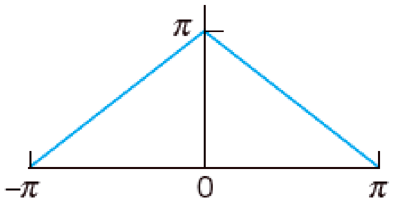
§11.1

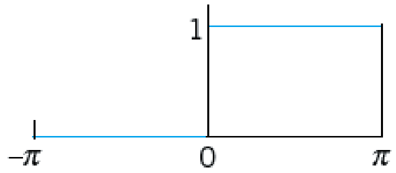
1. (a) If $f(x)$ and $g(x)$ have period p, show that $h(x) = af(x)+bg(x)(a, b$, constant) has the period p.
(b) If $f(x)$ has period p, show that $f(bx)$, $b \neq 0$, is a periodic function of x of period p/b.

2. Find the Fourier series of the given periodic function $f(x)$ of period 2π:

(a)



(b)



(c) $f(x) = x^2 \ (-\pi < x < \pi)$

(d) $f(x) = \begin{cases} -4x & \text{if} \quad -\pi < x < 0 \\ 4x & \text{if} \quad 0 < x < \pi \end{cases}$.

3. Using Problem 2, find the sum of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{2n-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

§11.2

1. Is the given function even or odd? Find its Fourier series.
(a) $f(x) = -1(-2 < x < 0), \ f(x) = 1(0 < x < 2)$ with the period $p = 4$.
(b) $f(x) = \cos \pi x(-\frac{1}{2} < x < \frac{1}{2})$ with the period $p = 1$.
(c) $f(x) = x^2(-1 \leq x \leq 1)$ with the period $p = 2$.
(d) $f(x) = x + 1(-1 < x < 0), \ f(x) = 1 - x(0 < x < 1)$ with the period $p = 2$.

2. Calculate from the problem 1

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)(2n+1)}$

3. Show that the identity $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ can be interpreted as the Fourier series expansion. Develop $\cos^4 x$.

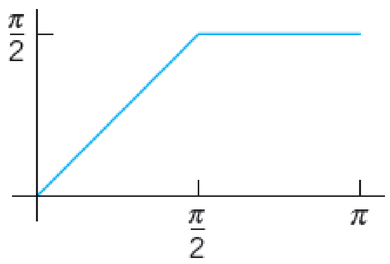
4. Find (I) the Fourier cosine series, (II) the Fourier sine series.

(a) $f(x) = 2 - x \ (0 < x < 2)$ (b) $f(x) = \begin{cases} 0 & \text{if} \quad 0 < x < 2 \\ 1 & \text{if} \quad 2 < x < 4 \end{cases}$.

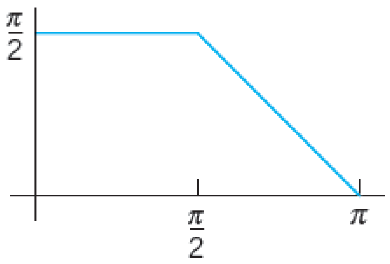
(c) $f(x) = x \ (0 < x < L)$ (d) $f(x) = \sin x \ (0 < x < \pi)$

5. Find the Fourier cosine series.

(a)



(b)



(c) Obtain the Fourier cosine series of (a) from that of (b).

§11.4

1. Find the trigonometric polynomial

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx)$$

for which the square error with respect to the function $f(x)$ on the interval $-\pi \le x \le \pi$ is minimum, and compute the minimum value for $N = 3$.

(a) $f(x) = x \; (-\pi < x < \pi)$ (b) $f(x) = |x| \; (-\pi < x < \pi)$

2. Using Parseval's identity and the given function $f(x)$ of period 2π, show that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$ (Use $f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x < -\pi/2,\ \pi/2 < x < \pi \\ 1 & \text{if} \quad -\pi/2 < x < \pi/2 \end{cases}$)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$ (Use $f(x) = \begin{cases} x+\pi & \text{if} \quad -\pi < x < 0 \\ \pi - x & \text{if} \quad 0 < x < \pi \end{cases}$)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 \cdot (2n+1)^2} = \frac{\pi^2}{16} - \frac{1}{2}$ (Use $f(x) = |\sin x| (-\pi < x < \pi)$.)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ (Use $f(x) = x^2 (-\pi < x < \pi)$)

(e) $\displaystyle\int_{-\pi}^{\pi} \sin^4 x dx = \frac{3}{4}\pi$ (Use $f(x) = \sin^2 x = \frac{1-\cos 2x}{2}$)

(f) $\displaystyle\int_{-\pi}^{\pi} \cos^6 x dx = \frac{5}{8}\pi$ (Use $f(x) = \cos^3 x = \frac{3\cos x + \cos 3x}{4}$)

§11.5

1. Show that the functions

$$P_n(\cos\theta),\ n = 0, 1, \cdots,$$

form an orthogonal set on the interval $0 \le \theta \le \pi$ with respect to the weight function $\sin\theta$. Here, P_n is a Legendre polynomial.

2. Show that

$$y'' + fy' + (g + \lambda h)y = 0$$

takes the Sturm-Liouville form if you set $p = \exp(\int f dx)$, $q = pg$, $r = hp$.

3. Find the eigenvalues and eigenfunctions. Verify orthogonality.
(a) $y'' - 2y' + (\lambda + 1)y = 0, \quad y(0) = 0,\ y(1) = 0$
(b) $y'' + \lambda y = 0, \quad y(0) = y(1),\ y'(0) = y'(1)$
(c) $(x^{-1}y')' + (\lambda + 1)x^{-3}y = 0, \quad y(1) = 0,\ y(e^{\pi}) = 0 \qquad$ (Set $x = e^t$.)

§11.6

1. Find the Fourier-Legendre series of the following functions.
(a) $f(x) = x$ (b) $g(x) = x^2$ (c) $h(x) = x^3$ (d) $h(x) = 7x^4 - 6x^2$

2. Prove that if $f(x)$ is even, its Fourier-Legendre series contains only $P_m(x)$ with even m.

§11.7

1. Show that

(a) $$\int_0^\infty \frac{\cos xw + w\sin xw}{1 + w^2} dw = \begin{cases} 0 & \text{if} \quad x < 0 \\ \pi/2 & \text{if} \quad x = 0 \\ \pi e^{-x} & \text{if} \quad x > 0 \end{cases}.$$

(b) $$\int_0^\infty \frac{(\sin w - w\cos w)\sin xw}{w^2} dw = \begin{cases} \pi x/2 & \text{if} \quad 0 < x < 1 \\ \pi/4 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x > 1 \end{cases}.$$

(c) $$\int_0^\infty \frac{\cos(\pi w/2)\cos xw}{1 - w^2} dw = \begin{cases} \frac{\pi}{2}\cos x & \text{if} \quad |x| < \pi/2 \\ 0 & \text{if} \quad |x| \ge \pi/2 \end{cases}.$$

(d) $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4} dw = \begin{cases} \frac{1}{2}\pi e^{-x}\cos x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \le 0 \end{cases}.$$

2. Find the Fourier cosine integral.

(a) $f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$ (b) $f(x) = \begin{cases} \sin x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

3. Find the Fourier sine integral.

(a) $f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$ (b) $f(x) = \begin{cases} \cos x & \text{if} \quad 0 < x < \pi \\ 0 & \text{if} \quad x > \pi \end{cases}$

4. Let $f(x) = \int_0^\infty A(w)\cos wx\,dw$, where $A(w) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv$. Show that

(a) $f(ax) = \dfrac{1}{a}\displaystyle\int_0^\infty A\left(\frac{w}{a}\right)\cos wx\,dw \ (a > 0)$ (b) $x^2 f(x) = \displaystyle\int_0^\infty \left(-\frac{d^2A}{dw^2}\right)\cos xw\,dw.$

§11.9

1. Find the Fourier transform of $f(x)$.

(a) $f(x) = \begin{cases} e^{2ix} & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$ (b) $f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} e^{kx} & \text{if} \quad x < 0 \quad (k > 0) \\ 0 & \text{if} \quad x > 0 \end{cases}$ (d) $f(x) = \begin{cases} x & \text{if} \quad -1 < x < 1 \\ 0 & \quad \text{otherwise} \end{cases}$

2. (a) Find the Fourier transform of $g(x) = e^{-x}$ if $x > 0$ and 0 otherwise.
(b) Find the Fourier transform of $f(x) = xe^{-x}$ if $x > 0$ and 0 otherwise, using $f'(x) = g(x) - f(x)$.

3. (a) Show that if $f(x)$ has a Fourier transform, so does $f(x-a)$, and $\mathcal{F}[f(x-a)] = e^{-iwa}\mathcal{F}[f(x)]$.
(b) Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w-a)$ is the Fourier transform of $e^{iax}f(x)$.
(c) Using $\hat{f}(w) = \frac{1}{\sqrt{2a}}e^{-\frac{w^2}{4a}}$, where $f(x) = e^{-ax^2}$, $a > 0$, find the Fourier transform of $f(x) = (2x-3)e^{-2x^2+4x+5}$.

4. (a) Find the discrete Fourier transform of the signal $\mathbf{f} = [1, 2, 6, 4]^T$.
(b) Using the fast Fourier transform, solve (a).