PROBLEM SET 5

Due: Dec 10, noon

100 credits + 50 bonus

Problem 1 (10 credits)**.** Let $i : Y \to X$ be a quasi-compact locally closed immersion and $\overline{Y}$ be the scheme theoretic closure of Y in X. Show that the canonical morphism $j : Y \to \overline{Y}$ has a dense image.

Problem 2 (10 credits)**.** Let X be a scheme and $U \subseteq X$ be an open subset. Show that $\mathcal{O}_{X_{\mathsf{red}}}(U) \simeq \mathcal{O}_X(U)_{\mathsf{red}}$.

Problem 3 (10 credits)**.** Let $f : X \to Y$ be a morphism between schemes. Suppose X is reduced. Show that f uniquely factors through Y_{red}.

Problem 4 (15 bonus credits)**.** Show that a morphsim $f : X \to Y$ of schemes is quasi-compact (resp. separated, quasi-separated) iff f_{red} is so.

Problem 5 (10 credits)**.** Let X be a scheme and $f \in \mathcal{O}_X(X)$. Consider the canonical morphism $\phi : X \to \mathsf{Spec}(\mathcal{O}_X(X))$. Show that $\phi^{-1}(U(f)) = X_f$.

Problem 6 (10 bonus credits)**.** Let k be a field and

$$A = k[x_1, x_2, \cdots][\frac{x_1}{x_2}, \frac{x_1}{x_2^2}, \cdots][\frac{x_2}{x_3}, \frac{x_2}{x_3^2}, \cdots]\cdots$$

be the sub-k-algebra of $k[x_1, x_2^{\pm}, x_3^{\pm}, \cdots]$ generated by $(x_i/x_{i+1}^m)_{i\geq 1, m\geq 0}$. Let $\mathfrak{m} \subseteq A$ be the kernel of the homomorphism $A \to k$, $x_i \mapsto 0$. Consider $X := \mathsf{Spec}(A_{\mathfrak{m}})$ and consider the unique closed point $x \in X$. Show that $U := X \setminus x$ is a scheme with no closed points.

Problem 7 (10 credits)**.** Let $X \to Y$ be a morphism between quasi-affine schemes. Show that the following diagram is Cartesian

$$\begin{tikzcd} X \arrow[r, "\subseteq"] \arrow[d] & \overline{X} \arrow[d] \\ Y \arrow[r, "\subseteq"] & \overline{Y}, \end{tikzcd}$$

where for a quasi-affine scheme Z, the scheme $\overline{Z}$ is the affine closure of Z.

Problem 8 (10 credits)**.** Let X be a scheme and $x \in X$ is a point. The following conditions are equivalent:

- There is a unique irreducible component of X that contains x.
- For any point $x \in X$, the nilpotent radical of $\mathcal{O}_{X,x}$ is a prime ideal.

Problem 9 (10 credits)**.** Let X be a scheme. The following conditions are equivalent:

(1) The scheme X is integral.

(2) The scheme X is nonempty and for any open subscheme $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.
(3) The scheme X is nonempty and for any affine open subscheme $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Problem 10 (10 credits)**.** Let X be a locally Noetherian scheme and $x \in X$ be a point such that $\mathsf{Spec}(\mathcal{O}_{X,x})$ is irreducible. Show that there exists an open neighborhood U of x such that U is irreducible.

Problem 11 (20 bonus credits)**.** Let P be a property on morphisms in CRing such that:

(i) For any $A \to B$ and $a \in A$, we have $P(A \to B) \Rightarrow P(A_f \to B_f)$;
(ii) For any $A, B \in \mathsf{CRing}$, $a \in A$, $b \in B$ and $A_a \to B$, we have $P(A_a \to B) \Rightarrow P(A \to B_b)$;
(iii) For any $A \to B$ and $b_1, \cdots, b_n \in B$ such that $(b_1, \cdots, b_n) = B$, we have $\bigcap_i P(A \to B_{b_i}) \Rightarrow P(A \to B)$.

Then for a morphism $f : X \to Y$ between schemes, the following conditions are equivalent:

(1) For any affine open scheme $U \subseteq X$ and $V \subseteq Y$ with $f(U) \subseteq V$, we have $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.
(2) For any point $x \in X$, there exists an affine open neighborhood U of x and an affine open subscheme $V \subseteq Y$ with $f(U) \subseteq V$ such that $P(\mathcal{O}_Y(V) \to \mathcal{O}_X(U))$.

Problem 12 (20+10 credits)**.** Let $(X, \mathcal{O}_X)$ be a ringed space and $0 \to \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F}_2 \xrightarrow{\beta} \mathcal{F}_3 \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules.

(1) (10 credits) If $\mathcal{F}_1$ is of finite type and $\mathcal{F}_2$ is of finite presentation, then $\mathcal{F}_3$ is of finite presentation.
(2) (10 bonus credits) If $\mathcal{F}_1$ and $\mathcal{F}_3$ are of finite presentation, so is $\mathcal{F}_2$.
(3) (10 credits) If $\mathcal{F}_2$ is of finite type and $\mathcal{F}_3$ is of finite presentation, then $\mathcal{F}_1$ is of finite type.