(5)

Q3 More Linear Algebra:

(a) Given $A = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}$

Lets find Eigen Values & Eigen Vectors of A.

$Ax = \lambda x$

x = Eigen Vector (or) Characteristic Vector

λ = Eigen value (or) Charecteristic Root

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

then

$$\begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$-2x_1 + 2x_2 = \lambda x_1$$

$$-6x_1 + 5x_2 = \lambda x_2$$

$$-(2+\lambda)x_1 + 2x_2 = 0 \rightarrow V_1 \qquad -6x_1 + (5-\lambda)x_2 = 0 \rightarrow V_2$$

$$\begin{bmatrix} -(2+\lambda) & 2 \\ -6 & (5-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow 1$$

$$\left[\underbrace{\begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}}_{A} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left[A - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(A - \lambda I)x = 0 \rightarrow 2$$

as $x \neq 0$ $\therefore$ $|A - \lambda I| = 0 \rightarrow 3$

Comparing 1 & 2

$$(A - \lambda I) = \begin{bmatrix} -(2+\lambda) & 2 \\ -6 & (5-\lambda) \end{bmatrix}$$

$$\det(A - \lambda I) = -(2+\lambda)(5-\lambda) - (-6)(2)$$

$$= -[10 - 2\lambda + 5\lambda - \lambda^2] - (-12)$$

$$= -10 - 3\lambda + \lambda^2 + 12$$

$$= \lambda^2 - 3\lambda + 2$$

$$= \lambda^2 - \lambda - 2\lambda + 2$$

$$= \lambda(\lambda - 1) - 2(\lambda - 1)$$

$$= (\lambda - 1)(\lambda - 2)$$

According to eqn 3

$$0 = (\lambda - 1)(\lambda - 2)$$

The Eigen values of given matrix A are $\lambda_1 = 1$, $\lambda_2 = 2$

Eigen Vectors

Plug the Eigen Values in Eqn (V_1) & (V_2) to get Eigen vectors.

For $\lambda_1 = 1$

(V_1) $\Rightarrow (-2+1)x_1 + 2x_2 = 0 \Rightarrow -3x_1 + 2x_2 = 0 \Rightarrow \boxed{2x_2 = 3x_1}$

(V_2) $\Rightarrow (-6x_1) + (5-1)x_2 = 0 \qquad -6x_1 + 4x_2 = 0 \Rightarrow \boxed{3x_1 = 2x_2}$

$\Rightarrow x_1 = \frac{2}{3}x_2$

$\therefore$ (V_2) $\Rightarrow$ $\cancel{-6x_1 + 4(3x_1) = 0}$

$\cancel{-6x_1 + 12x_1 = 0}$

$$\therefore \text{Eigen Vector } x_{\lambda=1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}x_2 \\ x_2 \end{bmatrix}$$

$$\boxed{x_{\lambda=1} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}}$$

3(a)(contd):-

7

For $\lambda_2 = 2$

(v1) $\Rightarrow -(2+2)x_1 + 2x_2 = 0 \Rightarrow -4x_1 + 2x_2 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}x_2}$

(v2) $\Rightarrow -6x_1 + (5-2)x_2 = 0 \Rightarrow -6x_1 - 3x_2 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}x_2}$

$\therefore$ Eigen Vector $x_{\lambda=2} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$ $\boxed{x_{\lambda=2} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}}$

Q3(b)

Λ is considered to be a diagonal matrix as ~~Eigen~~ Eigen values of A as its diagonal entries.

$\therefore \Lambda = \text{diag}\begin{bmatrix} 1 & \\ & 2 \end{bmatrix} \Rightarrow \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

we observe that the Eigen values of A are 1, 2 which are distinct, the matrix is diagonalizable and the Eigen vectors are linearly independent.

$\therefore [x_{\lambda=1} \;\; x_{\lambda=2}] = \begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}$

Lets call this matrix $S = [x_{\lambda=1} \;\; x_{\lambda=2}] = \begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}$

The inverse of S is $S^{-1} = \frac{1}{\frac{2}{3}\times 1 - \frac{1}{2}\times 1} \begin{bmatrix} 1 & -1/2 \\ -1 & 2/3 \end{bmatrix} = \frac{1}{\frac{2}{3} - \frac{1}{2}} \begin{bmatrix} 1 & -1/2 \\ -1 & 2/3 \end{bmatrix}$

$\therefore S^{-1} = \frac{1}{\frac{1}{6}} \begin{bmatrix} 1 & -1/2 \\ -1 & 2/3 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -6 & 4 \end{bmatrix}$

As we determined A is diagonalizable, lets calculate $S^{-1}AS$

$$S^{-1}AS = \begin{bmatrix} 6 & -3 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}\cdot\begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+18 & 12-15 \\ 12-24 & -12+20 \end{bmatrix}\begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -3 \\ -12 & 8 \end{bmatrix}\begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \overset{2}{\cancel{6}}\times\frac{2}{\cancel{3}} - 3\times 1 & \overset{3}{\cancel{6}}\times\frac{1}{\cancel{2}} - 3\times 1 \\ -\overset{4}{\cancel{12}}\times\frac{2}{\cancel{3}} + 8 & -\overset{6}{\cancel{12}}\times\frac{1}{\cancel{2}} + 8\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 3-3 \\ -8+8 & -6+8 \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S^{-1}AS = \Lambda$$

$\therefore$ $S = U$ which satisfies the Similarity Transformation of matrix

$\therefore$ $\boxed{U = \begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}}$ which satisfies $AU = U\Lambda$

Verification

$$AU = \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix}\begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \overset{\frac{6-4}{3}}{-\frac{4}{3}+2} & -1+2 \\ -4+5 & -3+5 \end{bmatrix}$$

$$AU = \begin{bmatrix} +\frac{2}{3} & 1 \\ 1 & 2 \end{bmatrix}$$

$$U\Lambda = \begin{bmatrix} 2/3 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 1 \\ 1 & 2 \end{bmatrix}$$

$\therefore$ $\boxed{AU = U\Lambda}$

3(c)

As we can write $A = U \Lambda U^{-1}$

$$U^{-1} = \frac{1}{\frac{2}{3} - \frac{1}{2}} \begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & +\frac{2}{3} \end{bmatrix}$$

$$\boxed{U^{-1} = \begin{bmatrix} 6 & -3 \\ -6 & 4 \end{bmatrix}}$$

<u>Verification</u> $A = U \Lambda U^{-1}$

$$U \Lambda U^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} \times 1 + 0 & 0 + 1 \\ 1 + 0 & 0 + 2 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 6 & -2 + 4 \\ 6 - 12 & -3 + 8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 \\ -6 & 5 \end{bmatrix} = A$$

$$\boxed{\therefore \; U \Lambda U^{-1} = A}$$