The numbers are the sections of the book. We will expect a self-contained exposition, making the appropiate definitions that are needed it, but going deeper on the important results of the topic. It will be good if you could prove some of the results. You will be able to bring some notes with you.

Preliminaries (2.1-2.3):

Ideal: Always two-sided, closed. Invertible elements in A are denoted $\mathrm{GL}(A)$.

1. **Gelfand-Naimark:** Every C^*-algebra is isomorphic to a sub-C^*-algebra of $B(H)$ for some H. If A is separable H can be chosen to be separable.

2. A sequence (finite or infinite):

$$\cdots \longrightarrow A_n \xrightarrow{\varphi_n} A_{n+1} \xrightarrow{\varphi_{n+1}} A_{n+2} \longrightarrow \cdots \tag{0.1}$$

   is **exact** if $\mathrm{Im}(\varphi_n) = \mathrm{Ker}(\varphi_{n+1})$. If the sequence is of the form:

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0 \tag{0.2}$$

   is called **short exact**. For example for I an ideal and $B = A/I$. If there is a map $\pi : B \to A$ such that $\psi \circ \pi = \mathrm{id}_B$, then π is called a **lift of** ψ and the sequence is called **split exact**.

3. The direct sum $A \oplus B$ of two C^*-algebras A and B is defined as all pairs (a, b) with entry-wise operations and the norm $\|(a, b)\| = \max\{\|a\|, \|b\|\}$. Can be visualised as a split exact sequence with inclusion and projection.

4. Let $\pi : \tilde{A} \to \mathbb{C}$ be the quotient, $\iota : A \to \tilde{A}$ be the inclusion and $\lambda : \mathbb{C} \to \tilde{A}$ be defined by $\alpha \mapsto \alpha 1_{\tilde{A}}$. Then the following sequence is split exact and $\tilde{A} = \{a + \alpha 1_{\tilde{A}} : a \in A, \alpha \in \mathbb{C}\}$. This is the **unitization of** A, also $\tilde{A}/A \simeq \mathbb{C}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \end{tikzcd} \tag{0.3}$$

   If A is unital, then $\tilde{A} = \{a + \alpha(1_{\tilde{A}} - 1_A) : a \in A, \alpha \in \mathbb{C}\} \simeq A \oplus \mathbb{C}$. If A is not unital, then this does not hold.

5. The **spectrum of** $a \in A$:

$$\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda \cdot 1 \notin \mathrm{GL}(A)\} \tag{0.4}$$

   Closed subset of $\mathbb{C}$. The **spectral radius**:

$$r(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a)\} \leq \|a\| \tag{0.5}$$

hence $\mathrm{Sp}(a)$ is a compact subset of $\mathbb{C}$. Also non-empty, and

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \tag{0.6}$$

If A not unital, view the spectrum of an element as embedded in the unitization. If A is not unital, then also $0 \in \mathrm{Sp}(a)$ for all $a \in A$.

6. 1. Self-adjoint: $a^* = a$

   2. Normal: $aa^* = a^*a$

   3. Positive: a normal and $\mathrm{Sp}(a) \subset \mathbb{R}^+$. Denoted by A^+.

7. 1. a self-adjoint $\Rightarrow \mathrm{Sp}(a) \subset \mathbb{R}$.

   2. a positive $\Leftrightarrow$ there exists some $x \in A$ such that $a = x^*x$.

   3. For normal element a, have $r(a) = \|a\|$ (show that normal implies $\|a^n\|^2 = \|a\|^{2n}$ equivalent to $\|(a^*a)^n\| = \|a^*a\|^n$, so enough to do for s.a. elements, but then $\|a\|^2 = \|a^2\| \Rightarrow \|a^{2^k}\| = \|a\|^{2^k}$, choose $m : 2^k = m+n$ and squeeze so $\|a^n\| = \|a\|^n$)

8. **Gelfand:** Every abelian C^*-algebra is isometriccaly $*$-isomorphic to the C^*-algebra $C_0(X)$ for some locally compact Hausdorff space X.

9. **Continuous function calculus:** (unital) A. For each normal $a \in A$ there is one and only one $*$-isomorphism:

$$C(\mathrm{Sp}(a)) \to C^*(a, 1) \subset A, \qquad f \mapsto f(a) \tag{0.7}$$

   which maps ι to a where $\iota(z) = z$ for all $z \in \mathrm{Sp}(a)$. Also respects polynomials and adjoints.

10. **Spectral mapping theorem:** Says that $\mathrm{Sp}(f(a)) = f(\mathrm{Sp}(a))$ for every normal a and continuous f on $\mathrm{Sp}(a)$.

Projections and unitaries (2.1-2.3):

Let A be a $C*$-algebra and X a topological space throughout.

Homotopy classes of unitary elements (2.1):

Definitions:

1. *Two points $a, b \in X$ are* ***homotopic*** *($a \sim_h b$) if there exists a continuous function $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$.*

   Can be seen to be an equivalence relation, depending on X. Always true for $C*$-algebras.

2. *An element $u \in A$ (unital) is unitary if $u^*u = uu^* = 1$ and we denote the group of unitaries by $\mathcal{U}(A) \subset A$. For $u, v \in \mathcal{U}(A)$ we see that $t \mapsto (1-t)u+tv$ is a continuous path and we can consider $\sim_h$ on $\mathcal{U}_0(A)$. Set $\mathcal{U}(A) = \{u \in \mathcal{U}(A) : u \sim_h 1\}$.*

3. Denote by $\mathrm{GL}_0(A) = \{a \in \mathrm{GL}(A) : a \sim_h 1\}$. Then $\mathcal{U}(A) \subset \mathrm{GL}(A)$. Denote by $|a| = (a^*a)^{1/2}$, this is the **absolute value**.

Results:

1. Unital A.

   1. $h \in A$ self-adjoint $\Rightarrow \exp(ih) \in \mathcal{U}_0(A)$.

   Proof. Let $f : \mathbb{R} \to \mathbb{T}$ be continuous. Since a self-adjoint element h is normal and $\mathrm{Sp}(h) \subset \mathbb{R}$, the CFC gives us an element $f(h) \in \mathbb{T}$ so $f(h)^* = f(h)^{-1} = \overline{f(h)}$ which shows that $f(h)$ is a unitary. In particular $f(h) = \exp(ih)$ is such a function.

   To see that $f(h) \sim_h 1$ define the map $f_t : \mathrm{Sp}(h) \to \mathbb{T}$ given by $f_t(x) = \exp(itx)$. The assignment of t to f_t is continuous and contained in $\mathcal{U}(A)$, so is $t \mapsto f_t(h)$. Finally $\exp(ih) = f_1(h) \sim_h f_0(h) = 1$. □

   2. $u \in \mathcal{U}(A)$ with $\mathrm{Sp}(u) \neq \mathbb{T} \Rightarrow u \in \mathcal{U}_0(A)$.

   Proof. Since $\mathrm{Sp}(u) \neq \mathbb{T}$, then there is some $\theta \in \mathbb{R}$ such that $\exp(i\theta) \notin \mathrm{Sp}(u)$. Define φ on $\mathrm{Sp}(u)$ as $\varphi(\exp(it)) = t$ where $t \in (\theta, \theta + 2\pi)$, then φ is continuous. For $z \in \mathrm{Sp}(u)$ there is some $s \in \mathbb{R}$ such that $z = \exp(is)$ and we have $\exp(i\varphi(z)) = \exp(is) = z$.

   Set $h = \varphi(u)$, then h is self-adjoint (real-valued) and $u = \exp(ih)$ and by 1.1. $u \in \mathcal{U}_0(A)$. □

   3. $u, v \in \mathcal{U}(A)$ with $\|u - v\| < 2 \Rightarrow u \sim_h v$.

Proof. By assumption $\|v^*u - 1\| = \|v^*(u-v)\| < 2$, so $-2 \notin \mathrm{Sp}(v^*u - 1) \Rightarrow -1 \notin \mathrm{Sp}(v^*u)$ so by 1.2. $v^*u \sim_h 1 \Leftrightarrow u \sim_h v$. $\square$

Corollary: $\mathcal{U}(M_n(\mathbb{C}))$ is connected; I.e. $\mathcal{U}(M_n(\mathbb{C})) = \mathcal{U}_0(M_n(\mathbb{C}))$.

Proof. Since $\mathcal{U}_0(A) \subset \mathcal{U}(A)$ always and the spectrum of any matrix is finite and therefore not the whole $\mathbb{T}$ it follows from 1.2. above. $\square$

2. **Whitehead:** (A unital), $u, v \in \mathcal{U}(A)$, then:

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}, \quad \text{in } \mathcal{U}(M_2(A)) \tag{0.8}$$

Proof. Since:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{0.9}$$

has eigenvalues $-1, 1$ it follows from 1.2. that:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.10}$$

By matrix multiplication, then:

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \tag{0.11}$$

the others follow similarly.

Corollary:

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.12}$$

$\square$

3. A unital.

1. $\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$.

Proof. Follows from the properties of path continuity. $\square$

2. $\mathcal{U}_0(A)$ is open and closed relative to $\mathcal{U}(A)$. An element $u \in A$ belongs to $\mathcal{U}_0(A)$ if and only if $u = \exp(ih_1) \cdots \exp(ih_n)$ for some $n \in \mathbb{N}$ and self-adjoint $h_1, \dots h_n \in A$.

Proof. Let

$$G = \{a \in A : a = \exp(ih_1) \cdots \exp(ih_n)\} \tag{0.13}$$

Then $G \subset \mathcal{U}_0(A)$ from 3.1 and 1.1. From the proof of 1.1. $\exp(ih)^{-1} = \exp(-ih)$ so G is a group.

For $u \in \mathcal{U}(A)$ and $v \in G$ with $\|u - v\| < 2$ we see that by 1.3. for some $h \in A_{sa}$ we can write $uv^* = \exp(ih) \Leftrightarrow u = \exp(ih)v \in G$. Hence G is open relative to $\mathcal{U}(A)$. If $u \in \mathcal{U}(A)$, then the set Gu is homeomorphic to G and therefore open relative to $\mathcal{U}(A)$. Furthermore the disjoint union of Gu is therefore open, but this is precisely the set $\mathcal{U}(A) \setminus G$ which shows that G is closed as the complement is open. Since $\mathcal{U}_0(A)$ is connected and $G \subset \mathcal{U}_0(A)$ is open and closed in $\mathcal{U}(A)$ this implies that $G = \mathcal{U}_0(A)$. ☐

4. A, B unital and $\varphi : A \to B$ surjective (and hence unit-preserving) $*$-homomorhism. Then $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(A)$.

   Proof. We have $\varphi(\mathcal{U}_0(A)) \subset \mathcal{U}_0(B)$ by the $*$-homomorphism structure. Let $u \in \mathcal{U}_0(B)$, then $u = \exp(ih_1) \cdots \exp(ih_n)$ for $h_j \in B_{sa}$ by 3.2. By surjectivity there are $x_j \in A$ such that $\varphi(x_j) = b_j$. By setting $k_j = (x_j + x_j^*)/2$ we get $k_j \in A_{sa}$ such that $\varphi(k_j) = h_j$, so set:

   $$v = \exp(ik_1) \cdots \exp(ik_n) \tag{0.14}$$

   then $v \in \mathcal{U}_0(A)$ and $\varphi(v) = u$. ☐

5. A unital. Recall $|a| = (a^*a)^{1/2}$ which is self-adjoint.

   1. If $z \in \mathrm{GL}(A)$, then $|z|, w(z) = z|z|^{-1} \in \mathcal{U}(A)$. Note $z = w(z)|z|$.

   Proof. $z \in \mathrm{GL}(A) \Rightarrow z^*, z^*z \in \mathrm{GL}(A) \Rightarrow |z| \in \mathrm{GL}(A)$ with inverse $((z^*z)^{-1})^{1/2}$. For $u = z|z|^{-1} = w(z)$ we have $z = u|z|$ and $u \in \mathrm{GL}(A)$ with $u^{-1} = u^*$ since $u^*u = (z|z|^{-1})^* z|z|^{-1} = |z|^{-1}z^*z|z|^{-1} = |z|^{-1}|z|^2|z|^{-1} = 1$. ☐

   2. The map $w : \mathrm{GL}(A) \to \mathcal{U}(A)$ in 1. is continuous, $w(u) = u$ for $u \in \mathcal{U}(A)$ and $w(z) \sim_h z$ in $\mathrm{GL}(A)$ for every $z \in \mathrm{GL}(A)$.

   Proof. Technical. ☐

   3. $u, v \in \mathcal{U}(A)$ and $u \sim_h v$ in $\mathrm{GL}(A)$, then $u \sim_h v$ in $\mathcal{U}(A)$.

   Proof. By 5.2. w is continuous, so a continuous path in $\mathrm{GL}(A)$ is sent to a continuous path in $\mathcal{U}(A)$. ☐

   The factorization $z = w(z)|z|$ is called the **polar decomposition for** z, we often write $z = u|z|$ with the emphasis being that we can decompose $z \in \mathrm{GL}(A)$ as a unique unitary and an absolute value.

6. A unital. If $\|1_A - a\| < 1$, then $a \in \mathrm{GL}(A)$ with inverse given by the **Neumann series** $a^{-1} = 1_A + (1_A - a) + (1_A - a)^2 + \cdots$, so $\|a^{-1}\| \leq (1 - \|1_A - a\|)^{-1}$.

7. A unital. Let $a \in \mathrm{GL}(A)$ and $b \in A$ with $\|a - b\| < \|a^{-1}\|^{-1}$, then $b \in \mathrm{GL}(A)$, and $\|b^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\|$ and $a \sim_h b$ in $\mathrm{GL}(A)$.

   Proof. Since

   $$\|1_A - a^{-1}b\| = \|a^{-1}(a - b)\| \leq \|a^{-1}\| \cdot \|a - b\| < 1 \tag{0.15}$$

   it follows that $a^{-1}b$ is invertible by the Neumann series. Therefore b is invertible with $b^{-1} = (a^{-1}b)^{-1}a^{-1}$ and $\|(a^{-1}b)^{-1}\| \leq (1 - \|1_A - a^{-1}b\|)^{-1}$. Furthermore

   $$\|b^{-1}\|^{-1} \geq \|(a^{-1}b)^{-1}\|^{-1}\|a^{-1}\|^{-1} \geq (1 - \|1_A - a^{-1}b\|)\|a^{-1}\|^{-1} \geq \|a^{-1}\|^{-1} - \|a - b\| \tag{0.16}$$

   Since for $c_t = (1-t)a + tb$ for $t \in [0,1]$ we have $\|a - c_t\| = t\|a - b\| \leq \|a^{-1}\|^{-1}$ which implies that $c_t \in \mathrm{GL}(A)$ it follows that $a \sim_h b$. □

Equivalence of projections (2.2):

Definitions:

1. *A projection in A is an element such that $p = p^* = p^2$. The set of all projections in A is denoted $\mathcal{P}(A)$.*

2. *Three relations on $\mathcal{P}(A)$:*

   Murray-Von Neumann equivalence:

   If there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$, denoted $p \sim q$.

   Unitary equivalence:

   If there exists a $u \in \mathcal{U}(\tilde{A})$ such that $q = upu^*$, denoted $p \sim_u q$.

   Homotopy equivalence

   If there exists a continuous function $f : [0,1] \to X$ such that $f(0) = p$ and $f(1) = q$, denoted $p \sim_h q$.

3. *An element $v \in A$ such that $v^*v \in \mathcal{P}(A)$ is called a **partial isometry** (implies $vv^* \in \mathcal{P}(A)$ as well).*

Results:

1. If $p = v^*v$ and $q = vv^*$, then $v = qv = vp = qvp$ (to show $v = vp$ use trick: $z = (1 - vv^*)v$ and calculate z^*z).

   Guarantees that MvN is a transitive relation.

2. A unital, $p, q \in \mathcal{P}(A)$. TFAE:

   1. $p \sim_u q$ (property in $\mathcal{U}(\tilde{A})$).
   2. $q = upu^*$ for some $u \in \mathcal{U}(A)$.
   3. $p \sim q$ and $1_A - p \sim 1_A - q$.

   Proof. Let $f = 1_{\tilde{A}} - 1_A = (-1_A, 1)$, then $\tilde{A} = A + \mathbb{C}f$ and $fa = af = 0$ for all $a \in A$. Recall that $(a, \alpha)(b, \beta) = (ab + a\beta + \alpha b, \alpha\beta)$.

   $1 \Rightarrow 2)$ Suppose $p \sim_u q$, then there is some $z \in \mathcal{U}(\tilde{A})$ such that $p = zqz^*$. We can write $z = u + \alpha f = (u - \alpha \cdot 1_A, \alpha)$ for $u \in A$, then:

$$
\begin{aligned}
(0,1) = z^*z &= (u - \alpha \cdot 1_A, \alpha)^*(u - \alpha \cdot 1_A, \alpha) \\
&= ((u - \alpha \cdot 1_A)^*(u - \alpha \cdot 1_A) + (u - \alpha \cdot 1_A)^*\alpha + \alpha^*(u - \alpha \cdot 1_A), |\alpha|^2) \\
&\Rightarrow \alpha = 1 \\
&\Rightarrow (u - 1_A)^*(u - 1_A) + (u - 1_A)^* + (u - 1_A) = 0 \\
&= u^*u - u^* - u + 1_A + u^* - 1_A + u - 1_A = 0 \\
&\Rightarrow u^*u = 1_A
\end{aligned}
$$

   and it is seen that $u \in \mathcal{U}(A)$ and also that $p = uqu^*$.

   $2 \Rightarrow 3)$ Suppose $p = uqu^*$ for some $u \in \mathcal{U}(A)$. Set $v = up$ and $w = u(1_A - p)$, then

$$
v^*v = p, \quad vv^* = q, \quad w^*w = 1_A - p, \quad ww^* = 1_A - q \tag{0.17}
$$

   $3 \Rightarrow 1)$ Suppose $p \sim q$ and $1_A - p \sim 1_A - q$. See that $z = v + w + f$ is unitary in $\tilde{A}$ and that $zpz^* = vpv^* = vv^* = q$. □

3. Let $p \in \mathcal{P}(A)$ and let $a \in A_{sa}$. If $\|p - a\| = \delta$, then $\mathrm{Sp}(a) \subset [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$.

   Proof. a is self-adjoint, so $\mathrm{Sp}(a) \subset \mathbb{R}$, and $\mathrm{Sp}(p) \subset \{0, 1\}$. Prove that if $t \in \mathbb{R}$ such that $d = \mathrm{dist}(t, \{0, 1\}) > \delta$ then $t \notin \mathrm{Sp}(a)$. We have $p - t \cdot 1 \in \mathrm{GL}(\tilde{A})$ and $\|(p - t \cdot 1)^{-1}\| = \max\{|-t|^{-1}, |1 - t|^{-1}\} = d^{-1}$. Using this

$$
\|(p - t \cdot 1)^{-1}(a - t \cdot 1) - 1\| = \|(p - t \cdot 1)^{-1}(a - p)\| \leq d^{-1}\delta < 1 \tag{0.18}
$$

   since $(p - t \cdot 1)^{-1}(p - t \cdot 1) = 1$. This proves that $(p - t \cdot 1)^{-1}(a - t \cdot 1)$ is invertible, and therefore $(a - t \cdot 1)$ is invertible and $t \notin \mathrm{Sp}(a)$. □

4. $p, q \in \mathcal{P}(A)$ with $\|p - q\| < 1$, then $p \sim_h q$.

$$
\textit{Technical.} \tag{0.19}
$$

5. A unital. $a, b \in A_{sa}$ and suppose that $z \in \mathrm{GL}(A)$ such that $b = zaz^{-1}$. Then $b = uau^*$ with $z = u|z|$ being the polar decomposition ($u \in \mathcal{U}(A)$).

Proof. $b = zaz^{-1} \Rightarrow bz = za$ and $z^*b = az^*$. This implies

$$|z|^2 a = (z^*z)a = z^*bz = az^*z = a|z|^2 \tag{0.20}$$

This implies that a commutes with $C^*(1, |z|^2)$ and therefore with $|z|^{-1}$ so $uau^* = z|z|^{-1}au^* = za|z|^{-1}u^* = bz|z|^{-1}u^* = buu^* = b$. □

6. $p, q \in \mathcal{P}(A)$. Then $p \sim_h q$ in $\mathcal{P}(A) \Leftrightarrow$ there is a $u \in \mathcal{U}_0(\tilde{A})$ such that $q = upu^*$.

   Proof. $\Leftarrow$) If $q = upu^*$ for $u \in \mathcal{U}_0(\tilde{A})$ and $t \mapsto u_t$ a continuous path of unitaries in $\tilde{A}$ from 1 to u. Since A is an ideal in $\tilde{A}$ it is closed and therefore $t \mapsto u_t p u_t^*$ is a continuous path of projections in A from p to q.

   $\Rightarrow$) suppose that $p \sim_h q$, then we can split the path to $n+1$ projections p_i such that $p_0 = p$ and $p_n = q$ and such that $\|p_{j+1} - p_j\| < 1/2$. It is therefore sufficient to prove the implication under the assumption that $\|p - q\| < 1/2$. Set $z = pq + (1_{\tilde{A}} - p)(1_{\tilde{A}} - q) \in \tilde{A}$, then $pz = pq = zq$ since $p \perp 1_{\tilde{A}} - p$. Further

$$\|z - 1\| = \|p(q-p) + (1_{\tilde{A}} - p)((1_{\tilde{A}} - q) - (1_{\tilde{A}} - p))\| \le 2\|p - q\| < 1 \tag{0.21}$$

   and $z \in \mathrm{GL}(\tilde{A})$ and $z \sim_h 1_{\tilde{A}}$ by 2.1.7. Set now $z = u|z|$, then $p = uqu^*$ by 2.2.5 since $pz = pq = zq$. By 2.1.5 it follows that $u \sim_h z \sim_h 1_{\tilde{A}}$ and therefore $u \in \mathcal{U}_0(A)$. □

7. $p, q \in \mathcal{P}(A)$, then

$$p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q \tag{0.22}$$

   Proof. The first implication follows 2.2.6. If $p \sim_u q$ then $q = upu^*$ for some unitary $u \in \mathcal{U}(\tilde{A})$. Since A is an ideal in $\tilde{A}$ it follows that $v = up \in A$ and it is easy to see that $p = v^*v$ and $q = vv^*$. □

   When passing to matrix algebras the relations are equal! Using Whitehead and matrix trickery.

   To see that this is not the case in general. To see that $p \sim q \not\Rightarrow p \sim_u q$ consider the unilateral shift on $\ell^2(\mathbb{N})$ to see that non-unitary isometries s exists. By def. $s^*s \sim ss^*$, but since the zero projection is not MvN to a non-zero projection it follows that $0 = 1 - s^*s \not\sim 1 - ss^* \neq 0$. The other case is more complicated.

Semigroups of projections (2.3):

Definitions:

1. *Set*

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)), \qquad \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A), \quad n \in \mathbb{N} \tag{0.23}$$

with convention that $\mathcal{P}_n(A)$ are pairwise disjoint.

Define relation $\sim_0$ as modified MvN-relation. If $p, q \in \mathcal{P}_\infty(A)$ then there exists $m, n \in \mathbb{N}$ such that $p \in \mathcal{P}_n(A)$ and $q \in \mathcal{P}_m(A)$, we say $p \sim_0 q$ if there exists $v \in M_{m,n}(A)$ such that $p = v^*v$ and $q = vv^*$.

Define further addition $\oplus$ on $\mathcal{P}_\infty(A)$ by $p \oplus q = \operatorname{diag}(p, q)$.

2. *Set*

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0 \tag{0.24}$$

and denote for each $p \in \mathcal{P}_\infty(A)$ the equivalence class containing p by $[p]_\mathcal{D} \in \mathcal{D}(A)$. Define addition on $\mathcal{D}(A)$ by

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}, \quad p, q \in \mathcal{P}_\infty(A) \tag{0.25}$$

This defines an Abelian semigroup $(\mathcal{D}(A), +)$.

Results:

1. $p, q, r, p', q' \in \mathcal{P}_\infty(A)$, then

   1. $p \sim_0 p \oplus 0_n$ for every $n \in \mathbb{N}$ and 0_n is the zero element of $M_n(A)$.

   Proof. (Utilize matrix mult.) Let $m, n \in \mathbb{N}$ and assume $p \in \mathcal{P}_m(A)$. Set $u_1 = \begin{pmatrix} p \\ 0 \end{pmatrix} \in M_{m+n,m}(A)$. Then $p = u_1^*u_1 \sim_0 u_1u_1^* = p \oplus 0_n$. □

   2. $p \sim_0 p'$ and $q \sim_0 q' \Rightarrow p \oplus q \sim_0 p' \oplus q'$.

   Proof. By assumption there exists suitable v, w such that $p = v^*v, p' = vv^*$ and $q = w^*w, q' = ww^*$, so if $u_2 = \operatorname{diag}(v, w)$ then $p \oplus q = u_2^*u_2 \sim_0 u_2u_2^* = p' \oplus q'$. □

   3. $p \oplus q \sim_0 q \oplus p$.

   Proof. Suppose $p \in \mathcal{P}_n(A), q \in \mathcal{P}_m(A)$ and set $u_3 = \begin{pmatrix} 0_{n,m} & q \\ p & 0_{m,n} \end{pmatrix} \in M_{m+n}(A)$. Then $p \oplus q = u_3^*u_3 \sim_0 u_3u_3^* = q \oplus p$ □

   4. $p, q \in \mathcal{P}_n(A)$ such that $pq = 0$, then $p + q \in \mathcal{P}(A)$ and $p + q \sim_0 p \oplus q$.

   Proof. If $pq = 0$ then $(p+q)^2 = p^2+q^2 = p+q$ and obviously $(p+q)^* = p+q$ so $p+q$ is a projection. Set $u_4 = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(A)$. Then $p+q = u_4^*u_4 \sim_0 u_4u_4^* = p \oplus q$. □

   5. $(p \oplus q) \oplus r = p \oplus (q \oplus r)$.

   Proof. Diagonal remains the same. □

Description of K_0*:*

The unital case (3.1):

Grothendieck group:

Let $(S, +)$ be an Abelian semigroup and define $\sim$ on $S \times S$ by $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $z \in S$ such that $x_1 + y_2 + z = x_2 + y_1 + z$. Need the z for transitivity to work so that $\sim$ indeed is an equivalence relation.

Write $G(S) = S \times S/\sim$ and denote by $\langle x, y\rangle$ the equivalence classes in $G(S)$ containing $(x, y) \in S \times S$. Define addition by

$$\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle \tag{0.26}$$

and note that then $(G(S), +)$ becomes an Abelian group with $-\langle x, y\rangle = \langle y, x\rangle$ and $0 = \langle x, x\rangle$. The group $G(S)$ is called the **Grothendieck group of** S. Think of it as a way of introducing inverses to a semigroup. For instance one can make $\mathbb{N}$ into $\mathbb{Z}$, or $\mathbb{N} \times \mathbb{N}$ into $\mathbb{Q}^+$.

The group $G(S)$ is said to have the **cancellation property** if $x + z = y + z \Rightarrow x = y$.

Fix an $y \in S$ and define the **Grothendieck map**:

$$\gamma_S : S \to G(S), \qquad x \mapsto \langle x + y, x\rangle \tag{0.27}$$

It is an additive map independent of y.

1. (**Standard picture**) $G(S) = \{\gamma_S(x) - \gamma_S(y) : x, y \in S\}$.

   Proof. Every element in $G(S)$ has the form $\langle x, y\rangle$ for some $x, y \in S$, and:

   $$\langle x, y\rangle = \langle x + y, y\rangle - \langle x + y, x\rangle = \gamma_S(x) - \gamma_S(y) \tag{0.28}$$

   □

2. () $x, y \in S$. Then $\gamma_S(x) = \gamma_S(y) \Leftrightarrow x + z = y + z$ for some $z \in S$.

   Proof. $\Leftarrow$) If $x + z = y + z$ then $\gamma_S(x + z) = \gamma_S(y + z) \Rightarrow \gamma_S(x) = \gamma_S(y)$ since γ_S is additive and since $G(S)$ is a group (and has cancellation).

   $\Rightarrow$) $\gamma_S(x) = \gamma_S(y) \Rightarrow \langle x + y, y\rangle = \langle y + x, x\rangle \Rightarrow \exists w \in S : (x + y) + x + w = (y + x) + y + w$ so by setting $z = x + y + w$ we see that $x + z = y + z$. □

3. () The Grothendieck map is injective if and only of S has the cancellation property.

   Proof. Follows from the previous. □

4. (**The universal property**) If H is an abelian group and $\varphi : S \to H$ an additive map, then there exists a unique group homomorphism $\psi : G(S) \to H$ such that the diagram commutes:

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "\gamma_S"'] & & H \\ & G(S) \arrow[ur, dashed, "\psi"'] & \end{tikzcd} \tag{0.29}$$

Proof. If $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle$ then for some $z \in S$ we have $x_1 + y_2 + z = x_2 + y_1 + z$ so by applying φ we get $\varphi(x_1) - \varphi(y_1) = \varphi(x_2) - \varphi(y_2)$ since H is a group and φ a group homomorphism. This implies that the map $\psi : G(S) \to H$ given by $\langle x, y\rangle \mapsto \varphi(x) - \varphi(y)$ is well-defined, since ψ preserves the equivalence classes. The additivity of ψ follows from that of φ and uniqueness from the standard picture of $G(S)$. ☐

5. (**Functorality**) To every additive map $\varphi : S \to T$ between semigroups S, T there exists a unique group homomorphism $G(\varphi) : G(S) \to G(T)$ such that the diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, "\gamma_S"'] & T \arrow[d, "\gamma_T"] \\ G(S) \arrow[r, dashed, "G(\varphi)"'] & G(T) \end{tikzcd} \tag{0.30}$$

Proof. By the universal property the additive map $\gamma_T \circ \varphi : S \to G(T)$ factors uniquely through a group homomorphism $G(\varphi) : G(S) \to G(T)$:

$$\begin{tikzcd} S \arrow[rr, "\gamma_T \circ \varphi"] \arrow[dr, "\gamma_S"'] & & G(T) \\ & G(S) \arrow[ur, dashed, "G(\varphi)"'] & \end{tikzcd} \tag{0.31}$$

☐

6. () Let $(H, +)$ be an Abelian group and S a non-empty subset of H. If S is closed under addition, then $(S, +)$ is an Abelian subgroup with the cancellation property, $G(S) \simeq H_0$ where H_0 is the subgroup generated by S, and $H_0 = \{x - y : x, y \in S\}$

Proof. A closed (under addition) non-empty subset S of an Abelian group H is automatically an Abelian semigroup with the cancellation property. So by using universality with the inclusion $\iota : S \to H$ we get a group homomorphism $\psi : G(S) \to H$ such that $\psi(\gamma_S(x)) = x$ for all $x \in S$. But by how ψ is defined, the image is $H_0 = \{x - y : x, y \in S\}$ by the standard picture. Finally if $\psi(\gamma_S(x) - \gamma_S(y)) = 0$, then $x - y = 0$ and $\gamma_S(x) - \gamma_S(y) = 0$. □

To define the K_0 we apply the Grothendieck construction to the semigroup of projections $\mathcal{D}(A)$. That is $K_0(A) = G(\mathcal{D}(A))$, and denote the Grothendieck map by $\gamma : \mathcal{D} \to K_0(A)$. Define:

$$\begin{aligned}[\cdot]_0 : \mathcal{P}_\infty(A) &\to K_0(A) \\ p &\mapsto \gamma([p]_{\mathcal{D}})\end{aligned}$$

We will see that we can express the $K_0(A)$ using $[\cdot]_0$. Define $\sim_s$ on $\mathcal{P}_\infty(A)$: If $p, q \in \mathcal{P}_\infty(A)$, then $p \sim_s q$ if and only if $p \oplus r \sim_0 q \oplus r$ for some $r \in \mathcal{P}_\infty(A)$. This is called **stable equivalence**.

Note that $p \sim_0 q \Rightarrow [p]_0 = [q]_0$. And we shall see that the reverse holds only when the stronger relation of stable equivalence is enforced.

For unital A we have $p \sim_s q \Leftrightarrow p \oplus 1_n \sim_0 q \oplus 1_n$ for some $n \in \mathbb{N}$. This since if there is some $r \in \mathcal{P}_n(A)$ such that $p \oplus r \sim_0 q \oplus r$, then $p \oplus 1_n \sim_0 p \oplus r \oplus (1_n - r) = q \oplus r \oplus (1_n - r) \sim_0 q \oplus 1_n$ since $r \perp 1_n - r$.

1. **The standard picture of $K_0(A)$ (unital).** Let A be unital. Then:

$$K_0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} \tag{0.32}$$
$$= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\} \tag{0.33}$$

Proof. The first equality follows by the standard picture of the Grothendieck group. For the second one if $g \in K_0(A)$, then $g = [p']_0 - [q']_0$ for some $p' \in \mathcal{P}_k(A), q' \in \mathcal{P}_l(A)$ for some $k, l \in \mathbb{N}$. Choose n greater than k, l and set $p = p' \oplus 0_{n-k}$ and $q = q' \oplus 0_{n-l}$, then $p, q \in \mathcal{P}_n(A)$ and $p \sim_0 p', q \sim_0 q'$ from which it follows that $g = [p]_0 - [q]_0$. □

2. $[p \oplus q]_0 = [p]_0 + [q]_0$ for all $p, q \in \mathcal{P}_\infty(A)$.

Proof. We have $[p \oplus q]_0 = \gamma([p \oplus q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}} + [q]_{\mathcal{D}}) = \gamma([p]_{\mathcal{D}}) + \gamma([q]_{\mathcal{D}}) = [p]_0 + [q]_0$. □

3. $[0_A]_0 = 0$ where 0_A is the zero projection in A.

Proof. Since $0_A \oplus 0_A \sim_0 0_A$ the above gives $[0_A]_0 = [0_A]_0 + [0_A]_0 \Rightarrow [0_A]_0 = 0$. □

4. If $p, q \in \mathcal{P}_n(A)$ for some n and $p \sim_h q$ in $\mathcal{P}_n(A)$ then $[p]_0 = [q]_0$.

Proof. If $p \sim_h q$ then $p \sim q \Rightarrow p \sim_0 q \Leftrightarrow [p]_{\mathcal{D}} = [q]_{\mathcal{D}} \to [p]_0 = [q]_0$. Need both $p, q \in \mathcal{P}_n(A)$ for the first two relations to be defined. □

5. If p, q are mutually orthogonal projections in $\mathcal{P}_n(A)$ then $[p+q]_0 = [p]_0 + [q]_0$.

   Proof. Since $p + q \sim_0 p \oplus q$ it follows that $[p+q]_0 = [p \oplus q]_0 = [p]_0 + [q]_0$. □

6. For all $p, q \in \mathcal{P}_\infty(A)$ we have $[p]_0 = [q]_0 \Leftrightarrow p \sim_s q$.

   Proof. If $[p]_0 = [q]_0 \Leftrightarrow \gamma([p]_{\mathcal{D}}) = \gamma([q]_{\mathcal{D}})$, then there is some $r \in \mathcal{P}_\infty(A)$ with $[p]_{\mathcal{D}} + [r]_{\mathcal{D}} = [q]_{\mathcal{D}} + [r]_{\mathcal{D}} \Leftrightarrow [p \oplus r]_{\mathcal{D}} = [q \oplus r]_{\mathcal{D}} \Rightarrow p \oplus r \sim_0 q \oplus r \Leftrightarrow p \sim_s q$.
   If $p \sim_s q$ then for some $r \in \mathcal{P}_\infty(A)$ we have $p \oplus r \sim_0 q \oplus r \Rightarrow [p]_0 + [r]_0 = [q]_0 + [r]_0 \Rightarrow [p]_0 = [q]_0$ since $K_0(A)$ is a group. □

The next result states the "essensial" properties of K_0. Let A be unital, G an Abelian group and suppose that $\nu : \mathcal{P}_\infty(A) \to G$ is a map such that

1. $\nu(p \oplus q) = \nu(p) + \nu(q)$
2. $\nu(0_A) = 0$
3. If $p, q \in \mathcal{P}_n(A)$ for some n and $p \sim_h q$ in $\mathcal{P}_n(A)$, then $\nu(p) = \nu(q)$

Then there is a unique group homomorphism $\alpha : K_0(A) \to G$ such that the diagram commutes:

$$\begin{tikzcd} \mathcal{P}_\infty(A) \arrow[rr, "\nu"] \arrow[dr, "{[\cdot]_0}"'] & & G \\ & K_0(A) \arrow[ur, dashed, "\alpha"'] & \end{tikzcd} \tag{0.34}$$

Proof. We argue using the universality of the Grothendieck group. First, if $p, q \in \mathcal{P}_\infty(A) \Rightarrow \exists k, l \in \mathbb{N} : p \in \mathcal{P}_k(A), q \in \mathcal{P}_l(A)$. Assume $p \sim_0 q$. Let $n > k, l$ and put $p' = p \oplus 0_{n-k}$ and $q' = q \oplus 0_{n-l}$, then $p', q' \in \mathcal{P}_n(A)$ and $p' \sim_0 p \sim_0 q \sim_0 q'$. Since passing to matrix algebra gives us homotopy in this case we get $p' \oplus 0_{3n} \sim_h q' \oplus 0_{3n}$ in $\mathcal{P}_{4n}(A)$. This implies that:

$$\nu(p) = \nu(p) + \nu(0) + \cdots \nu(0) = \nu(p' \oplus 0_{3n}) = \nu(q' \oplus 0_{3n}) = \nu(q) \tag{0.35}$$

so the map defined $\beta : \mathcal{D} \to G$ defined by $\beta([p]_{\mathcal{D}}) = \nu(p)$ is well-defined. Additivity follows readily and by universality of $G(S)$ there exists a unique group homomorphism $K_0(A) \to G$ such that the diagram commutes:. □

The non-unital case (4.1-4.2):

1. Motivated by wanting functor to preserve exactness. Therefore to extend to non-unital we do this:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \end{tikzcd} \tag{0.36}$$

   Induces:

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r] & K_0(\tilde{A}) \arrow[r, "K_0(\pi)"] & K_0(\mathbb{C}) \arrow[r] & 0 \end{tikzcd} \tag{0.37}$$

   Where $K_0(A)$ is defined to be $\mathrm{Ker}(K_0(\pi))$ which is an Abelian group, since subgroup of $K_0(\tilde{A})$. For $\mathcal{P}_\infty(A)$ consider $[p]_0 \in K_0(\tilde{A})$, since $K_0(\pi)([p]_0) = [\pi(p)]_0 = 0$ so $[p]_0 \in K_0(A)$. Induces map $[\cdot]_0 : \mathcal{P}_\infty(A) \to K_0(A)$.

   If A is unital we have that the map from $K_0(A)$ to $K_0(\tilde{A})$ is $K_0(\iota)$, exactness, $K_0(A) \simeq K_0(\tilde{A})$ under $K_0(\iota)$ and $K_0(\iota)([p]_0) = [p]_0 \in K_0(\tilde{A})$ for $[p]_0 \in K_0(A)$ and $p \in \mathcal{P}_\infty(A)$.

   When A is not unital the map from $K_0(A)$ to $K_0(\tilde{A})$ is the inclusion.

   For both we have that $\mathrm{Im}(K_0(\iota)) = \mathrm{Ker}(K_0(\pi))$, so $K_0(A) = \mathrm{Ker}(K_0(\pi))$ holds in both cases.

K_0 *as a functor (3.2):*

A **covariant functor** F **between categories** $\mathcal{E}$ **and** $\mathcal{F}$ is a map $A \to F(A)$ from $\mathrm{Obj}(\mathcal{E})$ to $\mathrm{Obj}(\mathcal{F})$ and a collection of maps $\varphi \mapsto F(\varphi)$ from $\mathrm{Mor}(A,B)$ to $\mathrm{Mor}(F(A),F(B))$ for each $A,B \in \mathrm{Obj}(\mathcal{E})$ such that:

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all Objects A in $\mathrm{Obj}(\mathcal{E})$
2. $F(\psi \circ \varphi) = F(\psi) \circ F(\varphi)$ for all objects A,B,C in $\mathrm{Obj}(\mathcal{E})$ and all morphisms $\varphi \in \mathrm{Mor}(A,B)$ abd $\psi \in \mathrm{Mor}(B,C)$.

Zero object: $\{0\}$ if $\mathrm{Mor}(A,\{0\})$ and $\mathrm{Mor}(\{0\},A)$ contains one object.

Zero morphism: $0_{B,A} : A \to B$ send all elements in A to $\{0\}$.

Definition of K_0 as a functor:

Let A,B be unital and $\varphi : A \to B$ a $*$-homomorphism. We can extend φ to a map $\varphi : M_n(A) \to M_n(B)$ for each n and since $*$-homomorphisms preserve projections φ maps $\mathcal{P}_\infty(A)$ to $\mathcal{P}_\infty(B)$.

Define $\nu : \mathcal{P}_\infty(A) \to K_0(B)$ by $\nu(p) = [\varphi(p)]_0$. Since K_0 is universal it follows that ν factors uniquely through a group homomorphism $K_0(\varphi) : K_0(A) \to K_0(B)$ given by $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ for $p \in \mathcal{P}_\infty(A)$.

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "{[\cdot]_0}"'] & B \arrow[d, "{[\cdot]_0}"] \\ K_0(A) \arrow[r, dashed, "K_0(\varphi)"'] & K_0(B) \end{tikzcd} \tag{0.38}$$

Functorality of K_0 for unital $C*$-algebras:

1. For each A, $K_0(\mathrm{id}_A) = \mathrm{id}_{K_0(A)}$.
2. For every A,B,C, if $\varphi : A \to B$ and $\psi : B \to C$ are $*$-hom, then $K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$.

   Proof. These two follow from $K_0(\varphi)([p]_0) = [\varphi(p)]_0$ and the standard picture of $K_0(A)$. □

3. $K_0(\{0\}) = \{0\}$.

   Proof. Since $\mathcal{P}_n(\{0\}) = \{0_n\}$ and all 0_n are equivalent under $\sim_0$ with $0 = 0_1$ it follows that $\mathcal{D}(\{0\}) = \{[0]_{\mathcal{D}}\}$ and that $K_0(\{0\}) = G(\{0\}) = \{0\}$. □

4. For every A,B we have $K_0(0_{B,A}) = 0_{K_0(B),K_0(A)}$.

Proof. Since $0_{B,A} : A \to B$ by $a \mapsto 0$ we can write $0_{B,A} = 0_{B,0} \circ 0_{0,A} : B \to \{0\} \to A$, the first and third property induces:

$$K_0(0_{B,A}) = K_0(0_{B,0}) \circ K_0(0_{0,A}) = 0_{K_0(B),0} \circ 0_{0,K_0(A)} = 0_{K_0(B),K_0(A)} \tag{0.39}$$

□

Two $*$-homomorphisms $\varphi, \psi : A \to B$ are **homotopic** ($\varphi \sim_h \psi$) if there is a continuous path of $*$-homs $\varphi_t : A \to B$ for $t \in [0,1]$ such that $t \mapsto \varphi_t(a)$ is continuous for each $a \in A$ and $\varphi_0 = \varphi, \varphi_1 = \psi$. Say $t \mapsto \varphi_t$ is pointwise cont.

A and B are **homotopy equivalent** if there are $*$-homs $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$, we say:

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} A \tag{0.40}$$

is a **homotopy** between A and B.

1. **Homotopy invariance of** K_0: A, B unital. If $\varphi, \psi : A \to B$ are homotopic $*$-hom, then $K_0(\varphi) = K_0(\psi)$.

   Proof. Extend φ_t to $\varphi_t : M_n(A) \to M_n(B)$ for each $n \in \mathbb{N}$. If $p \in \mathcal{P}_n(A)$ then $t \mapsto \varphi_t(p)$ is continuous and $\varphi(p) \sim_h \psi(p)$ which implies that:

   $$K_0(\varphi)([p]_0) = [\varphi(p)]_0 = [\psi(p)]_0 = K_0(\psi)([p]_0) \tag{0.41}$$

   by standard picture it therefore follows that $K_0(\varphi) = K_0(\psi)$. □

2. If A, B are homotopy equivalent, then $K_0(A) \simeq K_0(B)$.

   Proof. Since A, B homotopy equivalent there are $*$-homs $\varphi : A \to B$ and $\psi : B \to A$ such that $\psi \circ \varphi \sim_h \mathrm{id}_A$ and $\varphi \circ \psi \sim_h \mathrm{id}_B$. By the above:

   $$\mathrm{id}_{K_0(A)} = K_0(\mathrm{id}_A) = K_0(\psi \circ \varphi) = K_0(\psi) \circ K_0(\varphi)$$
   $$\mathrm{id}_{K_0(B)} = K_0(\mathrm{id}_B) = K_0(\varphi \circ \psi) = K_0(\varphi) \circ K_0(\psi)$$

   which shows that $K_0(\varphi)^{-1} = K_0(\psi)$ and hence that $K_0(\psi)$ and $K_0(\varphi)$ are $*$-isomorphisms. □

K_0 preserves exactness of the short exact sequence resulting from adjoining a unit to A. For unital A the split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{A} \arrow[r, shift left, "\pi"] & \mathbb{C} \arrow[l, shift left, "\lambda"] \arrow[r] & 0 \end{tikzcd} \tag{0.42}$$

induces a split exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & K_0(A) \arrow[r, "K_0(\iota)"] & K_0(\tilde{A}) \arrow[r, shift left, "K_0(\pi)"] & K_0(\mathbb{C}) \arrow[l, shift left, "K_0(\lambda)"] \arrow[r] & 0 \end{tikzcd} \tag{0.43}$$

Proof. Uses that if $\varphi \perp \psi (\varphi(x)\psi(y) = 0 \, \forall x, y \in A)$ then $K_0(\varphi + \psi) = K_0(\varphi) + K_0(\psi)$.

Define $*$-hom $\mu : \tilde{A} to A$ and $\lambda' : \mathbb{C} \to \tilde{A}$ by $\mu(a + \alpha f) = a$ and $\lambda'(\alpha) = \alpha f$, then:

$$\mathrm{id}_A = \mu \circ \iota, \quad \mathrm{id}_{\tilde{A}} = \iota \circ \mu + \lambda' \circ \pi, \quad \pi \circ \iota = 0, \quad \pi \circ \lambda = \mathrm{id}_{\mathbb{C}} \tag{0.44}$$

and $\iota \circ \mu \perp \lambda' \circ \pi$, so:

$$\begin{aligned} 0 &= K_0(0) = K_0(\pi \circ \iota) = K_0(\pi) \circ K_0(\iota) \\ \mathrm{id}_{K_0(\mathbb{C})} &= K_0(\mathrm{id}_{\mathbb{C}}) = K_0(\pi \circ \lambda) = K_0(\pi) \circ K_0(\lambda) \\ \mathrm{id}_{K_0(A)} &= K_0(\mathrm{id}_A) = K_0(\mu \circ \iota) = K_0(\mu) \circ K_0(\iota) \\ \mathrm{id}_{K_0(\tilde{A})} &= K_0(\mathrm{id}_{\tilde{A}}) = K_0(\iota \circ \mu + \lambda' \circ \pi) = K_0(\iota) \circ K_0(\mu) + K_0(\lambda') \circ K_0(\pi) \end{aligned}$$

The first implies that $\mathrm{Im}(K_0(\iota)) \subset \mathrm{Ker}(K_0(\pi))$. The second implies that the sequence has a lift. The third implies injectivity of $K_0(\iota)$. The fourth implies that if $g \in \mathrm{Ker}(K_0(\pi))$, then $g = K_0(\iota)(K_0(\mu)(g)) \Rightarrow g \in \mathrm{Im}(K_0(\iota))$

□

Examples of K_0 (3.3.1 - 3.3.6):

1. **Traces and K_0:**

   A **bounded trace** is a bounded linear map $\tau : A \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$. Can be extended to unique trace τ_n on $M_n(\mathbb{C})$ such that $\tau_n(\text{diag}(a, 0, \dots, 0)) = \tau(a)$ given by $\tau_n(V) = \sum \text{diag}(V)$ where $V \in M_n(\mathbb{C})$.

   A trace therefore gives rise to a function $\tau : \mathcal{P}_\infty(A) \to \mathbb{C}$ which satisfy the "essential" properties (universal) and therefore can be factored through a unique group homomorphism $K_0(\tau) : K_0(A) \to \mathbb{C}$ such that $\tau(p) = K_0(\tau)([p]_0)$, where the last universal property follows by $p \sim_h q \Rightarrow p \sim q \Rightarrow \tau(p) = \tau(q)$.

   If $\tau(a) \geq 0$ for every $a \geq 0$ (positive), then $K_0(\tau)([p]_0) \geq 0$ for every $p \in \mathcal{P}_\infty(A)$ and by the standard picture of $K_0(A)$ we get $K_0(\tau) : K_0(A) \to \mathbb{R}$.

   We denote the standard trace on $M_n(\mathbb{C})$ by Tr (sum of diagonals).

2. **$K_0(M_n(\mathbb{C})) \simeq \mathbb{Z}$ for every $n \in \mathbb{N}$:**

   By the standard picture, every $g \in K_0(M_n(\mathbb{C}))$ can be written as $g = [p]_0 - [q]_0$ for $p, q \in M_k(M_n(\mathbb{C})) = M_{kn}(\mathbb{C})$ for some $k \in \mathbb{N}$. By the above:

   $$K_0(\text{Tr})(g) = \text{Tr}(p) - \text{Tr}(q) = \dim(p(\mathbb{C}^{kn})) - \dim(q(\mathbb{C}^{kn})) \in \mathbb{Z} \tag{0.45}$$

   and if $K_0(\text{Tr})(g) = 0$ then $\dim(p(\mathbb{C}^{kn})) = \dim(q(\mathbb{C}^{kn}))$ which makes $p \sim q$ (by the isomorphism this induces) and therefore $g = 0$ and $K_0(\text{Tr})$ is injective. Since $\text{Im}(K_0(\text{Tr}))$ is a subgroup of $\mathbb{Z}$ we have that $\text{Im}(K_0(\text{Tr})) = \mathbb{Z} \Leftrightarrow 1 \in \text{Im}(K_0(\text{Tr}))$ and since $K_0(\text{Tr})([e]_0) = 1$ we are done.

3. **$K_0(B(H)) = 0$ for infinite dimensional (separable) Hilbert space H:**

   By considering the orthonormal basis it can be seen that $p \sim q \Leftrightarrow \dim(p(H)) = \dim(q(H)) \Leftrightarrow p(H) \simeq q(H)$. Denote $H^n = H \oplus \cdots \oplus H$ n times and identify $M_n(B(H)) = B(H^n)$, by using $x_1 \oplus x_2 \oplus \cdots \oplus x_n \mapsto \text{diag}(x_1, \dots, x_n)$ for $x_i \in H$.

   Then the map $\dim : \mathcal{P}_\infty(B(H)) \to \mathbb{N} \cup \{\infty\}$ given by $p \mapsto \dim(p(H^n))$ for $p \in \mathcal{P}_n(B(H)) = \mathcal{P}(B(H^n))$ is surjective. Furthermore $p \sim q \Leftrightarrow \dim(p) = \dim(q)$, and since $\dim(p \oplus 0) = \dim(p)$ we have $p \sim_0 q \Leftrightarrow \dim(p) = \dim(q)$. The dimension is also additive by the way $\oplus$ is defined, and therefore the map $d : \mathcal{D}(B(H)) \to \mathbb{N} \cup \{\infty\}$ given by $[p]_\mathcal{D} \mapsto \dim(p)$ is a well-defined semigroup isomorphism. This implies by functorality of the Grothendieck construction that $K_0(B(H)) \simeq G(\mathbb{N} \cup \{\infty\}) = 0$.

4. **$K_{00}(C_0(X)) = 0$ for X connected, loc.comp. non.comp. Hausdorff:**

   Identify $M_n(C_0(X))$ with $C_0(X, M_n(\mathbb{C}))$ which implies that $\mathcal{P}_n(C_0(X))$ is identified with $\mathcal{P}(C_0(X, M_n(\mathbb{C})))$. Let $p \in \mathcal{P}_n(C_0(X))$, then

   $$x \mapsto \text{Tr}(p(x)) \in C_0(X, \mathbb{Z}) \tag{0.46}$$

but since X is connected, every function in $C_0(X,\mathbb{Z})$ must be constant and since X is non-compact it must be 0. Hence $\mathrm{Tr}(p(x)) = 0$ for every $x \in X$ which implies that $p = 0$ and so $\mathcal{P}_n(C_0(X)) = \{0\} \Rightarrow K_{00}(C_0(X)) = 0$.

5. $\dim : K_0(C(X)) \to \mathbb{Z}$ **surjective for X connected, compact, Hausdorff:**

   Consider again $x \mapsto \mathrm{Tr}(p(x)) \in C(X,\mathbb{Z})$, again X is connected so every $x \mapsto \mathrm{Tr}(p(x))$ is constant for every $p \in \mathcal{P}_\infty(C(X))$.

   The map $\tau_x : C(X) \to \mathbb{C}$ given by $\tau_x(f) = f(x)$ is a trace on $C(X)$ for each $x \in X$. There is therefore a group homomorphism $K_0(\tau_x) : K_0(C(X)) \to \mathbb{C}$ such that $K_0(\tau_x)([p]_0) = \tau_x(p)$. If $p \in \mathcal{P}_n(C(X))$, then $\tau_x(p) = \mathrm{Tr}(p(x))$ from extension of trace to matrix algebra. And since $\mathrm{Tr}(p(x))$ is constant it follows that $\tau_x(p)$, and in turn $K_0(\tau_x)$ is independent of x. Further it has range in $\mathbb{Z}$ since $\mathrm{Tr}(p(x))$ is an integer for each $p \in \mathcal{P}_\infty(C(X))$. By setting $K_0(\tau_x) = \dim$ and noting that $1 = 1(x) = K_0(\tau_x)([1]_0)$ wee see that dim is surjective.

6. $K_0(C(X)) \simeq \mathbb{Z}$ **for contractible, compact, Hausdorff X:**

Topological K-theory of spaces (3.3.7):

A **complex vector bundle** over a topological space X is a triple $\xi = (E, \pi, X)$ where:

1. E is a topological space and $\pi : E \to X$ is a continuous surjection.
2. For every $x \in X$ the fiber $E_x = \pi^{-1}(x) \subset E$ has the structure of a finite-dimensional complex vector space.
3. (**Triviality**) Furthermore every $x \in X$ (locally trivial) has an open neighbourhood $U \subset X$, an $n \in \mathbb{N}$ and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{C}^n$ (a local trivialization) such that the diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "h"] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^n \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd} \tag{0.47}$$

where π_1 is the projection to the first factor; And such that for each $x \in U$, the restriction of h to E_x yields a vector space isomorphism $E_x \to \{x\} \times \mathbb{C}^n \simeq \mathbb{C}^n$.

If each fiber E_x in ξ has dimension n, then the vector bundle is said to be of dimension n. The archetypical example of a vector bundle is the **trivial vector bundle over X with dimension** n which is $\theta_n = (X \times \mathbb{C}^n, \pi, X)$ where $\pi(x, v) = x$.

Two vector bundles $\xi = (E, \pi, X)$ and $\eta = (F, \rho, X)$ over X are isomorphic if there exists a homeomorphism $h : E \to F$ such that:

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\rho"] \\ & X & \end{tikzcd} \tag{0.48}$$

commutes and the restriction of h to E_x is a vector space isomorphism $E_x \to F_x$ for each $x \in X$. Denote by $\langle \xi \rangle$ the equivalence class of all vector bundles isomorphic to ξ. Define the direct sum $\xi \oplus \eta$ as the triple (G, η, X) where

$$G = \{(v, w) \in E \times F : \pi(v) = \rho(w)\}, \tag{0.49}$$

$\eta(v, w) = \pi(v) = \rho(w)$ and where the fibers $G_x = E_x \times F_x$ are given the direct sum vector space structure. Define further $\xi \otimes \eta$ as the vector bundle whose fiber over x is the tensor product of the fiber of ξ over x and the fiber of η over x.

Define $\mathrm{Vect}(X)$ to be the set of all isomorphism classes $\langle \xi \rangle$ of vector bundles ξ over X. Define addition and multiplication by $\langle \xi \rangle + \langle \eta \rangle = \langle \xi + \eta \rangle$ and $\langle \xi \rangle \cdot \langle \eta \rangle = \langle \xi \otimes \eta \rangle$. Then

$(\mathrm{Vect}(X), +)$ is an abelian semigroup, multiplication is commutative and the distributive law holds.

Let $K^0(X)$ be the Grothendieck group of $(\mathrm{Vect}(X), +)$ and extend multiplication to $K^0(X)$. Let $[\xi]$ be the image of $\langle\xi\rangle$ in $K^0(X)$ under the Grothendieck map, then $K^0(X)$ is a commutative ring with unit $[\theta_1]$.

Properly infinite projections and the Cuntz algebra:

Exercise 4.5:

Let $n \geq 2$ be an integer, and let H be a separable, infinite dimensional Hilbert space (therefore $H \simeq \ell^2(\mathbb{N})$.

(i) Let (e_k) be a basis for H. Split the Hilbert space into n pieces H_i determined by the basis elements:

$$(e'_k)_i = \begin{cases} e_m, & \text{for } m = nk - (n-i) \\ 0, & \text{else} \end{cases} \tag{0.50}$$

Then $H_i \perp H_j$ for $i \neq j$ and if we define $s_i \in B(H)$ as:

$$s_i((e_k)) = (e_{nk-(n-i)}) \tag{0.51}$$

we see that

$$s_i^*((e_k)) = (e'_k)_i \tag{0.52}$$

and that $s_i^* s_i = 1$ for all i. Furthermore, since H_i partitions H and is pairwise orthogonal we get:

$$\sum_{i=1}^{n} s_i s_i^* = 1 \tag{0.53}$$

Denote by $\mathcal{O}_n = C^*(s_1, s_2, \ldots, s_n) \subset B(H)$. This is the Cuntz-algebra of order n. It is simple and has the property that if A is a unital algebra containing elements $t_1, t_2, \ldots, t_n$ satisfying:

$$t_j^* t_j = 1 = \sum_{i=1}^{n} t_i t_i^* \tag{0.54}$$

Then there is a unique $*$-homomorphism $\varphi : \mathcal{O}_n \to A$ such that $\varphi(s_j) = t_j$.

(ii) Let $u \in \mathcal{O}_n$ be unitary, then $x_j = us_j \in \mathcal{O}_n$ is a family of sets such that $x_j^* x_j = s_j^* u^* u s_j = 1$ and $x_j x_j^* = us_j s_j^* u^*$, so:

$$\sum_{j=1}^{n} us_j s_j^* u^* = u \left(\sum_{j=1}^{n} us_j s_j^* \right) u^* = 1 \tag{0.55}$$

so by the universality of $\mathcal{O}_n$ there is a unique $\varphi_u \in \mathrm{End}(\mathcal{O}_n)$ such that $\varphi_u(s_j) = us_j$.

$$\sum_{j=1}^{n} \varphi_u(s_j) s_j^* = \sum_{j=1}^{n} us_j s_j^* = u \tag{0.56}$$

(iii) Let φ be a unital endomorphism on $\mathcal{O}_n$. Set:

$$u = \sum_{j=1}^{n} \varphi(s_j) s_j^* \tag{0.57}$$

we show that u is a unitary and therefore $\varphi_u = \varphi$ since φ_u is unique.

$$u^* u = \left(\sum_{j=1}^{n} \varphi(s_j) s_j^* \right)^* \left(\sum_{j=1}^{n} \varphi(s_j) s_j^* \right) = \cdots = \sum_{j=1}^{n} s_i \varphi(s_i^* s_i) s_i^* = 1 \tag{0.58}$$

(iv) Let $\lambda : \mathcal{O}_n \to \mathcal{O}_n$ be given by

$$\lambda(x) = \sum_{j=1}^{n} s_j x s_j^* := \sum_{j=1}^{n} \lambda_j(x) \tag{0.59}$$

That λ is a (unital) endomorphism follows easily from the fact that λ_j is an endomorphism. Further λ_i is a projection and $\lambda_i \perp \lambda_j$. From Exercise 4.3. we also have that $K_0(\lambda_j) = \mathrm{id}$, so for $g \in K_0(\mathcal{O}_n)$ we have

$$K_0(\lambda)(g) = [\lambda(g)]_0 = \left[\sum_{j=1}^{n} \lambda_j(g) \right]_0 = \sum_{j=1}^{n} [\lambda_i(g)]_0 = \sum_{j=1}^{n} K_0(\lambda_i)(g) = \sum_{j=1}^{n} \mathrm{id}(g) = ng \tag{0.60}$$

(v) Since λ is a unital endomorphism on $\mathcal{O}$ there is some u such that $\lambda = \varphi_u$. φ_u is determined by the action on s_i so

$$\varphi_u(s_i) = \lambda(s_i) \Rightarrow u s_i = \sum_{j=1}^{n} s_j s_i s_j^* = s_i s_i s_i^* \Rightarrow u = s_i s_i s_i^* s_i^* \tag{0.61}$$

It follows that $u = u^*$ so u is unitary and self-adjoint which implies that $\mathrm{Sp}(u) \subset \{-1, 1\}$ and by previous theorem $u \sim_h 1$ in $\mathcal{U}_0(\mathcal{O}_n)$. So since $\lambda = \varphi_u$ we have $\lambda(s_i) = \varphi_u(s_i) = u s_i \sim_h s_i$ and $\lambda \sim_h \mathrm{id}$ by exercise 3.7 this extends to $\lambda \sim_h \mathrm{id}$ for $\mathcal{O}_n$, so:

$$K_0(\lambda)(g) = K_0(\mathrm{id})(g) = \mathrm{id}_{K_0(\mathcal{O}_n)}(g) = g \tag{0.62}$$

(vi) Since $ng = K_0(\lambda)(g) = g$ we see that $ng = g \Rightarrow (n-1)g = 0$ for all $g \in K_0(\mathcal{O}_n)$ and in the case $n = 2$ we get $g = 0$ and $K_0(\mathcal{O}_2) = 0$.

Exercise 4.6:

A non-zero projection in a C^*-algebra A is said to be **properly infinite** if there are mutually orthogonal projections $e, f \in A$ such that $e \leq p$ and $f \leq p$ and $p \sim e \sim f$. A unital $C*$-algebra A is said to be properly infinite if 1_A is a property infinite projection.

Assume that A is a properly infinite unital C^*-algebra.

(i) Since 1_A is a properly infinite projection there are mutually orthogonal projections e, f such that $e \leq 1_A$, $f \leq 1_A$ and $e \sim 1_A \sim f$. By the last property there are projections s_1, s_2 such that:

$$1_A = s_1^* s_1, \quad e = s_1 s_1^*, \quad 1_A = s_2^* s_2, \quad f = s_2 s_2^*, \tag{0.63}$$

so s_1, s_2 are isometries and since $e \perp f$ we have $s_1 s_1^* \perp s_2 s_2^*$

(ii) Following the hint, consider $t_i = s_2^{i-1} s_1$ for $i \geq 1$. Then we have

$$t_i^* t_i = (s_2^{i-1} s_1)^* s_2^{i-1} s_1 = 1_A, \quad \wedge \quad (t_i^* t_i)^2 = t_i^* t_i \tag{0.64}$$

so t_i are isometries (and projections) for all i. Since $t_i^* t_i$ is a projection it is positive. Further:

$$\|t_i^* t_j\|^2 =$$

(iii) Let $v_n \in M_{1,n}(A)$ be the row matrix with entries $t_1, \ldots, t_n$.

$$v_n^* v_n = \begin{pmatrix} t_1^* t_1 & t_1^* t_2 & \cdots & t_1^* t_n \\ t_2^* t_1 & t_2^* t_2 & \cdots & t_2^* t_1 \\ \vdots & \vdots & & \vdots \\ t_n^* t_1 & t_n^* t_2 & \cdots & t_n^* t_n \end{pmatrix} = \begin{pmatrix} 1_A & t_1^* t_2 & \cdots & t_1^* t_n \\ t_2^* t_1 & 1_A & \cdots & t_2^* t_1 \\ \vdots & \vdots & & \vdots \\ t_n^* t_1 & t_n^* t_2 & \cdots & 1_A \end{pmatrix} \tag{0.65}$$

Exercises:

Exercise 1.4:

p a projection, u a unitary (unital algebra).

(i) $\mathrm{Sp}(p) \subset \{0, 1\}$.

Proof. By definition:

$$\mathrm{Sp}(p) = \{\lambda \in \mathbb{C} : p - \lambda \cdot 1 \notin \mathrm{GL}(A)\} \tag{0.66}$$

So assume that $\lambda \notin \mathrm{Sp}(p)$. Then there exists some $a \in A$ such that $(p - \lambda \cdot 1)a = a(p - \lambda \cdot 1) = 1$ (by going to the unitization if necessary). To find an expression for a assume that a is of the form $\alpha p + \beta$, then $(p - \lambda \cdot 1)a = 1$ implies that $(p - \lambda \cdot 1)(\alpha p + \beta) = 1$, or:

$$\begin{aligned} 1 = (p - \lambda \cdot 1)(\alpha p + \beta) &= \alpha p + \beta p - \lambda \cdot 1 \alpha p - \lambda \cdot 1 \beta \\ &= p(\alpha + \beta - \alpha\lambda \cdot 1) - \beta\lambda \cdot 1 \\ &\Rightarrow \beta = 1/\lambda \quad \wedge \quad \alpha + \beta - \alpha\lambda \cdot 1 = 0 \\ &\Rightarrow \beta = 1/\lambda \quad \wedge \quad \alpha(1 - \lambda \cdot 1) = -1/\lambda \\ &\Rightarrow \beta = 1/\lambda \quad \wedge \quad \alpha = -1/(\lambda(1 - \lambda)) \end{aligned} \tag{0.67}$$

which gives:

$$(p - \lambda \cdot 1)^{-1} = (\lambda(1 - \lambda))^{-1} p + \lambda^{-1} \tag{0.68}$$

And so $\mathrm{Sp}(p) \subset \{0, 1\}$ since it is non-empty and the inverse exists for any other value of λ. □

(ii) p normal and $\mathrm{Sp}(p) \subset \{0, 1\} \Rightarrow p$ is a projection.

Proof. First p is self-adjoint since $\{0, 1\} \subset \mathbb{R}$. Since p is normal we can use continuous function calculus. Since $\mathrm{Sp}(p) \subset \{0, 1\}$ it follows that $f(p) = p(p - 1)$ is zero on $\mathrm{Sp}(p)$. That is $\mathrm{Sp}(f(p)) = f(\mathrm{Sp}(p)) = 0$. Since $f(p)$ is normal we have $0 = r(f(p)) = \|f(p)\| \Rightarrow p(p - 1) = 0 \Rightarrow p^2 = p$. □

(iii) $\mathrm{Sp}(u) \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Proof. Since u is unitary we have $\|u\|^2 = \|u^* u\| = 1$ and hence $\|u\| = \|u^*\| = 1$. If $\lambda \in \mathrm{Sp}(u)$, then $|\lambda| \leq \|u\| = 1$ and since $\mathrm{Sp}(u^{-1}) = \{\lambda^{-1} : \lambda \in \mathrm{Sp}(u)\}$ it follows that $|\lambda^{-1}| \leq \|u^{-1}\| = \|u^*\| = 1$ which shows that $|\lambda| = 1$ and $\lambda \in \mathbb{T}$. □

(iv) u normal and $\mathrm{Sp}(u) \subset \mathbb{T} \Rightarrow u$ is unitary.

Proof. Since u is normal we can again use the continuous function calculus: There exists a unique $*$-isomorphism $\pi : C(\mathrm{Sp}(u)) \to C^*(u, 1) \subset A$ given by $\pi(f) = f(u)$ such that $\iota : \mathrm{Sp}(u) \to \mathrm{Sp}(u)$ s.t. $\iota(z) = z$ gets mapped to u; i.e. $\pi(\iota) = u$. This implies that u is unitary if and only if ι is unitary on $\mathrm{Sp}(u)$. For any $z \in \mathrm{Sp}(u) \subset \mathbb{T}$ we have $1 = z\bar{z} = \iota(z)(\iota(z))^*$ and thus u is unitary. □

Exercise 1.8:

Let A and B be C^*-algebras and $\varphi : A \to B$ be a unital $*$-homomorphism.

(i) For each $a \in A$ we have $\mathrm{Sp}(\varphi(a)) \subset \mathrm{Sp}(a)$ with equality if and only if φ is injective.

Proof. Since $a \in A$, we have $\varphi(a) \in \varphi(A) \subset B$. We prove this by proving the equivalent proposition that $a - \lambda \cdot 1_A \in \mathrm{GL}(A) \Rightarrow \varphi(a) - \lambda \cdot 1_B \in \mathrm{GL}(\varphi(A))$ for $\lambda \in \mathbb{C} \setminus \mathrm{Sp}(a)$. If $a - \lambda \cdot 1_A \in \mathrm{GL}(A)$ there exists an element $c \in A$ such that $(a - \lambda \cdot 1_A)c = 1_A = c(a - \lambda \cdot 1_A)$, so by applying (the unital) φ we get:

$$1_B = \varphi(1_A) = \varphi((a - \lambda \cdot 1_A)c) = (\varphi(a) - \lambda \cdot 1_B)\varphi(c) \tag{0.69}$$

which proves that $\varphi(a) - \lambda \cdot 1_B \in \mathrm{GL}(\varphi(A))$.

Assume now that φ is injective (and therefore isomorphic on $\varphi(A)$) and that $\varphi(a) - \lambda \cdot 1_B \in \mathrm{GL}(\varphi(A))$, then there exists a $b \in A$ such that $(\varphi(a) - \lambda \cdot 1_B)\varphi(b) = 1_B = \varphi(b)(\varphi(a) - \lambda \cdot 1_B)$. Now since φ is a $*$-homomorphism we have $1_B = \varphi(b)(\varphi(a) - \lambda \cdot 1_B) = \varphi(b(a - \lambda \cdot 1_A))$, so by taking φ^{-1} we have $1_A = b(a - \lambda \cdot 1_A)$ and similarly for the other side which proves that $a - \lambda \cdot 1_A \in \mathrm{GL}(A)$. □

(ii) For all $a \in A$ we have $\|\varphi(a)\| \leq \|a\|$ with equality if φ is injective.

Proof. Apply (i) to $c = a^*a$ (which is a normal element in A) to get $\mathrm{Sp}(\varphi(a^*a)) \subset \mathrm{Sp}(a^*a)$ with equality when φ is injective. Since φ is a $*$-homomorphism $\varphi(a^*a)$ is normal too. By the definition of the spectral radius we have $r(\varphi(a^*a)) \leq r(a^*a)$ (with equality when φ is injective) and for any normal element c we have $r(c) = \|c\|$, so $\|\varphi(a^*a)\| \leq \|a^*a\|$. The C^*-property then implies that $\|\varphi(a)\| \leq \|a\|$ with equality when φ is injective. □

Exercise 2.9:

The standard trace $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$ is defined as the sum over diagonal. If $p, q \in M_n(\mathbb{C})$ is projections, then:

$$p \sim q \Leftrightarrow \mathrm{Tr}(p) = \mathrm{Tr}(q) \Leftrightarrow \dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n)).$$

Furthermore $\mathcal{D}(\mathbb{C}) \simeq \mathbb{Z}^+$.

Proof. Assume $p \sim q$, then there exists $v \in M_n(\mathbb{C})$ such that $p = v^*v$ and $q = vv^*$, so $\mathrm{Tr}(p) = \mathrm{Tr}(v^*v) = \mathrm{Tr}(vv^*) = \mathrm{Tr}(q)$ by definition of the matrix product.

Assume $\mathrm{Tr}(p) = \mathrm{Tr}(q)$, then since the spectrum (and hence eigenvalue) of any projection is contained in $\{0, 1\}$ it follows that the rank of p and q is equal.

Assume $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$, then since $\mathbb{C}^n$ is finite-dimensional we have $p(\mathbb{C}^n) \simeq q(\mathbb{C}^n)$ so there is an isomorphism $u : p(\mathbb{C}^n) \to q(\mathbb{C}^n)$. Any isomorphism is a partial isometry so it follows that $p = u^*u$ and $q = u^*u$.

To see that $\mathcal{D}(\mathbb{C}) \simeq \mathbb{Z}^+$ take an element $[p]_{\mathcal{D}} \in \mathcal{D}(\mathbb{C})$, then $p \in \mathcal{P}_n(\mathbb{C})$ for some n. Note that $p \sim_0 p \oplus 0_k$ for any $k \in \mathbb{N}$. Since adding zeros along the diagonal does not change the trace, it does not change the dimension of $p(\mathbb{C}^n)$ either by the above. This implies that $q \in [p]_{\mathcal{D}}$ if and only if $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$, and we can therefore identify $\mathcal{D}(\mathbb{C})$ with $\mathbb{N}$ (or $\mathbb{Z}^+$). $\square$