LECTURE 16

In this lecture, we introduce t-structures on stable ∞-categories.

1. Definitions

1.1. Let C be a stable ∞-category. A t-structure on C is just a t-structure on the triangulated category hC.

Definition 1.2. *Let C be a stable ∞-category. A **t-structure** on C is defined to be a pair of full sub-∞-categories $(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0})$ of C such that when we write*

$$\mathsf{C}^{\le n} := \mathsf{C}^{\le 0}[-n],\ \mathsf{C}^{\ge n} := \mathsf{C}^{\ge 0}[-n]$$

we have:

(0) *Both $\mathsf{C}^{\le 0}$ and $\mathsf{C}^{\ge 0}$ are stable under isomorphisms in C.*
(1) *For $X \in \mathsf{C}^{\le 0}$ and $Y \in \mathsf{C}^{\ge 1}$, we have $\mathsf{Maps}_{\mathsf{C}}(X, Y) \simeq \{0\}$.*
(2) *We have inclusions $\mathsf{C}^{\le -1} \subseteq \mathsf{C}^{\le 0}$ and $\mathsf{C}^{\ge 1} \subseteq \mathsf{C}^{\ge 0}$.*
(3) *For any $X \in \mathsf{C}$, there exists a fiber-cofiber sequence $X' \to X \to X''$ such that $X' \in \mathsf{C}^{\le 0}$ and $X'' \in \mathsf{C}^{\ge 1}$.*

Warning 1.3. *We use the **cohomological convention**. To compare with notations in the homological convention, let*

$$\mathsf{C}_{\le n} := \mathsf{C}^{\ge -n},\ \mathsf{C}_{\ge n} := \mathsf{C}^{\le -n}$$

Exercise 1.4. *The assignment*

$$(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0}) \mapsto (\mathsf{hC}^{\le 0}, \mathsf{hC}^{\ge 0})$$

gives a bijection between t-structures on C with t-structures on the triangulated category hC.

Exercise 1.5. *The assignment*

$$(\mathsf{C}^{\le 0}, \mathsf{C}^{\ge 0}) \mapsto ((\mathsf{C}^{\ge 0})^{\mathsf{op}}, (\mathsf{C}^{\le 0})^{\mathsf{op}})$$

gives a bijection between t-structures on C with t-structures on C^{op}.

Exercise 1.6. *Show that for $m < n$, $\mathsf{C}^{\le m} \cap \mathsf{C}^{\ge n} \simeq \{0\}$.*

Lemma 1.7. *Let $n \ge 0$. For any $X \in \mathsf{C}^{\le 0}$ and $Y \in \mathsf{C}^{\ge -n}$, the mapping space $\mathsf{Maps}_{\mathsf{C}}(X, Y)$ is a homotopy n-type.*

Sketch. First note that the connected components of $\mathsf{Maps}_{\mathsf{C}}(X, Y)$ are weakly homotopy equivalent to each other because it is the loop space of $\mathsf{Maps}_{\mathsf{C}}(X, \Sigma Y)$. Hence we only need to show $\pi_m \mathsf{Maps}_{\mathsf{C}}(X, Y) \simeq 0$ for $m > n$ and the base point $0 \in \mathsf{Maps}_{\mathsf{C}}(X, Y)$. This follows from observation that $\Omega^{n+1}\mathsf{Maps}_{\mathsf{C}}(X, Y) \simeq \mathsf{Maps}_{\mathsf{C}}(X, \Omega^{n+1} Y)$ is weakly contractible because $\Omega^{n+1} Y \in \mathsf{C}^{\ge 1}$. $\square$

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Definition 1.8. *Let C be a stable ∞-category equipped with a t-structure. The* ***heart*** *of this t-structure is defined to be*

$$\mathsf{C}^{\heartsuit} := \mathsf{C}^{\le 0} \cap \mathsf{C}^{\ge 0}.$$

Theorem 1.9. *The ∞-category $\mathsf{C}^{\heartsuit}$ is an ordinary abelian category. Moreover, $0 \to X'' \to X \to X' \to 0$ is a short exact sequence in $\mathsf{C}^{\heartsuit}$ iff $X'' \to X \to X'$ is a fiber-cofiber sequence of C contained in $\mathsf{C}^{\heartsuit}$.*

Sketch. By Lemma 1.7, $\mathsf{C}^{\heartsuit}$ is an ordinary category. It follows it can be identified with $\mathsf{hC}^{\le 0} \cap \mathsf{hC}^{\ge 0}$, which is the heart of the triangulated category hC. Now the claims follow from the corresponding well-known claims for triangulated categories. □

Warning 1.10. *A t-structure is not determined by its heart. Most information in a stable ∞-category cannot be recovered from the heart. For instance, for $X, Y \in \mathsf{C}^{\heartsuit}$, the groups $\mathsf{Ext}^{\bullet}_{\mathsf{C}}(X,Y)$ and $\mathsf{Ext}^{\bullet}_{\mathsf{C}^{\heartsuit}}(X,Y)$ are not isomorphic in general.*

Definition 1.11. *Let C and D be stable ∞-categories equipped with t-structures. For an exact functor $F : \mathsf{C} \to \mathsf{D}$, we say*

- *the functor F is* ***left t-exact*** *if it sends $\mathsf{C}^{\ge 0}$ into $\mathsf{D}^{\ge 0}$*
- *the functor F is* ***right t-exact*** *if it sends $\mathsf{C}^{\le 0}$ into $\mathsf{D}^{\le 0}$*
- *the functor F is* ***t-exact*** *if it is both left t-exact and right t-exact.*

Exercise 1.12. *Show that the left adjoint of a left t-exact functor is right t-exact.*

Exercise 1.13. *Construct an exact functor $F : \mathsf{C} \to \mathsf{D}$ that preserves the hearts but is not t-exact.*

2. Cohomologies

Proposition-Definition 2.1. *Let C be a stable ∞-category equipped with a t-structure. For any n, the inclusion functor $\mathsf{C}^{\le n} \to \mathsf{C}$ admits a right adjoint $\tau^{\le n}$, and the inclusion functor $\mathsf{C}^{\ge n} \to \mathsf{C}$ admits a left adjoint $\tau^{\ge n}$. These functors are called the* ***truncation functors*** *for the t-structure.*

Remark 2.2. *One can memorize the above handedness as $\mathsf{C}^{\le 0} \rightleftarrows \mathsf{C} \rightleftarrows \mathsf{C}^{\ge 1}$.*

Exercise 2.3. *Prove the above proposition by verifying the following claim. Let $X \in \mathsf{C}$ and $X' \to X \to X''$ be any fiber-cofiber sequence with $X' \in \mathsf{C}^{\le 0}$ and $X'' \in \mathsf{C}^{\ge 1}$, then*

1. *For any $Y \in \mathsf{C}^{\le 0}$, the morphism $X' \to X$ induces equivalences*
$$\mathsf{Maps}_{\mathsf{C}}(Y, X') \xrightarrow{\simeq} \mathsf{Maps}_{\mathsf{C}}(Y, X).$$
2. *For any $Z \in \mathsf{C}^{\ge 1}$, the morphism $X \to X''$ induces equivalences*
$$\mathsf{Maps}_{\mathsf{C}}(X, Z) \xrightarrow{\simeq} \mathsf{Maps}_{\mathsf{C}}(X'', Z).$$

Deduce that $X' \simeq \tau^{\le 0} X$ and $X'' \simeq \tau^{\ge 1} X$.

Exercise 2.4. *We have Cartesian squares*

$$\begin{tikzcd} \mathsf{C}^{\le 0} \arrow[r, "\subseteq"] \arrow[d] & \mathsf{C} \arrow[d, "\tau^{\ge 1}"] \\ \{0\} \arrow[r, "\subseteq"] & \mathsf{C}^{\ge 1} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}^{\ge 0} \arrow[r, "\subseteq"] \arrow[d] & \mathsf{C} \arrow[d, "\tau^{\le -1}"] \\ \{0\} \arrow[r, "\subseteq"] & \mathsf{C}^{\le -1} \end{tikzcd}$$

In particular, $\mathsf{C}^{\le 0}$ and $\mathsf{C}^{\ge 0}$ determine each other.

Exercise 2.5. *The full sub-∞-category $\mathsf{C}^{\le 0} \subseteq \mathsf{C}$ is stable under colimits, while $\mathsf{C}^{\le 0} \subseteq \mathsf{C}$ is stable under limits.*

Exercise 2.6. *A limit in $\mathsf{C}^{\le 0}$ is isomorphic to the $\tau^{\le 0}$-truncation of the corresponding limit in C. Dually, a colimit in $\mathsf{C}^{\ge 0}$ is isomorphic to the $\tau^{\ge 0}$-truncation of the corresponding colimit in C.*

2.7. The following exercises say that for a fiber-cofiber sequence $X' \to X \to X''$, *amplitude estimations* of two terms give an estimation for the third one.

Exercise 2.8. *Let $X' \to X \to X''$ be a fiber-cofiber sequence.*

1. *If $X', X'' \in \mathsf{C}^{\le 0}$, then $X \in \mathsf{C}^{\le 0}$.*
2. *If $X \in \mathsf{C}^{\le 0}$ and $X'' \in \mathsf{C}^{\le -1}$, then $X' \in \mathsf{C}^{\le 0}$.*
3. *If $X \in \mathsf{C}^{\le 0}$ and $X' \in \mathsf{C}^{\le 1}$, then $X'' \in \mathsf{C}^{\le 0}$.*

In particular, the full sub-∞-categories $\mathsf{C}^{\le 0} \subseteq \mathsf{C} \supseteq \mathsf{C}^{\ge 0}$ are stable under extensions.

Exercise 2.9. *Give examples to show the above estimations are optimal.*

Exercise 2.10. *The truncation functors $\tau_{\le\bullet}$ and $\tau_{\ge\bullet}$ commute with each other.*

Remark 2.11. *The precise meaning of the above exercise consists of the following. For m and n, we have*

- $\tau_{\ge m} \circ \tau_{\ge n} \simeq \tau_{\ge \max\{m,n\}}, \ \ \tau_{\le m} \circ \tau_{\le n} \simeq \tau_{\le \max\{m,n\}}.$
- *The commutative square*

$$\begin{tikzcd} \mathsf{C}^{\le m} \cap \mathsf{C}^{\ge n} \arrow[r, "\subseteq"'] \arrow[d, "\subseteq"] & \mathsf{C}^{\le m} \arrow[d, "\subseteq"] \\ \mathsf{C}^{\ge n} \arrow[r, "\subseteq"'] & \mathsf{C} \end{tikzcd}$$

is left adjointable along the horizontal direction, and the induced commutative square

$$\begin{tikzcd} \mathsf{C}^{\le m} \cap \mathsf{C}^{\ge n} \arrow[d, "\subseteq"] & \mathsf{C}^{\le m} \arrow[l, "\tau^{\ge n}"'] \arrow[d, "\subseteq"] \\ \mathsf{C}^{\ge n} & \mathsf{C} \arrow[l, "\tau^{\ge n}"'] \end{tikzcd}$$

is right adjointable along the vertical direction, i.e., induces a commutative square

$$\begin{tikzcd} \mathsf{C}^{\le m} \cap \mathsf{C}^{\ge n} & \mathsf{C}^{\le m} \arrow[l, "\tau^{\ge n}"'] \\ \mathsf{C}^{\ge n} \arrow[u, "\tau^{\le m}"] & \mathsf{C} \arrow[l, "\tau^{\ge n}"'] \arrow[u, "\tau^{\le m}"'] \end{tikzcd}$$

Definition 2.12. *Let C be a stable ∞-category equipped with a t-structure. Consider the functor*

$$\mathsf{H}^n : \mathsf{C} \xrightarrow{\tau^{\le n} \circ \tau^{\ge n}} \mathsf{C}^{\le n} \cap \mathsf{C}^{\ge n} \xrightarrow{[-n]} \mathsf{C}^{\heartsuit}.$$

For $X \in \mathsf{C}$, we call $\mathsf{H}^n(X) \in \mathsf{C}^{\heartsuit}$ the ***n-th cohomology of*** *X.*

Warning 2.13. *It may happen that $\mathsf{H}^n(X) \simeq 0$ while X is not isomorphic to 0. For instance, $\mathsf{C}^{\le 0} := \mathsf{C}$ and $\mathsf{C}^{\ge 0} := \{0\}$ give a t-structure with $\mathsf{C}^{\heartsuit} \simeq \{0\}$. There are lots of interesting examples in*

- non-regular *algebraic geometry*
- infinite type *algebraic geometry*
- infinite dimensional *representation theory*
- *...*

2.14. The following result follows from the corresponding well-known result for triangulated categories.

Proposition 2.15. *Let* C *be a stable* ∞*-category equipped with a t-structure. For any fiber-cofiber sequence* $X' \to X \to X''$*, we have a long exact sequence in* $\mathsf{C}^\heartsuit$

$$\cdots \to \mathsf{H}^n(X') \to \mathsf{H}^n(X) \to \mathsf{H}^n(X'') \xrightarrow{\delta} \mathsf{H}^{n+1}(X') \to \cdots,$$

where the connecting morphism δ *is induced by the morphism* $X'' \to X'[1]$.

2.16. The following result relates t-exact functors with exact functors between the hearts.

Exercise 2.17. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a left t-exact functor. Show that the composition*

$$\mathsf{H}^0 F : \mathsf{C}^\heartsuit \to \mathsf{C}^{\geq 0} \xrightarrow{F} \mathsf{D}^{\geq 0} \xrightarrow{\tau^{\leq 0}} \mathsf{D}^\heartsuit$$

is left exact.

Warning 2.18. *As in Warning 1.10, even a t-exact functor* F *cannot be recovered from* $\mathsf{H}^0 F$.

Remark 2.19. *Next time, we will define various versions of* derived ∞-categories *of an abelian categories* A*, which are equipped with t-structures whose hearts can be identified with* A*. By definition, all information about these derived* ∞*-categories can be recovered from* A.

Under certain assumptions, a left/right t-exact functor F *out of these derived* ∞*-categories can be identified with the right/left derived functor of* $\mathsf{H}^0 F$.

3. Bounded, separated and complete

Definition 3.1. *Let* C *be a stable* ∞*-category equipped with a t-structure. For an object* $X \in \mathsf{C}$,

- *we say* X *is* ***connective*** *if* $X \in \mathsf{C}^{\leq 0}$;
- *we say* X *is* ***coconnective*** *if* $X \in \mathsf{C}^{\geq 0}$;
- *we say* X *is* n***-connective*** *if* $X \in \mathsf{C}^{\leq -n}$;
- *we say* X *is* n***-coconnective*** *if* $X \in \mathsf{C}^{\geq -n}$;
- *we say* X *is* ***eventually connective****, or* ***right bounded*** *if*
$$X \in \mathsf{C}^- := \bigcup \mathsf{C}^{\leq n}$$
- *we say* X *is* ***eventually coconnective****, or* ***left bounded*** *if*
$$X \in \mathsf{C}^+ := \bigcup \mathsf{C}^{\geq n}$$
- *we say* X *is* ***bounded*** *if* X *is both left bounded and right bounded:*
$$X \in \mathsf{C}^{\mathsf{b}} := \mathsf{C}^+ \bigcap \mathsf{C}^-.$$

Warning 3.2. *As in Warning 2.13, the above properties cannot be tested via the cohomologies.*

Exercise 3.3. *Show that if* X *is left bounded, then* $X \in \mathsf{C}^{\geq 0}$ *iff* $\mathsf{H}^i(X) \simeq 0$ *for* $i < 0$.

Definition 3.4. *Let C be a stable ∞-category equipped with a t-structure.*

- *We say C is* ***right bounded*** *if $\mathsf{C} \simeq \mathsf{C}^-$.*
- *We say C is* ***left bounded*** *if $\mathsf{C} \simeq \mathsf{C}^+$.*
- *We say C is* ***bounded*** *if $\mathsf{C} \simeq \mathsf{C}^{\mathsf{b}}$.*
- *We say C is* ***right separated*** *if*

$$\mathsf{C}^{\geq\infty} := \bigcap_n \mathsf{C}^{\geq n} \simeq \{0\}.$$

- *We say C is* ***left separated*** *if*

$$\mathsf{C}^{\leq-\infty} := \bigcap_n \mathsf{C}^{\leq n} \simeq \{0\}.$$

Definition 3.5. *Let C be a stable ∞-category equipped with a t-structure. The* ***left completion*** *$\widehat{\mathsf{C}}$ of C is defined to be the limit of the following diagram*

$$\cdots \xrightarrow{\tau} \mathsf{C}^{\geq-2} \xrightarrow{\tau} \mathsf{C}^{\geq-1} \xrightarrow{\tau} \mathsf{C}^{\geq0} \xrightarrow{\tau} \cdots.$$

We say C is ***left complete*** *if the functor $\mathsf{C} \to \widehat{\mathsf{C}}$ is an equivalence.*

Dually, the ***right completion*** *is defined to be the limit of*

$$\cdots \xrightarrow{\tau} \mathsf{C}^{\leq2} \xrightarrow{\tau} \mathsf{C}^{\leq1} \xrightarrow{\tau} \mathsf{C}^{\leq0} \xrightarrow{\tau} \cdots.$$

Exercise 3.6. *Show that the right completion of C is equivalent to $\mathsf{Sptr}(\mathsf{C}^{\leq0})$.*

Exercise 3.7. *Show that the left completion $\widehat{\mathsf{C}}$ is stable and the functor $\mathsf{C} \to \widehat{\mathsf{C}}$ is exact.*

Exercise 3.8. *Show that the left completion $\widehat{\mathsf{C}}$ admits an essential unique t-structure such that $\mathsf{C}^{\geq0} \xrightarrow{\simeq} (\widehat{\mathsf{C}})^{\geq0}$.*

Exercise 3.9. *Show that C is left separated iff the functor $\mathsf{C} \to \widehat{\mathsf{C}}$ detects zero objects. In particular, C is left separated if it is left complete.*

Warning 3.10. *A left separated t-structure may not be left complete. For example, any left bounded t-structure is left separated, but is almost never left complete.*

Exercise 3.11. *Suppose C admits countable products and $\mathsf{C}^{\leq0} \subseteq \mathsf{C}$ is stable under countable products*[1]*. Then C is left separated iff it is left complete.*

Exercise 3.12. *Show that C is left complete iff it is* ***Postnikov complete****. The latter means*

- *Every object $X \in \mathsf{C}$ is the limit of its* ***Postnikov tower****:*

$$X \simeq \lim\left[\cdots \to \tau^{\geq n-1}X \to \tau^{\geq n}X \to \tau^{\geq n+1}X \to \cdots\right]$$

- *Any* ***Postnikov tower*** *in C converges. In other words, any collection $X_n \in \mathsf{C}^{\geq n}$ equipped with isomorphisms $\tau^{\geq n}X_{n-1} \xrightarrow{\simeq} X_n$ is the Postnikov tower of*

$$X := \lim\left[\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots\right] \in \mathsf{C}$$

[1]These conditions are often referred as: taking countable products is right t-exact.

4. t-structures on presentable stable ∞-categories

Proposition-Definition 4.1 (HA.1.4.4.13). *Let C be a presentable stable ∞-category equipped with a t-structure. The following conditions are equivalent:*

- *The ∞-category $\mathsf{C}^{\leq 0}$ is presentable.*
- *The ∞-category $\mathsf{C}^{\leq 0}$ is accessible.*
- *The ∞-category $\mathsf{C}^{\geq 0}$ is presentable.*
- *The ∞-category $\mathsf{C}^{\geq 0}$ is accessible.*
- *The composition $\mathsf{C} \xrightarrow{\tau_{\leq 0}} \mathsf{C}^{\leq 0} \to \mathsf{C}$ is accessible.*
- *The composition $\mathsf{C} \xrightarrow{\tau_{\geq 0}} \mathsf{C}^{\geq 0} \to \mathsf{C}$ is accessible.*

*We say a t-structure on C is **accessible** if it satisfies the above conditions.*

Proposition 4.2 (HA.1.4.4.11). *Let C be a presentable stable ∞-category. A full sub-∞-category $\mathsf{C}' \subseteq \mathsf{C}$ determines an accessible t-structure with $\mathsf{C}^{\leq 0} \simeq \mathsf{C}'$ iff*

- *C' is accessible*
- *$\mathsf{C}' \subseteq \mathsf{C}$ is closed under colimits and extensions.*

Remark 4.3. *In practice, one can apply the above proposition to the smallest full sub-∞-category C' generated by a* small *collection of objects under small colimits and extensions. Such C' is always accessible.*

Definition 4.4. *Let C be a presentable stable ∞-category. We say a t-structure on C is **compatible with filtered colimits** if taking filtered colimits is t-exact.*

Exercise 4.5. *Let C be a presentable stable ∞-category equipped with a t-structure. Show that the following are equivalent:*

- *The t-structure is compatible with filtered colimits.*
- *The embedding $\mathsf{C}^{\geq 0} \subseteq \mathsf{C}$ is stable under filtered colimits.*
- *$\tau^{\leq 0}$ preserves filtered colimits.*

Remark 4.6. *Note that $\tau^{\geq 0}$ and $\mathsf{C}^{\leq 0} \subseteq \mathsf{C}$* always *preserve any colimits because they admit right adjoints.*

Definition 4.7. *Let C be a presentable stable ∞-category. We say a t-structure on C is **compactly generated** if $\mathsf{C}^{\leq 0}$ is compactly generated, and the functor $\mathsf{C}^{\leq 0} \to \mathsf{C}$ preserves compact objects.*

Exercise 4.8. *Show that any compactly generated t-structure is accessible and compatible with filtered colimits.*

Exercise 4.9. *Let C be a presentable stable ∞-category equipped with a* right complete *t-structure. Show that the following is equivalent:*

- *The t-structure is compactly generated.*
- *The ∞-category $\mathsf{C}^{\leq 0}$ is compactly generated.*

Moreover, under these conditions, C is compactly generated, and compact objects in C are given by $Y[n]$ for some compact object Y in $\mathsf{C}^{\leq 0}$ and integer n.

Exercise 4.10. *Let C be a compactly generated stable ∞-category equipped with a t-structure, such that compact objects in C are right bounded and closed under truncations. Show that the t-structure is compactly generated and right complete.*

Exercise 4.11. *Let C_0 be a small stable ∞-category equipped with a bounded t-structure. Show that $\mathsf{C} := \mathsf{Ind}(\mathsf{C}_0)$ has a compactly generated t-structure with $\mathsf{C}^{\leq 0} := \mathsf{Ind}(\mathsf{C}_0^{\leq 0})$.*

Warning 4.12. *Compact objects in C may not be closed under truncations. Hence not all compactly generated t-structures come from ind-completion. There are lots of interesting examples in the settings listed in Warning 2.13.*

Remark 4.13. *In practice, most t-structures on presentable stable ∞-categories are right complete and compatible with filtered colimits.*

5. t-structure on Sptr

Proposition-Construction 5.1 (HA.1.4.3.4)**.** *Let C be a pointed presentable ∞-category. We have an accessible t-structure on $\mathsf{Sptr}(\mathsf{C})$ given by the following.*

- *Let $\mathsf{Sptr}(\mathsf{C})^{\geq 1}$ be the full sub-∞-category of objects X such that $\Omega^\infty(X) \simeq 0$.*
- *Let $\mathsf{Sptr}(\mathsf{C})^{\leq 0}$ be the full sub-∞-category generated by the essential image of the functor Σ^∞ under extensions and small colimits.*

Proposition 5.2 (HA.1.4.3.6)**.** *Let Sptr be equipped with the above t-structure. Then*

(1) *This t-structure is compactly generated[2], left complete and right complete.*
(2) *The Eilenberg–Maclane spectrum functor $A \mapsto \mathbb{H}A$ gives an equivalence $\mathsf{Ab} \xrightarrow{\simeq} \mathsf{Sptr}^\heartsuit$.*

Exercise 5.3. *Show that the cohomology functor $\mathsf{H}^i : \mathsf{Sptr} \to \mathsf{Sptr}^\heartsuit$ can be identified with π_{-i} via the equivalence $\mathsf{Ab} \xrightarrow{\simeq} \mathsf{Sptr}^\heartsuit$.*

Warning 5.4. *Let A_1 and A_2 be abelian groups. In general, the extension groups $\mathsf{Ext}^i_{\mathsf{Sptr}}(\mathbb{H}A_1, \mathbb{H}A_2)$ and $\mathsf{Ext}^i(A_1, A_2)$ are not isomorphic. For instance, the graded ring $\mathsf{Ext}^\bullet_{\mathsf{Sptr}}(\mathbb{H}\mathbb{F}_q, \mathbb{H}\mathbb{F}_q)$ is the **mod p Steenrod algebra**.*

5.5. Next time, we will construct a natural homomorphism

$$\mathsf{Ext}^i(A_1, A_2) \to \mathsf{Ext}^i_{\mathsf{Sptr}}(\mathbb{H}A_1, \mathbb{H}A_2)$$

which should be viewed as derived direct images along $\mathsf{Spec}\,\mathbb{Z} \to \mathsf{Spec}\,\mathbb{S}$.

Appendix A. Prestable ∞-categories

A.1. Let C be a stable ∞-category equipped with a t-structure. Sometimes it is more convenient to study $\mathsf{C}^{\leq 0}$ rather then C. It is possible to find several axioms that characterize ∞-categories of the form $\mathsf{C}^{\leq 0}$. Such ∞-categories are called **prestable ∞-categories**.

A.2. **Suggested readings.** SAG.C.

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[2]Compact generation is not proved in HA, but it follows from the fact that Spc_* is compactly generated.