LECTURE 3

The goal of this lecture is to introduce a combinatorial approach to the homotopy theory of topological spaces, known as **Kan–Quillen model category of simplicial sets**.

1. Simplicial sets

Definition 1.1. *For $n \in \mathbb{Z}_{\geq 0}$, let*
$$[n] := \{0 < 1 < \cdots < n\}$$
be the totally ordered set with $(n+1)$ objects. We view it as a category in the standard way.

Definition 1.2. *The **simplex category** Δ is the full subcategory of Cat consisting of $[n] \in \mathsf{Cat}$, $n \in \mathbb{Z}_{\geq 0}$, i.e.,*
$$\mathsf{Hom}_{\Delta}([m],[n]) = \{\textit{nondecreasing functions } [m] \to [n]\}.$$
Let Δ_{inj} and Δ_{surj} be the subcategories of Δ consisting of injective and surjective morphisms respectively.

1.3. Let $\mathcal{C}$ be a category. We have
$$\mathsf{Hom}_{\mathsf{Cat}}([n],\mathcal{C}) \simeq \{\text{chains in } \mathcal{C} \text{ of length } n\}.$$
The ubiquitous role of the simplex category Δ in category theory can be explained by the following result.

Proposition 1.4. *The functor*
$$\mathsf{Cat} \to \mathsf{Fun}(\Delta^{\mathsf{op}},\mathsf{Set}),\ \mathcal{C} \mapsto \mathsf{Hom}_{\mathsf{Cat}}(-,\mathcal{C}) \tag{1.1}$$
is fully faithful.

Definition 1.5. *Let $\mathcal{D}$ be a category.*
- *A **simplicial object of** $\mathcal{D}$ is a functor $\Delta^{\mathsf{op}} \to \mathcal{D}$.*
- *A **cosimplicial object of** $\mathcal{D}$ is a functor $\Delta \to \mathcal{D}$.*

1.6. Let X be a simplicial object, we often write $X_n := X([n])$, and therefore denote this simplicial object also by $X_\bullet$. Similarly, a cosimplicial object is often denoted by $Y^\bullet$.

Definition 1.7. *Write*
$$\mathsf{Set}_{\Delta} := \mathsf{Fun}(\Delta^{\mathsf{op}},\mathsf{Set})$$
*for the category of **simplicial sets**.*

Example 1.8. *The representable functor*
$$\mathsf{Hom}_{\Delta}(-,[n]) : \Delta^{\mathsf{op}} \to \mathsf{Set}$$
*defines a simplicial set Δ^n, called the n-**simplex**.*

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1.9. Let X be a simplicial set. By the Yoneda lemma, we have

$$\mathsf{Hom}_{\mathsf{Set}_\Delta}(\Delta^n, X) \simeq X_n.$$

This motivates the following definition:

Definition 1.10. *Let $X_\bullet$ be a simplicial set. An element $x \in X_n$ is called an **n-simplex of** X.*

Definition 1.11. *Let*

$$\mathsf{N}_\bullet(-) : \mathsf{Cat} \to \mathsf{Set}_\Delta$$

be the functor (1.1)*. For a category $\mathcal{C}$, the simplicial set $\mathsf{N}_\bullet(\mathcal{C})$ is called the **nerve of** $\mathcal{C}$.*

1.12. Proposition 1.4 says the theory of categories can be embedded into the theory of simplicial sets via the construction $\mathcal{C} \mapsto \mathsf{N}_\bullet(\mathcal{C})$. Therefore,

Slogan 1.13. *Simplicial sets generalize* categories.

2. Faces and degeneracies

Definition 2.1. *For $n > 0$, let*

$$\delta_n^i : [n-1] \to [n]$$

be the unique functor such that $i \in [n]$ is not in the image.

*Let X be a simplicial object in a category $\mathcal{D}$. The **i-th face operator** on X_n is the morphism*

$$d_i^n \overset{\text{def}}{=\!=} X(\delta_n^i) : X_n \to X_{n-1}.$$

Definition 2.2. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. The $(n-1)$-simplex $d_i^n(x)$ is called the **i-th face** of x.*

*More generally, for any injective functor $\iota : [m] \to [n]$, the m-simplex $X(\iota)(x)$ is called the **ι-face** of x.*

Definition 2.3. *For $n \geq 0$, let*

$$\sigma_n^i : [n+1] \to [n]$$

be the unique surjective functor that is constant on $\{i, i+1\}$.

*Let X be a simplicial object in a category $\mathcal{D}$. The **i-th degeneracy operator** on X_n is the morphism*

$$s_i^n \overset{\text{def}}{=\!=} X(\sigma_n^i) : X_n \to X_{n+1}.$$

Definition 2.4. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. The $(n+1)$-simplex $s_i^n(x)$ is called the **i-th degeneracy** of x.*

*More generally, for any surjective functor $\pi : [m] \to [n]$, the m-simplex $X(\pi)(x)$ is called the **π-degeneracy** of x.*

Definition 2.5. *Let X be a simplicial set and $x \in X_n$ be an n-simplex. We say x is **non-degenerate** if it is not a degeneracy of any m-simplex with $m < n$.*

2.6. The proof of the following result is elementary and left to the readers.

Lemma 2.7. *Any degenerate simplex is a π-degeneracy of some non-degenerate simplex x, and the pair (π, x) is unique.*

Exercise 2.8. *Any morphism in Δ is equal to a composition of δ's and σ's.*

2.9. It follows that knowing a simplicial object $X_\bullet$ is equivalent to knowing objects X_n and morphisms d_i^n, s_i^n satisfying certain relations. These relations, known as **simplicial identities**, can be written down explicitly:

$$d_i \circ d_j = d_{j-1} \circ d_i, \text{ if } i < j;$$

$$s_i \circ s_j = s_j \circ s_{i-1}, \text{ if } i > j;$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i, & \text{if } i < j \\ \mathsf{id}, & \text{if } i = j, j+1 \\ s_j \circ d_{i-1}, & \text{if } i > j+1. \end{cases}$$

Here we omit the superscipts from the notations.

2.10. We can depict the face and degeneracy morphisms as a diagram

$$\begin{tikzcd} X_0 \arrow[r, shift left=1] & X_1 \arrow[l, shift left=1] \arrow[l, shift left=3] \arrow[r, shift left=1] \arrow[r, shift left=3] & X_2 \arrow[l, shift left=1] \arrow[l, shift left=3] \arrow[l, shift right=1] \cdots . \end{tikzcd}$$

Sometimes people omit the degeneracy morphisms, and use

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left=1] \arrow[l, shift right=1] & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \cdots . \end{tikzcd}$$

to indicate a simplicial object, especially when they study the *colimit* of this diagram[1].

3. Dimension and skeletons

Definition 3.1. *Let X be a simplicial set and $k \in \mathbb{Z}$, we say X has* ***dimension*** $\leq \boldsymbol{k}$*, or* $\mathsf{dim}(X) \leq k$*, if every n-simplex of X is degenerate for $n > k$.*

Example 3.2. $\mathsf{dim}(\Delta^n) = n$.

Definition 3.3. *Let X be a simplicial set and $k \leq \mathbb{Z}$. The* k***-skeleton*** $\mathsf{sk}_k(X)$ *of X is the largest simplicial subset of X with dimension $\leq k$.*

4. Examples

4.1. We first introduce a standard way to draw a simplicial set $X_\bullet$, which is also how people actually think about them.

- Only non-degenerate simplexes are drawn. Degenerate simplexes "collapse" onto the non-degenerate ones that correspond to them in the sense of Lemma 2.7.

(0) For each $v \in X_0$, draw a vertex labelled by v.

(1) For each non-degenerate $e \in X_1$, draw an arrow labelled by e from $d_0^1(e)$ to $d_1^1(e)$.

(n) For each non-degenerate $\sigma \in X_n$, draw a *filled* n-simplex labelled by σ, with boundary given by $d_0^n(\sigma), d_1^n(\sigma), \ldots, d_n^n(\sigma)$[2].

- When necessary, put symbols inside the simplexes to indicate the order of its vertices.

Exercise 4.2. *Find all the non-degenerate simplexes in Δ^n.*

Exercise 4.3. *Let $\Delta^1 \times \Delta^1$ be the product taken in* Set_Δ*. Find all the non-degenerate simplexes in $\Delta^1 \times \Delta^1$.*

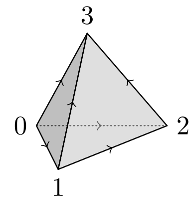

FIGURE 1. The 3-simplex Δ^3

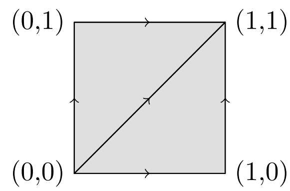

FIGURE 2. The product $\Delta^1 \times \Delta^1$

Exercise 4.4. *In the above example, what would happen if we disallow degeneracies in the definition of simplicial sets, i.e., replacing Δ with Δ_{inj}?*

Example 4.5. *Let*

$$\partial\Delta^n \overset{\text{def}}{=} \mathsf{sk}_{n-1}(\Delta^n)$$

be the $(n-1)$-skeleton of Δ^n. We call it the ***boundary of*** *Δ^n.*

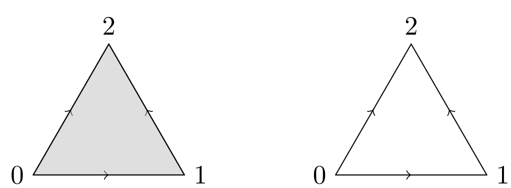

FIGURE 3. The 2-simplex Δ^2 and its boundary $\partial\Delta^2$

Example 4.6. *Let Λ^n_i be the largest simplicial subset of Δ^n that does not contain the i-th face of the unique non-degenerate n-simplex. We call it the* ***i-th horn of of*** *Δ^n.*

4.7. The readers might entertain themselves with the following exercise to check their understanding about the definitions.

Exercise 4.8. *Classify all simplicial quotient sets of Δ^2, i.e., simplicial sets X equipped with an epimorphism $\Delta^2 \to X$ in Set_Δ. Hint: there are 31 of them.*

[1] In future lectures, we will show that the degeneracy morphisms do not affect the ∞-colimit/homotopy colimit of a simplicial object.

[2] Note that these faces can be degenerate.

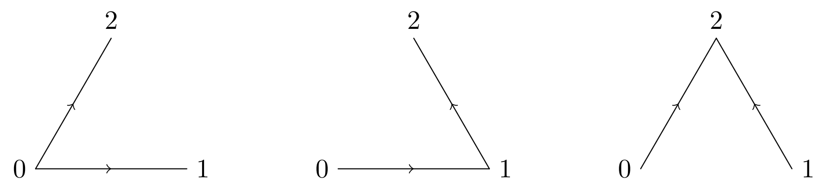

FIGURE 4. The horns Λ^2_0, Λ^2_1 and Λ^2_2

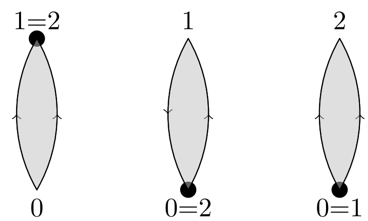

FIGURE 5. Some quotients of Δ^2

5. Simplicial sets and topological spaces

5.1. The simplex category Δ can be realized as a subcategory of Top as follows

Construction 5.2. *Let*

$$|\Delta^n| \overset{\text{def}}{=\!=} \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \,|\, x_0 + \cdots + x_n = 1\}$$

be the ***standard (topological)*** n***-simplex****. We have a functor*

$$\Delta \to \mathsf{Top},\ [n] \mapsto |\Delta^n| \tag{5.1}$$

sending a functor $f : [m] \to [n]$ *to the continuous map*

$$|\Delta^m| \to |\Delta^n|,\ (y_0, \cdots, y_m) \mapsto (\sum_{j \in f^{-1}(0)} y_j, \cdots, \sum_{j \in f^{-1}(n)} y_j).$$

Proposition 5.3. *We view* Δ *as a full subcategory of* Set_Δ *via the Yoneda embedding. Then the functor* (5.1) *can be extended to a colimit-preserving functor*

$$|-| : \mathsf{Set}_\Delta \to \mathsf{Top},$$

which is unique up to a unique equivalence*. We call it the* ***geometric realization functor.***

This functor admits a right adjoint given by

$$\mathsf{Sing} : \mathsf{Top} \to \mathsf{Set}_\Delta,\ X \mapsto \mathsf{Hom}_{\mathsf{Top}}(|\Delta^\bullet|, X).$$

We call it the ***singlular simplicial complex***[3] ***functor****.*

[3]A better name would be singular simplicial *set* functor because we allow degeneracies.

5.4. The adjoint pair
$$|-|:\mathsf{Set}_\Delta \rightleftarrows \mathsf{Top}:\mathsf{Sing}$$
belongs to the following paradigm in category theory.

Exercise 5.5. *Let $\mathcal{C}_0$ be a small category. Define the* ***category of presheaves on*** *$\mathcal{C}_0$ to be*
$$\mathsf{PShv}(\mathcal{C}_0) \overset{\text{def}}{=} \mathsf{Fun}(\mathcal{C}_0^{\mathsf{op}}, \mathsf{Set}).$$

(1) *Prove that $\mathsf{PShv}(\mathcal{C}_0)$ is the category* ***freely generated under small colimits*** *by $\mathcal{C}_0$. In other words, for any category $\mathcal{D}$ containing all small colimits, the Yoneda embedding $\mathcal{C}_0 \to \mathsf{PShv}(\mathcal{C}_0)$ induces an equivalence*
$$\mathsf{LFun}(\mathsf{PShv}(\mathcal{C}_0), \mathcal{D}) \simeq \mathsf{Fun}(\mathcal{C}_0, \mathcal{D}),$$
where $\mathsf{LFun}(-,-) \subset \mathsf{Fun}(-,-)$ consists of colimit-preserving functors.

(2) *Let $F : \mathsf{PShv}(\mathcal{C}_0) \to \mathcal{D}$ be a colimit-preserving functor extending $F_0 : \mathcal{C}_0 \to \mathcal{D}$. Prove that F admits a right adjoint given by*
$$G : \mathcal{D} \to \mathsf{PShv}(\mathcal{C}_0),\ d \mapsto \mathsf{Hom}_{\mathcal{D}}(F_0(-), d).$$

Exercise 5.6. *Challenge: construct an adjoint pair $L : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Cat} : R$ by applying the above paradigm to the functor $\Delta \to \mathsf{Cat}$. Describe the images of the simplicial sets in §4 under the functor.*

5.7. Unlike the nerve functor $\mathsf{N}_\bullet : \mathsf{Cat} \to \mathsf{Set}_\Delta$, the functor $\mathsf{Sing} : \mathsf{Top} \to \mathsf{Set}_\Delta$ is not fully faithful. Hence we cannot embed the theory of topological spaces into the theory of simplicial sets. Nevertheless, the following result, established by Quillen in the 1960s (see [Qui06]), says the *homotopy theories* of them are the same.

Theorem 5.8 (Quillen)**.** *The adjoint pair*
$$|-|:\mathsf{Set}_\Delta \rightleftarrows \mathsf{Top}:\mathsf{Sing}$$
induces an equivalence between the homotopy theories of topological spaces and simplicial sets.

6. Classical model structure on Set_Δ

6.1. To explain Quillen's result, we need first define a model structure on Set_Δ.

Theorem-Definition 6.2. *There exists a model structure on Set_Δ given by*

(W) *A* ***weak homotopy equivalence*** *is a morphism $f : X \to Y$ such that $|f| : |X_\bullet| \to |Y_\bullet|$ is a weak homotopy equivalence.*

(C) *A* ***cofibration*** *is a monomorphism.*

(F) *A* ***Kan fibration*** *is a morphism $f : X \to Y$ that has the right lifting property against all horn inclusions $\Lambda^n_i \to \Delta^n$ for $0 \le i \le n$:*

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y. \end{tikzcd}$$

We call it the ***classical****, or* ***Kan–Quillen*****,** ***model structure*** *on Set_Δ. Fibrant objects in this model category are called* ***Kan complexes****.*

Exercise 6.3. *Find a fibrant replacement of Δ^1, i.e., a weak homotopy equivalence $\Delta^1 \to X$ such that X is a Kan complex.*

Exercise 6.4. *Find a fibrant replacement of $\partial\Delta^2$.*

7. Equivalence between homotopy theories

7.1. In this section we explain the meaning of an equivalence between two homotopy theories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair between model categories[4]. We need to answer the following question:

> *When does this adjoint pair induce an equivalence between the homotopy theories underlying $\mathcal{C}$ and $\mathcal{D}$?*

The answer would be

- The adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ should induce an adjoint pair
$$F' : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : G'$$
such that F' and G' are inverse[5] to each other.

However, to make sense of this, we need to articulate the definition of the functors F' and G'. A touchstone for such a definition is the following example from homological algebra.

Example 7.2. *Let $\mathcal{A}_i$ be abelian categories such that the projective model structures on $\mathsf{Ch}^{\leq 0}(\mathcal{A}_i)$ is well-defined (see [Lecture 2, Example 2.7]). We have*
$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_i)[W^{-1}] \simeq \mathsf{D}^{\leq 0}(\mathcal{A}_i),$$
where $\mathsf{D}^{\leq 0}(\mathcal{A}_i)$ is the connective (= non-positive) part of the derived category of $\mathcal{A}_i$. Let $F_0 : \mathcal{A}_1 \to \mathcal{A}_2$ be an additive functor and consider the functor
$$F : \mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \to \mathsf{Ch}^{\leq 0}(\mathcal{A}_2)$$
induced by F_0. Then we want F' to be the ***left derived functor*** *$\mathbb{L}F$, which can be calculated by*
$$\mathbb{L}F(M^\bullet) \simeq F(P^\bullet),$$
where $P^\bullet \to M^\bullet$ is a cofibrant replacement, a.k.a. a projective resolution, of $M^\bullet$.

One needs additional assumptions on the functor F_0 to guarantee that $F(P^\bullet) \in \mathsf{D}^{\leq 0}(\mathcal{A}_2)$ does not depend on the choice of $P^\bullet$.

Example 7.3. *Dually, for the* injective *model categories $\mathsf{Ch}^{\geq 0}(\mathcal{A}_i)$, we want to recover the definition of* ***right derived functors****.*

7.4. Note that in the above example, the functor F does not preserve quasi-isomorphisms (because F_0 is not exact). Hence in the general setting of §7.1, we should not ask F or G to preserve weak equivalences between *all* objects. In particular, we *cannot* expect the following diagram to commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, "F'"] \\ \mathcal{D} \arrow[r] & \mathcal{D}[W^{-1}] \end{tikzcd}$$

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[4] We do not require F or G to preserve (W) or (C) or (F).

[5] The classical terminology would be *quasi-inverse*, i.e., we require $\mathsf{Id} \to G' \circ F'$ and $F' \circ G' \to \mathsf{Id}$ to be equivalences rather than equalities. The latter requirement violates the principle of equivalence hence does not make sense in higher category theory. Therefore we omit the prefix *quasi*.

Nevertheless, in homological algebra, derived functors provide best approximations to such a commutative square. To explain what this means, we need some definitions.

Definition 7.5. *Let $\pi : \mathcal{C} \to \mathcal{C}'$ be a functor between categories. For any category $\mathcal{E}$, we have a functor*

$$\pi^* : \mathsf{Fun}(\mathcal{C}', \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}, \mathcal{E})$$

*given by precomposing with π. The left (resp. right) adjoint of this functor, when exists, is called the **left** (resp. **right**) **Kan extension** along π, and is denoted by*

$$\mathsf{LKE}_\pi : \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}', \mathcal{E}),$$
$$\mathsf{RKE}_\pi : \mathsf{Fun}(\mathcal{C}, \mathcal{E}) \to \mathsf{Fun}(\mathcal{C}', \mathcal{E}).$$

Exercise 7.6. *For a functor $f : \mathcal{C} \to \mathcal{E}$, find the universal properties that characterize the functors $\mathsf{LKE}_\pi f$ and $\mathsf{RKE}_\pi f$. Hint:*

$$\begin{tikzcd} & \mathcal{C}' \arrow[dr, dotted, "\mathsf{LKE}_\pi f"] \arrow[d, phantom, "\Uparrow"] & \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[rr, "f"'] & {} & \mathcal{D}, \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}' \arrow[dr, dotted, "\mathsf{RKE}_\pi f"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[rr, "f"'] & {} & \mathcal{D} \end{tikzcd}$$

are closest to be commutative.

Example 7.7. *Let $F_0 : \mathcal{A}_1 \to \mathcal{A}_2$ be as in Example 7.2. Then the left derived functor*

$$\mathbb{L}F : \mathsf{D}^{\leq 0}(\mathcal{A}_1) \to \mathsf{D}^{\leq 0}(\mathcal{A}_2)$$

is defined as the right *Kan extension*[6] *of the functor*

$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \xrightarrow{F} \mathsf{Ch}^{\leq 0}(\mathcal{A}_2) \xrightarrow{\pi_2} \mathsf{D}^{\leq 0}(\mathcal{A}_2)$$

along

$$\mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \xrightarrow{\pi_1} \mathsf{D}^{\leq 0}(\mathcal{A}_1).$$

In diagram:

$$\begin{tikzcd} \mathsf{Ch}^{\leq 0}(\mathcal{A}_1) \arrow[r, "\pi_1"] \arrow[d, "F"'] & \mathsf{D}^{\leq 0}(\mathcal{A}_1) \arrow[d, dotted, "\mathbb{L}F \overset{\mathrm{def}}{=} \mathsf{RKE}_{\pi_1}(\pi_2 \circ F)"] \arrow[dl, phantom, "\Swarrow"] \\ \mathsf{Ch}^{\leq 0}(\mathcal{A}_2) \arrow[r, "\pi_2"'] & \mathsf{D}^{\leq 0}(\mathcal{A}_2). \end{tikzcd} \tag{7.1}$$

Similarly, the right derived functor is defined as a left *Kan extension.*

Exercise 7.8. *Convince yourself that left derived functor should be a right Kan extension rather than a left one by evaluating (7.1) on a complex $M^\bullet$.*

[6]This reversal of handedness is unfortunate but unavoidable.

7.9. Motivated by the above, we can define derived functors in *homotopical* algebra.

Definition 7.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a (plain) functor between model categories. The* ***left derived functor*** *of F:*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \to \mathcal{D}[W^{-1}]$$

is defined to be the following right *Kan extension*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dotted, "\mathbb{L}F \overset{\mathrm{def}}{=} \mathsf{RKE}"] \arrow[dl, dotted, Rightarrow] \\ \mathcal{D} \arrow[r] & \mathcal{D}[W^{-1}]. \end{tikzcd} \tag{7.2}$$

Similarly, the ***right derived functor*** *is defined as a* left *Kan extension.*

7.11. We can abuse notation and write the natural transformation (7.2) as morphisms

$$\mathbb{L}F(X) \to F(X) \text{ in } \mathcal{D}[W^{-1}]$$

that are functorial for X in $\mathcal{C}$.

Similarly, for a functor $G : \mathcal{D} \to \mathcal{C}$, we have

$$G(Y) \to \mathbb{R}G(Y) \text{ in } \mathcal{C}[W^{-1}]$$

that are functorial for Y in $\mathcal{D}$.

7.12. Note that the definition of derived functors does not use the classes of cofibrations and fibrations. The following result provides a convenient tool to calculate derived functors using these morphisms. See e.g. [DS95, §9].

Theorem 7.13. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between model categories.*

1. *Suppose F sends weak equivalences between cofibrant objects to weak equivalences. Then the left derived functor $\mathbb{L}F$ exists. Moreover, for any cofibrant object X in $\mathcal{C}$, we have $\mathbb{L}F(X) \xrightarrow{\simeq} F(X)$.*
2. *Suppose G sends weak equivalences between fibrant objects to weak equivalences. Then the right derived functor $\mathbb{R}G$ exists. Moreover, for any fibrant object Y in $\mathcal{D}$, we have $G(Y) \xrightarrow{\simeq} \mathbb{R}G(Y)$.*

Exercise 7.14. *Explain how to calculate the left/right derived functor using cofibrant/fibrant replacements. What do you get for Example 7.2?*

7.15. Motivated by the above, we make the following definition.

Proposition-Definition 7.16. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair between model categories. The following conditions are equivalent:*

(i) *F preserves cofibrations and acyclic cofibrations;*
(ii) *G preserves fibrations and acyclic fibrations;*
(iii) *F preserves cofibrations and G preserves fibrations;*
(iv) *F preserves acyclic cofibrations and G preserves acyclic fibrations.*

When these conditions hold, we say $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a ***Quillen adjunction****, and call F (resp. G) a* ***left*** *(resp.* ***right****)* ***Quillen functor****.*

Exercise 7.17. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. Show that conditions in Theorem 7.13 hold.*

Proposition 7.18. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. We have a natural adjunction*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : \mathbb{R}G.$$

7.19. We are finally ready to give a precise definition to equivalences between homotopy theories.

Definition 7.20. *A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is called a* ***Quillen equivalence*** *if the induced adjunction*

$$\mathbb{L}F : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : \mathbb{R}G$$

is an equivalence between categories.

8. Conclusion

Theorem 8.1 (Quillen)**.** *The adjoint pair*

$$|-| : \mathsf{Set}_\Delta \rightleftarrows \mathsf{Top} : \mathsf{Sing}$$

is a Quillen equivalence between the classical model structures on both sides.

8.2. Next time, we will construct a Quillen equivalence

$$\mathfrak{C} : \mathsf{Set}_\Delta^{\mathsf{Joyal}} \rightleftarrows \mathsf{Cat}_\Delta : \mathfrak{N}$$

where

- $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ is **Joyal model structure** on the category of simplicial sets;
- Cat_Δ is the category of small **simplicial categories**, i.e., Set_Δ-enriched categories. We will equip it with the model structure induced from the *classical* model structure on Set_Δ.

This will identify the homotopy theories underlying these model categories.

On the other hand, Quillen's Theorem 8.1 implies the homotopy theories of Cat_Δ and $\mathsf{Cat}_{\mathsf{Top}}$ are equivalent. Combining with the homotopy hypothesis ([Lecture 2, Slogan 0.1]), we will obtain strong evidences for the following:

theory of $(\infty,1)$-categories = homotopy theory underlying $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$.

Appendix A. More on Quillen adjunctions

Exercise A.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. Suppose we have an adjoint pair $F' : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{D}[W^{-1}] : G'$, is it always possible to lift it to a Quillen adjunction? How about Quillen equivalences?*

Appendix B. Quillen's cotangent complex

Exercise B.1. *People say:*

> The cotangent complex functor is the left derived functor of the Kähler differentials functor.

Make sense of this statement using derived functors for model categories.

References

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