LECTURE 4

1. Quasi-categories

1.1. Last time, we constructed a Quillen equivalence

$$|-| : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Top} : \mathsf{Sing}.$$

This implies:

- Any ∞-groupoid, which can be identified with a homotopy type by the Homotopy Hypothesis, can also be realized as a bifibrant object in Set_Δ, i.e., a **Kan complex**.
- Any functor between ∞-groupoids can be realized as a *homotopy class* of morphisms between the corresponding Kan complexes. Here two morphisms $f, g : X \to Y$ are homotopic iff there exists a morphism $H : X \times \Delta^1 \to Y$ with $H(-,0) = f$ and $H(-,1) = g$.

1.2. Recall a simplicial set X is a Kan complex iff it has the right lifting property against *all* horn inclusions with $n \geq 2$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

Let us inspect these lifting properties for different choices of (n, i) using the *language* of higher category theory.

1.3. For $n = 2, i = 0$, the lifting property says that for a given morphism $f : x_0 \to x_1$, any morphism of the form $x_0 \to x_2$ can factor through f.

For $n = 3, i = 0$, the lifting property says that for a given morphism $f : x_0 \to x_1$, any commutative diagram

$$\begin{tikzcd} & x_2 \arrow[dr] & \\ x_0 \arrow[ur] \arrow[rr] & & x_3, \end{tikzcd}$$

or more precisely, an invertible 2-morphism *witnessing* such a diagram, can factor through f. This means the above 2-morphism is equivalent to the horizontal composition of a 2-morphism

$$\begin{tikzcd} & x_2 \arrow[dr] & \\ x_1 \arrow[ur] \arrow[rr] & & x_3 \end{tikzcd}$$

with the identity 2-morphism id_f.

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In general, the lifting properties for $i = 0$ imply that for a given morphism $f : x \to y$, any *homotopy coherent* diagram with *initial* vertex x can factor through f. Note that this is a feature of ∞-groupoids rather than general ∞-categories: it implies the given morphism f has a left inverse.

1.4. Dually, the lifting properties for $i = n$ imply that for a given morphism $f : x \to y$, any homotopy coherent diagram with *final* vertex y can factor through f. Again, this is a feature of ∞-groupoids rather than general ∞-categories: it implies the given morphism f has a right inverse.

1.5. On the other hand, the lifting properties for $0 < i < n$ imply that for a given object x, one can compose any homotopy coherent diagram with final vertex x with another homotopy coherent diagram with initial vertex x. This requirement should be satisfied by any category-like entity in mathematics.

1.6. The above discussion motivates the following definition:

Definition 1.7 (Boardman–Vogt, 1973)**.** *A simplicial set X is called a **quasi-category** if it has the right lifting property against* inner *horn inclusions:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \qquad \textit{for } 0 < i < n.$$

*Let X be a quasi-category. We call a 0-simplex $x \in X_0$ an **object** of X, and a 1-simplex $e \in X_0$ a **morphism** from $d^1_0(e)$ to $d^1_1(e)$.*

*Let X and Y be quasi-categories, we call a morphism $F : X \to Y$ in Set_Δ a **functor** from X to Y.*

Let $\mathsf{QCat} \subset \mathsf{Set}_\Delta$ be the full subcategory of quasi-categories.

Example 1.8. *Kan complexes are quasi-categories.*

Exercise 1.9. *Nerves of categories are quasi-categories.*

1.10. In the 1980s, Joyal proposed:

Quasi-categories are models for $(\infty, 1)$-categories.

In this lecture, we take the above proposal as acknowledged, and use it to develop basic languages about $(\infty, 1)$-categories. We will justify this proposal in the next lecture.

1.11. For the purpose of this course, we do not give proofs in this lecture, nor present the results in the logical order.

2. Composition of morphisms

Definition 2.1. *Let $\mathcal{C}$ be a quasi-category. For a 2-simplex σ of the form*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

*we say σ **witnesses** h **as a composition of** f **and** g.*

Remark 2.2. *Note that in the above definition, the morphism h is not determined by f and g, nor is σ determined by f, g and h. We may say:*

Composition of morphisms in quasi-categories is not *concrete.*

Nevertheless, we will soon see that composition of morphisms in the ∞-category represented by $\mathcal{C}$ is well-defined, as long as we interpret well-definedness correctly.

3. Homotopy between morphisms

Definition 3.1. *Let $\mathcal{C}$ be a quasi-category. For a 2-simplex σ of the form*

$$\begin{tikzcd} & y \arrow[dr, "\mathsf{id}"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y, \end{tikzcd}$$

we say σ is a ***left pinched homotopy***[1]*, or* ***left pinched 2-morphism****, from f to g.*

Dually, a 2-simplex of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathsf{id}"] \arrow[rr, "g"'] & & y. \end{tikzcd}$$

is called a ***right pinched homotopy****, or* ***right pinched 2-morphism****, from f to g.*

Exercise 3.2. *The following conditions are equivalent:*

- *There exists a left pinched homotopy from f to g,*
- *There exists a right pinched homotopy from f to g,*

Moreover, these conditions define an equivalent relation on the set of morphisms from x to y. Hint: horn inclusions $\Lambda^3_1 \to \Delta^3$ and $\Lambda^3_2 \to \Delta^3$.

Definition 3.3. *Let $\mathcal{C}$ be a quasi-category and $f, g : x \to y$ be morphisms. We say f is* ***homotopic*** *to g if they satisfy the conditions in the above exercise.*

4. Homotopy category

Exercise 4.1. *Let $\mathcal{C}$ be a quasi-category. Show that the following construction defines a category $\mathsf{h}\mathcal{C}$:*

- *Objects of $\mathsf{h}\mathcal{C}$ are objects of $\mathcal{C}$;*
- *Morphisms of $\mathsf{h}\mathcal{C}$ are homotopy classes of morphisms of $\mathcal{C}$;*
- *Composition of $[f]$ and $[g]$ is equal to $[h]$ in $\mathsf{h}\mathcal{C}$ iff h is* a *composition of f and g in $\mathcal{C}$.*

Definition 4.2. *Let $\mathcal{C}$ be a quasi-category. We call the category in the above exercise the* ***homotopy category of*** *$\mathcal{C}$, and denote it by $\mathsf{h}\mathcal{C}$.*

Exercise 4.3. *We have an adjoint pair*

$$\mathsf{h} : \mathsf{QCat} \rightleftarrows \mathsf{Cat} : \mathsf{N}_\bullet.$$

[1] This terminology is not standard. For instance, [Rez22] called it a *right* homotopy.

Remark 4.4. *The above exercise says that* $\mathsf{h}\mathcal{C}$ *is the category obtained by forcing all the higher morphisms in the* ∞*-category represented by* $\mathcal{C}$ *to be identities. Note that this is not the same as* abandoning *all the higher morphisms. The latter construction is evil because it violates the principle of isomorphism.*

4.5. As indicated by their names, homotopy categories of quasi-categories are related to homotopy categories of model categories. We will explain such relations in future lectures.

5. Isomorphisms

Definition 5.1. *Let* $\mathcal{C}$ *be a quasi-category. We say a morphism* f *is an* ***isomorphism*** *iff the morphism* $[f]$ *in* $\mathsf{h}\mathcal{C}$ *is an isomorphism.*

5.2. In other words, $f : x \to y$ is an isomorphism iff there exists a morphism $g : y \to x$ such that $g \circ f$ is homotopic to id_x and $f \circ g$ is homotopic to id_y.

Exercise 5.3. *A quasi-category is a Kan complex iff all morphisms of it are isomorphisms.*

Exercise 5.4. *For a quasi-category* $\mathcal{C}$*, there is a unique Kan complex* $\mathcal{C}^{\simeq} \subset \mathcal{C}$*, called the* ***core of*** $\mathcal{C}$*, that contains all the isomorphisms of* $\mathcal{C}$*. The obtained functor* $(-)^{\simeq} : \mathsf{QCat} \to \mathsf{Kan}$ *is right adjoint to the embedding functor* $\mathsf{Kan} \to \mathsf{QCat}$*.*

6. Morphism spaces: idea

6.1. As a sanity check for Joyal's proposal, for a pair of objects x, y in a quasi-category $\mathcal{C}$, we should construct an ∞-groupoid $\mathsf{Maps}_{\mathcal{C}}(x, y)$ such that:

- Objects in $\mathsf{Maps}_{\mathcal{C}}(x, y)$ are morphisms from x to y;
- Morphisms in $\mathsf{Maps}_{\mathcal{C}}(x, y)$ are 2-morphisms between morphisms from x to y;
- ...

In this lecture, we provide three constructions of Kan complexes that represent this ∞-groupoid (see [Lecture 3, Definition 8.3] for what this means). These three Kan complexes are *weak homotopy equivalent* but not *isomorphic* to each other.

6.2. In a *model-independent* or *axiomatic* theory[2] of $(\infty, 1)$-categories, these three constructions should correspond respectively to the following ∞-groupoids:

(B) The fiber product
$$\{x\} \underset{\mathsf{Fun}(\{0\},\mathcal{C})}{\times} \mathsf{Fun}([1], \mathcal{C}) \underset{\mathsf{Fun}(\{1\},\mathcal{C})}{\times} \{y\},$$
where $\mathsf{Fun}(\mathcal{D}, \mathcal{C})$ is the $(\infty, 1)$**-category of functors from** $\mathcal{D}$ **to** $\mathcal{C}$.

(L) The fiber product
$$\mathcal{C}_{x/} \underset{\mathcal{C}}{\times} \{y\},$$
where $\mathcal{C}_{x/}$ is the **coslice** $(\infty, 1)$**-category of** $\mathcal{C}$ **under** x;

(R) The fiber product
$$\{x\} \underset{\mathcal{C}}{\times} \mathcal{C}_{/y},$$
where $\mathcal{C}_{/y}$ is the **slice** $(\infty, 1)$**-category of** $\mathcal{C}$ **over** y;

[2] I am not aware of the existence of such a theory, hence the discussion below only serves as motivations.

Exercise 6.3. *Convince yourself that the above three ∞-groupoids should be canonically equivalent.*

7. Functor quasi-categories

7.1. To realize (B) via quasi-categories, we need to define the quasi-category of functors between two quasi-categories.

Definition 7.2. *We say a category $\mathcal{C}$ is **Cartesian closed** if it satisfies the following conditions:*

(1) *Finite products exist in $\mathcal{C}$;*
(2) *For any objects X and Y in $\mathcal{C}$, the functor*
$$\mathsf{Hom}(X \times -, Y) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$$
*is corepresentable, i.e., there exists an object Y^X, called the **exponential object**, satisfying the following universal property:*
$$\mathsf{Hom}(X \times Z, Y) \simeq \mathsf{Hom}(Z, Y^X).$$

Remark 7.3. *Condition (2) is equivalent to*

(2′) *For any object X, the functor $X \times - : \mathcal{C} \to \mathcal{C}$ has a right adjoint.*

Indeed, this right adjoint is given by $(-)^X : \mathcal{C} \to \mathcal{C}$.

Proposition-Definition 7.4. *The category Set_Δ is Cartesian closed. For simplicial sets X and Y, we denote the exponential object by*
$$\mathsf{Fun}(X, Y) \overset{\text{def}}{=\!=} Y^X \in \mathsf{Set}_\Delta.$$

Exercise 7.5. *Prove that the nerve functor $\mathsf{N}_\bullet : \mathcal{C} \to \mathsf{Set}_\Delta$ is compatible with products and exponentials, i.e.,*
$$\mathsf{N}_\bullet(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} \mathsf{N}_\bullet(\mathcal{C}) \times \mathsf{N}_\bullet(\mathcal{D}),$$
$$\mathsf{N}_\bullet(\mathsf{Fun}(\mathcal{C}, \mathcal{D})) \xrightarrow{\simeq} \mathsf{Fun}(\mathsf{N}_\bullet(\mathcal{C}), \mathsf{N}_\bullet(\mathcal{D})).$$

7.6. The following result, due to Joyal, is combinatorial but fundamental. See Ker.0066] for a proof.

Proposition 7.7. *For $X \in \mathsf{Set}_\Delta$ and $Y \in \mathsf{QCat}$, we have $\mathsf{Fun}(X, Y) \in \mathsf{QCat}$.*

Corollary 7.8. *The category QCat is Cartesian closed.*

Proposition-Definition 7.9 (Ker.01JC)**.** *Let $\mathcal{C}$ be a quasi-category and x, y be be objects in $\mathcal{C}$. Then the simplicial set*
$$\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y) \overset{\text{def}}{=\!=} \{x\} \underset{\mathsf{Fun}(\{0\}, \mathcal{C})}{\times} \mathsf{Fun}(\Delta^1, \mathcal{C}) \underset{\mathsf{Fun}(\{1\}, \mathcal{C})}{\times} \{y\}$$
*is a Kan complex, which is called the **balanced complex of morphisms**[3] from x to y.*

7.10. By definition, for a testing simplicial set K, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x, y)$
- morphisms $K \times \Delta^1 \to \mathcal{C}$ with constant value x on $K \times \{0\}$ and constant value y on $K \times \{1\}$.

[3]This terminology is not standard. For instance, Lurie called it the *space of morphisms*.

7.11. In particular, a 1-simplex in $\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,y)$ corresponds to a diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathsf{id}"'] \arrow[dr] & y \arrow[d, "\mathsf{id}"] \\ x \arrow[r, "g"'] & y. \end{tikzcd}$$

We call such a diagram a **balanced homotopy**[4], or **balanced 2-morphism**, from f to g.

Exercise 7.12. *Let $\mathcal{C}$ be a quasi-category and $f, g : x \to y$ be morphisms. Show that f is homotopic to g iff there exists a balanced homotopy from f to g.*

8. Slice and coslice

8.1. To realize (L) and (R) via quasi-categories, we need to generalize the slice/coslice categories to quasi-categories.

Definition 8.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The **join** of $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \star \mathcal{D}$, is the category defined by*

$$\mathsf{Ob}(\mathcal{C} \star \mathcal{D}) :\overset{\text{def}}{=} \mathsf{Ob}(\mathcal{C}) \sqcup \mathsf{Ob}(\mathcal{D})$$

$$\mathsf{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) \overset{\text{def}}{=} \begin{cases} \mathsf{Hom}_{\mathcal{C}}(x,y) \ \text{if } x, y \in \mathcal{C} \\ \mathsf{Hom}_{\mathcal{D}}(x,y) \ \text{if } x, y \in \mathcal{D} \\ \{*\} \ \text{if } x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing \ \text{if } x \in \mathcal{D}, y \in \mathcal{C}. \end{cases}$$

Example 8.3. *We have $[i] \star [n-i-1] \simeq [n]$.*

8.4. The above definition can be generalized to simplicial sets.

Definition 8.5. *Let X and Y be simplicial sets. The **join** of X and Y, denoted by $X \star Y$, is the simplicial set defined by*

$$(X \star Y)([n]) \overset{\text{def}}{=} \bigsqcup_{-1 \le i \le n} (X([i]) \times Y([n-i-1])),$$

where we use the convention $[-1] = \varnothing$ and thereby $X[-1] = Y[-1] = \{*\}$. In the above definition, the functoriality in $[n]$ is provided by the equivalences $[i] \star [n-i-1] \simeq [n]$.

Example 8.6. *We have $\Delta^i \star \Delta^{n-i-1} \simeq \Delta^n$.*

Definition 8.7. *Let X be a simplicial set. We define the **left cone** and **right cone of** X to be*

$$X^{\triangleleft} \overset{\text{def}}{=} \Delta^0 \star X, \ X^{\triangleright} \overset{\text{def}}{=} X \star \Delta^0.$$

Definition 8.8. *Let X be a simplicial set and x be a 0-simplex in X. The **slice simplicial set of** X **over** x is the simplicial set $X_{/x}$ defined by*

$$\mathsf{Hom}(Y, X_{/x}) \simeq \mathsf{Hom}(Y^{\triangleright}, X) \underset{\mathsf{Hom}(\Delta^0, X)}{\times} \{x\}.$$

[4]This terminology is not standard.

Dually, the ***coslice simplicial set of*** X ***under*** x *is defined to be the simplicial set* $X_{x/}$ *satisfying the following universal property*

$$\mathsf{Hom}(Y, X_{x/}) \simeq \mathsf{Hom}(Y^{\triangleleft}, X) \underset{\mathsf{Hom}(\Delta^0, X)}{\times} \{x\}.$$

Exercise 8.9. *Show that* $X_{x/}$ *and* $X_{/x}$ *are quasi-categories if* X *is so.*

8.10. The embeddings $Y \to Y^{\triangleleft}$ and $Y \to Y^{\triangleright}$ induce forgetful morphisms

$$X_{x/} \to X,\ X_{/x} \to X.$$

Proposition-Definition 8.11 (Ker.01L0)**.** *Let* $\mathcal{C}$ *be a quasi-category and* x, y *be objects in* X*. The simplicial sets*[5]

$$\mathsf{Hom}^{\mathsf{L}}_{\mathcal{C}}(x,y) \overset{\text{def}}{=\!=} \mathcal{C}_{x/} \underset{\mathcal{C}}{\times} \{y\}$$

$$\mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(x,y) \overset{\text{def}}{=\!=} \{x\} \underset{\mathcal{C}}{\times} \mathcal{C}_{/y}$$

are Kan complexes, which are called the ***left/right pinched complex of morphisms*** *from* x *to* y*.*

8.12. By definition, for a testing simplicial set K, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}^{\mathsf{L}}_{\mathcal{C}}(x,y)$
- morphisms $K^{\triangleleft} \to \mathcal{C}$ with value x on the apex $\Delta^0 \subset K^{\triangleleft}$ and constant value y on the base $K \subset K^{\triangleleft}$.

Dually, we have a bijection between the sets of

- morphisms $K \to \mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(x,y)$
- morphisms $K^{\triangleright} \to \mathcal{C}$ with value y on the apex $\Delta^0 \subset K^{\triangleright}$ and constant value x on the base $K \subset K^{\triangleright}$.

In particular, morphisms in $\mathsf{Hom}^{\mathsf{L}}_{\mathcal{C}}(x,y)$ and $\mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(x,y)$ represents respectively left/right pinched 2-morphisms in $\mathcal{C}$.

9. Morphism spaces

9.1. For a testing simplicial set K, we have natural maps

$$K^{\triangleleft} \leftarrow K \times \Delta^1 \to K^{\triangleright},$$

where the leftward (resp. rightward) morphism collapse $K \times \{0\}$ (resp. $K \times \{1\}$) onto the apex of the left (resp right) cone. By §8.12 and §7.10, we obtain morphisms

$$\mathsf{Hom}^{\mathsf{L}}_{\mathcal{C}}(x,y) \to \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,y) \leftarrow \mathsf{Hom}^{\mathsf{R}}_{\mathcal{C}}(x,y). \tag{9.1}$$

Proposition 9.2 (Ker.01L5)**.** *Morphisms in* (9.1) *are weak homotopy equivalences.*

9.3. It follows that the ∞-groupoids/spaces represented by the Kan complexes in (9.1) are *canonically* equivalent.

Definition 9.4. *Let* $\mathcal{C}$ *be a quasi-category and* x, y *be objects in* X*. Let*

$$\mathsf{Maps}_{\mathcal{C}}(x,y) \in \mathsf{hGrpd}_{\infty} \overset{\text{def}}{=\!=} \mathsf{Set}_{\Delta}[W^{-1}]$$

be the *object represented by* (9.1)[6]*. We call it the* ***mapping space*** *from* x *to* y*.*

[5]In below, $\{x\}$ stands for a simplicial set Δ^0 with the 0-simplex labelled by x.

[6]See §11 for what this actually means.

9.5. Whatever the following means, we call a morphism in the ∞-groupoid $\mathsf{Maps}_{\mathcal{C}}(x,y)$ a **2-morphism in the ∞-category represented by $\mathcal{C}$.**

10. Composition in ∞-categories

10.1. The following result, due to Joyal, is combinatorial but fundamental. See Ker.0079 for a proof.

Theorem 10.2. *A simplicial set X is a quasi-category iff*

$$\mathsf{Fun}(\Delta^2, X) \to \mathsf{Fun}(\Lambda^2_1, X)$$

is an acyclic Kan fibration in Set_Δ.

10.3. Let $\mathcal{C}$ be a quasi-category. By definition, a 0-simplex in $\mathsf{Fun}(\Lambda^2_1, \mathcal{C})$ is a pair of composable morphisms in $\mathcal{C}$:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] & & z, \end{tikzcd}$$

and a lift of this 0-simplex to $\mathsf{Fun}(\Delta^2, \mathcal{C})$ is a 2-simplex σ

(10.1)
$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

witnessing h as a composition of f and g.

Now the above theorem implies the simplicial set of such lifts is a **weakly contractible Kan complex**. In other words, a composition of f and g, **when understood as a 2-simplex**, is defined up to a contractible space of choices. By Slogan 11.4 below, we can say:

> *The composition $g \circ f$ is well-defind as an object in the mapping ∞-groupoid* $\mathsf{Maps}_{\mathcal{C}}(x,y)$.

10.4. To obtain the above result, it is crucial that we incorporate the witness 2-simplex as part of the data in a composition. We can say:

Slogan 10.5. *In an ∞-category, composition is a* structure *rather than a* property.

10.6. Since this is a quite important point, let us rephrase it as follows. In an ∞-category, when saying h is the composition of $g \circ f$, we *always* mean we supply a 2-simplex (10.1). Otherwise, the words *the composition* do not make sense.

Exercise 10.7. *Alternatively, one can justify the above claim as follows. Let $\mathcal{C}$ be a quasi-category.*

(1) *Use Theorem 10.2 to deduce that*

$$\mathsf{Fun}(\Delta^2, \mathcal{C}) \underset{\mathcal{C}^{\times 3}}{\times} \{(x,y,z)\} \to \mathsf{Fun}(\Lambda^2_1, X) \underset{\mathcal{C}^{\times 3}}{\times} \{(x,y,z)\}$$

is an acyclic Kan fibration in Set_Δ.

(2) *Identify the target of the above morphism with*

$$\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,y) \times \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(y,z)$$

(3) *Construct a morphism*

$$\mathsf{Fun}(\Delta^2, \mathcal{C}) \underset{\mathcal{C}^{\times 3}}{\times} \{(x,y,z)\} \to \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,z)$$

in Set_Δ.

(4) *Construct a morphism*

$$\mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,y) \times \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(y,z) \to \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,z)$$

in $\mathsf{hGrpd}_\infty \overset{\text{def}}{=\!=} \mathsf{Set}_\Delta[W^{-1}]$.

(5) *Show that the composition in (4) satisfies* strict *identity and associativity axioms.*

Exercise 10.8. *In an* ∞*-category* C*, composing with an isomorphism induces equivalence betwen mapping spaces. In other words, for a quasi-category* $\mathcal{C}$*,*

(1) *If* $f : x \to y$ *is an isomorphism, then it induces a weak homotopy equivalence between Kan complexes:*

$$- \circ f : \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(y,z) \xrightarrow{\sim} \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,z).$$

(2) *If* $g : y \to z$ *is an isomorphism, then it induces a weak homotopy equivalence between Kan complexes:*

$$g \circ - : \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,y) \xrightarrow{\sim} \mathsf{Hom}^{\mathsf{B}}_{\mathcal{C}}(x,z).$$

11. When is an object well-defined?

11.1. This section serves to answer the following questions:

Why is $g \circ f$ *well-defined as an object in the* ∞*-groupoid* $\mathsf{Maps}_{\mathsf{C}}(x,y)$?

11.2. First, let us make it clear: we do not claim $g \circ f$ is a well-defined 1-simplex in the quasi-category $\mathcal{C}$, nor a well-defined 0-simplex in any of the Kan complexes in (9.1). Instead, we claim it is well-defined as a morphism in the ∞-category modeled by $\mathcal{C}$, or equivalently, well-defined as an object in the ∞-groupoid modeled by (9.1).

Since we have already sketched a *mathematical proof* of this claim in Exercise 10.7, now we will provide a more metaphysical/ideological argument for this claim.

11.3. In classical category theory, *construction* often means

- specifying an *element in a given set*;

However, in infinity category theory, we want to keep track of automorphisms between objects, as well as automorphisms between automorphisms, etc.. Hence *construction* should mean

- specifying an *object in a given* ∞*-groupoid*,

or equivalently

- specifying a *point of a homotopy type*.

Note that we cannot distinguish the homotopy type of a point with that of any contractible space. Hence we may say:

Slogan 11.4. *In an* ∞*-groupoid, an object is well-defined if it is defined up to a contractible space of choices*[7].

[7]A radical consequence of this philosophy is the so-called *univalent foundations* of mathematics.

Appendix A. Inverse

A.1. The lifting property in §1.3 actually implies f is an isomorphism (rather than just has a left inverse), as long as the simplicial set is a quasi-category. In more precise words:

Exercise A.2. *Let $\mathcal{C}$ be a quasi-category and f be a morphism. Suppose*

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\phi"] \arrow[d] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \qquad n \geq 2$$

has a solution for any ϕ such that $\phi|_{\Delta^{\{0,1\}}} = f$. Show that f is an isomorphism.

A.3. The following exercises say the left inverse, right inverse, and inverse of an isomorphism f is well-defined, i.e., defined up to a contractible space of choices. However, one has to be careful about the meanings of these words.

Exercise A.4. *Let $\mathcal{C}$ be a quasi-category and f be a morphism. Find the correct meaning of the words*

- *A 2-simplex σ* witnesses *g as a left/right inverse of f.*

If f is an isomorphism, show that the *left/right inverse of f is well-defined by imitating §10.*

Exercise A.5. *Let $\mathcal{C}$ be a quasi-category and f be an isomorphism. Find a simplicial set $\mathcal{I}$ such that the words*

- *A morphism $\mathcal{I} \to \mathcal{C}$* witnesses *$g$ as an inverse of f*

is the correct notion. Show that the *inverse of f is well-defined by imitating §10. Hint: $\mathcal{I}$ should be the nerve of an ordinary category.*

Exercise A.6. *Make sense of the following words: in an ∞-category, the left inverse and right inverse of an isomorphism are* canonically *equivalent.*

References

[Rez22] Charles Rezk. Introduction to quasicategories. *Lecture Notes for course at University of Illinois at Urbana-Champaign*, 2022.