2024 Spring — Geometric Representation Theory (1)

# HOMEWORK PROBLEMS

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## Homework 1 (Due on March 18)

**Problem 1.1** (Lecture 2, Exercise 13)**.** Prove the following.

(1) The map

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \xrightarrow{\mathsf{mult}} U(\mathfrak{g})$$

is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules.

(2) As an $\mathfrak{n}^-$-module, $M_\lambda$ is freely generated by $v_\lambda$, i.e.,

$$U(\mathfrak{n}^-) \longrightarrow M_\lambda, \quad x \longmapsto x \cdot v_\lambda$$

is an isomorphism.

*Solution.* (1) It is clear from the construction that $\mathsf{mult}$ is a homomorphism of left $U(\mathfrak{n}^-)$-modules and right $U(\mathfrak{b})$-modules. Note that the target $U(\mathfrak{g})$ is regarded as a $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodule via restricting the $U(\mathfrak{g})$-action on itself to $U(\mathfrak{b})$ and $U(\mathfrak{n}^-)$, respectively. It remains to show that $\mathsf{mult}$ is an isomorphism of vector spaces by PBW theorem [Lecture 2, Theorem 5]. For this, according to the context of [Lecture 2, Corollary 6], pick $\{x_1, \ldots, x_n\}$ as a basis of $\mathfrak{n}^-$ and $\{y_1, \ldots, y_m\}$ as a basis of $\mathfrak{b}$. Then the bases of $k$-vector spaces $U(\mathfrak{n}^-)$, $U(\mathfrak{b})$, $U(\mathfrak{g})$ are $\{x_1^{k_1} \cdots x_n^{k_n}\}_{k_i \geqslant 0}$, $\{y_1^{l_1} \cdots y_m^{l_m}\}_{l_j \geqslant 0}$, $\{x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geqslant 0}$, respectively; it follows that $U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b})$ has a basis $\{x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m}\}_{k_i, l_j \geqslant 0}$. Moreover,

$$\mathsf{mult}\colon x_1^{k_1} \cdots x_n^{k_n} \otimes y_1^{l_1} \cdots y_m^{l_m} \longmapsto x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_m^{l_m}.$$

This completes the proof that $\mathsf{mult}$ is an isomorphism between $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules.

(2) Using (1), we have that

$$U(\mathfrak{n}^-) \otimes_k U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} k_\lambda \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda = M_\lambda.$$

Here note that each $x \cdot v_\lambda \in M_\lambda$ can be identified with $x \otimes v_\lambda \in U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$. On the other hand, the left-hand side is isomorphic to $U(\mathfrak{n}^-) \otimes_k k_\lambda$, which is further isomorphic to $U(\mathfrak{n}^-)$. This completes the proof. $\square$

**Problem 1.2** (Lecture 2, Exercise 24)**.** In the case $\mathfrak{g} = \mathfrak{sl}_2$, show the Verma module $M_l$ is irreducible unless $l \in \mathbb{Z}_{\geqslant 0}$. In the latter case, show there is a non-split short exact sequence

$$0 \longrightarrow M_{-l-2} \longrightarrow M_l \longrightarrow L_l \longrightarrow 0$$

such that $L_l$ is a finite-dimensional irreducible $\mathfrak{sl}_2$-module with highest weight $l$.

*Solution.* Suppose $M_l$ is not irreducible. Then there is a nonzero proper submodule $N \subset M_l$ of highest weight $l'$; denote by $v_{l'}$ the highest weight vector. In this case $l'$ is also regarded as a weight of $M_l$. Recall that for $\mathfrak{g} = \mathfrak{sl}_2$, any weight of $M_l$ is of form $l - 2n$ with $n \geqslant 0$. So we may assume $l' = l - 2n$. Since $v_{l'}$ generates $N$ via $\mathfrak{g}$-action, the hypothesis $N \subsetneq M_l$ implies $l' \neq l$ (otherwise $M_l = N$), or equivalently $n > 0$. Let $e, f, g \in \mathfrak{sl}_2$ be the standard generators. The weight of $e \cdot v_{l'}$ is either 0 or $l'+2$. Since $l$ is the highest weight of $M_l$, we must be in the former case that $e \cdot v_l = 0$. Thus, we obtain from $[e,f] = h$ and $h \cdot f^j - f^j \cdot h = -2jf^j$ that[1]

$$\begin{aligned} e \cdot f^n \cdot v_l &= \sum_{1 \leqslant i \leqslant n} f^{n-i} \cdot [e,f] \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leqslant i \leqslant n} f^{n-i} \cdot h \cdot f^{i-1} \cdot v_l + f^n \cdot e \cdot v_l \\ &= \sum_{1 \leqslant i \leqslant n} (l - 2(i-1)) \cdot f^{n-1} \cdot v_l \\ &= n(l-(n-1)) \cdot f^{n-1} \cdot v_l. \end{aligned}$$

Since the above is 0 whereas $f^{n-1} \cdot v_l \neq 0$, it implies that $l = n-1 \geqslant 0$. This shows the irreducibility of $M_l$ for $l < 0$.

If $l = n-1$ for $n > 0$, the same computation shows that $e \cdot f^n \cdot v_l = e \cdot v_{l-2n} = 0$, with $l - 2n = -l-2$, i.e. the vector $v_{l-2n} \in M_l$ generates a submodule of $M_l$ that is isomorphic to $M_{-l-2}$. By fixing an isomorphism $L_l \simeq M_l/M_{-l-2}$, we get a quotient module $L_l$ of $M_l$ as well as the desired short exact sequence $0 \to M_{-l-2} \to M_l \to L_l \to 0$. Such $L_l$ is clearly finite-dimensional of highest weight $l$. If the sequence splits, then there is a nonzero section map $M_l \to M_{-l-2}$, and hence a nonzero vector in $M_{-l-2}$ of weight $l$, which is impossible. So the sequence is non-split.

It remains to show the irreducibility of $L_l$. For this, note that whenever $l = n-1$ the highest weight of a proper submodule $N$ of $M_l$ must be $-l-2$, so there is no proper submodule of $M_l$ containing $M_{-l-2}$. Correspondingly, the quotient $L_l$ must be irreducible. □

**Problem 1.3** (Lecture 2, Exercise 34)**.** Recall for any $V_1, V_2 \in \mathfrak{g}\text{-}\mathsf{mod}$, the tensor product $V_1 \otimes V_2$ of the underlying vector spaces has a natural $\mathfrak{g}$-module structure defined by $x \cdot (v_1 \otimes v_2) \coloneqq (x \cdot v_1) \otimes v_2 + v_1 \otimes (x \cdot v_2)$.

(1) Prove that if $V_1$ and $V_2$ are weight modules, so is $V_1 \otimes V_2$. Determine the weights and weight spaces of $V_1 \otimes V_2$ in terms of those for $V_1$ and $V_2$.
(2) Consider the case $\mathfrak{g} = \mathfrak{sl}_2$. Prove that the tensor product of two Verma modules is not contained in $\mathcal{O}$.

*Solution.* (1) If $V_1, V_2$ are weight modules, then we can respectively take $V_{1,\lambda} \subset V_1$ and $V_{2,\nu} \subset V_2$ to be $\lambda$- and $\nu$-eigenspaces, where $\lambda, \nu \in \mathfrak{t}^*$. For all $t \in \mathfrak{t}$ together with $v_1 \in V_{1,\lambda}$ and $v_2 \in V_{2,\nu}$, we have $t \cdot (v_1 \otimes v_2) = (t \cdot v_1) \otimes v_2 + v_1 \otimes (t \cdot v_2) = \lambda(t) v_1 \otimes v_2 + \nu(t) v_1 \otimes v_2 = (\lambda(t) + \nu(t)) \cdot v_1 \otimes v_2$. It follows that $v_1 \otimes v_2 \in V_1 \otimes V_2$ is of weight $\lambda + \nu$. This proves that $V_1 \otimes V_2$ is a weight module; each weight space of $V_1 \otimes V_2$ is of form $\bigoplus_{\lambda+\nu=\mu} V_{1,\lambda} \otimes V_{2,\nu}$ for some fixed $\mu \in \mathfrak{t}^*$.

(2) Let $M_\lambda, M_\nu$ be two Verma modules. We prove $M_\lambda \otimes M_\nu \notin \mathcal{O}$ by showing that it is not finitely generated. Suppose for the sake of contradiction that $M_\lambda \otimes M_\nu$ is generated by $m_1, \ldots, m_n$ for some $n \in \mathbb{Z}$. Indeed, $M_\lambda \otimes M_\nu$ is isomorphic to a quotient module of an extension of finitely many Verma modules. When $\mathfrak{g} = \mathfrak{sl}_2$, the weight spaces of each of these Verma modules are all 1-dimensional. (Note that this is not true for general $\mathfrak{g}$.) After taking the quotient towards $M_\lambda \otimes M_\nu$, it follows that the dimension of any weight space is at most $n$. Also, since $\mathfrak{g} = \mathfrak{sl}_2$, there turns out to be a weight space of $M_\lambda \otimes M_\nu$ of weight $\lambda + \nu - 2n$ and dimension $n+1$. This is impossible, and thus $M_\lambda \otimes M_\nu$ cannot be finitely generated. □

**Problem 1.4** (Lecture 3, Exercise 48)**.**

(1) Find all maps between $k$-schemes $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$.
(2) Find all 1-dimensional representations of the additive group $\mathbb{G}_{\mathrm{a}}$.
(3) Find all maps between $k$-schemes $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$.
(4) Find all 1-dimensional representations of the multiplicative group $\mathbb{G}_{\mathrm{m}}$.

[1]There is a typo in the proof of [Gai05, Proposition 1.9], c.f. the third line of the computation.

*Solution.* (1) It suffices to recognize all homomorphisms $k[x]_x \to k[x]$ between $k$-algebras; here $k[x]_x$ is the localization of $k[x]$ at the point with coordinate $x$. Since $x$ is invertible in $k[x]_x$, its image in $k[x]$ must be invertible as well, which implies that the image of $x$ must be an element of $k^\times$. It follows that all $k[x]_x \to k[x]$ (with $1 \mapsto 1$ and $x \mapsto t \in k^\times$) are parameterized by $k^\times$. Therefore, all maps between $k$-schemes $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$ are exactly parametrized by $k^\times$, which are constant maps sending all points on $\mathbb{A}^1$ to some point on $\mathbb{A}^1 \backslash 0$.

(2) Any 1-dimensional representation of $\mathbb{G}_\mathrm{a}$ is given by the map $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ of group schemes over some (algebraically closed) field, say $k$. At the level of $k$-schemes, the data of $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ can be specialized to the data of $\mathbb{A}^1 \to \mathbb{A}^1 \backslash 0$. But the result of (1) forces $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{m}$ to be a constant map, which can only be the trivial representation of $\mathbb{G}_\mathrm{a}$. In other words, the 1-dimensional representation of $\mathbb{G}_\mathrm{a}$ can only be trivial.

(3) As in (1), we aim to figure out all homomorphisms $k[x]_x \to k[x]_x$ between $k$-algebras. It suffices to determine the image of $x$. Note that all invertible elements of $k[x]_x$ are of form $tx^n$ with $t \in k^\times$ and $n \in \mathbb{Z}$. We thus conclude that each map $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$ is parametrized by some $tx^n \in k^\times x^{\mathbb{Z}}$, sending $P \in \mathbb{A}^1 \backslash 0$ to $tP^n$.

(4) As in (2), consider the map $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$ of group schemes over $k$. Note that this is determined by $\mathbb{A}^1 \backslash 0 \to \mathbb{A}^1 \backslash 0$, which must be of form $P \mapsto tP^n$ by (3). Moreover, as a group homomorphism, we must have $\mathbb{1} \mapsto \mathbb{1}$ where $\mathbb{1}$ is the identity element of $\mathbb{G}_\mathrm{m}$; it hence implies $t = 1 \in k^\times$. Therefore, the desired 1-dimensional representation of $\mathbb{G}_\mathrm{m}$ must be $\mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}$, $g \mapsto g^k$ for some $k \in \mathbb{Z}$. □

**Problem 1.5** (Lecture 3, Exercise 48)**.** Let $G$ be any semisimple algebraic group with Lie algebra $\mathfrak{g}$. Prove any Verma module of $\mathfrak{g}$ is not $G$-integrable.

*Solution.* Suppose $M_\lambda$ is a $G$-integrable Verma module of $\mathfrak{g}$, i.e. $M_\lambda$ is a $\mathfrak{g}$-module coming from a representation of $G$ through the functor $\mathsf{Rep}(G) \to \mathfrak{g}\text{-}\mathsf{mod}$. By [Lecture 3, Proposition 35], if this is the case, then the $G$-action on $M_\lambda$ is locally finite, i.e. $M_\lambda$ is a union of finite-dimensional subrepresentations.

Let $v_\lambda$ be the highest weight vector in $M_\lambda$. Note that $v_\lambda$ generates $M_\lambda$ through the orbit of $\mathfrak{g}$-action. Since $M_\lambda$ is locally finite, there exists a finite-dimensional $\mathfrak{g}$-submodule containing $v_\lambda$ that is also a $G$-invariant subspace; it must equal to $M_\lambda$ for the prescribed reason. In particular, in this case $M_\lambda$ is finite-dimensional, which is a contradiction. □

## References

[Gai05] Dennis Gaitsgory. *Course Notes for Geometric Representation Theory*. 2005. available at https://people.mpim-bonn.mpg.de/gaitsgde/267y/cat0.pdf.