bigcup_{i\in I} f^{-1}(U'_i)$.

□

Example 3.2.3. Let $x \in X$ be a point and write $i : \{x\} \to X$ for the emebdding map. Let $\mathcal{C}$ be a category admitting a final object $*$. For any object $A \in \mathcal{C}$, we have

$$i_*(A) \simeq \delta_{x,A},$$

where we abuse notations and use A to denote the unique $\mathcal{C}$-valued sheaf on $\{x\}$ whose object of global sections is A.

Example 3.2.4. Let $p : X \to \{*\}$ be the obvious projection map. For any *sheaf* $\mathcal{F}$, the direct image $p_*\mathcal{F}$ is uniquely determined by $p_*\mathcal{F}(\{*\})$, which is $\mathcal{F}(X)$ by definition. Hence in this case, we also call p_* is **taking global sections functor**.

Warning 3.2.5. Direct image functors do *not* commute with sheafifications. In other words $f_*(\mathcal{F}^\sharp)$ and $(f_*\mathcal{F})^\sharp$ are in general not isomorphic. For a counterexample, take $\mathcal{F}$ to be a constant *presheaf*.

3.3. **Inverse images for presheaves.**

Construction 3.3.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We define a presheaf $f^{-1}_{\mathsf{PShv}}\mathcal{F}' \in \mathsf{PShv}(X, \mathsf{Set})$ by the following formula

$$f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U) \coloneqq \operatorname*{colim}_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V),$$

where $\mathfrak{U}(X', f(U)) \subseteq \mathfrak{U}(X')$ is the full subcategory of open neighborhoods of $f(U)$ inside X', and the restriction maps for $f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ are induced by those for $\mathcal{F}'$.

The construction $\mathcal{F}' \to f^{-1}_{\mathsf{PShv}}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for presheaves of sets.

Note that $\mathfrak{U}(X', f(U))^{\mathsf{op}}$ is the category associated to a direct set. Hence $f^{-1}_{\mathsf{PShv}}\mathcal{F}'(U)$ can be calculated as a quotient of

$$\coprod_{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}} \mathcal{F}'(V).$$

Example 3.3.2. Let X be a topological space and x be a point. Write $i : \{x\} \to X$ for the embedding. We have

$$\big(i^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(\{x\}) \simeq \mathcal{F}'_x.$$

Remark 3.3.3. The functor $f^{-1}_{\mathsf{PShv}}\mathcal{F}' : \mathfrak{U}(X)^{\mathsf{op}} \to \mathsf{Set}$ is the left Kan extension of $\mathcal{F}' : \mathfrak{U}(X')^{\mathsf{op}} \to \mathsf{Set}$ along the pullback functor $\mathfrak{U}(X')^{\mathsf{op}} \to \mathfrak{U}(X)^{\mathsf{op}}$.

Remark 3.3.4. The functor f^{-1}_{PShv} commutes with finite limits because filtered colimits commute with finite limits in Set.

Lemma 3.3.5. *Let X be a topological space and $U \subseteq X$ be an open subset. Write $j : U \to X$ for the embedding map. Then j^{-1}_{PShv} sends sheaves to sheaves.*

Proof. For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V \subseteq U$, unwinding the definitions, we have

$$\big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V) \simeq \mathcal{F}(V).$$

Hence the sheaf condition for $j^{-1}_{\mathsf{PShv}}(\mathcal{F})$ follows from that for $\mathcal{F}$.

□

Warning 3.3.6. For general continuous map $f : X \to X'$, the functor f^{-1}_{PShv} does not send sheaves to sheaves. To see this, consider the projection map $p : X \to \{*\}$.

Construction 3.3.7. Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$ be a presheaf. We construct a morphism

$$\mathcal{F}' \to f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}') \tag{3.2}$$

as follows. For any open subest $U' \subseteq X'$, by definition,

$$\big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U') \simeq \big(f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(f^{-1}(U')) \simeq \underset{V \in \mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}}{\mathsf{colim}} \mathcal{F}'(V).$$

Note that U' is an object in $\mathfrak{U}(X', f(f^{-1}(U')))^{\mathsf{op}}$. Hence we have a canonical map

$$\mathcal{F}'(U') \to \big(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}')\big)(U').$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.2).

Moreover, we can upgrade these morphisms to a natural transformation

$$\mathsf{Id} \to f_* \circ f^{-1}_{\mathsf{PShv}}. \tag{3.3}$$

Construction 3.3.8. Dually, let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ be a presheaf. We construct a morphism

$$f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F}) \to \mathcal{F}. \tag{3.4}$$

as follows. For any open subest $U \subseteq X$, by definition,

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \simeq \underset{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}}{\mathsf{colim}} \big(f_*(\mathcal{F})\big)(V) \simeq \underset{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}}{\mathsf{colim}} \mathcal{F}(f^{-1}(V)).$$

Note that for any $V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}$, we have $U \subseteq f^{-1}(V)$, which gives a restriction map $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$. One can check these maps are functorial in V and give a map

$$\underset{V \in \mathfrak{U}(X', f(U))^{\mathsf{op}}}{\mathsf{colim}} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U).$$

Hence we obtain a map

$$\big(f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(U) \to \mathcal{F}(U).$$

One can check these maps are compatible with restrictions, and therefore gives a morphism (3.4).

Moreover, we can upgrade these morphisms to a natural transformation

$$f^{-1}_{\mathsf{PShv}} \circ f_* \to \mathsf{Id}. \tag{3.5}$$

The following proposition follows from a boring diagram chasing. We omit the details.

Proposition 3.3.9. *Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. The following compositions are inverse to each other:*

$$\begin{aligned}\mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F}) \xrightarrow{f_*} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f_* \circ f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f_*\mathcal{F}) \\ \xrightarrow{-\circ(3.2)} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F})\end{aligned}$$

and

$$\mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) \xrightarrow{f^{-1}_{\mathsf{PShv}}} \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), f^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F}))$$
$$\xrightarrow{(3.4)\circ-} \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F})$$

Corollary 3.3.10. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(X', \mathsf{Set}) \to \mathsf{PShv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{PShv}(X, \mathsf{Set}) \to \mathsf{PShv}(X', \mathsf{Set}).$$

Corollary 3.3.11. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1}_{\mathsf{PShv}} \simeq f^{-1}_{\mathsf{PShv}} \circ g^{-1}_{\mathsf{PShv}}$.*

Corollary 3.3.12. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any presheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}_{\mathsf{PShv}}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Remark 3.3.13. Let $\mathcal{C}$ be a category admitting direct colimits. One can define the functor f^{-1}_{PShv} for $\mathcal{C}$-valued presheaves using the same formula, and f^{-1}_{PShv} is canonically left adjoint to f_*.

3.4. **Inverse images for sheaves.**

Construction 3.4.1. Let $f : X \to X'$ be a continuous map between topological spaces. Let $\mathcal{F} \in \mathsf{Shv}(X', \mathsf{Set})$ be a sheaf. We define

$$f^{-1}\mathcal{F} := (f^{-1}_{\mathsf{PShv}}\mathcal{F}')^\sharp$$

to be the sheafification of the presheaf-theoretic inverse image of $\mathcal{F}$.

The construction $\mathcal{F}' \to f^{-1}\mathcal{F}'$ can be obviously upgraded to a functor

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set}).$$

We call it the **inverse image functor** (or **pullback functor**) along f for sheaves of sets.

Let $f : X \to X'$ be a continuous map between topological spaces and $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$, $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$. We have canonical bijections:

$$\mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{Set})}(f^{-1}(\mathcal{F}'), \mathcal{F}) \simeq \mathsf{Hom}_{\mathsf{PShv}(X,\mathsf{Set})}(f^{-1}_{\mathsf{PShv}}(\mathcal{F}'), \mathcal{F})$$
$$\simeq \mathsf{Hom}_{\mathsf{PShv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}) \simeq \mathsf{Hom}_{\mathsf{Shv}(X',\mathsf{Set})}(\mathcal{F}', f_*\mathcal{F}),$$

where

- the first bijection is due to the definition of sheafifications;
- the second bijection is that in Proposition 3.3.9;
- the last bijection is due to the fully faithful embedding $\mathsf{Shv}(X', \mathsf{Set}) \subseteq \mathsf{PShv}(X, \mathsf{Set})$.

Corollary 3.4.2. *Let $f : X \to X'$ be a continuous map between topological spaces. The functor*

$$f^{-1} : \mathsf{Shv}(X', \mathsf{Set}) \to \mathsf{Shv}(X, \mathsf{Set})$$

is canonically left adjoint to

$$f_* : \mathsf{Shv}(X, \mathsf{Set}) \to \mathsf{Shv}(X', \mathsf{Set}).$$

Corollary 3.4.3. *For continuous maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have a canonical natural isomorphism $(g \circ f)^{-1} \simeq f^{-1} \circ g^{-1}$.*

Corollary 3.4.4. *Let $f : X \to X'$ be a continuous map and $x \in X$ be a point. Write $x' := f(x)$. Then for any sheaf $\mathcal{F}' \in \mathsf{PShv}(X', \mathsf{Set})$, we have a canonical isomorphism*

$$f^{-1}(\mathcal{F}')_x \simeq \mathcal{F}'_{x'}.$$

Exercise 3.4.5. The following diagram commutes:

$$\begin{tikzcd} \mathsf{PShv}(X', \mathsf{Set}) \arrow[r, "f^{-1}_{\mathsf{PShv}}"] \arrow[d, "(-)^\sharp"] & \mathsf{PShv}(X, \mathsf{Set}) \arrow[d, "(-)^\sharp"] \\ \mathsf{Shv}(X', \mathsf{Set}) \arrow[r, "f^{-1}"] & \mathsf{Shv}(X, \mathsf{Set}). \end{tikzcd}$$

Exercise 3.4.6. Show that f^{-1} sends a constant sheaf to the constant sheaf associated to the same set.

Example 3.4.7. Let X be a topological space and x be a point. Write $i : \{x\} \to X$ for the embedding. For $\mathcal{F} \in \mathsf{Shv}(X, \mathsf{Set})$, we have

$$i^{-1}(\mathcal{F}) \simeq \mathcal{F}_x,$$

where in the RHS we abuse notations by identifying a sheaf on $\{x\}$ with its set of global sections (see Example 1.2.6).

Remark 3.4.8. Let $\mathcal{C}$ be a category admitting small limits and filtered colimits. One can define the functor f^{-1} for $\mathcal{C}$-valued sheaves using the same formula, and f^{-1} is canonically left adjoint to f_*.

3.5. **Open base-change.**

Construction 3.5.1. Given a commutative square of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & X' \arrow[d, "v"] \\ Y \arrow[r, "g"] & Y', \end{tikzcd} \tag{3.6}$$

consider the canonical natural isomorphism $v_* \circ f_* \simeq g_* \circ u_*$. Using the adjunctions $(g^{-1}_{\mathsf{PShv}}, g_*)$ and $(f^{-1}_{\mathsf{PShv}}, f_*)$, we obtain natural transformations

$$g^{-1}_{\mathsf{PShv}} \circ v_* \to g^{-1}_{\mathsf{PShv}} \circ v_* \circ f_* \circ f^{-1}_{\mathsf{PShv}} \simeq g^{-1}_{\mathsf{PShv}} \circ g_* \circ u_* \circ f^{-1}_{\mathsf{PShv}} \to u_* \circ f^{-1}_{\mathsf{PShv}},$$

where the first arrow is induced by $\mathsf{Id} \to f_* \circ f^{-1}_{\mathsf{PShv}}$ (see (3.3)), while the last arrow is induced by $g^{-1}_{\mathsf{PShv}} \circ g_* \to \mathsf{Id}$ (see (3.5)).

We call the above composition the **base-change natural transformation**[7] for presheaves associated to the square (3.6).

Similarly, we have the base-change natural transformation for sheaves

$$g^{-1} \circ v_* \to u_* \circ f^{-1}.$$

[7]Other name: Bech–Chevalley natural transformations.

Proposition 3.5.2. *Let $f : X \to X'$ be a continuous map between topological spaces and $U' \subseteq X'$ be an open subset. Write $U := f^{-1}(U')$ can consider the following diagram*

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ U' \arrow[r, "j'"] & X'. \end{tikzcd}$$

Then both

$$(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$$

and

$$(j')^{-1} \circ f_* \to g_* \circ j^{-1}$$

are natural isomorphisms.

Proof. We will prove the claim for presheaves. That for sheaves follow from Lemma 3.3.5.

For any $\mathcal{F} \in \mathsf{PShv}(X, \mathsf{Set})$ and open subset $V' \subseteq U'$, unwinding the definitions, we have

$$\big((j')^{-1}_{\mathsf{PShv}} \circ f_*(\mathcal{F})\big)(V') \simeq \big(f_*(\mathcal{F})\big)(V') \simeq \mathcal{F}(f^{-1}(V'))$$

and

$$\big(g_* \circ j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(V') \simeq \big(j^{-1}_{\mathsf{PShv}}(\mathcal{F})\big)(g^{-1}(V')) \simeq \mathcal{F}(f^{-1}(V')).$$

One can check that via these identifications, the value of $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ at $\mathcal{F}$ and V' is given by the identity map on $\mathcal{F}(f^{-1}(V'))$. In particular, $(j')^{-1}_{\mathsf{PShv}} \circ f_* \to g_* \circ j^{-1}_{\mathsf{PShv}}$ is a natural isomorphism.

□

Remark 3.5.3. Informally, we say: *open* pullbacks commute with pushforwards.

Warning 3.5.4. In the setting of Proposition 3.5.2, one can also consider the natural transformations

$$f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$$

and

$$f^{-1} \circ j'_* \to j_* \circ g^{-1}.$$

However, they are *not* invertible in general.

Exercise 3.5.5. Let $X' = \{s, b\}$ be the topological space with two points whose open subsets are exactly given by $\varnothing, \{b\}$ and X'. Consider the following diagram

$$\begin{tikzcd} \varnothing \arrow[r, "j"] \arrow[d, "g"] & \{s\} \arrow[d, "f"] \\ \{b\} \arrow[r, "j'"] & X'. \end{tikzcd}$$

Show that $f^{-1}_{\mathsf{PShv}} \circ j'_* \to j_* \circ g^{-1}_{\mathsf{PShv}}$ and $f^{-1} \circ j'_* \to j_* \circ g^{-1}$ are not invertible.

Part II. Definition of schemes

4. $\mathsf{Spec}(R)$

4.1. Zariski topology.

Definition 4.1.1. Let R be a (unital) commutative ring. Write $\mathsf{Spec}(R)$ for the set of prime ideals of R. We equip it with the **Zariski topology** so that the subsets

$$U(f) := \{\mathfrak{p} \in \mathsf{Spec}(R) \mid f \notin \mathfrak{p}\},\ f \in R$$

form a topological base. The obtained topological space is called the **prime spectrum** of R. The open subsets of the form $U(f)$ are called the **standard open subsets**[8].

Note that $U(f) \cap U(g) = U(fg)$.

Construction 4.1.2. For any ideal $I \subseteq R$, consider $Z(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$. By definition,

$$Z(I) \simeq \mathsf{Spec}(R) \smallsetminus \bigcup_{f \in I} U(f).$$

This implies the following result.

Lemma 4.1.3. *A subset Z of $\mathsf{Spec}(R)$ is closed iff it is of the form $Z(I)$ for some ideal $I \subseteq R$.*

Lemma 4.1.4. *Let $I, J \subseteq R$ be ideals. Then $Z(I) \subseteq Z(J)$ iff $J \subseteq \sqrt{I}$.*

Proof. Recall the radical $\sqrt{I}$ is equal to the intersection of prime ideals containing I, i.e.,

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in Z(I)} \mathfrak{p}. \tag{4.1}$$

For the "if" statement, suppose $J \subseteq \sqrt{I}$. Then $J \subseteq \mathfrak{p}$ and therefore $\mathfrak{p} \in Z(J)$ for any $\mathfrak{p} \in Z(I)$. Hence we have $Z(I) \subset Z(J)$ as desired.

For the "only if" statement, suppose $Z(I) \subseteq Z(J)$. By (4.1), $\sqrt{J} \subseteq \sqrt{I}$. In particular, $J \subseteq \sqrt{I}$ as desired.

□

Corollary 4.1.5. *Let $I, J \subseteq R$ be ideals. Then $Z(I) = Z(J)$ iff $\sqrt{J} = \sqrt{I}$.*

Corollary 4.1.6. *A point $\mathfrak{p} \in \mathsf{Spec}(R)$ is closed iff $\mathfrak{p}$ is maximal.*

Corollary 4.1.7. *Let x and $y \in \mathsf{Spec}(R)$ be points given by prime ideals $\mathfrak{p}$ and $\mathfrak{q}$. Then $x \in \overline{\{y\}}$ iff $\mathfrak{p} \supset \mathfrak{q}$.*

In above, we say x is a **specialization** of y, and y is a **generalization** of x.

Corollary 4.1.8. *The topological space $\mathsf{Spec}(R)$ is Kolmogorov, i.e., for any pair of distinct points, at least one of them has an open neighborhood not containing the other point.*

Remark 4.1.9. The space $\mathsf{Spec}(R)$ is in general not Haussdorff. Indeed, it is so iff the Krull dimension of R is zero.

[8]Other name: elementary open subsets.

Example 4.1.10. The points in $\mathsf{Spec}(\mathbb{Z})$ are listed as below:

(i) For each prime number p, there is a point $(p) \in \mathsf{Spec}(\mathbb{Z})$.
(ii) There is a point $(0) \in \mathsf{Spec}(\mathbb{Z})$.

A subset of $\mathsf{Spec}(\mathbb{Z})$ is closed iff it is finite collection of points in (i), or it is the entire space.

Note that points (i) are closed, while the point in (ii) is not closed. In fact, the closure of the latter is the entire space.

Example 4.1.11. For any field k, $\mathsf{Spec}(k)$ is a point.

Example 4.1.12. For any discrete valuation ring R, $\mathsf{Spec}(R)$ consists of two points: a closed point corresponding to its ideal of definition, and an open point corresponding to the zero ideal.

Exercise 4.1.13. Let k be an algebraically closed field. Describe the topological space $\mathsf{Spec}(k[x,y]/(xy))$.

Lemma 4.1.14. *The topological space* $\mathsf{Spec}(R)$ *is quasi-compact. In other words, any open covering of it admits a finite sub-covering.*

Proof. It is enough to show any open covering of the form $\mathsf{Spec}(R) = \bigcup_{f\in S} U(f)$ admits a finite sub-covering. Let $\langle S\rangle$ be the ideal generated by S. We obtain $Z(\langle S\rangle) = \varnothing$ and therefore $\langle S\rangle = R$. Hence there exists a finite subset $S' \subseteq S$ such that $1 \in \langle S'\rangle$ and therefore $R = \langle S'\rangle$. Hence we have

$$\varnothing = Z(\langle S'\rangle) = \mathsf{Spec}(R) \smallsetminus \bigcup_{f\in S'} U(f).$$

In other words, we have found a finite sub-covering given by $U(f)$, $f \in S'$. □

4.2. **Structure sheaf.** We are going to construct a canonical sheaf on $\mathsf{Spec}(R)$. For this purpose, we need to associate a set to any standard open subset. Note that a standard open subset $U(f)$ does *not* uniquely determine the element f. However, we have the following results.

Lemma 4.2.1. *For $f, f' \in R$, $U(f) \subseteq U(f')$ iff $R \to R_f$ (uniquely) factors through $R \to R_{f'}$.*

Proof. By definition, $U(f) \subseteq U(f')$ iff $Z(\langle f\rangle) \supset Z(\langle f'\rangle)$. By Lemma 4.1.4, this happens iff $f^n \in \langle f'\rangle$ for some $n \geq 0$. The latter condition is equivalent to f' being an unit under the map $R \to R_f$. By definition, this is equivalent to the condition that $R \to R_f$ factors through $R \to R_{f'}$. □

Corollary 4.2.2. *The open subsets $U(f)$ and $U(f')$ of* $\mathsf{Spec}(R)$ *are equal iff R_f and $R_{f'}$ are isomorphic as R-algebras.*

Proposition-Definition 4.2.3. *There exists an essentially unique*[9] *sheaf $\mathcal{O}$ of commutative rings on* $\mathsf{Spec}(R)$ *equipped with an isomorphism $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ such that for any $f \in R$, the R-algebra $\mathcal{O}(U(f))$ given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}(U(f))$$

[9]This means the pair $(\mathcal{O}, \phi)$ is unique up to a unique isomorphism.

is isomorphic to R_f.

The sheaf $\mathcal{O}_{\mathsf{Spec}(R)} \coloneqq \mathcal{O}$ *is called the* ***structure sheaf*** *on* $\mathsf{Spec}(R)$. *When using this terminology, we treat the isomorphism* $R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ *as implicit.*

Remark 4.2.4. Note that for an R-algebra A, being isomorphic to R_f is a *property rather than a structure*. Namely, there is at most one R-homomorphism from R_f to A.

Proof of Proposition-Definition 4.2.3. Let $\mathfrak{B}$ be the category of standard open subsets in $\mathsf{Spec}(R)$. Since a sheaf is uniquely determined by its restriction on a topological base, we only need to show there is a unique functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ equipped with an isomorphism $\varphi : R \xrightarrow{\simeq} \mathcal{O}(\mathsf{Spec}(R))$ such that:

(a) The functor $\mathcal{O} : \mathfrak{B}^{\mathsf{op}} \to \mathsf{CRing}$ satisfies the sheaf condition in Proposition 1.2.10.
(b) For any $f \in R$, $\mathcal{O}(U(f))$ is isomorphic to R_f as R-algebras.

By Lemma 4.2.1, there is a unique pair $(\mathcal{O}, \varphi)$ satisfying condition (b). Hence we only need to check condition (a). Unwinding the definitions, this amounts to the following fact in commutative algebra (applied to the case $M = A$). We leave the proof of it to the readers. □

Lemma 4.2.5. *Let* A *be a commutative ring and* f, $(f_i)_{i\in I}$ *be elements in* R *such that* $U(f) = \bigcup_{i\in I} U(f_i)$. *For any* A*-module* M, *the following sequence is exact:*

$$0 \to M_f \to \prod_{i\in I} M_{f_i} \to \prod_{(i,j)\in I^2} M_{f_i f_j}$$

is exact. Here the second map is induced by the canonical maps $M \to M_{f_i}$, *and the third map is*

$$(s_i)_{i\in I} \mapsto (s_j - s_i)_{(i,j)\in I^2}.$$

Exercise 4.2.6. Let k be a field and $R = k[x,y]$. Consider the point $0 \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x,y). Let $U \coloneqq \mathsf{Spec}(R) \setminus 0$ be the complementary open subset. Find $\mathcal{O}(U)$.

Definition 4.2.7. An **affine scheme** is a topological space X equipped with a sheaf $\mathcal{O}$ of commutative rings on X such that $(X, \mathcal{O}) \simeq (\mathsf{Spec}R, \mathcal{O}_{\mathsf{Spec}(R)})$ for some commutative ring R.

Proposition 4.2.8. *Let* $x \in \mathsf{Spec}(R)$ *be the point corresponding to a prime ideal* $\mathfrak{p} \subseteq R$. *Then the* R*-algebra* $\mathcal{O}_x$ *given by*

$$R \simeq \mathcal{O}(\mathsf{Spec}(R)) \to \mathcal{O}_x$$

is (uniquely) isomorphic to $R_{\mathfrak{p}}$ *as* R*-algebras. In particular,* $\mathcal{O}_x$ *is a local ring.*

Proof. By definition, we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{U}(\mathsf{Spec}(R),x)^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\mathfrak{B}_x \subseteq \mathfrak{U}(\mathsf{Spec}(R), x)$ be the full subcategory of standard open neighborhoods of x in $\mathsf{Spec}(R)$. By the definition of Zariski topology, $\mathfrak{B}_x^{\mathsf{op}} \to \mathfrak{U}(\mathsf{Spec}(R), x)^{\mathsf{op}}$ is (co)final. Hence we have

$$\mathcal{O}_x \simeq \operatorname*{colim}_{U\in\mathfrak{B}_x^{\mathsf{op}}} \mathcal{O}(U).$$

Let $\phi : R \to A$ be any test R-algebra. We have

$$\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \lim_{U \in \mathfrak{B}_x} \mathsf{Hom}_R(\mathcal{O}(U), A).$$

Since $\mathcal{O}(U)$ is a localization of R for each $U \in \mathfrak{B}_x$, we have

- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \varnothing$ if $U = U(f)$ and $\phi(f)$ is not a unit;
- $\mathsf{Hom}_R(\mathcal{O}(U), A) \simeq \{*\}$ if $U = U(f)$ and $\phi(f)$ is a unit.

It follows that

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $U(f) \in \mathfrak{B}_x$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $U(f) \in \mathfrak{B}_x$.

Note that for an element $f \in R$, the standard open $U(f)$ is a neighborhood of x iff $f \notin \mathfrak{p}$. Hence we have

- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \varnothing$ if $\phi(f)$ is not a unit for some $f \in R \smallsetminus \mathfrak{p}$;
- $\mathsf{Hom}_R(\mathcal{O}_x, A) \simeq \{*\}$ if $\phi(f)$ is a unit for all $f \in R \smallsetminus \mathfrak{p}$.

Note that $\mathsf{Hom}_R(R_\mathfrak{p}, A)$ has the same description. Hence by Yoneda lemma, there is a unique isomophism $\mathcal{O}_x \simeq R_\mathfrak{p}$ as R-algerbas.

□

4.3. **Functoriality.** Throughout this subsection, we fix the following notations:

- Let R and R' be commutative rings.
- Write $X := \mathsf{Spec}(R)$ and $X' := \mathsf{Spec}(R')$.
- Write $\mathcal{O}$ and $\mathcal{O}'$ respectively for the structure sheaves on X and X'.

Construction 4.3.1. Let $h : R \to R'$ be a homomorphism between commuative algebras. Consider the map

$$\phi : X' \to X,\ \mathfrak{p}' \mapsto h^{-1}(\mathfrak{p}').$$

By definition, for any $f \in R$,

$$\phi^{-1}(U(f)) = U(h(f)).$$

It follows that ϕ is a continuous map with respect to the Zariski topology.

Note that the assignment $h \mapsto \phi$ loses information: h *cannot* be reconstructed from ϕ.

Proposition 4.3.2. *Let $h : R \to R'$ be a homomorphism and $\phi : X' \to X$ be the corresponding continuous map. Then there exists a unique morphism in* $\mathsf{Shv}(X, \mathsf{CRing})$

$$\alpha : \mathcal{O} \to \phi_*(\mathcal{O}')$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_*\mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\
R \arrow[u, "\simeq"] \arrow[rr, "h"] & & R'. \arrow[u, "\simeq"']
\end{tikzcd}$$

Proof. Let $\mathfrak{B} \subseteq \mathfrak{U}(\mathsf{Spec}(R))$ be the full subcategory of standard open subsets. By Exercise 1.2.7, it is enough to show that there exists a unique natural transformation

$$\alpha : \mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}} \to \phi_*(\mathcal{O}')|_{\mathfrak{B}^{\mathrm{op}}}$$

that makes the diagram commute.

For any $U \in \mathfrak{B}^{\mathsf{op}}$, we claim there is a unique dotted homomorphism α_U making the following diagram commute

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r, dotted, "\alpha_U"] & \phi_*\mathcal{O}'(U) \arrow[r, "\simeq"] & \mathcal{O}'(\phi^{-1}(U)) \\ R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u] \end{tikzcd}$$

Indeed, choose $f \in R$ such that $U = U(f)$. Then $\phi^{-1}(U) = U(h(f))$. Hence $\mathcal{O}(U) \simeq R_f$ and $\mathcal{O}'(\phi^{-1}(U)) \simeq R'_{h(f)}$. Via these identications, the claim becomes obvious.

It follows that α_U can be assembled into a natural transformation α satisfying the desired property. Moreover, such α is unique because each α_U is unique. □

By adjunction, we obtain the following result.

Corollary 4.3.3. *Let $h : R \to R'$ be a homomorphism and $\phi : \mathsf{Spec}(R') \to \mathsf{Spec}(R)$ be the corresponding continuous map. Then there exists a unique morphism in* $\mathsf{Shv}(X', \mathsf{CRing})$

$$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$$

such that the following diagram commutes

$$\begin{tikzcd} \phi^{-1}\mathcal{O}(X') \arrow[r, "\beta_{X'}"] & \mathcal{O}'(X') \\ R \arrow[u] \arrow[r, "h"] & R'. \arrow[u, "\simeq"'] \end{tikzcd}$$

Moreover, for any point $x' \in X'$ and $x := \phi(x')$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

is a local homomorphism between local rings.

Proof. The first claim follows from Proposition 4.3.2 and the adjunction $\phi^{-1} \vdash \phi_*$. For the second claim, let $\mathfrak{p}' \subseteq R'$ be the prime ideal corresponding to x' and $\mathfrak{p} := \phi^{-1}(\mathfrak{p}')$. By Proposition 4.2.8, we can identify $\mathcal{O}_x \to \mathcal{O}'_{x'}$ with the unique R-homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$, which makes the desired claim manifest. □

The following result says knowing h is equivalent to knowing a pair (ϕ, β).

Proposition-Construction 4.3.4. *There is a canonical bijection between the following sets:*

(i) *The set $\mathsf{Hom}_{\mathsf{CRing}}(R, R')$ of homomorphisms from R to R'.*
(ii) *The set of pairs (ϕ, β), where*
  - *$\phi : X' \to X$ is a continuous map,*
  - *$\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ is a morphism in $\mathsf{Shv}(X', \mathsf{CRing})$*

  such that for any $x = \phi(x')$, $x' \in X'$, the homomorphism

$$\mathcal{O}_x \simeq (\phi^{-1}\mathcal{O})_{x'} \xrightarrow{\beta_{x'}} \mathcal{O}'_{x'}$$

  is a local homomorphism between local rings.

Proof. For any pair (ϕ, β) in (ii), let $\alpha : \mathcal{O} \to \beta_* \mathcal{O}'$ be the morphism corresponding to β via adjunction. There is a unique dotted homomorphism h that makes the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[r, "\alpha_X"] & \phi_*\mathcal{O}'(X) \arrow[r, "\simeq"] & \mathcal{O}'(X') \\
R \arrow[u, "\simeq"] \arrow[rr, dotted, "h"] & & R'. \arrow[u, "\simeq"']
\end{tikzcd}$$

This defines a map (ii)→(i). We have seen this map is surjective (Corollary 4.3.3). It remains to check it is injective.

Suppose (ϕ_1, β_1) and (ϕ_2, β_2) produce the same homomorphism $h : R \to R'$.

We first show $\phi_1 = \phi_2$. Let $x' \in X'$ be a point corresponding to a prime ideal $\mathfrak{p}' \subseteq R'$, consider $x_i := \phi_i(x_i)$. We will show $x_1 = x_2$. Let $\mathfrak{p}_i \subseteq R$ be the prime ideal corresponding to x_i. For $i = 1, 2$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{x_i} \arrow[r, "\simeq"] & (\phi_i^{-1}\mathcal{O})_{x'} \arrow[r, "(\beta_i)_{x'}"] & \mathcal{O}'_{x'} \\
R \arrow[u] \arrow[rr, "h"] & & R' \arrow[u]
\end{tikzcd}$$

By Proposition 4.2.8, $\mathcal{O}_{x_i} \simeq R_{\mathfrak{p}_i}$ and $\mathcal{O}'_{x'} \simeq R'_{\mathfrak{p}'}$. Hence the commutative diagram implies $h^{-1}(\mathfrak{p}') \subseteq \mathfrak{p}_i$. Moreover, since by assumption the top horizontal arrow is a local homomorphism, we must have $h^{-1}(\mathfrak{p}') = \mathfrak{p}_i$. In particular, $\mathfrak{p}_1 = \mathfrak{p}_2$ and therefore $x_1 = x_2$ as desired.

Now write $\phi = \phi_1 = \phi_2$. It remains to show $\beta_1 = \beta_2$. By the last paragraph, for any $x' \in X'$, we have $(\beta_1)_{x'} = (\beta_2)_{x'}$ because it can be identified with the *unique* homomorphism $R_{\mathfrak{p}} \to R'_{\mathfrak{p}'}$ compatible with $h : R \to R'$. Now by Corollary 2.2.3, we obtain $\beta_1 = \beta_2$ as desired.

□

Exercise 4.3.5. Show that the conclusion of Proposition-Construction 4.3.4 would be false if we do not require $\beta_{x'}$ to be a local homomorphism. In other words, show that there exists a continuous map $\phi : X' \to X$ together with a morphism $\beta : \phi^{-1}\mathcal{O} \to \mathcal{O}'$ such that $\beta_{x'}$ is not a local homomorphism for some point $x' \in X'$.

5. Schemes as locally ringed spaces

5.1. **Locally ringed spaces.** Motivated by the construction $(\mathsf{Spec}(R), \mathcal{O})$, we make the following definition.

Definition 5.1.1. A **ringed space** is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of commutative rings on X.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between ringed spaces** is a pair $\phi = (\phi, \beta)$, where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves.

Let $\mathsf{Top}_{\mathsf{CRing}}$ be the category of ringed spaces and morphisms between them.

Remark 5.1.2. Equivalently, we can replace β by a morphism $\alpha : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$.

Construction 5.1.3. By definition, any morphism $(\phi, \beta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between ringed spaces factors as

$$(X, \mathcal{O}_X) \xrightarrow{(\mathrm{id}, \beta)} (X, \phi^{-1}(\mathcal{O}_Y)) \xrightarrow{(\phi, \mathrm{id})} (Y, \mathcal{O}_Y).$$

Definition 5.1.4. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that for any point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.

Let $(X, \mathcal{O}_X)$ be a locally ringed space and $x \in X$ be a point. The **residue field of** $(X, \mathcal{O}_X)$ **at** x is the field

$$\kappa_x := \mathcal{O}_{X,x}/\mathfrak{m}_x,$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$.

A **morphism** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ **between locally ringed spaces** is a pair (ϕ, β), where $\phi : X \to Y$ is a continous map and $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism between sheaves such that for any point $x \in X$, the homomorphism

$$\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$$

is a local homomorphism.

Let $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the category of locally ringed spaces and morphisms between them.

Construction 5.1.5. Let $(\phi, \beta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a homomorphism between locally ringed spaces. For any $x \in X$, the local homomorphism $\beta_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ induces a homomorphism

$$\kappa_{Y,\phi(x)} \to \kappa_{X,x}.$$

Warning 5.1.6. The functor $\mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}} \to \mathsf{Top}_{\mathsf{CRing}}$ is faithful but not full.

Example 5.1.7. For any commutative ring R, we obtain a locally ringed space $(\mathsf{Spec}(R), \mathcal{O})$. We will abuse notation and denote this locally ringed space just by $\mathsf{Spec}(R)$, and treat its structure sheaf as implicit.

For any homomorphism $h : R \to R'$, we obtain a morphism $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ between locally ringed spaces as in Proposition-Construction 4.3.4. Moreover, the information of h is exactly encoded by this morphism.

Definition 5.1.8. An **affine scheme** is a locally ringed space that is isomorphic to $\mathsf{Spec}(R)$ for some R. A **morphism between affine schemes** is a morphism between locally ringed spaces. Let $\mathsf{Aff} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory of affine schemes.

Using these new terminilogies, we can reformulate Proposition-Construction 4.3.4 as follows.

Proposition-Construction 5.1.9. *The following functors are inverse to each other:*

$$\begin{array}{rcl} \mathsf{CRing}^{\mathsf{op}} & \simeq & \mathsf{Aff} \\ R & \mapsto & \mathsf{Spec}(R) \\ \mathcal{O}_X(X) & \leftarrow\!\shortmid & X. \end{array}$$

Construction 5.1.10. Let $(X, \mathcal{O}_X)$ be a (locally) ringed space. For any $f : Y \to X$, the pair $(Y, f^{-1}\mathcal{O}_X)$ defines a (locally) ringed space, and we have a canonical morphism

$$(Y, f^{-1}\mathcal{O}_X) \to (X, \mathcal{O}_X)$$

given by the pair $(f, \mathsf{id}_{f^{-1}\mathcal{O}_X})$.

When $f : Y \subseteq X$ is a subspace, we write $\mathcal{O}_X|_Y := f^{-1}\mathcal{O}_X$ and call the obtained (locally) ringed space $(Y, \mathcal{O}_X|_Y)$ the **restriction of** $(X, \mathcal{O}_X)$ **to** Y (or the **locally ringed subspace of** $(X, \mathcal{O}_X)$ **associated to** Y).

Example 5.1.11. Let R be a commutative ring and consider the locally ringed space $\mathsf{Spec}(R)$. For any element $f \in R$, by Construction 5.1.10, we obtain a locally ringed subspace of $\mathsf{Spec}(R)$ associated to $U(f)$. By construction, it can be identified with $\mathsf{Spec}(R_f)$. In particular, it is an affine scheme.

5.2. **Schemes.**

Definition 5.2.1. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open covering $X = \bigcup_{i \in I} U_i$ with each $(U_i, \mathcal{O}_X|_{U_i})$ being an affine scheme.

A **morphism** between schemes is a morphism between locally ringed spaces. Let $\mathsf{Sch} \subseteq \mathsf{Top}^{\mathsf{loc}}_{\mathsf{CRing}}$ be the full subcategory consisting of schemes.

Notation 5.2.2. We often abuse notation by writing X for a scheme $(X, \mathcal{O}_X)$ and treating its structure sheaf as implicit. Similarly, we often abuse notation by writing $\phi : X \to Y$ for a morphism between schemes and treating the morphism $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ as implicit.

Warning 5.2.3. Nevertheless, one should keep in their minds that schemes are not determined by their underlying topological spaces; similarly morphisms between schemes are not determined by the underlying continuous maps.

Exercise 5.2.4. Let X be a scheme over $\mathsf{Spec}(\mathbb{F}_q)$.

(1) For any open subset $U \subseteq X$, show that the map $\beta_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$, $f \mapsto f^q$ is a homomorphism, and these maps give an endomorphism $\beta : \mathcal{O}_X \to \mathcal{O}_X$ of the structure sheaf.
(2) Show that $\mathsf{Frob}_{X,q} := (\mathsf{id}_X, \beta)$ is an endomorphism of the scheme X defined over $\mathsf{Spec}(\mathbb{F}_q)$.

The morphism $\mathsf{Frob}_{X,q}$ is known as the **absolute** q**-Frobenius morphism of** X, and plays a central role in the study of schemes over finite fields.

The following results follow from their counterparts for affine schemes.

Lemma 5.2.5. *Let X be a scheme.*

- *The affine open subsets of X form a base for its topology.*
- *The space X is Kolmogorov.*

The following exercise provides examples of locally ringed spaces that are not schemes.

Exercise 5.2.6. Let X be a topological space. For any open subset $U \subseteq X$, let $\mathcal{C}_X(U)$ be the commutative ring of $\mathbb{R}$-valued[10] continuous functions on U. Note that $U \mapsto \mathcal{C}_X(U)$ defines a sheaf of commutative rings on X.

(1) Show that $(X, \mathcal{C}_X)$ is a locally ringed space.
(2) Show that a continuous map $X \to X'$ induces a morphism $(X, \mathcal{C}_X) \to (X', \mathcal{C}_{X'})$ between locally ringed spaces.
(3) Show that $(\mathbb{R}, \mathcal{C}_{\mathbb{R}})$ is not a scheme.

5.3. **Open immersions.**

Proposition-Definition 5.3.1. *Let X be a scheme. For any open subspace $U \subseteq X$, the corresponding locally ringed subspace is a scheme. We call it the* ***open subscheme of X associated to U****.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an open covering such that each U_i is an affine scheme. We only need to show the locally ringed subspace associated to each $U_i \cap U$ can be covered by affine schemes. Without lose of generality, we can replace X with U_i and U with $U_i \cap U$, and therefore assume $X \simeq \mathsf{Spec}(R)$ is affine. Now by the definition of the Zariski topology, we can find elements $(f_j)_{j \in J}$ in R such that $U = \bigcup_{j \in J} U(f_j)$. By Example 5.1.11, each $U(f_j)$ is an affine scheme isomorphic to $\mathsf{Spec}(R_{f_j})$. Hence U is a scheme as desired.

□

Definition 5.3.2. We say a morphism $f : Y \to X$ is an **open immersion** if there exists an (unique) open subscheme $U \subseteq X$ such that f factors as $Y \xrightarrow{\simeq} U \to X$.

Warning 5.3.3. An open subscheme U of X may fail to be affine even if X is affine. Also, an *affine* open subset of an affine scheme may fail to be a standard subset.

Exercise 5.3.4. Let k be a field and $R = k[x, y]$. Consider the point $(0, 0) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal (x, y). Let $U \coloneqq \mathsf{Spec}(R) \smallsetminus \{(0, 0)\}$ be the complementary open subset. Show that the scheme U is not affine.

Exercise 5.3.5. Let k be a field of characteristic 0 and $R \coloneqq k[x, y]/(y^2 - x^3)$. Consider the point $(1, 1) \in \mathsf{Spec}(R)$ corresponding to the maximal ideal $(x-1, y-1)$. $U \coloneqq \mathsf{Spec}(R) \smallsetminus \{(1, 1)\}$ be the complementary open subset. Show that the scheme U is affine but it is not a standard open subset of $\mathsf{Spec}(R)$.

Warning 5.3.6. Let X be a scheme and $Y \subseteq X$ be a subspace. The locally ringed subspace associated to Y is in general not a scheme.

Exercise 5.3.7. Let R be a local ring and $X \coloneqq \mathsf{Spec}(R)$. Consider the unique closed point $x \in \mathsf{Spec}(R)$. Show that $(\{x\}, \mathcal{O}_X|_{\{x\}})$ is not a scheme unless R is a field.

[10] We equip $\mathbb{R}$ with the usual topology.

Exercise 5.3.8. An open immersion $f : Y \to X$ is a monomorphism in Sch. In other words, for any $Z \in \mathsf{Sch}$, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(Z, Y) \xrightarrow{f \circ -} \mathsf{Hom}_{\mathsf{Sch}}(Z, X)$$

is injective. Moreover, a morphism $h : Z \to X$ is contained in the image iff the underlying continuous map factors through the open subset $f(Y) \subset X$.

6. Gluing schemes

6.1. **Statement.**

Definition 6.1.1. A **gluing data of schemes** is a collection

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

where

- I is a set;
- For each $i \in I$, X_i is a scheme;
- For any pair $(i,j) \in I^2$, U_{ij} is an open subscheme of X_i;
- For any pair $(i,j) \in I^2$,

$$\phi_{ij} : U_{ij} \to U_{ji},$$

  is an isomorphism between schemes.

The above data should satisfy the following conditions:

- For any $i \in I$, $U_{ii} = X_i$ and $\phi_{ii} = \mathsf{id}_{X_i}$.
- For any triple $(i,j,k) \in I^3$,

$$\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

  as open subsets of U_{ji}.
- For any triple $(i,j,k) \in I^3$, the following **cocycle condition** holds:

$$\phi_{jk}|_{U_{ji}\cap U_{jk}} \circ \phi_{ij}|_{U_{ij}\cap U_{ik}} = \phi_{ik}|_{U_{ij}\cap U_{ik}},$$

  i.e., the following diagram commutes:

$$\begin{tikzcd} & U_{ij}\cap U_{ik} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\phi_{ik}", "\simeq"'] & \\ U_{ji}\cap U_{jk} \arrow[rr, "\phi_{jk}", "\simeq"'] & & U_{ki}\cap U_{kj}. \end{tikzcd}$$

Proposition-Definition 6.1.2. *Given a gluing data of schemes*

$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big)$$

there exists an essentially unique collection

$$\big(X, (X_i')_{i\in I}, (\varphi_i)_{i\in I}\big)$$

where

- *X is a scheme;*
- *For each $i \in I$, X_i' is an open subscheme of X;*
- *For each $i \in I$,*

$$\varphi_i : X_i \xrightarrow{\simeq} X_i',$$

  is an isomorphism;

such that

- *$X = \bigcup_{i\in I} X_i'$ as topological spaces;*
- *For any pair $(i,j) \in I^2$,*

$$\varphi_i(U_{ij}) = X_i' \cap X_j'$$

  as open subsets of X_i';

- *For any pair $(i,j) \in I^2$, we have*
$$\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ji}} \circ \phi_{ij},$$
i.e., the following diagram commutes
$$\begin{tikzcd} & U_{ij} \arrow[dl, "\phi_{ij}"', "\simeq"] \arrow[dr, "\varphi_i", "\simeq"'] & \\ U_{ji} \arrow[rr, "\varphi_j", "\simeq"'] & & X'_i \cap X'_j. \end{tikzcd}$$

We say the scheme X is ***glued*** *from the given gluing data, and treat $\big((X'_i)_{i\in I}, (\varphi_i)_{i\in I}\big)$ as implicit.*

Proof. It is an exercise in point-set topology that the similar claim for topological spaces is correct. In other words, the gluing data gives an essentially unique topological space X equipped with open subspaces X'_i and homeomorphisms $\varphi_i : X_i \to X'_i$ satisfying the *topological* conditions listed in the statement.

Hence we only show there is an essentially unique $\mathcal{O}_X \in \mathsf{Shv}(X, \mathsf{CRing})$ equipped with isomorphisms $\varphi_i^{-1}(\mathcal{O}_X|_{X'_i}) \simeq \mathcal{O}_{X_i}$, that satisfies the remaining *sheaf-theoretic* conditions. Note that such a ringed space $(X, \mathcal{O}_X)$ will automatically be a scheme because its restriction to each X'_i is a scheme isomorphic to X_i.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X'_i$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. It is easy to see there exists an essentially unique functor
$$\mathcal{O}_{\mathfrak{B}^{\mathrm{op}}} : \mathfrak{B}^{\mathrm{op}} \to \mathsf{CRing}$$
equipped with isomorphisms
$$\beta_i : \varphi_i^{-1}((\mathcal{O}_{\mathfrak{B}^{\mathrm{op}}})|_{X'_i}) \xrightarrow{\simeq} \mathcal{O}_i$$
satisying the desired conditions. Here $(\mathcal{O}_{\mathfrak{B}^{\mathrm{op}}})|_{X'_i}$ is the restriction of $\mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}}$ along the fully faithful embedding $\mathfrak{U}(X'_i)^{\mathrm{op}} \to \mathfrak{B}^{\mathrm{op}}$. Moreover, one can check that the obtained $\mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}}$ satisfies the sheaf condition in Proposition 1.2.10. Hence there is an essentially unique extension of $\mathcal{O}|_{\mathfrak{B}^{\mathrm{op}}}$ to a CRing-valued sheaf $\mathcal{O}_X$ on X, which fulfills our goal.

□

The following proposition describes how to construct morphisms out of a glued space.

Proposition 6.1.3. *Let*
$$\big(I, (X_i)_{i\in I}, (U_{ij})_{(i,j)\in I^2}, (\phi_{ij})_{(i,j)\in I^2}\big),$$
be a gluing data of schemes and
$$\big(X, (X'_i)_{i\in I}, (\varphi_i)_{i\in I}\big)$$
be its gluing output. For any scheme Y, the map
$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Sch}}(X,Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Sch}}(X_i, Y) \\ f & \mapsto & (f|_{X'_i} \circ \varphi_i)_{i\in I} \end{array}$$
is injective, and a collection of morphisms $(g_i : X_i \to Y)_{i\in I}$ is contained in the image iff $g_i|_{U_{ij}} = g_j|_{U_{ji}} \circ \phi_{ij}$ for any pair $(i,j) \in I^2$.

Proof. To simplify the notations, we identify X_i with X_i' and identify U_{ij} with the intersection $X_i \cap X_j$ inside X. Consequently, ϕ_{ij} and φ_i are identity morphisms.

We first prove the similar claim for topological spaces. Indeed, since $X = \bigcup_{i\in X} X_i$ as a topological space, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Top}}(X,Y) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Top}}(X_i,Y) \\ f & \mapsto & (f|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of continuous map $(f_i : X_i \to Y)_{i\in I}$ is contained in the image iff $f_i|_{X_i\cap X_j} = f_j|_{X_i\cap X_j}$ for any pair $(i,j) \in I^2$.

It follows that we only need to show that for a given continuous map $f : X \to Y$ and $f_i := f|_{X_i}$, the map

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \to & \prod_{i\in I} \mathsf{Hom}_{\mathsf{Shv}(X_i,\mathsf{CRing})}(f_i^{-1}\mathcal{O}_Y,\mathcal{O}_{X_i}) \\ \beta & \mapsto & (\beta|_{X_i})_{i\in I} \end{array}$$

is injective, and a collection of morphisms $(\beta_i : f_i^{-1}\mathcal{O}_Y \to \mathcal{O}_{X_i})_{i\in I}$ is contained in the image iff $\beta_i|_{X_i\cap X_j} = \beta_j|_{X_i\cap X_j}$.

Let $\mathfrak{B} \subseteq \mathfrak{U}(X)$ be the full subcategory consisting of open subsets $V \subseteq X$ such that $V \subseteq X_i$ for some $i \in I$. Note that objects in $\mathfrak{B}$ form a base for the topology of X. By Exercise 1.2.7, we have a bijection

$$\begin{array}{rcl} \mathsf{Hom}_{\mathsf{Shv}(X,\mathsf{CRing})}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X) & \xrightarrow{\simeq} & \mathsf{Hom}_{\mathsf{Fun}(\mathfrak{B}^{\mathsf{op}},\mathsf{CRing})}((f^{-1}\mathcal{O}_Y)|_{\mathfrak{B}^{\mathsf{op}}},(\mathcal{O}_X)|_{\mathfrak{B}^{\mathsf{op}}}) \\ \beta & \mapsto & \beta|_{\mathfrak{B}^{\mathsf{op}}}. \end{array}$$

Now the desired claim follows from the fact that the category $\mathfrak{B}$ can be covered by its full subcategories $\mathfrak{U}(X_i)$, and $\mathfrak{U}(X_i)\cap\mathfrak{U}(X_j) \simeq \mathfrak{U}(X_i\cap X_j)$. □

We can also describe morphisms *into* a glued space.

Proposition-Construction 6.1.4. *Let*

$$\big(I,(X_i)_{i\in I},(U_{ij})_{(i,j)\in I^2},(\phi_{ij})_{(i,j)\in I^2}\big),$$

be a gluing data of schemes and

$$\big(X,(X_i')_{i\in I},(\varphi_i)_{i\in I}\big)$$

be its gluing output. For any scheme Y, there is a canonical bijection between the following sets:

(i) *The set $\mathsf{Hom}_{\mathsf{Sch}}(Y,X)$ of morphisms $f : Y \to X$*
(ii) *The set of collections*

$$(f_i : Y_i \to X_i)_{i\in I},$$

where
- *Each Y_i is an open subscheme of Y and $Y = \bigcup_{i\in I} Y_i$ is an open covering;*
- *Each $f_i : Y_i \to X_i$ is a morphism;*

such that
- *For each pair $(i,j) \in I^2$, $f_i^{-1}(U_{ij}) = Y_i \cap Y_j$;*
- *For each pair $(i,j) \in I^2$, $f_j|_{Y_i\cap Y_j} = \phi_{ij} \circ f_i|_{Y_i\cap Y_j}$.*

Sketch. We first construct a map (i)→(ii). Give a morphism $f : Y \to X$, we declare Y_i to be the open subscheme associated to the open subset $f^{-1}(X_i')$. In particular, $f|_{Y_i}$ gives a morphism $Y_i \to X_i'$. We declare f_i to be the composition

$$Y_i \to X_i' \xrightarrow{\varphi_i^{-1}} X_i.$$

One can verify the collection $(f_i : Y_i \to X_i)_{i\in I}$ satisfies the desired requirements. This gives a map (i)→(ii).

Now we construct a map (ii)→(i). Given a collection $(f_i : Y_i \to X_i)_{i\in I}$. Consider the compositions

$$g_i : Y_i \xrightarrow{f_i} X_i \xrightarrow{\varphi_i} X_i' \to X.$$

One can check $g_i|_{Y_i\cap Y_j} = g_j|_{Y_i\cap Y_j}$. It follows that there is a unique morphism $f : Y \to X$ such that $f|_{Y_i} = g_i$. This gives a map (ii)→(i).

Now one can check the above two maps are inverse to each other.

□

Remark 6.1.5. Results in this subsection also works for general (locally) ringed spaces.

6.2. **Examples.**

Example 6.2.1. Let $(X_i)_{i\in I}$ be a set of schemes, U_{ij} be the empty scheme for $i \neq j$, and ϕ_{ij} be the identity morphisms. This is obviously a gluing data. The scheme X glued from this gluing data is called the **disjoint union of** $(X_i)_{i\in I}$, and we denote it by $\bigsqcup_{i\in I} X_i$. By Proposition 6.1.3, $\bigsqcup_{i\in I} X_i$ is also the coproduct of $(X_i)_{i\in I}$ inside the category Sch.

Example 6.2.2. As one would expect, the n-dimensional projective space can be glued from $(n+1)$ affine spaces of dimension n. Below are the details.

Let R be any commutative ring. For $n \geq 0$, let $I := \{0, 1, \cdots, n\}$ and

$$X_i := \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]/(x_i^{(i)} - 1)).$$

Let

$$U_{ij} := U(x_j^{(i)}) \subseteq X_i.$$

Then we have

$$U_{ij} \simeq \mathsf{Spec}(R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1)).$$

Note that we have an isomorphism

$$R[x_0^{(i)}, \cdots, x_n^{(i)}]_{x_j^{(i)}}/(x_i^{(i)} - 1) \simeq R[x_0^{(j)}, \cdots, x_n^{(j)}]_{x_i^{(j)}}/(x_j^{(j)} - 1)$$

that sends $x_k^{(i)}$ to $x_k^{(j)}/x_i^{(j)}$. This gives an isomorphism

$$U_{ij} \xrightarrow{\simeq} U_{ji}.$$

One can check the above gives a gluing data, hence we obtain an essentially unique scheme X glued from it.

We write $\mathbb{P}^n_R$ for the gluing result and call it the n**-dimensional projective space over** R.

Exercise 6.2.3. Let R be a commutative ring and k be an algebraically closed field.

(1) Find $\mathcal{O}_{\mathbb{P}^n_R}(\mathbb{P}^n_R)$. Deduce that $\mathbb{P}^n_R$ is not affine for $n \geq 1$.
(2) Show that the closed points of $\mathbb{P}^n_k$ can be canonically identified with elements in $(k^{n+1} \smallsetminus 0)/k^\times$, where $k^\times$ acts on the vector space k^{n+1} via scaler multiplication.

Exercise 6.2.4. Let R be any commutative ring and $I = \{1,2\}$. Let

$$X_1 = X_2 \coloneqq \mathbb{A}^1_R \coloneqq \mathsf{Spec}(R[t])$$

and

$$U_{12} = U_{21} \coloneqq U(t), U_{11} \coloneqq X_1, U_{22} \coloneqq X_2.$$

Let ϕ_{ij} be the identity morphisms. Consider the scheme X glued from the above gluing data. Show that X is not affine.

7. Morphisms to affine schemes

7.1. A criterion for being affine.

Construction 7.1.1. Let X and Y be schemes. For any morphism $f : X \to Y$, by definition we have a morphism $\alpha : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Taking global sections, we obtain a homomorphism

$$\alpha_Y : \mathcal{O}_Y(Y) \to (f_*\mathcal{O}_X)(Y) \simeq \mathcal{O}_X(X).$$

One can check this defines a functor

$$\mathsf{Sch} \to \mathsf{CRing}^{\mathsf{op}},\ X \mapsto \mathcal{O}_X(X)$$

that sends a morphism $f : X \to Y$ to α_Y as above.

Theorem 7.1.2. *A scheme Y is affine iff for any scheme X, the natural map*

$$\mathsf{Hom}_{\mathsf{Sch}}(X, Y) \to \mathsf{Hom}_{\mathsf{CRing}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{7.1}$$

is a bijection.

Proof. We first prove the "only if" statement. Let $Y \simeq \mathsf{Spec}(R)$ be an affine scheme, we need to show the natural map

$$\mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \tag{7.2}$$

is a bijection.

We construct a map of the inverse direction as follows. Let $h : R \to \mathcal{O}_X(X)$ be a homomorphism. For any affine open subscheme $U \subseteq X$, by Proposition-Definition 5.1.9, the map

$$\mathsf{Hom}_{\mathsf{Sch}}(U, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(U))$$

is a bijection. Let $f_U : U \to \mathsf{Spec}(R)$ be the morphism corresponding to the composition

$$R \xrightarrow{h} \mathcal{O}_X(X) \xrightarrow{(-)|_U} \mathcal{O}_X(U).$$

Unwinding the constructions, one can check for affine open subschemes $U \subseteq V \subseteq X$, we have $p_U = p_V|_U$. Recall affine open subsets form a base for the topology of X. Using Exercise 1.2.7, one can show there is a unique morphism $p : X \to \mathsf{Spec}(R)$ such that $p_U = p|_U$ for any affine open subscheme $U \subseteq X$. The construction $h \mapsto p$ as above gives a map

$$\mathsf{Hom}_{\mathsf{CRing}}(R, \mathcal{O}_X(X)) \to \mathsf{Hom}_{\mathsf{Sch}}(X, \mathsf{Spec}(R)). \tag{7.3}$$

One can check that (7.2) and (7.3) are inverse to each other. Namely, (7.3) $\circ$ (7.2) = id follows from the uniqueness property about the morphism p, while (7.2) $\circ$ (7.3) = id follows from the fact that an element in $\mathcal{O}_X(X)$ is determined by its restrictions in $\mathcal{O}_X(U)$'s.

Now we deduce the "if" statement from the "only if" one. Let Y be a scheme such that (7.1) is bijective for any scheme X. Write $R \coloneqq \mathcal{O}_Y(Y)$. By the "if" statement, we have

$$\mathsf{Hom}_{\mathsf{Sch}}(Y, \mathsf{Spec}(R)) \to \mathsf{Hom}_{\mathsf{CRing}}(R, R).$$

In particular, there is a canonical morphism $q : Y \to \mathsf{Spec}(R)$ corresponding to id_R. Moreover, by construction, for any scheme X the following diagram commutes:

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Sch}}(X,Y) \arrow[r, "\simeq"] \arrow[d, "q\circ -"] & \mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X)) \arrow[d, equal] \\ \mathsf{Hom}_{\mathsf{Sch}}(X,\mathsf{Spec}(R)) \arrow[r, "\simeq"] & \mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X)), \end{tikzcd}$$

where the horizontal maps are (7.1) applied to Y and $\mathsf{Spec}(R)$ respectively, and they are bijective either by the "if" statement or by assumption. It follows that composing with q induces a bijection

$$\mathsf{Hom}_{\mathsf{Sch}}(X,Y) \xrightarrow{\simeq} \mathsf{Hom}_{\mathsf{Sch}}(X,\mathsf{Spec}(R))$$

for any $X \in \mathsf{Sch}$. By Yoneda lemma, q is an isomorphism and therefore Y is affine as desired.

□

7.2. Applications.

Corollary 7.2.1. *For any scheme X, there is a unique morphism $X \to \mathsf{Spec}(\mathbb{Z})$. In other words, $\mathsf{Spec}(\mathbb{Z}) \in \mathsf{Sch}$ is a final object.*

Corollary 7.2.2. *For any scheme X, there is a unique morphism*

$$q : X \to \mathsf{Spec}(\mathcal{O}_X(X))$$

that induces the identity homomorphism between global sections.

Corollary 7.2.3. *The embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ admits a canonical left adjoint given by*

$$\mathsf{Sch} \to \mathsf{Aff},\ X \mapsto \mathsf{Spec}(\mathcal{O}_X(X)).$$

Exercise 7.2.4. Show that the embedding functor $\mathsf{Aff} \to \mathsf{Sch}$ does not admit a right adjoint.

8. Functor of points

The main goal of this section is to construct a fully faithful functor $\mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ and describe its essential image.

8.1. R-points.

Definition 8.1.1. Let X be a scheme and R be a commutative ring. An **R-point of X** is a morphism $\mathsf{Spec}(R) \to X$. Let

$$X(R) := \mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), X)$$

be the set of R-points of X.

A **field-valued point of** X is a k-point of X for some field k.

A **geometric point of** X is a k-point of X for some *separably closed* field k.

Proposition-Construction 8.1.2. *Let X be a scheme and k be a field. There is a bijection between*

- *the set $X(k)$ of k-points of X;*
- *the set of pairs (x, i), where $x \in X$ is a topological point*[11] *and $i : \kappa_x \to k$ is a homomorphism.*

Proof. By definition, a k-point of X is given by a pair (ϕ, β). Since $\mathsf{Spec}(k)$ has only one topological point, the continuous map ϕ is given by a topological point $x \in X$. Now the morphism β is given by a *local* homomorphism $\mathcal{O}_{X,x} \to k$, where we identify a sheaf $\mathcal{F}$ of commutative rings on the one-point space $\{*\}$ with $\mathcal{F}(\{*\})$. Now a local homomorphism $\mathcal{O}_{X,x} \to k$ uniquely factors as $\mathcal{O}_{X,x} \to \kappa_x \to k$. This gives the desired bijection.

□

Remark 8.1.3. For a field-valued point $s : \mathsf{Spec}(k) \to X$, we sometimes abuse notations and write it as $s \to X$, where s is understood as $\mathsf{Spec}(k)$. Similarly, for a topological point $x \in X$, we sometimes abuse notations and write the κ_x-point $\mathsf{Spec}(\kappa_s) \to X$ as $x \to X$.

Note however that a scheme is *not* determined by its field-valued points.

Exercise 8.1.4. Let k be a field, and consider $X := \mathsf{Spec}(k)$ and $X' := \mathsf{Spec}(k[\epsilon]/(\epsilon^2))$. Let $f : X \to X'$ be the morphism corresponding to the obvious homomorphism $k[\epsilon]/(\epsilon^2) \to k$. Show that f induces a bijection between the set of field-valued points of X and that of X'.

Construction 8.1.5. Let X be a scheme and $x \in X$ be a topological point. Let $U \subseteq X$ be an affine open subset containing x. Note that we have $U \simeq \mathsf{Spec}(\mathcal{O}_X(U))$. The canonical homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$ induces a morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U))$. Now the composition

$$\mathsf{Spec}(\mathcal{O}_{X,x}) \to \mathsf{Spec}(\mathcal{O}_X(U)) \simeq U \to X$$

gives an $\mathcal{O}_{X,x}$-point of X.

Exercise 8.1.6. Show that:

(1) The above $\mathcal{O}_{X,x}$-point of X does not depend on the choice of U.
(2) The morphism $\mathsf{Spec}(\mathcal{O}_{X,x}) \to X$ identifies the locally ringed space $\mathsf{Spec}(\mathcal{O}_{X,x})$ as the restriction of X to a certain subspace.

[11] A point in the underlying topological space.

8.2. **Schemes as a functor.**

Definition 8.2.1. Let X be a scheme. The functor

$$\mathsf{h}_X : \mathsf{CRing} \to \mathsf{Set},\ R \mapsto X(R)$$

is called the **functor of points of** X.

Note that the construction $X \mapsto \mathsf{h}_X$ is functorial. Hence we have a canonical functor

$$\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

Remark 8.2.2. By construction, the composition

$$\mathsf{CRing}^{\mathsf{op}} \simeq \mathsf{Aff} \to \mathsf{Sch} \xrightarrow{\mathsf{h}} \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$$

can be identified with the Yoneda embedding $R \mapsto \mathsf{Hom}_{\mathsf{CRing}}(R, -)$.

Theorem 8.2.3. *The functor* $\mathsf{h} : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$ *is fully faithful.*

Remark 8.2.4. For a description of the essential image of this functor[12], see [EH00, Chapter VI].

Remark 8.2.5. The theorem suggests another way to develop scheme theory *without* using locally ringed spaces. Namely, one can *define* a scheme as a functor $\mathsf{CRing} \to \mathsf{Set}$ satisfying certain properties. In fact, in the 1970s, Grothendieck himself radically urged to abandon his earlier definition of schemes in favor of the functorial point of view. In my opinion, this approach at least has the following advantages.

- It makes a lot of constructions about schemes formal and therefore easier.
- It provides a more direct way to deal with moduli problems and deformation theory.
- It allows one to define more exotic geometric objects, such as algebraic spaces, stacks, indschemes...

No matter how much I love this functorial approach, however, I do *not* believe a learner should ignore the classical view of schemes as a structured topological space[13].

Definition 8.2.6. We say a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is **represented by a scheme** X if it is equipped with a natural isomorphism $F \simeq \mathsf{h}_X$.

Example 8.2.7. The functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to its underlying set is represented by the affine scheme $\mathbb{A}^1_{\mathbb{Z}} \coloneqq \mathsf{Spec}(\mathbb{Z}[t])$.

Exercise 8.2.8. Show that the functor $\mathsf{CRing} \to \mathsf{Set}$ that sends R to the set $\mathsf{GL}_n(R)$ of $n \times n$ invertible matrices over R is represented by an affine scheme.

Exercise 8.2.9. Show that the constant functor $\mathsf{CRing} \to \mathsf{Set}$, $R \mapsto I$ is not represented by a scheme unless $I \simeq \{*\}$. What is the functor represented by the disjoint union $\bigsqcup_{i \in I} \mathsf{Spec}(\mathbb{Z})$?

---

[12] In standard terminology, a functor $F : \mathsf{CRing} \to \mathsf{Set}$ is contained in the essential image iff F satisfies Zariski descents and admits an open covering by representable functors.

[13] Just imagine learning the projective space $\mathbb{P}^n_{\mathbb{Z}}$ for the first time using the following definition: it is the functor sending a commutative algebra R to the isomorphism classes of surjections from the free R-module $R^{\oplus(n+1)}$ to a rank 1 projective R-module P.

Proof of Theorem 8.2.3. For any pair of schemes (X,Y), we need to show

$$\mathsf{Hom}_{\mathsf{Sch}}(X,Y) \to \mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X,\mathsf{h}_Y) \tag{8.1}$$

is bijective.

We first construct a map of the inverse direction as follows. Let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation. For any affine open subscheme $U \subseteq X$, by definition we have identifications

$$\mathsf{Hom}_{\mathsf{Sch}}(U,Z) \simeq \mathsf{h}_Z(\mathcal{O}_X(U))$$

functorial in Z. Let $j_U \in \mathsf{h}_X(\mathcal{O}_X(U))$ be the element corresponding to the canonical immersion $U \to X$. Consider the morphism $f_U : U \to Y$ corresponding to the element $\theta(j_U) \in \mathsf{h}_Y(\mathcal{O}_X(U))$. One can check that for affine open subschemes $U \subseteq V \subseteq X$, we have $f_U = f_V|_U$. Using Exercise 1.2.7, one can show there is a unique morphism $f : X \to Y$ such that $f_U = f|_U$ for any affine open subscheme $U \subseteq X$. The construction $\theta \mapsto f$ as above gives a map

$$\mathsf{Hom}_{\mathsf{Fun}(\mathsf{CRing},\mathsf{Set})}(\mathsf{h}_X,\mathsf{h}_Y) \to \mathsf{Hom}_{\mathsf{Sch}}(X,Y). \tag{8.2}$$

It remains to check (8.1) and (8.2) are inverse to each other. Using the uniqueness property about the morphism f, it is easy to see $(8.2) \circ (8.1) = \mathsf{id}$.

It remains to show $(8.1) \circ (8.2) = \mathsf{id}$. For this, let $\theta : \mathsf{h}_X \to \mathsf{h}_Y$ be a natural transformation and $f : X \to Y$ be the morphism constructed as above. Let $\mathsf{h}_f : \mathsf{h}_X \to \mathsf{h}_Y$ be the natural transformation induced by f via functoriality. We only need to show $\theta = \mathsf{h}_f$. In other words, for any R-point $x \in X(R)$, we need to show

$$\theta(x) = \mathsf{h}_f(x). \tag{8.3}$$

Note that by definition x is a morphism $x : \mathsf{Spec}(R) \to X$, and both sides in (8.3) are morphisms from $\mathsf{Spec}(R)$ to Y.

Unwinding the constructions, it is easy to see for any affine open subscheme $j_U : U \subseteq X$, we have

$$\theta \circ \mathsf{h}_{j_U} = \mathsf{h}_{\theta(j_U)} = \mathsf{h}_{f_U} = \mathsf{h}_f \circ \mathsf{h}_{j_U}.$$

In other words, we know (8.3) is true if x is contained in the image of $U(R) \to X(R)$ for some affine open subscheme U.

Now for general x, we can find a covering of $\mathsf{Spec}(R) = \bigcup_{i \in I} V_i$ by its affine open subschemes such that $x|_{V_i} : V_i \to X$ factors through some affine open subscheme of X. By the last paragraph, we see

$$\theta(x|_{V_i}) = \mathsf{h}_f(x|_{V_i}).$$

In other words, the restrictions of the morphisms $\theta(x)$ and $\mathsf{h}_f(x) : \mathsf{Spec}(R) \to Y$ to each V_i are equal. Now Exercise 1.2.7 implies these two morphisms are equal as desired.

□

Part III. Language of schemes

9. Fiber products

9.1. **Definition of fiber products.** Recall we have the notion of fiber products in any category.

Definition 9.1.1. Let $\mathcal{C}$ be a category. We say a commutative square in $\mathcal{C}$

$$\begin{tikzcd} d \arrow[r, "g'"] \arrow[d, "f'"] & a \arrow[d, "f"] \\ b \arrow[r, "g"] & c \end{tikzcd} \tag{9.1}$$

is **Cartesian**, if for any object $x \in \mathcal{C}$, the commutative square

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{C}}(x,d) \arrow[r, "g'\circ-"] \arrow[d, "f'\circ-"] & \mathsf{Hom}_{\mathcal{C}}(x,a) \arrow[d, "f\circ-"] \\ \mathsf{Hom}_{\mathcal{C}}(x,b) \arrow[r, "g\circ-"] & \mathsf{Hom}_{\mathcal{C}}(x,c) \end{tikzcd}$$

is a Cartesian square in Set. In other words, if it induces a bijection

$$\mathsf{Hom}_{\mathcal{C}}(x,d) \to \mathsf{Hom}_{\mathcal{C}}(x,a) \underset{\mathsf{Hom}_{\mathcal{C}}(x,c)}{\times} \mathsf{Hom}_{\mathcal{C}}(x,b).$$

In this case, we also say (9.1) is a **pullback square** and say (9.1) **exhibits** d **as the pullback of the diagram** $a \xrightarrow{f} c \xleftarrow{g} b$. We also say d is the **fiber product** of $a \xrightarrow{f} c \xleftarrow{g} b$.

Remark 9.1.2. By Yoneda's lemma, the object d, equipped with the morphisms f' and g', is essentially unique. We often write

$$d \simeq a \underset{c}{\times} b$$

when the morphisms are clear from the context.

Remark 9.1.3. By Yoneda's lemma, the construction

$$[a \xrightarrow{f} c \xleftarrow{g} b] \mapsto a \underset{c}{\times} b$$

is functorial. In other words, for a commutative diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, "p"] & c \arrow[d, "r"] & b \arrow[l] \arrow[d, "q"] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd},$$

there is a unique dotted morphism $a \times_c b \to a' \times_c{}' b'$ that make the following diagram commute

$$\begin{tikzcd} a \arrow[d, "p"] & a \times_c b \arrow[l] \arrow[r] \arrow[d, dotted] & b \arrow[d, "q"] \\ a' & a' \times_{c'} b' \arrow[l] \arrow[r] & b'. \end{tikzcd}$$

We often abuse notation and denote this morphism

$$(p,q) : a \underset{c}{\times} b \to a' \underset{c}{\times'} b'$$

(but it also depends on other morphisms in the diagram).

Example 9.1.4. If c is a final object, then $a \times_c b \simeq a \times b$.

Example 9.1.5. Fiber products exist in Ab and the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$ preserves fiber products. Given a diagram $A \xrightarrow{f} C \xleftarrow{g} B$ in Ab, we have

$$A \underset{C}{\times} B \simeq \mathsf{ker}(A \bigoplus B \xrightarrow{(f,-g)} C).$$

Example 9.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose fiber products exist in $\mathcal{C}$. Then fiber products exist in $\mathsf{Fun}(\mathcal{D}, \mathcal{C})$, and we have

$$(F_1 \underset{F_3}{\times} F_2)(d) \simeq F_1(d) \underset{F_3(d)}{\times} F_2(d)$$

for functors $F_i : \mathcal{D} \to \mathcal{C}$ and objects $d \in \mathcal{D}$.

Definition 9.1.7. Let $\mathcal{C}$ be a category. The **pushout of a diagram** $a \xleftarrow{f} c \xrightarrow{g} b$ is defined to be the pullback of the corresponding diagram in $\mathcal{C}^{\mathsf{op}}$. We also call it the **fiber coproduct of** $a \xleftarrow{f} c \xrightarrow{g} b$ and denote it by

$$a \coprod_c b.$$

9.2. Fiber products of affine schemes.

Exercise 9.2.1. Fiber coproducts exist in CRing and we have

$$A \coprod_C B \simeq A \underset{C}{\otimes} B.$$

Corollary 9.2.2. *Fiber products exist in* Aff. *Given a diagram* $A \leftarrow C \to B$, *we have*

$$\mathsf{Spec}(A \underset{C}{\otimes} B) \simeq \mathsf{Spec}(A) \underset{\mathsf{Spec}(C)}{\times} \mathsf{Spec}(B).$$

Warning 9.2.3. The underlying topological space of $\mathsf{Spec}(A \otimes_C B)$ is in general not the fiber product of the corresponding topological spaces. In other words, the forgetful functor $\mathsf{Aff} \to \mathsf{Top}$ does not preserve fiber products.

Exercise 9.2.4. Let $k \to k'$ be a finite *separable* extension of degree d and $\overline{k}$ be a algebraic closure of k. Show that

$$\mathsf{Spec}(k') \underset{\mathsf{Spec}(k)}{\times} \mathsf{Spec}(\overline{k}) \simeq \sqcup_d \mathsf{Spec}(\overline{k}).$$

Note that we also have the following formal corollary of Corollary 7.2.3:

Corollary 9.2.5. *The functor* $\mathsf{Aff} \to \mathsf{Sch}$ *preserves fiber products.*

9.3. **Fiber product of schemes.**

Theorem 9.3.1. *The category* Sch *admits fiber products.*

We will give a constructive proof of the theorem at the end of this section. For now, we prove a particular case of it.

Lemma 9.3.2. *Let $f : X \to Y$ be a morphism between schemes and $U \subseteq Y$ be an open subscheme. Then the fiber product $X \times_Y U$ exists, and the canonical morphism $X \times_Y U \to X$ is an open immersion onto the open subscheme $f^{-1}(U)$.*

Proof. We only need to show the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f'"] \arrow[d, "j'"] & U \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in Sch is Cartesian. Let $Z \in \mathsf{Sch}$ be a test object. We only need to show the commutative diagram

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Sch}}(Z, f^{-1}(U)) \arrow[r, "f'\circ-"] \arrow[d, "j'\circ-"] & \mathsf{Hom}_{\mathsf{Sch}}(Z, U) \arrow[d, "j\circ-"] \\ \mathsf{Hom}_{\mathsf{Sch}}(Z, X) \arrow[r, "f\circ-"] & \mathsf{Hom}_{\mathsf{Sch}}(Z, Y) \end{tikzcd}$$

in Set is Cartesian. Using the definition of open subschemes, the vertical maps are injective, and a morphism $g : Z \to X$ (resp. $h : Z \to Y$) is contained in the image iff the subspace $g(Z)$ (resp. $h(Z)$) is contained in $f^{-1}(U)$ (resp. U). It follows that $g : Z \to X$ is contained in the image of the left vertical map iff $f \circ g$ is contained in the image of the right vertical map. In other words, the above square is Cartesian as desired.

□

Corollary 9.3.3. *Let $X \to Y \xleftarrow{j} U$ be a diagram in* Sch *such that j is an open immersion. Then the forgetful functor* Sch → Top *preserves the fiber product of this diagram.*

Exercise 9.3.4. Let X be a scheme.

(1) Suppose X is affine. Show that the intersection of two affine open subsets in X is still affine.
(2) Show that (1) may fail for general X.

Proposition 9.3.5. *A diagram in* Sch

(9.2)
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

is Cartesian iff it induces a Cartesian diagram

$$\begin{tikzcd} \mathsf{h}_W \arrow[r] \arrow[d] & \mathsf{h}_X \arrow[d] \\ \mathsf{h}_Y \arrow[r] & \mathsf{h}_Z \end{tikzcd} \tag{9.3}$$

in $\mathsf{Fun}(\mathsf{CRing}, \mathsf{Set})$. *In particular, we have*

$$\mathsf{h}_{X \times_Z Y} \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y. \tag{9.4}$$

Proof. For the "only if" claim, let (9.2) be a Cartesian square. By definition, for any $R \in \mathsf{CRing}$, the functor $\mathsf{Hom}_{\mathsf{Sch}}(\mathsf{Spec}(R), -)$ sends (9.2) to a Cartesian square in Set. In other words, the values of the functors in (9.3) at $R \in \mathsf{CRing}$ form a Cartesian square in Set. This formally implies (9.3) itself is a Cartesian square as desired (Example 9.1.6).

Note that the "only if" claim itself implies the isomorphism (9.4).

For the "if" claim, let (9.2) be a commutative square such that (9.3) is Cartesian. By definition, (9.2) corresponds to a morphism $f : W \to X \times_Z Y$ and (7.3) corresponds to an isomorphism

$$\mathsf{h}_W \simeq \mathsf{h}_X \underset{\mathsf{h}_Z}{\times} \mathsf{h}_Y. \tag{9.5}$$

Moreover, unwinding the definitions, we see the following diagram commute

$$\begin{tikzcd} \mathsf{h}_W \arrow[r, "(9.5)"] \arrow[d, "\mathsf{h}_f"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y \arrow[d, equal] \\ \mathsf{h}_{X \times_Z Y} \arrow[r, "(9.4)"] & \mathsf{h}_X \times_{\mathsf{h}_Z} \mathsf{h}_Y. \end{tikzcd}$$

It follows that h_f is also an isomorphism. By Theorem 8.2.3, f is an isomorphism as desired.

□

9.4. **Existence of fiber products.** In this subsection, we prove Theorem 9.3.1. We first deduce the theorem from the following lemma.

Lemma 9.4.1. *Let* $X \to S \leftarrow Y$ *be a diagram in* Sch. *Let* $X = \bigcup_{i \in I} X_i$ *and* $Y = \bigcup_{j \in J} Y_j$ *be coverings by open subschemes. Suppose for each pair* $(i, j) \in I \times J$, *the fiber product* $X_i \times_S Y_j$ *exists in* Sch. *Then* $X \times_S Y$ *exists.*

Proof of Theorem 9.3.1. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram in Sch. We will show $X \times_S Y$ exists.

We first reduce to the case when S is affine. Let $S = \bigcup_{i \in I} S_i$ be an open covering by affine open subschemes. For each $i \in I$, let $X_i := f^{-1}(S_i) \subseteq X$ and $Y_i := g^{-1}(S_i)$ be the corresponding open subschemes. By Lemma 9.4.1, we only need to show $X_i \times_S Y_j$ exists for any pair $(i, j) \in I^2$. By Lemma 9.3.2, we have the following *Cartesian* square

$$\begin{tikzcd} & Y_i \cap Y_j \arrow[r] \arrow[d] & Y_j \arrow[d] \\ X_i \arrow[r] & S_i \arrow[r] & S. \end{tikzcd}$$

A diagram chasing shows that $X_i \times_S Y_j$ exists iff

$$X_i \underset{S_i}{\times} (Y_i \cap Y_j)$$

exists, and these two fiber products are canonically isomorphic. Note that S_i is affine by assumption. Hence we can reduce to the case when S is affine.

Apply Lemma 9.4.1 again, we can reduce to the case when X and Y are both affine. Now the claim follows from Corollary 9.2.5 and Corollary 9.2.2.

□[Theorem 9.3.1]

Sketch of Lemma 9.4.1. We will construct the desired fiber product using *gluing of schemes*. Write $P \coloneqq I \times J$ and $W_\alpha \coloneqq X_i \times_S Y_j$ for $\alpha = (i,j) \in P$. For $(i,k) \in I^2$, write $X_{ik} \coloneqq X_i \cap X_k$ and similarly $Y_{jl} \coloneqq Y_j \cap Y_l$.

For each pair $(\alpha,\beta) \in P^2$, we define an open subscheme $W_{\alpha\beta} \subseteq W_\alpha$ as follows. Write $\alpha = (i,j)$ and $\beta = (k,l)$. Since $X_{ik} \to X_i$ and $Y_{jl} \to Y_j$ are open subschemes, applying Lemma 9.3.2 twice, we see that

$$W_{\alpha\beta} \coloneqq X_{ik} \underset{X_i}{\times} W_\alpha \underset{Y_j}{\times} Y_{jl}$$

exists and can be identified with an open subscheme of W_α. Note that we have a canonical isomorphism

$$\phi_{\alpha\beta} : W_{\alpha\beta} \simeq X_{ik} \underset{X_i}{\times} (X_i \underset{S}{\times} Y_j) \underset{Y_j}{\times} Y_{jl} \simeq X_{ik} \underset{S}{\times} Y_{jl} \simeq X_{ki} \underset{S}{\times} Y_{lj} \simeq W_{\beta\alpha}.$$

One can check

$$(P, (W_\alpha)_{\alpha\in P}, (W_{\alpha\beta})_{(\alpha,\beta)\in P^2}, (\phi_{\alpha\beta})_{(\alpha,\beta)\in P^2})$$

is a gluing data of schemes (Definition 6.1.1). Let

$$(W, (W'_\alpha)_{\alpha\in P}, (\varphi_\alpha)_{\alpha\in P})$$

be the gluing output.

Now we construct a canonical morphism $p : W \to X$. By Proposition 6.1.3, we only need to construct morphisms $p_\alpha : W_\alpha \to X$ such that

$$p_\alpha|_{W_{\alpha\beta}} = p_\beta|_{W_{\beta\alpha}} \circ \phi_{\alpha\beta}.$$

We declare p_α to be the composition

$$W_\alpha = X_i \underset{S}{\times} Y_j \to X_i \to X.$$

One can check the collcetion $(p_\alpha)_{\alpha\in P}$ satisfies the above equations. Therefore we obtain a unique morphism $p : W \to X$ such that $p|_{W'_\alpha} \circ \varphi_\alpha = p_\alpha$.

Similarly, we use Proposition 6.1.3 to construct a canonical morphism $q : W \to Y$. By construction, the following diagram commutes:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S. \end{tikzcd}$$

It remains to show this diagram is Cartesian. Let $Z \in \mathsf{Sch}$ be a test object. We only need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z, -)$ sends the above diagram to a Cartesian square in Set. One can check this by applying Proposition 6.1.4.

□[Lemma 9.4.1]

10. Change of base

10.1. S-schemes.

Definition 10.1.1. Let S be a scheme. An **S-scheme is a scheme X equipped with a morphism** $X \to S$. When $S = \mathsf{Spec}(R)$, we also say X is a **R-scheme**.

Definition 10.1.2. Let X be an S-scheme. The **base-change of X along a morphism** $S' \to S$ is the S'-scheme

$$X_{S'} := X \underset{X}{\times} S'$$

equipped with its canonical projection to S'.

Remark 10.1.3. The reason for introducing the above terminology is to encourage the readers to view an S-scheme X as a *family of schemes over S*.

Exercise 10.1.4. Show that the base-change of an open immersion is still an open immersion. In other words, let $f : X \to Y$ be an open immersion between S-schemes and $S' \to S$ be any morphism. Then $f_{S'} : X_{S'} \to Y_{S'}$ is an open immersion, where $f_{S'}$ is the morphism

$$(f, \mathsf{id}_{S'}) : X \underset{S}{\times} S' \to Y \underset{S}{\times} S'.$$

10.2. **Fibers.**

Definition 10.2.1. Let X be an S-scheme and $s \in S$ be a topological point. The **fiber of X at s** is the κ_s-scheme

$$X_s := X \underset{S}{\times} s := X \underset{S}{\times} \mathsf{Spec}(\kappa_s),$$

where $\mathsf{Spec}(\kappa_s) \to S$ is the canonical κ_s-point lying over s.

Proposition 10.2.2. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. The* continuous map $X_s \to X$ *induces a homeomorphism*

$$X_s \xrightarrow{\simeq} p^{-1}(s)$$

between topological spaces.

Remark 10.2.3. One can reformulate the proposition as: the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ preserves the fiber product of $X \to S \leftarrow s$, where s is understood as $\mathsf{Spec}(\kappa_s)$.

Proof of Proposition 10.2.2. Note that the continuous map $q : X_s \to X$ indeed factors through the subspace $p^{-1}(s) \subseteq X$. This follows from applying the forgetful functor $\mathsf{Sch} \to \mathsf{Top}$ to the following commutative diagram

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathsf{Spec}(\kappa_s) \arrow[r] & S \end{tikzcd}$$

We first show $X_s \to p^{-1}(s)$ is surjective. Let $x \in p^{-1}(S)$ be a topological point. We have a commutative diagram in Sch

$$\begin{tikzcd} \mathsf{Spec}(\kappa_x) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathsf{Spec}(\kappa_s) \arrow[r] & S \end{tikzcd}$$

which by definition corresponds to a morphism $\mathsf{Spec}(\kappa_x) \to X_s$. By construction, the composition $\mathsf{Spec}(\kappa_x) \to X_s \to X$ is the canonical κ_x-point at $x \in X$. This shows $x \in p^{-1}(S)$ is in the image of the map $X_s \to p^{-1}(S)$.

Now we reduce to the case when S is affine. In other words, we will show the claim is *local on* S. To do this, let $U \subseteq S$ be an affine open subscheme containing the point s. Let $X_U := X \times_S U$ be the base-change of X to U. Consider the Cartesian squares

$$\begin{tikzcd} (X_U)_s \arrow[r] \arrow[d] & X_U \arrow[r, "j_X"] \arrow[d, "p_U"] & X \arrow[d, "p"] \\ s \arrow[r] & U \arrow[r, "j"] & S. \end{tikzcd}$$

It follows formally that the outer square is also Cartesian. In particular, we have

$$(X_U)_s \simeq X_s.$$

On the other hand, by Exercise 10.1.4, j_X is also an open immersion and its image is the open subset $p^{-1}(U) \subseteq X$. This implies j_X induces a homeomorphism

$$p_U^{-1}(s) \simeq p^{-1}(s).$$

Moreover, it is easy to see the diagram

$$\begin{tikzcd} (X_U)_s \arrow[r] \arrow[d, "\simeq"] & p_U^{-1}(s) \arrow[d, "\simeq"] \\ X_s \arrow[r] & p^{-1}(s). \end{tikzcd}$$

Hence to show the bottom horizontal map is a homeomorphism, we only need to show the top horizontal is a homeomorphism. This allows us to replace S with U (and therefore X with X_U) thereby assume S to be affine.

Now we reduce to the case when X is affine. In other words, we will show the claim is *local on* X. To do this, let $X = \bigcup_{i\in I} U_i$ be an open covering such that each U_i is affine. We only need to show the continuous map

$$X_s \cap q^{-1}(U_i) \to p^{-1}(s) \cap U_i$$

is a homeomorphism. Write p_i for the composition $U_i \to X \to S$. We have

$$p^{-1}(s) \cap U_i = p_i^{-1}(s)$$

as subspaces of U_i. On the other hand, by Lemma 9.3.2, $X_s \cap q^{-1}(U_i)$ is the underlying topological space of the fiber product

$$X_s \underset{X}{\times} U_i \simeq (s \underset{S}{\times} X) \underset{X}{\times} U_i \simeq s \underset{S}{\times} U_i \simeq (U_i)_s.$$

Hence we only need to show $(U_i)_s \to p_i^{-1}(s)$ is a homeomorphism. This allows us to replace X with U_i thereby assume X to be affine.

By the previous discussion, we can assume $S = \mathsf{Spec}(A)$ and $X = \mathsf{Spec}(B)$. Let $f : A \to B$ be the homomorphism corresponding to $p : X \to S$. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to the point $x \in S$. Recall $\kappa_{\mathfrak{p}} \simeq A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. By Corollary 9.2.2 and Corollary 9.2.5, we have

$$X_s \simeq \mathsf{Spec}(B \underset{A}{\otimes} \kappa_{\mathfrak{p}})$$

and the morphism $X_s \to X$ is induced by the homomorphism $B \simeq B \otimes_A A \to B \otimes_A \kappa_{\mathfrak{p}}$. We need to show $\mathsf{Spec}(B \otimes_A \kappa_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces a homeomorphism onto its image. This follows from the following exercise.

Exercise 10.2.4. Let $h : R \to R'$ be a homomorphism. Suppose any element $r' \in R'$ can be written as $h(r)u$ for some $r \in R$ and *invertible* element $u \in R'$. Then $\mathsf{Spec}(R') \to \mathsf{Spec}(R)$ induces a homeomorphism onto its image.

Namely, any element in $B \otimes_A \kappa_{\mathfrak{p}}$ can be written as

$$\sum_{i \in I} b_i \otimes \overline{a_i/c} = \Big((\sum_{i \in I} b_i f(a_i)) \otimes 1 \Big) \cdot (1 \otimes \overline{1/c})$$

where $a_i \in A$, $b_i \in B$, $c \in A \smallsetminus \mathfrak{p}$, and $\overline{b_i/c}$ is the image of b_i/c under the homomorphism $A_{\mathfrak{p}} \to \kappa_{\mathfrak{p}}$. Note that $(1 \otimes \overline{1/c})$ is invertible as desired.

□[Proposition 10.2.2]

Warning 10.2.5. In Proposition 10.2.2, X_s is in general *not* isomorphic to the locally ringed subspace $(p^{-1}(s), \mathcal{O}_X|_{p^{-1}(s)})$ associated to $p^{-1}(s)$. See Warning 5.3.6.

In particular, for a topological points $x \in X_s$, the homomorphism $\mathcal{O}_{X,i(x)} \to \mathcal{O}_{X_s,x}$ is in general not an isomorphism, where $i : X_s \to X$ is the canonical morphism.

That said, we have the following result:

Proposition 10.2.6. *The morphism $i : X_s \to X$ induces an isomorphism $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$ for any topological point $x \in X_s$.*

Proof. We claim i induces a bijection between the following sets:

- The set of field-valued points of X_s;
- The set of field-valued points of X contained in $p^{-1}(s)$.

To prove the claim, let k be any field. By the definition of fiber,

$$X_s(k) \simeq X(k) \underset{S(k)}{\times} s(k).$$

Using Proposition-Construction 8.1.2, it is easy to see $s(k) \to S(k)$ is injective, and its image contains k-points of S lying over s. It follows that $X_s(k) \to X(k)$ is also injective, and its image contains k-points of X whose underlying topological point is sent to s by p. This proves the desired claim.

Since the underlying topological space of X_s is homeomorphic to $p^{-1}(s)$, the claim implies i induces a bijection between the following sets:

- The set of field-valued points of X_s lying over x;
- The set of field-valued points of X lying over $i(x)$.

Applying Proposition-Construction 8.1.2 again, we see $\kappa_{i(x)} \to \kappa_x$ induces a bijection between the sets of field extensions of $\kappa_{i(x)}$ and κ_x. This is possible only if $\kappa_{i(x)} \xrightarrow{\simeq} \kappa_x$.

□

Exercise 10.2.7. Let $X \xrightarrow{f} S \xleftarrow{g} Y$ be a diagram of schemes. Show that the morphism

$$\bigsqcup_{(x,y,s)} \mathsf{Spec}(\kappa_x) \underset{\mathsf{Spec}(\kappa_s)}{\times} \mathsf{Spec}(\kappa_y) \to X \underset{S}{\times} Y$$

induces a bijection between topological points. Here $x \in X$, $y \in Y$ and $s \in S$ are topological points such that $f(x) = g(y) = s$.

10.3. **Base-change to local scheme.** We also have the following result about base-change along $\mathsf{Spec}(\mathcal{O}_{S,s}) \to S$.

Proposition 10.3.1. *Let $p : X \to S$ be an S-scheme and $s \in S$ be a topological point. Write $S' := \mathsf{Spec}(\mathcal{O}_{S,s})$. The* morphism

$$X_{S'} := X \underset{S}{\times} S' \to X$$

induces an isomorphism between locally ringed spaces

$$X_{S'} \xrightarrow{\simeq} p^{-1}(S').$$

In particular, the locally ringed subspace $p^{-1}(S')$ of X is a scheme.

Proof. As in the proof of Proposition 10.2.2, we have a surjective morphism $X_{S'} \to p^{-1}(S')$ between locally ringed spaces, and we can assume X and S are affine. Let $f : A \to B$ and $\mathfrak{p} \subset A$ be as in the proof of Proposition 10.2.2. We need to show $\phi : \mathsf{Spec}(B \otimes_A A_{\mathfrak{p}}) \to \mathsf{Spec}(B)$ induces an isomorphism

$$\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}}) \xrightarrow{\simeq} \phi(\mathsf{Spec}(B \underset{A}{\otimes} A_{\mathfrak{p}})),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$. This follows from the following exercise:

Exercise 10.3.2. Let R be a commutative ring and $S \subseteq R$ be a multiplicative subset. Consider the canonical homomorphism $R \to R[S^{-1}]$. Show that $\phi : \mathsf{Spec}(R[S^{-1}]) \to \mathsf{Spec}(R)$ induces an isomorphism

$$\mathsf{Spec}(R[S^{-1}]) \xrightarrow{\simeq} \phi(\mathsf{Spec}(R)),$$

where the target is viewed as a locally ringed subspace of $\mathsf{Spec}(R)$.

□

11. Subschemes and immersions

We have studied open subschemes and open immersions in Sect. 5.3. In this section, we introduce general subschemes and immersions.

11.1. **Monomorphisms and epimorphisms.**

Recall in any category $\mathcal{C}$, we can define monomorphisms and epimorphisms.

Definition 11.1.1. A morphism $f : x \to y$ in $\mathcal{C}$ is a **monomorphism** if for any test object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(z,x) \xrightarrow{f\circ -} \mathsf{Hom}_{\mathcal{C}}(z,y)$$

is an injection.

A morphism $f : x \to y$ in $\mathcal{C}$ is a **epimorphism** if the corresponding morphism in $\mathcal{C}^{\mathsf{op}}$ is a monomorphism, i.e., for any test object z, the map

$$\mathsf{Hom}_{\mathcal{C}}(y,z) \xrightarrow{-\circ f} \mathsf{Hom}_{\mathcal{C}}(x,z)$$

is an injection.

Example 11.1.2. In Set, monomorphisms are exactly injections, while epimorphisms are exactly surjections.

Exercise 11.1.3. Let $f : U \to X$ be an open immersion.

- Show that f is a monomorphism.
- Show that f is an epimorphism iff it is an isomorphism.

Warning 11.1.4. A morphism can simultaneously be a monomorphism and an epimorphism, but fail to be an isomorphism.

Exercise 11.1.5. Let $R \in \mathsf{CRing}$ and $f \in R$ be an element that is not a zero-divisor. Show that $R \to R_f$ is a monomorphism and an epimorphism in CRing.

Exercise 11.1.6. Show that the functor $\mathsf{Aff} \to \mathsf{Sch}$ sends monomorphisms to monomorphisms, but may fail to send epimorphisms to epimorphisms.

11.2. **Digression: epimorphisms between sheaves.**

Proposition 11.2.1. *Let X be a topological space and $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism in $\mathsf{Shv}(X,\mathsf{Set})$. The following conditions are equivalent:*

(i) *The morphism α is an epimorphism in $\mathsf{Shv}(X,\mathsf{Set})$.*
(ii) *For any point $x \in X$, the map $\alpha_x : \mathcal{F}_x \to \mathcal{F}'_x$ is a surjection.*

Proof. (i)⇒(ii): let α be an epimorphism in $\mathsf{Shv}(X,\mathsf{Set})$. Suppose α_x is not a surjection for some point $x \in X$. We can find a set A and maps $f,g : \mathcal{F}'_x \to A$ such that $f \neq g$ but $f \circ \alpha_x = g \circ \alpha_x$. By Proposition 2.3.4, f,g correspond to morphisms $\phi,\varphi : \mathcal{F}' \to \delta_{x,A}$ such that $\phi \neq \varphi$ but $\phi \circ \alpha = \varphi \circ \alpha$. But this contradicts the assumption that α is an epimorphism.

(ii)⇒(i): suppose α is a morphism that induces surjections between stalks. We will show α is an epimorphism. Let $\phi,\varphi : \mathcal{F}' \to \mathcal{F}''$ be morphisms in $\mathsf{Shv}(X,\mathsf{Set})$ such that $\phi \circ \alpha = \varphi \circ \alpha$. We only need to show $\phi_U = \varphi_U : \mathcal{F}'(U) \to \mathcal{F}''(U)$ for any open subset $U \subseteq X$. For any morphism $\psi : \mathcal{F}' \to \mathcal{F}''$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}'(U) \arrow[r] \arrow[d, "\psi_U"] & \prod_{x\in U} \mathcal{F}'_x \arrow[d, "(\psi_x)_{x\in U}"] \\
\mathcal{F}''(U) \arrow[r] & \prod_{x\in U} \mathcal{F}''_x
\end{tikzcd}$$

such that the horizontal arrows are injections (Lemma 2.2.1). Hence to show $\phi_U = \varphi_U$, we only need to show $\phi_x = \varphi_x$ for any $x \in U$. However, this follows from $\phi_x \circ \alpha_x = \varphi_x \circ \alpha_x$ and the assumption that α_x is a surjection.

□

Warning 11.2.2. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$. The map $\alpha_X : \mathcal{F}(X) \to \mathcal{F}'(X)$ is in general not a surjection.

11.3. **Closed immersions.**

Definition 11.3.1. Let $i : Y \to X$ be a morphism between schemes. We say i is a **closed immersion** if

(1) It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;
(2) The morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(Y, \mathsf{Set})$.

Let X be a scheme. A **closed subscheme** is an isomorphism class of closed immersions into X.

Remark 11.3.2. Condition (2) is equivalent to

(2Ab) The morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(Y, \mathsf{Ab})$.

Epimorphisms in $\mathsf{Shv}(Y, \mathsf{Ab})$ (or more generally in any abelian category) are often called *surjections*.

Proposition 11.3.3. *A morphism $i : Y \to X$ between schemes is a closed immersion iff*

(1) *It induces a homeomorphism $Y \xrightarrow{\simeq} i(Y)$ onto a closed subspace of X;*
(2') *For any point $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection.*

Proof. We only need to show conditions (2) and (2') are equivalent if (1) holds. Under the latter assumption, it is easy to see:

- If $x \notin i(Y)$, $(i_*\mathcal{O}_Y)_x \simeq 0$;
- If $x = i(y)$, $(i_*\mathcal{O}_Y)_x \simeq \mathcal{O}_{Y,y}$.

Now the claim follows from Proposition 11.2.1.

□

Remark 11.3.4. By Proposition 11.2.1, we can also replace condition (2') by

(2") The morphism $i^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is an epimorphism in $\mathsf{Shv}(X, \mathsf{Set})$.

Proposition 11.3.5. *Let $R \in \mathsf{CRing}$ and $I \subseteq R$ be an ideal. Let $i : \mathsf{Spec}(R/I) \to \mathsf{Spec}(R)$ be the morphism corresponding to the canonical surjection $\pi : R \to R/I$. Then i is a closed immersion.*

Proof. It is easy to see i induces a continuous bijective map from $\mathsf{Spec}(R/I)$ to the closed subspace $Z(I)$ of $\mathsf{Spec}(R)$. Moreover, for any standard open subset $U(\overline{f}) \in \mathsf{Spec}(R/I)$, $\overline{f} \in R/I$, its image is equal to the open subset $U(f) \cap Z(I) \subseteq Z(I)$, where $f \in R$ is any lifting of $\overline{f}$. This implies i induces a homeomorphism from $\mathsf{Spec}(R/I)$ to $Z(I)$.

Let $y \in \mathsf{Spec}(R/I)$ be a topological point and $\overline{\mathfrak{p}} \subseteq R/I$ be the corresponding prime ideal. By definition, $i(y)$ corresponds to the prime ideal $\mathfrak{p} := \pi^{-1}(\overline{\mathfrak{p}})$ and we have $\overline{\mathfrak{p}} = \mathfrak{p}/I$. The homomorphism $\mathcal{O}_{\mathsf{Spec}(R),i(y)} \to \mathcal{O}_{\mathsf{Spec}(R/I),y}$ can be identified with $R_{\mathfrak{p}} \to (R/I)_{\mathfrak{p}/I}$, which is obviously surjective.

□

Exercise 11.3.6. Let k be a field. Consider the closed immersion $i : \mathbb{A}^1_k \to \mathbb{A}^2_k$ corresponding to the surjection $k[x,y] \to k[x]$. Show that $\mathcal{O}_{\mathbb{A}^2_k}(U) \to (i_*\mathcal{O}_{\mathbb{A}^1_k})(U)$ is not surjective for general open subset $U \subseteq \mathbb{A}^2_k$.

Proposition 11.3.7. *A closed immersion $i : Y \to X$ is a monomorphism in* Sch*.*

Proof. Let Z be any test scheme. We need to show $\mathsf{Hom}_{\mathsf{Sch}}(Z,Y) \to \mathsf{Hom}_{\mathsf{Sch}}(Z,X)$ is injective.

Suppose $f, g : Z \to Y$ are morphisms such that $i \circ f = i \circ g$. It is clear that the underlying continuous maps of f and g are equal. Write ϕ for this continuous map. Now the morphisms f and g are given by morphisms

$$\alpha, \beta : \mathcal{O}_Y \to \phi_*\mathcal{O}_Z.$$

We only need to show $\alpha = \beta$.

Let $\gamma : \mathcal{O}_X \to i_*\mathcal{O}_Y$ be the canonical morphism. The assumption $i \circ f = i \circ g$ implies

$$i_*(\alpha) \circ \gamma = i_*(\beta) \circ \gamma : \mathcal{O}_X \to i_* \circ \phi_*\mathcal{O}_X$$

Since γ is an epimorphism, we obtain

$$i_*(\alpha) = i_*(\beta)$$

In particular

$$i^{-1} \circ i_*(\alpha) = i^{-1} \circ i_*(\alpha).$$

Now the desired claim follows from the fact that $i^{-1} \circ i_* \simeq \mathrm{Id}$, which can be checked by unwinding the definitions.

□

11.4. **Ideal of definition.**

Construction 11.4.1. Let $i : Y \to X$ be a closed immersion. Consider the kernel

$$\mathcal{I}_Y := \mathsf{ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y),$$

which is defined to be the fiber product of $\mathcal{O}_X \to i_*\mathcal{O}_Y \leftarrow 0$ inside $\mathsf{Shv}(X, \mathsf{Ab})$. We call it the **ideal of definition** for the closed immersion $i : Y \to X$.

For any open subset $U \subseteq X$, it is easy to show that the functor

$$(-)(U) : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$$

preserves fiber products (in fact arbitrary small limits). It follows that

$$\mathcal{I}_Y(U) \simeq \mathsf{ker}(\mathcal{O}_X(U) \to (i_*\mathcal{O}_Y)(U)) \simeq \mathsf{ker}(\mathcal{O}_X(U) \to \mathcal{O}_Y(i^{-1}U)).$$

In particular, $\mathcal{I}_Y(U) \subseteq \mathcal{O}_X(U)$ is an ideal, and therefore $\mathcal{I}_Y$ is a **sheaf of ideals for $\mathcal{O}_X$**[14].

[14]Other name: ideal sheaf of $\mathcal{O}_X$.

Note however that $\mathcal{O}_X(U) \to (i_*\mathcal{O}_Y)(U)$ may fail to be a surjection (see Exercise 11.3.6).

For any point $x \in X$, it is easy to show that the functor

$$(-)_x : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$$

preserves fiber products (in fact arbitrary *finite* limits). It follows that

$$\mathcal{I}_{Y,x} \simeq \mathsf{ker}(\mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x).$$

Hence

- If $x \notin i(Y)$, $\mathcal{I}_{Y,x} \simeq \mathcal{O}_{X,x}$;
- If $x = i(y)$, $\mathcal{I}_{Y,x} \simeq \mathsf{ker}(\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y})$.

Since $\mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x$ is surjective, we obtain a short exact sequence in Ab:

$$0 \to \mathcal{I}_{Y,x} \to \mathcal{O}_{X,x} \to (i_*\mathcal{O}_Y)_x \to 0$$

Lemma 11.4.2. *Let $i : Y \to X$ be a closed immersion and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. The canonical morphism*

$$\mathcal{O}_X/\mathcal{I}_Y := \mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X) \to i_*\mathcal{O}_Y$$

is an isomorphism, where the cokernel is taken in $\mathsf{Shv}(X, \mathsf{Ab})$.

Proof. Taking stalks commute with pushouts (in fact arbitrary small colimits), hence we have

$$\mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X)_x \simeq \mathsf{coker}(\mathcal{I}_{Y,x} \to \mathcal{O}_{X,x}) \simeq (i_*\mathcal{O}_Y)_x$$

for any point $x \in X$. This implies $\mathsf{coker}(\mathcal{I}_Y \to \mathcal{O}_X) \to i_*\mathcal{O}_Y$.

□

Remark 11.4.3. In fact, for any surjection $\phi : \mathcal{F} \to \mathcal{F}'$ in $\mathsf{Shv}(X, \mathsf{Ab})$, we have $\mathsf{coker}(\mathsf{ker}(\phi) \to \mathcal{F}) \simeq \mathcal{F}'$. We will return to this fact in future lectures.

Exercise 11.4.4. Let $i : Y \to X$ be a closed immersion and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. Show that $\mathcal{O}_X/\mathcal{I}_Y$ is the sheafification of the presheaf

$$U \mapsto \mathcal{O}_X(U)/\mathcal{I}_Y(U).$$

Give an example where this sheafification step is necessary.

Lemma 11.4.5. *A closed immersion $i : Y \to X$ is determined by its ideal of definition up to isomorphism.*

Proof. Let $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be the ideal of definition. We can reconstruct the corresponding closed subscheme as follows. Consider $\mathcal{Q} := \mathcal{O}_X/\mathcal{I}_Y$. We have $\mathcal{Q} \simeq i_*\mathcal{O}_Y$, hence $\mathcal{Q}_x \simeq 0$ if $x \notin i(Y)$ while $\mathcal{Q}_x \simeq \mathcal{O}_{Y,y} \neq 0$ if $x = i(y)$. This implies $i(Y)$ is equal to the *support* of $\mathcal{Q}$.

$$\mathrm{supp}(\mathcal{Q}) := \{x \in X \mid \mathcal{Q}_x \neq 0\}.$$

In particular, the underlying topological space of Y can be identified with the subspace $\mathrm{supp}(\mathcal{Q}) \simeq \mathcal{O}$. Via this identification, the structure sheaf $\mathcal{O}_Y$ is given by $\mathcal{Q}|_{\mathrm{supp}(\mathcal{Q})}$, and the morphism is $Y \to X$ is isomorphic to the following composition

$$(\mathrm{supp}(\mathcal{Q}), \mathcal{Q}|_{\mathrm{supp}(\mathcal{Q})}) \to (X, \mathcal{Q}) \to (X, \mathcal{O})$$

of morphisms between locally ringed spaces.

□

Note however that not every sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ can be realized an ideal of definiton for a closed immersion.

Exercise 11.4.6. Let k be a field and $X := \mathbb{A}^1_k$ and $0 \in X$ be the zero point. Show that

$$\mathcal{I}(U) := \begin{cases} \mathcal{O}(U) & \text{for} \quad 0 \notin U \\ 0 & \text{for} \quad 0 \in U \end{cases}$$

defines an sheaf of ideals $\mathcal{I}$ of $\mathcal{O}_X$, but it is not an ideal of definition for any closed immersion into X.

In future lectures, we will give a necessary and sufficient condition for a sheaf $\mathcal{I}$ of ideals to be an ideal of definition. Namely, we need $\mathcal{I}$ to be *quasi-coherent* as an $\mathcal{O}_X$-module.

11.5. **Locally closed immersions.**

Definition 11.5.1. Let $f : Y \to X$ be a morphism between schemes. We say f is a **locally closed immersion** if it can be written as a composition $Y \xrightarrow{i} Z \xrightarrow{j} X$ such that i is a closed immersion while j is an open immersion.

Proposition 11.5.2. *A morphism $f : Y \to X$ between schemes is a locally closed immersion iff*

(1) *It induces a homeomorphism $Y \xrightarrow{\simeq} f(Y)$ onto a locally closed subspace*[15] *of X;*
(2) *For any point $y \in Y$, the map $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection.*

Proof. The "only if" claim: let $f : Y \xrightarrow{i} Z \xrightarrow{j} X$ be a factorization of f such that i is a closed immersion and j is an open immersion. Then Y is homeomorphic to the closed subspace $i(Y) \subseteq Z$, and Z is homeomorphic to the open subspace $j(Z)$. Hence Y is homeomorphic to $j(i(Y)) = f(Y)$, which is a locally closed subspace of X. Moreover, for any point $y \in Y$, we have, the homomorphism $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ factors as

$$\mathcal{O}_{X,f(y)} = \mathcal{O}_{X,j\circ i(y)} \to \mathcal{O}_{Z,i(y)} \to \mathcal{O}_{Y,y}.$$

Since j is an open immersion, the second map is an isomorphism; since i is a closed immersion, the last map is a surjection. Hence $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a surjection as desired.

The "if" claim: by (1), we can find an open subset $Z \subseteq X$ such that $f(Y)$ is a closed subset of Z. Consider the open subscheme of X given by Z. Write $j : Z \to X$ for the corresponding open embedding. By Exercise 5.3.8, there is a unique morphism $i : Y \to Z$ such that $f = j \circ i$. It remains to show i is a closed immersion. Condition (1) implies i induces a homeomorphism $Y \simeq i(Y) = f(Y)$ onto the closed subspace $i(Y) \subseteq Z$. Condition (2) implies the composition

$$\mathcal{O}_{X,f(y)} = \mathcal{O}_{X,j\circ i(y)} \to \mathcal{O}_{Z,i(y)} \to \mathcal{O}_{Y,y}$$

is surjective. Since the second map is an isomorphism, the last map is a surjection as desired.

□

Warning 11.5.3. A locally closed immersion may fail to be written as an open immersion *followed* by a closed immersion. We will provide a counterexample in future lectures.

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[15] A subspace is locally closed iff it can be written as the intersection of an open subspace and a closed subspace.

Part IV. (Quasi-)coherent sheaves

12. $\mathcal{O}_X$-module sheaves

12.1. **Definition.** A study of commutative rings cannot be complete without mentioning their *modules*.

Definition 12.1.1. Let $(X,\mathcal{O}_X)$ be a ringed space. An **$\mathcal{O}_X$-module presheaf**[16] is an object $\mathcal{M} \in \mathsf{PShv}(X,\mathsf{Ab})$ equipped with the following structure:

- For any open subset $U \subset X$, the abelian group $\mathcal{M}(U)$ is equipped with an $\mathcal{O}_X(U)$-module structure,

such that

- For any open subsets $U \subseteq V \subseteq X$, the restriction map
$$\mathcal{M}(V) \to \mathcal{M}(U)$$
intertwines the action of the homomorphism
$$\mathcal{O}_X(V) \to \mathcal{O}_X(U).$$

An **$\mathcal{O}_X$-module (sheaf)** is an $\mathcal{O}_X$-module presheaf $\mathcal{M}$ such that the underlying object $\mathcal{M} \in \mathsf{PShv}(X,\mathsf{Ab})$ is a sheaf. When there is no danger of ambiguity, we just call it an **$\mathcal{O}_X$-module**.

Example 12.1.2. The structure sheaf $\mathcal{O}_X$ itself is an $\mathcal{O}_X$-module (sheaf).

Definition 12.1.3. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_X$-module presheaves. An **$\mathcal{O}_X$-linear morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a morphism $f : \mathcal{M} \to \mathcal{N}$ between sheaves such that:

- For any open subset $U \subset X$, the map $f_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is $\mathcal{O}_X(U)$-linear.

We can define a category
$$\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}},$$
where objects are $\mathcal{O}_X$-module presheaves, and morphisms are $\mathcal{O}_X$-linear morphisms. Let
$$\mathcal{O}_X\text{–}\mathsf{mod} \subseteq \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$$
be the full subcategory of $\mathcal{O}_X$-module (sheaves).

Example 12.1.4. A sheaf $\mathcal{I}$ of ideals of $\mathcal{O}_X$ has an obvious $\mathcal{O}_X$-module structure. The embedding $\mathcal{I} \to \mathcal{O}_X$ is an $\mathcal{O}_X$-linear morphism.

12.2. **Sheafification.**

Proposition 12.2.1. *The embedding functor $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ admits a left adjoint*
$$\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{–}\mathsf{mod}$$
compatible with the sheafification functor
$$(-)^\sharp : \mathsf{PShv}(X,\mathsf{Ab}) \to \mathsf{Shv}(X,\mathsf{Ab})$$

Sketch. For $\mathcal{M} \in \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$, consider its sheafification $\mathcal{M}^\sharp$, viewed as an object in $\mathsf{Shv}(X,\mathsf{Ab})$. We have an action of $\mathcal{O}_X$ on $\mathcal{M}^\sharp$ given by the formula
$$\mathcal{O}_X \times \mathcal{M}^\sharp \simeq \mathcal{O}_X^\sharp \times \mathcal{M}^\sharp \simeq (\mathcal{O}_X \times \mathcal{M})^\sharp \to \mathcal{M}^\sharp,$$
where

[16] Other name: presheaf of modules for $\mathcal{O}_X$.

- the first isomorphism is because $\mathcal{O}_X$ is a sheaf;
- the second isomorphism is because sheafification commutes with finite products (see Remark 3.1.5);
- the third morphism is provided with the action of $\mathcal{O}_X$ on $\mathcal{M}$.

One can check this defines an $\mathcal{O}_X$-module sheaf $\mathcal{M}^\sharp$, and the morphism $\phi : \mathcal{M} \to \mathcal{M}^\sharp$ is $\mathcal{O}_X$-linear. Moreover, for any test object $\mathcal{N} \in \mathcal{O}_X\text{-}\mathsf{mod}$, the following composition

$$\mathsf{Hom}_{\mathcal{O}_X-\mathsf{mod}}(\mathcal{M}^\sharp, \mathcal{N}) \xrightarrow{-\circ\phi} \mathsf{Hom}_{\mathcal{O}_X-\mathsf{mod}_{\mathsf{PShv}}}(\mathcal{M}, \mathcal{N})$$

is a bijection. This implies the desired statement.

□

12.3. **Pushforward and pullback.**

Construction 12.3.1. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Recall we have a morphism $\alpha : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ in $\mathsf{Shv}(Y, \mathsf{CRing})$.

For any $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}}$, one can check the object

$$\phi_*(\mathcal{M}) \in \mathsf{PShv}(Y, \mathsf{Ab})$$

has a structure of $\phi_*(\mathcal{O}_X)$-module presheaf given by

$$\phi_*(\mathcal{O}_X) \times \phi_*(\mathcal{M}) \simeq \phi_*(\mathcal{O}_X \times \mathcal{M}) \to \phi_*(\mathcal{O}_X).$$

Restricting along α, we obtain a structure of $\mathcal{O}_Y$-module presheaf on $\phi_*(\mathcal{O}_X)$. We call the obtained object

$$\phi_*(\mathcal{O}_X) \in \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}}$$

the **pushforward (= direct image) of the $\mathcal{O}_X$-module presheaf $\mathcal{N}$ along ϕ.**

One can upgrade this construction to a functor

$$\phi_* : \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}}$$

compatible with the direct image functor for abelian sheaves. Moreover, it restricts to a functor

$$\phi_* : \mathcal{O}_X\text{-}\mathsf{mod} \to \mathcal{O}_Y\text{-}\mathsf{mod}$$

Construction 12.3.2. For a chain $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y) \xrightarrow{\varphi} (Z, \mathcal{O}_Z)$, the composition

$$\mathcal{O}_X\text{-}\mathsf{mod}_{(\mathsf{PShv})} \xrightarrow{\phi_*} \mathcal{O}_Y\text{-}\mathsf{mod}_{(\mathsf{PShv})} \xrightarrow{\varphi_*} \mathcal{O}_Z\text{-}\mathsf{mod}_{(\mathsf{PShv})}$$

is canonically identified with $(\varphi \circ \phi)_*$.

Construction 12.3.3. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Recall we have a morphism $\beta : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ in $\mathsf{Shv}(X, \mathsf{CRing})$. In particular, there is a morphism

$$\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y \to \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$$

in $\mathsf{PShv}(X, \mathsf{CRing})$.

For any $\mathcal{N} \in \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{PShv}}$, one can check the object

$$\phi^{-1}_{\mathsf{PShv}}(\mathcal{N}) \in \mathsf{PShv}(X, \mathsf{Ab})$$

has a structure of $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y)$-module given by[17]

$$\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y) \times \phi^{-1}_{\mathsf{PShv}}(\mathcal{N}) \simeq \phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y \times \mathcal{N}) \to \phi^{-1}_{\mathsf{PShv}}(\mathcal{N}).$$

[17]See Remark 3.3.4 for the first isomorphism.

Consider the $\mathcal{O}_X$-module presheaf on X

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U).$$

We denote it by

$$\phi^*_{\mathsf{PShv}}(\mathcal{N}) \in \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}}$$

and call it the **pullback (=inverse image) of the $\mathcal{O}_Y$-module presheaf $\mathcal{M}$ along ϕ.**

One can upgrade this construction to a functor

$$\phi^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{−}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}}.$$

Proposition 12.3.4. *Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. The functor*

$$\phi^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{−}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}}$$

is canonically left adjoint to

$$\phi_* : \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{−}\mathsf{mod}_{\mathsf{PShv}}.$$

Sketch. Let $\mathcal{M} \in \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}}$ and $\mathcal{N} \in \mathcal{O}_Y\text{−}\mathsf{mod}_{\mathsf{PShv}}$. By definition, knowing a morphism $\phi^*_{\mathsf{PShv}}(\mathcal{N}) \to \mathcal{M}$ is equivalent to knowing $\mathcal{O}_X(U)$-linear maps

$$\mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U) \to \mathcal{M}(U)$$

compatible with the restriction maps. The latter is equivalent to knowing $(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)$-linear maps

$$(\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U) \to \mathcal{M}(U)$$

compatible with the restriction maps, i.e., a $\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y$-linear morphism

$$\phi^{-1}_{\mathsf{PShv}}\mathcal{N} \to \mathcal{M},$$

where $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y)$ acts on $\mathcal{M}$ via the homomorphism $\phi^{-1}_{\mathsf{PShv}}(\mathcal{O}_Y) \to \mathcal{O}_X$. Using the adjunction $(\phi^{-1}_{\mathsf{PShv}}, \phi_*)$, the above data is equivalent to an $\mathcal{O}_Y$-linear morphism

$$\mathcal{N} \to \phi_*\mathcal{M}$$

as desired.

□

We now define pullback of $\mathcal{O}$-module *sheaves*.

Construction 12.3.5. Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Define ϕ^* to be the following composition

$$\phi^* : \mathcal{O}_Y\text{−}\mathsf{mod} \xrightarrow{\subseteq} \mathcal{O}_Y\text{−}\mathsf{mod}_{\mathsf{PShv}} \xrightarrow{\phi^*_{\mathsf{PShv}}} \mathcal{O}_X\text{−}\mathsf{mod}_{\mathsf{PShv}} \xrightarrow{\sharp} \mathcal{O}_X\text{−}\mathsf{mod}.$$

For $\mathcal{N} \in \mathcal{O}_Y\text{−}\mathsf{mod}$, we call $\phi^*\mathcal{N}$ the **pullback (=inverse image) of $\mathcal{N}$ along ϕ.**

In other words, $\phi^*\mathcal{N}$ is the sheafification of

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}_{\mathsf{PShv}}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}_{\mathsf{PShv}}\mathcal{N})(U).$$

Exercise 12.3.6. Show that $\phi^*\mathcal{N}$ can be identified with the *sheafification* of

$$U \mapsto \mathcal{O}_X(U) \underset{(\phi^{-1}\mathcal{O}_Y)(U)}{\otimes} (\phi^{-1}\mathcal{N})(U). \tag{12.1}$$

As a corollary of the above exercise, we have:

Lemma 12.3.7. *Let $j : U \to X$ be an open immersion of schemes. For $\mathcal{M} \in \mathcal{O}_X\text{–}\mathsf{mod}$, we have $j^*\mathcal{M} \simeq j^{-1}\mathcal{M}$.*

Warning 12.3.8. The assignement (12.1) is in general *not* a sheaf. In other words, to define ϕ^*, we need to sheafify after tensoring up.

Exercise 12.3.9. Let k be a field, and consider the morphism $\phi : X \to Y$ between schemes given by

$$\bigsqcup_I \mathsf{Spec}(k[t]) \to \mathsf{Spec}(k),$$

where I is an infinite set. Let $\mathcal{N} \in \mathcal{O}_Y\text{–}\mathsf{mod} \simeq k\text{–}\mathsf{mod}$ be an infinite-dimensional object. Show that (12.1) is not a sheaf.

Exercise 12.3.10. Let k be a field. Let $\phi : \mathbb{A}^2_k \to \mathbb{A}^1_k$ be the projection map given by $k[x] \to k[x,y]$ and $i : \mathsf{Spec}(k) \to \mathbb{A}^1_k$ be the closed immersion given by $k[x] \to k[x]/x \simeq k$. Consider $\mathcal{N} := i_*\mathcal{O}_{\mathsf{Spec}(k)}$. Show that (12.1) is not a sheaf.

The following result follows formally from Proposition 12.2.1 and Proposition 12.3.4.

Corollary 12.3.11. *Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. The functor*

$$\phi^* : \mathcal{O}_Y\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

is canonically left adjoint to

$$\phi_* : \mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_Y\text{–}\mathsf{mod}.$$

Construction 12.3.12. For a chain $(X, \mathcal{O}_X) \xrightarrow{\phi} (Y, \mathcal{O}_Y) \xrightarrow{\varphi} (Z, \mathcal{O}_Z)$, the canonical natural isomorphism

$$\varphi_* \circ \phi_* \simeq (\varphi \circ \phi)_*$$

induces canonical natural isomorphisms

$$\phi^*_{\mathsf{PShv}} \circ \varphi^*_{\mathsf{PShv}} \simeq (\varphi \circ \phi)^*_{\mathsf{PShv}}, \;\; \phi^* \circ \varphi^* \simeq (\varphi \circ \phi)^*.$$

12.4. **Tensor products and inner homs.**

Construction 12.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. For $\mathcal{M}, \mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$, we define

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$$

to be the *sheafification* of the $\mathcal{O}_X$-module presheaf

$$U \mapsto \mathcal{M}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{N}(U).$$

Example 12.4.2. We have $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X \simeq \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \mathcal{M}$.

Lemma 12.4.3. *Let $x \in X$ be a point. We have*

$$(\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N})_x \simeq \mathcal{M}_x \underset{\mathcal{O}_{X,x}}{\otimes} \mathcal{N}_x.$$

Proof. Follows by combining the following two facts:

- Sheafification does not change the stalks.
- Filtered colimits commute with tensor products.

□

Remark 12.4.4. In fact, one can define a *symmetric monoidal structure* on $\mathcal{O}_X\text{-}\mathsf{mod}$ such that the tensor product functor is given by the above construction.

Construction 12.4.5. Let $(X, \mathcal{O}_X)$ be a ringed space. For $\mathcal{M}, \mathcal{N} \in \mathcal{O}_X\text{-}\mathsf{mod}$, one can check

$$U \mapsto \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$$

is an $\mathcal{O}_X$-module sheaf. Here a section $f \in \mathcal{O}_X(U)$ acts on $\mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U)$ by precomposing with $\mathcal{M}|_U \xrightarrow{f} \mathcal{M}|_U$, or equivalently, by postcomposing with $\mathcal{N}|_U \xrightarrow{f} \mathcal{N}|_U$.

We denote the obtained object by

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathcal{O}_X\text{-}\mathsf{mod}.$$

Example 12.4.6. We have $\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}) \simeq \mathcal{N}$.

Construction 12.4.7. Let $x \in X$ be a point. For any open neighborhood $U \subseteq X$ of x, we have a canonical map

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})(U) = \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{M}|_U, \mathcal{N}|_U) \to \mathsf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x).$$

This indues a morphism

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})_x \to \mathsf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x). \tag{12.2}$$

Warning 12.4.8. The canonical morphism (12.2) is in general not an isomorphism. For example, for $\mathcal{M} = \bigoplus_I \mathcal{O}_X$, this map is given by

$$(\prod_I \mathcal{N})_x \to \prod_I (\mathcal{N}_x).$$

Construction 12.4.9. Let $(X, \mathcal{O}_X)$ be a ringed space. For $\mathcal{M}_i \in \mathcal{O}_X\text{-}\mathsf{mod}$, $i = 1, 2, 3$, we have a canonical morphism

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2) \underset{\mathcal{O}_X}{\otimes} \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}_2, \mathcal{M}_3) \to \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_3).$$

Taking $\mathcal{M}_1 = \mathcal{O}_X$, $\mathcal{M}_2 = \mathcal{M}$ and $\mathcal{M}_3 = \mathcal{N}$, we obtain a morphism

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \to \mathcal{N}$$

functorial in $\mathcal{N}$. In other words, we have a natural transformation

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -) \to \mathsf{id} \tag{12.3}$$

The proof of the following result is similar to its analogue in commutative algebra.

Proposition 12.4.10. *For any $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}$, the natural transformation* (12.3) *gives an adjoint pair*

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} - : \mathcal{O}_X\text{-}\mathsf{mod} \rightleftarrows \mathcal{O}_X\text{-}\mathsf{mod} : \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -).$$

Corollary 12.4.11. *For any $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}$, we have:*

- *The functor $\mathcal{M} \otimes_{\mathcal{O}_X} -$ commutes with colimits. In particular, it is right exact.*
- *The functor $\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -)$ commutes with limits. In particular, it is left exact.*

In fact, it is easy to prove the following result:

Lemma 12.4.12. *For any $\mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$, the functor*

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(-, \mathcal{N}) : \mathcal{O}_X\text{–}\mathsf{mod}^{\mathsf{op}} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

commutes with limits. In particular, it is left exact.

13. Quasi-coherent modules

13.1. $\widetilde{M}$. In this subsection, let $A \in \mathsf{CRing}$ be a commutative ring and $X := \mathsf{Spec}(A)$ be the corresponding affine scheme. Consider the ringed space $X' := (*, A)$ whose underlying topological space is a singleton, and whose structure sheaf is given by $A \in \mathsf{CRing} \simeq \mathsf{Shv}(*, \mathsf{CRing})$. Let

$$\pi : X \to (*, A) = X'$$

be the morphism such that $\alpha : \mathcal{O}_{X'} \to \pi_* \mathcal{O}_X$ corresponds to the canonical isomorphism $A \simeq \mathcal{O}_X(X)$. Note that we can make the following identificaiton

$$A\text{–}\mathsf{mod} \simeq \mathcal{O}_{X'}\text{–}\mathsf{mod}.$$

Definition 13.1.1. Consider the functor

$$\pi^* : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}.$$

For $M \in A\text{–}\mathsf{mod}$, we write

$$\widetilde{M} := \pi^*(\mathcal{M}) \in \mathcal{O}_X\text{–}\mathsf{mod}$$

and call it the **$\mathcal{O}_X$-module associated to the A-module** M.

By Corollary 12.3.11, we have:

Corollary 13.1.2. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. We have a canonical adjunction*

$$\begin{array}{ccc} A\text{–}\mathsf{mod} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod} \\ M & \mapsto & \widetilde{M} \\ \mathcal{F}(X) & \leftarrow\!\shortmid & \mathcal{F}. \end{array}$$

Construction 13.1.3. Let $A \to B$ be a homomorphism in CRing and $f : Y \to X$ be the corresponding morphism between affine schemes. We have a canonical commutative diagram of ringed spaces:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & (*, B) \arrow[d] \\ X \arrow[r] & (*, A). \end{tikzcd}$$

By Construction 12.3.12, we have a canonical commutative diagram of functors

$$\begin{tikzcd} A\text{–}\mathsf{mod} & \mathcal{O}_X\text{–}\mathsf{mod} \arrow[l, "(-)(X)"'] \\ B\text{–}\mathsf{mod} \arrow[u, "\mathsf{res}"] & \mathcal{O}_Y\text{–}\mathsf{mod}, \arrow[l, "(-)(Y)"'] \arrow[u, "f_*"'] \end{tikzcd} \tag{13.1}$$

and a commutative diagram of the left adjoint functors

$$\begin{tikzcd} A\text{–}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] \arrow[d, "B\otimes_A -"'] & \mathcal{O}_X\text{–}\mathsf{mod} \arrow[d, "f^*"] \\ B\text{–}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_Y\text{–}\mathsf{mod}. \end{tikzcd} \tag{13.2}$$

On the other hand, using the method in the proof of Proposition-Definition 4.2.3, one can prove the following result:

Proposition 13.1.4. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. For any $M \in A\text{–}\mathsf{mod}$, there is an essentially unique $\mathcal{O}_X$-module $\mathcal{M}$ equipped with an A-linear isomorphism $M \xrightarrow{\simeq} \mathcal{M}(X)$ such that for any $f \in A$, the A-linear map $M \simeq \mathcal{M}(X) \to \mathcal{M}(U(f))$ induces an isomorphism*

$$M_f \xrightarrow{\simeq} \mathcal{M}(U(f)).$$

Let $\mathcal{M}$ and $\widetilde{M}$ be the $\mathcal{O}_X$-module sheaves as above. By definition, $\widetilde{M}$ is the *sheafification* of $\pi^*_{\mathsf{PShv}}(M)$. In particular, for any $f \in A$, there is a canonical $\mathcal{O}_X(U(f))$-linear map

$$\mathcal{O}_X(U(f)) \underset{A}{\otimes} M \simeq \big(\pi^*_{\mathsf{PShv}}(M)\big)(U(f)) \to \widetilde{M}(U(f)).$$

Recall that we have $A_f \simeq \mathcal{O}_X(U(f))$ and

$$\mathcal{M}(U(f)) \simeq M_f \simeq A_f \underset{A}{\otimes} M.$$

This gives a canonical A_f-linear map

$$\mathcal{M}(U(f)) \to \widetilde{M}(U(f)).$$

By Exercise 1.2.7, there is a unique morphism $\mathcal{M} \to \widetilde{M}$ in $\mathcal{O}_X\text{–}\mathsf{mod}$ whose evaluation on each $U(f)$ is given by the above map.

Proposition 13.1.5. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. For $M \in A\text{–}\mathsf{mod}$, let $\widetilde{M}$ and $\mathcal{M}$ be the $\mathcal{O}_X$-modules defined as above. Then the canonical morphism $\mathcal{M} \to \widetilde{M}$ is an isomorphism. In particular, for any $f \in A$, we have*

$$M_f \to \widetilde{M}(U(f)).$$

Proof. Let $\mathcal{N} \in \mathcal{O}_X\text{–}\mathsf{mod}$ be a testing object, we only need to show the map

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\widetilde{M},\mathcal{N}) \to \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\mathcal{M},\mathcal{N}) \tag{13.3}$$

is a bijection. By the definition of sheafification, we have

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\widetilde{M},\mathcal{N}) \xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}}(\pi^*_{\mathsf{PShv}}(M),\mathcal{N}). \tag{13.4}$$

Note that the values of $\pi^*_{\mathsf{PShv}}(M)$ and $\mathcal{M}$ on $U(f)$ are both canonically identified with M_f. By Exercise 1.2.7, we obtain an identification

$$\mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}}(\mathcal{M},\mathcal{N}) \simeq \mathsf{Hom}_{\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}}(\pi^*_{\mathsf{PShv}}(M),\mathcal{N}). \tag{13.5}$$

One can check (13.4) is equal to the composition (13.5) $\circ$ (13.3). Hence (13.3) is also a bijection.

□

Corollary 13.1.6. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The functor*

$$\widetilde{(-)} : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

is fully faithful.

Proof. This functor admits a right adjoint $\mathcal{F} \mapsto \mathcal{F}(X)$, and the unit natural transformation $M \to \widetilde{M}(X)$ is invertible (by Proposition 13.1.5). This formally implies the left adjoint is fully faithful.

□

Warning 13.1.7. The functor $\widetilde{(-)} : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$ is not essentially surjective. For example, the sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ in Exercise 11.4.6 is not isomorphic to $\widetilde{M}$ for any $M \in k[t]\text{–}\mathsf{mod}$, because $\mathcal{I}(X) \simeq 0$.

Corollary 13.1.8. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The functor*

$$\widetilde{(-)} : A\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$$

commutes with finite limits and small colimits.

Proof. Follows from Proposition 13.1.5 and the fact that $M \mapsto M_f$ commutes with finite limits and small colimits.

□

Corollary 13.1.9. *Let $\phi : A \to B$ be a homomorphism in CRing and $f : Y \to X$ be the corresponding morphism between affine schemes. The commutative diagram 13.1 induces a natural isomorphism $\widetilde{\mathsf{res}(-)} \xrightarrow{\simeq} f_*(\widetilde{-})$, i.e., a commutative diagram*

$$\begin{tikzcd} A\text{–}\mathsf{mod} \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_X\text{–}\mathsf{mod} \\ B\text{–}\mathsf{mod} \arrow[u, "\mathsf{res}"] \arrow[r, "\widetilde{(-)}"] & \mathcal{O}_Y\text{–}\mathsf{mod}. \arrow[u, "f_*"'] \end{tikzcd} \tag{13.6}$$

Proof. For $M \in A\text{–}\mathsf{mod}$ and $f \in A$, Proposition 13.1.5 implies

$$\widetilde{\mathsf{res}(M)}(U(f)) \simeq \mathsf{res}(M)_f \simeq M_{\phi(f)} \simeq \widetilde{M}(U(\phi(f))) \simeq f_*\widetilde{M}(U(f)).$$

One can check theis composition is the value of the Bech–Chevalley natural transformation $\widetilde{\mathsf{res}(-)} \to f_*(\widetilde{-})$ at $U(f)$. This implies the desired claim.

□

13.2. **Definition of quasi-coherent sheaves.**

Definition 13.2.1. Let X be a scheme. We say an $\mathcal{O}_X$-module is **quasi-coherent** if for any open immersion $j : U = \mathsf{Spec}(A) \to X$, the $\mathcal{O}_U$-module $\mathcal{F}|_U := j^*\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some $M \in A\text{–}\mathsf{mod}$.

We write

$$\mathsf{QCoh}(X) = \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qcoh}} \subseteq \mathcal{O}_X\text{–}\mathsf{mod}$$

for the full subcategory of quasi-coherent $\mathcal{O}_X$-modules.

Example 13.2.2. The structure sheaf $\mathcal{O}_X$ is quasi-coherent when viewed as a module of itself.

13.3. **Affine case.**

Proposition 13.3.1. *Let $A \in \mathsf{CRing}$ and $X := \mathsf{Spec}(A)$. The following functors are well-defined and inverse to each other*

$$\begin{array}{ccc} A\text{–}\mathsf{mod} & \rightleftarrows & \mathsf{QCoh}(X) \\ M & \mapsto & \widetilde{M} \\ \mathcal{F}(X) & \leftarrow & \mathcal{F}. \end{array}$$

Proof. For well-definedness, we only need to show $\widetilde{M}$ is quasi-coherent for $M \in A\text{–}\mathsf{mod}$. Let $U = \mathsf{Spec}(B) \to X$ be an open immersion given by a homomorphism $A \to B$. By Constuction 13.1.3, we have $\widetilde{M}|_U \simeq \widetilde{B \otimes_A M}$. By definition, this implies $\widetilde{M}$ is quasi-coherent.

By Corollary 13.1.2, we only need to show the functor $A\text{–}\mathsf{mod} \to \mathsf{QCoh}(X)$, $M \mapsto \widetilde{M}$ is an equivalence. By Corollary 13.1.6, we only need to show it is essentially surjective. But this follows from the definition.

□

Combining Proposition 13.3.1 with Corollary 13.1.8, we obtain:

Corollary 13.3.2. *Let X be an affine scheme. Finite limits and small colimits of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Combining Proposition 13.3.1 with (13.2), we obtain:

Corollary 13.3.3. *Let $f : X \to Y$ be a morphism between affine schemes. The functor f^* sends quasi-coherent $\mathcal{O}_Y$-modules to quasi-coherent $\mathcal{O}_X$-modules.*

Combining Proposition 13.3.1 with Corollary 13.1.9, we obtain:

Corollary 13.3.4. *Let $f : X \to Y$ be a morphism between affine schemes. The functor f_* sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules.*

13.4. **General case.**

Theorem 13.4.1. *Being quasi-coherent is a local condition. In other words, for a scheme X and an open covering $X = \bigcup_{i\in I} X_i$, an object $\mathcal{F} \in \mathcal{O}_X\text{–}\mathsf{mod}$ is quasi-coherent iff $\mathcal{F}|_{X_i}$ is quasi-coherent for each $i \in I$.*

Proof. The "only if" claim is obvious,

For the "if" claim, let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that each $\mathcal{F}|_{X_i}$ is quasi-coherent.

We only need to show $\mathcal{F}|_U$ is quasi-coherent for any affine open subscheme $U \subseteq X$. Let $U = \bigcup_{j\in J} U_j$ be a covering of U by affine open subschemes such that for any $j \in J$, $U_j \subseteq X_i$ for some $i \in I$. The assumption implies each $\mathcal{F}|_{U_j} \simeq (\mathcal{F}|_{X_i})|_{U_j}$ is quasi-coherent. Hence we can replace $(X, \{X_i\}, \mathcal{F})$ with $(U, \{U_j\}, \mathcal{F}|_U)$ and reduce to the case when X and X_i are affine.

In the case when X is affine, since its underlying topological space is quasi-compact, we can assume I is a finite set. Write $X_{ij} := X_i \cap X_j$, and let $f_i : X_i \to X$, $f_{ij} : X_{ij} \to X$ be the open immersions. Since $\mathcal{F}$ is a sheaf, by Exercise 1.3.4, we have an isomorphism

$$\mathcal{F} \simeq \mathsf{ker}(\prod_{i\in I} f_{i,*}(\mathcal{F}|_{X_i}) \to \prod_{(i,j)\in I^2} f_{ij,*}(\mathcal{F}|_{X_ij})).$$

By assumption, each $\mathcal{F}|_{X_i}$ is quasi-coherent and therefore so is $\mathcal{F}|_{X_ij}$. By Corollary 13.3.4, $f_{i,*}(\mathcal{F}|_{X_i})$ and $f_{ij,*}(\mathcal{F}|_{X_ij})$ are quasi-coherent. By Corollary 13.3.2, $\mathcal{F}$ is quasi-coherent as desired.

□

Corollary 13.4.2. *Let $f : X \to Y$ be a morphism between schemes. The functor f^* sends quasi-coherent $\mathcal{O}_Y$-modules to quasi-coherent $\mathcal{O}_X$-modules.*

Proof. Using Theorem 13.4.1, we can assume Y is affine by replacing $X \to Y$ with its base-change $X \times_Y U \to U$, where U ranges over all affine open subschemes of Y. Using Theorem 13.4.1 again, we can assume X is affine by replacing X with V, where V ranges over all affine open subschemes of X. Now the desired claim follows from Corollary 13.3.3.

□

Corollary 13.4.3. *Let X be a scheme. Finite limits and small colimits of quasi-coherent $\mathcal{O}_X$-modules are quasi-coherent.*

Proof. Follows from Corollary 13.3.2 and the fact that j^{-1} commutes with finite limits and small colimits for any open immersion j.

□

Lemma 13.4.4. *Let $f : X \to Y$ be a closed immersion. The functor f_* sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules.*

Proof. The problem is local on Y, so we can assume Y is affine and therefore quasi-compact. This implies X is quasi-compact because f is a closed immersion. Let $X = \sqcup_{i\in I} X_i$ be a finite covering of X by affine open subschemes. We have

$$X_{ij} \coloneqq X_i \cap X_j \simeq X_i \underset{X}{\times} X_j \xrightarrow{\simeq} X_i \underset{Y}{\times} X_j,$$

where the last isomorphism is because $f : X \to Y$ is a monomorphism (Proposition 11.3.7). In particular X_{ij} is affine. Let $g_i : X_i \to X$ and $g_{ij} : X_{ij} \to X$ be the open immersions.

Let $\mathcal{F} \in \mathsf{QCoh}(X)$. We only need to show $f_*\mathcal{F}$ is quasi-coherent. As in the proof of Theorem 13.4.1, we have

$$\mathcal{F} \simeq \mathsf{ker}\big(\prod_{i\in I} g_{i,*}(\mathcal{F}|_{X_i}) \to \prod_{(i,j)\in I^2} g_{ij,*}(\mathcal{F}|_{X_{ij}})\big).$$

Since f_* is a right adjoint, it commutes with limits. This implies

$$f_*\mathcal{F} \simeq \mathsf{ker}\big(\prod_{i\in I} (f\circ g_i)_*(\mathcal{F}|_{X_i}) \to \prod_{(i,j)\in I^2} (f\circ g_{ij})_*(\mathcal{F}|_{X_{ij}})\big).$$

Now the desired claim follows from Corollary 13.4.3 and Corollary 13.3.4.

□

Remark 13.4.5. In the future lectures, we will show f_* preserves quasi-coherent modules if f is (relatively) *quasi-compact and quasi-separated.*

Warning 13.4.6. For general morphism $f : X \to Y$, f_* does not preserve quasi-coherent modules.

Exercise 13.4.7. Let k be a field and $X \coloneqq \mathbb{A}^1_k$.

(1) Show that infinite products of quasi-coherent $\mathcal{O}_X$-modules may fail to be quasi-coherent.
(2) Let I be an infinite set and consider the obvious morphism

$$(\mathsf{id})_{i\in I} : \bigsqcup_{i\in I} \mathbb{A}^1_k \to \mathbb{A}^1_k.$$

Show that pushforward along this morphism does not preserve quasi-coherent modules.

Exercise 13.4.8. Let X be a scheme and $\mathcal{M}, \mathcal{N} \in \mathsf{QCoh}(X)$. Show that $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N} \in \mathsf{QCoh}(X)$.

14. Quasi-coherent algebras

14.1. Definition.

Definition 14.1.1. Let $(X,\mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-algebra (sheaf) is an object $\mathcal{A}\in\mathsf{Shv}(X,\mathsf{CRing})$ equipped with a morphism $\mathcal{O}_X\to\mathcal{A}$. Let

$$\mathcal{O}_X\text{–}\mathsf{alg}:=\mathsf{Shv}(X,\mathsf{CRing})_{\mathcal{O}_X/}$$

be the category of $\mathcal{O}_X$-algebras.

Construction 14.1.2. Let X be a scheme and $\mathcal{I}\subseteq\mathcal{O}_X$ be an ideal sheaf. Consider the $\mathcal{O}_X$-module

$$\mathcal{O}_X/\mathcal{I}:=\mathsf{coker}(\mathcal{I}\to\mathcal{O}_X)\in\mathcal{O}_X\text{–}\mathsf{mod}.$$

Unwinding the definitions, it is isomorphic to the sheafification of the $\mathcal{O}_X$-module presheaf $(\mathcal{O}_X/\mathcal{I})_{\mathsf{PShv}}$, which sends U to $\mathcal{O}_X(U)/\mathcal{I}(U)$. Note that $\mathcal{O}_X(U)/\mathcal{I}(U)$ has an obvious $\mathcal{O}_X(U)$-algebra structure. It follows that $\mathcal{O}_X/\mathcal{I}$ inherits an $\mathcal{O}_X$-algebra (sheaf) structure.

Construction 14.1.3. There is a forgetful functor

$$\mathcal{O}_X\text{–}\mathsf{alg}\to\mathcal{O}_X\text{–}\mathsf{mod}$$

sending an $\mathcal{O}_X$-algebra $\mathcal{A}$ to the underlying abelian sheaf $\mathcal{A}$ equipped with the $\mathcal{O}_X$-module structure given by

$$\mathcal{O}_X(U)\times\mathcal{A}(U)\to\mathcal{A}(U)\times\mathcal{A}(U)\to\mathcal{A}(U).$$

Definition 14.1.4. Let X be a scheme. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is **quasi-coherent** if it is quasi-coherent when viewed as an $\mathcal{O}_X$-module. Let

$$\mathcal{O}_X\text{–}\mathsf{alg}_{\mathsf{qcoh}}\subseteq\mathcal{O}_X\text{–}\mathsf{alg}$$

be the full subcategory of quasi-coherent $\mathcal{O}_X$-algebras.

Definition 14.1.5. Let X be a scheme. A **quasi-coherent ideal of** $\mathcal{O}_X$ is a sheaf of ideals $\mathcal{I}$ that is quasi-coherent when viewed as an $\mathcal{O}_X$-module.

Example 14.1.6. Let X be a scheme and $\mathcal{I}\subseteq\mathcal{O}_X$ be a quasi-coherent ideal. Then $\mathcal{O}_X/\mathcal{I}$ is a quasi-coherent $\mathcal{O}_X$-algebra.

Example 14.1.7. Let $A\in\mathsf{CRing}$ and $X:=\mathsf{Spec}(A)$. For any $B\in A\text{–}\mathsf{alg}$, we can view it as an A-module and consider the quasi-coherent $\mathcal{O}_X$-module $\widetilde{B}$. Recall $\widetilde{B}$ is the sheafification of the $\mathcal{O}_X$-module presheaf

$$U\mapsto\mathcal{O}_X(U)\underset{A}{\otimes}B.$$

Note that $\mathcal{O}_X(U)\otimes_A B$ has an obvious $\mathcal{O}_X(U)$-algebra structure. It follows that $\widetilde{B}$ inherits an $\mathcal{O}_X$-algebra (sheaf) structure.

Exercise 14.1.8. Let $A\in\mathsf{CRing}$ and $X:=\mathsf{Spec}(A)$. Show that the following functors are inverse to each other:

$$\begin{array}{ccc} A\text{–}\mathsf{alg} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{alg}_{\mathsf{qcoh}} \\ B & \mapsto & \widetilde{B} \\ \mathcal{B}(X) & \leftarrow\!\shortmid & \mathcal{B}. \end{array}$$

14.2. The relative Spec construction.

Construction 14.2.1. Let S be a scheme and $p: X \to S$ be an S-scheme. The object $p_*\mathcal{O}_X \in \mathsf{Shv}(S, \mathsf{CRing})$ has an $\mathcal{O}_S$-algebra structure given by the canonical morphism $\mathcal{O}_X \to p_*\mathcal{O}_X$.

Example 14.2.2. Let $p: X \to S$ be a closed immersion of schemes and $\mathcal{I}_X \subseteq \mathcal{O}_S$ be its ideal of definition. Recall we have an equivalence between abelian sheaves:

$$\mathcal{O}_S/\mathcal{I}_X \simeq p_*\mathcal{O}_X.$$

One can check this is also an equivalence between $\mathcal{O}_X$-algebras. Note that they are quasi-coherent by Lemma 13.4.4. In particular,

$$\mathcal{I}_X \simeq \mathsf{ker}(\mathcal{O}_S \to p_*\mathcal{O}_X)$$

is a quasi-coherent ideal of $\mathcal{O}_S$.

Construction 14.2.3. Let S be a scheme, and $p: X \to S$, $q: Y \to S$ be S-schemes. Let $f: X \to Y$ be an S-morphism, i.e., a morphism such that $p = q \circ f$. We have a canonical $\mathcal{O}_S$-homomorphism

$$q_*\mathcal{O}_Y \to q_*(f_*\mathcal{O}_X) \simeq p_*\mathcal{O}_X.$$

One can check this defines a *contravariant* functor

$$\begin{array}{ccc} (\mathsf{Sch}_S)^{\mathsf{op}} & \to & \mathcal{O}_S\text{--}\mathsf{alg} \\ (X \xrightarrow{p} S) & \mapsto & p_*\mathcal{O}_X. \end{array}$$

The following theorem says the partially defined left adjoint to this functor is well defined on *quasi-coherent* $\mathcal{O}_S$-algebras.

Theorem-Definition 14.2.4. *Let S be a scheme. For any $\mathcal{A} \in \mathcal{O}_S\text{--}\mathsf{alg}_{\mathsf{qcoh}}$, there is an essentially unique S-scheme $q: Y \to S$ equipped with an isomorphism $\mathcal{A} \xrightarrow{\simeq} q_*\mathcal{O}_Y$ such that for any testing S-scheme $p: X \to S$ with $p_*\mathcal{O}_X$ being quasi-coherent, the composition*

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, Y) \to \mathsf{Hom}_{\mathcal{O}_S\text{--}\mathsf{alg}}(q_*\mathcal{O}_Y, p_*\mathcal{O}_X) \xrightarrow{\simeq} \mathsf{Hom}_{\mathcal{O}_S\text{--}\mathsf{alg}}(\mathcal{A}, p_*\mathcal{O}_X) \tag{14.1}$$

is a bijection. We denote this S-scheme by

$$\mathsf{Spec}_S(\mathcal{A}) \in \mathsf{Sch}_S.$$

Sketch. The (essential) uniqueness follows from Yoneda's lemma. We only need to prove the claim about existence.

We first treat the case when $S = \mathsf{Spec}(R)$ is affine.

Let $A := \mathcal{A}(S)$ be the R-algebra corresponding to $\mathcal{A}$. Write $Y := \mathsf{Spec}(A)$ and view it as an S-scheme via the morphism $q: \mathsf{Spec}(A) \to \mathsf{Spec}(R)$ that corresponds to $A \to R$. By Exercise 14.1.8, we have $\mathcal{A} \simeq q_*\mathcal{O}_Y$ because $\mathcal{A}(S) \simeq A \simeq (q_*\mathcal{O}_Y)(S)$.

Let $p: X \to S$ be a testing S-scheme such that $p_*\mathcal{O}_X$ is quasi-coherent, we need to verify (14.1) is a bijection. Let $\phi: R \simeq \mathcal{O}_S(S) \to \mathcal{O}_X(X)$ be the homomorphism induced by p. Unwinding the definitions, we have

$$\begin{aligned}
& \mathsf{Hom}_{\mathsf{Sch}_S}(X,Y) \\
\simeq\ & \mathsf{Hom}_{\mathsf{Sch}}(X,Y) \underset{\mathsf{Hom}_{\mathsf{Sch}}(X,S)}{\times} \{p\} \\
\simeq\ & \mathsf{Hom}_{\mathsf{CRing}}(A,\mathcal{O}_X(X)) \underset{\mathsf{Hom}_{\mathsf{CRing}}(R,\mathcal{O}_X(X))}{\times} \{\phi\} \\
\simeq\ & \mathsf{Hom}_{R\text{-}\mathsf{alg}}(A,\mathcal{O}_X(X)) \\
\simeq\ & \mathsf{Hom}_{\mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}}}(\mathcal{A},p_*\mathcal{O}_X)
\end{aligned}$$

where the second bijection is due to Theorem 7.1.2, and the fourth bijection is due to Exercise 14.1.8. One can check the above composition is equal to (14.1), hence the latter is also a bijection as desired.

Now we treat the case when S is a general scheme. Let $S = \bigcup_{i\in I} S_i$ be an covering of S_i by affine open subschemes and write $S_{ij} := S_i \cap S_j$. Note that $\mathcal{A}_i := \mathcal{A}|_{S_i}$ is a quasi-coherent $\mathcal{O}_{S_i}$-algebra. By the previous discussion, there exists an essentailly unique S_i-scheme $q_i : Y_i \to S_i$ equipped with an isomorphism $\mathcal{A}_i \simeq q_{i,*}\mathcal{O}_{Y_i}$ such that

$$\mathsf{Hom}_{\mathsf{Sch}_{S_i}}(X,Y_i) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p_*\mathcal{O}_X)$$

for any S_i-scheme $p : X \to S_i$ such that $p_*\mathcal{O}_{X_i}$ is quasi-coherent. For $(i,j) \in I^2$, consider the S_{ij}-scheme

$$q_{ij} : Y_{ij} := Y_i \underset{S_i}{\times} S_{ij} \to S_{ij}.$$

It is easy to check

$$q_{ij,*}\mathcal{O}_{Y_{ij}} \simeq (q_{i,*}\mathcal{O}_{Y_i})|_{S_{ij}} \simeq \mathcal{A}|_{S_{ij}}.$$

In particular, $q_{ij,*}\mathcal{O}_{Y_{ij}}$ is quasi-coherent.

For any testing S_{ij}-scheme $p : X \to S_{ij}$, we have

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathsf{Sch}_{S_i}}(X,Y_i) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p'_*\mathcal{O}_X),$$

where p' is the composition $X \xrightarrow{p} S_{ij} \xrightarrow{u_{ij}} S_i$. We have

$$\mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,p'_*\mathcal{O}_X) \simeq \mathsf{Hom}_{\mathcal{O}_{S_i}\text{-}\mathsf{alg}}(\mathcal{A}_i,s_{ij,*}\circ p_*\mathcal{O}_X) \simeq \mathsf{Hom}_{\mathcal{O}_{S_{ij}}\text{-}\mathsf{alg}}(\mathcal{A}|_{S_{ij}},p_*\mathcal{O}_X).$$

This implies

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathcal{O}_{S_{ij}}\text{-}\mathsf{alg}}(\mathcal{A}|_{S_{ij}},p_*\mathcal{O}_X).$$

Note that the RHS is invariant after switching i with j. Hence we obtain a canonical bijection

$$\mathsf{Hom}_{\mathsf{Sch}_{S_{ij}}}(X,Y_{ij}) \simeq \mathsf{Hom}_{\mathsf{Sch}_{S_{ji}}}(X,Y_{ji}).$$

One can check this bijection is functorial in X. Hence by Yoneda's lemma, we obtain an isomorphism

$$\phi_{ij} : Y_{ij} \to Y_{ji}$$

defined over $S_{ij} = S_{ji}$.

One can check

$$(I,(Y_i)_{i\in I},(Y_{ij})_{(i,j)\in I^2},(\phi_{ij})_{(i,j)\in I^2})$$

is a gluing data of schemes (see Definition 6.1.1). Let

$$(Y,(Y'_i)_{i\in I},(\varphi_i)_{i\in I})$$

be its gluing output (see Proposition-Definition 6.1.2). By Proposition 6.1.3, there is a canonical morphism

$$q : Y \to S$$

glued from the morphisms $Y_i \xrightarrow{q_i} S_i \to S$. By construction, we have an isomorphism

$$\alpha_i : \mathcal{A}|_{S_i} \simeq q_{i,*}\mathcal{O}_{Y_i} \simeq q_*\mathcal{O}_Y|_{S_i}.$$

Moreover, the restriction of α_i to S_{ij} is equal to the restriction of α_j to $S_{ji} = S_{ij}$. Hence by Exercise 1.2.7, there is a unique isomorphism

$$\alpha : \mathcal{A} \simeq q_*\mathcal{O}_Y$$

such that its restriction to each S_i is equal to α_i.

For $q : Y \to S$ and $\mathcal{A} \simeq q_*\mathcal{O}_Y$ constructed as above, using Proposition-Construction 6.1.4, one can check (14.1) is bijective whenever $p_*\mathcal{O}_X$ is quasi-coherent.

□

Corollary 14.2.5. *Let S be a scheme and $\mathsf{Sch}'_S \subseteq \mathsf{Sch}_S$ be the full subcategory consisting of S-schemes $p : X \to S$ such that $p_*\mathcal{O}_X$ is quasi-coherent. Then the functor*

$$(\mathsf{Sch}'_S)^{\mathsf{op}} \to \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}, \ \ (p : X \to S) \mapsto p_*\mathcal{O}_X$$

admits a fully faithful left adjoint

$$\mathsf{Spec}_S : \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}} \to (\mathsf{Sch}'_S)^{\mathsf{op}}.$$

Exercise 14.2.6. Let $f : S' \to S$ be a morphism between schemes. For $\mathcal{A} \in \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}$, consider

$$\mathcal{A}' \coloneqq f^*\mathcal{A} \in \mathcal{O}_{S'}\text{–}\mathsf{alg}_{\mathsf{qcoh}}.$$

Show that

$$\mathsf{Spec}_S(\mathcal{A}) \underset{S}{\times} S' \simeq \mathsf{Spec}_{S'}(\mathcal{A}').$$

Exercise 14.2.7. Let S be a scheme and $\mathcal{A} \in \mathcal{O}_S\text{–}\mathsf{alg}_{\mathsf{qcoh}}$. Show that for *any* S-scheme $p : X \to S$, the canonical map

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, \mathsf{Spec}_S(\mathcal{A})) \to \mathsf{Hom}_{\mathcal{O}_S\text{–}\mathsf{alg}}(\mathcal{A}, p_*\mathcal{O}_X)$$

is a bijection. Hint: base-change along $X \to S$.

14.3. **Vector bundles.**

Construction 14.3.1. Let $(S, \mathcal{O}_S)$ be a ringed space and $\mathcal{E}$ be an $\mathcal{O}_S$-module. The **symmetric algebra** $\mathsf{Sym}_{\mathcal{O}_S}(\mathcal{E})$ for $\mathcal{E}$ is defined to be the sheafification of the presheaf

$$U \mapsto \mathsf{Sym}_{\mathcal{O}_S(U)}(\mathcal{E}(U)).$$

We leave the proof of the following result to the readers.

Lemma 14.3.2. *Let S be a scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. Then $\mathsf{Sym}_{\mathcal{O}_S}(\mathcal{E})$ is a quasi-coherent $\mathcal{O}_S$-algebra. Moreover, for any affine open subscheme $U \subseteq S$, we have*

$$\big(\mathsf{Sym}_{\mathcal{O}_S}(\mathcal{E})\big)(U) \simeq \mathsf{Sym}_{\mathcal{O}_S(U)}(\mathcal{E}(U)).$$

Construction 14.3.3. Let S be a scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. We call

$$\mathsf{V}_S(\mathcal{E}) \coloneqq \mathsf{Spec}_S(\mathsf{Sym}_{\mathcal{O}_S}(\mathcal{E}))$$

the vector bundle associated to S.

Example 14.3.4. We have

$$\mathsf{V}_S(\mathcal{O}_S^{\oplus n}) \simeq \mathbb{A}_S^n := \mathbb{A}_{\mathbb{Z}}^n \times S.$$

In particular, if $\mathcal{E}$ is locally free of rank n, then the S-schemes $\mathsf{V}_S(\mathcal{E})$ and $\mathbb{A}_S^n$ become isomorphic after base-change to some open covering of S.

By Exercise 14.2.7, we have the following result.

Lemma 14.3.5. *Let S be a scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_S$-module. For any S-scheme $p : X \to S$, the canonical map*

$$\mathsf{Hom}_{\mathsf{Sch}_S}(X, \mathsf{V}_S(\mathcal{E})) \to \mathsf{Hom}_{\mathcal{O}_S\text{-}\mathsf{mod}}(\mathcal{E}, p_*\mathcal{O}_X)$$

is a bijection.

15. Application: classification of closed immersions

15.1. The classification.

Theorem 15.1.1. *Let X be a scheme. The following functors are well-defined and inverse to each other*

$$\begin{array}{ccc} \{\textit{closed immersions into } X\}^{\mathsf{op}} & \rightleftarrows & \{\textit{quasi-coherent ideals of } \mathcal{O}_X\} \\ (i : Y \to X) & \mapsto & \ker(\mathcal{O}_X \to i_*\mathcal{O}_Y) \\ \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) & \leftarrow\!\shortmid & \mathcal{I} \subseteq \mathcal{O}_X. \end{array}$$

Proof. For the claim of well-definedness, we only need to show for any quasi-coherent ideal $\mathcal{I}$, the structural morphism $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \to X$ is a closed immersion. By Exercise 14.2.6, this claim is local on X, hence we can assume $X = \mathsf{Spec}(A)$ is affine. Let $I \subseteq A$ be the ideal such that $\mathcal{I} \simeq \widetilde{I}$. By construction, $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \simeq \mathsf{Spec}(A/I)$ and the structural morphism $\mathsf{Spec}(A/I) \to \mathsf{Spec}(A)$ corresponds to the canonical homomorphism $A \to A/I$, which is indeed a closed immersion.

It remains to check these two functors are inverse to each other.

Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal of $\mathcal{O}_X$ and consider $i : Y = \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \to X$. By definition, we have a canonical isomorphism $\mathcal{O}_X/\mathcal{I} \simeq i_*\mathcal{O}_Y$ of quasi-coherent $\mathcal{O}_X$-algebras. This implies

$$\ker(\mathcal{O}_X \to i_*\mathcal{O}_Y) \simeq \ker(\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}) \simeq \mathcal{I}.$$

In other words, the composition of the leftwards functor followed by the rightwards functor is equivalent to the identity functor.

On ther other hand, let $i : Y \to X$ be a closed immersion. By definition of Spec_X, the identity homomorphism $i_*\mathcal{O}_Y \to i_*\mathcal{O}_Y$ corresponds to an X-morphism $\phi : Y \to \mathsf{Spec}_X(i_*\mathcal{O}_Y)$. It remains to show this is an isomorphism. By construction, ϕ induces an isomorphism between the corresponding ideals of definition. Hence by Lemma 11.4.5, ϕ is an isomorphism as desired.

□

Corollary 15.1.2. *Let $X = \mathsf{Spec}(A)$ be an affine scheme. Any closed immersion into X is isomorphic to $\mathsf{Spec}(A/I) \to \mathsf{Spec}(A)$ for a unique ideal $I \subseteq X$.*

Corollary 15.1.3. *Closed immersions are stable under base-change.*

Proof. Let $i : Y \to X$ be a closed immersion between S-schemes and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be its ideal of definition. We identify Y with $\mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I})$. For any $S' \to S$, consider $X' := X \times_S S'$ and $Y' := Y \times_S S'$. We have

$$Y' \simeq \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \underset{S}{\times} S' \simeq \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}) \underset{X}{\times} X' \simeq \mathsf{Spec}_{X'}(f^*(\mathcal{O}_X/\mathcal{I})),$$

where $f : X' \to X$ is the canonical projection. Note that we have $f^*(\mathcal{O}_X/\mathcal{I}) \simeq \mathcal{O}_{X'}/f^*\mathcal{I}$ and $f^*\mathcal{I}$ is a quasi-coherent ideal of $\mathcal{O}_{X'}$. Hence $Y' \to X'$ is a closed immersion as desired.

□

Combining with Exercise 10.1.4, we obtain

Corollary 15.1.4. *Locally closed immersions are stable under base-change.*

16. Coherent modules

16.1. **Modules of finite type.**

Definition 16.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F} \in \mathcal{O}_X-\mathsf{mod}$. We say $\mathcal{F}$ is **generated by sections** $s_i \in \mathcal{F}(X)$**,** $i \in I$ if the morphism

$$\bigoplus_{i\in I} \mathcal{O}_X \xrightarrow{(s_i)_{i\in I}} \mathcal{F}$$

is a surjection in the abelian category $\mathcal{O}_X-\mathsf{mod}$.

Definition 16.1.2. Let $(X, \mathcal{O}_X)$ be a ringed space. We say $\mathcal{F} \in \mathcal{O}_X-\mathsf{mod}$ is **of finite type** if for any point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x such that $\mathcal{F}|_U$ is generated by finitely many sections.

Lemma 16.1.3. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces. Then f^* sends $\mathcal{O}_Y$-modules of finite type to $\mathcal{O}_X$-modules of finite type.*

Proof. Follows from the fact that f^* is right exact. □

Lemma 16.1.4. *Let $(X, \mathcal{O}_X)$ be a ringed space and $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules.*

(1) *If $\mathcal{F}_2$ is of finite type, so is $\mathcal{F}_3$.*
(2) *If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite type, so is $\mathcal{F}_2$.*

Proof. (1) is obvious. To prove (2), let $x \in X$ be a point and $U \subseteq X$ be an open neighborhood of x such that $\mathcal{F}_1|_U$ and $\mathcal{F}_3|_U$ are generated respectively by sections $a_i \in \mathcal{F}_1(U)$, $i \in I$ and $c_j \in \mathcal{F}_3(U)$, $j \in J$ with $|I|, |J| < \infty$. Since $\mathcal{F}_2 \to \mathcal{F}_3$ is a surjection, for each $j \in J$, there exists an open neighborhood $U_i \subseteq U$ of x and a section $b_j \in \mathcal{F}_2(U_j)$ such that $\beta(b_j) = c_j|_{U_j}$. Write $V := \bigcap_{j\in J} U_j$. It is easy to see $\mathcal{F}_2|_V$ is generated by $\alpha(a_i)|_V$, $i \in I$ and $b_j|_V$, $j \in J$.

□

Warning 16.1.5. The category of $\mathcal{O}_X$-modules that are of finite type may not form an abelian category.

Lemma 16.1.6. *let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module that is of finite type. Then $\mathsf{supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}$ is a closed subset of X.*

Proof. We only need to show that if $\mathcal{F}_x = 0$ for a point $x \in X$, then there exists an open neighborhood $V \subseteq X$ of x such that $\mathcal{F}|_V \simeq 0$. Choose U such that $\mathcal{F}|_U$ is generated by sections $s_i \in \mathcal{F}(U)$, $i \in I$ with $|I| < \infty$. Since $\mathcal{F}_x = 0$, we have $(s_i)_x = 0$. Hence for each $i \in I$, there exists an open neighborhood $U_i \subseteq U$ of x such that $s_i|_{U_i} = 0$. Let $V := \bigcap_{i\in I} U_i$. Since $\mathcal{F}|_U$ is generated by s_i, the $\mathcal{O}_V$-module $\mathcal{F}|_V$ is generated by $s_i|_V$, $i \in I$. Now $s_i|_V = 0$ implies $\mathcal{F}|_V = 0$ as desired.

□

Lemma 16.1.7. *Let X be scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. Then for any affine open subscheme $U \subseteq X$, $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.*

Proof. We can assume $X = U = \mathsf{Spec}(A)$ is affine. Then the claim follows from the following Exercise 16.1.8 in commutative algebra.

□

Exercise 16.1.8. Let A be a commutative ring and $f_1, \cdots, f_n \in A$ be elements such that $(f_1, \cdots, f_n) = A$. If M is an A-module such that each M_{f_i} is a finitely generated A_{f_i}-module, then M is finitely generated as an A-module.

Corollary 16.1.9. *Let X be scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. Then for any $x \in X$, $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module.*

Using Nakayama's lemma, we obtain the following result.

Corollary 16.1.10. *Let X be scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. Then $x \in \mathsf{supp}(\mathcal{F})$ iff the **fiber** $\mathcal{F}_x \otimes_{\mathcal{O}_{x,x}} \kappa_x$ of $\mathcal{F}$ at x is nonzero.*

Construction 16.1.11. Let X be scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. For any affine open subscheme $U \subseteq X$, let

$$\mathsf{Ann}_{\mathcal{O}_X(U)}(\mathcal{F}(U)) \subseteq \mathcal{O}_X(U)$$

be the annihilator of $\mathcal{F}(U)$. By Lemma 16.1.7, $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module. It follows that for any $f \in \mathcal{O}_X(U)$, we have

$$\mathsf{Ann}_{\mathcal{O}_X(U)}(\mathcal{F}(U))_f \simeq \mathsf{Ann}_{\mathcal{O}_X(U_f)}(\mathcal{F}(U_f)).$$

This implies there is a unique quasi-coherent ideal

$$\mathsf{Ann}_{\mathcal{O}_X}(\mathcal{F}) \subseteq \mathcal{O}_X$$

such that

$$\big(\mathsf{Ann}_{\mathcal{O}_X}(\mathcal{F})\big)(U) = \mathsf{Ann}_{\mathcal{O}_X(U)}(\mathcal{F}(U))$$

for any affine open subscheme $U \subseteq X$. We call $\mathsf{Ann}_{\mathcal{O}_X}(\mathcal{F})$ the **annihilator** of $\mathcal{F}$.

Proposition-Definition 16.1.12. *Let X be scheme and $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. Let $i : Y \to X$ be the closed immersion corresponding to $\mathsf{Ann}_{\mathcal{O}_X}(\mathcal{F})$. Then there exists an essentially unique quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$ equipped with an isomorphism $\alpha : \mathcal{F} \xrightarrow{\simeq} i_*\mathcal{M}$. Moreover, we have:*

1. *The quasi-coherent module $\mathcal{M}$ is of finite type.*
2. *We have $\mathsf{supp}(\mathcal{M}) = Y$ and therefore $\mathsf{supp}(\mathcal{F}) = i(Y)$.*
3. *For any closed immersion $i' : Y' \to X$ and $\mathcal{M}' \in \mathsf{QCoh}(Y')$ equipped with an isomorphism $\alpha' : \mathcal{F} \xrightarrow{\simeq} i'_*\mathcal{M}'$, there exists a unique X-morphism $f : Y \to Y'$ and a unique isomorphism $\beta : \mathcal{M}' \xrightarrow{\simeq} f_*\mathcal{M}$ such that $i'_*(\beta) \circ \alpha' = \alpha$.*

*We call Y the **scheme theoretic support** of $\mathcal{F}$.*

Proof. For any $(i', \mathcal{M}', \alpha')$ as in (3), we have a canonical isomorphism

$$(i')^*(\mathcal{F}) \xrightarrow{(i')^*(\alpha)} (i')^* \circ i'_*(\mathcal{M}') \xrightarrow{\simeq} \mathcal{M}'. \tag{16.1}$$

Hence $(\mathcal{M}, \alpha)$ is essentially unique if exists, and we must have $\mathcal{M} \simeq i^*\mathcal{F}$ with α given by the canonical morphism

$$\mathcal{F} \to i_* \circ i^*(\mathcal{F}). \tag{16.2}$$

Hence to show that $(\mathcal{M}, \alpha)$ exists, we only need to show (16.2) is an isomorphism. By Construction 16.1.11, for any affine open subscheme $U \subseteq X$, we have

$$\mathcal{F}(U) \simeq \mathcal{O}_X(U)/\mathsf{Ann}_{\mathcal{O}_X(U)}(\mathcal{F}(U)) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{F}(U) \simeq$$
$$\simeq \mathcal{O}_Y(U \cap Y) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{F}(U) \simeq \big(i_* \circ i^*(\mathcal{F})\big)(U).$$

This implies (16.2) is indeed an isomorphism.

Claim (1) follows from Lemma 16.1.3.

To prove claim (2), we can assume that $Y = \mathsf{Spec}(A)$ is affine, $\mathcal{M} = \widetilde{M}$ and $\mathsf{Ann}_A(M) = 0$. Using the fact that M is finitely generated as an A-module, it is easy to see that for any prime ideal $\mathfrak{p} \subseteq A$, $M_{\mathfrak{p}} \neq 0$. Therefore $\mathsf{supp}(\mathcal{M}) = Y$ as desired.

To prove claim (3), let $\mathcal{I}'$ be the ideal of definition for $i' : Y' \to X$. It is easy to see that

$$\mathsf{Ann}_{\mathcal{O}_X}(\mathcal{F}) \simeq \mathsf{Ann}_{\mathcal{O}_X}(i'_*\mathcal{M}') \supseteq \mathcal{I}'.$$

Hence by the classification of closed subschemes, we obtain a unique X-morphism $f : Y \to Y'$. Moreover, we have a canonical isomorphism

$$\mathcal{M}' \underset{(16.1)}{\xrightarrow{\simeq}} (i')^*(\mathcal{F}) \simeq (i')^* \circ i_*(\mathcal{F}) \simeq (i')^* \circ i'_* \circ f_*(\mathcal{F}) \simeq f_*(\mathcal{F}).$$

This gives the desired isomorphism β, which is unique by a diagram chasing.

□

16.2. Modules of finite presentation.

Definition 16.2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. We say $\mathcal{F} \in \mathcal{O}_X\text{–}\mathsf{mod}$ is **of finite presentation** if for any point $x \in X$, there exists an open neighborhood $U \subseteq X$ of x and an exact sequence

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

with $|I|, |J| < \infty$.

We have the following obvious result.

Lemma 16.2.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module that is of finite presentation. Then $\mathcal{F}$ is of finite type.*

Lemma 16.2.3. *Let X be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module that is of finite presentation. Then $\mathcal{F}$ is quasi-coherent.*

Proof. For each $x \in X$, let U be an open neighborhood of x such that there is an exact sequence

$$\bigoplus_{i \in I} \mathcal{O}_U \to \bigoplus_{j \in J} \mathcal{O}_U \to \mathcal{F}|_U \to 0$$

with $|I|, |J| < \infty$. Then $\mathcal{F}|_U$ is quasi-coherent because it is a finite colimit of quasi-coherent $\mathcal{O}_U$-modules. This implies $\mathcal{F}$ is quasi-coherent because being quasi-coherent is local on the schemes.

□

Warning 16.2.4. Let X be a scheme. An $\mathcal{O}_X$-module that is of finite type may fail to be quasi-coherent. Indeed, for any non-quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, the $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{I}$ is of finite type but not quasi-coherent.

We have the following analogue of Lemma 16.1.4.

Exercise 16.2.5. Let $(X, \mathcal{O}_X)$ be a ringed space and $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules.

(1) If $\mathcal{F}_1$ is of finite type and $\mathcal{F}_2$ is of finite presentation, then $\mathcal{F}_3$ is of finite presentation.
(2) If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite presentation, so is $\mathcal{F}_2$.
(3) If $\mathcal{F}_2$ is of finite type and $\mathcal{F}_3$ is of finite presentation, then $\mathcal{F}_1$ is of finite type.

Warning 16.2.6. The category of $\mathcal{O}_X$-modules that are of finite presentation may not form an abelian category.

16.3. **Modules of finite presentation and inner Homs.** The slogan is: the inner Hom functor $\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -)$ behaves well when $\mathcal{M}$ is of finite presentation.

Lemma 16.3.1. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{M}, \mathcal{N} \in \mathcal{O}_X\text{-}\mathsf{mod}$. If $\mathcal{M}$ is of finite presentation, then the canonical map*

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})_x \to \underline{\mathsf{Hom}}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x)$$

is an isomorphism.

Proof. By shrinking X, we may assume $\mathcal{M}$ fits into an exact sequence

$$\bigoplus_I \mathcal{O}_X \to \bigoplus_J \mathcal{O}_X \to \mathcal{M} \to 0$$

such that I and J are finite sets. It follows that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})_x \arrow[r] \arrow[d] & \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\oplus_I \mathcal{O}_X, \mathcal{N})_x \arrow[r] \arrow[d] & \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\oplus_J \mathcal{O}_X, \mathcal{N})_x \arrow[d] \\
0 \arrow[r] & \underline{\mathsf{Hom}}_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{N}_x) \arrow[r] & \underline{\mathsf{Hom}}_{\mathcal{O}_{X,x}}((\oplus_I \mathcal{O}_X)_x, \mathcal{N}_x) \arrow[r] & \underline{\mathsf{Hom}}_{\mathcal{O}_{X,x}}((\oplus_J \mathcal{O}_X)_x, \mathcal{N}_x)
\end{tikzcd}$$

such that both rows are exact. Hence we only need to show the second and third vertical maps are isomorphisms. But this is obvious. Indeed, for the second vertical map, both the source and the target can be identified with $\oplus_I \mathcal{N}_x$. □

Similarly, one can prove the following results.

Lemma 16.3.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{M}$ be an $\mathcal{O}_X$-module that is of finite presentation. Then the functor*

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-}\mathsf{mod} \to \mathcal{O}_X\text{-}\mathsf{mod}$$

preserves filtered colimits.

Lemma 16.3.3. *Let X be a scheme and $\mathcal{M}$ be an $\mathcal{O}_X$-module that is of finite presentation. Then the functor*

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -) : \mathcal{O}_X\text{-}\mathsf{mod} \to \mathcal{O}_X\text{-}\mathsf{mod}$$

preserves quasi-coherent modules. The obtained functor

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{M}, -) : \mathsf{QCoh}(X) \to \mathsf{QCoh}(X)$$

preserves filtered colimits.

Exercise 16.3.4. Let $X = \mathsf{Spec}(A)$ be an affine scheme and M, N be A-modules. If M if of finite presentation, show that $\widetilde{M}$ is of finite presentation and

$$\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$$

is the quasi-coherent $\mathcal{O}_X$-module associated to the A-module $\mathsf{Hom}_A(M, N)$.

16.4. **Coherent modules.**

Definition 16.4.1. Let $(X, \mathcal{O}_X)$ be a ringed space. We say an $\mathcal{O}_X$-module $\mathcal{F}$ is **coherent** if $\mathcal{F}$ is of finite type and for any open subset $U \subseteq X$ and a finite collection of sections $s_i \in \mathcal{F}(U)$, $i \in I$, the kernel

$$\ker(\bigoplus_{i \in I} \mathcal{O}_U \xrightarrow{(s_i)_{i \in I}} \mathcal{F}|_U)$$

is of finite type.

The above notion is generally ill-behaved. In this course, we will only consider coherent modules on *locally Noetherian schemes*.

17. Locally free modules

17.1. Locally free modules.

Definition 17.1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. We say an $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free** if for any $x \in X$, there exists an open neighborhood $U \subseteq X$ of x such that $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^{\oplus I}$ for some set I.

We say a locally free $\mathcal{O}_X$-module $\mathcal{F}$ is **of finite rank** if in above we can choose I to be a finite set.

We say a locally free $\mathcal{O}_X$-module $\mathcal{F}$ is **of rank** n if in above we can choose I such that $|I| = n$.

A locally free $\mathcal{O}_X$-module of rank 1 is also called a **line bundle**.

The following result is obvious.

Lemma 17.1.2. *Let X be a scheme. A locally free $\mathcal{O}_X$-module is quasi-coherent.*

Lemma 17.1.3. *Let X be a ringed space and $\mathcal{L}$ be a finitely presented $\mathcal{O}_X$-module. Let $x \in X$ be a point such that $\mathcal{L}_x \simeq \mathcal{O}_{X,x}^{\oplus r}$ for some $r \geq 0$. Then there exists an open neighborhood $U \subseteq X$ of x such that $\mathcal{L}|_U \simeq \mathcal{O}_U^{\oplus r}$.*

Proof. Choose a basis $s_{1,x}, \cdots, s_{r,x}$ of $\mathcal{L}_x$ as a free $\mathcal{O}_{X,x}$-module. Choose an open neighborhood U of x such that each $s_{i,x}$ is the germ of a section $s_i \in \mathcal{L}(U)$. Consider the obtained morphism

$$\phi : \mathcal{O}_U^{\oplus r} \xrightarrow{(s_1, \cdots, s_r)} \mathcal{L}|_U.$$

By construction, ϕ_x is an isomorphism. It follows that both $\mathsf{ker}(\phi)_x$ and $\mathsf{coker}(\phi)_x$ are zero.

By Lemma 16.1.4(1), $\mathsf{coker}(\phi)$ is of finite type. Hence by Lemma 16.1.6, its support $\mathsf{supp}(\mathsf{coker}(\phi))$ is a closed subset of U. Since $x \notin \mathsf{supp}(\mathsf{coker}(\phi))$, we can shrink U and assume $\mathsf{coker}(\phi) = 0$.

On the other hand, by Lemma 16.2.5(3), $\mathsf{ker}(\phi)$ is of finite type. By the same reason, we can shrink U and assume $\mathsf{ker}(\phi) = 0$.

This proves that there exists an open neighborhood U of x such that ϕ is an isomorphism.

□

Proposition 17.1.4. *Let X be a locally ringed space and $\mathcal{L} \in \mathcal{O}_X\text{–}\mathsf{mod}$. The following conditions are equivalent:*

(i) *The $\mathcal{O}_X$-module $\mathcal{L}$ is locally free of rank 1.*
(ii) *The $\mathcal{O}_X$-module $\mathcal{L}$ is **invertible**, i.e., there exists an $\mathcal{O}_X$-module $\mathcal{L}'$ such that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \simeq \mathcal{O}_X$.*

Moreover, if these conditions are satisfied, $\mathcal{L}'$ in (ii) is isomorphic to $\underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Proof. (i)⇒(ii): we only need to show the canonical morphism

$$\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \to \mathcal{O}_X$$

is an isomorphism. This claim is local on X, hence we can assume $\mathcal{L}$ is free of rank 1. Then the above morphism is clearly an isomorphism.

(ii)⇒(i): Let $\phi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \to \mathcal{O}_X$ and $\varphi : \mathcal{O}_X \to \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{L}$ be isomorphisms that are inverse to each other. Note that φ corresponds to a global section of $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{L}$. Since (i) is local on X, we can assume this section is in the image of

$$\mathcal{L}'(X) \underset{\mathcal{O}_X(X)}{\otimes} \mathcal{L}(X) \to (\mathcal{L}' \underset{\mathcal{O}_X}{\otimes} \mathcal{L})(X).$$

In other words, there exist finitely many sections

$$s_i \in \mathcal{L}(X),\ s_i' \in \mathcal{L}'(X)$$

such that

$$\varphi = \sum_{i=1}^{n} s_i' \otimes s_i, \tag{17.1}$$

where we view s_i (resp. s_i') as a morphism $\mathcal{O}_X \to \mathcal{L}$ (resp. $\mathcal{O}_X \to \mathcal{L}'$).

Consider the isomorphisms

$$\mathcal{L} \xrightarrow{\mathrm{id}_{\mathcal{L}} \otimes \varphi} \mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}' \underset{\mathcal{O}_X}{\otimes} \mathcal{L} \xrightarrow{\phi \otimes \mathrm{id}_{\mathcal{L}}} \mathcal{L}.$$

By (17.1), the above composition is equal to

$$\mathcal{L} \xrightarrow{(\phi(-,s_1'),\cdots,\phi(-,s_n'))} \mathcal{O}_X^{\oplus n} \xrightarrow{(s_1,\cdots,s_n)} \mathcal{L}. \tag{17.2}$$

It follows that

$$\mathcal{O}_X^{\oplus n} \simeq \mathcal{L} \oplus \mathcal{M},$$

where $\mathcal{M}$ is the kernel of the second map in (17.2). This implies we have an exact sequence

$$\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n} \to \mathcal{L} \to 0,$$

where the first map is the composition $\mathcal{O}_X^{\oplus n} \to \mathcal{M} \to \mathcal{O}_X^{\oplus n}$. In particular, we obtain that $\mathcal{L}$ is of finite presentation.

Now for any point $x \in X$, we have $\mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{L}_x' \simeq \mathcal{O}_{X,x}$. This implies $\mathcal{L}_x$ is an invertible module for the local ring $\mathcal{O}_{X,x}$. It is a well-known fact in commuatitive algebra that this implies $\mathcal{L}_x$ is a free $\mathcal{O}_{X,x}$-module of rank 1. By Lemma 17.1.3, there exists an open neighborhood $U \subseteq X$ of x such that $\mathcal{L}|_U$ is a free $\mathcal{O}_U$-module of rank 1. This proves (ii).

It remains to show when (i) and (ii) are satisfied, $\mathcal{L}' \simeq \underline{\mathsf{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O})$. This follows from Proposition 12.4.10.

□

Part V. Basic properties of schemes and morphisms

18. Quasi-compact morphisms and (quasi-)separated morphisms

18.1. Quasi-compact morphisms.

Definition 18.1.1. We say a scheme X is **quasi-compact** if its underlying topological space is quasi-compact, i.e., any open covering of X admits a finite subcovering.

Definition 18.1.2. Let $f : X \to Y$ be a morphism between schemes. We say f is **quasi-compact** if for any quasi-compact open subset $U \subseteq Y$, its inverse image $f^{-1}(U)$ is quasi-compact.

Lemma 18.1.3. *Let $f : X \to Y$ be a morphism such that Y is affine. Then f is quasi-compact iff X is quasi-compact.*

Proof. The "only if" claim is obvious because Y is quasi-compact. For the "if" claim, suppose X is quasi-compact. For any quasi-compact open subset U, we only need to show $X \times_Y U$ is quasi-compact. Choose a finite open covering $X = \bigcup_{i\in I} X_i$ by affine open subscheme X_i of X, and similarly choose $U = \bigcup_{j\in J} U_j$. Then

$$X \underset{Y}{\times} U \simeq \bigcup_{(i,j)\in I\times J} X_i \underset{Y}{\times} U_j.$$

Note that each $X_i \times_Y U_j$ is affine and therefore quasi-compact. This implies $X \times_Y U$ is quasi-compact. . $\square$

Lemma 18.1.4. *Consider the class of quasi-compact morphisms.*

(i) *Quasi-compact morphisms are stable under compositions.*
(ii) *Quasi-compact morphisms are stable under base-changes.*
(iii) *Being a quasi-compact morphism is local on the target.*

Proof. (i) is obvious.

To prove (iii), let $f : X \to Y$ be a morphism and $Y = \sqcup_{i\in I} Y_i$ be an open covering such that each $f_i : X \times_Y Y_i \to Y_i$ is quasi-compact. We only need to show f is quasi-compact. Let $U \subseteq Y$ be a quasi-compact open subset. We can choose a finite covering $U = \bigcup_{j\in J} U_j$ of U by its affine open subschemes such that for any $j \in J$, $U_j \subseteq Y_i$ for some $i \in I$. Since f_i is quasi-compact and U_j is a quasi-compact open subset of Y_i, its inverse image

$$f_i^{-1}(U_j) \simeq (X \underset{Y}{\times} Y_i) \underset{Y_i}{\times} U_j \simeq X \underset{Y}{\times} U_j$$

is quasi-compact. It follows that

$$f^{-1}(U) \simeq X \underset{Y}{\times} U \simeq \bigcup_{j\in J} X \underset{Y}{\times} U_j$$

is also quasi-compact as desired.

To prove (ii), let $f : X \to Y$ be a quasi-compact morphism and $f' : X' \to Y'$ be its base-change. We only need to show f' is quasi-compact. By Lemma 18.1.3, for any affine open subscheme $U \subseteq Y$, the base-change $X \times_Y U \to U$ is quasi-compact. Hence by (iii), we can reduce to the case when Y and Y' are affine. Using Lemma 18.1.3 again, we see that X is quasi-compact. It follows that $X' \simeq X \times_Y Y'$ has a *finite* covering by its affine open subsets. This implies X' is quasi-compact. By Lemma 18.1.3 again, we see that f' is quasi-compact as desired.

$\square$

Exercise 18.1.5. Show that a closed immersion is quasi-compact.

18.2. **Separated morphisms.**

Definition 18.2.1. Let $f : X \to Y$ be a morphism between schemes. The **(relative) diagonal morphism**

$$\Delta_f : X \to X \underset{Y}{\times} X$$

is defined to be the morphism corresponding to the commutative square

$$\begin{tikzcd} X \arrow[r, "="] \arrow[d, "="'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Let $p : X \to S$ be an S-scheme. We also write

$$\Delta_X := \Delta_p : X \to X \underset{S}{\times} X$$

and call it the **diagonal morphism of the S-scheme X.**

Lemma 18.2.2. *Let $f : X \to Y$ be a morphism between schemes. The moprhism $\Delta_f : X \to X \times_Y X$ is a locally closed immersion.*

Proof. Consider the projection map $p_1 : X \times_Y X \to X$. Note that $p_1 \circ \Delta_f = \mathsf{id}_X$. This implies Δ_f induces a homeomorphism from X to its image. Moreover, for any point $x \in X$, the identity homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ factors as

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X \times_Y X, \Delta_f(x)} \to \mathcal{O}_{X,x},$$

where the first homomorphism is induced by p_1 while the second one is induced by Δ_f. In particular, the second homomorphism is a surjection.

By Proposition 11.5.2, we only need to show that $\Delta_f(X)$ is a locally closed subset of $X \times_Y X$. The question is local on Y, hence we can assume $Y = \mathsf{Spec}(B)$ is affine. Let $x \in X$ be a point and $U \subseteq X$ be an affine open neighborhood of x. Note that $U \times_Y U \subseteq X \times_Y X$ is an open neighborhood of $\Delta(x)$. Hence we only need to show that $\Delta_f(X) \cap (U \times_Y U)$ is a closed subset of $(U \times_Y U)$. Note that we have

$$\Delta_f(X) \cap (U \underset{Y}{\times} U) = \Delta_f(U).$$

Hence we can replace X with U, and therefore assume $X = \mathsf{Spec}(A)$ is affine. Now Δ_f corresponds to the ring homomorphism $A \otimes_B A \to A$, $a \otimes a' \mapsto aa'$, which is a surjection. Hence Δ_f is a closed immersion as desired.

□

Exercise 18.2.3. Show that $f : X \to Y$ is a monomorphism iff Δ_f is an isomorphism.

Definition 18.2.4. Let $f : X \to Y$ be a morphism between schemes. We say f is **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a *closed* immersion. We say a scheme X is **separted** if $X \to \mathsf{Spec}(\mathbb{Z})$ is separated.

By Lemma 18.2.2, we have:

Corollary 18.2.5. *A morphism $f : X \to Y$ is separated iff $\Delta_f(X)$ is a closed subset of $X \times_Y X$.*

By Exercise 18.2.3, we have:

Corollary 18.2.6. *Any monomorphism between schemes is separated. In particular, a locally closed immersion is separated.*

Lemma 18.2.7. *Consider the class of separated morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are separated. Then $g \circ f$ is separated.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is separted. Then f is separated.*
(iii) *Separated morphisms are stable under base-changes.*
(iv) *Being separated is local on the targets.*

Proof. For (i) and (ii), we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[dr, "\Delta_{g\circ f}"'] & X \times_Y X \arrow[r] \arrow[d, "\Delta'_g"] & Y \arrow[d, "\Delta_g"] \\ & X \times_Z X \arrow[r] & Y \times_Z Y. \end{tikzcd}$$

Moreover, one can check the right square is Cartesian.

If f and g are separated, then Δ_f and Δ_g are closed immersions. By Corollary 15.1.3, Δ'_g is also a closed immersion. Hence $\Delta_{g\circ f} \simeq \Delta'_g \circ \Delta_f$ is also a closed immersion. This proves (i).

If $g \circ f$ is separated, then $\Delta_{g\circ f} \simeq \Delta'_g \circ \Delta_f$ is a closed immersion. In particular $\Delta'_g(\Delta_f(X))$ is a closed subset of $X \times_Z X$.

Note that Δ'_g is a locally closed immersion by Corollary 15.1.4 and Lemma 18.2.2. Hence Δ'_g induces a homeomorphism from $X \times_Y X$ to its image. Combining with the last paragraph, we obtain that $\Delta_f(X)$ is a closed subset of $X \times_Y X$. By Corollary 18.2.5, this implies (ii).

For (iii) and (iv), note that for a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Delta_{f'}"] \arrow[d] & X' \times_{Y'} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \arrow[r] & Y \end{tikzcd}$$

such that the right square and the outer square are both Cartesian. Hence the left square is also Cartesian. Now (iii) follows from Corollary 15.1.3, while (iv) follows from the fact that being a closed immersion is local on the target.

□

Exercise 18.2.8. A morphism out of an affine scheme is separated.

Exercise 18.2.9. Let X be a separated scheme. Show that the intersection of two affine open subsets of X is affine.

Definition 18.2.10. Let $f : X \to Y$ be a morphism between S-schemes. The *graph morphism (relative to S)*

$$\Gamma_f : X \to X \underset{S}{\times} Y$$

is defined to be the morphism corresponding to the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="] & Y \arrow[d] \\ X \arrow[r] & S. \end{tikzcd}$$

Exercise 18.2.11. Let $f : X \to Y$ be a morphism between S-schemes.

(1) Show that the graph morphism $\Gamma_f : X \to X \times_S Y$ is a locally closed immersion.
(2) If $q : Y \to S$ is separated, show that Γ_f is a closed immersion. In particular, any section $S \to Y$ to q is a closed immersion.

18.3. **Quasi-separated morphisms.**

Definition 18.3.1. Let $f : X \to Y$ be a morphism between schemes. We say f is **quasi-separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. We say a scheme X is **quasi-separated** if $X \to \mathsf{Spec}(\mathbb{Z})$ is quasi-separated.

By Exercise 18.1.5, we have

Corollary 18.3.2. *A separated morphism is quasi-separated.*

Lemma 18.3.3. *Let X be a scheme. The following conditions are equivalent:*

(i) *The scheme X is quasi-separated.*
(ii) *The intersection of any pair of quasi-compact open subsets of X is quasi-compact.*
(iii) *The intersection of any pair of affine open subsets of X is quasi-compact.*

Proof. For (i)⇒(ii), suppose X is quasi-separated. Let U_1 and U_2 be quasi-compact open subsets of X. By Lemma 18.1.4, $U_1 \times U_2$ is also quasi-compact. Note that by assumption $\Delta : X \to X \times X$ is quasi-compact, hence $U_1 \cap U_2 \simeq \Delta^{-1}(U_1 \times U_2)$ is quasi-compact as desired.

(ii)⇒(iii) is obvious.

For (iii)⇒(i), suppose the intersection of any pair of quasi-compact open subsets of X is quasi-compact. Let $X = \bigcup_{i \in I} U_i$ be a covering of X by affine open subschemes. Then $X \times X$ can be covered by its affine open subschemes $U_i \times U_j$, $(i, j) \in I^2$. By Lemma 18.1.4, we only need to show

$$U_i \cap U_j \simeq X \underset{X \times X}{\times} (U_i \times U_j) \to U_i \times U_j$$

is quasi-compact. But this follows from the assumption that $U_i \cap U_j$ is quasi-compact (Lemma 18.1.3).

□

The proof of the following results is similar to that of Lemma 18.2.7. We leave it to the readers.

Lemma 18.3.4. *Consider the class of quasi-separated morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are quasi-separated. Then $g \circ f$ is quasi-separated.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is quasi-separted. Then f is quasi-separated.*
(iii) *Quasi-separated morphisms are stable under base-changes.*
(iv) *Being quasi-separated is local on the targets.*

Exercise 18.3.5. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is quasi-separated and $g \circ f$ is quasi-compact. Show that f is quasi-compact.

18.4. **Direct image along qcqs morphisms.**

Theorem 18.4.1. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then the functor $f_* : \mathcal{O}_X\text{−}\mathsf{mod} \to \mathcal{O}_Y\text{−}\mathsf{mod}$ sends quasi-coherent $\mathcal{O}_X$-modules to quasi-coherent $\mathcal{O}_Y$-modules. Moreover, the obtained functor*

$$f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y).$$

commutes with (small) coproducts and filtered colimits.

Proof. We first show the claims are local on Y. This follows from the following observations:

- The class of quasi-compact quasi-separated morphisms are closed under base-changes.
- For any $\mathcal{M} \in \mathcal{O}_X\text{−}\mathsf{mod}$ and open subscheme $U \in Y$, $f_*(\mathcal{M})|_U \simeq f'_*(\mathcal{M}|_V)$, where $f' : V \to U$ is the base-change of $f : X \to Y$.
- For an open covering $Z = \bigcup_{i\in I} Z_i$ of schemes, the functors $(-)|_{Z_i} : \mathcal{O}_Z\text{−}\mathsf{mod} \to \mathcal{O}_{Z_i}\text{−}\mathsf{mod}$ preserve and detect quasi-coherent modules.
- For an open covering $Z = \bigcup_{i\in I} Z_i$ of schemes, the functors $(-)|_{Z_i} : \mathsf{QCoh}(Z) \to \mathsf{QCoh}(Z_i)$ preserve and detect small coproducts (in fact, any small colimits).

As a consequence, we can assume $Y = \mathsf{Spec}(A)$ is affine, and therefore X is quasi-compact and quasi-separated.

We now reduce to the case when X is quasi-compact and *separated*. For this purpose, let $X = \bigcup_{i\in I} U_i$ be a finite covering of X by its affine open subschemes. Write $U_{ij} \coloneqq U_i \cap U_j$, which is quasi-compact by Lemma 18.3.3. Note that U_{ij} is also separated because U_i is separated (Exercise 18.2.8) and $U_{ij} \to U_i$ is separated (Corollary 18.2.6). Let $\mathcal{M} \in \mathsf{QCoh}(X)$ be a quasi-coherent $\mathcal{O}_X$-module. Recall we have

$$\mathcal{M} \simeq \mathsf{ker}\Big(\prod_{i\in I} g_{i,*}(\mathcal{M}|_{U_i}) \to \prod_{(i,j)\in I^2} g_{ij,*}(\mathcal{M}|_{U_{ij}})\Big),$$

where $g_i : U_i \to X$ and $g_{ij} : U_{ij} \to X$ are the open immersions. It follows that

$$f_*\mathcal{M} \simeq \mathsf{ker}\Big(\prod_{i\in I} (f \circ g_i)_*(\mathcal{M}|_{U_i}) \to \prod_{(i,j)\in I^2} (f \circ g_{ij})_*(\mathcal{M}|_{U_{ij}})\Big).$$

By Corollary 13.4.3, to show $f_*\mathcal{M}$ is quasi-coherent, we only need to show each $(f \circ g_i)_*(\mathcal{M}|_{U_i})$ and $(f \circ g_{ij})_*(\mathcal{M}|_{U_{ij}}))$ is quasi-coherent. Moreover, Corollary 13.4.3 also implies coproducts (resp. filtered colimits) commute with finite limits in $\mathsf{QCoh}(Z)$. Hence to show the obtained functor $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ commutes with coproducts (resp. filtered colimits), we only need to show each $(f \circ g_i)_*$ and

$(f \circ g_{ij})_*$ does. As a consequence, we can replace X with U_i or U_{ij}, and therefore assume X is quasi-compact and separated.

We now repeat the above paragraph. Note that under the new assumption that X is separated, $U_{ij} \simeq U_i \times_X U_j \to U_i \times U_j$ is a closed immersion because it is a base-change of $\Delta : X \to X \times X$. Since $U_i \times U_j$ is affine, so is U_{ij} (Corollary 15.1.2). As a consequence, after replacing X with U_i or U_{ij}, we can assume X is *affine*.

Now the desired claim follows from Corollary 13.3.4 and the fact that for any homomorphism $\phi : A \to B$, the restriction functor $B\text{-}\mathsf{mod} \to A\text{-}\mathsf{mod}$ commutes with small coproducts and filtered colimits.

□

Exercise 18.4.2. Consider $\mathbb{A}^\infty_{\mathbb{Z}} := \mathsf{Spec}(\mathbb{Z}[t_1, t_2, \cdots])$ and its closed subscheme Z corresponding to the ideal $(t_1, t_2, \cdots)$. Let $U := \mathbb{A}^\infty_{\mathbb{A}} \smallsetminus Z$ be the complementary open subscheme. Define X to be the scheme glued from two pieces of $\mathbb{A}^\infty_{\mathbb{Z}}$ via the identity morphism on U.

(1) Show that X is quasi-compact, but is not quasi-separated.
(2) Consider the unique morphism $p : X \to \mathsf{Spec}(\mathbb{Z})$. Show that p_* does not preserve quasi-coherent modules.

Corollary 18.4.3. *Let X be a quasi-compact and quasi-separated scheme. Then the functor*

$$\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}$$

commutes with (small) coproducts and filtered colimits.

Exercise 18.4.4. Let X be a quasi-compact and quasi-separated scheme, and $\mathcal{M}$ be an $\mathcal{O}_X$-module that is of finite presentation. Then the functor

$$\mathsf{Hom}_{\mathcal{O}_X}(\mathcal{M}, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}$$

preserves filtered colimits.

18.5. **Application: closure of subschemes.**

Lemma 18.5.1. *Let $i : Y \to X$ be a* quasi-compact *locally closed immersion. Then $i_*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By Corollary 18.2.6, the morphism i is separated. Now the claim follows from Theorem 18.4.1.

□

Construction 18.5.2. Let $i : Y \to X$ be a *quasi-compact* locally closed immersion. Consider the quasi-coherent ideal $\mathcal{I} := \mathsf{ker}(\mathcal{O}_X \to i_*\mathcal{O}_Y)$ and

$$\overline{Y} := \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{I}).$$

Note that $\overline{Y}$ is a closed subscheme of X. By Theorem 14.2.4, the homomorphism $\mathcal{O}_X/\mathcal{I} \to i_*\mathcal{O}_Y$ induces an X-morphism

$$j : Y \to \overline{Y}.$$

Proposition-Definition 18.5.3. *Let $i : Y \to X$ be a* quasi-compact *locally closed immersion. Then*

(1) *The canonical morphism $j : Y \to \overline{Y}$ is an open immersion.*
(2) *If $Z \to X$ is a closed immersion such that $i : Y \to X$ factors through Z, then $\overline{i} : \overline{Y} \to X$ also factors through Z.*

We call $\overline{Y}$ the ***(scheme theoretic) closure of*** Y ***in*** X.

Proof. For (1), choose an open subscheme $U \subseteq X$ such that $i(Y)$ is a closed subset of U. Write $i' : Y \to U$ for the corresponding closed immersion. The exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_* \mathcal{O}_Y$$

indcues an exact sequence

$$0 \to \mathcal{I}|_U \to \mathcal{O}_X|_U \to (i_* \mathcal{O}_Y)|_U.$$

Note that we have $\mathcal{O}_X|_U \simeq \mathcal{O}_U$ and $(i_*\mathcal{O}_Y)|_U \simeq i'_*\mathcal{O}_Y$. Hence the above exact sequence gives a short exact sequence

$$0 \to \mathcal{I}|_U \to \mathcal{O}_U \to i'_*\mathcal{O}_Y \to 0,$$

where we used the fact that i'_* is a closed immersion. In other words, we have

$$(\mathcal{O}_X/\mathcal{I})|_U \simeq \mathcal{O}_U/\mathcal{I}|_U \simeq i'_*\mathcal{O}_Y.$$

Hence

$$\overline{Y} \underset{X}{\times} U \simeq \mathsf{Spec}_U((\mathcal{O}_X/\mathcal{I})|_U) \simeq \mathsf{Spec}_U(i'_*\mathcal{O}_Y) \simeq Y.$$

In other words, the morphism $j : Y \to \overline{Y}$ can be identified with the base-change of $U \to X$ along $\overline{Y} \to X$. This implies j is an open immersion.

For (2), let $\mathcal{J}$ be the ideal of definition for $Z \to X$ such that $Z \simeq \mathsf{Spec}_X(\mathcal{O}_X/\mathcal{J})$. By Theorem 14.2.4, the X-morphism $Y \to Z$ corresponds to an $\mathcal{O}_X$-homomorphism $\mathcal{O}_X/\mathcal{J} \to i_*\mathcal{O}_Y$. In particular,

$$\mathcal{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Y) \subseteq \mathcal{J}.$$

By Theorem 15.1.1, this gives an X-morphism $\overline{Y} \to Z$.

□

Exercise 18.5.4. Show that $j(Y)$ is dense in $\overline{Y}$. Hint: consider $\mathcal{I}|_{X \smallsetminus Z}$, where Z is the *set-theoretic closure of* $i(Y)$.

Warning 18.5.5. Proposition-Definition 18.5.3(1) may fail if $i : Y \to X$ is not quasi-compact.

18.6. Extension of modules of finite type.

Theorem 18.6.1. *Let X be a quasi-compact and quasi-separated scheme and $j : U \to X$ be a quasi-compact open immersion. Then the following statement is true:*

- (*) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, and any quasi-coherent $\mathcal{O}_U$-submodule $\mathcal{M} \subseteq j^*\mathcal{F}$ that is of finite type, there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{M}' \subseteq \mathcal{F}$ that is of finite type, such that $\mathcal{M} \simeq j^*\mathcal{M}'$.*

To prove the theorem, we need some preparations.

Lemma 18.6.2. *Let X be a scheme and $\mathcal{F} \in \mathsf{QCoh}(X)$. Let $\mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}$ be the category of subobjects of $\mathcal{F}$ that are quasi-coherent and of finite type. Then $\mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}$ is filtered.*

Proof. Note that $\mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}$ admits finite coproducts. Indeed, for a finite set I, the coproducts of $\mathcal{F}_i \subseteq \mathcal{F}$, $i \in I$ can be calculated as the image of

$$\bigoplus_{i \in I} \mathcal{F}_i \to \mathcal{F},$$

which is indeed of finite type because $\bigoplus_{i\in I}\mathcal{F}_i$ is so. Now the claim follows from the general fact that if a category admits finite coproducts, then it is filtered.

□

Lemma 18.6.3. *Let X be a scheme and $\mathcal{F}\in\mathsf{QCoh}(X)$. The canonical morphism*

$$\operatorname*{colim}_{\mathcal{F}'\in\mathsf{Sub}(\mathcal{F})_{\mathrm{ft}}}\mathcal{F}'\to\mathcal{F} \tag{18.1}$$

is an injection. If X is affine, this morphism is an isomorphism.

Proof. The second claim is obvious. For the first one, we only need to show that for any affine open subset $U\subseteq X$,

$$(\operatorname*{colim}_{\mathcal{F}'\in\mathsf{Sub}(\mathcal{F})_{\mathrm{ft}}}\mathcal{F}')(U)\to\mathcal{F}(U) \tag{18.2}$$

is an injection. Note that we have

$$(\operatorname*{colim}_{\mathcal{F}'\in\mathsf{Sub}(\mathcal{F})_{\mathrm{ft}}}\mathcal{F}')(U)\simeq\operatorname*{colim}_{\mathcal{F}'\in\mathsf{Sub}(\mathcal{F})_{\mathrm{ft}}}\mathcal{F}'(U)$$

and

$$\mathcal{F}(U)\simeq\operatorname*{colim}_{\mathcal{F}'\in\mathsf{Sub}(\mathcal{F})_{\mathrm{ft}}}\mathcal{F}(U).$$

Hence (18.2) can be identified with the filtered colimit of the injective morphisms $\mathcal{F}'(U)\to\mathcal{F}(U)$. This implies (18.2) is injective as desired.

□

Lemma 18.6.4. *Let X be a scheme such that for any $\mathcal{F}\in\mathsf{QCoh}(X)$, the canonical morphism* (18.1) *is an isomorphism. Let $j:U\to X$ be a quasi-compact open immersion such that U is quasi-compact. Then (*) in Theorem 18.6.1 is true.*

Proof. Note that j_* preserves quasi-coherent modules by Theorem 18.4.1. Let $\mathcal{N}\in\mathsf{QCoh}(X)$ be the following fiber product:

$$\begin{tikzcd} \mathcal{N} \arrow[r, dotted] \arrow[d, dotted] & j_*\mathcal{M} \arrow[d] \\ \mathcal{F} \arrow[r] & j_*j^*\mathcal{F}, \end{tikzcd} \tag{18.3}$$

where the right vertical morphism is induced by the injection $\mathcal{M}\to j^*\mathcal{F}$. Since j_* is left exact, $j_*\mathcal{M}\to j_*j^*\mathcal{F}$ is also an injection. It follows that $\mathcal{N}\to\mathcal{F}$ is an injection.

Applying j^* to (18.3), we obtain a Cartesian square

$$\begin{tikzcd} j^*\mathcal{N} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ j^*\mathcal{F} \arrow[r] & j^*\mathcal{F}. \end{tikzcd}$$

This implies $j^*\mathcal{N}\to\mathcal{M}$ is an isomorphism. Now the desired claim follows from Lemma 18.6.5 below.

□

Lemma 18.6.5. *Let X be a scheme such that for any $\mathcal{F}\in\mathsf{QCoh}(X)$, the canonical morphism* (18.1) *is an isomorphism. Let $j:U\to X$ be an open immersion such that U is quasi-compact. Then the following statement is true:*

(*) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{N}$ such that $j^*\mathcal{N}$ is of finite type, there exists a quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{M}' \subseteq \mathcal{N}$ that is of finite type such that $j^*\mathcal{M}' \simeq j^*\mathcal{N}$.*

Proof. By assumption, we have

$$\operatorname*{colim}_{\mathcal{N}' \in \mathsf{Sub}(\mathcal{N})_{\mathsf{ft}}} \mathcal{N}' \xrightarrow{\simeq} \mathcal{N}.$$

This implies that for any affine open subset $V \subseteq U$, and for any section

$$s \in (j^*\mathcal{N})(V) \simeq \mathcal{N}(V),$$

there exists $\mathcal{N}' \in \mathsf{Sub}(\mathcal{N})_{\mathsf{ft}}$ such that s is in the image of

$$(j^*\mathcal{N}')(V) \to (j^*\mathcal{N})(V).$$

Since $\mathsf{Sub}(\mathcal{N})_{\mathsf{ft}}$ is filtered, and since U is quasi-compact and $\mathcal{N}$ is of finite type, we see that there exists a large enough object $\mathcal{M}' \in \mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}$ such that $j^*\mathcal{M}' \to j^*\mathcal{N}$ is a surjection. It is also an injection because j^* is exact.

□

Proof of Theorem 18.6.1. Let $X = \bigcup_{i=1}^n X_i$ be a finite covering of X by affine open subsets $X_i \subseteq X$. For any $i \in \{0, 1 \cdots, n\}$, write

$$X_{\leq i} := U \cup (\bigcup_{j=1}^{i} X_j)$$

so that we have

$$U = X_{\leq 0} \subseteq X_{\leq 1} \subseteq \cdots \subseteq X_{\leq n} = X.$$

Write $\mathcal{M}'_0 := \mathcal{M}$. We will use induction to construct

$$\mathcal{M}'_i \in \mathsf{Sub}(\mathcal{F}|_{X_{\leq i}})_{\mathsf{ft}}$$

such that

$$\mathcal{M}'_{i+1}|_{X_{\leq i}} = \mathcal{M}'_i.$$

Then $\mathcal{M}' := \mathcal{M}'_n$ satisfies the desired requirements.

Suppose that $\mathcal{M}'_i$ is constructed for $0 \leq i < n$. To construct $\mathcal{M}'_{i+1}$, we only need to extend the object

$$\mathcal{M}'_i|_{X_{i+1} \cap X_{\leq i}} \in \mathsf{Sub}(\mathcal{F}|_{X_{i+1} \cap X_{\leq i}})_{\mathsf{ft}}$$

to an object in

$$\mathsf{Sub}(\mathcal{F}|_{X_{i+1}})_{\mathsf{ft}}.$$

In other words, we only need to prove the statement (*) for the affine scheme X_{i+1} and the open immersion

$$X_{i+1} \cap X_{\leq i} \to X_{i+1}.$$

Note that

$$X_{i+1} \cap X_{\leq i} = (X_{i+1} \cap U) \cup (\bigcup_{j=1}^{i} (X_{i+1} \cap X_j))$$

is quasi-compact becasue

- U is quasi-compact,
- each X_j is quasi-compact,
- X is quasi-separated.

Hence Lemma 18.6.4 implies the desired claim.

□

Theorem 18.6.6. *Let X be a quasi-compact and quasi-separated scheme and $\mathcal{F} \in \mathsf{QCoh}(X)$. The canonical morphism*

$$\operatorname*{colim}_{\mathcal{F}' \in \mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}} \mathcal{F}' \to \mathcal{F} \tag{18.4}$$

is an isomorphism.

Proof. By Lemma 18.6.3, we only need to show that (18.4) is a surjection. Hence we only need to show that for any affine open subset $U \subseteq X$,

$$\operatorname*{colim}_{\mathcal{F}' \in \mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}} \mathcal{F}'|_U \to \mathcal{F}|_U$$

is a surjection. Since U is affine, we have

$$\operatorname*{colim}_{\mathcal{M} \in \mathsf{Sub}(\mathcal{F}|_U)_{\mathsf{ft}}} \mathcal{M} \simeq \mathcal{F}|_U.$$

Now the claim follows from Theorem 18.6.1, which says any $\mathcal{M} \in \mathsf{Sub}(\mathcal{F}|_U)_{\mathsf{ft}}$ can be extended to an object $\mathcal{F}' \in \mathsf{Sub}(\mathcal{F})_{\mathsf{ft}}$.

□

19. Reduced schemes

19.1. Definition.

Definition 19.1.1. Let X be a scheme. We say X is reduced if for any open subset $U \subseteq X$, the commutative ring $\mathcal{O}_X(U)$ is reduced.

The following result follows from the sheaf condition for $\mathcal{O}_X$.

Lemma 19.1.2. *A scheme X is reduced iff for some topological base $\mathfrak{B}$ of X, the commutative ring $\mathcal{O}_X(U)$ is reduced for any $U \in \mathfrak{B}$.*

Corollary 19.1.3. *An affine scheme $\mathsf{Spec}(A)$ is reduced iff A is reduced.*

Lemma 19.1.4. *A scheme X is reduced iff the local ring $\mathcal{O}_{X,x}$ is reduced for any $x \in X$.*

Proof. The claim is local on X and the affine case is well-known. □

19.2. **Reduced subschemes.** Let X be a scheme and $Z \subseteq X$ be a locally closed subset. Consider the category $\mathsf{SubSch}_{X,Z}$ of locally closed immersions $i : Y \to X$ such that $i(Y) = Z$.

Proposition-Definition 19.2.1. *There is an essentially unique object $Z_{\mathsf{red}} \to X$ in $\mathsf{SubSch}_{X,Z}$ such that Z_{red} is reduced. Moreover, it is an initial object in this category. We call Z_{red} the* ***reduced subscheme of** X **on** Z.*

Proof. Let $U \subseteq X$ be an open subscheme such that Z is a closed subset of U. Note that the obvious functor $\mathsf{SubSch}_{U,Z} \to \mathsf{SubSch}_{X,Z}$ is an equivalence. Hence we can replace X with U and therefore assume Z is closed in X.

For the case $X = \mathsf{Spec}(A)$ is affine, choose an ideal $J \subseteq A$ such that $Z = Z(J)$. By Theorem 15.1.1, the category $\mathsf{SubSch}_{X,Z}$ is equivalent to the opposite category of ideals $I \subseteq A$ such that $Z(I) = Z(J)$. The last condition is equivalent to $\sqrt{I} = \sqrt{J}$. Hence $\mathsf{Spec}(A/\sqrt{J}) \to \mathsf{Spec}(A)$ is the unique reduced object in this category, moreover it is an initial object because any I as above satisfies $I \subseteq \sqrt{J}$.

We now prove the general case. For any affine open subscheme $U \subseteq X$, consider the reduced closed subscheme $(U \cap Z)_{\mathsf{red}} \to U$. If $U \subseteq V$ are two affine open subschemes, one can check

$$(V \cap Z)_{\mathsf{red}} \underset{V}{\times} U \to U$$

is a reduced closed subscheme whose image is the topological space $U \cap Z$. Hence we have an isomorphism between closed subschemes of U:

$$(V \cap Z)_{\mathsf{red}} \underset{V}{\times} U \simeq (U \cap Z)_{\mathsf{red}}.$$

This implies there is a unique closed subscheme $Z_{\mathsf{red}} \to X$ such that

$$Z_{\mathsf{red}} \underset{X}{\times} U \simeq (U \cap Z)_{\mathsf{red}}$$

as closed subschemes of U. By Lemma 19.1.2, Z_{red} is indeed reduced.

It remains to show the object $Z_{\mathsf{red}} \to X$ constructed as above satisfies the desired properties. Let $Y \to X$ be any closed immersion with image Z. We only need to show that there is a unique X-morphism $Z_{\mathsf{red}} \to Y$, and this morphism is an isomorphism if Y is reduced. One can check that this claim is local on X, hence it follows from the affine case, which has been proved.

□

In particular, there is a unique reduced subscheme $X_{\mathsf{red}} \to X$ whose underlying topological space is X.

In the proof of Proposition 19.2.1, we have actually shown the following result.

Lemma 19.2.2. *Let X be a scheme and $U \subseteq X$ be an affine open subset. Then the canonical morphism $\mathcal{O}_X \to \mathcal{O}_{X_{\mathsf{red}}}$ induces an isomorphism*

$$\mathcal{O}_X(U)_{\mathsf{red}} \xrightarrow{\simeq} \mathcal{O}_{X_{\mathsf{red}}}(U).$$

Warning 19.2.3. The claim of the above lemma may fail for general open subset U. For example, let k be a field and $X = U = \sqcup_{n>0}\mathsf{Spec}(k[t]/(t^n))$. Then $X_{\mathsf{red}} \simeq \sqcup_{n>0}\mathsf{Spec}(k)$ and therefore

$$\mathcal{O}_{X_{\mathsf{red}}}(X) \simeq \prod_{n>0} k.$$

On the other hand

$$\mathcal{O}_X(U)_{\mathsf{red}} \simeq (\prod_{n>0} k[t]/(t^n))_{\mathsf{red}},$$

and the canonical morphism $\mathcal{O}_X(U)_{\mathsf{red}} \to \mathcal{O}_{X_{\mathsf{red}}}(X)$ is given by

$$(\prod_{n>0} k[t]/(t^n))_{\mathsf{red}} \xrightarrow{t \mapsto 0} \prod_{n>0} k.$$

Note however that this is not an isomorphism. Indeed, $(t, t, \cdots)$ is a nonzero element in the kernel.

Lemma 19.2.4. *Let X be a scheme and $x \in X$ be a point. We have $\mathcal{O}_{X_{\mathsf{red}},x} \simeq (\mathcal{O}_{X,x})_{\mathsf{red}}$.*

Proof. The claim in local on X hence we can assume $X = \mathsf{Spec}(A)$ is affine. Now the claim is a well-known fact in commutative algebra.

□

Lemma 19.2.5. *Let X and Y be S-schemes. The canonical morphism*

$$X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y$$

is a closed immersion and induces a homeomorphism between the topological spaces. In particular

$$(X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}})_{\mathsf{red}} \simeq (X \underset{S}{\times} Y)_{\mathsf{red}}.$$

Proof. The morphism is a closed immersion because it can be written as the composition

$$X_{\mathsf{red}} \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y_{\mathsf{red}} \to X \underset{S}{\times} Y,$$

and each of the two morphisms is a base-change of a closed immersion. To show this morphism induces a homeomorphism, note that this claim is local in X, Y and S. Hence we can assume $X = \mathsf{Spec}(A)$, $Y = \mathsf{Spec}(B)$ and $S = \mathsf{Spec}(R)$. Then the desired morphism is given by $A \otimes_R B \to A_{\mathsf{red}} \otimes_R B_{\mathsf{red}}$, which is a surjection with a nilpotent kernel. It follows that $\mathsf{Spec}(A_{\mathsf{red}} \otimes_R B_{\mathsf{red}}) \to \mathsf{Spec}(A \otimes_R B)$ is a homeomorphism.

□

Warning 19.2.6. Products of reduced schemes may fail to be reduced. For example, $\mathbb{Z}[t]/(t^2 - p) \otimes \mathbb{F}_p \simeq \mathbb{F}_p[t]/(t^2)$ is not reduced.

19.3. Morphisms between reduced subschemes.

Exercise 19.3.1. Let $f : X \to Y$ be a morphism between schemes. If X is a reduced scheme, then f uniquely factors through Y_{red}. Hint: reduce to the case when Y is affine.

Corollary 19.3.2. *Let $f : X \to Y$ be a morphism between schemes. There is a unique moprhism $f_{\mathsf{red}} : X_{\mathsf{red}} \to Y_{\mathsf{red}}$ such that the following diagram commutes:*

$$\begin{tikzcd} X_{\mathsf{red}} \arrow[r, dotted, "f_{\mathsf{red}}"] \arrow[d] & Y_{\mathsf{red}} \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Exercise 19.3.3. If $f : X \to Y$ be an open (resp. closed, locally closed) immersion, so is $f_{\mathsf{red}} : X_{\mathsf{red}} \to Y_{\mathsf{red}}$.

Exercise 19.3.4. Show that f is quasi-compact (resp. separated, quasi-separated) iff f_{red} is so.

20. (Quasi-)affine morphisms

20.1. **Affine morphisms.**

Definition 20.1.1. We say a morphism $f : X \to Y$ between schemes is **affine** if for any affine open subscheme $U \subseteq Y$, its inverse image $X \times_Y U$ is affine.

Example 20.1.2. Let S be an affine scheme. A morphism $f : X \to S$ is affine iff X is affine.

Lemma 20.1.3. *An affine morphism $f : X \to Y$ is quasi-compact and separated.*

Proof. By Lemma 18.1.4(iii) and Lemma 18.2.7(iv), we can assume Y is affine. Since f is affine, we see that X is affine. Now f is quasi-compact by Lemma 18.1.3 and is separated by Exercise 18.2.8.

□

Proposition 20.1.4. *Let S be a scheme. The following functors are well-defined and inverse to each other:*

$$\begin{array}{ccc} \mathcal{O}_S\text{-}\mathsf{alg}_{\mathsf{qcoh}} & \rightleftarrows & \{\textit{affine morphisms } p : X \to S\}^{\mathsf{op}} \\ \mathcal{A} & \mapsto & (\mathsf{Spec}_S(\mathcal{A}) \to S) \\ p_*\mathcal{O}_X & \leftarrow\!\shortmid & (p : X \to S) \end{array} \tag{20.1}$$

Proof. The leftward functor is well-defined by Lemma 20.1.3 and Theorem 18.4.1.

To show the rightward functor is well-defined, we only need to show $\mathsf{Spec}_S(\mathcal{A}) \to S$ is affine. Let $U \subseteq S$ be an affine open subscheme, we have

$$\mathsf{Spec}_S(\mathcal{A}) \underset{S}{\times} U \simeq \mathsf{Spec}_U(\mathcal{A}|_U) \simeq \mathsf{Spec}(\mathcal{A}(U)),$$

which is indeed an affine scheme.

By Corollary 14.2.5, (20.1) is an adjoint pair, and the left adjoint is fully faihtful. Hence we only need to show that for any affine morphism $p : X \to S$, the S-morphism

$$X \to \mathsf{Spec}_S(p_*\mathcal{O}_X)$$

that corresponds to the identity morphism $p_*\mathcal{O}_X$ is an isomorphism. This claim is local on S, hence we can assume S is affine, and thereby X is affine. Now the above morphism becomes the canonical morphism

$$X \to \mathsf{Spec}(\mathcal{O}_X(X)),$$

which is indeed an isomorphism.

□

Corollary 20.1.5. *Any closed immersion is affine.*

Lemma 20.1.6. *Consider the class of affine morphisms.*

- (i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are affine. Then $g \circ f$ is affine.*
- (ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is separated and $g \circ f$ is affine. Then f is affine.*
- (iii) *Affine morphisms are stable under base-changes.*
- (iv) *Being affine is local on the targets.*

Proof. (i) is obvious.

(iii) follows from Proposition 20.1.4 and Exercise 14.2.6.

To prove (ii), we factor f as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{\mathsf{pr}_2} Y$, where the first morphism is the graph of f relative to Z and pr_2 is the obvious projection. By Exercise 18.2.11(2), Γ_f is a closed immersion and therefore affine (Corollary 20.1.5). On the other hand, pr_2 is the base-change of $g \circ f : X \to Z$ along $Y \to Z$. By (iii), pr_2 is affine. It follows that f is affine because it is the composition of two affine morphisms.

To prove (iv), let $f : X \to Y$ be a moprhism and $Y = \bigcup_{i \in I} Y_i$ be an open covering such that each $f_i : X \times_Y Y_i \to Y_i$ is affine. By Lemma 20.1.3, each f_i is quasi-compact and separated. By Lemma 18.1.4(iii) and Lemma 18.2.7(iv), f is quasi-compact and separated. In particular, $f_* \mathcal{O}_X$ is quasi-coherent. By Corollary 14.2.5 and Proposition 20.1.4, we only need to show the Y-morphism

$$X \to \mathsf{Spec}_Y(f_* \mathcal{O}_X) \tag{20.2}$$

that corresponds to the identity morphism on $f_* \mathcal{O}_X$ is an isomorphism. The assumption implies the base-change of (20.2) to each Y_i is an isomorphism. It follows that (20.2) is indeed an isomorphism.

□

20.2. **Quasi-coherent modules and affine morphisms.**

Construction 20.2.1. Let S be a scheme and $f : X \to S$ be an affine morphism. Write $\mathcal{A} := f_* \mathcal{O}_X$, which is a quasi-coherent $\mathcal{O}_S$-algebra. Consider the morphism $(X, \mathcal{O}_X) \to (S, \mathcal{A})$ between ringed spaces. Pullback and pushforward along this morphism provide an adjoint pair

$$\begin{array}{ccc} \mathcal{A}\text{–}\mathsf{mod} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod} \\ \mathcal{M} & \mapsto & \widetilde{\mathcal{M}} \\ f_* \mathcal{F} & \leftarrow\!\shortmid & \mathcal{F}, \end{array}$$

where recall

$$\widetilde{\mathcal{M}} \simeq \mathcal{O}_X \underset{f^{-1}\mathcal{A}}{\otimes} f^{-1}\mathcal{M}.$$

Let

$$\mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} \subseteq \mathcal{A}\text{–}\mathsf{mod}$$

be the full subcategory consisting of $\mathcal{F} \in \mathcal{A}\text{–}\mathsf{mod}$ such that $\mathcal{F}$ is quasi-coherent as an $\mathcal{O}_S$-module.

Proposition 20.2.2. *In the above setting, the following functors are well-defined and inverse to each other:*

$$\begin{array}{ccc} \mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} & \rightleftarrows & \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qcoh}} \\ \mathcal{M} & \mapsto & \widetilde{\mathcal{M}} \\ f_* \mathcal{F} & \leftarrow\!\shortmid & \mathcal{F}, \end{array}$$

Proof. The claim is local on S and we can assume $S = \mathsf{Spec}(R)$. Recall that $\mathsf{QCoh}(S) \simeq R\text{–}\mathsf{mod}$. Write $A := \mathcal{A}(S)$ and view it as a commutative algebra object in $R\text{–}\mathsf{mod}$. It follows that $\mathcal{A}\text{–}\mathsf{mod}_{\mathsf{qcoh}} \simeq A\text{–}\mathsf{mod}(R\text{–}\mathsf{mod}) \simeq A\text{–}\mathsf{mod}$. Now the desired equivalence becomes that in Corollary 13.1.2.

□

Exercise 20.2.3. Let S be a scheme and $f : X \to S$ be an affine morphism. Write $\mathcal{A} := f_*\mathcal{O}_X$. Show that

$$f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{F}') \simeq f_*(\mathcal{F}) \underset{\mathcal{A}}{\otimes} f_*(\mathcal{F}').$$

20.3. **Serre's criterion: statement.**

Theorem 20.3.1 (Serre)**.** *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism between schemes. The following conditions are equivalent:*

(1) *The morphism f is affine.*
(2) *The functor*

$$f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$$

is exact.

Proof. We claim both conditions are local on Y. For (1), this is Lemma 20.1.6(iv). For (2), let $Y = \sqcup_{i \in I} Y_i$ be an open covering and $f_i : X_i \to Y_i$ be the base-change of f along $Y_i \to Y$. For $\mathcal{F} \in \mathsf{QCoh}(X)$, it is easy to see that

$$f_*(\mathcal{F})|_{Y_i} \simeq f_{i,*}(\mathcal{F}|_{X_i}).$$

Note that the restriction functors $\mathsf{QCoh}(Y) \to \mathsf{QCoh}(Y_i)$ and $\mathsf{QCoh}(X) \to \mathsf{QCoh}(X_i)$ $(i \in I)$ preserve and detect exact sequences of quasi-coherent modules. It follows that f_* is exact iff each $f_{i,*}$ is exact. In other words, condition (2) is local on Y.

By the previous discussion, we can assume $Y = \mathsf{Spec}(R)$ is affine and therefore X is quasi-compact and quasi-separated. Recall we have $\mathsf{QCoh}(Y) \simeq R\text{-}\mathsf{mod}$ and the forgetful functor $R\text{-}\mathsf{mod} \to \mathsf{Ab}$ preserves and detect exact sequences. It follows that f_* is exact iff the functor

$$\mathsf{QCoh}(X) \xrightarrow{f_*} \mathsf{QCoh}(Y) \simeq R\text{-}\mathsf{mod} \to \mathsf{Ab}$$

is exact. By definition, this functor is just

$$\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}.$$

Now the claim follows from Theorem 20.3.2 below.

□

Theorem 20.3.2 (Serre)**.** *Let X be a quasi-compact scheme. The following conditions are equivalent:*

(1) *The scheme X is affine.*
(2) *The functor*

$$\Gamma(X, -) : \mathsf{QCoh}(X) \to \mathsf{Ab}$$

is exact.

20.4. **Principle open subschemes.**

Proposition-Definition 20.4.1. *Let X be a scheme and $f \in \mathcal{L}(X)$ be a global section. For any $x \in X$, consider the germ $f_x \in \mathcal{O}_{X,x}$ of f at x. Consider*

$$X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x\},$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the unique maximal ideal. Then X_f is an open subset of X. We call it the ***principle open subset*** *corresponding to f.*

Proof. The claim in local in X. Hence we can reduce to the case when X is affine, where the claim is obvious.

□

Remark 20.4.2. When $X = \mathsf{Spec}(A)$ is affine, we have $X_f = U(f)$.

Exercise 20.4.3. Let X be a scheme and $f \in \mathcal{O}_X(X)$. Consider the canonical morphism $\phi : X \to \mathsf{Spec}(\mathcal{O}_X(X))$. Show that $\phi^{-1}(U(f)) = X_f$.

Lemma 20.4.4. *Let X be a scheme and $f \in \mathcal{L}(X)$ be a global section. Then X_f is the open locus where $f : \mathcal{O}_X \to \mathcal{O}_X$ is invertible. More precisely:*

(1) *The restriction*
$$f|_{X_f} : \mathcal{O}_X|_{X_f} \to \mathcal{O}_X|_{X_f}$$
to X_f is an isomorphism.

(2) *If $V \subseteq X$ is a subset such that $f|_V$ is an isomorphism, then $V \subseteq X_f$.*

Proof. The claim in local in X. Hence we can reduce to the case when X is affine, where the claim is obvious. □

The following result says that sections of quasi-coherent sheaves on principle open subsets have only meromorphic singularities.

Lemma 20.4.5. *Let X be a quasi-compact and quasi-separated scheme and let $f \in \mathcal{O}_X(X)$. For any $\mathcal{F} \in \mathsf{QCoh}(X)$, the $\mathcal{O}_X(X)$-linear map $\mathcal{F}(X) \to \mathcal{F}(X_f)$ induces an isomorphism*
$$\mathcal{F}(X)_f \to \mathcal{F}(X_f).$$

Proof. Consider the canonical morphism $\phi : X \to Y$, wherer $Y := \mathsf{Spec}(\mathcal{O}_X(X))$. Note that ϕ is quasi-compact and quasi-separated. By Theorem 18.4.1, $\phi_*\mathcal{F}$ is quasi-coherent. By Exercise 20.4.3, the map $\mathcal{F}(X) \to \mathcal{F}(X_f)$ can be identified with
$$(\phi_*\mathcal{P})(Y) \to (\phi_*\mathcal{P})(U(f)).$$
Now the claim becomes obvious because Y is affine. □

20.5. Proof of Theorem 20.3.2.

20.5.1. (1)⇒(2): if $X = \mathsf{Spec}(A)$ is affine, then $\mathsf{QCoh}(X) \simeq A\text{−}\mathsf{mod}$ and the functor $\Gamma(X, -)$ can be identified with the forgetful functor $A\text{−}\mathsf{mod} \to \mathsf{Ab}$, which is clearly exact.

20.5.2. To show (2)⇒(1): let $W \subseteq X$ be the union of all *affine* principle open subsets X_f. Note that W is an open subset.

20.5.3. We first show $X = W$. To prove this claim, suppose that on the contrary $X \setminus W \neq \varnothing$.

Since X is quasi-compact and Kolmogorov, so is its closed subspace $X \setminus W$. By Exercise 20.5.7 below, $X \setminus W$ contains a closed point $x \in X \setminus W$. Note that x is also a closed point of X.

Choose an affine open neighborhood $\mathsf{Spec}(A) = V$ of x in X. Let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to $x \in V$. Consider the closed subsets
$$Z := X \setminus V, \;\; Z' := Z \sqcup \{x\}$$
of X. Let $\mathcal{I}$ and $\mathcal{I}'$ be the quasi-coherent ideals corresponding to Z_{red} and Z'_{red}. We have $\mathcal{I}' \subseteq \mathcal{I}$ since Z_{red} is a closed subscheme of Z'_{red}.

Consider the short exact sequence

$$0 \to \mathcal{I}' \to \mathcal{I} \to \mathcal{I}/\mathcal{I}' \to 0.$$

Since $Z \cap (X \smallsetminus x) = Z' \cap (X \smallsetminus x)$, we have $\mathcal{I}|_{X \smallsetminus x} \simeq \mathcal{I}'|_{X \smallsetminus x}$. Hence

$$\mathsf{supp}(\mathcal{I}/\mathcal{I}') \subseteq \{x\}.$$

On the other hand, since $Z \cap V = \varnothing$ and $Z' \cap V = \{x\}$, we have $\mathcal{I}|_V \simeq \widetilde{A}$ while $\mathcal{I}'|_V \simeq \widetilde{\mathfrak{m}}$. It follows that

$$\Gamma(X, \mathcal{I}/\mathcal{I}') \simeq \Gamma(V, (\mathcal{I}/\mathcal{I}')|_V) \simeq A/\mathfrak{m}.$$

Since $\Gamma(X, -)$ is assumed to be exact, there exists a section $f \in \Gamma(X, \mathcal{I})$ whose image under

$$\Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}/\mathcal{I}') \simeq A/\mathfrak{m}$$

is the unit element $1 \in A/\mathfrak{m}$. By definition, we have $x \in U(f) \subseteq V$. It follows that $U(f) = U(f|_V)$, where $f|_V$ is a global section of $\mathcal{I}|_V$. Since V is affine, so is $U(f|_V)$. Hence $U(f)$ is an affine open subscheme that contains x, which implies $x \in U(f) \subseteq W$. But this contradicts the assumption that $x \in X \smallsetminus W$.

20.5.4. We have shown $X = \bigcup_{i \in I} X_{f_i}$ for some *affine* principle open subsets. Since X is quasi-compact, we can assume I is finite.

20.5.5. We claim $(f_1, \cdots, f_n) = \mathcal{O}_X(X)$. To prove the claim, consider the morphism

$$\mathcal{O}_X^{\oplus n} \xrightarrow{(f_1, \cdots, f_n)} \mathcal{O}_X. \tag{20.3}$$

Since $f_i|_{X_{f_i}} : \mathcal{O}_X|_{X_{f_i}} \to \mathcal{O}_X|_{X_{f_i}}$ is an isomorphism and X is covered by X_{f_i}, $i \in I$, we see that (20.3) is a surjection. Taking $\Gamma(X, -)$, we obtain a surjection

$$\Gamma(X, \mathcal{O}_X)^{\oplus n} \xrightarrow{(f_1, \cdots, f_n)} \Gamma(X, \mathcal{O}_X).$$

This proves the claim.

20.5.6. Finally, consider the canonical morphism $\phi : X \to \mathsf{Spec}(\mathcal{O}_X(X))$. We only need to show ϕ is affine. By the claim in §20.5.5, the affine scheme $\mathsf{Spec}(\mathcal{O}_X(X))$ is covered by $U(f_i)$. Hence we only need to show $\phi^{-1}(U(f_i)) \to U(f_i)$ is affine. This follows from the fact that $\phi^{-1}(U(f_i)) = X_{f_i}$ (Exercise 20.4.3) and the assumption that $U(f_i)$ is affine.

$\square$[Theorem 20.3.2]

Exercise 20.5.7. Any non-empty quasi-compact Kolmogorov topological space contains at least one closed point.

Warning 20.5.8. A scheme may contain no closed points.

Exercise 20.5.9. Let k be a field and

$$A = k[x_1, x_2, \cdots][\frac{x_1}{x_2}, \frac{x_1}{x_2^2}, \cdots][\frac{x_2}{x_3}, \frac{x_2}{x_3^2}, \cdots]\cdots$$

be the sub-k-algebra of $k[x_1, x_2^{\pm}, x_3^{\pm}, \cdots]$ generated by $(x_i/x_{i+1}^m)_{i \geq 1, m \geq 0}$. Let $\mathfrak{m} \subseteq A$ be the kernel of the homomorphism $A \to k$, $x_i \mapsto 0$. Consider $X := \mathsf{Spec}(A_\mathfrak{m})$ and the unique closed point $x \in X$. Show that $U := X \smallsetminus x$ is a scheme with no closed points.

20.6. **Quasi-affine morphisms.**

Definition 20.6.1. We say a scheme X is **quasi-affine** if it is quasi-compact and isomorphic to an open subscheme of an affine scheme.

Proposition 20.6.2. *A scheme X is quasi-affine iff the canonical morphism $f : X \to \mathsf{Spec}(\mathcal{O}_X(X))$ is a quasi-compact open immersion.*

Proof. The "if" claim is obvious. For the "only if" claim, let X be a quasi-affine scheme and $j : X \to \mathsf{Spec}(B)$ be a quasi-affine open immersion. Write $A := \mathcal{O}_X(X)$. By Theorem 7.1.2, j factors as

$$X \xrightarrow{f} \mathsf{Spec}(A) \xrightarrow{g} \mathsf{Spec}(B).$$

Let $\phi : B \to A$ be the homomorphism corresponding to g. Since j is injective, so is f. Hence we only need to show that for any point $x \in X$, there exists an open neighborhood $x \in U \subseteq X$ such that $f|_U$ is an open immersion. Let $U(b) \subseteq j(X)$ be a standard open neighborhood of $j(x)$ that is contained in $j(X)$. Since j is an open immersion, we have $U := j^{-1}(U(b)) \simeq U(b)$. In particular, U is affine. On the other hand, we have

$$U = j^{-1}(U(b)) = f^{-1}(U(\phi(b))) = X_{\phi(b)},$$

where the last identity is due to Exercise 20.4.3. By Lemma 20.4.5, we have $\mathcal{O}_X(X)_{\phi(b)} \simeq \mathcal{O}_X(X_{\phi(b)})$, i.e.,

$$A_{\phi(b)} \simeq \mathcal{O}_X(U).$$

Since U is affine, this implies $f|_U$ is given by the open imersion $\mathsf{Spec}(A_{\phi(b)}) \to \mathsf{Spec}(A)$.

□

Definition 20.6.3. Let X be a quasi-affine scheme. We call $\overline{X} := \mathsf{Spec}(\mathcal{O}_X(X))$ the **affine closure** of X.

Definition 20.6.4. We say a morphism $f : X \to Y$ of schemes is **quasi-affine** if for any affine open subscheme $U \subseteq Y$, its inverse image $X \times_Y U$ is quasi-affine.

We have the following analog of Lemma 20.1.3.

Lemma 20.6.5. *A quasi-affine morphism $f : X \to Y$ is quasi-compact and separated.*

The following result follows easily from Proposition 20.6.2.

Proposition 20.6.6. *A quasi-compact and quasi-separated morphism $p : X \to S$ is quasi-affine iff the canonical morphism $f : X \to \mathsf{Spec}_S(p_*\mathcal{O}_X)$ is an open immersion.*

As in the affine case, we have the following results:

Lemma 20.6.7. *Any quasi-compact locally closed immersion is quasi-affine.*

Lemma 20.6.8. *Consider the class of affine morphisms.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are quasi-affine. Then $g \circ f$ is quasi-affine.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that g is quasi-separated and $g \circ f$ is quasi-affine. Then f is quasi-affine.*

(iii) *Quasi-affine morphisms are stable under base-changes.*
(iv) *Being quasi-affine is local on the targets.*

Exercise 20.6.9. Let $f : X \to Y$ be an affine morphism between quasi-affine schemes. Let $\overline{X}$ and $\overline{Y}$ respectively be the affine closure of X and Y. Show that the canonical commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "f"] & \overline{X} \arrow[d, "\overline{f}"] \\ Y \arrow[r, "j_Y"] & \overline{Y}. \end{tikzcd}$$

is Cartesian. Hint: show that $\overline{X} \simeq \mathsf{Spec}_{\overline{Y}}(j_{Y,*} \circ f_*(\mathcal{O}_X))$.

21. Irreducible schemes

21.1. Irreducible spaces.

Definition 21.1.1. We say a topological space X is **irreducible** if it is nonempty and is not a union of two closed proper subspaces.

Exercise 21.1.2. Let X be an irreducible space. Then
(1) The space X is connected.
(2) Any nonempty open subset of X is irreducible.
(3) Any nonempty open subset of X is dense in X.
(4) The intersection of two nonempty open subsets of X is nonempty.

Exercise 21.1.3. The closure of an irreducible subspace in a topologicial space is irreducible.

Exercise 21.1.4. The image of an irreducible space under a continuous map is irreducible.

Definition 21.1.5. We say a topological space X admits a generic point $x \in X$ if $X = \overline{\{x\}}$.

Exercise 21.1.6. If a space X admits a generic point, then X is irreducible.

Exercise 21.1.7. If X is a Kolmogorov, then any two generic points of X are equal.

21.2. Irreducible components.

Definition 21.2.1. Let X be a topological space. An **irreducible component** of X is a maximal irreducible subspace of X.

Exercise 21.2.2. Any irreducible component of X is closed.

Exercise 21.2.3. The union of all irreducible components of X is equal to X.

21.3. Irreducible schemes.

Definition 21.3.1. We say a scheme X is **irreducible** if its underlying topological space is irreducible.

Lemma 21.3.2. *Let X be a scheme. Any closed irreducible subspace of X admits a unique generic point.*

Proof. The uniqueness follows from Exercise 21.1.7.

For the existence, let $Y \subseteq X$ be a closed irreducible subspace. Choose an affine open subset $U \subseteq X$ such that $Y \cap U$ is nonempty. Since Y is irreducible, we have $Y = \overline{Y \cap U}$. Hence we only need to show $Y \cap U$ admits a generic point. Hence we can replace X with U, and therefore assume $X = \mathsf{Spec}(A)$ is affine.

Choose a *radical* ideal $I \subseteq A$ such that $Y = Z(I)$. We only need to show I is a prime ideal, because then the corresponding point will be a generic point of $Z(I)$. Replacing X by $\mathsf{Spec}(A/I)$, we can assume $I = (0)$ and therefore $\mathsf{Spec}(A)$ is reduced and irreducible.

Now for $xy \in (0)$, note that $\mathsf{Spec}(A)$ is covered by $Z(x)$ and $Z(y)$. Since $\mathsf{Spec}(A)$ is irreducible, without lose of generality, we can assume $Z(x) = \mathsf{Spec}(A)$. This implies $\sqrt{(x)} = \sqrt{(0)}$. Since $\mathsf{Spec}(A)$ is reduced, we must have $x = 0$. This shows (0) is a prime ideal as desired.

□

Corollary 21.3.3. *Let X be a scheme. Then $x \mapsto \overline{\{x\}}$ gives a bijection between the underlying set of X and the set of closed irreducible subspaces of X.*

Corollary 21.3.4. *The set of irreducible components of $\mathsf{Spec}(A)$ is in bijection with the set of minimal prime ideals of A.*

Corollary 21.3.5. *An affine sheme $\mathsf{Spec}(A)$ is irreducible iff the nilpotent radical of A is a prime ideal.*

Exercise 21.3.6. Let X be a scheme and $x \in X$ is a point. The following conditions are equivalent:

- There is a unique irreducible component of X that contains x.
- The nilpotent radical of $\mathcal{O}_{X,x}$ is a prime ideal.

Lemma 21.3.7. *Let X be a scheme. The following conditions are equivalent:*

1. *The scheme X is irreducible.*
2. *The scheme X is nonempty and any nonempty affine open subscheme U of X is irreducible.*
3. *There exists a nonempty covering $X = \bigcup_{i \in I} U_i$ such that each U_i is an irreducible affine open subscheme, and each intersection $U_i \cap U_j$ is nonempty.*

Proof. (1)⇒(2) follows from Exercise 21.1.2(2).

(2)⇒(3): let $X = \bigcup_{i \in I} U_i$ be *any* covering such that each U_i is a nonempty affine open subscheme. By assumption, each U_i is irreducible. Suppose for some i, j, the intersection $U_i \cap U_j$ is empty, then the disjoint union $U_i \sqcup U_j$ is also a nonempty affine open subscheme of X, but it is not irreducible because both U_i and U_j are closed subspaces of $U_i \sqcup U_j$. This contradicts the assumption in (2). Hence $U_i \cap U_j$ is nonempty as desired.

(3)⇒(1): suppose the underlying topological space X is the union $Z_1 \cup Z_2$ of two closed subspaces. We only need to show either $X = Z_1$ or $X = Z_2$. Since each U_i is irreducible, we see that either $U_i \subseteq Z_1$ or $U_i \subseteq Z_2$. Without lose of generality, pick $i \in I$ such that $U_i \in Z_1$. For each $j \in J$, note that $U_i \cap U_j$ is dense in U_j (Exercise 21.1.2(3)) and is contained in the closed subset $Z_1 \cap U_j$. Hence we must have $U_j = Z_1 \cap U_j$, i.e. $U_j \subseteq Z_1$. This implies $X = Z_1$ as desired.

□

21.4. **Integral schemes.**

Definition 21.4.1. We say a scheme X is **integral** if it is irreducible and reduced.

Combining Corollary 21.3.5 and Corollary 19.1.3, we obtain the following result:

Corollary 21.4.2. *An affine scheme $\mathsf{Spec}(A)$ is integral iff A is an integral domain.*

Exercise 21.4.3. Let X be a scheme. The following conditions are equivalent:

1. The scheme X is integral.
2. The scheme X is nonempty and for any open subscheme $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

(3) The scheme X is nonempty and for any affine open subscheme $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

22. (Locally) Noetherian schemes

22.1. Definition.

Definition 22.1.1. We say a scheme X is **locally Noetherian** if for any affine open subscheme $\mathsf{Spec}(A) \subseteq X$, the commutative ring A is Noetherian. We say X is **Noetherian** if it is locally Noetherian and quasi-compact.

Lemma 22.1.2. *Let X be a scheme. Suppose there is a covering $X = \bigcup_{i \in I} X_i$ of X by affine open subschemes such that $\mathcal{O}_X(X_i)$ is Noetherian for each $i \in I$. Then X is locally Noetherian.*

Proof. Let $U \subseteq X$ be an affine open subscheme. Choose an open covering $U = \bigcup_{j \in J} U_j$ such that each U_j is a *standard* affine open subscheme for some X_i. Hence $\mathcal{O}_X(U_j)$ is Noetherian because it is a localization of a Noetherian ring. Therefore we can replace X with U, and assume X is affine. Since X is quasi-compact, we can also assume I is finite.

Write $A = \mathcal{O}_X(X)$ and $A_i = \mathcal{O}_X(X_i)$. Let $J_1 \subseteq J_2 \subseteq \cdots$ be an ascending chain of ideals of A, and

$$\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots \tag{22.1}$$

be the corresponding chain of quasi-coherent ideals of $\mathcal{O}_X$. We only need to show this chain stablizes.

For any $i \in I$, the restriction of (22.1)

$$\mathcal{J}_1|_{X_i} \subseteq \mathcal{J}_2|_{X_i} \subseteq \cdots$$

is an ascending chain of quasi-coherent ideals of $\mathcal{O}_{X_i}$. Since $X_i \simeq \mathsf{Spec}(A_i)$ and A_i is Noetherian, this chain must stablizes. Since I is finite, the chain (22.1) also stablizes.

□

Corollary 22.1.3. *An affine scheme $\mathsf{Spec}(A)$ is Noetherian iff A is Noetherian.*

Corollary 22.1.4. *Being locally Noetherian is local on the scheme.*

Warning 22.1.5. Fiber products of Noetherian schemes may fail to be locally Noetherian. See Example 0.2.3.

22.2. Noetherian spaces.

Definition 22.2.1. We say a topological space X is **Noetherian** if any descending chain of closed subsets of X stablizes. We say X is **locally Noetherian** if for any point $x \in X$, there is a Noetherian open neighborhood of x in X.

Exercise 22.2.2. Consider the class of (locally) Noetherian topological spaces.

1. A space is Noetherian iff it is locally Noetherian and quasi-compact.
2. A subspace of (locally) Noetherian space is (locally) Noetherian.
3. A finite union of (locally) Noetherian subspaces is (locally) Noetherian.
4. A space is Noetherian iff any open subspace of it is quasi-compact.

Lemma 22.2.3. *A Noetherian topological space X has finitely many irreducible components, and each irreducible component contains a nonempty open subset of X.*

Proof. For the first claim, let I be the collection of closed subspaces of X that have infinitely many irreducible components. We only need to show $I =$. Suppose I was nonempty. Since X is Noetherian, I contains a minimal element Y. By assumption, Y is not irreducible. Hence we can write $Y = Y_1 \cup Y_2$ such that $Y_1, Y_2 \neq Y$. By assumption, Y_i is a finite union of irreducible subspaces, hence so is Y. It is easy to see this implies Y has finitely many irreducible components. A contradiction.

For the second claim, let $Z_1, \cdots, Z_n$ be the irreducible components of X. It is easy to see $X \setminus (Z_2 \cup \cdots \cup Z_n)$ is a nonempty open subset of X contained in Z_1. □

Lemma 22.2.4. *The underlying topological space of a (locally) Noetherian scheme is (locally) Noetherian.*

Proof. By Exercise 22.2.2, we only need to show the underlying topological space of a Noetherian scheme is locally Noetherian. Hence we only need to show the underlying topological space of $\mathsf{Spec}(A)$ is Noetherian. But this is obvious. □

Exercise 22.2.5. Let X be a locally Noetherian scheme and $x \in X$ be a point. Show that if $\mathsf{Spec}(\mathcal{O}_{X,x})$ is irreducible, then there exists an open neighborhood U of x such that U is irreducible.

22.3. Immersions of Noetherian schemes.

Proposition 22.3.1. *Let $f : X \to Y$ be a locally closed immersion of schemes. If Y is (locally) Noetherian, so is X.*

Proof. We only need to prove the claim when f is either a closed immersion or an open immersion. Moreover, the claims in both cases are local on Y, hence we can assume $Y = \mathsf{Spec}(B)$ with B being Noetherian.

The case of closed immersions is obvious because any quotient ring of B is Noetherian. For the case of open immersions, let $f : X \to Y$ be an open immersion. By Corollary 22.1.4, X is locally Noetherian. By Lemma 22.2.4 and Exercise 22.2.2, the underlying topological space of X is quasi-compact. Hence X is a Noetherian scheme as desired. □

Lemma 22.3.2. *Let X be a locally Noetherian scheme. Any morphism $f : X \to Y$ is quasi-separated.*

Proof. By Lemma 18.3.4(ii), we only need to show X is quasi-separated. By Lemma 18.3.3, we only need to show that for any affine open subsets $U, V \subseteq X$, the intersection $U \cap V$ is quasi-compact. By assumption, U is Noetherian. By Proposition 22.3.1, $U \cap V$ is Noetherian because it is an open subscheme of U. In particular, $U \cap V$ is quasi-compact as desired. □

Lemma 22.3.3. *Let X be a Noetherian scheme and Y be a locally Noetherian scheme. Any morphism $f : X \to Y$ is quasi-compact.*

Proof. Follows from Lemma 18.1.4(ii) and Lemma 22.3.2. □

By Proposition-Definition 18.5.3, we obtain:

Corollary 22.3.4. *Let X be a locally Noetherian scheme and $i : Y \to X$ be a locally closed immersion of schemes. Then the closure $\overline{Y}$ of Y is well-defined, and the canonical morphism $j : Y \to \overline{Y}$ is an open immersion.*

22.4. Coherent modules on locally Noetherian schemes.

Lemma 22.4.1. *Let X be a locally Noetherian scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. The following conditions are equivalent:*

(1) *The $\mathcal{O}_X$-module $\mathcal{F}$ is coherent.*
(2) *The $\mathcal{O}_X$-module $\mathcal{F}$ is of finite presentation.*
(3) *The $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent and of finite type.*
(4) *For any affine open subscheme $U \subseteq X$, we have $\mathcal{F}|_U \simeq \widetilde{M}$ for some finitely generated $\mathcal{O}_X(U)$-module M.*

Proof. (1)⇒(2) follows from Exercise 16.2.5(1).

(2)⇒(3) follows from Lemma 16.2.2 and Lemma 16.2.3.

(3)⇒(4): we can assume $X = U = \mathsf{Spec}(A)$ is affine and $\mathcal{F} \simeq \widetilde{M}$ for some A-module M. It remains to show M is finitely generated. Since $\mathcal{F}$ is of finite type, we can cover X by standard affine open subsets $X = \bigcup_{i \in I} U(f_i)$ such that $\mathcal{F}|_{U(f_i)}$ is generated by finitely many sections. This implies M_{f_i} is a finitely generated A_{f_i}-module. By Exercise 16.1.8, M is finitely generated as desired.

(4)⇒(1): to verify the condition in Definition 16.4.1, it is enough to treat open subsets U that belong to a topological basis of X. Hence we can assume U is affine. Now the condition follows immediately from (4) because $\mathcal{O}_X(U)$ is Noetherian. □

Corollary 22.4.2. *Being a coherent module is local on locally Noetherian schemes.*

Proof. Condition (3) in Lemma 22.4.1 is local on schemes. □

Corollary 22.4.3. *Let $f : X \to Y$ be a morphism between locally Noetherian schemes. Then f^* preserves coherent modules.*

Corollary 22.4.4. *Let X be a locally Noetherian scheme. Coherent $\mathcal{O}_X$-modules are stable under taking kernals, cokernals and extensions. In particular, they form an abelian category $\mathsf{Coh}(X)$, and the embedding functor $\mathsf{Coh}(X) \subseteq \mathsf{QCoh}(X)$ is exact.*

Remark 22.4.5. If X is Noetherian, then $\mathsf{Coh}(X) \subseteq \mathsf{QCoh}(X)$ is the full subcategory of compact objects.

Exercise 22.4.6. Let X be a locally Noetherian scheme and $\mathcal{M}, \mathcal{N} \in \mathsf{Coh}(X)$. Then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathsf{Coh}(X)$.

23. Morphisms of finite type and finite presentation

23.1. **Morphisms of finite type.** Recall a ring homomorphism $A \to B$ is **of finite type** if B is **finite generated** over A, i.e., B is isomorphic to a quotient of $A[x_1, \cdots, x_n]$ as an A-algebra.

Definition 23.1.1. Let $f : X \to Y$ be a morphism of schemes. We say that f is **locally of finite type** if for any affine open subschemes $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the induced ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite type.

We say f is **of finite type** if it is locally of finite type and quasi-compact.

Lemma 23.1.2. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent:*

1. *The morphism f is locally of finite type.*
2. *For any point $x \in X$, there exists an affine open neighborhood U of x and an affine open subscheme $V \subseteq Y$ with $f(U) \subseteq V$ such that the induced ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite type.*

The lemma follows from the following paradigm:

Exercise 23.1.3. Let P be a property on morphisms in CRing such that:

- (i) For any $A \to B$ and $a \in A$, we have $P(A \to B) \Rightarrow P(A_a \to B_a)$;
- (ii) For any $A, B \in \mathsf{CRing}$, $a \in A$, $b \in B$ and $A_a \to B$, we have $P(A_a \to B) \Rightarrow P(A \to B_b)$;
- (iii) For any $A \to B$ and $b_1, \cdots, b_n \in B$ such that $(b_1, \cdots, b_n) = B$, we have $\bigcap_i P(A \to B_{b_i}) \Rightarrow P(A \to B)$.

Then for a morphism $f : X \to Y$ between schemes, the following conditions are equivalent:

1. For any affine open scheme $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, we have $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.
2. For any point $x \in X$, there exists an affine open neighborhood U of x and an affine open subscheme $V \subseteq Y$ with $f(U) \subseteq V$ such that $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.

Corollary 23.1.4. *A morphism $\mathsf{Spec}(B) \to \mathsf{Spec}(A)$ is of finite type iff $A \to B$ is of finite type.*

The following results follow easily from Lemma 23.1.2.

Corollary 23.1.5. *A (locally) closed immersion is (locally) of finite type.*

Lemma 23.1.6. *Consider the collection of (locally) finite type morphisms of schemes.*

- (i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are (locally) of finite type. Then so is $g \circ f$.*
- (ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is locally of finite type. Then f is locally of finite type.*
- (ii') *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is of finite type and g is quasi-separated. Then f is of finite type.*
- (iii) *Morphisms (locally) of finite type are stable under base-changes.*
- (iv) *Being (locally) of finite type is local on the targets.*
- (v) *Being locally of finite type is local on the sources.*

The following result follows from Hilbert's basis theorem.

Lemma 23.1.7. *Let $f : X \to Y$ be a morphism of schemes. If Y is (locally) Noetherian and f is (locally) of finite type, then X is (locally) Noetherian.*

Corollary 23.1.8. *Let $\mathcal{C}$ be the subcategory of* Sch *consisting of (locally) Noetherian schemes and morphisms that are (locally) of finite type. Then $\mathcal{C}$ admits fiber products.*

23.2. **Morphisms of finite presentation.** Recall a ring homomorphism $A \to B$ is **of finite presentation** if B is isomorphic to a quotient of $A[x_1, \cdots, x_n]$ by a finite generated ideal.

Definition 23.2.1. Let $f : X \to Y$ be a morphism of schemes. We say that f is **locally of finite presentation** if for any affine open subschemes $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, the induced ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

We say f is **of finite presentation** if it is locally of finite presentation, quasi-compact and quasi-separated.

We have the following analogue of Lemma 23.1.2.

Lemma 23.2.2. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent:*

(1) *The morphism f is locally of finite presentation.*
(2) *For any point $x \in X$, there exists an affine open neighborhood U of x and an affine open subscheme $V \subseteq Y$ with $f(U) \subseteq V$ such that the induced ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.*

Corollary 23.2.3. *A morphism* $\mathsf{Spec}(B) \to \mathsf{Spec}(A)$ *is of finite presentation iff $A \to B$ is of finite presentation.*

Corollary 23.2.4. *An open immersion is locally of finite presentation. A quasi-compact open immersion is of finite presentation.*

Warning 23.2.5. A closed immersion may fail to be locally of finite presentation.

We have the following analogue of Lemma 23.1.6.

Lemma 23.2.6. *Consider the collection of (locally) finitely presented morphisms of schemes.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are (locally) of finite presentation. Then so is $g \circ f$.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is locally of finite presentation and g is locally of finite type. Then f is locally of finite presentation.*
(ii') *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is of finite presentation and g is locally of finite type and quasi-separated. Then f is of finite presentation.*
(iii) *Morphisms (locally) of finite presentation are stable under base-changes.*
(iv) *Being (locally) of finite presentation is local on the targets.*
(v) *Being locally of finite presentation is local on the sources.*

We also have the following obvious results about morphisms of (locally) finite type versus morphisms of (locally) finite presentation.

Lemma 23.2.7. *Let $f : X \to Y$ be a morphism.*

(1) *If f is (locally) of finite presentation, then f is (locally) of finite type.*
(2) *If f is (locally) of finite type and Y is locally Noetherian, then f is (locally) of finite presentation.*

23.3. **Noetherian approximation.** *Noetherian approximation* is a technique that allows one to reduce problems about a general base scheme S to the case when S is Noetherian, as long as one restricts to those S-schemes that are of *finite presenetation*. This technique was implicit in [Gro66, Chapter 8], and was articulated in [TT13, Appendix C]. For simplicity, we only state the following baby version of it.

Theorem 23.3.1. *Let $A \simeq \mathsf{colim}_{i\in I} A_i$ be a filtered colimit in CRing. For any $B \in \mathsf{CRing}$, let $\mathsf{Sch}_B^{\mathsf{fp}} \subseteq \mathsf{Sch}_B$ be the full subcategory of schemes that are of finite presentation over $\mathsf{Spec}(B)$. Then the functors*

$$\mathsf{Sch}_{A_i}^{\mathsf{fp}} \to \mathsf{Sch}_{A}^{\mathsf{fp}},\ Y \mapsto Y \underset{\mathsf{Spec}(A_i)}{\times} \mathsf{Spec}(A)$$

induces an equivalence

$$\underset{i\in I}{\mathsf{colim}}\, \mathsf{Sch}_{A_i}^{\mathsf{fp}} \xrightarrow{\simeq} \mathsf{Sch}_{A}^{\mathsf{fp}}.$$

In concrete, the above theorem says the following.

Corollary 23.3.2. *Let $A \simeq \mathsf{colim}_{i\in I} A_i$ be a filtered colimit in CRing.*

(1) *For any scheme $X \in \mathsf{Sch}_{A}^{\mathsf{fp}}$, there exists an index $i \in I$ and $X_i \in \mathsf{Sch}_{A_i}^{\mathsf{fp}}$ such that*
$$X \simeq X_i \underset{\mathsf{Spec}(A_i)}{\times} \mathsf{Spec}(A).$$
(2) *For any morphism $f : X \to X'$ in $\mathsf{Sch}_{A}^{\mathsf{fp}}$, there exists an index $i \in I$ and a morphism $f_i : X_i \to X_i'$ in $\mathsf{Sch}_{A_i}^{\mathsf{fp}}$ such that f can be identified with the base-change of f_i along $\mathsf{Spec}(A) \to \mathsf{Spec}(A_i)$.*

Example 23.3.3. Note that any commutative ring R can be written as a filtered colimit of finitely generated $\mathbb{Z}$-algebras. In practice, one applies the above Theorem to such a colimit and reduce problems about finitely presented R-schemes to those about finite type $\mathbb{Z}$-schemes.

Example 23.3.4. Let S be a scheme and $s \in S$ be a point. Note that $\mathcal{O}_{S,s} \simeq \mathsf{colim}\mathcal{O}_S(U)$, where U ranges over affine open neighborhood of s. In practice, one applies the above Theorem to such a colimit and extend constructions about finitely presented $\mathcal{O}_{S,s}$-schemes to an open neighborhood of s.

24. Proper morphisms

24.1. Closed morphisms.

Definition 24.1.1. We say a continuous map $f : X \to Y$ of topological spaces is **closed** if it sends closed subsets of X to closed subsets of Y.

Example 24.1.2. A closed immersion of schemes is a closed morphism.

Lemma 24.1.3. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. The following conditions are equivalent:*

(i) *The subset $f(X)$ is closed in Y;*
(ii) *The subset $f(X)$ is stable under specializations.*

Proof. (i)⇒(ii) is obvious.

For (ii)⇒(i), note that the question is local on Y. Hence we can assume Y is affine.

Since f is quasi-compact, X is quasi-compact. Choose a finite covering $X = \bigcup_{i \in I} U_i$ of X by its affine open subschemes. Consider the *disjoint* union $X' := \sqcup_{i \in I} U_i$ and the composition $f' : X' \to X \to Y$. Note that $f(X') = f(X)$. Hence we can replace X with X', and assume X is affine.

Let $\phi : A \to B$ be the homomorphism corresponding to $f : X \to Y$. We can replace A with $A/\mathsf{ker}(\phi)$, and assume ϕ is injective. It follows that $f(X)$ contains all the points that correspond to the minimal primes of $\mathsf{Spec}(A)$. Now (ii) implies $f(X) = Y$. In particular, (i) holds.

□

Corollary 24.1.4. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. The following conditions are equivalent:*

(i) *The f is closed.*
(ii) *Specializations lift along f. In other words, for $y, y' \in Y$ and $x \in X$ such that $y' \in \overline{\{y\}}$ and $f(x) = y$, there exists $x' \in \overline{\{x\}}$ such that $f(x') = y'$.*

Proof. Follows from Lemma 24.1.3 and the fact that any closed subset of X underlies some closed subschemes of X.

□

The proof of the following results is left to the readers.

Lemma 24.1.5. *Consider the collection of closed morphisms of schemes.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are closed. Then so is $g \circ f$.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is closed, and g is separated. Then f is closed.*
(iv) *Being closed is local on the targets.*

24.2. Universally closed morphisms.

Definition 24.2.1. We say a morphism $f : X \to Y$ of schemes is **universally closed** if for any scheme Y' over Y, the base-change $f' : X \times_Y Y' \to Y'$ is closed.

Example 24.2.2. Let k be a field. The morphism $f : \mathbb{A}^1 \to \mathsf{Spec}(k)$ is closed but its base-change $\mathbb{A}^2 \to \mathbb{A}^1$ is not closed. Indeed, the image of the closed subset $Z((xy-1)) \subseteq \mathbb{A}^2$ is $\mathbb{A}^1 \smallsetminus 0 \subseteq \mathbb{A}^1$.

Lemma 24.2.3. *A universally closed morphism $f : X \to Y$ of schemes is quasi-compact.*

Proof. See [Sta25, Lemma 04XU]. $\square$

The proof of the following results is left to the readers.

Lemma 24.2.4. *Consider the collection of closed morphisms of schemes.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are universally closed. Then so is $g \circ f$.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is universally closed, and g is separated. Then f is universally closed.*
(iii) *Universally closed morphisms are stable under base-changes.*
(iv) *Being universally closed is local on the targets.*

24.3. **Valuative criteria.** Recall the following definition.

Definition 24.3.1. A valuation ring is an integral domain A such that for any non-zero element x of its field of factions $K := \mathsf{Frac}(A)$, at least one of x or x^{-1} belongs to A.

Exercise 24.3.2. A valuation ring A is a local ring.

Also recall the following fact in commutative algebra:

Lemma 24.3.3. *Let K be a field and $\mathcal{C}$ be the set of local subrings of K. For $A, B \in \mathcal{C}$, we say $B \leq A$ if $B \subseteq A$ and $B \to A$ is a local homomorphism. Then for $A \in \mathcal{C}$, the following conditions are equivalent:*

- *The element $A \in \mathcal{C}$ is a maximal element in $(\mathcal{C}, \leq)$;*
- *The ring A is a valuation ring and the canonical map $\mathsf{Frac}(A) \to K$ is an isomorphism.*

Definition 24.3.4. Let $f : X \to Y$ be a morphism of schemes. We say f satisfies **the existence part of the valuative criterion** if for any valuation ring A with $K = \mathsf{Frac}(A)$ and for any solid commutative square

(24.1)
$$\begin{tikzcd} \mathsf{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathsf{Spec}(A) \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

there exists a dotted morphism making the two triangles commute.

We say f satisfies **the uniqueness part of the valuative criterion** if in above there is at most one dotted morphism making the two triangles commute.

We say f satisfies **the valuative criterion** if it satisfies both the existence part and uniqueness part of it.

Proposition 24.3.5. *Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then f is universally closed iff it satisfies the existence part of the valuative criterion.*

Proof. We first prove the "only if" claim. Suppose f is universally closed. For a given solid commutative diagram (24.1), consider $X_A := X \times_Y \mathsf{Spec}(A)$. It is easy to see that knowing a dotted morphism making (24.1) commute is equivalent to knowing a dotted morphism making the following diagram commute:

$$\begin{tikzcd} \mathsf{Spec}(K) \arrow[r] \arrow[d] & X_A \arrow[d, "f_A"] \\ \mathsf{Spec}(A) \arrow[r, equal] \arrow[ur, dotted] & \mathsf{Spec}(A). \end{tikzcd}$$

Hence we can replace f with f_A, and therefore assume $\mathsf{Spec}(A) \to Y$ is the identity morphism.

Let $x \in X$ be the image of $\mathsf{Spec}(K) \to X$. Note that $f(x)$ is the generic point of $\mathsf{Spec}(A)$. By Corollary 24.1.4, there exists $x' \in \overline{\{x\}}$ such that $f(x')$ is the closed point of $\mathsf{Spec}(A)$. We only need to show there exists a dotted morphism making the following diagram commute

$$\begin{tikzcd} \mathsf{Spec}(K) \arrow[r] \arrow[d] & \mathsf{Spec}(\mathcal{O}_{X,x'}) \arrow[d] \\ \mathsf{Spec}(A) \arrow[r, equal] \arrow[ur, dotted] & \mathsf{Spec}(A). \end{tikzcd}$$

Note that the right vertical morphism preserves the closed points, hence the right vertical homomorphism in the following diagram is a local homomorphism

$$\begin{tikzcd} K & \mathcal{O}_{X,x'} \arrow[l] \arrow[dl, dotted] \\ A \arrow[u] \arrow[r, equal] & A. \arrow[u] \end{tikzcd}$$

Let $B := \mathsf{Im}(\mathcal{O}_{X,x'} \to K)$. It is easy to see B is a local ring and the composition $A \to \mathcal{O}_{X,x'} \to B$ is a local homomorphism. By Lemma 24.3.3, the morphism $A \to B$ is an isomorphism. Now the composition $\mathcal{O}_{X,x'} \to B \simeq A$ gives the desired dotted arrow.

We now prove the "if" claim. Suppose f satisfies the existence part of the valuative criterion. It follows formally that any base-change of f also satisfies the same property. Hence we only need to show f is closed. By Corollary 24.1.4, we only need to show specializations lift along f. Let $y, y' \in Y$, $x \in X$ such that $y' \in \overline{\{y\}}$ and $f(x) = y$.

Note that we have a canonical commutative diagram

$$\begin{tikzcd} \mathsf{Spec}(\kappa_x) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathsf{Spec}(\mathcal{O}_{Y,y'}) \arrow[r] & Y, \end{tikzcd}$$

where the left vertical morphism is given by the homomorphism

$$\mathcal{O}_{Y,y'} \to \mathcal{O}_{Y,y} \to \kappa_y \to \kappa_x. \tag{24.2}$$

Write $K := \kappa_x$ and let $B \subseteq K$ be the image of (24.2). Note that B is a local ring. By Lemma 24.3.3, we can find a valuation subring $A \subseteq K$ with $\mathsf{Frac}(A) \xrightarrow{\simeq} K$ such that $B \subseteq A$ and $B \to A$ is a local homomorphism. In particular, (24.2) is equal to the composition

$$\mathcal{O}_{Y,y'} \to A \to K,$$

with the first morphism being a local homomorphism. This gives a commutative diagram

$$\begin{tikzcd}
\mathsf{Spec}(K) \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\
\mathsf{Spec}(A) \arrow[r] \arrow[urr, dotted, "g"] & \mathsf{Spec}(\mathcal{O}_{Y,y'}) \arrow[r] & Y.
\end{tikzcd}$$

By construction, the bottom horizontal composition sends the closed point to y' and the generic point to y. By assumption, there exists a dotted arrow g making the diagram commute. It follows that g sends the closed point to a point $x' \in X$ such that $x' \in \overline{\{x\}}$ and $f(x') = y'$. Hence specializations lift along f as desired. □

Proposition 24.3.6. *Let $f : X \to Y$ be a quasi-separated morphism of schemes. Then f is separated iff it satisfies the uniqueness part of the valuative criterion.*

Proof. Since f is quasi-separated, $\Delta_f : X \to X \times_Y X$ is a quasi-compact immersion. It follows that we have the followimg implications

f is separated;
$\Leftrightarrow$ Δ_f is a closed immersion;
$\Leftrightarrow$ Any base-change of Δ_f is a closed immersion;
$\Leftrightarrow$ Δ_f is universally closed;
$\Leftrightarrow$ $X \to X \times_Y X$ satisfies the existence part of the valuative criterion;
$\Leftrightarrow$ $X \to Y$ satisfies the uniqueness part of the valuative criterion.

□

Using Noetherian approximation, one can prove the following variants of Proposition 24.3.5 and Proposition 24.3.6

Proposition 24.3.7. *Let $f : X \to Y$ be a morphism of schemes that is of finite type. Suppose Y is locally Noetherian. Then f is universally closed iff it satisfies the existence part of the valuative criterion for all* discrete *valuation rings.*

Proof. See [Sta25, Lemma 05JY]. □

Proposition 24.3.8. *Let $f : X \to Y$ be a morphism of schemes that is locally of finite type. Suppose Y is locally Noetherian. Then f is separated iff it satisfies the uniqueness part of the valuative criterion for all* discrete *valuation rings.*

Proof. See [Sta25, Lemma 0207]

□

Remark 24.3.9. A valuation ring is Noetherian iff it is a discrete valuation ring.

24.4. **Proper morphisms.**

Definition 24.4.1. We say a morphism $f : X \to Y$ of schemes is **proper** if it is separated, of finite type, and universally closed.

Remark 24.4.2. By Lemma 24.2.3, we can replace the second condition by "locally of finite type".

Example 24.4.3. Any closed immersion of schemes is proper.

Exercise 24.4.4. Show that $\mathbb{P}^n_{\mathbb{Z}} \to \mathsf{Spec}(\mathbb{Z})$ is proper.

The following result follows by combining Proposition 24.3.5 and Proposition 24.3.6.

Corollary 24.4.5. *Let $f : X \to Y$ be a quasi-separated morphism of schemes that is of finite type. Then f is proper iff it satisfies the valuative criterion.*

We also have the following variant in the Noetherian case.

Corollary 24.4.6. *Let $f : X \to Y$ be a morphism of schemes that is of finite type. Suppose Y is locally Noetherian. Then f is proper iff it satisfies the valuative criterion for all* discrete *valuation rings.*

The proof of the following results is left to the readers.

Lemma 24.4.7. *Consider the collection of proper morphisms of schemes.*

(i) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that f and g are proper. Then so is $g \circ f$.*
(ii) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a chain of morphisms such that $g \circ f$ is proper, and g is separated. Then f is proper.*
(iii) *Proper morphisms are stable under base-changes.*
(iv) *Being proper is local on the targets.*

The following theorem will be proved in *Algebraic Geometry II*:

Theorem 24.4.8. *Let $f : X \to Y$ be a proper morphism that is of finite presentation. Then f_* preserves coherent modules.*

Remark 24.4.9. In fact, the derived functors $\mathsf{R}^i f_*$ also preserves coherent modules.

Warning 24.4.10. In the above theorem, the condition "of finite presentation" is necessary.

Exercise 24.4.11. Let k be a field. Show that the direct image functor along the closed immersion $0 \to \mathbb{A}^\infty_k$ does not preserve coherent modules.

Part VI. (Quasi-)projective morphisms

25. $\mathsf{Proj}(S)$

25.1. The space $\mathsf{Proj}(S)$. Throughout this subsection, S is a **$\mathbb{Z}_{\geq 0}$-graded (commutative) ring**. In other words, S is a commutative ring equipped with a direct sum decomposition $S = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} S_n$ as abelian groups, such that:

- The unit element 1 is of degree 0, i.e., $1 \in S_0$;
- The multiplication map is compatible with the grading, i.e., $xy \in S_{m+n}$ for $x \in S_m$ and $y \in S_n$.

Recall an element $x \in S$ is **homogeneous** if $x \in S_n$ for some $n \in \mathbb{Z}$. An ideal $I \subseteq S$ is **homogeneous** if it can be generated by homogeneous elements, equivalently, I is homogeneous if $I \simeq \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (I \cap S_n)$.

Example 25.1.1. The augmentation ideal

$$S_+ := \bigoplus_{n>0} S_n = \mathsf{ker}(S \to S_0)$$

is a homogeneous ideal.

Definition 25.1.2. We define the **homogeneous (prime) spectrum** of S to be the *subspace*

$$\mathsf{Proj}(S) \subseteq \mathsf{Spec}(S)$$

consisting of points that correspond to homogeneous prime ideals $\mathfrak{p} \subseteq S$ such that $S_+ \not\subseteq \mathfrak{p}$.

Exercise 25.1.3. Let k be an algebraically closed field and $n \geq 0$ be an integer. Let $S = k[x_1, \cdots, x_n]$ be the $\mathbb{Z}_{\geq 0}$-graded ring such that $S_0 = k$ and $x_i \in S_1$. Show that for any point $y \in \mathsf{Spec}(S) = \mathbb{A}^n_k$, the following conditions are equivalent:

(1) The point y is contained in $\mathsf{Proj}(S)$.
(2) The point y is not closed, and $\overline{\{y\}} \cap \mathbb{A}^n_k(k)$ is a conical subset of $\mathbb{A}^n_k(k) \simeq k^{\oplus n}$.

Notation 25.1.4. We use the following notations:

- For an element $f \in S$, we define

$$U_+(f) := \mathsf{Proj}(S) \cap U(f),$$

which is an open subset of $\mathsf{Proj}(S)$.
- For an ideal $I \subseteq S$, we define

$$Z_+(I) := \mathsf{Proj}(S) \cap Z(I),$$

which is a closed subset of $\mathsf{Proj}(S)$.

Unwinding the definitions, we have the following results.

Lemma 25.1.5. *Let S be a $\mathbb{Z}_{\geq 0}$-graded ring.*

- *For any element $f = \sum_{i=0}^n f_i \in S$ with $f_i \in S_i$, we have*

$$U_+(f) = \bigcup_{i=0}^n U_+(f_i)$$

- *For any element $f_0 \in S_0$, we have*

$$U_+(f_0) = \bigcup_{g \in S_n, n>0} U_+(f_0 g).$$

- *For any ideal $I \subseteq S$, we have*
$$Z_+(I) = Z_+\Big(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} (I \cap S_n)\Big).$$
- *For any ideal $I \subseteq S$ and $n \geq 0$, we have*
$$Z_+(I) = Z_+(\sqrt{I} \cap S_+).$$
Moreover, if I is homogeneous, so is $\sqrt{I} \cap S^+$.

As in the affine case, we have the following results.

Corollary 25.1.6. *The open subsets $U_+(f)$ form a basis for the topology of $\mathsf{Proj}(S)$, where f ranges over all homogeneous elements in S_+. We call them the **standard open subsets** of $\mathsf{Proj}(S)$.*

Corollary 25.1.7. *The closed subsets of $\mathsf{Proj}(S)$ correspond bijectively to homogeneous ideals $I \subseteq S_+$ such that $I = \sqrt{I} \cap S^+$.*

Notation 25.1.8. Let $f \in S_+$ be a homogeneous element. The ring S_f has a natural $\mathbb{Z}$-grading. Write $S_{(f)} \subseteq S_f$ for the subring consisting of elements of grading zero.

Remark 25.1.9. For $f \in S_n$, we have
$$S_{(f)} \simeq \Big(\bigoplus_{d\in\mathbb{Z}_{\geq 0}} S_{dn}\Big)\Big/(f-1).$$

Warning 25.1.10. The ring $S_{(f)}$ is *not* the localization of S with respect to the system $S \smallsetminus (f)$. In fact, (f) is in general not a prime ideal.

Exercise 25.1.11. Show that the composition
$$U_+(f) \subseteq U(f) \simeq \mathsf{Spec}(S_f) \to \mathsf{Spec}(S_{(f)})$$
is a homeomorphism.

25.2. **The scheme $\mathsf{Proj}(S)$.** Throughout this subsection, S is a $\mathbb{Z}_{\geq 0}$-graded ring. Write
$$\mathsf{Spec}(S)^\circ := \mathsf{Spec}(S) \smallsetminus \mathsf{Spec}(S_0) = \bigcup_{f\in\oplus_{n>0} S_n} U(f)$$
where we view $\mathsf{Spec}(S_0)$ as a closed subscheme of $\mathsf{Spec}(S)$ via the projection $S \to S_0$. Note that points of $\mathsf{Spec}(S)^\circ$ are (possibly non-homogenous) prime ideals $\mathfrak{p} \subseteq S$ that does not contain S_+.

Exercise 25.2.1. The map
$$\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S),\ \mathfrak{p} \mapsto \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\mathfrak{p} \cap S_n)$$
is well-defined and continuous. Moreover, for any homogeneous $f \in S_+$,
$$\pi^{-1}(U_+(f)) = U(f).$$

Construction 25.2.2. Consider the object
$$\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ} \in \mathsf{Shv}(\mathsf{Proj}(S), \mathsf{CRing}).$$
For any standard open subset $V \subseteq \mathsf{Proj}(S)$, choose a homogeneous element $f \in S_+$ such that $V = U_+(f)$. We have
$$(\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})(V) \simeq \mathcal{O}_{\mathsf{Spec}(S)^\circ}(U(f)) \simeq S_f.$$

This gives a $\mathbb{Z}$-grading on $(\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})(V)$. One can check this grading does not depend on the choice of f. Moreover, for standard open subsets $V_1 \subseteq V_2$, the restriction map

$$(\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})(V_2) \to (\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})(V_1)$$

is a $\mathbb{Z}$-graded homomorphism. Therefore we obtain an upgrade

$$\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ} \in \mathsf{Shv}(\mathsf{Proj}(S), \mathsf{CRing}_{\mathsf{gr}}),$$

where $\mathsf{CRing}_{\mathsf{gr}}$ is the category of $\mathbb{Z}$-graded rings. We now define

$$\mathcal{O}_{\mathsf{Proj}(S)} \coloneqq (\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})_0 \in \mathsf{Shv}(\mathsf{Proj}(S), \mathsf{CRing})$$

to be the degree zero piece of this object. We call it the **structure sheaf** of $\mathsf{Proj}(S)$.

We abuse notations and write $\mathsf{Proj}(S)$ for the ringed space $(\mathsf{Proj}(S), \mathcal{O}_{\mathsf{Proj}(S)})$. Note that we have a morphism $\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S)$ of ringed spaces given by the morphism

$$\mathcal{O}_{\mathsf{Proj}(S)} = (\pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ})_0 \to \pi_*\mathcal{O}_{\mathsf{Spec}(S)^\circ}.$$

Remark 25.2.3. By construction, on any standard open subset $U_+(f) \subseteq \mathsf{Proj}(S)$, we have

$$\mathcal{O}_{\mathsf{Proj}(S)}(U_+(f)) \simeq S_{(f)}.$$

Let $x \in \mathsf{Proj}(S)$ be a point and $\mathfrak{p} \subseteq S$ be the corresponding homogeneous ideal, we have

$$\mathcal{O}_{\mathsf{Proj}(S),x} \simeq S_{(\mathfrak{p})},$$

which is the degree zero piece of $S_{\mathfrak{p}}$.

Proposition 25.2.4. *The ringed space* $(\mathsf{Proj}(S), \mathcal{O}_{\mathsf{Proj}(S)})$ *is a scheme and* $\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S)$ *is an affine morphism of schemes.*

Proof. It is easy to see that for any standard open subset $U_+(f)$, the ringed space $(U_+(f), \mathcal{O}_{\mathsf{Proj}(S)}|_{U_+(f)})$ is isomorphic to the scheme $\mathsf{Spec}(S_{(f)})$. Moreover, over this open subspace, π can be identified with the morphism $\mathsf{Spec}(S_f) \to \mathsf{Spec}(S_{(f)})$ corresponding to the ring homomorphism $S_{(f)} \to S_f$.

□

Warning 25.2.5. The continuous map $\mathsf{Proj}(S) \to \mathsf{Spec}(S)$ in general cannot be upgraded to a morphism of schemes.

Remark 25.2.6. One should view $\mathsf{Proj}(S)$ as the quotient of $\mathsf{Spec}(S)^\circ$ by a dilation action.

Exercise 25.2.7. Let $\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S)$ be the canonical morphism.

(1) Show that π is an epimorphism of schemes.
(2) Show that $\mathsf{Spec}(S)^\circ \xrightarrow{\subseteq} \mathsf{Spec}(S) \to \mathsf{Spec}(S_0)$ uniquely factors through $\mathsf{Proj}(S)$.

In particular, if S is a $\mathbb{Z}_{\geq 0}$-graded A-algebra, then $\mathsf{Proj}(S)$ is canonically an A-scheme.

Exercise 25.2.8. Let R be a commutative ring and $n \geq 0$. Let $S = R[x_0, \cdots, x_n]$ be the $\mathbb{Z}_{\geq 0}$-graded ring such that $S_0 = R$ and $x_i \in S_1$. Show that $\mathsf{Proj}(S) \simeq \mathbb{P}^n_R$.

Warning 25.2.9. The $\mathbb{Z}_{\geq 0}$-graded commutative ring S cannot be recovered from $\mathsf{Proj}(S)$.

Exercise 25.2.10. Let d be a positive integer and $S' = \bigoplus_{n\geq 0} S'_{nd}$ be the $\mathbb{Z}_{\geq 0}$-graded ring such that $S'_{nd} \coloneqq S_n$. Show that $\mathsf{Proj}(S') \simeq \mathsf{Proj}(S)$.

25.3. **Functoriality.** Throughout this subsection, $\phi : S \to S'$ is a graded homomorphism of $\mathbb{Z}_{\geq 0}$-graded rings.

In general, we should not expect to have a morphism $\mathsf{Proj}(S') \to \mathsf{Proj}(S)$ because the morphism $\mathsf{Spec}(S') \to \mathsf{Spec}(S)$ does not send $\mathsf{Spec}(S')^\circ$ into $\mathsf{Spec}(S)^\circ$.

Notation 25.3.1. Consider

$$U(\phi) \coloneqq \mathsf{Spec}(S') \underset{\mathsf{Spec}(S)}{\times} \mathsf{Spec}(S)^\circ \simeq \bigcup_{f \in \bigoplus_{n>0} S_n} U(\phi(f)).$$

Note that $U(\phi)$ is contained in $\mathsf{Spec}(S')^\circ$. Moreover, it is the inverse image of

$$U_+(\phi) \coloneqq \bigcup_{f \in \bigoplus_{n>0} S_n} U_+(\phi(f)) \subseteq \mathsf{Proj}(S')$$

under the canonical morphism $\mathsf{Spec}(S')^\circ \to \mathsf{Proj}(S')$.

The following result follows from Construction 25.2.2.

Proposition 25.3.2. *There is a unique dotted morphism that makes the following diagram commute:*

$$\begin{tikzcd}
\mathsf{Spec}(S')^\circ \arrow[d, "\pi'"] & U(\phi) \arrow[l, "\supseteq"'] \arrow[r] \arrow[d] & \mathsf{Spec}(S)^\circ \arrow[d, "\pi"] \\
\mathsf{Proj}(S') & U_+(\phi) \arrow[l, "\supseteq"'] \arrow[r, dotted] & \mathsf{Proj}(S).
\end{tikzcd}$$

Exercise 25.3.3. Let S be a $\mathbb{Z}_{\geq 0}$-graded ring and d be a positive integer. Let $S' \subseteq S$ be the subring generated by elements $f \in S_{nd}$, $n \geq 0$. Show that $j : S' \to S$ induces isomorphisms $\mathsf{Proj}(S) \simeq U_+(j) \simeq \mathsf{Proj}(S')$.

Exercise 25.3.4. Let $\phi : S \to S'$ be a graded homomorphism such that $\phi_n : S_n \to S'_n$ is an isomorphism for large enough n. Show that it induces isomorphisms $\mathsf{Proj}(S) \simeq U_+(\phi) \simeq \mathsf{Proj}(S')$.

Lemma 25.3.5. *Let $A \to B$ be a homomorphism in* CRing. *Let S be a $\mathbb{Z}_{\geq 0}$-graded A-algebra. Note that $S \otimes_A B$ is naturally a $\mathbb{Z}_{\geq 0}$-graded B-algebra. Consider the $\mathbb{Z}_{\geq 0}$-graded homomorphism $\phi : S \to S \otimes_A B$. Then $U_+(\phi) = \mathsf{Proj}(S \otimes_A B)$ and the following diagram is Caretsian*

$$\begin{tikzcd}
\mathsf{Proj}(S \otimes_A B) \arrow[r] \arrow[d] & \mathsf{Proj}(S) \arrow[d] \\
\mathsf{Spec}(B) \arrow[r] & \mathsf{Spec}(A).
\end{tikzcd}$$

Proof. The first claim follows from the fact that the ideal $(S \otimes_A B)_+$ is generated by $\phi(S_+)$.

For any standard open subscheme $U_+(f)$ of $\mathsf{Proj}(S)$, we have a Cartesian square

$$\begin{tikzcd}
U_+(f \otimes 1) \arrow[r] \arrow[d] & U_+(f) \arrow[d] \\
\mathsf{Proj}(S \otimes_A B) \arrow[r] & \mathsf{Proj}(S).
\end{tikzcd}$$

Hence we only need to show

$$\begin{tikzcd} U_+(f\otimes 1) \arrow[r] \arrow[d] & U_+(f) \arrow[d] \\ \mathsf{Spec}(B) \arrow[r] & \mathsf{Spec}(A) \end{tikzcd}$$

is Cartesian. Indeed, this is true because

$$U_+(f\otimes 1) \simeq \mathsf{Spec}((S \underset{A}{\otimes} B)_{(f\otimes 1)}) \simeq \mathsf{Spec}(S_{(f)} \underset{A}{\otimes} B) \simeq U_+(f) \underset{\mathsf{Spec}(A)}{\otimes} \mathsf{Spec}(B).$$

□

25.4. **Properties of $\mathsf{Proj}(S)$.** Throughout this subsection, S is a $\mathbb{Z}_{\geq 0}$-graded ring.

The following results can be proved by inspecting the standard open subsets. We leave them to the readers.

Lemma 25.4.1. *Suppose S_+ contains no nilpotent elements. Then:*

1. *The scheme $\mathsf{Proj}(S)$ is reduced.*
2. *The scheme $\mathsf{Proj}(S)$ is integral iff S_+ contains no zero divisors.*

Lemma 25.4.2. *The scheme $\mathsf{Proj}(S)$ is quasi-compact iff there exists a finitely generated homogenous ideal $I \subseteq S_+$ such that $S_+ \subseteq \sqrt{I}$.*

Lemma 25.4.3. *The scheme $\mathsf{Proj}(S)$ is separated.*

Lemma 25.4.4. *If S is Noetherian, $\mathsf{Proj}(S)$ is a Noetherian scheme.*

Remark 25.4.5. It is known that S is Noetherian iff S_0 is Noetherian and S_+ is finitely generated as an ideal of S.

Lemma 25.4.6. *If S is finitely generated over S_0, the morphism $\mathsf{Proj}(S) \to \mathsf{Spec}(S_0)$ is of finite type.*

26. Quasi-coherent modules on $\mathsf{Proj}(S)$

Throughout this section, S is a $\mathbb{Z}_{\geq 0}$-graded ring. We write $\mathcal{O} := \mathcal{O}_{\mathsf{Proj}(S)}$.

26.1. $\widetilde{M}$.

Construction 26.1.1. Let M be a $\mathbb{Z}$-graded S-module. Consider the quasi-coherent module $\widetilde{M} \in \mathsf{QCoh}(\mathsf{Spec}(S))$ and its restriction

$$\widetilde{M}|_{\mathsf{Spec}(S)^{\circ}} \in \mathsf{QCoh}(\mathsf{Spec}(S)^{\circ}).$$

Recall the canonical affine morphism $\pi : \mathsf{Spec}(S)^{\circ} \to \mathsf{Proj}(S) = X$. It follows that we have a quasi-coherent module

$$\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^{\circ}}) \in \mathsf{QCoh}(\mathsf{Proj}(S)). \tag{26.1}$$

For any standard open subset $V \subseteq \mathsf{Proj}(S)$, choose a homogeneous element $f \in S_+$ such taht $V = U_+(f)$. We have

$$(\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^{\circ}}))(V) \simeq \widetilde{M}(U(f)) \simeq M_f.$$

This gives a $\mathbb{Z}$-grading on $(\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^{\circ}}))(V)$. One can check this grading does not depend on the choice of f, and is compatible with the restriction maps. It follows that we obtain a $\mathbb{Z}$-grading on (26.1). We abuse notations and define

$$\widetilde{M} := (\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^{\circ}}))_0 \in \mathsf{QCoh}(\mathsf{Proj}(S))$$

to be the degree zero piece of (26.1). We call it the **quasi-coherent module on** $\mathsf{Proj}(S)$ **associated to the graded module** M.

Remark 26.1.2. By construction, on any standard open subset $U_+(f) \subseteq \mathsf{Proj}(S)$, we have

$$\widetilde{M}|_{U_+(f)} \simeq \widetilde{M_{(f)}} \in \mathsf{QCoh}(U_+(f))$$

where $M_{(f)} := (M_f)_0$ is a module of $S_{(f)}$.

For $f \in S_n$, we have

$$M_{(f)} \simeq (\bigoplus_{d \in \mathbb{Z}} M_{dn})\big/(f-1).$$

Example 26.1.3. For $M = S$, we have $\widetilde{M} \simeq \mathcal{O}$.

Note that $\mathbb{Z}$-graded S-modules form an abelian category. The above construction is functorial in M, i.e., we have a functor

$$S\text{–}\mathsf{mod}_{\mathsf{gr}} \to \mathsf{QCoh}(\mathsf{Proj}(S)) \tag{26.2}$$

sending a $\mathbb{Z}$-graded S-module M to the quasi-coherent $\mathcal{O}$-module $\widetilde{M}$.

The following result follows from Remark 26.1.2.

Lemma 26.1.4. *The functor* (26.2) *is exact and commutes with (small) colimits.*

Warning 26.1.5. The functor (26.2) is not an equivalence. In fact, it is not even conservative.

Definition 26.1.6. We say a $\mathbb{Z}$-graded S-module M is **eventually zero** if $M_n = 0$ for $n >> 0$.

Lemma 26.1.7. *If M is eventually zero, then $\widetilde{M} = 0$.*

Proof. For any homogeneous $f \in S_+$, we have $M_{(f)} = 0$. It follows that $\widetilde{M}|_{U_+(f)} = 0$, which implies $\widetilde{M} = 0$.

□

Notation 26.1.8. Let M be a $\mathbb{Z}$-graded S-module and n be an integer. Write $M(n)$ for the $\mathbb{Z}$-graded S-module defined by the following formula:

$$M(n)_m := M_{m+n}.$$

Remark 26.1.9. By construction, we have

$$\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^\circ}) \simeq \bigoplus_{n\in\mathbb{Z}} \widetilde{M(n)}$$

as $\mathbb{Z}$-graded quasi-coherent $\mathcal{O}$-modules.

Lemma 26.1.10. *Let $U_+(f) \subseteq \mathsf{Proj}(S)$ be a standard open subset. Suppose M be a $\mathbb{Z}$-graded S-module such that $M \xrightarrow{f} M$ is a bijection. Then $\widetilde{M}$ is an $*$-extension along the open embedding $U_+(f) \to \mathsf{Proj}(S)$.*

Proof. The assumption implies M is an S_f-module. It follows that the object $\widetilde{M} \in \mathsf{QCoh}(\mathsf{Spec}(S)^\circ)$ is an $*$-extension along the open embedding $U(f) \to \mathsf{Spec}(S)^\circ$. Recall the image of $U(f)$ under $\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S)$ is $U_+(f)$. It follows that the object $\widetilde{M} \in \mathsf{QCoh}(\mathsf{Proj}(S))$ is an $*$-extension along the open embedding $U_+(f) \to \mathsf{Proj}(S)$.

□

26.2. **Tautological line bundle.** Throughout this subsection, we assume the ideal S_+ is generated by elements in S_1.

Warning 26.2.1. All the results in this subsection may fail if S_+ is not generated by S_1.

Proposition-Definition 26.2.2. *For any $n \in \mathbb{Z}$, the $\mathcal{O}$-module*

$$\mathcal{O}(n) := \widetilde{S(n)}$$

is invertible. We call $\mathcal{O}(1)$ the ***tautological line bundle*** *on $\mathsf{Proj}(S)$.*

Proof. The assumption on S implies $\mathsf{Proj}(S)$ can be covered by $U_+(f)$ with $f \in S_1$. Hence we only need to show $\widetilde{S(n)}|_{U_+(f)}$ is locally free of rank 1. By definition, it is the quasi-coherent $\mathcal{O}_{U_+(f)}$ associated to the $S_{(f)}$-module $S(n)_{(f)}$.

Since f is of degree 1, for any $d \in \mathbb{Z}$, the maps $f : (S_f)_d \to (S_f)_{d+1}$ and $f^{-1} : (S_f)_{d+1} \to (S_f)_d$ are inverse to each other. It follows that

$$S(n)_{(f)} \simeq (S_f)_n \simeq (S_f)_0 \simeq S_(f).$$

This implies $\widetilde{S(n)}|_{U_+(f)} \simeq \mathcal{O}_{U_+(f)}$.

□

Corollary 26.2.3. *For any $n \in \mathbb{Z}$, we have an automorphism*

$$\mathsf{QCoh}(\mathsf{Proj}(S)) \to \mathsf{QCoh}(\mathsf{Proj}(S)),\ \mathcal{F} \mapsto \mathcal{F}(n) := \mathcal{F} \underset{\mathcal{O}}{\otimes} \mathcal{O}(n).$$

Exercise 26.2.4. For any $\mathbb{Z}$-graded S-module M, we have

$$\pi^*(\widetilde{M}) \simeq \widetilde{M}|_{\mathsf{Spec}(S)^\circ}$$

as objects in $\mathsf{QCoh}(\mathsf{Spec}(S)^\circ)$.

Construction 26.2.5. Let M and N be a $\mathbb{Z}$-graded S-modules. Recall $\pi : \mathsf{Spec}(S)^\circ \to \mathsf{Proj}(S)$ is affine. By Exercise 20.2.3, we have isomorphisms

(26.3)
$$\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^\circ}) \underset{\pi_*(\mathcal{O}_{\mathsf{Spec}(S)^\circ})}{\otimes} \pi_*(\widetilde{N}|_{\mathsf{Spec}(S)^\circ}) \simeq \pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^\circ} \underset{\mathcal{O}_{\mathsf{Spec}(S)^\circ}}{\otimes} \widetilde{N}|_{\mathsf{Spec}(S)^\circ}) \simeq \simeq \pi_*(\widetilde{M \underset{S}{\otimes} N}|_{\mathsf{Spec}(S)^\circ})$$

in $\mathsf{QCoh}(\mathsf{Proj}(S))$. Moreover, one can check these isomorphisms are compatible with the natural $\mathbb{Z}$-gradings on these objects. It follows that we obtain a *morphism*

(26.4)
$$\pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^\circ})_0 \underset{\pi_*(\mathcal{O}_{\mathsf{Spec}(S)^\circ})_0}{\otimes} \pi_*(\widetilde{N}|_{\mathsf{Spec}(S)^\circ})_0 \to \to \pi_*(\widetilde{M}|_{\mathsf{Spec}(S)^\circ}) \underset{\pi_*(\mathcal{O}_{\mathsf{Spec}(S)^\circ})}{\otimes} \pi_*(\widetilde{N}|_{\mathsf{Spec}(S)^\circ})_0 \simeq \pi_*(\widetilde{M \underset{S}{\otimes} N}|_{\mathsf{Spec}(S)^\circ})_0$$

in $\mathsf{QCoh}(\mathsf{Proj}(S))$. In other words, there is a canonical morphism

(26.5)
$$\widetilde{M} \underset{\mathcal{O}}{\otimes} \widetilde{N} \to \widetilde{M \underset{S}{\otimes} N}$$

in $\mathsf{QCoh}(\mathsf{Proj}(S))$.

Lemma 26.2.6. *The canonical morphism* (26.5) *is an isomorphism.*

Proof. We can restrict to a standard open subset $U_+(f)$ with $f \in S_1$. Then (26.5) is the morphism associated to

$$M_{(f)} \underset{S_{(f)}}{\otimes} N_{(f)} \to (M \underset{S}{\otimes} N)_{(f)},$$

which can be checked to be an isomorphism. □

Remark 26.2.7. In fact, one can show

$$S\text{−}\mathsf{mod}_{\mathsf{gr}} \to \mathsf{QCoh}(\mathsf{Proj}(S)),\ M \mapsto \widetilde{M}$$

is a symmetric monoidal functor.

Corollary 26.2.8. *We have*

$$\mathcal{O}(m) \underset{\mathcal{O}}{\otimes} \mathcal{O}(n) \simeq \mathcal{O}(m+n).$$

In particular, for $\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}(S))$, we have

$$(\mathcal{F}(m))(n) \simeq \mathcal{F}(m+n).$$

Corollary 26.2.9. *Let M be a $\mathbb{Z}$-graded S-module. We have*

$$\widetilde{M(n)} \simeq \widetilde{M}(n).$$

Definition 26.2.10. Let M be a $\mathbb{Z}$-graded S-module. We say M is **of (graded) finite presentation** if there exists an exact sequence

$$\bigoplus_{i=1}^{m} S(d_i) \to \bigoplus_{j=1}^{n} S(e_j) \to M \to 0$$

of $\mathbb{Z}$-graded S-modules.

Exercise 26.2.11. Let M be a $\mathbb{Z}$-graded S-module that is of finite presentation. Show that:

(1) The $\mathcal{O}$-module $\widetilde{M}$ is of finite presentation.

(2) For any $\mathbb{Z}$-graded S-module N, we have
$$\underline{\mathsf{Hom}}_{\mathcal{O}}(\widetilde{M},\widetilde{N}) \simeq \widetilde{\mathsf{Hom}^{\mathsf{gr}}_S(M,N)},$$
where $\mathsf{Hom}^{\mathsf{gr}}_S(M,N)$ is the $\mathbb{Z}$-graded S-module such that
$$\mathsf{Hom}^{\mathsf{gr}}_S(M,N)_d := \mathsf{Hom}_{S-\mathsf{mod}_{\mathsf{gr}}}(M,N(d)).$$

26.3. **Twisted global sections.** Throughout this subsection, we assume the ideal S_+ is generated by elements in S_1.

Construction 26.3.1. Note that $\bigoplus_{n\in\mathbb{Z}}\mathcal{O}(n)$ is naturally a $\mathbb{Z}$-graded sheaf of commutative rings on $\mathsf{Proj}(S)$. It follows that
$$\Gamma(\mathcal{O})_{\mathsf{gr}} := \bigoplus_{n\in\mathbb{Z}}\Gamma(\mathsf{Proj}(S),\mathcal{O}(n))$$
is naturally a $\mathbb{Z}$-graded commutative ring.

For any $\mathcal{O}$-module $\mathcal{F}$,
$$\bigoplus_{n\in\mathbb{Z}}\mathcal{F}(n)$$
is naturally a $\mathbb{Z}$-graded module for $\bigoplus_{n\in\mathbb{Z}}\mathcal{O}(n)$. It follows that
$$\Gamma(\mathcal{F})_{\mathsf{gr}} := \bigoplus_{n\in\mathbb{Z}}\Gamma(\mathsf{Proj}(S),\mathcal{F}(n))$$
is naturally a $\mathbb{Z}$-graded $\Gamma(\mathcal{O})_{\mathsf{gr}}$-module.

The above construction is functorial in $\mathcal{F}$. In other words, we have a functor
$$\Gamma(-)_{\mathsf{gr}} : \mathcal{O}\text{−}\mathsf{mod} \to \Gamma(\mathcal{O})_{\mathsf{gr}}\text{−}\mathsf{mod}_{\mathsf{gr}}.$$

Construction 26.3.2. Let M be a $\mathbb{Z}$-graded S-module M.

For any standard open subset $V \subseteq \mathsf{Proj}(S)$, by Remark 26.1.9, we have a canonical morphism of abelian groups:
$$M \to \Gamma(\pi^{-1}(V),\widetilde{M}) \simeq \Gamma(V,\bigoplus_{n\in\mathbb{Z}}\widetilde{M}(n)) \simeq \bigoplus_{n\in\mathbb{Z}}\Gamma(V,\widetilde{M}(n)). \tag{26.6}$$
We can choose a homogeneous element $f \in S_+$ such that $V = U_+(f)$. Then (26.6) can be identified with the canonical morphism $M \to M_f$. It follows that (26.6) is compatible with the $\mathbb{Z}$-gradings on both sides.

By construction, (26.6) is compatible with the restriction maps. Hence we obtain a $\mathbb{Z}$-graded morphism
$$M \to \bigoplus_{n\in\mathbb{Z}}\Gamma(\mathsf{Proj}(S),\widetilde{M}(n)) = \Gamma(\widetilde{M})_{\mathsf{gr}}. \tag{26.7}$$
In the particular case when $M = S$, one can check the obtained map
$$S \to \Gamma(\mathcal{O})_{\mathsf{gr}} \tag{26.8}$$
is a homomoprhism between $\mathbb{Z}$-graded commutative rings. Moreover, (26.7) is compatible with the action of (26.8) on both sides. In other words, (26.7) is a morphism between $\mathbb{Z}$-graded S-modules, where S acts on $\Gamma(\widetilde{M})_{\mathsf{gr}}$ via the homomorphism (26.8).

The morphism (26.7) is functorial in M. In other words, we obtain a natural transformation
$$\mathsf{id} \to \Gamma(\widetilde{-})_{\mathsf{gr}} \tag{26.9}$$
between endomorphisms on $S\text{−}\mathsf{mod}_{\mathsf{gr}}$.

Proposition 26.3.3. *The natural transformation* (26.9) *induces an adjoint pair*

$$\begin{array}{ccc} S\text{–}\mathsf{mod}_{\mathsf{gr}} & \rightleftarrows & \mathcal{O}\text{–}\mathsf{mod} \\ M & \mapsto & \widetilde{M} \\ \Gamma(\mathcal{F})_{\mathsf{gr}} & \leftarrow\!\shortmid & \mathcal{F}. \end{array}$$

Proof. For any $\mathbb{Z}$-graded S-module M and any $\mathcal{O}$-module $\mathcal{F}$, the following data are equivalent:

(i) A morphism
$$M \to \Gamma(\mathcal{F})_{\mathsf{gr}}$$
between $\mathbb{Z}$-graded S-modules.

(ii) For any standard open subset $V \subseteq \mathsf{Proj}(S)$, a morphism
$$M \to \bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{F}(n))$$
between $\mathbb{Z}$-graded S-modules, such that these morphisms are compatible with restriction maps for V.

Since $\mathsf{Proj}(S)$ is covered by $U_+(f)$ with $f \in S_1$, in (ii), we can assume $V = U_+(f)$ for some $f \in S_1$.

Unwinding the definitions, S acts on $\bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{F}(n))$ via the homomorphism
$$S \to \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathsf{Proj}(S), \mathcal{O}(n)) \xrightarrow{\mathsf{res}} \bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{O}(n)).$$

It follows that (ii) is equivalent to the following data:

(iii) For any standard open subset $V \subseteq \mathsf{Proj}(S)$, a morphism
$$\Big(\bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{O}(n))\Big) \underset{S}{\otimes} M \to \bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{F}(n))$$
between $\mathbb{Z}$-graded $\bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{F}(n))$-modules, such that these morphisms are compatible with restriction maps for V.

We claim there is a canonical isomorphism
$$\Big(\bigoplus_{n\in\mathbb{Z}} \Gamma(V, \mathcal{O}(n))\Big) \underset{S}{\otimes} M \simeq \bigoplus_{n\in\mathbb{Z}} \Gamma(V, \widetilde{M}(n)).$$

Indeed, choose $f \in S_1$ such that $V = U_+(f)$, the isomorphism is given by
$$\Big(\bigoplus_{n\in\mathbb{Z}} S(n)_{(f)}\Big) \underset{S}{\otimes} M \simeq S_f \underset{S}{\otimes} M \simeq M_f \simeq \bigoplus_{n\in\mathbb{Z}} M(n)_{(f)}.$$

Since V is affine, this implies (iii) is equivalent to the following data:

(iv) For any standard open subset $V \subseteq \mathsf{Proj}(S)$, a morphism
$$\bigoplus_{n\in\mathbb{Z}} \widetilde{M}(n)|_V \to \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)|_V$$
between $\mathbb{Z}$-graded $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)|_V$-modules, such that these morphisms are compatible with restriction maps for V.

This is equivalent to

(v) A morphism
$$\bigoplus_{n\in\mathbb{Z}} \widetilde{M}(n) \to \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)$$
between $\mathbb{Z}$-graded $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}(n)$-modules.

Since $\mathcal{O}(1)$ is invertible, this is equivalent to

(vi) A morphism
$$\widetilde{M} \to \mathcal{F}$$
between $\mathcal{O}$-modules.

□

26.4. **Finite type case.**

Definition 26.4.1. Let M be a $\mathbb{Z}$-graded S-module. We say M is **eventually of finite type** if there exists an integer N such that
$$\bigoplus_{n\geq N} M_n$$
is an S-module of finite type.

Lemma 26.4.2. *Suppose S_+ is generated by finitely many elements and M is a $\mathbb{Z}$-graded S-module that is eventually of finite type. Then $\widetilde{M}$ is of finite type.*

Proof. By Lemma 26.1.7 and Lemma 26.1.4, we can replace M by $\oplus_{n\geq N} M_n$ for large enough N, and therefore assume M is of finite type.

We only need to show that $\widetilde{M}|_{U_+(f)}$ is of finite type for any standard open $U_+(f) \subseteq \mathsf{Proj}(S)$. For this purpose, we only need to show that $M_{(f)}$ is a finitely generated $S_{(f)}$-module. This is an easy exercise in commutative algebra.

□

Exercise 26.4.3. Suppose S_+ is generated by finitely many elements in S_1 and M is a $\mathbb{Z}$-graded S-module that is eventually of finite type. Then $\widetilde{M} = 0$ iff M is eventually zero.

Warning 26.4.4. Exercise 26.4.3 would be false if we do not assume S_+ is generated by S_1.

Example 26.4.5. Let S be a $\mathbb{Z}_{\geq 0}$-graded ring that is concentrated in even degrees. For any $\mathbb{Z}$-graded S-module M that is concentrated in odd degrees, $\widetilde{M} \simeq 0$ in $\mathsf{QCoh}(\mathsf{Proj}(S))$.

Theorem 26.4.6. *Suppose S_+ is generated by finitely many elements in S_1. Then the functor*
$$\Gamma(-)_{\mathsf{gr}} : \mathsf{QCoh}(\mathsf{Proj}(S)) \to S\text{-}\mathsf{mod}_{\mathsf{gr}}$$
is fully faithful.

Proof. By Proposition 26.3.3, we only need to show that for any $\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}(S))$, the canonical natural transformation
$$\widetilde{\Gamma(-)_{\mathsf{gr}}} \to \mathsf{id} \tag{26.10}$$
sends $\mathcal{F}$ to an isomorphism.

Indeed, By Lemma 26.1.4, the functor $\widetilde{(-)}$ is exact. By Proposition 26.3.3, as a right adjoint, the functor $\Gamma(-)_{\mathsf{gr}}$ is left exact. It follows that the composition $\widetilde{\Gamma(-)_{\mathsf{gr}}}$ is left-exact.

By assumption, we can choose finitely many elements $f_i \in S_1$, $i \in I$ such that
$$\mathsf{Proj}(S) = \bigcup_{i\in I} U_+(f_i).$$

Write $U_i := U_+(f_i)$ and let $g_i : U_i \to \mathsf{Proj}(S)$ and $g_{ij} : U_i \cap U_j \to \mathsf{Proj}(S)$ for the open immersions. For any $\mathcal{O}$-module $\mathcal{F}$, we have an exact sequence

$$0 \to \mathcal{F} \to \bigoplus_{i \in I} g_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus_{(i,j) \in I^2} g_{ij,*}(\mathcal{F}|_{U_i \cap U_j})$$

that is functorial in $\mathcal{F}$. Hence we only need to show that (26.10) sends each $g_{i,*}(\mathcal{F}|_{U_i})$ and each $g_{ij,*}(\mathcal{F}|_{U_i \cap U_j})$ to an isomorphism.

Let $f \in S_1$, $U := U_+(f)$ and $g : U \to \mathsf{Proj}(S)$ be the open immersion. We only need to show that for any $\mathcal{M} \in \mathsf{QCoh}(U_+(f))$, the natural transformation (26.10) sends $\mathcal{M}$ to an isomorphism. Recall that $U \simeq \mathsf{Spec}(S_{(f)})$ is affine. Let

$$M = \Gamma(U, \mathcal{M})$$

be the $S_{(f)}$-module associated to $\mathcal{M}$.

We first show $(\Gamma(g_*\mathcal{M})_{\mathsf{gr}})^\sim$ is a $*$-extension along $g : U \to \mathsf{Proj}(S)$. By definition, we have

$$\begin{aligned}\Gamma(g_*\mathcal{M})_{\mathsf{gr}} := \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathsf{Proj}(S), (g_*\mathcal{M})(n)) \simeq \bigoplus_{n \in \mathbb{Z}} \Gamma(U, \mathcal{M} \underset{\mathcal{O}_U}{\otimes} \mathcal{O}(n)|_U) \simeq \\ \simeq \bigoplus_{n \in \mathbb{Z}} M \underset{S_{(f)}}{\otimes} S(n)_{(f)} \simeq M \underset{S_{(f)}}{\otimes} S_f.\end{aligned} \tag{26.11}$$

In particular, $\Gamma(\mathcal{M})_{\mathsf{gr}}$ is an S_f-module. Hence by Lemma 26.1.10, $(\Gamma(g_*\mathcal{M})_{\mathsf{gr}})^\sim$ is a $*$-extension along $g : U \to \mathsf{Proj}(S)$.

It remains to show that the canonical morphism

$$(\Gamma(g_*\mathcal{M})_{\mathsf{gr}})^\sim|_U \to \mathcal{M}$$

is an isomorphism. Since $U \simeq \mathsf{Spec}(S_{(f)})$ is affine, we only need to show the corresponding morphism between $S_{(f)}$-modules is an isomorphism. By (26.11), this morphism is just the obvious isomorphism

$$(M \underset{S_{(f)}}{\otimes} S_f)_0 \simeq M.$$

□

Corollary 26.4.7. *Suppose S_+ is generated by finitely many elements in S_1. Then any $\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}(S))$ is of the form $\widetilde{M}$. Moreover, if $\mathcal{F}$ is of finite type, we can choose M such that M is of finite type.*

Proof. The first claim follows immediately from Theorem 26.4.6. For the second claim, choose a $\mathbb{Z}$-graded S-module such that $\mathcal{F} \simeq \widetilde{M}$. Write $M = \bigcup M_\alpha$ as a union of $\mathbb{Z}$-graded submodules such that each M_α is generated by finitely many homogeneous elements. We have $\widetilde{M} \simeq \bigcup \widetilde{M_\alpha}$. Using the fact that $\mathsf{Proj}(S)$ is quasi-compact and $\widetilde{M}$ is of finite type, it is easy to see for large enough index α, $\widetilde{M} = \widetilde{M_\alpha}$.

□

Proposition 26.4.8. *Suppose S_+ is generated by finitely many elements in S_1. For any $\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}(S))$ that is finite type, there exists an integer N such that for $n \geq N$, the $\mathcal{O}_X$-module $\mathcal{F}(n)$ is generated by finitely many global sections, i.e., there exists a surjection of the form $\mathcal{O}^{\oplus m} \to \mathcal{F}(n)$.*

Proof. By Corollary 26.4.7, we can asssume $\mathcal{F} = \widetilde{M}$ for some $\mathbb{Z}$-graded S-module M that is of finite type. Note that there exists a surjection of the form $\oplus_{i \in I} S(n_i) \to M$

with $|I| < \infty$. Hence we only need to prove the claim for each $\widetilde{S(n_i)}$. We leave this as an exercise for the readers.

$\square$

Exercise 26.4.9. Suppose S_+ is generated by S_1. Show that for any $n \geq 0$, the $\mathcal{O}$-module $\mathcal{O}(n)$ is generated by its global sections that belong to the image of the canonical map

$$S_n \to \Gamma(\mathsf{Proj}(S), \mathcal{O}(n)).$$

26.5. **Application: closed subschemes and properness.**

Theorem 26.5.1. *Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring.*

(1) *For any graded ideal $I \subseteq S$, the homomorphism $\phi : S \to S/I$ induces a closed immersion*

$$\mathsf{Proj}(S/I) \to \mathsf{Proj}(S), \tag{26.12}$$

whose ideal of definition is given by $\widetilde{I}$.

(2) *Suppose S_+ is generated by finitely many elements in S_1. Then any closed immersion $i : X \to \mathsf{Proj}(S)$ is of the form* (26.12) *for*

$$I := \ker(S \to \Gamma(\mathcal{O})_{\mathsf{gr}} \to \Gamma(i_*\mathcal{O}_X)_{\mathsf{gr}}).$$

Proof. For (1), note that $\phi(S_+)$ generates $(S/I)_+$. This implies $U_+(\phi) = \mathsf{Proj}(S/I)$. Hence we have a well-defined morphism $\mathsf{Proj}(S/I) \to \mathsf{Proj}(S)$. By definition, over a standard open subset $U_+(f) \subseteq \mathsf{Proj}(S)$, this morphism is given by $\mathsf{Spec}(-)$ of the homomorphism $S_{(f)} \to (S/I)_{(f)}$, which is a surjection with kernel equal to $I_{(f)}$. This implies the claim in (1).

For (2), let $\mathcal{I} := \ker(\mathcal{O} \to i_*\mathcal{O}_X)$ be the ideal of definition for $i : X \to \mathsf{Proj}(S)$. By Theorem 26.4.6, we have

$$\big(\Gamma(i_*\mathcal{O}_X)_{\mathsf{gr}}\big)^{\sim} \xrightarrow{\simeq} i_*\mathcal{O}_X.$$

Recall we also have $\widetilde{S} \simeq \mathcal{O}$. It follows that

$$\mathcal{I} \simeq \ker(\widetilde{S} \to \big(\Gamma(i_*\mathcal{O}_X)_{\mathsf{gr}}\big)^{\sim}).$$

Unwinding the definitions, the morphism in the RHS is obtained by applying $\widetilde{(-)}$ to the composition

$$S \to \Gamma(\mathcal{O})_{\mathsf{gr}} \to \Gamma(i_*\mathcal{O}_X)_{\mathsf{gr}}.$$

Since $\widetilde{(-)}$ is exact, we obtain that $\mathcal{I} \simeq \widetilde{I}$. Therefore $\mathsf{Proj}(S/I) \simeq X$ by (1).

$\square$

Exercise 26.5.2. Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by S_1. Let $I \subseteq S$ be a graded ideal and $i : \mathsf{Proj}(S/I) \to \mathsf{Proj}(S)$ be the corresponding closed immersion. Then

$$i^*(\mathcal{O}_{\mathsf{Proj}(S)}(n)) \simeq \mathcal{O}_{\mathsf{Proj}(S/I)}(n).$$

Corollary 26.5.3. *Suppose S_+ is generated by finitely many elements in S_1. Then $p : \mathsf{Proj}(S) \to \mathsf{Spec}(S_0)$ is proper.*

Proof. By Lemma 25.4.6 and Lemma 25.4.3, it remains to show that p is universally closed. By Lemma 25.3.5, we only need to show that p is closed.

Let $X \subseteq \mathsf{Proj}(S)$ be a closed subset and X_{red} be the corresponding reduced closed subscheme. By Theorem 26.5.1, we have $X \simeq \mathsf{Proj}(S/I)$ for some graded ideal $I \subseteq S$.

Hence we can replace S with S/I, and therefore reduce to show that $p(\mathsf{Proj}(S))$ is a closed subset of $\mathsf{Spec}(S_0)$.

By definition, a point $x \in \mathsf{Spec}(S_0)$ is not in the image of p iff $p^{-1}(x) = \varnothing$. We have

$$\mathsf{Proj}(S) \underset{\mathsf{Spec}(S_0)}{\times} \mathsf{Spec}(\kappa_x) \simeq \mathsf{Proj}(S \underset{S_0}{\otimes} \kappa_x).$$

Hence $x \notin p(\mathsf{Proj}(S))$ iff $\mathcal{O}_{S \otimes_{S_0} \kappa_x} = 0$. By 26.4.3, this is equivalent to the condition that $S \otimes_{S_0} \kappa_x$ is eventually zero, i.e. $S_n \otimes_{S_0} \kappa_x = 0$ for $n >> 0$. Note that S_n is a finitely generated S_0-module. Hence by Nakayama's lemma, the above condition is equivalent to $(S_n)_{\mathfrak{p}} = 0$ for $n >> 0$, where $\mathfrak{p}_x \subseteq S_0$ is the prime ideal corresponding to x, i.e.,

$$\mathsf{Ann}_{S_0}(S_n) \nsubseteq \mathfrak{p}_x,\ n >> 0.$$

Since S_+ is generated by S_1, we have $\mathsf{Ann}_{S_0}(S_n) \subseteq \mathsf{Ann}_{S_0}(S_{n+1})$. Let $\mathfrak{a}$ be the union of these annihilators. We obtain that $x \notin \mathsf{im}(p)$ iff $\mathfrak{a} \notin \mathfrak{p}_x$. In other words, $\mathsf{im}(p) = Z(\mathfrak{a})$ is a closed subset as desired.

□

27. Relative Proj

Throughout this section, Y is a scheme and $\mathcal{S}$ be a $\mathbb{Z}_{\geq 0}$-graded quasi-coherent $\mathcal{O}_Y$-algebra. By definition, this means we have a quasi-coherent $\mathcal{O}_Y$-algebra $\mathcal{S}$ equipped with a decomposition of $\mathcal{S}$ as $\mathcal{O}_Y$-modules

$$\mathcal{S} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{S}_n$$

that is compatible with the multiplication structure on $\mathcal{S}$.

27.1. **The relative Proj construction.** The following construction generalizes $\mathsf{Proj}(S)$ to the relative case.

Construction 27.1.1. For any affine open subset $U \subseteq Y$, the restriction $\mathcal{S}|_U$ corresponds to the $\mathbb{Z}_{\geq 0}$-graded $\mathcal{O}_Y(U)$-algebra $\mathcal{S}(U)$. Consider the scheme $\mathsf{Proj}(\mathcal{S}(U))$ and the canonical morphism

$$p_U : \mathsf{Proj}(\mathcal{S}(U)) \to \mathsf{Spec}(\mathcal{S}(U)) \simeq U.$$

For affine open subsets $U \subseteq V \subseteq Y$, since $\mathcal{S}$ is quasi-coherent, we have

$$\mathcal{S}(U) \simeq \mathcal{S}(V) \underset{\mathcal{O}_Y(V)}{\otimes} \mathcal{O}_Y(U).$$

Hence by Lemma 25.3.5, we have a canonical Cartesian square

$$\begin{tikzcd} \mathsf{Proj}(\mathcal{S}(U)) \arrow[r] \arrow[d, "p_U"] & \mathsf{Proj}(\mathcal{S}(V)) \arrow[d, "p_V"] \\ U \arrow[r] & V. \end{tikzcd}$$

It follows that there is an essentially unique Y-scheme, which we denote by

$$p : \mathsf{Proj}_Y(\mathcal{S}) \to Y,$$

equipped with an U-isomorphism

$$\mathsf{Proj}_Y(\mathcal{S})_U \simeq \mathsf{Proj}(\mathcal{S}(U))$$

for any affine open subset $U \subseteq Y$, such that for affine open subsets $U \subseteq V \subseteq Y$, the following diagram commutes

$$\begin{tikzcd} \mathsf{Proj}_Y(\mathcal{S})_U \arrow[r] \arrow[d, "\simeq"] & \mathsf{Proj}_Y(\mathcal{S})_V \arrow[d, "\simeq"] \\ \mathsf{Proj}(\mathcal{S}(U)) \arrow[r] & \mathsf{Proj}(\mathcal{S}(V)). \end{tikzcd}$$

The morphism $p : \mathsf{Proj}_Y(\mathcal{S}) \to Y$ is separated because each $\mathsf{Proj}_Y(\mathcal{S})_U$ us separated.

Lemma 27.1.2. *Let $f : Y' \to Y$ be a morphism of schemes. There is a canonical isomorphism*

$$\mathsf{Proj}_Y(\mathcal{S}) \underset{Y}{\times} Y' \simeq \mathsf{Proj}_{Y'}(f^*\mathcal{S}).$$

Proof. Follows from Lemma 25.3.5

□

The following construction generalizes the standard open subsets of $\mathsf{Proj}(S)$ to the relative case:

Construction 27.1.3. For any $f \in \mathcal{S}_n(Y)$ with $n > 0$, there is a unique open subset $U_+(f) \subseteq \mathsf{Proj}_Y(\mathcal{S})$ such that for any open subset $V \subseteq Y$, we have

$$U_+(f)_V \simeq U_+(f|_V)$$

as open subsets of $\mathsf{Proj}_Y(\mathcal{S})_V \simeq \mathsf{Proj}(\mathcal{S}(V))$.

By construction, $U_+(f) \to Y$ is affine, and we have

$$U_+(f) \simeq \mathsf{Spec}_Y\Big(\big(\bigoplus_{d\in\mathbb{Z}_{\geq 0}} \mathcal{S}_{nd}\big)\big/(f-1)\Big).$$

The following construction generalizes the functoriality of $\mathsf{Proj}(S)$ in S.

Construction 27.1.4. Let $\phi : \mathcal{S} \to \mathcal{S}'$ be a graded homomorphism between $\mathbb{Z}_{\geq 0}$-graded quasi-coherent $\mathcal{O}_Y$-algebras.

There is a unique open susbet $U_+(\phi) \subseteq \mathsf{Proj}_Y(\mathcal{S}')$ such that for any open subset $V \subseteq Y$, we have

$$U_+(\phi)_V \simeq U_+(\phi|_V)$$

as open subsets of $\mathsf{Proj}_Y(\mathcal{S}')_V \simeq \mathsf{Proj}(\mathcal{S}(V))$.

There is a unique morphism $U_+(\phi) \to \mathsf{Proj}_Y(\mathcal{S})$ such that its restriction over any open subset $V \subseteq Y$ is equal to the canonical morphism $U_+(\phi)_V \to \mathsf{Proj}(\mathcal{S}(V))$ via the identifications in the last paragraph.

27.2. **Quasi-coherent modules.** The following construction generalizes $\widetilde{M}$ to the relative case.

Construction 27.2.1. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded quasi-coherent $\mathcal{S}$-module. There is a unique object

$$\widetilde{\mathcal{M}} \in \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S}))$$

equipped with an isomorphism

$$\widetilde{\mathcal{M}}|_{\mathsf{Proj}(\mathcal{S}(U))} \simeq \widetilde{\mathcal{M}(U)} \in \mathsf{QCoh}(\mathsf{Proj}(\mathcal{S}(U)))$$

for any affine open subset $U \subseteq Y$, such that these isomorphisms are functorial in U.

We have a functor

$$\mathcal{S}\text{–}\mathsf{mod}_{\mathsf{gr,qcoh}} \to \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S})),\ \mathcal{M} \to \widetilde{\mathcal{M}}, \tag{27.1}$$

which is exact and commutes with small colimits.

Definition 27.2.2. We say the augmentation ideal $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ if there exists a covering $Y = \bigcup_{i\in I} U_i$ by affine open subsets such that for each $i \in I$, the augmentation ideal $\mathcal{S}_+(U_i)$ is generated by $\mathcal{S}_1(U_i)$.

Remark 27.2.3. It is easy to see that $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ as an ideal of $\mathcal{S}$ iff $\mathcal{S}$ is generated by $\mathcal{S}_1$ as an $\mathcal{S}_0$-algebra.

Corollary 27.2.4. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$. Then*

$$\mathcal{O}_{\mathsf{Proj}_Y(\mathcal{S})}(1) := \widetilde{\mathcal{S}(1)} \in \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S}))$$

is a line bundle on $\mathsf{Proj}_Y(\mathcal{S})$.

Proof. Follows from Proposition-Definition 26.2.2. □

The following construction generalizes $\Gamma(\mathcal{F})_{\mathsf{gr}}$ to the relative case.

Construction 27.2.5. Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$.

The functor (27.1) admits a right adjoint given by

$$\begin{array}{rcl} \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S})) & \to & \mathcal{S}\text{−}\mathsf{mod}_{\mathsf{gr,qcoh}} \\ \mathcal{F} & \mapsto & p_*(\mathcal{F})_{\mathsf{gr}} := \bigoplus_{n\in\mathbb{Z}} p_*(\mathcal{F}(n)). \end{array} \tag{27.2}$$

27.3. **Finite type case.**

Corollary 27.3.1. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type. Then the morphism $\mathsf{Proj}_Y(\mathcal{S}) \to Y$ is of finite type.*

Proof. Follows from Lemma 25.4.6. ∎

Definition 27.3.2. Let $\mathcal{M}$ be a $\mathbb{Z}$-graded quasi-coherent $\mathcal{S}$-module.

- We say $\mathcal{M}$ is **locally eventually of finite type** if there exists a covering $Y = \bigcup U_i$ of Y by affine open subsets such that for each i, there exists N_i such that
$$\bigoplus_{n\geq N_i} \mathcal{M}_n(U_i)$$
is a finitely generated $\mathcal{S}(U_i)$-module.
- We say $\mathcal{M}$ is **locally eventually zero** if there exists a covering $Y = \bigcup U_i$ of Y by affine open subsets such that for each i, there exists N_i such that $\mathcal{M}_n(U_i) = 0$ for $n \geq N_i$.

Corollary 27.3.3. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type.*

(i) *Let $\mathcal{M}$ be a $\mathbb{Z}$-graded quasi-coherent $\mathcal{S}$-module that it is locally eventually of finite type. Then $\widetilde{\mathcal{M}}$ is of finite type.*
(ii) *Under the assumption in (i), $\widetilde{\mathcal{M}} = 0$ iff $\mathcal{M}$ is locally eventually zero.*

Proof. (i) follows from Lemma 26.4.2; (ii) follows from Exercise 26.4.3. ∎

Corollary 27.3.4. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type. Then the functor (27.2) is fully faithful.*

Proof. Follows from Theorem 26.4.6. ∎

Corollary 27.3.5. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type. Then any object*
$$\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S}))$$
is of the form $\widetilde{\mathcal{M}}$ for some $\mathcal{M} \in \mathcal{S}\text{−}\mathsf{mod}_{\mathsf{gr,qcoh}}$. Moreover, suppose that:

- *Y is quasi-compact and quasi-separated;*
- *$\mathcal{F}$ is of finite type.*

Then we can choose $\mathcal{M}$ to be of finite type.

Proof. Only the last claim requires a proof. Let $\mathcal{M}$ be any quasi-coherent $\mathbb{Z}$-graded $\mathcal{S}$-module such that $\mathcal{F} \simeq \widetilde{\mathcal{M}}$. By Theorem 18.6.6, we can write $\mathcal{M}$ as a filtered colimit
$$\mathcal{M} \simeq \operatorname{colim}_{\alpha} \mathcal{M}_\alpha$$

such that each $\mathcal{M}_\alpha \subseteq \mathcal{M}$ is a quasi-coherent $\mathbb{Z}$-graded $\mathcal{S}$-submodule that is of finite type. We have

$$\mathcal{F} \simeq \widetilde{\mathcal{M}} \simeq \operatorname*{colim}_{\alpha} \widetilde{\mathcal{M}_\alpha}.$$

Since $\mathcal{F}$ is of finite type and $\mathsf{Proj}_Y(S)$ is quasi-compact, we see that for large enough index α, $\mathcal{F} \simeq \widetilde{\mathcal{M}_\alpha}$.

□

Corollary 27.3.6. *Suppose that:*

- *The scheme Y is quasi-compact;*
- *The ideal $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type.*

Then for any $\mathcal{F} \in \mathsf{QCoh}(\mathsf{Proj}_Y(\mathcal{S}))$ that is of finite type, the canonical morphism

$$p^* p_* \mathcal{F}(n) \to \mathcal{F}(n)$$

is a surjection for $n >> 0$. If moreover Y is quasi-separated, then for $n >> 0$, there exists a surjection

$$p^* \mathcal{N} \to \mathcal{F}(n)$$

where $\mathcal{N}$ is a quasi-coherent $\mathcal{O}_Y$-module that is of finite type.

Proof. The first claim follows from Proposition 26.4.8. The second claim follows from the first one by using the argument in the proof of Corollary 27.3.5.

□

In the particular case when $\mathcal{F}$ is the structure sheaf, we have the following stronger result:

Corollary 27.3.7. *Suppose that $\mathcal{S}_+$ is generated by $\mathcal{S}_1$. Then for any $n \geq 0$, the canonical morphism*

$$\mathcal{S}_n \to p_*(\mathcal{O}_{\mathsf{Proj}_Y(\mathcal{S})}(n))$$

induces a surjection

$$\phi_n : p^*(\mathcal{S}_n) \to \mathcal{O}_{\mathsf{Proj}_Y(\mathcal{S})}(n).$$

Proof. Follows from Exercise 26.4.9.

□

Remark 27.3.8. Note that ϕ_0 is obviously a surjection, while ϕ_1 being a surjection implies ϕ_n being so for any $n > 0$.

Corollary 27.3.9. *For any graded ideal $\mathcal{I} \subseteq \mathcal{S}$, the homomorphism $\mathcal{S} \to \mathcal{S}/\mathcal{I}$ induces a closed immersion*

$$\mathsf{Proj}_Y(\mathcal{S}/\mathcal{I}) \to \mathsf{Proj}_Y(\mathcal{S}), \tag{27.3}$$

whose ideal of definition is given by $\widetilde{\mathcal{I}}$. Moreover, we have:

1. *If $\mathcal{S}_+$ is generated by $\mathcal{S}_1$, then the pullback functor along (27.3) sends $\mathcal{O}(n)$ to $\mathcal{O}(n)$.*
2. *If $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type, then any closed immersion $i : X \to \mathsf{Proj}_Y(\mathcal{S})$ is of the form (27.3) for*

$$\mathcal{I} := \ker(\mathcal{S} \to p_*(\mathcal{O}_{\mathsf{Proj}_Y(\mathcal{S})})_{\mathsf{gr}} \to p_*(i_* \mathcal{O}_X)_{\mathsf{gr}}).$$

Proof. Follows from Theorem 26.5.1.

□

Corollary 27.3.10. *Suppose that $\mathcal{S}_+$ is generated by $\mathcal{S}_1$ and $\mathcal{S}_1$ is of finite type. Then the canonical morphism*

$$\mathsf{Proj}_Y(\mathcal{S}) \to \mathsf{Spec}_Y(\mathcal{S}_0)$$

is proper.

Proof. Follows from Corollary 26.5.3. □

27.4. Projective bundles.

Construction 27.4.1. Let Y be a scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_Y$-module. We equip $\mathsf{Sym}_{\mathcal{O}_Y}(\mathcal{E})$ with the natural $\mathbb{Z}_{\geq 0}$-grading such that $\mathsf{Sym}_{\mathcal{O}_Y}(\mathcal{E})_1 \simeq \mathcal{E}$. The scheme

$$\mathsf{P}_Y(\mathcal{E}) \coloneqq \mathsf{Proj}_Y(\mathsf{Sym}_{\mathcal{O}_Y}(\mathcal{E}))$$

is called the **projective bundle on Y associated to $\mathcal{E}$.**

Example 27.4.2. We have

$$\mathsf{P}_Y(\mathcal{O}_Y^{\oplus n}) \simeq \mathbb{P}^n_S \coloneqq \mathbb{P}^n_{\mathbb{Z}} \times S.$$

Remark 27.4.3. By definition, there is a canonical morphism

$$\pi : \mathsf{V}_Y(\mathcal{E}) \smallsetminus Y \to \mathsf{P}_Y(\mathcal{E}),$$

where $Y \to \mathsf{V}_Y(\mathcal{E}) = \mathsf{Spec}_Y(\mathsf{Sym}_{\mathcal{O}_Y}(\mathcal{E}))$ is the closed immersion corresponding to the homomorphism $\mathsf{Sym}_{\mathcal{O}_Y}(\mathcal{E}) \to \mathcal{O}_Y$ that sends $\mathcal{E}$ to zero.

Exercise 27.4.4. Let $\mathcal{L}$ be a line bundle on Y. Show that:

(1) The canonical morphism $p : \mathsf{P}_Y(\mathcal{L}) \to Y$ is an isomorphism.
(2) The canonical morphism $p^*\mathcal{L} \to \mathcal{O}(1)$ is an isomorphims.

27.5. **Morphisms into relative Proj.** Throughout this subsection, let Y be a scheme and $\mathcal{S}$ be a $\mathbb{Z}_{\geq 0}$-graded quasi-coherent $\mathcal{O}_Y$-algebra. We write $\mathcal{O} \coloneqq \mathcal{O}_{\mathsf{Proj}_Y(\mathcal{S})}$.

Recall that if $\mathcal{S}_+$ is generated by $\mathcal{S}_1$, we have a tautological line bundle $\mathcal{O}(1)$ and a canonical homomorphism

$$\mathcal{S} \to \bigoplus_{n \geq 0} p_*(\mathcal{O}(n)).$$

Using the adjoint pair (p^*, p_*), we obtain a graded $\mathcal{O}$-homomorphism

$$\phi : p^*(\mathcal{S}) \to \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{O}(n) \simeq \mathsf{Sym}_{\mathcal{O}}(\mathcal{O}(1)) \tag{27.4}$$

such that

$$\phi_1 : p^*(\mathcal{S}_1) \to \mathcal{O}(1)$$

is a surjection (Corollary 27.3.7). It turns out the pair $(\mathcal{O}(1), \phi)$ characterizes $\mathsf{Proj}_Y(\mathcal{S})$.

Construction 27.5.1. For any Y-scheme $q : X \to Y$, let $F_{Y,\mathcal{S}}(X)$ be the set of isomorphism classes of pairs $(\mathcal{L}, \psi)$, where $\mathcal{L}$ is a line bundle on X and

$$\psi : q^*(\mathcal{S}) \to \mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L})$$

is a graded $\mathcal{O}_X$-homomorphism such that $\psi_1 : q^*(\mathcal{S}_1) \to \mathcal{L}$ is a surjection.

For a morphism $f : X' \to X$ between Y-schemes, we have a canonical map

$$F_{Y,\mathcal{S}}(X) \to F_{Y,\mathcal{S}}(X'),\ (\mathcal{L}, \psi) \mapsto (f^*(\mathcal{L}), f^*(\psi)),$$

where we above notation and use $f^*(\psi)$ to denote the following composition

$$(q')^*(\mathcal{S}) \simeq f^* \circ q^*(\mathcal{S}) \xrightarrow{f^*(\psi)} f^*(\mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L})) \simeq \mathsf{Sym}_{\mathcal{O}_{X'}}(f^*(\mathcal{L})).$$

This defines a functor

$$F_{Y,\mathcal{S}} : \mathsf{Sch}_Y^{\mathsf{op}} \to \mathsf{Set}.$$

Theorem 27.5.2. *Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$. Then the functor $F_{Y,\mathcal{S}}$ is represented by $\mathsf{Proj}_Y(\mathcal{S})$, i.e., we have a natural isomorphism*

$$F_{Y,\mathcal{S}} \simeq \mathsf{Hom}_Y(-, \mathsf{Proj}_Y(\mathcal{S})).$$

Moreover, the identity morphism $\mathsf{id}_{\mathsf{Proj}_Y(\mathcal{S})}$ corresponds to the element

$$(\mathcal{O}(1), \phi) \in F_{Y,\mathcal{S}}(\mathsf{Proj}_Y(\mathcal{S})),$$

where ϕ is the canonical homomorphism (27.4).

To prove the theorem, we will construct natural transformations

$$\mathsf{Hom}_Y(-, \mathsf{Proj}_Y(\mathcal{S})) \rightleftarrows F_{Y,\mathcal{S}}$$

that are inverse to each other.

Construction 27.5.3. Suppose $\mathcal{S}_+$ is generated by $\mathcal{S}_1$. The rightward natural transformation is given by the following composition

$$\mathsf{Hom}_Y(-, \mathsf{Proj}_Y(\mathcal{S})) \to \mathsf{Hom}(F_{Y,\mathcal{S}}(\mathsf{Proj}_Y(\mathcal{S})), F_{Y,\mathcal{S}}(-)) \xrightarrow{\mathsf{ev}_{(\mathcal{O}(1),\phi)}} F_{Y,\mathcal{S}},$$

where the last map is evaluation at the canonical element

$$(\mathcal{O}(1), \phi) \in F_{Y,\mathcal{S}}.$$

Explicitly, for a Y-scheme $q : X \to Y$ and a Y-morphism $r : X \to \mathsf{Proj}_Y(\mathcal{S})$, the above natural transformation sends r to

$$(r^*(\mathcal{O}(1)), r^*(\phi)) \in F_{Y,\mathcal{S}}(X),$$

where we abuse notation and use $r^*(\phi)$ to denote the following composition

$$q^*\mathcal{S} \simeq r^*p^*\mathcal{S} \xrightarrow{r^*(\phi)} r^*\mathsf{Sym}_{\mathcal{O}}(\mathcal{O}(1)) \simeq \mathsf{Sym}_{\mathcal{O}_X}(r^*(\mathcal{O}(1))).$$

Construction 27.5.4. We now define the leftward natural transformation. For future reference, we first drop the assumptions that $\mathcal{S}_+$ is generated by $\mathcal{S}_1$.

Let $q : X \to Y$ be a Y-scheme and $\mathcal{L}$ be a line bundle on X. For *any* graded homomorphism

$$\psi : q^*\mathcal{S} \to \mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L}),$$

we have a morphism

$$\mathfrak{r}_{\mathcal{L},\psi} : U_+(\psi) \to \mathsf{Proj}_X(q^*\mathcal{S}) \simeq \mathsf{Proj}_Y(\mathcal{S}) \underset{Y}{\times} X \to \mathsf{Proj}_Y(\mathcal{S}),$$

where

$$U_+(\psi) \subseteq \mathsf{Proj}_X(\mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L})) \simeq X$$

is viewed as an open subscheme of X. One can check that this open subscheme, as well as the morphism $\mathfrak{r}_{\mathcal{L},\psi}$, only depends on the isomorphism class of $(\mathcal{L}, \psi)$.

When $(\mathcal{L}, \psi) \in F_{Y,\mathcal{S}}(X)$, i.e., when ψ_1 is a surjection, we see that $U_+(\psi) = X$. Hence we obtain a map

$$F_{Y,\mathcal{S}}(X) \to \mathsf{Hom}_Y(X, \mathsf{Proj}_Y(\mathcal{S})), \;\; (\mathcal{L}, \psi) \mapsto \mathfrak{r}_{\mathcal{L},\psi}.$$

The above construction is functorial in X, hence we obtain a natural transformation

$$F_{Y,\mathcal{S}} \to \mathsf{Hom}_Y(-, \mathsf{Proj}_Y(\mathcal{S})).$$

Proof of Theorem 27.5.2. Let $q : X \to Y$ be a Y-scheme. We only need to show the maps

$$\mathsf{Hom}_Y(X, \mathsf{Proj}_Y(\mathcal{S})) \rightleftarrows F_{Y,\mathcal{S}}(X),$$

constructed as above are inverse to each other.

Note that we have canonical identifications

$$\mathsf{Hom}_Y(X, \mathsf{Proj}_Y(\mathcal{S})) \simeq \mathsf{Hom}_X(X, \mathsf{Proj}_Y(\mathcal{S}) \underset{Y}{\times} X) \simeq \mathsf{Hom}_X(X, \mathsf{Proj}_X(q^*\mathcal{S}))$$

and

$$F_{Y,\mathcal{S}}(X) \simeq F_{X,q^*\mathcal{S}}(X).$$

Hence we can replace $(Y, \mathcal{S})$ with $(X, q^*\mathcal{S})$, and therefore reduce to the case $X = Y$. In other words, we only need to show that

$$\mathsf{Hom}_Y(Y, \mathsf{Proj}_Y(\mathcal{S})) \to F_{Y,\mathcal{S}}(Y),\ r \mapsto (r^*(\mathcal{O}(1)), r^*(\phi)) \tag{27.5}$$

and

$$F_{Y,\mathcal{S}}(Y) \to \mathsf{Hom}_Y(Y, \mathsf{Proj}_Y(\mathcal{S})),\ (\mathcal{L}, \psi) \mapsto \mathfrak{r}_{(\mathcal{L},\psi)} \tag{27.6}$$

are inverse to each other.

We first show

$$(27.6) \circ (27.5) = \mathsf{id}.$$

Any element $r \in \mathsf{Hom}_Y(Y, \mathsf{Proj}_Y(\mathcal{S}))$ is a closed immersion because $\mathsf{Proj}_Y(\mathcal{S}) \to Y$ is separated (Exercise 18.2.11(2)). Hence we only need to identify the surjections

$$\mathcal{O} \to r_*\mathcal{O}_Y \ \text{ and } \ \mathcal{O} \to \mathfrak{r}_{(r^*\mathcal{O}(1), r^*\phi),*}\mathcal{O}_Y.$$

Unwinding the definitions, the second surjection is given by the composition

$$\mathcal{O} \simeq \widetilde{\mathcal{S}} \simeq \widetilde{r^*p^*(\mathcal{S})} \xrightarrow{\widetilde{r^*(\phi)}} \big(r^*\mathsf{Sym}_{\mathcal{O}}(\mathcal{O}(1))\big)^{\sim}.$$

Now the claim follows from the following isomorphisms:

$$r_*\mathcal{O}_Y \simeq \big(p_*(r_*\mathcal{O}_Y)_{\mathsf{gr}}\big)^{\sim} \simeq \Big(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} p_*(r_*\mathcal{O}_Y \underset{\mathcal{O}}{\otimes} \mathcal{O}(n))\Big)^{\sim} \simeq$$
$$\simeq \Big(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} p_*(r_*(\mathcal{O}_Y \underset{\mathcal{O}_Y}{\otimes} r^*\mathcal{O}(n)))\Big)^{\sim} \simeq \big(r^*\mathsf{Sym}_{\mathcal{O}}(\mathcal{O}(1))\big)^{\sim},$$

where

- the first isomorphism is due to Corollary 27.3.4;
- the second isomorphism follows from the construction of $\widetilde{(-)}$;
- the third isomorphism is due to Exercise 27.5.5 below;
- the last isomorphism is obvious.

It remains to show

$$(27.5) \circ (27.6) = \mathsf{id}.$$

This follows from Corollary 27.3.9(1) and Exercise 27.4.4.

□

Exercise 27.5.5. Let $r : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ be a morphism between ringed spaces, and $\mathcal{M}$ (resp. $\mathcal{N}$) be an $\mathcal{O}_Y$-module (resp. $\mathcal{O}_Z$-module).

(1) Construct a canonical $\mathcal{O}_Z$-linear morphism
$$r_*\mathcal{M} \underset{\mathcal{O}_Z}{\otimes} \mathcal{N} \to r_*(\mathcal{M} \underset{\mathcal{O}_Y}{\otimes} r^*\mathcal{N}).$$
(2) Suppose $\mathcal{N}$ is a locally free $\mathcal{O}_Z$-module. Show that the above morphism is an isomorphism. This isomorphism is known as (a special case of) the **projection formula**.

Corollary 27.5.6. *Let Y be a scheme and $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_Y$-module. For any Y-scheme $q : X \to Y$, there is a canonical bijection between the following two sets:*

(a) *The set of Y-morphisms $X \to \mathsf{P}_Y(\mathcal{E})$.*
(b) *The set of isomorphism classes of pairs $(\mathcal{L}, \psi)$, where $\mathcal{L}$ is a line bundle on X and*
$$q^*(\mathcal{E}) \to \mathcal{L}$$
is a surjection between $\mathcal{O}_X$-modules.

27.6. **Segre embedding.**

Construction 27.6.1. Let Y be a scheme and $(\mathcal{E}_i)_{i=1}^n$ be a finite collection of quasi-coherent $\mathcal{O}_Y$-modules. For each i, write $\mathsf{P}_i := \mathsf{P}_Y(\mathcal{E}_i)$ and let $p_i : \mathsf{P}_i \to Y$ be the canonical morphism. Let
$$\varphi_i : p_i^*\mathcal{E}_i \to \mathcal{O}_{\mathsf{P}_i}(1)$$
be the canonical surjection. Consider the fiber product
$$\mathsf{P} := \mathsf{P}_1 \underset{Y}{\times} \cdots \underset{Y}{\times} \mathsf{P}_n$$
and the canonical projections $\mathsf{pr}_i : \mathsf{P} \to \mathsf{P}_i$, $r : \mathsf{P} \to Y$. Let φ be the following composition
$$r^*(\mathcal{E}_1 \underset{\mathcal{O}_Y}{\otimes} \cdots \underset{\mathcal{O}_Y}{\otimes} \mathcal{E}_n) \simeq \mathsf{pr}_1^*(p_1^*\mathcal{E}_1) \underset{\mathcal{O}_\mathsf{P}}{\otimes} \cdots \underset{\mathcal{O}_\mathsf{P}}{\otimes} \mathsf{pr}_n^*(p_n^*\mathcal{E}_n) \to$$
$$\xrightarrow{\otimes \mathsf{pr}_i^*(\varphi_i)} \mathsf{pr}_1^*(\mathcal{O}_{\mathsf{P}_1}(1)) \underset{\mathcal{O}_\mathsf{P}}{\otimes} \cdots \underset{\mathcal{O}_\mathsf{P}}{\otimes} \mathsf{pr}_n^*(\mathcal{O}_{\mathsf{P}_n}(1)) =: \mathcal{L}.$$
Note that $\mathcal{L}$ is a line bundle on P.

Exercise 27.6.2. Show that:

(1) The morphism φ is a surjection.
(2) The pair $(\mathcal{L}, \varphi)$ corresponds to a closed immersion
$$\mathsf{P}_Y(\mathcal{E}_1) \underset{Y}{\times} \cdots \underset{Y}{\times} \mathsf{P}_Y(\mathcal{E}_n) \to \mathsf{P}_Y(\mathcal{E}_1 \underset{\mathcal{O}_Y}{\otimes} \cdots \underset{\mathcal{O}_Y}{\otimes} \mathcal{E}_n).$$

This closed immersion is known as the **Segre embedding**.

28. (Very) ample line bundles

28.1. **Principle open subsets for line bundles.** The following results can be easily reduced to the case when $\mathcal{L} = \mathcal{O}_X$.

Proposition-Definition 28.1.1. *Let X be a scheme and $\mathcal{L}$ be a line bundle on X. Let $f \in \mathcal{L}(X)$ be a global section. For any $x \in X$, consider the germ $f_x \in \mathcal{L}_x$ of f at x. Consider*

$$X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x\},$$

*where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the unique maximal ideal. Then X_f is an open subset of X. We call it the **principle open subset** corresponding to f.*

Lemma 28.1.2. *Let X be a scheme and $\mathcal{L}$ be a line bundle on X. Let $f \in \mathcal{L}(X)$ be a global section. Then X_f is the open locus where $f : \mathcal{O}_X \to \mathcal{L}$ is invertible. More precisely:*

(1) *The restriction*
$$f|_{X_f} : \mathcal{O}_X|_{X_f} \to \mathcal{L}|_{X_f}$$
to X_f is an isomorphism.
(2) *If $V \subseteq X$ is a subset such that $f|_V$ is an isomorphism, then $V \subseteq X_f$.*

Exercise 28.1.3. Let $\mathcal{L}$ be a line bundle on X. Show that the canonical isomorphism $p : \mathsf{P}_X(\mathcal{L}) \to X$ sends $U_+(f)$ to X_f.

Construction 28.1.4. Let X be a quasi-compact and quasi-separated scheme, and $\mathcal{L}$ be a line bundle on X. Let $f \in \mathcal{L}(X)$ be a global section. For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, consider

$$M := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n}),$$

which is a graded module for

$$S := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}).$$

Consider the restriction map

$$M \to \bigoplus_{n \in \mathbb{Z}} \Gamma(X_f, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n}).$$

Note that S acts on the target and the above map is S-linear. Moreover, by Lemma 28.1.2, $f \in S$ acts invetibly on the target. Hence we have an S_f-linear map

$$M_f \to \bigoplus_{n \in \mathbb{Z}} \Gamma(X_f, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n})$$

and in particular an $S_{(f)}$-linear map

$$M_{(f)} \to \Gamma(X_f, \mathcal{F}). \tag{28.1}$$

The following result generalizes Lemma 20.4.5.

Theorem 28.1.5. *The canonical map* (28.1) *is an isomorphism.*

Remark 28.1.6. Explicitly, Theorem 28.1.5 says that:

(1) If $s \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is such that $s|_{X_f} = 0$, then for $m >> 0$, $s \otimes f^m = 0$.
(2) If $u \in \Gamma(X_f, \mathcal{F})$, then for $m >> 0$, $u \otimes (f|_{X_f})^m$ can be extended to a section over X.

Indeed, (1) is equivalent to (28.1) being injective; while (2) is equivalent to (28.1) being surjective.

Proof. We first prove (1). Let
$$s \in \Gamma(X, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n})$$
be a section such that $s|_{X_f} = 0$. For any affine open subset $U \subseteq X$ such that $\mathcal{L}|_U$ is trivial, note that $X_f \cap U = U_{f|_U}$ is a standard open subset of U. Hence $s|_{X_f \cap U} = 0$ implies
$$s|_U \otimes (f|_U)^m = 0$$
for $m >> 0$. Since X is quasi-compact, we obtain that $s \otimes f^m = 0$ for $m >> 0$.

We now prove (2). Let $u \in \Gamma(X_f, \mathcal{F})$. Choose a finite open covering $X = \bigcup_{i \in I} U_i$ such that each U_i is affine and $\mathcal{L}|_{U_i}$ is trivial. By the same logic in (1), for $m >> 0$,
$$u \otimes (f|_{X_f \cap U_i})^m$$
can be extended to a section
$$u_i \in \Gamma(U_i, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes m})$$
for any $i \in I$. By construction, for $(i,j) \in I^2$ and $U_{ij} := U_i \cap U_j$, the restriction of
$$u_i|_{U_{ij}} - u_j|_{U_{ij}}$$
to $X_f \cap U_{ij}$ is zero. Hence by (1) (applied to the scheme U_{ij} and the section $f|_{U_{ij}}$), for $m' >> 0$,
$$(u_i|_{U_{ij}} - u_j|_{U_{ij}}) \otimes (f|_{U_{ij}})^{m'} = 0$$
for any pair $(i,j) \in I$. This implies we can glue the sections $u_i \otimes (f|_{U_i})^{m'}$ to obtain a global section
$$s \in \Gamma(X, \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes (m+m')}).$$
By construction, s extends $u \otimes (f|_{X_f})^{m+m'}$.

□

28.2. **Ample line bundles.** Throughout this subsection, let X be a scheme and $\mathcal{L}$ be a line bundle on X. Define
$$S := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n}),$$
which is a $\mathbb{Z}_{\geq 0}$-graded commutative ring. For any $\mathcal{O}_X$-module $\mathcal{F}$, we write
$$\mathcal{F}(n) := \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^{\otimes n}.$$

Construction 28.2.1. By adjunction, we have a graded homomorphism
$$\psi : q^*(\widetilde{S}) \to \mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L}),$$
where $q : X \to \mathsf{Spec}(\mathbb{Z})$ is the unique morphism. By Construction 27.5.4, we obtain a canonical morphism
$$\mathfrak{r}_{\mathcal{L}} : U_{\mathcal{L}} \to \mathsf{Proj}(S),$$
where
$$U_{\mathcal{L}} := U_+(\psi) \subseteq \mathsf{Proj}_X(\mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L})) \simeq X$$
is viewed as an open subscheme of X.

Exercise 28.2.2. For any section $f \in S_n$, $n > 0$, we have
$$\mathfrak{r}_{\mathcal{L}}^{-1}(U_+(f)) = X_f.$$

Proposition-Definition 28.2.3. *Suppose X is quasi-compact. The following conditions are equivalent:*

(a) *The principle open subsets X_f, where f ranges over homogeneous elements in S_+, form a basis for the topology of X.*
(a') *The principle open subsets X_f, where f ranges over homogeneous elements in S_+ such that X_f is affine, cover X.*
(b) *We have $U_{\mathcal{L}} = X$, and the morphism $\mathfrak{r}_{\mathcal{L}} : X \to \mathsf{Proj}(S)$ is an open immersion with dense image.*
(b') *We have $U_{\mathcal{L}} = X$, and the morphism $\mathfrak{r}_{\mathcal{L}} : X \to \mathsf{Proj}(S)$ induces a homeomorphism onto its image.*
(c) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the canonical morphisms*

$$\mathcal{O}_X \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X, \mathcal{F}(n))} \to \mathcal{F}(n)$$

induce a surjection

$$\bigoplus_{n \geq N} \mathcal{O}_X(-n) \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X, \mathcal{F}(n))} \to \mathcal{F}.$$

for any integer N.
(c') *For any quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, the canonical morphisms*

$$\mathcal{O}_X \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X, \mathcal{I}(n))} \to \mathcal{I}(n)$$

induce a surjection

$$\bigoplus_{n > 0} \mathcal{O}_X(-n) \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X, \mathcal{I}(n))} \to \mathcal{I}.$$

We say $\mathcal{L}$ is ***ample*** *if it satisfies the above conditions.*

Proof. (b)⇒(b') is obvious.

(b')⇒(a) follows from Exercise 28.2.2.

To show (a)⇒(a'), let $x \in X$ be a point and $U \subseteq X$ be an affine neighborhood of x such that $\mathcal{L}|_U \simeq \mathcal{O}_U$. By (a), there exists a homogeneous element

$$f \in S_n = \Gamma(X, \mathcal{L}^{\otimes n}), \ \ n > 0$$

such that $X_f \subseteq U$. It follows that $X_f = U_{f|_U}$ is the principle open subset for the section

$$f|_U \in \Gamma(U, \mathcal{L}|_U^{\otimes n}) \simeq \Gamma(U, \mathcal{O}_U).$$

Since U is affine, so is $U_{f|_U}$. Hence X_f is an affine principle open subset that contains x. In other words, (a') holds.

To show (a')⇒(b), let $f \in S_+$ be a homogeneous element such that X_f is affine. By Exercise 28.2.2, we have

$$\mathfrak{r}_{\mathcal{L}}^{-1}(U_+(f)) = X_f.$$

We claim that the obtained morphism $X_f \to U_+(f)$ is an isomorphism. Note that this claim indeed implies $\mathfrak{r}_{\mathcal{L}}$ is an open immersion with *dense* image[18].

To prove the claim, we first show that X is quasi-separated. Let $X = \bigcup_{i \in I} X_{f_i}$ be a finite covering of X such that each X_{f_i} is affine. Using the assumption that

[18]Denseness follows from the observation that an empty scheme is affine.

X is quasi-compact, it is easy to show that any principle open subset X_g is quasi-compact. In particular, for each pair $(i,j) \in I^2$, the intersection

$$X_{f_i} \cap X_{f_j} = X_{f_i f_j}$$

is quasi-compact. This implies X is quasi-separated.

Since X is quasi-compact and quasi-separated, by Theorem 28.1.5, we have a canonical isomorphism

$$S_{(f_i)} \xrightarrow{\simeq} \Gamma(X_{f_i}, \mathcal{O}_X)$$

is an isomorphism. It follows from the constructions that this gives the desired isomorphism $X_f \to U_+(f)$.

To show (a')⇒(c), let

$$X = \bigcup_{i \in I} X_{f_i}$$

be an open covering such that $f_i \in S_{d_i}$ and X_{f_i} is affine. We only need to show that for each $i \in I$,

$$\bigoplus_{n \geq 0} \mathcal{O}_{X_{f_i}}(-n) \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X, \mathcal{F}(n))} \to \mathcal{F}|_{X_{f_i}}$$

is a surjection. Since X_{f_i} is affine, we only need to show that any section $u \in \mathcal{F}(X_{f_i})$ is contained in the image. But this is exactly claim (2) in Remark 28.1.6.

(c)⇒(c') is obvious.

It remains to show (c')⇒(a). Let $x \in X$ be a point and $U \subseteq X$ be an affine neighborhood of x. We construct a principle open $X_f \subseteq U$ such that $x \in X_f$ as follows. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the quasi-coherent ideal corresponding to the closed subscheme $(X \smallsetminus U)_{\mathsf{red}}$. Note that $\mathcal{I}|U \simeq \mathcal{O}_U$. By (c'), there exists $n \in \mathbb{Z}$ and

$$f \in \Gamma(X, \mathcal{I}(n)) \subseteq \Gamma(X, \mathcal{O}(n)) = S_n$$

such that the morphism

$$\mathcal{O}_X \xrightarrow{f} \mathcal{I}(n)$$

is nonzero at x. We view f as a section of the line bundle $\mathcal{O}(n)$ and consider the corresponding principle open X_f. The above non-vanishing property implies $x \in U_{f|_U} \subseteq X_f$. On the other hand, for any point $y \in X \smallsetminus U$, we have

$$f_y \in \mathcal{I}(n)_y \simeq \mathcal{I}_y \underset{\mathcal{O}_{X,y}}{\otimes} \mathcal{L}_y^{\otimes n} \subseteq \mathfrak{m}_y \underset{\mathcal{O}_{X,y}}{\otimes} \mathcal{L}_y^{\otimes n}.$$

Hence $y \notin X_f$, and therefore $X_f \subseteq U$ as desired.

□

Lemma 28.2.4. *Let X be a quasi-compact scheme that admits an ample line bundle. Then X is separated.*

Proof. Follows from the fact that X is an open subscheme of $\mathsf{Proj}(S)$, which is a separated scheme. □

Proposition 28.2.5. *Suppose X is quasi-compact and quasi-separated. The following conditions are equivalent:*

- *the line bundle $\mathcal{L}$ is ample.*
- (d) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ that is of finite type, $\mathcal{F}(n)$ is generated by finitely many global sections for $n >> 0$.*
- (d') *For any quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ that is of finite type, there exists $n > 0$ such that $\mathcal{I}(n)$ is generated by global sections.*

Proof. (d)⇒(d') is obvious.

To show (d') implies $\mathcal{L}$ is ample, we verify condition (c') in Proposition-Definition 28.2.3. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a quasi-coherent ideal. By Theorem 18.6.6, $\mathcal{I}$ can be written as a filtered colimit $\mathcal{I} \simeq \mathsf{colim}\mathcal{I}_\alpha$ such that each $\mathcal{I}_\alpha$ is a quasi-coherent ideal of finite type. By (d'), the image of

$$\bigoplus_{n>0} \mathcal{O}_X(-n) \underset{\underline{\mathbb{Z}}}{\otimes} \underline{\Gamma(X,\mathcal{I}(n))} \to \mathcal{I}$$

contains each $\mathcal{I}_\alpha$. It follows that this is a surjection as desired.

It remains to show that if $\mathcal{L}$ is ample, then (d) is true. Since $\mathcal{L}$ is ample, there exists a finite covering

$$X = \bigcup_{i \in I} X_{f_i}$$

such that $f_i \in S_+$ is a homogeneous element and X_{f_i} is affine. Note that X_{f_i} does not change if we replace f_i with a positive power of it. Hence we may assume that f_i, $i \in I$ are of the same degree d, where d is a positive integer that only depends on X and $\mathcal{L}$.

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module that is of finite type. For each $i \in I$, since X_{f_i} is affine, the restriction $\mathcal{F}|_{X_{f_i}}$ is generated by finitely many sections

$$u_{ij} \in \Gamma(X_{f_i}, \mathcal{F}), \;\; j \in J_i.$$

By Remark 28.1.6(2), for $k >> 0$, each section

$$u_{ij} \otimes (f_i|_{X_{f_i}})^{\otimes k} \in \Gamma(X_{f_i}, \mathcal{F}(kd))$$

can be extended to a section

$$s_{ij} \in \Gamma(X, \mathcal{F}(kd)).$$

By construction, $\mathcal{F}(kd)$ is generated by these sections.

We have shown that for a fixed $\mathcal{F}$, the sheaf $\mathcal{F}(kd)$ is generated by finitely many global sections for $k >> 0$. Applying this to $\mathcal{F}, \mathcal{F}(1), \cdots, \mathcal{F}(d-1)$, we obtain that $\mathcal{F}(n)$ is generated by finitely many global sections for $n >> 0$ as desired.

□

Example 28.2.6. Any line bundle on a quasi-affine scheme is ample because condition (b') in Propoisition-Definition 28.2.3 is satisfied.

Conversely, if X is a quasi-compact scheme such that $\mathcal{O}_X$ is ample, then X is quasi-affine (Proposition 20.6.2).

Example 28.2.7. Let S be a $\mathbb{Z}_{\geq 0}$-graded commutative ring such that S_+ is generated by finitely many elements in S_1. Then the tautological line bundle $\mathcal{O}_{\mathsf{Proj}(S)}(1)$ is ample. See Proposition 26.4.8.

Exercise 28.2.8. Let X be a quasi-compact scheme and $\mathcal{L}$ be a line bundle on X. Let d be a positive integer, show that $\mathcal{L}$ is ample iff $\mathcal{L}^{\otimes d}$ is ample.

Exercise 28.2.9. Let X be a quasi-compact scheme and $\mathcal{L}_1, \mathcal{L}_2$ be ample line bundles on X. Show that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample.

Exercise 28.2.10. Let $n > 0$ and d be intergers. Show that $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(d)$ is ample iff $d > 0$.

28.3. **Relative version.**

Definition 28.3.1. Let $f : X \to Y$ be a quasi-compact morphism and $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is **f-ample**[19] if for any affine open subscheme $U \subseteq Y$, the restriction $\mathcal{L}|_{X_U}$ is an ample line bundle on $X_U := X \times_Y U$.

Corollary 28.3.2. *Let $f : X \to Y$ be a quasi-compact morphism that admits an f-ample line bundle. Then f is separated.*

Proof. Follows from Lemma 28.2.4. □

Construction 28.3.3. Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism and $\mathcal{L}$ be a line bundle on X. Write

$$\mathcal{S} := \bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n})$$

and

$$\psi : f^*\mathcal{S} \to \mathsf{Sym}_{\mathcal{O}_X}(\mathcal{L})$$

for the canonical $\mathcal{O}_X$-homomorphism. Let

$$\mathfrak{r}_{\mathcal{L}/Y} : U_{\mathcal{L}/Y} \to \mathsf{Proj}_Y(\mathcal{S})$$

be the morphism associated to $(\mathcal{L}, \psi)$ by Construction 27.5.4.

Corollary 28.3.4. *Using the above notations, the following conditions are equivalent:*

(a) *The line bundle $\mathcal{L}$ is f-ample.*
(b) *We have $U_{\mathcal{L}/Y} = X$, and the morphism $\mathfrak{r}_{\mathcal{L}/Y} : X \to \mathsf{Proj}_Y(\mathcal{S})$ is an open immersion with dense image.*
(b') *We have $U_{\mathcal{L}/Y} = X$, and the morphism $\mathfrak{r}_{\mathcal{L}/Y} : X \to \mathsf{Proj}_Y(\mathcal{S})$ induces a homeomorphism onto its image.*

If X is quasi-separated, then the above conditions are also equivalent to the following ones:

(c) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the canonical morphism*

$$\bigoplus_{n \geq N} \big(f^* f_*(\mathcal{F}(n))\big)(-n) \to \mathcal{F}$$

is surjective for any integer N.
(c') *For any quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$, the canonical morphism*

$$\bigoplus_{n > 0} \big(f^* f_*(\mathcal{F}(n))\big)(-n) \to \mathcal{F}$$

is a surjection.

If X is quasi-compact and quasi-separated, then the above conditions are also equivalent to the following ones

(d) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ that is of finite type, there exists N such that for any $n \geq N$, there exists a submodule $\mathcal{M} \subseteq f_*(\mathcal{F}(n))$ that is of finite type such that*

$$f^*\mathcal{M} \to f^* f_* \mathcal{F}(n) \to \mathcal{F}(n)$$

is surjective.

[19] Other terminologies: relatively ample for f; ample on X over Y.

(d') *For any quasi-coherent ideal $\mathcal{I} \subseteq \mathcal{O}_X$ that is of finite type, there exists $n > 0$ such that*
$$f^* f_* \mathcal{I}(n) \to \mathcal{I}(n)$$
is surjective.

Sketch. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(b') follows from Proposition-Definition 28.2.3.

When X is quasi-separated, note that for any affine open subscheme $U \subseteq Y$, the open immersion $j : X_U \to X$ is quasi-compact and quasi-separated, and therefore j_* preserves quasi-coherent modules. This allows us to apply Proposition-Definition 28.2.3 and obtain (a)$\Leftrightarrow$(c)$\Leftrightarrow$(c').

When X is quasi-compact and quasi-separated, we can apply Proposition 28.2.5, Theorem 18.6.1 and Theorem 18.6.6 to obtain (a)$\Leftrightarrow$(d)$\Leftrightarrow$(d'). □

Lemma 28.3.5. *We have the following standard properties for relatively ample line bundles.*

(1) *Let $f : X \to Y$ be a quasi-affine morphism. Then any line bundle $\mathcal{L}$ on X is f-ample.*
(2) *Let $f : X \to Y$ be a quasi-compact morphism and $d > 0$ be a positive integer. Then $\mathcal{L}$ is f-ample iff $\mathcal{L}^{\otimes d}$ is so.*
(3) *Let $f : X \to Y$ be a quasi-compact morphism and $\mathcal{L}$ (resp. $\mathcal{K}$) be a line bundle on X (resp. Y). Then $\mathcal{L}$ is f-ample iff $\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{K}$ is f-ample.*
(4) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be quasi-compact morphisms such that f induces a homeomorphism onto its image. Let $\mathcal{K}$ be a g-ample line bundle. Then $f^*\mathcal{K}$ is $(g \circ f)$-ample.*
(4') *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be quasi-compact morphisms such that Z be quasi-compact. Let $\mathcal{L}$ be an f-ample line bundle and $\mathcal{K}$ be a g-ample line bundle. Then $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K}^{\otimes l})$ is $(g \circ f)$-ample for $l >> 0$.*
(5) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms between schemes such that $g \circ f$ is quasi-compact and g is quasi-separated. Let $\mathcal{L}$ be a $(g \circ f)$-ample line bundle. Then $\mathcal{L}$ is an f-ample line bundle.*
(6) *Relatively ample line bundles are stable under tensor products.*
(7) *Relatively ample line bundles are preserved by base-change.*
(8) *Being relatively ample is local on the target scheme.*

Proof. (1) follows from Lemma 28.2.6. (2) follows from Exercise 28.2.8. (5) follows by verifying condition (d') in Corollary 28.3.4. (6) follows from Exercise 28.2.9. (7) follows by verifying condition (b') in Corollary 28.3.4. (8) follows obviously from the definition. (3) follows from (8) because we can trivialize $\mathcal{K}$ locally on Y.

To prove (4), we can assume Z is affine and $\mathcal{K}$ is ample. Then one can check condition (a) in Proposition-Definition 28.2.3 to obtain that $f^*\mathcal{K}$ is ample.

It remains to prove (4'). By (8), we can assume Z is affine and $\mathcal{K}$ is ample. As in the proof of Proposition 28.2.5, we can choose $d > 0$ and finitely many sections
$$v_i \in \Gamma(Y, \mathcal{K}^{\otimes d}), \ \ i \in I$$
such that
$$Y = \bigcup_{i \in I} Y_{v_i}$$

and each Y_{v_i} is affine. Note that d only depends on Y and $\mathcal{K}$.

Let
$$u_i \in \Gamma(X, f^*(\mathcal{K})^{\otimes d})$$
be the pullback of the section v_i. Then we have
$$X_{u_i} \simeq X \underset{Y}{\times} Y_{v_i}$$
and therefore
$$X = \bigcup_{i \in I} X_{u_i}.$$
Moreover, by definition, $\mathcal{L}|_{X_{u_i}}$ is an ample line bundle on X_{u_i}. Hence as before, we can choose $n > 0$ and finitely many sections
$$w_{ij} \in \Gamma(X_{u_i}, \mathcal{L}^{\otimes n}), \;\; j \in J_i$$
such that
$$X_{u_i} = \bigcup_{j \in J_i} (X_{u_i})_{w_{ij}}$$
and each $(X_{u_i})_{w_{ij}}$ is affine. Note that X is quasi-compact and separated because both f and g are so. Hence by Theorem 28.1.5, for $m >> 0$, the section
$$w_{ij} \otimes (u_i|_{X_{u_i}})^{\otimes m} \in \Gamma(X_{u_i}, \mathcal{L}^{\otimes n} \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{K})^{\otimes md})$$
can be extended to a section
$$s_{ij} \in \Gamma(X, \mathcal{L}^{\otimes n} \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{K})^{\otimes md}).$$
By construction, we have
$$X_{s_{ij}} = (X_{u_i})_{w_{ij}}.$$
Hence
$$X = \bigcup_{i \in I, j \in I_j} X_{s_{ij}}$$
is a covering of X such that each $X_{s_{ij}}$ is affine. Let $m = nr$, we obtain that
$$\mathcal{L} \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{K})^{\otimes rd}$$
is ample for $r >> 0$ because it satisfies condition (a') in Proposition-Definition 28.2.3.

We have proved that

(*) For *any* f-ample line bunlde $\mathcal{L}$, the line bundle $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K})^{\otimes rd}$ is ample for $r >> 0$.

Note that $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K})^{\otimes e}$ is f-ample for any integer e by (3). Hence we can apply (*) to $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K})^{\otimes e}$ for $e \in \{0, 1, \cdots, d-1\}$ and obtain that $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K})^{\otimes l}$ is ample for $l >> 0$.

□

Warning 28.3.6. In Lemma 28.3.5(4'), $\mathcal{L} \otimes_{\mathcal{O}_X} f^*\mathcal{K}$ is in general not $(g \circ f)$-ample. Indeed, suppose this was *always* true, then
$$\mathcal{L} \simeq (\mathcal{L} \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{K}^{-1})) \underset{\mathcal{O}_X}{\otimes} f^*(\mathcal{K})$$
would be $(g \circ f)$-ample because $\mathcal{L} \otimes_{\mathcal{O}_X} f^*(\mathcal{K}^{-1})$ is also f-ample by Lemma 28.3.5(3). In other words, we would obtain that any f-ample line bundle is $(g \circ f)$-ample, which is riduculous for $f = \mathsf{id}$.

Appendix A. Abelian categories

A.1. **Additive category.** In most textbooks, an additive category is defined as a category equipped with an *extra structure*[20] that admits finite products. However, it is a remarkable fact that such an extra structure is actually unique. In other words, being additive is *a property rather than a structure.*

Definition A.1.1. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ **admits** a zero object if it admits a final object $*$ and an initial object $\varnothing$, and the unique morphism $\varnothing \to *$ is an isomorphism.

If $\mathcal{A}$ admits a zero object, then any final object is also an initial object and vice versa. We often denote such an object by $0 \in \mathcal{A}$.

Example A.1.2. The category Ab of abelian groups admits a zero object, known as the zero abelian group.

Example A.1.3. The category Top of topological spaces admits both a final and an initial object, but these objects are not isomorphic. On the other hand, the category Top_* of *pointed* topological spaces admits a zero object.

Lemma A.1.4. *Let $\mathcal{A}$ be a category that admits a zero object. For any pair of objects $X, Y \in \mathcal{A}$, there exists a unique morphism $f : X \to Y$ that can factor through a zero object.*

Proof. Existence is obvious. For uniqueness, suppose f and f' are two such morphisms. We can write them as the following compositions:

$$f : X \xrightarrow{f_1} 0 \xrightarrow{f_2} Y,\ f' : X \xrightarrow{f_1'} 0' \xrightarrow{f_2'} Y,$$

where 0 and $0'$ are both zero objects. Let $h : 0 \to 0'$ be the unique morphism from 0 to $0'$. Since $0'$ is a final object, $f_1' = h \circ f_1$. Since 0 is an initial object, $f_2 = f_2' \circ h$. It follows that

$$f = f_2 \circ f_1 = f_2' \circ h \circ f_1 = f_2' \circ f_1' = f'$$

as desired. □

Definition A.1.5. Let $\mathcal{A}$ be a category that admits a zero object, and X, Y be a pair of objects. The unique morphism $X \to Y$ that factors through a zero object is called the **zero morphism**. We often denote this morphism by $0 : X \to Y$.

Definition A.1.6. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ **admits finite biproducts** if it satisfies the following conditions.

(i) The category $\mathcal{A}$ admits finite products, finite coproducts and a zero object.
(ii) For any pair of objects X, Y, the map $X \coprod Y \to X \times Y$ given by the matrix $\begin{pmatrix} \mathsf{id}_X & 0 \\ 0 & \mathsf{id}_Y \end{pmatrix}$ is an isomorphism.

If $\mathcal{A}$ is a category that admits finite biproducts, then any finite product of $(X_i)_{i\in I}$ is also a finite coproduct and vice versa. We often denote such object by $\bigoplus_{i\in I} X_i$.

[20]Namely, an enrichment over the category Ab of abelian groups. In other words, all Hom-sets are upgraded to an abelian group, and the composition maps are bilinear maps between abelian groups.

Construction A.1.7. Let $\mathcal{A}$ be a category that admits finite biproducts. For any pair of objects X, Y, and any pair of morphisms $f, g : X \to Y$, let $f + g$ be the composition

$$X \xrightarrow{\begin{pmatrix} \mathsf{id}_X & \mathsf{id}_X \end{pmatrix}} X \bigoplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \bigoplus Y \xrightarrow{\begin{pmatrix} \mathsf{id}_Y \\ \mathsf{id}_Y \end{pmatrix}} Y.$$

One can show this endows $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ with the structure of a abelian semigroup[21] (i.e. the binary operation $+$ is unital, commutative and associative), whose identity element is the zero morphism $0 : X \to Y$. Moreover, one can check that the composition map

$$\mathsf{Hom}_{\mathcal{A}}(X, Y) \times \mathsf{Hom}_{\mathcal{A}}(Y, Z) \to \mathsf{Hom}_{\mathcal{A}}(X, Z)$$

is *bilinear*. In other words: $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ and $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$.

Definition A.1.8. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ is **additive** if it satisfies the following conditions:

(i) The category $\mathcal{A}$ admits finite biproducts.
(ii) For any pair of objects X, Y, the abelian semigroup $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ in Construction A.1.7 is an abelian group, i.e., any element $f \in \mathsf{Hom}_{\mathcal{A}}(X, Y)$ admits an inverse $-f$ such that $f + (-f) = 0$.

Example A.1.9. The category Ab of abelian groups is an additive category.

The following result implies our definition of additive category coincides with other approaches in the literature.

Exercise A.1.10. Let $\mathcal{A}$ be a category that admits finite products. Suppose for any pair of objects X, Y, there is a binary operation $+'$ on $\mathsf{Hom}_{\mathcal{A}}(X, Y)$ such that:

(i) The pair $(\mathsf{Hom}_{\mathcal{A}}(X, Y), +')$ is an abelian group.
(ii) For objects X, Y, Z, the composition map

$$\mathsf{Hom}_{\mathcal{A}}(X, Y) \times \mathsf{Hom}_{\mathcal{A}}(Y, Z) \to \mathsf{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear with respect to the above abelian group structures.

Then $\mathcal{A}$ is additive and $+'$ is equal to the operation in Construction A.1.7.

A.2. **Kernel and cokernel.**

Recall (co)kernels can be defined as fiber (co)products.

Definition A.2.1. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism.

- The **kernel** of f, often denoted by $\mathsf{ker}(f)$, is defined to be the fiber product of the span $X \xrightarrow{f} Y \xleftarrow{0} 0$.
- The **cokernel** of f, often denoted by $\mathsf{coker}(f)$, is defined to be the fiber coproduct of the cospan $Y \xleftarrow{f} X \xrightarrow{0} 0$.

Lemma A.2.2. *Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism that admits a kernel. Then the canonical morphism $i : \mathsf{ker}(f) \to X$ is a monomorphism.*

Proof. By definition, we only need to show for any test object $T \in \mathcal{A}$, post-composing with i induces an injection between sets:

$$\mathsf{Hom}_{\mathcal{A}}(T, \mathsf{ker}(f)) \to \mathsf{Hom}_{\mathcal{A}}(T, X).$$

[21] Other name: commutative monoid.

By definition of $\mathsf{ker}(-)$, we have a Cartesian square of sets

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{A}}(T, \mathsf{ker}(f)) \arrow[r, "i\circ-"] \arrow[d, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, X) \arrow[d, "f\circ-"] \\ \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[r, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, Y). \end{tikzcd}$$

By the definitions of zero objects and zero morphisms, $\mathsf{Hom}_{\mathcal{A}}(T, 0)$ is a singleton and the bottom horizontal arrow is an injection. It follows that the top horizontal arrow is an injection as desired.

□

Remark A.2.3. Dually, the canonical map $p : Y \to \mathsf{coker}(f)$ is an *epimorphism*, i.e., pre-composing with p induces an injection

$$\mathsf{Hom}_{\mathcal{A}}(\mathsf{coker}(f), T) \to \mathsf{Hom}_{\mathcal{A}}(Y, T)$$

for any test object $T \in \mathcal{A}$.

Example A.2.4. In Ab, any morphism admits a kernel and a cokernel.

Example A.2.5. The canonical morphism $\mathsf{ker}(X \xrightarrow{0} Y) \to X$ is an isomorphism, while $Y \to \mathsf{coker}(X \xrightarrow{0} Y)$ is an isomorphism.

Lemma A.2.6. *Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism. The following conditions are equivalent:*

(i) *The morphism f is a monomorphism.*
(ii) *The kernel $\mathsf{ker}(f)$ exists and is a zero object.*

Proof. By definition, condition (ii) is equivalent to:

(ii') For any test object $T \in \mathcal{A}$, the following diagram is a Cartesian square of sets:

$$\begin{tikzcd} \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[r, "="] \arrow[d, "0\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, 0) \arrow[d, "0\circ-"] \\ \mathsf{Hom}_{\mathcal{A}}(T, X) \arrow[r, "f\circ-"] & \mathsf{Hom}_{\mathcal{A}}(T, Y). \end{tikzcd}$$

By the definitions of zero objects and zero morphisms, ths condition is equivalent to

(ii") For any test object $T \in \mathcal{A}$, the map

$$\mathsf{Hom}_{\mathcal{A}}(T, X) \to \mathsf{Hom}_{\mathcal{A}}(T, Y)$$

sends nonzero morphisms to nonzero morphisms.

Since $\mathcal{A}$ is additive, the above map is a homomorphism of abelian groups. Recall zero morphisms are the identity elements in these groups. It follows that condition (ii") is equivalent to $\mathsf{Hom}_{\mathcal{A}}(T, X) \to \mathsf{Hom}_{\mathcal{A}}(T, Y)$ being injective, which is exactly condition (i).

□

Warning A.2.7. In general, being a monomorphism and an epimorphism does not imply being an isomorphism. Hence vanishing of kernel and cokernel in an *additive* category does not imply a morphism is an isomorphism. See the exercise below for a counterexample. However, we will see abelian categories do not have this caveat.

Exercise A.2.8. Let $\mathsf{Ab}_{\mathsf{tf}} \subseteq \mathsf{Ab}$ be the full subcategory of torsion free abelian groups, i.e., those abelian groups containing no finite order elements.

(1) Show that $\mathsf{Ab}_{\mathsf{tf}}$ is an additive category and any morphism in $\mathsf{Ab}_{\mathsf{tf}}$ admits a kernel and a cokernel.
(2) Find a morphism f in $\mathsf{Ab}_{\mathsf{tf}}$ such that $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects but f is not an isomorphism.

A.3. **Abelian category.**

Definition A.3.1. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism.

- If $\mathsf{coker}(f)$ exists, the **image** of f is defined to be
$$\mathsf{im}(f) := \mathsf{ker}(Y \to \mathsf{coker}(f)).$$
- If $\mathsf{ker}(f)$ exists, the **coimage** of f is defined to be
$$\mathsf{coim}(f) := \mathsf{coker}(\mathsf{ker}(f) \to X).$$

Remark A.3.2. Let $f' \in \mathsf{Hom}_{\mathcal{A}^{\mathsf{op}}}(Y, X)$ be the morphism corresponding to f. Note that $\mathcal{A}^{\mathsf{op}}$ is also an additive category and the monomorphism $\mathsf{ker}(f) \to X$ in $\mathcal{A}$ corresponds to the epimorphism $X \to \mathsf{coker}(f')$ in $\mathcal{A}^{\mathsf{op}}$. It follows that the monomorphism $\mathsf{im}(f) \to Y$ in $\mathcal{A}$ corresponds to the epimorphism $Y \to \mathsf{coim}(f')$ in $\mathcal{A}^{\mathsf{op}}$.

Exercise A.3.3. Let $\mathcal{A}$ be an additive category and $f : X \to Y$ be a morphism such that $\mathsf{ker}(f)$, $\mathsf{coker}(f)$, $\mathsf{im}(f)$ and $\mathsf{coim}(f)$ all exists. Let $p : X \to \mathsf{coim}(f)$ and $i : \mathsf{im}(f) \to Y$ be the canonical morphisms. Show that f admits a unique factorization as
$$X \xrightarrow{p} \mathsf{coim}(f) \to \mathsf{im}(f) \xrightarrow{i} Y.$$

In particular, we obtain a *canonical* morphism $\mathsf{coim}(f) \to \mathsf{im}(f)$.

Definition A.3.4. Let $\mathcal{A}$ be a category. We say $\mathcal{A}$ is **abelian** if it satisfies the following conditions:

(i) The category $\mathcal{A}$ is additive.
(ii) Any morphism in $\mathcal{A}$ admits a kernel and a cokernel.
(iii) For any morphism f in $\mathcal{A}$, the canonical morphism $\mathsf{coim}(f) \to \mathsf{im}(f)$ is an isomorphism.

Example A.3.5. The category Ab is an abelian category.

Remark A.3.6. By Remark A.3.2, $\mathcal{A}$ is abelian iff $\mathcal{A}^{\mathsf{op}}$ is so.

Lemma A.3.7. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ be a morphism. The following conditions are equivalent:*

(i) *The morphism f is a monomorphism.*
(ii) *The kernel $\mathsf{ker}(f)$ exists and is a zero object.*
(iii) *The morphism f is a kernel, can be written as $\mathsf{ker}(g) \to X$ for some morphism $g : Y \to Z$.*

Proof. We have seen (i)⇔(ii) in Lemma A.2.6, and (iii)⇒(i) in Lemma A.2.2. It remains to show (ii)⇒(iii).

Now suppose condition (ii) holds. By Example A.2.5, the canonical map from X to $\mathsf{coim}(f) = \mathsf{coker}(\mathsf{ker}(f) \to X)$ is an isomorphism. Since $\mathcal{A}$ is abelian, we obtain isomorphisms
$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f).$$

By definition, the canonical map $\mathsf{im}(f) \to Y$ is a kernel of the canonical map $Y \to \mathsf{coker}(f)$. It follwos that the composition

$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f) \to Y$$

is also a kernel of the canonical map $Y \to \mathsf{coker}(f)$. But by definition, this composition is just f. Hence f is a kernel as desired.

□

Proposition A.3.8. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ be a morphism. Then f is an isomorphism iff both $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects.*

Proof. The "only if" statement is obvious. For the "if" statement, suppose both $\mathsf{ker}(f)$ and $\mathsf{coker}(f)$ are zero objects. By Example A.2.5, the canonical maps $\mathsf{ker}(Y \to \mathsf{coker}(f)) \to Y$ and $X \to \mathsf{coker}(\mathsf{ker}(f) \to X)$ are both isomorphisms. In other words, $\mathsf{im}(f) \to Y$ and $X \to \mathsf{coim}(f)$ are both isomorphisms. On the other hand, the canonical map $\mathsf{coim}(f) \to \mathsf{im}(f)$ is an isomorphism because $\mathcal{A}$ is abelian. It follows that the composition

$$X \xrightarrow{\simeq} \mathsf{coim}(f) \xrightarrow{\simeq} \mathsf{im}(f) \xrightarrow{\simeq} Y$$

is an isomorphism. But by definition, this composition is just f.

□

Construction A.3.9. Let $\mathcal{A}$ be an additive category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence such that $g \circ f = 0$. Since the canonical morphism $X \to \mathsf{coim}(f)$ is an epimorphism, the composition $\mathsf{coim}(f) \to Y \xrightarrow{g} Z$ is also zero. Hence the morphism $\mathsf{coim}(f) \to Y$ admits a unique factorization as $\mathsf{coim}(f) \to \mathsf{ker}(g) \to Y$ such that the second map is the canonical monomorphism. In particular, we obtain a canonical morphism

$$\mathsf{coim}(f) \to \mathsf{ker}(g).$$

Definition A.3.10. Let $\mathcal{A}$ be an abelian category. We say a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is **exact** if it satisfies the following conditions.

- The composition $g \circ f$ is zero.
- the canonical morphism $\mathsf{coim}(f) \to \mathsf{ker}(g)$ in Construction A.3.9 is an isomorphism.

We say a sequence $X_1 \to X_2 \to \cdots \to X_n$ is **exact** if any three consequential terms from an exact sequence.

A.4. **Additive functors.**

Proposition-Definition A.4.1. *Let $F : \mathcal{A} \to \mathcal{A}'$ be a functor between additive categories. The following conditions are equivalent:*

(i) *The functor F preserves finite products.*
(ii) *The functor F preserves finite coproducts.*

We say F is ***additive*** *if it satisfies the above conditions.*

Proof. We will show (i)⇒(ii). The other implication follows by duality. Suppose F preserves finite products. To verify (ii), we only need to check F preserves initial objects and binary coproducts. The claim for initial objects follow from the axiom that additive categories admit zero objects. The claim for binary coproducts follow from the axoim that the canonical map $X \coprod Y \to X \times Y$ is an isomorphism.

□

The following result follows immediately from Construction A.1.7.

Lemma A.4.2. *Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between additive categories. Then for any pair of objects $X, Y \in \mathcal{A}$, the map*

$$\mathsf{Hom}_{\mathcal{A}}(X,Y) \to \mathsf{Hom}'_{\mathcal{A}}(F(X),F(Y))$$

is a homomorphism between abelian groups.

Note that addtive functors in general do not preserve (co)kernels.

Definition A.4.3. Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between abelian categories.

- We say F is **left exact** if F preserves kernels.
- We say F is **right exact** if F preserves cokernels.
- We say F is **exact** if F is both left exact and right exact.

Remark A.4.4. It is easy to see F is left exact iff it preserves exact sequences of the form $0 \to X_1 \to X_2 \to X_3$, while F is right exact iff it preserves exact sequences of the form $X_1 \to X_2 \to X_3 \to 0$. Also, F is exact iff it preserves all exact sequences.

Example A.4.5. Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$ be an object. Note that we have functors

$$\mathsf{Hom}_{\mathcal{A}}(X,-) : \mathcal{A} \to \mathsf{Ab},\ \mathsf{Hom}_{\mathcal{A}}(-,X) : \mathcal{A}^{\mathsf{op}} \to \mathsf{Ab}$$

One can check both functors are left exact.

Lemma A.4.6. *An additive functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories is left exact iff it preserves fiber products. Dually, F is right exact iff it preserves fiber coproducts.*

Proof. The "if" part is obvious because any kernel is a fiber product. For the "only if" part, let $X \xrightarrow{f} Y \xleftarrow{g} Z$ be any span diagram. Consider

$$W := \mathsf{ker}(X \bigoplus Z \xrightarrow{\begin{pmatrix} f \\ -g \end{pmatrix}} Y).$$

One can check that the morphisms $W \to X \oplus Z \to X$ and $W \to X \oplus Z \to Z$ exhibits W as the fiber product of $X \xrightarrow{f} Y \xleftarrow{g} Z$. Since F preserves biprodcuts and kernels, it also preserves fiber products.

□

Remark A.4.7. Using the language of category theory, a functor $F : \mathcal{A} \to \mathcal{A}'$ between abelian categories is left (resp. right) exact iff it preserves finite limits (resp. colimits).

A.5. **Abelian categories in theses notes.** The following claims are either proved in the notes, or follow immediately from the definitions.

Proposition A.5.1. *Let X be a topological space.*

- *The category $\mathsf{PShv}(X,\mathsf{Ab})$ is an abelian category.*
- *For any open subset $U \subseteq X$, the functor $(-)(U) : \mathsf{PShv}(X,\mathsf{Ab}) \to \mathsf{Ab}$ is exact.*
- *For any point $x \in X$, the functor $(-)_x : \mathsf{PShv}(X,\mathsf{Ab}) \to \mathsf{Ab}$ is exact.*

Let $f : X \to Y$ be a continuous map between topological spaces.

- *The functor* $f_* : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{PShv}(Y, \mathsf{Ab})$ *is exact.*
- *The functor* $f^{-1}_{\mathsf{PShv}} : \mathsf{PShv}(Y, \mathsf{Ab}) \to \mathsf{PShv}(X, \mathsf{Ab})$ *is exact.*

Proposition A.5.2. *Let X be a topological space.*

- *The category* $\mathsf{Shv}(X, \mathsf{Ab})$ *is an abelian category.*
- *For any open subset $U \subseteq X$, the functor* $(-)(U) : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ *is left exact.*
- *For any point $x \in X$, the functor* $(-)_x : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ *is exact.*
- *The fully faithful functor* $\mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{PShv}(X, \mathsf{Ab})$ *is left exact.*
- *The sheafification functor* $\sharp : \mathsf{PShv}(X, \mathsf{Ab}) \to \mathsf{Shv}(X, \mathsf{Ab})$ *is exact.*

Let $f : X \to Y$ be a continuous map between topological spaces.

- *The functor* $f_* : \mathsf{Shv}(X, \mathsf{Ab}) \to \mathsf{Shv}(Y, \mathsf{Ab})$ *is left exact.*
- *The functor* $f^{-1} : \mathsf{Shv}(Y, \mathsf{Ab}) \to \mathsf{Shv}(X, \mathsf{Ab})$ *is exact.*

Proposition A.5.3. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

- *The categories* $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ *and* $\mathcal{O}_X\text{–}\mathsf{mod}$ *are abelian categories.*
- *The forgetful functors* $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathsf{PShv}(X, \mathsf{Ab})$ *and* $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathsf{Shv}(X, \mathsf{Ab})$ *are exact.*
- *The fully faithful functor* $\mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ *is left exact.*
- *The sheafification functor* $\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{–}\mathsf{mod}$ *is exact.*

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between ringed spaces.

- *The functor* $f_* : \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}}$ *is exact.*
- *The functor* $f_* : \mathcal{O}_X\text{–}\mathsf{mod} \to \mathcal{O}_Y\text{–}\mathsf{mod}$ *is left exact.*
- *The functor* $f^*_{\mathsf{PShv}} : \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{PShv}} \to \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{PShv}}$ *is right exact.*
- *The functor* $f^* : \mathcal{O}_Y\text{–}\mathsf{mod} \to \mathcal{O}_X\text{–}\mathsf{mod}$ *is right exact.*

Proposition A.5.4. *Let X be a scheme.*

- *The category* $\mathsf{QCoh}(X)$ *is an abelian category.*
- *The fully faithful functor* $\mathsf{QCoh}(X) \to \mathcal{O}_X\text{–}\mathsf{mod}$ *is exact.*

Proposition A.5.5. *Let $f : X \to Y$ be a morphism between schemes. The functor* $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ *is right exact. If f is an open immersion, it is exact.*

Remark A.5.6. In fact, f^* is exact iff f is *flat*.

Proposition A.5.7. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism between schemes. The functor* $f_* : \mathsf{QCoh}(X) \to \mathsf{QCoh}(Y)$ *is left exact. It is exact iff f is affine.*

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