

LECTURE 7

1. WHEN IS $[M_\lambda : L_\mu] \neq 0$?

We start with the following basis observation.

Lemma 1. *Let $\lambda \in \mathfrak{t}^*$. If $[M_\lambda : L_\mu] \neq 0$, then $\mu \in W \cdot \lambda$ and $\mu \leq \lambda$.*

Proof. The modules M_λ and L_μ should be contained in the same block; the weights of L_μ should be a subset of those of M_λ . □

Recall in [Lem. 4, Lect. 5], we proved that for a weight $\lambda \in \mathfrak{t}^*$ and a *simple* root $\alpha \in \Delta$, if $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0}$, then $M_{s_\alpha \cdot \lambda} \subset M_\lambda$. Note that this condition is equivalent to $s_\alpha \cdot \lambda \leq \lambda$. It follows that we have a partial inverse of Lemma 1:

Lemma 2. *Let $\lambda \in \mathfrak{t}^*$ be a weight and $\alpha \in \Delta$ be a simple root. If $s_\alpha \cdot \lambda \leq \lambda$, then $M_{s_\alpha \cdot \lambda} \subset M_\lambda$ and in particular, $[M_\lambda : L_{s_\alpha \cdot \lambda}] \neq 0$.*

It turns out this result remains true for *any* positive root $\alpha \in \Phi^+$.

Theorem 3 (Verma). *Let $\lambda \in \mathfrak{t}^*$ be a weight and $\alpha \in \Phi^+$ be any positive¹ root. If $s_\alpha \cdot \lambda \leq \lambda$, then $M_{s_\alpha \cdot \lambda} \subset M_\lambda$ and in particular, $[M_\lambda : L_{s_\alpha \cdot \lambda}] \neq 0$.*

Remark 4. As indicated by the \mathfrak{sl}_3 -case, the proof of Verma's theorem for general roots should be more involved than the proof for simple roots. Unfortunately we do not have enough time to discuss this proof. We refer the interested readers to [H, Sect. 4.5-4.7].

Verma's theorem gives a sufficient condition for $[M_\lambda : L_\mu] \neq 0$. Namely, if there is a chain

$$(1.1) \quad \mu \leq s_{\alpha_1} \cdot \mu \leq (s_{\alpha_2} s_{\alpha_1}) \cdot \mu \leq \cdots \leq (s_{\alpha_n} \cdots s_{\alpha_2} s_{\alpha_1}) \cdot \mu = \lambda,$$

with $s_{\alpha_i} \in \Phi^+$, then

$$M_\mu \subset M_{s_{\alpha_1} \cdot \mu} \subset \cdots \subset M_\lambda,$$

and in particular $[M_\lambda : L_\mu] \neq 0$. It turns out this condition is also necessary.

Definition 5. Let $\lambda, \mu \in \mathfrak{t}^*$. We write $\mu \leq_c \lambda$ ² if there exists a chain like (1.1).

Remark 6. It is easy to see (\mathfrak{t}^*, \leq_c) is a partially ordered set. Moreover, if μ and λ are \leq_c -comparable, then $\mu \in W \cdot \lambda$.

Theorem 7 (BGG). *Let $\lambda, \mu \in \mathfrak{t}^*$. If $[M_\lambda : L_\mu] \neq 0$, then $\mu \leq_c \lambda$.*

Remark 8. Unfortunately we do not have enough time to discuss the algebraic proofs of the BGG theorem. We refer the interested readers to [H, Sect. 5] for a proof using Jantzen sum formula. However, we will provide a geometric proof in future lectures, at least when the stabilizer of W_\bullet at λ is trivial.

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¹Since $s_\alpha = s_{-\alpha}$, the statement is true for any root. However, the equivalence $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^{\geq 0} \Leftrightarrow s_\alpha \cdot \lambda \leq \lambda$ needs α to be positive.

²This notation is not standard. Some authors prefer to denote the usual partial order on \mathfrak{t}^* (defined by positive roots) by \leq (rather than \preceq) and leave symbol \preceq for the partial order in this definition. But it is hard to distinguish these notations on blackboards.

Corollary 9 (Verma, BGG). *For $\lambda, \mu \in \mathfrak{t}^*$, the following conditions are equivalent:*

- (i) $\mu \leq_c \lambda$;
- (ii) $M_\mu \subset M_\lambda$;
- (iii) $[M_\lambda : L_\mu] \neq 0$.

Our main goal for this lecture is to explain the content of the above theorems.

Warning 10. *Note that $\mu \leq_c \lambda$ implies $\mu \leq \lambda$ and $\mu \in W \cdot \lambda$. The contrary happens to be true for \mathfrak{sl}_2 and \mathfrak{sl}_3 , but is false already for \mathfrak{sl}_4 . This can be seen via Example 53 below.*

$$2. \dim \operatorname{Hom}_{\mathcal{O}}(M_\mu, M_\lambda) \leq 1$$

Let me first explain why I wrote $M_\mu \subset M_\lambda$ rather than “ M_λ contains M_μ as a submodule”. Such notation would not be appropriate unless such embedding is unique if exists. This is indeed the case.

Proposition 11. *For $\lambda, \mu \in \mathfrak{t}^*$, we have*

- (1) *Any nonzero morphism $\phi : M_\mu \rightarrow M_\lambda$ is injective;*
- (2) $\dim \operatorname{Hom}_{\mathcal{O}}(M_\mu, M_\lambda) \leq 1$.

To prove the proposition, we need the following result:

Proposition 12. *For $\lambda \in \mathfrak{t}^*$, the Verma module M_λ contains a unique irreducible submodule.*

Proof. Suppose M_λ has two distinct irreducible submodules L and L' . We must have $L \cap L' = 0$. Recall $M_\lambda \simeq U(\mathfrak{n}^-)$ is \mathfrak{n}^- -modules. Hence these submodules correspond to nonzero left ideals of $R = U(\mathfrak{n}^-)$. However, this contradicts with the following elementary lemma.

Lemma 13. *Let R be a left Noetherian ring that contains no zero-divisor. Then any nonzero left ideals have nontrivial intersection.*

Proof. We can assume the two ideals are I and Rx for some nonzero element $x \in R$. Suppose $I \cap Rx = 0$. Consider $I_n := I + Ix + \dots + Ix^n$. Using induction, it is easy to show $I_{n-1} \subset I_n$ but $I_{n-1} \neq I_n$. This contradicts with the assumption that R is left Noetherian. □

□[Proposition 12]

Example 14. For $\mathfrak{g} = \mathfrak{sl}_2$, the unique irreducible submodule of M_l is isomorphic to M_{-l-2} if $l \in \mathbb{Z}^{\geq 0}$, and is M_l itself otherwise.

Proof of Proposition 11. (1) follows from the facts that $U(\mathfrak{n}^-)$ has no zero-divisor. Let $\phi_1, \phi_2 : M_\mu \rightarrow M_\lambda$ be two nonzero morphisms. We only need to show $\phi_1 = c\phi_2$ for some scalar. By (1), both morphisms are injective. Let L be the unique irreducible submodule of M_μ . We obtain two injective morphisms $L \rightarrow M_\mu \xrightarrow{\phi_i} M_\lambda$. Since L is irreducible, the image of each morphism must be the unique irreducible submodule L' of M_λ . Recall $\dim \operatorname{Hom}_{\mathcal{O}}(L, L') \leq 1$ ([Lem. 29, Lect. 6]). Hence there exists a scalar c such that $\phi_1|_L = c\phi_2|_L$. But this implies $\phi_1 = c\phi_2$ because otherwise $\phi_1 - c\phi_2$ would be injective by (1). □

□[Proposition 11]

Combining the two propositions, we actually have:

Corollary 15. *For $\lambda \in \mathfrak{t}^*$, the unique irreducible submodule of M_λ is also a Verma module.*

Proof. Let $L_\mu \hookrightarrow M_\lambda$ be the unique irreducible submodule. The composition $M_\mu \twoheadrightarrow L_\mu \hookrightarrow M_\lambda$ must be injective by Proposition 11(1). It follows that $M_\mu \simeq L_\mu$. □

3. DOMINANT AND ANTIDOMINANT WEIGHTS

The following combinatorial results characterize the maximal and minimal elements for the poset (\mathfrak{t}^*, \leq_c) .

Proposition-Definition 16. *For $\lambda \in \mathfrak{t}^*$, the following conditions are equivalent:*

- (i) λ is a maximal element with respect to the partial order \leq_c ;
- (ii) For any positive root $\alpha \in \Phi^+$, $\langle \lambda + \rho, \check{\alpha} \rangle \notin \mathbb{Z}^{<0}$.
- (iii) For any positive root $\alpha \in \Phi^+$, $\lambda \not\prec s_\alpha \cdot \lambda$;
- (iv) For any $w \in W$, $\lambda \not\prec w \cdot \lambda$.

We say λ is **dot-dominant**³ if it satisfies the above conditions.

It is easy to see (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftarrow (iv). For a complete proof, see [H, Sect. 3.4-3.5]. Also see [G, Sect. 3.7] for a quick proof when λ is either integral or generic.

Dually, we have

Proposition-Definition 17. *For $\lambda \in \mathfrak{t}^*$, the following conditions are equivalent:*

- (i) λ is a minimal element with respect to the partial order \leq_c ;
- (ii) For any positive root $\alpha \in \Phi^+$, $\langle \lambda + \rho, \check{\alpha} \rangle \notin \mathbb{Z}^{>0}$.
- (iii) For any positive root $\alpha \in \Phi^+$, $\lambda \not\prec s_\alpha \cdot \lambda$;
- (iv) For any $w \in W$, $\lambda \not\prec w \cdot \lambda$.

We say λ is **dot-antidominant** if it satisfies the above conditions.

Example 18. For $\mathfrak{g} = \mathfrak{sl}_2$ and the coordinate $l = \langle \lambda, \check{\alpha} \rangle$, λ is dot-dominant iff $l \neq -2, -3, \dots$, while it is dot-antidominant iff $l \neq 0, 1, \dots$.

Example 19. For $\mathfrak{g} = \mathfrak{sl}_3$ and the positive simple roots α_1, α_2 , a weight $\lambda = c_1\alpha_1 + c_2\alpha_2$ is dot-dominant iff $2c_1 - c_2, 2c_2 - c_1 \neq -2, -3, \dots$, while it is dot-antidominant iff $2c_1 - c_2, 2c_2 - c_1 \neq 0, 1, \dots$.

Note that the Verma–BGG theorem implies the following result, which we can prove directly.

Proposition 20. *If $\lambda \in \mathfrak{t}^*$ is dot-antidominant, then M_λ is irreducible.*

Proof. By Lemma 1, if $[M_\lambda : L_\mu] \neq 0$, then $\mu = w \cdot \lambda \leq \lambda$ for some $w \in W$. Since λ is dot-antidominant, we have $w \cdot \lambda \not\prec \lambda$. Hence we must have $w \cdot \lambda = \lambda$. In this case, we have $[M_\lambda : L_\lambda] = 1$ by considering the highest weight subspaces. □

Dot-dominant Verma modules also have categorical meaning:

Proposition 21. *If $\lambda \in \mathfrak{t}^*$ is dot-dominant, then M_λ is projective, i.e., $\text{Hom}_{\mathcal{O}}(M_\lambda, -)$ is exact.*

Proof. We only need to prove M_λ is a projective object in the block $\mathcal{O}_{\varpi(\lambda)}$. This follows from the following calculation:

Lemma 22. *If $\lambda \in \mathfrak{t}^*$ is dot-dominant, then $\text{Hom}_{\mathcal{O}_{\varpi(\lambda)}}(M_\lambda, N) \rightarrow N_{\text{wt}=\lambda}$ is an isomorphism.*

Proof. The map in question is the composition

$$\text{Hom}_{\mathcal{O}_{\varpi(\lambda)}}(M_\lambda, N) \simeq \text{Hom}_{\mathfrak{b}}(k_\lambda, N) \rightarrow \text{Hom}_{\mathfrak{t}}(k_\lambda, N) \simeq N_{\text{wt}=\lambda}.$$

Hence we only need to show any \mathfrak{t} -linear map $k_\lambda \rightarrow N$ is \mathfrak{b} -linear, i.e., any λ -weight vector v of N is annihilated by \mathfrak{n} . By the linkage-principle, any weight μ of N satisfies $\mu \leq w \cdot \lambda$ for some

³This terminology is also not standard. Some authors (including [H]) just say λ is *dominant*, while others (including BGG and [G]) say $\lambda + \rho$ is *dominant*. To avoid ambiguity, many people advocate to adopt the second convention and use the equivalent words “ λ is dot-dominant”.

$w \in W$. Since λ is dot-dominant, we have $w \cdot \lambda \not\succeq \lambda$ and therefore $\mu \not\succeq \lambda$. This implies $\mathbf{n} \cdot v = 0$ because weights of $\mathbf{n} \cdot v$ would be higher than $\lambda = \mathbf{wt}(v)$.

□

□[Proposition 21]

4. STRUCTURE OF \mathcal{O}_χ

The blocks $\mathcal{O}_{\varpi(\lambda)}$ behave differently when λ varies. To describe this phenomenon, we need to review some conditions on λ .

Definition 23. For $\lambda \in \mathfrak{t}^*$, we say:

- λ is **integral** if $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ for any $\alpha \in \Phi$.
- λ is **dot-regular** if $w \cdot \lambda \neq \lambda$ for any $w \neq \text{Id}$.
- λ is **dot-singular** if it is not dot-regular.
- λ is **generic** if $\langle \lambda, \check{\alpha} \rangle \notin \mathbb{Z}$ for any $\alpha \in \Phi$.

Example 24. For $\mathfrak{g} = \mathfrak{sl}_2$ and the coordinate $l = \langle \lambda, \check{\alpha} \rangle$, λ is integral iff l is an integer; it is dot-regular iff $l \neq -1$; it is generic iff l is not an integer.

Example 25. For $\mathfrak{g} = \mathfrak{sl}_3$ and the positive simple roots α_1, α_2 , a weight $\lambda = c_1\alpha_1 + c_2\alpha_2$ is integral iff $2c_1 - c_2$ and $2c_2 - c_1$ are integers; it is dot-regular iff $2c_1 - c_2$ and $2c_2 - c_1$ are nonzero; it is generic iff $2c_1 - c_2$ and $2c_2 - c_1$ are not integers.

Remark 26. By the theory of Weyl chambers, a weight λ is dot-regular iff $s_\alpha \cdot \lambda \neq \lambda$ for any $\alpha \in \Phi$, which is equivalent to $\langle \lambda, \check{\alpha} \rangle \neq 0$. In particular any generic weight is dot-regular.

Remark 27. One can show $\lambda \in \mathfrak{t}^*$ is dot-regular iff the morphism $\varpi : \mathfrak{t}^* \rightarrow \text{Spec}(Z(\mathfrak{g})) \simeq \mathfrak{t}^* // W$ is regular (=smooth) at the point λ .

Remark 28. Note that for any $w \in W$, λ is integral (resp. dot-regular, dot-singular, generic) iff $w(\lambda)$ is so. Indeed, this follows from $\langle \lambda, \check{\alpha} \rangle = \langle w(\lambda), w(\check{\alpha}) \rangle$ and $\langle \rho, \check{\alpha} \rangle = 1$.

Note that by definition each linkage class $W \cdot \lambda$ contains at least a dot-(anti)dominant element. However, the \mathfrak{sl}_2 -case implies such element is not unique. As we will soon see, if λ is integral, such dot-(anti)dominant element is unique. In general, the number of dot-(anti)dominant elements in $W \cdot \lambda$ depends on the residue class of λ .

Definition 29. We define⁴:

- Let $\Lambda \subset \mathfrak{t}^*$ be the subset of integral weights.
- Let $\Lambda_r := \mathbb{Z}\Phi \subset \mathfrak{t}^*$ be the subset of \mathbb{Z} -spans of roots.

Remark 30. Using the axioms of root systems, it is easy to see $\Lambda_r \subset \Lambda$, and they are lattices in \mathfrak{t}^* . In other words, they are free abelian groups of rank $\dim(\mathfrak{t}^*)$ and $\Lambda_r \otimes_{\mathbb{Z}} k \simeq \Lambda \otimes_{\mathbb{Z}} k \simeq \mathfrak{t}^*$.

In general, these two lattices are different. For example, for $\mathfrak{g} = \mathfrak{sl}_2$ and the coordinate $l = \langle \lambda, \check{\alpha} \rangle$, $\Lambda_r = 2\mathbb{Z}$ while $\Lambda = \mathbb{Z}$.

Notation 31. For $\lambda \in \mathfrak{t}^*$, we write $[\lambda] \in \mathfrak{t}^* / \Lambda$ for its residue class.

We have the following combinatorial results, see [H, Sect. 3.4-3.5] for their proofs.

⁴Some authors use the notations Λ_{sc} for Λ and Λ_{ad} for Λ_r because they are exactly the lattices appearing in the root data of G_{sc} and G_{ad} .

Theorem 32. For $[\lambda] \in \mathfrak{t}^*/\Lambda$, let $\Phi_{[\lambda]} \subset \Phi$ be the subset of roots α such that $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ and $W_{[\lambda]} \subset W$ be the subgroup of elements w such that $w(\lambda) - \lambda \in \Lambda_r$ ⁵. Then $(\mathbb{R}\Phi_{[\lambda]}, \Phi_{[\lambda]})$ is a root system and $W_{[\lambda]}$ is the Weyl group of it.

Lemma 33. For $\lambda \in \mathfrak{t}^*$, if $\mu \in W_{[\lambda]} \cdot \lambda$, then $\Phi_{[\mu]} = \Phi_{[\lambda]}$ and $W_{[\mu]} = W_{[\lambda]}$.

Lemma 34. For $\lambda \in \mathfrak{t}^*$, the orbit $W_{[\lambda]} \cdot \lambda$ contains exactly one dot-dominant (resp. dot-antidominant) element.

Remark 35. Note that $\Phi_{[\lambda]} = \Phi$ and $W_{[\lambda]} = W$ if λ is integral.

Remark 36. It is easy to see if $\lambda, \mu \in \mathfrak{t}^*$ are \leq_c -comparable, then $\mu \in W_{[\lambda]} \cdot \lambda$. Hence Lemma 34 implies for any $\lambda \in \mathfrak{t}^*$, there is a unique dot-dominant (resp. dot-antidominant) weight λ^+ (resp. λ^-) such that $\lambda \leq_c \lambda^+$ (resp. $\lambda^- \leq_c \lambda$).

Corollary 37. We have:

- (1) If $\lambda \in \mathfrak{t}^*$ is dot-dominant, then $M_\mu \subset M_\lambda$ for any $\mu \in W_{[\lambda]} \cdot \lambda$.
- (2) If $\lambda \in \mathfrak{t}^*$ is dot-antidominant, then $M_\mu \supset M_\lambda$ for any $\mu \in W_{[\lambda]} \cdot \lambda$.

Let us look at some examples.

Example 38. The negative half sum $-\rho$ is integral and dot-singular, and is both dot-dominant and dot-antidominant. It follows that $M_{-\rho}$ is irreducible and projective. Note that $M_{-\rho}$ is the only irreducible object in the block $\mathcal{O}_{\varpi(-\rho)}$ and $\text{Hom}(M_{-\rho}, M_{-\rho}) = k \cdot \text{Id}$ ([Lem. 29, Lect. 6]). These formally imply

$$\text{Hom}_{\mathcal{O}_{\varpi(-\rho)}}(M_{-\rho}, -) : \mathcal{O}_{\varpi(-\rho)} \rightarrow \text{Vect}_{\text{fd}}$$

is an equivalence.

The block $\mathcal{O}_{\varpi(-\rho)}$ is the so-called **most singular block**. The structure of the most singular block is boring.

Remark 39. Any generic weight $\lambda \in \mathfrak{t}^*$ is dot-regular, and is both dot-dominant and dot-antidominant. It follows that there are $\#W$ Verma modules in a generic block $\mathcal{O}_{\varpi(\lambda)}$ and each of them is both irreducible and projective. These formally imply $\mathcal{O}_{\varpi(\lambda)}$ is semisimple and contains $\#W$ irreducible objects.

The structure of any generic block is boring.

Remark 40. The 0 weight is integral and dot-regular. It is the unique dot-dominant element in the orbit $W \cdot 0$. The unique dot-antidominant element is

$$w_0 \cdot 0 = w_0(\rho) - \rho = -2\rho \in W \cdot 0.$$

Here recall w_0 is the **longest element** in W , which is the unique element sending Φ^+ to Φ^- .

The block $\mathcal{O}_{\varpi(0)}$ is the so-called **principle block**. As we have seen in the \mathfrak{sl}_3 -case, the structure of the principle block is interesting.

Proposition 41. If λ is integral, then $\mathcal{O}_{\varpi(\lambda)}$ is indecomposable.

Proof. We can assume λ is dot-antidominant. Suppose $\mathcal{O}_{\varpi(\lambda)} \simeq \mathcal{O}_1 \oplus \mathcal{O}_2$. Recall any Verma module M_μ is indecomposable because it has a *unique* irreducible quotient. It follows that each M_μ in the block $\mathcal{O}_{\varpi(\lambda)}$ is contained either in \mathcal{O}_1 or \mathcal{O}_2 . However, by Corollary 37, and M_λ is contained in each M_μ as a submodule. It follows that all Verma modules in the block are contained either in \mathcal{O}_1 or \mathcal{O}_2 . This implies either $\mathcal{O}_2 \simeq 0$ or $\mathcal{O}_1 \simeq 0$. □

⁵It is clear that $\Phi_{[\lambda]}$ only depends on the residue class $[\lambda]$. For $W_{[\lambda]}$, it is enough to show $w(\lambda) - \lambda \in \Lambda_r$ whenever $\lambda \in \Lambda$. This can be reduced to the case $w = s_\alpha$, and a direct calculation shows the claim is true.

Exercise 42. This is **Homework 3, Problem 5**. For any weight $\lambda \in \mathfrak{t}^*$, let $\mathcal{O}_\lambda \subset \mathcal{O}$ be the full subcategory containing those objects M whose composition factors are of the form L_μ for $\mu \in W_{[\lambda]} \cdot \lambda$. Prove:

- (1) Each \mathcal{O}_λ is indecomposable.
- (2) For $M \in \mathcal{O}$, suppose we have a decomposition $M \simeq M_1 \oplus M_2$ as \mathfrak{t} -modules such that the set $\text{wt}(M_1) - \text{wt}(M_2) := \{\lambda_1 - \lambda_2 \mid \lambda_i \in \text{wt}(M_i)\}$ has empty intersection with Λ_r , then this is also a decomposition of \mathfrak{g} -modules.
- (3) For any central character χ of $Z(\mathfrak{g})$, we have a direct sum decomposition

$$\mathcal{O}_\chi \simeq \bigoplus_{\lambda \in \varpi^{-1}(\chi) \text{ is dot-antidominant}} \mathcal{O}_\lambda.$$

- (4) Conclude that

$$\mathcal{O} \simeq \bigoplus_{\lambda \text{ is dot-antidominant}} \mathcal{O}_\lambda.$$

Remark 43. Some authors (including [H]) prefer to call the full subcategories \mathcal{O}_λ the blocks of \mathcal{O} . This is different from our convention when λ is not integral. We made this choice because we will give geometric incarnations to $\mathcal{O}_{\varpi(\lambda)}$ even for non-integral λ .

5. THE BRUHAT ORDER

Let λ be an integral, dot-regular and dot-antidominant weight. As suggested by the \mathfrak{sl}_3 -case, the block $\mathcal{O}_{\varpi(\lambda)} = \mathcal{O}_\lambda$ is interesting. This block contains $\#W$ Verma modules, labelled by weights $W \cdot \lambda$. In this subsection, we can give a more combinatorial description of the partial order \leq_c when restricted to $W \cdot \lambda$.

Recall the following definition:

Definition 44. For $w \in W$, its **length** is defined to be

$$\ell(w) := \#\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}.$$

Example 45. We have $\ell(\text{id}) = 0$.

The following results are well-known (see. e.g. [H, Sect. 0.3-0.4]).

Proposition-Definition 46. For $w \in W$, the length $\ell(w)$ is the length of the shortest presentation $w = s_1 s_2 \cdots s_\ell$ such that each s_i is a simple reflection (corresponding to a simple positive root). In particular $\ell(w) = \ell(w^{-1})$. Any such presentation with $\ell = \ell(w)$ is called a **reduced presentation** of w .

Lemma 47. For any reflection s_α , $\alpha \in \Phi$, the number $\ell(ws_\alpha) - \ell(w)$ is positive (resp. negative) iff $w(\alpha)$ is positive (resp. negative).

Lemma 48. There exists a **longest element** $w_0 \in W$ such that $w_0(\Phi^+) = \Phi^-$. By definition, $w_0 = w_0^{-1}$ and $\ell(w_0) = \#\Phi^+ = \dim(\mathfrak{n})$.

Proposition-Definition 49. For $w, w' \in W$, the following conditions are equivalent:

- (a) There exists a chain $w' = w_{[0]}, w_{[1]}, \dots, w_{[m]} = w$ such that $w_{[i+1]} = s_{\alpha_i} w_{[i]}$ for $\alpha_i \in \Phi$ and $\ell(w_{[i+1]}) > \ell(w_{[i]})$;
- (b) There exists a chain $w' = w_{[0]}, w_{[1]}, \dots, w_{[m]} = w$ such that $w_{[i+1]} = w_{[i]} s_{\alpha_i}$ for $\alpha_i \in \Phi$ and $\ell(w_{[i+1]}) > \ell(w_{[i]})$;
- (c) There exists a reduced presentation $w = s_1 s_2 \cdots s_\ell$ such that $w' = s_{i_1} \cdots s_{i_m}$ for a substring $1 \leq i_1 < \cdots < i_m \leq \ell$;

We write $w' \leq w$ if they satisfy the above conditions. This defines a partial order on W , which is called the **Bruhat order**.

Remark 50. By definition, for any $w \in W$, we have $1 \leq w \leq w_0$.

We will review more properties of the Burhat order when we need them. For now, we state the following combinatorial result. For a proof, see [H, Sect. 5.2].

Proposition 51. *Let λ be a integral, dot-regular and dot-antidominant weight. Then $w' \cdot \lambda \leq_c w \cdot \lambda$ iff $w' \leq w$.*

Warning 52. *There is a critical typo in [H, Sect. 5.2]. See the online erratum to the book.*

Example 53. For $\mathfrak{g} = \mathfrak{sl}_4$ and $W \simeq \Sigma_4$, consider $w' = (1423)$ and $w = (2341)$. It is easy to check $w' \leq w$ is false while $w' \cdot \lambda \leq w \cdot \lambda$ is true. In particular, \leq and \leq_c are different partial orders.

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