

# 14

## The Plane Stress Problem

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### §14.1. Introduction

We now pass to the variational formulation of two-dimensional *continuum* finite elements. The problem of *plane stress* will serve as the vehicle for illustrating such formulations. As narrated in Appendix O, continuum-based structural finite elements were invented in the aircraft industry (at Boeing during the early 1950s) to solve this kind of problem when it arose in the design and analysis of delta wing panels [?].

The problem is presented here within the framework of the linear theory of elasticity.

### §14.2. Plate in Plane Stress

In structural mechanics, a flat thin sheet of material is called a *plate*.<sup>1</sup> The distance between the plate faces is the *thickness*, denoted by  $h$ . The *midplane* lies halfway between the two faces.

The direction normal to the midplane is the *transverse* direction. Directions parallel to the midplane are called *in-plane* directions. The global axis  $z$  is oriented along the transverse direction. Axes  $x$  and  $y$  are placed in the midplane, forming a right-handed Rectangular Cartesian Coordinate (RCC) system. Thus the equation of the midplane is  $z = 0$ . The  $+z$  axis conventionally defines the *top surface* of the plate as the one that it intersects, whereas the opposite surface is called the *bottom surface*. See Figure 14.1(a).

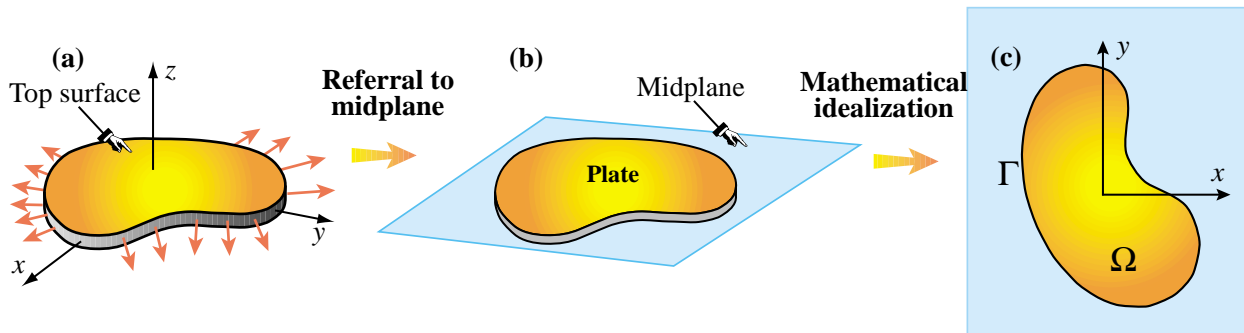


FIGURE 14.1. A plate structure in plane stress: (a) configuration; (b) referral to its midplane; (c) 2D mathematical idealization as boundary value problem.

#### §14.2.1. Behavioral Assumptions

A plate loaded in its midplane is said to be in a state of *plane stress*, or a *membrane state*, if the following assumptions hold:

1. All loads applied to the plate act in the midplane direction, and are symmetric with respect to the midplane.
2. All support conditions are symmetric about the midplane.
3. In-plane displacements, strains and stresses can be taken to be uniform through the thickness.
4. The normal and shear stress components in the  $z$  direction are zero or negligible.

<sup>1</sup> If it is relatively thick, as in concrete pavements or Argentinian beefsteaks, the term *slab* is also used but not usually for plane stress conditions.

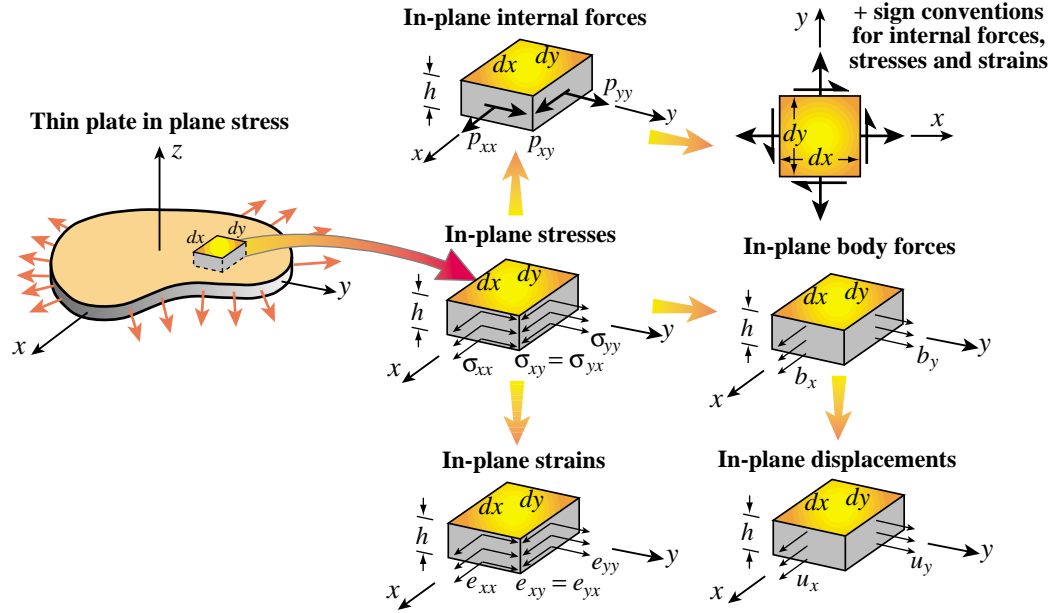


FIGURE 14.2. Notational conventions for in-plane stresses, strains, displacements and internal forces of a thin plate in plane stress.

The last two assumptions are not necessarily consequences of the first two. For the latter to hold, the thickness  $h$  should be small, typically 10% or less, than the shortest in-plane dimension. If the plate thickness varies it should do so gradually. Finally, the plate fabrication must exhibit symmetry with respect to the midplane.

To these four assumptions we add the following restriction:

5. The plate is fabricated of the same material through the thickness. Such plates are called *transversely homogeneous* or (in aerospace) *monocoque* plates.

The last assumption excludes wall constructions of importance in aerospace, in particular composite and honeycomb sandwich plates. The development of mathematical models for such configurations requires a more complicated integration over the thickness as well as the ability to handle coupled bending and stretching effects, and will not be considered here.

**Remark 14.1.** Selective relaxation from assumption 4 leads to the so-called *generalized plane stress state*, in which  $z$  stresses are accepted. The *plane strain state* is obtained if strains in the  $z$  direction are precluded. Although the construction of finite element models for those states has many common points with plane stress, we shall not consider those models here. For isotropic materials the plane stress and plane strain problems can be mapped into each other through a fictitious-property technique; see Exercise 14.1.

**Remark 14.2.** Transverse loading on a plate produces *plate bending*, which is associated with a more complex configuration of internal forces and deformations. This subject is studied in [271].

### §14.2.2. Mathematical Model

The mathematical model of the plate in plane stress is set up as a two-dimensional boundary value problem (BVP), in which the plate is projected onto its midplane; see Figure 14.1(b). This allows to formulate the BVP over a plane domain  $\Omega$  with a boundary  $\Gamma$ , as illustrated in Figure 14.1(c).

In this idealization the third dimension is represented as functions of  $x$  and  $y$  that are *integrated through the plate thickness*. Engineers often work with internal plate forces, which result from integrating the in-plane stresses through the thickness. See Figure 14.2.

### §14.2.3. Problem Data

The following summarizes the givens in the plate stress problem.

*Domain geometry.* This is defined by the boundary  $\Gamma$  illustrated in Figure 14.1(c).

*Thickness.* Most plates used as structural components have constant thickness. If the thickness does vary, in which case  $h = h(x, y)$ , it should do so gradually to maintain the plane stress state. Sudden changes in thickness may lead to stress concentrations.

*Material data.* This is defined by the constitutive equations. Here we shall assume that the plate material is linearly elastic but not necessarily isotropic.

*Specified Interior Forces.* These are known forces that act in the interior  $\Omega$  of the plate. There are of two types. *Body forces* or *volume forces* are forces specified per unit of plate volume; for example the plate weight. *Face forces* act tangentially to the plate faces and are transported to the midplane. For example, the friction or drag force on an airplane skin is of this type if the skin is modeled to be in plane stress.

*Specified Surface Forces.* These are known forces that act on the boundary  $\Gamma$  of the plate. In elasticity they are called *surface tractions*. In actual applications it is important to know whether these forces are specified per unit of surface area or per unit length. The former may be converted to the latter by multiplying through the appropriate thickness value.

*Displacement Boundary Conditions.* These specify how the plate is supported. Points subject to support conditions may be fixed, allowed to move in one direction, or subject to multipoint constraints. Also symmetry and antisymmetry lines may be identified as discussed in Chapter 8 of IFEM [273].

If no displacement boundary conditions are imposed, the plate is said to be *free-free* or *floating*.

### §14.2.4. Problem Unknowns

The unknown fields are displacements, strains and stresses. Because of the assumed wall fabrication homogeneity the in-plane components are assumed to be *uniform through the plate thickness*. Thus the dependence on  $z$  disappears and all such components become functions of  $x$  and  $y$  only.

*Displacements.* The in-plane displacement field is defined by two components:

$$\mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} \quad (14.1)$$

The transverse displacement component  $u_z(x, y, z)$  component is generally nonzero because of Poisson's ratio effects, and depends on  $z$ . However, this displacement does not appear in the governing equations.

*Strains.* The in-plane strain field forms a tensor defined by three independent components:  $e_{xx}$ ,  $e_{yy}$  and  $e_{xy}$ . To allow stating the FE equations in matrix form, these components are cast to form a

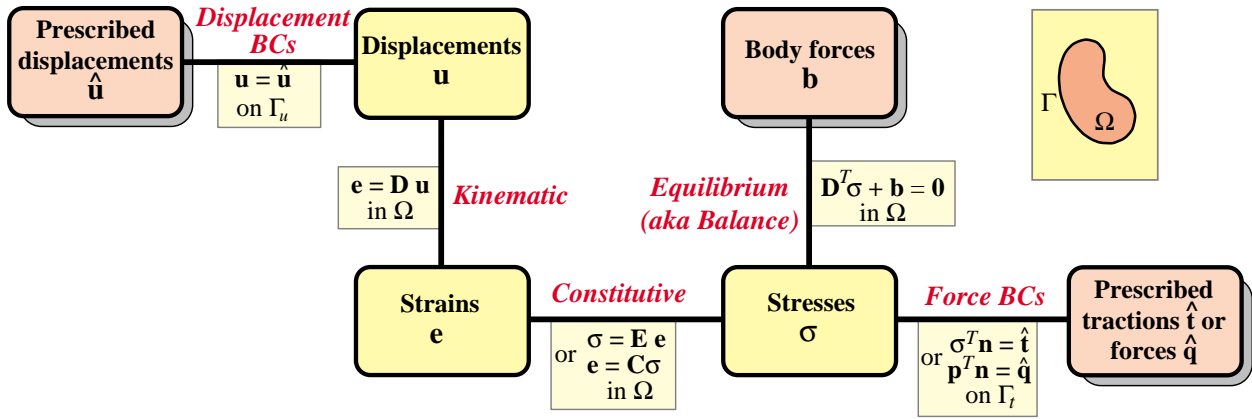


FIGURE 14.3. The Strong Form of the plane stress equations of linear elastostatics displayed as a Tonti diagram. Yellow boxes identify prescribed fields whereas orange boxes denote unknown fields. The distinction between Strong and Weak Forms is explained in §14.3.3.

3-component “strain vector”

$$\mathbf{e}(x, y) = \begin{bmatrix} e_{xx}(x, y) \\ e_{yy}(x, y) \\ 2e_{xy}(x, y) \end{bmatrix} \quad (14.2)$$

The factor of 2 in  $e_{xy}$  shortens strain energy expressions. The shear strain components  $e_{xz}$  and  $e_{yz}$  vanish. The transverse normal strain  $e_{zz}$  is generally nonzero because of Poisson’s ratio effects. This strain does not enter the governing equations as unknown, however, because the associated stress  $\sigma_{zz}$  is zero. This eliminates the contribution of  $\sigma_{zz}e_{zz}$  to the internal energy.

*Stresses.* The in-plane stress field forms a tensor defined by three independent components:  $\sigma_{xx}$ ,  $\sigma_{yy}$  and  $\sigma_{xy}$ . As in the case of strains, to allow stating the FE equations in matrix form, these components are cast to form a 3-component “stress vector”

$$\boldsymbol{\sigma}(x, y) = \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{bmatrix} \quad (14.3)$$

The remaining three stress components:  $\sigma_{zz}$ ,  $\sigma_{xz}$  and  $\sigma_{yz}$ , are assumed to vanish.

The *plate internal forces* are obtained on integrating the stresses through the thickness. Under the assumption of uniform stress distribution,

$$p_{xx} = \sigma_{xx}h, \quad p_{yy} = \sigma_{yy}h, \quad p_{xy} = \sigma_{xy}h. \quad (14.4)$$

These  $p$ ’s also form a tensor. They are called *membrane forces* in the literature. See Figure 14.2.

### §14.3. Plane Stress Governing Equations

We shall develop plane stress finite elements in the framework of classical linear elasticity. The necessary governing equations are presented below. They are graphically represented in the Strong Form Tonti diagram of Figure 14.3.

### §14.3.1. Governing Equations

The three internal fields: displacements, strains and stresses (14.1)–(14.3) are connected by three field equations: kinematic, constitutive and internal-equilibrium equations. If initial strain effects are ignored, these equations read

$$\begin{aligned} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} &= \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \\ \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \\ \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (14.5)$$

The compact matrix version of (14.5) is

$$\boxed{\mathbf{e} = \mathbf{D}\mathbf{u}, \quad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e}, \quad \mathbf{D}^T \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0},} \quad (14.6)$$

Here  $\mathbf{E} = \mathbf{E}^T$  is the  $3 \times 3$  stress-strain matrix of plane stress elastic moduli,  $\mathbf{D}$  is the  $3 \times 2$  symmetric-gradient operator and its transpose the  $2 \times 3$  tensor-divergence operator.<sup>2</sup>

If the plate material is isotropic with elastic modulus  $E$  and Poisson's ratio  $\nu$ , the moduli in the constitutive matrix  $\mathbf{E}$  reduce to  $E_{11} = E_{22} = E/(1 - \nu^2)$ ,  $E_{33} = \frac{1}{2}E/(1 + \nu) = G$ ,  $E_{12} = \nu E_{11}$  and  $E_{13} = E_{23} = 0$ . See also Exercise 14.1.

### §14.3.2. Boundary Conditions

Boundary conditions prescribed on  $\Gamma$  may be of two types: displacement BC or force BC (the latter is also called stress BC or traction BC). To write down those conditions it is conceptually convenient to break up  $\Gamma$  into two subsets:  $\Gamma_u$  and  $\Gamma_t$ , over which displacements and force or stresses, respectively, are specified. See Figure 14.4.

Displacement boundary conditions are prescribed on  $\Gamma_u$  in the form

$$\boxed{\mathbf{u} = \hat{\mathbf{u}}.} \quad (14.7)$$

Here  $\hat{\mathbf{u}}$  are prescribed displacements. Often  $\hat{\mathbf{u}} = \mathbf{0}$ . This happens in fixed portions of the boundary, as the ones illustrated in Figure 14.4.

Force boundary conditions (also called stress BCs and traction BCs in the literature) are specified on  $\Gamma_t$ . They take the form

$$\boxed{\boldsymbol{\sigma}_n = \hat{\mathbf{t}}.} \quad (14.8)$$

Here  $\hat{\mathbf{t}}$  are prescribed surface tractions specified as a force per unit area (that is, not integrated through the thickness), and  $\boldsymbol{\sigma}_n$  is the stress vector shown in Figure 14.4.

<sup>2</sup> The dependence on  $(x, y)$  has been omitted to reduce clutter.

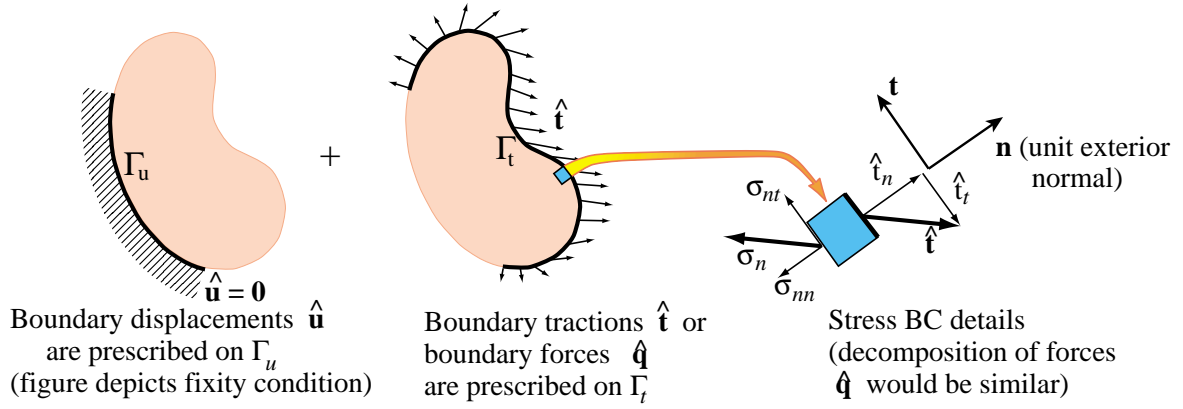


FIGURE 14.4. Displacement and force (stress, traction) boundary conditions for the plane stress problem.

An alternative form of (14.8) uses the internal plate forces:

$$\mathbf{p}_n = \hat{\mathbf{q}}. \quad (14.9)$$

Here  $\mathbf{p}_n = \boldsymbol{\sigma}_n h$  and  $\hat{\mathbf{q}} = \hat{\mathbf{t}} h$ . This form is used more often than (14.8) in structural design, particularly when the plate wall construction is inhomogeneous.

The components of  $\boldsymbol{\sigma}_n$  in Cartesian coordinates follow from Cauchy's stress transformation formula

$$\boldsymbol{\sigma}_n = \begin{bmatrix} \sigma_{xx}n_x + \sigma_{xy}n_y \\ \sigma_{xy}n_x + \sigma_{yy}n_y \end{bmatrix} = \begin{bmatrix} n_x & 0 & n_y \\ 0 & n_y & n_x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}, \quad (14.10)$$

in which  $n_x$  and  $n_y$  denote the Cartesian components of the unit normal vector  $\mathbf{n}^e$  (also called the direction cosines of the normal). Thus (14.8) splits into two scalar conditions:  $\hat{t}_x = \sigma_{nx}$  and  $\hat{t}_y = \sigma_{ny}$ . The derivation of (14.10) is the subject of Exercise 14.4.

It is sometimes convenient to write the condition (14.8) in terms of normal  $n$  and tangential  $t$  directions:

$$\sigma_{nn} = \hat{t}_n, \quad \sigma_{nt} = \hat{t}_t \quad (14.11)$$

in which  $\sigma_{nn} = \sigma_{nx}n_x + \sigma_{ny}n_y$  and  $\sigma_{nt} = -\sigma_{nx}n_y + \sigma_{ny}n_x$ . See Figure 14.4.

**Remark 14.3.** The separation of  $\Gamma$  into  $\Gamma_u$  and  $\Gamma_t$  is useful for conciseness in the mathematical formulation, such as the energy integrals presented below. It does not exhaust, however, all BC possibilities. Frequently at points of  $\Gamma$  one specifies a displacement in one direction and a force (or stress) in the other. An example of these are roller and sliding conditions as well as lines of symmetry and antisymmetry. These are called *mixed displacement-traction* BC. To cover these situations one needs either a generalization of the boundary split, in which  $\Gamma_u$  and  $\Gamma_t$  are permitted to overlap, or to define another portion  $\Gamma_m$  for 'mixed conditions'. Such generalizations will not be presented here, as they become unimportant once the FE discretization is done.



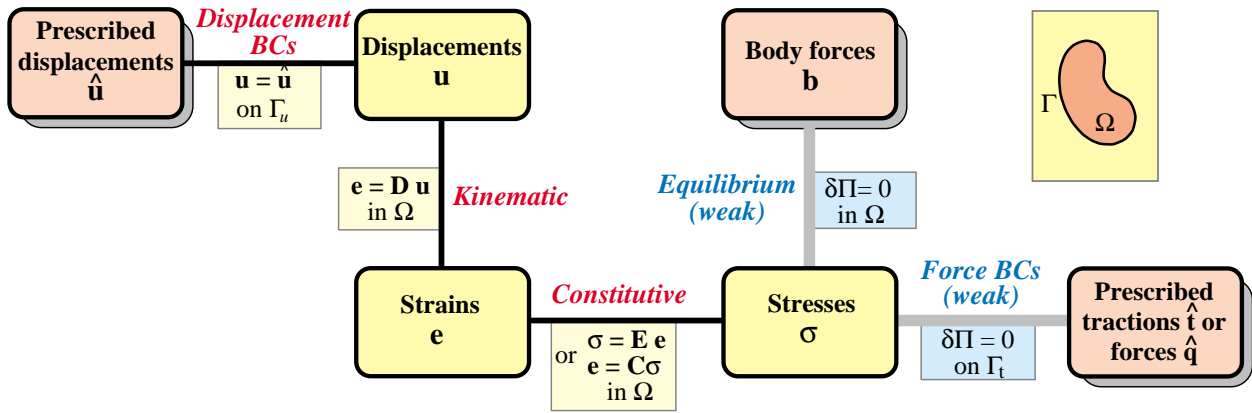


FIGURE 14.5. The TPE-based Weak Form of the plane stress equations of linear elastostatics. Weak links are marked with grey lines.

### §14.3.3. Weak versus Strong Form

We introduce now some further terminology from variational calculus. The Tonti diagram of Figure 14.3 is said to display the *Strong Form* of the governing equations because all relations are verified point by point. These relations, called *strong links*, are shown in the diagram with black lines.

A Weak Form is obtained by *relaxing* one or more strong links, as briefly described in Chapter 11. Those are replaced by *weak links*, which enforce relations in an average or integral sense rather than point by point. The weak links are then provided by the variational formulation chosen for the problem. Because in general many variational forms of the same problem are possible, there are many possible Weak Forms. On the other hand the Strong Form is unique.

The Weak Form associated with the Total Potential Energy (TPE) variational form is illustrated in Figure 14.5. The internal equilibrium equations and stress BC become weak links, which are drawn by gray lines. These equations are given by the variational statement  $\delta\Pi = 0$ , where the TPE functional  $\Pi$  is given in the next subsection. The FEM displacement formulation discussed below is based on this particular Weak Form.

### §14.3.4. Total Potential Energy

The Total Potential Energy functional for the plane stress problem is given by

$$\Pi = U - W. \quad (14.12)$$

The internal energy can be expressed in terms of the strains only as

$$U = \frac{1}{2} \int_{\Omega} h \boldsymbol{\sigma}^T \mathbf{e} d\Omega = \frac{1}{2} \int_{\Omega} h \mathbf{e}^T \mathbf{E} \mathbf{e} d\Omega. \quad (14.13)$$

in which  $\frac{1}{2} \mathbf{e}^T \mathbf{E} \mathbf{e}$  is the *strain energy density*. The derivation details are relegated to Exercise 14.5. The external energy (potential of the applied forces) is the sum of contributions from the given

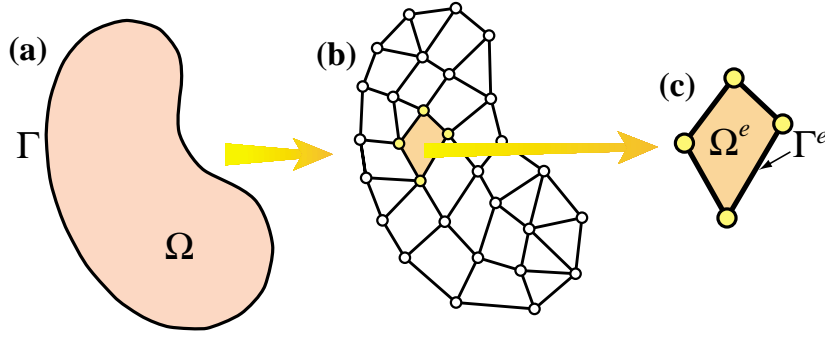


FIGURE 14.6. Finite element discretization and extraction of generic element.

interior (body) and exterior (boundary) forces:

$$W = \int_{\Omega} h \mathbf{u}^T \mathbf{b} d\Omega + \int_{\Gamma_t} h \mathbf{u}^T \hat{\mathbf{t}} d\Gamma. \quad (14.14)$$

Note that the boundary integral over  $\Gamma$  is taken only over  $\Gamma_t$ . That is, the portion of the boundary over which tractions or forces are specified.

#### §14.4. Finite Element Equations

The necessary equations to apply the finite element method to the plane stress problem are collected here and expressed in matrix form. The domain of Figure 14.6(a) is discretized by a finite element mesh as illustrated in Figure 14.6(b). From this mesh we extract a generic element labeled  $e$  with  $n \geq 3$  node points. In subsequent derivations the number  $n$  is kept *arbitrary*. Therefore, the formulation is applicable to arbitrary two-dimensional elements, for example those sketched in Figure 14.7.

To comfortably accommodate general element types, the node points will be labeled 1 through  $n$ . These are called *local node numbers*. Numbering will always start with corners.

The element domain and boundary are denoted by  $\Omega^e$  and  $\Gamma^e$ , respectively. The element has  $2n$  degrees of freedom. These are collected in the element node displacement vector in a node by node arrangement:

$$\mathbf{u}^e = [u_{x1} \quad u_{y1} \quad u_{x2} \quad \dots \quad u_{xn} \quad u_{yn}]^T. \quad (14.15)$$

##### §14.4.1. Displacement Interpolation

The displacement field  $\mathbf{u}^e(x, y)$  over the element is interpolated from the node displacements. We shall assume that the same interpolation functions are used for both displacement components.<sup>3</sup> Thus

$$u_x(x, y) = \sum_{i=1}^n N_i^e(x, y) u_{xi}, \quad u_y(x, y) = \sum_{i=1}^n N_i^e(x, y) u_{yi}, \quad (14.16)$$

<sup>3</sup> This is the so called *element isotropy* condition, which is studied and justified in advanced FEM courses.

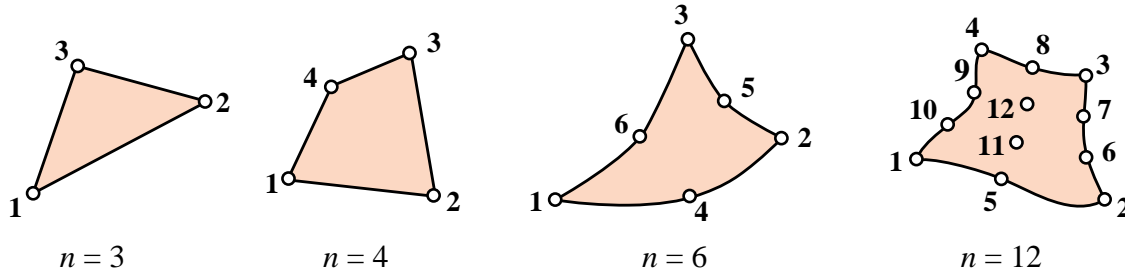


FIGURE 14.7. Example plane stress finite elements, characterized by their number of nodes  $n$ .

in which  $N_i^e(x, y)$  are the element shape functions. In matrix form:

$$\mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \dots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \dots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e = \mathbf{N} \mathbf{u}^e. \quad (14.17)$$

This  $\mathbf{N}$  (with superscript  $e$  omitted to reduce clutter) is called the *shape function matrix*. It has dimensions  $2 \times 2n$ . For example, if the element has 4 nodes,  $\mathbf{N}$  is  $2 \times 8$ .

The *interpolation condition* on the element shape function  $N_i^e(x, y)$  states that it must take the value one at the  $i^{\text{th}}$  node and zero at all others. This ensures that the interpolation (14.17) is correct at the nodes. Additional requirements on the shape functions are stated in later Chapters.

Differentiating the finite element displacement field yields the strain-displacement relations:

$$\mathbf{e}(x, y) = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \dots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \dots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \dots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \mathbf{u}^e = \mathbf{B} \mathbf{u}^e. \quad (14.18)$$

This  $\mathbf{B} = \mathbf{D} \mathbf{N}$  is called the *strain-displacement matrix*. It is dimensioned  $3 \times 2n$ . For example, if the element has 6 nodes,  $\mathbf{B}$  is  $3 \times 12$ . The stresses are given in terms of strains and displacements by  $\boldsymbol{\sigma} = \mathbf{E} \mathbf{e} = \mathbf{E} \mathbf{B} \mathbf{u}^e$ , which is assumed to hold at all points of the element.

#### §14.4.2. Element Energy

To obtain finite element stiffness equations, the variation of the TPE functional is decomposed into contributions from individual elements:

$$\delta \Pi^e = \delta U^e - \delta W^e = 0. \quad (14.19)$$

in which

$$U^e = \frac{1}{2} \int_{\Omega^e} h \boldsymbol{\sigma}^T \mathbf{e} d\Omega^e = \frac{1}{2} \int_{\Omega^e} h \mathbf{e}^T \mathbf{E} \mathbf{e} d\Omega^e \quad (14.20)$$

and

$$W^e = \int_{\Omega^e} h \mathbf{u}^T \mathbf{b} d\Omega^e + \int_{\Gamma^e} h \mathbf{u}^T \hat{\mathbf{t}} d\Gamma^e \quad (14.21)$$

Note that in (14.21)  $\Gamma_l^e$  has been taken equal to the *complete boundary*  $\Gamma^e$  of the element. This is a consequence of the fact that displacement boundary conditions are applied *after* assembly, to a free-free structure. Consequently it does not harm to assume that all boundary conditions are of stress type insofar as forming the element equations.

### §14.4.3. Element Stiffness Equations

Inserting the relations  $\mathbf{u} = \mathbf{N}\mathbf{u}^e$ ,  $\mathbf{e} = \mathbf{B}\mathbf{u}^e$  and  $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$  into  $\Pi^e$  yields the quadratic form in the nodal displacements

$$\Pi^e = \frac{1}{2} \mathbf{u}^{eT} \mathbf{K}^e \mathbf{u}^e - \mathbf{u}^{eT} \mathbf{f}^e. \quad (14.22)$$

Here the element stiffness matrix is

$$\mathbf{K}^e = \int_{\Omega^e} h \mathbf{B}^T \mathbf{E} \mathbf{B} d\Omega^e, \quad (14.23)$$

and the consistent element nodal force vector is

$$\mathbf{f}^e = \int_{\Omega^e} h \mathbf{N}^T \mathbf{b} d\Omega^e + \int_{\Gamma^e} h \mathbf{N}^T \hat{\mathbf{t}} d\Gamma^e. \quad (14.24)$$

In the second integral of (14.24) the matrix  $\mathbf{N}$  is evaluated on the element boundary only.

The calculation of the entries of  $\mathbf{K}^e$  and  $\mathbf{f}^e$  for several elements of historical or practical interest is described in subsequent Chapters.

### Notes and Bibliography

The plane stress problem is well suited for introducing continuum finite elements, from both historical and technical standpoints. Some books use the Poisson equation for this purpose, but problems such as heat conduction cannot illustrate features such as vector-mixed boundary conditions and shear effects.

The first continuum structural finite elements were developed at Boeing in the early 1950s to model delta-wing skin panels [155,834]. A plane stress model was naturally chosen for the panels. The paper that gave the method its name [145] used the plane stress problem as application driver.

The technical aspects of plane stress can be found in any book on elasticity. A particularly readable one is the excellent textbook by Fung [309], which is unfortunately out of print.

### References

Referenced items have been moved to Appendix R.

### Homework Exercises for Chapter 14

#### The Plane Stress Problem

**EXERCISE 14.1** [A+C:15] Suppose that the structural material is isotropic, with elastic modulus  $E$  and Poisson's ratio  $\nu$ . The in-plane stress-strain relations for plane stress ( $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ ) and plane strain ( $e_{zz} = e_{xz} = e_{yz} = 0$ ) as given in any textbook on elasticity, are

$$\begin{aligned} \text{plane stress: } \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}, \\ \text{plane strain: } \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} &= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}. \end{aligned} \quad (\text{E14.1})$$

Show that the constitutive matrix of plane strain can be formally obtained by replacing  $E$  by a fictitious modulus  $E^*$  and  $\nu$  by a fictitious Poisson's ratio  $\nu^*$  in the plane stress constitutive matrix. Find the expression of  $E^*$  and  $\nu^*$  in terms of  $E$  and  $\nu$ .

You may also chose to answer this exercise by doing the inverse process: go from plane strain to plain stress by replacing a fictitious modulus and Poisson's ratio in the plane strain constitutive matrix.

This device permits “reusing” a plane stress FEM program to do plane strain, or vice-versa, as long as the material is isotropic.

Partial answer to go from plane stress to plane strain:  $\nu^* = \nu/(1-\nu)$ .

**EXERCISE 14.2** [A:25] In the finite element formulation of near incompressible isotropic materials (as well as plasticity and viscoelasticity) it is convenient to use the so-called *Lamé constants*  $\lambda$  and  $\mu$  instead of  $E$  and  $\nu$  in the constitutive equations. Both  $\lambda$  and  $\mu$  have the physical dimension of stress and are related to  $E$  and  $\nu$  by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)}. \quad (\text{E14.2})$$

Conversely

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}. \quad (\text{E14.3})$$

Substitute (E14.3) into both of (E14.1) to express the two stress-strain matrices in terms of  $\lambda$  and  $\mu$ . Then split the stress-strain matrix  $\mathbf{E}$  of plane strain as

$$\mathbf{E} = \mathbf{E}_\mu + \mathbf{E}_\lambda \quad (\text{E14.4})$$

in which  $\mathbf{E}_\mu$  and  $\mathbf{E}_\lambda$  contain only  $\mu$  and  $\lambda$ , respectively, with  $\mathbf{E}_\mu$  diagonal and  $E_{\lambda 33} = 0$ . This is the Lamé or  $\{\lambda, \mu\}$  splitting of the plane strain constitutive equations, which leads to the so-called **B**-bar formulation of near-incompressible finite elements.<sup>4</sup> Express  $\mathbf{E}_\mu$  and  $\mathbf{E}_\lambda$  also in terms of  $E$  and  $\nu$ .

For the plane stress case perform a similar splitting in which where  $\mathbf{E}_\lambda$  contains only  $\bar{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$  with  $E_{\lambda 33} = 0$ , and  $\mathbf{E}_\mu$  is a diagonal matrix function of  $\mu$  and  $\bar{\lambda}$ .<sup>5</sup> Express  $\mathbf{E}_\mu$  and  $\mathbf{E}_\lambda$  also in terms of  $E$  and  $\nu$ .

<sup>4</sup> Equation (E14.4) is sometimes referred to as the deviatoric+volumetric splitting of the stress-strain law, on account of its physical meaning in plane strain. That interpretation is not fully accurate, however, for plane stress.

<sup>5</sup> For the physical significance of  $\bar{\lambda}$  see [?, pp. 254ff].

**EXERCISE 14.3** [A:20] Include thermoelastic effects in the plane stress constitutive field equations, assuming a thermally isotropic material with coefficient of linear expansion  $\alpha$ . Hint: start from the two-dimensional Hooke's law including temperature:

$$e_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) + \alpha \Delta T, \quad e_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) + \alpha \Delta T, \quad 2e_{xy} = \sigma_{xy}/G, \quad (\text{E14.5})$$

in which  $\Delta T = \Delta T(x, y)$  and  $G = \frac{1}{2}E/(1 + \nu)$ . Solve for stresses and collect effects of  $\Delta T$  in one vector of “thermal stresses.”

**EXERCISE 14.4** [A:15] Derive the Cauchy stress-to-traction equations (14.10) using force equilibrium along  $x$  and  $y$  and the geometric relations shown in Figure E14.1. (This is the “wedge method” in Mechanics of Materials.)

Hint:  $t_x ds = \sigma_{xx} dy + \sigma_{xy} dx$ , etc.

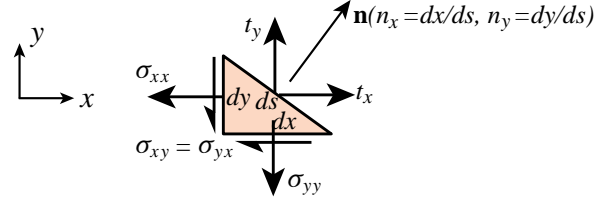


FIGURE E14.1. Geometry for deriving (?).

**EXERCISE 14.5** [A:25=5+5+15] A linearly elastic plate is in plane stress. It is shown in courses in elasticity that the internal strain energy density stored per unit volume of the plate expressed in terms of stresses and strains is the bilinear form

$$\mathcal{U} = \frac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{xy}e_{xy} + \sigma_{yx}e_{yx}) = \frac{1}{2}(\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + 2\sigma_{xy}e_{xy}) = \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{e}. \quad (\text{E14.6})$$

(a) Show that (E14.6) can be written in terms of strains only as

$$\mathcal{U} = \frac{1}{2} \mathbf{e}^T \mathbf{E} \mathbf{e}, \quad (\text{E14.7})$$

thus justifying the strain energy density expression given in (14.13) for the plane stress problem.

(b) Show that (E14.6) can be written in terms of stresses only as

$$\mathcal{U} = \frac{1}{2} \boldsymbol{\sigma}^T \mathbf{C} \boldsymbol{\sigma}, \quad (\text{E14.8})$$

in which  $\mathbf{C} = \mathbf{E}^{-1}$  is the elastic compliance (strain-stress) matrix.

(c) Suppose you want to write (E14.6) in terms of the extensional strains  $\{e_{xx}, e_{yy}\}$  and of the shear stress  $\sigma_{xy} = \sigma_{yx}$ . This is known as a *mixed* representation, which is used in finite elements formulated with mixed variational principles. Show that

$$\mathcal{U} = \frac{1}{2} \begin{bmatrix} e_{xx} \\ e_{yy} \\ \sigma_{xy} \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ \sigma_{xy} \end{bmatrix}, \quad (\text{E14.9})$$

and explain how the entries  $A_{ij}$  of the kernel matrix  $\mathbf{A}$  that appears in (E14.9) can be calculated<sup>6</sup> in terms of the elastic moduli  $E_{ij}$ .

*Hint.* Parts (a,b) are easy. Part (c) is more difficult. It can be symbolically done by the *Mathematica* script

<sup>6</sup> The process of computing  $\mathbf{A}$  is an instance of “partial inversion” of the elasticity matrix  $\mathbf{E}$ . It is closely related to the *Schur complement* concept covered in Appendix P.

```

ClearAll[exx,eyy,gxy,sxx,syy,sxy,E11,E22,E33,E12,E13,E23];
Emat={{E11,E12,E13},{E12,E22,E23},{E13,E23,E33}};
s={sxx,syy,sxy}; e={exx,eyy,gxy}; m={exx,eyy,sxy};
eqs={sxx==E11*exx+E12*eyy+E13*gxy,syy==E12*exx+E22*eyy+E23*gxy,
      sxy==E13*exx+E23*eyy+E33*gxy};
sol=Simplify[Simplify[Solve[eqs,{sxx,syy,gxy}]]];
Print[sol]; U=Simplify[(e.Emat.e/2)/.sol[[1]]];
fac[i_,j_]:=If[i==j,1,1/2];
A=Table[fac[i,j]*Coefficient[U,m[[i]]*m[[j]]],{i,1,3},{j,1,3}];
Print["A=",A//MatrixForm];

```

If you use this solution, make sure to explain what is going on.

Note: the following Table list relations between commonly used moduli for isotropic linear elastic material. Here  $K$  is the bulk modulus whereas  $M$  is the P-wave modulus used in seismology. The table is useful for Exercise 14.2.

	$(\lambda, \mu)$	$(E, \mu)$	$(K, \lambda)$	$(K, \mu)$	$(\lambda, \nu)$	$(\mu, \nu)$	$(E, \nu)$	$(K, \nu)$	$(K, E)$
$K =$	$\lambda + \frac{2\mu}{3}$	$\frac{E\mu}{3(3\mu-E)}$			$\lambda \frac{1+\nu}{3\nu}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{E}{3(1-2\nu)}$		
$E =$	$\mu \frac{3\lambda+2\mu}{\lambda+\nu}$		$9K \frac{K-\lambda}{3K-\lambda}$	$\frac{9K\mu}{3K+\mu}$		$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$2\mu(1+\nu)$		$3K(1-2\nu)$
$\lambda =$		$\mu \frac{E-2\mu}{3\mu-E}$		$K - \frac{2\mu}{3}$		$\frac{2\mu\nu}{1-2\nu}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{3K}{1+\nu}$	$\frac{3K(3K-E)}{9K-E}$
$\mu = G =$			$\mu \frac{K-\lambda}{2}$		$\lambda \frac{K-\lambda}{3K-\lambda}$	$\frac{9K\mu}{3K+\mu}$	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{E}{2(1+\nu)}$	$3K \frac{(1-2\nu)}{2(1+\nu)}$
$\nu =$		$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{E}{2\mu} - 1$	$\frac{\lambda}{3K-\lambda}$	$\frac{3K-2\mu}{2(3K+\mu)}$				$\frac{3K-E}{6K}$
$M =$	$\lambda+2\mu$	$\mu \frac{4\mu-E}{3\mu-E}$	$3K-2\lambda$	$K + \frac{4\mu}{3}$	$\lambda \frac{1-\nu}{\nu}$	$\mu \frac{2-2\nu}{1-2\nu}$	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$3K \frac{1-\nu}{1+\nu}$	$3K \frac{3K+E}{9K-E}$

(E14.10)