All rings are Noetherian, all schemes are locally Noetherian.

Zariski Main Thm. If X is quasi-compact (Noetherian), then any separated, quasi-finite morphism $f: Y \to X$ factors as

Recall, finite type, fiber discrete.

$$Y \xrightarrow{f'} Y' \xrightarrow{g} X$$

where f' is an open immersion and g is finite.

Proof of Zariski Thm. $f: Y \to X$

By Nagata compactification: Since X is quasi-compact. quasi-separated (since X is assume to be Noetherian). and f is separated, there exists $Y \xrightarrow{j} Y' \xrightarrow{\bar{f}} X$

where j is an open immersion. and $\bar{f}$ is proper.

Then $\bar{f}$ is proper, thus $\bar{f}_* \mathcal{O}_{Y'}$ is a coherent $\mathcal{O}_X$-sheaf.

We consider the stein factorization:

$$Y' \xrightarrow{f'} \operatorname{Spec}_X \bar{f}_* \mathcal{O}_{Y'} \xrightarrow{g} X$$

The Latter one is a finite morphism.

Then we just need to show that $Y' \xrightarrow{f' \circ j} \operatorname{Spec}_X \bar{f}_* \mathcal{O}_{Y'}$ is an open immersion.

By prop 2.5.13. in Lei Fu's book Algebraic Geometry.

Proposition 2.5.13. *Let $f : X \to Y$ be a proper morphism of noetherian schemes and let X' be the subset of X consisting of those points x which are isolated in $f^{-1}(f(x))$. Then X' is open in X. Let $X \xrightarrow{f'} Y' \xrightarrow{g} Y$ be the Stein factorization of f. Then $f'|_{X'} : X' \to Y'$ is an open immersion.*

(Note. one should also read Theorem 2.5.10, the Theorem of Connectness).

Since $f: Y \to X$ is quasi-finite, $\Rightarrow$ If let $Y'' \subseteq Y'$ denote all y, s.t. y is isolated in $\bar{f}^{-1}(\bar{f}(y))$. then $Y \subseteq Y''$, and $Y \to Y'' \hookrightarrow Y'$ is an open immersion.

and $Y'' \to \mathrm{Spec}_X f_* \mathcal{O}_Y$ is an open immersion.

Hence $Y \to \mathrm{Spec}_X f_* \mathcal{O}_Y \to X$ is the desired decomposition.