# Project 1 - CEE 6755 - Tomas Schmieder

## Step 1: 2D FEM vs. 1D solution with one sink in non-deformable medium

We are considering a reservoir of thickness $H$ and horizontal permeability $k$, fully penetrated by a vertical well of radius $R_w$. The reservoir is sandwiched between two impermeable layers, such that it can be assumed that the fluid flow is purely radial around the well. We assume that at some radius $R_0$ the pressure remains at its undistributed value, $P_0$. Fluid is pumped from this well at a rate of $Q$. We seek to calculate the steady-state pressure distributed in the reservoir. For a non-deformable porous host material, a rigid skeleton, the analytical solutions for the 1D problem is,

$$P(R) = P_0 - \frac{\mu Q}{2\pi k H}\ln\left(\frac{R}{R_0}\right)$$

where $\mu$ is the viscosity of the fluid. We must first propose a one-dimensional FEM model to solve the 1D problem, and then check the numerical results against the analytical solution above. As this is a fluid flow problem through porous media we can think about using mass conservation and its relation to Darcy's Law. Our general form for this is,

$$\nabla \cdot \mathbf{q} = 0, \;\; \mathbf{q} = -\frac{k}{\mu} \cdot \nabla P$$

In cylindrical coordinates $(r, \theta, z)$ with only radial flow, our equation will simplify to this,

$$\frac{1}{R}\frac{d}{dR}(R \cdot q_R) = 0$$

If we then substitute Darcy's law in,

$$q_R = -\frac{k}{\mu}\frac{dP}{dR}$$

We get,

$$\frac{1}{R}\frac{d}{dR}(-R\frac{k}{\mu}\frac{dP}{dR}) = 0$$

Which fully simplifies to,

$$\frac{d}{dR}(R\frac{dP}{dR}) = 0$$

Now we can convert this equation into its weak form so that we can implement FEM. We start by multiplying it by a test function $v(R)$ that integrating over the domain from $R_w \leq R \leq R_0$,

$$\int_{R_w}^{R_0} v(R)\frac{d}{dR}\left(R\frac{dP}{dR}\right)dR = 0$$

After integration by parts,

$$\int_{R_w}^{R_0} R\frac{dP}{dR}\frac{dv}{dR}dR = \left[v(r)R\frac{dP}{dR}\right]_{R_w}^{R_0}$$

Our boundaries are first, pressure at the outer boundary is fixed: $P(R_0) = P_0$, and second, at the well boundary $R = R_w$, there is a volumetric flow rate of $Q_w$ being extracted.
At the radius of $R = R_0$, we have that,

$$P(R_0) = P_0$$

This is our Dirichlet boundary condition, and implies that our test function term will disappear,

$$v(R_0)R_0\frac{dP}{dR}|_{R_0} = 0$$

Our other boundary condition is represented as at $R = R_w$, our flow rate can be found as,

$$Q = q_r(R_w)A = q_r(R_w)2\pi R_w H$$

We can then substitute Darcy's law into this and find that,

$$Q = -\frac{k}{\mu}\frac{dP}{dR}|_{R_w}2\pi R_w H$$

Therefore for our pressure gradient at $R_w$ we have,

$$\frac{dP}{dR}|_{R_w} = -\frac{\mu Q}{2\pi k H R_w}$$

This is our Neumann boundary condition and causes the boundary term to become,

$$v(R_w) R_w \frac{dP}{dR}|_{R_w} = -v(R_w) R_w \frac{\mu Q}{2\pi k H R_w}$$

Now with both of our boundary conditions, we can apply them to our weak form and find that the final weak form is given as,

$$\int_{R_w}^{R_0} R \frac{dP}{dR} \frac{dv}{dR} dR = -v(R_w) \frac{\mu Q}{2\pi k H}$$

Now applying this FEM model to solve the 1D problem we use the PorousFlow module in MOOSE. We assume that the host material is very stiff to mimic a quasi-non-deformable medium, we assume that the fluid is water with a viscosity of $10^{-3}\ Pa.s$, we use a typical well pumping rate of $Q = 5\ \frac{gallons}{minute} = 315 \cdot 10^{-6}\ \frac{m^3}{s}$, we have chosen $P_0$ to be $1\ MPa = 1 \cdot 10^6\ Pa$, we have chosen a reservoir height of $H = 20\ m$. Now we can run our simulation to assess the sensitivity of the pressure distribution to the permeability of the reservoir, we will vary $k$ in the range of $10^{-12}\ m^2$ (sandstones, chalk) to $10^{-8}\ m^2$ (coarse gravel), with values of $10^{-12},\ 10^{-11},\ 10^{-10},\ 10^{-9},\ 10^{-8}$.

To calculate the mass flux value that will be used in our MOOSE code we use the equation,

$$q = \frac{\rho_w Q}{2\pi R_w H}$$

This flux value will have units of $\frac{kg}{m^2 \cdot s}$, our value of flux, we have chosen an inner well radius of $R_w = 1$ m and a height of $H = 20$ m. We get,

$$q = \frac{(1000)(315 \cdot 10^{-6})}{(2)(\pi)(1)(20)} = 0.02506690354$$

We have also chosen a value for the outer radius of our well, as $R_0 = 20\ m$.

Now looking at our MOOSE model for step 1, the mesh looks like the one pictured below,

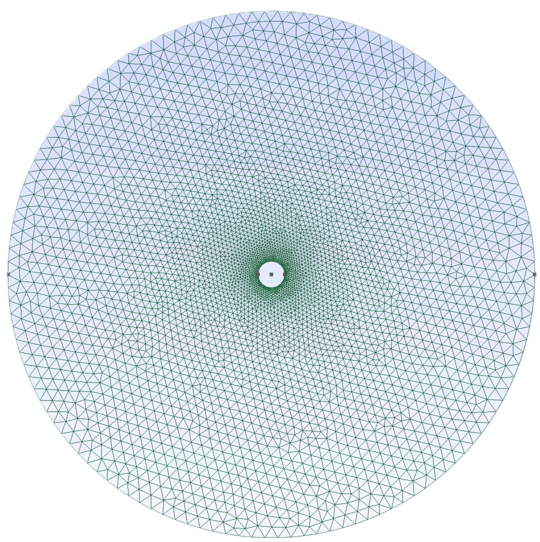

Figure 1. 2D Mesh for Step 1

This mesh represents a 2D axisymmetric domain in the radial plane, it is a horizontal slice taken out of our well. The domain is circular, at the center of our well there is a circular cut-out of radius $R_w = 1\ m$, this cut out represents our well-bore. The outer boundary is the extent of the reservoir domain and consists of radius $R_0 = 20\ m$. Our hydraulic boundary conditions are represented by a Dirichlet boundary condition at our outer radius $R_0$ where the pressure is fixed at the value of $P_0$. As well there is a Neumann boundary condition at the inner radius $R_w$ where we have a constant mass flux being extracted with the value of $q$.

The results of our MOOSE simulation and plotting our analytical vs. numerical calculated values with varying permeability can be seen below,

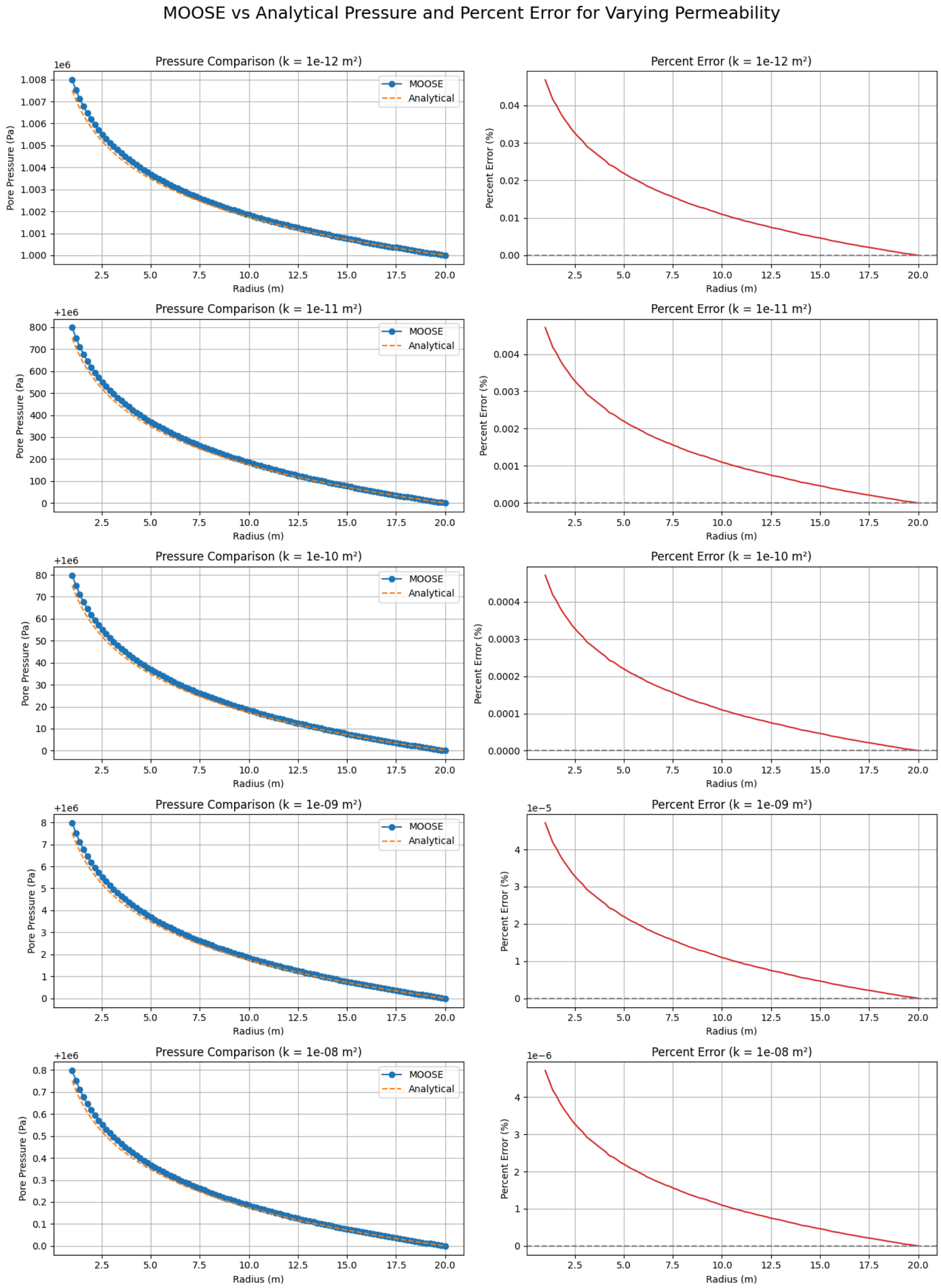


Figure 2. Plot of MOOSE vs. Analytical Pressure and Percent Error for Varying Permeability

Looking at the results of our model we find that, with higher permeability values our percent error between the MOOSE model and the analytical solution decreases, or lower permeabilities have higher error. When we look at Darcy's law we see that for the same flux value, when permeability is smaller, the system must then generate a steep pressure gradient, especially closer to the borehole. The FEM model in MOOSE is likely having difficulty dealing with the steep pressure gradient created with these lower permeabilities, causing it to have more percent error.

## Step 2: 2D model with one source or sink

The next step is to model a problem of injection and withdrawal in 2D, with an account for deformability of the porous skeleton. The system of partial differential equations that we are solving for consists of:
Governing equation for the solid phase:

$$-\alpha\nabla(p_w) + \mathbf{D_e} : \nabla\epsilon = 0$$

Governing equation for the liquid phase:

$$\left(\frac{\alpha-\phi}{K_s}+\frac{\phi}{K_w}\right)\frac{D^s p_w}{Dt}+\alpha\frac{\partial\epsilon_V}{\partial t}=\frac{1}{\mu_k}\mathbf{k}:\nabla^2 p_w$$

We begin modeling our problem by writing our governing equation in 2D, using cylindrical coordinates. Looking at the governing equation for the solid phase, assuming axial symmetry and cylindrical coordinates we will only take into account the variables $(r, z)$ as $\theta$ will have no variation in its direction. To begin, our pressure gradient can be said to equal,

$$\nabla(p_w)=\frac{\partial p_w}{\partial r}\hat{r}+\frac{\partial p_w}{\partial z}\hat{z}$$

Our other term can be viewed as the divergence of our stress tensor and can be seen as this in cylindrical coordinates,

$$\mathbf{D_e}:\nabla\epsilon=\nabla\cdot\sigma=\begin{bmatrix}\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}\\ \frac{\partial\sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}+\frac{\partial\sigma_{zz}}{\partial z}\end{bmatrix}=\begin{bmatrix}(\nabla\cdot\sigma)_z\\(\nabla\cdot\sigma)_r\end{bmatrix}$$

Therefore our governing equation for the solid phase in cylindrical coordinates is,

$$-\alpha\left(\frac{\partial p_w}{\partial r}\hat{r}+\frac{\partial p_w}{\partial z}\hat{z}\right)+\begin{bmatrix}(\nabla\cdot\sigma)_r\\(\nabla\cdot\sigma)_z\end{bmatrix}=0$$

or also,

$$-\alpha\frac{\partial p_w}{\partial r}+\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{\partial\sigma_{rz}}{\partial z}=0$$

$$-\alpha\frac{\partial p_w}{\partial z}+\frac{\partial\sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}+\frac{\partial\sigma_{zz}}{\partial z}=0$$

Looking at the governing equation for the liquid phase we have,

$$\left(\frac{\alpha-\phi}{K_s}+\frac{\phi}{K_w}\right)\frac{D^s p_w}{Dt}+\alpha\frac{\partial\epsilon_V}{\partial t}=\frac{1}{\mu_k}\mathbf{k}:\nabla^2 p_w$$

First let us note that, $S=\left(\frac{\alpha-\phi}{K_s}+\frac{\phi}{K_w}\right)$, we can also change our gradient term, on the right-hand side. We see that $\mathbf{k}:\nabla^2 p_w$, is a double contraction between the permeability tensor $k$ and the second gradient of pressure. We can evaluate this in cylindrical coordinates as,

$$\mathbf{k}:\nabla^2 p_w=k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right)+\frac{\partial^2 p_w}{\partial z^2}\right)$$

Now looking at our $\frac{D^s p_w}{Dt}$, if we assume small deformations then we can assume that,

$$\frac{D^s p_w}{Dt}\approx\frac{\partial p_w}{\partial t}$$

So now for our liquid phase in cylindrical coordinates, we have,

$$S\frac{\partial p_w}{\partial t}+\alpha\frac{\partial\epsilon_V}{\partial t}=\frac{k}{\mu_k}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right)+\frac{\partial^2 p_w}{\partial z^2}\right)$$

Therefore our governing equations in cylindrical coordinates for our 2D model are,

$$-\alpha\left(\frac{\partial p_w}{\partial r}+\frac{\partial p_w}{\partial z}\right)+\nabla_{rz}\cdot\sigma=0$$

$$S\frac{\partial p_w}{\partial t}+\alpha\frac{\partial\epsilon_V}{\partial t}=\frac{k}{\mu_k}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_w}{\partial r}\right)+\frac{\partial^2 p_w}{\partial z^2}\right)$$

Now to simulate the 2D problem using MOOSE, we assume the same reservoir thickness, boundary conditions, and material properties as in Step 1, however, the properties of the reservoir are those of a rock. We begin by analyzing the sensitivity of the response to the value of $Q$ varying it between 2-6 $\frac{gallons}{minute}$, where for each value of $Q$ we convert the value to a mass flux.

$$Q=6\ \frac{gallons}{minute}=0.000378\ \frac{m^3}{s},\ \ q=0.00301233356\ \frac{kg}{m^2\cdot s}$$

$$Q=5\ \frac{gallons}{minute}=0.000315\ \frac{m^3}{s},\ \ q=0.00250995303\ \frac{kg}{m^2\cdot s}$$

$$Q=4\ \frac{gallons}{minute}=0.000252\ \frac{m^3}{s},\ \ q=0.00200822503\ \frac{kg}{m^2\cdot s}$$

$$Q=3\ \frac{gallons}{minute}=0.000189\ \frac{m^3}{s},\ \ q=0.001506170762\ \frac{kg}{m^2\cdot s}$$

$$Q=2\ \frac{gallons}{minute}=0.000126\ \frac{m^3}{s},\ \ q=0.001004108536\ \frac{kg}{m^2\cdot s}$$

Now looking at our MOOSE model for step 2, the mesh looks like the one pictured below,

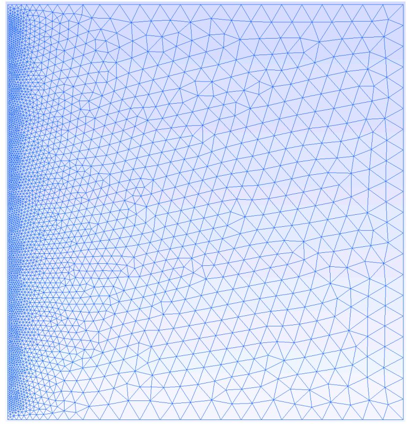

Figure 3. 2D Mesh in RZ for Step 3

This mesh represents a 2D vertical slice out of the cylindrical well, in the R-Z coordinate system, the left side of the mesh is where our well is and the right side is the outer part. The mesh spans from $R_w$ to $R_0$ in the R direction, where the value 1 is the radius of our well in the middle. The Z direction spans from the values of 0 to $H = 20\ m$. If we look at what happens with constant flux for different values we find a graph similar to that of the one produced in step 1,

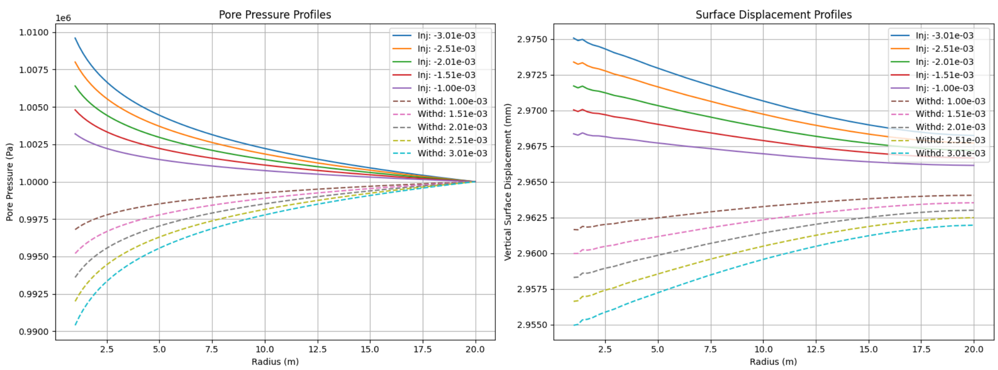


Figure 4. Pore Pressure and Surface Displacement vs. Radius Graphs for Varying Flux Values

If we simulate a cycle of equal injection and withdrawal we get results in the figure below, our mesh and hydro-mechanical boundary conditions were the same as Step 2, except for the flux. This was done by using our original flux value and applying a sinusoidal function to it so that the flux would vary from the value it currently is from -1 to 1 over a certain time step, after 1,000 seconds 2 full cycles occurred with the setup,

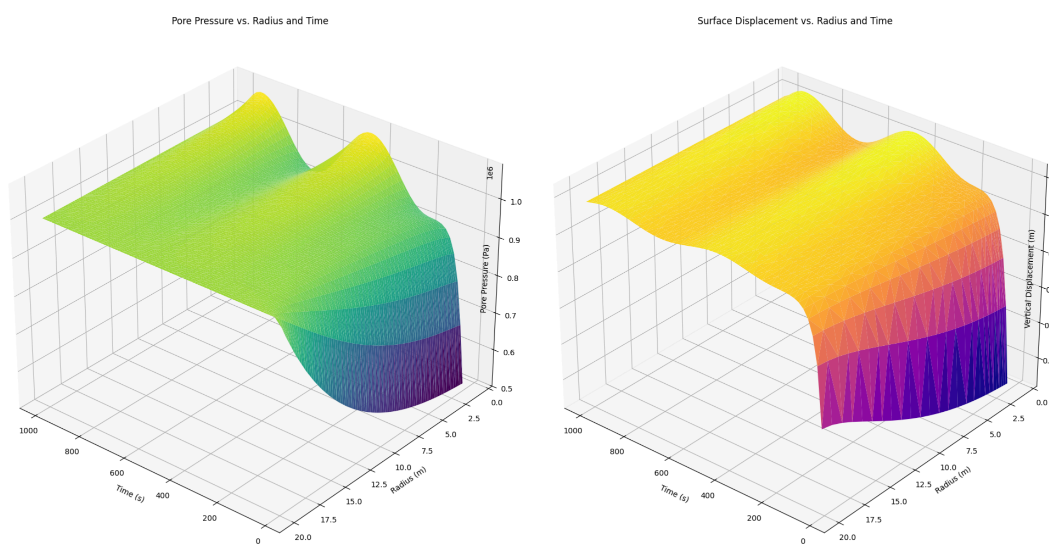


Figure 5. Time vs. Radius vs. Pore Pressure for Cycles of Injection and Withdrawal

The well starts with a uniform pressure of 1 $MPa$ across the entire domain, the sinusoidal flux is then applied at the well, as water is withdrawn from the well overtime, the pressure near the well drops sharply, during injection, the pressure rises again, this can be seen in the figure from the bumps on the plot. As time goes on, fluid flow and the stress redistribute the pressure, causing the system to approach an equilibrium and the pressure variation becomes consistent, which makes the radial pressure gradient flatten out, except when its close to the well. Similarly, vertical displacement at the beginning near the borehole is low and then steeply increases until it becomes a cycle overtime, where as radially it also flattens out. If we were to study the influence of uneven injection and pumping operations it would results in similar graphs but ones that do not end up in an cycle, rather there would be unnatural pressure gradients. If the injection flux was larger than that of the withdrawal flux then over time we would expect to see an accumulation of pore pressure in the domain. Since more fluid is being added than removed, the average pressure would gradually increase rather than stabilize or return to its original value. As this pore pressure would build up, there would be increase in surface displacement, the displacement curves would continue to increase. Similarly, if the withdrawal was larger than the injection then we would expect to observe a decrease in average pore pressure. This would lead to the surface displacement decreases over time with each cycle.

## Step 3: 2D model with a source and sink

Our last step is to model a pair of two wells, here we will focus on the stress and pressure distribution in the horizontal plane at a depth $z$, the operation of both wells being used simultaneously will be modeled where one is injected fluid and the other is withdrawing fluid. Now looking at our MOOSE model for step 3, the mesh looks like the one pictured below,

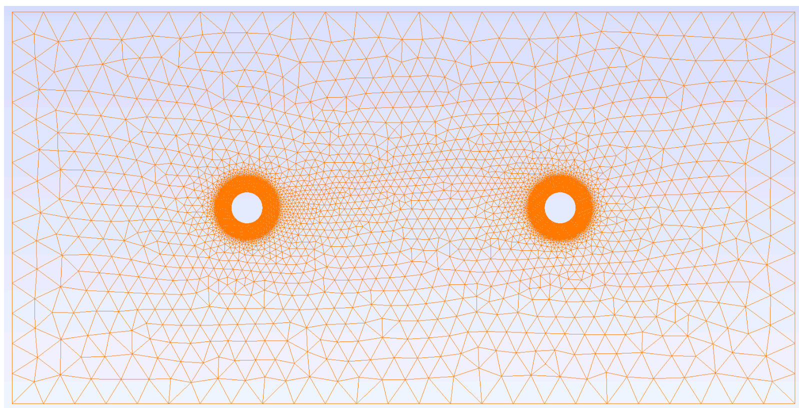

Figure 6. 2D Mesh in XY for Step 3

This mesh spans the XY plane, in X it goes from values of 0 to 50, in Y it goes from values of 0 to 25. The mesh acts as an axisymmetric representation of two boreholes where one borehole is subjected to injection and the other withdrawal, these boreholes are located at the points (15, 12.5) and (35, 12.5) on their radius spans 1 meter. Note that in our MOOSE analysis we must jump over these gaps. Lets first analyze the sensitivity of the response for both pore pressure and surface displacement to the ratio of $\frac{Q_{injection}}{|Q_{withdrawal}|}$,

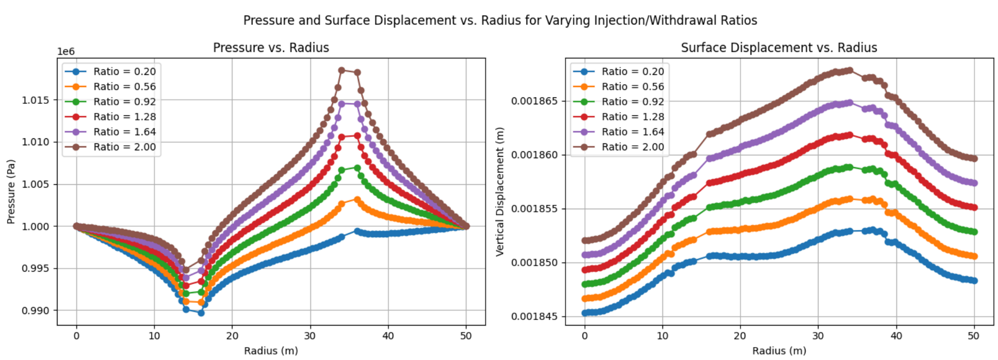


Figure 7. Pressure and Displacement vs. Radius for Varying Injection/Withdrawal Ratios

We see from the graphs above that as the ratio of $\frac{Q_{injection}}{|Q_{withdrawal}|}$ increases then our pore pressure also increases, however as the ratio increases then our vertical displacement will decrease. Now, lets analyze for a given ratio of $\frac{Q_{injection}}{|Q_{withdrawal}|}$, say 1, lets compare the response of the host medium in drained vs. undrained conditions,

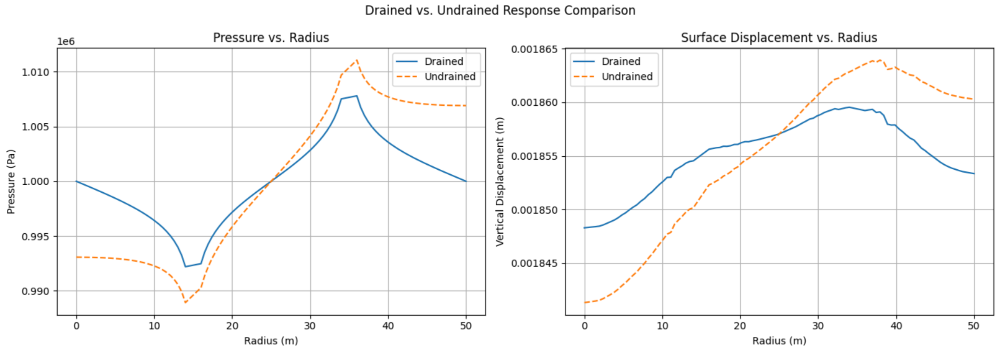


Figure 8. Pressure and Displacement vs. Radius for Drained and Undrained Medium

In the drained conditions the left and right boundaries are held at a constant pressure of 1 MPa, allowing for fluid to leave the domain. For the undrained conditions the pressure at the boundaries is not constrained, allowing for fluid to build up internally, not letting any fluid leave the domain. In the drained plots we see that the pressure drains to the boundaries causing for there to be a dip in pressure near the wells and then a return to 1 MPa at the edge. In the undrained conditions since pressure isn't allowed to leave, fluid ends up accumulating more, which results in higher pore pressure. For vertical displacement the undrained case shows more an increase in surface displacement, this is due to their being more fluid trapped in the pores which is causing the medium to expand. In contrast, the drained case allows fluid to leave, relieving pressure and stopping the deformation from happening. Now lastly, for a given $\frac{Q_{injection}}{|Q_{withdrawal}|}$, lets again say 1, lets analyze the sensitivity of the FE predictions to the well spacing in drained conditions, here we will analyze two cases, a case where the spacing between the wells is 20 m and another where the spacing is 5 m,

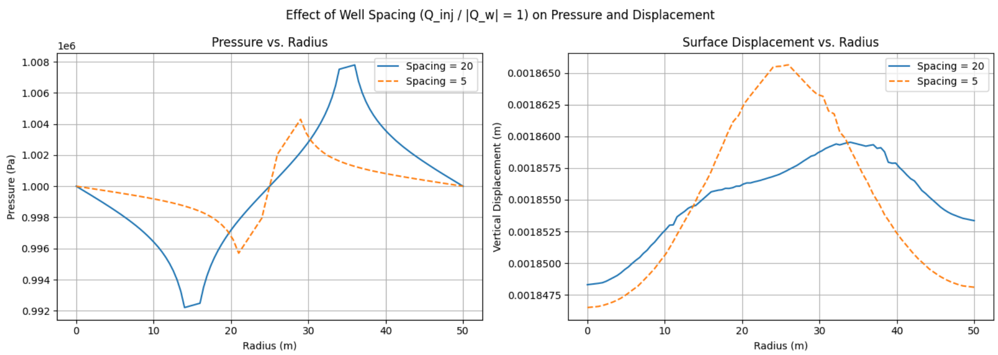


Figure 9. Pressure and Displacement vs. Radius for Different Well Spacing

Looking at our differences in pressure, we see that when the spacing is smaller the interaction between the two wells is stronger, so the individual effect of each well overlaps with each other, this ultimately causes the pressure to not increase to as high levels were the spacings larger. Less space means more pressure buildup and drawdown interfering with each other. Looking at the surface displacement differences, we see that when the spacing is smaller the deformation becomes much larger and more apparent at a single point. This happens because the effects of the injection and withdrawal are interacting more with each other, causing stress to be concentrated in a more focused area.

## Works Cited

All of the MOOSE resources posted on canvas that were created by Chloe Arson and Alec Tristani.
"Radial Flow from an Injection Well." *MOOSE*, mooseframework.inl.gov/modules/porous*flow/1Dradial.html.*
*"Porous Flow Tutorial Page 11. Two-phase THM borehole injection." _MOOSE*, https://mooseframework.inl.gov/modules/porous_flow/tutorial_11.html.
"Porous Flow Tutorial Page 04. Adding solid mechanics" *MOOSE*, https://mooseframework.inl.gov/modules/porous_flow/tutorial_04.html.
User GiudGiud on Github helped me install MOOSE answering my questions on a Github thread
Acknowledgments: Portions of the code troubleshooting were assisted by *OpenAI's ChatGPT*, a large language model. The tool was used for debugging and mesh generation for use in Python and MOOSE. All code and decisions were reviewed and verified by the author.

## Note

All of my files were uploaded into a github repository found at this link, please look at that for access to my code as I couldn't upload all of it onto Canvas, open up the readme file to see what files are actually relevant to be looked at or graded.
https://github.com/windy-schmieder/CEE-6755-Project1