FS Seminar: Banach-Colmez Spaces

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1 Intuition

Let's consider the curve $\mathbb{P}^1$ over $\mathbb{C}$. Let us recall some facts from the study of Bridgeland stability:

- Let Q denote the Kronecker quiver ($\bullet \rightrightarrows \bullet$), and $\mathrm{Rep}^{\heartsuit}(Q)$ the representation category of this quiver. There is a derived equivalence (due to Beilinson)

$$\mathrm{Coh}(\mathbb{P}^1) \simeq \mathrm{Rep}(Q) \quad \mathcal{F} \mapsto \mathrm{Hom}(\mathcal{T}, \mathcal{F})$$

  where $\mathcal{T} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathcal{P}^1}(1)$ and $\mathrm{End}(\mathcal{T})$ is the path algebra of Q. This equivalence induces a *tilting* t-structure on the LHS; we denote the new heart (i.e. image of $\mathrm{Rep}^{\heartsuit}(Q)$) by $\mathrm{Coh}^{\spadesuit}(\mathbb{P}^1)$. The basic fact is that

$$\mathcal{O}_{\mathbb{P}^1} \mapsto (k \rightrightarrows 0) \quad \mathcal{O}_{\mathbb{P}^1}(-1)[1] \mapsto (0 \rightrightarrows k);$$

  Note the skyscrapers are still in this new heart, namely, if $y = [a : b] \in \mathbb{P}^1$, then $(i_y)_*(k) \mapsto (k \rightrightarrows k)$ where the two arrows are multiplication by a and b respectively. This category can also be described as

$$\mathrm{Coh}^{\spadesuit}(\mathbb{P}^1) \simeq \{\mathcal{F} \in \mathrm{Coh}(\mathbb{P}^1) \mid H^i(\mathcal{F}) = 0, i \neq 0, -1; H^0(\mathcal{F}) \geq 0, H^{-1}(\mathcal{F}) < 0\},$$

  where $H^0(\mathcal{F}) \geq 0$ notation means that the graded pieces of Harder-Narasimhan filtration of $H^0(\mathcal{F})$ all have slope ≥ 0, and similarly for $H^{-1}(\mathcal{F}) < 0$;

- Let us fix a point $x \in \mathbb{P}^1$, and fix the absolute base point $b := \mathrm{Spec}(\mathbb{C})$ over which $\mathbb{P}^1$ lives (via a map $\pi_b : \mathbb{P}^1 \to b$), then we have two embeddings

$$\mathrm{Coh}^{\heartsuit}(x) \simeq \mathrm{Vect}^{\heartsuit,\mathrm{fin.dim}} \hookrightarrow \mathrm{Coh}^{\spadesuit}(\mathbb{P}^1) \quad W \mapsto W^{\mathrm{an}} := (i_x)_*(W)$$

$$\mathrm{Coh}^{\heartsuit}(b) \simeq \mathrm{Vect}^{\heartsuit,\mathrm{fin.dim}} \hookrightarrow \mathrm{Coh}^{\spadesuit}(\mathbb{P}^1) \quad V \mapsto V^{\mathrm{et}} := (\pi_b)^*(V)$$

  and the latter category is generated by these two types of objects (the meaning of the notations will become clear later). In this sense, one is justified in saying that $\mathrm{Coh}^{\spadesuit}(\mathbb{P}^1)$ is a mixture of $\mathrm{Coh}^{\heartsuit}(x)$ and $\mathrm{Coh}^{\heartsuit}(b)$.

Of course, this last statement is a bit silly in the $\mathbb{C}$-world; but let us now try to tell the same story in the (algebraic or adic)[1] Fargues-Fontaine curve setting. For uniformity of presentation we consider the adic curve. Here we have to be careful:

[1]The GAGA principle proved last time concerns only vector bundles, so one has to be careful when talking about replacing "algebraic" with "adic" here; the correct thing to cite is Kedlaya-Liu, "Relative p-adic Hodge Theory, II", where GAGA for coherent sheaves was proven.

- Recall that the diamond formula tells us that we should think of the Fargues-Fontaine curve as a geometrization of $\mathbb{Q}_p$. In other words, though we don't have a base point b, the essential image of the second embedding above still has a good candidate, namely that of finite-dimensional $\mathbb{Q}_p$-vector spaces;
- What is a closed point of the FF curve? Recall there is no such thing as an absolute FF curve, so we can only answer this question when looking at X_S. For simplicity, set $S = \mathrm{Spa}(C^\flat)$ for $C^\flat$ a complete algebraically closed non-archimedean field over $\mathbb{F}_p$. Then we know that classical points of X_S are untilts of $C^\flat$ over $\mathbb{Q}_p$ (which are also algebraically closed). Moreover, if C is such an untilt, then it should be regarded as the residue field at the corresponding point. In other words, the essential image of the first embedding should be that of finite-dimensional C-vector spaces.

Punchline: the category of Banach-Colmez spaces will be the FF-curve version of $\mathrm{Coh}^{\spadesuit}(\mathbb{P}^1)$.

Definition 1.0.1. Fix some C as above. The category $\mathcal{BC} = \mathcal{BC}(C)$ of Banach-Colmez spaces for C is the smallest abelian subcategory of sheaves of topological $\mathbb{Q}_p$-vector spaces on $\mathrm{Perfd}_{/\mathrm{Spa}(C)}$ (equipped with the pro-etale topology) that contains the following two objects:

- $\mathbb{G}_{a,C}^{\diamond}$, the diamondification of the adic space $(\mathbb{G}_a)_C^{\mathrm{an}}$;
- $\underline{\mathbb{Q}_p}$, the constant sheaf given by $\underline{\mathbb{Q}_p}(R, R^+) = C(|\mathrm{Spa}(R, R^+)|, \mathbb{Q}_p)$ (continuous maps of topological spaces).

It follows that the category of finite dimensional $\mathbb{Q}_p$-vector spaces (resp. C-vector spaces) embeds into $\mathcal{BC}$ via the map $W \mapsto W^{\mathrm{an}}(R, R^+) = C(|\mathrm{Spa}(R, R^+)|, W)$ (resp. $V \mapsto V^{\mathrm{et}}(R, R^+) = R \otimes_C V$).

Remark 1.0.1. This is not the original definition by Colmez; it is a result of Le Bras that this agrees with the original one.

We have the following interpretation:

Proposition 1.0.1 (Le Bras)**.** *Using the GAGA principle one can define the notion of slope for any coherent sheaf on the adic curve, so that one can define* $\mathrm{Coh}^{\spadesuit}(X_{\mathrm{Spa}(C^\flat)})$ *using the same slope definition from the previous page. Then the map*

$$\mathbb{H}^0 := \mathcal{F} \mapsto \mathbb{H}^0(\mathcal{F})(S) := \mathbb{H}^0(X_S, \mathcal{F}|_{X_S})$$

induces an equivalence $\mathrm{Coh}^{\spadesuit}(X_{\mathrm{Spa}(C^\flat)}) \simeq \mathcal{BC}$*.*

Remark 1.0.2. Here I'm already cheating: one needs to know that $\mathbb{H}^0(X_{C^\flat}, \mathcal{O}) \simeq \mathbb{Q}_p$ to even parse this claim. This will follow from our discussion below.

Remark 1.0.3. It follows that the category $\mathcal{BC}$ actually only depends on $C^\flat$, and not on a particular choice of C. In other words, the category $\mathcal{BC}$ is intrinsically defined for the geometric point $\mathrm{Spa}(C^\flat)$. One can ask what is the correct definition for a general S; this doesn't seem to exist in literature.

To orient ourselves, let us write down some correspondences (which we'll prove later today):

$$\mathcal{O}_X \mapsto \underline{\mathbb{Q}_p} \quad (i_C)_*(C) \mapsto \mathbb{G}_{a,C}^{\diamond} \quad \mathcal{O}_X(-1)[1] \mapsto \mathbb{G}_{a,C}^{\diamond}/\underline{\mathbb{Q}_p}.$$

Remark 1.0.4. Regarding Remark 15.2.2 of Berkeley notes: in Milne's paper the category being considered is $\mathscr{G}(p)$ appearing on page 181.

Remark 1.0.5. As we will see later in the seminar, the Hilbert diamond Div^d and local charts of Bun_G are all built out of BC spaces.

2 p-divisible Groups

Let us now discuss a class (in fact, the only non-trivial ones I know) of examples of BC spaces. Lin already covered this story in dimension 1, so let me quickly remind you of it.

Fix G a p-divisible group over $\mathcal{O}_C$ and take $\mathcal{G}$ to be the formal completion near 1, which is a formal scheme over $\mathrm{Spf}(\mathcal{O}_C)$. We have an exact sequence of adic spaces

$$0 \to G_C^{\mathrm{ad}}[\pi^\infty] \to G_C^{\mathrm{ad}} \xrightarrow{\log_G} \mathrm{Lie}(G) \otimes (\mathbb{G}_a)_C^{\mathrm{an}} \to 0$$

here G_C^{ad} is the adic generic fiber of $\mathcal{G}$, and the first term is its torsion part. We introduce the universal cover

$$\widetilde{G_C^{\mathrm{ad}}} := \lim_p G_C^{\mathrm{ad}}$$

then the above pulls back (along the evident projection map $\widetilde{G_C^{\mathrm{ad}}} \to G_C^{\mathrm{ad}}$) to the following SES

$$0 \to \underline{T(G)[p^{-1}]} \to \widetilde{G_C^{\mathrm{ad}}} \to \mathrm{Lie}(G) \otimes (\mathbb{G}_a)_C^{\mathrm{an}} \to 0$$

here $T(G)$ is the Tate module of G. This can be found in e.g. [Scholze-Weinstein Proposition 3.4.2], and it follows that any G gives rise to a BC space $\widetilde{G_C^{\mathrm{ad}}}$)

Example 2.0.1. For $G = \mathbb{Q}_p/\mathbb{Z}_p$, the SES reads

$$0 \to \underline{\mathbb{Q}_p} \to \underline{\mathbb{Q}_p} \to 0 \to 0;$$

for $G = \mu_{p^\infty}$, this reads

$$0 \to \underline{\mathbb{Q}_p} \to \widetilde{\mu_{p^\infty,C}^{\mathrm{ad}}} \to \mathbb{G}_{a,C}^\diamond \to 0.$$

Claim 2.0.1. *Each $\widetilde{G_C^{\mathrm{ad}}}$ is a represented by a perfectoid space.*

Corollary 2.0.1. *All objects in $\mathcal{BC}$ are diamonds over $C^\flat$.*

Sketch. By playing with short exact sequences for coherent sheaves, we can express any $\mathcal{F} \in \mathrm{Coh}^{\spadesuit}(X_{\mathrm{Spa}(C^\flat)})$ as a quotient

$$0 \to \mathcal{O}_X \otimes V \to \mathcal{F}' \to \mathcal{F} \to 0$$

where V is a finite-dimensional $\mathbb{Q}_p$-vector space and $\mathcal{F}'$ is a vector bundle of slope in $[0, 1]$; by the last point of our main statement below, $\mathbb{H}^0(\mathcal{F}')$ is given by the universal cover of a p-divisible group, hence a perfectoid. □

3 Absolute BC Spaces

Notation 3.0.1. We let $*$ denote the final object in $\mathrm{Sh}_{\text{pro-etale}}(\mathrm{Perfd})$. This is the functor $\mathrm{Spd}(\mathbb{F}_p)$ classifying charateristic p untilts.

Warning 3.0.1. Recall that Perfd_p has no final object; indeed, $*$ is only an *absolute* perfectoid spaces, in the sense that $* \times S$ is a perfectoid space for any $S \in \mathrm{Perfd}_p$.

Definition 3.0.1. We define an *absolute diamond* to be a pro-etale sheaf $\mathcal{F}$ on Perfd_p such that $\mathcal{F} \times D$ is a diamond for any diamond D. Equivalently, $\mathcal{F}$ is an absolute diamond if $\mathcal{F} \to *$ is representable by a diamond.

Example 3.0.1. $\mathrm{Spd}(\mathbb{Z}_p)$ is an absolute diamond but not a diamond. To see the first claim, note that it has a non-analytic point, but perfectoids/diamonds can't have such (recall that perfectoid rings are required to be Tate). The second claim is [Berkeley 11.2.1].

From now on we fix a general $S \in \mathrm{Perfd}_p$, not necessarily an algebraically closed field. We also switch to FF curve defined over a general local field E of residue field $\mathbb{F}_p$ instead of $\mathbb{Q}_p$. We remind that for every $\lambda = r/s \in \mathbb{Q}$ (where r, s are coprimes), we have constructed a line bundle $\mathcal{O}(\lambda)$ on the adic FF curve (using isocrystals and GAGA for the FF curve). Let us introduce the notations

$$\mathcal{BC}(\mathcal{O}(\lambda)) := S \mapsto H^0(X_S, \mathcal{O}(\lambda)) \quad \mathcal{BC}(\mathcal{O}(\lambda)[1]) := S \mapsto H^1(X_S, \mathcal{O}(\lambda));$$

a priori, these are just presheaves; however,

Lemma 3.0.1. *$\mathcal{BC}(\mathcal{O}(\lambda))$ is a v- (thus pro-etale-) sheaf. If $H^0(X_S, \mathcal{O}(\lambda))$ is zero for all S, then $\mathcal{BC}(\mathcal{O}(\lambda)[1])$ is a v-sheaf.*

This requires proof and boils down to checking that the presheaf $S \mapsto \mathcal{O}_S(S)$ is a v-sheaf; this is [EtCohDiamonds, Theorem 8.7].

We can now state the main goal of today's talk:

Proposition 3.0.1 (FS II.2.5)**.** *We have:*

1. *If $\lambda < 0$, then $H^0(X_S, \mathcal{O}(\lambda)) = 0$, and $\mathcal{BC}(\mathcal{O}(\lambda)[1])$ is a nice absolute diamond;*
2. *If $\lambda > 0$, then $H^1(X_S, \mathcal{O}(\lambda)) = 0$, and $\mathcal{BC}(\mathcal{O}(\lambda))$ is a nice absolute diamond;*
3. *If $\lambda = 0$, then $\mathcal{BC}(\mathcal{O}[1])$ vanishes after pro-etale sheafification, and $\mathcal{BC}(\mathcal{O}) \simeq \underline{E}$;*
4. *If $0 < \lambda \leq [E : \mathbb{Q}_p]$, then $\mathcal{BC}(\mathcal{O}(\lambda)) \simeq \mathrm{Spd}(\overline{\mathbb{F}_p}[[x_1^{1/p^\infty}, \ldots, x_r^{1/p^\infty}]])$.*

Here $\mathcal{F}$ being a "nice absolute diamond" means the structural map $\mathcal{F} \to *$ is representable in locally spatial diamonds, partially proper, and cohomologically smooth.

A few remarks are in order:

- Note that these BC spaces are not diamonds themselves (as they also contain non-analytic points); it is true, however, the analytical locus (i.e. after removing the unique non-analytic point) will actually be a diamond (this is [FS II.3.7]);
- Note that in particular we see that $H^1(X_S, \mathcal{O}(-1)[1]) \neq 0$, contrary[2] to the case of $\mathbb{P}^1$;
- It will follow from the proof that after base changing to $\mathrm{Spa}(C)$, these absolute BC spaces become BC spaces in the sense discussed above. However, the following distinction is important: in the case $0 < \lambda \leq [E : \mathbb{Q}_p]$, for instance, the absolute BC space is a qcqs perfectoid space which is simply connected when $r > 1$, but its base change to $\mathrm{Spa}(C)$ (which is the punctured unit ball) is neither qc nor simply connected[3].

Sketch of Proof. By replacing E with a ramified extension of degree s we reduce to the case $s = 1$. The vanishing of H^1 for $\lambda > 0$ is an explicit computation using period rings and we refer readers to [FS].

The fourth point is one of the main results of [Scholze-Weinstein], which equates the Frobenius fixed point $(B^+_{\mathrm{cris}}(R^\sharp/\pi))^{\varphi^r = \pi}$ of the Dieudonne module of G with $\widetilde{G}(R^+)$ (when $S = \mathrm{Spa}(R, R^+)$ and

[2] Daniel during the talk pointed out that this sometimes goes under the slogan that the FF curve is "halfway between a projective line and an elliptic curve".

[3] Recall it is the union of $\mathrm{Spa}(C\langle T, T^n/\varpi\rangle)$.

$R^\sharp$ is an untilt). The global section of $\mathcal{O}(\lambda)$ is, by construction, given by $B_{R,[1,\infty]}^{\varphi^r=\pi}$ (we follow [FS]'s notation), and we have[4]

$$(B_{R,[1,\infty]})^{\varphi^r=\pi} \simeq (B_R^+)^{\varphi^r=\pi} \simeq (B_{\mathrm{cris}}^+(R^\sharp/\pi))^{\varphi^r=\pi}$$

(this is what [FS] calls "contracting property of Frobenius").

We have a short exact sequence (proven last time)

$$0 \to \mathcal{O}(n) \to \mathcal{O}(n+1) \to \delta_{S^\sharp} \to 0 \tag{A}$$

where $S^\sharp$ is any chosen untilt of S and $\delta_{S^\sharp}$ is the pushforward of the structure sheaf along the corresponding closed embedding. Using this we can inductively prove part (2) using closure properties of "nice absolute diamonds" proved in [EtCohDiamonds] (which we shall blackbox), starting from $\lambda = 1$ which is handled by part (4) above.

The same SES (A), in the case $n = 0$, gives the following LES

$$0 \to H^0(X_S, \mathcal{O}) \to H^0(X_S, \mathcal{O}(1)) \xrightarrow{\log_{LT}} R^\sharp \to H^1(X_S, \mathcal{O}) \to 0$$

where, converting to the p-divisible language, $\log_{LT}$ is the log map for the Lubin-Tate formal group ([FS II.2.2], we also talked about this last time). Claim (via explicit formula): pro-etale locally, we have an SES[5]

$$0 \to \underline{E}(R) \to \widetilde{LT}(R^+) \xrightarrow{\log_{LT}} R^\sharp \to 0 \tag{B}$$

which gives part (3). Finally, part (1) again follows from the SES (A) above and the results proven in [EtCohDiamonds]. □

[4] [1801.00422, Remarkque 6.6]

[5] This SES, when we use $\widetilde{LT}(R^+) \simeq (B_{\mathrm{cris}}^+)^{\varphi=\pi}$, is the *fundamental exact sequence of p-adic Hodge theory.*