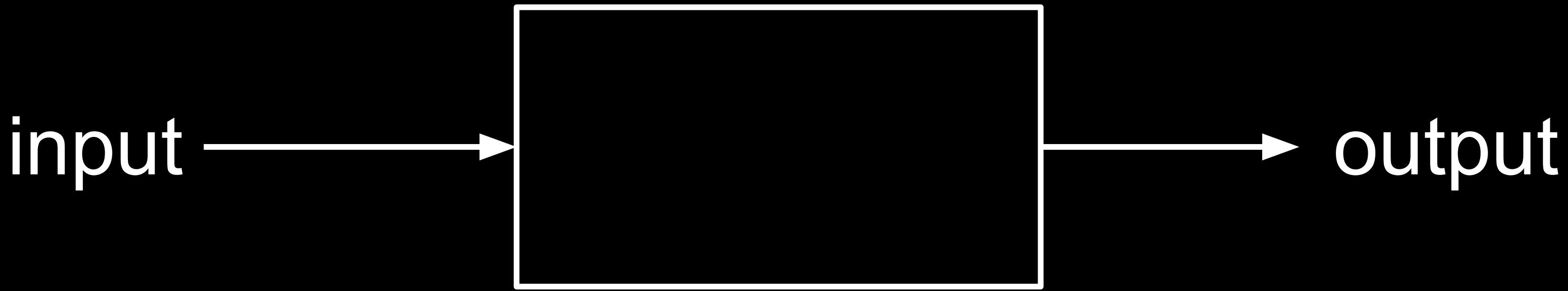
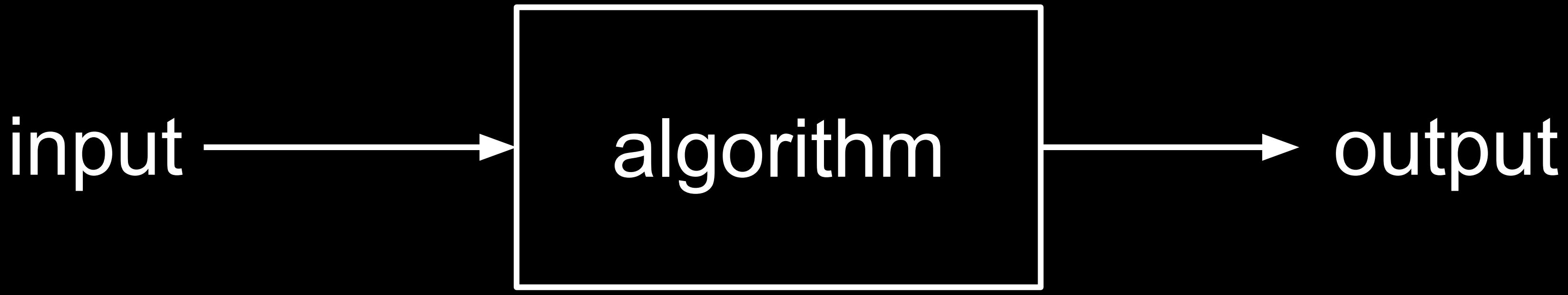
ALGORITHMS

BACK

who solves the problem?



%%algorithms_computational_schema_blackbox%%



%%algorithms_computational_schema_algorithm%%

algorithm, program, process

"A finitely long rule consisting of individual instructions is called an algorithm."

Source: Vornberger, O., Algorithmen und Datenstrukturen, Lecture notes: http://www-lehre.inf.uos.de/~ainf/2013/PDF/skript.pdf

"A program is an algorithm expressed in a programming language."

Source: Vornberger, O., Algorithmen und Datenstrukturen, Lecture notes: http://www-lehre.inf.uos.de/~ainf/2013/PDF/skript.pdf

"A process is a program that is currently executed by a computer."

Source: Vornberger, O., Algorithmen und Datenstrukturen, Lecture notes: http://www-lehre.inf.uos.de/~ainf/2013/PDF/skript.pdf

$$\text{input} \longrightarrow \boxed{\;?\;} \longrightarrow \text{output}$$

greatest common divisor

euclidean algorithm

Identify the larger number. If $a < b$, swap numbers so that $a > b$

Subtract b from a and replace a with the result

Repeat until one of the numbers becomes 0

Return the number that is not zero

%%algorithms_euclidean_textual%%

Loop 1:

a = 18, b = 48 → swap

Loop 1:
a = 18, b = 48 → swap → a = 48, b = 18
a = 48 - 18 = 30

Loop 1:

```
a = 18, b = 48 → swap → a = 48, b = 18
a = 48 - 18 = 30
```

Loop 2:

```
a = 30, b = 18 → no swap
a = 30 - 18 = 12
```

Loop 1:
a = 18, b = 48 ⟶ swap ⟶ a = 48, b = 18
a = 48 - 18 = 30

Loop 2:
a = 30, b = 18 ⟶ no swap
a = 30 - 18 = 12

Loop 3:
a = 12, b = 18 ⟶ swap ⟶ a = 18, b = 12
a = 18 - 12 = 6

```
Loop 1:
a = 18, b = 48 ⟶ swap ⟶ a = 48, b = 18
a = 48 - 18 = 30

Loop 2:
a = 30, b = 18 ⟶ no swap
a = 30 - 18 = 12

Loop 3:
a = 12, b = 18 ⟶ swap ⟶ a = 18, b = 12
a = 18 - 12 = 6

Loop 4:
a = 6, b = 12 ⟶ swap ⟶ a = 12, b = 6
a = 12 - 6 = 6
```

Loop 1:
a = 18, b = 48 → swap → a = 48, b = 18
a = 48 - 18 = 30

Loop 2:
a = 30, b = 18 → no swap
a = 30 - 18 = 12

Loop 3:
a = 12, b = 18 → swap → a = 18, b = 12
a = 18 - 12 = 6

Loop 4:
a = 6, b = 12 → swap → a = 12, b = 6
a = 12 - 6 = 6

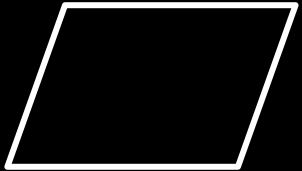
Loop 5:
a = 6, b = 6 → no swap
a = 6 - 6 = 0

```
Loop 1:
a = 18, b = 48 → swap → a = 48, b = 18
a = 48 - 18 = 30

Loop 2:
a = 30, b = 18 → no swap
a = 30 - 18 = 12

Loop 3:
a = 12, b = 18 → swap → a = 18, b = 12
a = 18 - 12 = 6

Loop 4:
a = 6, b = 12 → swap → a = 12, b = 6
a = 12 - 6 = 6

Loop 5:
a = 6, b = 6 → no swap
a = 6 - 6 = 0

return b = 6
```

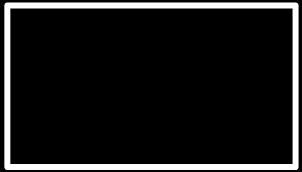
%%algorithms_euclidean_example%%

flow diagrams

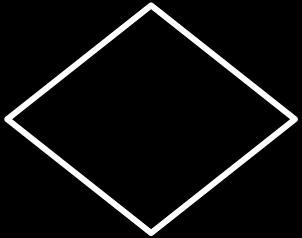
start / end of algorithm



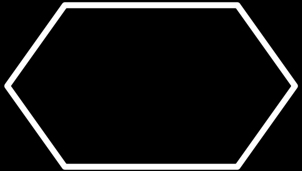
input / output



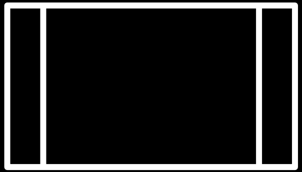
command / assignment



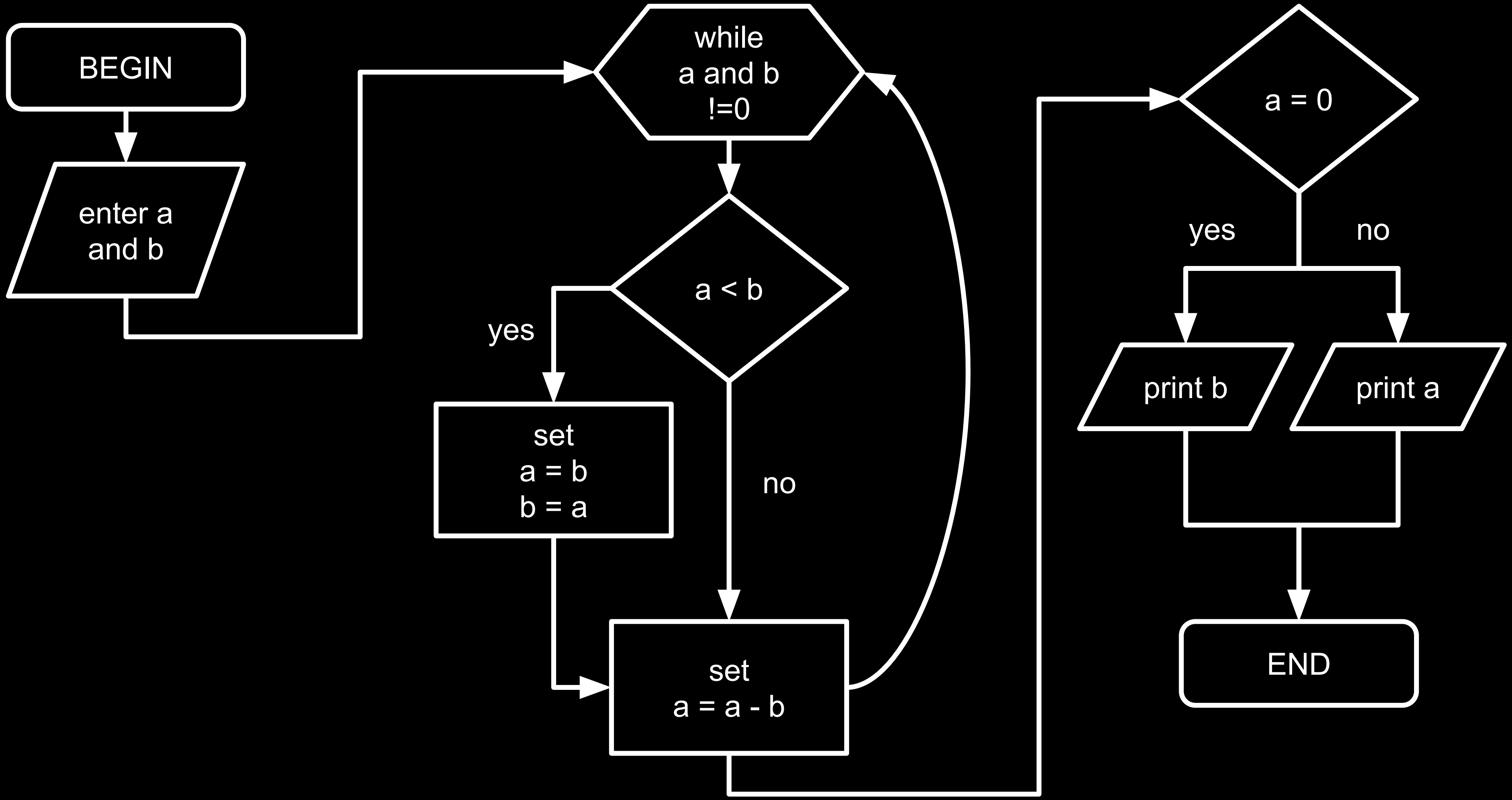
decision



repetition



external routine


BEGIN
enter a
and b
while
a and b
!=0
a < b
yes
set
a = b
b = a
no
set
a = a - b
a = 0
yes
no
print b
print a
END

square roots

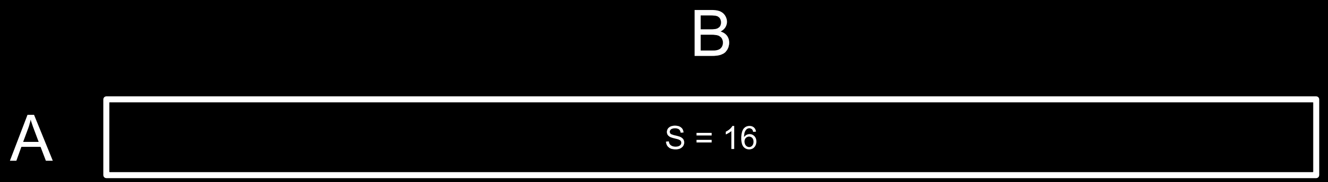
babylonian method

calculate square root of

$x = 16$

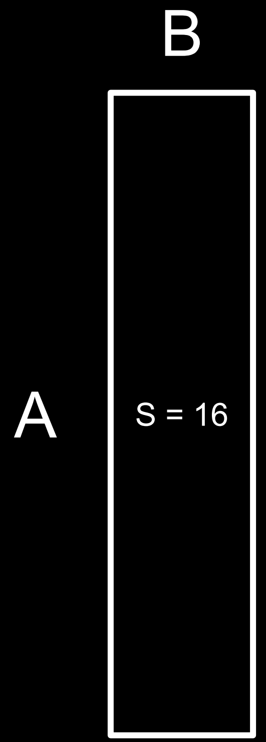
$A = 1$

$B = X / A = 16$



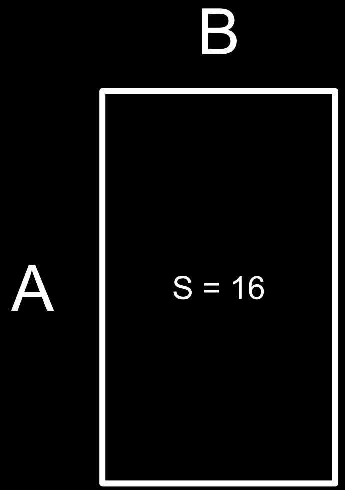
$$A = (A + B) / 2 = 17 / 2 = 8.5$$

$$B = X / A = 16 / 8.5 \approx 1.88$$



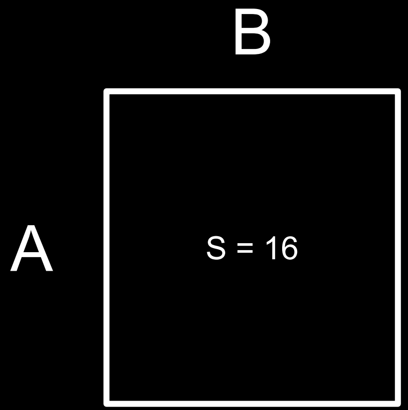
$$A = (A + B) / 2 \approx 10.38 / 2 \approx 5.19$$

$$B = X / A \approx 16 / 5.19 \approx 3.08$$



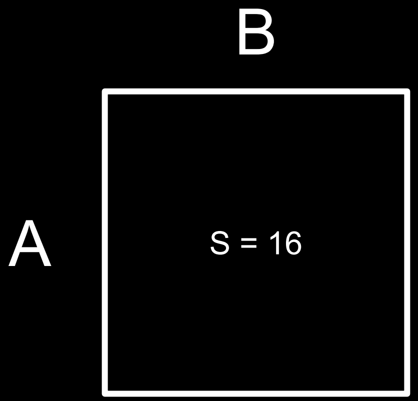
$$A = (A + B) / 2 \approx 8.27 / 2 \approx 4.14$$

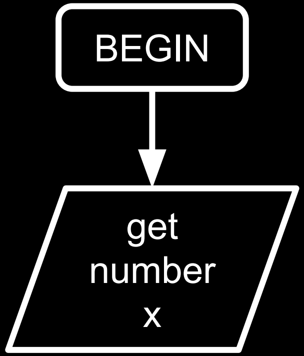
$$B = X / A \approx 16 / 4.14 \approx 3.86$$

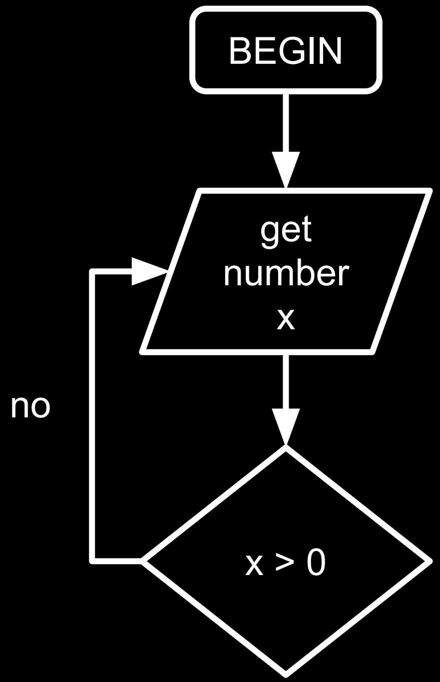


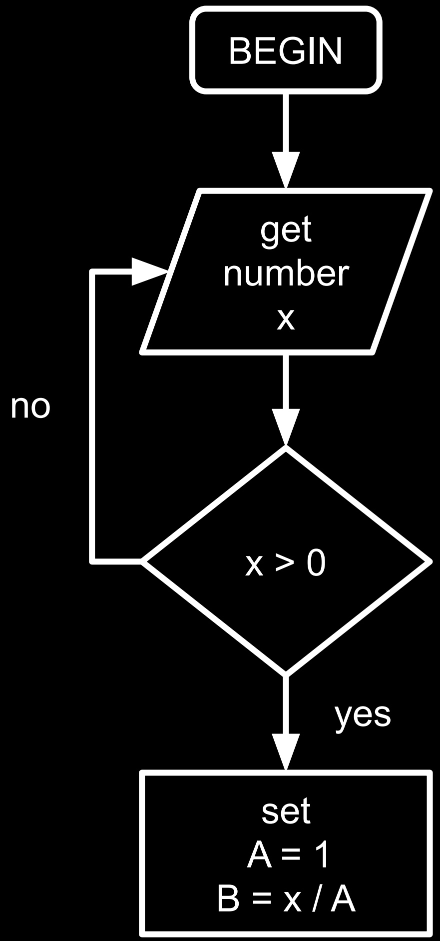
$$A = (A + B) / 2 = 8 / 2 = \mathbf{4}$$

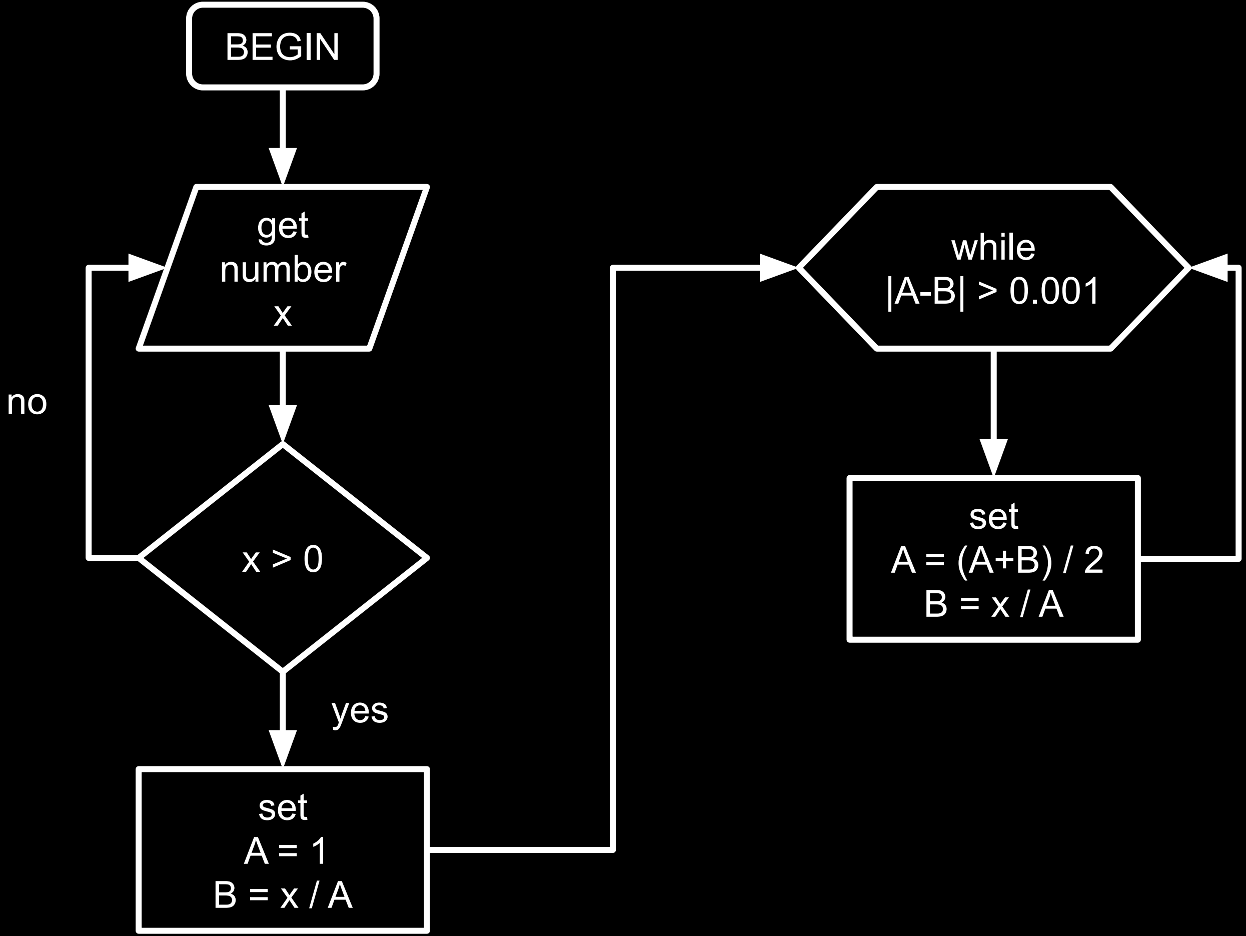
$$B = X / A = 16 / 4 = \mathbf{4}$$

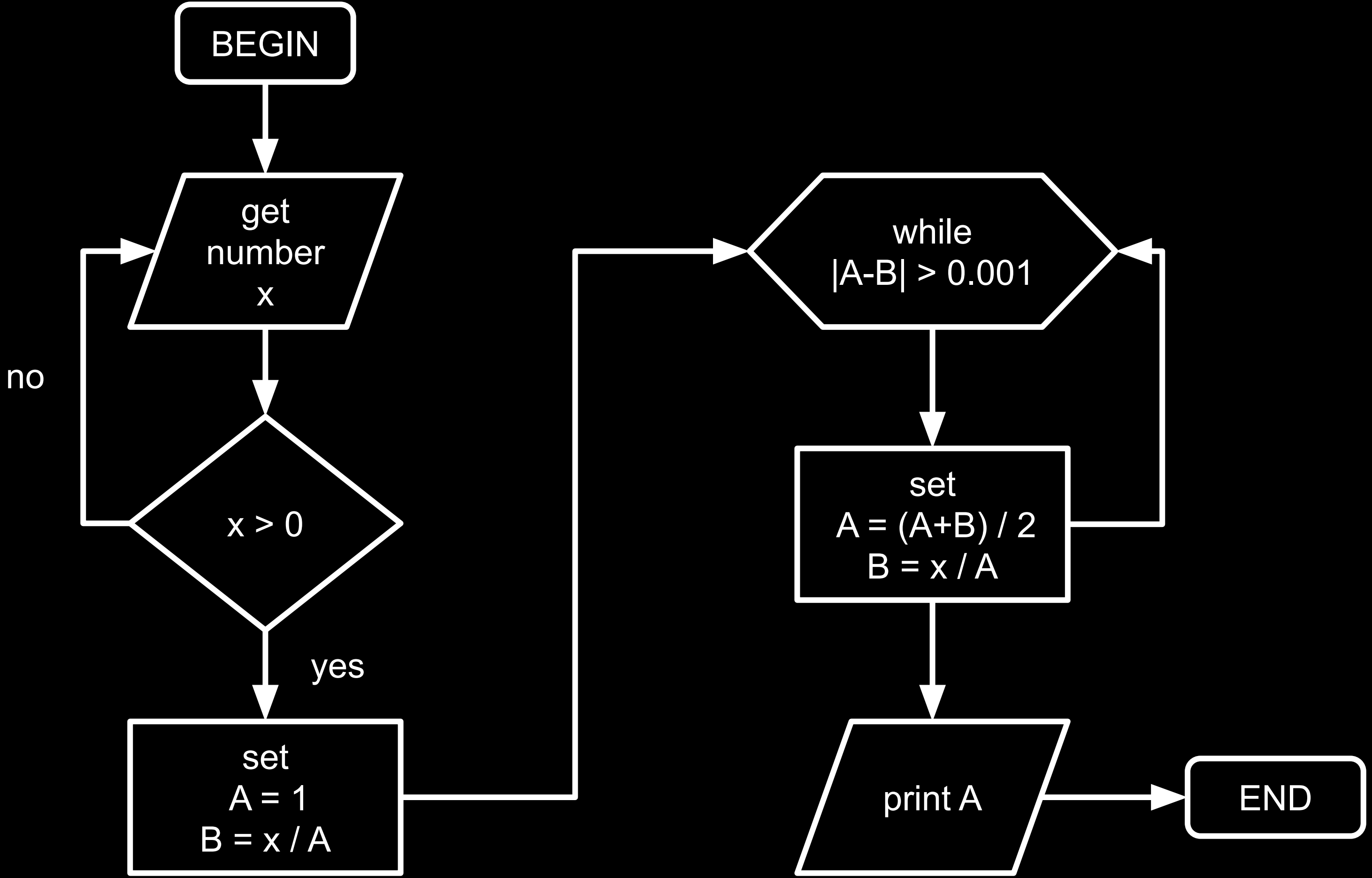



BEGIN
get number x

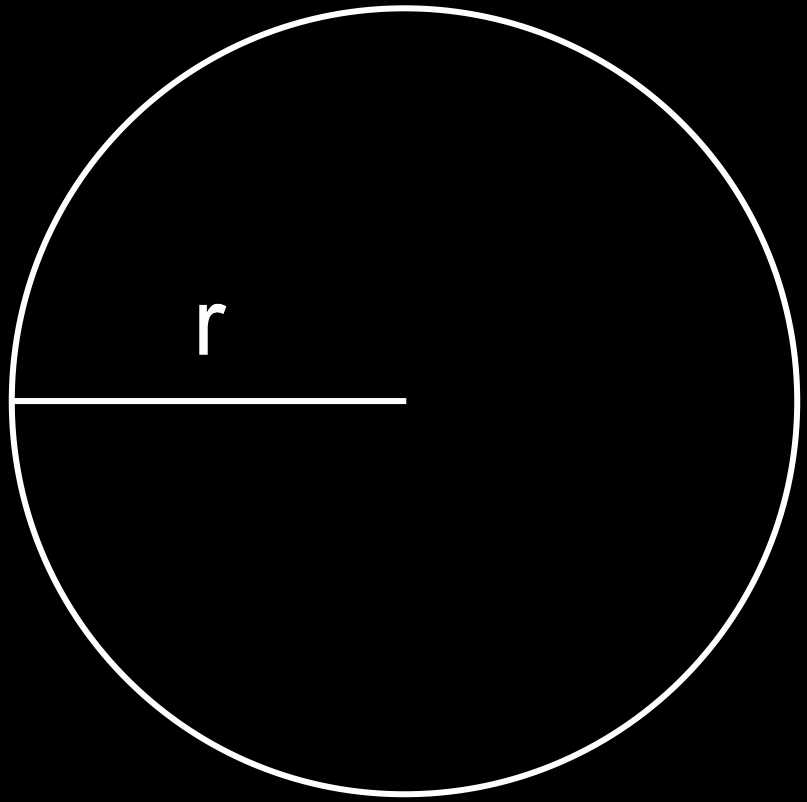

BEGIN
get
number
x
no
x > 0

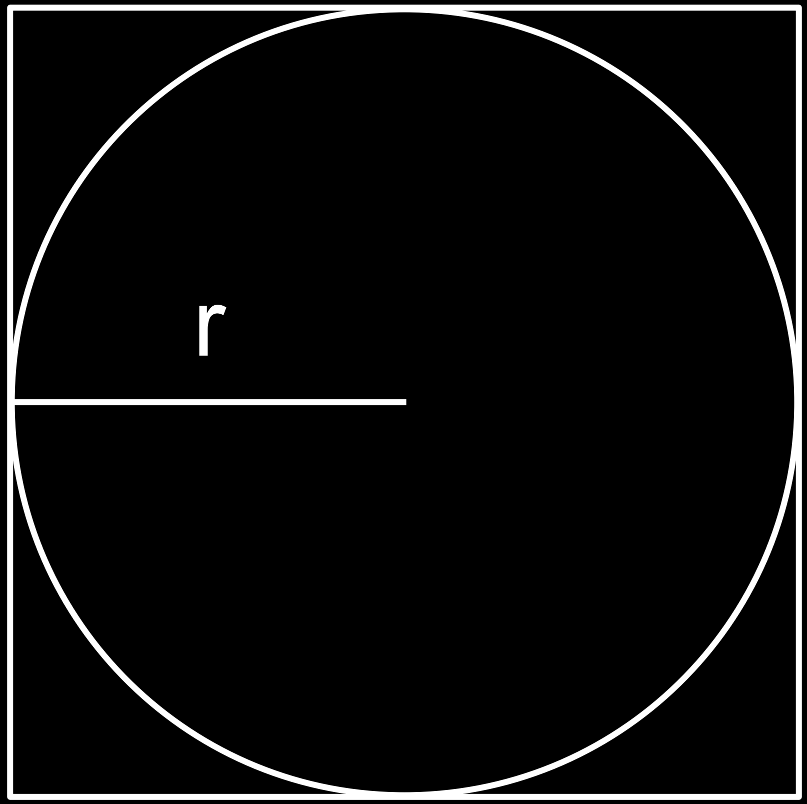

BEGIN
get
number
x
no
x > 0
yes
set
A = 1
B = x / A

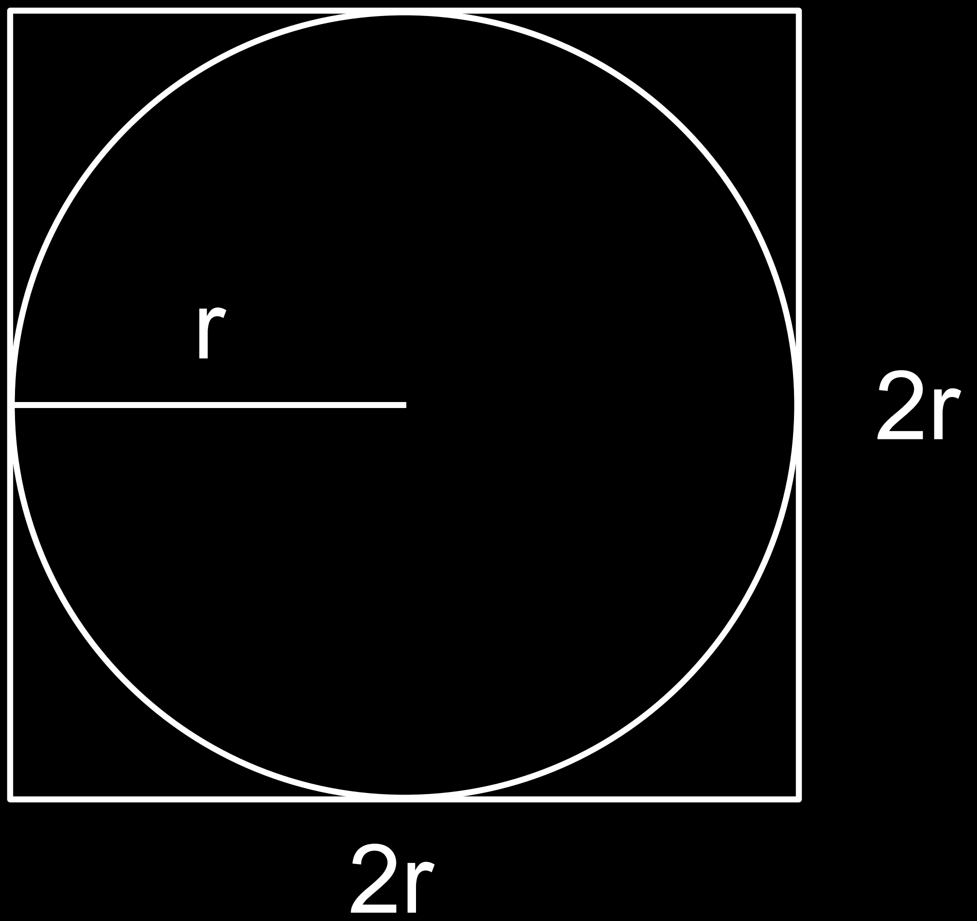

BEGIN
get number x
x > 0
no
yes
set A = 1 B = x / A
while |A-B| > 0.001
set A = (A+B) / 2 B = x / A

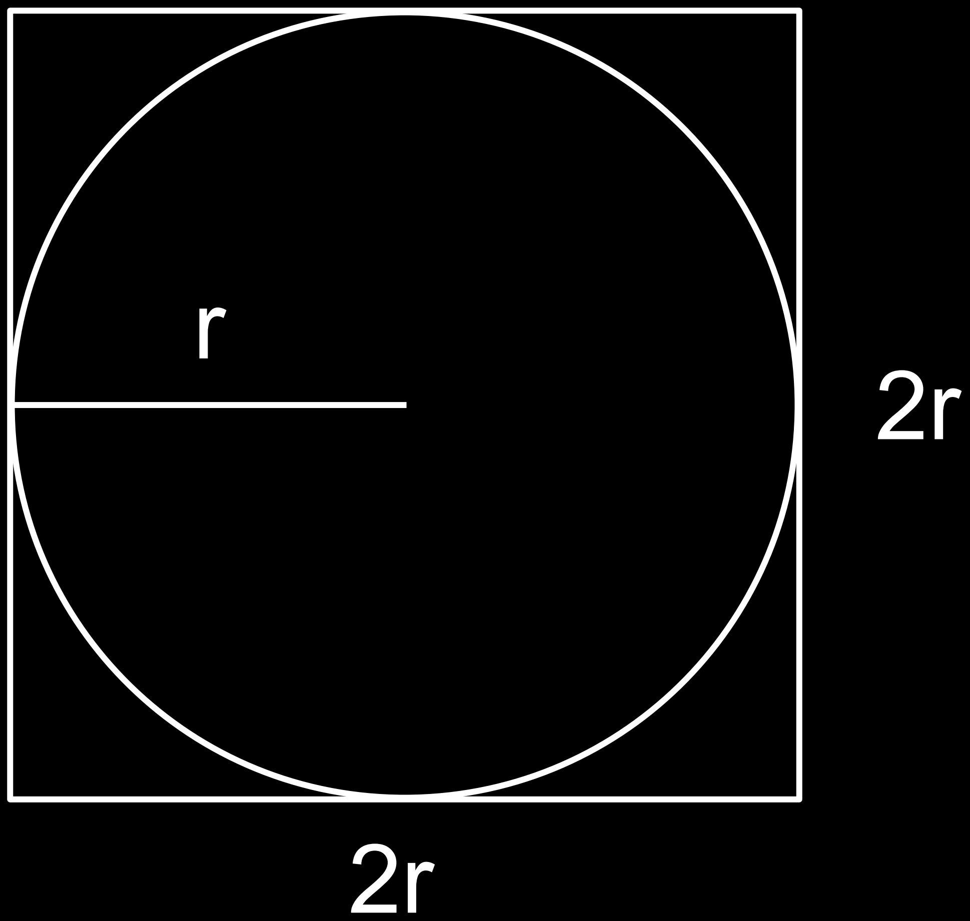

BEGIN
get number x
no
x > 0
yes
set A = 1 B = x / A
while |A-B| > 0.001
set A = (A+B) / 2 B = x / A
print A
END

estimating π

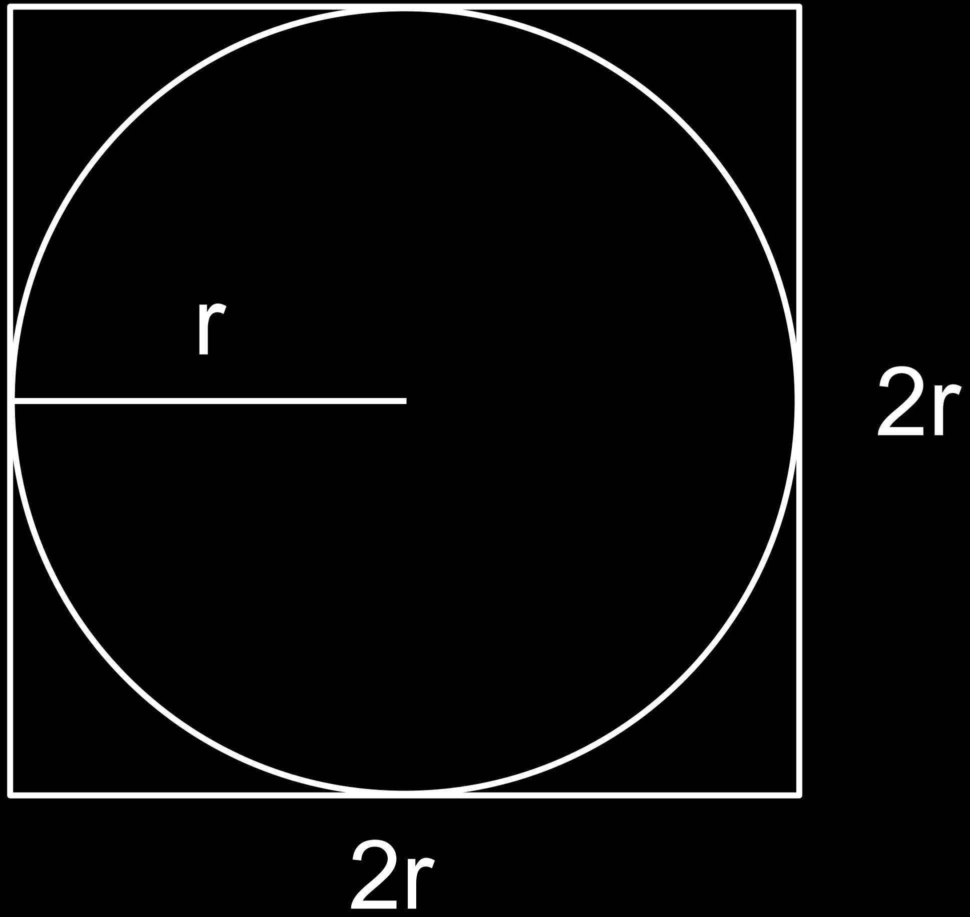

r


r

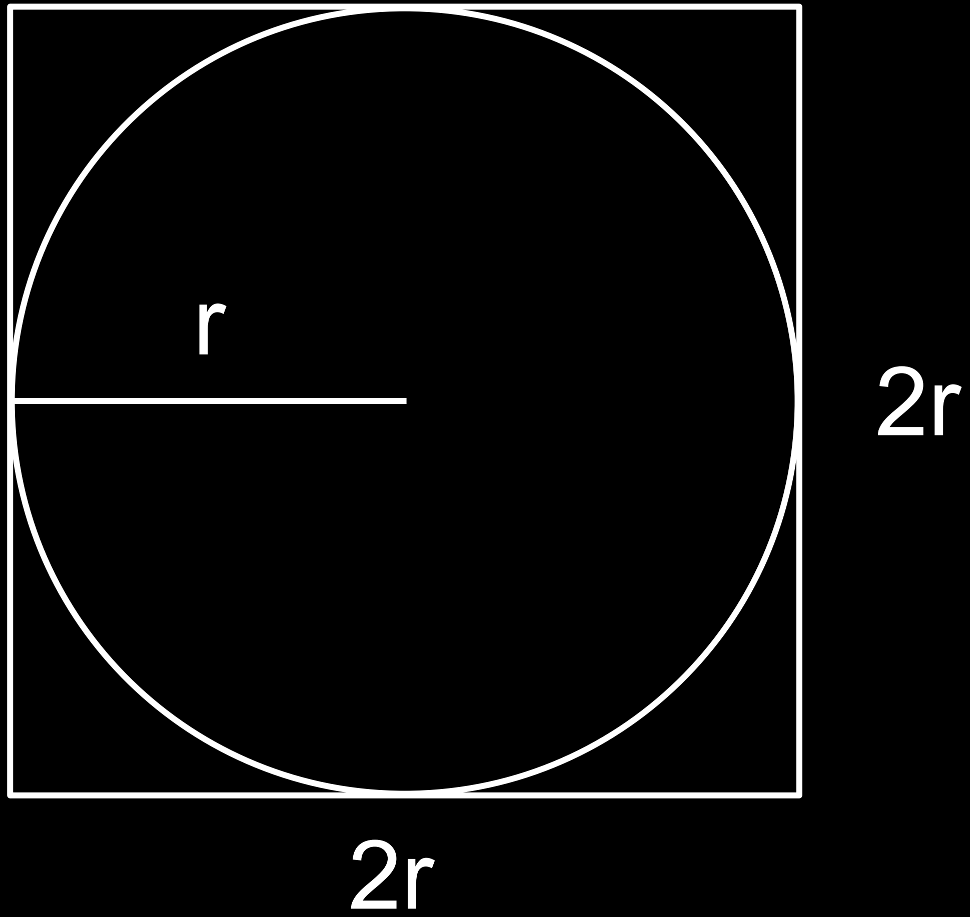

r
2r
2r



$$\frac{\bigcirc}{\square} = \frac{\pi r^2}{4r^2}$$

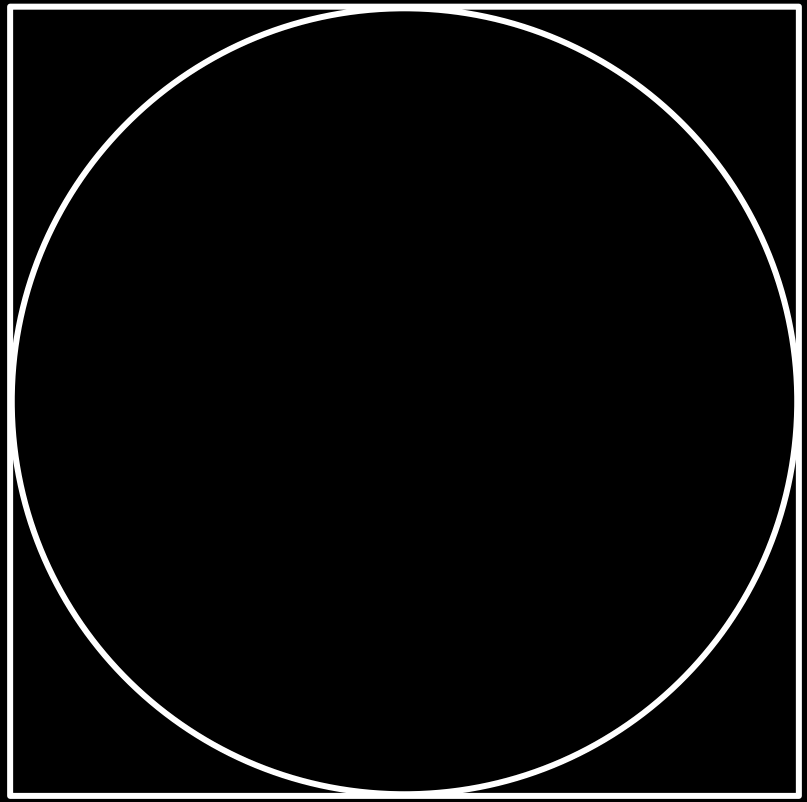


$$\frac{\bigcirc}{\square} = \frac{\pi \cancel{r^2}}{4\cancel{r^2}}$$

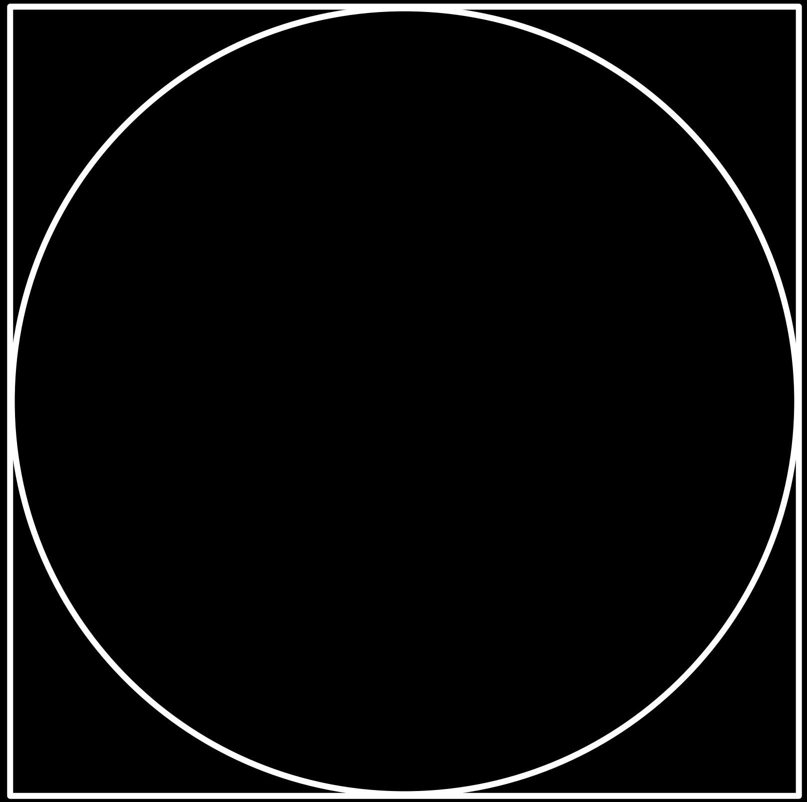


$$\frac{\bigcirc}{\square} = \frac{\pi}{4}$$

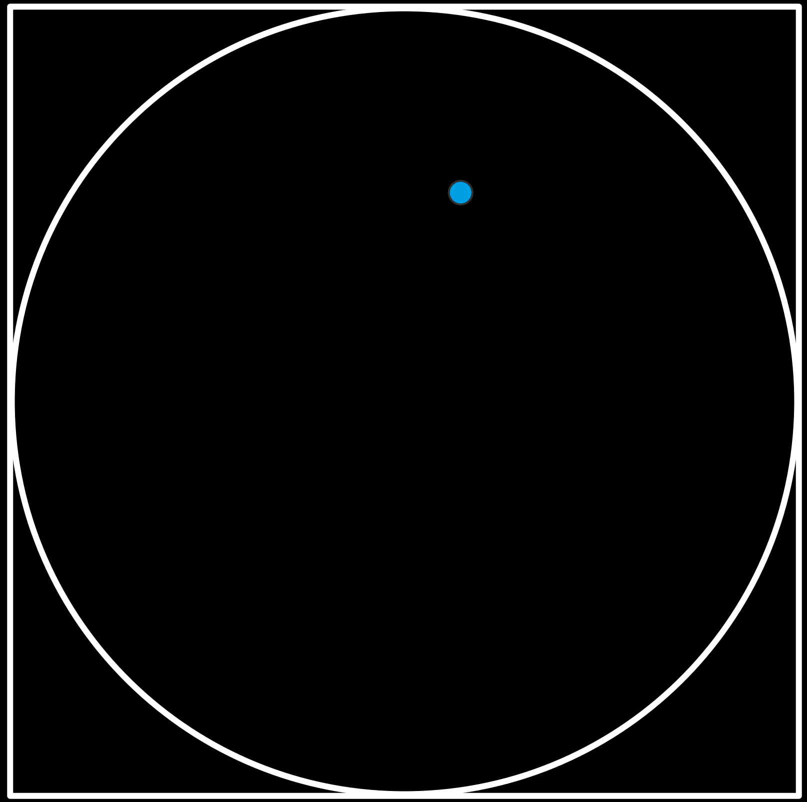
monte carlo simulation



$$\frac{\bigcirc}{\square} = \frac{\pi}{4}$$

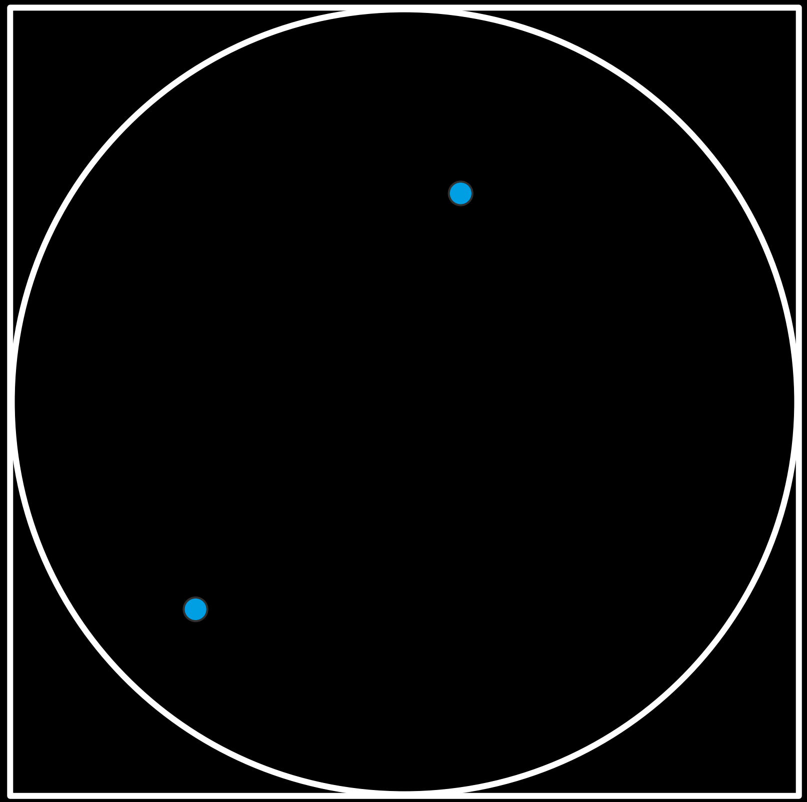


$$4\,\frac{\bigcirc}{\square} = \pi$$



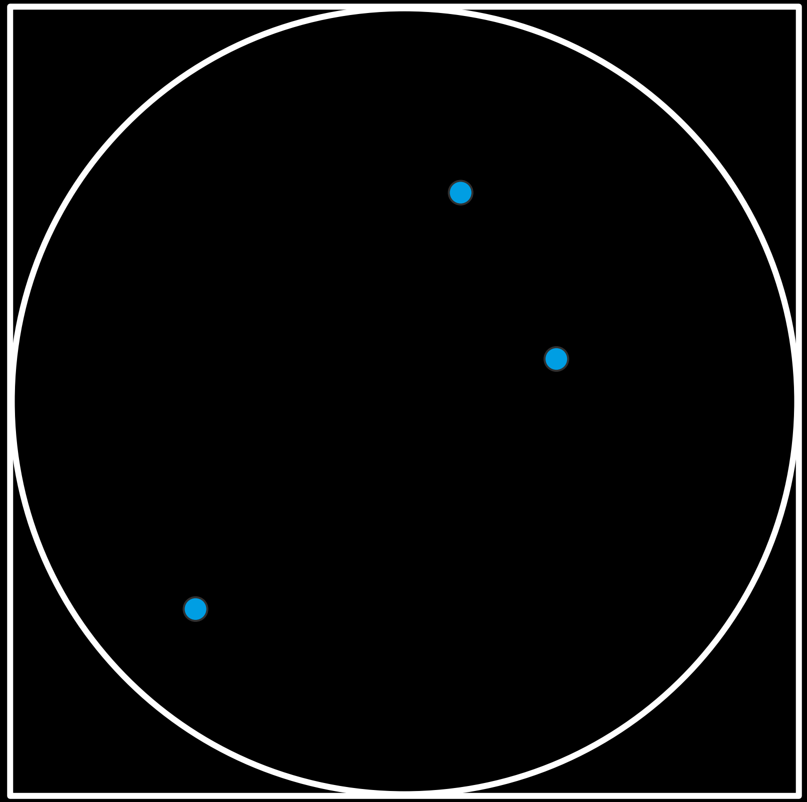
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 4$$



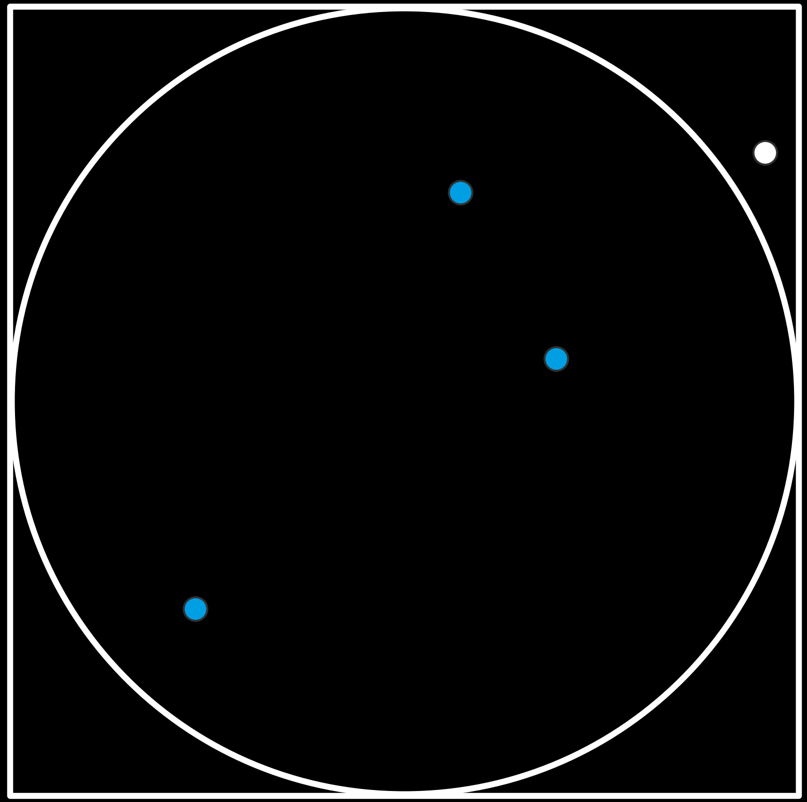
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 4$$



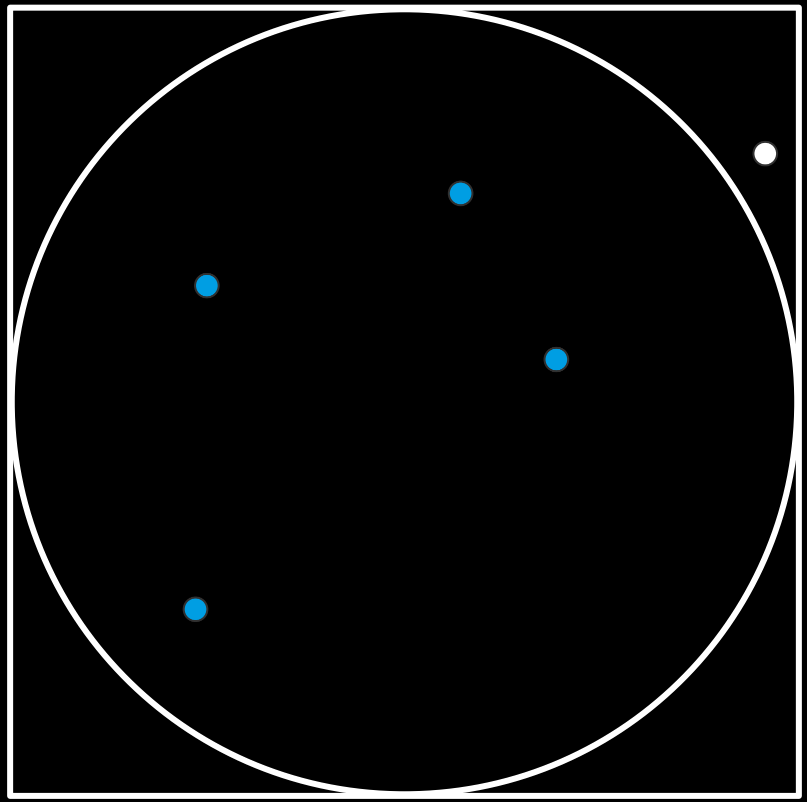
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 4$$



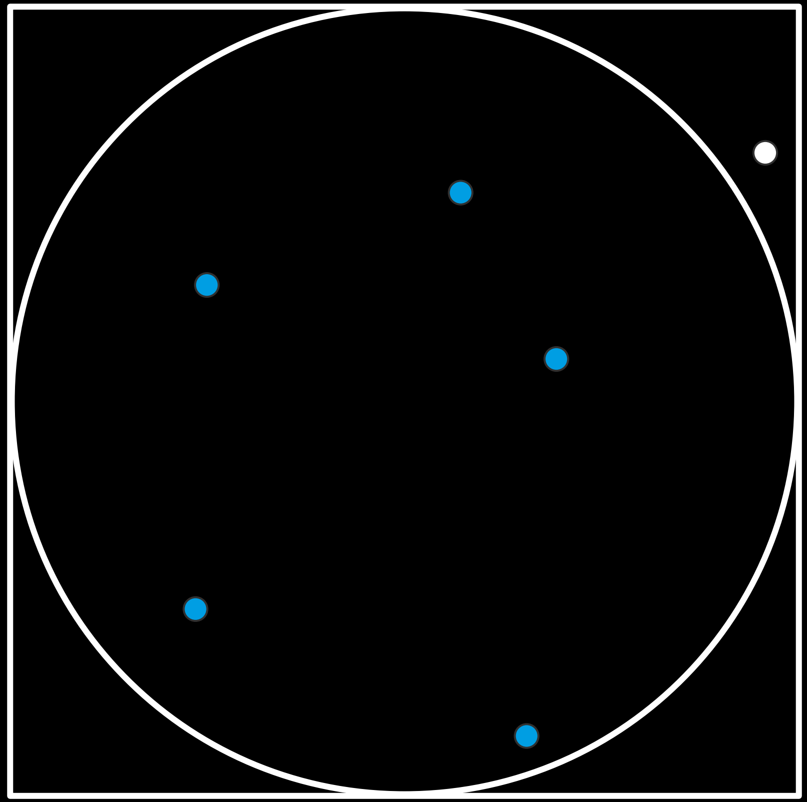
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 3$$



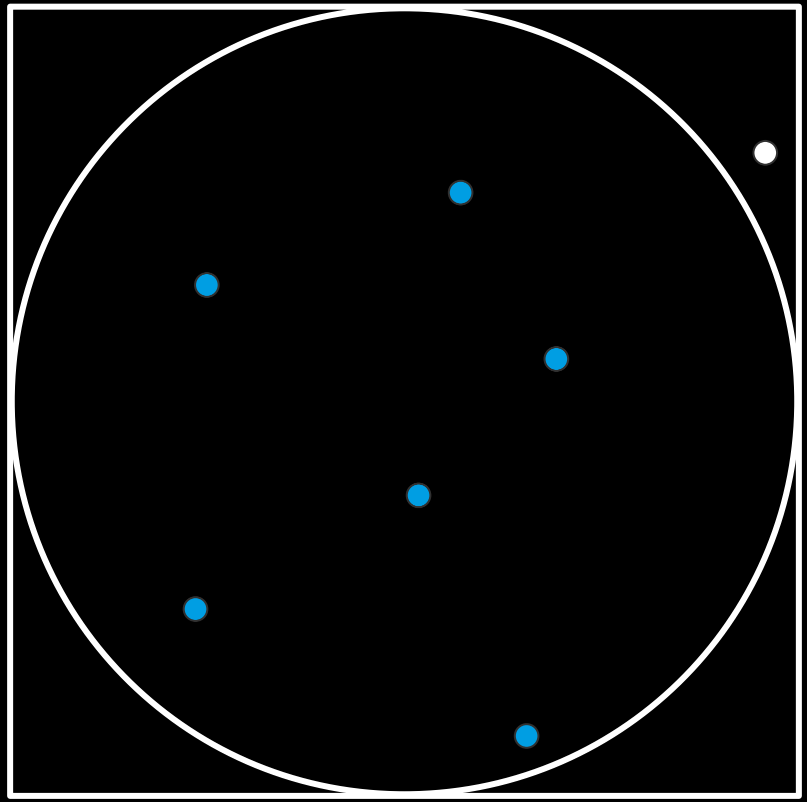
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 3{,}2$$



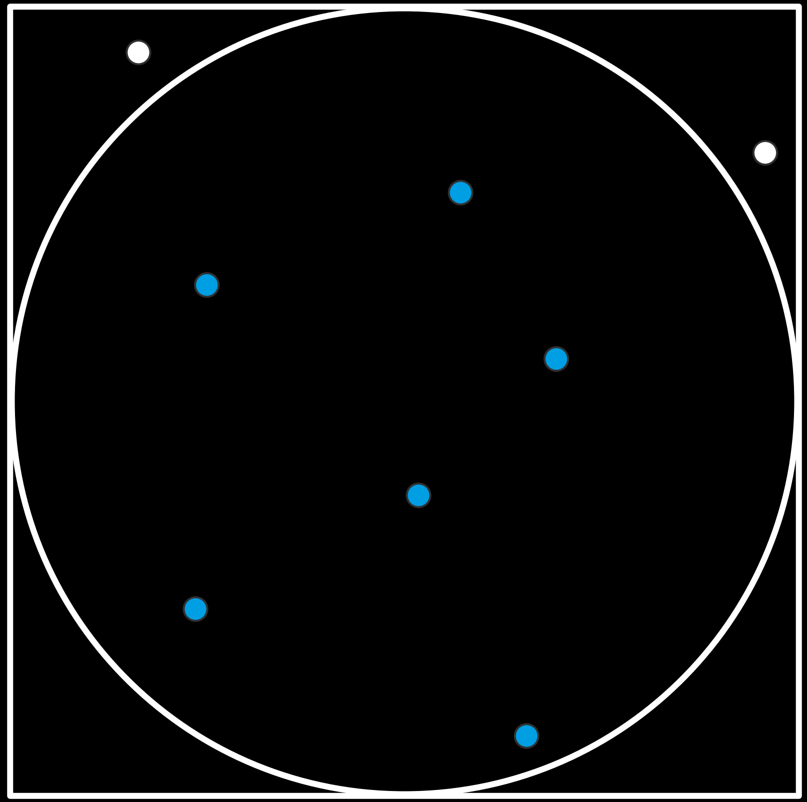
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 3{,}33$$



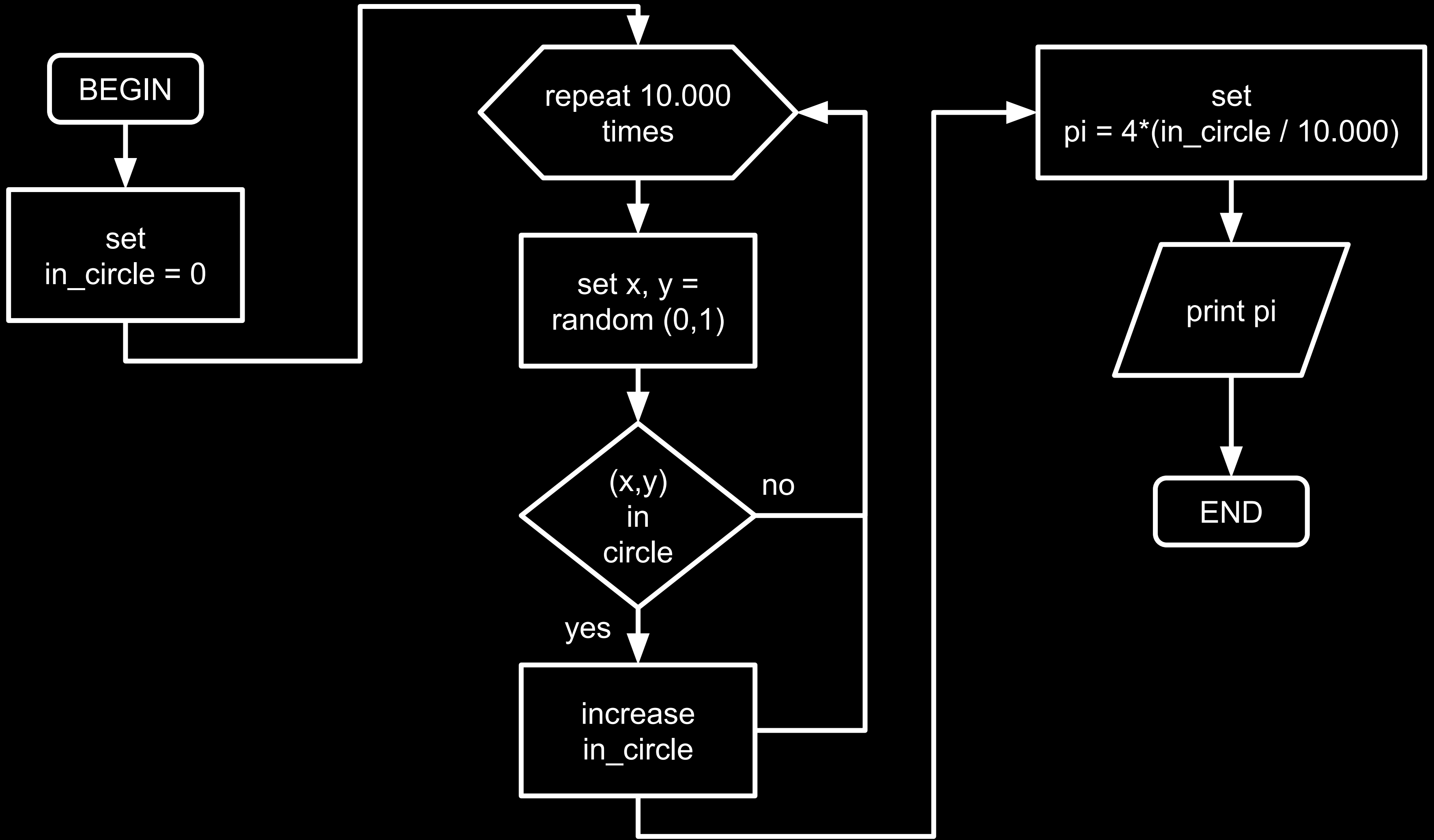
$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 3{,}43$$



$$4\,\frac{\bigcirc}{\square} = \pi$$

$$= 3$$


BEGIN
set
in_circle = 0
repeat 10.000
times
set x, y =
random (0,1)
(x,y)
in
circle
no
yes
increase
in_circle
set
pi = 4*(in_circle / 10.000)
print pi
END

gregory-leibniz series

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)$$

sorting

[9, 5, 2, 1, 4, 7]

bubble sort

repeatedly compare and swap elements until done.

[9, 5, 2, 1, 4, 7]

[9, 5, 2, 1, 4, 7] ⟶ 9 > 5 ?

$$[\,9, 5, 2, 1, 4, 7\,] \longrightarrow 9 > 5\,? \xrightarrow{\text{yes}} [\,5, 9, 2, 1, 4, 7\,]$$

$$[\,9, 5, 2, 1, 4, 7\,] \longrightarrow 9 > 5\,? \xrightarrow{\text{yes}} [\,5, 9, 2, 1, 4, 7\,]$$

$$[\,5, 9, 2, 1, 4, 7\,] \longrightarrow 9 > 2\,? \xrightarrow{\text{yes}} [\,5, 2, 9, 1, 4, 7\,]$$

$$[\,9, 5, 2, 1, 4, 7\,] \longrightarrow 9 > 5\ ? \xrightarrow{\text{yes}} [\,5, 9, 2, 1, 4, 7\,]$$

$$[\,5, 9, 2, 1, 4, 7\,] \longrightarrow 9 > 2\ ? \xrightarrow{\text{yes}} [\,5, 2, 9, 1, 4, 7\,]$$

$$[\,5, 2, 9, 1, 4, 7\,] \longrightarrow 9 > 1\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 9, 4, 7\,]$$

$$[\,9, 5, 2, 1, 4, 7\,] \longrightarrow 9 > 5\ ? \xrightarrow{\text{yes}} [\,5, 9, 2, 1, 4, 7\,]$$

$$[\,5, 9, 2, 1, 4, 7\,] \longrightarrow 9 > 2\ ? \xrightarrow{\text{yes}} [\,5, 2, 9, 1, 4, 7\,]$$

$$[\,5, 2, 9, 1, 4, 7\,] \longrightarrow 9 > 1\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 9, 4, 7\,]$$

$$[\,5, 2, 1, 9, 4, 7\,] \longrightarrow 9 > 4\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 4, 9, 7\,]$$

$$[\,9, 5, 2, 1, 4, 7\,] \longrightarrow 9 > 5\ ? \xrightarrow{\text{yes}} [\,5, 9, 2, 1, 4, 7\,]$$

$$[\,5, 9, 2, 1, 4, 7\,] \longrightarrow 9 > 2\ ? \xrightarrow{\text{yes}} [\,5, 2, 9, 1, 4, 7\,]$$

$$[\,5, 2, 9, 1, 4, 7\,] \longrightarrow 9 > 1\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 9, 4, 7\,]$$

$$[\,5, 2, 1, 9, 4, 7\,] \longrightarrow 9 > 4\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 4, 9, 7\,]$$

$$[\,5, 2, 1, 4, 9, 7\,] \longrightarrow 9 > 7\ ? \xrightarrow{\text{yes}} [\,5, 2, 1, 4, 7, 9\,]$$

$$[\,5, 2, 1, 4, 7, 9\,] \longrightarrow 5 > 2\ ? \xrightarrow{\text{yes}} [\,2, 5, 1, 4, 7, 9\,]$$

[5, 2, 1, 4, 7, 9] $\longrightarrow$ 5 > 2 ? $\xrightarrow{\text{yes}}$ [2, 5, 1, 4, 7, 9]

[2, 5, 1, 4, 7, 9] $\longrightarrow$ 5 > 1 ? $\xrightarrow{\text{yes}}$ [2, 1, 5, 4, 7, 9]

[5, 2, 1, 4, 7, 9] $\longrightarrow$ 5 > 2 ? $\xrightarrow{\text{yes}}$ [2, 5, 1, 4, 7, 9]

[2, 5, 1, 4, 7, 9] $\longrightarrow$ 5 > 1 ? $\xrightarrow{\text{yes}}$ [2, 1, 5, 4, 7, 9]

[2, 1, 5, 4, 7, 9] $\longrightarrow$ 5 > 4 ? $\xrightarrow{\text{yes}}$ [2, 1, 4, 5, 7, 9]

[5, 2, 1, 4, 7, 9] $\longrightarrow$ 5 > 2 ? $\xrightarrow{\text{yes}}$ [2, 5, 1, 4, 7, 9]

[2, 5, 1, 4, 7, 9] $\longrightarrow$ 5 > 1 ? $\xrightarrow{\text{yes}}$ [2, 1, 5, 4, 7, 9]

[2, 1, 5, 4, 7, 9] $\longrightarrow$ 5 > 4 ? $\xrightarrow{\text{yes}}$ [2, 1, 4, 5, 7, 9]

[2, 1, 4, 5, 7, 9] $\longrightarrow$ 5 > 7 ? $\xrightarrow{\text{no}}$ [2, 1, 4, 5, 7, 9]

[2, 1, 4, 5, 7, 9] ⟶ 2 > 1 ? $\xrightarrow{\text{yes}}$ [1, 2, 4, 5, 7, 9]

$[\ 2, 1, 4, 5, 7, 9\ ] \longrightarrow 2 > 1\ ? \xrightarrow{\text{yes}} [\ 1, 2, 4, 5, 7, 9\ ]$

$[\ 1, 2, 4, 5, 7, 9\ ] \longrightarrow 2 > 4\ ? \xrightarrow{\text{no}} [\ 1, 2, 4, 5, 7, 9\ ]$

[2, 1, 4, 5, 7, 9] $\longrightarrow$ 2 > 1 ? $\xrightarrow{\text{yes}}$ [1, 2, 4, 5, 7, 9]

[1, 2, 4, 5, 7, 9] $\longrightarrow$ 2 > 4 ? $\xrightarrow{\text{no}}$ [1, 2, 4, 5, 7, 9]

[1, 2, 4, 5, 7, 9] $\longrightarrow$ 4 > 5 ? $\xrightarrow{\text{no}}$ [1, 2, 4, 5, 7, 9]

$[\,1, 2, 4, 5, 7, 9\,] \longrightarrow 1 > 2\ ? \xrightarrow{\text{no}} [\,1, 2, 4, 5, 7, 9\,]$

$[\,1, 2, 4, 5, 7, 9\,] \longrightarrow 1 > 2\ ? \xrightarrow{\text{no}} [\,1, 2, 4, 5, 7, 9\,]$

$[\,1, 2, 4, 5, 7, 9\,] \longrightarrow 2 > 4\ ? \xrightarrow{\text{no}} [\,1, 2, 4, 5, 7, 9\,]$

$$[\,1, 2, 4, 5, 7, 9\,] \longrightarrow 1 > 2\,? \xrightarrow{\text{no}} [\,1, 2, 4, 5, 7, 9\,]$$

[1, 2, 4, 5, 7, 9] $\longrightarrow$ 1 > 2 ? $\xrightarrow{\text{no}}$ [1, 2, 4, 5, 7, 9]

[1, 2, 4, 5, 7, 9] DONE!

selection sort

find the smallest element and move it to front. repeat for the rest of the elements.

move to front

[9, 5, 2, 1, 4, 7] $\longrightarrow$ [1, 5, 9, 2, 4, 7]

move to front

[9, 5, 2, 1, 4, 7] ⟶ [1, 5, 9, 2, 4, 7]

[1, 5, 9, 2, 4, 7] ⟶ [1, 2, 5, 9, 4, 7]

move to front

[9, 5, 2, 1, 4, 7] $\longrightarrow$ [1, 5, 9, 2, 4, 7]

[1, 5, 9, 2, 4, 7] $\longrightarrow$ [1, 2, 5, 9, 4, 7]

[1, 2, 5, 9, 4, 7] $\longrightarrow$ [1, 2, 4, 5, 9, 7]

move to front

[9, 5, 2, 1, 4, 7] $\longrightarrow$ [1, 5, 9, 2, 4, 7]

[1, 5, 9, 2, 4, 7] $\longrightarrow$ [1, 2, 5, 9, 4, 7]

[1, 2, 5, 9, 4, 7] $\longrightarrow$ [1, 2, 4, 5, 9, 7]

[1, 2, 4, 5, 9, 7] $\longrightarrow$ [1, 2, 4, 5, 9, 7]

move to front

[9, 5, 2, 1, 4, 7] → [1, 5, 9, 2, 4, 7]

[1, 5, 9, 2, 4, 7] → [1, 2, 5, 9, 4, 7]

[1, 2, 5, 9, 4, 7] → [1, 2, 4, 5, 9, 7]

[1, 2, 4, 5, 9, 7] → [1, 2, 4, 5, 9, 7]

[1, 2, 4, 5, 9, 7] → [1, 2, 4, 5, 7, 9]

move to front

[9, 5, 2, 1, 4, 7] → [1, 5, 9, 2, 4, 7]

[1, 5, 9, 2, 4, 7] → [1, 2, 5, 9, 4, 7]

[1, 2, 5, 9, 4, 7] → [1, 2, 4, 5, 9, 7]

[1, 2, 4, 5, 9, 7] → [1, 2, 4, 5, 9, 7]

[1, 2, 4, 5, 9, 7] → [1, 2, 4, 5, 7, 9]

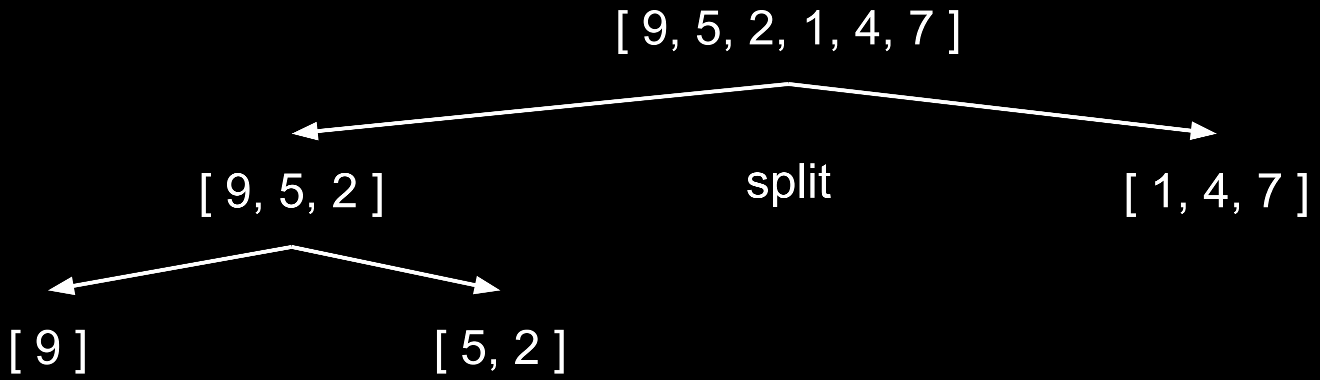
[1, 2, 4, 5, 7, 9] → [1, 2, 4, 5, 7, 9] merge sort

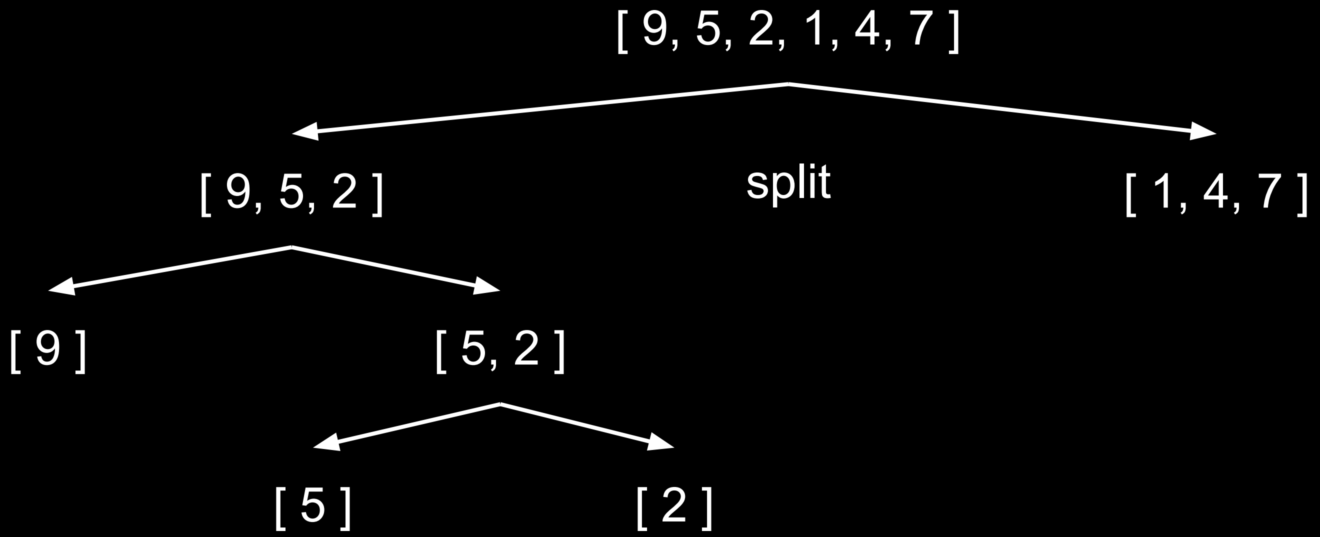
divide the elements recursively in two
halves until only one element is left.
then merge the sorted halves back together.

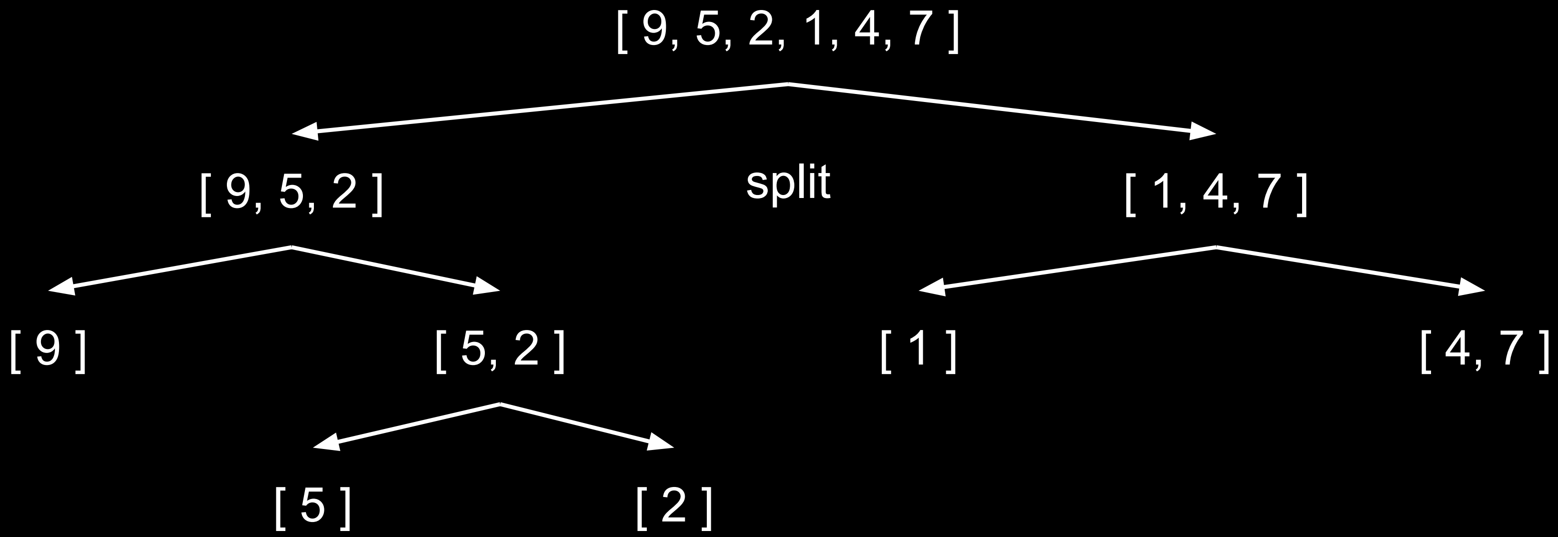
divide the elements recursively in two
halves until only one element is left.
then merge the sorted halves back together.

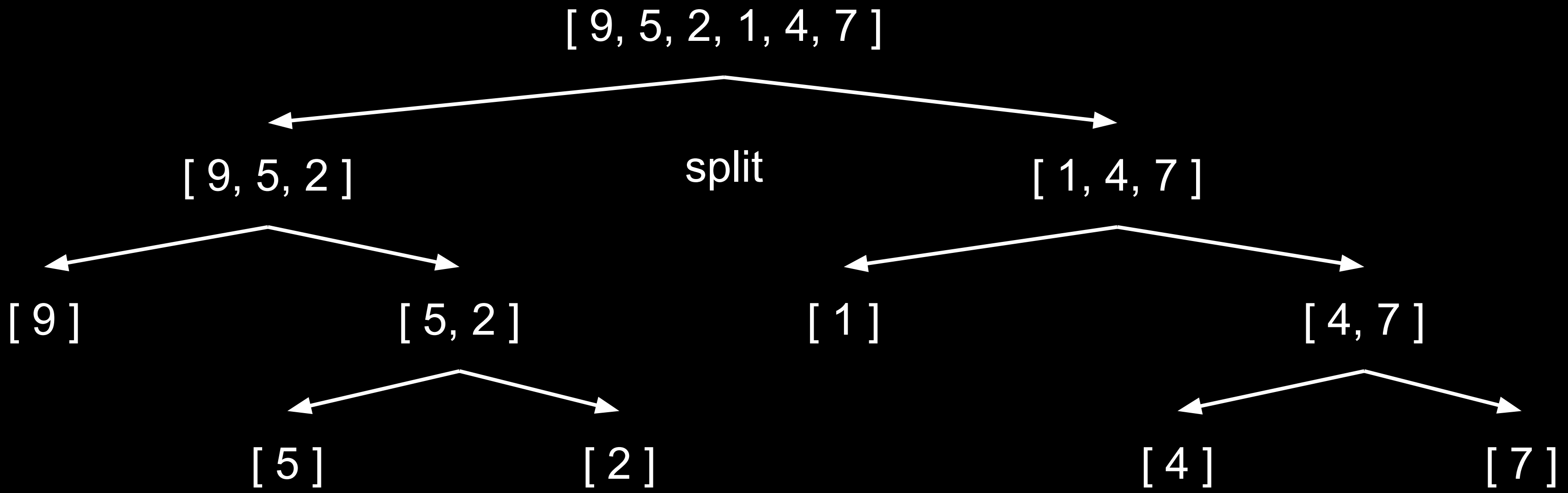
[9, 5, 2, 1, 4, 7]

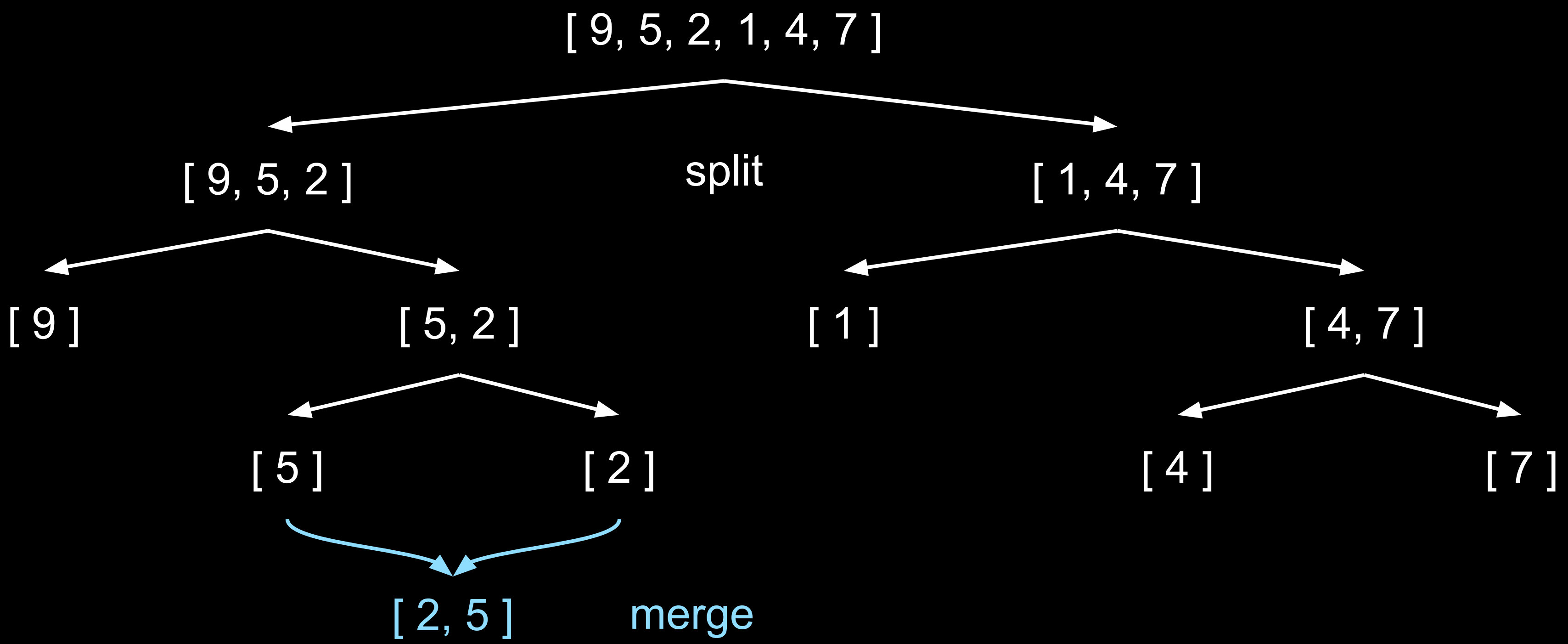
[9, 5, 2] split [1, 4, 7]

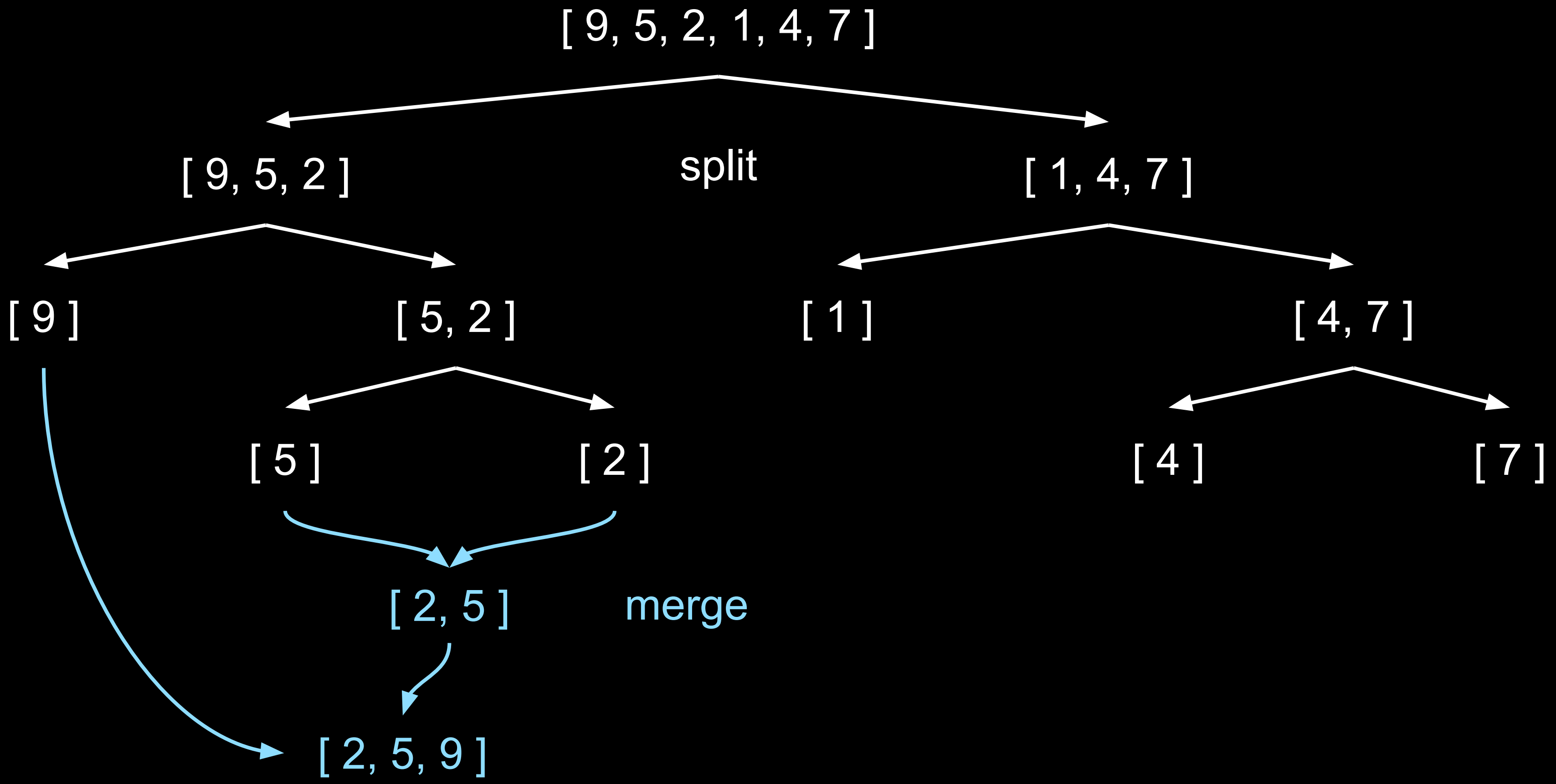

[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]

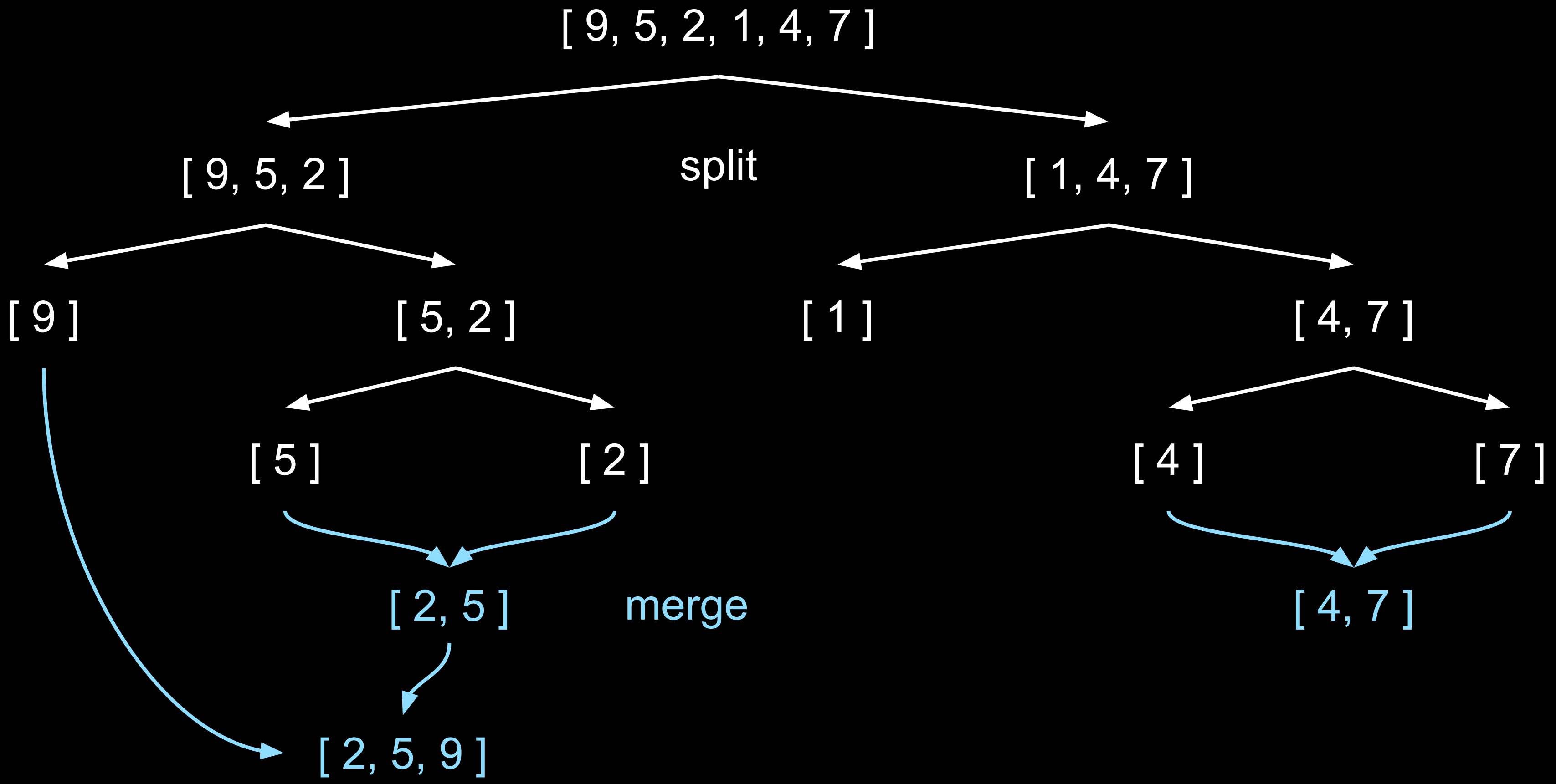

[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[5]
[2]


[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]

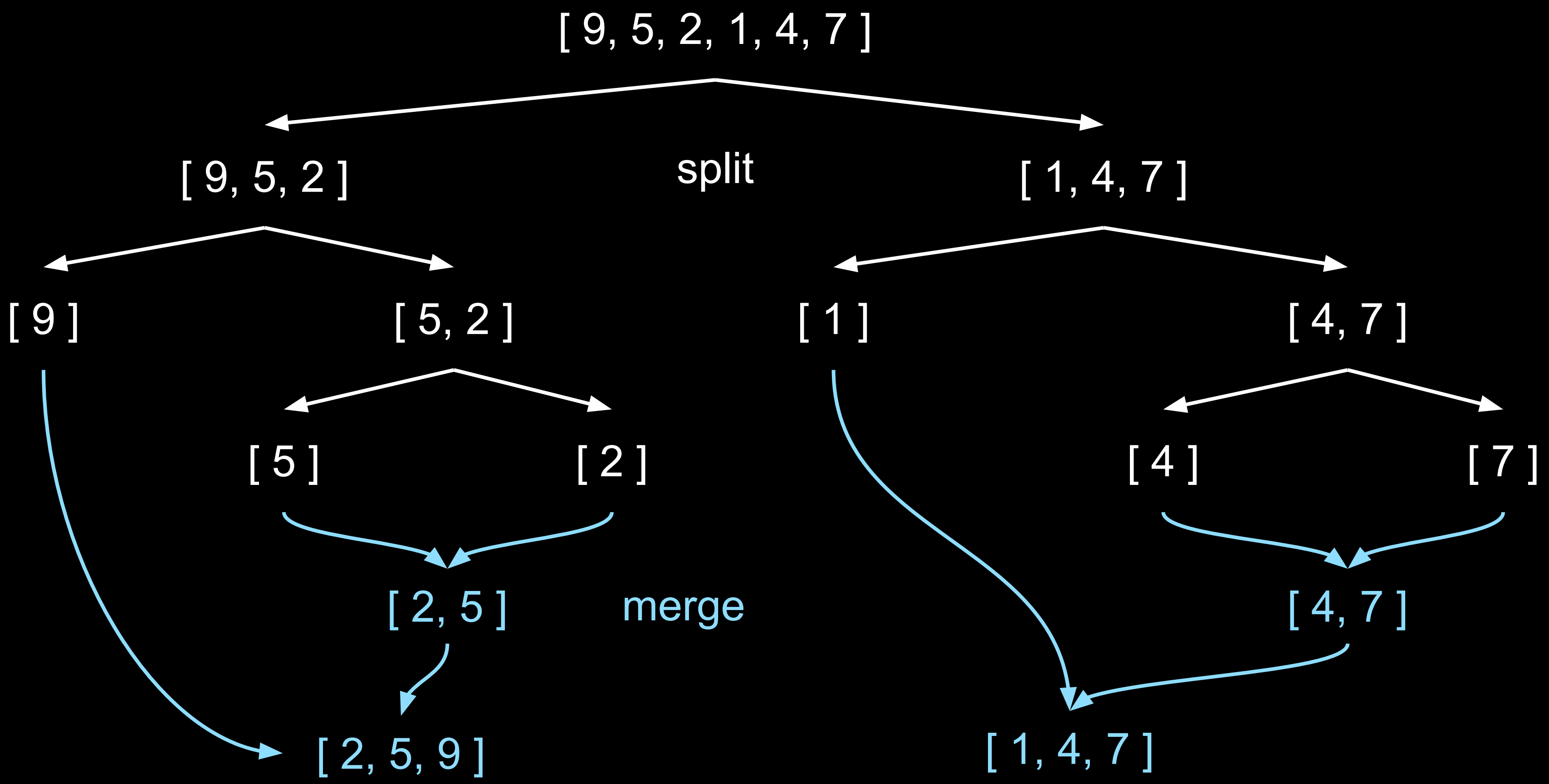

[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]

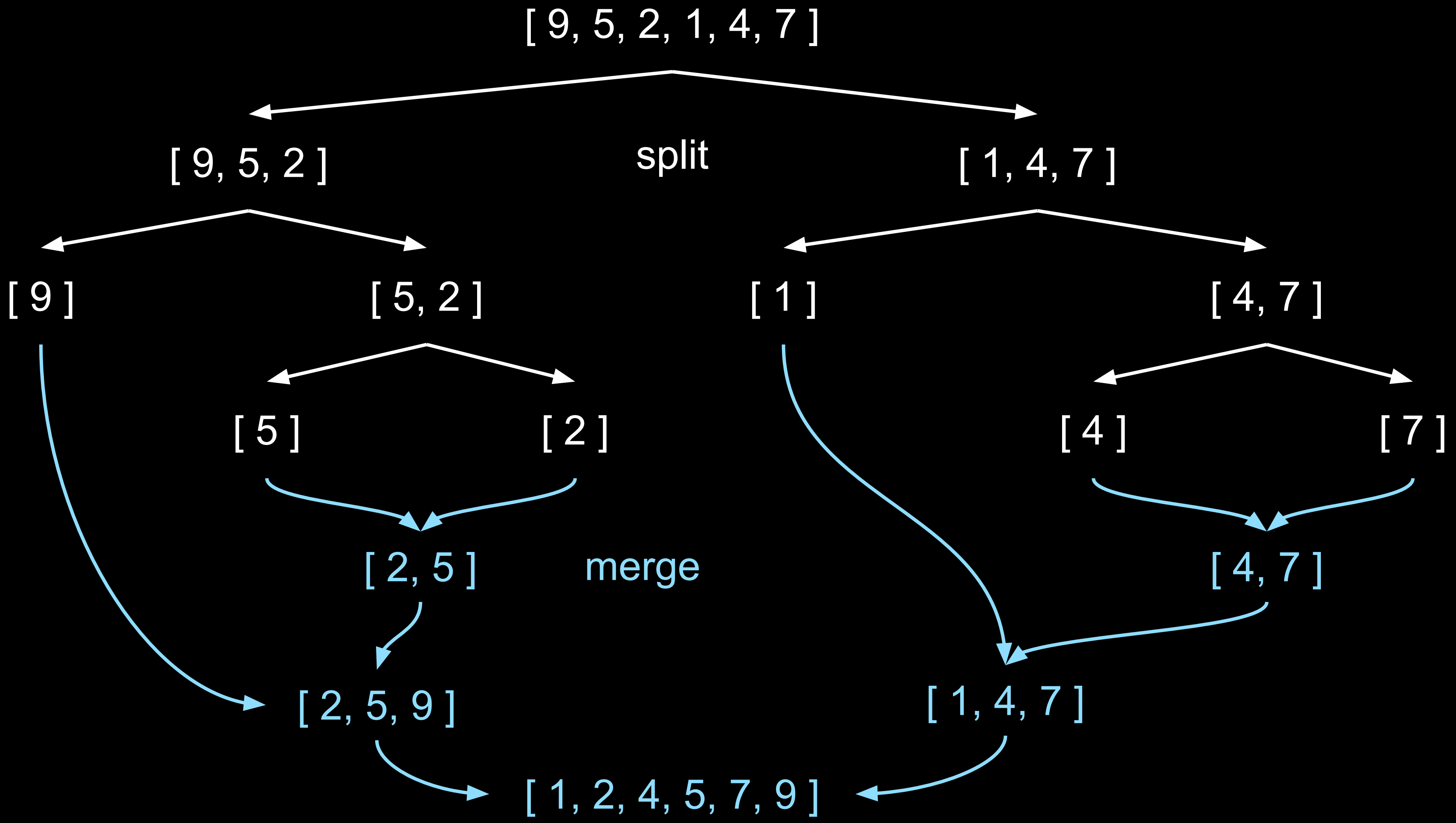

[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]
[2, 5]
merge


[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]
[2, 5]
merge
[2, 5, 9]

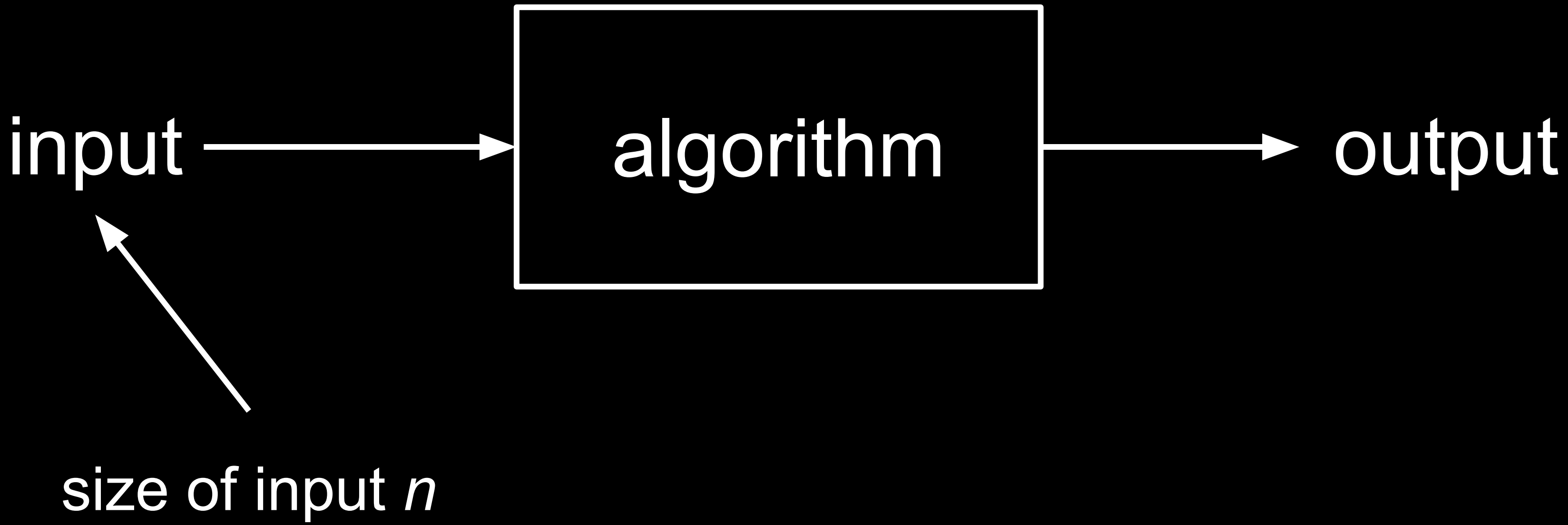


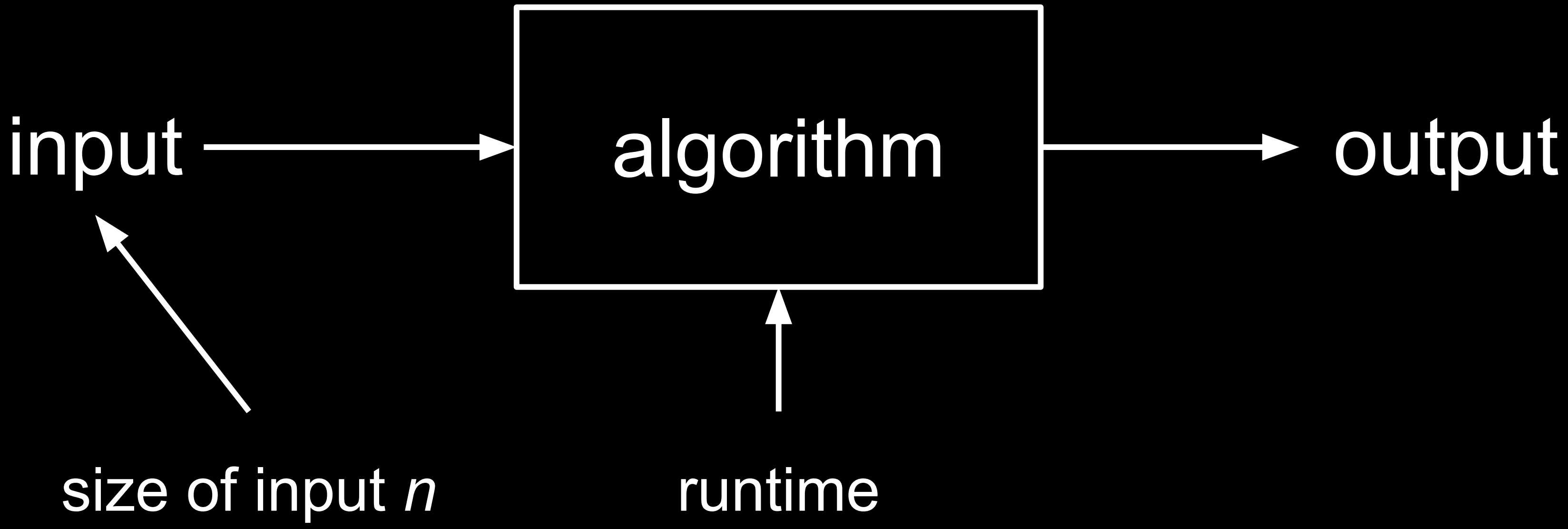
[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]
[2, 5]
merge
[4, 7]
[2, 5, 9]

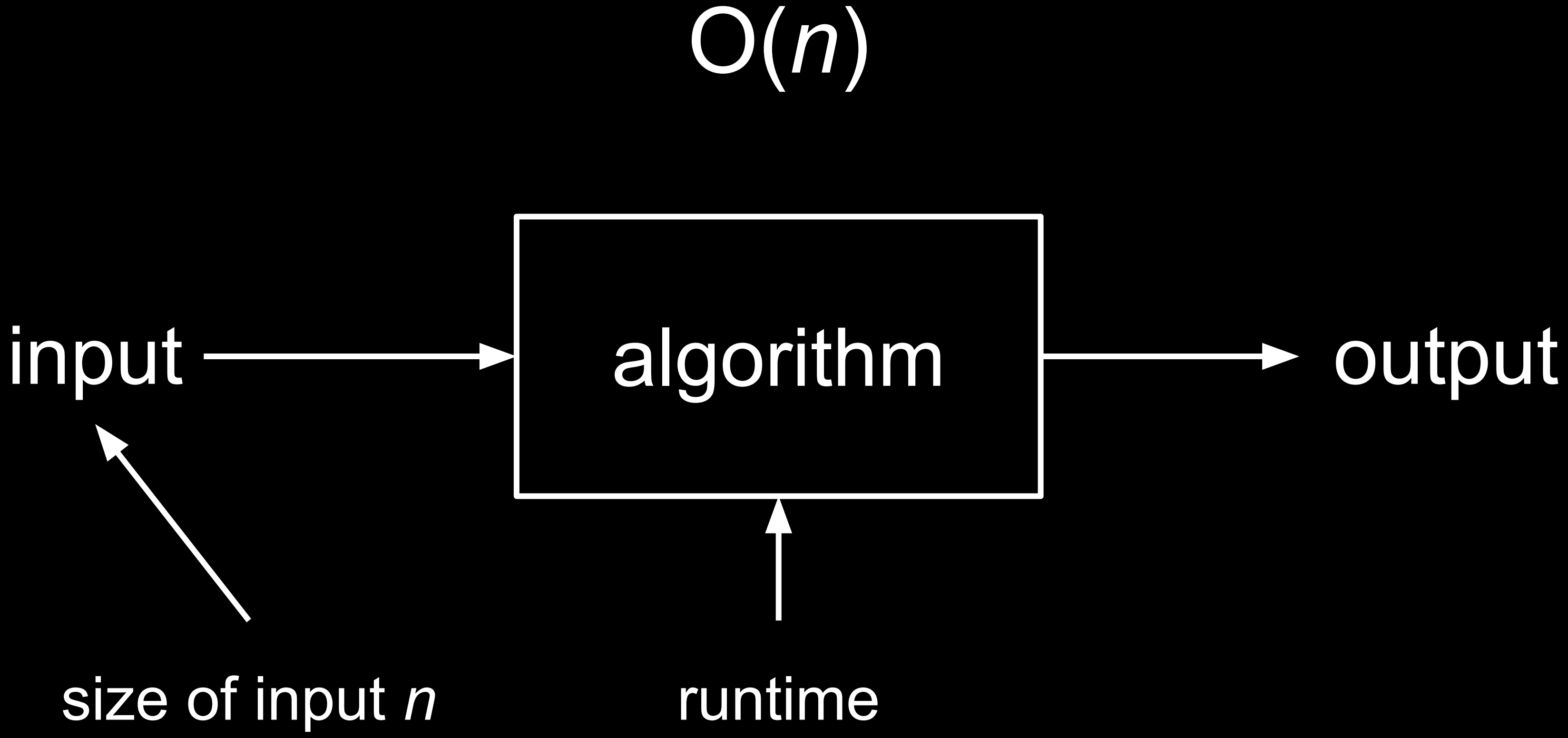

[9, 5, 2, 1, 4, 7]
split
[9, 5, 2]
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]
[2, 5]
merge
[4, 7]
[2, 5, 9]
[1, 4, 7]


[9, 5, 2, 1, 4, 7]
[9, 5, 2]
split
[1, 4, 7]
[9]
[5, 2]
[1]
[4, 7]
[5]
[2]
[4]
[7]
[2, 5]
merge
[4, 7]
[2, 5, 9]
[1, 4, 7]
[1, 2, 4, 5, 7, 9]

complexity

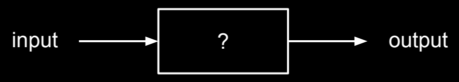

input
algorithm
output
size of input *n*


input
algorithm
output
size of input *n*
runtime


O(n)
input
algorithm
output
size of input n
runtime

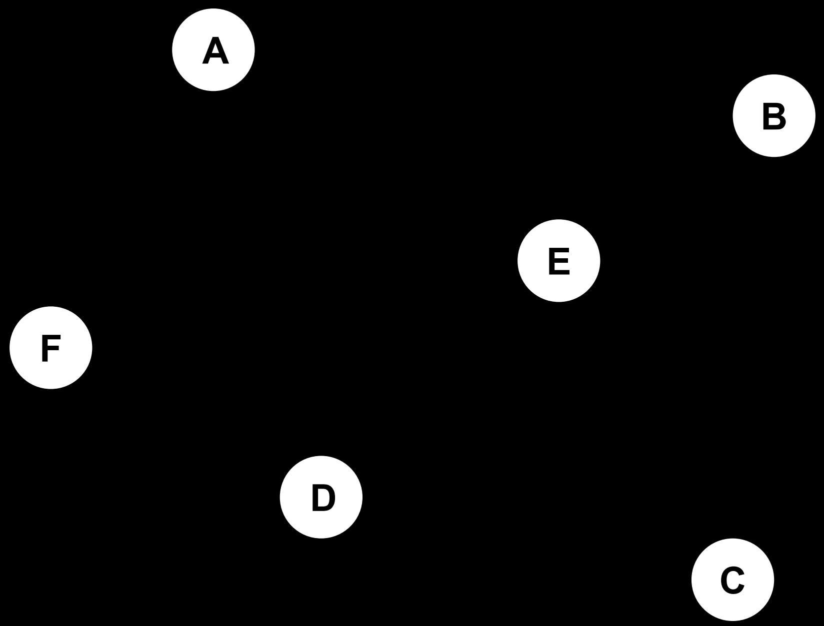
O(1)	runtime is constant and independent of problem size
$O(\log_2 n)$	runtime is determined by the logarithm of problem size
O(n)	runtime is linear to problem size
$O(n^2)$	runtime grows quadratically with the size of the problem
$O(n^3)$	runtime grows cubically with the size of the problem
$O(2^n)$	runtime grows exponentially with the size of the problem
O(n!)	runtime grows factorially with the size of the problem

optimization

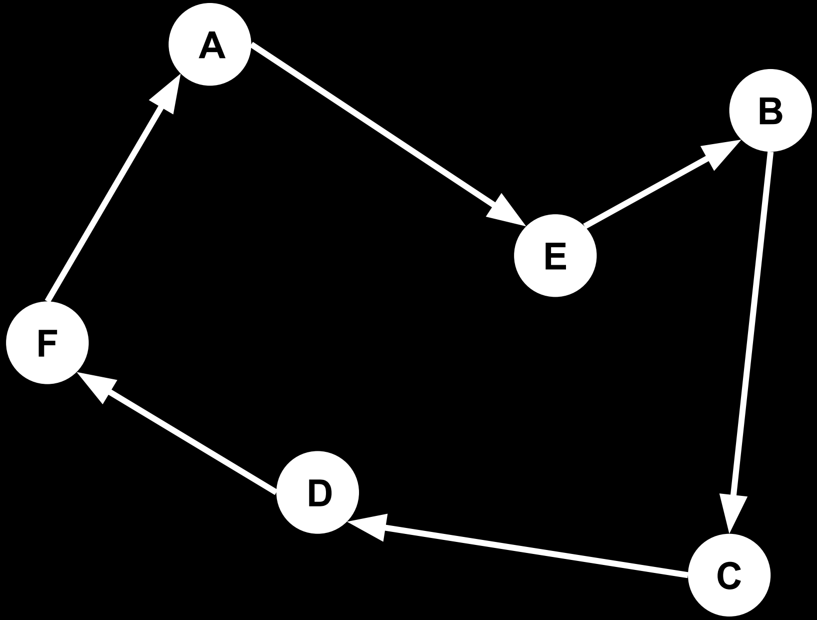


traveling salesmen

shortest tour?



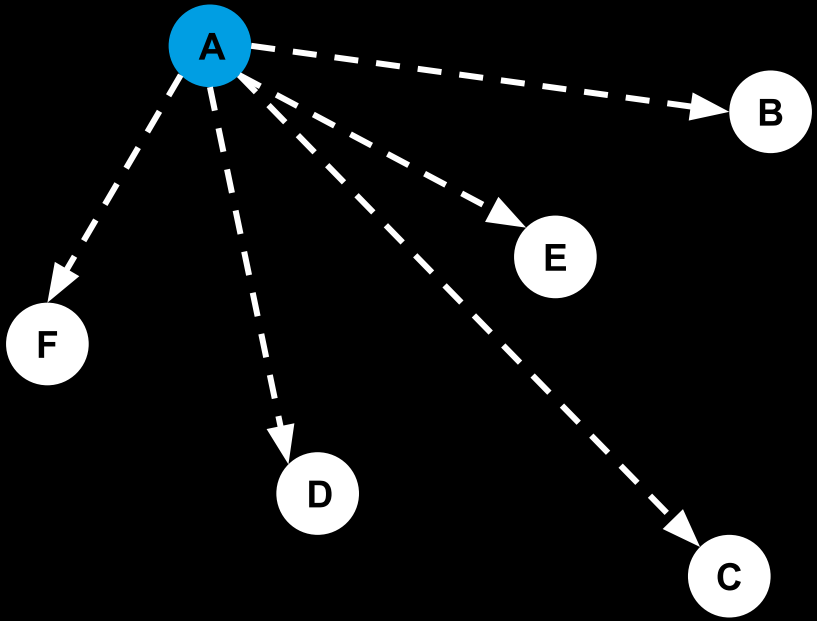
shortest tour?



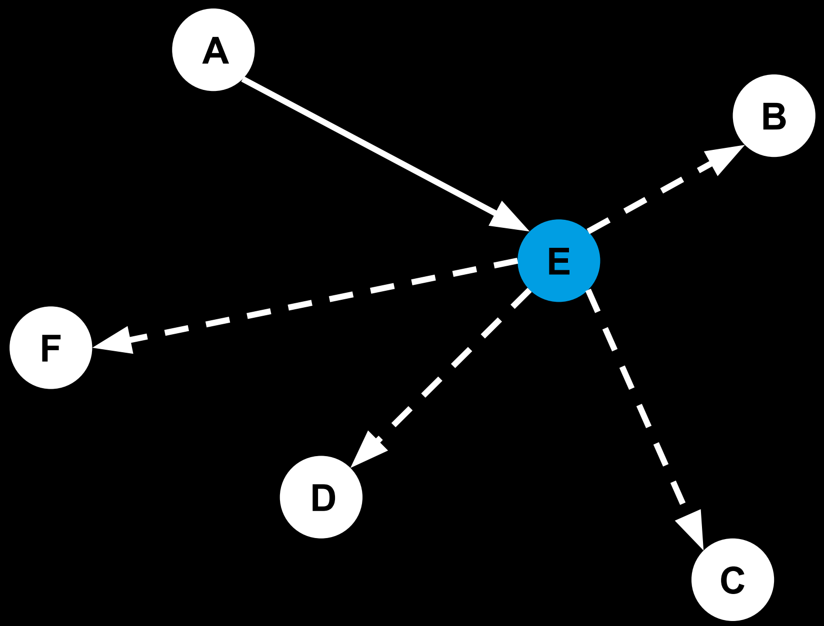
brute force

$$O(n) = n!$$

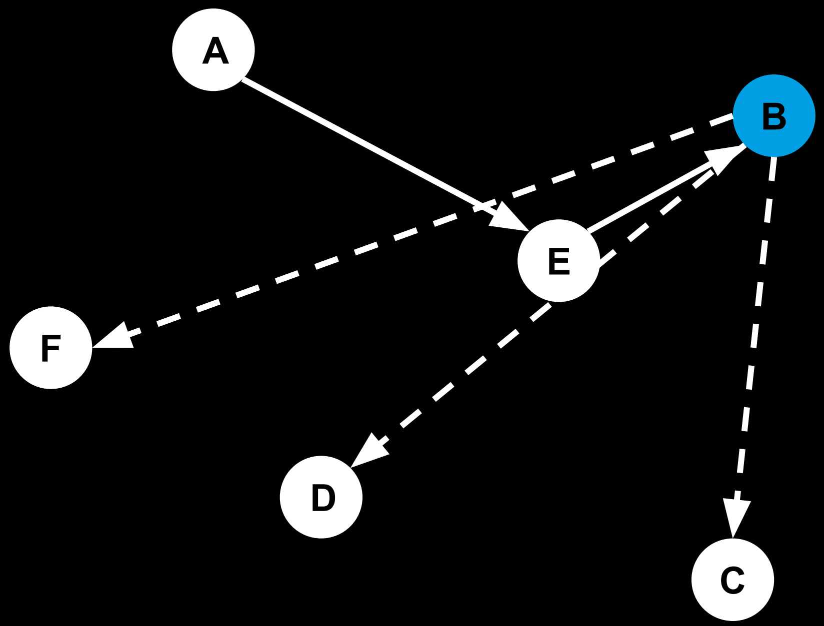
5 possible cities



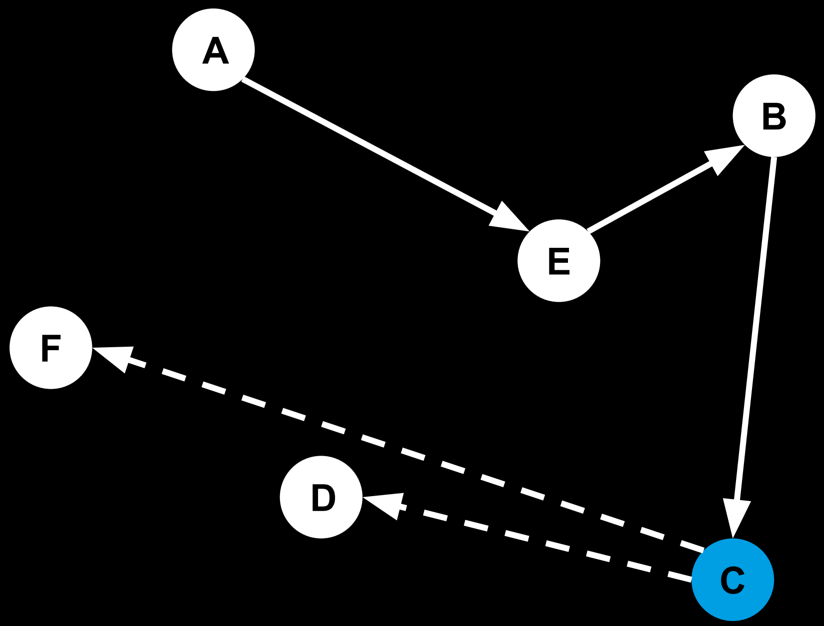
~~5~~ 4 possible cities



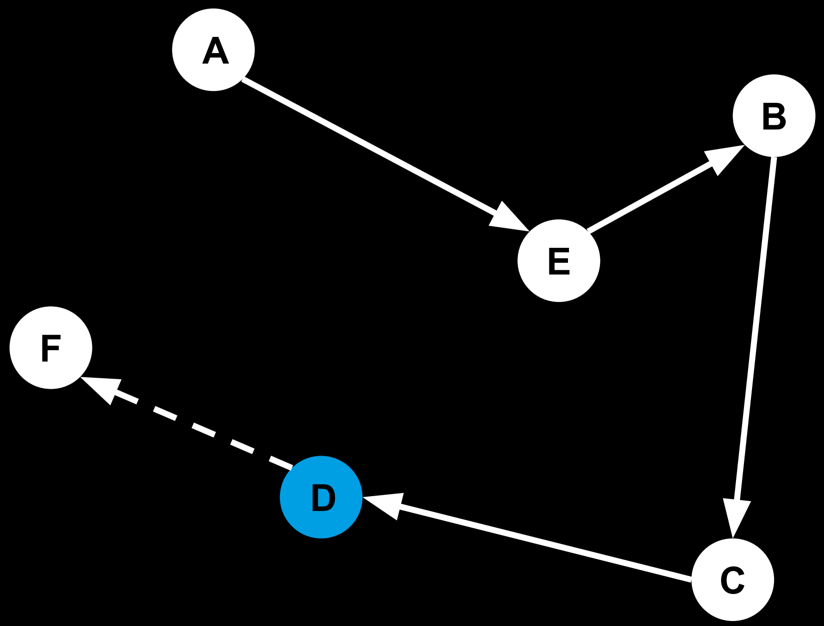
~~5~~ ~~4~~ 3 possible cities



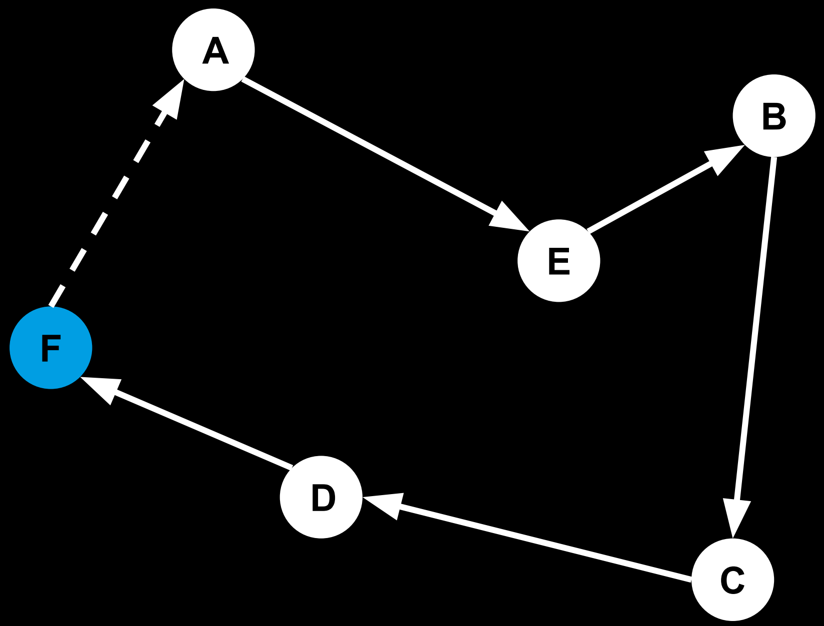
~~5~~ ~~4~~ ~~3~~ 2 possible cities



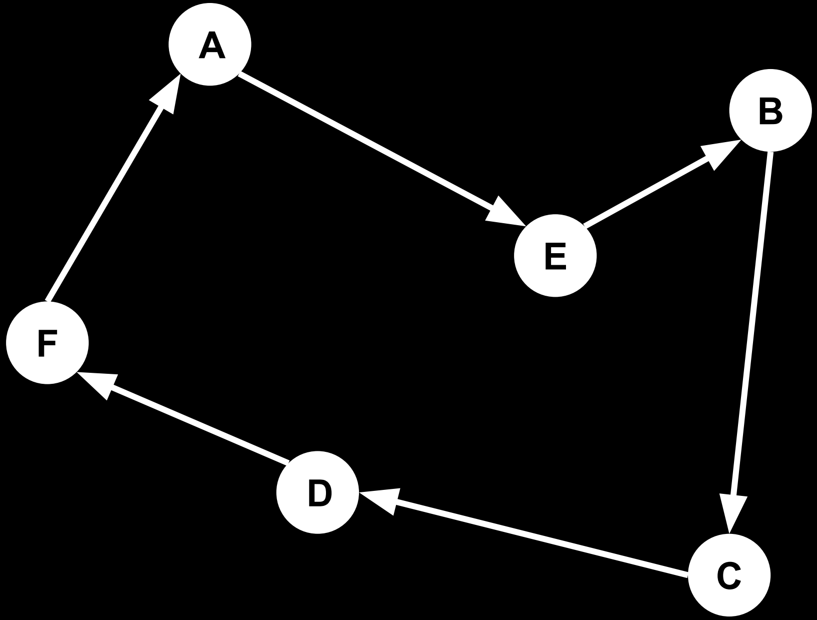
~~5~~ ~~4~~ ~~3~~ ~~2~~ 1 possible city



~~5~~ ~~4~~ ~~3~~ ~~2~~ ~~1~~ return nome



~~5~~ ~~4~~ ~~3~~ ~~2~~ ~~1~~ return nome



$$O(n) = n!$$

$$n = 5$$

$$O(n) = n!$$

$$n = 5$$

$$N = 5 * 4 * 3 * 2 * 1$$

$$O(n) = n!$$

$$n = 5$$

$$N = 5 * 4 * 3 * 2 * 1$$

$$= 120$$

$n = 10$

$n = 20$

$n = 30$

$n = 25$

brute force takes longer than the universe is old

$n = 60$

more possible routes than atoms in the universe

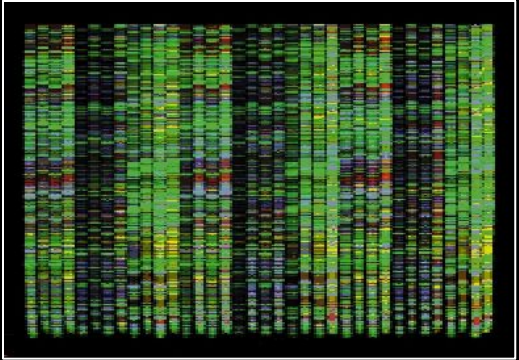


Image source: IEEE

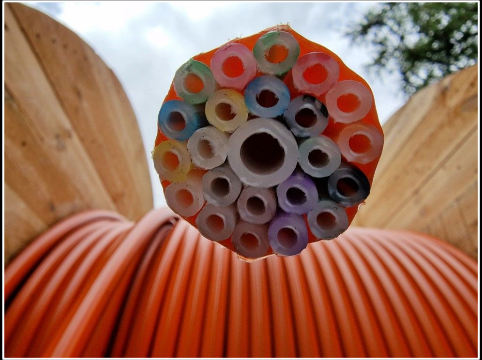


Image source: VDI Nachrichten

Image source: Wikimedia

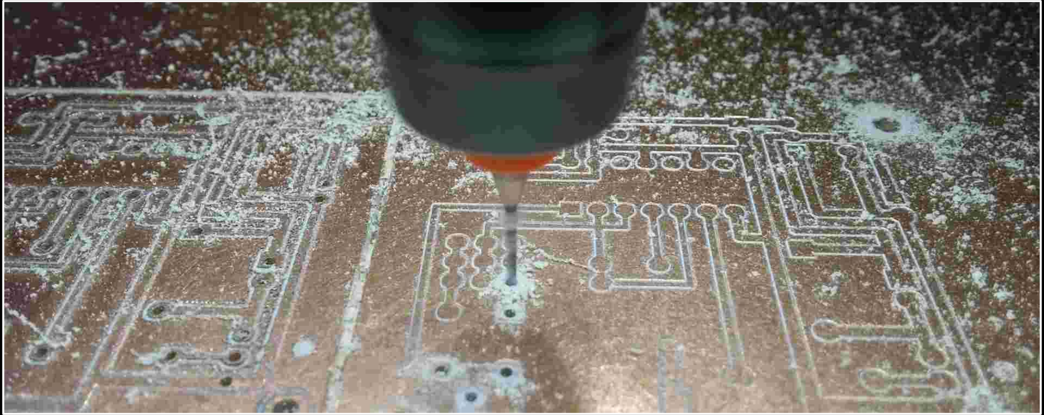
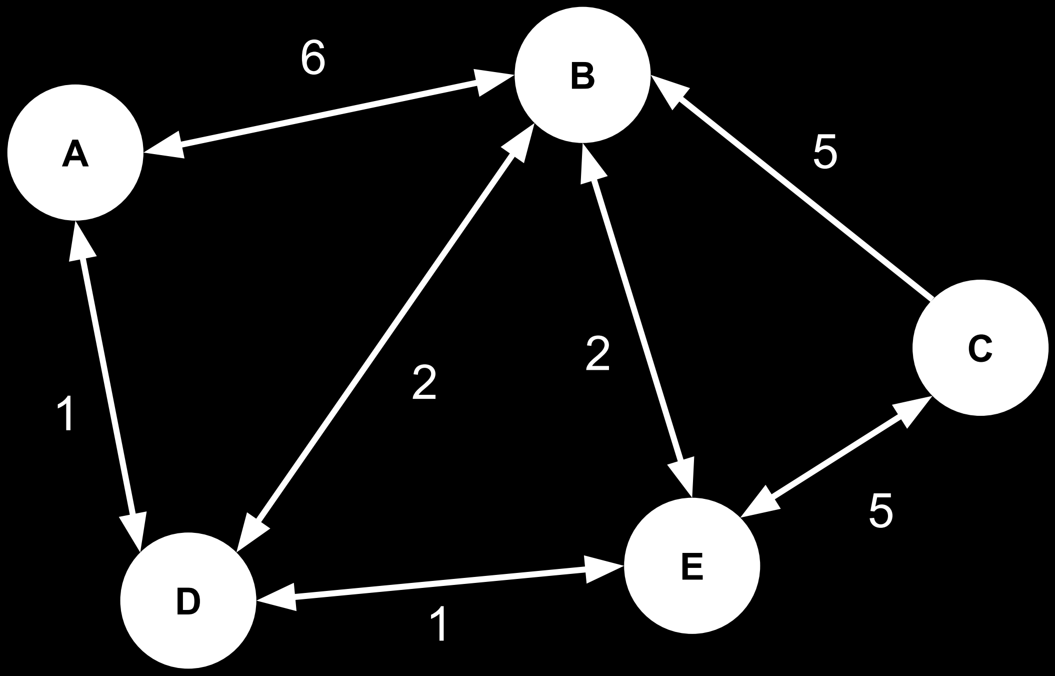


Image source: https://github.com/meiyi1986/tutorials/blob/master/notebooks/img/pcb-drilling.jpeg

Image source: IAS Observatory

shortest paths

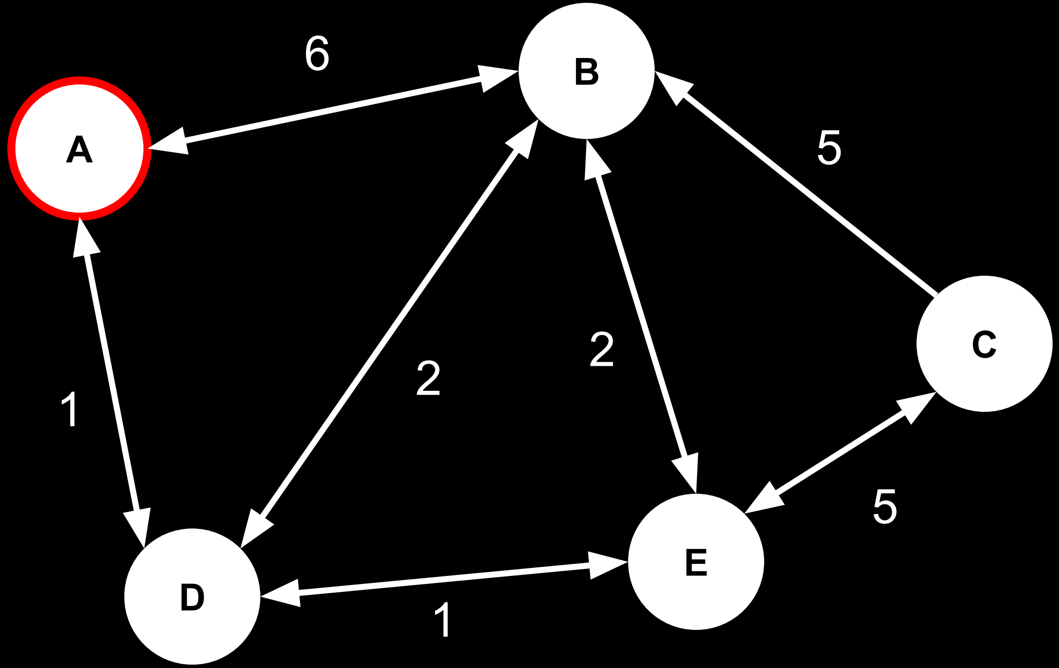


A
B
C
D
E
6
5
1
2
2
5
1

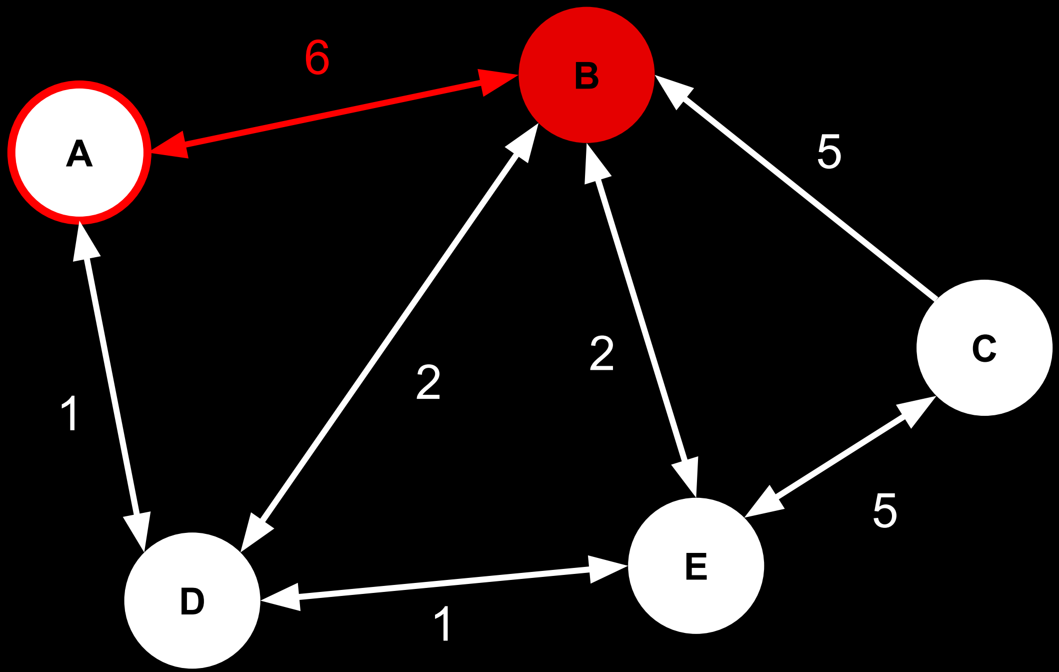
dijkstra's algorithm

Distances: A = 0, B = ∞, C = ∞, D = ∞, E = ∞

Set A as first location

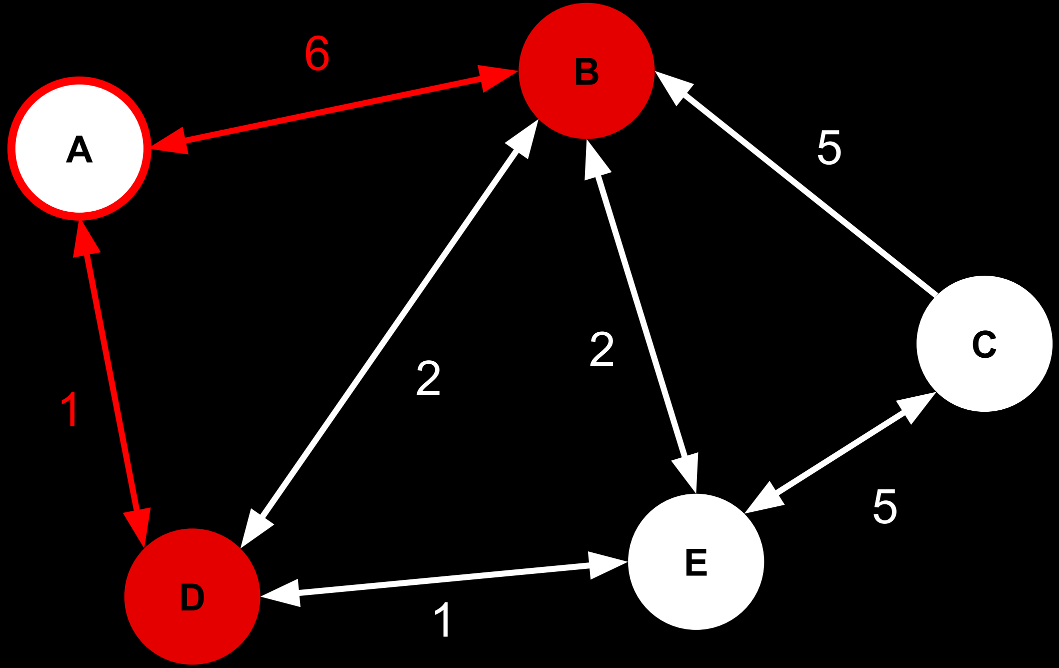


Distances: A = 0, B = 6, C = ∞, D = ∞, E = ∞



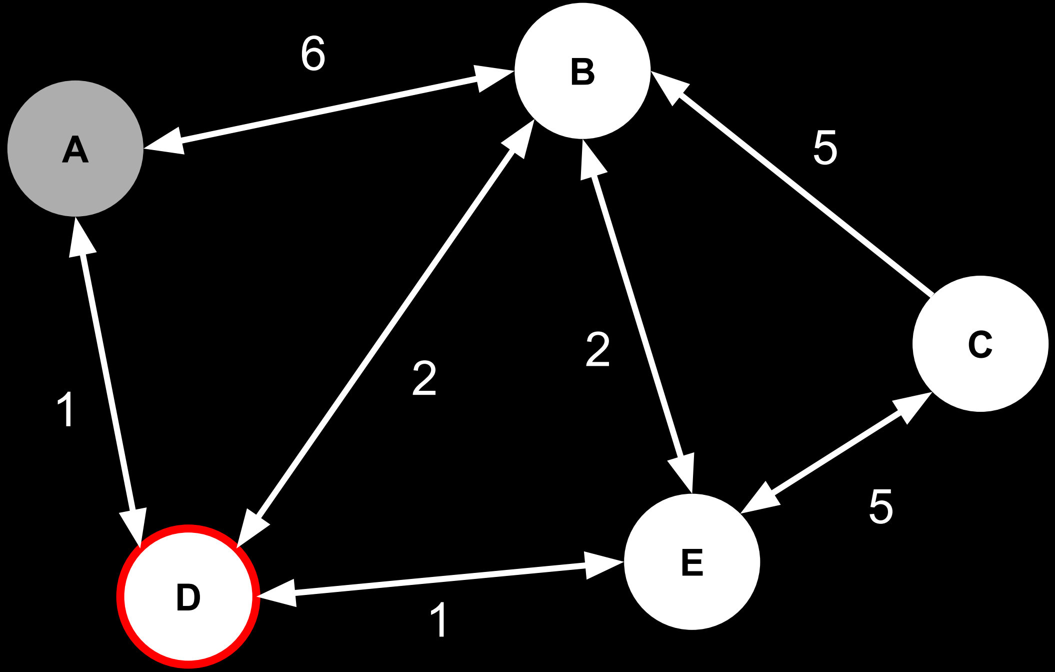
Update distance to B to 6

Distances: A = 0, B = 6, C = ∞, D = 1, E = ∞



Update distance
to D to 1

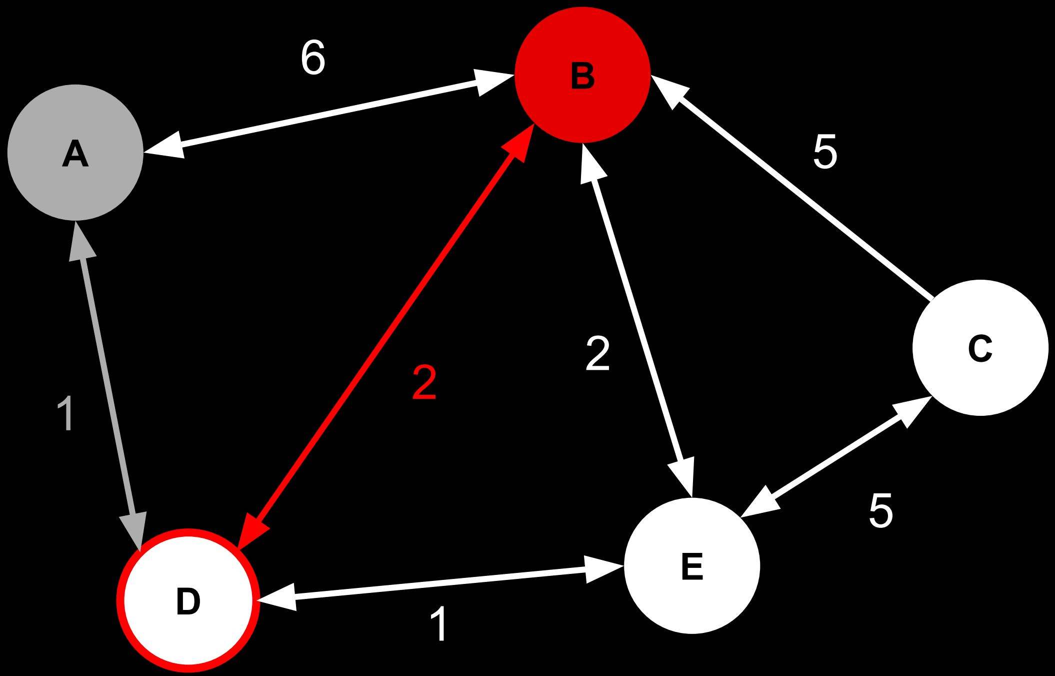
Distances: A = 0, B = 6, C = ∞, D = 1, E = ∞



Move to D as next location

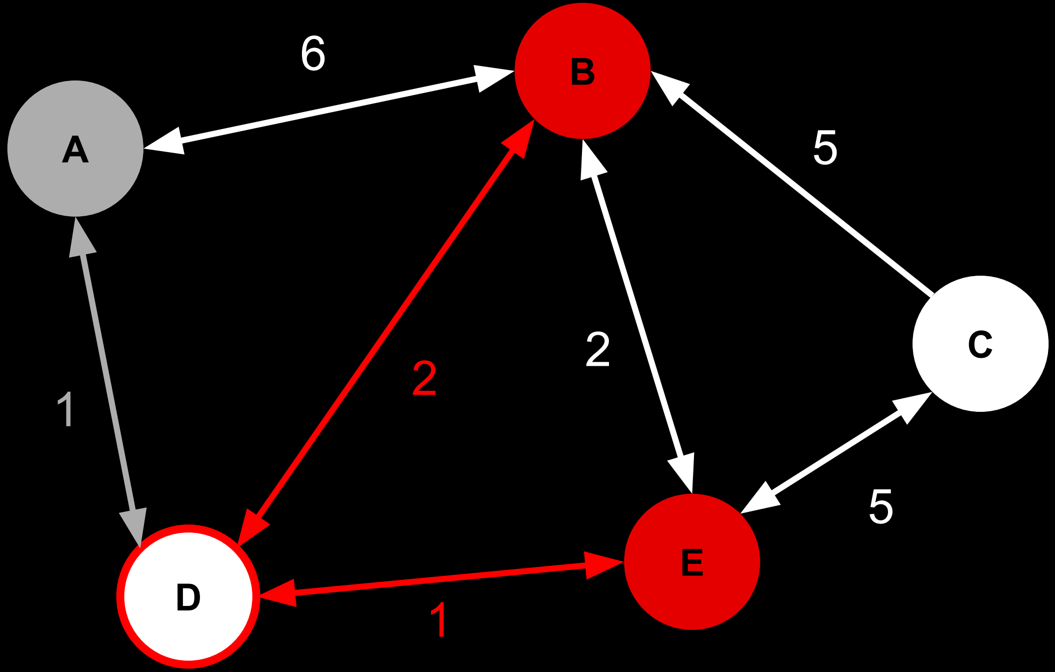
Mark A as visited

Distances: A = 0, B = 3, C = ∞, D = 1, E = ∞



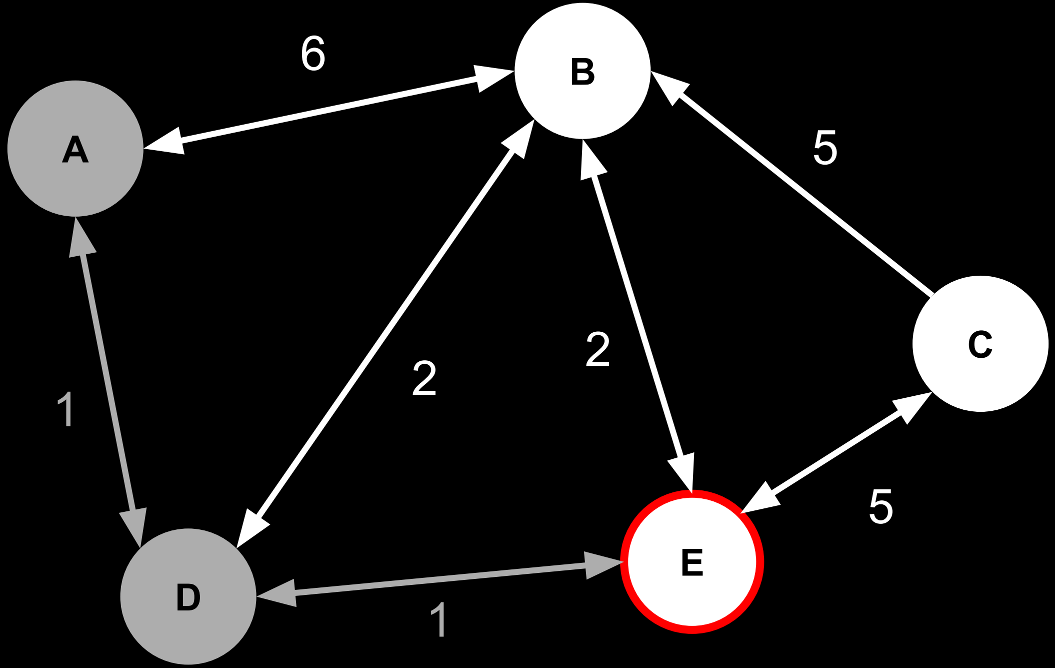
Update distance to B to 3

Distances: A = 0, B = 3, C = ∞, D = 1, E = 2



Update distance
to E to 2

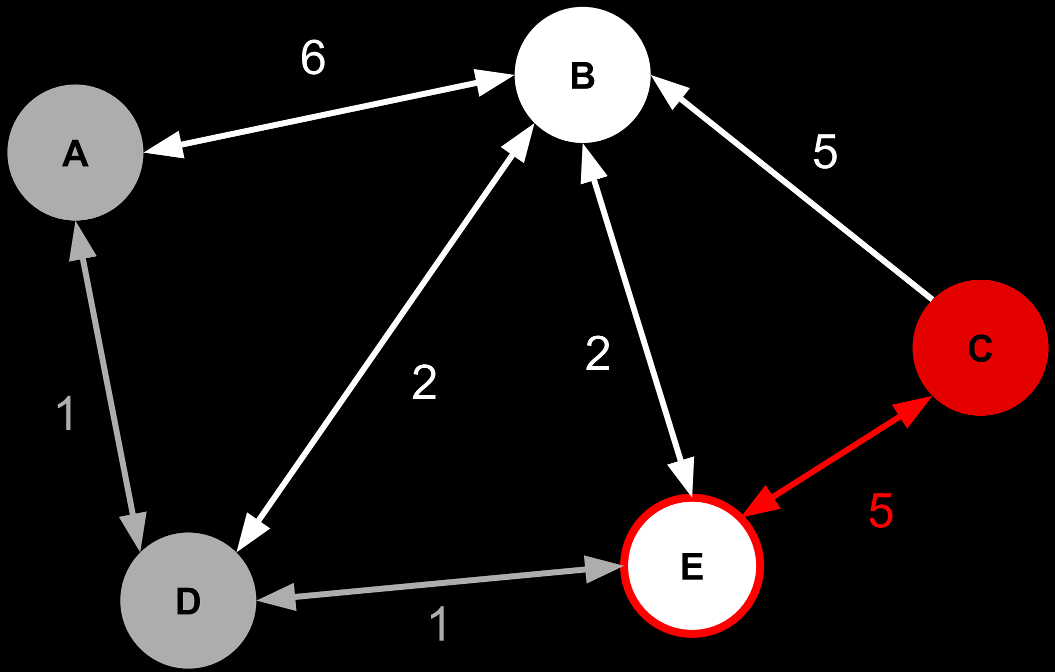
Distances: A = 0, B = 3, C = ∞, D = 1, E = 2



Move to E as next location

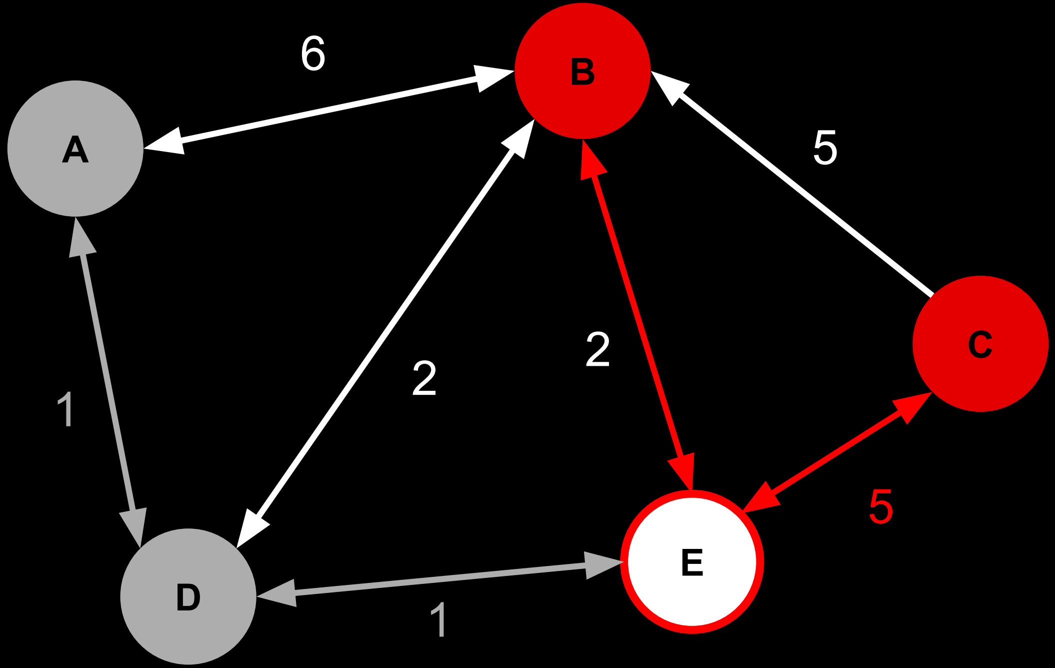
Mark D as visited

Distances: A = 0, B = 3, C = 7, D = 1, E = 2



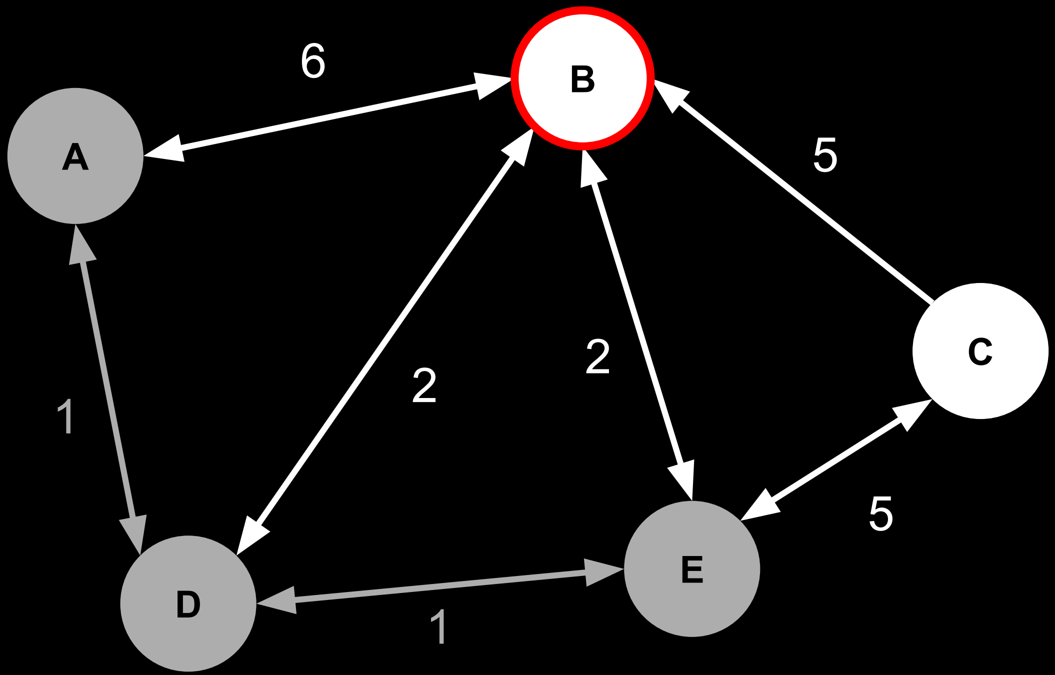
Update distance to C to 7

Distances: A = 0, B = 3, C = 7, D = 1, E = 2



No update for B, shorter path exists

Distances: A = 0, B = 3, C = 7, D = 1, E = 2

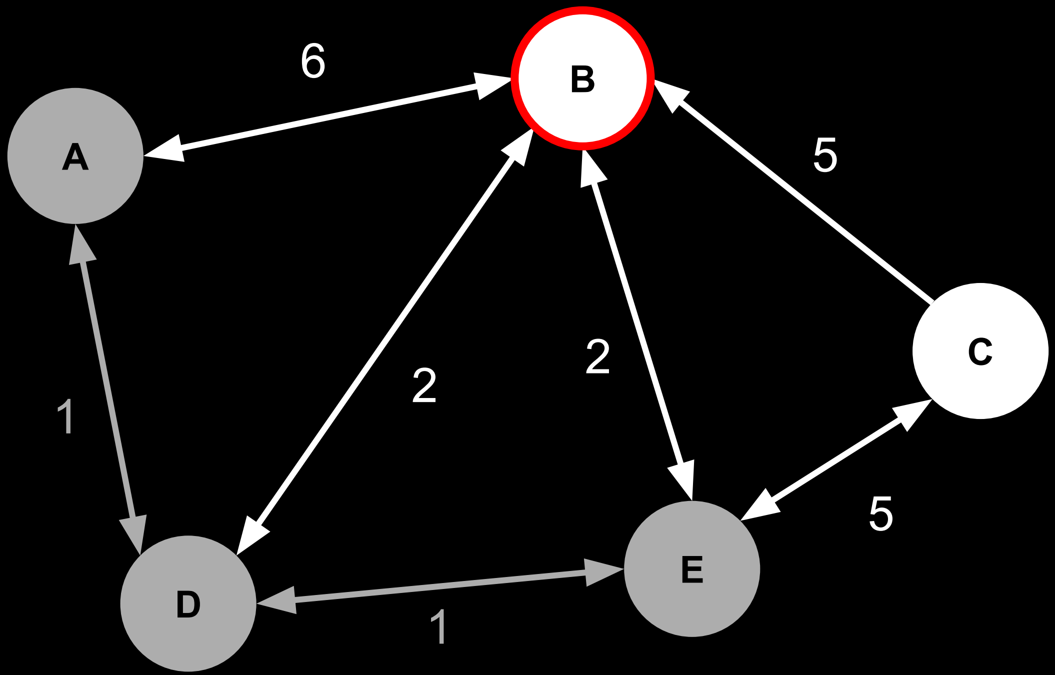


Move to B as new location

Mark E as visited

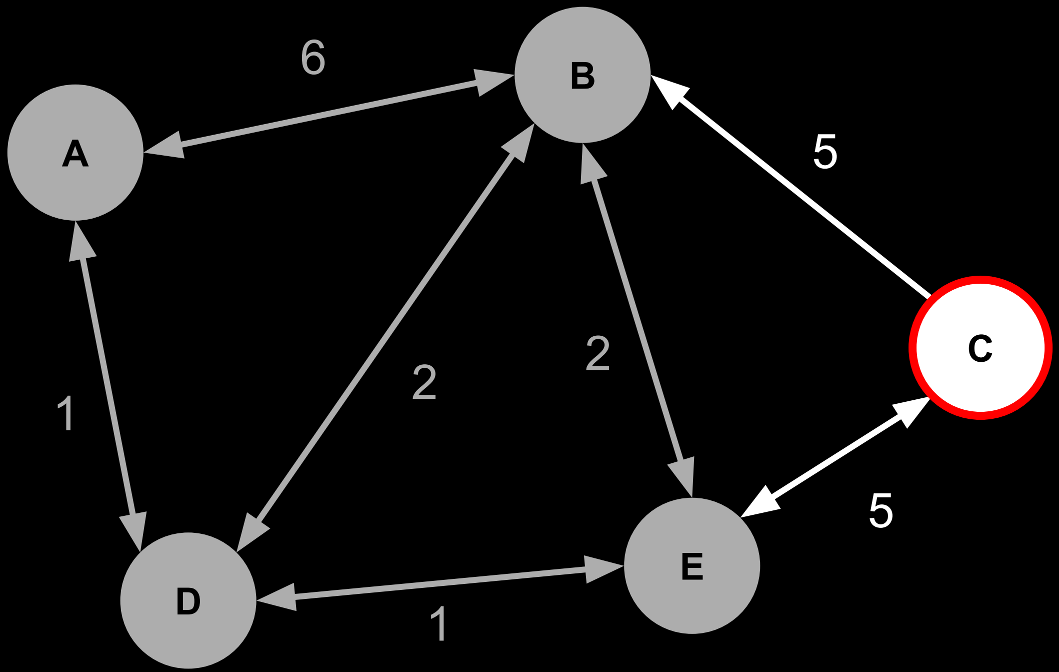
Distances: A = 0, B = 3, C = 7, D = 1, E = 2

No further locations reachable from B



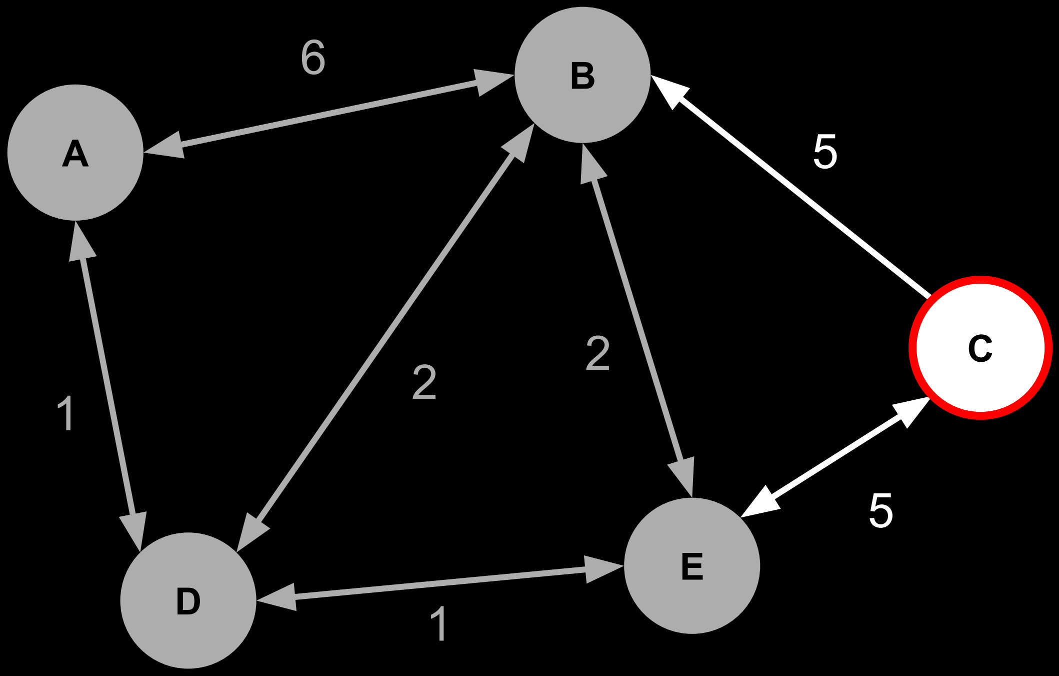
Distances: A = 0, B = 3, C = 7, D = 1, E = 2

Move to C as new location



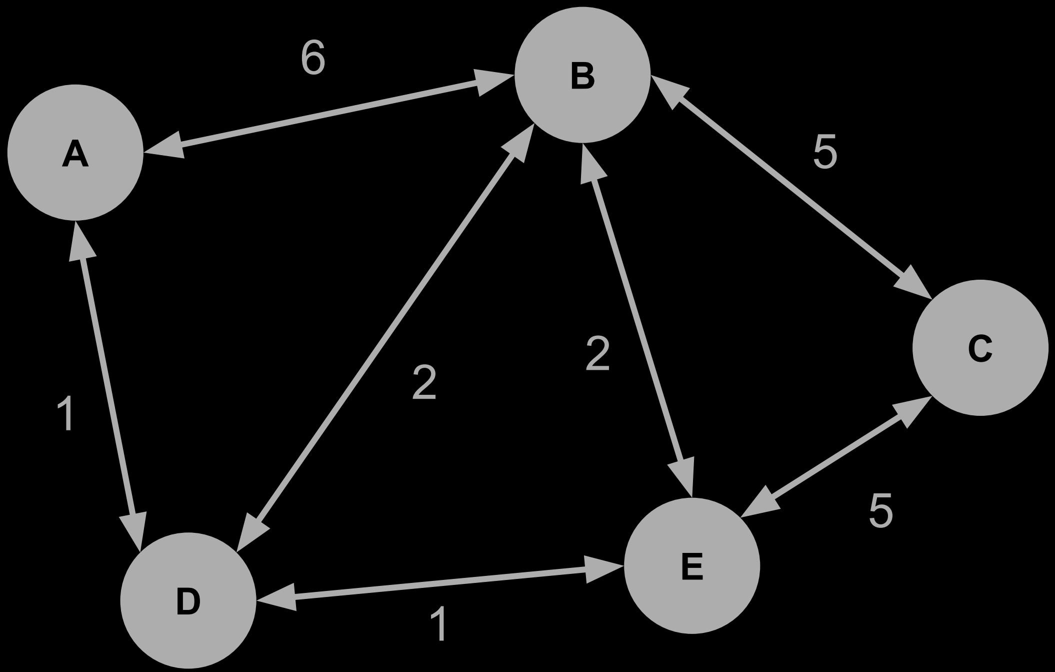
Distances: A = 0, B = 3, C = 7, D = 1, E = 2

No further locations reachable from C

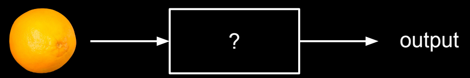


Distances: A = 0, B = 3, C = 7, D = 1, E = 2

All nodes visited, we're done!



spam emails



finding oranges in images

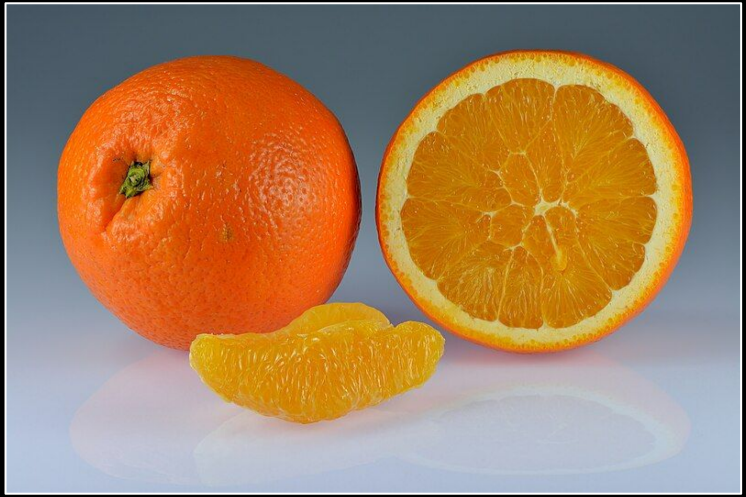


Image source: Wikimedia

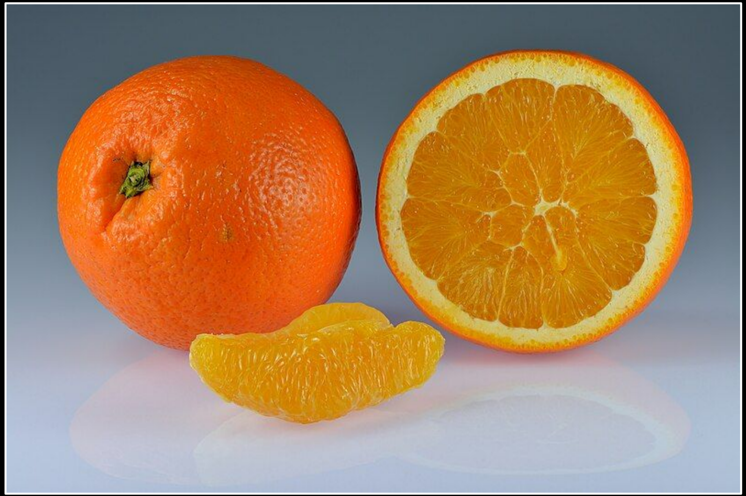


Image source: Wikimedia

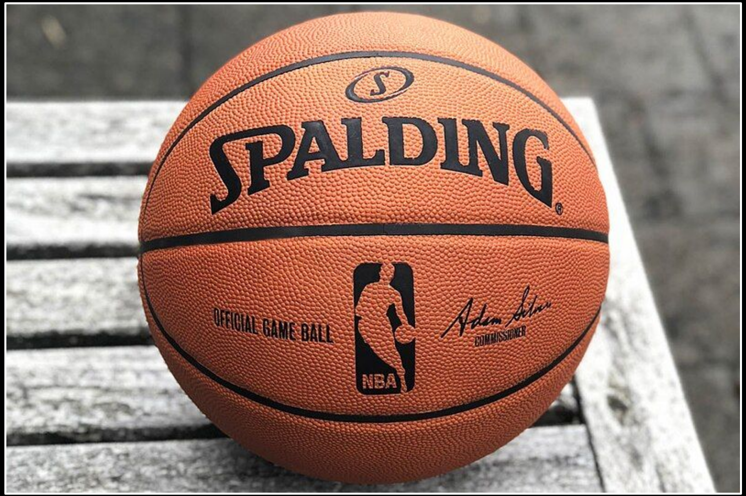
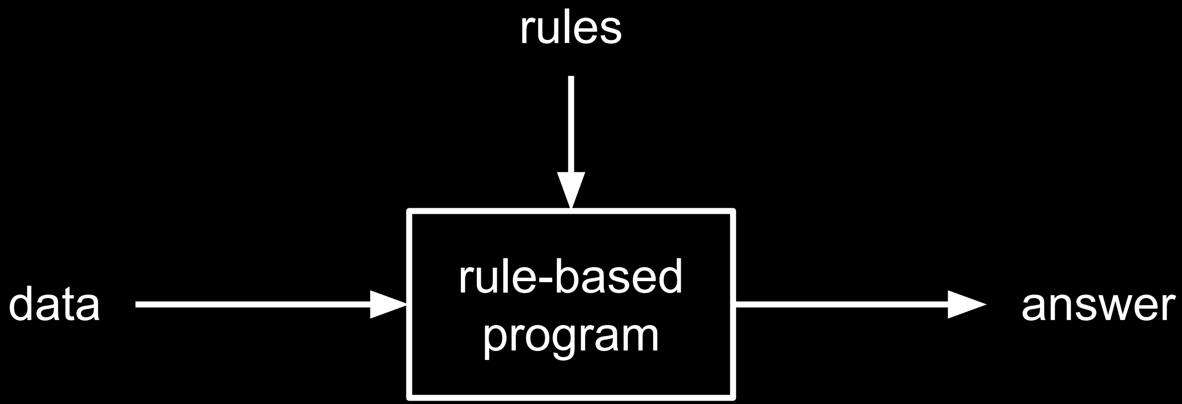
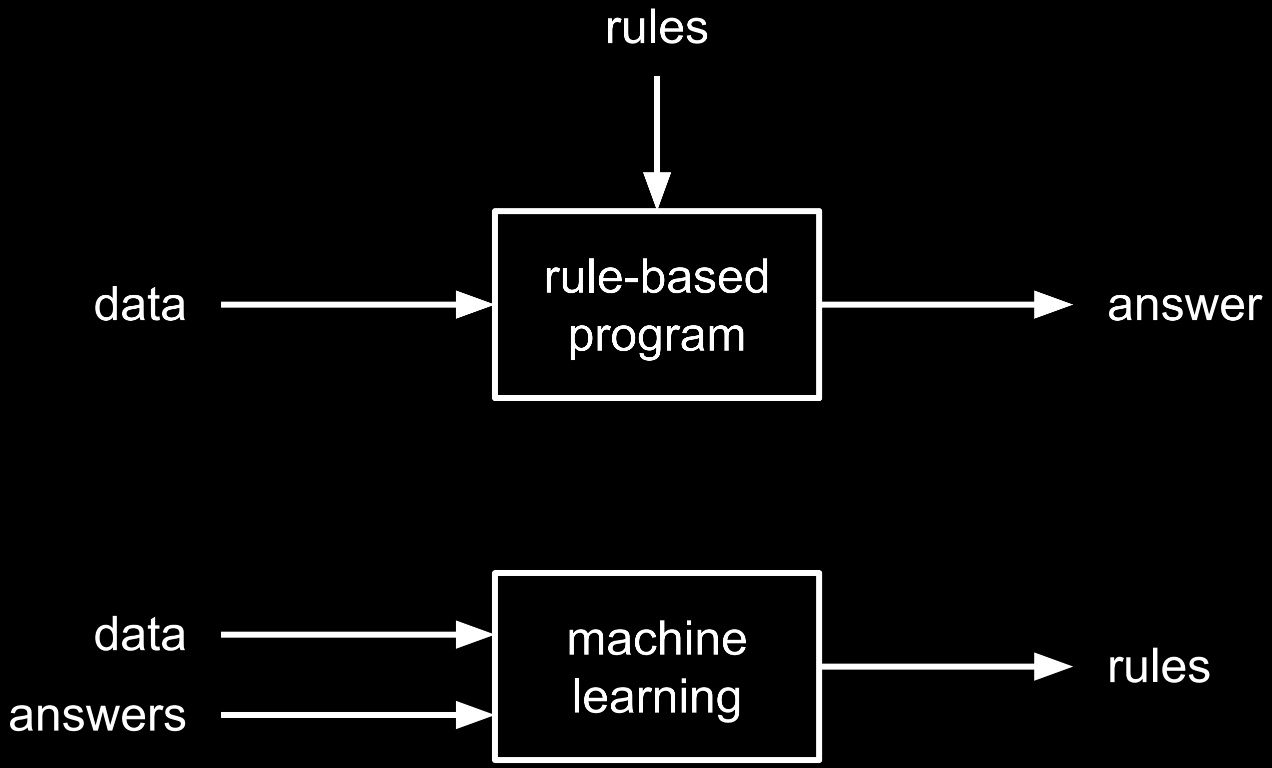


Image source: Wikimedia

what set of rules can solve this?

machine learning algorithms


rules
data
rule-based program
answer


rules
data
rule-based program
answer
data
answers
machine learning
rules