

LECTURE 4

1. WHAT IS THIS COURSE ACTUALLY ABOUT?

After the preparations of the first three lectures, we can finally say a few words about what this course is actually about.

From now on, we fix the following notations. Let G be a connected semisimple algebraic group with Lie algebra \mathfrak{g} . We fixed $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ and let $G \supset B \supset T$ be the corresponding closed connected algebraic subgroups^{1,2} of G .

We can consider the right multiplication action of B on G . This is a free action and thereby the quotient G/B exists³. This quotient is a projective smooth k -scheme, and actually does not depend on the choice of B . It is known as the **flag variety**⁴ of G , usually denoted by Fl_G .

Now the main goal of this course is to exhibit the following phenomenon: representations of the Lie algebra \mathfrak{g} is closely related to the geometry of Fl_G . In particular, we will see that the structure of the category \mathcal{O} is related to the orbits of the *left* B -action on $\mathrm{Fl}_G \simeq G/B$.

2. BLOCKS

Let's be back to the study of the category \mathcal{O} . The main goal of this and the next lectures is to split the category \mathcal{O} into blocks

$$\mathcal{O} \simeq \bigoplus_{\chi \in \mathrm{Spec}(Z(\mathfrak{g}))} \mathcal{O}_\chi$$

and study each block. Here $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ is the center of the associative algebra $U(\mathfrak{g})$.

Note that any $M \in \mathfrak{g}\text{-mod}$ can be viewed as a $Z(\mathfrak{g})$ -module, which is the same as a quasi-coherent sheaf on $\mathrm{Spec}(Z(\mathfrak{g}))$. We have:

Theorem 1. *Let $M \in \mathcal{O}$, then the corresponding quasi-coherent sheaf on $\mathrm{Spec}(Z(\mathfrak{g}))$ is supported on a 0-dimensional subvariety.*

Remark 2. Equivalently, the theorem says the annihilator $I := \{f \in Z(\mathfrak{g}) \mid f \cdot M = 0\}$ of M has a finite codimension, i.e., $\dim(Z(\mathfrak{g})/I) < \infty$.

Before we prove the above theorem, we make the following definition and state a corollary of it.

Definition 3. For any closed point $\chi \in \mathrm{Spec}(Z(\mathfrak{g}))$, let $\mathcal{O}_\chi \subset \mathcal{O}$ be the full subcategory containing those $M \in \mathcal{O}$ that is set-theoretically supported on χ . We call \mathcal{O}_χ a **block** of \mathcal{O} .

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¹In general, for an algebraic group G and $\mathfrak{g} = \mathrm{Lie}(G)$, there is a 1-1 correspondence between the set of closed connected algebraic subgroups of G and the set of Lie subalgebras of \mathfrak{g} . Warning: This is only true in characteristic 0.

²As in the study of semisimple Lie algebras, B is a **Borel subgroup** of G , which is defined to be a maximal connected solvable subgroup of G ; T is called **Cantan subgroup** of G , which is defined to be a maximal connected torus inside G .

³In future lectures, we will revisit its definition.

⁴Example: for $G = \mathrm{SL}_2$, Fl_G is isomorphic to the projective line \mathbb{P}^1 .

Remark 4. In a more representation-theoretic language: a closed point $\text{Spec}(Z(\mathfrak{g}))$ is the same as a character $\chi : Z(\mathfrak{g}) \rightarrow k$. Then for $M \in \mathcal{O}$, it is set-theoretically supported on χ iff $\ker(\chi)^n \cdot M = 0$ for $n \gg 0$.

Corollary 5. *We have*

$$\mathcal{O} \simeq \bigoplus_{\chi \in \text{Spec}(Z(\mathfrak{g}))} \mathcal{O}_\chi.$$

In other words, any $M \in \mathcal{O}$ admits a unique finite decomposition $M = \bigoplus_\chi M_\chi$ such that $M_\chi \in \mathcal{O}_\chi$. Also, for $M = \bigoplus_\chi M_\chi$ and $N = \bigoplus_\chi N_\chi$, we have

$$\text{Hom}_{\mathcal{O}}(M, N) \simeq \bigoplus_\chi \text{Hom}_{\mathcal{O}_\chi}(M_\chi, N_\chi).$$

Proof. The decomposition $M = \bigoplus_\chi M_\chi$ follows from the theorem because any 0-dimensional subvariety contains finitely many points.

For $\chi \neq \chi'$, there is no nonzero \mathfrak{g} -linear map $M_\chi \rightarrow N_{\chi'}$ because there is no nonzero $Z(\mathfrak{g})$ -linear map between them. \square

Remark 6. In the future, we will see the decomposition $\mathcal{O} \simeq \bigoplus_{\chi \in \text{Spec}(Z(\mathfrak{g}))} \mathcal{O}_\chi$ also holds for the derived categories. In other words, we can replace $\text{Hom}_\bullet(-, -)$ in above by $\text{Ext}_\bullet^i(-, -)$.

Proof of Theorem 1. Note that if an ideal $I \subset Z(\mathfrak{g})$ annihilates two \mathfrak{g} -modules, then I^2 annihilates any extension of them. By [Prop. 31, Lec. 2], any $M \in \mathcal{O}$ is a quotient of a finite successive extension of Verma modules. It follows that we can assume $M = M_\lambda$ is a Verma module.

We will use the following lemma.

Lemma 7. *For any element $z \in Z(\mathfrak{g})$ and Verma module M_λ , the action of z on M_λ is given by a scalar. In other words, there exists a unique scalar $\xi_{z,\lambda} \in k$ such that*

$$z \cdot v = \xi_{z,\lambda} v$$

for any $v \in M_\lambda$.

Let us first finish the proof using the lemma. By the definition of actions, the map

$$\chi := \xi_{-, \lambda} : Z(\mathfrak{g}) \rightarrow k$$

has to be a homomorphism between k -algebras. In other words, χ is a character of Z and the lemma says $M_\lambda \in \mathcal{O}_\chi$, i.e., M_λ is annihilated by the 1-codimensional ideal $\ker(\chi)$.

Lemma 7. Since z is an element in the center, the map $z \cdot - : M_\lambda \rightarrow M_\lambda$ is a $U(\mathfrak{g})$ -linear map. In particular, it is $U(\mathfrak{t})$ -linear and thereby preserves the weight subspaces of M_λ . Consider the highest weight subspace, which is spanned by the vector $v_\lambda \in M_\lambda$. This subspace is preserved by the action of z , hence there exists a unique scalar $\xi_{z,\lambda} \in k$ such that

$$z \cdot v_\lambda = \xi_{z,\lambda} v_\lambda.$$

Recall M_λ is generated by v_λ . Hence any $v \in M_\lambda$ can be written as $u \cdot v_\lambda$ for some $u \in U(\mathfrak{g})$. Then we have

$$z \cdot v = z \cdot (u \cdot v_\lambda) = u \cdot (z \cdot v_\lambda) = \xi_{z,\lambda} u \cdot v_\lambda = \xi_{z,\lambda} v$$

as desired. Here we used the assumption that z is contained in the center.

\square [Lemma 7]

\square [Theorem 1]

3. THE HARISH-CHANDRA ISOMORPHISM: PREPARATION

The scalar $\xi_{z,\lambda}$ in Lemma 7 can be calculated as follows.

Construction 8. The trivial modules define homomorphisms between associative algebras $U(\mathfrak{n}^\pm) \twoheadrightarrow k$. Consider the k -linear map

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} U(\mathfrak{n}) \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} k$$

induced by these homomorphisms. By the PBW theorem, the RHS can be identified as $U(\mathfrak{t}) = \text{Sym}(\mathfrak{t})$. In other words, we obtain a k -linear surjection

$$(3.1) \quad U(\mathfrak{g}) \twoheadrightarrow \text{Sym}(\mathfrak{t}).$$

To have a better feeling about this projection, please do the following exercise.

Exercise 9. This is **Homework 2, Problem 1**. Let $\mathfrak{g} = \mathfrak{sl}_2$ and e, h, f be the standard basis. Consider $\Omega := ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2)$. Calculate its image in $\text{Sym}(\mathfrak{t}) = k[h]$ under the map (3.1).

Lemma 10. The composition

$$\phi : Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow \text{Sym}(\mathfrak{t})$$

is a homomorphism and makes the following diagram commute

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\phi} & \text{Sym}(\mathfrak{t}) \\ & \searrow \chi = \xi_{-, \lambda} & \downarrow \text{ev}_\lambda \\ & & k_\lambda, \end{array}$$

where ev_λ is taking the value of a function $f \in \text{Sym}(\mathfrak{t}) \simeq \text{Fun}(\mathfrak{t}^*)$ at $\lambda \in \mathfrak{t}^*$.

Proof. Note that χ and ev_λ are homomorphisms. Also, the map $\prod_{\lambda \in \mathfrak{t}^*} \text{ev}_\lambda : \text{Sym}(\mathfrak{t}) \rightarrow \prod_{\lambda \in \mathfrak{t}^*} k_\lambda$ is injective. Hence we only need to show the above diagram commutes as *vector spaces*.

Note that the PBW theorem also implies

$$U(\mathfrak{b}) \otimes_{U(\mathfrak{n})} k \simeq \text{Sym}(\mathfrak{t})$$

as $U(\mathfrak{b})$ -modules, where the $U(\mathfrak{b})$ -module on the RHS is provided by the homomorphism $U(\mathfrak{b}) \rightarrow U(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t})$. It follows that the map (3.1) is equal to

$$U(\mathfrak{g}) \rightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \text{Sym}(\mathfrak{t}) \simeq \text{Sym}(\mathfrak{t}).$$

Then $\text{ev}_\lambda \circ \phi$ is given by

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \simeq k_\lambda,$$

where as before we view k_λ as a $U(\mathfrak{b})$ -module via $U(\mathfrak{b}) \twoheadrightarrow \text{Sym}(\mathfrak{t})$.

On the other hand, by definition, χ is the composition

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \xrightarrow{-v_\lambda} M_\lambda \xrightarrow{\text{pr}} k_\lambda$$

where pr is the projection to the highest weight subspace. Recall

$$M_\lambda \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$$

and via this identification, the projection pr is

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \simeq k_\lambda.$$

It follows χ is also given by

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \twoheadrightarrow k \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \simeq k_\lambda.$$

□

Notation 11. The homomorphism ϕ corresponds to a morphism between k -schemes, which we denote by

$$\varpi : \mathfrak{t}^* \rightarrow \mathrm{Spec}(Z(\mathfrak{g})).$$

Using this notation, Lemma 7 says

$$M_\lambda \in \mathcal{O}_{\varpi(\lambda)}.$$

In fact, it implies the schematic support of M_λ is the point $\varpi(\lambda) \in \mathfrak{t}^*$.

Remark 12. The map $\varpi : \mathfrak{t}^* \rightarrow \mathrm{Spec}(Z(\mathfrak{g}))$ is not an isomorphism because two Verma modules can belong to the same block.

Warning 13. The embedding $\mathfrak{t} \rightarrow \mathfrak{g}$ also induces an injection $\mathrm{Sym}(\mathfrak{t}) \hookrightarrow U(\mathfrak{g})$, but the center $Z(\mathfrak{g})$ is not contained in the image of it. This can be seen e.g. from the following exercise.

Exercise 14. This is **Homework 2, Problem 2**. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ be any nondegenerate symmetric invariant bilinear form on \mathfrak{g} . For any basis x_1, \dots, x_n of \mathfrak{g} and its dual basis x_1^*, \dots, x_n^* with respect to the form κ ⁵, consider the **Casimir element**

$$\Omega_\kappa = \sum_{i=1}^n x_i \cdot x_i^* \in U(\mathfrak{g}).$$

- (1) Prove: the Casimir element Ω_κ does not depend on the choice of the basis, and is contained in the center $Z(\mathfrak{g})$.
- (2) For $\mathfrak{g} = \mathfrak{sl}_2$, $\kappa = \mathrm{Kil}$ and the canonical basis e, h, f , find Ω_{Kil} and prove it is not contained in $\mathrm{Sym}(\mathfrak{t}) \subset U(\mathfrak{g})$.

Example 15. On the other hand, recall there is a nontrivial map $M_{-l-2} \rightarrow M_l$ for $l \in \mathbb{Z}^{\geq 0}$. It follows that $\varpi(-l-2) = \varpi(l)$ where we identify \mathfrak{t}^* with $\mathrm{Spec}(k[h])$. Since $\mathbb{Z}^{\geq 0} \subset \mathrm{Spec}(k[h])$ is a dense subset⁶, we obtain $\varpi(-l-2) = \varpi(l)$ for any $l \in \mathfrak{t}^*$. In other words, the image of $\phi : Z(\mathfrak{sl}_2) \rightarrow \mathrm{Sym}(\mathfrak{t}) \simeq k[h]$ is invariant under the change of variable $h \mapsto -h-2$.

Note that this can serve as a sanity-check for your calculation of Ω_{Kil} in the above exercise.

4. RECOLLECTION: THE WEYL GROUP

In the above example, $Z(\mathfrak{sl}_2)$ is actually isomorphic to the subalgebra of $k[h]$ containing exactly those polynomials that are invariant under $h \mapsto -h-2$. We will prove this for general \mathfrak{g} . Namely $Z(\mathfrak{g})$ is isomorphic to a certain invariance of $\mathrm{Sym}(\mathfrak{t})$. Let us first define the desired symmetry on $\mathrm{Sym}(\mathfrak{t})$.

Recall the following definition.

Definition 16. Let (E, Φ) be a root system. The **Weyl group** $W \subset \mathrm{GL}_E$ is the subgroup generated by the reflections s_α , where $s_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha$.

If (E, Φ) corresponds to a semisimple Lie algebra \mathfrak{g} , we also say W is the Weyl group of \mathfrak{g} (and even of the algebraic group G).

⁵By definition, this means $\kappa(x_i, x_j^*)_{i,j}$ is the unit matrix.

⁶Note that we are using the Zariski topology on $\mathrm{Spec}(k[h])$. Equivalently, this means any polynomial in $k[h]$ that vanishes at $h \in \mathbb{Z}^{\geq 0}$ is 0.

Remark 17. By definition, W preserves Φ (this is an axiom of root systems) and any element $w \in W$ is determined by its action on Φ (because Φ spans E). It follows that W is a finite subgroup because Φ is finite.

Remark 18. It also follows that W preserves $E_{\mathbb{Q}} \subset E$ where recall $E_{\mathbb{Q}}$ is the \mathbb{Q} -vector space spanned by Φ . Hence we obtain a linear W -action on $\mathfrak{t}^* \simeq k \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$. This action is often denoted as

$$W \times \mathfrak{t}^* \rightarrow \mathfrak{t}^*, (w, \lambda) \mapsto w(\lambda).$$

Passing to dual vector spaces, we obtained a linear W -action on \mathfrak{t} .

Remark 19. The Weyl groups of (E, Φ) and $(E^*, \check{\Phi})$ are isomorphic such that s_{α} corresponds to $s_{\check{\alpha}}$.

Proposition 20. *Let $N_G(T)$ be the normalizer of T inside G . Then we have a canonical isomorphism*

$$W \simeq N_G(T)/T$$

such that the W -action on \mathfrak{t} is identified with the adjoint action of $N_G(T)/T$ on $\text{Lie}(T)$ ⁷.

Remark 21. Weyl groups are (*crystallographic*) *Coxeter groups*. Namely, if we choose positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subset \Phi^+$. Then

$$W = \langle s_{\alpha}, \alpha \in \Delta \mid s_{\alpha}^2 = 1, (s_{\alpha} s_{\beta})^{m_{\alpha\beta}} = 1, \alpha \neq \beta \rangle$$

where $m_{\alpha\beta} \in \{2, 3, 4, 6\}$. Elements s_{α} , $\alpha \in \Delta$ are called **simple reflections**.

Example 22. For $\mathfrak{g} = \mathfrak{sl}_n$, W is isomorphic to the symmetric group Σ_n and the simple reflections are $(i, i+1)$ for $i = 1, \dots, n-1$. The Σ_n -action on \mathfrak{t} is the standard one.

Example 23. For $\mathfrak{g} = \mathfrak{sl}_2$, the nontrivial element $s \in W = \Sigma_2$ acts on $\mathfrak{t}^* = \text{Spec}(k[h])$ as $h \mapsto -h$. Note that this is not the reflection in Example 15.

Definition 24. Let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

be the half-sum of all positive roots. The **dotted action** of W on \mathfrak{t}^* , denoted by

$$W \times \mathfrak{t}^* \rightarrow \mathfrak{t}^*, (w, \lambda) \mapsto w \cdot \lambda,$$

is defined to be $w \cdot \lambda := w(\lambda + \rho) - \rho$.

Construction 25. *The dotted W -action on \mathfrak{t}^* is not k -linear (e.g. the fixed point is $-\rho \in \mathfrak{t}^*$). Instead, each $w \cdot - : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ is an affine transformation. In particular, it is a morphism between k -schemes and corresponds to a homomorphism $\text{Sym}(\mathfrak{t}) \leftarrow \text{Sym}(\mathfrak{t})$. This defines a right action of W on $\text{Sym}(\mathfrak{t})$. The standard **dotted** W -action on $\text{Sym}(\mathfrak{t})$ is obtained by taking inverse.*

5. THE HARISH-CHANDRA ISOMORPHISM: STATEMENT

Theorem 26 (Harish-Chandra). *The homomorphism $\phi : Z(\mathfrak{g}) \rightarrow \text{Sym}(\mathfrak{t})$ induces an isomorphism*

$$\phi_{\text{HC}} : Z(\mathfrak{g}) \xrightarrow{\sim} \text{Sym}(\mathfrak{t})^{W \bullet}$$

from $Z(\mathfrak{g})$ to the invariance of $\text{Sym}(\mathfrak{t})$ with respect to the dotted W -action.

We will prove this theorem next time. For now, let us state some corollaries of it.

⁷We haven't defined normalizers for algebraic groups, nor the quotient group $N_G(T)/T$ and its action on $\text{Lie}(T)$. But nothing surprising happens here.

Corollary 27. *The map $\varpi : \mathfrak{t}^* \rightarrow \mathrm{Spec}(Z(\mathfrak{g}))$ induces an isomorphism*

$$\mathfrak{t}^* // W_{\bullet} \xrightarrow{\sim} \mathrm{Spec}(Z(\mathfrak{g}))$$

from the (GIT) quotient of \mathfrak{t}^ by W_{\bullet} to $\mathrm{Spec}(Z(\mathfrak{g}))$.*

Remark 28. In algebraic geometry, the quotient of a scheme X by an algebraic group H , defined to be the coequalizer $H \times X \rightrightarrows X$, does not always exist as a scheme. Mostly for this reason, people defined the notion of *stacks* such that quotients always exist as stacks.

However, the quotients inside the category of schemes and the category of stacks are different whenever the action is not free. Hence people call the former *categorical quotient* and the latter *quotient stack*.

On the other hand, the *GIT quotient*, or *geometric invariant theory quotient* is a variant of the categorical quotient when we restrict to the categories of *affine schemes*. For $X = \mathrm{Spec}(A)$ is affine, the GIT quotient is defined to be $X // H := \mathrm{Spec}(A^H)$. Therefore the above corollary is just a rephrasing of the theorem.

Note however in general GIT quotient is *not* the categorical quotient⁸. For example, the categorical quotient G/B is projective rather than affine.

It is known that $\mathfrak{t}^* // W_{\bullet}$ is also the categorical quotient⁹. But we keep this notation because $\mathfrak{t}^* / W_{\bullet}$ is reserved for the quotient stack (at least by modern mathematicians).

Corollary 29 (Linkage principle). *Verma modules M_{λ} and M_{μ} belong to the same block iff $\mu = w \cdot \lambda$ for some $w \in W$. In particular, there are only finitely many Verma modules and irreducible objects¹⁰ in each block.*

Remark 30. Next time, we will actually first prove the “if” part of the corollary and use it to show ϕ factors through the W_{\bullet} -invariance.

Exercise 31. This is **Homework 2, Problem 3**. Let $\mathfrak{g} = \mathfrak{sl}_2$. Prove $Z(\mathfrak{sl}_2) \simeq k[\Omega_{\mathrm{Kil}}]$ where Ω_{Kil} is the Casimir element. You can use Theorem 26 for this exercise.

⁸Warning: Wikipedia falsely claims this. Maybe they mean the categorical quotient considered in the category of *affine* schemes. But this is not the standard terminology.

⁹If you know some algebraic geometry, try proving this by yourself.

¹⁰Recall M_{λ} has a unique irreducible quotient L_{λ} , and all irreducible objects of \mathcal{O} are of this form. See [Thm. 25, Prop. 31, Lec. 2].