PREDICATE LOGIC

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Contents

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1. Formal languages

In this section we define what is a first order language, usually denoted by $\mathscr{L}$. $\mathscr{L}$ will consist of an alphabet and a set of finite sequences (strings) of elements of that alphabet, built according to certain rules; these string will be called formulas.

The intention is to capture formal expression in mathematics like the following:

$$\forall n\ n \geq 2 \to \exists p \left(2 \leq p \wedge p|n \wedge \forall k \forall m\ (p = k \cdot m \to p = k \vee p = m)\right)$$

1.1. **The alphabet of a language.**

In order to capture formulas as the one above we will choose a definition of an *alphabet* that allows us to cover a wide range of expressions.

1.1.1. **Definition.** (Alphabet)
The **alphabet** of a language $\mathscr{L}$ consists of the following data:

(I) A set of **logical symbols**, which are present in every language:
- $\neg$ ('not'), $\to$ ('implies'), $\forall$ ('for all')
- The equality symbol: $\doteq$
- Brackets:) (
- Comma: ,
- Symbols to denote **variables**: $v_0, v_1, v_2, ...$ Notice that each v_i is considered as a single symbol (and not as a concatenation of two symbols).

(II)
- Three mutually disjoint sets $\mathscr{R}$ (called the set of **relation symbols** or **predicate symbols**), $\mathscr{F}$ (called the set of **function symbols**) and $\mathscr{C}$ (called the set of **constant symbols**). Further, none of these sets contains a logical symbol.
- Maps

$$\lambda : \mathscr{R} \longrightarrow \mathbb{N} \text{ called the “}\textbf{arity of relation symbols}\text{”}$$
$$\mu : \mathscr{F} \longrightarrow \mathbb{N} \text{ called the “}\textbf{arity of function symbols}\text{”}$$

For $R \in \mathscr{R}$ and $F \in \mathscr{F}$, the numbers $\lambda(R)$ and $\mu(F)$ are called the **arity** of R, F respectively. We say that R, F is n**-ary**, if $\lambda(R) = n$, $\mu(F) = n$, respectively.

Every logical symbol and every element from $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C}$ is called an $\mathscr{L}$-symbol or simply a **symbol** whenever $\mathscr{L}$ is clear from the context. We shall also use the term **($\mathscr{L}$-)letter** instead of ($\mathscr{L}$)-symbol.

We define the **set of variables** as

$$\text{Vbl} := \{v_n \mid n \in \mathbb{N}_0\}.$$

The alphabet of a language $\mathscr{L}$ is called **finite** if $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$ are finite. Otherwise the alphabet of $\mathscr{L}$ is called **infinite**

The alphabet of a language $\mathscr{L}$ is called **countable** if $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$ are countable or finite. Otherwise the alphabet of $\mathscr{L}$ is called **uncountable**

Notice: we still are in the process of defining a language $\mathscr{L}$, and the alphabet is the first thing we need to know. However, we will see shortly that the other parts of the language (finite sequences of elements of the alphabet, built according to certain rules) only depend on the alphabet - in contrast to natural languages.

1.1.2. *Remark.* Strictly speaking, the set of symbols of an alphabet is always infinite, since there are infinitely many variables. One might insist to work with finite languages only: In this case an option is to introduce only two new symbols v and $'$ and to treat v_n as

$$v\overbrace{'\cdots'}^{n\text{-times}}.$$

Notation. Obviously, the alphabet of a language is uniquely determined by the data in item II of definition 1.1.1. These data are called the **similarity type** of $\mathscr{L}$. Hence the similarity type of $\mathscr{L}$ is given by

$$(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$$

1.1.3. *Examples.*

(i) The empty similarity type. Here $\mathscr{R} = \mathscr{F} = \mathscr{C} = \emptyset$.
(ii) The similarity type of a composition (or of an operation): $(\emptyset, \mu : \{\circ\} \longrightarrow \{2\}, \emptyset)$. This means: $\mathscr{R} = \mathscr{C} = \emptyset$ and $\mathscr{F}$ consist of a single element $\circ$ of arity 2: $\mu(\circ) = 2$.
(iii) The similarity type of groups: $(\emptyset, \mu : \{\circ, {}^{-1}\} \longrightarrow \mathbb{N}, \{e\})$ where $\mu(\circ) = 2$ and $\mu({}^{-1}) = 1$; hence $\circ$ is a **binary function symbol** (i.e. of arity 2), ${}^{-1}$ is a function symbol of arity 1 and e is a constant symbol.
(iv) The similarity type of unital rings: $(\emptyset, \mu : \{+, -, \cdot\} \longrightarrow \mathbb{N}, \{0, 1\})$, where $\mu(+) = \mu(\cdot) = 2$ and $\mu(-) = 1$. Hence $-$ is a **unary** (i.e. 1-ary) and $+, \cdot$ are binary function symbols. 0 and 1 are constant symbols.
(v) The similarity type of set theory: $(\lambda : \{\in\} \longrightarrow \{2\}, \emptyset, \emptyset)$. Here $\in$ is a **binary predicate symbol**. Sometimes this similarity type also contains a constant symbol (denoting the empty set).
(vi) The similarity type of partially ordered sets: $(\lambda : \{\leq\} \longrightarrow \{2\}, \emptyset, \emptyset)$. Here $\leq$ is a binary predicate symbol.
(vii) The similarity type of ordered groups: $(\lambda : \{\leq\} \longrightarrow \{2\}, \mu : \{\circ, {}^{-1}\} \longrightarrow \mathbb{N}, \{e\})$. Here $\leq$ is a binary relation symbol.

1.1.4. **Definition.** ($\mathscr{L}$-string)
A **string** of an alphabet of a language $\mathscr{L}$ (or simply "a string of $\mathscr{L}$" or an $\mathscr{L}$-string) is a finite sequence of logical symbols and elements from $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$. The **length of a string** is the length of that sequence. Every logical symbol and each element from $\mathscr{R}$, $\mathscr{F}$ and $\mathscr{C}$ is also considered to be a string (a sequence of length 1). We shall denote strings by concatenation of letters, e.g. we write $\forall v_0 \neg, v_0 \rightarrow, \doteq v_0$ instead of the sequence $(\forall, v_0, \neg, ,, v_0, \rightarrow, ,, \doteq, v_0)$.

In formal arguments we still make use of the notation of strings as sequences. Here an example:

A **string** $(z'_1, ..., z'_m)$ **occurs in a string** $(z_1, ..., z_n)$ **at position** k if $m + k \leq n + 1$ and $z_k = z'_1, ..., z_{k+m-1} = z'_m$ (i.e. $z_{k+i-1} = z'_i$ $(1 \leq i \leq m)$). In a picture:

$$\begin{array}{cccccccc} z_1 & \cdots & z_{k-1} & z_k & \cdots & z_{k+m-1} & z_{k+m} & \cdots & z_n \\ & & & \| & & \| & & & \\ & & & z'_1 & \cdots & z'_m & & & \end{array}$$

In this case, $(z'_1, ..., z'_m)$ is called a **substring** of $(z_1, ..., z_n)$. If in addition $k = 1$, then $(z'_1, ..., z'_m)$ is called an **initial segment** or an **initial substring** of $(z_1, ..., z_n)$. Similarly, if $k = n - m + 1$, then $(z'_1, ..., z'_m)$ is called a **terminal segment** or a **terminal substring** of $(z_1, ..., z_n)$.

1.1.5. *Remark.* Obviously the substring relation is transitive. We need to record this in more detail: If r, s, t are strings such that

- r is a substring of s occurring at position p in s and
- s is a substring of t occurring at position k in t,

then r is a substring of t occurring at position $k + p - 1$ in t.

1.2. **Terms.**

1.2.1. **Definition.** ($\mathscr{L}$-term)
Given the similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ of $\mathscr{L}$, we define subsets $\mathrm{tm}_k(\mathscr{L})$ of strings of the alphabet of $\mathscr{L}$ by induction on $k \in \mathbb{N}_0$ as follows:

$$\mathrm{tm}_0(\mathscr{L}) = \mathrm{Vbl} \cup \mathscr{C} \text{ and}$$

$$\mathrm{tm}_{k+1}(\mathscr{L}) = \mathrm{tm}_k(\mathscr{L}) \cup \{F(t_1,t_2,...,t_n) \mid F \in \mathscr{F}, n = \mu(F),\ t_1, ..., t_n \in \mathrm{tm}_k(\mathscr{L})\}.$$

Notice that

$$\mathrm{tm}_0(\mathscr{L}) \subseteq \mathrm{tm}_1(\mathscr{L}) \subseteq \mathrm{tm}_2(\mathscr{L}) \subseteq \dots$$

is an increasing chain of sets of $\mathscr{L}$-strings. The set of $\mathscr{L}$**-terms** is defined as

$$\mathrm{tm}(\mathscr{L}) := \bigcup_{k\in\mathbb{N}_0} \mathrm{tm}_k(\mathscr{L}).$$

The elements of $\mathrm{tm}(\mathscr{L})$ are called $\mathscr{L}$**-terms** or simply 'terms' if $\mathscr{L}$ is clear from the context. Notice that by induction on k we see that the number of opening brackets occurring in a term $t \in \mathrm{tm}_k(\mathscr{L})$, is equal to the number of closing brackets occurring in t.

The **complexity of an** $\mathscr{L}$**-term** t - denoted by $c(t)$ - is the least $k \in \mathbb{N}_0$ such that $t \in \mathrm{tm}_k(\mathscr{L})$. Notice that for $t \in \mathrm{tm}(\mathscr{L})$ and $k \in \mathbb{N}_0$ we have by definition $c(t) \le k \iff t \in \mathrm{tm}_k(\mathscr{L})$.

Warning. The expression $F(t_1,t_2,...,t_n)$ in the definition of $\mathrm{tm}_{k+1}(\mathscr{L})$ above denotes the concatenation of the strings $F, (, t_1, ,, t_2, ,, ..., ,, t_n,$ and $)$. It is **not** the application of a function to $(t_1, ..., t_n)$. F here is only a letter, not a function!

1.2.2. **Lemma.** *Let t be an $\mathscr{L}$-term. Then*

(i) t is a variable if and only if the first letter of t is a variable.
(ii) t is a constant symbol if and only if the first letter of t is a constant symbol.
(iii) $t \in \mathrm{tm}(\mathscr{L}) \setminus \mathrm{tm}_0(\mathscr{L})$ if and only if the first letter of t is a function symbol.
(iv) $c(t) = 0 \iff t \in \mathrm{tm}_0(\mathscr{L})$.

Proof. This will be done in Question 3 of the examples. □

1.2.3. **Lemma.** *If s, t are $\mathscr{L}$-terms and t is an initial segment of s then $s = t$.*

Proof. We pick $t \in \mathrm{tm}_k(\mathscr{L})$ for some $k \in \mathbb{N}_0$ and we show the following assertion by induction on k:

($*$) If s is an $\mathscr{L}$-term and t is an initial segment of s <u>or vice versa</u> then $s = t$.

If $k = 0$, then t is a variable or a constant symbol and ($*$) follows immediately from 1.2.2.

If $t \in \mathrm{tm}_{k+1}(\mathscr{L})$ and $t \notin \mathrm{tm}_k(\mathscr{L})$, then pick $n \in \mathbb{N}$, $F \in \mathscr{F}$ and $t_1, ..., t_n \in \mathrm{tm}_k(\mathscr{L})$ with $t = F(t_1,t_2,...,t_n)$. Suppose t is an initial segment of $G(s_1,s_2,...,s_r)$ for some $G \in \mathscr{F}, r \in \mathbb{N}$ and $s_1, ..., s_r \in \mathrm{tm}(\mathscr{L})$ - or vice versa. Then obviously $F = G$, which implies $n = \mu(F) = \mu(G) = r$, and the string $t_1,t_2,...,t_n)$ is an initial segment of $s_1,s_2,...,s_n)$ or vice versa. Depending on the length of the strings t_1 and s_1, it follows that t_1 is an initial segment of s_1 or vice versa. Since $t_1 \in \mathrm{tm}_k(\mathscr{L})$ the induction hypothesis gives $t_1 = s_1$. It follows that $t_2,t_2,...,t_n)$ is an initial segment of $s_2,s_2,...,s_n)$ or vice versa.

Consequently t_2 is an initial segment of s_2 or vice versa. Since $t_2 \in \mathrm{tm}_k(\mathscr{L})$ the induction hypothesis gives $t_2 = s_2$. Continuing in this way we see that $t_1 = s_1, ..., t_n = s_n$. □

1.2.4. **Theorem.** *(Unique readability theorem for terms) If t is an $\mathscr{L}$-term, then either t is a variable or t is a constant symbol or there are uniquely determined $n \in \mathbb{N}$, $F \in \mathscr{F}$ of arity n and $t_1, ..., t_n \in \mathrm{tm}(\mathscr{L})$ such that*

$$t = F(t_1,t_2,...,t_n).$$

Proof. That these cases are mutually exclusive follows by comparing the first letters. The uniqueness statement is clear if $t \in \mathrm{Vbl} \cup \mathscr{C}$. Suppose $t = F(t_1,t_2,...,t_n)$ and $t = G(s_1,s_2,...,s_r)$ for some $G \in \mathscr{F}, r \in \mathbb{N}$ and $s_1, ..., s_r \in \mathrm{tm}(\mathscr{L})$. Then $F = G$, hence $n = r$ and the string $t_1,t_2,...,t_n)$ is equal to $s_1,s_2,...,s_n)$. It follows that t_1 is an initial segment of s_1 or vice versa. By 1.2.3 we get $t_1 = s_1$ and so $t_2,...,t_n)$ is equal to $s_2,...,s_n)$. Continuing in this way we get $s_i = t_i$ for each $i \in \{1, ..., n\}$. □

1.2.5. **Corollary.** *For all $n \in \mathbb{N}$, all $\mathscr{L}$-terms $t_1, ..., t_n$ and each n-ary function symbol F of $\mathscr{L}$ we have*

$$c(F(t_1, ..., t_n)) = 1 + \max\{c(t_1), ..., c(t_n)\}.$$

Proof. (**level 4/MSc**) Let $t = F(t_1,...,t_n)$ and let $k = \max\{c(t_1), ..., c(t_n)\}$. Then $t_1, ..., t_n \in \mathrm{tm}_k(\mathscr{L})$ by definition of the complexity and so $t \in \mathrm{tm}_{k+1}(\mathscr{L})$. By definition of the complexity again, this means $c(t) \leq 1+k$. Suppose $c(t) < 1+k$. We write $l := c(t)$. Then $t \in \mathrm{tm}_l(\mathscr{L})$. Since the first letter of t is F, we know $c(t) > 0$, thus $l > 0$. By definition of $\mathrm{tm}_l(\mathscr{L})$ there are $G \in \mathscr{F}$, $r \in \mathbb{N}$ and $s_1, ..., s_r \in \mathrm{tm}_{l-1}(\mathscr{L})$ such that $t = G(s_1,...,s_r)$. By 1.2.4 we have $G = F$, $r = n$ and $s_i = t_i$ $(1 \leq i \leq n)$. It follows that $t_i = s_i \in \mathrm{tm}_{l-1}(\mathscr{L})$ $(1 \leq i \leq n)$, hence by definition of the complexity, $\max\{c(t_1), ..., c(t_n)\} \leq l - 1$. Since $\max\{c(t_1), ..., c(t_n)\} = k$ we get the contradiction $k \leq l - 1 = c(t) - 1 < (1 + k) - 1 = k$. □

1.2.6. **Definition.** A term s is called a **subterm** of a term t, if s occurs in t; see 1.1.4 for the definition of "occurs in".

We want to show that the subterm relation is reasonable, which means the following A term t is built up (or 'constructed') starting with constant symbols and variables by repeatedly concatenating function symbols with brackets, commas and previously build terms in the form $F(t_1, \ldots, t_n)$. Every term that is used in this construction process, clearly is a subterm of t in the sense of definition 1.2.6. However, it is a priori not clear why a term s that occurs as a string in t was indeed used to construct t. This is the content of 1.2.9 and of 1.2.8. We need a preparation.

1.2.7. **Lemma.** (***level* 4/MSc**) *Let t be a term and let s be a nonempty substring of t with $s \neq t$.*

(i) If s is an initial substring of t, then either s is in $\mathcal{C} \cup \mathcal{F} \cup \mathrm{Vbl}$, or, the number of opening brackets occurring in s is greater than the number of closing brackets occurring in s.

In either case, the number of opening brackets occurring in s is greater than or equal to the number of closing brackets occurring in s.

(ii) If s is a terminal substring of t, then either $t = Fs$ for some function symbol F, or the number of opening brackets occurring in s is less than the number of closing brackets occurring in s.

In either case, the number of opening brackets occurring in s is less than or equal to the number of closing brackets occurring in s.

Proof. (**level 4/MSc**) Recall that the number of opening brackets occurring in a term is equal to the number of closing brackets occurring in that term.

(i) By induction on $c(t)$, where $c(t) = 0$ is clear by 1.2.2(iv). So assume $t = F(t_1, \dots, t_n)$ with $F \in \mathcal{F}$, $\lambda(F) = n$ and terms $t_1, \dots, t_n$ of lower complexity than $c(t)$. Let s be an initial segment of t. If s is of length ≤ 2, the assertion is clear by 1.2.2. Let s_0 be the string obtained from s by removing the first two letters. Since $s \neq t$, s_0 is an initial segment of the string

$$t_1, \dots, t_n$$

Using the induction hypothesis, we see that the number of opening brackets in s_0 is greater or equal to the the number of closing brackets in s_0. Since s is the string

$$F(s_0$$

we see that the number of opening brackets in s is bigger than the number of closing brackets in s.

(ii) follows from (i) applied to the initial substring s_0 of t for which t is $s_0 s$. $\square$

1.2.8. **Proposition.** (***level* 4/MSc**) *Let $t_1, \dots, t_n$ be terms and let l_i be the length of t_i. Let F be a function symbol of arity n and suppose s is a subterm of $t = F(t_1, \dots, t_n)$ occurring at position p. For $i \in \{1, \dots, n-1\}$, let $p_i = l_1 + \ldots + l_i + i + 2$. Further, let $p_0 = 2$ and let p_n be the length of t. Here is a picture of the positions $p_0, \dots, p_n$ in t.*

$$\begin{array}{ccccccccccc}
F & (& t_1 & , & t_2 & , & \cdots & , & t_n &) \\
\uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & p_0 & & p_1 & & p_2 & \cdots & p_{n-1} & & p_n
\end{array}$$

(Notice that the seemingly more intuitive definition of the $p_1, \dots, p_n$ as the position "after" $t_1, \dots, t_n$, would not be correct: for example, t_2 could occur in t_1, so "after t_2" is ambiguous) .

Then, either $p = 1$ and $s = t$, or, there is some $i \in \{1, \dots, n\}$ such that $p_{i-1} < p < p_i$ and s is a subterm of t_i occurring at position $p - p_{i-1}$ in t_i.

Proof. (**level 4/MSc**) If $p = 1$, then s is an initial segment of t and $s = t$ by 1.2.3.

So assume $p > 1$. Since the first letter of s is in $\mathrm{Vbl} \cup \mathcal{C} \cup \mathcal{F}$, we know $p \notin \{p_0, \dots, p_n\}$. As $p > 1$, there is a unique $i \in \{1, \dots, n\}$ with $p_{i-1} < p < p_i$ and it remains to show that s is a subterm of t_i occurring at position $p - p_{i-1}$ in t_i. Suppose this is not the case. Then the substring s_0 of t starting at position p and ending at position $p_i - 1$ is a nonempty, initial segment of s with $s_0 \neq s$. It follows that s is neither a variable nor a constant symbol, hence s_0 is neither a variable nor a constant symbol either. If s_0 were a function symbol, then this function symbol would be the last letter of t_i, which is impossible by the definition of 'term'. Thus, s_0 is not in $\mathcal{C} \cup \mathcal{F} \cup \mathrm{Vbl}$ and 1.2.7(i) implies that the number of opening brackets occurring in s_0 is greater than the number of closing brackets occurring in s_0. On the other hand, s_0 is also a terminal substring of t_i and this contradicts 1.2.7(ii). $\square$

1.2.9. **Proposition.** *Let s be a subterm of t. Then $s = t$, or there are $n \in \mathbb{N}$, $F \in \mathscr{F}$ of arity n and $t_1, ..., t_n \in \mathrm{tm}(\mathscr{L})$ such that $t = F(t_1,t_2,...,t_n)$ and s is a subterm of one of the t_i.* □

Proof. (**level 4/MSc**) This is an immediate consequence of 1.2.8. □

The assertion of proposition 1.2.9 could thus be used as an inductive definition of the subterm relation. Our reasoning in 1.2.8 and 1.2.7 would then prove that any term that is a substring of a term t, necessarily must be used in any construction of t. In fact, more is true. With the aid of 1.2.8 one can show easily that every occurrence of a term as a substring necessarily appears in any construction of t at some stage. To see the strength of 1.2.8 over its consequence 1.2.9, also see Question 5 of the example sheets.

1.2.10. **Proposition.** *Let s be a subterm of a term t occurring at position p in t. Let r be another term. We define a new string u by replacing the substring s of t occurring at position p with the string r. Then u is again a term.*

Proof. (**level 4/MSc**) This is done in Question 5 of the example sheets. □

1.3. **Formulas.**

1.3.1. **Definition.** (formulas)
Given a similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ of a language $\mathscr{L}$, an **atomic $\mathscr{L}$-formula** is a string of the alphabet of $\mathscr{L}$ of the form

$$t_1 \doteq t_2,$$

where t_1, t_2 are $\mathscr{L}$-terms or

$$R(t_1,...,t_n),$$

where R is a relation symbol of arity $n \in \mathbb{N}$ and $t_1, ..., t_n$ are $\mathscr{L}$-terms. The set of atomic $\mathscr{L}$-formulas is denoted by at-Fml$(\mathscr{L})$.

We define

$$\begin{aligned} \mathrm{Fml}_0(\mathscr{L}) &= \text{at-Fml}(\mathscr{L}) \text{ and inductively for each } k \in \mathbb{N}_0 : \\ \mathrm{Fml}_{k+1}(\mathscr{L}) &= \mathrm{Fml}_k(\mathscr{L}) \cup \{(\neg\varphi), (\varphi \rightarrow \psi), (\forall x \varphi) \mid \varphi, \psi \in \mathrm{Fml}_k(\mathscr{L}),\ x \in \mathrm{Vbl}\}. \end{aligned}$$

Notice that $\mathrm{Fml}_0(\mathscr{L}) \subseteq \mathrm{Fml}_1(\mathscr{L}) \subseteq \mathrm{Fml}_2(\mathscr{L}) \subseteq \ldots$. The set of **$\mathscr{L}$-formulas** is defined as

$$\mathrm{Fml}(\mathscr{L}) := \bigcup_{k \in \mathbb{N}_0} \mathrm{Fml}_k(\mathscr{L}).$$

As for terms, the number of opening brackets in a formula is equal to the number of closing brackets in that formula.

If the letter $\forall$ does not occur in the $\mathscr{L}$-formula φ, then φ is called **quantifier free**.

In the literature one can also find the expression **well formed formula** (abbreviated 'wff') instead of 'formula'.

Warning. Not every formula that has (obvious) meaning in mathematics is a formula in our sense. This is in particular important after we have proved significant theorems involving formulas. Here an example:

$$\forall n \in \mathbb{N}\ \exists r, q \in \mathbb{N}_0\ n = q \cdot m + r \wedge r < m.$$

There is no language (according to our definition) such that the above is a formula in that language.

Notice that the quantifier introduced in the definition of $\mathrm{Fml}_{k+1}(\mathscr{L})$ (cf. 1.3.1) is always applied in a nonrestricted way, e.g.

$$\forall n \exists r, q\ n \doteq q \cdot m + r \wedge r < m$$

will be a formula in the language of rings after we have introduced the appropriate abbreviations (concerning the symbols $\exists$ and $\wedge$)in subsection 1.4 below.

The **language** or **signature** $\mathscr{L}$ is the triple consisting of the alphabet of $\mathscr{L}$, the set of $\mathscr{L}$-terms and the set of $\mathscr{L}$-formulas. Obviously, tm$(\mathscr{L})$ and Fml$(\mathscr{L})$ are uniquely determined by the similarity type of $\mathscr{L}$ and we shall simply communicate languages by their similarity type.

Hence the expression 'let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language' stands for 'let $\mathscr{L}$ be the language with similarity type $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$'.

We say that a language is **finite**, **infinite**, **countable** or **uncountable** if the alphabet of that language has this property.

As for terms we have a unique readability theorem and we need two preparations.

1.3.2. **Lemma.** *Let* $\varphi \in \mathrm{Fml}(\mathscr{L})$. *Then*

(i) φ *is atomic if and only if the first letter of* φ *is in* $\mathscr{R} \cup \mathscr{F} \cup \mathscr{C} \cup \mathrm{Vbl}$. *Otherwise, the first letter of* φ *is an opening bracket.*

(ii) *Let* $\varphi, \psi, \delta, \varepsilon \in \mathrm{Fml}(\mathscr{L})$ *and let* γ_1, γ_2 *and* γ_3 *be defined as follows:*

$$\gamma_1 \text{ is } (\neg\delta),$$
$$\gamma_2 \text{ is } (\varphi{\rightarrow}\psi) \text{ and}$$
$$\gamma_3 \text{ is } (\forall x \varepsilon) \text{ for some variable } x.$$

Furthermore, let $\gamma_0 \in \text{at-Fml}(\mathscr{L})$ *be another formula.*

If $i, j \in \{0, ..., 3\}$ *with* $i \neq j$, *then* γ_i *is* <u>not</u> *an initial segment of* γ_j.

Remark: For terms we had to prove a corresponding statement: see 1.2.2.

Proof. This will be treated in question 8 of the example sheets. □

1.3.3. **Lemma.** *If* $\varphi, \psi \in \mathrm{Fml}(\mathscr{L})$ *and* φ *is an initial segment of* ψ *then* $\varphi = \psi$.

Remark: For terms the corresponding statement is 1.2.3.

Proof. We choose k such that $\varphi \in \mathrm{Fml}_k(\mathscr{L})$ and we show the following assertion by induction on $k \in \mathbb{N}_0$:

($*$) If $\psi \in \mathrm{Fml}(\mathscr{L})$ and φ is an initial segment of ψ <u>or vice versa</u> then $\varphi = \psi$.

If $k = 0$, then φ is atomic and by 1.3.2(i), also ψ is atomic. If φ is a term equality, i.e. of the form $t_1 \doteq t_2$, then also ψ has to be a term equality (compare the first letter). Then, using 1.2.3, we see that φ is equal to ψ. Hence φ and ψ both are of the form $R(t_1,...,t_n)$ and again by using 1.2.3, it follows easily that φ is equal to ψ.

For the induction step $k \Rightarrow k+1$ we may assume that $\varphi \in \mathrm{Fml}_{k+1}(\mathscr{L})$ is not atomic. By 1.3.2(ii) there are exactly three cases possible:

<u>Case 1.</u> $\varphi = (\neg\varphi_1)$ and $\psi = (\neg\psi_1)$, where $\varphi_1 \in \mathrm{Fml}_k(\mathscr{L})$ and $\psi_1 \in \mathrm{Fml}(\mathscr{L})$.

<u>Case 2.</u> $\varphi = (\varphi_1{\rightarrow}\varphi_2)$ and $\psi = (\psi_1{\rightarrow}\psi_2)$ where $\varphi_1, \varphi_2 \in \mathrm{Fml}_k(\mathscr{L})$ and $\psi_1, \psi_2 \in \mathrm{Fml}(\mathscr{L})$.

<u>Case 3.</u> $\varphi = (\forall x \varphi_1)$ and $\psi = (\forall y \psi_1)$, where $x, y \in \mathrm{Vbl}$, $\varphi_1 \in \mathrm{Fml}_k(\mathscr{L})$ and $\psi_1 \in \mathrm{Fml}(\mathscr{L})$.

In all three cases it is clear that φ_1 is an initial segment of ψ_1 or vice versa. As $\varphi_1 \in \mathrm{Fml}_k(\mathscr{L})$ the induction hypothesis implies $\varphi_1 = \psi_1$. Hence in case 1 and case 3 we get $\varphi = \psi$. In case 2 it follows that the string φ_2 is an initial segment of ψ_2 or vice versa. So again $\varphi_2 = \psi_2$ by induction and $\varphi = \psi$. □

1.3.4. **Theorem.** *(Unique readability theorem for formulas)*
Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ *be a language and let* φ *be an* $\mathscr{L}$*-formula. Then exactly one of the following holds true:*

(i) φ *is atomic and there are uniquely determined* $t_1, t_2 \in \mathrm{tm}(\mathscr{L})$ *such that* φ *is* $t_1 \doteq t_2$, *or*

(ii) φ *is atomic and there are a unique* $n \in \mathbb{N}$, *a unique* $R \in \mathscr{R}$ *and uniquely determined* $\mathscr{L}$*-terms* $t_1, ..., t_n$ *such that* φ *is* $R(t_1,...,t_n)$, *or*

(iii) φ *is equal to a string of the form* $(\neg\psi)$ *for a uniquely determined* $\psi \in \mathrm{Fml}(\mathscr{L})$, *or*

(iv) *φ is equal to a string of the form* $(\varphi_1 \to \varphi_2)$ *for uniquely determined* $\varphi_1, \varphi_2 \in \mathrm{Fml}(\mathscr{L})$, *or*

(v) *φ is equal to a string of the form* $(\forall x \psi)$ *for uniquely determined* $\psi \in \mathrm{Fml}(\mathscr{L})$ *and* $x \in \mathrm{Vbl}$.

Proof. By 1.3.2(ii) it is clear that φ is equal to exactly one of the strings stated in (i)-(v). Now the uniqueness statements in (i)-(v) follow easily from 1.3.3. For example: Suppose φ is of the form $(\varphi_1 \to \varphi_2)$ as in (iv). If φ is also of the form $(\psi_1 \to \psi_2)$, then φ_1 is an initial segment of ψ_1 or vice versa. Hence by 1.3.3, $\varphi_1 = \psi_1$, which then implies $\varphi_2 = \psi_2$, thus $\varphi = \psi$. $\square$

1.4. **Domestication of the notation.**

- To increase readability we will omit brackets if this does not lead to ambiguity. For example, we will generally omit the outer brackets of non-atomic formulas.
- We use the following abbreviation for $\mathscr{L}$-formulas φ, ψ: $\varphi \vee \psi := (\neg\varphi) \rightarrow \psi$, $\varphi \wedge \psi := \neg(\varphi \rightarrow (\neg\psi))$, $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\exists x\varphi := \neg\forall x(\neg\varphi)$ where x is a variable.
- We shall not color the symbols of our languages anymore.
- We write

  $\forall x_1, ..., x_n\ \varphi$ instead of $\forall x_1 ... \forall x_n \varphi$ and $\exists x_1, ..., x_n\ \varphi$ instead of $\exists x_1 ... \exists x_n \varphi$

  where each x_i is a variable.

  The strings $\forall x$ and $\exists x$ are called **quantifiers**. A string of quantifiers is a string of the form $Q_1 x_1 ... Q_n x_n$, where each Q_i is either $\forall$ or $\exists$ and each x_i is a variable.
- We write

$$\bigwedge_{i=1}^{n} \varphi_i \text{ instead of } \overbrace{(...(}^{n\text{-times}} \varphi_1 \wedge \varphi_2) \wedge \varphi_3) ... \wedge \varphi_n) \text{ and}$$

$$\bigvee_{i=1}^{n} \varphi_i \text{ instead of } \overbrace{(...(}^{n\text{-times}} \varphi_1 \vee \varphi_2) \vee \varphi_3) ... \vee \varphi_n).$$

- We write $t_1 \neq t_2$ instead of $(\neg t_1 \doteq t_2)$.
- If R is a binary relation, we write $t_1 R t_2$ instead of $R(t_1, t_2)$.

It should be understood that these conventions are meta theoretic conventions and they are meant to increase readability in the notes. When we are arguing about syntactic manipulations with terms and formulas we still refer to the rigorous definitions 1.2.1 and 1.3.1.

1.5. **Complexity and subformulas.** The unique readability theorems 1.2.4 and 1.3.4 allow us to define new objects from formulas, and to prove statements about formulas. This will be done via induction on the construction depth (or the 'complexity') of terms and formulas:

1.5.1. **Definition.** The **complexity of an $\mathscr{L}$-formula** φ - denoted by $c(\varphi)$ - is the least $k \in \mathbb{N}_0$ such that $\varphi \in \mathrm{Fml}_k(\mathscr{L})$.

Notice that this is not in conflict with the definition of the complexity of $\mathscr{L}$-terms (cf. 1.2.1), since the set of $\mathscr{L}$-terms is disjoint from the set of $\mathscr{L}$-formulas. Notice also that for any given terms $t_1, t_2, ..., t_n$ and each n-ary relation symbol R of $\mathscr{L}$, $c(R(t_1, ..., t_n)) = 0$. Similarly $c(t_1 \doteq t_2) = 0$.
By definition, for every $\mathscr{L}$-formula φ and each $k \in \mathbb{N}_0$ we have

$$c(\varphi) \leq k \iff \varphi \in \mathrm{Fml}_k(\mathscr{L}).$$

1.5.2. **Lemma.** *For all $\mathscr{L}$-formulas φ, ψ and every variable x we have*

$$c(\neg\varphi) = 1 + c(\varphi),\ c(\varphi \to \psi) = 1 + \max\{c(\varphi), c(\psi)\} \text{ and } c(\forall x\varphi) = 1 + c(\varphi).$$

Proof. (**level 4/MSc**) The proof is done in Question 9 of the example sheets. □

1.5.3. **Definition.** (subformula)
We say that a formula φ is a **subformula** of a formula ψ if φ occurs in the string ψ; see 1.1.4 for the definition of "occurs in".

As for terms, we need to verify that the subformula relation is reasonable, i.e. that every subformula of a formula φ is indeed used when φ is "constructed"; this is the content of 1.5.6. We need a preparation, similar to 1.2.7 for terms.

1.5.4. **Lemma.** (***level* 4/MSc**) *Let φ be a formula that is not atomic and let s be a nonempty substring of φ with $s \neq \varphi$.*

(i) If s is an initial substring of φ, then the number of opening brackets occurring in s is greater than the number of closing brackets occurring in s.
(ii) If s is a terminal substring of t, then the number of opening brackets occurring in s is less than the number of closing brackets occurring in s.

Proof. (**level 4/MSc**) Recall that the number of opening brackets occurring in a formula is equal to the number of closing brackets occurring in that formula.
(i) By induction on $c(\varphi)$, where $c(\varphi) = 0$ is trivial. So assume φ is equal to $(\neg\psi)$, or equal to $(\forall x\ \psi)$, or equal to $(\psi \to \delta)$, where ψ, δ are of lower complexity than $c(\varphi)$. By the induction hypothesis we know the lemma for ψ and δ instead of φ.
Let s_0 be the substring of s obtained by removing the first letter.
<u>Case 1.</u> φ is equal to $(\neg\psi)$.
Since s is an initial segment of φ and s is different from φ, the string s_0 is an initial segment of the string $\neg\psi$. Hence the substring s_1 of s_0 obtained by removing the first letter is an initial substring of ψ. Since we have removed the opening bracket and the $\neg$ - symbol from s to obtain s_1 it is obviously enough to show that the number of opening brackets occurring in s_1 is greater than or equal to the number of closing brackets occurring in s_1.
However, either s_1 is ψ, in which case this is true, or, s_1 is an initial segment of ψ with $s_1 \neq \psi$ and we may use the induction hypothesis (when s_1 is empty, or when ψ is atomic we don't need induction of course).

This shows the lemma in case 1.

Case 2. φ is equal to $(\forall x\ \psi)$.

This is shown as in case 1; just work with s_1 as being the substring of s_0 obtained by removing the first two letters (observe that s_0 is an initial segment of $\forall x\ \psi$ in case 2).

Case 3. φ is equal to $(\psi \to \delta)$.

Since s is an initial segment of φ and s is different from φ, the string s_0 is an initial segment of the string $\psi \to \delta$. If the length of s_0 is less than or equal to the length of ψ, then using the induction hypothesis we know that the number of opening brackets occurring in s_0 is greater than or equal to the number of closing brackets occurring in s_0; this proves assertion (i).

If the length of s_0 is greater than the length of ψ, then let s_1 be the terminal segment of s_0 such that the string $\psi \to\ \ s_1$ is equal to $\psi \to \delta$. Then s_1 is an initial segment of δ and using the induction hypothesis we know that the number of opening brackets occurring in s_1 is greater than or equal to the number of closing brackets occurring in s_1. This again implies assertion (i).

(ii) follows from (i) applied to the initial substring s_0 of φ for which φ is $s_0 s$. □

1.5.5. **Lemma.** (***level* 4/MSc**) *Let ψ, δ be formulas and let φ be a subformula of $(\psi \to \delta)$ occurring at position p. Let l be the length of ψ. Then φ is $(\psi \to \delta)$, or, $2 \leq p \leq l+1$ and φ is a subformula of ψ occurring at position $p-1$, or, $l+3 \leq p$ and φ is a subformula of δ occurring at position $p-l-2$.*

Proof. (**level 4/MSc**) If $l+3 \leq p$, then φ is a substring of the string

$$\delta)$$

Since φ is a formula, the number of opening brackets in φ is equal to the number of closing brackets in φ. Applying 1.5.4(ii) this entails that φ must be a subformula of δ.

Hence we may assume that $p \leq l+2$. Since φ is a formula, the first letter of φ is in $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \cup \mathrm{Vbl}$, or it is an opening bracket. But this is not possible if p would be $l+2$ (which is the position of a $\to$ - symbol in $(\psi \to \delta)$).

Hence we may assume that $p \leq l+1$. If $p = 1$, then φ is an initial segment of $(\psi \to \delta)$ and φ is equal to $(\psi \to \delta)$ by 1.3.3.

Thus we may assume that $2 \leq p \leq l+1$ and it remains to show that φ is a subformula of ψ. Suppose this is not the case. Then the substring s of $(\psi \to \delta)$ starting at position p and ending at position $l-1$ is a nonempty initial segment of φ with $s \neq \varphi$. Further, φ is not atomic, since the symbol $\neg$ occurs in φ. By 1.5.4(i), the number of opening brackets occurring in s is greater than the number of closing brackets occurring in s. On the other hand, s is also a terminal substring of ψ and this contradicts 1.5.4(ii) (ψ is not atomic since s contains an opening bracket). □

1.5.6. **Proposition.** *Suppose φ is a subformula of ψ.*

(i) If $c(\psi) = 0$ (equivalently: ψ is atomic), then $\varphi = \psi$.

(ii) Now suppose $c(\psi) = k+1$.

- *(a) If $\psi = (\forall x \vartheta)$ or $\psi = (\neg \vartheta)$, then φ is ψ or φ is a subformula of ϑ.*
- *(b) If $\psi = (\psi_1 \to \psi_2)$, then $\varphi = \psi$, or φ is a subformula of ψ_1 or φ is a subformula of ψ_2.*

Proof. (**level 4/MSc**) (i). Firstly assume that ψ is atomic. Then no bracket appears in ψ. Hence no bracket appears in φ either and so φ is atomic, too. If φ is of the form $R(t_1, ..., t_n)$ then φ must occur at position 1 in ψ, since the only position at which a relation symbol can occur in an atomic formula is 1. By 1.3.3, we see that φ is ψ.

On the other hand if φ is of the form $t_1 \doteq t_2$, then also ψ is of this form (see the definition of "atomic formula). Say ψ is $s_1 \doteq s_2$. Then, t_2 is an initial substring of s_2 and by 1.2.3 we know $t_2 = s_2$.

Further, t_1 is a terminal segment of s_1 and 1.2.7(ii) easily implies $s_1 = t_1$.

(ii)(a). Both cases a similar, we only do the case $\psi = (\forall x\vartheta)$. If φ is an initial segment of ψ, then φ is ψ by 1.3.3. Hence we may assume that φ is a substring of

$$\forall x\vartheta)$$

By 1.3.2(ii), the first letter of φ is in $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C} \cup \mathrm{Vbl}$, or an opening bracket. Hence φ is a substring of

$$\vartheta)$$

and by 1.2.7(ii), φ cannot be a terminating segment of that string. Thus, φ is a substring of ϑ as required.

(ii)(b) is an immediate consequence of 1.5.5. $\square$

1.5.7. **Theorem.** *Let ψ be a formula.*

(i) Let φ be a subformula of ψ occurring at position p in ψ. Let δ be another formula. We define a new string γ by replacing the substring φ of ψ occurring at position p with the string δ. Then γ is again a formula.

(ii) Let t be a term occurring in ψ at position p. Let s be another term. We define a new string γ by replacing the substring t of ψ occurring at position p with the string s. Then γ is again a formula.

Proof. (**level 4/MSc**) This is done in question 11 of the example sheets. $\square$

1.6. **Free and bound occurrences of variables.**

In the next section we will write down formulas that we would like to consider as absolutely true (they will be called <u>axioms</u>). Instances of such formulas could be formulas of the form

$$(*) \qquad (\forall x \varphi) \to \varphi(x/y),$$

where φ is a formula of some language and $\varphi(x/y)$ stands for the formula obtained from φ by replacing x with y. It seems plausible that $(*)$ should be an axiom, because a priori $(\forall x\varphi)$ seems to say something stronger than $\varphi(x/y)$. However, there is a glitch here: For an example, consider the formula φ defined as $\exists y\ x \not\doteq y$. If we replace x by y we obtain $\exists y\ y \not\doteq y$ and hence $(*)$ is

$$(\forall x \exists y\ x \not\doteq y) \to (\exists y\ y \not\doteq y).$$

But obviously we do not want this formula to be a universal axiom. The problem here is that we replace x in a position of φ within the *scope of the quantifier* $\exists y$ and this has to be avoided if we want to include formulas of the shape $(*)$ as axioms. So we need to clarify the situation and first address the clause "*scope of the quantifier*".

Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language.

1.6.1. **Definition.** (scope of a quantifier)
The **scope** of a quantifier $\forall x$ in an $\mathscr{L}$-formula φ is the set of all positions of letters in φ, which are captured in a subformula of the form $(\forall x\psi)$ of φ.

More formally: the scope of $\forall x$ in φ is the set of all $k \in \mathbb{N}$ such that there is a subformula of the form $(\forall x\psi)$ of φ, of length $l \in \mathbb{N}$ that occurs at a position p in φ with $p \le k < p + l$.

1.6.2. *Example.* For example, look at the formula φ in the language of ordered groups (cf. 1.1.3):

$$(\forall v_2((\forall v_1\ \circ(e, v_1) \doteq v_5) \to \le(\ ^{-1}(v_1), e))).$$

Here the scope of the quantifier $\forall v_1$ in φ:

$$(\forall v_2(\overbrace{(\forall v_1\ \circ(e, v_1) \doteq v_5)}^{\text{scope of } \forall v_1} \to \le(\ ^{-1}(v_1), e))).$$

1.6.3. **Definition.** (free and bound occurrence of variables)
Let φ be an $\mathscr{L}$-formula and let x be a variable.

(i) If x occurs in φ at position $k \in \mathbb{N}$ and if k is <u>not</u> in the scope of the quantifier $\forall x$ in φ, then we say x occurs **free** in φ at position k.
(ii) If x occurs in φ at position $k \in \mathbb{N}$ and if k is in the scope of the quantifier $\forall x$ in φ, then we say x occurs **bound** in φ at position k.
(iii) x is a **free variable** of φ if there is some $k \in \mathbb{N}$ such that x occurs freely in φ at position k.

The set of free variables of φ is denoted by $\mathrm{Fr}(\varphi)$. It is convenient to extend the notation to terms:

(4) If t is an $\mathscr{L}$-term, then we define $\mathrm{Fr}(t)$ to be the set of all variables occurring in t and we will also say that x is free in t instead of $x \in \mathrm{Fr}(t)$. Notice that there are no variables that are possibly bound in t. If $\mathrm{Fr}(t) = \emptyset$, then t is called a **closed term** or a **constant term**.

Different occurrences of a given variable in a formula may be free or bound, depending on where they are. In example 1.6.2 above, v_1 occurs bound at two positions in φ and free at one position.

$$(\forall v_2((\forall \overbrace{v_1}^{\text{bound occurrence}} \circ(e, \overbrace{v_1}^{\text{bound occurrence}}) \doteq v_5) \to \leq ({}^{-1}(\overbrace{v_1}^{\text{free occurrence}}), e))).$$

We have $\mathrm{Fr}(\varphi) = \{v_1, v_5\}$.

1.6.4. **Lemma.** *Let φ, ψ be $\mathscr{L}$-formulas.*

(i) If φ is quantifier free then $\mathrm{Fr}(\varphi)$ *is the set of variables occurring in φ.*
(ii) $\mathrm{Fr}(\neg\varphi) = \mathrm{Fr}(\varphi)$
(iii) $\mathrm{Fr}((\varphi \to \psi)) = \mathrm{Fr}(\varphi) \cup \mathrm{Fr}(\psi)$.
(iv) $\mathrm{Fr}(\forall x\varphi) = \mathrm{Fr}(\varphi) \setminus \{x\}$ *for all* $x \in \mathrm{Vbl}$.

Proof. (**level 4/MSc**) (i). If φ is quantifier free, then by definition of 'free occurrence', every occurrence of a variable in φ is a free occurrence.

(ii). By definition of $\mathrm{Fr}(\varphi)$ it is enough to show for each $k \in \mathbb{N}$ and each variable x the following:

x occurs in φ at position k and k is in the scope of $\forall x$ in φ if and only if
x occurs in $(\neg\varphi)$ at position $k+2$ and $k+2$ is in the scope of $\forall x$ in $(\neg\varphi)$.

$\Rightarrow$. If x occurs in φ at position k and k is in the scope of $\forall x$ in φ, then there is a subformula of the form $(\forall x\delta)$ of φ, of length $l \in \mathbb{N}$ that occurs at a position p in φ with $p \leq k < p + l$. Obviously x occurs in $(\neg\varphi)$ at position $k+2$ and by using 1.1.5 we see that $k+2$ is in the scope of $\forall x$ in $(\neg\varphi)$.

$\Leftarrow$. If x occurs in $(\neg\varphi)$ at position $k+2$ and $k+2$ is in the scope of $\forall x$ in $(\neg\varphi)$, then there is a subformula of the form $(\forall x\delta)$ of $(\neg\varphi)$, of length $l \in \mathbb{N}$ that occurs at a position p in $(\neg\varphi)$ with $p \leq k+2 < p+l$. Obviously $(\forall x\delta)$ is different from $(\neg\varphi)$ and so $(\forall x\delta)$ is a subformula of φ. This shows the assertion.

(iii) is left as question 13.

(iv). By definition, x is not in $\mathrm{Fr}(\forall x\varphi)$. Hence it is enough to show for each variable $y \neq x$ and every $k \in \mathbb{N}$ the following:

y occurs in φ at position k and k is in the scope of $\forall y$ in φ if and only if
y occurs in $(\forall x\varphi)$ at position $k+3$ and $k+3$ is in the scope of $\forall y$ in $(\forall x\varphi)$.

This is shown similarly to the proof of (ii) using $y \neq x$ for the implication "$\Leftarrow$":

$\Rightarrow$. If y occurs in φ at position k and k is in the scope of $\forall y$ in φ, then there is a subformula of the form $(\forall y\delta)$ of φ, of length $l \in \mathbb{N}$ that occurs at a position p in φ with $p \leq k < p+l$. Obviously y occurs in $(\forall x\varphi)$ at position $k+3$ and by using 1.1.5 we see that $k+3$ is in the scope of $\forall y$ in $(\forall x\varphi)$.

$\Leftarrow$. If y occurs in $(\forall x\varphi)$ at position $k+3$ and $k+3$ is in the scope of $\forall y$ in $(\forall x\varphi)$, then there is a subformula of the form $(\forall y\delta)$ of φ, of length $l \in \mathbb{N}$ that occurs at a position p in $(\forall x\varphi)$ with $p \leq k+3 < p+l$. Since $y \neq x$, $(\forall y\delta)$ is different from $(\forall x\varphi)$ and so $(\forall y\delta)$ is a subformula of φ. Now we use 1.1.5 again to show the assertion. □

1.6.5. *Notation.*

- The expressions '$t(x_1,...,x_n) \in \mathrm{tm}(\mathscr{L})$' or 'let $t(x_1,...,x_n)$ be an $\mathscr{L}$-term' are shorthand for

  "$t \in \mathrm{tm}(\mathscr{L}), x_1,...,x_n \in \mathrm{Vbl}$ with $x_i \neq x_j$ $(i \neq j)$ and $\mathrm{Fr}(t) \subseteq \{x_1,...,x_n\}$".

  This is common practice in mathematics, for example a polynomial in two variables is also considered as a polynomial in three variables.
- The expressions '$\varphi(x_1,...,x_n) \in \mathrm{Fml}(\mathscr{L})$' or 'let $\varphi(x_1,...,x_n)$ be an $\mathscr{L}$-formula' are shorthand for

  "$\varphi \in \mathrm{Fml}(\mathscr{L}), x_1,...,x_n \in \mathrm{Vbl}$ with $x_i \neq x_j$ $(i \neq j)$ and $\mathrm{Fr}(\varphi) \subseteq \{x_1,...,x_n\}$".

1.6.6. **Definition.** Let φ be an $\mathscr{L}$-formula.

(i) Let x, y be variables. We define

x **is free in** φ **for** y or y **is substitutable for** x **in** φ

if no position of φ at which x occurs freely in φ, is in the scope of the quantifier $\forall y$ in φ.

(ii) Let t be an $\mathscr{L}$-term. We define

x **is free in** φ **for** t or t **is substitutable for** x **in** φ

if x is free in φ for every variable that occurs in t.

So by definition, each variable x is free for x in φ and each variable which does not occur in φ is free in φ for every term.

In example 1.6.2, i.e.

$$\varphi = (\forall v_2((\forall v_1 \ \circ(e, v_1) \doteq v_5) \to \leq (\ ^{-1}(v_1), e))),$$

v_1 is free for v_5 but not free for v_2 in φ; v_5 is not free for the term $\circ(v_2, v_5)$.

1.6.7. **Definition.** Let $\varphi \in \mathrm{Fml}(\mathscr{L})$, $t_1,...,t_n, t \in \mathrm{tm}(\mathscr{L})$ and let $x_1,...,x_n$ be n distinct variables.

(i) The expression $t(x_1/t_1,...,x_n/t_n)$ denotes the string obtained from t by replacing simultaneously every occurrence of x_i in t with the string t_i $(1 \leq i \leq n)$.

(ii) If for each $i \in \{1,...,n\}$ the variable x_i is free in φ for t_i then the expression $\varphi(x_1/t_1,...,x_n/t_n)$ denotes the string obtained from φ by simultaneously replacing every <u>free</u> occurrence of x_i in φ with the string t_i $(1 \leq i \leq n)$. We call $\varphi(x_1/t_1,...,x_n/t_n)$ the **substitution** of $x_1,...,x_n$ by $t_1,...,t_n$ in φ.

Warning. Notice that we replace the variables x_i by the terms t_i <u>simultaneously</u> and not consecutively: For example if φ is $(\forall x_2 \ x_1 \doteq x_2) \to x_2 \doteq x_3$, then $\varphi(x_1/t_1, x_2/t_2)$ is $(\forall x_2 \ t_1 \doteq x_2) \to t_2 \doteq x_3$.

However, in general $\varphi(x_1/t_1, x_2/t_2)$ is NOT the same as $\varphi(x_1/t_1)(x_2/t_2)$. Why?

1.6.8. **Lemma.** *Let $\varphi \in \mathrm{Fml}(\mathscr{L})$, $t_1,...,t_n, t \in \mathrm{tm}(\mathscr{L})$ and let $x_1,...,x_n$ be n distinct variables.*

(i) $t(x_1/t_1,...,x_n/t_n)$ is an $\mathscr{L}$-Term and if $\mathrm{Fr}(t) \subseteq \{x_1,...,x_n\}$, then

$$\mathrm{Fr}(t(x_1/t_1,...,x_n/t_n)) \subseteq \mathrm{Fr}(t_1) \cup ... \cup \mathrm{Fr}(t_n).$$

(ii) If for each $i \in \{1,...,n\}$ the variable x_i is free in φ for t_i then the string $\varphi(x_1/t_1,...,x_n/t_n)$ is an $\mathscr{L}$-formula and in the case $\mathrm{Fr}(\varphi) \subseteq \{x_1,...,x_n\}$ we have

$$\mathrm{Fr}(\varphi(x_1/t_1,...,x_n/t_n)) \subseteq \mathrm{Fr}(t_1) \cup ... \cup \mathrm{Fr}(t_n).$$

Proof. (**level 4/MSc**) This is done in Question 14 of the example sheets. □

2. Formal Proofs

Throughout, $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ denotes a formal language.

2.1. Logical axioms and the definition of a formal proof.

2.1.1. **Definition.** Each of the following $\mathscr{L}$-formulas are called **logical Axioms** (of $\mathscr{L}$), where φ, ψ and γ are $\mathscr{L}$-formulas:

(AxProp):

(a) $\varphi \to (\psi \to \varphi)$
(b) $(\varphi \to (\psi \to \gamma)) \to ((\varphi \to \psi) \to (\varphi \to \gamma))$
(c) $((\neg\psi) \to (\neg\varphi)) \to (((\neg\psi) \to \varphi) \to \psi)$

(Ax$\forall\to$): $\forall x(\varphi \to \psi) \ \to \ (\forall x\varphi \ \to \ \forall x\psi)$

(AxSubst): $(\forall x\varphi) \ \to \ \varphi(x/t)$, where x is free in φ for $t \in \mathrm{tm}(\mathscr{L})$.

(AxGen): $\varphi \to \forall x\varphi$, where x is <u>not</u> free in φ.

(AxEq): For every $\mathscr{L}$-term t, every n-ary relation symbol R and all variables $x_1, ..., x_n, x, y, z$ the axioms

(a) $x \doteq x$
(b) $x \doteq y \wedge y \doteq z \ \to \ z \doteq x$
(c) $x \doteq y \to t(z/y) \doteq t(z/x)$
(d) $x \doteq y \to \Big(R(x_1, ..., x_n)(z/x) \leftrightarrow (R(x_1, ..., x_n)(z/y))\Big)$

(Ax$\forall$): Any formula of the form

$$\forall x_1 ... x_n \ \varphi,$$

where φ is one of the formulas introduced by the other logical axiom schemes above and $x_1, ..., x_n \in \mathrm{Vbl}$.

2.1.2. **Definition.** Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$. A **formal proof** or a **deduction** from Σ (in $\mathscr{L}$) is a finite sequence $(\varphi_1, ..., \varphi_n)$ of $\mathscr{L}$-formulas such that for each $k \in \{1, ..., n\}$ one of the following conditions hold:

(PR1): φ_k is a logical axiom (of $\mathscr{L}$) or

(PR2): $\varphi_k \in \Sigma$ or

(PR3): (Modus Ponens) There are $i, j < k$ such that

$$\varphi_j \text{ is the formula } \varphi_i \to \varphi_k.$$

If $\Phi \subseteq \mathrm{Fml}(\mathscr{L})$, then we say Σ **proves** Φ and write

$$\Sigma \vdash_{\mathscr{L}} \Phi \text{ or } \Sigma \vdash \Phi \text{ when } \mathscr{L} \text{ is clear from the context,}$$

if every $\varphi \in \Phi$ is an entry of a proof from Σ. If $\Phi = \{\varphi\}$ we just write $\Sigma \vdash \varphi$. If $\Sigma = \emptyset$ we just write $\vdash \Phi$.

2.1.3. *Remark.* The following are immediate consequences of definition 2.1.2:

(i) If $(\varphi_1, ..., \varphi_n)$ is a proof from Σ, then also $(\varphi_1, ..., \varphi_m)$ is a proof from Σ for every $m \leq n$. Moreover $\varphi_1 \in \Sigma$ or φ_1 is a logical axiom.

(ii) If $(\varphi_1, ..., \varphi_n)$ and $(\psi_1, ..., \psi_m)$ are proofs from Σ, then also $(\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m)$ is a proof from Σ.

(iii) $\Sigma \vdash \Phi \iff \Sigma \vdash \varphi$ for all $\varphi \in \Phi$.

(iv) $\Sigma \vdash \Phi$ for all $\Phi \subseteq \Sigma$.

Proof. To get used to definition 2.1.2, please carry this out; see Question 15 of the example sheets. □

2.1.4. **Corollary.** *(Modus Ponens for proofs)*

$$\textit{If } \Sigma \vdash \varphi \to \psi \textit{ and } \Sigma \vdash \varphi \textit{ then } \Sigma \vdash \psi.$$

Proof. By 2.1.3(ii) and Modus Ponens. □

We say $(\varphi_1, ..., \varphi_n)$ is a **proof of** φ **from** Σ if $(\varphi_1, ..., \varphi_n)$ is a proof from Σ and $\varphi = \varphi_n$.

The following theorem follows again immediately from definition 2.1.2 of a formal proof. It is of central importance to Predicate Logic and used in many places later on.

2.1.5. **Theorem.** *"proofs are finite"*
For every $\varphi \in \mathrm{Fml}(\mathscr{L})$ and all subsets $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ the following are equivalent:

(i) $\Sigma \vdash \varphi$

(ii) there is a finite subset $\Sigma_0 \subseteq \Sigma$ with $\Sigma_0 \vdash \varphi$

(iii) there is a proof $(\varphi_1, ..., \varphi_n)$ of φ from Σ.

Proof. Again, please carry this out and use it to for practicing definition 2.1.2; see Question 15 of the example sheets. □

2.1.6. **Lemma.** *(Transitivity of proofs)*

$$\textit{If } \Sigma, \Phi, \Psi \subseteq \mathrm{Fml}(\mathscr{L}) \textit{ with } \Sigma \vdash \Phi \textit{ and } \Phi \vdash \Psi \textit{ then } \Sigma \vdash \Psi.$$

Proof. Since $\Phi \vdash \Psi$, it suffices to show by induction on n the following:

$$\text{If } (\psi_1, ..., \psi_n) \text{ is a proof from } \Phi, \text{ then } \Sigma \vdash \psi_n.$$

If $n = 1$, then ψ_1 is a logical axiom or $\psi_1 \in \Phi$. In each case $\Sigma \vdash \psi_1$, using our assumption $\Sigma \vdash \Phi$.

For the induction step, suppose we are given a proof $(\psi_1, ..., \psi_{n+1})$ from Φ. If $\psi_{n+1} \in \Phi$ or ψ_{n+1} is a logical axiom then $\Sigma \vdash \psi_{n+1}$ again, using our assumption $\Sigma \vdash \Phi$. Otherwise there are $i, j \leq n$ such that ψ_j is $\psi_i \to \psi_{n+1}$. By the induction hypothesis and because $(\psi_1, ..., \psi_i)$ and $(\psi_1, ..., \psi_j)$ are proofs from Φ, we know $\Sigma \vdash \psi_i$ and $\Sigma \vdash \psi_j$. As ψ_j is $\psi_i \to \psi_{n+1}$ we may apply 2.1.4 to obtain $\Sigma \vdash \psi_{n+1}$. □

2.1.7. **Proposition.** *(Propositional Tautologies)*
Let α be a formula of propositional logic (cf. [PropLog, 1.2]*) and let $n \in \mathbb{N}$ be such that every atomic formula of the propositional calculus (cf.* [PropLog, 1.1]*) that occurs in α is among $A_0, ..., A_n$.*

Let $\varphi_0, ..., \varphi_n$ be $\mathscr{L}$-formulas and let ψ be the $\mathscr{L}$-string obtained from α by replacing for each $i \in \{0, ..., n\}$ the letter A_i in α with the string φ_i.

Then ψ is again an $\mathscr{L}$-formula and if α is a tautology of propositional logic (cf. [PropLog, 2.6]*), then*

$$\vdash_{\mathscr{L}} \psi.$$

Sketch of the proof. That $\psi \in \mathrm{Fml}(\mathscr{L})$ follows by induction from the definition of formulas in propositional logic (see [PropLog, 1.2]).

If α is a tautology, then by the completeness theorem for propositional logic (cf. [PropLog, 4.14]) we know that there is a formal proof of α in propositional logic (cf. [PropLog, 3.5]) from $\emptyset$. Now by inspection we see that each formal proof from $\emptyset$ in propositional logic translates into a formal proof from $\emptyset$ from predicate logic when we replace propositional variables by $\mathscr{L}$-formulas. □

2.1.8. *Example.* If $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and $\varphi_1, ..., \varphi_n \in \mathrm{Fml}(\mathscr{L})$, then

$$\Sigma \vdash \varphi_1 \wedge ... \wedge \varphi_n \iff \Sigma \vdash \varphi_1 \text{ and ... and } \Sigma \vdash \varphi_n.$$

The implication "$\Rightarrow$" is obtained from 2.1.7 and the fact that $A_1 \wedge ... \wedge A_n \to A_i$ is a propositional tautology. The implication "$\Leftarrow$" is left as question 17 using 2.1.7 after the Deduction theorem 2.2.1 below.

2.2. **Deduction and Generalisation.**

2.2.1. **Theorem.** *(Deduction Theorem)*
Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and let $\varphi, \psi \in \mathrm{Fml}(\mathscr{L})$. Then

$$\Sigma \vdash \varphi \to \psi \quad \iff \quad \Sigma \cup \{\varphi\} \vdash \psi.$$

Proof. The implication $\Rightarrow$ holds by Modus Ponens for proofs (cf. 2.1.4).
In order to prove the implication $\Leftarrow$ it suffices to show by induction on n the following:

> If $(\varphi_1, ..., \varphi_n)$ is a proof from $\Sigma \cup \{\varphi\}$, then the sequence
> $(\varphi \to \varphi_1, ..., \varphi \to \varphi_n)$ can be completed to a proof from Σ.

If $n = 1$ and (φ_1) is a proof from $\Sigma \cup \{\varphi\}$, then φ_1 is a logical axiom or $\varphi_1 \in \Sigma \cup \{\varphi\}$. If $\varphi_1 = \varphi$, then $\vdash \varphi_1 \to \varphi$ by 2.1.7, which implies the assertion. If φ_1 is a logical axiom or $\varphi_1 \in \Sigma$, then

$$(\varphi_1 \to (\varphi \to \varphi_1), \varphi_1, \varphi \to \varphi_1)$$

is a proof from Σ, since $\varphi_1 \to (\varphi \to \varphi_1)$ is an instance of 2.1.1, (**AxProp**)(a).

For the induction step, suppose $(\varphi_1, ..., \varphi_{n+1})$ is a proof from $\Sigma \cup \{\varphi\}$. By the induction hypothesis there is a proof $(\psi_1, ..., \psi_m)$ from Σ which contains $(\varphi \to \varphi_1, ..., \varphi \to \varphi_n)$ (in this order) as a subsequence.

<u>Case 1.</u> φ_{n+1} is a logical axiom or $\varphi_{n+1} \in \Sigma \cup \{\varphi\}$.

Then (φ_{n+1}) is already a proof from $\Sigma \cup \{\varphi\}$ and as seen in the case $n = 1$, $(\psi_1, ..., \psi_m)$ can be extended to a proof from Σ with last entry $\varphi \to \varphi_{n+1}$.

<u>Case 2.</u> There are $k, l \leq n$ such that φ_l is $\varphi_k \to \varphi_{n+1}$.

Then the sequence $(\psi_1, ..., \psi_m)$ has $\varphi \to \varphi_k$ and $\varphi \to (\varphi_k \to \varphi_{n+1})$ as entries. Let ϑ be the formula

$$(\varphi \to (\varphi_k \to \varphi_{n+1})) \;\to\; ((\varphi \to \varphi_k) \to (\varphi \to \varphi_{n+1})).$$

This is an instance of 2.1.1, (**AxProp**)(b). Hence ϑ is a logical axiom and

$$(\psi_1, ..., \psi_m, \vartheta, (\varphi \to \varphi_k) \to (\varphi \to \varphi_{n+1}), \varphi \to \varphi_{n+1})$$

is a proof from Σ (Modus Ponens is applied twice). $\square$

2.2.2. **Theorem.** *(Generalisation Theorem)*
Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ *and* $\varphi \in \mathrm{Fml}(\mathscr{L})$. *If* $x \in \mathrm{Vbl}$ *does not occur free in any formula from* Σ *then*

$$\Sigma \vdash \varphi \iff \Sigma \vdash \forall x \varphi.$$

Proof. The implication $\Leftarrow$ holds even without the assumption that x does not occur free in any formula from Σ by 2.1.1 (**AxSubst**) applied with $x = t$ and Modus Ponens for proofs. The crucial statement here is the implication $\Rightarrow$, which we prove by induction on the length of a proof of φ from Σ:

If φ is a logical axiom, then by 2.1.1, (**Ax**$\forall$) we know that also $\forall x \varphi$ is a logical axiom, thus $\Sigma \vdash \forall x \varphi$. If $\varphi \in \Sigma$, then by assumption, x does not occur free in φ. Hence by 2.1.1, (**AxGen**), $\varphi \to \forall x \varphi$ is a logical axiom. Applying Modus Ponens for proofs we get $\Sigma \vdash \forall x \varphi$.

For the induction step, suppose $(\varphi_1, ..., \varphi_{n+1})$ is a proof from Σ with $\varphi = \varphi_{n+1}$. From what we have seen above we may assume that φ is neither a logical axiom nor an element of Σ. Hence there are $i, j \leq n$ such that φ_j is $\varphi_i \to \varphi_{n+1}$. By 2.1.1, (**Ax**$\forall \to$) we have that

$$\forall x(\varphi_i \to \varphi_{n+1}) \ \to \ (\forall x \varphi_i \ \to \ \forall x \varphi_{n+1})$$

is a logical axiom. By the induction hypothesis we know $\Sigma \vdash \forall x \varphi_j$, in other words $\Sigma \vdash \forall x(\varphi_i \to \varphi_{n+1})$. So by Modus Ponens for proofs we get $\Sigma \vdash \forall x \varphi_i \ \to \ \forall x \varphi_{n+1}$. Since also $\Sigma \vdash \forall x \varphi_i$ by induction we may apply Modus Ponens for proofs again to obtain $\Sigma \vdash \forall x \varphi_{n+1}$. $\square$

3. Structures

3.1. **Definition of $\mathscr{L}$-structures and validity of formulas.**

Throughout, $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ denotes (the similarity type of) a formal language.

3.1.1. **Definition.** An **$\mathscr{L}$-structure** is a tuple

$$\mathscr{M} = \Big(M \ , \ (R^{\mathscr{M}} \mid R \in \mathscr{R}) \ , \ (F^{\mathscr{M}} \mid F \in \mathscr{F}) \ , \ (c^{\mathscr{M}}, c \in \mathscr{C})\Big)$$

consisting of

(S1) A nonempty set M, called the **universe** or the **domain** or the **carrier** of $\mathscr{M}$. We shall also write $|\mathscr{M}|$ instead of M.

(S2) A family $(R^{\mathscr{M}} \mid R \in \mathscr{R})$ of relations of M such that for $R \in \mathscr{R}$, $R^{\mathscr{M}} \subseteq M^{\lambda(R)}$. Hence $R^{\mathscr{M}}$ is a $\lambda(R)$-ary relation of M, called the **interpretation of R in $\mathscr{M}$**. Observe that for different $R_1, R_2 \in \mathscr{R}$ we may have $R_1^{\mathscr{M}} = R_2^{\mathscr{M}}$.
Formally, $(R^{\mathscr{M}} \mid R \in \mathscr{R})$ is a map $\mathscr{R} \longrightarrow \bigcup_{n\in\mathbb{N}} \mathcal{P}(M^n)$ such that the image $R^{\mathscr{M}}$ of $R \in \mathscr{R}$ under this map is a subset of $M^{\lambda(R)}$.

(S3) A family $(F^{\mathscr{M}} \mid F \in \mathscr{F})$ of functions, where for $F \in \mathscr{F}$, $F^{\mathscr{M}} : M^{\mu(F)} \longrightarrow M$. Hence $F^{\mathscr{M}}$ is a $\mu(F)$-ary function of M, called the **interpretation of F in $\mathscr{M}$**. Observe that for different $F_1, F_2 \in \mathscr{F}$ we may have $F_1^{\mathscr{M}} = F_2^{\mathscr{M}}$.
Formally, $(F^{\mathscr{M}} \mid F \in \mathscr{F})$ is a map $\mathscr{F} \longrightarrow \bigcup_{n\in\mathbb{N}} \mathrm{Maps}(M^n, M)$ such that the image $F^{\mathscr{M}}$ of $F \in \mathscr{F}$ under this map is a function $M^{\mu(F)} \longrightarrow M$.

(S4) A family $(c^{\mathscr{M}} \mid c \in \mathscr{C})$ of elements of M. Hence $c^{\mathscr{M}}$ is an element of M, called the **interpretation of c in $\mathscr{M}$**. Observe that for different $c_1, c_2 \in \mathscr{C}$ we may have $c_1^{\mathscr{M}} = c_2^{\mathscr{M}}$.
Formally, $(c^{\mathscr{M}} \mid c \in \mathscr{C})$ is simply a map $\mathscr{C} \longrightarrow M$.

$\mathscr{M}$ is called **finite/countable/uncountable/infinite** if its universe $|\mathscr{M}|$ is finite/countable/uncountable/infinite. When $\mathscr{M}$ is finite, we say $\mathscr{M}$ is **of size** $k \in N$ if $|\mathscr{M}|$ is of size k.

3.1.2. **Definition.** An **assignment** or a **valuation** of an $\mathscr{L}$-structure $\mathscr{M}$ is a map

$$h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|.$$

In the literature, also the pair (M, h) is called a valuation. Given an assignment h of $\mathscr{M}$, a variable x and an element $a \in |\mathscr{M}|$ we denote by $h(\begin{smallmatrix} x \\ a \end{smallmatrix})$ the assignment of $\mathscr{M}$ which differs from h only at the variable x, with value a at x:

$$h(\begin{matrix} x \\ a \end{matrix})(y) = \begin{cases} h(y) & \text{if } y \neq x \\ a & \text{if } y = x. \end{cases}$$

3.1.3. **Definition.** Let $\mathscr{M}$ be an $\mathscr{L}$-structure with domain M.

(A) We define by induction on the complexity of an $\mathscr{L}$-term t an element $t^{\mathscr{M}}[h] \in M$ for each assignment h of $\mathscr{M}$ as follows:

(i) If $c(t) = 0$, then

$$t^{\mathscr{M}}[h] = \begin{cases} t^{\mathscr{M}} & \text{if } t \in \mathscr{C} \\ h(t) & \text{if } t \in \mathrm{Vbl}\,. \end{cases}$$

(ii) If $t_1, ..., t_n$ are $\mathscr{L}$-terms and $F \in \mathscr{F}$ with $\mu(F) = n$, then we define

$$F(t_1, ..., t_n)^{\mathscr{M}}[h] := F^{\mathscr{M}}(t_1^{\mathscr{M}}[h], ..., t_n^{\mathscr{M}}[h]).$$

Notice that this is well defined by the unique readability theorem 1.2.4 for terms.

(B) We define by induction on the complexity of an $\mathscr{L}$-formula φ and each assignment h of $\mathscr{M}$, the expression φ **holds in** $\mathscr{M}$ **at** h, or φ **is valid in** $\mathscr{M}$ **at** h, or $\mathscr{M}$ **satisfies** φ **at** h, denoted by

$$\mathscr{M} \models \varphi[h],$$

as follows:

(i) If φ is of the form $t_1 \doteq t_2$ with $\mathscr{L}$-terms t_1, t_2 then

$$\mathscr{M} \models t_1 \doteq t_2\ [h] \iff t_1^{\mathscr{M}}[h] = t_2^{\mathscr{M}}[h].$$

If φ is of the form $R(t_1, ..., t_n)$ with $R \in \mathscr{R}$ of arity n and $t_1, ..., t_n \in \mathrm{tm}(\mathscr{L})$ then

$$\mathscr{M} \models R(t_1, ..., t_n)\ [h] \iff (t_1^{\mathscr{M}}[h], ..., t_n^{\mathscr{M}}[h]) \in R^{\mathscr{M}}.$$

Notice that this is well defined by the unique readability theorem 1.3.4 for formulas.

(ii) For the induction step we take $\varphi, \psi \in \mathrm{Fml}(\mathscr{L})$, $x \in \mathrm{Vbl}$ and define

- $\mathscr{M} \models (\varphi \to \psi)[h] \iff$ if $\mathscr{M} \models \varphi[h]$ then $\mathscr{M} \models \psi[h]$,

- $\mathscr{M} \models (\neg\varphi)[h] \iff \mathscr{M} \not\models \varphi[h]$ i.e. $\mathscr{M} \models \varphi[h]$ does not hold

and

- $\mathscr{M} \models (\forall x \varphi)[h] \iff$ for all $a \in |\mathscr{M}|$ we have $\mathscr{M} \models \varphi[h(\begin{smallmatrix} x \\ a \end{smallmatrix})]$.

Again notice that these definitions are correct by the unique readability theorem 1.3.4 for formulas.

(C) Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and let h be an assignment of $\mathscr{M}$. $\mathscr{M}$ is called a **model of** Σ **at** h if

$$\mathscr{M} \models \sigma[h] \text{ for all } \sigma \in \Sigma.$$

We denote this by

$$\mathscr{M} \models \Sigma[h].$$

Some authors also use $\mathscr{M} \models_h \Sigma$ instead of $\mathscr{M} \models \Sigma[h]$. We say that Σ has a model if it has a model at some assignment. In this case, Σ is called **satisfiable**.

Before giving examples we include a lemma that says that the expression $\mathscr{M} \models \varphi[h]$ only depends on the value of the assignment h at the free variables of φ.

3.1.4. **Lemma.** *Let $\varphi(x_1,...,x_n) \in \mathrm{Fml}(\mathscr{L})$, $t(x_1,...,x_n) \in \mathrm{tm}(\mathscr{L})$ and let $\mathscr{M}$ be an $\mathscr{L}$-structure. If h and h' are assignments of $\mathscr{M}$ with $h(x_i) = h'(x_i)$ $(1 \le i \le n)$ then $t^{\mathscr{M}}[h] = t^{\mathscr{M}}[h']$ and*

$$\mathscr{M} \models \varphi[h] \iff \mathscr{M} \models \varphi[h'].$$

Proof. $t^{\mathscr{M}}[h] = t^{\mathscr{M}}[h']$ follows from a straightforward induction on the complexity of t (please verify this as part of question 26). The assertion $\mathscr{M} \models \varphi[h] \;\Rightarrow\; \mathscr{M} \models \varphi[h']$ follows by a straightforward induction on the complexity of φ using 1.6.4, except for one case (please verify the other cases as part of question 26):

Suppose φ is $\forall y\psi$ and we know already the implication $\mathscr{M} \models \psi[h] \;\Rightarrow\; \mathscr{M} \models \psi[h']$ for all assignments h, h' of $\mathscr{M}$ that are identical on $\mathrm{Fr}(\psi)$. We show $\mathscr{M} \models \varphi[h] \;\Rightarrow\; \mathscr{M} \models \varphi[h']$ for assignments $h.h'$ that are identical on $\{x_1,...,x_n\}$. If $\mathscr{M} \models \varphi[h]$ then $\mathscr{M} \models \psi[h\binom{y}{a}]$ for all $a \in |\mathscr{M}|$. Now $h\binom{y}{a}(x_i) = h'\binom{y}{a}(x_i)$ $(1 \le i \le n)$ and both assignments are identical on y. Hence $h\binom{y}{a}(z) = h'\binom{y}{a}(z)$ for each free variable z of ψ (cf. 1.6.4(iv)) and from the induction hypothesis we get $\mathscr{M} \models \psi[h'\binom{y}{a}]$ for all $a \in |\mathscr{M}|$. This in turn shows $\mathscr{M} \models \varphi[h']$. $\square$

Notation. If no variable occurs in $t \in \mathrm{tm}(\mathscr{L})$ then by 3.1.4, for any $\mathscr{L}$-structure $\mathscr{M}$ we may denote by $t^{\mathscr{M}}$ the element $t^{\mathscr{M}}[h]$ of $|\mathscr{M}|$, where h is some assignment of $\mathscr{M}$.

Similarly, if $\varphi \in \mathrm{Fml}(\mathscr{L})$ with $\mathrm{Fr}\,\varphi = \emptyset$, we may write $\mathscr{M} \models \varphi$ for $\mathscr{M} \models \varphi[h]$ for some assignment h of $\mathscr{M}$.

3.1.5. **Definition.** ($\mathscr{L}$-sentence)
An $\mathscr{L}$-formula φ is called an **$\mathscr{L}$-sentence**, if $\mathrm{Fr}(\varphi) = \emptyset$, i.e. no variable occurs freely in φ. The set of all $\mathscr{L}$-sentences is denoted by $\mathrm{Sen}(\mathscr{L})$.

If $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ and $\mathscr{M}$ is an $\mathscr{L}$-structure, then by 3.1.4 the expression $\mathscr{M} \models \Sigma[h]$ is independent of the assignment h of $\mathscr{M}$. Hence in this case we may simply write

$$\mathscr{M} \models \Sigma.$$

and call $\mathscr{M}$ a **model of** Σ.

3.1.6**. Examples.** Note that in order to state an example of a structure we have to do three things:

(a) First we have to specify the language we are working in.
(b) We need to specify what the universe of the structure is.
(c) We need to specify what the interpretation of the non-logical symbols in the language are supposed to be.

1. An example (denoted by $\mathscr{M}$) in the language of a binary relation.
  (a) Here the language has exactly one binary relation symbol R and no other non-logical symbols.
  (b) We choose the universe of $\mathscr{M}$ to be the powerset of $\mathbb{C}$, denoted by $\mathfrak{P}(\mathbb{C})$.
  (c) We interpret R in $\mathscr{M}$ as the inclusion between subsets of $\mathbb{C}$. More formally if A, B are in the universe of $\mathscr{M}$ we define

$$R^{\mathscr{M}} = \{(A,B) \in \mathfrak{P}(\mathbb{C}) \times \mathfrak{P}(\mathbb{C}) \mid A \subseteq B\}$$

The sentence φ defined as $\forall x\forall y\forall z(R(x,y) \wedge R(y,z) \to R(x,z))$, is true in $\mathscr{M}$ - this is just saying that the relation $R^{\mathscr{M}}$ on $\mathfrak{P}(\mathbb{C})$ is transitive. In fact $\mathscr{M}$ is a partially ordered set, i.e. $\mathscr{M}$ satisfies the three axioms of a partially ordered set. (Recall that when we want to check if a sentence is true in a structure, all quantifiers range over the universe of that structure.)

2. Another example in the language of a binary relation.
(a) Again the language is the language of a binary relation symbol R.
(b) We choose the universe of $\mathscr{M}$ to be the set $\mathbb{Z}$ of integers.
(c) We interpret R in $\mathscr{M}$ as the set

$$R^{\mathscr{M}} = \{(k, k+1) \in \mathbb{Z} \mid k \in \mathbb{Z}\}.$$

The sentence φ from the previous example is not true in $\mathscr{M}$. But the sentence

$$\Big(\forall x \exists y R(x,y)\Big) \wedge \Big(\forall x \forall y \forall z (R(x,y) \wedge R(x,z) \rightarrow y = z)\Big)$$

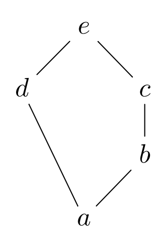
is true. It says that R is interpreted as the graph of a function from the universe of the structure to the universe of the structure.

3. An example in the language of a unary function symbol.
(a) Here the language has no relation symbols, no constant symbols and exactly one function symbol f and this is unary (i.e., $\mu(f) = 1$).
(b) We choose the universe of $\mathscr{M}$ to be the set $\mathbb{Z}$ of integers.
(c) We interpret f in $\mathscr{M}$ as the function $f^{\mathscr{M}} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by

$$f^{\mathscr{M}}(k) = k + 1.$$

This structure is very similar to the structure in item **2.**, but notice it is a structure in a different language, and so it is indeed different from the structure in the previous item.

4. An example of a finite lattice
(a) Here the language has the non-logical symbols $R, \top$ and $\bot$, where R is a binary relation symbol and $\top$ (called "top") and $\bot$ (called "bottom") are constant symbols.
(b) We choose five elements a, b, c, d, e (these could be the numbers 1,2,3,4,5) and let the universe of $\mathscr{M}$ consist of these 5 elements.
(c) We interpret $\top^{\mathscr{M}} = e$, $\bot^{\mathscr{M}} = a$ and we interpret R as the partially ordered set (aka poset) given by the Hasse diagram



Recall that by definition of a poset, this also forces the pairs $(a,c), (a,e)$ and (b,e) to be in $\mathscr{R}^M$. The structure $\mathscr{M}$ is a poset with a largest and a smallest element. Further, in $\mathscr{M}$ the following sentence is true:

$$\forall x \forall y \exists z \Big(R(z,x) \wedge R(z,y) \wedge \forall u (R(u,x) \wedge R(u,y) \rightarrow R(u,z)) \Big)$$

This says that any two elements in the (universe of the) poset $\mathscr{M}$ have a greatest lower bound for the binary relation $R^{\mathscr{M}}$ (such posets are called "lattices").

5. An example of a group and a designated subgroup with a constant symbol.
(a) Here the language "is" $\mathscr{L} = \{f, R, c\}$, where f is a binary function symbol, R is a unary relation symbol and c is a constant symbol.
(b) We choose the universe of $\mathscr{M}$ to be the set $\mathbb{R}^{>0}$ of positive real numbers.
(c) We interpret f in $\mathscr{M}$ as the multiplication function $f^{\mathscr{M}} : \mathbb{R}^{>0} \times \mathbb{R}^{>0} \longrightarrow \mathbb{R}^{>0}$. The relation symbol R is interpreted in $\mathscr{M}$ as $R^{\mathscr{M}} = \mathbb{Q}^{>0}$ The constant symbol c is interpreted in $\mathscr{M}$ as $c^{\mathscr{M}} = 1$.

In $\mathscr{M}$ the following sentences are true:

$$\forall x \forall y \Big((R(x) \wedge R(y)) \to R(f(x,y)) \Big)$$

$$\forall x \Big(R(x) \to \exists y (R(y) \wedge f(x,y) \doteq 1) \Big).$$

This just says that $R^{\mathscr{M}}$ is the (universe of) a subgroup of $(\mathbb{R}^{>0}, \cdot)$.

6. An example of an ordered ring and a designated subring with two constant symbols.
(a) Here the language is $\mathscr{L} = \{f, g, R, c, d\}$, where f, g are binary function symbols, R is a unary relation symbol and c, d are constant symbols.
(b) We choose the universe of $\mathscr{M}$ to be the set $\mathbb{R}$ of real numbers.
(c) We interpret f in $\mathscr{M}$ as the addition function $f^{\mathscr{M}} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and g as the multiplication function $g^{\mathscr{M}} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. The relation symbol R is interpreted in $\mathscr{M}$ as the nonnegative, rational numbers. The constant symbol c is interpreted in $\mathscr{M}$ as $c^{\mathscr{M}} = 0$ and d is interpreted as $d^{\mathscr{M}} = 1$.

Consider the following formula φ of $\mathscr{L}$:

$$\begin{aligned} &\exists z \ (z \not\doteq c \wedge f(x, g(z,z)) \doteq y) \ \longrightarrow \\ &\exists u \ \Big(R(u) \wedge (\exists z f(x, g(z,z)) \doteq u) \wedge (\exists z f(u, g(z,z)) \doteq y) \Big). \end{aligned}$$

In question 25 you are supposed to work out all assignments h of $\mathscr{M}$ for which $\mathscr{M} \models \varphi[h]$.

7. (a) Here the language is $\mathscr{L} = \{f, R\}$, where f is a unary function symbol and R is a ternary (i.e. "of arity 3") relation symbol.
(b) We choose the universe of $\mathscr{M}$ to be the unit interval $[0,1]$ of the real numbers.
(c) We interpret f in $\mathscr{M}$ as the function $f^{\mathscr{M}} : [0,1] \longrightarrow [0,1]$ defined by

$$f^{\mathscr{M}}(t) = e^{t-1}.$$

(note that we may **not** define $f^{\mathscr{M}}(t) = e^t$ - why?) and the relation symbol R as the betweeness relation of $[0,1]$:

$$R^{\mathscr{M}} = \{(a,b,c) \in [0,1]^3 \mid b \text{ is between } a \text{ and } c\}$$

In $\mathscr{M}$ the following sentence is true:

$$\forall x \forall y \forall z \Big(R(f(x), y, f(z)) \to \exists u \ (R(x,u,z) \wedge f(u) = y) \Big).$$

This is saying that the intermediate value theorem is true for the function $f^{\mathscr{M}}$.

3.2. **Correctness of logical axioms and formal proofs.**

We will now show that our logical axioms hold at all assignments of all structures (see 3.2.2). Furthermore the soundness theorem 3.2.4 shows that our logical proofs do not lead to contradictions. (Formally, 3.2.4 actually implies 3.2.2). This constitutes one part of a justification of our choice of axioms. The second part, namely that we did not state more axioms, will be justified later, by the so called Completeness Theorem (cf. 4.4.2): This theorem implies that, whenever φ is an $\mathscr{L}$-formula, which is valid in all assignments of all $\mathscr{L}$-structures, then $\vdash_{\mathscr{L}} \varphi$, in other words there is a formal proof of φ from our logical axioms. In this sense our choice of logical axioms, together with the rules of a formalized proof is complete. We start with a preparation:

3.2.1. **Lemma.** *Let $t, t' \in \mathrm{tm}(\mathscr{L})$, $\varphi \in \mathrm{Fml}(\mathscr{L})$ and $x \in \mathrm{Vbl}$. Let h be an assignment of an $\mathscr{L}$-structure $\mathscr{M}$ and let $a = t^{\mathscr{M}}[h]$.*
(i) $(t'(x/t))^{\mathscr{M}}[h] = t'^{\mathscr{M}}[h\binom{x}{a}]$.
(ii) If x is free in φ for t then $\mathscr{M} \models \varphi(x/t)\ [h] \iff \mathscr{M} \models \varphi[h\binom{x}{a}]$.

Proof. (i) is a straightforward induction on the complexity of t' (please verify this as part of question 26). Item (ii) is also a straightforward induction on the complexity of φ except for one case (please verify the other cases as part of question 26): Suppose we know (ii) already for $\psi \in \mathrm{Fml}(\mathscr{L})$ and φ is $\forall y\psi$ with $y \in \mathrm{Vbl}$. We show (ii) for φ.

Case 1. x does not occur free in φ.

Then $\varphi(x/t)$ is equal to φ and h is identical to $h\binom{x}{a}$ at the free variables of φ. By 3.1.4 we have $\mathscr{M} \models \varphi(x/t)\ [h] \iff \mathscr{M} \models \varphi[h] \iff \mathscr{M} \models \varphi[h\binom{x}{a}]$.

Case 2. x does occur free in φ.
Since φ is $\forall y\ \psi$, we must have $x \neq y$.

Claim. x is free in ψ for t.

Proof of the claim. Let z be a variable occurring in t. We have to show that x is free in ψ for z. As x is free in φ for t and x occurs in φ by assumption, y does not occur in t, which shows that $z \neq y$. Since $x \neq y$, each free occurrence of x in ψ is also a free occurrence of x in φ (where the position is shifted by 3). Thus, as $z \neq y$ and x is free in φ for z, x is also free in ψ for z. □ claim

The claim implies that the formula $(\forall y\psi)(x/t)$ is equal to $\forall y(\psi(x/t))$ and we get

$$\begin{aligned}
\mathscr{M} \models \varphi(x/t)\ [h] &\iff \mathscr{M} \models (\forall y\psi)(x/t)\ [h] \\
&\iff \mathscr{M} \models \forall y(\psi(x/t))\ [h] \\
&\iff \mathscr{M} \models \psi(x/t)\ [h(\genfrac{}{}{0pt}{}{y}{b})] \text{ for all } b \in M, \text{ by definition} \\
&\iff \mathscr{M} \models \psi\ [h(\genfrac{}{}{0pt}{}{y}{b})(\genfrac{}{}{0pt}{}{x}{a})] \text{ for all } b \in M, \text{ by the induction} \\
&\qquad\qquad\qquad\qquad \text{hypothesis using the claim} \\
&\iff \mathscr{M} \models \psi\ [h(\genfrac{}{}{0pt}{}{x}{a})(\genfrac{}{}{0pt}{}{y}{b})] \text{ for all } b \in M, \text{ since } x \neq y \\
&\iff \mathscr{M} \models (\forall y\psi)\ [h(\genfrac{}{}{0pt}{}{x}{a})], \text{ by definition} \\
&\iff \mathscr{M} \models \varphi[h(\genfrac{}{}{0pt}{}{x}{a})]
\end{aligned}$$

as desired. □

3.2.2. **Lemma.** *For every logical axiom φ and each assignment h of every $\mathscr{L}$-structure $\mathscr{M}$ we have $\mathscr{M} \models \varphi[h]$.*

Proof. For the axiom schemes (**AxProp**), (**Ax**$\forall \rightarrow$) and (**AxEq**) this follows immediately from the definition of $\mathscr{M} \models \varphi[h]$ (please make sure you can actually do this). We show the universal validity of the logical axioms built by the schemes (**AxSubst**), (**AxGen**) and (**Ax**$\forall$). Let h be an assignment of an $\mathscr{L}$-structure $\mathscr{M}$.

Case 1. φ is an instance of (**AxSubst**), i.e. φ is of the form $(\forall x\psi) \rightarrow \psi(x/t)$, where x is free in ψ for $t \in \mathrm{tm}(\mathscr{L})$.

We have to assume $\mathscr{M} \models (\forall x\psi)[h]$ and we are supposed to show $\mathscr{M} \models \psi(x/t)[h]$. By 3.2.1(ii), it suffices to show $\mathscr{M} \models \psi[h(\genfrac{}{}{0pt}{}{x}{a})]$ with $a = t^{\mathscr{M}}[h]$. This clearly holds, as $\mathscr{M} \models (\forall x\psi)[h]$.

Case 2. φ is an instance of (**AxGen**), i.e. φ is of the form $\psi \rightarrow \forall x\psi$, where x is not free in ψ.

We have to assume $\mathscr{M} \models \psi[h]$ and we are supposed to show $\mathscr{M} \models \forall x\psi[h]$. This means we have to pick some $a \in |\mathscr{M}|$ and to prove $\mathscr{M} \models \psi[h(\genfrac{}{}{0pt}{}{x}{a})]$. Since x is assumed to be not free in ψ, the assignments h and $h(\genfrac{}{}{0pt}{}{x}{a})$ have the same values on all free variables of ψ. Hence with $\mathscr{M} \models \psi[h]$ we may apply 3.1.4 to obtain $\mathscr{M} \models \psi[h(\genfrac{}{}{0pt}{}{x}{a})]$.

Case 3. φ is an instance of (**Ax**$\forall$), i.e. φ is of the form $\forall x_1 ... x_n\ \psi$, where ψ is a logical axiom defined under one of the schemes (**AxProp**), (**Ax**$\forall \rightarrow$), (**AxEq**), (**AxSubst**) or (**AxGen**).

Since we have already shown the lemma for these logical axioms we know that $\mathscr{M} \models \psi[h']$ for all assignments h'. But now it is clear that $\mathscr{M} \models \forall x_1 ... x_n\ \psi[h]$. □

3.2.3. **Definition.** Let $\Sigma, \Phi \subseteq \mathrm{Fml}(\mathscr{L})$. We say that Σ **logically implies** Φ if for every assignment h of every $\mathscr{L}$-structure $\mathscr{M}$ we have:

$$\mathscr{M} \models \Sigma[h] \Rightarrow \mathscr{M} \models \Phi[h].$$

If this is the case we write

$$\Sigma \models \Phi.$$

3.2.4. **Theorem.** *(Soundness theorem) For all* $\Sigma, \Phi \subseteq \mathrm{Fml}(\mathscr{L})$,

$$\text{if } \Sigma \vdash \Phi \text{ then } \Sigma \models \Phi.$$

Proof. It is enough to show by induction on n the following: If $(\varphi_1, ..., \varphi_n)$ is a proof from Σ then $\mathscr{M} \models \varphi_n[h]$ for every assignment h of every $\mathscr{L}$-structure $\mathscr{M}$ satisfying $\mathscr{M} \models \Sigma[h]$.

If $\varphi_n \in \Sigma$ or φ_n is a logical axiom, then this holds true by 3.2.2. In particular the claim holds for $n = 1$. Moreover for the induction step "$n-1 \Rightarrow n$" it is clear that we only need to show the claim in the case where φ_n is the result of applying Modus Ponens to two entries of $(\varphi_1, ..., \varphi_{n-1})$. Hence there are $k, j < n$ such that $\varphi_k = \varphi_j \to \varphi_n$.

If h is an assignment of $\mathscr{M}$ with $\mathscr{M} \models \Sigma[h]$, then by induction we know $\mathscr{M} \models \varphi_j[h]$ and $\mathscr{M} \models \varphi_j \to \varphi_n\ [h]$. By 3.1.3(B)(ii) we get $\mathscr{M} \models \varphi_n[h]$ as desired. □

Frequently the Soundness theorem is used in its contrapositive form, e.g., when we want to show that $\Sigma \nvdash \varphi$ we can establish this by finding an $\mathscr{L}$-structure $\mathscr{M}$ and an assignment h of $\mathscr{M}$ such that $\mathscr{M} \models \Sigma[h]$ and $\mathscr{M} \not\models \varphi[h]$.

3.2.5. **Definition.** A subset $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ is called **consistent** if there is no $\mathscr{L}$-formula φ with $\Sigma \vdash \varphi \wedge \neg\varphi$. Otherwise Σ is called **contradictory** or **inconsistent**. A formula φ is called consistent, inconsistent resp., if $\{\varphi\}$ has this property.

3.2.6. **Corollary.** *If* $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ *and if there is a model of* Σ *at some assignment, then* Σ *is consistent.*

Proof. This follows immediately from 3.2.4 since $\varphi \wedge \neg\varphi$ does not hold for any $\mathscr{L}$-formula φ in any $\mathscr{L}$-structure at any assignment. □

3.3. **Term algebras.**

The main aim in this course is to prove Gödel's completeness theorem which says that the converse of 3.2.6 holds, i.e. every consistent set of $\mathscr{L}$-formulas has a model at some assignment (this statement will then easily also show the converse of the Soundness Theorem 3.2.6). The first problem is, how could we possibly produce $\mathscr{L}$-structures from a given consistent set of $\mathscr{L}$-formulas?

In this subsection we introduce a key tool how to do this, namely via term algebras of (consistent) sets. These will be used in the proof of the completeness theorem. Independently of this application, the construction of term algebras is omnipresent and highly important in mathematics.

3.3.1. *Notation.* Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language. The set of all closed terms of $\mathscr{L}$ is denoted by $\mathrm{ctm}(\mathscr{L})$. Recall from 1.6.3(iv) that an $\mathscr{L}$-term is closed (or constant) if no variable occurs in it.

Notice that $\mathscr{C} \subseteq \mathrm{tm}(\mathscr{L})$, in particular $\mathrm{ctm}(\mathscr{L}) \neq \emptyset$ if $\mathscr{L}$ has a constant symbol.

Before we give the definition of term algebras, more precisely of the domain of term algebras we have to understand the following relation $\sim_\Sigma$ on $\mathrm{ctm}(\mathscr{L})$ for a given subset Σ of $\mathrm{Fml}(\mathscr{L})$. We define

$$t_1 \sim_\Sigma t_2 \iff \Sigma \vdash t_1 \doteq t_2 \quad (t_1, t_2 \in \mathrm{ctm}(\mathscr{L})).$$

3.3.2. **Lemma.** *Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$.*

(i) $\sim_\Sigma$ is an equivalence relation on $\mathrm{ctm}(\mathscr{L})$.
(ii) If $F \in \mathscr{F}$ is a function symbol of arity n and $t_1, s_1, ..., t_n, s_n \in \mathrm{ctm}(\mathscr{L})$ with $t_i \sim_\Sigma s_i$ for each i, then $F(t_1, ..., t_n) \sim_\Sigma F(s_1, ..., s_n)$.
(iii) If $R \in \mathscr{R}$ is a relation symbol of arity n and $t_1, s_1, ..., t_n, s_n \in \mathrm{ctm}(\mathscr{L})$ with $t_i \sim_\Sigma s_i$ for each i, then $\Sigma \vdash R(t_1, ..., t_n) \leftrightarrow R(s_1, ..., s_n)$.

Proof. (i). If $t \in \mathrm{ctm}(\mathscr{L})$, then $t \sim_\Sigma t$: To see this, apply (**AxEq**)(a) and (**Ax**$\forall$) to obtain $\vdash \forall x\; x \doteq x$. Then using (**AxSubst**) and Modus Ponens shows $\vdash t \doteq t$.

Now let $t_1, t_2, t_3 \in \mathrm{ctm}(\mathscr{L})$ with $t_1 \sim_\Sigma t_2$ and $t_2 \sim_\Sigma t_3$. We have to show $t_3 \sim_\Sigma t_1$. Let x, y, z be three distinct variables. By (**AxEq**)(b) and (**Ax**$\forall$) we know $\vdash \forall xyz\; (x \doteq y \wedge y \doteq z \;\rightarrow\; z \doteq x)$. Applying (**AxSubst**) three times we get $\vdash t_1 \doteq t_2 \wedge t_2 \doteq t_3 \;\rightarrow\; t_3 \doteq t_1$. By 2.1.7 we get

$$\vdash t_1 \doteq t_2 \rightarrow (t_2 \doteq t_3 \rightarrow t_3 \doteq t_1).$$

Now, using Modus Ponens and the assumption $\Sigma \vdash t_1 \doteq t_2$ and $\Sigma \vdash t_2 \doteq t_3$ shows $\Sigma \vdash t_3 \doteq t_1$ as desired.

(ii). Let $x_1, y_1, ..., x_n, y_n$ be mutually distinct variables. The following are instances of (**AxEq**)(c):

$$\begin{array}{rcl} x_1 \doteq y_1 & \rightarrow & F(x_1, ..., x_n) \doteq F(y_1, x_2, ..., x_n) \\ x_2 \doteq y_2 & \rightarrow & F(y_1, x_2, ..., x_n) \doteq F(y_1, y_2, x_3, ..., x_n) \\ & \vdots & \\ x_n \doteq y_n & \rightarrow & F(y_1, ..., y_{n-1}, x_n) \doteq F(y_1, ..., y_n) \end{array}$$

Together with 2.1.7 and (**AxEq**)(a),(b) we get

$$\vdash (x_1 \doteq y_1 \wedge ... \wedge x_n \doteq y_n) \rightarrow F(x_1, ..., x_n) \doteq F(y_1, ..., y_n)$$

By the generalisation theorem 2.2.2 applied $2n$ times we get

$$\vdash \forall x_1 y_1 ... x_n y_n \left((x_1 \doteq y_1 \wedge ... \wedge x_n \doteq y_n) \rightarrow F(x_1, ..., x_n) \doteq F(y_1, ..., y_n) \right).$$

By applying (**AxSubst**) $2n$-times we obtain

$$\vdash (t_1 \doteq s_1 \wedge ... \wedge t_n \doteq s_n) \rightarrow F(t_1, ..., t_n) \doteq F(s_1, ..., s_n)$$

and from 2.1.7 we get

$$\vdash (t_1 \doteq s_1 \rightarrow (t_2 \doteq s_2 \rightarrow ...(t_n \doteq s_n \rightarrow F(t_1, ..., t_n) \doteq F(s_1, ..., s_n))...)$$

Since $\Sigma \vdash t_i \doteq s_i$ for all i we may now apply Modus Ponens for proofs n-times to obtain $F(t_1, ..., t_n) \sim_\Sigma F(s_1, ..., s_n)$.

(iii). Let $x_1, y_1, ..., x_n, y_n$ be mutually distinct variables.

The following are instances of (**AxEq**)(d):

$$\begin{array}{lcl} x_1 \doteq y_1 & \rightarrow & (R(x_1, ..., x_n) \leftrightarrow R(y_1, x_2, ..., x_n)) \\ x_2 \doteq y_2 & \rightarrow & (R(y_1, x_2, ..., x_n) \leftrightarrow R(y_1, y_2, x_3, ..., x_n)) \\ & \vdots & \\ x_n \doteq y_n & \rightarrow & (R(y_1, ..., y_{n-1}, x_n) \leftrightarrow R(y_1, ..., y_n)) \end{array}$$

Together with 2.1.7 we get

$$\vdash (x_1 \doteq y_1 \wedge ... \wedge x_n \doteq y_n) \rightarrow (R(x_1, ..., x_n) \leftrightarrow R(y_1, ..., y_n)).$$

By the generalisation theorem 2.2.2 applied $2n$ times we get

$$\vdash \forall x_1 y_1 ... x_n y_n \left((x_1 \doteq y_1 \wedge ... \wedge x_n \doteq y_n) \rightarrow (R(x_1, ..., x_n) \leftrightarrow R(y_1, ..., y_n)) \right).$$

By applying (**AxSubst**) $2n$-times we obtain

$$\vdash (t_1 \doteq s_1 \wedge ... \wedge t_n \doteq s_n) \rightarrow (R(t_1, ..., t_n) \leftrightarrow R(s_1, ..., s_n)).$$

Since $\Sigma \vdash t_i \doteq s_i$ for all i me may again apply 2.1.7 and Modus Ponens for proofs n-times to obtain $R(t_1, ..., t_n) \leftrightarrow R(s_1, ..., s_n)$. □

3.3.3. **Definition.** (term algebra)
Let $\mathscr{L}$ be a language with at least one constant symbol and let Σ be a set of $\mathscr{L}$-formulas. We will now define a certain $\mathscr{L}$-structure attached to Σ, called the **term algebra of** Σ and denoted by $\mathrm{TmAlg}(\Sigma)$.

Warning: Before giving the definition, it has to be mentioned that the term algebra of Σ will in general **NOT** be a model of Σ (at any assignment).

Now the definition of $\mathrm{TmAlg}(\Sigma)$.

(i) The domain of $\mathrm{TmAlg}(\Sigma)$ is defined to be the set of equivalence classes of constant $\mathscr{L}$-terms with respect to the equivalence relation $\sim_\Sigma$ (cf. 3.3.2(i)):

$$|\,\mathrm{TmAlg}(\Sigma)| := \mathrm{ctm}(\mathscr{L}) / \sim_\Sigma .$$

We denote the equivalence class of $t \in \mathrm{ctm}(\mathscr{L})$ with respect to $\sim_\Sigma$ by $t/\sim_\Sigma$.

(ii) For $R \in \mathscr{R}$ of arity n we define

$$R^{\mathrm{TmAlg}(\Sigma)} = \{(t_1/\sim_\Sigma, ..., t_n/\sim_\Sigma) \mid t_i \in \mathrm{ctm}(\mathscr{L}) \text{ and } \Sigma \vdash R(t_1, ..., t_n)\}.$$

This is correct by 3.3.2(iii)

(iii) For $F \in \mathscr{F}$ of arity n we define the function $F^{\mathrm{TmAlg}(\Sigma)} : |\,\mathrm{TmAlg}(\Sigma)|^n \longrightarrow |\,\mathrm{TmAlg}(\Sigma)|$ by

$$F^{\mathrm{TmAlg}(\Sigma)}(t_1/\sim_\Sigma, ..., t_n/\sim_\Sigma) := F(t_1, ..., t_n)/\sim_\Sigma .$$

This is correct by 3.3.2(ii)

(iv) For $c \in \mathcal{C}$ we define

$$c^{\mathrm{TmAlg}(\Sigma)} := c/\sim_\Sigma \quad (\in |\,\mathrm{TmAlg}(\Sigma)|).$$

The **term algebra of a formula** σ is defined as the term algebra of $\{\sigma\}$.

3.3.4. *Example.* Here is an example of a consistent set Σ such that $\mathrm{TmAlg}(\Sigma)$ is not a model of Σ: Let $\mathscr{L}$ be a language that has exactly one constant symbol c and no function symbols. Let $\sigma = \exists x\ x \neq c$. Since every $\mathscr{L}$-structure with more than one element is a model of σ, σ is consistent (Soundness!). Moreover, the models of σ are precisely the $\mathscr{L}$-structures with more than one element. However, the term algebra of σ only has one element, since already $\mathrm{ctm}(\mathscr{L})$ only has one element!

3.3.5. *Example.* Let us compute another term algebra. This time, suppose $\mathscr{L}$ is a language that has exactly three constant symbols c, d, e and no function symbols (observe that $\mathscr{L}$ might have relation symbols). Let σ be the $\mathscr{L}$-sentence $c \doteq d \vee c \doteq e$. Since $\mathscr{L}$ does not have function symbols, $\mathrm{ctm}(\mathscr{L}) = \{c, d, e\}$. Hence $\mathrm{TmAlg}(\sigma)$ has <u>at most</u> three elements: $|\,\mathrm{TmAlg}(\sigma)| = \{c/\sim_\sigma, d/\sim_\sigma, e/\sim_\sigma\}$. What is the size of $\mathrm{TmAlg}(\sigma)$? We need to check which of the elements $c/\sim_\sigma, d/\sim_\sigma, e/\sim_\sigma$ of $|\,\mathrm{TmAlg}(\sigma)|$ are in fact different. By definition of the equivalence relation $\sim_\sigma$, this means we have to check whether or not $\sigma \vdash c \doteq d$, $\sigma \vdash c \doteq e$ and $\sigma \vdash d \doteq e$. Since σ is the sentence $c \doteq d \vee c \doteq e$ one could have a suspicion that $\sigma \vdash c \doteq d$ or $\sigma \vdash c \doteq e$. However, both fail and also $\sigma \vdash d \doteq e$ fails. In order to see that $\sigma \nvdash c \doteq d$, take an $\mathscr{L}$-structure $\mathscr{M}$ with universe $|\mathscr{M}| := \{1, 2\}$,

$$c^{\mathscr{M}} := 1,\ d^{\mathscr{M}} := 2 \text{ and } e^{\mathscr{M}} := 1.$$

The relation symbols of $\mathscr{L}$ can be assigned arbitrarily in $\mathscr{M}$. Then $\mathscr{M}$ is a model of σ, as $\mathscr{M} \models c \doteq d \vee c \doteq e$. But obviously $\mathscr{M}$ is not a model of $c \doteq d$. Hence by Soundness, we cannot have $\sigma \vdash c \doteq d$, i.e. we have $\sigma \nvdash c \doteq d$. Be aware, that this does NOT mean $\sigma \vdash \neg c \doteq d$, because every $\mathscr{L}$ structure with exactly one element satisfies σ and $c \doteq d$.

Thus $\sigma \nvdash c \doteq d$ and similarly $\sigma \nvdash c \doteq e$ and $\sigma \nvdash d \doteq e$. It follows that $\mathrm{TmAlg}(\sigma)$ has three elements and by definition

$$c^{\mathrm{TmAlg}(\sigma)} := c/\sim_\sigma,\ d^{\mathrm{TmAlg}(\sigma)} := d/\sim_\sigma \text{ and } e^{\mathrm{TmAlg}(\sigma)} := e/\sim_\sigma .$$

It remains to see, how the relation symbols of $\mathscr{L}$ are interpreted in $\mathrm{TmAlg}(\sigma)$: Let R be an n-ary relation symbol of $\mathscr{L}$. We claim that

$$(*) \qquad R^{\mathrm{TmAlg}(\sigma)} = \emptyset.$$

In order to see this, we pick $t_1, ..., t_n \in \mathrm{ctm}(\mathscr{L})$ and we have to show $\sigma \nvdash R(t_1, ..., t_n)$ (convince yourself that this is precisely the meaning of $(*)$). We may use the soundness theorem again and so it is enough to find a model $\mathscr{M}$ of σ with $\mathscr{M} \not\models$

$R(t_1, ..., t_n)$. We choose $\mathscr{M}$ of size 1 (then we automatically have $\mathscr{M} \models \sigma$) and let $R^{\mathscr{M}} := \emptyset$.

3.3.6. **Proposition.** *Let $\mathscr{L}$ be a language with at least one constant symbol and let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$. Let $n \in \mathbb{N}_0$, $t_1, ..., t_n \in \mathrm{ctm}(\mathscr{L})$, $x_1, ..., x_n \in \mathrm{Vbl}$ pairwise distinct and let h be an assignment of $\mathrm{TmAlg}(\Sigma)$ with $h(x_i) = t_i/\sim$ $(1 \le i \le n)$.*

(i) If $t(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$ then $t^{\mathrm{TmAlg}(\Sigma)}[h] = t(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma$.
(ii) If $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ is atomic then

$$\mathrm{TmAlg}(\Sigma) \models \varphi[h] \iff \Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n).$$

Proof. (i). We show $t^{\mathrm{TmAlg}(\Sigma)}[h] = t(x_1/t_1, ..., x_n/t_n)/\sim$ by induction on the complexity of t. If $t = x_i$ then $t^{\mathrm{TmAlg}(\Sigma)}[h] = h(x_i) = t_i/\sim_\Sigma$ as desired. If $t = c$ is a constant symbol then $t^{\mathrm{TmAlg}(\Sigma)}[h] = c^{\mathrm{TmAlg}(\Sigma)}$ and $t(x_1/t_1, ..., x_n/t_n) = c$ as desired.

For the induction step we may assume that

$$t = F(t'_1(x_1, ..., x_n), ..., t'_k(x_1, ..., x_n))$$

with $t'_i \in \mathrm{tm}(\mathscr{L})$ and $F \in \mathscr{F}$ of arity k. Then $t^{\mathrm{TmAlg}(\Sigma)}[h] =$

$$\begin{aligned} &= F^{\mathrm{TmAlg}(\Sigma)}({t'_1}^{\mathrm{TmAlg}(\Sigma)}[h], ..., {t'_k}^{\mathrm{TmAlg}(\Sigma)}[h]) \overset{\text{by ind. hyp.}}{=} \\ &= F^{\mathrm{TmAlg}(\Sigma)}(t'_1(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma, ..., t'_k(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma) \overset{\text{by def.}}{=} \\ &= F(t'_1(x_1/t_1, ..., x_n/t_n), ..., t'_k(x_1/t_1, ..., x_n/t_n))/\sim_\Sigma = t(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma . \end{aligned}$$

(ii). In order to prove the equivalence we have two cases:
<u>Case 1.</u> φ is of the form $t \doteq s$ with $t(x_1, ..., x_n)$, $s(x_1, .., x_n) \in \mathrm{tm}(\mathscr{L})$.

Then

$$\begin{aligned} \mathrm{TmAlg}(\Sigma) \models \varphi[h] &\overset{\text{by def.}}{\iff} t^{\mathrm{TmAlg}(\Sigma)}[h] = s^{\mathrm{TmAlg}(\Sigma)}[h] \\ &\overset{\text{by (i)}}{\iff} t(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma = s(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma \\ &\overset{\text{by def.}}{\iff} \Sigma \vdash t(x_1/t_1, ..., x_n/t_n) \doteq s(x_1/t_1, ..., x_n/t_n) \\ &\iff \Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n). \end{aligned}$$

<u>Case 2.</u> φ is of the form $R(t'_1, ..., t'_k)$ with $t'_i(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$ and $R \in \mathscr{R}$ of arity k.

Then

$$\begin{aligned} \mathrm{TmAlg}(\Sigma) \models \varphi[h] &\overset{\text{by def.}}{\iff} ({t'_1}^{\mathrm{TmAlg}(\Sigma)}[h], ..., {t'_k}^{\mathrm{TmAlg}(\Sigma)}[h]) \in R^{\mathrm{TmAlg}(\Sigma)} \\ &\overset{\text{by (i)}}{\iff} (t'_1(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma, ..., t'_k(x_1/t_1, ..., x_n/t_n)/\sim_\Sigma) \in R^{\mathrm{TmAlg}(\Sigma)} \\ &\overset{\text{by def.}}{\iff} \Sigma \vdash R(t'_1(x_1/t_1, ..., x_n/t_n), ..., t'_k(x_1/t_1, ..., x_n/t_n)) \\ &\iff \Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n). \end{aligned}$$

$\square$

3.3.7. **Corollary.** *If $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and $\varphi_1, ..., \varphi_n$ are atomic $\mathscr{L}$-sentences (i.e. formulas from $\mathrm{Sen}(\mathscr{L}) \cap \text{at-}\mathrm{Fml}(\mathscr{L})$), then*

$$\Sigma \vdash \varphi_1 \wedge ... \wedge \varphi_n \iff \mathrm{TmAlg}(\Sigma) \models \varphi_1 \wedge ... \wedge \varphi_n.$$

Proof. By 3.3.6(ii) with question 17. $\square$

3.3.8. **Proposition.** (***level* 4/MSc**) *Let $\mathscr{L}$ be a language with at least one constant symbol and let $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$. Let $\varphi_1, ..., \varphi_n, \psi$ be atomic $\mathscr{L}$-formulas with free variables among $x_1, ..., x_k$.*
If

$$\Sigma \vdash (\varphi_1 \wedge ... \wedge \varphi_n) \to \psi$$

then

$$\mathrm{TmAlg}(\Sigma) \models \forall x_1 ... x_k((\varphi_1 \wedge ... \wedge \varphi_n) \to \psi).$$

Proof. Take $t_1, ..., t_k \in \mathrm{ctm}(\mathscr{L})$ and let h be an assignment of $\mathrm{TmAlg}(\Sigma)$ with $h(x_i) = t_i/\sim$. Suppose $\mathrm{TmAlg}(\Sigma) \models \bigwedge_i \varphi_i[h]$. We have to show $\mathrm{TmAlg}(\Sigma) \models \psi[h]$. Define

$$\varepsilon_i := \varphi_i(x_1/t_1, ..., x_k/t_k) \text{ and } \delta := \psi(x_1/t_1, ..., x_k/t_k).$$

By 3.3.6 we know $\Sigma \vdash \varepsilon_i$ for each i, consequently $\Sigma \vdash \varepsilon_1 \wedge ... \wedge \varepsilon_n$ (cf. question 17). Since $\Sigma \vdash (\varphi_1 \wedge ... \wedge \varphi_n) \to \psi$ and none of the x_i occurs freely in any formula from Σ we may apply the Generalisation Theorem 2.2.2 to obtain

$$\Sigma \vdash \forall x_1 ... x_k \ ((\varphi_1 \wedge ... \wedge \varphi_n) \to \psi).$$

Now (**AxSubst**) (applied k-times) and Modus Ponens gives

$$\Sigma \vdash (\varepsilon_1 \wedge ... \wedge \varepsilon_n) \to \delta.$$

Thus we get $\Sigma \vdash \delta$, which means $\mathrm{TmAlg}(\Sigma) \models \psi[h]$ by 3.3.6 again. □

4. Completeness

In this section we will show Gödel's theorem for first order logic which says that every consistent set of $\mathscr{L}$-sentences has a model. From this it is then easy to derive the so-called Completeness Theorem (cf. 4.4.2) which says that the converse of the Soundness Theorem 3.2.4 holds (for sets of formulas). The explanation of the term "Completeness Theorem" can be found at the beginning of subsection 3.2.

4.1. **Outline of the proof of the completeness theorem.** For a given consistent set Σ of $\mathscr{L}$-sentences we have to "construct" a model of Σ. In fact we shall not construct such a model, instead we only show the existence of a model. In section 3.3 we have introduced a method how to attach an $\mathscr{L}$-structure to Σ, namely the structure $\mathrm{TmAlg}(\Sigma)$. In general $\mathrm{TmAlg}(\Sigma)$ is not a model of Σ. In this subsection we show that under two assumptions on Σ, namely "Σ is complete" (defined in 4.1.1 below) and "Σ possesses a system of witnesses" (defined in 4.1.3 below), the term algebra of Σ indeed is a model of Σ. In subsequent sections we will show how to use this fact for a general Σ. This is explained in more details in 4.1.5 below.

4.1.1. **Definition.** A subset $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ is called **complete** if for all $\varphi \in \mathrm{Sen}(\mathscr{L})$ we have

$$\Sigma \nvdash \varphi \iff \Sigma \vdash \neg\varphi$$

Notice that Σ is consistent if Σ is complete (use the implication $\Leftarrow$ of the definition). The main example of a complete set is the following: We take an $\mathscr{L}$-structure $\mathscr{M}$ and define

$$\mathrm{Th}(\mathscr{M}) := \{\varphi \in \mathrm{Sen}(\mathscr{L}) \mid \mathscr{M} \models \varphi\}.$$

This is called the **theory** of $\mathscr{M}$ and denoted by $\mathrm{Th}(\mathscr{M})$.

In question 32 you are asked to show that $\mathrm{Th}(\mathscr{M}) \vdash \varphi \iff \varphi \in \mathrm{Th}(\mathscr{M})$ for every $\varphi \in \mathrm{Sen}(\mathscr{L})$. It follows that the theory of $\mathscr{M}$ is complete. Indeed we shall see soon, that all complete sets are of this form (this is a version of the completeness theorem!)

4.1.2. *Notation.* Given a language $\mathscr{L}$ we define $\mathrm{Fml}(\mathscr{L})(1)$ as the set of all $\mathscr{L}$-formulas with at most one free variable.

4.1.3. **Definition.** Let $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$. A **system of witnesses** for Σ is a map

$$\zeta : \mathrm{Fml}(\mathscr{L})(1) \longrightarrow \mathscr{C}$$

such that for all $x \in \mathrm{Vbl}$ and all $\varphi(x) \in \mathrm{Fml}(\mathscr{L})(1)$ we have

$$\Sigma \vdash (\exists x\varphi) \ \rightarrow\ \varphi(x/\zeta(\varphi)).$$

4.1.4. **Proposition.** *If $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ is complete and possesses a system of witnesses, then the term algebra of Σ is a model of Σ.*

Proof. (**level 4/MSc**)
It suffices to prove by induction on the complexity of $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ the following:

For all $t_1, ..., t_n \in \mathrm{ctm}(\mathscr{L})$,

$$(\dagger) \qquad \Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n) \iff \mathrm{TmAlg}(\Sigma) \models \varphi(x_1/t_1, ..., x_n/t_n).$$

If φ is atomic, then we know this from 3.3.6 (without any assumption on Σ).

If $\varphi = \neg\psi$, then

$$\begin{array}{rcl} \Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n) & \iff & \Sigma \vdash \neg\psi(x_1/t_1, ..., x_n/t_n) \\ & \overset{\text{as } \Sigma \text{ is complete}}{\iff} & \Sigma \nvdash \psi(x_1/t_1, ..., x_n/t_n) \\ & \overset{\text{by ind. hypoth.}}{\iff} & \mathrm{TmAlg}(\Sigma) \not\models \psi(x_1/t_1, ..., x_n/t_n) \\ & \iff & \mathrm{TmAlg}(\Sigma) \models \varphi(x_1/t_1, ..., x_n/t_n) \end{array}$$

If $\varphi = \varphi_1 \wedge \varphi_2$ then (†) holds if and only if (†) for both φ_1 and φ_2. Therefore, if φ is $\psi \rightarrow \gamma$, and (†) holds for ψ and γ, then (†) also holds for φ: φ is provably equivalent to $\neg(\psi \wedge \neg\gamma)$.

It remains to do the case $\varphi = \forall y\psi$, where $y \in \mathrm{Vbl}$ is arbitrary. As $\mathrm{Fr}(\varphi) \subseteq \{x_1, ..., x_n\}$ and $y \notin \mathrm{Fr}(\varphi)$ we may assume that $y \neq x_i$ $(1 \leq i \leq n)$.

"$\Rightarrow$": $\Sigma \vdash \varphi(x_1/t_1, ..., x_n/t_n)$ means $\Sigma \vdash \forall y\psi(x_1/t_1, ..., x_n/t_n)$, hence by **AxSubst** we have

$$\Sigma \vdash \psi(x_1/t_1, ..., x_n/t_n, y/t) \text{ for all } t \in \mathrm{ctm}(\mathscr{L}).$$

From the induction hypothesis we get $\mathrm{TmAlg}(\Sigma) \models \psi(x_1/t_1, ..., x_n/t_n, y/t)$ for all $t \in \mathrm{ctm}(\mathscr{L})$. By 3.2.1 this means

$$(+) \qquad \mathrm{TmAlg}(\Sigma) \models \psi(x_1/t_1, ..., x_n/t_n)[h]$$

for every assignment h of $\mathrm{TmAlg}(\Sigma)$ with $h(y) = t/\sim_\Sigma$ and all $t \in \mathrm{ctm}(\mathscr{L})$. Since $|\mathrm{TmAlg}(\Sigma)|$ is the set of all $t/\sim_\Sigma$ with $t \in \mathrm{ctm}(\mathscr{L})$, we get

$$\mathrm{TmAlg}(\Sigma) \models \forall y\ \psi(x_1/t_1, ..., x_n/t_n)$$

as required.

"$\Leftarrow$": Here and only here we need that Σ possesses a system of witnesses $\zeta : \mathrm{Fml}(\mathscr{L})(1) \longrightarrow \mathscr{C}$.

Suppose $\mathrm{TmAlg}(\Sigma) \models \forall y\psi(x_1/t_1, ..., x_n/t_n)$, but

$$\Sigma \nvdash \forall y\psi(x_1/t_1, ..., x_n/t_n).$$

Since Σ is complete we have

$$\Sigma \vdash \exists y\neg\psi(x_1/t_1, ..., x_n/t_n).$$

Let $c = \zeta(\neg\psi(x_1/t_1, ..., x_n/t_n))$.

Since ζ is a system of witnesses for Σ we have

$$\Sigma \vdash (\exists y\neg\psi(x_1/t_1, ..., x_n/t_n)) \ \rightarrow\ \neg\psi(x_1/t_1, ..., x_n/t_n)(y/c).$$

With Modus Ponens we get

$$\Sigma \vdash\ \neg\psi(x_1/t_1, ..., x_n/t_n, y/c).$$

In particular $\Sigma \nvdash \psi(x_1/t_1, ..., x_n/t_n, y/c)$. Using the induction hypothesis we obtain

$$\mathrm{TmAlg}(\Sigma) \not\models \psi(x_1/t_1, ..., x_n/t_n, y/c).$$

However, using 3.2.1 again, this contradicts our assumption

$$\mathrm{TmAlg}(\Sigma) \models \forall y\psi(x_1/t_1, ..., x_n/t_n).$$

$\square$

4.1.5. *Strategy.* Let Σ be an arbitrary consistent set of $\mathscr{L}$-sentences. We will apply 4.1.4 to "construct" a model of Σ, in the following way:

In subsection 4.3 we will show that there is a language $\mathscr{L}^*$ that has the same relation and function symbols as $\mathscr{L}$, but a huge set of constant symbols $\mathscr{C}^*$ (containing the constant symbols of $\mathscr{L}$), together with a consistent set Σ^* of $\mathscr{L}^*$-sentences containing Σ such that Σ^* possesses a system of witnesses (cf. 4.3.11).

In subsection 4.2 we will show that every consistent set of sentences in every language is contained in a complete consistent set (cf. 4.2.6). Applying this to Σ^* gives a complete set $\Sigma^\dagger$ of $\mathscr{L}^*$-sentences containing Σ^*. Now observe that any system of witnesses for Σ^* also is a system of witnesses of $\Sigma^\dagger$.

Hence $\Sigma^\dagger$ is a complete set of $\mathscr{L}^*$-sentences containing Σ, and $\Sigma^\dagger$ possesses a system of witnesses. Thus we may apply 4.1.4 to obtain a model of $\Sigma^\dagger$, namely $\mathrm{TmAlg}(\Sigma^\dagger) \models \Sigma^\dagger$. Finally if we forget the interpretations of the new constants from $\mathscr{L}^*$ in $\mathrm{TmAlg}(\Sigma^\dagger)$ we obtain an $\mathscr{L}$-structure $\mathscr{M}$ which is a model of Σ (see 4.3.3).

4.2. Theories and complete theories.

4.2.1. **Definition.** Let $\mathscr{L}$ be a language and let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$.

(i) The **deductive closure** of Σ is the set

$$\mathrm{Ded}_{\mathscr{L}}(\Sigma) := \{\varphi \in \mathrm{Fml}(\mathscr{L}) \mid \Sigma \vdash \varphi\}.$$

If $\mathscr{L}$ is clear from the context we will write $\mathrm{Ded}(\Sigma)$ instead of $\mathrm{Ded}_{\mathscr{L}}\,\Sigma$.

(ii) If $\Sigma = \mathrm{Ded}(\Sigma)$, then Σ is called **deductively closed**.

(iii) Σ is called a **deductively closed set of ($\mathscr{L}$-)sentences** or **deductively closed in** $\mathrm{Sen}(\mathscr{L})$ if $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ and $\Sigma = \mathrm{Sen}(\mathscr{L}) \cap \mathrm{Ded}(\Sigma)$; in other words if

$$\Sigma \vdash \varphi \in \mathrm{Sen}(\mathscr{L}) \text{ implies } \varphi \in \Sigma.$$

Observe that by 2.1.6, $\mathrm{Ded}(\Sigma)$ is deductively closed.

4.2.2. **Definition.** A subset T of $\mathrm{Sen}(\mathscr{L})$ is called an **$\mathscr{L}$-theory** or simply a **theory**, if T is consistent and a deductively closed set of ($\mathscr{L}$-)sentences. An $\mathscr{L}$-theory T which is also complete is called a **complete theory** .

4.2.3. **Observations.** *Let $\mathscr{L}$ be a language and let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$.*

(i) *Σ is consistent if and only if $\mathrm{Ded}(\Sigma) \neq \mathrm{Fml}(\mathscr{L})$.*

(ii) $\mathrm{Ded}(\Sigma)$ *is deductively closed.*

(iii) $\mathrm{Sen}(\mathscr{L}) \cap \mathrm{Ded}(\Sigma)$ *is deductively closed in* $\mathrm{Sen}(\mathscr{L})$.

(iv) An arbitrary (nonempty) intersection of $\mathscr{L}$-theories is again an $\mathscr{L}$-theory.

(v) If Σ is a complete set of $\mathscr{L}$-sentences, then $\mathrm{Sen}(\mathscr{L}) \cap \mathrm{Ded}(\Sigma)$ *is a complete theory.*

4.2.4. **Lemma.** *Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and let $\varphi \in \mathrm{Fml}(\mathscr{L})$. Then*

(i) $\Sigma \vdash \varphi \iff \Sigma \cup \{\neg\varphi\}$ *is inconsistent.*

(ii) If Σ is consistent, then $\Sigma \cup \{\varphi\}$ is consistent or $\Sigma \cup \{\neg\varphi\}$ is consistent.

Proof. (i). This has been done in question 28.

(ii). If both $\Sigma \cup \{\varphi\}$ and $\Sigma \cup \{\neg\varphi\}$ are inconsistent, then by (i), $\Sigma \vdash \neg\varphi$ and $\Sigma \vdash \varphi$. Hence Σ is inconsistent as well. □

4.2.5. **Proposition.** *For every subset $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ the following are equivalent:*

(i) Σ is a complete $\mathscr{L}$-theory.

(ii) Σ is a maximally consistent set of $\mathscr{L}$-sentences, i.e. Σ is a consistent set of $\mathscr{L}$-sentences and each of its proper supersets of $\mathscr{L}$-sentences is inconsistent.

Proof. (i)⇒(ii). First suppose Σ is complete and an $\mathscr{L}$-theory. Then Σ is consistent. Let Σ' be a proper superset of Σ consisting of $\mathscr{L}$-sentences. Take $\sigma \in \Sigma' \setminus \Sigma$. As Σ is deductively closed in $\mathrm{Sen}(\mathscr{L})$, we have $\Sigma \nvdash \sigma$. As Σ is complete we get $\Sigma \vdash \neg\sigma$. Since $\Sigma \subseteq \Sigma'$ we get $\Sigma' \vdash \neg\sigma$. Hence $\Sigma' \vdash \sigma \wedge \neg\sigma$ which shows that Σ' is inconsistent as desired.

(ii)⇒(i). Conversely suppose Σ is a maximally consistent set of $\mathscr{L}$-sentences. Then also $\mathrm{Sen}(\mathscr{L}) \cap \mathrm{Ded}(\Sigma)$ is consistent, hence the maximality gives $\Sigma = \mathrm{Sen}(\mathscr{L}) \cap \mathrm{Ded}(\Sigma)$, in other words, Σ is deductively closed in $\mathrm{Sen}(\mathscr{L})$. It remains to show that Σ is complete. Take $\varphi \in \mathrm{Sen}(\mathscr{L})$. As Σ is consistent we know that $\Sigma \vdash \neg\varphi$ implies $\Sigma \nvdash \varphi$. Conversely suppose $\Sigma \nvdash \varphi$. By 4.2.4(i), $\Sigma \cup \{\neg\varphi\}$ is consistent. Hence the maximality assumption shows $\Sigma = \Sigma \cup \{\neg\varphi\}$, in other words $\neg\varphi \in \Sigma$. □

4.2.6. **Theorem.** *Every consistent set of $\mathscr{L}$-sentences is contained in a complete $\mathscr{L}$-theory.*

Proof using the Lemma of Zorn. (**level 4/MSc**) Let $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ be our consistent set. Let $\mathfrak{S}$ be the set of all consistent subsets of $\mathrm{Sen}(\mathscr{L})$ containing Σ. By 4.2.5 it is enough to show that $\mathfrak{S}$ contains elements that are maximal with respect to inclusion. However, the partial ordered set $(\mathfrak{S}, \subseteq)$ satisfies the requirement of Zorn's Lemma, because the union of a chain of $(\mathfrak{S}, \subseteq)$ is again in $\mathfrak{S}$ (as follows from the fact that formal proofs are finite). Hence by Zorn's lemma **??**, $\mathfrak{S}$ indeed contains elements that are maximal with respect to inclusion. $\square$

Proof in the countable case without Zorn. Let $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ be our consistent set. Suppose $\mathscr{L}$ is countable (or finite). Then the set of finite strings is also countable, in particular $\mathrm{Sen}(\mathscr{L})$ is countable. Let $\varphi_1, \varphi_2, ...$ be an enumeration of $\mathrm{Sen}(\mathscr{L})$. We define by induction a subset Σ_n of $\mathrm{Sen}(\mathscr{L})$ as follows: $\Sigma_0 = \Sigma$ and

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_{n+1}\} & \text{if } \Sigma_n \cup \{\varphi_{n+1}\} \text{ is consistent} \\ \Sigma_n \cup \{\neg\varphi_{n+1}\} & \text{otherwise.} \end{cases}$$

We claim that $T := \bigcup_{n \in \mathbb{N}_0} \Sigma_n$ is a complete theory (observe that $\Sigma = \Sigma_0 \subseteq T$. By definition of Σ_{n+1} we know that $\varphi_{n+1} \in T$ or $\neg\varphi_{n+1} \in T$. Since each $\varphi \in \mathrm{Sen}(\mathscr{L})$ is one of the φ_{n+1} for some $n \geq 0$ we see that $T \nvdash \varphi$ implies $\varphi_{n+1} \notin \Sigma_{n+1}$, hence $\neg\varphi_{n+1} \in \Sigma_{n+1}$, and so $T \vdash \neg\varphi$ for every such φ. It remains to show that T is consistent. We show by induction that Σ_n is consistent: The base case $n = 0$ holds by assumption and the induction step holds by 4.2.4(ii). $\square$

4.3. **Extensions by constants.**

4.3.1. **Definition.** Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ and $\mathscr{L}^+ = (\lambda^+ : \mathscr{R}^+ \longrightarrow \mathbb{N}, \mu^+ : \mathscr{F}^+ \longrightarrow \mathbb{N}, \mathscr{C}^+)$ be languages. $\mathscr{L}^+$ is called an **extension of** $\mathscr{L}$ and $\mathscr{L}$ is called a **sublanguage** of $\mathscr{L}^+$ if the following conditions hold:

- $\mathscr{R} \subseteq \mathscr{R}^+$ and $\lambda^+ \restriction \mathscr{R} = \lambda$.
- $\mathscr{F} \subseteq \mathscr{F}^+$ and $\mu^+ \restriction \mathscr{F} = \mu$.
- $\mathscr{C} \subseteq \mathscr{C}^+$.

If $\mathscr{L}^+$ is an extension of $\mathscr{L}$ with $\mathscr{R}^+ = \mathscr{R}$ and $\mathscr{F}^+ = \mathscr{F}$ (hence also $\lambda^+ = \lambda$ and $\mu^+ = \mu$), then $\mathscr{L}^+$ is called an **extension by constants** of $\mathscr{L}$. In this case we also write $\mathscr{L}^+ = \mathscr{L}(\mathscr{D})$, where $\mathscr{D} = \mathscr{C}^+ \setminus \mathscr{C}$. .

4.3.2. *Remark and Definition.* Let $\mathscr{L}^+ = (\lambda^+ : \mathscr{R}^+ \longrightarrow \mathbb{N}, \mu^+ : \mathscr{F}^+ \longrightarrow \mathbb{N}, \mathscr{C}^+)$ be an extension of the language $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$.

(i) The set of $\mathscr{L}$-terms is the set of all $\mathscr{L}^+$-terms that are $\mathscr{L}$-strings. The set of $\mathscr{L}$-formulas is the set of all $\mathscr{L}^+$-formulas that are $\mathscr{L}$-strings. (see question 35)

(ii) If $\mathscr{M}^+$ is an $\mathscr{L}^+$-structure then there is obviously a unique $\mathscr{L}$ structure $\mathscr{M}$ with $|\mathscr{M}| = |\mathscr{M}^+|$, $R^{\mathscr{M}} = R^{\mathscr{M}^+}$ $(R \in \mathscr{R})$, $F^{\mathscr{M}} = F^{\mathscr{M}^+}$ $(F \in \mathscr{F})$ and $c^{\mathscr{M}} = c^{\mathscr{M}^+}$ $(c \in \mathscr{C})$. $\mathscr{M}$ is called the **restriction of** $\mathscr{M}^+$ **to** $\mathscr{L}$ and $\mathscr{M}^+$ is called an **expansion** of $\mathscr{M}$ to $\mathscr{L}^+$. The structure $\mathscr{M}$ is also called a **reduct of** $\mathscr{M}^+$. We write $\mathscr{M} = \mathscr{M}^+ \restriction \mathscr{L}$.

(iii) If $\mathscr{M}$ is an $\mathscr{L}$ structure then $\mathscr{M}$ can be expanded to $\mathscr{L}^+$ (in several ways if $\mathscr{L}^+ \neq \mathscr{L}$). Simply choose an arbitrary interpretation of the symbols from $\mathscr{L}^+$ that are not symbols of $\mathscr{L}$.

4.3.3. **Proposition.** *Let $\mathscr{L}^+$ be an extension of the language $\mathscr{L}$. If $\mathscr{M}$ is the restriction of the $\mathscr{L}^+$-structure $\mathscr{M}^+$ to $\mathscr{L}$, $t \in \mathrm{tm}(\mathscr{L})$, $\varphi \in \mathrm{Fml}(\mathscr{L})$ and $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ (hence h is an assignment of $\mathscr{M}$ and of $\mathscr{M}^+$), then*

(i) $t^{\mathscr{M}}[h] = t^{\mathscr{M}^+}[h]$ *and*
(ii) $\mathscr{M} \models \varphi[h] \iff \mathscr{M}^+ \models \varphi[h]$.

Proof. (i) We show by induction on the complexity of t that $t^{\mathscr{M}}[h] = t^{\mathscr{M}^+}[h]$: If t is a variable, then $t^{\mathscr{M}}[h] = h(t) = t^{\mathscr{M}^+}[h]$ by definition 3.1.3(A)(i). If t is a constant symbol $c \in \mathscr{C}$, then

$$t^{\mathscr{M}}[h] = c^{\mathscr{M}} = c^{\mathscr{M}^+} = t^{\mathscr{M}^+}[h],$$

by definition 3.1.3(A)(i) and as $\mathscr{M}$ is the restriction of $\mathscr{M}^+$ to $\mathscr{L}$ (see 4.3.2(ii)).

If $F \in \mathscr{F}$ is an n-ary function symbol and $t_1, \dots, t_n$ are $\mathscr{L}$-terms for which we already know our assertion, and t is $F(t_1, \dots, t_n)$, then

$$\begin{aligned} t^{\mathscr{M}}[h] &= F^{\mathscr{M}}(t_1^{\mathscr{M}}[h], \dots, t_n^{\mathscr{M}}[h]), \text{ by definition 3.1.3(A)(ii)} \\ &= F^{\mathscr{M}^+}(t_1^{\mathscr{M}}[h], \dots, t_n^{\mathscr{M}}[h]), \text{ since } \mathscr{M} \text{ is the restriction of } \mathscr{M}^+ \text{ to } \mathscr{L} \\ &= F^{\mathscr{M}^+}(t_1^{\mathscr{M}^+}[h], \dots, t_n^{\mathscr{M}^+}[h]), \text{ by induction} \\ &= t^{\mathscr{M}^+}[h], \text{ by definition 3.1.3(A)(ii).} \end{aligned}$$

(ii) We show $\mathscr{M} \models \varphi[h] \iff \mathscr{M}^+ \models \varphi[h]$ by induction on the complexity of the $\mathscr{L}$-formula φ. If φ is atomic of the form $s \doteq t$ for $\mathscr{L}$-terms s, t, then

$$\begin{aligned}
\mathscr{M} \models \varphi[h] &\iff s^{\mathscr{M}}[h] = t^{\mathscr{M}}[h], \text{ by definition 3.1.3(B)(i)} \\
&\iff s^{\mathscr{M}^+}[h] = t^{\mathscr{M}^+}[h], \text{ by (i)} \\
&\iff \mathscr{M}^+ \models \varphi[h], \text{ by definition 3.1.3(B)(i)}
\end{aligned}$$

If φ is atomic of the form $R(t_1, \dots, t_n)$ for some n-ary relation symbol $R \in \mathscr{R}$ and some $t_1, \dots, t_n \in \mathrm{tm}(\mathscr{L})$, then

$$\begin{aligned}
\mathscr{M} \models \varphi[h] &\iff (t_1^{\mathscr{M}}[h], \dots, t_n^{\mathscr{M}}[h]) \in R^{\mathscr{M}}, \text{ by definition 3.1.3(B)(i)} \\
&\iff (t_1^{\mathscr{M}}[h], \dots, t_n^{\mathscr{M}}[h]) \in R^{\mathscr{M}^+}, \text{ since } \mathscr{M} = \mathscr{M}^+ \upharpoonright \mathscr{L} \\
&\iff (t_1^{\mathscr{M}^+}[h], \dots, t_n^{\mathscr{M}^+}[h]) \in R^{\mathscr{M}^+}, \text{ by (i)} \\
&\iff \mathscr{M}^+ \models \varphi[h], \text{ by definition 3.1.3(B)(i)}
\end{aligned}$$

This shows (ii) when φ is atomic. For the induction step we have to process three cases, as usual.

<u>Case 1.</u> φ is $\neg\psi$. Then

$$\begin{aligned}
\mathscr{M} \models \varphi[h] &\iff \mathscr{M} \not\models \psi[h], \text{ by definition 3.1.3(B)(ii)} \\
&\iff \mathscr{M}^+ \not\models \psi[h], \text{ by induction} \\
&\iff \mathscr{M}^+ \models \varphi[h], \text{ by definition 3.1.3(B)(ii)}
\end{aligned}$$

<u>Case 2.</u> φ is $\psi \to \delta$. Then

$$\begin{aligned}
\mathscr{M} \models \varphi[h] &\iff \text{if } \mathscr{M} \models \psi[h], \text{ then } \mathscr{M} \models \delta[h], \text{ by definition 3.1.3(B)(ii)} \\
&\iff \text{if } \mathscr{M}^+ \models \psi[h], \text{ then } \mathscr{M}^+ \models \delta[h], \text{ by induction} \\
&\iff \mathscr{M}^+ \models \varphi[h], \text{ by definition 3.1.3(B)(ii)}
\end{aligned}$$

<u>Case 3.</u> φ is $\forall x\ \psi$. Then

$$\begin{aligned}
\mathscr{M} \models \varphi[h] &\iff \mathscr{M} \models \psi[h(\begin{smallmatrix} x \\ a \end{smallmatrix})] \text{ for all } a \in |\mathscr{M}|, \text{ by definition 3.1.3(B)(ii)} \\
&\iff \mathscr{M}^+ \models \psi[h(\begin{smallmatrix} x \\ a \end{smallmatrix})] \text{ for all } a \in |\mathscr{M}|, \text{ by induction} \\
&\iff \mathscr{M}^+ \models \psi[h(\begin{smallmatrix} x \\ a \end{smallmatrix})] \text{ for all } a \in |\mathscr{M}^+|, \text{ since } |\mathscr{M}^+| = |\mathscr{M}| \\
&\iff \mathscr{M}^+ \models \varphi[h], \text{ by definition 3.1.3(B)(ii)}
\end{aligned}$$

□

4.3.4. **Corollary.** *Let $\mathscr{L}^+$ be an extension of $\mathscr{L}$ and let $\mathscr{M}$ be the restriction of the $\mathscr{L}^+$-structure $\mathscr{M}^+$ to $\mathscr{L}$. Let $t(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$, $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ and let $d_1, ..., d_n$ be pairwise distinct constants of $\mathscr{L}^+$. If $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ with $h(x_i) = d_i^{\mathscr{M}^+}$, then*

(i) $t^{\mathscr{M}}[h] = t(x_1/d_1, ..., x_n/d_n)^{\mathscr{M}^+}$.
(ii) $\mathscr{M} \models \varphi[h] \iff \mathscr{M}^+ \models \varphi(x_1/d_1, ..., x_n/d_n)$.

Proof. This follows from 4.3.3 and 3.2.1 and is left as question 36. □

4.3.5. **Lemma.** (***level* 4/MSc**)
Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language and let $c \in \mathscr{C}$. Let $\mathscr{L}_0$ be the language $(\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C} \setminus \{c\})$, in other words $\mathscr{L}_0$ has the same alphabet as $\mathscr{L}$, except the constant symbol c is missing. In other words, $\mathscr{L}$ is the extension $\mathscr{L}_0(c)$ by constants of $\mathscr{L}_0$. Notice that by the very definition of terms and formulas we have $\mathrm{tm}(\mathscr{L}_0) \subseteq \mathrm{tm}(\mathscr{L})$ and $\mathrm{Fml}(\mathscr{L}_0) \subseteq \mathrm{Fml}(\mathscr{L})$.

For each variable x and every $\mathscr{L}$-string s let s^c_x be the result of replacing the symbol c by the variable x. Then

(i) (a) If t is an $\mathscr{L}$-term, then t^c_x is an $\mathscr{L}_0$-term and $\mathrm{Fr}(t^c_x) \subseteq \mathrm{Fr}(t) \cup \{x\}$.
(b) If φ is an $\mathscr{L}$-formula, then φ^c_x is an $\mathscr{L}_0$-formula and $\mathrm{Fr}(\varphi^c_x) \subseteq \mathrm{Fr}(\varphi) \cup \{x\}$.

(ii) (a) If $\varphi \in \mathrm{Fml}(\mathscr{L}_0)$ then $\varphi(x/c)^c_x = \varphi$.
(b) If x does not occur in $\varphi \in \mathrm{Fml}(\mathscr{L})$ then $\varphi^c_x(x/c) = \varphi$.

(iii) If φ is a logical axiom of $\mathscr{L}$ and c does not occur in the scope of the quantifier $\forall x$ in φ then $\vdash_{\mathscr{L}_0} \varphi^c_x$.

(iv) If $\Sigma \subseteq \mathrm{Fml}(\mathscr{L}_0)$, c does not occur in the scope of the quantifier $\forall x$ in $\varphi \in \mathrm{Fml}(\mathscr{L})$ and if $\Sigma \vdash_{\mathscr{L}} \varphi$ then $\Sigma \vdash_{\mathscr{L}_0} \varphi^c_x$

Proof. (i) is a straightforward induction on the complexity of terms (to obtain (a)) and on the complexity of formulas (to obtain (b)).

(ii) is simply a statement about replacing letters in strings.

(a) It is obvious that the replacement operations $x \mapsto c$ and $c \mapsto x$ are inverse to each other, provided c does not occur in φ.

(b) It is obvious that the replacement operations $c \mapsto x$ and $x \mapsto c$ are inverse to each other, provided x does not occur in φ.

(iii). For the axiom schemes (**AxProp**) and (**Ax**$\forall \rightarrow$) this is immediate from (i) even if c occurs in the scope of the quantifier $\forall x$ in φ.

Suppose φ is an instance of (**AxSubst**). Hence there are $y \in \mathrm{Vbl}$, $t \in \mathrm{tm}(\mathscr{L})$ and $\psi \in \mathrm{Fml}(\mathscr{L})$ such that φ is $\forall y \psi \rightarrow \psi(y/t)$.

<u>Case 1.</u> $y = x$.

Then c does not occur in ψ since by assumption c does not occur in the scope of the quantifier $\forall x$ in φ. Clearly then $\psi(y/t)^c_x = \psi(y/t^c_x)$ (observe that $y = x$ is free in ψ for t^c_x, since this is true for all variables occurring in t by assumption and for $x = y$ anyway). Hence φ^c_x is $\forall y \psi \rightarrow \psi(y/t^c_x)$ which is a logical axiom of $\mathscr{L}_0$ (and of $\mathscr{L}$) by (i).

<u>Case 2.</u> $y \neq x$.

Firstly, y is free in ψ^c_x for t^c_x, since either c does not occur in t and there is nothing to do, or c does occur in t and no free occurrence of y in ψ is in the scope of $\forall x$ in ψ (otherwise c would occur in the scope of $\forall x$ in $\psi(y/t)$). Clearly $\psi(y/t)^c_x = \psi^c_x(y/t^c_x)$ and so φ^c_x is $\forall y \psi^c_x \rightarrow \psi^c_x(y/t^c_x)$ which is a logical axiom of $\mathscr{L}_0$ (and of $\mathscr{L}$) by (i).

Suppose φ is an instance of (**AxGen**). Hence there are $\psi \in \mathrm{Fml}(\mathscr{L})$ and $y \in \mathrm{Vbl}$ which does not occur free in ψ such that φ is $\psi \rightarrow \forall y \psi$.
<u>Case 1.</u> $y = x$.

Then c does not occur in ψ since by assumption c does not occur in the scope of the quantifier $\forall x$ in φ. But then c does not occur in φ either and therefore $\varphi^c_x = \varphi$ is a logical axiom of $\mathscr{L}_0$ (and of $\mathscr{L}$) by (i).

Case 2. $y \neq x$.

Then y is not a free variable of ψ^c_x by (i)(b). Hence $\psi^c_x \to \forall y\ \psi^c_x$ is a logical axiom, too. Since $\psi^c_x \to \forall y\ \psi^c_x$ is φ^c_x this shows the claim.

Suppose φ is an instance of (**AxEq**). If φ is an instance of (a),(b) or (d) of (**AxEq**), then c does not occur in φ and there is nothing to do.

So we may assume that φ is an instance of (**AxEq**)(c). This means, φ is $y \doteq z \to t(w/z) \doteq t(w/y)$ for some $t \in \mathrm{tm}(\mathscr{L})$ and some $w, y, z \in \mathrm{Vbl}$.

Case 1. $x \neq w$.

Then $t(w/y)^c_x = t^c_x(w/y)$ and $t(w/z)^c_x = t^c_x(w/z)$. Using (i), φ^c_x is $y \doteq z \to t^c_x(w/z) \doteq t^c_x(w/y)$, which is a logical axiom of $\mathscr{L}_0$ (and of $\mathscr{L}$) by (i).

Case 2. $x = w$.

Let u be a variable, different from x, y, z which does not occur in t and let $s = t^c_u$. By (ii), s is an $\mathscr{L}_0$-term and as $u \notin \mathrm{Fr}(t)$, $t = s(u/c)$. Hence $t(w/z) = t(x/z) = s(u/c)(x/z) \overset{\text{as } u \neq z,x}{=} s(x/z)(u/c)$ and

$$(*) \qquad t(w/z)^c_x = s(x/z)(u/c)^c_x \overset{\text{as } c \text{ does not occur in } s(x/z)}{=} s(x/z)(u/x).$$

(observe that $s(x/z)(u/x)$ is not $s(u/x)(x/z)$ in general). The same reasoning with z replaced by y shows

$$(+) \qquad t(w/y)^c_x = s(x/y)(u/x).$$

By (**AxEq**)(c) and (**Ax**$\forall$) we have $\vdash \forall u\ y \doteq z \to s(x/z) \doteq s(x/y)$. Hence by (**AxSubst**) we obtain $\vdash y \doteq z \to s(x/z)(u/x) \doteq s(x/y)(u/x)$. Since φ^c_x is $y \doteq z \to t(w/z)^c_x \doteq t(w/y)^c_x$, $(*)$ and $(+)$ show that $\vdash_{\mathscr{L}_0} \varphi^c_x$.

Finally suppose φ is an instance of (**Ax**$\forall$),i.e. φ is of the form $\forall x_1 ... x_k\ \psi$ for some instance ψ of the other logical axioms of $\mathscr{L}$. We have shown that $\vdash_{\mathscr{L}_0} \psi^c_x$ (since c also is not in the scope of $\forall x$ in ψ). By the Generalisation Theorem 2.2.2 we get $\vdash_{\mathscr{L}_0} \forall x_1 ... x_n\ \psi^c_x$ as desired.

(iv) We start with a

Claim. If φ is a logical axiom of $\mathscr{L}$ or $\varphi \in \Sigma$, then $\Sigma \vdash_{\mathscr{L}_0} \varphi^c_x$.

Proof of the claim. If φ is a logical axiom of $\mathscr{L}$, then by (iii) $\vdash_{\mathscr{L}_0} \varphi^c_x$. If $\varphi \in \Sigma$, then as $\Sigma \subseteq \mathrm{Fml}(\mathscr{L}_0)$, $\varphi = \varphi^c_x$ and $\Sigma \vdash_{\mathscr{L}_0} \varphi^c_x$. $\square$

Now we prove (iv) by induction on the length k of a proof $(\psi_1, ..., \psi_k)$ of φ from Σ (recall that $\psi_k = \varphi$ By definition).

If $k = 1$, then we may apply the claim. For the induction step $k \Rightarrow k+1$ we also may apply the claim, provided ψ_{k+1} is a logical axiom of $\mathscr{L}$ or $\psi_{k+1} \in \Sigma$. It remains to show the assertion if ψ_{k+1} is obtained from $\psi_1, ..., \psi_k$ by Modus Ponens. Pick $i, j \leq n$ such that ψ_j is $\psi_i \to \psi_{k+1}$. Since $\Sigma \vdash_{\mathscr{L}} \psi_i$ and $\Sigma \vdash_{\mathscr{L}} \psi_j$ there is a finite subset Σ_0 of Σ with $\Sigma_0 \vdash_{\mathscr{L}} \psi_i$ and $\Sigma_0 \vdash_{\mathscr{L}} \psi_j$.

Let y be a variable, not occurring in $\Sigma_0, \psi_1, ..., \psi_{k+1}$. By the induction hypothesis we get $\Sigma_0 \vdash_{\mathscr{L}_0} (\psi_i)^c_y$ and $\Sigma \vdash_{\mathscr{L}_0} (\psi_j)^c_y$. Since $(\psi_j)^c_y$ is $(\psi_i)^c_y \to (\psi_{k+1})^c_y$ we may apply Modus Ponens to obtain $\Sigma_0 \vdash_{\mathscr{L}_0} (\psi_{k+1})^c_y$. Since y does not occur in Σ_0 we get $\Sigma_0 \vdash_{\mathscr{L}_0} \forall y (\psi_{k+1})^c_y$ from the Generalisation Theorem 2.2.2. Since c is not in the scope of $\forall x$ in $\varphi = \psi_{k+1}$, x is not in the scope of $\forall x$ in $\varphi = (\psi_{k+1})^c_x$. So the substitution axiom (**AxSubst**) gives $\Sigma_0 \vdash_{\mathscr{L}_0} (\psi_{k+1})^c_y(y/x)$. Since y does not occur in ψ_{k+1}, $(\psi_{k+1})^c_y(y/x)$ is $(\psi_{k+1})^c_x$ as desired. □

4.3.6. **Proposition.** *Let $\mathscr{L}^+ = \mathscr{L}(\mathscr{D})$ be an extension by constants of $\mathscr{L}$, let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$, $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ and suppose no x_i occurs freely in any formula from Σ. Let $d_1, ..., d_n \in \mathscr{D}$ with $d_i \neq d_j$ for all $i \neq j$. Then*

$$\Sigma \vdash_{\mathscr{L}} \varphi \iff \Sigma \vdash_{\mathscr{L}^+} \varphi(x_1/d_1, ..., x_n/d_n).$$

Proof. (**level 4/MSc**)
Since proofs are finite, we may assume that $\mathscr{L}^+ = \mathscr{L}(d_1, ..., d_n)$. The implication "$\Rightarrow$" follows from the generalisation theorem 2.2.2 and (**AxSubst**).

The implication "$\Leftarrow$" in the case $n = 1$ follows from 4.3.5. A trivial induction for $n > 1$ then shows the claim: Observe that $\mathscr{L}(d_1, ..., d_n) = \mathscr{L}(d_1)(d_2)...(d_n)$ and $\varphi(x_1/d_1, ..., x_n/d_n) = \varphi(x_1/d_1)...(x_n/d_n)$. □

With the aid of 4.3.6 we can now translate deduction of formulas from formulas entirely into deduction of sentences from sentences:

4.3.7. **Theorem.** *Let $\mathscr{L}$ be language and let $\mathscr{D} = \{d_0, d_1, ...\}$ be a countable set of new constants. For $\varphi(v_0, ..., v_n) \in \mathrm{Fml}(\mathscr{L})$ let $\varphi_{\mathscr{D}}$ be the $\mathscr{L}(\mathscr{D})$-sentence*

$$\varphi_{\mathscr{D}} \ = \ \varphi(v_0/d_0, ..., v_n/d_n).$$

Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and $\varphi \in \mathrm{Fml}(\mathscr{L})$. Then

$$\Sigma \vdash_{\mathscr{L}} \varphi \iff \Sigma_{\mathscr{D}} \vdash_{\mathscr{L}(\mathscr{D})} \varphi_{\mathscr{D}},$$

where $\Sigma_{\mathscr{D}} = \{\sigma_{\mathscr{D}} \mid \sigma \in \Sigma\}$.

Proof. Since proofs are finite we may assume that $\Sigma = \{\sigma_1, ..., \sigma_k\}$. Then

$$\begin{array}{rcl} \Sigma \vdash_{\mathscr{L}} \varphi & \overset{\text{Deduction Theorem}}{\iff} & \vdash_{\mathscr{L}} (\sigma_1 \wedge ... \wedge \sigma_k) \to \varphi \\ & \overset{4.3.6}{\iff} & \vdash_{\mathscr{L}(\mathscr{D})} ((\sigma_1)_{\mathscr{D}} \wedge ... \wedge (\sigma_k)_{\mathscr{D}}) \to \varphi_{\mathscr{D}} \\ & \overset{\text{Deduction Theorem}}{\iff} & \Sigma_{\mathscr{D}} \vdash_{\mathscr{L}(\mathscr{D})} \varphi_{\mathscr{D}} \end{array}$$

□

4.3.8. **Corollary.** *In the notation of 4.3.7, a set Σ of $\mathscr{L}$-formulas is consistent (in $\mathscr{L}$) if and only if $\Sigma_{\mathscr{D}}$ is a consistent set of $\mathscr{L}(\mathscr{D})$-sentences.*

Proof. Obvious from 4.3.7. □

4.3.9. **Lemma.**

(i) If y does not occur free in $\varphi \in \mathrm{Fml}(\mathscr{L})$ then $\vdash \forall x\varphi \to \forall y(\varphi(x/y))$.

(ii) If $\varphi, \psi \in \mathrm{Fml}(\mathscr{L})$ then

$$\vdash (\varphi \to \exists x\psi) \to \exists x(\varphi \to \psi).$$

Proof. (i) This has been done in question 20: By (**AxSubst**) we have $\vdash \forall x\varphi \to (\varphi(x/y))$, hence $\forall x\varphi \vdash (\varphi(x/y))$, too. Since y does not occur free in $\forall x\varphi$, the generalisation theorem 2.2.2 gives $\forall x\varphi \vdash \forall y(\varphi(x/y))$. Applying the deduction theorem 2.2.1 yields (i).

(ii). By contraposition (and 2.1.7) it suffices to show

$$\vdash \neg\neg\forall x\ \neg(\varphi \to \psi) \to \neg(\varphi \to \neg\forall x\ \neg\psi).$$

Using 2.1.7 and the propositional tautology

$$\Big(\neg\neg A_0\ \to\ \neg(A_1 \to \neg A_2)\Big)\ \leftrightarrow\ \Big(A_0 \to (A_1 \wedge A_2)\Big)$$

it suffices to show

$$\vdash \forall x\ \neg(\varphi \to \psi) \to (\varphi \wedge \forall x\ \neg\psi).$$

By the deduction theorem 2.2.1 it suffices to show

$$(*) \qquad \forall x\ \neg(\varphi \to \psi) \vdash \varphi \wedge \forall x\ \neg\psi$$

By question 17 it remains to show

(a) $\forall x\ \neg(\varphi \to \psi) \vdash \varphi$ and
(b) $\forall x\ \neg(\varphi \to \psi) \vdash \forall x\ \neg\psi$.

Now by (**AxSubst**) applied to the term x we know $\forall x\ \neg(\varphi \to \psi) \vdash \neg(\varphi \to \psi)$. By propositional logic and 2.1.7 we know that $\vdash \neg(\varphi \to \psi) \to \varphi$ and $\vdash \neg(\varphi \to \psi) \to \neg\psi$. Hence by MP we get $\forall x\ \neg(\varphi \to \psi) \vdash \varphi$ (which shows (a)) and $\forall x\ \neg(\varphi \to \psi) \vdash \neg\psi$. Since x is not a free variable of $\forall x\ \neg(\varphi \to \psi)$ we can apply the generalisation theorem to obtain (b). □

4.3.10. **Proposition.** *Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language and let $\zeta : \mathrm{Fml}(\mathscr{L})(1) \longrightarrow \mathscr{D}$ be a bijection onto a set $\mathscr{D}$ which is disjoint from the alphabet of $\mathscr{L}$. Let $\mathscr{L}'$ be the language $\mathscr{L}' = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C} \cup \mathscr{D})$.*

If $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ is consistent, then also

$$\Sigma' := \Sigma \cup \{(\exists x\varphi) \to \varphi(x/\zeta(\varphi)) \mid \varphi(x) \in \mathrm{Fml}(\mathscr{L}), x \in \mathrm{Vbl}\} \subseteq \mathrm{Sen}(\mathscr{L}')$$

is consistent.

Proof. (**level 4/MSc**)
Firstly, notice that by 4.3.6, it is not important to explicitly mention whether Σ is consistent in $\mathscr{L}$ or in $\mathscr{L}'$. We will make use of 4.3.6 in the proof below, without mentioning the language: If we say "is a proof" or "is inconsistent", then this always means w.r.t. $\mathscr{L}'$.

We have to show that no contradiction is provable from Σ' in $\mathscr{L}'$. Since every proof from Σ' is also a proof of a finite subset of Σ' we may assume that Σ is finite and it suffices to show by induction on $k \in \mathbb{N}_0$ the following:

If $\varphi_1(x_1), ..., \varphi_k(x_k) \in \mathrm{Fml}(\mathscr{L}), x_1, ..., x_k \in \mathrm{Vbl}$ (**NOT** necessarily $x_i \neq x_j$ for $i \neq j$) then

$$\Sigma \cup \{(\exists x_i\varphi_i) \to \varphi_i(x_i/\zeta(\varphi_i)) \mid i \in \{1, ..., k\}\}$$

is consistent in $\mathscr{L}'$.

If $k = 0$ then this holds by assumption. For the induction step $k-1 \Rightarrow k$ we take $\varphi_1(x_1), ..., \varphi_k(x_k) \in \mathrm{Fml}(\mathscr{L})$ with $\varphi_i \neq \varphi_j$ for $i \neq j$. Let $d_i := \zeta(\varphi_i)$ $(1 \leq i \leq k)$. Since $\varphi_i \neq \varphi_j$ for $i \neq j$ we have $d_i \neq d_j$ for $i \neq j$, too. For $i \in \{1, ..., k\}$ let σ_i be the $\mathscr{L}'$-sentence

$$\sigma_i\ :=\ (\exists x_i\varphi_i) \to \varphi_i(x_i/d_i).$$

We assume that $\Sigma \cup \{\sigma_1, ..., \sigma_k\}$ is inconsistent and we show that also $\Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\}$ is inconsistent.

We have $\Sigma \cup \{\sigma_1, ..., \sigma_k\} \vdash \gamma \wedge \neg\gamma$ for some $\gamma \in \mathrm{Fml}(\mathscr{L})$. By the deduction theorem 2.2.1 we have $\Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\} \vdash \sigma_k \to \gamma \wedge \neg\gamma$. As $\sigma_k \to \gamma \wedge \neg\gamma$ is provably equivalent to $\neg\sigma_k$ we obtain

$$(\dagger) \qquad \Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\} \vdash \neg\sigma_k.$$

Since Σ is finite there is a variable which does not occur in $\Sigma \cup \{\sigma_1, ..., \sigma_k\}$. We pick such a variable y and define

$$\psi := (\exists x_k \varphi_k(x_k)) \to \varphi_k(x_k/y).$$

Notice: since y does not occur in σ_k we have $(\sigma_k)_y^{d_k} = \psi$, hence $(\neg\sigma_k)_y^{d_k} = (\neg\psi)$ as well.

Since $\Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\} \subseteq \mathrm{Sen}(\mathscr{L}(d_1, ..., d_{k-1}))$ and y does not occur in σ_k, we may apply 4.3.5(iv) to obtain

$$(\ddagger) \qquad \Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\} \vdash (\neg\sigma_k)_y^{d_k}.$$

Since y does not occur in $\Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\}$, we may apply the generalisation theorem 2.2.2 to ($\ddagger$) and obtain (using $(\neg\sigma_k)_y^{d_k} = (\neg\psi)$)

$$(*) \qquad \Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\} \vdash \forall y \neg\psi.$$

Thus, in order to reach our goal (i.e. to show that $\Sigma \cup \{\sigma_1, ..., \sigma_{k-1}\}$ is inconsistent) it suffices to prove $\vdash \exists y \psi$:

Since x_k does not occur free in $\neg\varphi_k(x_k/y)$ we may apply 4.3.9(i) and get $\vdash \forall y(\neg\varphi_k(x_k/y)) \to \forall x_k \neg\varphi_k$ (observe that $\varphi_k(x_k/y)(y/x_k)$ is φ_k). Applying contraposition we obtain

$$\vdash \exists x_k \varphi_k \to \exists y(\varphi_k(x_k/y)).$$

Applying 4.3.9(ii) to $\exists x_k \varphi_k$ and $\varphi_k(x_k/y)$ gives

$$\vdash \exists y(\exists x_k \varphi_k \to \varphi_k(x_k/y)).$$

But $\exists x_k \varphi_k \to \varphi_k(x_k/y)$ is ψ as desired. □

4.3.11. **Theorem.** *Let $\mathscr{L} = (\lambda : \mathscr{R} \longrightarrow \mathbb{N}, \mu : \mathscr{F} \longrightarrow \mathbb{N}, \mathscr{C})$ be a language and let $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ be consistent. The there is an extension by constants $\mathscr{L}^*$ of $\mathscr{L}$ and a consistent subset Σ^* of $\mathscr{L}^*$-sentences containing Σ such that Σ^* possess a system of witnesses (in $\mathscr{L}^*$).*

Proof. By induction we define constant extensions $\mathscr{L}^n$ and a set $\Sigma^n \subseteq \mathrm{Sen}(\mathscr{L})^n$ as follows:

$\mathscr{L}^0 := \mathscr{L}$ and $\Sigma^0 = \Sigma$. For the induction step we apply 4.3.10 to $\mathscr{L}^n$ and Σ^n to obtain

$$\mathscr{L}^{n+1} = (\mathscr{L}^n)' \text{ and } \Sigma^{n+1} = (\Sigma^n)'$$

in the notation of 4.3.10. By 4.3.10 we see inductively, that Σ^{n+1} is a consistent set of $\mathscr{L}^{n+1}$-sentences containing Σ^n. Hence also $\Sigma^* := \bigcup_{n \in \mathbb{N}} \Sigma^n$ is a consistent set of $\mathscr{L}^*$-sentences containing Σ, where $\mathscr{L}^*$ is the extension by constants of $\mathscr{L}$ obtained by adding all constants symbols from all languages $\mathscr{L}^n$. By construction, Σ^* possess a system of witnesses (in $\mathscr{L}^*$). □

4.4. **The completeness and the compactness theorem.**

4.4.1. **Theorem.** *(Gödel, 1929)*
Every consistent set of $\mathscr{L}$-sentences is satisfiable (i.e. has a model).

Proof. See 4.1.5. We have filled in the details by 4.2.6 and 4.3.11. □

4.4.2. **Theorem.** *(Completeness Theorem)*
For all $\Sigma, \Phi \subseteq \mathrm{Fml}(\mathscr{L})$ we have

$$\Sigma \models \Phi \Rightarrow \Sigma \vdash \Phi.$$

(Recall that the implication $\Leftarrow$ also holds by the Soundness Theorem 3.2.4.)

Proof. It is obviously enough to show that

$$\Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi$$

for every $\mathscr{L}$-formula φ. Firstly assume that $\Sigma \subseteq \mathrm{Sen}(\mathscr{L})$ and $\varphi \in \mathrm{Sen}(\mathscr{L})$. Suppose $\Sigma \nvdash \varphi$. By 4.2.4(i), $\Sigma \cup \{\neg\varphi\}$ is consistent. Hence by 4.4.1, $\Sigma \cup \{\neg\varphi\}$ has a model $\mathscr{M}$. So $\mathscr{M} \models \Sigma$ and $\mathscr{M} \not\models \varphi$, which implies $\Sigma \not\models \varphi$.

Now let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and $\varphi \in \mathrm{Fml}(\mathscr{L})$ with $\Sigma \nvdash \varphi$. Let $\mathscr{D}$ be a countable (infinite) set of constant symbols, new w.r.t. $\mathscr{L}$. By 4.3.7 we know $\Sigma_{\mathscr{D}} \nvdash_{\mathscr{L}(\mathscr{D})} \varphi_{\mathscr{D}}$. By what we have shown above there is a model $\mathscr{N}$ of $\Sigma_{\mathscr{D}}$ with $\mathscr{N} \not\models \varphi_{\mathscr{D}}$. Let $\mathscr{M}$ be the restriction of $\mathscr{N}$ to $\mathscr{L}$ and let $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ be defined by $h(v_i) = d_i^{\mathscr{N}}$. Then by 4.3.4, $\mathscr{M} \models \Sigma[h]$ and $\mathscr{M} \not\models \varphi[h]$. Thus $\Sigma \not\models \varphi$, too. □

4.4.3. **Corollary.** *Every consistent set of $\mathscr{L}$-formulas is satisfiable (i.e. has a model at some assignment).*

Proof. This is done in question 37 □

4.4.4. **Theorem.** *(Compactness Theorem)*
If $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and every finite subset of Σ has a model at some assignment, then also Σ has a model at some assignment.

Proof. By 4.4.3 it is enough to show that Σ is consistent. Since proofs are finite, this can be checked by looking into finite subsets of Σ. But our assumption and the Soundness Theorem imply that every finite subset of Σ is consistent. □

4.4.5. *Example.* (Finite and infinite structures) Let $\mathscr{L}$ be any language. For $n \in \mathbb{N}$, let ε_n be the $\mathscr{L}$-sentence

$$\exists x_1 \ldots x_n \ \forall y \bigvee_{1 \le i \le n} y \doteq x_i$$

and let σ_n be the $\mathscr{L}$-sentence

$$\exists x_1 \ldots x_n \bigwedge_{1 \le i < j \le n} x_i \not\doteq x_j.$$

(i) Obviously an $\mathscr{L}$-structure $\mathscr{M}$ is a model of ε_n if and only if $\mathscr{M}$ has size at most n and $\mathscr{M}$ is a model of σ_n if and only if $\mathscr{M}$ has size at least n. Consequently $\mathscr{M}$ is a model of $\varepsilon_n \wedge \sigma_n$ if and only if the size of $\mathscr{M}$ is n.

(ii) It follows from (i) that an $\mathscr{L}$-structure $\mathscr{M}$ is infinite if and only is $\mathscr{M}$ is a model of the set

$$\Sigma_\infty = \{\sigma_n \mid n \in \mathbb{N}\}.$$

(iii) In spite of (ii) there is no **single** sentence ψ in any language $\mathscr{L}$ such that the infinite $\mathscr{L}$-structures are **precisely** the models of ψ; informally: there is no way to express that a structure is infinite, using only a single sentence.

To see this we assume there is such a sentence ψ and we use the compactness theorem to derive a contradiction: Consider the set

$$\Gamma = \{\neg\psi\} \cup \Sigma_\infty.$$

Let $\Gamma_0 \subseteq \Gamma$ be finite and choose $n \in \mathbb{N}$ with

$$\Gamma_0 \subseteq \{\neg\psi, \sigma_1, \ldots, \sigma_n\}$$

Then any finite $\mathscr{L}$-structure with at least n-elements (in its universe) satisfies Γ_0, because by assumption it satisfies $\neg\psi$ and the σ_i with $i \leq n$ hold in any structure of size n (see (i)). By the compactness theorem, Γ has a model $\mathscr{M}$. Since $\mathscr{M} \models \Sigma_\infty$, $\mathscr{M}$ is infinite. However by assumption, every model of $\neg\psi$ is finite, a contradiction.

(iv) There are sentences in some languages that are consistent and that have only infinite models. One such example will be done in question 38.

4.4.6. *Example.* There is an **infinite** group G that has the following surprising property: Whenever φ is a sentence in the language of groups that is true in **all finite** groups, then φ is true in G.

This can be seen with the aid of the compactness theorem. Let Σ be the set of all sentences in the language of groups that are true in all finite groups and let

$$\Gamma = \Sigma \cup \Sigma_\infty.$$

(Σ_∞ defined as in 4.4.5) The example just claims that Γ has a model and by the compactness theorem we just need to show that every finite subset Γ_0 of Γ has a model. Since Γ_0 is finite, there is some $n \in \mathbb{N}$ such that

$$(*) \qquad \Gamma_0 \subseteq \Sigma \cup \{\sigma_1, \ldots, \sigma_n\},$$

where σ_n is as in 4.4.5. The group $(\mathbb{Z}/n\mathbb{Z}, +, 0)$ is a model of Σ by definition of Σ. Since $\mathbb{Z}/n\mathbb{Z}$ has size n, this group also satisfies $\sigma_1, \ldots, \sigma_n$. By $(*)$, this group is a model of Γ_0, as required.

5. Comparing structures

5.1. **Formulas preserved by maps.**

Firstly we simplify notation. Let $\mathscr{M}$ be an $\mathscr{L}$-structure, $t(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$, $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ and let $a_1, ..., a_n \in |\mathscr{M}|$ (not necessarily $a_i \neq a_j$ for $i \neq j$). We write $t^{\mathscr{M}}[a_1, ..., a_n]$ instead of $t^{\mathscr{M}}[h]$ and $\mathscr{M} \models \varphi[a_1, ..., a_n]$ instead of $\mathscr{M} \models \varphi[h]$, where $h : \mathrm{Vbl} \longrightarrow |\mathscr{M}|$ with $h(x_1) = a_1, ..., h(x_n) = a_n$. Recall from 3.1.4 that this is well defined.

5.1.1. **Definition.** Let $\mathscr{M}, \mathscr{N}$ be $\mathscr{L}$-structures. A **map between $\mathscr{M}$ and $\mathscr{N}$** is a map $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$. We write $f : \mathscr{M} \longrightarrow \mathscr{N}$ instead of $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$.

A formula $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ is **preserved by a map** $f : \mathscr{M} \longrightarrow \mathscr{N}$, or f **respects** φ, if for all $a_1, ..., a_n$ we have

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \Rightarrow \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

5.1.2. *Examples.*

(i) If x, y are different variables, then the formula $\neg x \doteq y$ is preserved by $f : \mathscr{M} \longrightarrow \mathscr{N}$ if and only if f is injective.

(ii) If $f : \mathscr{M} \longrightarrow \mathscr{N}$ is bijective (i.e. $f : |\mathscr{M}| \longrightarrow |\mathscr{N}|$ is bijective), then we also have a map $f^{-1} : \mathscr{N} \longrightarrow \mathscr{M}$ and for every formula φ,

$$\varphi \text{ is preserved by } f \iff \neg\varphi \text{ is preserved by } f^{-1}.$$

(iii) If $\mathscr{L} = \{\emptyset, \{\circ\}, \{e\}\}$ is the language of groups, then a map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between groups preserves the formulas $y = x_1 \cdot x_2$ if and only if $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$. I.e. we are describing group homomorphisms here.

5.1.3. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between $\mathscr{L}$-structures is called an **$\mathscr{L}$-homomorphism** if f respects all atomic formulas.

5.1.4. **Lemma.** *Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a map between $\mathscr{L}$-structures. The following are equivalent:*

(i) f is an $\mathscr{L}$-homomorphism.

(ii) f satisfies each of the following conditions:

(a) For all $R \in \mathscr{R}$ of arity n and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$(a_1, ..., a_n) \in \mathscr{R}^{\mathscr{M}} \Rightarrow (f(a_1), ..., f(a_n)) \in \mathscr{R}^{\mathscr{N}}.$$

(b) For all $F \in \mathscr{F}$ of arity n and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$f(F^{\mathscr{M}}(a_1, ..., a_n)) = F^{\mathscr{N}}(f(a_1), ..., f(a_n)).$$

(c) For all $c \in \mathscr{C}$ we have

$$f(c^{\mathscr{M}}) = c^{\mathscr{N}}.$$

(iii) f respects each of the following formulas:

(a) all formulas of the form $R(v_1, ..., v_n)$, where $R \in \mathscr{R}$ is a relation symbol of $\mathscr{L}$ of arity n.

(b) all formulas of the form $v_0 \doteq F(v_1, ..., v_n)$, where $F \in \mathscr{F}$ is a function symbol of $\mathscr{L}$ of arity n.

(c) all formulas of the form $v_0 \doteq c$, where $c \in \mathscr{C}$ is a constant symbol of $\mathscr{L}$

If this is the case, then for every $\mathscr{L}$-term $t(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$ and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$f(t^{\mathscr{M}}[a_1, ..., a_n]) = t^{\mathscr{N}}(f(a_1), ..., f(a_n)).$$

Proof. (i)⇒(iii) and (iii)⇒(ii) are trivial. We assume now (ii) and show (i) as well as the additional statement. A straightforward induction on the complexity of $t(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$ using (ii)(b) and (ii)(c) shows that $f(t^{\mathscr{M}}[a_1, ..., a_n]) = t^{\mathscr{N}}(f(a_1), ..., f(a_n))$ for all $a_1, ..., a_n \in |\mathscr{M}|$.

Clearly this, together with (ii)(a) proves (i). □

5.1.5. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between $\mathscr{L}$-structures is called an **embedding** if f preserves every quantifier free formula. If $|\mathscr{M}| \subseteq |\mathscr{N}|$ and the inclusion $|\mathscr{M}| \longrightarrow |\mathscr{N}|$ is an embedding then $\mathscr{M}$ is called a **substructure** of $\mathscr{N}$. If in addition $|\mathscr{M}| \neq |\mathscr{N}|$, then $\mathscr{M}$ is called a **proper substructure** of $\mathscr{N}$.

5.1.6. **Lemma.** *Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be a map between $\mathscr{L}$-structures. The following are equivalent:*

(i) f is an embedding.

(ii) f is an <u>injective</u> $\mathscr{L}$-homomorphism such that for all $R \in \mathscr{R}$ of arity n and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$(a_1, ..., a_n) \in \mathscr{R}^{\mathscr{M}} \Leftrightarrow (f(a_1), ..., f(a_n)) \in \mathscr{R}^{\mathscr{N}}.$$

(iii) For all $\varphi(x_1, ..., x_n) \in \text{at-Fml}(\mathscr{L})$ and all $a_1, ..., a_n \in |\mathscr{M}|$ we have

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \iff \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

Proof. (i)⇒(ii). By 5.1.4 and 5.1.2(i), f is an injective $\mathscr{L}$-homomorphism. If $R \in \mathscr{R}$ of arity n and $a_1, ..., a_n \in |\mathscr{M}|$ then we know

$$(a_1, ..., a_n) \in \mathscr{R}^{\mathscr{M}} \Leftarrow (f(a_1), ..., f(a_n)) \in \mathscr{R}^{\mathscr{N}},$$

since f respects the quantifier free formula $\neg R(v_1, ..., v_n)$. Since f also respects $R(v_1, ..., v_n)$, this proves (ii).

(ii)⇒(iii). Let $\varphi(x_1, ..., x_n) \in \text{at-Fml}(\mathscr{L})$ and $a_1, ..., a_n \in |\mathscr{M}|$.

<u>Case 1.</u> φ is of the form $t \doteq s$ with $t(x_1, ..., x_n), s(x_1, ..., x_n) \in \mathrm{tm}(\mathscr{L})$.

Since f is an $\mathscr{L}$-homomorphism, f respects φ. Furthermore, by 5.1.4 , we have

$$f(t^{\mathscr{M}}[a_1, ..., a_n]) = t^{\mathscr{N}}(f(a_1), ..., f(a_n)) \text{ and}$$

$$f(s^{\mathscr{M}}[a_1, ..., a_n]) = s^{\mathscr{N}}(f(a_1), ..., f(a_n)).$$

Hence if $\mathscr{N} \models \varphi[f(a_1), ..., f(a_n)]$, then

$$f(s^{\mathscr{M}}[a_1, ..., a_n]) = s^{\mathscr{N}}(f(a_1), ..., f(a_n)) = t^{\mathscr{N}}(f(a_1), ..., f(a_n)) = f(t^{\mathscr{M}}[a_1, ..., a_n])$$

and since f is assumed to be injective we obtain $\mathscr{M} \models \varphi[a_1, ..., a_n]$.

<u>Case 2.</u> φ is of the form $R(t_1, ..., t_k)$ where $R \in \mathscr{R}$ is of arity n and $t_1, ..., t_k \in \mathrm{tm}(\mathscr{L})$ with free variables in $\{x_1, ..., x_n\}$.

Again by 5.1.4, we have

$$f(t_i^{\mathscr{M}}[a_1, ..., a_n]) = t_i^{\mathscr{N}}(f(a_1), ..., f(a_n)) \ (1 \leq i \leq k).$$

Together with the equivalence assumed in (ii) this shows

$$\mathscr{M} \models R(t_1, ..., t_k)[a_1, ..., a_n] \iff \mathscr{N} \models R(t_1, ..., t_k)[f(a_1), ..., f(a_n)].$$

(iii)$\Rightarrow$(i) holds since the equivalence in (iii) is preserved by negation and conjunction (i.e. if the equivalence holds for φ and ψ then it also holds for $\neg\varphi$ and $\varphi\wedge\psi$). Since every quantifier free formula is provably equivalent to a formula that is built up from atomic formulas using negations and conjunctions, this shows (i). $\square$

5.1.7. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ between $\mathscr{L}$-structures is called an **elementary embedding** if f preserves all formulas. If $|\mathscr{M}| \subseteq |\mathscr{N}|$ and the inclusion $|\mathscr{M}| \longrightarrow |\mathscr{N}|$ is an elementary embedding then $\mathscr{M}$ is called a an **elementary substructure** of $\mathscr{N}$, denoted by $\mathscr{M} \prec \mathscr{N}$ and $\mathscr{N}$ is called an **elementary extension** of $\mathscr{M}$.

At the moment we have only one (rather trivial) class of examples of elementary embeddings:

5.1.8. **Definition.** A map $f : \mathscr{M} \longrightarrow \mathscr{N}$ is called an ($\mathscr{L}$-)**isomorphism** if f is a bijective embedding. Two $\mathscr{L}$-structures are called **isomorphic** if there is an isomorphism $\mathscr{M} \longrightarrow \mathscr{N}$.

5.1.9. *Warning.* In general, a bijective $\mathscr{L}$-homomorphism is not an embedding. E.g. if $\mathscr{L}$ has a unique non-logical symbol, namely a unary relation symbol R. Let $\mathscr{N}$ be an $\mathscr{L}$-structure with $R^{\mathscr{N}} = |\mathscr{N}|$ and let $\mathscr{M}$ be the $\mathscr{L}$-structure with universe $|\mathscr{N}|$ and $R^{\mathscr{M}} = \emptyset$. Then the identity map $|\mathscr{M}| \longrightarrow |\mathscr{N}|$ is a bijective $\mathscr{L}$-homomorphism. But this map is not an embedding!

On the other hand, a bijective $\mathscr{L}$-homomorphism $f : \mathscr{M} \longrightarrow \mathscr{N}$ is an isomorphism if and only if the inverse map $f^{-1} : \mathscr{N} \longrightarrow \mathscr{M}$ is a homomorphism (as follows from 5.1.6(i)$\Leftrightarrow$(iii)).

5.1.10. **Lemma.** *Every $\mathscr{L}$-isomorphism is an elementary embedding.*

Proof. Let $f : \mathscr{M} \longrightarrow \mathscr{N}$ be an $\mathscr{L}$-isomorphism. It is straightforward to show by induction on the complexity of $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ that

$$\mathscr{M} \models \varphi[a_1, ..., a_n] \iff \mathscr{N} \models \varphi[f(a_1), ..., f(a_n)].$$

The case of atomic formulas holds by 5.1.6. For quantification use the surjectivity of f. Also see question 41 of the example sheets for this argument. $\square$

5.1.11. **Proposition.** (***level* 4/MSc**)
Let $\Sigma \subseteq \mathrm{Fml}(\mathscr{L})$ and let $\mathscr{M}$ be a model of all atomic $\mathscr{L}$-sentences provable from Σ. Then there is a unique $\mathscr{L}$-homomorphism $f : \mathrm{TmAlg}(\Sigma) \longrightarrow \mathscr{M}$.

Proof. We define $f : |\,\mathrm{TmAlg}(\Sigma)| \longrightarrow |\mathscr{M}|$ by $f(t/\sim_\Sigma) := t^{\mathscr{M}}$. Recall from 3.3.3 that $|\,\mathrm{TmAlg}(\Sigma)| = \mathrm{ctm}(\mathscr{L})/\sim_\Sigma$. f is well defined since for $t, s \in \mathrm{ctm}(\mathscr{L})$ with $t/\sim_\Sigma = s/\sim_\Sigma$ we have $\Sigma \vdash t \doteq s$ (by definition), hence by assumption, $\mathscr{M} \models t \doteq s$ and $t^{\mathscr{M}} = s^{\mathscr{M}}$.

On the other hand, if $g : \mathrm{TmAlg}(\Sigma) \longrightarrow \mathscr{M}$ is an $\mathscr{L}$-homomorphism and $t \in \mathrm{ctm}(\mathscr{L})$ then $g(t/\sim_\Sigma) = f(t/\sim_\Sigma)$: g respects the $\mathscr{L}$-formula $\varphi(v_0)$ defined as $v_0 \doteq t$. Since $\mathrm{TmAlg}(\Sigma) \models \varphi[a]$ with $a = t/\sim_\Sigma \in |\,\mathrm{TmAlg}(\Sigma)|$ we have $\mathscr{M} \models \varphi[g(a)]$, i.e. $g(a) = t^{\mathscr{M}} = f(a)$. $\square$

5.1.12. *Examples.* (**level 4/MSc**)

(i) The universal property in 5.1.11 of $\operatorname{TmAlg}(\Sigma)$ does **not** determine $\operatorname{TmAlg}(\Sigma)$ among $\mathscr{L}$-structures. Look at the following example: Let T be the theory of commutative unital rings in the language $\mathscr{L} = \{+, \cdot, 0, 1\}$ (where we don't have a function symbol for $-$). Then $\operatorname{TmAlg} T = (\mathbb{N}_0, +, \cdot, 0, 1)$, but the structure $(\mathbb{Z}, +, \cdot, 0, 1)$ also has the universal property of 5.1.11.

On the other hand, if we work in the language $\mathscr{L} = \{+, -, \cdot, 0, 1\}$ (where we do have a function symbol for $-$), then $\operatorname{TmAlg} T = (\mathbb{Z}, +, \cdot, 0, 1)$ is a ring and in that case it is the only $\mathscr{L}$-structure having the universal property of 5.1.11.

(ii) Here is another example illustrating 5.1.11: Let $\mathscr{L} = \{+, -, \cdot, 0, 1\}$ be the language of unital rings and let Σ be the set of axioms of commutative unital rings in this language together with the sentence $\forall x\ x \doteq 0 \vee x^2 \doteq 1$. For example the field with 2 elements and the field with 3 elements are models of Σ. One can show that $\operatorname{TmAlg}(\Sigma) = \mathbb{Z}/6\mathbb{Z}$, but this ring is not a model of Σ, because for every model $\mathscr{M}$ of Σ we have $1 + 1 = 0$ or $(1 + 1)^2 = 1$, i.e. $1 + 1 + 1 = 0$.

5.2. **Elementary equivalent structures.**

5.2.1. **Definition.** Two $\mathscr{L}$-structures $\mathscr{M}$ and $\mathscr{N}$ are called **elementary equivalent** if $\mathrm{Th}(\mathscr{M}) = \mathrm{Th}(\mathscr{N})$. This is usually denoted by $\mathscr{M} \equiv \mathscr{N}$.

Hence $\mathscr{M}$ is elementary equivalent to $\mathscr{N}$ just if $\mathscr{M}$ and $\mathscr{N}$ satisfy the same $\mathscr{L}$-sentences.

5.2.2. **Lemma.** *If $f : \mathscr{M} \longrightarrow \mathscr{N}$ is an elementary embedding then $\mathscr{M} \equiv \mathscr{N}$. In particular isomorphic structures are elementary equivalent.*

Proof. We have to show $\mathscr{M} \models \varphi \iff \mathscr{N} \models \varphi$ for every $\mathscr{L}$-sentence φ. Since f is an elementary embedding, f preserves all $\mathscr{L}$-formulas, in particular we know the implication "$\Rightarrow$". Since f also preserves $\neg\varphi$ we also have $\mathscr{M} \models \varphi \Leftarrow \mathscr{N} \models \varphi$. $\square$

5.2.3. *Example.* The converse of 5.2.2 fails, more precisely if $f : \mathscr{M} \longrightarrow \mathscr{N}$ is an embedding and $\mathscr{M} \equiv \mathscr{N}$ then f is in general not an elementary embedding. For example let $\mathscr{N} = (\mathbb{N}, \leq)$, $\mathscr{M} = (\mathbb{N} \setminus \{1\}, \leq)$ and $f : \mathscr{M} \longrightarrow \mathscr{N}$ be the inclusion. Observe that the map $g : \mathscr{M} \longrightarrow \mathscr{N}$ given by $g(x) = x - 1$ is an isomorphism !

We will now produce a method to obtain elementary extensions. First let us illustrate what can be achieved by doing this: Let $\mathscr{L}_{\mathrm{OR}}$ be the language $\{\leq, +, -, \cdot, 0, 1\}$ of ordered rings; here $\leq$ is a binary relation symbol, $+$ and $\cdot$ are binary function symbols, $-$ is a unary function symbol and $0, 1$ are constant symbols. Let $\mathfrak{R}$ be the $\mathscr{L}_{\mathrm{OR}}$-structure with universe $\mathbb{R}$ and with the natural interpretation of the non-logical symbols from $\mathscr{L}_{\mathrm{OR}}$. Suppose we have produced a <u>proper</u> elementary extension $\mathfrak{R}^{\#}$ of $\mathfrak{R}$. Pick an element $\alpha \in |\mathfrak{R}^{\#}|$ which is not in $\mathbb{R}$. Let $A = \{a \in \mathbb{R} \mid a \leq^{\mathfrak{R}^{\#}} \alpha\}$ and $B = \{b \in \mathbb{R} \mid \alpha \leq^{\mathfrak{R}^{\#}} b\}$. Clearly $A \cup B = \mathbb{R}$ and as $\alpha \notin \mathbb{R}$ we have $A \cap B = \emptyset$. We have three cases: $A = \emptyset$ in which case $\alpha < \mathbb{R}$, or $B = \emptyset$ in which case $\mathbb{R} < \alpha$, or $A \neq \emptyset$ and $B \neq \emptyset$. In the latter case, since $(\mathbb{R}, \leq)$ is Dedekind complete, there is a real number $s \in \mathbb{R}$ such that $A \leq s \leq B$ (i.e. for all $a \in A$, $b \in B$ we have $a \leq s \leq b$). In all three cases we can produce a positive element μ of $|\mathfrak{R}^{\#}|$ with $\mu < r$ for all positive $r \in \mathbb{R}$: If $\alpha < -\mathbb{R}$, then take $\mu = -\frac{1}{\alpha}$, if $\alpha > \mathbb{R}$, then take $\mu = \frac{1}{\alpha}$, if $A, B \neq \emptyset$ then we take $\frac{1}{|\alpha - s|}$.

What have we achieved? We have a field (better: an ordered over-field of the real field) $\mathfrak{R}^{\#}$ which shares all the properties of the real field that are expressible in our language (note that most of the statements of elementary geometry are expressible in our language.) In addition, $\mathfrak{R}^{\#}$ contains an infinitesimal element μ (which is by far not unique, e.g. also 2μ and $\mu^3 + \mu$ are infinitesimal). In $\mathfrak{R}^{\#}$ it is now possible to give a rigorous explanation of fundamental expressions of calculus, involving "informal infinitesimal expression" like the chain rule

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{†}$$

More precisely, the informal proof of the chain rule using (†) in calculus can be transformed into a strict proof.

Here is the main method how to produce elementary extensions: Let $\mathscr{M}$ be an <u>infinite</u> structure in the language $\mathscr{L}$ with universe M. We will first add constant symbols to $\mathscr{L}$, in order to name elements of M. For each $m \in M$, let c_m be a constant symbol, new with respect to $\mathscr{L}$ such that $c_{m_1} \neq c_{m_2}$ for all $m_1, m_2 \in M$ with $m_1 \neq m_2$. The set of all these new constants is denoted by C_M.

Let $(\mathscr{M}, M)$ be the expansion of $\mathscr{M}$ to the $\mathscr{L}(C_M)$-structure which interprets the c_m as

$$(c_m)^{(\mathscr{M},M)} := m.$$

What are the models of $\mathrm{Th}((\mathscr{M}, M))$? Let us take one, call it $\mathscr{N}^+$. Firstly, $\mathscr{N}^+$ is an $\mathscr{L}(C_M)$-structure and $\mathscr{N}^+$ expands the $\mathscr{L}$-structure $\mathscr{N} := \mathscr{N}^+ \restriction \mathscr{L}$. For every $\mathscr{L}(C_M)$-sentence φ^+ with $(\mathscr{M}, M) \models \varphi^+$ we have $\mathscr{N}^+ \models \varphi^+$. In particular, for every $\mathscr{L}$-sentence φ with $(\mathscr{M}, M) \models \varphi$ we have $\mathscr{N}^+ \models \varphi$. Since φ is an $\mathscr{L}$-sentence, 4.3.3 says, $(\mathscr{M}, M) \models \varphi \iff \mathscr{M} \models \varphi$ and $\mathscr{N}^+ \models \varphi \iff \mathscr{N} \models \varphi$.

By question 39, $\mathscr{N} \equiv \mathscr{M}$. But we know more: For $m \in M$, the structure $\mathscr{N}^+$ comes equipped with an interpretation $(c_m)^{\mathscr{N}^+} \in |\mathscr{N}^+|$ of the constant symbol c_m. This can be viewed as a map $f : M \longrightarrow |\mathscr{N}|$ $(= |\mathscr{N}^+|)$ defined by

$$f(m) := (c_m)^{\mathscr{N}^+}.$$

Claim 1. f, when viewed as a map $\mathscr{M} \longrightarrow \mathscr{N}$ is an elementary embedding.

Proof. Let $\varphi(x_1, ..., x_n) \in \mathrm{Fml}(\mathscr{L})$ and let $m_1, ..., m_n \in M$ with

$$\mathscr{M} \models \varphi[m_1, ..., m_n].$$

We have to show $\mathscr{N} \models \varphi[f(m_1), ..., f(m_n)]$:

$$\begin{aligned}
\mathscr{M} \models \varphi[m_1, ..., m_n] &\iff (\mathscr{M}, M) \models \varphi(x_1/c_{m_1}, ..., x_1/c_{m_n}), \text{ by 4.3.4} \\
&\Longrightarrow \mathscr{N}^+ \models \varphi(x_1/c_{m_1}, ..., x_1/c_{m_n}), \text{ as } \mathscr{N}^+ \models \mathrm{Th}(\mathscr{M}, M) \\
&\iff \mathscr{N} \models \varphi[(c_{m_1})^{\mathscr{N}^+}, ..., (c_{m_n})^{\mathscr{N}^+}], \text{ by 4.3.4} \\
&\iff \mathscr{N} \models \varphi[f(m_1), ..., f(m_n)], \text{ by definition of } f
\end{aligned}$$

□

Claim 1 shows that the models of $\mathrm{Th}(\mathscr{M}, M)$ give rise to an elementary embedding from $\mathscr{M}$ into some $\mathscr{L}$-structure $\mathscr{N}$.

Claim 2. If $f : \mathscr{M} \longrightarrow \mathscr{N}$ is an elementary embedding of $\mathscr{L}$-structures, then the expansion $\mathscr{N}^+$ of $\mathscr{N}$ to an $\mathscr{L}(C_M)$-structure via

$$(c_m)^{\mathscr{N}^+} := f(m) \ (m \in M)$$

is a model of $\mathrm{Th}(\mathscr{M}, M)$.

Proof. Exercise. □

Claim 2 together with claim 1 says that an elementary embedding of $\mathscr{L}$-structures with domain $\mathscr{M}$ "is" the same as a model of $\mathrm{Th}(\mathscr{M}, M)$. This means we have now a method to produce elementary embeddings: Choose models of $\mathrm{Th}(\mathscr{M}, M)$! The completeness theorem 4.4.2 will help us of course:

5.2.4. **Proposition.** *Every infinite $\mathscr{L}$-structure $\mathscr{M}$ has proper elementary extensions.*

Proof. (**level 4/MSc**)
We continue to work with the setup above. It is enough to show that $\mathscr{M}$ has an elementary embedding $f : \mathscr{M} \longrightarrow \mathscr{N}$ such that f is **NOT** surjective. (Then upon replacing the elements of the universe M with their image in $\mathscr{N}$ we also get a proper elementary extension of $\mathscr{M}$ - this process is called identification.)

By what we have shown it is enough to find a model $\mathscr{N}^+$ of $\mathrm{Th}(\mathscr{M}, M)$ such that the underlying map f is not surjective. How to achieve this? Well, for $m \in M$, $f(m)$ was defined above as

$$f(m) = (c_m)^{\mathscr{N}^+}.$$

Therefore, an element $b \in |\mathscr{N}|$ is not in the image of f, if $b \neq (c_m)^{\mathscr{N}^+}$ for any $m \in M$. In first-order logic notation this reads as

$$(*) \qquad N^+ \models v_0 \neq c_m[b] \text{ for every } m \in M.$$

Combining $(*)$ with the characterization of elementary embeddings via models of $\mathrm{Th}(\mathscr{M}, M)$, we see that the only thing we need to verify is that the set Σ of $\mathscr{L}$-formulas (in the free variable v_0), defined as

$$\Sigma = \mathrm{Th}(\mathscr{M}, M) \cup \{v_0 \neq c_m \mid m \in M\},$$

has a model $\mathscr{N}^+$ at some assignment h (then $h(v_0)$ is not in the image of f).

We prove this by using the Compactness Theorem 4.4.4, which says it is enough to verify that every finite subset $\Sigma_0 \subseteq \Sigma$ has a model. Take $\Sigma_0 \subseteq \Sigma$ finite. Then there are $n \in \mathbb{N}$ and $m_1, ..., m_n \in M$ such that

$$\Sigma_0 \subseteq \mathrm{Th}(\mathscr{M}, M) \cup \{v_0 \neq c_{m_1}, ..., v_0 \neq c_{m_n}\}.$$

Since $\mathscr{M}$ is infinite there is some $m \in M$ different from $m_1, ..., m_n$, in other words.

$$(\mathscr{M}, M) \models v_0 \neq c_{m_i}[m] \text{ for every } i \in \{1, ..., n\}.$$

But this implies that $(\mathscr{M}, M)$ is a model of Σ_0 at some assignment. $\square$

6. Prenex normal form theorem

This section is reading assignment for level 4 and MSc students.

6.1. **Definition.** An $\mathscr{L}$-formula φ is said to be in **prenex normal form** if there are $n \in \mathbb{N}_0$, $x_1, ..., x_n \in \mathrm{Vbl}$, $Q_1, ..., Q_n \in \{\forall, \exists\}$ and a quantifier free formula χ such that φ is $Q_1 x_1 ... Q_n x_n\ \chi$.

We will show that every $\mathscr{L}$-formula is provably equivalent to a formula in prenex normal form.

6.2. **Lemma.** *If φ, ψ are $\mathscr{L}$-formulas and $x \notin \mathrm{Fr}\,\psi$, $y \notin \mathrm{Fr}\,\varphi$ then*

(i) $\vdash ((\forall x\varphi) \wedge (\forall y\psi)) \leftrightarrow \forall xy(\varphi \wedge \psi)$.
(ii) $\vdash ((\forall x\varphi) \vee (\forall y\psi)) \leftrightarrow \forall xy(\varphi \vee \psi)$.
(iii) $\vdash ((\exists x\varphi) \vee (\exists y\psi)) \leftrightarrow \exists xy(\varphi \vee \psi)$.
(iv) $\vdash ((\exists x\varphi) \wedge (\exists y\psi)) \leftrightarrow \exists xy(\varphi \wedge \psi)$.

Proof. In all statements we may assume that $x \neq y$. Otherwise, our assumption $x \notin \mathrm{Fr}\,\psi$ and $y \notin \mathrm{Fr}\,\varphi$ implies that neither x nor y occurs freely in any of the formulas under consideration; now observe that $\vdash \varphi \leftrightarrow \forall x\varphi$ for each formula φ and every variable $x \notin \mathrm{Fr}(\varphi)$.

(i) By the Completeness Theorem 4.4.2 it suffices to show $\models ((\forall x\varphi) \wedge (\forall y\psi)) \leftrightarrow \forall xy(\varphi \wedge \psi)$. Hence we have to take an $\mathscr{L}$-structure $\mathscr{M}$ and an assignment h of $\mathscr{M}$ and we have to show that $\mathscr{M} \models ((\forall x\varphi) \wedge (\forall y\psi))[h] \iff \mathscr{M} \models \forall xy(\varphi \wedge \psi)[h]$. The implication "$\Leftarrow$" is clear. For the implication "$\Rightarrow$" we use assumption $x \notin \mathrm{Fr}\,\psi$ and $y \notin \mathrm{Fr}\,\varphi$ and 3.1.4.

(ii). By the Completeness Theorem 4.4.2 it suffices to show $\models ((\forall x\varphi) \vee (\forall y\psi)) \leftrightarrow \forall xy(\varphi \vee \psi)$. Hence we have to take an $\mathscr{L}$-structure $\mathscr{M}$ and an assignment h of $\mathscr{M}$ and we have to show that

$$\mathscr{M} \models (\forall x\varphi) \vee (\forall y\psi)[h] \iff \mathscr{M} \models \forall xy(\varphi \vee \psi)[h].$$

"$\Rightarrow$": We may assume that $\mathscr{M} \models (\forall x\varphi)[h]$. Since y does not occur free in φ we get $\mathscr{M} \models (\forall xy\varphi)[h]$ from 3.1.4. But then $\mathscr{M} \models \forall xy(\varphi \vee \psi)[h]$ as well.

"$\Leftarrow$": Suppose $\mathscr{M} \not\models (\forall x\varphi) \vee (\forall y\psi)[h]$. Hence there are $a, b \in |\mathscr{M}|$ with $\mathscr{M} \not\models \varphi[h\binom{x}{a}]$ and $\mathscr{M} \not\models \psi[h\binom{y}{b}]$. Since $x \notin \mathrm{Fr}\,\psi$ and $y \notin \mathrm{Fr}\,\varphi$ and $x \neq y$ and by using 3.1.4 again, we have $\mathscr{M} \not\models \varphi \vee \psi[h\binom{x}{a}\binom{y}{b}]$. Hence $\mathscr{M} \not\models \forall xy(\varphi \vee \psi)[h]$.

(iii) and (iv) follow from (i) and (ii) by contraposition. □

$\not\models$

6.3. **Lemma.** *If φ is an $\mathscr{L}$-formula, $x, y \in \mathrm{Vbl}$ and x is free in φ for y then*

$$\vdash (\forall x\varphi) \leftrightarrow (\forall y\varphi(x/y)) \text{ and } \vdash (\exists x\varphi) \leftrightarrow (\exists y\varphi(x/y)).$$

Proof. From the Completeness Theorem 4.4.2 using 3.2.1. □

6.4. **Theorem.** *(Prenex Normal Form Theorem)*
Every $\mathscr{L}$-formula is provably equivalent to a formula in prenex normal form.

Proof. By induction on the complexity of φ, where φ is already in prenex normal form if φ is quantifier free.

If $\varphi = \neg\psi$ or $\varphi = \forall y\psi$ and ψ is provably equivalent to a formula in prenex normal form then clearly φ also is provably equivalent to a formula in prenex normal form.

It remains to show that $\varphi \wedge \psi$ is provably equivalent to a formula in prenex normal form provided φ and ψ have this property.

So assume $\vdash \varphi \leftrightarrow Q_1x_1...Q_nx_n\ \chi$ and $\vdash \psi \leftrightarrow P_1y_1...P_ky_k\ \delta$ with quantifier free formulas χ, δ and $Q_i, P_j \in \{\forall\exists\}$.

Using 6.3 n-times we may substitute all variables x_i in the string $Q_1x_1...Q_nx_n\ \chi$ by variables which do not occur in $P_1y_1...P_ky_k\ \delta$. Hence we may assume that no x_i occurs in $P_1y_1...P_ky_k\ \delta$. Applying this again to $P_1y_1...P_ky_k\ \delta$ we may also assume that no y_j occurs in $Q_1x_1...Q_nx_n\ \chi$. Moreover we may assume that $k = n$, otherwise fix this by placing quantifiers in front of one of the formulas containing a new variable. Now we can apply 6.2(i) and (iv) to obtain

$$\vdash Q_1x_1...Q_nx_n\ \chi \wedge P_1y_1...P_ky_n\ \delta \leftrightarrow Q_1x_1P_1y_1...Q_nx_nP_nx_n(\chi \wedge \delta).$$

Hence also $\varphi \wedge \psi$ is provably equivalent to $Q_1x_1P_1y_1...Q_nx_nP_nx_n(\chi \wedge \delta)$, which is in prenex normal form. $\square$

6.5. *Example.* Let $\varphi = Q_1x_1...Q_nx_n\ \chi$ and $\varphi' = Q'_1y_1...Q'_ky_k\ \chi'$ be formulas in prenex normal form. We want to find a formula ϑ in prenex normal form such that $\vdash (\varphi \to \varphi') \leftrightarrow \vartheta$.

Let $u_1, ..., u_n$ be variables neither occurring in φ nor in φ' such that $u_i = u_j$ for all $i, j \in \{1, ..., n\}$ with $x_i = x_j$. Applying n-times 6.3 we see that $\vdash \varphi \leftrightarrow Q_1u_1...Q_nu_n\ \chi(x_1/u_1)...(x_n/u_n)$. Hence we may replace χ by $\chi(x_1/u_1)...(x_n/u_n)$ and x_i by u_i if necessary and so we may assume that no x_i occurs in φ'. By applying the same argument to φ' instead of φ we may also assume that no y_j occurs in φ.

For $i \in \{1, ..., n\}$, let $Q_i^* := \begin{cases} \forall & \text{if } Q_i \text{ is } \exists, \\ \exists & \text{if } Q_i \text{ is } \forall. \end{cases}$

Let

$$\vartheta := Q_1^*x_1...Q_n^*x_nQ'_1y_1...Q'_ky_k(\chi \to \chi').$$

By completeness, $\vdash \neg\varphi \leftrightarrow Q_1^*x_1...Q_n^*x_n\ \neg\chi$. Since $\vdash (\varphi \to \varphi') \leftrightarrow (\neg\varphi \vee \varphi')$ we obtain $\vdash (\varphi \to \varphi') \leftrightarrow \vartheta$ from the completeness theorem (keeping in mind that none of the x_i occurs in φ' and none of the y_j occurs in φ).

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