Sparse Sampling: Theory, Algorithms and Applications

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Outline

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2. Signals with Finite Rate of Innovation (FRI)
3. Kernels Reproducing Polynomials
4. Sampling FRI Signals with Polynomial Reproducing Kernels
5. Noisy Scenario
6. Application: Image Super-Resolution and Neuroscience
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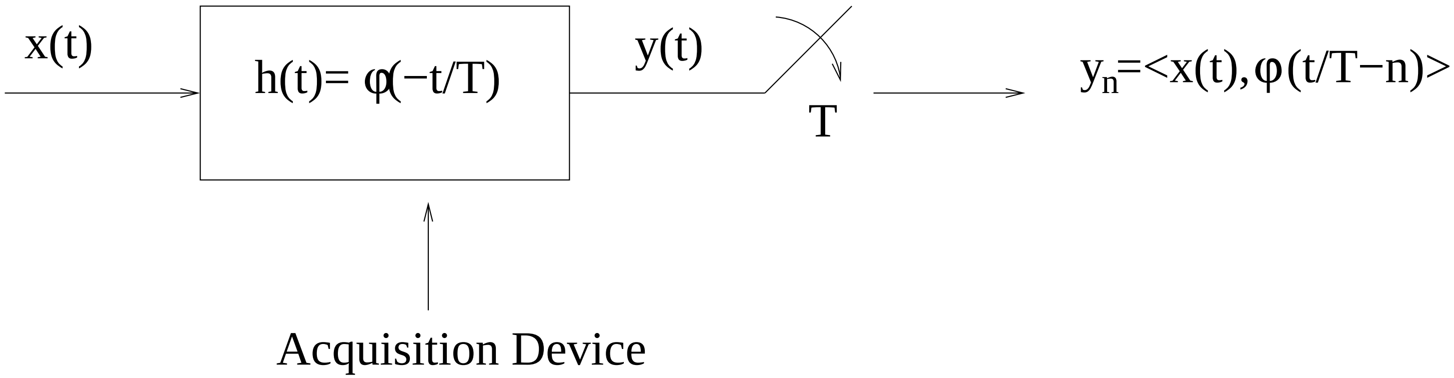
Problem Statement and Motivation

You are given a class of functions. You have a sampling device, typically, a low-pass filter. Given the measurements $y_n = \langle \varphi(t/T - n), x(t) \rangle$, you want to reconstruct $x(t)$.

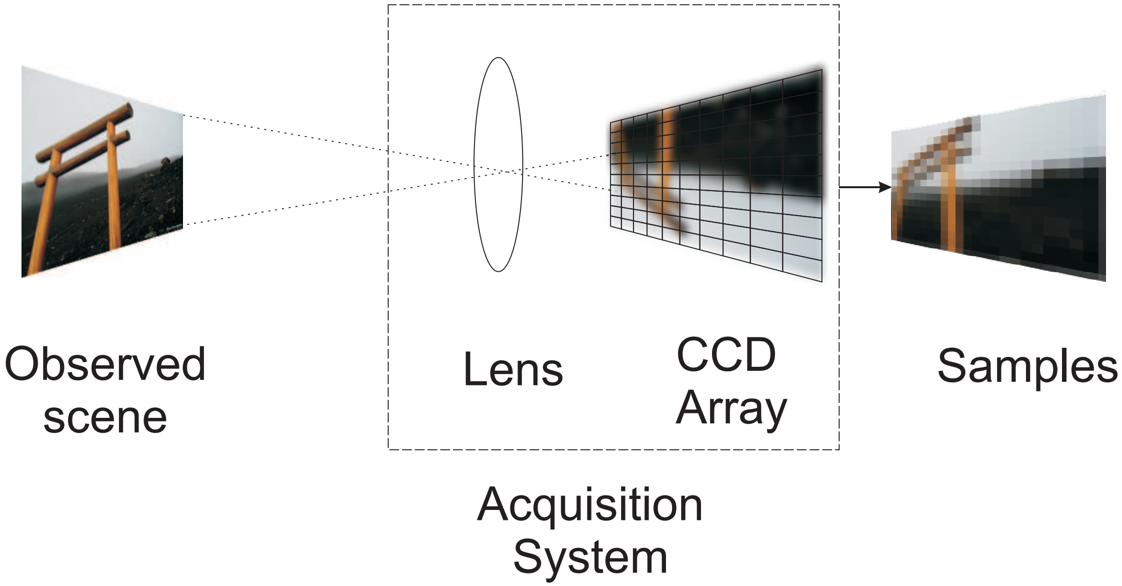


Natural questions:

- When is there a one-to-one mapping between $x(t)$ and y_n?
- What signals can be sampled and what kernels $\varphi(t)$ can be used?
- What reconstruction algorithms?

Problem Statement and Motivation

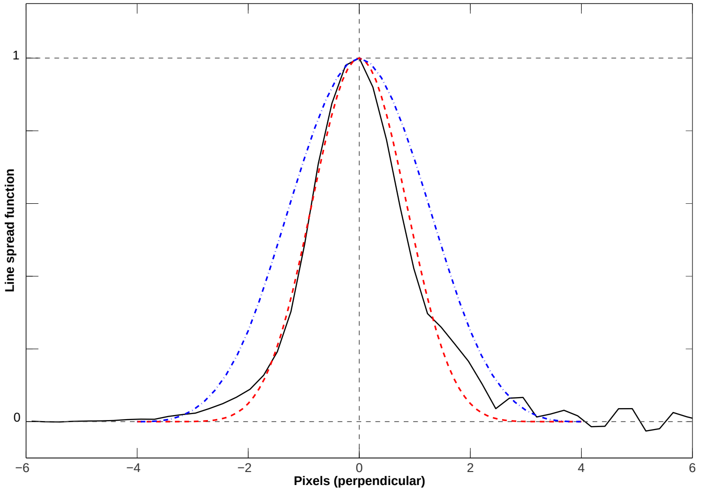
Digital Cameras:



- The low-quality lens blurs the images. Lenses are, to some extent, equivalent to the sampling kernel. Lenses do not necessarily behave like the sinc function.

Problem Statement and Motivation

(a)Original (2014×3039)

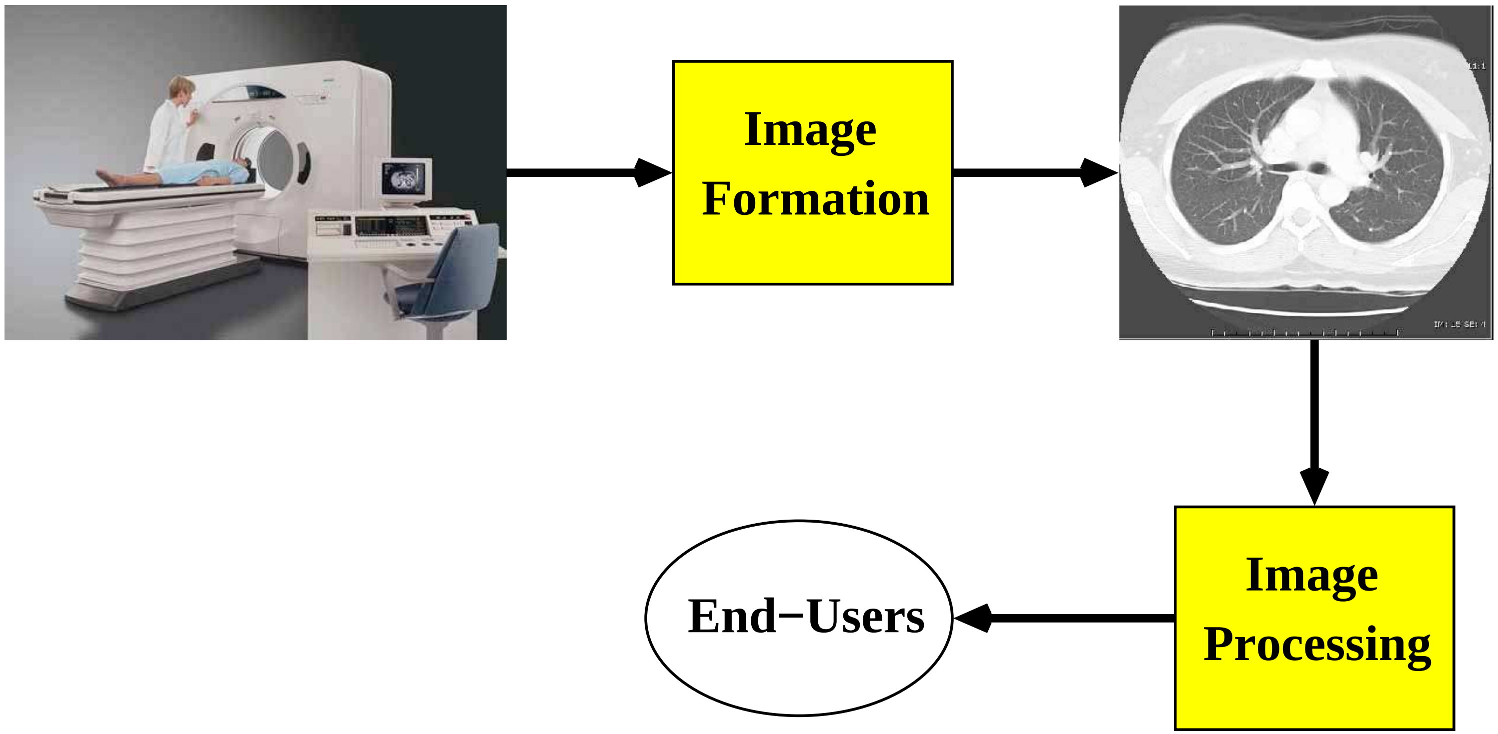


(b) Point Spread function in black

The distortion introduced by a lens can be well approximated with splines.

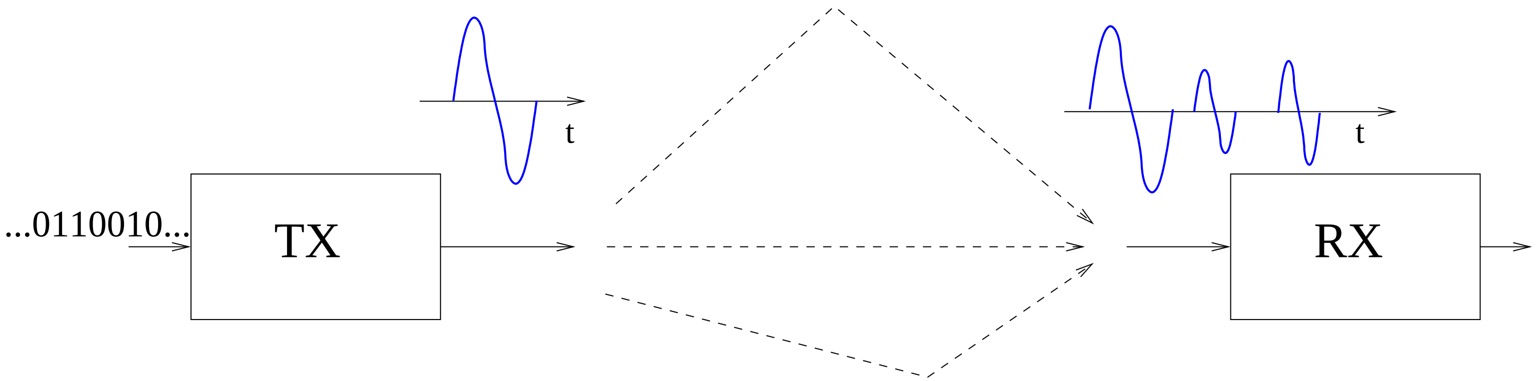
Problem Statement and Motivation

"In 2005, the U.S. spent 16% of its GDP on health care. It is projected that this will reach 20% by 2015." Goal: Individualized treatments based on low-cost and effective medical devices.



Problem Statement and Motivation

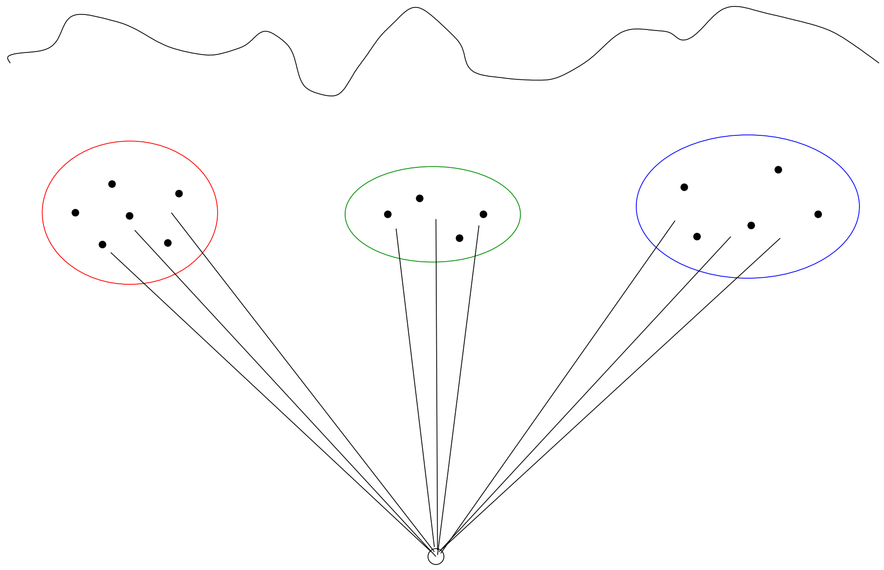
Wide-Band Communications:



Current A-to-D converters in UWB communications operate at several gigaherz.

This is a parametric estimation problem, only the location and amplitude of the pulses need to be estimated.

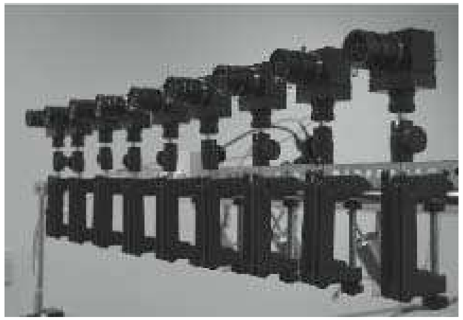
Problem Statement: Distributed Sensing of Diffusive Sources

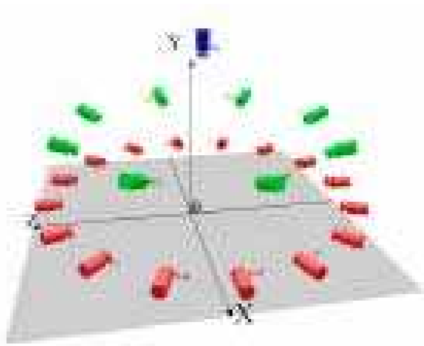


- The source (phenomenon) is distributed in space and time.
- The phenomenon is sampled in space (finite number of sensors) and time.
- Sensors need to be low-power.

Emerging Application: Free Viewpoint Video

Multiple cameras are used to record a dynamic scene. Users can freely choose an arbitrary viewpoint for 3D viewing.





- Standard approaches are supervised, model-based and use a relatively small number of high-resolution cameras (6-8 cameras). They do not achieve photorealistic rendering.
- This is a multi-dimensional sampling problem.

Classical Sampling Formulation

The Shannon sampling theorem provides sufficient but not necessary conditions for perfect reconstruction.

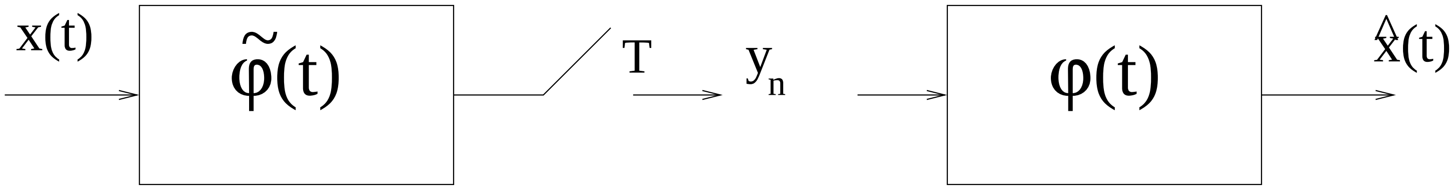
Moreover: How many real signals are bandlimited? How many realizable filters are ideal low-pass filters?

By the way, who discovered the sampling theorem? The list is long ;-)

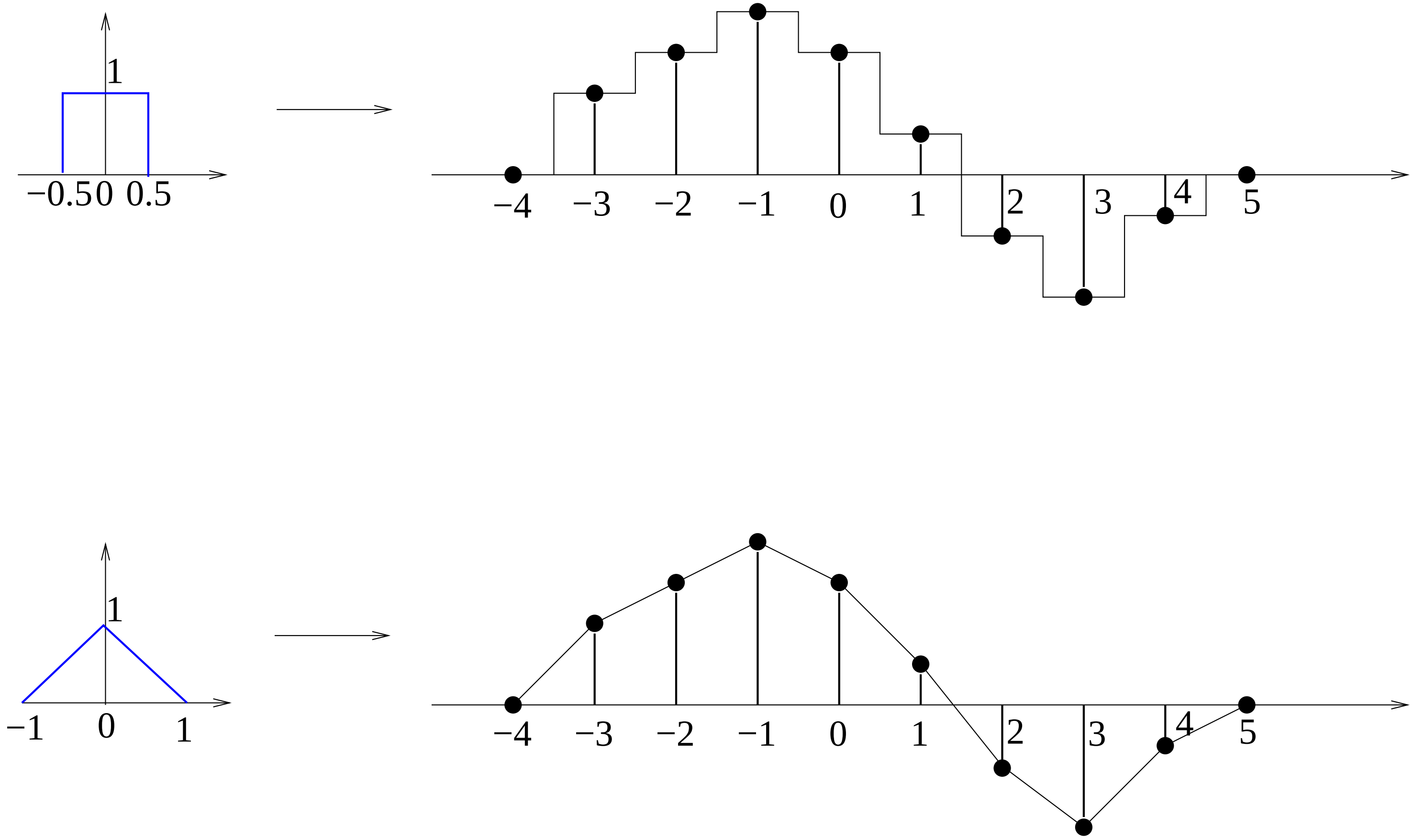
- Whittaker 1915, 1935
- Kotelnikov 1933
- Nyquist 1928
- Raabe 1938
- Gabor 1946
- Shannon 1948
- Someya 1948

Sub-Space Interpretation of the Sampling Theorem

- Sampling of $x(t)$ is equivalent to projecting $x(t)$ into the shift-invariant subspace $V = \text{span}\{\varphi(t/T - n)\}_{n\in\mathbb{Z}}$.
- If $x(t) \in V$, perfect reconstruction is possible.
- Reconstruction process is linear: $\hat{x}(t) = \sum_n y_n \varphi(t/T - n)$.
- For bandlimited signals $\varphi(t) = \text{sinc}(t)$.



Sampling as Projecting into Shift-Invariant Sub-Spaces



Signals with Finite Rate of Innovation

Can we go beyond this subspace interpretation and extend the family of signals we can sample?
The Shannon-Whittaker-Kotelnikov theorem is essentially a signal representation theorem.
What is so special about those signals?
The signal $x(t) = \sum_n y_n \varphi(t/T - n)$ is exactly specified by one parameter y_n every T seconds, $x(t)$ has a finite number $\rho = 1/T$ of degrees of freedom per unit of time.

In the classical formulation, innovation is uniform. How about signals where the rate of innovation is finite but non-uniform? E.g.

- Piecewise sinusoidal signals (Frequency Hopping modulation)
- Piecewise Constant Signals (CDMA)
- Pulse position modulation (UWB)
- Edges in images

Signals with Finite Rate of Innovation

Consider a signal of the form:

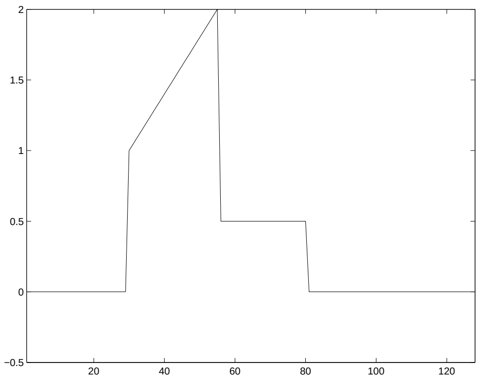
$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{r=0}^{R-1} \gamma_{k,r} g_r(t - t_k). \tag{1}$$

The rate of innovation of $x(t)$ is then defined as

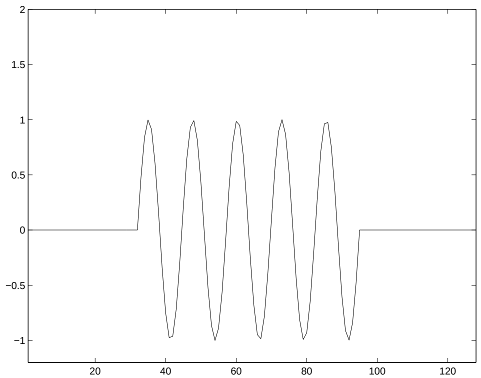
$$\rho = \lim_{\tau\to\infty} \frac{1}{\tau} C_x\left(-\frac{\tau}{2}, \frac{\tau}{2}\right). \tag{2}$$

Definition [VetterliMB:02] A signal with a finite rate of innovation is a signal whose parametric representation is given in (1) and with a finite ρ as defined in (2).

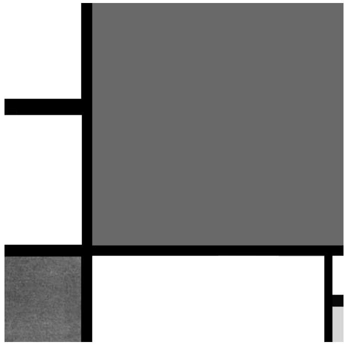
Examples of Signals with Finite Rate of Innovation



Piecewise Polynomial Signals

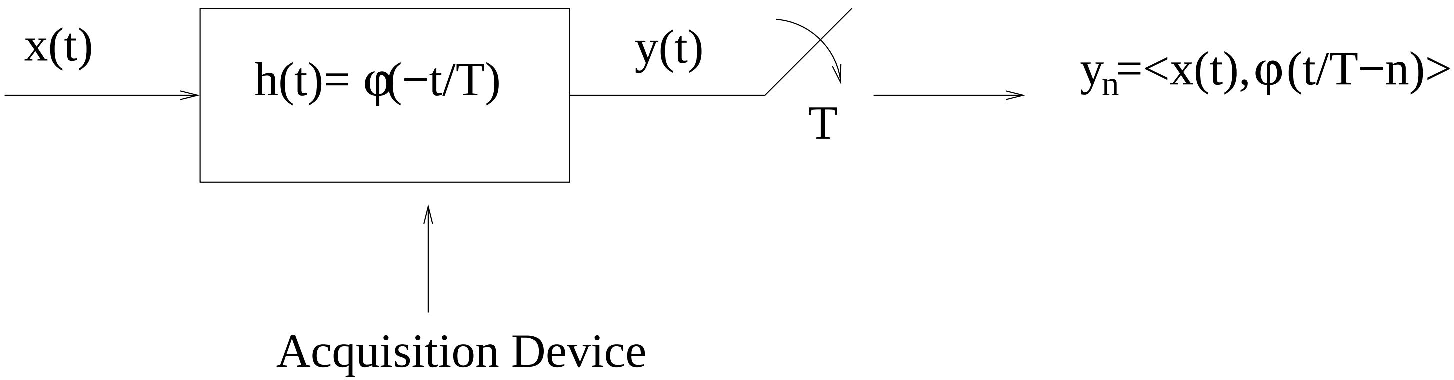


Piecewise Sinusoidal Signals



Mondrian paintings ;-)

The Observation Kernel



- Given by nature
  - Diffusion equation, Green function. Ex: sensor networks.
- Given by the set-up
  - Designed by somebody else. Ex: Hubble telescope, digital cameras.
- Given by design
  - Pick the best kernel. Ex: engineered systems.

About the sampling rate: Given by problem. Ex: sensor networks. Given by design (usually, as low as possible). Ex: UWB

Sampling Kernels

Possible classes of kernels (we want to be as general as possible):

1. Any kernel $\varphi(t)$ that can reproduce polynomials:

$$\sum_n c_{m,n}\varphi(t-n) = t^m \quad m = 0, 1, \dots, L,$$

   for a proper choice of coefficients $c_{m,n}$.

2. Any kernel $\varphi(t)$ that can reproduce exponentials:

$$\sum_n c_{m,n}\varphi(t-n) = e^{a_m t} \quad a_m = \alpha_0 + m\lambda \text{ and } m = 0, 1, \dots, L,$$

   for a proper choice of coefficients $c_{m,n}$.

3. Any kernel with rational Fourier transform:

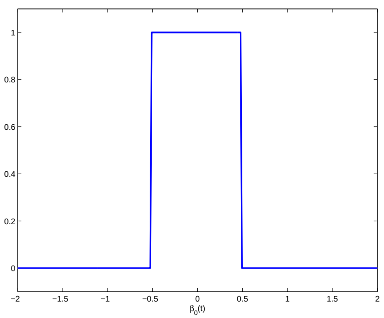
$$\hat{\varphi}(\omega) = \frac{\prod_i (j\omega - b_i)}{\prod_m (j\omega - a_m)} \quad a_m = \alpha_0 + m\lambda \text{ and } m = 0, 1, \dots, L.$$

Sampling Kernels

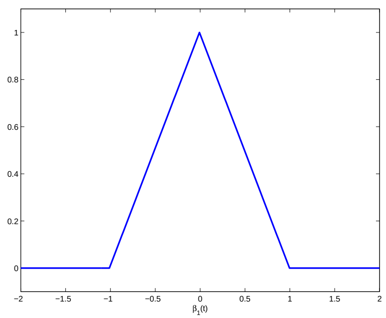
Class 1 is made of all functions $\varphi(t)$ satisfying Strang-Fix conditions. This includes any scaling function generating a wavelet with $L+1$ vanishing moments (e.g., Splines and Daubechies scaling functions).
Strang-Fix Conditions:

$$\hat{\varphi}(0) \neq 0 \text{ and } \hat{\varphi}^{(m)}(2n\pi) = 0 \text{ for } n \neq 0 \text{ and } m = 0, 1, .., L,$$

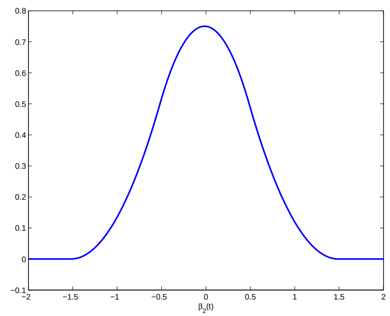
where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$.



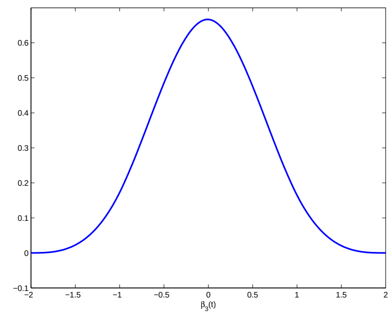
$\beta_0(t)$



$\beta_1(t)$



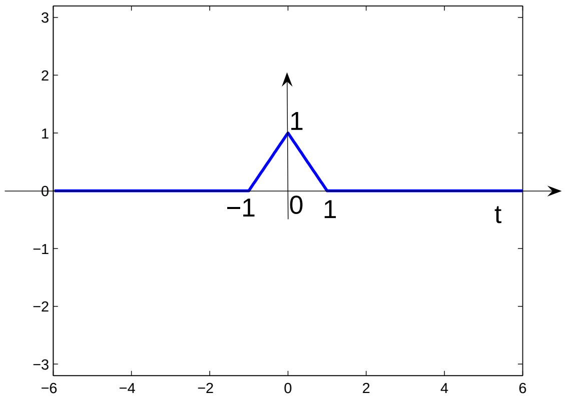
$\beta_2(t)$



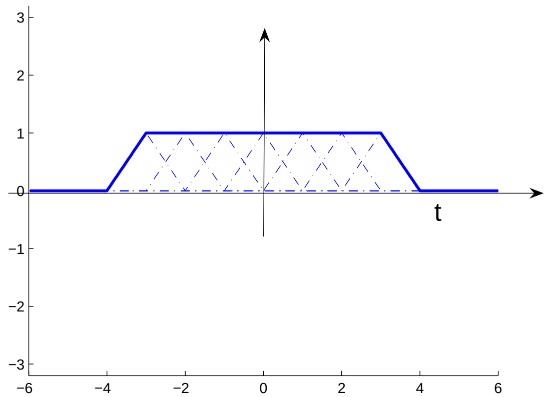
$\beta_3(t)$

B-splines satisfy Strang-Fix conditions since $\hat{\beta}_n(\omega) = \text{sinc}(\omega/2)^{n+1}$.

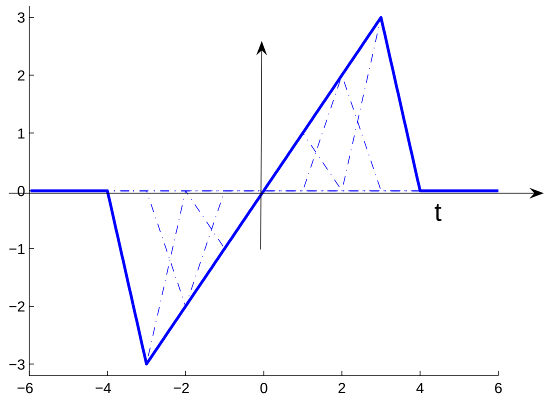
B-splines Reproduce Polynomials



$\beta_1(t)$



$c_{0,n} = (1, 1, 1, 1, 1, 1, 1)$

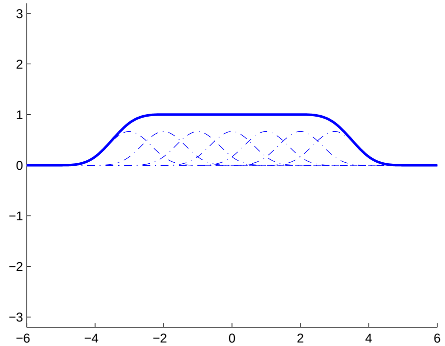


$c_{1,n} = (-3, -2, -1, 0, 1, 2, 3)$

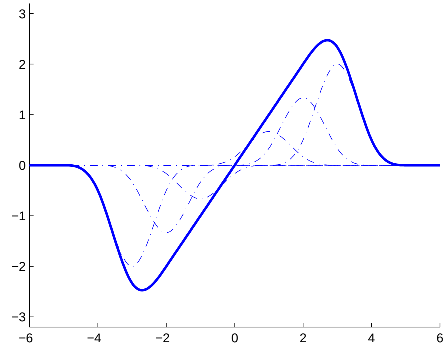
The linear spline reproduces polynomials up to degree L=1: $\sum_n c_{m,n}\beta_1(t-n) = t^m \quad m = 0, 1$, for a proper choice of coefficients $c_{m,n}$ (in this example $n = -3, -2, \ldots, 1, 2, 3$).

Notice: $c_{m,n} = \langle \tilde{\varphi}(t-n), t^m \rangle$ where $\tilde{\varphi}(t)$ is biorthogonal to $\varphi(t)$: $\langle \tilde{\varphi}(t), \varphi(t-n) \rangle = \delta_n$.

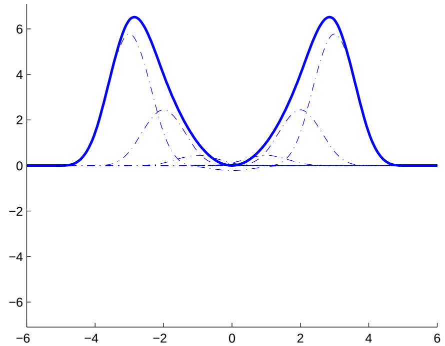
B-splines Reproduce Polynomials



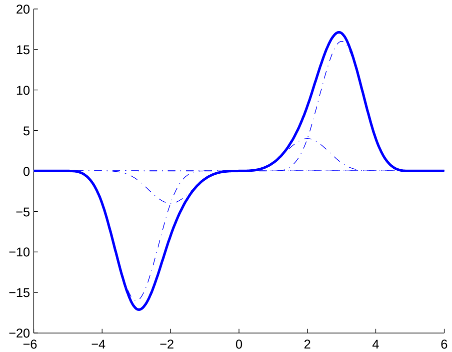
$$c_{0,n} = (1, 1, 1, 1, 1, 1, 1)$$



$$c_{1,n} = (-3, -2, -1, 0, 1, 2, 3)$$



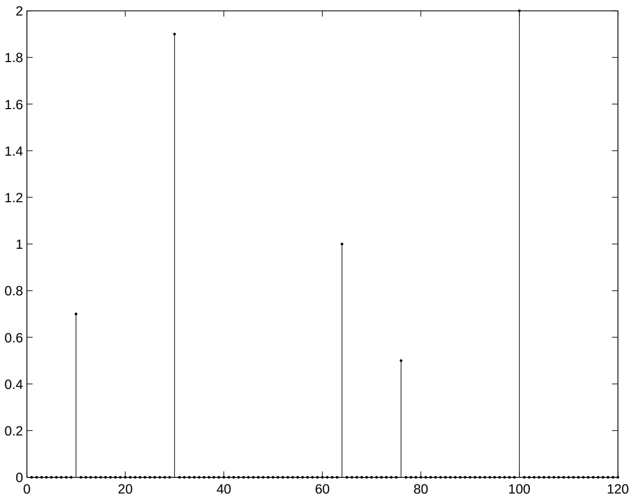
$$c_{2,n} \sim (8.7, 3.7, 0.7, -0.333, 0.7, 3.7, 8.7)$$



$$c_{3,n} \sim (-24, -6, -0.001, 0, 0.001, 6, 24)$$

The cubic spline reproduces polynomials up to degree L=3: $\sum_n c_{m,n} \beta_3(t-n) = t^m \quad m = 0, 1, 2, 3.$

Basic Set-up: Sampling Streams of Diracs



- Assume that $x(t)$ is a stream of K Diracs on the interval $[0, \tau[$:

$$x(t) = \sum_{k=0}^{K-1} x_k \delta(t - t_k).$$

  The signal has $2K$ degrees of freedom.

- Assume that $\varphi(t)$ is any function that can reproduce polynomials of maximum degree $L \geq 2K - 1$:

$$\sum_n c_{m,n} \varphi(t - n) = t^m, \quad m = 0, 1, ..., L,$$

  where $c_{m,n} = \langle \tilde{\varphi}(t - n), t^m \rangle$ and $\tilde{\varphi}(t)$ is biorthogonal to $\varphi(t)$.

- We want to retrieve $x(t)$, from the samples $y_n = \langle x(t), \varphi(t/T - n) \rangle$ with $n = 0, 1, ..., N - 1$ and $TN = \tau$.

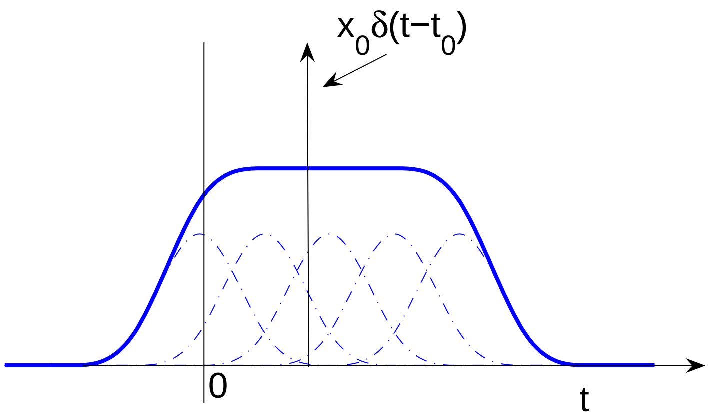
Sampling Streams of Diracs

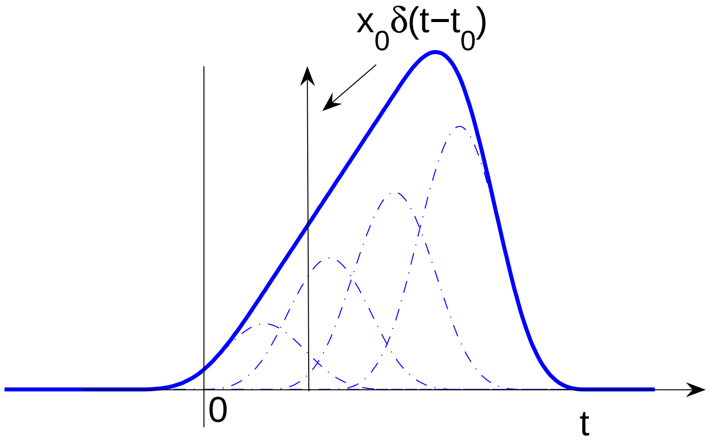
Assume for simplicity that $T = 1$ and define $s_m = \sum_n c_{m,n} y_n$, $m = 0, 1, ..., L$, we have that

$$\begin{aligned} s_m &= \sum_{n=0}^{N-1} c_{m,n} y_n \\ &= \langle x(t), \sum_{n=0}^{N-1} c_{m,n} \varphi(t-n) \rangle \\ &= \int_{-\infty}^{\infty} x(t) t^m dt \\ &= \sum_{k=0}^{K-1} x_k t_k^m \quad m = 0, 1, ..., L \end{aligned} \tag{3}$$

We thus observe $s_m = \sum_{k=0}^{K-1} x_k t_k^m$, $m = 0, 1, ..., L$.

Sampling Streams of Diracs

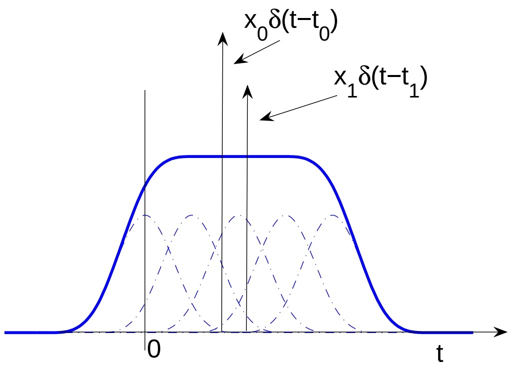


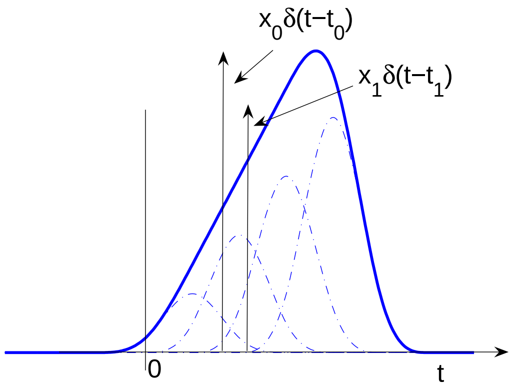


$$\sum_n y_n = \langle x_0\delta(t-t_0), \sum_n \varphi(t-n)\rangle = \int_{-\infty}^{\infty} x_0\delta(t-t_0)\sum_n \varphi(t-n)dt = x_0\sum_n \varphi(t_0-n) = x_0$$

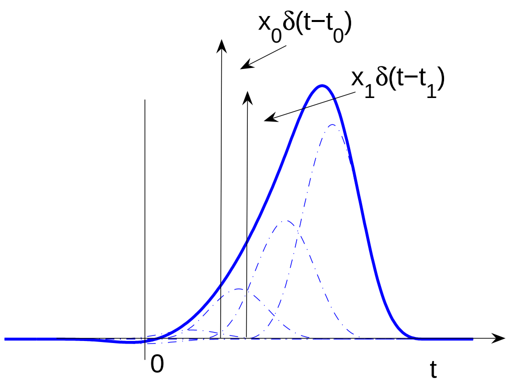
$$\sum_n c_{1,n}y_n = \langle x_0\delta(t-t_0), \sum_n c_{1,n}\varphi(t-n)\rangle = x_0\sum_n c_{1,n}\varphi(t_0-n) = x_0t_0$$

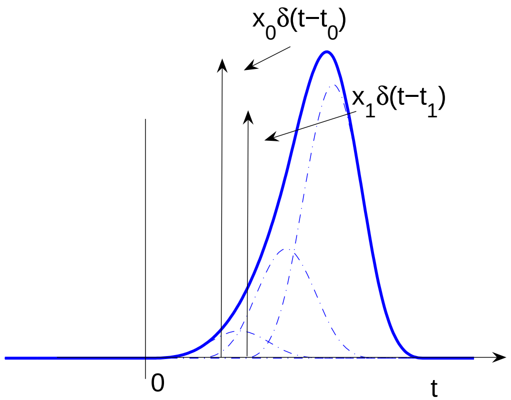
Sampling Streams of Diracs





$$s_0 = \sum_n y_n = x_0 + x_1 \qquad s_1 = \sum_n c_{1,n} y_n = x_0 t_0 + x_1 t_1$$

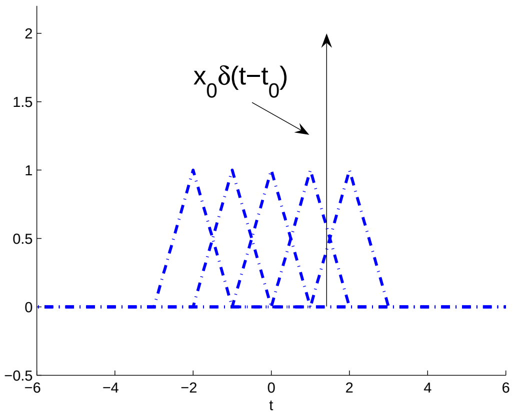


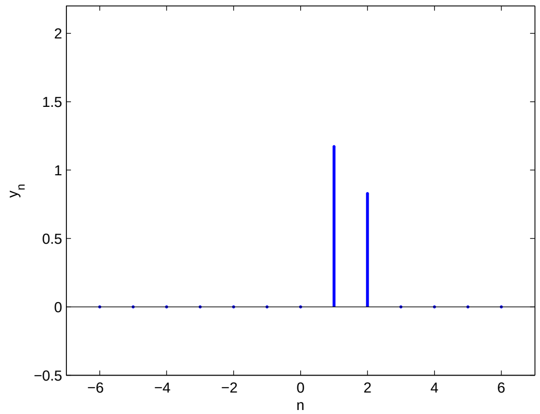


$$s_2 = \sum_n c_{2,n} y_n = x_0 t_0^2 + x_1 t_1^2 \qquad s_3 = \sum_n c_{3,n} y_n = x_0 t_0^3 + x_1 t_1^3$$

Sampling Streams of Diracs - Toy Examples

Example: $T = 1$, $x(t) = x_0 \delta(t - t_0)$ with $x_0 = 2$ and $t_0 = \sqrt{2}$.
The sampling kernel is a linear spline.





In this case

$$y_n = \langle x(t), \varphi(t - n) \rangle = x_0 \varphi(t_0 - n) = \begin{cases} 0 & n \neq 1, 2 \\ 2(2 - \sqrt{2}), & n = 1, \\ 2(\sqrt{2} - 1) & n = 2. \end{cases}$$

and

$$\begin{aligned} s_0 &= \sum_n c_{0,n} y_n = \sum_n y_n = y_1 + y_2 = x_0 = 2, \\ s_1 &= \sum_n c_{1,n} y_n = \sum_n n y_n = y_1 + 2y_2 = x_0 t_0 = 2\sqrt{2}. \end{aligned}$$

The Annihilating Filter Method

The quantity

$$s_m = \sum_{k=0}^{K-1} x_k t_k^m, \quad m = 0, 1, ..., L$$

is a power-sum series.

We can retrieve the locations t_k and the amplitudes x_k with the annihilating filter method (also known as Prony's method since it was discovered by Gaspard de Prony in 1795).

The Annihilating Filter Method

1. Call h_m the filter with z-transform $H(z) = \sum_{i=0}^{K} h_i z^{-i} = \prod_{k=0}^{K-1}(1 - t_k z^{-1})$.

We have that

$$h_m * s_m = \sum_{i=0}^{K} h_i s_{m-i} = \sum_{i=0}^{K} \sum_{k=0}^{K-1} x_k h_i t_k^{m-i} = \sum_{k=0}^{K-1} x_k t_k^m \underbrace{\sum_{i=0}^{K} h_i t_k^{-i}}_{0} = 0.$$

This filter is thus called the annihilating filter.
In matrix/vector form we have that $\mathbf{S}H = \mathbf{H}S = 0$ or alternatively using the fact that $h_0 = 1$

$$\begin{bmatrix} s_{K-1} & s_{K-2} & \cdots & s_0 \\ s_K & s_{K-1} & \cdots & s_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_{L-1} & s_{L-2} & \cdots & s_{L-K} \end{bmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_K \end{pmatrix} = - \begin{pmatrix} s_K \\ s_{K+1} \\ \vdots \\ s_L \end{pmatrix}.$$

Solve the above Toeplitz system to find the coefficient of the annihilating filter.

The Annihilating Filter Method

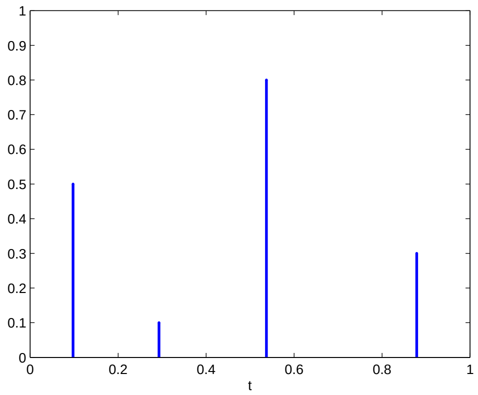
2. Given the coefficients $\{1, h_1, h_2, \ldots, h_k\}$, we get the locations t_k by finding the roots of $H(z)$.

3. Solve the first K equations in $s_m = \sum_{k=0}^{K-1} x_k t_k^m$ to find the amplitudes x_k. In matrix/vector form

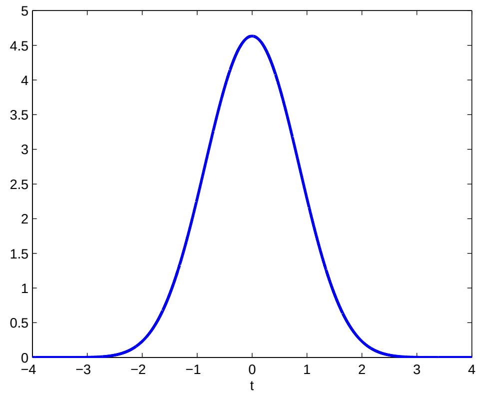
$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ t_0 & t_1 & \cdots & t_{K-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_0^{K-1} & t_1^{K-1} & \cdots & t_{K-1}^{K-1} \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{K-1} \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{K-1} \end{pmatrix}. \tag{4}$$

Classic Vandermonde system. Unique solution for distinct t_ks.

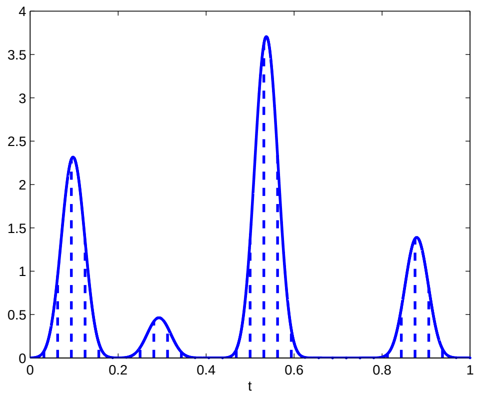
Sampling Streams of Diracs: Numerical Example



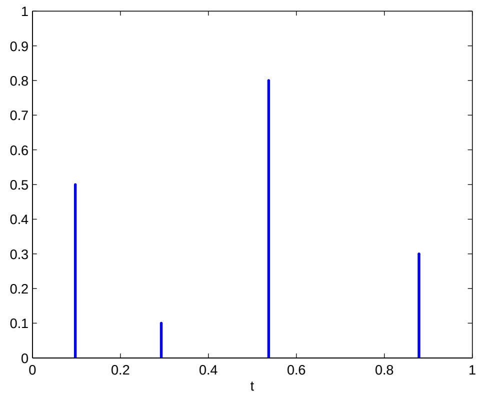
(a) Original Signal



(b) Sampling Kernel ($\beta_7(t)$)

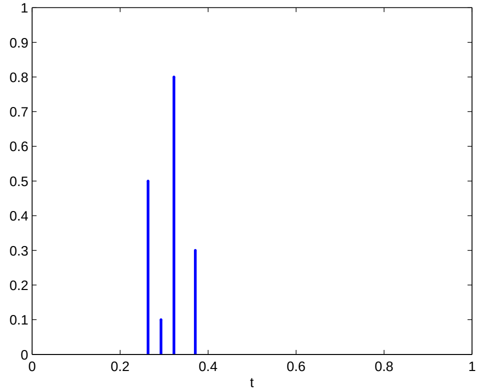


(c) Samples

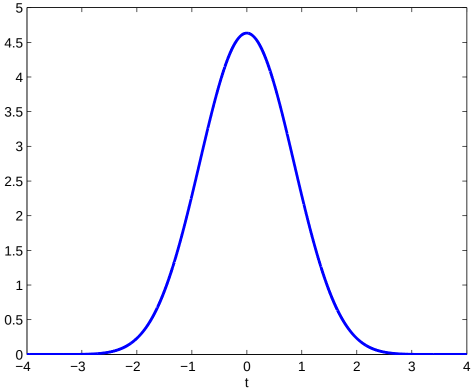


(d) Reconstructed Signal

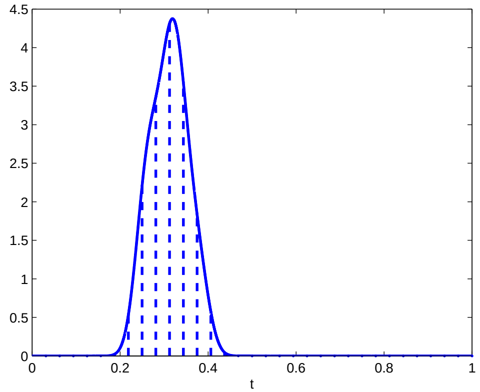
Sampling Streams of Diracs: Closely Spaced Diracs



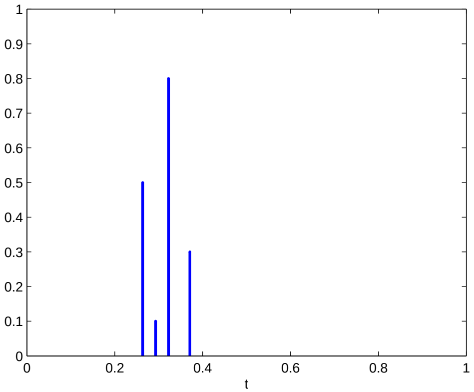
(a) Original Signal



(b) Sampling Kernel ($\beta_7(t)$)



(c) Samples



(d) Reconstructed Signal

Complexity

Notice the proof is based on a constructive algorithm:

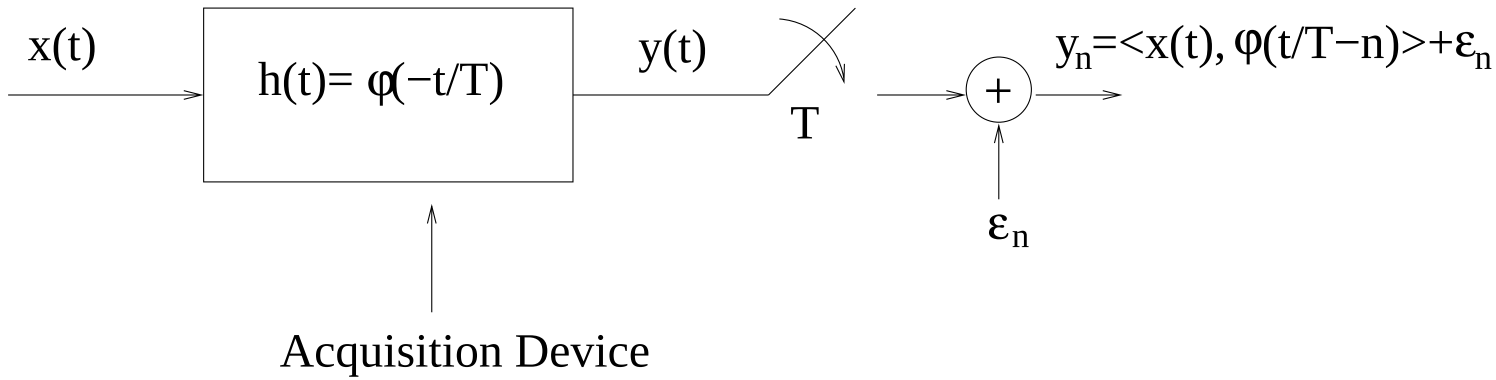
1. Given the N samples y_n, compute the moments s_m using the polynomial reproduction formula
2. Solve a $K \times K$ Toeplitz system to find $H(z)$
3. Find the roots of $H(z)$
4. Solve a $K \times K$ Vandermonde system to find the a_k

Complexity

1. $O(KN)$
2. $O(K^2)$
3. $O(K^3)$
4. $O(K^2)$

Thus, the algorithm complexity is polynomial with the signal innovation.

The Noisy Scenario



- The measurements are noisy
- The noise is additive and i.i.d. Gaussian
- The sampling kernel is the sinc function or B-splines or E-splines.

Total Least Square Solution

The annihilation equation

$$\mathbf{S}H = 0$$

is only approximately satisfied.

Minimize: $\|\mathbf{S}H\|_2$ under the constraint $\|H\|_2 = 1$.

This is achieved by performing an SVD of $\mathbf{S}$:

$$\mathbf{S} = \mathbf{U}\lambda\mathbf{V}^{\mathbf{T}}.$$

Then H is the last column of $\mathbf{V}$.

Notice: this is similar to Pisarenko's method in spectral estimation.

Cadzow's algorithm

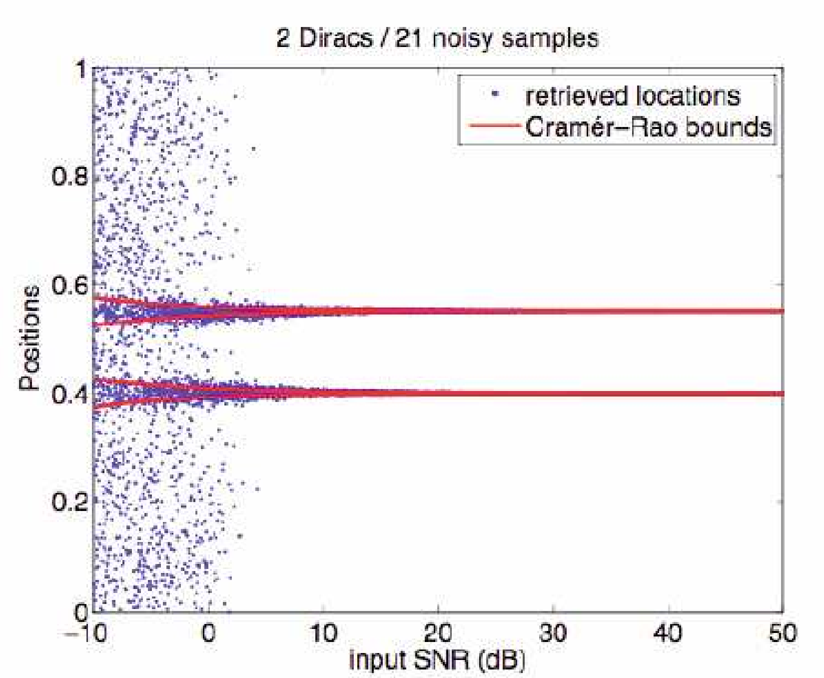
For small SNR use Cadzow's method to denoise $\mathbf{S}$ before applying TLS.

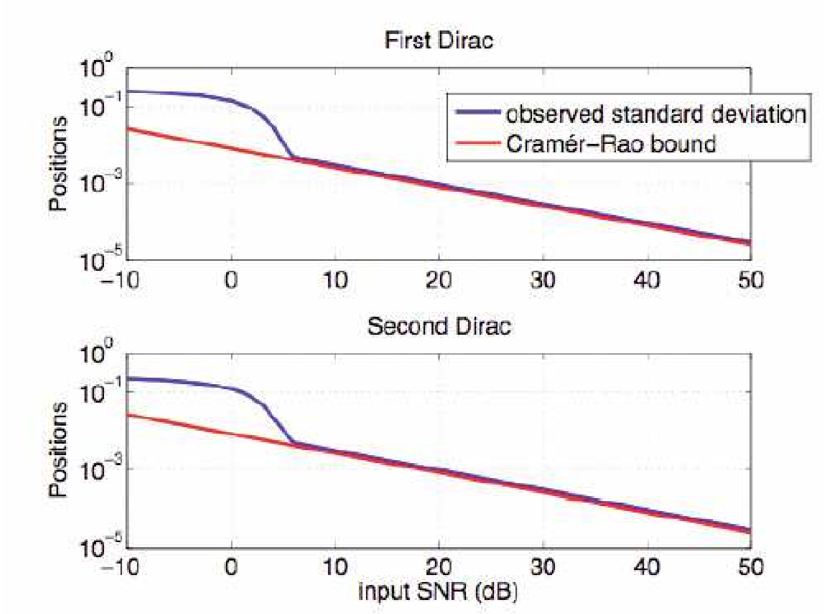
The basic intuition behind this method is that, in the noiseless case, $\mathbf{S}$ is rank deficient (rank K) and Toeplitz, while in the noisy case $\mathbf{S}$ is full rank.

Algorithm:

- SVD of $\mathbf{S} = \mathbf{U}\lambda\mathbf{V}^{\mathbf{T}}$.
- Keep the K largest diagonal coefficients of λ and set the others to zero.
- Reconstruct $\mathbf{S}' = \mathbf{U}\lambda'\mathbf{V}^{\mathbf{T}}$.
- This matrix is not Toeplitz, make it so by averaging along the diagonals.
- Iterate.

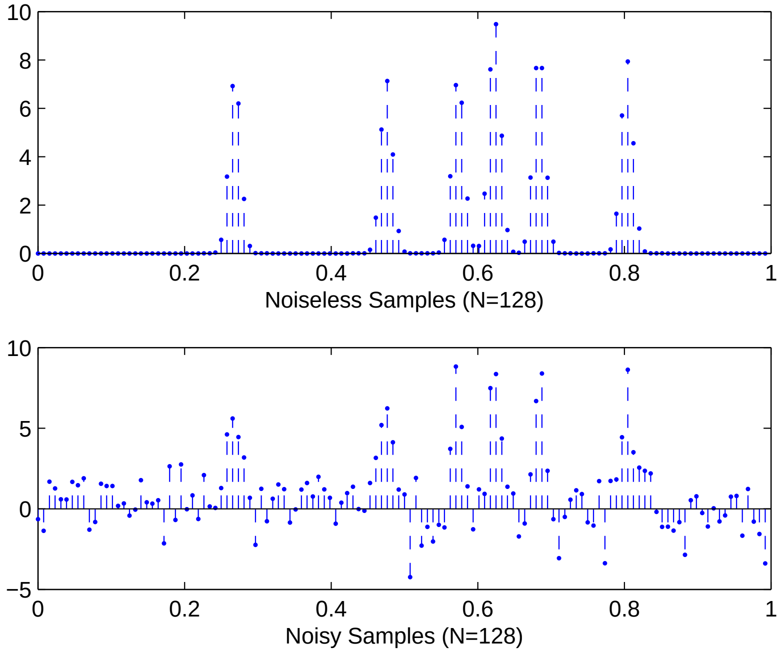
Simulation Results

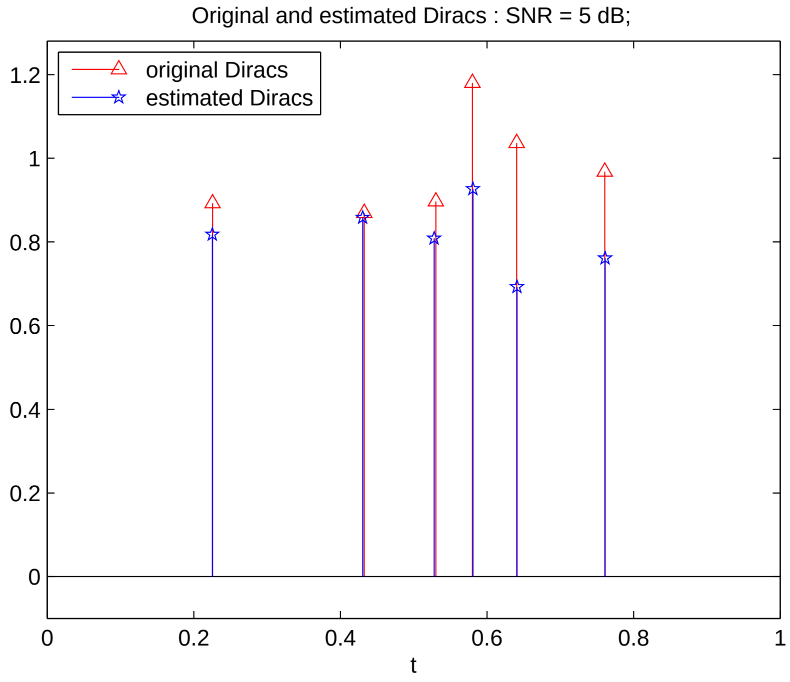




- This is a parametric estimation problem.
- Unbiased algorithms have a covariance matrix lower bounded by CRB.
- The proposed algorithm reaches CRB down to SNR of 5dB.

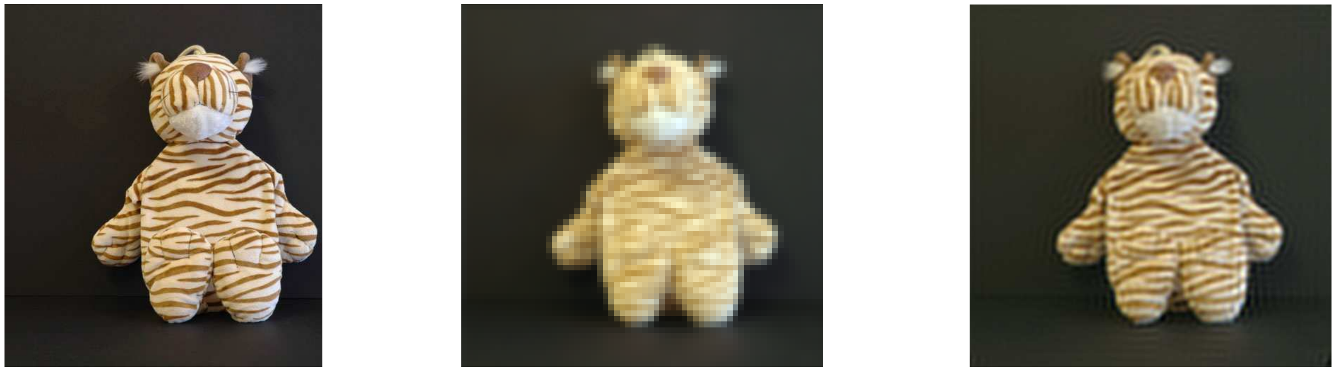
Simulation Results (cont'd)





Application: Image Super-Resolution

Super-Resolution is a multichannel sampling problem with unknown shifts. Use moments to retrieve the shifts or the geometric transformation between images.

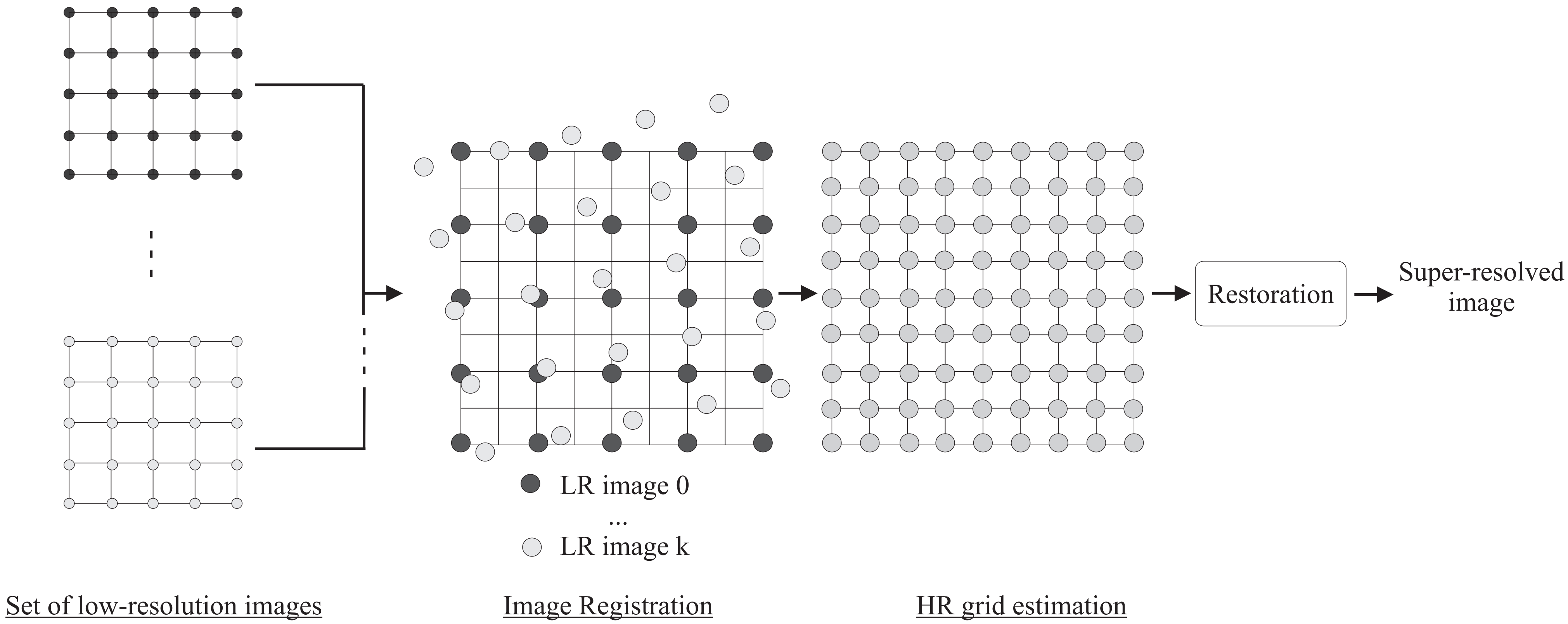


(a)Original (512×512) (b) Low-res. (64×64) (c) Super-res (PSNR=24.2dB)

- Forty low-resolution and shifted versions of the original.
- Intuition: The disparity between images has a finite rate of innovation and can be retrieved exactly.
- Accurate registration is achieved by retrieving the continuous moments of the 'Tiger' from the samples.
- The registered images are interpolated to achieve super-resolution.

Application: Image Super-Resolution

Image super-resolution basic building blocks



Application: Image Super-Resolution

- For each blurred image $I(x, y)$:
  - A sample $P_{m,n}$ in the blurred image is given by

$$P_{m,n} = \langle I(x, y), \varphi(x/T - n, y/T - m) \rangle,$$

    where $\varphi(t)$ represents the point spread function of the lens.
  - We assume $\varphi(t)$ is a spline that can reproduce polynomials:

$$\sum_n \sum_m c_{m,n}^{(l,j)} \varphi(x - n, y - m) = x^l y^j \quad l = 0, 1, ..., N; j = 0, 1, ..., N.$$

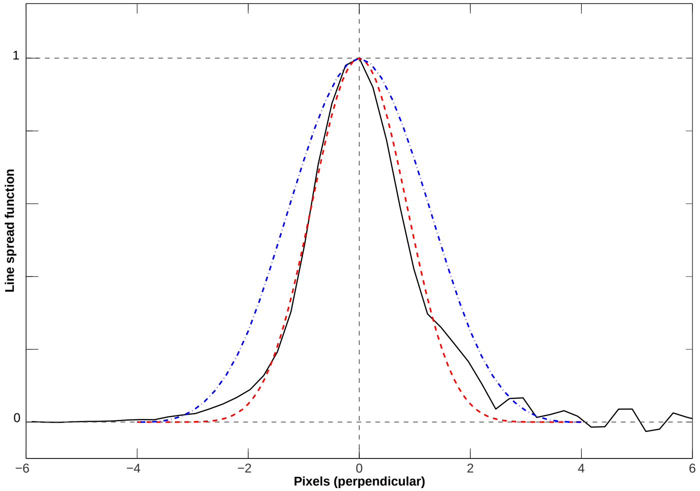
  - We retrieve the exact moments of $I(x, y)$ from $P_{m,n}$:

$$\tau_{l,j} = \sum_n \sum_m c_{m,n}^{(l,j)} P_{m,n} = \langle I(x, y), \sum_n \sum_m c_{m,n}^{(l,j)} \varphi(x/2^J - n, y/2^J - m) \rangle = \int \int I(x, y) x^l y^j dx dy.$$

- Given the moments from two or more images, we estimate the geometrical transformation and register them. Notice that moments of up to order three along the x and y coordinates allows the estimation of an affine transformation.

Application: Image Super-Resolution

(a)Original (2014×3039)



(b) Point Spread function

Application: Image Super-Resolution

(a)Original (128×128)

(b) Super-res (1024×1024)

Application: Image Super-Resolution

(a)Original (48×48)

(b) Super-res (480×480)

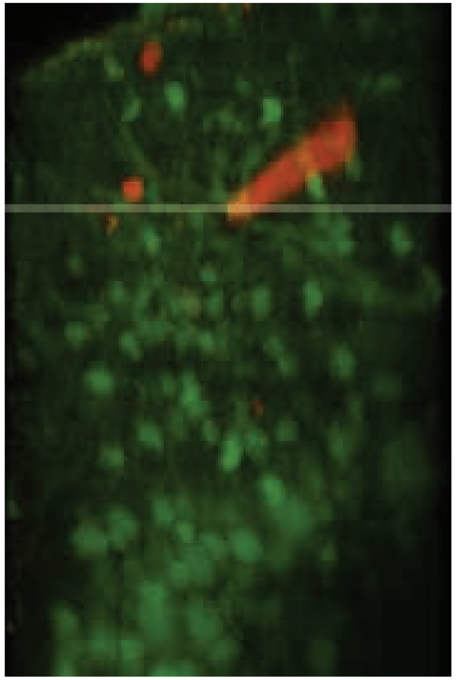
Application: Image Super-Resolution

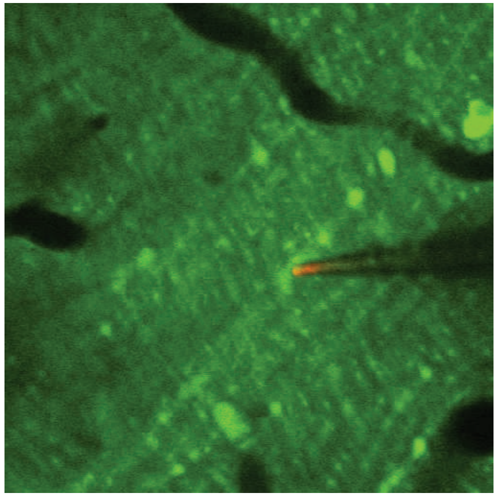
(c) Super-res using Harris (PSNR=14.9dB)

(d) Super-res using FRI (PSNR=16dB)

Local features (e.g., edges and corners) are estimated using FRI algorithms.

Application: Two-photon Microscopy for Neuroscience

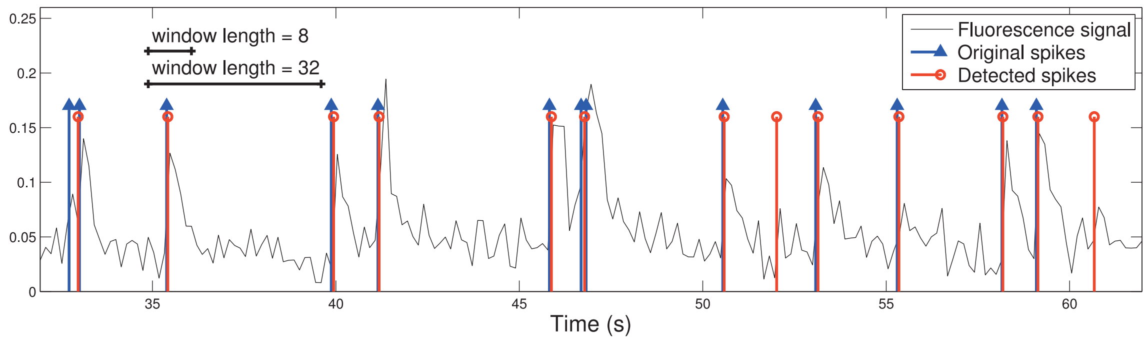




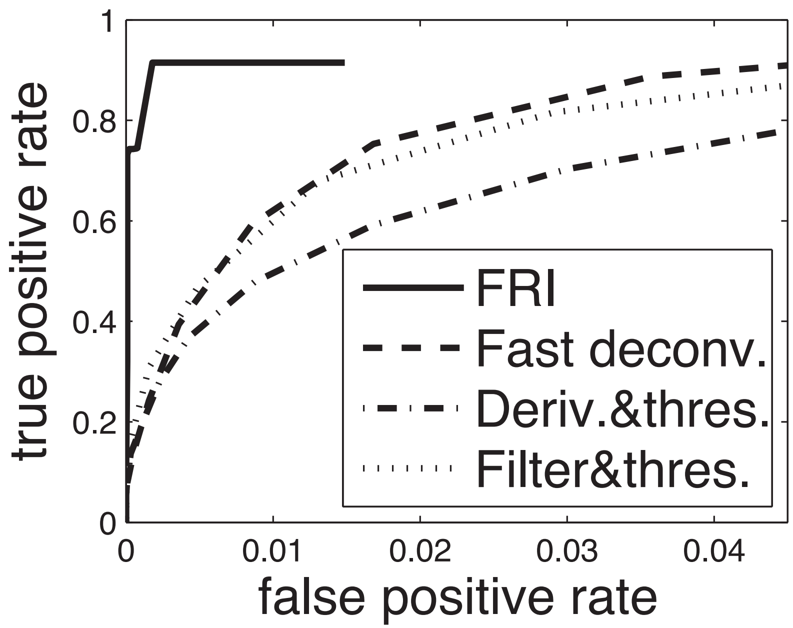
Application: Two-photon Microscopy for Neuroscience

Challenges:

- Increase temporal resolution
- Detect calcium transients



Application: Calcium Transient Detection



Conclusions

Sampling signals at their rate of innovation:

- New framework that allows the sampling and reconstruction of signals at a rate smaller than Nyquist rate.
- Robust and fast algorithms with complexity proportional to the number of degrees of freedom, i.e., $O(K^3)$.
- Provable optimality (i.e. CRB achieved over wide SNR ranges)
- Intriguing connections with multi-resolution analysis, Fourier theory and analogue circuit theory.

But also

- There is no such thing as a free lunch ;-)
- Core application is actually difficult.
- More work need to be done to fit the model and devise the right algorithm for the given application.
- Still many open questions from theory to practice.

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