LECTURE 11

Last time, for a smooth k-scheme X, we introduced the sheaf of differential operators $\mathcal{D}_X$ on X and defined quasi-coherent $\mathcal{D}_X$-modules, known as *$\mathcal{D}$-module on X*. In this lecture, we introduce operations on $\mathcal{D}$-modules. We will main focus on the formal aspect of this thoery, known as the *six functors formalism for $\mathcal{D}$-modules*. For more details, see [B] and [HTT].

1. Conventions on derived categories

For the purpose of this course, we do not need the full power of the derived categories of $\mathcal{D}$-modules. However, these categories are useful in other topics of geometric representation theory, hence I choose to include the results about them in this lecture.

When talking about derived categories of $\mathcal{D}$-modules, we always assume X is quasi-projective. In this case, the abelian category $\mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{qc}}^{l/r}$ has enough injective and locally projective objects, and any object admits a resolution by locally projective $\mathcal{D}_X$-modules with length $\le 2d_X$, where $d_X = \dim(X)$ is the dimension (function) of X. The latter implies $\mathsf{Ext}^i(-,-) \simeq 0$ for $i > 2d_X$.

We have the following triangulated categoies equipped with natural t-strcutures:

- $D(\mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{qc}}^{l/r})$, the derived category of quasi-coherent $\mathcal{D}_X$-modules. This can be identified with the full subcategory of the derived category $D(\mathcal{D}_X\text{–}\mathsf{mod}^{l/r})$ of *all* $\mathcal{D}_X$-modules containing those complices whose cohomologies are quasi-coherent.
- $D^b(\mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{c}}^{l/r})$, the bounded derived category of coherent $\mathcal{D}_X$-modules. This can be identified with the full subcategory of the bounded derived category $D^b(\mathcal{D}_X\text{–}\mathsf{mod}^{l/r})$ of *all* $\mathcal{D}_X$-modules containing those complices whose cohomologies are coherent.

When talking about functors between derived categories, even if such functors are left/right derived functors, we drop the decorations "L/R" from the notations. For example, $-\otimes-$ in derived categories would mean $-\otimes^L-$ in classical literatures. We choose to do so because we will enconter functors that are not derived functors.

Remark 1.1. The (essential) image of the fully faithful functor

$$D^b(\mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{c}}^{l/r}) \to D(\mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{qc}}^{l/r})$$

contains exactly the *compact* objects in the target, i.e., those objects $\mathcal{M}$ such that $\mathsf{Hom}(\mathcal{M},-)$ commutes with filtered (homotopy) colimits.

2. Forget and induce

Construction 2.1. *We have adjoint functors*

$$\mathsf{ind}^{l/r} : \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qc}} \rightleftarrows \mathcal{D}_X\text{–}\mathsf{mod}_{\mathsf{qc}}^{l/r} : \mathsf{oblv}^{l/r}$$

such that

$$\mathsf{ind}^l(\mathcal{F}) \coloneqq \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F},\ \mathsf{ind}^r(\mathcal{D}) \coloneqq \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X.$$

Both functors are exact.

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3. Tensor and Hom

Construction 3.1. *Let $\mathcal{M},\mathcal{N} \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ and $\mathcal{M}',\mathcal{N}' \in \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$. Then there are natural objects defined using the signed Lebniz rules:*

- $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ *defined by* $\partial \cdot (m \otimes n) = (\partial \cdot m) \otimes n + m \otimes (\partial \cdot n)$;
- $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$ *defined by* $(m' \otimes n) \cdot \partial = (m' \cdot \partial) \otimes n - m' \otimes (\partial \cdot n)$;
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ *defined by* $(\partial \cdot \phi)(m) = \partial \cdot \phi(m) - \phi(\partial \cdot m)$;
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}',\mathcal{N}') \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ *defined by* $(\partial \cdot \phi)(m') = -\phi(m') \cdot \partial + \phi(m \cdot \partial)$;
- $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}') \in \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$ *defined by* $(\phi \cdot \partial)(m) = \phi(m) \cdot \partial + \phi(m \cdot \partial)$.

Remark 3.2. One way to memorize the above rules for left vs. right is using the known objects $\mathcal{O}_X \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ and $\omega_X := \Omega^n_X \in \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$. For example, $\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\omega_X) \simeq \mathcal{O}_X$ has a left $\mathcal{D}$-module structure, while $\mathcal{H}om_{\mathcal{O}_X}(\omega_X,\mathcal{O}_X) \simeq \omega_X^{-1}$ in general has no $\mathcal{D}$-module structures.

Remark 3.3. One way to memorize the signed Lebniz rules: (i) put a minus sign when acting on a section of the source object in $\mathcal{H}om_{\mathcal{O}_X}(-,-)$; (ii) put a minus sign when moving ∂ from the one side of $\cdot$ to the other side.

Remark 3.4. The tensor operations make $\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ a symmetric monoidal category such that the forgetful functor $\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \to \mathcal{O}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ is naturally symmetric monoidal. The category $\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$ is a module category of it.

The following result follows by unwinding the definitions:

Lemma 3.5. *Let $\mathcal{M} \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ and $\mathcal{M}' \in \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$. We have adjoint functors*

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} - : \mathcal{D}_X\text{-}\mathsf{mod}^{l/r}_{\mathsf{qc}} \rightleftarrows \mathcal{D}_X\text{-}\mathsf{mod}^{l/r}_{\mathsf{qc}} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},-)$$
$$\mathcal{M}' \underset{\mathcal{O}_X}{\otimes} - : \mathcal{D}_X\text{-}\mathsf{mod}^{l}_{\mathsf{qc}} \rightleftarrows \mathcal{D}_X\text{-}\mathsf{mod}^{r}_{\mathsf{qc}} : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}',-)$$

compatible with the similar adjunction between $\mathcal{O}_X$-modules.

Remark 3.6. The above compatibility means e.g. the isomorphism between $\mathcal{D}_X$-modules

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N})$$

provided by the adjunction in the lemma gives the corresponding isomorphism between $\mathcal{O}_X$-modules provided by the adjunction between $\mathcal{O}_X$-modules.

Recall ω_X is a line bundle. It follows formally that:

Corollary 3.7. *The following functors are inverse to each other:*

$$\omega_X \underset{\mathcal{O}_X}{\otimes} - : \mathcal{D}_X\text{-}\mathsf{mod}^{l}_{\mathsf{qc}} \rightleftarrows \mathcal{D}_X\text{-}\mathsf{mod}^{r}_{\mathsf{qc}} : \mathcal{H}om_{\mathcal{O}_X}(\omega_X,-).$$

Remark 3.8. In particular, for any right $\mathcal{D}$-module $\mathcal{N}'$, we obtain a left $\mathcal{D}$-module structure on $\omega_X^{-1} \otimes \mathcal{N}'$.

Remark 3.9. We also have similar results for the derived category of $\mathcal{D}$-modules and the corresponding derived functors. The left derived functor $- \otimes_{\mathcal{O}_X} -$ has cohomological amplitude $[-d_X, 0]$ while the right derived functor $\mathcal{H}om_{\mathcal{O}_X}(-,-)$ has cohomological amplitude $[0, d_X]$.

Remark 3.10. When identifying the *derived* categories of left and right $\mathcal{D}$-modules, it is more convenient to use the complex $\omega_X[d_X]$, which is also known as the *dualizing complex*[1] on X.

[1]This notion makes sense even when X is singular.

In other words, we use the equivalences

$$\omega_X[d_X] \underset{\mathcal{O}_X}{\otimes} - : D(\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}) \rightleftarrows D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}) : \mathcal{H}om_{\mathcal{O}_X}(\omega_X[d_X], -).$$

Construction 3.11. *We define a symmetric monoidal structure* $-\otimes^! -$ *on* $D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}})$ *by translating the symmetric monoidal structure* $-\otimes_{\mathcal{O}_X} -$ *of* $D(\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}})$ *via the above equivalences. In other words, as* $\mathcal{O}_X$*-modules, we have*

$$\mathcal{M}' \overset{!}{\otimes} \mathcal{N}' \simeq \mathcal{M}' \underset{\mathcal{O}_X}{\otimes} \mathcal{N}' \underset{\mathcal{O}_X}{\otimes} \omega_X^{-1}[-d_X].$$

4. External tensor

Construction 4.1. *Let* $\mathcal{M}_i \in \mathcal{D}_{X_i}\text{-}\mathsf{mod}^{l/r}_{\mathsf{qc}}$. *Then there are natural objects*

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 := \mathcal{M}_1 \underset{k}{\otimes} \mathcal{M}_2 \in \mathcal{D}_{X_1\times X_2}\text{-}\mathsf{mod}^{l/r}_{\mathsf{qc}}$$

induced by the isomorphism $\mathcal{D}_{X_1} \otimes_k \mathcal{D}_{X_2} \simeq \mathcal{D}_{X_1\times X_2}$. *This is called the* ***external tensor product*** *of* $\mathcal{D}$*-modules.*

5. Pullbacks

Let $\phi : Y \to X$ be a map between smooth k-schemes.

Construction 5.1. *We will construct a commutative diagram*

$$\begin{tikzcd} \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \arrow[r, dotted, "\phi^*"] \arrow[d, "\mathsf{oblv}^l"] & \mathcal{D}_Y\text{-}\mathsf{mod}^l_{\mathsf{qc}} \arrow[d, "\mathsf{oblv}^l"] \\ \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{qc}} \arrow[r, "\phi^*"] & \mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}. \end{tikzcd}$$

The functor

$$\phi^* : \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}} \to \mathcal{D}_Y\text{-}\mathsf{mod}^l_{\mathsf{qc}}$$

is called the ***(∗-)pullback of left*** $\mathcal{D}$***-modules****.*

The construction is as follows. For $\mathcal{M} \in \mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{qc}}$, *recall* $\phi^*(\mathcal{M}) := \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{M}$. *Suppose* $\mathcal{M}$ *is equipped with a left* $\mathcal{D}_X$*-module structure, then there is a left* $\mathcal{D}_Y$*-module structure on* $\phi^*(\mathcal{M})$ *defined by the Lebniz rule:*

$$\partial \cdot (f \otimes s) := \partial(f) \otimes s + f\overline{\partial} \cdot s,$$

where

- ∂ *is a local section of* $\mathcal{T}_Y$ *and* $\overline{\partial}$ *is the image of it under the map* $\mathcal{T}_Y \to \phi^*\mathcal{T}_X = \mathcal{O}_Y \otimes_{\phi^{-1}\mathcal{O}_X} \phi^{-1}\mathcal{T}_X$;
- f *is a local section of* $\mathcal{O}_Y$;
- s *is a local section of* $\phi^{-1}\mathcal{M}$, *and* $\overline{\partial} \cdot s$ *is defined using the action of* $\phi^{-1}\mathcal{T}_X$ *on* $\phi^{-1}\mathcal{M}$.

Remark 5.2. One can show the pullback of left $\mathcal{D}$-modules are compatible with composition of maps between smooth k-schemes.

Example 5.3. The pullback of the object $\mathcal{O}_X \in \mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ is $\mathcal{O}_Y \in \mathcal{D}_Y\text{-}\mathsf{mod}^l_{\mathsf{qc}}$.

Construction 5.4. *We write:*

$$\mathcal{D}_{Y\to X} := \phi^*\mathcal{D}_X \simeq \mathcal{O}_Y \underset{\phi^{-1}\mathcal{O}_X}{\otimes} \phi^{-1}\mathcal{D}_X$$

and call it the ***transfer module****.*

The above construction gives a left $\mathcal{D}_Y$-module structure on $\mathcal{D}_{Y\to X}$. On the other hand, there is an obvious right *$\phi^{-1}\mathcal{D}_X$-module structure on $\mathcal{D}_{Y\to X}$. One can show there two actions commute. In other words, $\mathcal{D}_{Y\to X}$ is a $(\mathcal{D}_Y, \phi^{-1}\mathcal{D}_X)$-bimodule.*

Note that for $\mathcal{M} \in \mathcal{D}_X\text{–}\mathsf{mod}^l_{\mathsf{qc}}$, we have

$$\phi^*\mathcal{M} \simeq \mathcal{D}_{Y\to X} \underset{\phi^{-1}\mathcal{D}_X}{\otimes} \phi^{-1}\mathcal{M}.$$

Construction 5.5. *Note that the functor $\phi^* : \mathcal{D}_X\text{–}\mathsf{mod}^l_{\mathsf{qc}} \to \mathcal{D}_Y\text{–}\mathsf{mod}^l_{\mathsf{qc}}$ is right exact. We abuse notation and let*

$$\phi^* : D(\mathcal{D}_X\text{–}\mathsf{mod}^l_{\mathsf{qc}}) \to D(\mathcal{D}_Y\text{–}\mathsf{mod}^l_{\mathsf{qc}})$$

be the left derived functor of it. Note that it is compatible with the left derived functor $\phi^* : D(\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qc}}) \to D(\mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}})$ and the forgetful functors.

Remark 5.6. We have:

- If ϕ is flat, then ϕ^* is t-exact, i.e., preserves the heart.
- If ϕ is a closed embedding (which is automatically regular), then ϕ^* has cohomological amplitude $[-d_X + d_Y, 0]$.

Example 5.7. Let ϕ is a closed embedding. Since X and Y are smooth, ϕ is a regular immersion. For any closed point $p \in Y$, we can find an étale coordinate system $x_1, \cdots, x_m$ of X near p such that Y is locally cut out by the ideal $(x_{n+1}, \cdots, x_m)$ ($m = \dim(\mathcal{O}_{X,y})$ and $n = \dim(\mathcal{O}_{Y,y})$). Let $y_1, \cdots, y_n$ be the restriction of $x_1, \cdots, x_n$ on Y. They form an étale coordinate system of Y near p. Then near the point $p \in Y$, we have

$$\mathcal{D}_{Y\to X} \simeq \mathcal{D}_Y \underset{k}{\otimes} k[\partial_{n+1}, \cdots, \partial_m]$$

as left $\mathcal{D}_Y$-modules. In particular, $\mathcal{D}_{Y\to X}$ is a locally free left $\mathcal{D}_Y$-module.

Construction 5.8. *Let $\phi^!$ be the unique functor that makes the following diagram commute*

$$\begin{tikzcd}
D(\mathcal{D}_X\text{–}\mathsf{mod}^l_{\mathsf{qc}}) \arrow[r, "\phi^*"] \arrow[d, "\omega_X[d_X]"', "\simeq"] & D(\mathcal{D}_Y\text{–}\mathsf{mod}^l_{\mathsf{qc}}) \arrow[d, "\simeq"', "\omega_Y[d_Y]"] \\
D(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}}) \arrow[r, dotted, "\phi^!"] & D(\mathcal{D}_Y\text{–}\mathsf{mod}^r_{\mathsf{qc}}),
\end{tikzcd}$$

The obtained functor

$$\phi^! : D(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}}) \to D(\mathcal{D}_Y\text{–}\mathsf{mod}^r_{\mathsf{qc}})$$

is called the ***!-pullback of (complices of) right $\mathcal{D}$-modules****. It has cohomological amplitude $[-d_Y, d_X - d_Y]$ and in general is not a derived functor.*

Remark 5.9. We have:

- If ϕ is flat, then $\phi^!$ is t-exact up to a shift.
- If ϕ is a closed embedding, then $\phi^!$ has cohomological amplitude $[0, d_X - d_Y]$.

Example 5.10. By definition, $\phi^!(\omega_X[d_X]) \simeq \omega_Y[d_Y]$.

Example 5.11. If $j : U \to X$ is an open embedding, then $j^!$ is t-exact and the corresponding functor $\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}} \to \mathcal{D}_U\text{–}\mathsf{mod}^r_{\mathsf{qc}}$ is the restriction functor. Indeed, this follows from $\omega_X|_U \simeq \omega_U$.

Fact 5.12. *If $\phi : Y \to X$ is a closed embedding, then $\phi^!$ is equivalent to the right derived functor of a functor*
$$\phi^! : \mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}} \to \mathcal{D}_Y\text{–}\mathsf{mod}^r_{\mathsf{qc}}.$$

Construction 5.13. *Let $\phi : Y \to X$ be a closed embedding. The functor $\phi^! : \mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}} \to \mathcal{D}_Y\text{–}\mathsf{mod}^r_{\mathsf{qc}}$ can be described as follows.*

Recall we have adjoint functors
$$\phi_* : \mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}} \rightleftarrows \mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qc}} : \phi^!$$
where for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any open subset $U \subset X$, a section m of $\phi^!(\mathcal{M})$ on $U \cap Y$ corresponds to a section $\widetilde{m}$ of $\mathcal{M}$ on U annihilated by the ideal $\mathcal{I}_Y := \mathsf{ker}(\mathcal{O}_X \to \mathcal{O}_Y)$. Suppose $\mathcal{M}$ is equipped with a right $\mathcal{D}_X$-module structure. For any local section ∂ of $\mathcal{T}_Y$, we can extend it to a local section $\widetilde{\partial}$ of $\mathcal{T}_X$. Now for a local section m of $\phi^!(\mathcal{M})$, we define $m \cdot \partial$ such that
$$\widetilde{m \cdot \partial} = \widetilde{m} \cdot \widetilde{\partial}.$$
One can show the local section $m \cdot \partial$ is well-defined and does not depend on the choice of $\widetilde{\partial}$. Moreover, this defines a right $\mathcal{D}_Y$-module structure on $\phi^!(\mathcal{M})$. $\phi^!(\mathcal{M})$.

Remark 5.14. For any map $\phi : Y \to X$ between finite type k-schemes, one can define a functor
$$\phi^! : D(\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qc}}) \to D(\mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}})$$
as follows.

If ϕ is an open embedding, take $\phi^! := \phi^*$. If ϕ is proper, take $\phi^!$ to be the right adjoint of (the right derived functor) ϕ_*. For the general case, choose a Nagata compactification $Y \xrightarrow{j} \overline{Y} \xrightarrow{\overline{\phi}} X$ such that j is an open embedding and $\overline{\phi}$ is proper, and take $\phi^! := j^! \circ \overline{\phi}^!$. One can show the functor $\phi^!$ does not depend on the choice of the compactification, and these functors are compatible with compositions of maps. In fact, the construction $\phi \mapsto \phi^!$ can be uniquely characterized by these properties (if stated properly).

When X and Y are smooth, the !-pullback functors of $\mathcal{O}$-modules and right $\mathcal{D}$-modules are compatible via the forgeful functors. In other words, we have a commutative diagram

$$\begin{tikzcd} D(\mathcal{O}_Y\text{–}\mathsf{mod}_{\mathsf{qc}}) & D(\mathcal{O}_X\text{–}\mathsf{mod}_{\mathsf{qc}}) \arrow[l, "\phi^!"'] \\ D(\mathcal{D}_Y\text{–}\mathsf{mod}^r_{\mathsf{qc}}) \arrow[u, "\mathsf{oblv}^r"'] & D(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{qc}}) \arrow[l, "\phi^!"'] \arrow[u, "\mathsf{oblv}^r"'] \end{tikzcd} \tag{5.1}$$

Fact 5.15. *In the (derived) setting of Construction 3.1, we have*
$$\begin{aligned} \phi^*(\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{N}) &\simeq \phi^*(\mathcal{M}) \underset{\mathcal{O}_Y}{\otimes} \phi^*(\mathcal{N}), \\ \phi^!(\mathcal{M}' \underset{\mathcal{O}_X}{\otimes} \mathcal{N}) &\simeq \phi^!(\mathcal{M}') \underset{\mathcal{O}_Y}{\otimes} \phi^*(\mathcal{N}), \\ \phi^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}) &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^*\mathcal{M},\phi^*\mathcal{N}), \\ \phi^*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}',\mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^!\mathcal{M}',\phi^!\mathcal{N}'), \\ \phi^!\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{N}') &\simeq \mathcal{H}om_{\mathcal{O}_Y}(\phi^*\mathcal{M},\phi^!\mathcal{N}'). \end{aligned}$$

Fact 5.16. *For $\mathsf{pr}_i : X_1 \times X_2 \to X_i$, we have*
$$\begin{aligned} \mathcal{M}_1 \boxtimes \mathcal{M}_2 &\simeq \mathsf{pr}_1^*(\mathcal{M}_1) \underset{\mathcal{O}_{X_1 \times X_2}}{\otimes} \mathsf{pr}_2^*(\mathcal{M}_2) \\ \mathcal{M}_1' \boxtimes \mathcal{M}_2' &\simeq \mathsf{pr}_1^!(\mathcal{M}_1) \overset{!}{\otimes} \mathsf{pr}_2^!(\mathcal{M}_2). \end{aligned}$$

6. Pushforwards

Construction 6.1. *Let $\phi : Y \to X$ be a map between smooth k-schemes. Recall the transfer module*

$$\mathcal{D}_{Y\to X} := \phi^*\mathcal{D}_X \simeq \mathcal{O}_Y \underset{\phi^{-1}\mathcal{O}_X}{\otimes} \phi^{-1}\mathcal{D}_X$$

is a bimodule for $(\mathcal{D}_Y, \phi^{-1}\mathcal{D}_X)$. We define a functor

$$\phi_{*,\mathsf{dR}} : D(\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \to D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}),\ \mathcal{N} \mapsto \phi_*(\mathcal{N} \underset{\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\to X}),$$

where

- *The (left derived) tensor product functor $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ sends a complex of right $\mathcal{D}_Y$-modules to a complex of right $\phi^{-1}\mathcal{D}_X$-modules.*
- *The (right derived) functor ϕ_* sends a complex of right $\phi^{-1}\mathcal{D}_X$-modules to a complex of right $\mathcal{D}_X$-modules via the homomorphism $\mathcal{D}_X \to \phi_*(\phi^{-1}\mathcal{D}_X)$.*

*We call $\phi_{*,\mathsf{dR}}$ the **direct image functor**, or **de Rham pushforward functor**, of (complices) of right $\mathcal{D}$-modules.*

Remark 6.2. One can show the direct image functors of right $\mathcal{D}$-modules are compatible with composition of maps between smooth k-schemes.

Remark 6.3. The functor $\phi_{*,\mathsf{dR}}$ is called the *de Rham* pushforward functor because for $\pi : X \to \mathsf{pt}$, $\pi_{*,\mathsf{dR}}(\omega_X[-d_X])$ can be identified with the de Rham complex of X. For this reason, we also write

$$\Gamma_{\mathsf{dR}}(X, -) := \pi_{*,\mathsf{dR}}(-).$$

You are strongly encouraged to look at its proof in [G, Sect. 5.17].

Remark 6.4. Some authors use the notation $\phi_\star$ for $\phi_{*,\mathsf{dR}}$.

Remark 6.5. The cohomological amplitude of $\phi_{*,\mathsf{dR}}$ is $[-d_Y, d_Y]$. Better estimation exist in the following cases:

- If ϕ is affine, then the bounds can be $[-d_Y, 0]$.
- If ϕ is smooth, then the bounds can be $[-d_Y + d_X, d_Y]$.
- If ϕ is a closed embedding, then the functor is t-exact.

Warning 6.6. *One can define a functor between the abelian categories using the same formula. However, that functor would not be $\mathcal{H}^0(\phi_{*,\mathsf{dR}})$ and is of less interests.*

Exercise 6.7. This is Homework 5, Problem 4. Let $x : \mathsf{pt} \to X$ be a closed point of X. We write $\delta_x := x_{*,\mathsf{dR}}(k)$. Prove:

(1) $\delta_x \simeq k_x \otimes_{\mathcal{O}_X} \mathcal{D}_X$ as a right $\mathcal{D}_X$-module.
(2) δ_x is set-theoretically support on at x, i.e., for the complement open $U := X - x$, we have $\delta_x|_U = 0$.
(3) There exists a unique section Dirac_x of δ_x such that $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$ for any local section f of $\mathcal{O}_X$ defined near x.
(4) δ_x is generated by Dirac_x as a right $\mathcal{D}_X$-module.

Remark 6.8. The section Dirac_x should be viewed as the incarnation of the Dirac function in the theory of $\mathcal{D}$-modules.

Lemma 6.9. *The following diagram commutes:*

$$\begin{tikzcd} D(\mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}}) \arrow[r, "\phi_*"] \arrow[d, "\mathsf{ind}^r"'] & D(\mathcal{O}_X\text{-}\mathsf{mod}_{\mathsf{qc}}) \arrow[d, "\mathsf{ind}^r"'] \\ D(\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \arrow[r, "\phi_{*,\mathsf{dR}}"] & D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \end{tikzcd} \tag{6.1}$$

Sketch. For $\mathcal{F} \in D(\mathcal{O}_Y\text{-}\mathsf{mod}_{\mathsf{qc}})$, we have

$$\phi_{*,\mathsf{dR}} \circ \mathsf{ind}^r(\mathcal{F}) \simeq \phi_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y \underset{\mathcal{D}_Y}{\otimes} \mathcal{D}_{Y\to X}) \simeq \phi_*(\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \phi^*\mathcal{D}_X) \simeq \phi_*\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \simeq \mathsf{ind}^r \circ \phi_*(\mathcal{F})$$

where the last isomorphism is the (derived) projection formula.

□

We state the following results without proof.

Proposition 6.10. *If $\phi : Y \to X$ is proper, then we have adjoint functors*

$$\phi_{*,\mathsf{dR}} : D(\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \rightleftarrows D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}) : \phi^!.$$

Proposition 6.11. *If $\phi : Y \to X$ is smooth, then we have adjoint functors*

$$\phi^![-2d_Y + 2d_X] : D(\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \rightleftarrows D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}) : \phi_{*,\mathsf{dR}}.$$

Remark 6.12. If $\phi : Y \to X$ is proper, then the square (5.1) can be obtained from (6.1) by passing to right adjoints.

Construction 6.13. *As in the case of pullback functors, we can define the **direct image functor** of left $\mathcal{D}$-modules:*

$$\begin{tikzcd} D(\mathcal{D}_Y\text{-}\mathsf{mod}^l_{\mathsf{qc}}) \arrow[r, dotted, "\phi_{*,\mathsf{dR}}"] \arrow[d, "\omega_Y[d_Y]"', "\simeq"] & D(\mathcal{D}_X\text{-}\mathsf{mod}^l_{\mathsf{qc}}) \arrow[d, "\simeq"', "\omega_X[d_X]"] \\ D(\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \arrow[r, "\phi_{*,\mathsf{dR}}"] & D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}). \end{tikzcd}$$

7. Kashiwara's lemma

If $\phi : Y \to X$ is a closed embedding, then the tensor product functor $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is t-exact because $\mathcal{D}_{Y\to X}$ is locally free as a $\mathcal{D}_Y$-module. On the other hand, the functor ϕ_* is also t-exact because ϕ is affine. Therefore the functor $\phi_{*,\mathsf{dR}}$ is t-exact.

Theorem 7.1 (Kashiwara's lemma)**.** *Let $\phi : Y \to X$ be a closed embedding between smooth k-schemes, then the exact functor*

$$\phi_{*,\mathsf{dR}} : \mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}} \to \mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}$$

is fully faithful and its essential image contains exactly right $\mathcal{D}_X$-modules that are set-theoretically supported on Y.

Remark 7.2. Using Kashiwara's lemma, we can define $\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}$ even for finite type singular k-scheme Y. Namely, if Y is affine, we can embed Y into a smooth ambidient k-scheme X and define a right $\mathcal{D}$-module on Y to be a right $\mathcal{D}$-module on X that is set-theoretically supported on the image of Y. One can show the obtained abelian category does not depend on the choice of the embedding. When Y is not affine, we can define the category by gluing.

Moreover, all the previous constructions about right $\mathcal{D}$-modules can be generalized to the singular case.

A more canonical construction of $\mathcal{D}_Y\text{-}\mathsf{mod}^r_{\mathsf{qc}}$ or even $\mathcal{D}_Y\text{-}\mathsf{mod}^l_{\mathsf{qc}}$ for singular k-schemes is to use the theory of (Grothendieck's) crystals.

Another application of Kashiwara's lemma is the following result. See [G, Sect. 5.12] for a proof.

Corollary 7.3. *Let X be a smooth k-scheme, then any $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is locally free as an $\mathcal{O}_X$-module.*

8. Base-change isomorphism and projection formula

Fact 8.1. *Let*

$$\begin{tikzcd} Y' \arrow[r, "\phi'"] \arrow[d, "f"] & X' \arrow[d, "g"] \\ Y \arrow[r, "\phi"] & X \end{tikzcd}$$

be a Cartesian square of finite type k-schemes. Then we have equivalences

$$g^! \circ \phi_{*,\mathsf{dR}} \simeq \phi'_{*,\mathsf{dR}} \circ f^!$$

between functors $D(\mathcal{D}_Y\text{-}\mathsf{mod}_q c^r) \to D(\mathcal{D}_{X'}\text{-}\mathsf{mod}_q c^r)$.

Fact 8.2. *Let $\phi : Y \to X$ be any morphism between finite type k-schemes. Then we have*

$$\phi_{*,\mathsf{dR}}(-\overset{!}{\otimes}\phi^!(\bullet)) \simeq \phi_{*,\mathsf{dR}}(-)\overset{!}{\otimes}\bullet.$$

Exercise 8.3. This is Homework 5, Problem 5. Let $x : \mathsf{pt} \to X$ be a closed point of X. Prove[2] $\delta_x \otimes^! \delta_x \simeq \delta_x$.

9. Duality

The duality functor is only defined on *coherent* $\mathcal{D}$-modules.

Fact 9.1. *For any $\mathcal{M} \in D^b(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{c}})$, there exists a unique object $\mathbb{D}\mathcal{M} \in D^b(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{c}})$ such that*

$$\Gamma_{\mathsf{dR}}(X, \mathcal{M}\overset{!}{\otimes}-) \simeq \mathsf{Hom}(\mathbb{D}\mathcal{M}, -)$$

as functors $D(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{qc}}) \to \mathsf{Vect}$. The obtained functor

$$\mathbb{D} : D^b(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{c}})^{\mathsf{op}} \to D^b(\mathcal{D}_X\text{-}\mathsf{mod}^r_{\mathsf{c}})$$

is an anti-involution, i.e., $\mathbb{D} \circ \mathbb{D} \simeq \mathsf{Id}$.

Remark 9.2. The construction of $\mathbb{D}\mathcal{M}$ can be treated as a blackbox.

Example 9.3. We have $\mathbb{D}(\omega_X) \simeq \omega_X$.

Construction 9.4. *Let $\phi : Y \to X$ be a map between finite type k-schemes. The standard functors $\phi^!$ and $\phi_{*,\mathsf{dR}}$ in general do not preserve coherent complices. Hence we only have* partially defined *functors*

$$\phi_! := \mathbb{D} \circ \phi_{*,\mathsf{dR}} \circ \mathbb{D},\ \phi^*_{\mathsf{dR}} := \mathbb{D} \circ \phi^! \circ \mathbb{D}.$$

They are called the **!*-direct image functor*** *and the* ***de Rham pullback functor***.

[2] A formal proof exists, but you are encouraged to do some direct calculations to see $\mathcal{H}^i(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = 0$ unless $i = -d_X$ and $\mathcal{H}^{-d}(\delta_x \otimes_{\mathcal{O}_X} \delta_x) = \delta_x$.

Fact 9.5. *Let $\phi : Y \to X$ be a map between finite type k-schemes. Then $\phi_!$ is equivalent to the partially defined left adjoint of $\phi^!$. More precisely, we have*

$$\mathsf{Hom}(\phi_! \mathcal{M}, -) \simeq \mathsf{Hom}(\mathcal{M}, \phi^!(-))$$

whenever $\phi_! \mathcal{M}$ is well-defined. Similarly, ϕ^*_{dR} is equivalent to the partially defined left adjoint of $\phi_{*,\mathsf{dR}}$.

Remark 9.6. If ϕ is proper, then $\phi_! \simeq \phi_{*,\mathsf{dR}}$. If ϕ is smooth, then $\phi^*_{\mathsf{dR}} \simeq \phi^![-2d_Y + 2d_X]$.

10. Holonomic D-modules

We do not give the standard definition of holonomic D-modules. Instead, we characterize them as follows:

Fact 10.1. *Let $\mathcal{M} \in \mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{c}}$, then $\mathcal{M}$ is* ***holonomic*** *iff $\mathbb{D}\mathcal{M} \in \mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{c}}$ (rather than just in the derived category).*

Fact 10.2. *Let $\mathcal{M} \in D^b(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{c}})$, then $\mathcal{M}$ has* ***holonomic cohomologies****, i.e., $\mathcal{H}^\bullet(\mathcal{M})$ are holonomic, iff for any closed point $i : x \to X$, the complex $i^! \mathcal{M} \in D^b(\mathcal{D}_{\mathsf{pt}}\text{–}\mathsf{mod}^r_{\mathsf{c}}) \simeq D^b(\mathsf{Vect})$ has finite dimensional cohomologies.*

Notation 10.3. *Let $\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{hol}}$ be the abelian category of holonomic right $\mathcal{D}_X$-modules and $D^b(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{hol}})$ be the bounded derived category.*

Fact 10.4. *$D^b(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{hol}})$ is equivalent to the full subcategory of $D^b(\mathcal{D}_X\text{–}\mathsf{mod}^r_{\mathsf{c}})$ containing complices with holonomic cohomologies.*

Fact 10.5. *All the functors defined so far preserve bounded holonomic complices.*

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