LECTURE 13

In this lecture, we prove the first part of the localization theorem. Throughout this lecture, we write $X := \mathsf{Fl}_G$.

1. Fibers of the localization functor

In this section, we prove the following result.

Proposition 1.1. *For $M \in U(\mathfrak{g})\text{–}\mathsf{mod}$ and any closed point $x \in X$, we have*

$$\mathsf{Loc}(M)|_x \simeq M_{\mathsf{stab}_{\mathfrak{g}}(x)},$$

where $\mathsf{stab}_{\mathfrak{g}}(x)$ is the stabilizer of $\mathfrak{g}$ at x, i.e., $\mathsf{stab}_{\mathfrak{g}}(x) := \mathsf{ker}(\mathfrak{g} \to \mathcal{T}(X) \to \mathcal{T}_{X,x})$.

Remark 1.2. It is easy to see $\mathsf{stab}_{\mathfrak{g}}(x)$ is the Borel subalgebra of $\mathfrak{g}$ corresponding to the closed point $x \in X$ (see [Lecture 12, Construction 1.7]).

Proof. By definition,

$$\mathsf{Loc}(M)|_x \simeq \Gamma(X, k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \underset{\underline{U(\mathfrak{g})}}{\otimes} \underline{M}) \simeq \Gamma(X, \delta_x \underset{\underline{U(\mathfrak{g})}}{\otimes} \underline{M}) \simeq \Gamma(X, \delta_x) \underset{U(\mathfrak{g})}{\otimes} M,$$

where δ_x is the Delta right $\mathcal{D}$-module in [Lecture 11, Exercise 6.8]. By *loc.cit.*, δ_x has a unique global section $\mathsf{Dirac}_x \in \Gamma(X, \delta_x)$ such that $\mathsf{Dirac}_x \cdot f = f(x)\mathsf{Dirac}_x$ for any local section f of $\mathcal{O}_X$. It follows for any vector field ∂ with $\partial|_x = 0$, we have $\mathsf{Dirac}_x \cdot \partial = 0$ because locally we can write $\partial = \sum f_k \partial_k$ with $f_k(x) = 0$. In particular, the right $U(\mathfrak{g})$-action on Dirac_x annihilates $\mathsf{stab}_{\mathfrak{g}}(x) \subset \mathfrak{g} \subset U(\mathfrak{g})$. In other words, we have a right $U(\mathfrak{g})$-linear map

$$k \underset{U(\mathsf{stab}_{\mathfrak{g}}(x))}{\otimes} U(\mathfrak{g}) \to \Gamma(X, \delta_x),\ 1 \otimes u \mapsto \mathsf{Dirac}_x \cdot u.$$

It is easy to see both sides have natural filtrations induced respectively by the PBW filtrations on $U(\mathfrak{g})$ and $\mathcal{D}_X$, and the above map is compatible with the filtrations. Taking associated graded spaces, we only need to show the following obtained map is an isomorphism

$$\mathsf{Sym}^{\bullet}(\mathfrak{g}/\mathsf{stab}_{\mathfrak{g}}(x)) \to \mathsf{Sym}^{\bullet}(\mathcal{T}_{X,x}).$$

Unwinding the definitions, this map is induced by the isomorphism $\mathfrak{g}/\mathsf{stab}_{\mathfrak{g}}(x) \simeq \mathcal{T}_{X,x}$. □

Remark 1.3. As can be seen from the proof, Proposition 1.1 remains true if X is replaced by any homogenous space under G.

Remark 1.4. As can be seen from the proof, Proposition 1.1 remains true for derived categories and derived functors. In other words, the derived fiber of $\mathsf{Loc}(M)$ at x can be identified with the derived coinvariance of M for $\mathsf{stab}_{\mathfrak{g}}(x)$.

Let $e \in X \simeq G/B$ be the closed point corresponding to the chosen Borel subgroup B. In the above proof, we have shown

$$\Gamma(X, \delta_e) \simeq k \underset{U(\mathfrak{b})}{\otimes} U(\mathfrak{g})$$

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as right $U(\mathfrak{g})$-modules. Note that the RHS is the "*right* Verma module" with highest weight 0. As stated in the localization theorem, we can produce the (left) Verma module $M_{-2\rho}$ with highest weight -2ρ if using the left $\mathcal{D}$-module corresponding to δ_e. The following exercise gives a direct proof to this fact.

Exercise 1.5. This is Homework 6, Problem 4. In above, let $\delta_e^l \simeq \delta_e \otimes \omega_X^{-1}$ be the left $\mathcal{D}$-module corresponding to δ_e. Consider the left $U(\mathfrak{g})$-module $V := \Gamma(X, \delta_e^l)$.

(1) Prove: there is a *canonical* isomorphism
$$\delta_e^l \simeq \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \ell,$$
where ℓ is the fiber of ω_X^{-1} at e, viewed as a skyscrapter sheaf.

(2) Let $\ell \hookrightarrow V$ be the injection induced by taking global sections for the embedding $\mathcal{O}_X \otimes_{\mathcal{O}_X} \ell \hookrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \ell$. Prove: this line in V is a weight subspace of weight -2ρ[1].

(3) Prove: the subalgebra $\mathfrak{b} \subset \mathfrak{g}$ stabilizes the line $\ell \subset V$[2].

(4) Construct a $U(\mathfrak{g})$-linear map
$$M_{-2\rho} \to V$$
and prove it is an isomorphism.

2. The ring $\mathcal{D}(X)$

Proposition 2.1. *The homomorphism $a : U(\mathfrak{g}) \to \mathcal{D}(X)$ factors through $U(\mathfrak{g})_{\chi_0}$.*

Proof. We only need to show $a(z) = 0$ for any $z \in \mathsf{ker}(\chi_0) \subset Z(\mathfrak{g})$. We only need to show for any closed point $x \in X$, the composition
$$\mathsf{ker}(\chi_0) \to \mathcal{D}(X) \to \Gamma(X, k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)$$
is zero. By the proof of Proposition 1.1, this map can be identified with
$$\mathsf{ker}(\chi_0) \to k \underset{U(\mathfrak{b}_x)}{\otimes} U(\mathfrak{g}),$$
where $\mathfrak{b}_x = \mathsf{stab}_{\mathfrak{g}}(x)$ is the Borel subalgebra corresponding to x. Note that the ideal $\mathsf{ker}(\chi_0) \subset Z(\mathfrak{g})$ does not depond on the choice of any Borel subalgebra: it is the character for the trivial representation. Hence we only need to show $\mathsf{ker}(\chi_0) \to k \otimes_{U(\mathfrak{b}^-)} U(\mathfrak{g})$ is the zero map. But this follows from the Harish-Chandra embedding
$$Z(\mathfrak{g}) \to U(\mathfrak{g}) \to k \underset{U(\mathfrak{n}^-)}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} k \simeq U(\mathfrak{t}).$$
□

Remark 2.2. Alternatively, we can use left $\mathcal{D}$-modules and reduce to show
$$\mathsf{ker}(\chi_0) \to \mathcal{D}(X) \to \Gamma(X, \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} k_e)$$
is zero. By Exercise 1.5, the RHS is non-canonically isomorphic to $M_{-2\rho}$[3] and the above map can be identified with the action map on a highest weight vector. Then the claim follows from $\varpi(-2\rho) = \chi_0$.

[1]Hint: $\ell \simeq \wedge^d \mathcal{T}_{X,e}$ and $\mathcal{T}_{X,e} \simeq \mathfrak{n}^-$.

[2]Hint: consider the PBW filtration of $\mathcal{D}_X$ and the induced filtration on V. Show that $\mathfrak{b} \otimes \ell \to V$ factors through $\mathsf{F}^{\leq 1}V$ and the composition $\mathfrak{b} \otimes \ell \to \mathsf{F}^{\leq 1}V \to \mathsf{gr}^1 V$ is zero.

[3]Such an isomorphism depends on a trivialization of the line ℓ, i.e., a choice of vector in it.

To prove the obtained homomorphism

$$U(\mathfrak{g})_{\chi_0} \to \mathcal{D}(X)$$

is an isomorphism, we consider filtrations on both sides.

Construction 2.3. *The PBW filtration on* $U(\mathfrak{g})$ *induces a filtration on* $U(\mathfrak{g})_{\chi_0}$. *The surjection* $U(\mathfrak{g}) \to U(\mathfrak{g})_{\chi_0}$ *induces a surjection* $\mathsf{Sym}^\bullet(\mathfrak{g}) \to \mathsf{gr}^\bullet(U(\mathfrak{g})_{\chi_0})$ *which sends* $\ker(\mathsf{gr}^\bullet(Z(\mathfrak{g})) \to k)$ *to* 0. *Recall* $\mathsf{gr}^\bullet(Z(\mathfrak{g})) \simeq \mathsf{Sym}^\bullet(\mathfrak{g})^{\mathfrak{g}}$ *([Lecture 5, Lemma 3.2]). Hence we obtain a surjection*

$$\mathcal{O}(\mathfrak{g}^* \underset{\mathfrak{g}^*/\!/G}{\times} 0) \simeq \mathsf{Sym}^\bullet(\mathfrak{g}) \underset{\mathsf{Sym}^\bullet(\mathfrak{g})^{\mathfrak{g}}}{\otimes} k \twoheadrightarrow \mathsf{gr}^\bullet(U(\mathfrak{g})_{\chi_0}).$$

Note that a priori *we do not know this is an isomorphism.*

Construction 2.4. *On the other hand, the short exact sequences* $0 \to \mathsf{F}^{\leq k-1}\mathcal{D}_X \to \mathsf{F}^{\leq k}\mathcal{D}_X \to \mathsf{Sym}^k_{\mathcal{O}_X}\mathcal{T}_X \to 0$ *induce*

$$0 \to \Gamma(X, \mathsf{F}^{\leq k-1}\mathcal{D}_X) \to \Gamma(X, \mathsf{F}^{\leq k}\mathcal{D}_X) \to \Gamma(X, \mathsf{Sym}^k_{\mathcal{O}_X}\mathcal{T}_X)$$

and therefore an injection

$$\mathsf{gr}^\bullet\mathcal{D}(X) \hookrightarrow \Gamma(X, \mathsf{Sym}^\bullet_{\mathcal{O}_X}\mathcal{T}_X) \simeq \mathcal{O}(T^*X),$$

where $T^*X \simeq \mathsf{Spec}_X(\mathsf{Sym}^\bullet_{\mathcal{O}_X}\mathcal{T}_X)$ *is the cotangent bundle on* X. *Note that* a priori *we do not know this is an isomorphism.*

Combining the above constructions, we obtain homomorphisms

$$\mathcal{O}(\mathfrak{g}^* \underset{\mathfrak{g}^*/\!/G}{\times} 0) \twoheadrightarrow \mathsf{gr}^\bullet(U(\mathfrak{g})_{\chi_0}) \to \mathsf{gr}^\bullet\mathcal{D}(X) \hookrightarrow \mathcal{O}(T^*X).$$

We only need to show this composition is an isomorphism. This composition corresponds to a map

$$T^*X \to \mathfrak{g}^* \underset{\mathfrak{g}^*/\!/G}{\times} 0 \tag{2.1}$$

which will be studied in the next section.

Remark 2.5. The map $T^*X \to \mathfrak{g}^*$, which is the (algebro-geometric) dual of $\mathfrak{g} \to \mathcal{T}(X)$ is called the **moment map**.

3. Nilpotent cone and the Springer resolution

In this and the next sections, we study the map (2.1). I recommend [CG, Section 3] for these contents.

Recall we have an identification $\mathfrak{g} \simeq \mathfrak{g}^*$ provided by the Killing form. Also recall $\mathfrak{g}^*/\!/G \simeq \mathfrak{g}/\!/G \simeq \mathfrak{t}/\!/W$ are isomorphic to an affine space of dimension equal to $\dim(\mathfrak{t})$ (see [Lecture 6]).

We first describe the target of (2.1).

Definition 3.1. Define $\mathcal{N}$ to be the fiber product

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ 0 \arrow[r] & \mathfrak{g}/\!/G, \end{tikzcd}$$

and call it the **nilpotent cone** of $\mathfrak{g}$.

Remark 3.2. By Kostant's theorem ([Lecture 6, Corollary 1.15]), the projection map $\mathfrak{g} \to \mathfrak{g}/\!/G$ is flat. Recall regular immersions are closed under flat base-changes. Hence $\mathcal{N} \to \mathfrak{g}$ is a regular immersion. In particular, $\mathcal{N}$ is Cohen–Macaulay.

Remark 3.3. We have $\dim(\mathcal{N}) = \dim(\mathfrak{g}) - \dim(\mathfrak{t})$.

Warning 3.4. *The nilpotent cone $\mathcal{N}$ is always singular.*

The name "nilpotent cone" is justified by the following result:

Proposition 3.5. *A closed point of $\mathfrak{g}$ is contained in $\mathcal{N}$ iff it is an nilpotent element.*

Proof. Recall for any Borel pair $(\mathfrak{b}, \mathfrak{t})$, we have a commutative diagram (see [Lecture 5, (4.2)])

$$\begin{tikzcd} \mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g}/\!/G \\ \mathfrak{t} \arrow[r] & \mathfrak{t}/\!/W. \arrow[u, "\simeq"'] \end{tikzcd}$$

Also, W acts transitively on the fibers of the map $\mathfrak{t} \to \mathfrak{t}/\!/W$ at the closed points ([Lecture 6, Proposition 1.1]). It follows that a closed point $v \in \mathfrak{b}$ is sent to $0 \in \mathfrak{g}/\!/G$ iff it is sent to $0 \in \mathfrak{t}$. The latter condition is equivalent to v being nilpotent. Now the claim follows from the fact that any element of $\mathfrak{g}$ is contained in some Borel subalgebra.

□

Remark 3.6. We will see $\mathcal{N}$ is reduced (and even normal) and therefore it can be characterized by the above proposition.

Now we describe the source of (2.1). Note that T^*X is smooth because X is so.

Proposition 3.7. *Consider the obvious projection $T^*X \to X$ and the moment map $T^*X \to \mathfrak{g}^*$. The obtained map*

$$T^*X \to X \times \mathfrak{g}^* \tag{3.1}$$

is a closed embedding. Moreover, via the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, a closed points $(x, v) \in X \times \mathfrak{g}$ is contained in T^*X iff $v \in \mathfrak{n}_x := [\mathfrak{b}_x, \mathfrak{b}_x]$, where $\mathfrak{b}_x$ is the Borel subalgebra corresponding to x.

Proof. Let $x \in X$ be a closed point. We have a "realizing" map $X \simeq G/B_x$, $x \mapsto B_x/B_x$. It follows that $\mathcal{T}_{X,x} \simeq \mathfrak{g}/\mathfrak{b}_x$ and therefore $\mathcal{T}^*_{X,x} \simeq (\mathfrak{g}/\mathfrak{b}_x)^*$. By definition, the fiber of (3.1) at $x \in X$ is the obvious map $(\mathfrak{g}/\mathfrak{b}_x)^* \to \mathfrak{g}^*$ which is a closed embedding.

In general, a linear map between two vector bundles on X is a closed embedding iff its fiber at any closed point $x \in X$ is a closed embedding. Therefore (3.1) is a closed embedding.

Now the second claim follows from the isomorphism $(\mathfrak{g}/\mathfrak{b}_x)^* \simeq \mathfrak{n}_x$.

□

Remark 3.8. One can find local trivialization of the vector bundle $T^*X \to \mathsf{Fl}_G$ as follows. Let $x, x^- \in X$ be closed points such that the Borel subalgebras $\mathfrak{b}_x$ and $\mathfrak{b}_{x^-}$ intersect transversally. Using the Bruhat decomposition, the (N_{x^-})-orbit of x is open, and we denote it by U_{x,x^-}. It follows that the commposition

$$T^*X \to X \times \mathfrak{g}^* \to X \times \mathfrak{n}^*_{x^-} \simeq X \times \mathfrak{n}_x$$

is an isomorphism when restricted to the open subscheme $U_{x,x^-} \subset X$. Indeed, this follows fromm the fact that for any $y \in U_{x,x^-}$, the composition $\mathfrak{b}_{x^-} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{b}_y$ is an isomorphism.

Definition 3.9. We write $\widetilde{\mathcal{N}} := T^*X$ can call the map (2.1)

$$\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}.$$

the **Springer resolution of the nilpotent cone**.

Lemma 3.10. *The map $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$ is proper and surjective.*

Proof. We have a commutative diagram

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r] \arrow[d] & X \times \mathfrak{g} \arrow[d] \\ \mathcal{N} \arrow[r] & \mathfrak{g}. \end{tikzcd}$$

The top horizontal map is proper because it is a closed embedding. The right vertical map is proper because X is complete. Hence the composition $\widetilde{\mathcal{N}} \to \mathfrak{g}$ is proper. Since $\mathcal{N} \to \mathfrak{g}$ is separated, the map $\widetilde{\mathcal{N}} \to \mathcal{N}$ is also proper.

It remains to show $\mathfrak{p}$ is surjective on closed points. This follows from the fact that any (nilpotent) element in $\mathfrak{g}$ is contained in some Borel subalgebras.

□

Corollary 3.11. *The scheme $\mathcal{N}$ is irreducible.*

We will see $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities. For example, in the case of SL_2, we have:

Exercise 3.12. This is Homework 6, Problem 5. For $G = \mathsf{SL}_2$, prove $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the blow-up of $\mathcal{N}$ at the point $0 \in \mathcal{N}$.

Remark 3.13. The Springer resolution plays a central role in geometric representation theory.

4. Kostant's theorem

Our goal is to prove the following result.

Theorem 4.1 (Kostant)**.** *The map $\widetilde{\mathcal{N}} \to \mathcal{N}$ induces an isomorphism $\mathcal{O}(\mathcal{N}) \xrightarrow{\sim} \mathcal{O}(\widetilde{\mathcal{N}})$.*

Remark 4.2. In fact, one can show the *derived* direct image functor $\mathfrak{p}_* : D(\mathcal{O}_{\widetilde{\mathcal{N}}}\text{–}\mathsf{mod}_{\mathsf{qc}}) \to D(\mathcal{O}_{\mathcal{N}}\text{–}\mathsf{mod}_{\mathsf{qc}})$ sends $\mathcal{O}_{\widetilde{\mathcal{N}}}$ to $\mathcal{O}_{\mathcal{N}}$. The proof of this stronger result is an elaboration of the proof of Theorem 4.1 displayed below, with the help of the (derived non-flat) base-change isomorphisms. See [G, Section 7] for more details.

Note that the above theorem implies the first part of the localization theorem.

Corollary 4.3. *The homomorphism $U(\mathfrak{g})_{\chi_0} \to \mathcal{D}(X)$ is an isomorphism.*

Proof. By the discussion in previous sections, we only need to show $\mathcal{O}(\mathcal{N}) \to \mathcal{O}(\widetilde{\mathcal{N}})$ is an isomorphism, which is Kostant's theorem.

□

To prove Kostant's theorem, we need more geometric inputs.

Proposition-Definition 4.4. *There is a unique reduced closed subscheme $\widetilde{\mathfrak{g}}$ of $X \times \mathfrak{g}$, called the* ***Grothendieck's alteration****, whose closed points are those (x, v) satisfying $v \in \mathfrak{b}_x$. Moreover, $\widetilde{\mathfrak{g}}$ is smooth.*

Sketch. It is easy to show $v \in \mathfrak{b}_x$ is a closed condition and therefore defines a reduced closed subscheme $\widetilde{\mathfrak{g}}$. Also, as in Remark 3.8, the commposition

$$\widetilde{\mathfrak{g}} \to X \times \mathfrak{g} \to X \times \mathfrak{g}/\mathfrak{n}_{x^-} \simeq X \times \mathfrak{b}_x$$

is an isomorphism when restricted to the open subscheme $U_{x,x^-} \subset X$. This implies $\widetilde{\mathfrak{g}}$ is smooth.

□

Lemma 4.5. *There exists a Cartesian square*

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r] \arrow[d] & \widetilde{\mathfrak{g}} \arrow[d] \\ 0 \arrow[r] & \mathbf{t}, \end{tikzcd}$$

where $\mathbf{t}$ *is the* abstract *Cartan Lie algebra (see Appendix A). Moreover, the vertical maps are smooth.*

Sketch. We have an obvious injective map $\widetilde{\mathcal{N}} \to \widetilde{\mathfrak{g}}$ between vector bundles on X. By Remark 3.8 and the proof of Proposition-Definition 4.4, this map can be identified with $X \times \mathfrak{n}_x \to X \times \mathfrak{b}_x$ when restricted to the open subscheme $U_{x,x'} \subset X$. Hence the quotient bundle can be identified with $X \times \mathfrak{t}_x \simeq X \times \mathbf{t}$ over $U_{x,x'}$, where we used the realizing isomorphism $\mathbf{t} \to \mathfrak{t}_x$.

One can show these identifications can be glued into a short exact sequence of vector bundles over X:

$$0 \to \widetilde{\mathcal{N}} \to \widetilde{\mathfrak{g}} \to X \times \mathbf{t} \to 0,$$

which makes the desired claim manifest.

□

Notation 4.6. *Let* $\mathfrak{g}_{\mathsf{rss}} \subset \mathfrak{g}_{\mathsf{reg}} \subset \mathfrak{g}$ *be the open subschemes whose closed points are regular semisimple (resp. regular*[4]*) elements in* $\mathfrak{g}$*. Let* $\widetilde{\mathfrak{g}}_{\mathsf{rss}} \subset \widetilde{\mathfrak{g}}_{\mathsf{reg}} \subset \widetilde{\mathfrak{g}}$ *be their preimages.*

Let $\mathcal{N}_{\mathsf{reg}} := \mathfrak{g}_{\mathsf{reg}} \cap \mathcal{N}$ *be the open subscheme of* $\mathcal{N}$*. Its closed points are regular nilpotent elements.*

Let $\mathbf{t}_{\mathsf{reg}} \subset \mathbf{t}$ *be the open subscheme whose closed points are regular elements*[5]*.*

We have the following basic results. See e.g. [CG, Section 3.1] for a proof.

Proposition 4.7. *Consider the map* $\mathfrak{g} \to \mathfrak{g}/\!/G \to \mathbf{t}/\!/\mathbf{W}$ *given by the abstract Chevalley isomorphism (see Appendix A). We have:*

(1) *The following diagram commutes:*

$$\begin{tikzcd} \widetilde{\mathfrak{g}} \arrow[r] \arrow[d] & \mathbf{t} \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathbf{t}/\!/\mathbf{W} \end{tikzcd}$$

(2) *When restricted to the regular locus* $\widetilde{\mathfrak{g}}_{\mathsf{reg}}$*, the above diagram is Cartesian. In other words, the following diagram is Cartesian:*

$$\begin{tikzcd} \widetilde{\mathfrak{g}}_{\mathsf{reg}} \arrow[r] \arrow[d] & \mathbf{t} \arrow[d] \\ \mathfrak{g}_{\mathsf{reg}} \arrow[r] & \mathbf{t}/\!/\mathbf{W}. \end{tikzcd}$$

[4] Recall an element $v \in \mathfrak{g}$ is regular if its centralizer is of minimal dimension, which is $\dim(\mathbf{t})$.

[5] This means the realizations in any/all Cartan subalgebras are regular. Equivalently, this means $\mathbf{W}$ acts freely at these points.

(3) *When restricted to the regular semisimple locus* $\widetilde{\mathfrak{g}}_{\mathsf{rss}}$*, the following diagram is Cartesian, and the Vertical maps are finite étale covers with Galois group* $\mathbf{W}$*:*

$$\begin{tikzcd} \widetilde{\mathfrak{g}}_{\mathsf{rss}} \arrow[r] \arrow[d] & \mathbf{t}_{\mathsf{reg}} \arrow[d] \\ \mathfrak{g}_{\mathsf{rss}} \arrow[r] & \mathbf{t}_{\mathsf{reg}}/\mathbf{W}. \end{tikzcd}$$

Warning 4.8. *The map* $\widetilde{\mathfrak{g}} \to \mathbf{t}$ *does* not *send regular elements to regular elements. Indeed, it sends* $\mathcal{N}_{\mathsf{reg}}$ *to* 0*.*

Proposition 4.9. *The scheme* $\mathcal{N}$ *is normal.*

Sketch. We have seen $\mathcal{N}$ is Cohen–Macaulay (Remark 3.2). Hence by Serre's criterion, we only need to show $\mathcal{N}$ is regular in codimension 1.

By Lemma 4.5, the map $\widetilde{\mathfrak{g}} \to \mathbf{t}$ is smooth, hence so is $\widetilde{\mathfrak{g}}_{\mathsf{reg}} \to \mathbf{t}$. Recall $\mathbf{t} \to \mathbf{t}/\!/W$ is faithfully flat ([Lecture 6, Proposition 1.1 and Corollary 1.5]). Hence by Proposition 4.7(2), the map $\mathfrak{g}_{\mathsf{reg}} \to \mathbf{t}/\!/\mathbf{W}$ is smooth (by flat descent of smooth maps). By definition, we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{N}_{\mathsf{reg}} \arrow[r] \arrow[d] & \mathfrak{g}_{\mathsf{reg}} \arrow[d] \\ 0 \arrow[r] & \mathfrak{g}/\!/G \simeq \mathbf{t}/\!/\mathbf{W}. \end{tikzcd}$$

Hence $\mathcal{N}_{\mathsf{reg}}$ is smooth.

It remains to show the closed subset $\mathcal{N} - \mathcal{N}_{\mathsf{reg}}$ of $\mathcal{N}$ is of codimension ≥ 2. Since $\mathcal{N}$ is irreducible (Corollary 3.11), $\mathcal{N} - \mathcal{N}_{\mathsf{reg}}$ is of codimension ≥ 1. We need to use the following two basic facts:

(i) The adjoint action of G on $\mathcal{N}$ has only finitely many orbits[6] (see [CG, Proposition 3.2.9]);

(ii) Each G-orbit on $\mathfrak{g}$ has a symplectic structure (see [CG, Proposition 1.1.5]).

By (ii), each G-orbit has an even dimension. Hence each G-orbit in $\mathcal{N} - \mathcal{N}_{\mathsf{reg}}$ has even codimension. By (i), $\mathcal{N} - \mathcal{N}_{\mathsf{reg}}$ has codimension ≥ 2 as desired.

□

Corollary 4.10. *The map* $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$ *is a resolution of singularities, i.e., it is birational proper and surjective.*

Proof. We have already proved $\mathfrak{p}$ is proper and surjective (Lemma 3.10). It remains to show $\mathfrak{p}$ is birational. We claim its restriction on $\mathcal{N}_{\mathsf{reg}} \subset \mathcal{N}$ is an isomorphism. Since $\mathcal{N}$ is reduced, we only need to show any closed point of $\mathcal{N}_{\mathsf{reg}}$ has a unique preimage in $\widetilde{\mathcal{N}}_{\mathsf{reg}}$. Now the claim follows from Proposition 4.7(2) because $0 \in \mathbf{t}/\!/\mathbf{W}$ has a unique preimage in $\mathbf{t}$.

□

Remark 4.11. In fact, $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a *semismall* resolution, i.e., $\dim(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}) = \dim(\mathcal{N})$. This fact is crucial in the Springer theory. The fiber product $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$ is known as the **Steinberg variety**, which also plays a central role in geometric representation theory. For more information, see [CG].

[6]This can be viewed as a generalization of the theory of Jordan blocks.

Proof of Theorem 4.1. Follows by applying Zariski's main theorem to the projection $\mathfrak{p} : \widetilde{\mathcal{N}} \to \mathcal{N}$. Direct proof: $\mathfrak{p}_* \mathcal{O}_{\widetilde{\mathcal{N}}}$ is coherent because $\mathfrak{p}$ is proper. It is generically of rank 1 because $\mathfrak{p}$ is birational. Both the source and target of $\mathcal{O}_{\mathcal{N}} \to \mathfrak{p}_* \mathcal{O}_{\widetilde{\mathcal{N}}}$ are sheaves of integral domains, hence they have isomorphic sheaves of fractional fields. Then we win because $\mathcal{O}_{\mathcal{N}}$ is integrally closed. □

Appendix A. Abstract Cartan group and abstract Weyl group

Construction A.1. *Let B_x and B_y be two Borel subgroups of G. Let $T_x := B_x/[B_x, B_x]$ and $T_y := B_y/[B_y, B_y]$ be their abelianizations. Recall there exists $g \in G(k)$ such that Ad_g induces an isomorphism $\mathsf{Ad}_g : B_x \xrightarrow{\simeq} B_y$. Hence we obtain an isomorphism between the abelianizations $\overline{\mathsf{Ad}}_g : T_x \xrightarrow{\simeq} T_y$. The isomorphism $\overline{\mathsf{Ad}}_g$ does not depend on the choice of g because any other choice g' satisfies $g' \in gB_x(k)$ and the adjoint action of B_x on T_x is trivial. For this reason, we write the above isomorphism as*

$$\phi_{x,y} : T_x \xrightarrow{\sim} T_y.$$

It is easy to check $\phi_{x,x} = \mathsf{Id}$ and $\phi_{y,z} \circ \phi_{x,y} = \phi_{x,z}$. Hence there exists an algebraic group $\mathbf{T}$, equipped with isomorphisms

$$r_x : \mathbf{T} \xrightarrow{\sim} T_x,$$

such that $r_y = \phi_{x,y} \circ r_x$. The data $(\mathbf{T}, r_x)$ are unique up to an unique isomorphism[7].

*We call $\mathbf{T}$ the **abstract Cartan group** for G, and call r_x the **realizing isomorphisms**.*

*Similarly, the Lie algebra of $\mathbf{T}$ is denoted by $\mathfrak{t}$ and is called the **abstract Cartan algebra** for $\mathfrak{g}$.*

Warning A.2. *The algebraic group $\mathbf{T}$ is* not *a subgroup of G, at least not in a canonical way.*

Remark A.3. A Cartan subgroup $T_1 \hookrightarrow G$ of G does *not* give a realizing isomorphism $\mathbf{T} \to T_1$, at least not in a canonical way. Instead, if we further choose a Borel subgroup B_x that contains T_1, i.e., if we have a **Borel pair** (B_x, T_1), then there is a realizing isomorphism $\mathbf{T} \to T_1$ defined to be the composition

$$r_{(B_x,T_1)} : \mathbf{T} \xrightarrow{r_x} T_x \xleftarrow{\sim} T_1,$$

where the second isomorphism is given by $T_1 \hookrightarrow B_x \twoheadrightarrow T_x$.

Warning A.4. *One cannot define the abstract Borel group for G.*

Construction A.5. *Let (B_x, T_1) and (B_y, T_2) be two Borel pairs. Recall there is a unique element $g \in G(k)$ such that $\mathsf{Ad}_g : B_x \xrightarrow{\sim} B_y$ and $\mathsf{Ad}_g : T_1 \xrightarrow{\sim} T_2$. Hence we obtain an isomorphism between the normalizers $\mathsf{Ad}_g : N_G(T_1) \xrightarrow{\sim} N_G(T_2)$ and therefore an isomorphism between the corresponding Weyl groups. We denote this isomorphism by*

$$\varphi_{(B_x,T_1),(B_y,T_2)} : W_{T_1} \to W_{T_2}.$$

It is easy to check $\varphi_{(B_x,T_1),(B_x,T_1)} = \mathsf{Id}$ and $\varphi_{(B_y,T_2),(B_z,T_3)} \circ \varphi_{(B_x,T_1),(B_y,T_2)} = \varphi_{(B_x,T_1),(B_z,T_3)}$. Hence there exists a group $\mathbf{W}$ equipped with isomorphisms

$$r_{(B_x,T_1)} : \mathbf{W} \xrightarrow{\sim} W_{T_1}$$

such that $r_{(B_y,T_2)} = \varphi_{(B_x,T_1),(B_y,T_2)} \circ r_{(B_x,T_1)}$. The data $(\mathbf{W}, r_{(B_x,T_1)})$ are unique up to an unique isomorphism.

*We call $\mathbf{W}$ the **abstract Weyl group** for G, and call $r_{(B_x,T_1)}$ the **realizing isomorphisms**.*

Warning A.6. *The isomorphism $\varphi_{(B_x,T_1),(B_y,T_2)}$ depends on B_x and B_y.*

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[7]This means for $(\mathbf{T}, r_x)$ and $((\mathbf{T})', r'_x)$, there is a unique isomorphism $\alpha : \mathbf{T} \xrightarrow{\sim} (\mathbf{T})'$ such that $r_x = r'_x \circ \alpha$.

Warning A.7. *The group* $\mathbf{W}$ *is* not *a subgroup of* G*, at least not in a canonical way.*

Remark A.8. One can also define the abstract Weyl group by providing a group structure on $|G\backslash(X\times X)|$. This construction was introduced by Deligne–Lusztig when developing the theory named by them.

Construction A.9. *Let* (B_x, T_1) *be any Borel pair. The action of* W_{T_1} *on* T_1 *defines an action of* $\mathbf{W}$ *on* $\mathbf{T}$ *via the realizing isomorphisms* $r_{(B_x,T_1)} : \mathbf{T} \xrightarrow{\sim} T_1$ *and* $r_{(B_x,T_1)} : \mathbf{W} \xrightarrow{\sim} W_{T_1}$*. Unwinding the definitions, one can show this action does not depend on the choice of the Borel pair. Hence we obtain a* canonical *action of* $\mathbf{W}$ *on* $\mathbf{T}$*, which is called* ***the (abstract) action of*** $\mathbf{W}$ ***on*** $\mathbf{T}$.

Construction A.10. *Recall for any Borel pair* (B,T)*, we have the Chevalley isomorphism* $\mathfrak{t}/\!/W \xrightarrow{\sim} g/\!/G$ *characterized by the following commutative diagram (see [Lecture 5, (4.2)])*

$$\begin{tikzcd}
\mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g}/\!/G \\
\mathfrak{t} \arrow[r] & \mathfrak{t}/\!/W. \arrow[u, "\simeq"']
\end{tikzcd}$$

Via the realizing isomorphism $r_{(B,T)} : \mathbf{t}/\!/\mathbf{W} \xrightarrow{\sim} \mathfrak{t}/\!/W$*, we obtain an isomorphism*

$$\mathbf{t}/\!/\mathbf{W} \xrightarrow{\sim} \mathfrak{g}/\!/G$$

which can be shown to do not depend on the choice of the Borel pair. We call this isomorphism the ***abstract Chevalley isomorphism****.*

References

[CG] Chriss, Neil, and Victor Ginzburg. Representation theory and complex geometry. Vol. 42. Boston: Birkhäuser, 1997.

[G] Gaitsgory, Dennis. Course Notes for Geometric Representation Theory, 2005, available at `https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf`.