Eternal Inflation

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In generic models of cosmological inflation, the geometry of spacetime is highly inhomogeneous on scales of many Hubble sizes, consisting of infinitely many causally disconnected "pocket universes." The values of cosmological observables and even of the low-energy coupling constants and particle masses may vary among the pocket universes. String-theoretic landscape models present a similar structure of a "multiverse" where an infinite number of de Sitter, asymptotically flat (Minkowski), and anti-de Sitter pocket universes are nucleated via quantum tunneling. Since observers on Earth have no information about their location within the eternally inflating multiverse, the main question in this context has been that of obtaining *statistical* predictions for quantities observed at a "random" location.

In this book, I discuss the long-standing technical and conceptual problems arising within this statistical framework, known collectively as the "measure problem" in multiverse cosmology. After reviewing various existing approaches and mathematical techniques developed in the past two decades for studying these issues, I describe a new proposal for a measure in the multiverse, called the reheating-volume (RV) measure. The RV measure is based on approximating an infinite multiverse by a family of progressively larger but finite multiverses. Such multiverses occur seldom but are allowed by all cosmological "multiverse" models. I give a detailed description of the new measure and its applications to generic models of eternal inflation of random-walk type and to landscape scenarios. The RV prescription is formulated differently for scenarios with eternal inflation of the random walk type and for landscape scenarios. For models of random-walk inflation, the RV cutoff considers events where one has a finite (although large) total reheating volume to the future of an initial Hubble patch. For landscape scenarios, I propose to calculate the distribution of observable quantities in a landscape that is conditioned in probability to nucleate a finite total number of bubbles to the future of an initial bubble. In each case I show in a mathematically rigorous manner that the RV measure yields well-defined results that are invariant with respect to general coordinate transformations, independent of the initial conditions at the beginning of inflation, and free of the "youngness paradox" and the "Boltzmann brain" problems affecting some of the previously proposed measures. I derive analytic formulas for RV-regulated probability distributions that is suitable for numerical computations.

Contents

List of Figures

Preface

This book is an introduction to the theory of "multiverse" in inflationary cosmology and to the so-called "measure problem."

It is likely, given the present state of cosmological knowledge, that the universe may continue expanding forever and that local conditions in the universe are at least to some extent randomly produced. As a result, different and causally independent regions of the universe — that is to say, regions inaccessibly far removed from each other in space and time — may, by chance, acquire the hot homogeneous state (the "hot Big Bang") that is known to be the origin of our observed universe. It follows that, given unlimited time, the universe will generate an *infinite* number of regions similar to ours, where galaxies and stars may give rise to intelligent life capable of scientific observation. Each region where local conditions are approximately homogeneous is called a pocket universe, while the totality of (infinitely many) pocket universes is called a multiverse.

In almost all scenarios of multiverse cosmology (the most recent such scenario being the "landscape" of string theory vacua) it is virtually certain that the different pocket universes will have different physical properties, such as the observed values of physical constants, including the masses of elementary particles, the fine structure constant, and the effective value of the cosmological constant. Since our present location in the universe is (presumably) in no way special, it follows that the presently observed values of physical constants are randomly generated and cannot be predicted on the basis of a fundamental "theory of everything." In other words, the local values of "constants of nature" in each pocket universe are *environmental givens* rather than manifestations of global underlying physical properties.

A natural approach in this situation is to try predicting the probability distribution for the local values of physical constants observed in a randomly chosen pocket universe. This approach has been initiated in the late 1980s, and quite soon a difficulty (now known as the measure problem) presented itself. The root of the measure problem is the fact that it is impossible to compute statistical averages directly on the ensemble of infinitely many pocket universes. The notion of a "randomly chosen" pocket universe is mathematically ill-defined. One needs to introduce a cutoff on the infinite set of pocket universes; however, the results depend sensitively on the cutoff procedure.

The measure problem has resisted solution for more than two decades since its original formulation in the early 1990s, which was in the context of cosmological inflation driven by scalar fields. At present, the renewed interest in the measure problem is due to the discovery of inflationary models in the string-theoretic landscape, which allows a very large number of metastable vacua with different physical laws. The need to extract predictions from these models has stimulated significant research activity in this field, resulting in the appearance of several competing measure proposals and a deeper understanding of the issues involved.

This book gives an overview of the presently known approaches to the measure problem and proceeds to describe a novel measure proposal, which was developed in a series of recent publications. Currently the measure problem in multiverse cosmology is an active area of research, and more progress may be achieved in the near future. Rather than trying to anticipate the forthcoming results, I wish to concentrate on the exposition of concepts and mathematical methods that will probably remain useful for future research in this area. I give merely a brief expository presentation of inflationary cosmology, referring the reader to numerous available textbooks for the needed background material. To recover the necessary background, the reader is only obliged to follow an introductory course in inflationary cosmology and cosmological perturbations. Similarly, I include only a necessary minimum of facts regarding the landscape of string theory; no knowledge of string theory is assumed or needed in this book. On the other hand, I show some derivations and illuminate details not ordinarily given in the literature on eternal inflation. Terminology defined in this text is shown in **boldface**; I reserve *italics* for emphasis.

The first research papers on eternal inflation was published by Andrei Linde, Alexey Starobinsky, and Alexander Vilenkin between 1982 and 1986. I will attempt to give due credit by referencing substantially all contributors to the field. A recent book [1] tells the story of eternal inflation in a form accessible to the general public. The present book is an introduction to the technical side of that story.

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Sergei Winitzki, March 2009

1 Introduction

Cosmology is the branch of physics concerned with description of the universe "at large" as manifested by large-scale astronomical observations. Since the discovery of the expansion of the universe by Hubble and the development of the general relativistic model of non-stationary homogeneous spacetime by Friedmann and others, the accepted point of view has been that the universe is expanding, and that in the distant past the state of the universe was extremely different from its present state. How the universe began (if it had a beginning) and how it evolved to its present condition are among the main questions modern cosmology aims to answer.

Although the goal of describing the entire universe might seem to require a "theory of everything," which may or may not be ever constructed but in any case is not presently available, it turns out that many important cosmological observations could be explained in the framework of currently available physical theories, in particular the known high-energy physics and classical gravity. Gamow and others introduced a model of the universe expanding from an extremely hot and dense state (the "hot Big Bang"). In what concerns the evolution of the universe after the Big Bang, this scenario is in such good agreement with observations that it is now considered to be the "standard cosmology." However, the standard cosmology leaves several important questions unanswered. For instance, in the Big Bang scenario one *assumes* rather than derives the initial state of the universe — a state characterized for instance by high temperature, homogeneity, and isotropy over extremely large length scales — and the origins of this highly fine-tuned initial state still need to be adequately explained.

This is why one of the central problems of cosmology today is to describe the era *before* the onset of the hot Big Bang scenario. Because of the present lack of detailed knowledge of high-energy physics and quantum gravity, and due to insufficient precision and scope of available astrophysical observations, numerous cosmological models of the very early universe compete on more or less equal footing.

The currently popular and observationally well-supported models of the very early universe are *inflationary* models. The scenario of inflation assumes an epoch of an extremely fast expansion (**inflation**) of the universe, during which the universe becomes very cold and homogeneous over large scales. This epoch is followed by **reheating** to a hot thermal state, with the temperature being approximately homogeneous in space. The resulting hot state becomes the starting point of the standard "hot Big Bang" scenario.

Inflationary models have been extensively studied during the last three decades. It was found early on that in most inflationary models driven by scalar fields the process of inflation is strongly affected by quantum fluctuations, which may randomly postpone or advance the onset of reheating. One talks of a **random-walk inflation** in order to describe the evolution of the scalar fields. In regions where reheating has been delayed, the exponential expansion continues and produces more inflating regions. As a result, the universe viewed at extremely large distance scales presents a mosaic of domains with dramatically different physical properties: Some regions have already thermalized and developed matter structures such as stars and galaxies, while other regions still undergo an inflationary expansion and are cold and empty. Thus, a radical departure from homogeneity can be expected at extremely large distances. Generically, at *arbitrarily late* times there will still exist large domains that continue inflating. This phenomenon was called eternal inflation.[1]

In some models, the various thermalized regions may also differ from each other in observable cosmological parameters or even in values of the coupling constants observed in low-energy physics. There are two problems with such models: First, a model that allows a wide range of parameters to be observed in various regions of the universe has little predictive power, since we cannot determine which region we happen to inhabit. Second, the existence of a (typically infinite) multitude of regions of space that are too far from us to be ever observable and/or the assumption of the *a priori* unobservable many-universe ensemble make it impossible to test these theories directly by experimental observations.

A similar situation is found in the recently discovered "landscape of string theory." It was realized already in the late 1980s [2] that string theory admits an exceedingly large number (of order 10^{1000}) of disjoint, metastable vacuum states. Different vacua have different values of the effective cosmological constant, coupling constants of low-energy physics, and particle masses. Transitions between these states are possible through bubble nucleation. The interior of each bubble appears to the interior observers as an infinite homogeneous open universe (if one disregards rare bubble collisions). For this reason, the interior of a bubble has been called a **pocket universe**. The presently observed universe is situated within a bubble whose vacuum state is, in some sense, randomly chosen. Eternal inflation occurs generically in this setting and produces an infinite number of bubbles, each containing a pocket universe in a particular vacuum state. This bewildering array of "universes" is currently referred to as a "multiverse" in order to stress the fact that different pocket universes (as well as different regions within a single pocket universe) are causally disconnected and are seen by observers as separate universes. It appears to be impossible to predict with certainty the values of the cosmological parameters that we measure at our present position in the multiverse.

To overcome these problems, one may change focus and concentrate on obtaining the *probability distribution* of values for the measured cosmological observables, such as the cosmological constant, coupling constants, or particle

[1] More precisely one can refer to *future*-eternal inflation if the origin of inflation is also being discussed. In this book I will use the term **eternal inflation** only in the strict sense that inflating domains can be found at an arbitrarily late time to the future.

masses. Heuristically, one would like to compute probability distributions of the cosmological parameters as measured by an observer "randomly located" in the spacetime. This idea, sometimes called the "principle of mediocrity," has been first clearly formulated in the mid-1990s [3, 4, 5]. One hopes that this procedure not only provides results that are in principle testable by observation, but also indirectly confirms the existence of the otherwise unobservable regions of space. This was the program outlined in those early works on eternal inflation.

However, one runs into an immediate difficulty when one tries to extract statistical predictions for cosmological observables in this setting. The difficulty is due to an infinite volume of regions where an observer may be located. Indeed, an eternally inflating universe contains an infinite, inhomogeneous, and topologically complicated spacelike hypersurface (the reheating surface) near which observers may be expected to appear with a constant density per unit 3-volume. Hence, the total number of observers in the universe is infinite.

In the landscape scenarios, one encounters a kind of infinity that is in some sense even more ill-behaved than in the random-walk inflationary scenarios. Not only each pocket universe may contain infinitely many observers, but also the number of different pocket universes in the entire spacetime is infinite. Pocket universes of different types are not statistically equivalent to each other because they have different rates of nucleation of other pocket universes as well as different interior physics. There seems to be no natural ordering on the set of all pocket universes throughout the spacetime, since most of the pocket universes are spacelike separated.

In both these contexts — random-walk models and the landscape models — one can view eternal inflation as a stochastic process that generates a topologically complicated and noncompact locus of points where observers may appear. A "random location" of an observer within that locus is a mathematically undefined concept, similarly to the concept of an integer number "uniformly chosen" among all the integers, or a real number "uniformly chosen" among all the reals. This is the root cause of the technical and conceptual difficulties known collectively as the **measure problem** in multiverse cosmology. Nevertheless, one may try to formulate a *prescription* for calculating observer-weighted probabilities of events. Such a prescription, also called a **measure**, should in some sense correspond to the intuitive notion of probability of observation at a "random" location in the spacetime. These issues are discussed in Chapter 6.

Several measure prescriptions have been proposed in the literature. Below in Sec. 6.1 these proposals will be reviewed and their contrasting features will be characterized. Almost all of the existing prescriptions are based on cutting off the infinite volume of space by some geometric construction. Thus, one considers a *finite subset* of the total volume where observers may appear; the subset is characterized by a regulating parameter, such as the largest scale factor attained by the observers or another geometric parameter. Then one gathers statistics throughout the finite subset of the volume, and finally takes the limit as the regulating parameter tends to infinity. The limit usually exists and yields a certain probability distribution for cosmological observables. Unfortunately, it turned out that the limiting distribution depends sensitively on the choice of the regularizing procedure. Since a "natural" mathematically consistent definition of the measure is absent, one needs to be open to different possibilities and try different measure prescriptions. A measure prescription can be judged as viable if its predictions are not obviously pathological. Possible pathologies include the dependence on choice of spacetime coordinates, the "youngness paradox", and the "Boltzmann brain" problem, to be discussed in more detail below.

The main goal of this book is to summarize the known results and to report on a novel class of measure prescriptions that were introduced in my recent publications, which form the basis of Chapters 7-9. The new measures are not based on regulating the spacetime by any geometric construction but rather on manipulating the events in the probability space in order to obtain a well-defined sub-ensemble of finite multiverses. A finite multiverse can be generated by rare chance if inflation ends everywhere; the probability of this event is small but nonzero. One can characterize the "size" of a finite multiverse in some way, *e.g.* by specifying the total volume of the reheating regions (which will be finite), the total number of nucleated bubbles, or another such number considered as a regulating parameter. In this way, one obtains a family of finite multiverses that in a well-defined sense approximate the actual, infinite multiverse as the regulating parameter tends to infinity. It is then expected that the probability distribution of any observable will tend to a well-defined limit. That limit is the prediction of the new measure.

I work out in detail the mathematical formalism necessary for computations in the new measure, both in the case of random-walk inflation (Chapter 8) and in the case of a landscape model (Chapter 9). The computations turn out to be cumbersome since the possibility of creating a finite multiverse is difficult to describe explicitly, especially in the limit of a very large size of the finite multiverse. Nevertheless, it is possible to obtain direct results and general proofs for various properties of the new measure. This is achieved by different methods in random-walk eternal inflation and in landscape scenarios. In each case I derived explicit formulas for the predictions of the new measure so as to make the final computations more tractable if a specific model is given.

In the Conclusion, I summarize the results described in this book and discuss some problems and possibilities for future research.

Throughout most of the exposition I use the Planck units, $c = \hbar = G = 1$, which corresponds to measuring energy in units of the Planck mass $M_{\mathrm{P}} \approx 1.2 \cdot 10^{19}$GeV, time in the Planck times $\sim 10^{-43}$ sec, and so on. Sometimes for illustrative purposes I will leave the Planck mass M_{P} in the equations.

2 Inflationary cosmology

The hypothesis of cosmological inflation is a currently a dominant framework in theoretical cosmology. I briefly outline its origins and main postulates as far as is necessary to set the context for this book. Early reviews of inflation are found *e.g.* in [6] and in the book [7]. For more detailed textbook expositions of inflationary cosmology, I refer the reader to [8, 9, 10, 11].

We begin with the story of inflation. While the "hot Big Bang" cosmological scenario has been widely accepted after the detection of the cosmic microwave background (CMB) radiation and explanation of the nucleosynthesis, several poorly explained and contradictory facts remained. The main problems of the standard hot cosmology were the horizon problem and the flatness problem.

The horizon problem stems from the fact that the observed CMB is highly isotropic, with relative temperature variation $\sim 10^{-5}$ (see, *e.g.*, [12]). The CMB radiation is coming from the last scattering surface, which at the time of decoupling consisted of a large number of causally disconnected horizon-size regions, each region spanning about 2° of today's sky. However, observations of the high degree of homogeneity of the CMB temperature suggest that all these regions had had a nearly equal temperature at the time of last scattering. This absence of large-scale fluctuations is difficult to explain unless we assume rather unnatural-looking, extremely uniform initial conditions.

The flatness problem is also in a sense a problem of initial conditions: the universe must initially have been unnaturally close to flat. The density parameter Ω evolves in such a way that any deviation from $\Omega = 1$ grows with time,

$$|\Omega - 1| \propto t^{2-\frac{4}{3(1+w)}} \propto a^{1+3w}, \tag{2.1}$$

which can also be expressed through the temperature T as

$$|\Omega - 1| \propto T^{-(1+3w)}. \tag{2.2}$$

Since the value of Ω at present is of order 1, it must have been extremely close to 1 at early times. For instance, at Planck temperatures $T_\mathrm{P} \sim 10^{19}$ GeV with $w = 1/3$ and $T_\mathrm{now} = 3$ K one obtains the estimate

$$|\Omega_P - 1| = |\Omega_\mathrm{now} - 1| \left(\frac{T_\mathrm{now}}{T_\mathrm{P}}\right)^{-2} \lesssim 10^{-60}. \tag{2.3}$$

If one assumes a more natural initial condition for Ω, for example, that $\Omega \sim 1$ at Planck time, then one finds that the universe would have either collapsed within a few Planck times if $\Omega > 1$, or cooled down to the present temperature of 3 K within 10^{-11} sec if $\Omega < 1$ [13]. It is hard to explain this exceptional fine-tuning of the initial matter density.

These problems are not the only faults of the standard scenario. For instance, explanation of the origin of structure in the framework of the standard model is also problematic. The description of growth of density fluctuations of matter due to gravitational instability [14, 15, 16] would provide an explanation for the formation of stars and galaxies if the initial fluctuations had a scale-invariant power spectrum at horizon crossing [14, 17, 18]

$$P(k) \propto k^n, n \approx 1. \tag{2.4}$$

However, the wavelength of a galaxy-scale fluctuation must have been at early times larger than the horizon size. Such a perturbation is difficult to explain by a causal mechanism. The standard model simply assumes a homogeneous initial state and neither explains how the initial fluctuations occurred nor predicts their spectrum.

These and other shortcomings of the standard hot scenario were sufficiently compelling so that the concept of cosmological inflation was relatively quickly accepted when its advantages were first clearly advocated in [19].

2.1 The inflationary paradigm

For simplicity, we consider a homogeneous cosmology with flat spatial sections and a spacetime metric of the form

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right) \equiv dt^2 - a^2(t)d\mathbf{x}^2, \tag{2.5}$$

where $a(t)$ is called the scale factor. Cosmological inflation is a regime of fast expansion of the universe with the expansion rate $\dot{a}/a = H(t)$ given by a slowly changing function of t (so that the change of H during one Hubble time is negligible, $\dot{H}H^{-1} \ll H$). Then the scale factor is approximately exponentially growing with time,

$$a(t) \sim \exp\left(\int H(t)\,dt\right). \tag{2.6}$$

The spacetime with this scale factor is similar to the de Sitter space of constant curvature $R \approx 12H^2$. More precisely, **inflation** is defined as a period of accelerated expansion, $\ddot{a}(t) > 0$. This condition allows the Hubble rate $H(t)$ to decrease as long as it does not decrease too quickly.

The earliest working proposal of an inflationary model can be found in [20, 21]. In that scenario, a period of inflation was driven by a modification of gravity where the Einstein-Hilbert action term, $R\sqrt{-g}$, was replaced by $\left(R + \alpha R^2\right)\sqrt{-g}$ with a constant α. However, this pioneering work[1] remained largely unappreciated until Guth pointed out [19] that several major problems of standard cosmology would disappear if an epoch of accelerated expansion were to precede

[1] As well as several others, such as [22, 23, 24], that proposed various hypothetical scenarios that included a period of accelerated or de Sitter-like expansion.

the hot initial state of the standard scenario, provided that the duration of the inflationary epoch is large enough compared with the Hubble time scale, such that the total expansion factor during inflation is $a \gtrsim \exp(60)$. Since such a large expansion cools the universe to extremely low temperatures, a "reheating" must occur prior to the onset of the radiation-dominated epoch. We will now briefly explain how the problems outlined in the previous section are solved in this modified scenario and then describe how inflation was implemented in particular models.

The evolution of Ω during the inflationary epoch is given by

$$\Omega(t) - 1 = \left(\frac{\dot{a}_0}{\dot{a}(t)}\right)^2 (\Omega_0 - 1) \approx \frac{H_0^2}{H^2(t)} (\Omega_0 - 1) \exp\left(-2\int_{t_0}^{t} H(t)\, dt\right). \quad (2.7)$$

Unlike the power-law expansion $a \propto t^\gamma$ with $\gamma < 1$, inflation draws the value of Ω nearer to 1. If the total expansion factor a/a_0 (the "amount of inflation") is large enough, *i.e.* at least $\exp(60)$, which is called 60 *e*-**foldings**, then for a generic initial condition $\Omega_0 \sim 1$ the value of Ω at the end of inflation will be as close to 1 as the observational constraints (2.3) require. This solves the flatness problem. In fact, the amount of inflation in generic models is much larger than 60 *e*-foldings, and Ω would be typically driven so close to 1 by the end of inflation that it would not significantly deviate from $\Omega = 1$ afterwards, during the radiation-dominated and matter-dominated expansion. Thus, generic models of inflation predict that we should observe $\Omega \approx 1$ today; this is indeed in agreement with observations.

The horizon problem manifested by the observed homogeneity of the CMB is absent because, according to the inflationary scenario, the whole surface of last scattering has been before inflation a small patch well under horizon size, and one would expect inhomogeneities in a small region to be small. An alternate way to express this is to say that the initial inhomogeneities have been "inflated away." In this light, the homogeneous state at the beginning of the radiation era does not appear mysterious.

It has also been shown [25, 26, 27, 28, 29, 30] that vacuum fluctuations of matter fields during inflation give rise to an approximately scale-invariant ($n \approx 1$) spectrum of perturbations (2.4), as necessary to explain structure formation. Models of inflation that predict a significantly "tilted" (with n significantly different from 1) spectrum of density fluctuations exist as well (see *e.g.* [31]).

2.2 Slow-roll inflation and attractors

A large number of specific inflationary models have been proposed in the early 1980s. It is interesting to note that the R^2 model originally proposed by Starobinsky remains viable even in view of today's experimental data. The model of [19], sometimes called the "old" inflationary scenario, did not provide an adequate explanation of the exit from inflation (the "graceful exit" problem). Several "new" inflationary scenarios were subsequently introduced by Linde and others (*e.g.* [32, 33, 34, 35, 36]). A detailed review of inflationary models is beyond the scope of this work; we instead concentrate on the most general common features of inflationary models.

According to the standard results of Einstein's General Relativity, an accelerated (inflationary) expansion can be supported only by an unusual kind of matter having an equation of state $p < -\frac{1}{3}\rho$. The only alternative to the presence of such matter is a modification of Einstein's theory. Since neither a detectable modification of the Einstein gravity nor any matter with the required equation of state has been found in experiments, a specific model of inflation necessarily hypothesizes either a new matter field or a modified theory of gravity that coincides with Einstein's in every respect except for the behavior of gravity during the inflationary epoch. We will not need to dwell on the details of model-building since the main topic of this book concerns features common to all inflationary models.

The first major type of inflationary models considered in this book is a class of models where the Hubble rate $H(t)$ is a smooth function of time. Initially H is large although well below the Planck scale, $H \ll 1$, in order to justify the use of classical gravity. The process of inflation ends when H approaches zero.

The easiest way to model such evolution is to assume that a scalar-field ϕ dominates the energy density of the universe and drives inflation. Such a field is called the **inflaton**. We consider a prototypical model with the action

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\mathrm{P}}^2}{16\pi} R + \frac{1}{2} \left(\partial_\mu \phi\right)^2 - V\left(\phi\right) \right], \tag{2.8}$$

where the scalar field ϕ has canonical action with self-interaction potential $V(\phi)$ and is minimally coupled to the standard Einstein gravity. We will assume the Friedmann-Robertson-Walker (FRW) metric ansatz,

$$ds^2 = dt^2 - a^2\left(t\right) \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]. \tag{2.9}$$

By using this ansatz we assume that the metric is spatially homogeneous with constant curvature k. It is then natural to assume that the field ϕ is spatially homogeneous, $\phi = \phi(t)$. One can then derive Einstein's equations of motion for the scale factor $a(t)$ and the field $\phi(t)$:

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3M_{\mathrm{P}}^2} \left(V\left(\phi\right) + \frac{1}{2}\dot{\phi}^2 \right), \tag{2.10}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -\frac{dV\left(\phi\right)}{d\phi}. \tag{2.11}$$

Because of large value of the scale factor a, we can disregard the curvature term $\left(k/a^2\right)$ in Eq. (2.10) at late times, *i.e.* we set $k = 0$, which corresponds to assuming a spatially flat universe. An exact treatment of the resulting equations, including a procedure for constructing a potential $V\left(\phi\right)$ that would yield a given evolution $a(t)$ of the scale factor can be found in [37]. If the inflationary expansion history $a(t)$ could be fully reconstructed from observations, that procedure would fix the potential $V(\phi)$. However, it is impossible to observe directly the universe at times t corresponding to inflation because the exponential expansion pushes the relevant distances beyond the horizon. A reconstruction of the potential $V(\phi)$ or of the expansion history $a(t)$ can be performed only by observing the primordial cosmological perturbations, either through the large-scale distribution of matter or through the fluctuations in the cosmic microwave background (CMB) [8]. A detailed discussion of cosmological perturbations is beyond the scope of this book; the relevant results will be summarized in Sec. 4.1.

For now, we will continue the discussion of the spatially homogeneous solutions $a(t)$, $\phi(t)$ of Eqs. (2.10)–(2.11) in the case $k = 0$. These equations can be rewritten as a single, nonlinear second-order equation for $\phi(t)$,

$$\ddot{\phi} + \frac{\dot{\phi}}{M_{\mathrm{P}}}\sqrt{24\pi\left(V(\phi) + \tfrac{1}{2}\dot{\phi}^2\right)} + \frac{dV}{d\phi} = 0. \tag{2.12}$$

Since this equation does not contain t explicitly, it can be reduced to a first-order equation for $\dot{\phi} \equiv v(\phi)$,

$$vv' + \frac{v}{M_{\mathrm{P}}}\sqrt{24\pi\left(V(\phi) + \tfrac{1}{2}v^2\right)} + V'(\phi) = 0. \tag{2.13}$$

An analysis of solutions of this equation shows, for a wide class of potentials $V(\phi)$ with monotonic behavior at infinity, such as $V(\phi) = \lambda\phi^n$, that the solutions $v(\phi)$ starting at large values of ϕ will approach a certain attractor curve $v_0(\phi)$ exponentially quickly. The attractor curve can be defined as the solution $v_0(\phi)$ of Eq. (2.13) that does *not* grow exponentially at $\phi \to \infty$ (whereas all other solutions do, as a rule, grow exponentially at $\phi \to \infty$). It can be shown in many cases that the attractor solution exists; see [38] for details and rigorous proofs.

Since solutions approach an attractor curve exponentially quickly, we may disregard all other solutions except the attractor curve. Let $\phi_0(t)$ be a solution of Eq. (2.12) that corresponds to the attractor curve $v_0(t)$. (All such solutions $\phi_0(t)$ differ only by the choice of the initial time $t = t_0$.) We arrive at the picture that the inflationary process may have started in a generic initial point $(\phi, \dot{\phi})$ in the phase plane of Eq. (2.12) but evolves essentially along the attractor curve $\phi_0(t)$ after a relatively short time. Thus, inflation "forgets" its initial conditions; in order to study the inflationary evolution, it is sufficient to analyze the attractor solution $\phi_0(t)$. In effect, the two-dimensional phase space $(\phi, \dot{\phi})$ is reduced to a one-dimensional line $\phi_0(t)$ along which the evolution of the field during inflation will proceed. In order to justify this picture, we only need to assume that inflation started with a large enough value of ϕ while the initial value of $\dot{\phi}$ is not exponentially large, which appears to be a reasonable assumption.

It needs to be mentioned that the initial conditions for inflation is a remaining question in the inflationary paradigm (see *e.g.* [39] for a review). It is known that the initial state suitable for inflation must have a sufficiently homogeneous value of ϕ such that the potential energy $V(\phi)$ dominates over any gradients or kinetic energy in ϕ or in other fields throughout a region of several horizon sizes. This state might appear unlikely; there are nevertheless some models (*e.g.* [40]) where the inflationary state is generic. Also, we will see below that eternal inflation mitigates the problem of initial conditions for inflation.

While an exact expression for the trajectory $\phi_0(t)$ may be available for some $V(\phi)$, there is an easy approximation that allows one to represent $\phi_0(t)$ with sufficient accuracy in most cases of interest for generic $V(\phi)$. This approximation is known as the **slow-roll approximation**.

The slow-roll approximation is valid when the evolution of the field ϕ is such that the potential energy $V(\phi)$ dominates over the kinetic energy, and the value of $V(\phi)$ does not change appreciably on the Hubble time scale $\Delta t = H^{-1} \equiv a/\dot{a}$. More precisely, one assumes that, on the solution $\phi(t)$, the following inequalities hold,

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi), \quad |\ddot{\phi}| \ll \left| \frac{dV(\phi)}{d\phi} \right|, \tag{2.14}$$

and thus it is justified to disregard the kinetic term $\dot{\phi}^2/2$ in Eq. (2.10) and the $\ddot{\phi}$ term in Eq. (2.11). The latter assumption means that the "friction" term $3H\dot{\phi}$ in Eq. (2.11) balances the "force" term $-V'(\phi)$. These assumptions are valid in many (but not all) models of scalar-field inflation.

Under the slow-roll assumptions, the effective equations of motion become

$$\frac{\dot{a}^2}{a^2} \equiv H^2(\phi) = \frac{8\pi}{3M_{\mathrm{P}}^2} V(\phi), \tag{2.15}$$

$$\dot{\phi} = -\frac{M_{\mathrm{P}}^2}{4\pi} \frac{dH(\phi)}{d\phi}. \tag{2.16}$$

The last equation can be integrated as

$$t - t_0 = -\frac{4\pi}{M_{\mathrm{P}}^2} \int_{\phi_0}^{\phi} \frac{d\phi}{H'(\phi)}. \tag{2.17}$$

Here we denoted by $H(\phi)$ the function $\sqrt{8\pi V(\phi)/(3M_{\mathrm{P}}^2)}$, as is common in the literature.[2]

Equations (2.15)–(2.16) describe the slow-roll regime of the evolution of ϕ. Using Eqs. (2.15)–(2.16), we can express the conditions (2.14) through $H(\phi)$ and obtain equivalent conditions

$$M_{\mathrm{P}}^2 \left(\frac{H'}{16\pi H} \right)^2 \ll 1, \quad M_{\mathrm{P}}^2 \left| \frac{H''}{12\pi H} \right| \ll 1. \tag{2.18}$$

[2] Although in some inflationary models, notably in the "open" inflation [41, 42, 43, 44], *i.e.* inflation models predicting a universe with significant negative spatial curvature today, the function $H(\phi)$ does not actually approximate the Hubble expansion rate $\dot{a}/a$, we will keep the notation throughout this text. Models of "closed" inflation exist as well (*e.g.* [45]). These models fell out of favor when observational data indicated that today's universe is spatially flat within a couple of percent.

The quantities in Eq. (2.18) are called **slow-roll parameters**. The smallness of the slow-roll parameters is the condition on the potential $V(\phi)$ necessary for the slow roll approximation to be valid.[3] The first of the conditions (2.18) also guarantees that the relative change of $V(\phi)$ in one Hubble time is negligible:

$$H^{-1}\left|\frac{d}{dt}V(\phi)\right| \ll V(\phi). \tag{2.19}$$

It can be shown [38] that the slow-roll approximation corresponds to the first term of an asymptotic expansion of the attractor solution $\phi_0(t)$ in the powers of the slow-roll parameters. In most currently considered models of slow-roll inflation, the slow-roll parameters during inflation have values of the order of a few percent. Therefore, for our purposes the slow-roll approximation is an adequate description of the attractor trajectory far from the end of inflation. We will not further make the distinction between the attractor trajectory and the slow-roll trajectory $\phi_{\mathrm{sr}}(t)$. [4]

For a given potential there is usually a range of ϕ for which the slow roll conditions (2.18) are satisfied. It is usually the case that reheating begins when ϕ reaches values for which the slow roll condition is violated, so that the slow-roll parameters grow to values of order 1.[5] For simplicity we will assume that reheating begins at the value $\phi = \phi_*$ such that

$$H'(\phi_*) \sim \frac{16\pi}{M_{\mathrm{P}}}H(\phi_*). \tag{2.20}$$

Let us call the value $\phi = \phi_*$ the **end-of-inflation boundary**.

If the slow-roll trajectory starts at $\phi = \phi_0$ and ends at $\phi = \phi_*$, the total expansion factor $a(t_*)/a(t_0)$ can be found from

$$\ln\frac{a(t_*)}{a(t_0)} = \int_{t_0}^{t_*} H(t)\,dt = \int_{\phi_0}^{\phi_*} H(\phi)\,\frac{d\phi}{\dot{\phi}} = -\frac{4\pi}{M_{\mathrm{P}}^2}\int_{\phi_0}^{\phi_*}\frac{H(\phi)}{H'(\phi)}d\phi. \tag{2.21}$$

There are several typical shapes of the potential $V(\phi)$ that allow for inflation. In models of "new" inflation, the potential has a nearly flat top, and inflation ends when the field rolls down to the bottom (see Fig. 2.1). In "chaotic" models the potential is usually of the power-law form $V(\phi) \propto \phi^n$. Another possibility is a potential of an exponential form, $V(\phi) \propto e^{\lambda\phi}$. We will not need to specify the potential in what follows. The considerations of eternal inflation apply generally to every such model, although specific calculations of course require the knowledge of the inflaton potential.

[3] Another implied assumption is $V(\phi) \sim M_{\mathrm{P}}^2 H^2 \ll M_{\mathrm{P}}^4$, since any classical description is only valid for energies far below the Planck scale.

[4] Other, non-slow-roll models of inflation are certainly possible but we will not consider then in this book. In almost any inflationary model, a very similar attractor behavior is found, and the considerations developed in this book will still apply if we replace the slow-roll trajectory $\phi_{\mathrm{sr}}(t)$ by the relevant attractor trajectory $\phi_0(t)$.

[5] A detailed theory of reheating has been worked out in [46, 47].

An important caveat regarding chaotic models is the existence of the **Planck boundary** $\phi = \phi_P$ where the potential $V(\phi)$ reaches Planck scale, $V(\phi_P) \sim M_P^4$. It is inconsistent to consider slow-roll inflation with $\phi = \phi_P$ since classical gravity presumably breaks down at Planck energy densities. Therefore, in models of inflation where the potential reaches Planck scales, one needs to consider only the values of ϕ away from the Planck boundary. In case the evolution of the field ϕ pushes the value of ϕ towards ϕ_P, one needs to decide whether to discard such possibilities or to treat the field in some special manner, *e.g.* effectively change the potential $V(\phi)$ in such a way that $\phi = \phi_P$ is unreachable. This remains an *ad hoc* decision since the existing theory does not prescribe what happens near $\phi = \phi_P$.

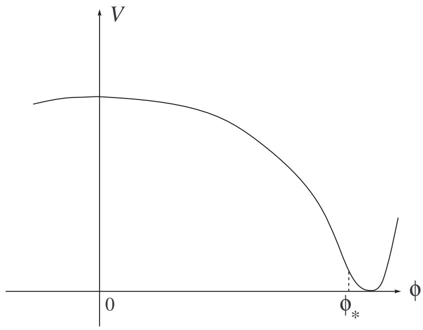

Figure 2.1: The inflaton potential for the "new" inflationary scenario at $T = 0$. The neighborhood of the maximum of the potential at $\phi = 0$ is very flat, so that the field ϕ changes very slowly. Thermalization occurs after the field ϕ finally rolls down to the "thermalization point" $\phi = \phi_*$.

Until now we have been considering single-field models of inflation. This is the simplest possible class of models that can be adjusted to agree with the present cosmological observations. While the exact nature of the field ϕ is still unknown (up to now, none of the observed fundamental elementary particles are scalar fields), the single-field model may be viewed as an adequate effective description of the yet unknown forces driving the inflationary process.

To accommodate future observations, one may wish to have a more general model of inflation than a single canonical scalar field. Possible venues of generalization include models with several minimally coupled, canonical scalar fields $\phi_1, ..., \phi_N$ and a potential $V(\phi_1, ..., \phi_N)$, or non-canonical scalar fields such as

$$S[\phi] = \int p(\tfrac{1}{2} (\partial_\mu \phi)^2, \phi) \sqrt{-g} d^4x, \tag{2.22}$$

where $p(X, \phi)$ is a nonlinear function of the canonical kinetic term

$$X \equiv \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi). \tag{2.23}$$

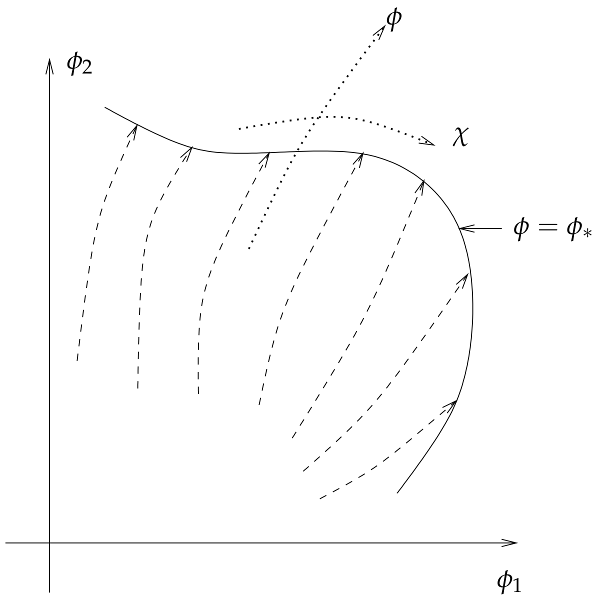

Figure 2.2: Reheating boundary in field space (solid line) in a fiducial model with two fields ϕ_1, ϕ_2. The slow-roll trajectories (dashed lines) approach the reheating boundary, which is a one-dimensional manifold in field space. In the coordinates (ϕ, χ) the reheating boundary is $\phi = \text{const}$, and different reheating possibilities have different values of χ.

In models with several scalar fields and/or with noncanonical fields, the equations of motion are of course quite different. Nevertheless, the attractor behavior persists: the $(2N)$-dimensional phase space of a model with N scalar fields is effectively reduced to an N-dimensional manifold spanned by the possible attractor curves. An analog of the slow-roll approximation can be developed to approximate the attractor curves. There is generically an $(N-1)$-dimensional locus of points in field space where inflation may end; we will call this locus the **end-of-inflation boundary** or **reheating boundary** in field space (see Fig. 2.2). For convenience, one may choose the coordinates in field space such that the reheating boundary is described by the equation $\phi = \text{const}$, where ϕ is one of the fields.

The endpoint of inflation in multi-field models is the only information remaining from the initial conditions for inflation. As long as these initial conditions are unknown, there can be no definite prediction for the physical state at reheating. For instance, the reheating temperature or other cosmological parameters could depend on the values of the fields ϕ_j at reheating. In some models, the potential $V(\phi_1, ..., \phi_N)$ is arranged so that the set of the possible endpoints of inflation are limited to a relatively small region in field space. In that case, the physical state at the end of inflation is approximately independent of the initial conditions for inflation.

For the purposes of further discussion in this book, we will assume a simplistic view that an inflationary model is specified such that the field ϕ "drives inflation," in the sense that $\phi = \phi_*$ is a reheating boundary in field space, while other fields (χ_i) may be present as well. While evolution of the universe during the last 60 *e*-foldings of inflation is somewhat constrained by CMB and large-scale structure observations, the epoch preceding the 60 last *e*-foldings and the conditions at the beginning of inflation (if any such beginning was globally present) are completely beyond observational possibilities.

3 Eternal inflation

Eternal inflation, or the property that inflation never ends in the whole spacetime, is a generic property of inflationary models. The mechanism of eternal inflation is based on the effect of quantum fluctuations on the average values of ϕ and on the metric in horizon-size volumes. This is essentially the same physical effect as that which is assumed to be responsible for generating the observed, *classical* cosmological perturbations out of *quantum* fluctuations in the vacuum state.

Let us begin by examining the main qualitative features of eternal inflation, following the review [48]. (Other reviews and discussions of eternal inflation are found in [49, 50, 51, 52, 53].) In the following chapter, we will describe eternal inflation in a precise, quantitative manner.

The general idea of eternally inflating spacetime was first introduced and developed in the context of slow-roll inflation [54, 55, 56, 57, 58]. A prototypical model contains a minimally coupled, canonical scalar field ϕ (the "inflaton") with an effective interaction potential $V(\phi)$ that is sufficiently flat in some range of ϕ. When the field ϕ has values in this range, the spacetime is approximately de Sitter with the Hubble rate

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3M_{\mathrm{P}}^2}V(\phi)} \equiv H(\phi). \tag{3.1}$$

The value of H remains approximately constant on timescales of several Hubble times ($\Delta t \gtrsim H^{-1}$), while the field ϕ follows the attractor trajectory, which is approximately given by the slow-roll trajectory $\phi_{\mathrm{sr}}(t)$. Quantum fluctuations of the scalar field ϕ in de Sitter background grow linearly with time [59, 34, 28],

$$\langle \hat{\phi}^2(t+\Delta t)\rangle - \langle \hat{\phi}^2(t)\rangle = \frac{H^3}{4\pi^2}\Delta t, \tag{3.2}$$

at least for time intervals Δt of order several H^{-1}. Due to the quasi-exponential expansion of spacetime during inflation, Fourier modes of the field ϕ are quickly stretched to super-Hubble length scales. However, quantum fluctuations with super-Hubble wavelengths cannot maintain quantum coherence and become essentially classical [34, 28, 60, 61, 62]; this issue is discussed in more detail in Sec. 3.2 below. The resulting field evolution $\phi(t)$ can be visualized [30, 55, 60] as a Brownian motion with a "random jump" of typical step size $\Delta\phi \sim H/(2\pi)$ during a time interval $\Delta t \sim H^{-1}$, superimposed onto the deterministic slow-roll trajectory $\phi_{\mathrm{sr}}(t)$. A statistical description of this "random walk"-type evolution $\phi(t)$ is reviewed in Sec. 4.1.

One can distinguish two possible regimes of evolution depending on whether the deterministic change of ϕ in one Hubble time is smaller or larger than a typical random fluctuation. If the deterministic change $\dot{\phi}_{\mathrm{sr}} H^{-1}$ dominates the fluctuations, the slow roll evolution proceeds essentially unmodified. In the opposite regime, $\dot{\phi}_{\mathrm{sr}} H^{-1} \ll \delta\phi$, the random walk dominates the evolution of ϕ, which means that steps toward larger and smaller values of ϕ are almost equally probable. A range of ϕ where the random walk dominates (called the **fluctuation-dominated range**) is present in many inflationary models. If the field ϕ is approximately homogeneous throughout a horizon-sized region (H-**region**) and has a value in the fluctuation-dominated range, the H-region will expand to form several horizon-sized regions, most of which would again contain values of the field ϕ within the fluctuation-dominated range. One may say that a fluctuation-dominated H-region "reproduces itself" [57]. It is important that the self-reproduction occurs regardless of the values of ϕ in neighbor H-regions because the neighbor regions are beyond the horizon (the well-known "no-hair" property of de Sitter space [63, 64, 65]). We will use interchangeably the terms **self-reproducing regime** and **fluctuation-dominated range** of ϕ.

The "jumps" at points separated in space by many Hubble distances are essentially uncorrelated; this is another manifestation of the "no-hair" property. Thus the field ϕ becomes extremely inhomogeneous on large (super-horizon) scales after many Hubble times, even though the field values remain approximately homogeneous within each Hubble volume (H-region).

In models of "new" inflation, the fluctuation-dominated range is near the flat top of the potential where the deterministic change of ϕ is small. In "chaotic" models it is usually a range of ϕ bounded from below by some value ϕ_{fluct}, and from above by the Planck boundary ϕ_{P}, which leads to the existence of the fluctuation-dominated range $\phi_{\mathrm{fluct}} < \phi < \phi_{\mathrm{P}}$ if $\phi_{\mathrm{fluct}} < \phi_{\mathrm{P}}$ and to the absence of the fluctuation-dominated range otherwise. For the power-law potential $V(\phi) = \lambda\phi^n$ with $\lambda \ll 1$ (in Planck units), one obtains $\phi_{\mathrm{fluct}} \sim \lambda^{-1/(n+2)} \ll \phi_{\mathrm{P}} \sim \lambda^{-1/n}$, so the fluctuation-dominated range is present in such models. Rather than consider a specific model of inflation, we will keep the discussion general and assume that a fluctuation-dominated range exists.

While the fluctuations of the field ϕ have ultimately quantum origin, their semi-classical character leads, through coupling to classical gravity, to the consequence that the local expansion rate $\dot{a}/a \equiv H(\phi)$ tracks the local value of the field $\phi(t, \mathbf{x})$ according to the Einstein equation (3.1). Here $a(t, \mathbf{x})$ is the scale factor function that varies with $\mathbf{x}$ only on superhorizon scales, $|\Delta x| \gtrsim a^{-1}H^{-1}$. Hence, the spacetime can be visualized as having a slowly varying, "locally de Sitter" metric, written in spatially flat coordinates $\mathbf{x}$ as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = dt^2 - a^2(t, \mathbf{x})d\mathbf{x}^2. \tag{3.3}$$

This picture requires that the metric remains everywhere *classical* despite the fact that it is *quantum* fluctuations that generate the inhomogeneity of the metric. It is assumed, in effect, that quantum fluctuations of the metric become classical metric perturbations when their wavelength significantly exceeds the horizon size. This assumption is not yet theoretically justified in a completely explicit manner and is still a subject of active research. Further discussion of this issue is given below in Sec. 3.2. Presently, let us emphasize that the same assumption — that of classicality of superhorizon-scale metric fluctuations resulting from quantum field fluctuations — is at the foundation of the accepted and experimentally successful theory of inflation-generated cosmological perturbations (see, for instance, the textbooks [8, 9, 10]). This appears to confirm the classicality of superhorizon perturbations underlying eternal inflation.[1]

It is important to visualize the evolution of values of $\phi(\mathbf{x}, t)$ at a single point $\mathbf{x} = \text{const}$, *i.e.* the evolution of ϕ along a single **comoving worldline**. Let us denote this evolution temporarily by $\phi(t)$. The function $\phi(t)$ is a superposition of the deterministic ("slow-roll" approximated) trajectory $\phi_{\text{sr}}(t)$ and a Brownian motion driven by quantum fluctuations. The deterministic trajectory $\phi_{\text{sr}}(t)$ eventually reaches the reheating boundary $\phi = \phi_*$ where the slow-roll inflationary regime ends and the reheating epoch (thermalization) begins. Note that fluctuations are very small (of order 10^{-5}) within 60 e-foldings before reheating, hence the trajectory $\phi(t)$ is very close to $\phi_{\text{sr}}(t)$ near $\phi = \phi_*$. However, far away from $\phi = \phi_*$ the trajectory $\phi(t)$ can fluctuate significantly, and the fluctuations are space-dependent. Thus different comoving points may reach the reheating regime $\phi = \phi_*$ at significantly different times, or may instead reach the Planck boundary ϕ_{P} if such a boundary is present in the model.

As noted in the previous chapter, the presently available physics does not prescribe how to treat the evolution of regions that reach the Planck boundary. The problem of the Planck boundary was considered *e.g.* in [68]. One possibility is to discard the regions reaching the Planck boundary since they presumably have an essentially quantum (not semiclassical) state of the metric and thus cannot be interpreted as regions of a classical spacetime. Another possibility is to introduce an artificial reflecting boundary, modifying the potential $V(\phi)$ near $\phi = \phi_{\text{P}}$ such that the field ϕ is reflected away from $\phi = \phi_P$ without reaching the Planck boundary. In either case, one essentially obtains the situation in which the Planck boundary is never reached, so that the evolution $\phi(t)$ at any point $\mathbf{x}$ definitely (with probability 1) reaches the reheating boundary $\phi = \phi_*$.

Even though any given comoving point $\mathbf{x}$ will eventually enter a thermalized region, the global behavior of the field ϕ is different. Since the random walk process will lead the value of ϕ away from $\phi = \phi_*$ in some regions, reheating will not begin everywhere at the same time. Moreover, regions where ϕ remains in the inflationary range will typically expand faster than regions near the end of inflation. Therefore, a delay of the onset of reheating will be rewarded by additional expansion of the 3-volume, thus generating proportionally more regions that are still inflating. Since each Hubble-size region evolves independently of other such regions, one may visualize the spacetime as an ensemble of inflating Hubble-size domains (Fig. 3.1).

The process of self-reproduction will be eternal (will not result in a global

[1]Let us mention that several authors [66, 67] have argued, on the grounds of the holographic principle, against the possibility of a global semiclassical description of the *entire* inflationary spacetime (rather than only a single horizon-size region) during eternal inflation.

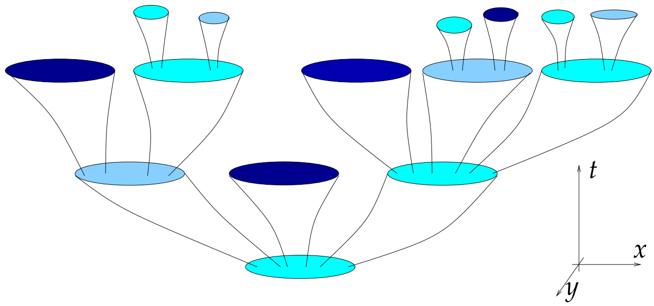

Figure 3.1: A qualitative diagram of self-reproduction during inflation. Shaded spacelike domains represent Hubble-size regions with different values of the inflaton field ϕ. The time step is of order H^{-1}. Dark-colored shades are regions undergoing reheating ($\phi = \phi_*$); lighter-colored shades are regions where inflation continues. On average, the number of inflating regions grows with time.

reheating) if the probability of jumping away from $\phi = \phi_*$ and the corresponding additional volume expansion factors are sufficiently large. The quantitative conditions for the presence of eternal self-reproduction are reviewed in Sec. 4.2. Under these conditions, the process of self-reproduction of inflating regions continues forever. At the same time, as we already argued, every given comoving worldline (except for a set of measure zero; see Sec. 4.2) will sooner or later reach the value $\phi = \phi_*$ and enter the reheating epoch. The resulting situation is known as "eternal inflation." More precisely, the term "eternal inflation" means future-eternal self-reproduction of inflating regions [69].[2] To emphasize the fact that the self-reproduction is due to random fluctuations of a field, one refers to this scenario as eternal inflation **of random-walk type**. Below we use the terms "eternal self-reproduction" and "eternal inflation" interchangeably.

Observers like us may appear only in regions where reheating already took place. Hence, it is useful to consider the locus of all reheating events in the entire spacetime; in the presently considered example, it is the set of spacetime points x there $\phi(x) = \phi_*$. This locus is called the **reheating surface** and is a noncompact, spacelike three-dimensional hypersurface [70, 5]. It is important to realize that a *finite*, initially inflating 3-volume of space may give rise to a reheating surface having an *infinite* 3-volume, and even to infinitely many causally disconnected pieces of the reheating surface, each having an infinite 3-volume. This feature of eternal inflation is at the root of several technical and conceptual difficulties, as will be discussed below.

Everywhere along the reheating surface, the reheating process is expected to provide appropriate initial conditions for the standard "hot Big Bang" cosmological evolution, including nucleosynthesis and structure formation. In other

[2] It is worth emphasizing that the term "eternal inflation" refers to future-eternity of inflation in the sense described above, but does not imply past-eternity. In fact, inflationary spacetimes are generically *not* past-eternal [70, 71].

words, the reheating surface may be visualized as the locus of the "hot Big Bang" events in the spacetime. It is thus natural to view the reheating surface as the initial equal-time surface for astrophysical observations in the post-inflationary epoch. Note that the observationally relevant range of the primordial density fluctuations is generated only during the last 60 e-foldings of inflation. Hence, the duration of the inflationary epoch that preceded reheating at a given point is not directly measurable beyond the last 60 e-foldings; the total number of past e-foldings can vary along the reheating surface and can be in principle arbitrarily large.[3]

Generically, in both "new" and "chaotic" inflation the regions in the self-reproducing stage expand at the fastest possible rate. An early attempt to quantify the effects of eternal inflation was to track the 3-volume of the inflating domain at a fixed proper time [74, 57]. As a result of self-reproduction, the volume of the inflating domain grows exponentially with time, even though any given co-moving point will eventually enter a thermalized region. This feature of self-reproduction helps solve the problem of initial conditions for inflation: Wherever a self-reproducing region is formed, its inflating offspring dominates the physical volume at fixed proper time, and all other regions (including those with initial conditions unsuitable for inflation) will occupy an exponentially small fraction of the total volume.

A problem with this argument is the fact that 3-volume is a quantity dependent on the choice of coordinates (the choice of "gauge") in spacetime. Indeed, one may choose coordinates in which the volume of the inflating domain appears to shrink to zero at late times despite the unambiguous presence of inflation; an explicit construction in de Sitter spacetime was given in [75]. Nevertheless, one can formulate the conditions for the presence of eternal inflation and the property of independence of initial conditions in a gauge-independent way (see Sec. 4.2).

The phenomenon of eternal inflation is also found in multi-field models of inflation [76, 77], as well as in scenarios based on Brans-Dicke theory [78, 79, 80], topological inflation [81, 40], braneworld inflation [82], "recycling universe" [83], and the string theory landscape [84]. In some of these models, quantum tunneling processes may generate "bubbles" of a different phase of the vacuum (see Sec. 5 for further details). Bubbles will be created randomly at various places and times, with a fixed rate per unit 4-volume. In the interior of some bubbles, additional inflation may take place, followed by a new reheating surface. The interior structure of such bubbles is sketched in Fig. 3.2. The nucleation event and the formation of bubble walls is followed by a period of additional inflation, which terminates by reheating. Standard cosmological evolution and structure formation eventually give way to a Λ-dominated universe. Infinitely many galaxies and possible civilizations may appear within a thin spacelike

[3]For instance, it was shown that holographic considerations do not place any bounds on the total number of e-foldings during eternal inflation [72]. For recent attempts to limit the number of e-foldings using a different approach, see *e.g.* [73]. Note also that the effects of "random jumps" on the total e-folding number are small during the last 60 e-foldings of inflation, since the produced perturbations must be of order 10^{-5} according to observations.

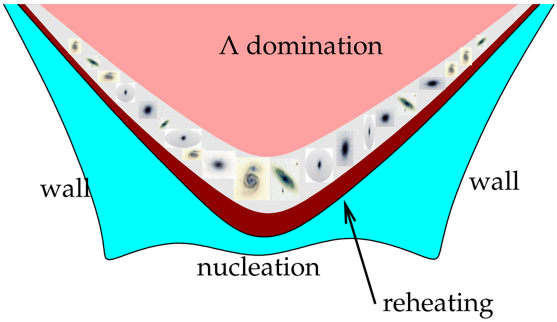

Figure 3.2: A spacetime diagram of a bubble interior. The infinite, spacelike reheating surface is shown in darker shade. Galaxy formation is possible within the spacetime region indicated.

slab running along the interior reheating surface. This reheating surface appears to interior observers as an infinite, spacelike hypersurface [85]. For this reason, such bubbles are called **pocket universes** and the entire spacetime is referred to as **multiverse**. Generally, the term "pocket universe" refers to a non-compact, connected component of the reheating surface [86].

In scenarios of the tunneling type, each bubble is causally disconnected from most other bubbles.[4] Hence, bubble nucleation events may generate infinitely many statistically inequivalent, causally disconnected patches of the reheating surface, every patch giving rise to a possibly infinite number of galaxies and observers. This feature significantly complicates the task of extracting physical predictions from these models. This class of models is referred to as "eternal inflation of tunneling type."

The global picture of the universe is significantly modified by the fact that eternal inflation is generic in many scenarios. It is likely that while inflation ended in *our* neighborhood of the universe approximately 10^{10} years ago, there still are very large domains far away from us where inflation goes on. In a sense, eternal inflation is a reversal of the cosmological principle, yielding a picture of the universe which is extremely inhomogeneous on the ultra-large scale but homogeneous on the presently observable scale.

In the following subsections, I discuss the motivation for studying eternal inflation as well as physical justifications for adopting the effective stochastic picture. A more detailed study of eternal inflation will begin in the next chapter.

3.1 Predictions in eternal inflation

The hypothesis of cosmological inflation was invoked to explain several outstanding puzzles in observational data [19]. However, some observed quanti-

[4] Collisions between bubbles are rare [87]; however, effects of bubble collisions are observable in principle [88].

ties (such as the cosmological constant Λ or elementary particle masses) may be expectation values of slowly-varying effective fields χ_a. Within the phenomenological approach, we are compelled to consider also the fluctuations of the fields χ_a during inflation, on the same footing as the fluctuations of the inflaton ϕ. Hence, in a generic scenario of eternal inflation, all the fields χ_a arrive at the reheating surface $\phi = \phi_*$ with values that can be determined only statistically. Observers appearing at different points in space may thus measure different values of the cosmological constant, elementary particle masses, spectra of primordial density fluctuations, and other cosmological parameters that may depend on χ_a.

It is important to note that inhomogeneities in observable quantities are created on scales far exceeding the Hubble horizon scale. Such inhomogeneities are not directly accessible to astrophysical experiments. Nevertheless, the study of the global structure of eternally inflating spacetime is not merely of academic interest. Fundamental questions regarding the cosmological singularities, the beginning of the universe and of its ultimate fate, as well as the issue of the cosmological initial conditions all depend on knowledge of the global structure of the spacetime as predicted by the theory, whether or not this global structure is directly observable (see *e.g.* [89, 90]). In other words, the fact that some theories predict eternal inflation influences our assessment of the viability of these theories. In particular, the problem of initial conditions for inflation is significantly alleviated when eternal inflation is present. For instance, it was noted early on that the presence of eternal self-reproduction in the "chaotic" inflationary scenario [91] essentially removes the need for the fine-tuning of the initial conditions [74, 57]. More recently, constraints on initial conditions were studied in the context of self-reproduction in models of quintessence [92] and k-inflation [38].

Since the values of the observable parameters χ_a are random, it is natural to ask for the probability distribution of χ_a that would be measured by a randomly chosen observer. Understandably, this question has been the main theme of much of the work on eternal inflation. Obtaining an answer to this question promises to establish a more direct contact between scenarios of eternal inflation and experiment. For instance, if the probability distribution for the cosmological constant Λ were peaked near the experimentally observed, puzzlingly small value (see *e.g.* [93] for a review of the cosmological constant problem), the smallness of Λ would be explained as due to observer selection effects rather than to fundamental physics. Considerations of this sort necessarily involve some anthropic reasoning; however, the relevant assumptions are minimal. The basic goal of theoretical cosmology is to select physical theories of the early universe that are most compatible with astrophysical observations, including the observation of our existence. It appears reasonable to assume that the civilization of Planet Earth evolved near a randomly chosen star compatible with the development of life, within a randomly chosen galaxy where such stars exist. Many models of inflation generically include eternal inflation and hence predict the formation of infinitely many galaxies where civilizations like ours may develop. It is then also reasonable to assume that our civilization is typical among all the civilizations that evolved in galaxies formed at any time anywhere in the universe. This assumption is called the **principle of mediocrity**.

To use the principle of mediocrity for extracting statistical predictions from a model of eternal inflation, one proceeds as follows [5, 94]. In the example with the fields χ_a described above, the question is to determine the probability distribution for the values of χ_a that a random observer will measure. Presumably, the values of the fields χ_a do not directly influence the emergence of intelligent life on planets, although they may affect the efficiency of structure formation or nucleosynthesis. Therefore, we may assume a fixed, χ_a-dependent mean number of civilizations $\nu_{\text{civ}}(\chi_a)$ per galaxy and proceed to ask for the probability distribution $P_G(\chi_a)$ of χ_a near a randomly chosen galaxy. The observed probability distribution of χ_a will then be

$$P(\chi_a) = P_G(\chi_a)\nu_{\text{civ}}(\chi_a). \tag{3.4}$$

One may use the standard "hot Big Bang" cosmology to determine the average number $\nu_G(\chi_a)$ of suitable galaxies per unit volume in a region where reheating occurred with given values of χ_a; in any case, this task does not appear to pose difficulties of principle. Then the computation of $P_G(\chi_a)$ is reduced to determining the volume-weighted probability distribution $\mathcal{V}(\chi_a)$ for the fields χ_a within a randomly chosen 3-volume along the reheating surface. The probability distribution of χ_a will be expressed as

$$P(\chi_a) = \mathcal{V}(\chi_a)\nu_G(\chi_a)\nu_{\text{civ}}(\chi_a). \tag{3.5}$$

However, defining $\mathcal{V}(\chi_a)$ turns out to be far from straightforward since the reheating surface in eternal inflation is an infinite 3-surface with a complicated geometry and topology. The lack of a natural, unambiguous, unbiased measure on the infinite reheating surface is known as the measure problem in eternal inflation. Existing approaches and measure prescriptions are discussed in Chapter 6 where two main alternatives (the volume-based and worldline-based measures) are presented (Sec. 6.1). I will give arguments in favor of using the volume-based measure for computing the probability distribution of values χ_a measured by a random observer. The volume-based measures have been applied to obtain statistical predictions for the gravitational constant in Brans-Dicke theories [78, 79], cosmological constant (dark energy) [95, 96, 97, 98, 99, 100, 101], particle physics parameters [102, 103, 104], and the amplitude of primordial density perturbations [105, 97, 106, 100]. The worldline-based measure was also used to predict the cosmological constant and other observables [107, 108, 109, 110]. A detailed exposition of these applications is beyond the scope of this introductory book; the reader is referred to the cited literature for further information.

The issue of statistical predictions has recently come to the fore in conjunction with the discovery of the string theory landscape. According to various estimates, one expects to have between 10^{500} and 10^{1500} possible vacuum states of string theory [2, 111, 84, 112, 113]. The string vacua differ in the geometry of spacetime compactification and have different values of the effective cosmological constant (or "dark energy" density). Transitions between vacua may happen via the well-known Coleman-De Luccia tunneling mechanism [85]. Once the dark energy dominates in a given region, the spacetime becomes locally de Sitter. Then the tunneling process will create infinitely many disconnected "daughter" bubbles of other vacua. Observers like us may appear within any of the habitable bubbles. Since the fundamental theory does not specify a single "preferred" vacuum, it remains to try determining the probability distribution of vacua as found by a randomly chosen observer. The volume-based and worldline-based measures can be extended to scenarios with multiple bubbles, as discussed in more detail below in Sec. 6.3. Some recent results obtained using these measures are reported in [114, 50, 115, 109]. Ultimately, one hopes to choose the correct measure prescription by analyzing the general features of measure prescriptions in judiciously chosen toy models, and also by computing specific predictions for observable parameters and comparing them with observed values. Future research will show whether this approach proves to be fruitful.

3.2 Physical justifications of the semiclassical picture

The standard framework of inflationary cosmology asserts that vacuum quantum fluctuations with super-horizon wavelengths become classical inhomogeneities of the field ϕ and of the metric. The calculations of cosmological density perturbations generated during inflation [25, 26, 27, 28, 29, 30, 116, 117] also assume that a "classicalization" of quantum fluctuations takes place via the same mechanism. This assumption, while not always made explicit, is manifest in the calculations because one obtains the statistical average $\langle \delta\phi^2 \rangle$ of *classical* fluctuations on super-Hubble scales by computing the *quantum* expectation value $\langle 0 | \hat{\phi}^2 | 0 \rangle$ in a suitable vacuum state $|0\rangle$. While this approach is widely accepted in the cosmology literature, a growing body of research is devoted to the explicit analysis of the quantum-to-classical transition during inflation (see *e.g.* [118] for an early review of that line of work). Since a detailed analysis would be beyond the scope of the present text, and since the quantum-to-classical transition in cosmology is still under active investigation, I merely outline the main ideas and arguments relevant to this issue.

A standard phenomenological explanation of the "classicalization" of the perturbations is as follows. For simplicity, we restrict our attention to a slow-roll inflationary scenario with one scalar field ϕ. In the slow roll regime, one can approximately regard ϕ as a massless scalar field in de Sitter background spacetime [58]. In a more fundamental approach, one can take into account the scalar metric fluctuations sourced by the fluctuations of the scalar field and again obtain an effective description of both fluctuations in terms of a single auxiliary scalar field (sometimes called the **Mukhanov-Sasaki field**) propagating in a fixed classical background.[5] Therefore, it is sufficient to consider a scalar field (here denoted by ϕ) in a *fixed* de Sitter-like background.

Due to the exponentially fast expansion of de Sitter spacetime, super-horizon Fourier modes of the field ϕ are in squeezed quantum states with exponentially large ($\sim e^{Ht}$) squeezing parameters [121, 122, 123, 124, 125, 126]. Such highly squeezed states have a macroscopically large uncertainty in the field value ϕ and thus quickly decohere due to interactions with gravity and with other fields. The resulting mixed state is equivalent to a statistical ensemble with a Gaussian distributed value of ϕ. Therefore one may compute the statistical average $\langle \delta\phi^2 \rangle$ as the quantum expectation value $\langle 0 | \hat{\phi}^2 | 0 \rangle$ and interpret the fluctuation $\delta\phi$ as a classical "noise."

Another heuristic description of the "classicalization" [58] is that the quantum commutators of the creation and annihilation operators of the field modes, $[\hat{a}, \hat{a}^\dagger] = 1$, are much smaller than the expectation values $\langle a^\dagger a \rangle \gg 1$ and are thus negligible.

The backreaction of fluctuations of the scalar field ϕ on the metric is treated in the same way.[6] Since the phenomenon of "classicalization" applies to the fluctuations of the Mukhanov-Sasaki field, the same phenomenon applies to perturbations of the metric effectively described by that field. At the same time, the metric perturbations can be viewed, in an appropriate local coordinate system, as fluctuations of the local expansion rate $H(\phi(\mathbf{x}, t))$ (see [28, 58, 134] for further details). One thus arrives at the picture of a "locally de Sitter" spacetime with the metric (3.3), where the Hubble rate $\dot{a}/a = H(\phi)$ fluctuates on super-horizon length scales and locally follows the fluctuating value of ϕ via the *classical* Einstein equation (3.1).

The picture as outlined is certainly phenomenological and does not provide a description of the quantum-to-classical transition in the field and metric perturbations at the level of field theory or quantum gravity. For instance, a fluctuation of ϕ leading to a local *increase* of the value of $H(\phi)$ necessarily violates the null energy condition (**NEC**) [135, 136, 137]. However, it is clear that the dynamics of a scalar field ϕ cannot violate the NEC either at classical level or at the level of the expectation value of the quantum energy-momentum tensor [137]. The cosmological implications of such NEC-violating fluctuations (see *e.g.* the scenario of "island cosmology" [138, 139, 140]) cannot be understood quantitatively within the framework of the phenomenological picture as outlined here.

A more fundamental approach to describing the quantum-to-classical transition of perturbations was developed using non-equilibrium quantum field theory and the influence functional formalism [141, 142, 143, 144]. In this approach, decoherence of a pure quantum state of ϕ into a mixed state is entirely due to the self-interaction of the field ϕ. In particular, it is predicted that no decoherence would occur for a free field with $V(\phi) = \frac{1}{2} m \phi^2$.

[5] This standard theory of cosmological perturbations can be found *e.g.* in the technical reviews [116, 117]; see also [119] for a more modern derivation, and the books [10, 120] for a more pedagogical exposition.

[6] The backreaction effects of the long-wavelength fluctuations of a scalar field during inflation have been investigated extensively (see *e.g.* [127, 128, 129, 130, 131, 132, 133]).

This result is dramatically at variance with the accepted paradigm of "classicalization" as outlined above. Firstly, due to the absence of decoherence as computed in the influence functional approach, the model of inflation with $V(\phi) = \frac{1}{2}m\phi^2$ would have generated *no* classical perturbations at all. This is in contrast with the present consensus where this model is a successful match with the observed CMB fluctuations [145, 146, 147, 148].

Secondly, if the source of the "noise" is the coupling between different perturbation modes of ϕ, the typical amplitude of the "noise" will be second-order in the perturbation. This is several orders of magnitude smaller than the amplitude of "noise" found in the standard approach. Accordingly, it is claimed [149, 150] that the amplitude of cosmological perturbations generated by inflation is several orders of magnitude smaller than the results currently accepted as standard, and that the shape of the perturbation spectrum depends on the details of the process of "classicalization" [151].

Thus, the results obtained so far via the influence functional techniques are in sharp contradiction with the widely accepted phenomenological picture of "classicalization" as outlined above. It is possible that the influence functional techniques were used so far to derive a real but subdominant, second-order effect, — an effect of decoherence due to self-interaction of the inflaton field, — while the main effect responsible for generating the classical cosmological perturbations is due to interaction with gravity, which has not yet been derived in the influence functional approach. The mismatch is presently not clarified in the literature, which emphasizes the need for a better understanding of the nature of the quantum-to-classical transition for cosmological perturbations.

Finally, let us mention a different line of work which supports the standard "classicalization" picture. In [60, 152, 153, 154, 155, 156], calculations of (renormalized) quantum expectation values such as $\langle\hat{\phi}^2\rangle$, $\langle\hat{\phi}^4\rangle$, etc., were performed for field operators $\hat{\phi}$ in a *fixed* de Sitter background (*i.e.* ignoring the backreaction on the metric). The results were compared with the statistical averages

$$\int P(\phi,t)\phi^2 d\phi, \quad \int P(\phi,t)\phi^4 d\phi, \quad \text{etc.,} \tag{3.6}$$

where the distribution $P(\phi,t)$ describes the "random walk" of the field ϕ in the Fokker–Planck approach (see Sec. 4.1). It was shown that the leading late-time asymptotics of the quantum expectation values coincide with the corresponding statistical averages (3.6). Furthermore, an explicit picture of decoherence due to interaction in a two-field model was obtained in [157], where an explicit expression of the density matrix for perturbation modes was computed. It was shown that decoherence of a perturbation mode occurs several (of order 3) *e*-foldings after the mode crosses the Hubble horizon scale.

These results appear to validate the "random walk" approach, albeit in a limited context, namely in the absence of backreaction on the metric and/or without a detailed underlying picture of quantum fluctuations of the metric. In the rest of this book, we assume the validity of this approach.

4 Stochastic approach to inflation

The **stochastic approach to inflation** is a semiclassical, statistical description of the spacetime resulting from quantum fluctuations of the inflaton field(s) and their backreaction on the metric. This approach originated in the early works [55, 56, 158, 159, 160, 161, 162, 163, 164, 165, 166, 68]. In this approach, the spacetime remains everywhere classical but its geometry is determined by a stochastic process. In the next subsections I review the main tools used in the stochastic approach for calculations in the context of random-walk type, slow-roll inflation. Models involving tunneling-type eternal inflation are considered later in Chapter 5.

4.1 Eternal inflation of random walk type

The salient features of random walk type eternal inflation can be understood by considering a simple slow-roll inflationary model with a single scalar field ϕ and a potential $V(\phi)$. The slow-roll evolution equation is

$$\dot{\phi} = -\frac{1}{3H}\frac{dV}{d\phi} = -\frac{1}{4\pi}\frac{dH}{d\phi} \equiv v(\phi), \tag{4.1}$$

where $H(\phi)$ is defined by Eq. (3.1) and $v(\phi)$ is a model-dependent function describing the "velocity" $\dot{\phi}$ of the deterministic evolution of the field ϕ, expressed as a function of ϕ. The slow-roll trajectory $\phi_{\mathrm{sr}}(t)$, which is a solution of Eq. (8.9), is (approximately) an attractor for trajectories starting with a wide range of initial conditions, as we described in the previous chapter. If a higher precision is desired, we may use the exact attractor curve $\phi_0(t)$ to determine the function $v(\phi)$.

According to the discussion in Sec. 3.2, the super-horizon modes of the field ϕ are assumed to undergo a rapid quantum-to-classical transition. Therefore one regards the spatial average of ϕ on scales of several H^{-1} as a *classical* field variable. The spatial averaging can be described with the help of a suitable window function,

$$\langle\phi(\mathbf{x})\rangle \equiv \int W(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})d^3\mathbf{y}. \tag{4.2}$$

It is implied that the window function $W(\mathbf{x})$ decays quickly on physical distances $a\,|\mathbf{x}|$ of order several H^{-1}. From now on, let us denote the volume-averaged field also by ϕ (no other field ϕ will be used).

The influence of quantum fluctuations leads to random "jumps" superimposed on top of the deterministic evolution of the volume-averaged field $\phi(t, \mathbf{x})$. This may be described by a **Langevin equation** of the form [56]

$$\dot{\phi}(t, \mathbf{x}) = v(\phi) + N(t, \mathbf{x}), \tag{4.3}$$

where $N(t, \mathbf{x})$ stands for "noise" and is assumed to be a Gaussian random function with known correlator [56, 167, 168, 169]

$$\langle N(t, \mathbf{x}) N(\tilde{t}, \tilde{\mathbf{x}}) \rangle = C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}| \,; \phi). \tag{4.4}$$

An explicit form of the correlator C depends on the specific window function W used for averaging the field ϕ on Hubble scales. However, the window function W is merely a phenomenological device used in lieu of a complete *ab initio* derivation of the stochastic inflation picture. One expects, therefore, that results of calculations should be robust with respect to the choice of W. In other words, any uncertainty due to the choice of the window function must be regarded as an imprecision inherent in the method. For instance, a robust result in this sense is an exponentially fast decay of correlations on time scales $\Delta t \gtrsim H^{-1}$,

$$C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}| \,; \phi) \propto \exp\left(-2H(\phi) \left| t - \tilde{t} \right|\right), \tag{4.5}$$

and similarly a fast decay at large distances; both results hold for a wide class of smooth window functions [169].

For the purposes of the present consideration, we only need to track the evolution of $\phi(t, \mathbf{x})$ along a single comoving worldline $\mathbf{x} = \text{const}$. Thus, we will not need an explicit form of $C(t, \tilde{t}, |\mathbf{x} - \tilde{\mathbf{x}}| \,; \phi)$ but merely the value at coincident points $t = \tilde{t}$, $\mathbf{x} = \tilde{\mathbf{x}}$, which is computed in the slow-roll inflationary scenario as [56]

$$C(t, t, 0; \phi) = \frac{H^2(\phi)}{4\pi^2} = \sqrt{2D(\phi) H(\phi)}. \tag{4.6}$$

(This represents the fluctuation (3.2) accumulated during one Hubble time, $\Delta t = H^{-1}$.) Due to the property (4.5), one may neglect correlations on time scales $\Delta t \gtrsim H^{-1}$ in the "noise" field.[1] Thus, the evolution of ϕ on time scales $\Delta t \gtrsim H^{-1}$ can be described by a finite-difference form of the Langevin equation (4.3),

$$\phi(t + \Delta t) - \phi(t) = v(\phi) \Delta t + \sqrt{2D(\phi) \Delta t}\, \xi(t), \tag{4.7}$$

where

$$D(\phi) \equiv \frac{H^3(\phi)}{8\pi^2} \tag{4.8}$$

and ξ is a normalized random variable representing "white noise,"

$$\langle \xi \rangle = 0, \quad \left\langle \xi^2 \right\rangle = 1, \tag{4.9}$$

$$\langle \xi(t) \xi(t + \Delta t) \rangle = 0 \quad \text{for} \quad \Delta t \gtrsim H^{-1}. \tag{4.10}$$

[1] Taking these correlations into account leads to a picture of "color noise" [170, 171]. In what follows, we only consider the simpler picture of "white noise" as an approximation adequate for the issues at hand.

Equation (4.7) is interpreted as describing a Brownian motion $\phi(t)$ with the systematic "drift" $v(\phi)$ and the "diffusion coefficient" $D(\phi)$. In a typical slow-roll inflationary scenario, there will be a range of ϕ where the noise dominates over the deterministic drift,

$$v(\phi)\Delta t \ll \sqrt{2D(\phi)\Delta t}, \quad \Delta t \equiv H^{-1}. \tag{4.11}$$

Such a range of ϕ is called the **diffusion-dominated regime**. For ϕ near the end of inflation, the opposite inequality holds because the amplitude of the noise is very small since the observational constraints indicate that the amplitude of cosmological density fluctuations is

$$\frac{\delta\rho}{\rho} \sim \frac{\sqrt{2D(\phi)H(\phi)}}{v(\phi)} \sim 10^{-5}. \tag{4.12}$$

Values of ϕ where the inequality (4.11) holds constitute the **deterministic regime** where the random jumps can be neglected and the field ϕ follows the slow-roll trajectory. The ranges of ϕ that correspond to the deterministic regime or to the fluctuation-dominated regime can be determined when a specific form of the potential $V(\phi)$ is given or, alternatively, when the coefficients $v(\phi)$ and $D(\phi)$ are known.

4.1.1 Fokker–Planck equations: comoving distribution

A useful description of the statistical properties of $\phi(t)$ is furnished by the probability density $P(\phi,t)d\phi$ of having a value ϕ at time t. As in the case of the Langevin equation, the values $\phi(t)$ are measured along a single, randomly chosen comoving worldline $\mathbf{x} = \text{const}$. The probability distribution $P(\phi,t)$ is called the **comoving** probability distribution and satisfies the Fokker–Planck (FP) equation [172, 173],

$$\partial_t P = \partial_\phi \left[-v(\phi)P + \partial_\phi \left(D(\phi)P \right) \right]. \tag{4.13}$$

The coefficients $v(\phi)$ and $D(\phi)$ are in general model-dependent and need to be calculated in each particular scenario. These calculations require only the knowledge of the deterministic inflationary evolution, *e.g.* in the slow-roll approximation ($v_{\text{sr}}(\phi) = \dot{\phi}$ where $\dot{\phi}$ must be expressed as a function of ϕ) and the mode functions of the quantized scalar perturbations for computing $D(\phi)$. For ordinary slow-roll inflation with an effective potential $V(\phi)$, the results are well-known expressions (8.9) and (7.8). The corresponding expressions for models of k-inflation were derived in [38] using the quantum theory of perturbations in k-inflation [174]. In general, we can determine D by computing the quantum average

$$2D\Delta t \equiv \langle 0 | \left\langle \delta\dot{\hat{\phi}}\Delta t \right\rangle_L^2 | 0 \rangle , \tag{4.14}$$

where $\left\langle \delta\dot{\hat{\phi}}\Delta t \right\rangle_L$ is the (quantum) fluctuation of the field ϕ averaged on distance scales L that correspond to effective "classicalization" of the fluctuations. In the case of minimally coupled field, $L \sim H^{-1}$ while for non-canonical fields having nontrivial speed of sound, $c_s \neq 1$, as has $L \sim c_s H^{-1}$ as shown in [38].

The FP equation similar to Eq. (4.13) first appeared in the paper [55], although the main topic of that paper was the creation of the universe in quantum cosmology. The formalism of FP equations was subsequently developed by many authors [56, 58, 175, 160, 162, 163, 164, 176, 166, 68, 177, 178].

A "physicist's" derivation of Eq. (4.13) is as follows.[2] Let us introduce the "propagator" $P(\phi_1, \phi_2, t_1, t_2)$ signifying the probability density of having the value $\phi = \phi_2$ at time $t = t_2$ if we had $\phi = \phi_1$ at time $t = t_1$. The distribution $P(\phi_1, \phi_2, t_1 - \Delta t, t_2)$ is expressed through $P(\phi_1, \phi_2, t_1, t_2)$ as

$$P(\phi_1, \phi_2, t_1 - \Delta t, t_2) = \int P(\tilde{\phi}, \phi_2, t_1, t_2) \, \mathrm{Prob}(\Delta\phi = \tilde{\phi} - \phi_1) d\tilde{\phi}, \tag{4.15}$$

where $\mathrm{Prob}(\Delta\phi = x)dx$ is the probability density of obtaining a value x for the random quantity $\Delta\phi$ expressed by Eq. (4.7). The quantity $\Delta\phi$ depends on the "white noise" variable $\xi \equiv \xi(t)$, for which we may introduce explicitly the normal Gaussian distribution $p(\xi)$. Then we may write

$$\begin{aligned} &\int P(\tilde{\phi}, \phi_2, t_1, t_2) \, \mathrm{Prob}(\Delta\phi = \tilde{\phi} - \phi_1) \, d\tilde{\phi} \\ &= \int P(\phi_1 + \Delta\phi(\xi), \phi_2, t_1, t_2) \, p(\xi) d\xi. \end{aligned} \tag{4.16}$$

Now we substitute Eq. (4.7) for $\Delta\phi$,

$$\Delta\phi(\xi) = v\Delta t + \xi\sqrt{2D\Delta t}. \tag{4.17}$$

Note that we evaluated $D(\phi)$, $v(\phi)$, and $\Delta t(\phi) \equiv H^{-1}$ at the "old" value, $\phi \equiv \phi(t)$, rather than at the new value $\tilde{\phi} \equiv \phi(t + \Delta t)$. For brevity we simply wrote D, v, and Δt above, implying $D(\phi)$, $v(\phi)$, and $H^{-1}(\phi)$. Now we transform Eq. (4.16) by expanding P in ϕ_1, keeping terms up to $O(\Delta t)$, and evaluate the integrals over $d\xi$ using Eq. (4.9). The result is

$$\begin{aligned} \int P(\phi_1 + \Delta\phi(\xi), \phi_2, t_1, t_2) \, p(\xi) d\xi &= P(\phi_1, \phi_2, t_1, t_2) \\ &+ v\Delta t \, \partial_{\phi_1} P + D\Delta t \, \partial_{\phi_1} \partial_{\phi_1} P + O(\Delta t^{3/2}). \end{aligned} \tag{4.18}$$

As usual in kinetic theory, we disregard higher-order terms in Δt, which would give higher derivatives in the kinetic equation. Finally we take the limit $\Delta t \to 0$ and arrive at the **backward** Fokker–Planck (or Kolmogorov) equation,

$$-\partial_{t_1} P(\phi_1, \phi_2, t_1, t_2) = \hat{L}_{\phi_1} P(\phi_1, \phi_2, t_1, t_2), \tag{4.19}$$

[2] The "physical level of rigor" in treating stochastic equations consists of treating the "white noise" variable ξ as a Gaussian random variable and using a "single step" of the Brownian motion, $\Delta\phi \propto \sqrt{\Delta t}\xi$. In other words, we explicitly introduce a quantity that is proportional to the square root of the time step Δt. Then we expand all expressions up to first order in Δt and take the limit $\Delta t \to 0$ at the end of the calculations. This procedure yields results that are equivalent to the Ito interpretation of the stochastic calculus.

where we have introduced the linear differential operator $\hat{L}$,

$$\hat{L}_\phi f \equiv \left[v(\phi)\partial_\phi + D(\phi)\partial_\phi\partial_\phi \right] f.$$

The backward FP equation describes the dependence of the probability P on the initial value ϕ_1. It is more useful to obtain a differential equation with respect to the final value ϕ_2. This is derived as follows. The distribution $P(\phi_1, \phi_2, t_1, t_2)$ satisfies the usual "propagator" identity

$$P(\phi_1, \phi_3, t_1, t_3) = \int P(\phi_1, \phi_2, t_1, t_2) P(\phi_2, \phi_3, t_2, t_3) d\phi_2. \tag{4.20}$$

Taking the derivative of both parts with respect to t_2, and denoting for brevity $P_{ij} \equiv P(\phi_i, \phi_j, t_i, t_j)$, we find

$$\begin{aligned} 0 &= \int \left(\partial_{t_2} P_{12}\right) P_{23} d\phi_2 + \int P_{12} \left(\partial_{t_2} P_{23}\right) d\phi_2 \\ &= \int \left(\partial_{t_2} P_{12}\right) P_{23} d\phi_2 + \int P_{12} \hat{L}_{\phi_2} P_{23} d\phi_2 \\ &= \int \left[\partial_{t_2} P_{12} - \hat{L}^\dagger_{\phi_2} P_{12}\right] P_{23} d\phi_2. \end{aligned} \tag{4.21}$$

Here the adjoint operator $\hat{L}^\dagger$ is defined by

$$\hat{L}^\dagger_\phi f = -\partial_\phi \left[v(\phi) f \right] + \partial_\phi\partial_\phi \left[D(\phi) f \right], \tag{4.22}$$

and the boundary terms are assumed to vanish; this is usually the case due to boundary conditions. Now we use the property

$$P(\phi_1, \phi_2, t, t) = \delta(\phi_1 - \phi_2) \tag{4.23}$$

in order to eliminate P_{23} in Eq. (4.21), and finally obtain

$$\partial_{t_2} P_{12} = \hat{L}^\dagger_{\phi_2} P_{12}. \tag{4.24}$$

This coincides with Eq. (4.13).

We note that the use of the Langevin equation (4.7) is justified only with finite values $\Delta t \sim H^{-1}$ rather than with infinitesimal steps $\Delta t \to 0$. The derivation of the FP equation proceeds by taking the limit $\Delta t \to 0$ in the Langevin equation (4.7), promoting it from a difference equation to a stochastic *differential* equation. Although the stochastic differential equation appears to describe fluctuations of ϕ that scale as $\sqrt{\Delta t}$ for arbitrarily small Δt, we know that these fluctuations are not actually semiclassical until a few e-folding times, that is, until $\Delta t \gtrsim H^{-1}$. Accordingly, the Langevin and the FP equations can be used only to describe features of the distribution at time scales at least $\Delta t \gtrsim H^{-1}$. It is, however, technically more convenient to consider continuous-time Brownian motion with infinitesimal steps, using either Langevin or FP equations. It is sufficient that these equations reproduce correct results for steps $\Delta t \gtrsim H^{-1}$. In order to quantify this artificial distinction between continuous and discrete

Brownian motion, we may introduce a parameter ε such that $\Delta t = (\varepsilon H)^{-1}$. (This parameter is related to the parameter ε introduced in the early derivations of the stochastic approach, *e.g.* [56, 175].) As explained above, fluctuations can be approximately treated as classical white noise only on timescales $\Delta t \gtrsim H^{-1}$, or more precisely, with values of ε such that $\exp(\varepsilon) \gg 1$. We find by examining the derivation of the FP equation that the resulting equation is independent of ε. For this reason, we conclude that a description in terms of the finite-difference Langevin equation is adequately translated into a FP equation.

It is well known that there exists a "factor ordering" ambiguity in translating the Langevin equation into the FP equation if the "diffusion coefficient" depends on ϕ. If the size of the typical fluctuation, $\delta\phi \sim \sqrt{2D\Delta t}$, depends on ϕ itself, it is not unambiguous at which value of ϕ we need to evaluate $D(\phi)$ and $H(\phi)$. We could evaluate $D(\phi)$ and $\Delta t(\phi) \equiv H^{-1}(\phi)$ at the initial value of ϕ, or at the final value $\phi + \Delta\phi$; more generally, the factors $D(\phi)$ and $v(\phi)$ in Eq. (4.7) may be replaced by $D(\phi + \theta\Delta t)$ and $v(\phi + \theta\Delta t)$, where $0 < \theta < 1$ is an arbitrary constant. The "correct" value of θ needs to come from additional physical information and is not specified by the phenomenological formulation of the Langevin equation. When the derivation of the FP equation is repeated with $\theta \neq 0$, the term $\partial_{\phi\phi}(DP)$ in Eq. (4.13) will be replaced by a different ordering of the factors, namely

$$\partial_\phi \partial_\phi (DP) \to \partial_\phi \left[D^\theta \partial_\phi \left(D^{1-\theta} P \right) \right]. \tag{4.25}$$

Popular choices $\theta = 0$ and $\theta = \frac{1}{2}$ are called the **Ito** and the **Stratonovich** factor ordering respectively.

Given the phenomenological nature of the Langevin equation (4.7), one expects that any ambiguity due to the choice of θ represents an imprecision inherent in the stochastic approach. Indeed, one obtains an imprecision, for instance, in the eigenvalues of the FP operator; this (relative) imprecision is typically of order H^2 in Planck units [179], which is necessarily much less than 1 (or else the considerations are inconsistent with the assumption of semiclassical gravity). A more fundamental description of the stochastic process could clarify this issue. Lacking such a description, we content ourselves with choosing a particular factor ordering and verifying that results do not depend sensitively on that choice. Below (at the end of Sec. 4.1.3) we will give an argument for choosing the Ito factor ordering ($\theta = 0$).

Another limitation of the FP approach has been recently uncovered [180] in the context of k-inflation models, or more generally models with noncanonical fields and a small speed of sound of perturbations, $c_s \ll 1$, such as DBI inflation [181]. It turns out that the fluctuation-dominated regime in such models is generally outside the limits of applicability of first-order perturbation theory, on which the calculations of the fluctuation amplitudes and the coefficient $D(\phi)$ rely. Further research is needed to obtain an adequate description of fluctuation-dominated regime in such models [182, 183].

The quantity $P(\phi, t)$ may be interpreted not only as the probability distribution of ϕ at a fixed comoving point, but also as the fraction of the **comoving**

volume (*i.e.* the "coordinate" volume computed as $\int d^3\mathbf{x}$ in a spatially flat coordinate system) occupied by the field value ϕ at time t.

4.1.2 Volume-weighted distribution

Another important characteristic is the **volume-weighted distribution** $P_V(\phi, t)d\phi$, which is defined as the physical 3-volume (computed as $\int \sqrt{-^{(3)}g}\, d^3\mathbf{x}$, as opposed to the comoving volume) of regions having the value ϕ at time t, *i.e.* within an equal-t hypersurface. To avoid considering infinite volumes, one may restrict one's attention to a finite comoving domain in the universe and normalize $P_V(\phi, t)$ to unit volume at some initial time $t = t_0$. The volume distribution satisfies a modified FP equation [58, 162, 164],

$$\partial_t P_V = \partial_\phi \left[-v(\phi)P_V + \partial_\phi \left(D(\phi)P_V\right)\right] + 3H(\phi)P_V, \tag{4.26}$$

which differs from Eq. (4.13) by the term $3HP_V$ that describes the exponential growth of 3-volume in inflating regions.

A derivation of Eq. (4.26) is as follows. We may consider the joint comoving distribution $P(\phi, a, t)$ of the field ϕ and the scale factor a. This distribution is interpreted as the fraction of the comoving volume where the field has the value ϕ and the scale factor has the value a, within an equal-t hypersurface. Alternatively, $P(\phi, a, t)$ is the probability of the field having the value ϕ and the scale factor having the value a at time t along a single, randomly chosen comoving trajectory. The distribution $P(\phi, a, t)$ is the solution of the FP equation

$$\partial_t P = \hat{L}_\phi^\dagger P - \partial_a \left(aHP\right), \tag{4.27}$$

where $\hat{L}_\phi^\dagger$ is the same differential operator entering Eq. (4.13).

To obtain the distribution $P_V(\phi, t)$, we can now use the procedure of **volume weighting**, which means that we average the stochastic quantities with weights proportional to the physical 3-volume of regions where the particular values of these quantities are measured. In the present case, the 3-volume is proportional to the factor a^3, so it suffices to define

$$P_V(\phi, t) \equiv \int_0^\infty da\, a^3 P(\phi, a, t), \tag{4.28}$$

which weighs the distribution of ϕ with the physical 3-volume by using the value of the volume expansion factor (a^3). It is easy to verify that the resulting (unnormalized) distribution $P_V(\phi, t)$ satisfies the FP equation (4.26). This establishes a link between the volume interpretation of the distribution $P_V(\phi, t)$ and properties of a single comoving trajectory embodied by the joint probability $P(\phi, a, t)$.

We note that the value of the function $P_V(\phi, t)$ has the meaning of *volume* rather than probability. Accordingly, $P_V(\phi, t)$ is not interpreted as a normalized probability distribution. If desired, one can normalize this volume distribution and consider the ratio

$$P_p(\phi, t) \equiv \frac{P_V(\phi, t)}{\int P_V(\phi, t) d\phi}. \tag{4.29}$$

A modified FP equation describing directly the normalized distribution was given in [162] and can be written in our notation as

$$\partial_t P_p = \hat{L}_\phi^\dagger P_p + 3\left(H - \langle H \rangle\right) P_p, \tag{4.30}$$

$$\langle H \rangle \equiv \int H(\phi) P_p(\phi, t) d\phi. \tag{4.31}$$

We will not consider this integrodifferential equation since it is much easier to treat the FP equation (4.26) and obtain equivalent results.

Presently we focus on scenarios with a single scalar field; however, the formalism of FP equations can be straightforwardly extended to multi-field models as well as to models with non-canonical kinetic terms (see *e.g.* [184, 185, 38]). For instance, the FP equation for a two-field model is

$$\partial_t P = \partial_{\phi\phi}\left(DP\right) + \partial_{\chi\chi}\left(DP\right) - \partial_\phi\left(v_\phi P\right) - \partial_\chi\left(v_\chi P\right), \tag{4.32}$$

where $D(\phi, \chi)$, $v_\phi(\phi, \chi)$, and $v_\chi(\phi, \chi)$ are appropriate coefficients. The volume distribution $P_V(\phi, \chi, t)$ is defined analogously.

To specify the solution uniquely, the FP equations must be supplemented by boundary conditions at both ends of the inflating range of ϕ (this issue was considered in [166, 68]). At the reheating boundary ($\phi = \phi_*$), one may imposes the "exit" boundary conditions,

$$\partial_\phi\left[D(\phi)P\right]_{\phi=\phi_*} = 0, \quad \partial_\phi\left[D(\phi)P_V\right]_{\phi=\phi_*} = 0. \tag{4.33}$$

These boundary conditions express the fact that random jumps are very small at the end of inflation and cannot move the value of ϕ away from $\phi = \phi_*$. Another, simpler possibility is to impose the absorbing boundary condition, $P(\phi_*, t) = 0$. It is usually the case that the solutions of the FP equations are insensitive to the precise choice of the boundary conditions at the reheating boundary. While a different choice of boundary conditions will distort the solution $P(\phi, t)$ near the boundary points, this distortion is negligible away from these points [68].

If the potential $V(\phi)$ reaches Planck energy scales at some Planck boundary $\phi = \phi_{\mathrm{P}}$, a boundary condition must be imposed also at $\phi = \phi_{\mathrm{P}}$. For instance, one can use the absorbing boundary condition,

$$P(\phi_{\mathrm{P}}) = 0, \tag{4.34}$$

which means that Planck-energy regions with $\phi = \phi_{\mathrm{P}}$ disappear from consideration [166, 68]. Since the Planck boundary condition is imposed *ad hoc* rather than derived from fundamental considerations, we would need to verify that the conclusions do not depend sensitively on the choice of boundary condition.

4.1.3 Gauge dependence issues

An important feature of the FP equations is their dependence on the choice of the time coordinate (or "gauge" as it is frequently called in this context). One can consider a replacement of the form

$$t \to \tau, \quad d\tau \equiv T(\phi)dt, \tag{4.35}$$

understood in the sense of integrating along comoving worldlines $\mathbf{x} = \text{const}$, where $T(\phi) > 0$ is an arbitrary function of the field. For instance, a possible choice is $T(\phi) \equiv H(\phi)$, which makes the new time variable dimensionless,

$$\tau = \int H dt = \ln a. \tag{4.36}$$

This time variable is called **scale factor time** or ***e*-folding time** since it measures the number of e-foldings along a comoving worldline.

The distributions $P(\phi, \tau)$ and $P_V(\phi, \tau)$ are defined as before, except that one considers the 3-volumes along hypersurfaces of equal τ. These distributions satisfy FP equations similar to Eqs. (4.13) and (4.26). With the replacement (4.35), the coefficients of the new FP equations are modified as follows [179],

$$D(\phi) \to \frac{D(\phi)}{T(\phi)}, \quad v(\phi) \to \frac{v(\phi)}{T(\phi)}, \tag{4.37}$$

while the "growth" term $3HP_V$ in Eq. (4.26) is replaced by $3HT^{-1}P_V$.

As an example, let us show the FP equations written in the e-folding time $\tau \equiv \ln a$. The comoving FP equation is

$$\partial_\tau P = \partial_{\phi\phi}\left[\frac{D}{H}P\right] - \partial_\phi\left[\frac{v}{H}P\right], \tag{4.38}$$

while the FP equation for the volume-weighted distribution is

$$\partial_\tau P_V = \partial_{\phi\phi}\left[\frac{D}{H}P_V\right] - \partial_\phi\left[\frac{v}{H}P_V\right] + 3P_V. \tag{4.39}$$

It is important to note that these two equations are essentially equivalent: If $P(\phi, \tau)$ is a solution of Eq. (4.38) then $P_V(\phi, \tau) = e^{3\tau}P(\phi, \tau)$ is a solution of Eq. (4.39), and *vice versa*. This equivalence is easily understood: The volume-weighted distribution on surfaces of equal τ, that is, of equal scale factor, is the same as the comoving distribution since the growth of volume along a surface of equal scale factor is everywhere the same by construction. However, comoving and volume-weighted distributions in other time gauges are not equivalent.

Changing the time gauge may significantly alter the qualitative behavior of the solutions of the FP equations. For instance, stationary distributions defined through the proper time t and the e-folding time $\tau = \ln a$ were found to have radically different behavior [68, 3, 5]. This sensitivity to the choice of the "time gauge" τ is unavoidable since hypersurfaces of equal τ may preferentially select regions with certain properties. It is easy to see that most of the physical volume in equal-t hypersurfaces is filled with regions that have gained expansion by remaining near the top of the potential $V(\phi)$, while hypersurfaces of equal scale factor will instead under-represent those regions. Thus, a statement such as "most of the volume in the universe has values of ϕ with high $V(\phi)$" is largely gauge-*dependent*. This is to be expected since the FP equation describes the distribution of properties along equal-τ hypersurfaces, while these hypersurfaces defined for different time gauges τ are not statistically equivalent but differ from each other in a systematic way. Hence, as a rule, any solution or any property of the FP equation is gauge-dependent. Exceptions are found only when one considers some results obtained by solving *time-independent* FP equations, more precisely, FP equations that do not contain the time variable and can be reformulated in terms of gauge-independent coefficients such as D/v and v/H. We will give examples of such cases below.

In the early works on eternal inflation [166, 68, 3, 79], the late-time asymptotic distribution of volume $P_V^{(\tilde{\gamma})}(\phi)$ along hypersurfaces of equal proper time [see Eq. (4.65)] was interpreted as the "stationary" distribution of field values in the universe. However, the high sensitivity of this distribution to the choice of the time variable makes this interpretation unsatisfactory. Also, it was noted [186] that equal-proper time volume distributions predict an unacceptably small probability for the currently observed CMB temperature. The reason for this result is the extreme bias of the proper-time gauge towards over-representing regions where reheating occurred very recently [187, 5]. This is a manifestation of the so-called "youngness paradox" (see Sec. 6.4). One might ask whether hypersurfaces of equal scale factor or some other choice of time gauge would provide less biased answers. However, it turns out that there exists no *a priori* choice of the time gauge τ that provides precisely unbiased equal-τ probability distributions for all potentials $V(\phi)$ in models of slow-roll inflation (see Sec. 6.2 for details).

Although the FP equations necessarily involve a dependence on gauge, they do provide a useful statistical picture of the distribution of fields in the universe. The FP techniques can also be used for deriving several gauge-independent results. For instance, the presence of eternal inflation is a gauge-independent statement (see also Sec. 4.2): if the largest eigenvalue $\tilde{\gamma}$ is positive in one gauge of the form (4.35), then $\tilde{\gamma} > 0$ in every other gauge (see [188] for a rigorous proof). Using the FP approach, one can also compute the fractal dimension of the inflating domain [189, 188] and the probability of exiting inflation through a particular point ϕ_* of the reheating boundary in the configuration space (in case there exists more than one such point, such as in multi-field models). These computations yield gauge-invariant results.

We postpone the consideration of the fractal structure of the spacetime to Sec. 4.2. Presently we will show how to determine the exit probabilities [184, 38]. Let us assume for simplicity that there are two possible exit points ϕ_* and

ϕ_E, and that the initial distribution is concentrated at $\phi = \phi_0$, *i.e.*

$$P(\phi, t=0) = \delta(\phi - \phi_0), \tag{4.40}$$

where $\phi_E < \phi_0 < \phi_*$. Since we are considering the probability distribution $P(\phi, t)$ along a single comoving worldline, the appropriate FP equation is Eq. (4.13). This equation can be rewritten as a "conservation law,"

$$\partial_t P = \partial_\phi J, \quad J(\phi) \equiv \partial_\phi (DP) - vP, \tag{4.41}$$

where J plays the role of the "current." Using the conservation law (4.41), one finds that the probability of exiting inflation through $\phi = \phi_E$ during a time interval $[t, t+dt]$ is

$$\begin{aligned} dp_{\text{exit}}(\phi_E) &= J(\phi_E)dt \equiv \left(\partial_\phi (DP) - vP\right)\big|_{\phi=\phi_E} dt \\ &= -v(\phi_E)P(\phi_E, t)dt. \end{aligned} \tag{4.42}$$

Note that $v(\phi_E) < 0$ and we used the boundary condition (4.33). Hence, the total probability of exiting through $\phi = \phi_E$ at any time is

$$p_{\text{exit}}(\phi_E) = \int_0^\infty dp_{\text{exit}}(\phi_E) = -v(\phi_E) \int_0^\infty P(\phi_E, t)dt. \tag{4.43}$$

Introducing an auxiliary function $F(\phi)$ as

$$F(\phi) \equiv -v(\phi) \int_0^\infty P(\phi, t)dt, \tag{4.44}$$

and integrating Eq. (4.41) in t one can show that $F(\phi)$ satisfies the gauge-invariant equation,

$$\partial_\phi \left[\partial_\phi \left(\frac{D}{v} F \right) - F \right] = \delta(\phi - \phi_0). \tag{4.45}$$

This is in accord with the fact that $p_{\text{exit}}(\phi_E) = F(\phi_E)$ is a gauge-invariant quantity. Equation (4.45) with the boundary conditions

$$F(\phi_*) = 0, \quad \partial_\phi \left(\frac{D}{v} F\right)\Big|_{\phi=\phi_E} = 0, \tag{4.46}$$

can be straightforwardly integrated and yields explicit expressions for the exit probability $p_{\text{exit}}(\phi_E)$ as a function of the initial value ϕ_0 (see [38] for some specific calculations). The exit probability $p_{\text{exit}}(\phi_*)$ can be determined similarly.

In the same way one can try to use the volume distribution $P_V(\phi, t)$ to compute the total volume $V_*(t_0)$ reheated up to a given time $t = t_0$. One finds

$$V_*(t_0) = -v(\phi_*) \int_0^{t_0} P_V(\phi_*, t)dt. \tag{4.47}$$

However, one generally finds that $P_V(\phi_*, t)$ grows exponentially at late times, so this integral cannot be evaluated in the limit $t_0 \to \infty$.

In the multi-field situation, *e.g.* with fields ϕ, χ_1, ..., χ_N, where the reheating boundary is $\phi = \phi_*$ with various possible values of χ_i, one may be interested in the total physical volume reheated at $\phi = \phi_*$ and with given values of the fields χ_i. One may also compute the comoving exit probability at various points of the reheating boundary. Those computations are analogous to the computations in one-field scenarios.

The consideration of the exit probability can motivate the choice of the Ito factor ordering for the FP equation [184]. Namely, it can be shown that only the Ito factor ordering gives gauge-independent results when one computes the exit probability. An arbitrary factor ordering can be described by introducing a parameter θ and by replacing

$$\partial_{\phi\phi}(DP) \to \partial_\phi(D^\theta \partial_\phi(D^{1-\theta}P)) \tag{4.48}$$

in the FP equation. With an arbitrary gauge, $d\tau \equiv T dt$, the coefficients D, v are replaced by $D/T, v/T$ and so Eq. (4.45) becomes

$$\partial_\phi \left[\left(\frac{D}{T}\right)^\theta \partial_\phi \left(\left(\frac{D}{T}\right)^{1-\theta} \frac{T}{v} F \right) - F \right] = \delta(\phi - \phi_0). \tag{4.49}$$

Now it is clear that the above equation depends on T (*i.e.* is gauge-dependent) as long as $\theta \neq 0$. However, we expect that the exit probability is gauge-independent since it is defined as the probability of finishing inflation at a given value $\phi = \phi_*$ for a randomly chosen comoving observer, which is clearly independent of the choice of coordinates. Motivated by this consideration, we choose the Ito factor ordering ($\theta = 0$) as shown in Eqs. (4.7) and (4.13).

4.1.4 Interpretation of FP equations

The Langevin and the Fokker–Planck equations are, in principle, equivalent descriptions of the same stochastic process, and we will focus on the FP description because it is more convenient to work with. It is important to note also that the boundary conditions (at the reheating and Planck boundaries) can be imposed straightforwardly in the FP description, but cannot be easily described in the framework of the Langevin equation.

The issue of gauge dependence is also most conveniently investigated in the FP formalism. Let us now consider the FP equations with an arbitrary time gauge τ.

First we comment on the backward FP equation (4.19). I would like to emphasize that this equation,

$$-\partial_t P(\phi, \phi_f, t, t_f) = \hat{L}_\phi P(\phi, \phi_f, t, t_f), \tag{4.50}$$

does *not* (despite its appearance) describe the evolution of the distribution $P(\phi, \phi_f, t, t_f)$ "backward in time t" with a fixed future value $\phi = \phi_f$ at $t = t_f$. As is well known, the backward diffusion equation is ill-defined because of instabilities;

an arbitrarily small uncertainty in the fixed future value ϕ_f will lead to an arbitrarily large variation in the solution P at earlier times. Therefore, one cannot actually solve the backward FP equation in a directly constructive way (*e.g.* numerically). One must always solve the *forward* FP equation, and then one may use the fact that its solution also satisfies the backward FP equation. The proper interpretation of the distribution $P(\phi, \phi_f, t, t_f)$ is always a "forward" one, *i.e.* P is the probability of future values ϕ_f given past values ϕ.

We now turn to the forward FP equations. As noted above, the comoving distribution $P(\phi, \tau)$ admits two interpretations. On the one hand, $P(\phi, \tau)$ is the probability of the field having a value ϕ at time τ along a single, randomly chosen comoving trajectory; on the other hand, $P(\phi, \tau)$ is the fraction of comoving volume where the field has the value ϕ, within an equal-τ hypersurface. (The "comoving 3-volume" of a subdomain S of an equal-τ hypersurface is defined as the 3-volume of the subdomain S_0 of the hypersurface $\tau = 0$, where S_0 consists of points whose comoving worldlines pass through S at time τ.)

The equivalence of these interpretations can be seen from the following argument. Consider a large but finite domain of volume V_0 within the hypersurface $\tau = 0$ and a very dense grid of comoving trajectories starting out from this domain. The total number N of these trajectories may be chosen as finite but very large, and the initial volume V_0 can be subdivided into N tiny regions of comoving volume V_0/N situated around each of the N comoving lines. If the initial volume V_0 is sufficiently large so that all the statistically possible histories are sufficiently well sampled, then the number of lines that reach a value ϕ at a time τ is very close to $NP(\phi, \tau)$. Thus the comoving volume of the subdomain of the equal-τ hypersurface having the value ϕ is $V_0 P(\phi, \tau)$. Therefore in the statistical limit $N \to \infty$ and $V_0 \to \infty$, the fraction of the comoving volume containing the field value ϕ at the time τ is equal to $P(\phi, \tau)$.

Conversely, if $P(\phi, \tau)$ is the statistical distribution of the comoving volume along an equal-τ hypersurface, then the comoving volume of the subdomain with the value ϕ will be $V_0 P(\phi, \tau)$. We denote this subdomain by $S(\phi, \tau)$ and consider a comoving trajectory randomly chosen from the total comoving volume V_0. The probability of this trajectory passing through the subdomain $S(\phi, \tau)$ is equal to $V_0 P(\phi, \tau)/V_0 = P(\phi, \tau)$. Therefore $P(\phi, \tau)$ is the probability of the field having the value ϕ at time τ along a randomly chosen trajectory.

We note that there *do* exist correlations between the values of ϕ at nearby comoving trajectories. However, these correlations are not reflected by the distribution $P(\phi, \tau)$ since it specifies only the total volume, but not the location, of regions with the field value ϕ. If the initial comoving volume V_0 is sufficiently large (many Hubble volumes), the distribution of the field ϕ at time τ along a randomly chosen comoving trajectory will be $P(\phi, \tau)$ despite the correlations between nearby trajectories.

Turning now to the distribution of the physical volume $P_V(\phi, \tau)$, we note that there is now only one interpretation, namely $P_V(\phi, \tau)$ is the total proper 3-volume of the subdomain with the field value ϕ within an equal-τ hypersurface. (As before, we consider only the comoving future of a large but finite initial volume V_0 at time $\tau = 0$.) It appears to be impossible to interpret $P_V(\phi, \tau)$ directly in terms of observations made on a single randomly chosen comoving trajectory. The distribution $P_V(\phi, \tau)$ describes a branching diffusion process with a possibly unlimited growth, and its interpretation necessarily has to involve a potentially very large ensemble of trajectories.

An interesting alternative description of the inflationary spacetime can be given using the path integral formalism [190]. The path integral is used to describe Brownian random walk, as is standard in the theory of stochastic processes. While the description of Brownian motion in terms of path integrals is entirely equivalent to that by FP equations, some aspects of the evolution may be more easily understood, for instance, by visualizing the "most likely" trajectory. One can compute the saddle point approximation to the path integral in hopes of achieving a qualitative understanding of the behavior of trajectories. It has been shown that the most likely trajectories during eternal inflation spend a long time in fluctuation-dominated regime and subsequently quickly reach reheating.[3] Such results cannot be obtained in the FP equation approach since the FP equation furnishes no global information about the values of ϕ along the entire trajectory.

Another kind of information not furnished by the distribution $P(\phi, t)$ is the information about correlations of field values between neighbor points, as already noted above. The pairwise distribution $P(\phi_1, \phi_2, t)$ of field values at *two* fixed comoving points $\mathbf{x}_1$ and $\mathbf{x}_2$ has been considered *e.g.* in [191, 192] and can be also obtained as a solution of a modified FP equation. Correlations between neighbor points may be used, for example, to determine the mean curvature of the reheating surface at superhorizon scales.[4] We will not treat these advanced applications here.

4.1.5 Methods of solution; approximations

In principle, one can solve the FP equations forward in time by a numerical procedure, starting from a given initial distribution at $t = t_0$. Let us thus concentrate on available analytical methods.

Once the initial and boundary conditions for the FP equation are specified, one may write the general solution of the FP equation (4.13) as

$$P(\phi, t) = \sum_\lambda C_\lambda P^{(\lambda)}(\phi)\, e^{\lambda t}, \tag{4.51}$$

where the sum is performed over all the eigenvalues λ of the differential operator

$$\hat{L}^\dagger P \equiv \partial_\phi \left[-v(\phi) P + \partial_\phi \left(D(\phi) P \right) \right], \tag{4.52}$$

and the corresponding eigenfunctions $P^{(\lambda)}$ are defined by

$$\hat{L}^\dagger P^{(\lambda)}(\phi) = \lambda P^{(\lambda)}(\phi). \tag{4.53}$$

[3] C. Armendariz-Picon and V. F. Mukhanov (1998), unpublished; R. Abramo, C. Armendariz-Picon, and V. F. Mukhanov (2000), unpublished.

[4] S. Winitzki (2002), unpublished.

The constants C_λ can be expressed through the initial distribution $P(\phi, t_0)$.

By an appropriate change of variables $\phi \to z$, $P(\phi) \to F(z)$, the operator $\hat{L}^\dagger$ may be brought into a manifestly self-adjoint form [160, 162, 193, 163, 164, 179],

$$\hat{L} \to \frac{d^2}{dz^2} - U(z). \tag{4.54}$$

Here the "potential" $U(z)$ and the new variable z are defined by

$$z \equiv z(\phi) \equiv \int^\phi \frac{d\phi}{\sqrt{D(\phi)}}, \quad \partial_\phi = \frac{1}{\sqrt{D}}\partial_z, \tag{4.55}$$

$$P^{(\lambda)}(\phi) \equiv \psi^{(\lambda)}(z) D^{-3/4}(\phi) \exp\left[\frac{1}{2}\int^\phi \frac{v_*(\phi)}{D(\phi)} d\phi\right], \tag{4.56}$$

where the replacement $\phi \to \phi(z)$ is implied in the last line. Under the replacements (4.55)-(4.56), Eq. (4.53) is transformed into

$$\psi_{,zz} - U(z)\psi = \lambda\psi, \tag{4.57}$$

where $U(z)$ is the "potential" defined as a function of ϕ through the relation

$$U(z) \equiv \frac{3}{16}\frac{\left(D_{,\phi}\right)^2}{D} - \frac{D_{,\phi\phi}}{4} - \frac{vD_{,\phi}}{2D} + \frac{v_{,\phi}}{2} + \frac{v^2}{4D}. \tag{4.58}$$

Equation (4.57) is the self-adjoint form of the stationary FP equation and is formally equivalent to a stationary Schrödinger equation for a particle in a one-dimensional "potential" $U(z)$ with the "energy" $E \equiv -\lambda$. Then one can show that all the eigenvalues λ of $\hat{L}^\dagger$ are nonpositive; in particular, the (algebraically) largest eigenvalue $\lambda_{\max} \equiv -\gamma < 0$ is nondegenerate and the corresponding eigenfunction $P^{(\lambda_{\max})}(\phi)$ is everywhere positive [179, 38]. Hence, this eigenfunction describes the late-time asymptotic distribution $P(\phi, t)$,

$$P(\phi, t) \propto P^{(\lambda_{\max})}(\phi)\, e^{-\gamma t}. \tag{4.59}$$

The distribution $P^{(\lambda_{\max})}(\phi)$ is the "stationary" distribution of ϕ per comoving volume at late times. The exponential decay of the distribution $P(\phi, t)$ means that at late times most of the comoving volume (except for an exponentially small fraction) has finished inflation and entered reheating.

Another, equivalent way to make the operator $\hat{L}^\dagger$ self-adjoint without changing variables is to introduce the scalar product

$$\langle f_1, f_2 \rangle = \int f_1(\phi) f_2(\phi) M(\phi) d\phi, \tag{4.60}$$

where

$$M(\phi) \equiv D_0(\phi) \exp\left[-\int^\phi \frac{v_0(\phi)}{D_0(\phi)} d\phi\right]. \tag{4.61}$$

One can then verify that the operator $\hat{L}^\dagger$ is formally self-adjoint with this scalar product,

$$\langle f_1, \hat{L}^\dagger f_2 \rangle = \langle \hat{L}^\dagger f_1, f_2 \rangle. \tag{4.62}$$

This points to another way of showing that the spectrum of $\hat{L}^\dagger$ is bounded from above.

Similar considerations apply to the volume-weighted distribution P_V. One can represent the general solution of Eq. (4.26) by

$$P_V(\phi, t) = \sum_{\tilde{\lambda}} C_{\tilde{\lambda}} P_V^{(\tilde{\lambda})}(\phi) e^{\tilde{\lambda} t}, \tag{4.63}$$

where

$$[\hat{L}^\dagger + 3H(\phi)] P_V^{(\tilde{\lambda})}(\phi) = \tilde{\lambda} P_V^{(\tilde{\lambda})}(\phi). \tag{4.64}$$

By the same method as for the operator $\hat{L}^\dagger$, it is possible to show that the spectrum of eigenvalues $\tilde{\lambda}$ of the operator $\hat{L}^\dagger + 3H(\phi)$ is bounded from above and that the largest eigenvalue $\tilde{\lambda}_{\max} \equiv \tilde{\gamma}$ admits a nondegenerate, everywhere positive eigenfunction $P_V^{(\tilde{\gamma})}(\phi)$. However, the largest eigenvalue $\tilde{\gamma}$ may be either positive or negative. If $\tilde{\gamma} > 0$, the late-time behavior of $P_V(\phi, t)$ is

$$P_V(\phi, t) \propto P_V^{(\tilde{\gamma})}(\phi) e^{\tilde{\gamma} t}, \tag{4.65}$$

which means that the total physical volume of all the inflating regions grows with time. This is the behavior expected in eternal inflation: the number of independently inflating domains increases without limit. Thus, the condition $\tilde{\gamma} > 0$ is a criterion for the presence of eternal self-reproduction of inflating domains. The corresponding distribution $P_V^{(\tilde{\gamma})}(\phi)$ is called the "stationary" distribution [166, 68, 3, 80].

If $\tilde{\gamma} \leq 0$, eternal inflation does not occur and the entire spatial slice will almost surely (*i.e.* with probability 1) enter the reheating stage after a finite time.

If the potential $V(\phi)$ is of "new" inflationary type and has a global maximum at say $\phi = \phi_0$, the eigenvalues γ and $\tilde{\gamma}$ can be estimated (under the usual slow-roll assumptions on V) as follows [179]. One approximates the potential $V(\phi)$ near its maximum by a parabola,

$$V(\phi) \approx V(\phi_0) - \frac{1}{2} \left| V''(\phi_0) \right| \phi^2 + O(\phi^3), \tag{4.66}$$

and reduces the FP equation to a Schrödinger equation with a harmonic potential near $\phi = \phi_0$. The known spectrum of this Schrödinger operator allows one to obtain an estimate (in Planck units)

$$\gamma \approx -\frac{|V''(\phi_0)|}{8\pi V(\phi_0)} H(\phi_0) < 0, \quad \tilde{\gamma} \approx 3H(\phi_0) > 0. \tag{4.67}$$

Numerical calculations confirm that these estimates are precise. Therefore, we find $\tilde{\gamma} > 0$, meaning that eternal inflation is generic in the "new" inflationary scenario.

Let us comment on the possibility of obtaining solutions $P(\phi, t)$ in practice. With the potential $V(\phi) = \lambda\phi^4$, the full time-dependent FP equation (4.13) can be solved analytically via a nonlinear change of variable $\phi \to \phi^{-2}$ [194, 193, 195]. This exact solution, as well as an approximate solution $P(\phi, t)$ for a general potential, can be also obtained using the saddle-point evaluation of a path-integral expression for $P(\phi, t)$ [190]. In some cases the eigenvalue equation $\hat{L}^\dagger P^{(\lambda)} = \lambda P^{(\lambda)}$ may be reduced to an exactly solvable Schrödinger equation. These cases include potentials of the form $V(\phi) = \lambda e^{\mu\phi}$, $V(\phi) = \lambda\phi^{-2}$, $V(\phi) = \lambda \cosh^{-2}(\mu\phi)$; see *e.g.* [179] for other examples of exactly solvable potentials.

While the reduction of the FP equation to a self-adjoint form (*i.e.* to the Schrödinger equation) is useful in the theoretical analysis, it is only rarely the case that a Schrödinger equation is exactly solvable. Except for such rare cases, one has to resort to approximations. We will briefly outline the available results.

One important approximation applies to the evolution in the deterministic regime where fluctuations are subdominant and the field approximately follows its deterministic, slow-roll trajectory. In that case, one may neglect the second-derivative term in the FP equation. For instance, Eq. (4.26) becomes

$$\partial_t P_V \approx -\partial_\phi \left[v(\phi) P_V \right] + 3H(\phi) P_V, \tag{4.68}$$

which may be solved by the method of characteristics. Let us consider the corresponding time-independent FP equation,

$$-\partial_\phi \left[v(\phi) P_V^{(\lambda)} \right] + 3H(\phi) P_V^{(\lambda)} = \lambda P_V^{(\lambda)}. \tag{4.69}$$

This linear equation can be integrated to

$$P_V^{(\lambda)}(\phi) = \frac{1}{v(\phi)} \exp\left[\int_{\phi_0}^{\phi} \frac{3H(\phi) - \lambda}{v(\phi)} d\phi \right], \tag{4.70}$$

where ϕ_0 is an integration constant. We note that the factors in the expression above have a clear physical interpretation:

$$\exp\left[\int_{\phi_0}^{\phi} \frac{3H(\phi)}{v(\phi)} d\phi \right] = \frac{[a(\phi)]^3}{[a(\phi_0)]^3}, \tag{4.71}$$

where

$$a(\phi) \equiv \exp\left[\int_{\phi_0}^{\phi} \frac{H(\phi)}{\dot{\phi}_{\mathrm{sr}}} d\phi \right] = \exp\left[\int H(t) dt \right] \tag{4.72}$$

is the scale factor accumulated during the slow-roll evolution from ϕ to ϕ_0, while the integral

$$\int_{\phi_0}^{\phi} \frac{1}{v(\phi)} d\phi = \int_{\phi_0}^{\phi} \frac{d\phi}{\dot{\phi}_{\mathrm{sr}}} = \int dt = t_{\mathrm{sr}}(\phi) - t_{\mathrm{sr}}(\phi_0) \tag{4.73}$$

yields the (deterministic) duration of time t_{sr} corresponding to the slow-roll evolution. Hence we may write approximately

$$P_V^{(\lambda)}(\phi) \approx \frac{\text{const}}{v(\phi)} [a(\phi)]^3 e^{-\lambda t_{sr}(\phi)}. \tag{4.74}$$

Note that the analogous approximation for the comoving-volume FP equation will not contain the volume expansion factor a^3.

So far we have entirely disregarded diffusion; in this approximation, the initial distribution $P_V(\phi, t = t_0)$ is shifted towards $\phi = \phi_*$ without any spreading. If the initial distribution is concentrated at $\phi = \phi_0$ so that $P_V(\phi, t_0) = \delta(\phi - \phi_0)$, the distribution $P_V(\phi, t)$ will remain a δ-function of ϕ at later times, meaning that the initial region with $\phi = \phi_0$ traverses the deterministic regime and reheats during a fixed time interval. In reality, fluctuations are small but present, so the traversal time $t(\phi_*) - t(\phi_0)$ is a random variable that fluctuates around a certain mean value. This mean value is only *approximately* equal to $t_{sr}(\phi_*) - t_{sr}(\phi_0)$.

A more precise approximate method for determining the *comoving* distribution $P(\phi, t)$ in the deterministic regime was used in [196, 197, 198] and consists of a perturbative expansion,

$$\phi(t) = \phi_0(t) + \delta\phi_1(t) + \delta\phi_2(t) + ..., \tag{4.75}$$

which is to be applied to the stochastic *Langevin* equation, rather than to the FP equation. The result is (at the lowest order) a Gaussian approximation with a time-dependent mean and variance [196],

$$P(\phi, t) \approx \frac{1}{\sqrt{2\pi\sigma^2(t)}} \exp\left[-\frac{(\phi - \phi_0(t))^2}{2\sigma^2(t)}\right], \tag{4.76}$$

$$\sigma^2(t) \equiv \frac{H'^2(\phi_{sr})}{\pi} \int_{\phi_{sr}}^{\phi_{in}} \frac{H^3}{H'^3} d\phi, \tag{4.77}$$

$$\phi_0(t) \equiv \phi_{sr}(t) + \frac{H''}{2H'}\sigma^2(t) + \frac{H'}{4\pi}\left[\frac{H_{in}^3}{H_{in}'^2} - \frac{H^3}{H'^2}\right], \tag{4.78}$$

where $\phi_{sr}(t)$ is the slow-roll trajectory and ϕ_{in} is the initial value of ϕ. While methods based on the Langevin equation do not take into account boundary conditions or volume weighting effects, the formula (4.76) provides an adequate approximation to the distribution $P(\phi, t)$ in a useful range of ϕ and t, as shown in [198]. This formula shows how the initial distribution spreads and also gives a correction to the slow-roll traversal time.

An analogous approximation for the deterministic regime can be developed also in the FP formalism, whose advantage over the Langevin description is that the boundary conditions as well as volume weighting can be taken into account. One notes that Eq. (4.26) in the deterministic regime contains a small parameter (which is not the same as the slow-roll parameter), $DH/v^2 \ll 1$, and that the small parameter multiplies the highest-derivative term in that equation. Therefore, a WKB-like asymptotic perturbation theory can be applied. We will illustrate this technique by computing the volume-weighted average of the traversal time $\langle t(\phi_*) - t(\phi_q) \rangle_V$ spent in the deterministic regime, where ϕ_q is the approximate beginning of the deterministic regime and ϕ_* is the reheating boundary.

The slow-roll conditions can be rewritten in terms of the coefficients $D(\phi)$, $v(\phi)$, $H(\phi)$ as

$$\left| \frac{v}{H} \right| \ll 1, \quad v' \ll H, \quad \frac{D'}{D} \ll \frac{H}{v}. \tag{4.79}$$

Note that the ratios H/v, D/v of the kinetic coefficients are independent of the choice of time parametrization. Therefore, below we will express all relevant results in terms of these gauge-independent ratios. In terms of the slow-roll parameters ε, η as usually introduced, we have

$$\varepsilon = \frac{1}{16\pi} \left(\frac{V'}{V} \right)^2 = 36\pi \left(\frac{v}{H} \right)^2, \quad \eta = \frac{1}{8\pi} \frac{V''}{V} = \frac{3v'}{H} + \varepsilon. \tag{4.80}$$

Thus, the small quantities h/H and v'/H are analogous to the slow-roll parameters.

In the deterministic regime where the random motion is negligible, we have additionally

$$\frac{DH}{v^2} \ll 1. \tag{4.81}$$

Indeed, the observed primordial cosmological inhomogeneity of order 10^{-5} can be expressed as

$$10^{-5} \sim \frac{H^2}{\dot{\phi}} \sim \sqrt{\frac{DH}{v^2}} \bigg|_{\phi_*}. \tag{4.82}$$

Thus, the gauge-invariant combination DH/v^2 characterizing the relative importance of diffusion is of order 10^{-10} during the last 60 e-foldings of inflation.

Diffusion dominates for the values of ϕ where

$$\frac{DH}{v^2} \gg 1. \tag{4.83}$$

Therefore, we may define the boundary ϕ_q of the diffusion-dominated regime by an order-of-magnitude relation,

$$\frac{DH}{v^2} \bigg|_{\phi_q} \sim 1. \tag{4.84}$$

Note that the reheating boundary ϕ_* is also defined through an order-of-magnitude relation,

$$\frac{v}{H} \bigg|_{\phi_*} \sim 1. \tag{4.85}$$

For definiteness we assume that $\phi_q < \phi_*$. Since we are interested in describing the evolution only within the range of ϕ where self-reproduction is absent, we may impose boundary conditions at $\phi = \phi_q$ and $\phi = \phi_*$ that express the absence of diffusion at these points,

$$\partial_\phi \left(DP_V\right)\big|_{\phi=\phi_q,\phi_*} = 0. \tag{4.86}$$

While these boundary conditions will distort the solution $P_V(\phi, t)$ near the points ϕ_q, ϕ_*, this distortion is negligible away from these points. Let us consider the Green's function $G(\phi, \phi_0, t)$ for Eq. (4.26) with the boundary conditions (4.86). The Green's function can be represented as the solution of Eqs. (4.26), (4.86) with the initial condition

$$G(\phi, \phi_0, t)|_{t=0} = \delta(\phi - \phi_0). \tag{4.87}$$

The Green's function $G(\phi, \phi_0, t)$ can be visualized as the time-dependent volume distribution that results from an initial unit volume with $\phi = \phi_0$ at time $t = 0$. For the purposes of the present calculation we assume that ϕ_0 is near ϕ_q but not equal to ϕ_q. The infinitesimal probability of reaching $\phi = \phi_*$ within the time interval $[t, t + dt]$ is

$$v(\phi_*)G(\phi_*, \phi_0, t)dt. \tag{4.88}$$

We can calculate the infinitesimal probability of reaching $\phi = \phi_*$ within the time interval $[t, t + dt]$ when starting within the time interval $[t_0, t_0 + dt_0]$ as

$$v(\phi_*)G(\phi_*, \phi_0, t - t_0)dt\, dt_0. \tag{4.89}$$

Thus, the average traversal time for regions that reached $\phi = \phi_*$ within the time interval $[t, t + dt]$ is

$$\langle \Delta t \rangle = \frac{\int_{-\infty}^{t} dt_0\, (t - t_0)\, v(\phi_*)G(\phi_*, \phi_0, t - t_0)}{\int_{-\infty}^{t} dt_0 v(\phi_*)G(\phi_*, \phi_0, t - t_0)}. \tag{4.90}$$

The mean traversal time (which we have denoted for brevity by $\langle \Delta t \rangle$) is actually independent of the arrival time t because, after the substitution $\tilde{t} \equiv t - t_0$, it can be identically rewritten as

$$\langle \Delta t \rangle = \frac{\int_0^\infty \tilde{t} d\tilde{t}\, G(\phi_*, \phi_0, \tilde{t})}{\int_0^\infty d\tilde{t}\, G(\phi_*, \phi_0, \tilde{t})}. \tag{4.91}$$

It is convenient to introduce the Laplace transform of G,

$$g(\phi, \phi_0; \omega) \equiv \int_0^\infty d\tilde{t}\, G(\phi, \phi_0, \tilde{t})e^{-\omega \tilde{t}}. \tag{4.92}$$

Then the expectation value of the traversal time is found as

$$\langle \Delta t \rangle = - \left.\frac{\partial}{\partial \omega}\right|_{\omega=0} \ln g(\phi_*, \phi_0; \omega). \tag{4.93}$$

The generating function g can be also used to estimate the spread of the traversal time,

$$\left\langle \Delta t^2 \right\rangle - \langle \Delta t \rangle^2 = \left. \frac{\partial^2}{\partial \omega^2} \right|_{\omega=0} \ln g(\phi_*, \phi_0; \omega). \tag{4.94}$$

By integrating Eq. (4.26) with respect to time, one can derive the differential equation for $g(\phi; \omega)$,

$$\partial_{\phi\phi} (Dg) - \partial_\phi (vg) + (3H - \omega) g = -\delta(\phi - \phi_0). \tag{4.95}$$

This equation is valid as long as the integral $\int_0^\infty G(\phi_*, \phi_0, t) e^{-\omega t} dt$ converges, which is expected to be the case at least for all positive ω since the range $\left[\phi_q, \phi_*\right]$ does not support self-reproduction.

We find that the logarithmic generating function of traversal time is the Green's function $g(\phi, \phi_0; \omega)$ of the stationary FP equation with eigenvalue ω. To obtain an analytic formula for $g(\phi, \phi_0; \omega)$, we apply the adiabatic approximation. It is convenient to change variables so that the term with first derivative vanishes from Eq. (8.44). To this end, we introduce the auxiliary function $f(\phi, \phi_0; \omega)$ as follows,

$$g(\phi, \phi_0; \omega) = \frac{1}{D(\phi)} \exp \left[\frac{1}{2} \int_{\phi_0}^{\phi} \frac{v}{D} d\phi \right] f(\phi, \phi_0; \omega). \tag{4.96}$$

The equation for $f(\phi, \phi_0; \omega)$ is derived from Eq. (8.44),

$$f'' - \frac{1}{2} \left(\frac{v}{D} \right)' f - \frac{v^2}{4D^2} f + \frac{(3H - \omega)}{D} f = -D(\phi_0) \delta(\phi - \phi_0), \tag{4.97}$$

where we denote $f' \equiv \partial_\phi f$ (the variables ϕ_0 and ω enter as parameters). We may rewrite Eq. (4.97) as

$$f'' - Q^2(\phi) f = -D(\phi_0) \delta(\phi - \phi_0), \tag{4.98}$$

$$Q(\phi) \equiv \sqrt{\frac{v^2}{4D^2} + \frac{1}{2} \left(\frac{v}{D} \right)' + \frac{\omega - 3H}{D}}. \tag{4.99}$$

Since Eq. (4.97) contains only terms f'' and f, we can now apply the WKB approximation. Due to Eq. (4.79), the dominant term in Eq. (4.97) is $v^2/(4D^2)$, at least for moderate values of ω; we recall that at the end of the calculation we will need to set $\omega = 0$ in order to compute $\langle \Delta t \rangle$ and $\left\langle \Delta t^2 \right\rangle$. Therefore, $Q^2 > 0$ in Eq. (4.99). The WKB approximation gives the general solutions of Eq. (4.97) away from $\phi = \phi_0$ as a linear combination of growing and decaying exponentials,

$$f(\phi) \approx \frac{C_1 \exp \left[\int Q(\phi) d\phi \right]}{\sqrt{Q(\phi)}} + \frac{C_2 \exp \left[- \int Q(\phi) d\phi \right]}{\sqrt{Q(\phi)}}. \tag{4.100}$$

The delta-function source in Eq. (4.97) means that

$$\lim_{\delta\phi \to 0} \left[f'(\phi_0 + \delta\phi) - f'(\phi_0 - \delta\phi) \right] = -D(\phi_0). \tag{4.101}$$

Since the boundary conditions fix the values of $f(\phi, \phi_0; \omega)$ at the endpoints $\phi = \phi_q$ and $\phi = \phi_*$, it is clear that the correct solution is represented (with an exponentially good precision) by a growing exponential for $\phi_q < \phi < \phi_0$ and by a decaying exponential for $\phi_0 < \phi < \phi_*$. Therefore we may write

$$f(\phi, \phi_0; \omega) \approx \frac{C_0}{\sqrt{Q(\phi)}} \exp\left[-\int_{\phi}^{\phi_0} Q d\phi \right], \quad \phi_q < \phi < \phi_0; \tag{4.102}$$

$$f(\phi, \phi_0; \omega) \approx \frac{C_0}{\sqrt{Q(\phi)}} \exp\left[-\int_{\phi_0}^{\phi} Q d\phi \right], \quad \phi_0 < \phi < \phi_*. \tag{4.103}$$

The value of the integration constant C_0 is fixed by the condition (4.101), which yields

$$C_0 = \frac{D(\phi_0)}{2\sqrt{Q(\phi_0)}}. \tag{4.104}$$

Using this value, the approximate solution $g(\phi, \phi_0; \omega)$ can be written as

$$\begin{aligned} & g(\phi, \phi_0; \omega) \\ & \approx \frac{1}{2} \frac{1}{\sqrt{Q(\phi_0) Q(\phi)}} \frac{D(\phi_0)}{D(\phi)} \exp\left[\frac{1}{2} \int_{\phi_0}^{\phi} \frac{v}{D} d\phi - \int_{\phi_0}^{\phi} Q d\phi \right]. \end{aligned} \tag{4.105}$$

We need to compute only the values $\partial_\omega \ln g$ and $\partial_\omega \partial_\omega \ln g$, so the result (8.122) can be simplified further. Since $Q(\phi)$ is dominated by the term $v^2/(4D^2)$, the exponential function in Eq. (8.122) contains a difference of almost equal terms. In order to compute the spread of the traversal time, we need to expand $\ln g(\phi, \phi_0; \omega)$ up to terms quadratic in ω. To make expanding $Q(\phi)$ from Eq. (4.99) easier, we rewrite

$$Q(\phi) = \frac{v}{2D} \sqrt{1 + \frac{4\omega D}{v^2} + 2W(\phi)}, \tag{4.106}$$

$$W(\phi) \equiv \frac{v' D - D' v - 6DH}{v^2}, \tag{4.107}$$

where we introduced an auxiliary function $W(\phi)$ for convenience. We have $|W| \ll 1$ due to the slow-roll conditions and the smallness of diffusion, $DH \ll v^2$. Neglecting terms quadratic in W and in D, but keeping terms of order ω^2, we obtain

$$Q(\phi) \approx \frac{v}{2D} (1 + W) + \frac{\omega}{v} (1 - W) - \frac{\omega^2 D}{v^3}; \tag{4.108}$$

$$\ln \sqrt{Q} \approx \frac{1}{2} \ln \frac{v}{2D} + \frac{W}{2} + \omega \frac{D}{v^2} (1 - 2W). \tag{4.109}$$

Finally, computing $\ln g$ from Eq. (8.122) and expanding up to terms quadratic in ω, we derive an approximate expression for $\ln g(\phi_*, \phi_0; \omega)$. In that expression, we may neglect also the linear terms of order D when evaluated at $\phi = \phi_*$

(but not at $\phi = \phi_0$). The result is

$$\ln g(\phi_*, \phi_0; \omega) \approx \ln \frac{D(\phi_0)}{v(\phi_*)} - \frac{W(\phi_0) + W(\phi_*)}{2} + \int_{\phi_0}^{\phi_*} \frac{3H}{v} d\phi$$
$$- \omega \frac{D(\phi_0)}{v^2(\phi_0)} - \omega \int_{\phi_0}^{\phi_*} \frac{d\phi}{v(\phi)} (1 - W) + \omega^2 \int_{\phi_0}^{\phi_*} \frac{D}{v^3} d\phi. \tag{4.110}$$

It follows that the expectation value of the traversal time is

$$\langle \Delta t \rangle = - \left. \frac{\partial}{\partial \omega} \right|_{\omega=0} \ln g$$
$$\approx \frac{D(\phi_0)}{v^2(\phi_0)} + \int_{\phi_0}^{\phi_*} \frac{d\phi}{v(\phi)} (1 - W), \tag{4.111}$$

while the spread in the traversal time is

$$\left\langle \Delta t^2 \right\rangle - \langle \Delta t \rangle^2 = 2 \int_{\phi_0}^{\phi_*} \frac{D}{v^3} d\phi. \tag{4.112}$$

Note that the deterministic slow-roll traversal time (disregarding diffusion) would be

$$\Delta t_{\mathrm{sr}} = \int_{\phi_0}^{\phi_*} \frac{d\phi}{v(\phi)}. \tag{4.113}$$

We have thus obtained $\langle \Delta t \rangle$ up to the leading-order correction to Δt_{sr}; a further development of adiabatic perturbation theory (using higher-order WKB approximation and further expansion of Q) yields higher-order corrections to Δt_{sr}.

The approximations considered so far are applicable to the deterministic regime. An adiabatic approximation can be constructed also for the regime of high diffusion. We proceed as in the construction of the WKB approximation. Replacing for convenience D by θD, where θ is a formal parameter, we substitute into Eq. (4.53) the ansatz

$$P_V^{(\lambda)}(\phi) = \exp \left[\frac{1}{\theta} \int^{\phi} \left(W_0(\phi) + \theta W_1(\phi) + \theta^2 W_2 + ... \right) d\phi \right] \tag{4.114}$$

and proceed to expand the result in powers of θ. For the lowest terms, we obtain the equations

$$DW_0^2 - vW_0 = 0, \tag{4.115}$$
$$2D'W_0 + DW_0' - v' + 3H - \lambda = (2DW_0 - v) W_1, \tag{4.116}$$
$$D'' + 2D'W_1 + D(W_1^2 + W_1') = (2DW_0 - v) W_2, \tag{4.117}$$

where the prime stands for ∂_ϕ. We find two solutions, $W_0 = 0$ and $W_0 = v/D$. These two solutions correspond to the two independent WKB branches. The other equations can be solved straightforwardly for W_1, W_2, etc., in each branch separately. For instance, in the $W_0 = 0$ branch we get

$$W_1 = \frac{3H - \lambda - v'}{v}, \tag{4.118}$$

which reproduces the approximate solution (4.70). This procedure will yield all other corrections in a systematic way.[5]

4.2 Presence of eternal inflation

In this section we use the method of FP equations to solve the problem of determining whether eternal inflation is present in a given model of inflation of random-walk type.

The hallmark of eternal inflation is the unbounded growth the total number of independent inflating regions. The total proper 3-volume of the inflating domain also grows without bound at late times, at least when computed along hypersurfaces of equal proper time or equal scale factor. However, the proper 3-volume is a gauge-dependent quantity, and one may construct complete spacelike foliations whose 3-volume decreases with time even in an everywhere expanding universe [75]. Clearly, the volume growth computed using an *arbitrary* family of equal-time hypersurfaces cannot be used as a criterion for the presence of eternal inflation. However, a weaker condition is adequate: Eternal inflation is present if and only if there *exists* a choice of time slicing with an unbounded growth of the 3-volume of inflating domains. Thus, the presence or absence of eternal inflation is a gauge-*independent* statement. One may, of course, use a particular gauge (such as the proper time or *e*-folding time) for calculations, since the result is known to be gauge-independent. For instance, it can be shown that eternal inflation is present if and only if the 3-volume grows in the *e*-folding time slicing [75].

Another way to formulate the condition for the presence of eternal inflation is that an infinite number of reheated H-regions may result from *finitely* many initially inflating H-regions. It is then clear that the initial conditions for inflation cease to play a significant role in the resulting universe. As long as the spacetime contains even a single H-region that supports inflation, no matter how unlikely it is to realize the favorable initial conditions for inflation, an *infinite* number of reheated (and potentially inhabitable) H-regions will be created to the future of the initial H-region. Assuming that there is a finite nonzero probability for the emergence of life in any reheated H-region, it follows that eternal inflation generates an infinite number of regions containing life. A non-inflating initial region in the spacetime will give rise at most to a finite number of inhabitable regions. In this sense, it may be said that the inhabitable parts of the universe are dominated by the products of eternal inflation, and that eternal

[5]However, as is well known, the WKB approximation produces only an asymptotic series whose precision is fundamentally limited. An estimate of the best possible precision of the WKB approximation can be found in [199].

inflation removes the problem of fine-tuning of the initial conditions for inflation.

The presence of eternal inflation has been analyzed in many specific scenarios. For instance, eternal inflation is generic in "chaotic" [200, 74, 57] and "new" [60] inflationary models. It is normally sufficient to establish the existence of a "diffusion-dominated" regime, that is, a range of ϕ where the typical amplitude $\delta\phi \sim H$ of "jumps" is larger than the typical slow-roll increment of the field, $\dot{\phi}_{\mathrm{sr}}\Delta t$, during one Hubble time $\Delta t = H^{-1}$. For models of scalar-field inflation, this condition is

$$H^2 \gg \left| \frac{dH}{d\phi} \right| . \tag{4.119}$$

Such a range of ϕ is present in most slow-roll models of inflation. (For an example of an inflationary scenario where eternal inflation is generically *not* present, see [201].) The range of ϕ where diffusion dominates must be sufficiently wide to support eternal inflation; heuristically, H-regions must not leave the diffusion-dominated range too efficiently. A strict formal criterion for the presence of eternal inflation is $\tilde{\gamma} > 0$, where $\tilde{\gamma}$ is the largest eigenvalue of the operator $\hat{L}^{\dagger} + 3H(\phi)$, as explained in Sec. 4.1.5.

The causal structure of the eternally inflating spacetime and the topology of the reheating surface can be visualized using the construction of **eternal points** [188]. Eternal points are comoving worldlines $\mathbf{x} = \text{const}$ that forever remain within the inflating domain, *i.e.* they strictly *never* enter the reheating epoch. These worldlines correspond to places where the reheating surface reaches $t = \infty$ in a spacetime diagram (see Fig. 7.1).

We will now show using topological arguments [188] that the presence of inflating domains at arbitrarily late times entails the existence of infinitely many eternal points.

There is a non-vanishing, if small, probability that the entire comoving region will thermalize at a certain finite time, because fluctuations of the scalar field ϕ could have accidentally cooperated to drive ϕ toward the end point of inflation ϕ_* everywhere in the region. One could say in that case that eternal inflation has not been realized in that region.[6] If, on the other hand, eternal inflation has been realized, then for an arbitrarily late time t there will be regions that are still inflating at that time.

In that case, there must exist some comoving worldlines that never enter a thermalized domain. To demonstrate this, one could choose a monotonically increasing sequence of time instances, $t_n = nH^{-1}$, $n = 0, 1, 2, ...$; for each n there must exist a point x_n surrounded by a horizon-sized domain that will be still inflating at time t_n. (As t_n grows, these domains become progressively smaller in comoving coordinates, due to expansion of space.) An infinite sequence of points x_n on a finite (compact) comoving volume of 3-dimensional space must have accumulation points, *i.e.* there must be at least one point x_* such that any

[6] Here we do not consider the possibility of spontaneous return of thermalized regions to inflation in "recycling universe" scenarios [83] or in the landscape of string theory. These features will be considered in the following chapter, in the framework of tunneling models.

arbitrarily small (comoving) neighborhood of x_* contains infinitely many points x_n from the sequence. It is clear that the comoving world-line at an accumulation point x_* cannot reach thermalization at any finite time: if it did, there would exist a comoving neighborhood around x_* which thermalized at that time, and this contradicts the construction of the point x_*. Therefore, accumulation points x_* are eternal points, and we may define the set E of all such points, as a set drawn on the spatial section of the comoving volume at initial time. (One could also imagine an infinitely dense grid of comoving world-lines starting at the initial surface, with the set E consisting of all world-lines that never reach thermalization.) As we have shown, eternal points exist, *i.e.* the set E for a comoving region is not empty, if eternal inflation has been realized in that region. However, it is important to note that the 3-volume of the set E is equal to zero (*i.e.* E has measure zero). This is the same as to say that a randomly chosen point $\mathbf{x}$ will reach reheating with probability 1.

Different histories and different choices of the initial comoving volume will generate different sets E. Since E is a stochastically generated set, one may characterize it probabilistically, by finding probabilities for the set E to have certain properties. The underlying stochastic process is the random walk of the inflaton ϕ and, since it is a stationary stochastic process that does not explicitly depend on time, probabilities of any properties of the set E depend only on ϕ_0. For instance, below we will denote by $X(\phi_0)$ the probability of the set E being non-empty if the initial comoving H-region has a given value $\phi = \phi_0$ of the inflaton field.

During stochastic evolution, there will be (infinitely many) times when the inflaton returns to the value ϕ_0 in some horizon-sized domain; each time the distribution of properties of the subset of E within those domains will be the same as that for the whole set E. In this way, the set E naturally acquires fractal structure. The fractal dimension of this set can be understood as the fractal dimension of the inflating domain [189, 188, 75] and is invariant under any *smooth* coordinate transformations in the spacetime [188].

The fractal set E can be thought of as an infinite-time limit of the (fractal) inflating domain considered in [189, 202, 68]. Since the 3-volume of E vanishes, we need a slightly different definition of fractal dimension than that of [189]. The definition of fractal dimension we would like to use is the so-called "box fractal dimension", which in our case is equivalent to the Hausdorff-Besicovitch fractal dimension [203]. The **box fractal dimension** is defined for a non-empty subset S of a finite region of a d-dimensional space. Rectangular coordinates are chosen in the region to divide it into a regular lattice of d-dimensional cubes with side length l. (The cubical shape of the lattice is not essential to the definition of the "box fractal dimension".) Then, the "coarse-grained volume" $V(l)$ of the set S is defined as the total volume of all cubes of the lattice that contain at least one point from the set S. If the set S has fractal dimension less than d, one expects $V(l)$ to decay at $l \to 0$ as some power of l. The fractal dimension is defined by

$$\dim S = d - \lim_{l \to 0} \frac{\ln V(l)}{\ln l}. \tag{4.120}$$

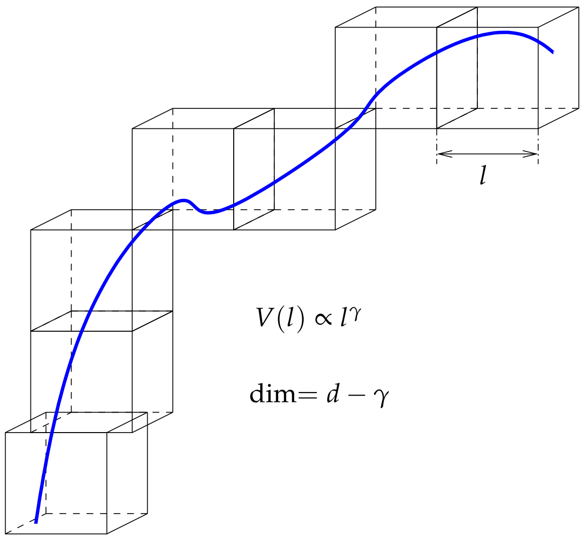

Figure 4.1: Definition of "box" fractal dimension. The given set (curved line) is approximated by cubes of size l; the total volume scales as $V(l) \propto l^{\gamma}$ for $l \to 0$, and the fractal dimension is then $d - \gamma$ where d is the dimension of space.

For example, if the set S consists of a finite number of line segments, then the coarse-grained volume at scale l will be equal to the volume of narrow tubes of width l surrounding the line segments (Fig. 4.1). Then $V(l) \sim l^{d-1}$ and the fractal dimension defined by Eq. (4.120) would come out to be 1, as expected for a set consisting of lines.

It was shown in [188] that the eternal set E has a well-defined fractal dimension dim$E$ which is independent of the choice of spacetime coordinates, and which coincides with the largest eigenvalue γ of the stationary FP equation $\hat{L}_{\phi}^{\dagger} P = -\gamma P$ in which one *must* use the e-folding time, $\tau = \ln a$. This yields a procedure to compute dim$E$ in any given inflationary model, and at the same time a coordinate-invariant interpretation of the eigenvalue γ. Note that the largest eigenvalue of the FP equation in other time parametrizations is different and will not have the same coordinate-invariant meaning.

The fractal structure of the eternal set E can be easily understood if we visualize E as a random Cantor set (Fig. 4.2). The Cantor set is constructed by starting from a line segment, say $[0, 1]$, and by removing portions of that line segment, such that the total length of all removed portions approaches 1. The points remaining after all removals will form a set of measure zero. In the present case, the removed portions are selected at random, hence the name *random* Cantor set. The remaining points are the eternal points. Regions between eternal points are reheated. Assuming for simplicity that reheated regions become asymptotically flat (that is, neglecting the influence of dark energy), we may visualize

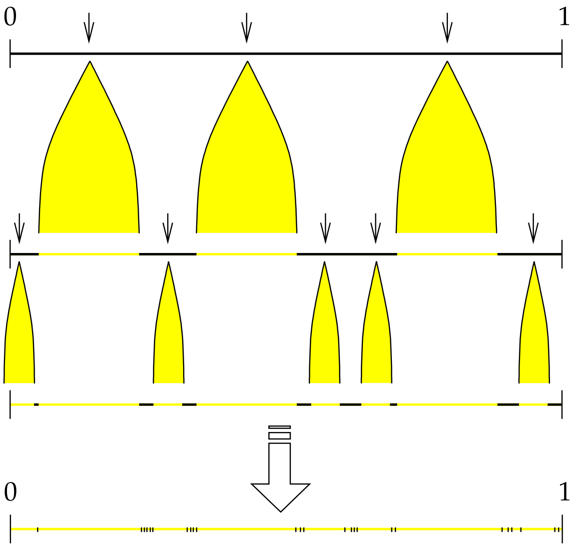

Figure 4.2: Construction of a random Cantor set. Portions of the comoving segment $[0, 1]$ are removed due to the appearance of reheated regions (yellow cones). At later times, the comoving size of removed regions shrinks. The set of points that remain after infinite time has measure zero and a fractal structure.

the causal structure of the resulting spacetime using the conformal diagram in Fig. 4.3 (this conformal diagram, showing only the future of an initial spacelike surface, was derived in [204]). Note that the conformal diagram has a fractal structure due to the fractal structure of the set E: No two asymptotically flat regions are causally connected because there always exist infinitely many eternal points separating any two given asymptotically flat regions (and hence, infinitely many other asymptotically flat regions).

Eternal points separate reheated regions forever, because the space surrounding an eternal point inflates faster than any signal could traverse it. If eternal points are sufficiently dense so that they form a continuous two-dimensional surface surrounding a reheated region (this is possible even though the set of eternal points is fractal), the reheated region will be forever causally separated from all other reheated regions. Deciding whether the set of eternal points is, in this sense, "sufficiently dense" turns out to be a complicated task related to the problem of fractal percolation [205, 206]. This question was raised in [207, 188] but no definite conclusion has been reached so far.

4.2.1 A nonlinear diffusion equation

Eternal points are generated by the same stochastic process that drives eternal inflation. The existence of eternal points can be used as another gauge-invariant

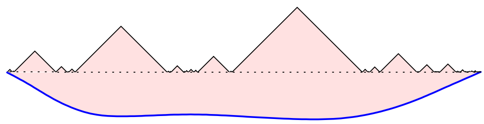

Figure 4.3: Conformal diagram of eternally inflating spacetime showing the future of an initial spacelike Cauchy hypersurface (bottom curve). There is a fractal arrangement of eternal points; regions between eternal points are reheated and are assumed to become asymptotically flat in the future.

criterion for the presence of eternal inflation. In this section we will give an introduction to the methods developed by the author for the analysis of this issue.

Our focus is the probability $X(\phi)$ of having at least one eternal point in an initial Hubble-size region where the field has value ϕ. As we will now show, this probability satisfies the equation[7]

$$\hat{L}_{\phi} X - 3H\left(1 - X\right)\ln\left(1 - X\right) = 0 \tag{4.121}$$

with zero boundary conditions, $X(\phi_*) = 0$ at the reheating boundary and $X(\phi_{\mathrm{P}} = 0)$ at the Planck boundary (if any). Here

$$\hat{L}_{\phi} \equiv D(\phi)\partial_{\phi\phi} + v(\phi)\partial_{\phi} \tag{4.122}$$

is the Fokker–Planck operator used in the *backward* FP equation. Note that Eq. (4.121) is nonlinear due to the presence of the logarithmic term; this equation is of reaction-diffusion type.

We remark that $X(\phi) \equiv 0$ is trivially a solution of Eq. (4.121). However, this solution is not physically relevant if eternal inflation is present, since in that case the probability of having eternal points is certainly nonzero. As will be demonstrated below, the presence of eternal inflation is *equivalent* to the existence of a nontrivial solution $X(\phi) \not\equiv 0$ of Eq. (4.121). We will begin by deriving that equation and showing its independence of time parametrization.

It will be convenient to work with the complementary probability $\bar{X}\left(\phi\right) = 1 - X\left(\phi\right)$ and to use the proper time t as the time variable. We recall that the random walk of the inflaton field ϕ is described by a Langevin equation which can be written as a difference equation (4.7),

$$\phi\left(t + \Delta t\right) = \phi\left(t\right) + v\left(\phi\right)\Delta t + \xi\sqrt{2D\left(\phi\right)\Delta t}. \tag{4.123}$$

The evolution of ϕ in a horizon-size domain with field value $\phi = \phi_0$ at time $t = 0$ gives rise to independent random walks of ϕ in each of the "daughter" horizon-size subdomains that were formed out of the original domain. After

[7] This equation was first derived in [188]. A similar equation was recently rederived using a different approach in [208].

one e-folding, *i.e.* after time $\Delta t = H^{-1}$, there are $N \equiv \exp(3H\Delta t) \approx 20$ daughter subdomains of approximately horizon size. The stochastic process corresponding to ξ assigns a probability density $P(\xi_1, ..., \xi_N)\, d\xi_1 ... d\xi_N$ for various sets of values of ξ_i $(i = 1, ..., N)$ in the N daughter subdomains.

The probability $\bar{X}(\phi_0)$ of having *no* eternal points in the region is equal to the probability of having no eternal points in any of its N daughter subdomains. The evolution in inflating daughter subdomains proceeds independently, so the probability of having no eternal points in each of them is described by the same function $\bar{X}(\phi)$ evaluated at new values $\phi(t + \Delta t)$. (Although the daughter subdomains have slightly varying $H(\phi)$ and are not exactly of horizon size, the correction due to this is negligible.) This argument gives an integral equation,

$$\bar{X}(\phi_0) = \int d\xi_1 ... d\xi_N P(\xi_1, ..., \xi_N) \times \prod_{i=1}^{N} \bar{X}\left(\phi_0 + v(\phi_0)\,\Delta t + \xi_i \sqrt{2D(\phi_0)\,\Delta t}\right). \tag{4.124}$$

Here we use the Ito interpretation of the Langevin equation, where v and D are evaluated at the initial point of the step, $\phi = \phi_0$.

Since correlations between different daughter subdomains are small, we can approximate the probability density of ξ_i by a product of independent identical distributions, $P(\xi_1, ..., \xi_N) = P(\xi_1) \cdots P(\xi_N)$. Then Eq. (4.124) becomes

$$\bar{X}(\phi_0) = \left[\int d\xi P(\xi)\, \bar{X}\left(\phi_0 + v\Delta t + \xi\sqrt{2D\Delta t}\right)\right]^{\exp(3H\Delta t)}. \tag{4.125}$$

Rather than trying to solve Eq. (4.125), we consider its limit for small values of $H\Delta t$. Although the Langevin description of the random walk given by Eq. (4.123) is valid only for times $H\Delta t \gtrsim 1$, we will formally consider it to be valid at all values of Δt and take a partial derivative $\partial/\partial\Delta t$ of Eq. (4.125) at $\Delta t = 0$. The same formal limit $\Delta t \to 0$ is used to derive the FP equations in the stochastic approach of inflation, and the same limits of validity apply to the new nonlinear diffusion equation that will result from the present argument.

We find

$$\frac{\partial \bar{X}(\phi_0)}{\partial \Delta t} = 0 = 3H\bar{X}\ln\bar{X} + \lim_{\Delta t \to 0} \int d\xi P(\xi)\, \frac{d\bar{X}\left(\phi_0 + v\Delta t + \xi\sqrt{2D\Delta t}\right)}{d\phi} \times \left(v + \frac{\xi\sqrt{2D}}{2\sqrt{\Delta t}}\right). \tag{4.126}$$

We now expand $d\bar{X}/d\phi$ in Taylor series around $\phi = \phi_0$ and use the fact that ξ is a normalized random variable,

$$\int d\xi P(\xi) = 1, \quad \langle \xi \rangle = 0, \quad \left\langle \xi^2 \right\rangle = 1, \tag{4.127}$$

to obtain an equation for $\bar{X}$,

$$D\frac{d^2\bar{X}}{d\phi^2}+v\frac{d\bar{X}}{d\phi}+3H\bar{X}\ln\bar{X}=0. \tag{4.128}$$

Equation (4.128) is equivalent to Eq. (4.121).

This equation needs to be supplemented with suitable boundary conditions at end of inflation and/or at Planck boundaries. For a region which is near the end of inflation $\phi=\phi_*$, the probability of having eternal points is zero, so $X(\phi_*)=0$. Domains reaching Planck boundaries ϕ_{P} disappear from consideration, so we may assume that they will not contain eternal points and set $X=0$ at those boundaries. Physically meaningful solutions $X(\phi)$ have the meaning of probability and so should vary between 0 and 1. Therefore, the additional requirements are

$$0\le X(\phi)\le 1;\quad X(\phi_*)=0;\quad X(\phi_{\mathrm{P}})=0. \tag{4.129}$$

It is straightforward to verify that Eq. (4.128), unlike previous FP equations, is a gauge-invariant equation. A change of time variable, $t\to t'$, with $dt'/dt=T(t,\phi)$, will divide the functions $D(\phi)$ and $v(\phi)$ as well as the factor $3H(\phi)$ in the "growth" term by $T(t,\phi)$. This extra factor $T(t,\phi)$ cancels and thus Eq. (4.128) remains unchanged. This feature is to be expected since the probability $\bar{X}(\phi)$ is defined in a gauge-invariant way and should be described by a gauge-invariant equation.

An analogous nonlinear diffusion equation can be derived for models of inflation with multiple scalar fields ϕ_k, namely

$$D\frac{\partial}{\partial\phi^k}\frac{\partial\bar{X}}{\partial\phi_k}+v_k\frac{\partial\bar{X}}{\partial\phi_k}+3H\bar{X}\ln\bar{X}=0, \tag{4.130}$$

where

$$v_k(\phi)\equiv-\frac{1}{4\pi}\frac{\partial H}{\partial\phi^k} \tag{4.131}$$

are the new drift coefficients (in the proper time gauge). Boundary conditions will be $\bar{X}(\phi)=1$ along thermalization boundaries and Planck boundaries (if any).

The qualitative behavior of the probability $\bar{X}(\phi)$ for a chaotic type inflationary model is illustrated in Fig. 4.4. An approximate WKB solution in the fluctuation-dominated regime can be obtained if we disregard the drift term $v(\phi)\bar{X}'$ and write an ansatz

$$\bar{X}(\phi)=Ae^{-W(\phi)}, \tag{4.132}$$

where A is approximately constant and $W(\phi)$ is a slowly changing function, $|W''|\ll(W')^2$. The function $W(\phi)$ then satisfies approximately the equation

$$D\left(W'\right)^2-3HW=0. \tag{4.133}$$

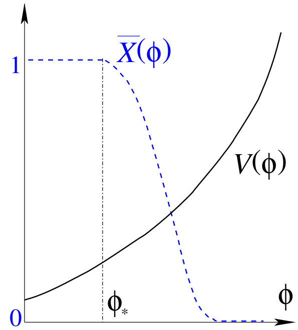

Figure 4.4: Probability $\bar{X}(\phi)$ of having no eternal points is very small in the inflating regime but almost equal to 1 near reheating, $\phi \approx \phi_*$.

The solution is[8]

$$W(\phi) = 6\pi^2 \left[\int_{\phi_q}^{\phi} \frac{d\phi}{H(\phi)} \right]^2, \tag{4.134}$$

where ϕ_q is the boundary of fluctuation-dominated range of ϕ. This calculation shows that the probability of having eternal points, $X(\phi) = 1 - \bar{X}(\phi)$, exponentially rapidly approaches 1 in the fluctuation-dominated regime.

Since $\bar{X} > 0$, there is always a certain (perhaps exceedingly small) probability of *not realizing* eternal inflation within a given initial horizon-sized region (H-region). This can happen if the field ϕ by rare chance fluctuates in the same direction everywhere and reaches reheating globally, everywhere to the future of the initial H-region. Also, an inflationary model could be arranged to entirely disallow eternal inflation, so that Eqs. (4.128)-(4.129) have no solutions other than the trivial solution $\bar{X} \equiv 1$. Note that Eqs. (4.128)-(4.129) alone do not provide enough information to choose between the trivial solution $\bar{X} \equiv 1$ and a nontrivial solution $\bar{X}(\phi)$. However, we have already shown that eternal points are *possible*, *i.e.* the probability of having eternal points is nonzero, if eternal inflation is allowed. Therefore, we must choose the trivial solution $\bar{X} \equiv 1$ only when no other solution of Eq. (4.128) that satisfies Eq. (4.129) can be found.

4.2.2 The existence of solutions of nonlinear equations

It is not easy to demonstrate directly the existence of nontrivial solutions of reaction-diffusion equations such as Eq. (4.128). However, there is a connection between solutions of such nonlinear equations and solutions of the linearized equations. Rigorous results are available in the mathematical literature on nonlinear functional analysis and bifurcation theory. In this and the following sections we will present the techniques that can be applied to Eq. (4.128).

For convenience, here we will use the e-folding time parametrization, which

[8] Note the factor $6\pi^2$, which corrects a misprint in [188].

is the same as rewriting Eq. (4.128) as

$$\frac{D}{H}\bar{X}_{,\phi\phi} + \frac{v}{H}\bar{X}_{,\phi} + 3\bar{X}\ln\bar{X} = 0. \tag{4.135}$$

We begin with a heuristic argument; a rigorous demonstration will follow. Let us consider the situation when eternal inflation is barely allowed. One will then expect that the probability of having eternal points is very small, *i.e.* there exists a nontrivial solution of Eq. (4.135) that approximately satisfies $\bar{X}(\phi) \approx 1$. The solution can be linearized in the neighborhood of $\bar{X} \approx 1$ as $\bar{X} = 1 - \chi(\phi)$. Then $\chi(\phi)$ satisfies the linear FP equation

$$\left[\hat{L} + 3\right]\chi = 0, \quad \hat{L} \equiv \frac{D}{H}\partial_{\phi\phi} + \frac{v}{H}\partial_{\phi}. \tag{4.136}$$

The FP operator $\hat{L} + 3$ is adjoint to the operator $\hat{L}^{\dagger}$ defined by

$$\left[\hat{L}^{\dagger} + 3\right] P \equiv \partial_{\phi\phi}\left(\frac{D}{H}P\right) - \partial_{\phi}(\frac{v}{H}P) + 3P, \tag{4.137}$$

which enters the FP equation for the 3-volume distribution $P(\phi, t)$ in the *e*-folding time parametrization. If eternal inflation is allowed in a given model, the operator $\hat{L}^{\dagger} + 3$ has a positive eigenvalue. The largest eigenvalue of that operator is zero in the borderline case when eternal inflation is just about to set in. The spectrum of the operator $\hat{L} + 3$ is the same as that of the adjoint operator $\hat{L}^{\dagger} + 3$. Hence, in the borderline case the largest eigenvalue of the operator $\hat{L} + 3$ will be zero, and there will exist a nontrivial, everywhere nonnegative solution χ of Eq. (4.136). Thus, heuristically one can expect that a nontrivial solution $\bar{X}(\phi) \not\equiv 1$ will exist away from the borderline case, *i.e.* when the operator $\hat{L} + 3$ has a positive eigenvalue.

Following the approach of [192], one can imagine a family of inflationary models parametrized by a label Ω, such that eternal inflation is allowed when $\Omega < 1$ while $\Omega = 1$ is a borderline case (called a "phase transition" point in [192]). Then Eq. (4.135) will have only the trivial solution, $\bar{X}(\phi) \equiv 1$, for $\Omega \geq 1$. The case $\Omega = 1$ where eternal inflation is on the borderline of existence is the *bifurcation point* for the solutions of Eq. (4.135). At the bifurcation point, a nontrivial solution $\bar{X}(\phi) \not\equiv 1$ appears, branching off from the trivial solution. A rigorous theory of bifurcation can be developed using methods of nonlinear functional analysis (see *e.g.* chapter 9 of the book [209]). In this chapter we will prove that a nontrivial solution of a nonlinear equation, such as Eq. (4.135), exists if and only if the dominant eigenvalue of the linearized operator $\hat{L} + 3$ with zero boundary conditions is positive.

First, we establish the necessary condition: The largest eigenvalue of $\hat{L} + 3$ with zero boundary conditions is positive if a nontrivial solution of Eq. (4.135) exists. The sufficient condition and the uniqueness of the solution of Eq. (4.135) requires a more involved analysis and will be proved in Sec. 4.2.3.

We assume that $\bar{X}(\phi)$ is a nontrivial solution of Eq. (4.135). The spectrum of $\hat{L}^{\dagger} + 3$ is bounded from above, and there exists an eigenfunction $\psi(\phi)$ of

$\hat{L}^{\dagger}\psi = \gamma\psi$ with zero boundary conditions $\psi(\phi_1) = 0$, $\psi(\phi_2) = 0$, where γ is the largest eigenvalue. We need to prove that $\gamma > 0$. It is known that the eigenfunction ψ is everywhere positive, $\psi(\phi) > 0$, as a ground state of a self-adjoint operator. Then the following integral must be positive,

$$I_1 \equiv \int_{\phi_1}^{\phi_2} \left(1 - \bar{X}\left(\phi\right)\right) \psi\left(\phi\right) d\phi > 0. \tag{4.138}$$

Here $\phi_{1,2}$ are left and right boundaries, $\phi_2 > \phi_1$. Since

$$\gamma = \frac{1}{I_1} \int_{\phi_1}^{\phi_2} \left(1 - \bar{X}\right) [\hat{L}^{\dagger} + 3] \psi \, d\phi, \tag{4.139}$$

we would obtain the desired inequality $\gamma > 0$ if we prove that

$$I_2 \equiv \int_{\phi_1}^{\phi_2} \left(1 - \bar{X}\right) [\hat{L}^{\dagger} + 3] \psi \, d\phi > 0. \tag{4.140}$$

Integrating Eq. (4.140) by parts and using boundary conditions $\bar{X}\left(\phi_{1,2}\right) = 1$ and $\psi(\phi_{1,2}) = 0$, we find

$$\begin{aligned} I_2 &= \int_{\phi_1}^{\phi_2} \psi \, [\hat{L} + 3] \left(1 - \bar{X}\right) d\phi + D\left(\phi\right) \psi \frac{d\bar{X}}{d\phi}\bigg|_{\phi_1}^{\phi_2} \\ &= \int_{\phi_1}^{\phi_2} \psi \, [\hat{L} + 3] \left(1 - \bar{X}\right) d\phi. \end{aligned} \tag{4.141}$$

Using Eq. (4.135), we obtain

$$\begin{aligned} & [\hat{L} + 3] \left(1 - \bar{X}\right) \\ & = -\frac{D}{H} \bar{X}_{,\phi\phi} - \frac{v}{H} \bar{X}_{,\phi} + 3\left(1 - \bar{X}\right) \\ & = 3\left(1 - \bar{X} + \bar{X} \ln \bar{X}\right) \geq 0. \end{aligned} \tag{4.142}$$

It follows that the integral $\int \psi \, [\hat{L} + 3] \left(1 - \bar{X}\right) d\phi$ is positive: its integrand is nonnegative and not everywhere zero. This proves Eq. (4.140), so γ must be positive. This concludes the proof of the necessary condition.

There remains a technical difference between the eigenvalue problem for the operator $\hat{L} + 3$ with *zero* boundary conditions and the same eigenvalue problem with the "no-diffusion" boundary conditions normally used in the stochastic approach,

$$\frac{\partial}{\partial\phi}\bigg|_{\phi_*} [D(\phi) P(\phi)] = 0. \tag{4.143}$$

It was also demonstrated in [188] that the eigenvalue of $\hat{L} + 3$ with the boundary conditions (4.143) is positive if a nontrivial solution of Eq. (4.121) exists. In principle, the eigenvalue of $\hat{L} + 3$ with zero boundary conditions is not the same as the eigenvalue of the same operator with the boundary conditions (4.143). One can have a borderline case when one of these two eigenvalues is positive while the other is negative. In this case, the two criteria for the presence of eternal inflation (based on the positivity of the two different eigenvalues) will disagree. However, the alternative boundary conditions are imposed at reheating, *i.e.* in the regime of very small fluctuations where the value of the eigenfunction $P(\phi)$ is exponentially small compared with its values in the fluctuation-dominated range of ϕ. Hence, the difference between the two eigenvalues is always exponentially small (it is suppressed at least by the factor e^{-3N}, where N is the number of e-foldings in the deterministic slow-roll regime before reheating). Therefore, we may interpret the discrepancy as a limitation inherent in the stochastic approach to inflation. In other words, one cannot use the stochastic approach to establish the presence of eternal inflation more precisely than with the accuracy e^{-3N}. Barring an extremely fine-tuned borderline case, this accuracy is perfectly adequate for establishing the presence or absence of eternal inflation.

4.2.3 Proof of the sufficient condition

Finding a sufficient condition for the existence of positive solutions of Eq. (4.121) requires a more detailed analysis. We will treat only the one-dimensional case, *i.e.* the case of a single scalar field ϕ, but the argument can be generalized to the case of multi-dimensional field ϕ without significant changes. We will show that a positive solution of Eq. (4.121) exists and is unique if the largest eigenvalue of the operator $\hat{L}^\dagger + 3$ (with zero boundary conditions at $\phi = \phi_*^{(1)}$ and $\phi = \phi_*^{(2)}$) is positive. This condition is equivalent to the condition that the smallest eigenvalue γ of the operator $-\hat{L}$ (with zero boundary conditions) satisfies $\gamma < 3$. In this section it will be convenient to work with the operator $-\hat{L}$ and its smallest eigenvalue $\gamma > 0$.

We begin by considering Eq. (4.121), which is of the form

$$\hat{L}X + K(X) = 0, \tag{4.144}$$

where $K(X)$ is a nonlinear function defined by

$$K(x) \equiv -3\,(1-x)\ln(1-x)\,. \tag{4.145}$$

The function $K(x)$ has an unbounded derivative as $x \to 1$. This will turn out to be a technical problem for the further analysis; this problem is circumvented as follows. We will will only look for solutions of Eq. (4.144) with zero boundary conditions that do not reach $X(\phi) = 1$ at any intermediate value of ϕ. For those solutions, $K'(X)$ is bounded for all relevant values of $X(\phi)$. This property can be equivalently expressed by saying that there exists a constant $A > 0$ such that $|K'(X(\phi))| < A$ for all ϕ. Then we use this value A to rewrite Eq. (4.144) equivalently as

$$[-\hat{L} + A]X = K(X) + AX \equiv h(X). \tag{4.146}$$

Note that the operator $-\hat{L}$ has all positive eigenvalues and $K'(x) \le 3$ for $0 \le x \le 1$. Therefore, the constituents of Eq. (4.146) have the following properties:

The operator $-\hat{L}+A$ has all positive eigenvalues, while the function $h(X)$ satisfies $h(0)=0$, $h(X)\geq 0$, and $0<h'(X)<3+A$ for all relevant X. Thus, $h(X)$ monotonically grows, and also $h(X)/X$ decreases monotonically for all relevant X, while

$$\sup_{X\in(0,1)} \frac{h(X)}{X} = 3+A. \tag{4.147}$$

Under these numerous conditions, we can demonstrate the existence and uniqueness of positive solutions using techniques of nonlinear functional analysis (see chapter 9 of the book [209]).

We first rewrite Eq. (4.146) as

$$X = [-\hat{L}+A]^{-1}h(X) \equiv \hat{T}[X], \tag{4.148}$$

where $[-\hat{L}+A]^{-1}$ is the Green's function of the operator $-\hat{L}+A$ with zero boundary conditions at $\phi=\phi_*^{(1,2)}$. We have introduced the nonlinear operator $\hat{T}$ in Eq. (4.148) in order to exhibit the iteration structure, $X=\hat{T}[X]$, of this equation. We will now demonstrate the existence and uniqueness of positive solutions to this equation.

The idea of the construction is to find an initial function $X_0>0$ such that the consecutive iterations of Eq. (4.148) yield an *increasing* sequence of functions, $X_{n+1}(\phi)=\hat{T}[X_n]$, such that $X_{n+1}\geq X_n$. To compare functions, let us write for brevity $f\geq g$ if $f(\phi)\geq g(\phi)$ for all $\phi\in(\phi_*^{(1)},\phi_*^{(2)})$. We will also show that the sequence X_n is bounded from above and hence has a limit, $X_\infty(\phi)$, which is a solution of Eq. (4.148). Then we will prove that this solution is unique.

First, we note that the operator $\hat{T}$ is "monotonically increasing" in the sense that $\hat{T}[f]\geq\hat{T}[g]$ if $f\geq g$. This property can be verified by considering

$$\hat{T}[f]-\hat{T}[g] = [-\hat{L}+A]^{-1}(h(f)-h(g)). \tag{4.149}$$

Since $h(x)$ monotonically grows (by construction) for all $x\in[0,1]$, we have $h(f)-h(g)>0$ if $f>g$. On the other hand, the Green's function $G_L(\phi,\phi')$ of the positive operator $-\hat{L}+A$ is positive, $G_L(\phi,\phi')>0$, so that

$$[-\hat{L}+A]^{-1}f(\phi) = \int d\phi' G_L(\phi,\phi')f(\phi') > 0 \tag{4.150}$$

if $f>0$. To demonstrate that $G_L>0$, we first show that G_L cannot be equal to zero except at the boundaries. Suppose that $G_L(\phi,\phi')=0$ for some $\phi=\phi_1$ and $\phi'=\phi_1'$ within the open interval $(\phi_*^{(1)},\phi_*^{(2)})$. Then $G_L(\phi,\phi_1')$ for fixed $\phi'=\phi_1'$ can be thought of as a solution of the equation

$$[-\hat{L}+A]\, G_L(\phi,\phi_1') = \delta(\phi-\phi') \tag{4.151}$$

with boundary conditions $G_L(\phi_*^{(1,2)},\phi_1')=0$ and $G_L(\phi_1,\phi_1')=0$. One of the intervals $I_1\equiv(\phi_*^{(1)},\phi_1)$ or $I_2\equiv(\phi_1,\phi_*^{(2)})$ does not contain ϕ_1'. Let us, without loss of generality, assume that it is I_1 that does not contain ϕ_1'. Then $G_L(\phi, \phi_1')$ satisfies a homogeneous equation,

$$[-\hat{L} + A]\, G_L(\phi, \phi_1') = 0, \tag{4.152}$$

with zero boundary conditions at the boundaries of the interval I_1. Any such solution would be an eigenfunction of $-\hat{L} + A$ with zero boundary conditions at the boundaries of I_1. However, the operator $-\hat{L} + A$ has all positive eigenvalues with zero boundary conditions on the interval $(\phi_*^{(1)}, \phi_*^{(2)})$ while its eigenvalues with zero boundary conditions on a smaller interval are strictly larger. Hence, $-\hat{L} + A$ has no zero eigenfunctions, so $G_L(\phi, \phi_1') = 0$ for all $\phi \in I_1$. However, $G_L(\phi, \phi_1')$ is a solution of second-order differential equation and so cannot be identically equal to zero throughout an *interval* of values of ϕ. Hence, G_L has the same sign and is nonzero everywhere. It remains to show that G_L is everywhere positive rather than everywhere negative. The positivity of G_L follows if we consider the eigenfunction $f_0(\phi)$ of $-\hat{L} + A$ with the smallest eigenvalue, which is equal to $\gamma + A$,

$$[-\hat{L} + A]\, f_0(\phi) = (\gamma + A)\, f_0(\phi). \tag{4.153}$$

Such an eigenfunction $f_0(\phi)$ can be chosen everywhere positive and is nondegenerate. Now we need to choose a different scalar product in the space of functions, in such a way that the operator $\hat{L}$ is self-adjoint,

$$\langle f_1, f_2 \rangle \equiv \int_{\phi_*^{(1)}}^{\phi_*^{(2)}} d\phi\, f_1(\phi) f_2(\phi) \frac{1}{M(\phi)}, \tag{4.154}$$

where $M(\phi)$ is defined by Eq. (4.61). Taking this scalar product of f_0 with $G_L(\phi, \phi')$ and using Eq. (4.151), we find

$$\begin{aligned}
\langle (\gamma + A)\, f_0, G_L \rangle &= \langle (-\hat{L} + A)\, f_0, G_L \rangle \\
&= \langle f_0, (-\hat{L} + A)\, G_L \rangle \\
&= \int_{\phi_*^{(1)}}^{\phi_*^{(2)}} f_0(\phi) \delta(\phi - \phi') d\phi = f_0(\phi') > 0.
\end{aligned} \tag{4.155}$$

Since the integral of G_L with a positive function $(\gamma + A) f_0$ is positive, it follows that $G_L(\phi, \phi') > 0$.

Second, we need to find an initial function $X_0(\phi)$ such that $\hat{T}[X_0] \geq X_0$. This function can be chosen as $\beta f_0(\phi)$, where $\beta > 0$ is a sufficiently small constant and $f_0(\phi)$ is the positive eigenfunction satisfying Eq. (4.153). Then we have

$$h(\beta f_0) = K(\beta f_0) + A\beta f_0 = \left[K'(0) + A\right] \beta f_0 + O(\beta^2). \tag{4.156}$$

We may choose β sufficiently small so that $\beta f_0 < 1$ and that the terms $O(\beta^2)$ in the above equation are negligible; then we obtain

$$\hat{T}[\beta f_0] = [-\hat{L} + A]^{-1} h(\beta f_0) \approx \frac{K'(0) + A}{\gamma + A} \beta f_0 > \beta f_0, \tag{4.157}$$

where we used the relationships

$$K'(0) = 3 = \sup_{x \in (0,1)} \frac{K(x)}{x} \equiv \gamma_* > \gamma. \tag{4.158}$$

Note that the coincidence of the values $K'(0)$ and γ_* is related to the fact that $K(x)/x$ is monotonically decreasing in x starting from $x = 0$.

Having found $X_0 \equiv \beta f_0$ such that $\hat{T}[X_0] > X_0$ and using the increasing property of $\hat{T}$, we find that the sequence $X_{n+1} \equiv \hat{T}[X_n]$ is increasing, $X_{n+1} > X_n$. We need to prove that the sequence X_n is bounded from above. Consider the constant function $X(\phi) = 1$; it is clear that $[-\hat{L} + A]c = Ac$ for any constant c, and hence $[-\hat{L} + A]^{-1}c = A^{-1}c$. Therefore

$$\hat{T}[1] = [-\hat{L} + A]^{-1} h(1) = 1. \tag{4.159}$$

Since $\hat{T}$ is an increasing operator, and since $X_0 < 1$, it follows by induction that all $X_n < 1$. Hence, the sequence X_n has a limit, $X_\infty(\phi)$, since that sequence is increasing and bounded from above. The function $X_\infty(\phi)$ is a solution of $X_\infty = \hat{T}[X_\infty]$ and hence is a positive solution of Eq. (4.144) that never reaches 1.

Uniqueness of this positive solution can be demonstrated as follows. First we note that the solution $X_\infty(\phi)$ is the *minimal* positive solution; for suppose that $f(\phi)$ is any solution, then $f(\phi)$ is positive since

$$f = [-\hat{L} + A]^{-1} h(f) \tag{4.160}$$

while $h(f) > 0$ and the Green's function is also positive. Then we can show that $f > \beta f_0$ for sufficiently small β. A positive solution $f(\phi)$ cannot have zero derivatives and at the same time zero values at the boundaries. Hence, the derivatives of f are nonzero at the boundaries. Then there exists a sufficiently small β such that $f > \beta f_0 \equiv X_0$. It follows from $f > X_0$ and from the increasing property of $\hat{T}$ that $\hat{T}[f] > \hat{T}[X_0] \equiv X_1$, and so by induction $f > X_n$ for any n. Hence $f \geq X_\infty$; in other words, X_∞ is the minimal positive solution.

It remains to show that there are no other positive solutions larger than $X_\infty(\phi)$. Suppose f_1 and f_2 are two different positive solutions such that $f_1 > f_2$. Since $\hat{L}$ is self-adjoint with respect to the scalar product (4.154), we can perform the following computation,

$$\begin{aligned} 0 &= \langle \hat{L} f_1, f_2 \rangle - \langle f_1, \hat{L} f_2 \rangle = - \langle K(f_1), f_2 \rangle + \langle f_1, K(f_2) \rangle \\ &= \int_{\phi_*^{(1)}}^{\phi_*^{(2)}} \frac{d\phi}{M(\phi)} f_1 f_2 \left(\frac{K(f_2)}{f_2} - \frac{K(f_1)}{f_1} \right). \end{aligned} \tag{4.161}$$

This is a contradiction since $f_2 > f_1$ while the function $K(x)/x$ is strictly monotonically decreasing for all relevant $x > 0$, so the last integrand is everywhere positive. Therefore, we cannot have another solution f such that $f > X_\infty$, so the positive solution X_∞ is unique.

The conclusion is that Eq. (4.144) has unique positive solutions not reaching $X = 1$ as long as the largest eigenvalue of the operator $\hat{L}^\dagger + 3$ with zero boundary conditions is positive. This eigenvalue is exponentially close to the lowest eigenvalue of the operator $\hat{L}^{\dagger} + 3$ with boundary conditions (4.143). The proof depends on certain technical properties of the function $K(X)$ defined by Eq. (4.145), namely

$$K(0) = K(1) = 0, \quad K'(0) = 3 = \sup_{x \in (0,1)} \frac{K(x)}{x}, \tag{4.162}$$

on the monotonic behavior of $K(x)/x$, and on the fact that solutions do not reach $X(\phi) = 1$ and so there exists a constant A such that $h(x) \equiv K(x) + Ax$ is a monotonically increasing function for all relevant values of x.

5 Models with bubble nucleation

Until now, we considered eternal self-reproduction due to random walk of a scalar field. Another important class of models includes self-reproduction due to bubble nucleation. Such scenarios of eternal inflation were studied in [41, 87, 42, 44, 210, 211]. Both processes (tunneling and random walk) may be combined in a single scenario called "recycling universe" [83], but we will consider them separately for clarity.

In a locally de Sitter universe dominated by dark energy, nucleation of bubbles of false vacuum may occur due to tunneling [212, 85, 65, 213]. The bubble nucleation rate Γ per unit 4-volume is very small [85, 214],

$$\Gamma = O(1) H^4 \exp\left[-S_I - \frac{\pi}{H^2}\right], \tag{5.1}$$

where S_I is the instanton action and H is the Hubble constant of the parent de Sitter background. Hence, bubbles will generically not merge into a single false-vacuum domain [87], and infinitely many bubbles will be nucleated at different places and times. The resulting "daughter" bubbles may again contain an asymptotically de Sitter, infinite universe, which again gives rise to infinitely many "grand-daughter" bubbles. This picture of eternal self-reproduction was called the "recycling universe" [83]. Some (or all) of the created bubbles may support a period of additional inflation followed by reheating, as illustrated in Fig. 3.2. We note that the presence of eternal self-reproduction in models of tunneling type is generically certain (unless the nucleation rate for terminal vacua is unusually high) because every bubble continues expanding for a long time before the next nucleation can occur.

In the model of [83], there were only two vacua that could tunnel into each other. A more recently developed paradigm of **string theory landscape** [111, 84, 112, 215] is a theory that involves a very large number of metastable vacua, corresponding to local minima of an effective potential in field space. In the "landscape" models, transitions between vacuum states are possible through bubble nucleation; the interior of a bubble appears as an infinite homogeneous open universe [85] if one disregards the small probability of bubble collisions.[1] The value of the potential at each minimum is the effective value of the cosmological constant Λ in the corresponding vacuum. Therefore, there exist many different types of bubbles, each having some value of Λ. The presently observed universe is situated within some bubble (called a **pocket universe**) having a certain value of Λ. Figure 5.1 shows a phenomenologist's view of the "landscape."

[1] See *e.g.* [88, 216, 217] for an analysis of bubble collisions.

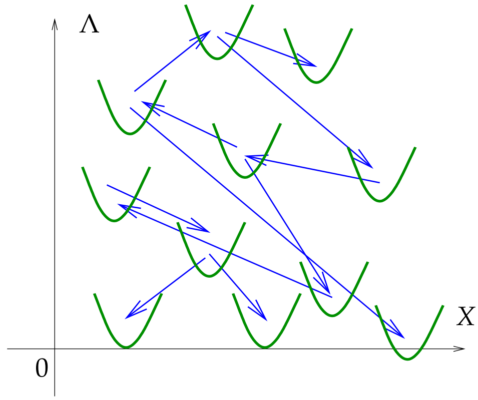

Figure 5.1: A schematic representation of the "landscape of string theory," consisting of a large number of local minima of an effective potential. The variable X collectively denotes various fields and Λ is the effective cosmological constant. Arrows show possible tunneling transitions between vacua.

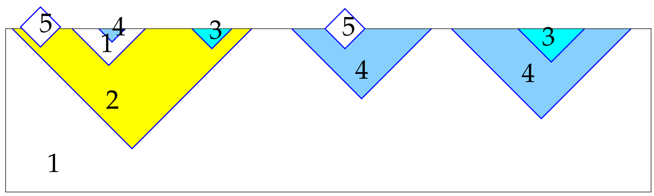

Figure 5.2: A conformal diagram of the spacetime where self-reproduction occurs via bubble nucleation. Regions labeled "5" are asymptotically flat ($\Lambda = 0$).

Vacua with $\Lambda \leq 0$ do not allow any further tunneling[2] and are called **terminal** in [222], while vacua with $\Lambda > 0$ are called **recyclable** or **transient** since they can tunnel to other vacua with $\Lambda > 0$ or $\Lambda \leq 0$. Recyclable bubbles will give rise to infinitely many nested "daughter" bubbles. Conformal diagrams of the resulting spacetime are shown in Figs. 5.2 and 5.3. Of course, only a finite number of bubbles can be drawn on a diagram; as we have mentioned, the diagram actually contains a fractal structure [204].

After nucleation, bubble walls move with the speed of light and rapidly approach horizon size (in comoving coordinates). While the geometry of bubbles may be complicated, especially considering the possible bubble collisions, a sufficiently precise picture can be obtained by simply regarding each bubble as an instantaneously nucleated horizon-size region, *i.e.* a sphere of radius H^{-1} where H is the Hubble rate in the parent spacetime. This approximation, implicitly

[2] Asymptotically flat $\Lambda = 0$ vacua cannot support tunneling [218, 219, 220]; vacua with $\Lambda < 0$ will quickly collapse to a "big crunch" singularity [85, 221].

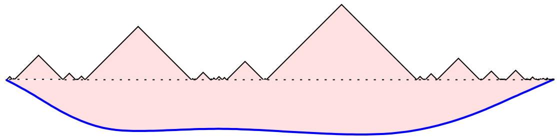

Figure 5.3: A conformal diagram of the future part of a landscape-driven spacetime (figure from [204]).

used in many calculations, was called the **square bubble** approximation and shown to be adequate in [223]. In this approximation, a statistical description of the "recycling" spacetime can be developed [83, 222] by considering a single comoving worldline $\mathbf{x} =$ const that passes through different bubbles at different times. (It is implied that the worldline is randomly chosen from an ensemble of infinitely many such worldlines passing through different points $\mathbf{x}$.) Let the index $\alpha = 1, ..., N$ label all the available types of bubbles. For calculations, it is convenient to use the e-folding time $\tau \equiv \ln a$. We are interested in the probability $f_\alpha(\tau)$ of passing through a bubble of type α at time τ. This probability distribution is normalized by $\sum_\alpha f_\alpha = 1$; the quantity $f_\alpha(\tau)$ can be also visualized as the fraction of the comoving volume occupied by bubbles of type α at time τ. Denoting by $\Gamma_{\beta\to\alpha}$ the nucleation rate for bubbles of type α within bubbles of type β, computed according to Eq. (5.1), and by $\kappa_{\beta\to\alpha}$ the corresponding rate per unit time, $\kappa_{\beta\to\alpha} \equiv H_\beta^{-3}\Gamma_{\beta\to\alpha}$, we write the "master equation" describing the evolution of $f_\alpha(\tau)$,

$$\frac{df_\beta}{d\tau} = \sum_\alpha \left(-\kappa_{\beta\to\alpha} f_\beta + \kappa_{\alpha\to\beta} f_\alpha\right) \equiv \sum_\alpha M_{\beta\alpha} f_\alpha, \tag{5.2}$$

where we introduced the auxiliary matrix $M_{\beta\alpha}$. Given a set of initial conditions $f_\alpha(0)$, one can evolve $f_\alpha(\tau)$ according to Eq. (5.2).

To proceed further, one may now distinguish the following two cases: Either terminal vacua exist (some β such that $\kappa_{\beta\to\alpha} = 0$ for all α), or all the vacua are recyclable. (Theory suggests that it is far more probable that terminal vacua exist[112].) If terminal vacua exist then a typical comoving worldline will eventually enter a bubble of terminal type, at which point no further transitions to any other type of bubble will be possible. For landscapes with terminal bubbles, the late-time asymptotic solution can be written as [222]

$$f_\alpha(\tau) \approx f_\alpha^{(0)} + s_\alpha e^{-q\tau}, \tag{5.3}$$

where $f_\alpha^{(0)}$ is a constant vector that depends on the initial conditions and has nonzero components only in terminal vacua, and s_α does not depend on initial conditions and is an eigenvector of $M_{\alpha\beta}$ such that $\sum_\alpha M_{\beta\alpha} s_\alpha = -q s_\beta$ with $q > 0$. It was shown in [222] that the eigenvector s_α is nondegenerate and that $q > 0$ under some natural technical constraints on the landscape. The solution (5.3) shows that all *comoving* volume reaches terminal vacua exponentially quickly.

As in the case of random-walk inflation, there are infinitely many "eternally recycling" points $\mathbf{x}$ that never enter any terminal vacua, and these points form a set of measure zero. Nevertheless, the physical volume of bubbles in nonterminal vacua as well as the total number of bubbles are growing exponentially with time.

If there are no terminal vacua, a typical comoving worldline will pass from one type of bubble to another for ever. In that case it is easy to show that the solution $f_\alpha(\tau)$ approaches a stationary distribution [224], as is to be expected for a Markov process,

$$\lim_{\tau\to\infty} f_\alpha(\tau) \approx f_\alpha^{(0)}, \quad \sum_\beta M_{\beta\alpha} f_\alpha^{(0)} = 0, \tag{5.4}$$

$$f_\alpha^{(0)} = \text{const} \cdot H_\alpha^4 \exp\left[\frac{\pi}{H_\alpha^2}\right]. \tag{5.5}$$

In this case, the quantities $f_\alpha^{(0)}$ are independent of initial conditions and are interpreted (with a suitable normalization factor) as the fractions of time spent by the randomly chosen comoving worldline in bubbles of type α.

We note that the description of spacetime in terms of the distribution $f_\alpha(\tau)$ satisfying Eq. (5.2) depends on the choice of the time variable τ in an essential way. A different choice (*e.g.* the proper time t) would lead to an inequivalent "master equation" and qualitatively different distributions $f_\alpha(t)$ [75].[3]

One may adopt another approach and ignore the duration of time spent by the worldline within each bubble. The duration of time spent in a bubble is perhaps irrelevant for the probability of being in the bubble, since civilizations can appear only for a (cosmologically) short time after reheating (see Fig. 3.2). Thus, one may restrict attention to merely the *sequence* of the bubbles encountered along a randomly chosen worldline [224, 107, 50]. If the worldline is initially in a bubble of type α, then the probability $\mu_{\beta\alpha}$ of entering the bubble of type β as the next bubble in the sequence after α is

$$\mu_{\beta\alpha} = \frac{\kappa_{\alpha\to\beta}}{\sum_\gamma \kappa_{\gamma\alpha}}. \tag{5.6}$$

(For terminal vacua α, we have $\kappa_{\alpha\to\gamma} = 0$ and so we may define $\mu_{\beta\alpha} = 0$ for convenience.) Once again we consider landscapes without terminal vacua separately from terminal landscapes. If there are no terminal vacua, then the matrix $\mu_{\alpha\beta}$ is normalized, $\sum_\beta \mu_{\beta\alpha} = 1$, and is thus a stochastic matrix [225] describing a Markov process of choosing the next visited vacuum. The sequence of visited vacua is infinite, so one can define the mean frequency $f_\alpha^{(\mathrm{mean})}$ of visiting bubbles of type α. If the probability distribution for the first element in the sequence is $f_{(0)\alpha}$, then the distribution of vacua after k steps is given (in the matrix

[3] In the case of a landscape without terminal vacua, different choices of time lead to a simple transformation of the probabilities f_α because these are the *fractions of time* spent by the worldline in the bubbles of type α.

notation) by the vector

$$\mathbf{f}_{(k)} = \boldsymbol{\mu}^k \mathbf{f}_{(0)}, \tag{5.7}$$

where $\boldsymbol{\mu}^k$ means the k-th power of the matrix $\boldsymbol{\mu} \equiv \mu_{\alpha\beta}$. Therefore, the mean frequency of visiting a vacuum α is computed as an average of $f_{(k)\alpha}$ over n consecutive steps in the limit of large n:

$$\mathbf{f}^{(\mathrm{mean})} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{f}_{(k)} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \boldsymbol{\mu}^k \mathbf{f}_{(0)}. \tag{5.8}$$

It is proved in the theory of Markov processes that the limit $f_\alpha^{(\mathrm{mean})}$ given by Eq. (5.8) almost surely coincides with the mean frequency of visiting the state α (see *e.g.* [226], chapter 5, Theorem 2.1, and [227], Theorem 3.5.9). It turns out that the distribution $\mathbf{f}^{(\mathrm{mean})}$ is independent of the initial state $\mathbf{f}_{(0)}$ and coincides with the distribution (5.4) found in the continuous-time description [224].

If there exist terminal vacua, then almost all sequences will have a finite length. The distribution of vacua in a randomly chosen sequence is still well-defined and can be computed using Eq. (5.8) without the normalizing factor $\frac{1}{n}$,

$$\mathbf{f}^{(\mathrm{mean})} = \left(\mathbf{1} - \boldsymbol{\mu}\right)^{-1} \boldsymbol{\mu} \mathbf{f}_{(0)}, \tag{5.9}$$

but now the resulting distribution depends on the initial state $\mathbf{f}_{(0)}$ [107, 50].

The formalism of the "master equation" plays the same role in the description of the spacetime in landscape models as the formalism of FP equations in inflationary models of random-walk type.[4] In the following chapters we will use these formalisms as we investigate the problem of predictions in eternal inflation.

[4]We note that the "master equation" describes Markov processes. If bubble nucleation were non-Markovian, *e.g.* dependent on bubble age [228], a different formalism needs to be used [229].

6 The measure problem and proposed solutions

6.1 Observer-based measures

A compelling question in the context of eternal inflation is how to make statistical predictions of observed parameters χ_a in theories where the values of these parameters are stochastically distributed in the spacetime. As discussed in Sec. 3.1, considerations of this type necessarily involve some form of the "principle of mediocrity." The approach taken so far is to attempt to predict the probability distribution $P(\chi_a)$ of measuring χ_a for a "randomly chosen" observer, and subsequently compare the observed actual value of χ_a with the distribution $P(\chi_a)$.

In inflationary cosmology, observers appear only near the reheating surface in spacetime. If eternal inflation is present, the reheating surface contains infinitely many causally disconnected and (possibly) statistically inequivalent domains. Thus, in the context of eternal inflation there is no naturally defined mathematical measure on the set of all observers since it is a countably infinite set. Hence, one needs an *ad hoc* prescription in order to *define* the probability distribution. We call such a prescription an **observer-based measure**, as is now common in the cosmological literature. On a more formal level, one needs to consider the ensemble of the "possible observers" and to define a probability distribution $P(\chi_a)$ for quantities χ_a that have different values on different observers from this ensemble.

Existing proposals for an observer-based measure fall in two major classes, which may be designated as **volume-based** vs. **worldline-based**.[1] The difference between these classes of measures is in the approach taken to construct the ensemble of observers. In the volume-based approach [68, 5, 94, 231, 222, 232], the ensemble contains every observer appearing in the universe, at any time or place. In the worldline-based approach [233, 107, 66, 234], the ensemble consists of observers appearing near a single, randomly selected comoving worldline $\mathbf{x} = \text{const}$; more generally, an arbitrary timelike geodesic could also be used. If the ensemble contains infinitely many observers (this is typically the case for volume-based ensembles and sometimes also the case for worldline-based ensembles), a regularization procedure is needed to obtain probability distributions. Finding and applying suitable regularization procedures is a separate technical issue explored in Secs. 6.2 and 6.3. I begin with a general discussion of

[1]To avoid confusion, let us note that the recent work [230] proposes a measure in the space of trajectories rather than an observer-based measure in the sense discussed here.

these measure prescriptions; particular prescriptions will be described in those sections.

There exist several measure proposals of volume-based type. These proposals differ by the manner in which the infinite ensemble of observers is cut off. A number of these proposals were found to be lacking in one aspect or another [68, 5, 179, 235, 236, 233]. One of the problems was the dependence on the choice of the time gauge [166, 68]. A prescription manifestly free from gauge dependence is the "spherical cutoff" measure [94]. This prescription provides unambiguous predictions for models of random-walk type eternal inflation if the reheating condition $\phi = \phi_*$ corresponds to a topologically compact and connected locus in the field space $\{\phi, \chi_a\}$ (see Sec. 6.2). For models where the reheating condition is met at several disconnected loci in field space (tunneling-type eternal inflation belongs to this class), one can use the recently proposed prescription of "comoving cutoff" [222, 232]. Recently, the equal-time cutoff with e-folding time (the "scale factor" cutoff) has received renewed attention [115, 237, 238].

Existing measure proposals of the "worldline" type appear to converge essentially to a single prescription [107, 50] (however, see [239, 240] for the most recent developments). This prescription in most cases does not require a cutoff since the number of observers along a single comoving worldline is almost surely finite. Therefore, we will refer to "the" worldline-based measure rather than distinguish between different versions.

A principal difference between the worldline-based and volume-based measures is in their dependence on the initial state. When considering the volume-based measure, one starts from a finite initial spacelike 3-volume and restricts attention to the comoving future of that initial region. (Final results are insensitive to the choice of this 3-surface in spacetime or to its geometry.) The initial state consists of the initial values of the fields χ_a within the initial volume, and possibly a label α corresponding to the type of the initial bubble.[2] When considering the worldline-based measure, one also assumes knowledge of these data at the initial point of a worldline. It turns out that the worldline-based measure always yields results that depend sharply on the initial state (except for the case of a "nonterminal" landscape, *i.e.* a landscape scenario without terminal vacua). A theory of initial conditions is necessary to obtain a specific prediction from the worldline-based measure. In contrast, probabilities obtained using any of the volume-based measures are always independent of the initial conditions. This agrees with the concept of the "stationarity" of a self-reproducing universe as discussed in the earlier works [166, 68, 3, 5] where it was proposed that the universe as a whole forgets the initial state and continues forever producing reheated regions and inflating in a self-similar way.

At this time, there is no consensus as to which of the two classes of measures is the physically relevant one. I regard the two measures as reasonable answers to *different* questions. The first question is to determine the probabil-

[2]Naturally, it is assumed that the initial state is in the self-reproduction regime: For random-walk type models, the initial 3-volume is undergoing inflation rather than reheating and, for tunneling models, the initial 3-volume is not situated within a terminal bubble.

ity distribution for observed values of χ_a, given that the observer is randomly chosen from all the observers present in the entire spacetime. Since we have no knowledge as to our position in the spacetime, the total duration of inflation in our past, or the total number of bubble nucleations preceding the most recent one, it appears reasonable to include in the ensemble all the observers that will ever appear anywhere in the spacetime. An answer to the first question is thus provided by a volume-based measure.

The second question is posed in a rather different manner. In the context of tunneling-type eternal inflation, upon discovering the type of our bubble we may wish to leave a message to a future civilization that may arise in our place after an unspecified number of consecutive bubble nucleations. The analogous situation in the context of random-walk type inflation is a hypothetical observer located within an inflating region of spacetime who wishes to communicate with future civilizations that will eventually appear when inflation is finished in that region. The only available means of communication is by leaving a message in a sealed box. The message might contain the probability distribution $P(\chi_a)$ for parameters χ_a that we expect the future civilization to observe. In this case, the initial state is *known* at the time of writing the message. It is clear that the message can be discovered only by future observers near the worldline of the box. It is then natural to consider the ensemble of observers appearing along that worldline; this ensemble certainly depends on the initial state, *e.g.* on the nucleation rates of various bubbles in the initial bubble. In that case, the correct message to be left in the box is $P(\chi_a)$ computed according to the worldline-based measure with the known initial state. It is natural in that case that $P(\chi_a)$ depends on the initial state.

Calculations using the worldline-based measure usually do not require regularization (except for the case of nonterminal landscapes) because the worldline-based ensemble of observers is almost surely finite [107]. For instance, in random-walk type models the worldline-based measure predicts the exit probability distribution p_{exit}, which can be computed by solving a suitable differential equation [see Eq. (4.45)]. In contrast, the ensemble used in the volume-based measure is infinite and always requires a regularization. For this reason, in the following exposition we will concentrate on volume-based measure proposals. The main regularization methods are reviewed in Sections 6.2 and 6.3.

A simple toy model [5] where the predictions of the various measures can be obtained analytically is a slow-roll scenario with a potential shown in Fig. 6.1. The potential is flat in the range $\phi_1 < \phi < \phi_2$ where the evolution is diffusion-dominated, while the evolution of regions with $\phi > \phi_2$ or $\phi < \phi_1$ is assumed to be completely deterministic (fluctuation-free). It is also assumed that the diffusion-dominated range $\phi_1 < \phi < \phi_2$ is sufficiently wide to support eternal self-reproduction of inflating regions. There are two thermalization points, $\phi = \phi_*^{(1)}$ and $\phi = \phi_*^{(2)}$, which may be associated to different types of true vacuum and thus to different observed values of cosmological parameters. If we are interested in a volume-based measure, the question is to compare the volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ of regions thermalized into these two vacua. Since there is an infinite

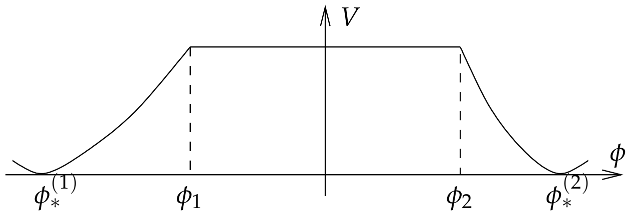

Figure 6.1: Illustrative inflationary potential with a flat self-reproduction regime $\phi_1 < \phi < \phi_2$ and deterministic regimes $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$.

volume thermalized into either vacuum, one looks for the volume ratio $\mathcal{V}_1/\mathcal{V}_2$.

The potential is symmetric in the range $\phi_1 < \phi < \phi_2$, so it is natural to assume that Hubble-size regions exiting the self-reproduction regime at $\phi = \phi_1$ and at $\phi = \phi_2$ are equally abundant.[3] Since the evolution within the ranges $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$ is deterministic, the regions exiting the self-reproduction regime at $\phi = \phi_1$ or $\phi = \phi_2$ will be expanded by fixed numbers of e-foldings, which we may denote N_1 and N_2 respectively,

$$N_j = -4\pi \int_{\phi_j}^{\phi_*^{(j)}} \frac{H(\phi)}{H'(\phi)} d\phi. \tag{6.1}$$

Therefore the volume of regions thermalized at $\phi = \phi_*^{(j)}$, where $j = 1, 2$, will be increased by the factors $\exp(3N_j)$. Hence, the volume ratio is

$$\frac{\mathcal{V}_1}{\mathcal{V}_2} = \exp\left(3N_1 - 3N_2\right). \tag{6.2}$$

This answer is found in several volume-based measures, except the scale-factor cutoff where one finds $V_1/V_2 = 1$ and the RV cutoff where the answer is more complicated (see Sec. 8.2.5). It is important to remark that in every volume-based proposal, the answer is independent of the initial state at the beginning of inflation.

The worldline-based measure gives a very different probability distribution in this toy model. One considers a single, randomly chosen comoving worldline and computes the probability of finishing at $\phi_*^{(1)}$ or at $\phi_*^{(2)}$ if the initial value of ϕ was $\phi = \phi_0$. It is clear that the result depends sensitively on the value of ϕ_0. For instance, if ϕ_0 is exactly in the middle of the self-reproduction interval $[\phi_1, \phi_2]$, *i.e.* if $\phi_0 = \frac{1}{2}(\phi_1 + \phi_2)$, the probability of first reaching ϕ_1 or ϕ_2 during random walk are exactly equal, hence the probabilities of reaching $\phi_*^{(1)}$ and $\phi_*^{(2)}$ are equal. If the initial point is moved away from $\phi_0 = \frac{1}{2}(\phi_1 + \phi_2)$, say towards

[3] This assumption, of course, needs to be verified in a given measure prescription.

$\phi = \phi_1$, the probability of first reaching ϕ_1 will be exponentially larger than that of first reaching ϕ_2. We find that the worldline measure gives a prediction that depends sharply on the initial value of the field ϕ; a value that is beyond observation and may be predicted only by a fundamental theory of initial conditions.

In general, the prediction of the worldline measure in a model of random-walk eternal inflation is identical with the comoving exit probability (see Sec. 4.1.3). In other words, it is the probability of a randomly chosen comoving point exiting through a given point of the reheating boundary in field space. This exit probability, of course, depends sensitively on the initial values of the fields at the beginning of inflation, which are unknown.

6.2 Regularization for a single reheating surface

We will now describe the main volume-based measure proposals applicable to models of random-walk inflation, and more specifically to models where there is a single type of reheating surface in spacetime. (Note that the reheating surface itself may have a very complicated geometry and topology. The important assumption is that all parts of the reheating surface are statistically equivalent.)

The task at hand is to define a measure that ascribes equal weight to each observer ever appearing anywhere in the universe. As discussed in Sec. 3.1, it is sufficient to construct a measure $\mathcal{V}(\chi_a)$ of the 3-volume along the reheating surface since observers appear only along that surface. Once a measure is constructed, the volume-weighted distribution $P(\chi_a)$ will then be given by Eq. (3.5).

In the presence of eternal inflation, the proper 3-volume of the reheating surface diverges even when we limit the spacetime domain under consideration to the comoving future of a finite initial spacelike 3-volume. In other words, a finite initial spacelike slice gives rise to an infinite reheating surface. Therefore, the reheating surface needs to be regularized.

In this section we consider the case when the reheating condition is met at a topologically compact and connected locus in the configuration space $\{\phi, \chi_a\}$. In this case, every connected component of the reheating surface in spacetime will contain all the possible values of the fields χ_a, and all such connected pieces are statistically equivalent. Hence, it suffices to consider a single connected piece of the reheating surface. A situation of this type is illustrated in Fig. 6.2. A sketch of the random walk in configuration space is shown in Fig. 6.3.

A simple regularization scheme is known as the **equal-time cutoff**. One considers the part of the reheating surface formed before a fixed time $t_{\max}$; that part is finite as long as $t_{\max}$ is finite. Then one can compute the distribution of the quantities of interest within that part of the reheating surface. Subsequently, one takes the limit $t_{\max} \to \infty$.

The equal-time cutoff yields a probability distribution for observed quantities that can be found from the solution $P_V^{(\tilde{\gamma})}(\phi)$ of the "stationary" FP equation,

$$[\hat{L}^{\dagger} + 3H(\phi)]P_V^{(\tilde{\gamma})} = \tilde{\gamma} P_V^{(\tilde{\gamma})}, \tag{6.3}$$

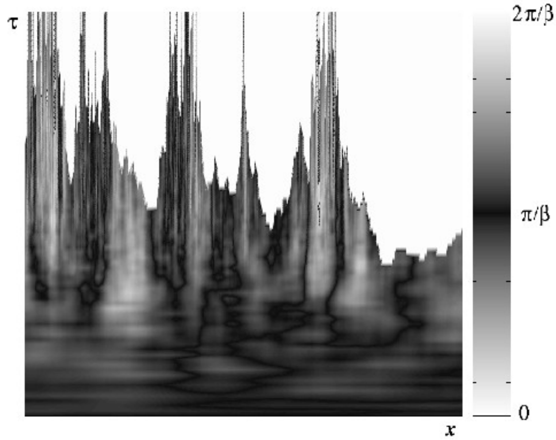

Figure 6.2: A 1+1-dimensional slice of spacetime in a two-field inflationary model (numerical simulation in [231]). Shades denote different values of the field χ, which takes values in the periodically identified interval $[0, 2\pi/\beta]$. The white region represents the thermalized domain. The boundary of the thermalized domain is the reheating surface in spacetime, which contains regions with all possible values of the field χ.

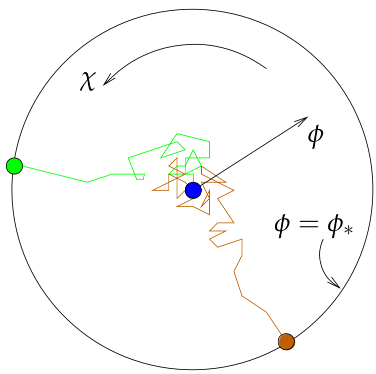

Figure 6.3: A random walk in field space for a two-field inflationary model considered in [231]. The center of the field space is a diffusion-dominated regime. The reheating condition, $\phi = \phi_*$, selects a compact and connected region (a circle) in field space, which is the exit boundary for the random walk. The problem is to determine the volume-weighted probability distribution for the values of χ at reheating.

with the largest eigenvalue $\tilde{\gamma}$ (see Sec. 4.1.5). This is easy to see by considering a fiducial two-field model (ϕ drives inflation, χ is an fluctuating parameter) and by computing the total 3-volume reheated until time $t = t_{\max}$ with a given value χ,

$$V(\chi; t_{\max}) = \left| v_{\phi}(\phi_*, \chi) \right| \int_0^{t_{\max}} P_V(\phi_*, \chi, t)\, dt$$
$$\approx \left| v(\phi_*, \chi) \right| P_V^{(\tilde{\gamma})}(\phi_*, \chi) \frac{e^{\tilde{\gamma} t_{\max}}}{\tilde{\gamma}}. \tag{6.4}$$

Here we used the late-time behavior of the volume distribution,

$$P_V(\phi, \chi, \tau) \approx P_V^{(\tilde{\gamma})}(\phi, \chi) e^{\tilde{\gamma}\tau}. \tag{6.5}$$

It follows that the distribution of χ tends to a limit as $t_{\max} \to \infty$; for instance we may compute the ratio

$$\lim_{t_{\max}\to\infty} \frac{V(\chi_1; t_{\max})}{V(\chi_2; t_{\max})} = \frac{\left| v(\phi_*, \chi_1) \right| P_V^{(\tilde{\gamma})}(\phi_*, \chi_1)}{\left| v(\phi_*, \chi_2) \right| P_V^{(\tilde{\gamma})}(\phi_*, \chi_2)}, \tag{6.6}$$

which predicts the relative probability of observing χ_1 or χ_2 at reheating in the equal-time cutoff. However, both the eigenvalue $\tilde{\gamma}$ and the distribution $P_V^{(\tilde{\gamma})}(\phi, \chi)$ depend rather sensitively on the choice of the equal-time hypersurfaces. Since there appears to be no preferred choice of the cutoff hypersurfaces in spacetime, the equal-time cutoff cannot serve as an unbiased measure. For instance, the result (6.2) cannot be obtained via an equal-time cutoff with any choice of the time gauge [5, 75]. This can be shown as follows. We consider the model potential shown in Fig. 6.1 and use an arbitrary time variable τ for the equal-time cutoff. The ratio of the volume of regions thermalized before a fixed time $\tau = \tau_{\max}$ in $\phi = \phi_*^{(1)}$ and $\phi = \phi_*^{(2)}$ can be expressed as

$$\frac{\mathcal{V}_1}{\mathcal{V}_2} = \frac{|v_1|\, Z_1^3}{|v_2|\, Z_2^3} \exp\left[-\tilde{\gamma}(\Delta\tau_1 - \Delta\tau_2) \right], \tag{6.7}$$

where $\Delta\tau_j$, $j = 1, 2$, are the time intervals elapsed in the time gauge τ during the deterministic evolution of the field from ϕ_j to $\phi_*^{(j)}$, and $Z_{1,2}$ are the respective volume factors accumulated during the deterministic evolution, while $v_j \equiv v(\phi_*^{(j)})$.

One can derive Eq. (6.7) from elementary considerations as follows. A Hubble region exiting the self-reproduction regime at $\phi = \phi_1$ will thermalize after a fixed duration of time $\Delta\tau_1$ and will accumulate the growth factor Z_1 during this time. Similarly, regions exiting at $\phi = \phi_2$ will reach the thermalization point $\phi = \phi_*^{(2)}$ after a time interval $\Delta\tau_2$ and accumulate the growth factor Z_2. The "arrival times" $\Delta\tau_j$ and the growth factors Z_j, $j = 1, 2$, can be computed using the particular shape of the potential $V(\phi)$ as explained in Sec. 4.1.5.

It follows that the regions thermalized at $\phi = \phi_*^{(j)}$ at time τ have previously exited the self-reproduction regime at time $\tau - \Delta\tau_j$. According to the late-time asymptotics given by Eq. (6.5) with χ playing the role of the label 1 or 2, the volume of regions exiting at $\phi = \phi_j$ at time $\tau - \Delta\tau_j$ is

$$\left|v_j\right| P_V^{(\tilde{\gamma})}(\phi_j) \exp\left[\tilde{\gamma}(\tau - \Delta\tau_j)\right] d\tau, \tag{6.8}$$

where $P_V^{(\tilde{\gamma})}(\phi_1) = P_V^{(\tilde{\gamma})}(\phi_2) \equiv P_V^{(\tilde{\gamma})}$ due to the symmetry of the potential in the self-reproduction regime and $v_j \equiv v(\phi_*^{(j)})$. The total volume of regions thermalized at $\phi = \phi_*^{(j)}$ up to a time $\tau_{\max}$ is then

$$\begin{aligned}\mathcal{V}_j(\tau_{\max}) &= \left|v_j\right| P_V^{(\tilde{\gamma})} Z_j^3 \int_0^{\tau_{\max}} \exp\left[\tilde{\gamma}(\tau - \Delta\tau_j)\right] d\tau \\ &\propto \left|v_j\right| Z_j^3 \exp\left[\tilde{\gamma}(\tau_{\max} - \Delta\tau_j)\right],\end{aligned} \tag{6.9}$$

where we have omitted j-independent factors. Thus we obtain the volume ratio as the limit

$$\frac{\mathcal{V}_1}{\mathcal{V}_2} = \lim_{\tau_{\max}\to\infty} \frac{\mathcal{V}_1(\tau_{\max})}{\mathcal{V}_2(\tau_{\max})} = \frac{|v_1|\, Z_1^3}{|v_2|\, Z_2^3} \exp\left[-\tilde{\gamma}(\Delta\tau_1 - \Delta\tau_2)\right]. \tag{6.10}$$

This coincides with Eq. (6.7).

The reason for the gauge dependence of the volume ratio (6.7) is the presence of the times $\Delta\tau_j$ which generally leads to an appreciable bias. The arrival times $\Delta\tau_j$ are generally different, $\Delta\tau_1 \neq \Delta\tau_2$, unless the inflaton potential $V(\phi)$ is completely symmetric.[4] Since $\gamma > 0$ in all gauges, it is impossible to choose a gauge τ in which the results (6.2) and (6.7) would coincide for all potentials.

A similar result is found in a two-field model where we are looking for the volume-weighted distribution $P(\chi)$ of the field χ. The arrival times $\Delta\tau(\chi)$ will be χ-dependent and also vary greatly with different choices of the time coordinate. We conclude that there exists no "correct" time coordinate τ that could always guarantee unbiased results for the equal-time cutoff procedure.

Recently, the equal-time cutoff with t chosen as the number of e-foldings (the "scale factor" time, $\tau = \ln a$) received new attention [115, 237, 241, 238]. Previously this cutoff was disfavored due to the arbitrariness of choosing the e-folding time rather than any other time coordinate. A tentative fundamental motivation of the scale factor cutoff was presented in [242]. The properties of the scale factor cutoff and its suitability as the multiverse measure is still under investigation. One attractive feature of the scale factor cutoff is that the bias due to the dependence on arrival times $\Delta\tau$ is small. This is manifest in the approximate absence of the youngness paradox (see Sec. 6.4).

In an attempt to cure the equal-time cutoff from the bias, two modifications appeared. In the first, called the ε-**prescription** [5, 235, 179, 236], one introduces

[4] The arrival times would be equal if it were possible to use the hypersurfaces of equal ϕ as the time slicing. However, a hypersurface $\phi =$ const is in general not spacelike because the evolution of ϕ is not monotonic in the self-reproduction regime.

the equal-time cutoff in a χ-dependent way. Namely, regions with a given value of χ are cut off at a χ-dependent cutoff time $\tau_c(\varepsilon;\chi)$, defined such that only a fraction ε of the total comoving volume has not yet reheated in these regions. Finally, the parameter ε is taken to zero, which corresponds to $\tau_c(\varepsilon;\chi) \to \infty$, and the limit ratio of cutoff volumes is computed. Since the late-time asymptotic of the comoving volume distribution is of the form

$$P(\phi,\chi,\tau) \approx P^{(\gamma)}(\phi,\chi)e^{-\gamma\tau}, \tag{6.11}$$

while the reheated comoving volume up to time τ is computed as

$$|v(\phi_*,\chi)| \int_0^\tau P(\phi_*,\chi,\tau)d\tau, \tag{6.12}$$

one finds that the χ-dependent cutoff time is

$$\tau_c(\varepsilon;\chi) = \frac{1}{\gamma}\ln\left[\frac{1}{\varepsilon\gamma}\frac{P^{(\gamma)}(\phi_*,\chi)}{\int_0^\infty P(\phi_*,\chi,\tau)d\tau}\right]. \tag{6.13}$$

The physical volume reheated up to time $\tau_c(\varepsilon;\chi)$ is

$$\begin{aligned} V(\varepsilon;\chi) &= |v(\phi_*,\chi)| \int_0^{\tau_c(\varepsilon;\chi)} P_V(\phi_*,\chi,\tau)d\tau \\ &\approx |v(\phi_*,\chi)|\, P_V^{(\gamma)}(\phi_*,\chi)\left[\frac{1}{\varepsilon\gamma}\frac{P^{(\gamma)}(\phi_*,\chi)}{\int_0^\infty P(\phi_*,\chi,\tau)d\tau}\right]^{\tilde\gamma/\gamma}. \end{aligned} \tag{6.14}$$

With $\varepsilon \to 0$, a volume ratio such as $V(\varepsilon;\chi_1)/V(\varepsilon;\chi_2)$ tends to a well-defined limit, which is the result of the ε-prescription. Note that the results include the *integral* of the comoving distribution through all times, which is related to the exit probability and depends on the initial values of ϕ and χ or on the initial distribution $P(\phi,\chi,\tau=0)$. The properties of the ε-prescription were analyzed in [235, 179] where it was found that the results depend weakly on the choice of the time parameter τ and on the initial distribution. Nevertheless, this dependence is not completely absent, which makes the ε-prescription less than satisfactory.

In the second modification of the equal-time cutoff, called the **stationary measure** [53, 223, 243], one also introduces a χ-dependent cutoff times $\tau_c(\varepsilon;\chi)$ in a way quite similar to the ε-prescription — except that the volume-weighted distribution is used instead of the comoving distribution. Namely, we define $\tau_c(\varepsilon;\chi)$ as the time at which the volume-weighted distribution $P_V(\phi,\chi,\tau)$ reaches the asymptotic late-time regime with relative precision ε,

$$\left|1 - \frac{P_V(\phi_*,\chi,\tau_c(\varepsilon;\chi))}{P_V^{(\tilde\gamma)}(\phi_*,\chi)e^{\tilde\gamma\tau_c(\varepsilon;\chi)}}\right| = \varepsilon. \tag{6.15}$$

Then a similar procedure yields the volume ratios such as $V(\varepsilon;\chi_1)/V(\varepsilon;\chi_2)$ in the limit $\varepsilon \to 0$. It was shown in [243] that the resulting procedure has attractive properties, although a full analysis of its dependence on the time parameter τ and on the initial conditions has not yet been performed.

The **spherical cutoff** measure prescription [94] regularizes the reheating surface in a different way. A finite region within the reheating surface is selected as a spherical region of radius R around a randomly chosen center point. (Since the reheating 3-surface is spacelike, the distance between points can be calculated as the length of the shortest path within the reheating 3-surface.) Then the distribution of the quantities of interest is computed within the spherical region. Subsequently, the limit $R \to \infty$ is evaluated. Since every portion of the reheating surface is statistically the same, the results are independent of the choice of the center point. The spherical cutoff is gauge-invariant since it is formulated entirely in terms of the intrinsic properties of the reheating surface.

While the spherical cutoff prescription is manifestly coordinate-invariant, there is no universally applicable analytic formula for the resulting distribution. Therefore, applying the spherical cutoff is possible only if one chooses specific models of random-walk inflation and performs a direct numerical simulation of the stochastic field dynamics in the inflationary spacetime. Such simulations were reported in [189, 244, 68, 245, 231] and used the Langevin equation (4.3) with a specific stochastic ansatz for the noise field $N(t, \mathbf{x})$.

Apart from numerical simulations, the results of the spherical cutoff method may be obtained analytically in a certain class of models [231]. One such case is a multi-field model where the potential $V(\phi, \chi_a)$ is independent of χ_a within the range of ϕ where the "diffusion" in ϕ dominates. Then the distribution in χ_a is flat when field ϕ exits the regime of self-reproduction and resumes the deterministic slow-roll evolution. One can derive a gauge-invariant Fokker–Planck equation for the volume distribution $P_V(\phi, \chi_a)$, using ϕ as the time variable [231],

$$\partial_\phi P_V = \partial_{\chi\chi} \left(\frac{D}{v_\phi} P_V \right) - \partial_\chi \left(\frac{v_\chi}{v_\phi} P_V \right) + \frac{3H}{v_\phi} P_V, \tag{6.16}$$

where D, v_ϕ, and v_χ are the coefficients of the FP equation (4.32). This equation is valid for the range of ϕ where the evolution of ϕ is free of fluctuations. By solving Eq. (6.16), one can calculate the volume-based distribution of χ_a predicted by the spherical cutoff method as $P_V(\phi_*, \chi)$. Note that the mentioned restriction on the potential $V(\phi, \chi_a)$ is important. In general, the inflaton field ϕ cannot be used as the time variable since the surfaces of constant ϕ are not everywhere spacelike due to large fluctuations of ϕ in the diffusion-dominated regime.

Finally, we note that all the regularization methods reviewed so far were based on restricting the reheating surface by a geometric construction, such as by hypersurfaces of equal time. The worldline-based measure can be also viewed as a geometric restriction of the reheating surface (to a neighborhood of a comoving worldline). In the next chapters I will describe a measure proposal (the RV measure) that does not use any geometric constructions for regularizing the volume.

6.3 Regularization for multiple types of reheating surfaces

The regularization proposals reviewed in the previous section are not applicable to models with bubble nucleation because in those models the reheating surface consists of infinitely many *statistically inequivalent* parts (the pieces of reheating surface in bubbles of different types). The same problem can be found in a simpler example: Consider an inflationary scenario with an *asymmetric* slow-roll potential $V(\phi)$ having two minima $\phi_*^{(1)}$, $\phi_*^{(2)}$ such as in Fig. 6.1. This scenario has two possibilities for thermalization, possibly differing in the observable parameters χ_a. More generally, one may consider a scenario with n different minima of the potential, possibly representing n distinct reheating scenarios. It is important that the minima $\phi_*^{(j)}$, $j = 1, ..., n$ are topologically disconnected *in the field space*. This precludes the possibility that different minima are reached within one connected component of the reheating hypersurface in spacetime. Additionally, the fields χ_a may fluctuate across each connected component in a way that depends on the minimum j. Thus, the distribution $P(\chi_a; j)$ of the fields χ_a at each connected component of the reheating surface may depend on j. In other words, the different components of the reheating surface may be *statistically inequivalent* with respect to the distribution of χ_a on them. To use the volume-based measure for making predictions in such models, one needs a regularization method that is applicable to situations with a large number of disconnected and statistically inequivalent components of the reheating hypersurface.

Although equal-time cutoff measures and their improvements are directly applicable to these situations, the drawbacks of these measures (mainly the dependence on the time coordinate and the initial conditions) force us to seek other alternatives.

The spherical cutoff prescription (see Sec. 6.2) cannot be applied to a topologically disconnected reheating surface because a sphere of a finite radius R in a region of type j will never reach parts of the reheating surface of different type $k \neq j$. Thus the spherical cutoff yields only the distribution of χ_a across one connected component and needs to be supplemented by a "weighting" prescription, which would assign a weight p_j to the component of the reheating surface labeled j.

In scenarios of tunneling type, such as the string theory landscape, observers may find themselves in bubbles of types $\alpha = 1, ..., N$. Again, a weighting prescription or, figuratively speaking, a bubble-counting prescription, is needed to determine the probabilities p_α of being in a bubble of type α.

Two different weighting prescriptions have been formulated, again using the volume-based [222, 232] and the worldline-based approach [107], respectively. The first prescription is called the **comoving cutoff** while the second is called the **worldline** or the **holographic** cutoff. For clarity, we illustrate these weighting prescriptions on models of tunneling type where infinitely many nested bubbles of types $\alpha = 1, ...N$ are created and where some bubbles are "termi-

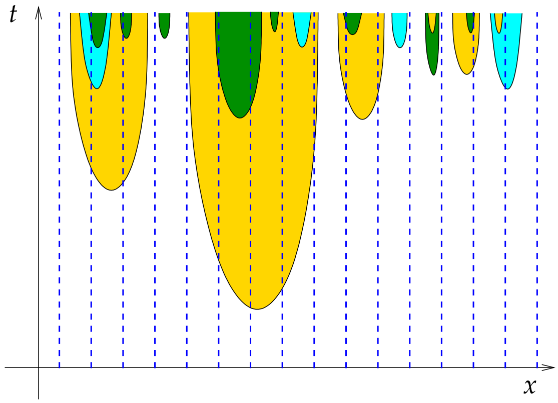

Figure 6.4: Weighting prescriptions for models of tunneling type. Shaded regions are bubbles of different types. Dashed vertical lines represent randomly chosen comoving geodesics used to define a finite subset of bubbles.

nal," *i.e.* can contain no "daughter" bubbles. In the volume-based approach, every bubble nucleated anywhere in the spacetime needs to be counted; in the worldline-based approach, only bubbles intersected by a selected worldline are counted. Let us now explain and examine these two prescriptions in more detail.

Since the set of all bubbles is infinite, one needs in any case to perform a "regularization," that is, one needs to select a very large but *finite* subset of bubbles. The weight p_α will be then calculated as the fraction of bubbles of type α within the selected subset; afterwards, the number of bubbles in the subset will be taken to infinity. Technically, the prescriptions differ in the choice of the finite subset of bubbles. The difference between the two prescriptions can be understood pictorially (Fig. 6.4). In a spacetime diagram, one draws a finite number of timelike comoving geodesic worldlines emitted from an initial 3-surface towards the future. (It can be shown that the results are independent of the choice of these lines, as long as that choice is uncorrelated with the bubble nucleation process [232].) Each of these lines will intersect only a finite number of bubbles, since the final state of any worldline is (almost surely) a terminal bubble. The subset of bubbles needed for the regularization procedure is defined as the set of all bubbles intersected by at least one line. At this point, the volume-based approach assigns equal weight to each bubble in the subset, while the worldline-based approach assigns equal weight to each bubble along each worldline. As a result, the volume-based measure counts each bubble in the subset only once, while the worldline-based measure counts each bubble as many times as it is intersected by some worldlines. After determining the weights p_α by counting the bubbles as described, one increases the number of worldlines to infinity and evaluates the limit values of p_α.

It is clear that the volume-based measure represents the counting of bubbles in the entire universe, and it is then appropriate that each bubble is being counted only once. On the other hand, the worldline-based measure counts bubbles occurring along a single worldline, ignoring the bubbles produced in other parts of the universe and introducing an unavoidable bias due to the initial conditions at the starting point of the worldline.

Explicit formulas for p_α were derived for a tunneling-type scenario (with terminal vacua) in the volume-based approach [222],

$$p_\alpha = \sum_\beta H_\beta^q \kappa_{\alpha\beta} s_\beta, \tag{6.17}$$

where $\kappa_{\alpha\beta}$ is the matrix of nucleation rates, q and s_α are the quantities defined by Eq. (5.3), and H_β is the Hubble parameter in bubbles of type β. The expressions for p_α obtained from the worldline-based measure are given by Eq. (5.9). As we have noted before, the volume-based measure assigns weights p_α that are independent of initial conditions, while the weights obtained from the worldline-based measure depend sensitively on the type of the initial bubble where the evolution begins.

It is interesting to note that in the case of a non-terminal landscape, *i.e.* in the absence of terminal types, both the volume-based and the worldline-based measures give identical results for p_α, which then coincide with the mean frequency (5.4) of visiting a bubble of type α by a randomly chosen comoving worldline [224, 107].

With the weighting prescriptions just described, the volume-based and the worldline-based measure proposals can be considered complete. In other words, we have two alternative prescriptions that can be applied (in principle) to arbitrary models of random-walk or tunneling-type eternal inflation. Further research is needed to reach a definite conclusion concerning the viability of these measure prescriptions.

6.4 Youngness paradox and Boltzmann brains

One way to choose between the competing measure prescriptions is to examine their predictions for various paradoxical answers. At least two sources of problems for the measure prescriptions have been uncovered (for more discussion, see [50]).

One possible problem is the so-called "youngness" paradox [187, 94, 186, 223]. It is manifested in some volume-based measures, notably in the equal-time cutoff by proper time t, by the predominance of universes where inflation ended very recently [186]. Indeed, a tiny and improbable fluctuation that delays reheating by a time δt is rewarded by a huge additional expansion factor, $\propto \exp\left(3H_{\text{big}}\delta t\right)$, where H_{big} is a fiducial value of the highest Hubble rate available in the model. While H_{big} must be below the Planck scale M_{P}, values such as $H_{\rm big} \approx 10^{-6} M_{\rm P}$ are already sufficient to predict that the most likely temperature of the CMB is much higher than the observed 2.7°K. For this reason, measure prescriptions exhibiting the "youngness" paradox are considered to be unsatisfactory.

Generically, every equal-time cutoff suffers from the youngness paradox because of the dependence on the delay time δt inherent in the probability distribution obtained from the FP equation. Interestingly, it was found that the scale factor cutoff has a negligible youngness bias [237, 238]. In order to cure the equal-time cutoff from the youngness paradox, a "stationarity measure" was proposed in [53]. The ε-prescription [5], the spherical cutoff [94] as well as the comoving cutoff [222] and the RV measure [246] are also free from the youngness paradox. Worldline-based measures [107] by construction cannot suffer from the youngness paradox.

Another pathology that appears in the results of several measure prescription is the so-called "Boltzmann brain" problem [247, 248, 249, 115, 250, 251, 252, 223, 253]. The root of the problem is the thermal property of de Sitter space that behaves as a thermal bath [63] with the Gibbons-Hawking temperature $T = \frac{1}{2\pi} H$. Thermal fluctuations are able to produce any classical system, for instance, an Earth-like planet with observers on it, with an exceedingly small but *nonzero* probability per unit spacetime 4-volume. Such rare events will nevertheless occur in eternally inflating de Sitter spacetime, somewhere in the infinite interior of every recyclable bubble. Hence, one can expect that infinitely many observers will be created who do not see a universe full of galaxies around them, but instead find themselves surrounded by empty de Sitter space. An extreme case is an isolated brain nucleating out of thermal fluctuations in empty space (a "Boltzmann brain"). It appears reasonable to reject a multiverse measure that predicts a large probability of us being Boltzmann brains rather than ordinary observers.

Recent works investigated these issues for many of the currently considered measure proposals [50, 115, 53, 223, 237, 241, 238]. It was found that the worldline-based measure as well as the scale factor cutoff disallow Boltzmann brains if the landscape satisfies certain technical conditions, which are however not trivially fulfilled. The "stationarity measure" [53] and the RV measure [254] are free from the Boltzmann brain problem. (In the following chapters I will introduce the RV measure and demonstrate its properties in detail.)

7 The RV measure: first look

In a series of publications [246, 255, 254] I proposed and studied a new class of volume-weighted probability measures applicable to any cosmological "multiverse" scenarios. The new measure, called the **reheating-volume** (RV) **cutoff**, calculates the distribution of observable quantities on an ensemble of multiverses that are *conditioned in probability* to be finite (with a finite reheating hypersurface), as will be fully explained below. In this chapter I introduce the RV measure and discuss its general properties. By considering slow-roll inflationary models with a scalar inflaton, I show that the RV measure is gauge-invariant, does not suffer from the "youngness paradox," and is independent of initial conditions at the beginning of inflation. The application of the RV cutoff to random-walk inflation will be treated more thoroughly in Chapter 8. Then in Chapter 9 I will apply the RV measure to eternal inflation of tunneling time, i.e. to landscape scenarios or generally to scenarios with bubble nucleation.

As I described in the previous chapters, eternal inflation gives rise to infinitely many causally disconnected regions of the spacetime where the cosmological observables may have significantly different values. One hopes to obtain probability distribution of the cosmological parameters as measured by an observer randomly located in the spacetime. The main difficulty in obtaining such probability distributions is due to the infinite volume of regions where an observer may be located.

Since the universe is cold and empty during inflationary evolution, observers may appear only after reheating. The standard cosmology after reheating is tightly constrained by current experimental knowledge. Hence, the average number of observers produced in any freshly-reheated spatial domain is a function of cosmological parameters in that domain. Calculating that function is, in principle, a well-defined astrophysical problem that does not involve any infinities. Therefore we focus on the problem of obtaining the probability distribution of cosmological observables at reheating.

The set of all spacetime points where reheating takes place is a spacelike three-dimensional hypersurface [70, 5, 192] called the **reheating surface** in spacetime. The hallmark feature of eternal inflation is that a *finite*, initially inflating spatial 3-volume typically gives rise to a reheating surface having an *infinite* 3-volume (see Fig. 7.1). The geometry and topology of the reheating surface is quite complicated. For instance, the reheating surface contains infinitely many future-directed spikes around never-thermalizing comoving worldlines or "eternal points" [188, 204, 240]. It is known that the set of "spikes" has a well-defined fractal dimension that can be computed in the stochastic approach. Since the reheating surface is a highly inhomogeneous, noncompact 3-manifold without any symmetries, the notion of "random location" on such a surface is

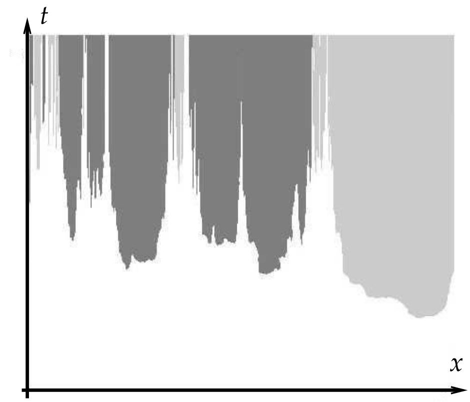

Figure 7.1: A 1+1-dimensional slice of the spacetime in an eternally inflating universe (numerical simulation in [231]). Shades of different color represent different regions where reheating took place. The reheating surface is the line separating the white (inflating) domain and the shaded domains.

mathematically ill-defined. This feature of eternal inflation is at the root of the technical and conceptual difficulties known collectively as the measure problem.

To motivate the RV measure, we first visualize the measure problem as follows. It is convenient to consider an initial inflating spacelike region S of horizon size (an "H-region") and the portion $R \equiv R(S)$ of the reheating surface that corresponds to the comoving future of S. If the 3-volume of R were finite, the volume-weighted average of any observable quantity Q at reheating would be defined simply by averaging Q over R,

$$\langle Q \rangle \equiv \frac{\int_R Q\sqrt{\gamma} d^3x}{\int_R \sqrt{\gamma} d^3x}, \tag{7.1}$$

where γ is the induced metric on the 3-surface R. This would have been the natural prescription for the observer-based average of Q; all higher moments of the distribution of Q, such as $\langle Q^2 \rangle$, $\langle Q^3 \rangle$, etc., would have been well-defined as well. However, in the presence of eternal inflation[1] the 3-volume of R is infinite with a nonzero probability $X(\phi_0)$, where $\phi = \phi_0$ is the initial value of the inflaton field at S. The function $X(\phi_0)$ has been computed in slow-roll inflationary models [188] where typically $X(\phi_0) \approx 1$ for ϕ_0 not too close to reheating. In other words, the volume of R is infinite with a probability close to 1. In that case, the straightforward average (7.1) of a fluctuating quantity $Q(x)$ over R is mathematically undefined since $\int_R \sqrt{\gamma} d^3x = \infty$ and $\int_R Q\sqrt{\gamma} d^3x = \infty$.

[1] Various equivalent conditions for the presence of eternal inflation were examined in more detail in Sec. 4.2. Here I adopt the condition that $X(\phi)$ is nonzero for all ϕ in the inflating range.

The average $\langle Q \rangle$ can be computed only after imposing a volume cutoff on the reheating surface, making its volume finite in a controlled way. What has become known in cosmology as the "measure problem" is the difficulty of coming up with a physically motivated cutoff prescription (informally called a "measure") that makes volume averages $\langle Q \rangle$ well-defined.

Volume cutoffs are usually implemented by restricting the infinite reheating domain R to a large but finite subdomain having a volume $\mathcal{V}$. Then one defines the "regularized" distribution $p(Q|\mathcal{V})$ of an observable Q by gathering statistics about the values of Q over the finite volume $\mathcal{V}$. More precisely, $p(Q|\mathcal{V})\mathcal{V}dQ$ is the 3-volume of regions (within the finite domain $\mathcal{V}$) where the observable Q has values in the interval $[Q, Q + dQ]$. The final probability distribution $p(Q)$ is then defined as

$$p(Q) \equiv \lim_{\mathcal{V}\to\infty} p(Q|\mathcal{V}), \tag{7.2}$$

provided that the limit exists.

A cutoff prescription is a specific choice of the compact subset $\mathcal{V}$ and of the way $\mathcal{V}$ approaches infinity when the cutoff is removed. Several cutoffs have been proposed in the literature, differing in the choice of the compact subset $\mathcal{V}$ and in the way $\mathcal{V}$ approaches infinity. It has been found early on (*e.g.* [68, 5]) that probability distributions, such as $p(Q)$, depend sensitively on the choice of the cutoff. This is the root of the measure problem. Since a "natural" mathematically consistent definition of the measure is absent, one judges a cutoff prescription viable if its predictions are not obviously pathological. Possible pathologies include the dependence on choice of spacetime coordinates [179, 235], the "youngness paradox" [187, 94], and the "Boltzmann brain" problem [247, 248, 249, 115, 251, 252, 223, 253].

The presently viable cutoff proposals fall into two rough classes that may be designated as "worldline-based" and "volume-based" measures (a more fine-grained classification of measure proposals can be found in [50, 240]). The "worldline" or the "holographic" measure [107, 66] avoids considering the infinite total 3-volume of the reheating surface in the entire spacetime. Instead it focuses only on the reheated 3-volume of one H-region surrounding a *single* randomly chosen comoving worldline. This measure, by construction, is sensitive to the initial conditions at the location where the worldline starts and is essentially equivalent to performing calculations with the comoving-volume probability distribution. Proponents of the "holographic" measure have argued that the infinite reheating surface cannot be considered because the spacetime beyond one H-region is not adequately described by semiclassical gravity [66]. However, the semiclassical approximation was recently shown to be valid in a large class of inflationary models [182]. In my view, an attempt to count the total volume of the reheating surface corresponds more closely to the goal of obtaining the probability distribution of observables in the entire universe, as measured by a "typical" observer (see [256, 257, 258, 259, 223] for recent discussions of "typicality" and accompanying issues). The sensitive dependence of "holographic" proposals on the conditions at the beginning of inflation also appears to be undesirable. Volume-based proposals are insensitive to the initial conditions because the 3-volume of the universe is, in a certain well-defined sense, dominated by regions that spent a long time in the inflationary regime.[2] Existing volume-based proposals include the equal-time cutoff [78, 3, 79], the "spherical cutoff" [94], the "comoving horizon cutoff" [222, 232, 224], the "stationary measure" [53, 101], the "no-boundary" measure with volume weighting [260, 261, 262, 263], the "pseudo-comoving" measure [115, 238]. Here I propose a new volume-based measure called the "reheating-volume (RV) cutoff" [246]. The RV cutoff is based on the notion of "finitely-produced volume distribution," which is explained in the next section.

7.1 Reheating-volume cutoff

In the RV cutoff, the reheating surface is not being restricted to an artificially chosen domain. Instead, one simply selects only those initial regions S that, *by rare chance*, evolve into compact reheating surfaces R having a finite, fixed volume $\mathrm{Vol}(R) = \mathcal{V}$. The ensemble $\mathcal{E}_{\mathcal{V}}$ of such initial regions S is a nonempty subset of the ensemble $\mathcal{E}$ of all initial regions S. The volume-weighted probability distribution $p(Q|\mathcal{E}_{\mathcal{V}})$ of a cosmological observable Q in the ensemble $\mathcal{E}_{\mathcal{V}}$ can be determined through ordinary sampling of the values of Q over the finite volume $\mathcal{V}$. The RV cutoff defines the probability distribution $p(Q)$ as the limit of $p(Q|\mathcal{E}_{\mathcal{V}})$ at $\mathcal{V} \to \infty$, provided that the limit exists.

To develop an approach for practical computations in the RV cutoff, let us first consider the probability density $\rho(\mathcal{V};\phi_0)d\mathcal{V}$ of having *finite* volume $\mathrm{Vol}(R) \in [\mathcal{V}, \mathcal{V} + d\mathcal{V}]$ of the reheating surface R that results from a single H-region with initial value $\phi = \phi_0$. This distribution is normalized to the probability of the event $\mathrm{Vol}(R) < \infty$, namely

$$\int_0^\infty \rho(\mathcal{V};\phi_0)d\mathcal{V} = \mathrm{Prob}\,(\mathrm{Vol}(R) < \infty) = 1 - X(\phi_0). \tag{7.3}$$

The probability density $\rho(\mathcal{V};\phi_0)$ is nonzero since $X(\phi_0) < 1$. I call this $\rho(\mathcal{V};\phi_0)$ the "finitely produced reheated volume" (FPRV) distribution. This and related distributions constitute the mathematical basis of the RV cutoff.

Below I will use the Fokker–Planck (or "diffusion") formalism to derive equations from which the FPRV distributions can be in principle computed for models of slow-roll inflation with a single scalar field. Generalizations of $\rho(\mathcal{V};\phi_0)$ to multiple-field or non-slow-roll models are straightforward since the Fokker–Planck formalism is already developed in those contexts (*e.g.* [38, 182]).

Let us now define the FPRV distribution for some cosmological observable Q at reheating. Consider the probability density $\rho(\mathcal{V}, \mathcal{V}_{Q_R};\phi_0, Q_0)$, where ϕ_0 and Q_0 are values of ϕ and Q in the initial H-region, $\mathcal{V}$ is the total reheating volume, and $\mathcal{V}_{Q_R}$ is the portion of the reheating volume where the observable Q has a

[2] It has been noted that 3-volume is a coordinate-dependent quantity, and hence statements involving 3-volume need to be formulated with care [197]. Indeed there exist time foliations where the 3-volume of inflationary space does not grow with time. The issues of coordinate dependence were analyzed in [75].

particular value Q_R. The distribution $\rho(\mathcal{V}, \mathcal{V}_{Q_R}; \phi_0, Q_0)$ as a function of $\mathcal{V}_{Q_R}$ at fixed and large $\mathcal{V}$ is sharply peaked around a mean value $\langle \mathcal{V}_{Q_R}|_{\mathcal{V}} \rangle$ corresponding to the average volume of regions with $Q = Q_R$ within the total reheated volume $\mathcal{V}$. Hence, although the full distribution $\rho(\mathcal{V}, \mathcal{V}_{Q_R}; \phi_0, Q_0)$ could be in principle determined, it suffices to compute the mean value $\langle \mathcal{V}_{Q_R}|_{\mathcal{V}} \rangle$. One can then expect that the limit

$$p(Q_R) \equiv \lim_{\mathcal{V}\to\infty} \frac{\langle \mathcal{V}_{Q_R}|_{\mathcal{V}} \rangle}{\mathcal{V}} = \lim_{\mathcal{V}\to\infty} \frac{\int_0^\infty \rho\left(\mathcal{V}, \mathcal{V}_Q; \phi_0, Q_0\right) \mathcal{V}_Q d\mathcal{V}_Q}{\mathcal{V}\, \mathrm{Prob}(\mathrm{Vol}(R) = \mathcal{V})} \tag{7.4}$$

exists and is independent of ϕ_0 and Q_0. (Below I will justify this statement more formally.) The function $p(Q_R)$ is then interpreted as the mean fraction of the reheated volume where $Q = Q_R$. In this way, the RV cutoff yields the volume-weighted distribution for any cosmological observable Q at reheating.

To obtain a more visual picture of the RV cutoff, consider a large number of initially identical H-regions having different evolution histories to the future. A small subset of these initial H-regions will generate *finite* reheating surfaces. An even smaller subset of H-regions will have the total reheated volume equal to a given value $\mathcal{V}$. Conditioning on a finite value $\mathcal{V}$ of the reheating volume, one obtains a well-defined statistical ensemble $E_{\mathcal{V}}$ of initial H-regions. For large $\mathcal{V}$, the ensemble $E_{\mathcal{V}}$ can be pictured as a set of initial H-regions that happen to be located very close to some eternally inflating worldlines but do not actually contain any such worldlines (see Fig. 7.2). In this way, the ensemble $E_{\mathcal{V}}$ samples the total reheating surface near the "spikes" where an infinite reheated 3-volume is generated from a finite initial 3-volume. It is precisely near these "spikes" that one would like to sample the distribution of observable quantities along the reheating surface. Therefore, one expects that the ensemble $E_{\mathcal{V}}$ (in the limit of large $\mathcal{V}$) provides a representative sample of the infinite reheating surface, despite the small probability of the event $\mathrm{Vol}(R) = \mathcal{V}$. In this sense, the ensemble $E_{\mathcal{V}}$ at large $\mathcal{V}$ is designed to yield a controlled approximation to the infinite reheating surfaces R.

The RV cutoff proposed here has several attractive features. By construction, the RV cutoff is coordinate-invariant; indeed, only the intrinsically defined 3-volume within the reheating surface R is used, rather than the 3-volume within a coordinate-dependent 3-surface. The results of the RV cutoff are also independent of initial conditions. This independence is demonstrated more formally below and can be understood heuristically as follows. The evolution of regions S conditioned on a large (but finite) value of $\mathrm{Vol}(R)$ is dominated by trajectories that spend a long time in the high-H inflationary regime and thereby gain a large volume. These trajectories forget about the conditions at their initial points and establish a certain equilibrium distribution of values of Q on the reheating surface. Hence, one can expect that the distribution of observables within the reheating domain R will be independent of the initial conditions in S.

The "youngness paradox" arises in some volume-based prescriptions because H-regions with delayed reheating are rewarded by an exponentially large ad-

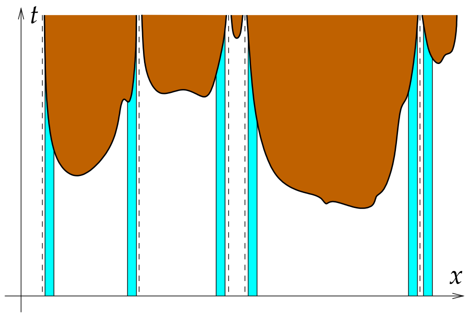

Figure 7.2: A schematic representation of the ensemble $E_{\mathcal{V}}$ in comoving coordinates (t, x). Lightly shaded vertical strips represent the comoving future of various initial H-regions from $E_{\mathcal{V}}$; dark shades represent reheated domains; the boundary of the dark-shaded domains is the reheating surface. Vertical dashed lines are the eternally inflating comoving worldlines that never cross the reheating surface. The 3-volumes of the reheating surfaces in the comoving future of the pictured H-regions are large but finite because these H-regions are located near eternal worldlines but do not contain any such worldlines.

ditional volume expansion. However, the RV measure groups together the H-regions that produce *equal* final reheated volume V; a delay in reheating is not rewarded but suppressed by the small probability of a quantum fluctuation at the end of inflation. Therefore, most of these H-regions have "normal" slow-roll evolution before reheating. For this reason, the youngness paradox is absent in the RV measure. A more explicit calculation confirming this conclusion will be given in Sec. 9.1.

7.2 RV cutoff in slow-roll inflation

As a first specific application, I implement the RV measure in a slow-roll inflationary model with a scalar inflaton ϕ and the action

$$S = \int \left[-\frac{R}{16\pi G} + \frac{1}{2} \left(\partial_\mu \phi \right)^2 - V(\phi) \right] \sqrt{-g} d^4 x. \tag{7.5}$$

We recapitulate the main assumptions of the stochastic approach to inflation. If the energy density is dominated by the potential energy $V(\phi)$, the Hubble expansion rate H is approximately given by

$$H \approx \sqrt{\frac{8\pi G}{3} V(\phi)}. \tag{7.6}$$

In the stochastic approach to inflation, the semiclassical dynamics of the field ϕ averaged over an H-region is a superposition of a deterministic motion with velocity

$$\dot{\phi} = v(\phi) \equiv -\frac{1}{4\pi G} H_{,\phi} \tag{7.7}$$

and a random walk with root-mean-squared step size

$$\sqrt{\langle \delta\phi \rangle^2} = \frac{H(\phi)}{2\pi} \equiv \sqrt{\frac{2D(\phi)}{H(\phi)}}, \quad D \equiv \frac{H^3}{8\pi^2}, \tag{7.8}$$

during time intervals $\delta t = H^{-1}$. A convenient description of the evolution of the field at time scales $\delta t \lesssim H^{-1}$ is

$$\phi(t + \delta t) = \phi(t) + v(\phi)\delta t + \xi(t)\sqrt{2D(\phi)\delta t}, \tag{7.9}$$

where $\xi(t)$ is (approximately) a "white noise" variable,

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{7.10}$$

which is statistically independent for different H-regions. This stochastic process describes the evolution $\phi(t)$ along a single comoving worldline. For convenience, we assume that inflation ends in a given H-region when ϕ reaches a fixed value $\phi = \phi_*$.

Consider an ensemble of initial H-regions $S_1, S_2, ...$, where the inflaton field ϕ is homogeneous and has value $\phi = \phi_0$ within the inflationary regime. Following the spacetime evolution of the field ϕ in each of the regions S_j along comoving geodesics, we arrive at reheating surfaces R_j where $\phi = \phi_*$. Most of the surfaces R_j will have infinite 3-volume; however, some (perhaps small) subset of S_j will have finite R_j. The ϕ_0-dependent probability, denoted $\bar{X}(\phi_0) \equiv 1 - X(\phi_0)$, of having a finite volume of R_j is a solution of the gauge-invariant equation (see Sec. 4.2)

$$D(\phi)\bar{X}_{,\phi\phi} + v(\phi)\bar{X}_{,\phi} + 3H(\phi)\bar{X}\ln \bar{X} = 0, \tag{7.11}$$

with the boundary conditions $\bar{X}(\phi_*) = 1$ and $\bar{X}(\phi_{\mathrm{Pl}}) = 1$ at reheating and at Planck boundaries. While $\bar{X}(\phi) \equiv 1$ is always a solution of Eq. (7.11), the existence of a nontrivial solution with $0 < \bar{X}(\phi) < 1$ indicates the possibility of eternal inflation. The gauge invariance of Eq. (7.11) is manifest since a change of the time variable, $\tau(t) \equiv \int^t T(\phi)dt$, results in dividing the three coefficients D, v, H by $T(\phi)$, which leaves Eq. (7.11) unchanged.

The probability distribution $\rho(\mathcal{V};\phi_0)$ can be found by considering a suitable generating function. Let us define the generating function $g(z;\phi_0)$ by

$$g(z;\phi_0) \equiv \left\langle e^{-z\mathcal{V}} \right\rangle_{\mathcal{V}<\infty} \equiv \int_0^\infty e^{-z\mathcal{V}} \rho(\mathcal{V};\phi_0)d\mathcal{V}. \tag{7.12}$$

(Note that the formal parameter z has the dimension of inverse volume. The parameter z can be made dimensionless by a trivial rescaling which we omit.) The function $g(z;\phi_0)$ is analytic in z and has no singularities for $\mathrm{Re}\, z \geq 0$. Moments of the distribution $\rho(\mathcal{V};\phi_0)$ are determined as usual through derivatives of $g(z;\phi_0)$ in z at $z=0$, while $\rho(\mathcal{V},\phi_0)$ itself can be reconstructed through the inverse Laplace transform of $g(z;\phi_0)$ in z.

The generating function $g(z;\phi)$ has the following multiplicative property: For two statistically independent H-regions that have initial values $\phi=\phi_1$ and $\phi=\phi_2$ respectively, the sum of the (finitely produced) reheating volumes $\mathcal{V}_1+\mathcal{V}_2$ is distributed with the generating function

$$\left\langle e^{-z(\mathcal{V}_1+\mathcal{V}_2)}\right\rangle = \left\langle e^{-z\mathcal{V}_1}\right\rangle\left\langle e^{-z\mathcal{V}_2}\right\rangle = g(z;\phi_1)g(z;\phi_2). \tag{7.13}$$

This multiplicative property is the only assumption in the derivation of Eq. (7.11). Hence, $g(z;\phi)$ satisfies the same equation (we drop the subscript 0 in ϕ_0),

$$Dg_{,\phi\phi}+vg_{,\phi}+3Hg\ln g=0. \tag{7.14}$$

The boundary condition at ϕ_* is $g(z;\phi_*)=e^{-zH^{-3}(\phi_*)}$ since an H-region with $\phi=\phi_*$ is already reheating and has volume $H^{-3}(\phi_*)$. The boundary condition at the Planck boundary ϕ_{Pl} (or other boundary where the effective field theory breaks down) is "absorbing," *i.e.* we assume that regions with $\phi=\phi_{\mathrm{Pl}}$ disappear and never generate any reheating volume: $g(z;\phi_{\mathrm{Pl}})=e^{z\cdot 0}=1$. Note that the variable z enters Eq. (8.44) as a parameter and only through the boundary conditions. At $z=0$ the solution of Eq. (8.44) is $g(0;\phi)=\bar{X}(\phi)$.

Let us now consider FPRV distributions of cosmological parameters Q. The generating function for the distribution $\rho(\mathcal{V},\mathcal{V}_{Q_R};\phi,Q)$ discussed above is

$$\tilde{g}(z,q;\phi,Q)\equiv\iint e^{-z\mathcal{V}-q\mathcal{V}_Q}\rho(\mathcal{V},\mathcal{V}_Q;\phi,Q)d\mathcal{V}d\mathcal{V}_Q. \tag{7.15}$$

The equation for $g(z,q;\phi,Q)$ is derived similarly to Eq. (8.44) and is of the form

$$D_\phi\tilde{g}_{,\phi\phi}+D_Q\tilde{g}_{,QQ}+v_\phi\tilde{g}_{,\phi}+v_Q\tilde{g}_{,Q}+3H\tilde{g}\ln\tilde{g}=0, \tag{7.16}$$

where D_ϕ, D_Q, v_ϕ, v_Q are the suitable kinetic coefficients representing the "diffusion" and the mean "drift velocity" of ϕ and Q. The boundary condition at $\phi=\phi_*$ is

$$\tilde{g}(z,q;\phi_*,Q)=\exp\left[-\left(z+q\delta_{QQ_R}\right)H^{-3}(\phi_*)\right], \tag{7.17}$$

where we use the delta-symbol defined by $\delta_{QQ_R}=1$ if Q belongs to a narrow interval $[Q_R,Q_R+dQ]$ and $\delta_{QQ_R}=0$ otherwise.

To obtain the distribution (7.4), we need to compute the average $\langle\,\mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle$ at fixed $\mathcal{V}$. We define the auxiliary generating function

$$h(z;\phi,Q)\equiv\left\langle\mathcal{V}_{Q_R}e^{-z\mathcal{V}}\right\rangle_{\mathcal{V}<\infty}=-\tilde{g}_{,q}(z,q=0;\phi,Q). \tag{7.18}$$

Note that $\tilde{g}(z,q=0;\phi,Q)=g(z;\phi)$. The equation for $h(z;\phi,Q)$ then follows from Eq. (7.16),

$$D_\phi h_{,\phi\phi}+D_Q h_{,QQ}+v_\phi h_{,\phi}+v_Q h_{,Q}+3H\left(\ln g+1\right)h=0. \tag{7.19}$$

This *linear* equation contains as a coefficient the function $g(z;\phi)$, which is the solution of Eq. (8.44). The boundary condition for Eq. (7.19) is

$$h(z;\phi_*,Q) = e^{-zH^{-3}(\phi_*)}H^{-3}(\phi_*)\delta(Q-Q_R). \tag{7.20}$$

Here we can use the ordinary δ-function instead of the symbol δ_{QQ_R} because the δ-function enters linearly into the boundary condition. An appropriate rescaling of the distribution h by the factor dQ is implied when we pass from δ_{QQ_R} to $\delta(Q-Q_R)$.

Finally, the expectation value $\left\langle \mathcal{V}_{Q_R}\middle|_{\mathcal{V}}\right\rangle$ at a fixed $\mathcal{V}$ and the limit (7.4) can be found using the inverse Laplace transform of $h(z;\phi,Q)$ in z.

The computation just outlined allows one, in principle, to obtain quantitative predictions from the RV measure. Further details and a more direct computational procedure will be given elsewhere [255]. Presently, let us analyze the limit $\mathcal{V}\to\infty$ in qualitative terms. The function $p(Q_R;\mathcal{V})$ is expressed as [cf. Eq. (7.4)]

$$p(Q_R;\mathcal{V}) \equiv \frac{\left\langle \mathcal{V}_{Q_R}\middle|_{\mathcal{V}}\right\rangle}{\mathcal{V}} = \frac{\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{z\mathcal{V}}h(z;\phi,Q)dz}{\mathcal{V}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{z\mathcal{V}}g(z;\phi)dz}. \tag{7.21}$$

The asymptotic behavior of the inverse Laplace transform of $h(z;\phi,Q)$ at large $\mathcal{V}$ is determined by the locations of the singularities of $h(z;\phi,Q)$ in the complex z plane. The dominant asymptotics of the inverse Laplace transform are of the form $\propto \exp(z_*\mathcal{V})$, where z_* is the singularity with the smallest $|\mathrm{Re}\, z_*|$. It can be shown that solutions of Eqs. (8.44) and (7.19) cannot diverge at *finite* values of ϕ or Q. Thus $g(z;\phi)$ and $h(z;\phi,Q)$ cannot have ϕ- or Q-dependent singularities in z. Moreover, the function $g(z;\phi)$ cannot have pole-like singularities in z; the only possible singularities are branch points where the function $g(z;\phi)$ is finite but a derivative with respect to z diverges. Furthermore, derivatives $\partial_z^n h$ satisfy linear equations with coefficients depending on the derivatives $\partial_z^{n-1}g(z;\phi)$, which diverge at the singularities of g. Hence the singularities of $\partial_z^n h$ in the z plane coincide with those of $g(z;\phi)$. For these reasons the limit in Eq. (7.4) exists and is independent of the initial values ϕ, Q. A more detailed analysis justifying these statements will be given in the next chapter.

8 The RV measure for random-walk inflation

The focus of this chapter is a more detailed study of the RV measure in the context of eternal inflation of random-walk type. As a typical generic model I choose a scenario where inflation is driven by the potential $V(\phi)$ of a minimally coupled scalar field ϕ. In this model, there exists a range of ϕ where large quantum fluctuations dominate over the deterministic slow-roll evolution, which gives rise to eternal self-reproduction of inflationary domains. I extensively use the stochastic approach to inflation, which is based on the Fokker–Planck or "diffusion" equations as developed in Chapter 4. The results can be straightforwardly generalized to multiple-field or non-slow-roll models are straightforward since the Fokker–Planck formalism is already developed in those contexts [38, 182]. Applications of the RV measure to "landscape" scenarios will be considered in Chapter 9.

8.1 Motivation

An attractive feature of the RV measure is that its construction lacks extraneous geometric elements that could introduce a bias. An example of a biased measure is the equal-time cutoff where one considers the subdomain of the reheating surface to the past of a hypersurface of fixed proper time $t = t_c$, subsequently letting $t_c \to \infty$. It is well known that the volume-weighted distribution of observables within a hypersurface of equal proper time is strongly dominated by regions where inflation ended very recently. A time delay δt in the onset of reheating due to a rare quantum fluctuation is overwhelmingly rewarded by an additional volume expansion factor $\propto \exp[3H_{\max}\delta t]$, where $H_{\max}$ is roughly the highest Hubble rate accessible to the inflaton. This is the essence of the youngness paradox that seems unavoidable in an equal-time cutoff. Moreover, the results of the equal-time cutoff are sensitive to the choice of the time gauge, as discussed in Sec. 4.1.3.

The "spherical cutoff" [94] and the "stationary measure" [53] prescriptions were motivated by the need to remove the bias inherent in the equal-time cutoff. In particular, the spherical cutoff selects as a compact subset $\mathcal{V}$ the interior of a large sphere drawn within the reheating surface R around a randomly chosen center. The spherical cutoff is manifestly gauge-invariant since its construction uses only the intrinsically defined 3-volume of the reheating surface rather than the spacetime coordinates (t, x). Some results were obtained in the spherical cutoff using numerical simulations [231]. A disadvantage of the spherical cutoff is that its direct implementation requires one to perform costly numerical simulations of random-walk inflation on a spacetime grid, for instance, using the techniques of [68, 231, 169]. Instead, one would prefer to obtain a generally valid analytic formula for the probability distribution of cosmological observables. For instance, one could ask whether the results of the spherical cutoff depend in an essential way on the spherical shape of the region, on the position of the center of the sphere, and on the initial conditions. Satisfactory answers to these questions (in the negative) were obtained in [94, 231] in some tractable cases where results could be obtained analytically. However, it is difficult to analyze these questions in full generality since one lacks a general analytic formula for the probability distribution in the spherical cutoff.

The RV measure is similar in spirit to the spherical cutoff because the RV cutoff uses only the intrinsic geometrical information defined by the reheating surface. It can be argued that the RV cutoff is "more natural" than other cutoffs in that it selects a finite portion $\mathcal{V}$ of the reheating surface without using artificial constant-time hypersurfaces, spheres, worldlines, or any other extraneous geometrical data. Instead, the selection of $\mathcal{V}$ in the RV cutoff is performed using a certain well-defined selection of subensemble in the probability space, which is determined by the stochastic evolution itself.

The central concept in the RV cutoff is the "finitely produced volume." The basic idea is that there is always a nonzero probability that a given initial H-region S *does not* give rise to an infinite reheating surface in its comoving future. For instance, it is possible that by a rare coincidence the inflaton field ϕ rolls towards reheating at approximately the same time everywhere in S. Moreover, there is a nonzero (if small) probability $\rho(\mathcal{V})d\mathcal{V}$ that the total volume $\mathrm{Vol}(R)$ of the reheating surface R to the future of S belongs to a given interval $[\mathcal{V}, \mathcal{V} + d\mathcal{V}]$,

$$\rho(\mathcal{V}) \equiv \lim_{d\mathcal{V} \to 0} \frac{\mathrm{Prob}\left\{\mathrm{Vol}(R) \in [\mathcal{V}, \mathcal{V} + d\mathcal{V}]\right\}}{d\mathcal{V}}. \tag{8.1}$$

I call $\rho(\mathcal{V})$ the "finitely produced volume distribution." This distribution is nontrivial because the probability of the event $\mathrm{Vol}(R) < \infty$ is nonzero, if small, for any given (non-reheated) initial region S. The distribution $\rho(\mathcal{V})$ is, by construction, normalized to that probability:

$$\int_0^\infty \rho(\mathcal{V}) d\mathcal{V} = \mathrm{Prob}\left\{\mathrm{Vol}(R) < \infty\right\} < 1. \tag{8.2}$$

The RV cutoff consists of a selection of a certain ensemble $E_{\mathcal{V}}$ of the histories that produce a total reheated volume equal to a given value $\mathcal{V}$ starting from an initial H-region. In the limit of large $\mathcal{V}$, the ensemble $E_{\mathcal{V}}$ consists of H-regions that evolve "almost" to the regime of eternal inflation. Thus, heuristically one can expect that the ensemble $E_{\mathcal{V}}$ provides a representative sample of the infinite reheating surface.[1] Given the ensemble $E_{\mathcal{V}}$, one can determine the volume-weighted probability distribution $p(Q|\mathcal{E}_{\mathcal{V}})$ of a cosmological parameter Q by

[1] Of course, this heuristic statement cannot be made rigorous since there exists no natural measure on the infinite reheating surface. We use this statement merely as an additional motivation for considering the RV measure.

ordinary sampling of the values of Q throughout the finite volume $\mathcal{V}$. Finally, the probability distribution $p(Q)$ is defined as the limit of $p(Q|\mathcal{E}_{\mathcal{V}})$ at $\mathcal{V} \to \infty$, provided that the limit exists.

To clarify the construction of the ensemble $E_{\mathcal{V}}$, it is helpful to begin by considering the distribution $\rho(\mathcal{V})$ in a model that does *not* permit eternal inflation. In that case, the volume of the reheating surface is finite with probability 1, so the distribution $\rho(\mathcal{V})$ is an ordinary probability distribution normalized to unity. In that context, the distribution $\rho(\mathcal{V})$ was introduced in the recent work [192] where the authors considered a family of inflationary models parametrized by a number Ω, such that eternal inflation is impossible in models where $\Omega > 1$. It was then found by a direct calculation that all the moments of the distribution $\rho(\mathcal{V})$ diverge at the value $\Omega = 1$ where the possibility of eternal inflation is first switched on. One can show that the *finitely produced* distribution $\rho(\mathcal{V})$ for $\Omega < 1$ is again well-behaved and has finite moments (see Sec. 8.2.3). This FPRV distribution $\rho(\mathcal{V})$ is the formal foundation of the RV cutoff. It is worth emphasizing that the RV cutoff does not regulate the volume of the reheating surface by modifying the dynamics of a given inflationary model and making eternal inflation impossible. Rather, finite volumes $\mathcal{V}$ are generated by rare chance (*i.e.* within the ensemble $E_{\mathcal{V}}$) through the unmodified dynamics of the model, directly in the regime of eternal inflation.

Below I compute the distribution $\rho(\mathcal{V})$ asymptotically for very large $\mathcal{V}$ in models of slow-roll inflation (Sec. 8.2.3). Specifically, I will compute the distribution $\rho(\mathcal{V};\phi_0)$, where ϕ_0 is the (homogeneous) value of the inflaton field in the initial region S. To implement the RV cutoff explicitly for predicting the distribution of a cosmological parameter Q, it is necessary to consider the joint finitely produced distribution $\rho(\mathcal{V}, \mathcal{V}_{Q_R};\phi_0, Q_0)$ for the reheating volume $\mathcal{V}(R)$ and the portion $\mathcal{V}_{Q_R}$ of the reheating volume in which $Q = Q_R$. (As before, ϕ_0 and Q_0 are the values in the initial H-region.) If the distribution $\rho(\mathcal{V}, \mathcal{V}_{Q_R};\phi_0, Q_0)$ is found, one can determine the mean volume $\langle \mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle$ while the total reheating volume $\mathcal{V}$ is held fixed,

$$\langle \mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle = \frac{\int \rho(\mathcal{V}, \mathcal{V}_{Q_R};\phi_0, Q_0)\mathcal{V}_{Q_R} d\mathcal{V}_{Q_R}}{\rho(\mathcal{V};\phi_0, Q_0)}. \tag{8.3}$$

Then one computes the probability of finding the value of Q within the interval $[Q_R, Q_R + dQ]$ at a random point in the volume $\mathcal{V}$,

$$p(Q = Q_R;\mathcal{V}) \equiv \frac{\langle \mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle}{\mathcal{V}}. \tag{8.4}$$

The RV cutoff *defines* the probability distribution $p(Q)$ for an observable Q as the limit of the distribution $p(Q;\mathcal{V})$ at large $\mathcal{V}$,

$$p(Q) \equiv \lim_{\mathcal{V}\to\infty} \frac{\langle \mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle}{\mathcal{V}}. \tag{8.5}$$

One expects that this limit is independent of the initial values ϕ_0, Q_0 because the large volume $\mathcal{V}$ is generated by regions that spent a very long time in the self-reproduction regime and forgot the initial conditions.

In the previous chapter I derived equations from which the distributions $\rho(\mathcal{V}, \mathcal{V}_{Q_R}; \phi_0, Q_0)$ and $\rho(\mathcal{V}; \phi_0, Q_0)$ can be in principle determined. However, a direct computation of the limit $\mathcal{V} \to \infty$ (for instance, by a numerical method) will be cumbersome since the relevant probabilities are exponentially small in that limit. One of the main results of the present chapter is an analytic evaluation of the limit $\mathcal{V} \to \infty$ and a derivation of a more explicit formula, Eq. (8.33), for the distribution $p(Q)$. The formula shows that the distribution $p(Q)$ can be computed as a ground-state eigenfunction of a certain modified Fokker–Planck equation. The explicit representation also proves that the limit (8.5) exists, is gauge-invariant, and is independent of the initial conditions ϕ_0 and Q_0.

It was argued qualitatively in the previous chapter that the RV measure does not suffer from the youngness paradox. In this chapter I demonstrate the absence of the youngness paradox in the RV measure by an explicit calculation. To this end, I will consider a toy model where every H-region starts in the fluctuation-dominated (or "self-reproduction") regime with a constant expansion rate H_0 and proceeds to reheating via two possible channels. The first channel consists of a short period δt_1 of deterministic slow-roll inflation, yielding N_1 e-foldings until reheating; the second channel has a different period $\delta t_2 \neq \delta t_1$ of deterministic inflation, yielding N_2 e-foldings. (For simplicity, in this model one neglects fluctuations that may return the field from the slow-roll regime to the self-reproduction regime, and thus the time periods δt_1 and δt_2 are sharply defined.) Thus there are two types of reheated regions corresponding to the two possible slow-roll channels. The task is to compute the relative volume-weighted probability $P(2)/P(1)$ of regions of these types within the reheating surface. (Essentially the same model was considered, *e.g.*, in [5, 94, 75, 53]. See Fig. 6.1 for a sketch of the potential $V(\phi)$ in this model.)

This toy model serves as a litmus test of measure prescriptions. The "holographic" or "worldline" prescription yields $P(2)/P(1)$ equal to the probability ratio of exiting through the two channels for a single comoving worldline. This probability ratio depends on the initial conditions. Thus, the worldline measure is (by design) blind to the volume growth during the slow-roll periods. On the other hand, the volume-weighted prescriptions of [94, 53] both yield

$$\frac{P(2)}{P(1)} = \frac{\exp(3N_2)}{\exp(3N_1)}, \tag{8.6}$$

rewarding the reheated H-regions that went through channel j by the additional volume factor $\exp(3N_j)$. This ratio is now independent of the initial conditions. For comparison, an equal-time cutoff gives

$$\frac{P(2)}{P(1)} = \frac{\exp\left[3N_2 - (3H_{\max} - \Gamma_1 - \Gamma_2)\,\delta t_2\right]}{\exp\left[3N_1 - (3H_{\max} - \Gamma_1 - \Gamma_2)\,\delta t_1\right]}. \tag{8.7}$$

The overwhelming exponential dependence on δt_1 and δt_2 manifests the youngness paradox: Even a small difference $\delta t_2 - \delta t_1$ in the duration of the slow-roll inflationary epoch leads to the exponential bias towards the "younger" universes. The bias persists regardless of the choice of the time gauge [75], essentially because the *presence* of δt_1 and δt_2 in the ratio $P(2)/P(1)$ cannot be eliminated by using a different time coordinate.[2] One expects that the RV measure will be free from this bias because the RV prescription does not involve the time coordinate t at all. Below (Sec. 8.2.5) I will show that the ratio $P(2)/P(1)$ computed using the RV cutoff is indeed independent of the slow-roll durations $\delta t_{1,2}$. The RV-regulated result [shown in Eq. (8.34) below] depends only on the gauge-invariant quantities such as N_1 and N_2 and is, in general, different from Eq. (8.6).

8.2 Overview of the results

In this section I describe the central results of this chapter; in particular, I develop simplified mathematical procedures for practical calculations in the RV measure. For convenience of the reader, the results are stated here without proof, while the somewhat lengthy derivations are given in Sec. 8.3.

8.2.1 Preliminaries

I consider a model of slow-roll inflation driven by an inflaton ϕ with the action

$$\int \left[\frac{R}{16\pi G} + \frac{1}{2} \left(\partial_\mu \phi\right)^2 - V(\phi) \right] \sqrt{-g} d^4 x. \tag{8.8}$$

As before, I use the semiclassical stochastic approach to inflation in which the semiclassical dynamics of the field ϕ averaged over an H-region is regarded as a superposition of a deterministic slow roll,

$$\dot{\phi} = v(\phi) \equiv -\frac{V_{,\phi}(\phi)}{3H(\phi)} = -\frac{H_{,\phi}}{4\pi G}, \tag{8.9}$$

and a random walk with root-mean-squared step size

$$\sqrt{\langle \delta\phi \rangle^2} = \frac{H(\phi)}{2\pi} \equiv \sqrt{\frac{2D(\phi)}{H(\phi)}}, \quad D \equiv \frac{H^3}{8\pi^2}, \tag{8.10}$$

during time intervals $\delta t = H^{-1}$, where $H(\phi)$ is the function defined by

$$H(\phi) \equiv \sqrt{\frac{8\pi G}{3} V(\phi)}. \tag{8.11}$$

A useful effective description of the evolution of the field at time scales $\delta t \lesssim H^{-1}$ can be given as

$$\phi(t + \delta t) = \phi(t) + v(\phi)\delta t + \xi(t)\sqrt{2D(\phi)\delta t}, \tag{8.12}$$

[2] It should be noted that the youngness bias becomes very small, possibly even negligible, if one uses the number N of inflationary e-foldings as the time variable rather than the proper time t. I am grateful to A. Linde and A. Vilenkin for bringing this to my attention.

where $\xi(t)$ is a normalized "white noise" function,

$$\langle \xi \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \delta(t - t'), \tag{8.13}$$

which is approximately statistically independent between different H-regions. This stochastic process describes the evolution $\phi(t)$ and the accompanying cosmological expansion of space along a single comoving worldline. For simplicity, we assume that inflation ends in a given horizon-size region when $\phi = \phi_*$, where ϕ_* is a fixed value such that the relative change of H during one Hubble time $\delta t = H^{-1}$ becomes of order 1, *i.e.*

$$\left| \frac{H_{,\phi} v H^{-1}}{H} \right|_{\phi=\phi_*} = \left| \frac{H_{,\phi}^2}{4\pi G H^2} \right|_{\phi=\phi_*} \sim 1. \tag{8.14}$$

From the point of view of the stochastic approach, an inflationary model is fully specified by the kinetic coefficients $D(\phi)$, $v(\phi)$, $H(\phi)$. These coefficients are found from Eqs. (8.9)–(8.11) in models of canonical slow-roll inflation and by suitable analogues in other models.

Dynamics of any fluctuating cosmological parameter Q is described in a similar way. One assumes that the value of Q is homogeneous in H-regions. The evolution of Q is described by an effective Langevin equation,

$$Q(t + \delta t) = Q(t) + v_Q(\phi, Q)\delta t + \xi_Q(t)\sqrt{2D_Q(\phi, Q)\delta t}, \tag{8.15}$$

where the kinetic coefficients D_Q and v_Q can be computed, similarly to D and v, from first principles.

Let us consider an initial H-region S where the inflaton field ϕ as well as the parameter Q are homogeneous and have values $\phi = \phi_0$ and $Q = Q_0$. For convenience we assume that reheating starts when $\phi = \phi_*$ and the Planck energy scales are reached at $\phi = \phi_{\text{Pl}}$ independently of the value of Q. (If necessary, the field variables ϕ, Q can be redefined to achieve this.)

Although eternal inflation to the future of S is almost always the case, it is possible that reheating is reached at a finite time everywhere to the future of S, due to a rare fluctuation. In that event, the total reheating volume $\mathcal{V}$ to the future of S is finite. The (small) probability of that event, denoted by

$$\text{Prob}\,(\mathcal{V} < \infty | \phi_0, Q_0) \equiv \bar{X}(\phi_0, Q_0), \tag{8.16}$$

can be found as the solution of the following nonlinear equation,

$$\frac{D}{H}\bar{X}_{,\phi\phi} + \frac{D_Q}{H}\bar{X}_{,QQ} + \frac{v}{H}\bar{X}_{,\phi} + \frac{v_Q}{H}\bar{X}_{,Q} + 3\bar{X}\ln\bar{X} = 0, \tag{8.17}$$

$$\bar{X}(\phi_{\text{Pl}}, Q) = 1, \quad \bar{X}(\phi_*, Q) = 1, \tag{8.18}$$

where for brevity we dropped the subscript 0 in ϕ_0 and Q_0.

While $\bar{X}(\phi, Q) \equiv 1$ is always a solution of Eq. (8.17), it is not the correct one for the case of eternal inflation. A nontrivial solution, $\bar{X}(\phi, Q) \not\equiv 1$, exists and has small values $\bar{X}(\phi, Q) \ll 1$ for ϕ, Q away from the thermalization boundary. If the coefficients D/H and v/H happen to be Q-independent, the solution of Eq. (8.17) will be also independent of Q, *i.e.* $\bar{X}(\phi, Q) = \bar{X}(\phi)$, and thus determined by a simpler equation obtained from Eq. (8.17) by omitting derivatives with respect to Q,

$$\frac{D}{H}\bar{X}_{,\phi\phi} + \frac{v}{H}\bar{X}_{,\phi} + 3\bar{X}\ln\bar{X} = 0. \tag{8.19}$$

As we have already seen, Eqs. (8.17) and (8.19) are manifestly gauge-invariant.

8.2.2 Finitely produced volume

In a scenario where eternal inflation is possible, we now consider the probability density $\rho(\mathcal{V}; \phi_0)$ of having a *finite* total reheating volume $\mathcal{V}$ to the comoving future of an initial H-region with homogeneous value $\phi = \phi_0$ (focusing attention at first on the case of inflation driven by a single scalar field). The distribution $\rho(\mathcal{V}; \phi_0)$ is normalized to the overall probability $\bar{X}(\phi_0)$ of having a finite total reheating volume,

$$\int_0^\infty \rho(\mathcal{V}; \phi_0) d\mathcal{V} = \bar{X}(\phi_0). \tag{8.20}$$

The distribution $\rho(\mathcal{V}; \phi_0)$ can be calculated by first determining the generating function $g(z; \phi_0)$, which is defined by

$$g(z; \phi_0) \equiv \left\langle e^{-z\mathcal{V}} \right\rangle_{\mathcal{V}<\infty} \equiv \int_0^\infty e^{-z\mathcal{V}} \rho(\mathcal{V}; \phi_0) d\mathcal{V}. \tag{8.21}$$

This generating function is a solution of the nonlinear Fokker–Planck equation,

$$\hat{L}g + 3g\ln g = 0, \tag{8.22}$$

where the differential operator $\hat{L}$ is defined by

$$\hat{L} \equiv \frac{D}{H}\partial_\phi\partial_\phi + \frac{v}{H}\partial_\phi. \tag{8.23}$$

In the case of several fields, say ϕ and Q, one needs to use the corresponding Fokker–Planck operator such as

$$\hat{L} = \frac{D_{\phi\phi}}{H}\partial_\phi\partial_\phi + \frac{D_{QQ}}{H}\partial_Q\partial_Q + \frac{v_\phi}{H}\partial_\phi + \frac{v_Q}{H}\partial_Q. \tag{8.24}$$

The boundary conditions for Eq. (8.22) are

$$g(z; \phi, Q) = 1 \text{ for } \{\phi, Q\} \in \text{Planck boundary}, \tag{8.25}$$

$$g(z; \phi, Q) = e^{-zH^{-3}(\phi,Q)}\Big|_{\{\phi,Q\}\in\text{reheating boundary}}. \tag{8.26}$$

Note that the parameter z enters the boundary conditions but is not explicitly involved in Eq. (8.22). Also, the operator $\hat{L}$ and Eq. (8.22) are manifestly gauge-invariant.

The generating function g plays a central role in the calculations of the RV cut-off. It will be shown below that the solution $g(z;\phi,Q)$ of Eq. (8.22) needs to be obtained only at an appropriately determined *negative* value of z. This solution can be obtained by a numerical method or through an analytic approximation if available.

8.2.3 Asymptotics of $\rho(\mathcal{V};\phi_0)$

The finitely produced distribution $\rho(\mathcal{V};\phi)$ can be found through the inverse Laplace transform of the function $g(z;\phi)$,

$$\rho(\mathcal{V};\phi) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dz\, e^{z\mathcal{V}} g(z;\phi), \tag{8.27}$$

where the integration contour in the complex z plane can be chosen along the imaginary axis. While a full analysis of the function $\rho(\mathcal{V},\phi)$ is complicated (see, however, the recent work [208] where this function was investigated in more detail), we will merely need to derive the asymptotic behavior of $\rho(\mathcal{V};\phi)$ at large $\mathcal{V}$. This asymptotic behavior is determined by the right-most singularity of $g(z;\phi)$ in the complex z plane. It turns out that the function $g(z;\phi)$ always has a singularity at a real, nonpositive $z = z_*$, and that the singularity is of the type

$$g(z;\phi) = g(z_*;\phi) + \sigma(\phi)\sqrt{z - z_*} + O(z - z_*), \tag{8.28}$$

where z_* and $\sigma(\phi)$ are determined as follows. One considers the (z-dependent) linear operator

$$\hat{\hat{L}} \equiv \hat{L} + 3(\ln g(z;\phi) + 1), \tag{8.29}$$

where $\hat{L}$ is the Fokker–Planck operator described above. For $z > 0$ this operator is invertible in the space of functions $f(\phi)$ satisfying zero boundary conditions. The value of z_* turns out to be the algebraically largest real number (in any case, $z_* \leq 0$) such that there exists an eigenfunction $\sigma(\phi)$ of $\hat{\hat{L}}$ with zero eigenvalue and zero boundary conditions,

$$\hat{\hat{L}}\sigma(\phi) = 0, \quad \sigma(\phi_*) = \sigma(\phi_{\mathrm{Pl}}) = 0. \tag{8.30}$$

The specific normalization of the eigenfunction $\sigma(\phi)$ can be derived analytically but is unimportant for the present calculations.

The singularity type shown in Eq. (8.28) determines the leading asymptotic of $\rho(\mathcal{V};\phi)$ at $\mathcal{V} \to \infty$:

$$\rho(\mathcal{V};\phi) \approx \frac{1}{2\sqrt{\pi}} \sigma(\phi) \mathcal{V}^{-3/2} e^{z_* \mathcal{V}}. \tag{8.31}$$

The explicit form (8.31) allows one to investigate the moments of the distribution $\rho(\mathcal{V};\phi)$. It is clear that all the moments are finite as long as $z_* < 0$. However, if $z_* = 0$ all the moments diverge, namely for $n \geq 1$ we have

$$\langle \mathcal{V}^n \rangle = \int_0^\infty \rho(\mathcal{V};\phi)\mathcal{V}^n d\mathcal{V} \propto \int_0^\infty \mathcal{V}^{n-3/2} d\mathcal{V} = \infty. \tag{8.32}$$

The case $z_* = 0$ corresponds to the "transition point" analyzed in [192], corresponding to $\Omega = 1$ in their notation. This is the borderline case between the presence and the absence of eternal inflation. The fact that $z_* = 0$ in the borderline case can be seen directly by noting that the Fokker–Planck operator $\hat{L} + 3$ has in that case a zero eigenvalue, meaning that the 3-volume of equal-time surfaces does not expand with time (reheating of some regions is perfectly compensated by inflationary expansion of other regions). In that case, the only solution $g(z = 0; \phi) = \bar{X}(\phi)$ of Eq. (8.19) is $\bar{X} \equiv 1$ because there are no eternally inflating comoving geodesics. Hence $\ln g(z = 0, \phi) = 0$, and so the operator $\hat{\hat{L}}$ is simply $\hat{\hat{L}} = \hat{L} + 3$. It follows that the operator $\hat{\hat{L}}$ also has a zero eigenvalue at $z = 0$, and thus $z = z_* = 0$ is the dominant singularity of $g(z; \phi)$. This argument reproduces and generalizes the results obtained in [192] where direct calculations of various moments of $\rho(\mathcal{V}; \phi)$ were performed for the case of the absence of eternal inflation.

We note that the only necessary ingredients in the computation of $\sigma(\phi)$ is the knowledge of the singularity point z_* and the corresponding function $g(z_*; \phi)$, which is a solution of the nonlinear reaction-diffusion equation (8.22). Determining z_* and $g(z_*; \phi)$ in a given inflationary model does not require extensive numerical simulations.

8.2.4 Distribution of a fluctuating field

Above we denoted by Q a cosmological parameter that fluctuates during inflation but is in principle observable after reheating. One of the main questions to be answered using a multiverse measure is to derive the probability distribution $p(Q)$ for the values of Q observed in a "typical" place in the multiverse. I will now present a formula for the distribution $p(Q)$ in the RV cutoff. This formula is significantly more explicit and lends itself more easily to practical calculations than the expressions first shown in [246].

As in the previous section, we assume that the dynamics of the inflaton field ϕ and the parameter Q is described by a suitable Fokker–Planck operator $\hat{L}$, *e.g.* of the form (8.24), and that reheating occurs at $\phi = \phi_*$ independently of the value of Q. We then consider Eq. (8.22) for the function $g(z; \phi, Q)$ and the operator $\hat{\hat{L}} \equiv \hat{L} + 3(\ln g + 1)$; we need to determine the value z_* at which $g(z; \phi, Q)$ has a singularity. The operator $\hat{\hat{L}}$ has an eigenfunction with zero eigenvalue for this value of z. This eigenfunction $f_0(z_*; \phi, Q)$ needs to be determined with zero boundary conditions (at reheating and Planck boundaries). Then the RV-regulated distribution of Q at reheating is

$$p(Q_R) = \text{const} \left[\frac{\partial f_0(z_*; \phi, Q)}{\partial \phi} \frac{D_{\phi\phi} e^{-z_* H^{-3}}}{H^4} \right]_{\phi = \phi_*, Q = Q_R}, \tag{8.33}$$

where the normalization constant needs to be chosen such that $\int p(Q_R) dQ_R = 1$. The derivation of this result occupies Sec. 8.3.3.

We note that f_0 is the eigenfunction f_0 of a gauge-invariant operator, and that the result in Eq. (8.33) depends on the kinetic coefficients only through the gauge-independent ratio D/H times the volume factor H^{-3}. The distribution $p(Q_R)$ is independent of the initial conditions, which is due to a specific asymptotic behavior of the finitely produced volume distributions, as shown in Sec. 8.3.3.

8.2.5 Toy model of inflation

We now apply the RV cutoff to the toy model described at the end of Sec. 8.1. We consider a model of inflation driven by a scalar field with a potential shown in Fig. 6.1. For the purposes of the present argument, we may assume that there is exactly zero "diffusion" in the deterministic regimes $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$, while the range $\phi_1 < \phi < \phi_2$ is sufficiently wide to allow for eternal self-reproduction. Thus there are two slow-roll channels that produce respectively N_1 and N_2 e-foldings of slow-roll inflation after exiting the self-reproduction regime. Since the self-reproduction range generates arbitrarily large volumes of space that enter both the slow-roll channels, the total reheating volume going through each channel is infinite. We apply the RV cutoff to the problem of computing the regularized ratio of the reheating volumes in regions of types 1 and 2.

In this toy model it is possible to obtain the results of the RV cutoff using analytic approximations. The required calculations are somewhat lengthy and can be found in Sec. 8.3.4. The result for a generic case where one of the slow-roll channels has many more e-foldings than the other (say, $N_2 \gg N_1$) can be written as

$$\frac{P(2)}{P(1)} \approx O(1) \frac{H^{-3}(\phi_*^{(2)})}{H^{-3}(\phi_*^{(1)})} \frac{\exp\left[3N_2\right]}{\exp\left[3N_1\right]} \exp\left[3N_{12}\right], \tag{8.34}$$

where we have defined

$$N_{12} \equiv \frac{\pi^2}{\sqrt{2}H_0^2}\left(\phi_2 - \phi_1\right)^2, \quad H_0^2 \equiv \frac{8\pi G}{3}V_0. \tag{8.35}$$

The pre-exponential factor $O(1)$ can be computed numerically, as outlined in Sec. 8.3.4.

We note that the ratio (8.34) is gauge-invariant and does not involve any spacetime coordinates. This result can be interpreted as the ratio of volumes e^{3N_1} and e^{3N_2} gained during the slow-roll regime in the two channels multiplied by a correction factor $e^{3N_{12}}$. The dimensionless number N_{12} can be suggestively interpreted (up to the factor $\sqrt{2}$) as the mean number of "steps" of size $\delta\phi \sim \frac{1}{2\pi}H_0$ required for a random walk to reach the boundary of the flat region $[\phi_1, \phi_2]$ starting from the middle point $\phi_0 \equiv \frac{1}{2}\left(\phi_1 + \phi_2\right)$. Since each of the "steps" of the random walk takes a Hubble time H_0^{-1} and corresponds to one e-folding of inflation, the volume factor gained during such a traversal will be $e^{3N_{12}}$. Note that the correction factor increases the probability of channel 2 that was *already* the dominant one due to the larger volume factor $e^{3N_2} \gg e^{3N_1}$. Depending on the model, this factor may be a significant modification of the ratio (8.6) obtained in previously used volume-based measures.

8.3 Derivations

8.3.1 Derivation of the equation for g

In this section I derive Eq. (8.22), closely following the derivation of the nonlinear diffusion equation shown in Sec. 4.2.1.

We begin by considering the case when inflation is driven by a single scalar field ϕ, such that reheating is reached at $\phi = \phi_*$. Let $\rho(\mathcal{V};\phi_0)$ be the probability density of obtaining the finite reheated volume $\mathcal{V}$. We will derive an equation for a generating function of the distribution of volume, rather than an equation directly for $\rho(\mathcal{V};\phi_0)$. Since the volume $\mathcal{V}$ is by definition nonnegative, it is convenient to define a generating function $g(z;\phi_0)$ through the expectation value of the expression $\exp(-z\mathcal{V})$, where $z > 0$ is the formal parameter of the generating function,

$$g(z;\phi_0) \equiv \left\langle e^{-z\mathcal{V}} \right\rangle_{\mathcal{V}<\infty} \equiv \int_0^\infty e^{-z\mathcal{V}} \rho(\mathcal{V};\phi_0) d\mathcal{V}. \tag{8.36}$$

Note that for any z such that $\operatorname{Re} z \geq 0$ the integral in Eq. (8.36) converges, and the events with $\mathcal{V} = +\infty$ are automatically excluded from consideration. However, we use the subscript "$\mathcal{V} < \infty$" to indicate explicitly that the statistical average is performed over the subensemble of all events with finite $\mathcal{V}$. The distribution $\rho(\mathcal{V};\phi_0)$ is not normalized to unity; instead, the normalization is given by Eq. (8.2).

The parameter z has the dimension of inverse 3-volume. Physically, this is the 3-volume measured along the reheating surface and hence is defined in a gauge-invariant manner. If desired for technical reasons, the variable z can be made dimensionless by a constant rescaling.

The generating function $g(z;\phi)$ has the following multiplicative property: For two statistically independent regions that have initial values $\phi = \phi_1$ and $\phi = \phi_2$ respectively, the sum of the (finitely produced) reheating volumes $\mathcal{V}_1 + \mathcal{V}_2$ is distributed with the generating function

$$\left\langle e^{-z(\mathcal{V}_1+\mathcal{V}_2)} \right\rangle = \left\langle e^{-z\mathcal{V}_1} \right\rangle \left\langle e^{-z\mathcal{V}_2} \right\rangle = g(z;\phi_1)g(z;\phi_2). \tag{8.37}$$

We now consider an H-region at some time t, having an arbitrary value $\phi(t)$ not yet in the reheating regime. Suppose that the finitely produced volume distribution for this H-region has the generating function $g(z;\phi)$. After time δt the initial H-region grows to $N \equiv e^{3H\delta t}$ statistically independent, "daughter" H-regions. The value of ϕ in the k-th daughter region ($k = 1, ..., N$) is found from Eq. (8.12),

$$\phi_k = \phi + v(\phi)\delta t + \xi_k \sqrt{2D(\phi)\delta t}, \tag{8.38}$$

where the "noise" variables ξ_k ($k = 1, ..., N$) are statistically independent because they describe the fluctuations of ϕ in causally disconnected H-regions. The finitely produced volume distribution for the k-th daughter region has the generating function $g(z; \phi_k)$. The combined reheating volume of the N daughter regions must be distributed with the same generating function as reheating volume of the original H-region. Hence, by the multiplicative property we obtain

$$g(z; \phi) = \prod_{k=1}^{N} g(z; \phi_k). \tag{8.39}$$

We can average both sides of this equation over the noise variables ξ_k to get

$$g(z; \phi) = \left\langle \prod_{k=1}^{N} g(z; \phi_k) \right\rangle_{\xi_1, ..., \xi_N}. \tag{8.40}$$

Since all the ξ_k are independent, the average splits into a product of N identical factors,

$$g(z; \phi) = \left[\left\langle g(z; \phi + v(\phi)\delta t + \sqrt{2D(\phi)}\xi) \right\rangle_{\xi} \right]^{N}. \tag{8.41}$$

The derivation now proceeds as in [188]. We first compute, to first order in δt,

$$\left\langle g(z; \phi + v(\phi)\delta t + \sqrt{2D(\phi)}\xi) \right\rangle_{\xi} = g + \left(v g_{,\phi} + D g_{,\phi\phi}\right) \delta t. \tag{8.42}$$

Substituting $N = e^{3H\delta t}$ and taking the logarithmic derivative of both sides of Eq. (8.41) with respect to δt at $\delta t = 0$, we then obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial \delta t} \ln g(z; \phi) \\ &= 3H \ln g + \frac{v g_{,\phi} + D g_{,\phi\phi}}{g}. \end{aligned} \tag{8.43}$$

The equation for $g(z; \phi)$ follows,

$$D g_{,\phi\phi} + v g_{,\phi} + 3H g \ln g = 0. \tag{8.44}$$

This is formally the same as Eq. (8.17). However, the boundary conditions for Eq. (8.44) are different. The condition at the end-of-inflation boundary $\phi = \phi_*$ is

$$g(z; \phi_*) = e^{-zH^{-3}(\phi_*)} \tag{8.45}$$

because an H-region starting with $\phi = \phi_*$ immediately reheats and produces the reheating volume $H^{-3}(\phi_*)$. The condition at Planck boundary ϕ_{Pl} (if present), or other boundary where the effective field theory breaks down, is "absorbing," *i.e.* regions that reach $\phi = \phi_{\mathrm{Pl}}$ do not generate any reheating volume:

$$g(z; \phi_{\mathrm{Pl}}) = 1. \tag{8.46}$$

The variable z enters Eq. (8.44) as a parameter and only through the boundary conditions. At $z = 0$ the solution is $g(0; \phi) = \bar{X}(\phi)$.

A fully analogous derivation can be given for the generating function $g(z; \phi_0, Q_0)$ in the case when additional fluctuating fields, denoted by Q, are present. The generating function $g(z; \phi_0, Q_0)$ is defined by

$$g(z; \phi_0, Q_0) = \int_0^\infty e^{-z\mathcal{V}} \rho(\mathcal{V}; \phi_0, Q_0) d\mathcal{V}, \tag{8.47}$$

where $\rho(\mathcal{V}; \phi_0, Q_0)$ is the probability density for achieving a total reheating volume $\mathcal{V}$ in the future of an H-region with initial values ϕ_0, Q_0 of the fields. In the general case, the fluctuations of the fields ϕ, Q can be described by the Langevin equations

$$\phi(t + \delta t) = \phi(t) + v_\phi \delta t + \xi_\phi \sqrt{2D_{\phi\phi}\delta t} + \xi_Q \sqrt{2D_{\phi Q}\delta t}, \tag{8.48}$$

$$Q(t + \delta t) = Q(t) + v_Q \delta t + \xi_\phi \sqrt{2D_{\phi Q}\delta t} + \xi_Q \sqrt{2D_{QQ}\delta t}, \tag{8.49}$$

where the "diffusion" coefficients $D_{\phi\phi}$, $D_{\phi Q}$, and D_{QQ} have been introduced, as well as the "slow roll" velocities v_ϕ and v_Q and the "noise" variables ξ_ϕ and ξ_Q. The resulting equation for $g(z; \phi_0, Q_0)$ is (dropping the subscript 0)

$$\hat{L}g + 3g \ln g = 0, \tag{8.50}$$

where the differential operator $\hat{L}$ is defined by

$$\hat{L} \equiv \frac{D_{\phi\phi}}{H}\partial_\phi\partial_\phi + \frac{2D_{\phi Q}}{H}\partial_\phi\partial_Q + \frac{D_{QQ}}{H}\partial_Q\partial_Q + \frac{v_\phi}{H}\partial_\phi + \frac{v_Q}{H}\partial_Q. \tag{8.51}$$

The ratios $D_{\phi\phi}/H$, etc., are manifestly gauge-invariant.

Performing a redefinition of the fields if needed, one may assume that reheating is reached when $\phi = \phi_*$ independently of the value of Q. Then the boundary conditions for Eq. (8.50) at the reheating boundary can be written as

$$g(z; \phi_*, Q) = e^{-zH^{-3}(\phi_*, Q)}. \tag{8.52}$$

The Planck boundary still has the boundary condition $g(z; \phi_{\mathrm{Pl}}) = 1$.

8.3.2 Singularities of $g(z)$

For simplicity we now focus attention on the case of single-field inflation; the generating function $g(z; \phi)$ then depends on the initial value of the inflaton field ϕ. The corresponding analysis for multiple fields is carried out as a straightforward generalization.

The main nonlinear equation in the calculations of the RV cutoff is Eq. (8.22) for the generating function $g(z;\phi)$. That equation differs from Eq. (8.19) mainly by the presence of the parameter z in the boundary conditions. Therefore, we may expect that solutions of Eq. (8.22) may exist for some values of z but not for other values. Note that $g(z=0;\phi) = \bar{X}(\phi)$; hence, nontrivial solutions $g(z;\phi)$ exist for $z = 0$ under the same conditions as nontrivial solutions $\bar{X}(\phi) \not\equiv 1$ of Eq. (8.19). While it is certain that solutions $g(z;\phi)$ exist for $z \geq 0$, there may be values $z < 0$ for which no real-valued solutions $g(z;\phi)$ exist at all. However, the calculations in the RV cutoff require only to compute $g(z_*;\phi)$ for a certain value $z_* < 0$, which is the algebraically largest value z where $g(z;\phi)$ has a singularity in the z plane. The structure of that singularity will be investigated in detail below, and it will be shown that $g(z_*;\phi)$ is finite while $\partial g/\partial z \propto (z - z_*)^{-1/2}$ diverges at $z = z_*$. Hence, the solution $g(z;\phi)$ remains well-defined at least for all real z in the interval $[z_*, +\infty]$. It follows that $g(z;\phi)$ may be obtained *e.g.* by a numerical solution of a well-conditioned problem with $z = z_* + \varepsilon$, where $\varepsilon > 0$ is a small real constant.

By definition, $g(z;\phi)$ is an integral of a probability distribution $\rho(\mathcal{V};\phi)$ times $e^{-z\mathcal{V}}$. It follows that $g(z;\phi)$ is analytic in z and has no singularities for $\operatorname{Re} z > 0$. Then the probability distribution $\rho(\mathcal{V};\phi)$ can be recovered from the generating function $g(z;\phi)$ through the inverse Laplace transform,

$$\rho(\mathcal{V};\phi) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} dz\, e^{z\mathcal{V}} g(z;\phi), \tag{8.53}$$

where the integration contour in the complex z plane can be chosen along the imaginary axis because all the singularities of $g(z;\phi)$ are to the left of that axis. The RV cutoff procedure depends on the limit of $\rho(\mathcal{V};\phi)$ and related distributions at $\mathcal{V} \to \infty$. The asymptotic behavior at $\mathcal{V} \to \infty$ is determined by the type and the location of the right-most singularity of $g(z;\phi)$ in the half-plane $\operatorname{Re} z < 0$. For instance, if $z = z_*$ is such a singularity, the asymptotic is $\rho(\mathcal{V};\phi) \propto \exp[-z_*\mathcal{V}]$. The prefactor in this expression needs to be determined; for this, a detailed analysis of the singularities of $g(z;\phi)$ will be carried out.

It is important to verify that the singularities of $g(z;\phi)$ are ϕ-independent. We first show that solutions of Eq. (8.50) cannot diverge at finite values of ϕ. If that were the case and say $g(z;\phi) \to \infty$ as $\phi \to \phi_1$, the function $\ln\ln g$ as well as derivatives $g_{,\phi}$ and $g_{,\phi\phi}$ would diverge as well. Then

$$\lim_{\phi\to\phi_1} \partial_\phi \left[\ln\ln g\right] = \lim_{\phi\to\phi_1} \frac{\partial_\phi g}{g \ln g} = \infty. \tag{8.54}$$

It follows that the term $g \ln g$ is negligible near $\phi = \phi_1$ in Eq. (8.50) compared with the term $\partial_\phi g$ and hence also with the term $\partial_\phi \partial_\phi g$. In a very small neighborhood of $\phi = \phi_1$, the operator $\hat{L}$ can be approximated by a linear operator $\hat{L}_1$ with constant coefficients, such as

$$\hat{L} \approx \hat{L}_1 \equiv A_1 \partial_\phi \partial_\phi + B_1 \partial_\phi. \tag{8.55}$$

Since at least one of the coefficients A_1, B_1 is nonzero at $\phi = \phi_1$, it follows that $g(z;\phi)$ is approximately a solution of the linear equation $\hat{L}_1 g = 0$ near $\phi = \phi_1$. However, solutions of linear equations cannot diverge at finite values of the argument. Hence, the function $g(z;\phi)$ cannot diverge at a finite value of ϕ.

The only remaining possibility is that the function $g(z;\phi)$ has singular points $z = z_*$ such that $g(z_*;\phi)$ remains finite while $\partial g/\partial z$, or a higher-order derivative, diverges at $z = z_*$. We will now investigate such divergences and show that $g(z;\phi)$ has a leading singularity of the form

$$g(z;\phi) = g(z_*;\phi) + \sigma(\phi)\sqrt{z - z_*} + O(z - z_*), \tag{8.56}$$

where z_* is a ϕ-independent location of the singularity such that $z_* \leq 0$, while the function $\sigma(\phi)$ is yet to be determined.

Denoting temporarily $g_1(z;\phi) \equiv \partial g/\partial z$, we find a *linear* equation for g_1,

$$\hat{L} g_1 + 3\left(\ln g + 1\right) g_1 = 0, \tag{8.57}$$

with inhomogeneous boundary conditions

$$g_1(\phi_*) = -H^{-3}(\phi_*) e^{-zH^{-3}(\phi_*)}, \quad g_1(\phi_{\text{Pl}}) = 0. \tag{8.58}$$

The solution $g_1(z;\phi)$ of this linear problem can be found using a standard method involving the Green's function. The problem with inhomogeneous boundary conditions is equivalent to the problem with zero boundary conditions but with an inhomogeneous equation. To be definite, let us consider the operator $\hat{L}$ of the form used in Eq. (8.44),

$$\hat{L} = \frac{D(\phi)}{H(\phi)} \partial_\phi \partial_\phi + \frac{v(\phi)}{H(\phi)} \partial_\phi. \tag{8.59}$$

Then Eqs. (8.57)–(8.58) are equivalent to the inhomogeneous problem with zero boundary conditions,

$$\hat{L} g_1 + 3\left(\ln g + 1\right) g_1 = DH^{-4} e^{-zH^{-3}} \delta'(\phi - \phi_*), \tag{8.60}$$

$$g_1(z;\phi_*) = g_1(z;\phi_{\text{Pl}}) = 0. \tag{8.61}$$

The solution of this inhomogeneous equation exists as long as the linear operator $\hat{L} + 3(\ln g + 1)$ does not have a zero eigenfunction with zero boundary conditions.

Note that the operator $\hat{L} + 3(\ln g + 1)$ is explicitly z-dependent through the coefficient $g(z;\phi)$. Note also that $g(z;\phi) \neq 0$ by definition (8.36) for values of z such that the integral in Eq. (8.36) converges; hence $\ln g$ is finite for those z. Let us denote by $G(z;\phi,\phi')$ the Green's function of that operator with zero boundary conditions,

$$\hat{L} G + 3\left(\ln g(z,\phi) + 1\right) G = \delta(\phi - \phi'), \tag{8.62}$$

$$G(z;\phi_*,\phi') = G(z;\phi_{\text{Pl}},\phi') = 0. \tag{8.63}$$

This Green's function is well-defined for values of z such that $\hat{L} + 3(\ln g(z;\phi) + 1)$ is invertible. For these z we may express the solution $g_1(z;\phi)$ of Eqs. (8.57)–(8.58) explicitly through the Green's function as

$$g_1(z;\phi) = -\left.\frac{D}{H^4} e^{-zH^{-3}}\right|_{\phi_*} \left.\frac{\partial G(z;\phi,\phi')}{\partial\phi'}\right|_{\phi'=\phi_*}. \tag{8.64}$$

Hence, for these z the function $g_1(z;\phi) \equiv \partial g/\partial z$ remains finite at every value of ϕ. A similar argument shows that all higher-order derivatives $\partial^n g/\partial z^n$ remain finite at every ϕ for these z. Therefore, the singularities of $g(z;\phi)$ can occur only at certain ϕ-independent points $z = z_*$, $z = z'_*$, etc.

Since the generating function $g(z;\phi)$ is nonsingular for all complex z with $\mathrm{Re}\, z > 0$, it is assured that $g_1(z;\phi)$ and $G(z;\phi,\phi')$ exist for such z. However, there will be values of z for which the operator $\hat{L} + 3(\ln g + 1)$ has a zero eigenfunction with zero boundary conditions, so the Green's function G is undefined. Denote by z_* such a value with the algebraically largest real part; we already know that $\mathrm{Re}\, z_* \leq 0$ in any case. Let us now show that the function $g_1(z;\phi)$ actually diverges when $z \to z_*$. In other words, $\lim_{z\to z_*} g_1(z;\phi) = \infty$ for every value of ϕ.

To show this, we need to use the decomposition of the Green's function in the eigenfunctions of the operator $\hat{L} + 3(\ln g + 1)$,

$$G(z;\phi,\phi') = \sum_{n=0}^{\infty} \frac{1}{\lambda_n(z)} f_n(\phi) f_n^*(\phi'), \tag{8.65}$$

where $f_n(z;\phi)$ are the (appropriately normalized) eigenfunctions with eigenvalues $\lambda_n(z)$ and zero boundary conditions,

$$\left[\hat{L} + 3(\ln g(z;\phi) + 1)\right] f_n(z;\phi) = \lambda_n(z) f_n(z;\phi), \tag{8.66}$$

$$f_n(z;\phi) = 0 \text{ for } \phi = \phi_*,\ \phi = \phi_{\mathrm{Pl}}. \tag{8.67}$$

The decomposition (8.65) is possible as long as the operator $\hat{L}$ is self-adjoint with an appropriate choice of the scalar product in the space of functions $f(\phi)$. The scalar product can be chosen in the following way,

$$\langle f_1, f_2 \rangle = \int f_1(\phi) f_2^*(\phi) M(\phi) d\phi, \tag{8.68}$$

where $M(\phi)$ is a weighting function. One can attempt to determine $M(\phi)$ such that the operator $\hat{L}$ is self-adjoint,

$$\langle f_1, \hat{L} f_2 \rangle = \langle \hat{L} f_1, f_2 \rangle. \tag{8.69}$$

In single-field models of inflation where the operator $\hat{L}$ has the form (8.59), it is always possible to choose $M(\phi)$ appropriately [179]. However, in multi-field models this is not necessarily possible.[3] One can show that in standard slow-roll models with K fields $\phi_1, ..., \phi_K$ and kinetic coefficients

$$D_{ij} = \frac{H^3}{8\pi^2}\delta_{ij}, \quad v_i = -\frac{1}{4\pi G}\frac{\partial H}{\partial \phi_i}, \quad H = H(\phi_1, ..., \phi_K), \tag{8.70}$$

there exists a suitable choice of $M(\phi)$, namely

$$M(\phi_1, ..., \phi_K) = \frac{\pi G}{H^2} \exp\left[\frac{\pi G}{H^2}\right], \tag{8.71}$$

such that the operator

$$\hat{L} = H^{-1}\sum_{i,j} D_{ij}\frac{\partial^2}{\partial\phi_i\partial\phi_j} + H^{-1}\sum_i v_i\frac{\partial}{\partial\phi_i} \tag{8.72}$$

is self-adjoint in the space of functions $f(\phi)$ with zero boundary conditions and the scalar product (8.68). However, the operator $\hat{L}$ may be non-self-adjoint in more general inflationary models where the kinetic coefficients are given by different expressions. We omit the formulation of precise conditions for self-adjointness of $\hat{L}$ because this property is not central to the present investigation. In non-self-adjoint cases a decomposition similar to Eq. (8.65) needs to be performed using the left and the right eigenfunctions of the non-self-adjoint operator $\hat{L} + 3(\ln g + 1)$. One expects that such a decomposition will still be possible because (heuristically) the nondiagonalizable operators are a set of measure zero among all operators. The requisite left and right eigenfunctions can be obtained numerically. We leave the detailed investigation of those cases for future work. Presently, let us focus on the case when the decomposition of the form (8.65) holds, with appropriately chosen scalar product and the normalized eigenfunctions

$$\langle f_m, f_n\rangle = \delta_{mn}. \tag{8.73}$$

The eigenfunctions $f_m(z;\phi)$ can be obtained *e.g.* numerically by solving the boundary value problem (8.66)–(8.67).

In the limit $z \to z_*$, one of the eigenvalues λ_n approaches zero. Since linear operators such as $\hat{L}$ always have a spectrum bounded from above [179], we may renumber the eigenvalues λ_n such that λ_0 is the largest one. Then we define z_* as the value with the (algebraically) largest real part, such that $\lambda_0(z_*) = 0$. By construction, for all z with $\operatorname{Re} z > \operatorname{Re} z_*$ all the eigenvalues λ_n are negative. Note that the (algebraically) largest eigenvalue $\lambda_0(z)$ is always nondegenerate, and the corresponding eigenfunction $f_0(z;\phi)$ can be chosen real and positive for all ϕ, except at the boundaries $\phi = \phi_*$ and $\phi = \phi_{\mathrm{Pl}}$ where f satisfies the zero boundary conditions.

For z near z_*, only the nondegenerate eigenvalue λ_0 will be near zero, so the decomposition (8.65) of the Green's function will be dominated by the term

[3] I am grateful to D. Podolsky for pointing this out to me. The hermiticity of operators of diffusion type in the context of eternal inflation was briefly discussed in [264].

$1/\lambda_0$. Hence, we can use Eqs. (8.64) and (8.65) to determine the function $g_1(z;\phi)$ approximately as

$$g_1(z;\phi) \approx -\frac{f_0(z;\phi)}{\lambda_0(z)} \frac{\partial f_0(z;\phi_*)}{\partial \phi} \left[\frac{D}{H^4} e^{-zH^{-3}} \right]_{\phi_*}. \tag{8.74}$$

It follows that indeed $g_1(z;\phi) \to \infty$ as $z \to z_*$ because $\lambda_0(z) \to 0$.

This detailed investigation allows us now to determine the behavior of $g(z;\phi)$ at the leading singularity $z = z_*$. We will consider the function $g(z;\phi)$ for z near z_* and show that the singularity indeed has the structure (8.56).

We have already shown that the function g itself does not diverge at $z = z_*$ but its derivative $g_1 \equiv \partial g/\partial z$ does. Hence, the function $g(z_*;\phi)$ is continuous, and the difference $g(z_*;\phi) - g(z;\phi)$ is small for $z \approx z_*$, so that we have the expansion

$$\delta g(z;\phi) \equiv g(z;\phi) - g(z_*;\phi) \tag{8.75}$$

$$\approx g_1(z;\phi)\,(z - z_*) + O\left[(z_* - z)^2 \right]. \tag{8.76}$$

(Note that we are using the *finite* value $g_1(z;\phi)$ rather than the divergent value $g_1(z_*;\phi)$ in the above equation.) On the other hand, we have the explicit representation (8.74). Let us examine the values of $\lambda_0(z)$ for $z \approx z_*$. At $z = z_*$ we have $\lambda_0(z_*) = 0$, so the (small) value $\lambda_0(z)$ for $z \approx z_*$ can be found using standard perturbation theory for linear operators. If we denote the change in the operator $\hat{L}$ by

$$\delta\hat{L} \equiv 3(\ln g(z;\phi) - \ln g(z_*;\phi)) \approx \frac{3\delta g(z;\phi)}{g(z_*;\phi)}, \tag{8.77}$$

we can write, to first order,

$$\lambda_0(z) \approx \langle f_0, \delta\hat{L} f_0 \rangle = \int |f_0(\phi)|^2 \frac{3\delta g(z;\phi)}{g(z_*;\phi)} d\phi. \tag{8.78}$$

Now, Eqs. (8.74) and (8.76) yield

$$\frac{\delta g(z;\phi)}{z - z_*} \approx -\frac{f_0(z;\phi)}{\lambda_0(z)} \left[\frac{\partial f_0}{\partial \phi} \frac{D}{H^4} e^{-zH^{-3}} \right]_{\phi_*}. \tag{8.79}$$

Integrating the above equation in ϕ with the prefactor

$$|f_0(\phi)|^2 \frac{3}{g(z_*;\phi)} d\phi \tag{8.80}$$

and using Eq. (8.78), we obtain a closed equation for $\lambda_0(z)$ in which terms of order $(z - z_*)^2$ have been omitted,

$$\frac{\lambda_0(z)}{z - z_*} \approx -\frac{1}{\lambda_0(z)} \left[\frac{\partial f_0}{\partial \phi} \frac{D}{H^4} e^{-zH^{-3}} \right]_{\phi_*} \times \int |f_0|^2 \frac{3\delta g(z;\phi)}{g(z_*;\phi)} f_0(\phi) d\phi. \tag{8.81}$$

It follows that $\lambda_0(z) \propto \sqrt{z - z_*}$ and $g_1(z;\phi) \propto (z - z_*)^{-1/2}$, confirming the leading asymptotic of the form (8.56).

Let us also obtain a more explicit form of the singularity structure of $g(z;\phi)$. We can rewrite Eq. (8.81) as

$$\lambda_0(z) \approx \sigma_0 \sqrt{z - z_*} + O(z - z_*), \tag{8.82}$$

where σ_0 is a constant that may be obtained explicitly. Then Eq. (8.74) yields

$$g_1(z;\phi) \approx \frac{f_0(z;\phi)}{\sqrt{z - z_*}} \sigma_1, \tag{8.83}$$

with a different constant σ_1. Finally, we can integrate this in z and obtain

$$g(z;\phi) = g(z_*;\phi) + 2\sigma_1 f_0(z;\phi)\sqrt{z - z_*} + O(z - z_*). \tag{8.84}$$

We may rewrite this by substituting $z = z_*$ into $f_0(z;\phi)$,

$$g(z;\phi) = g(z_*;\phi) + \sigma(\phi)\sqrt{z - z_*} + O(z - z_*), \tag{8.85}$$

$$\sigma(\phi) \equiv 2\sigma_1 f_0(z_*;\phi). \tag{8.86}$$

The result is now explicitly of the form (8.56). It will turn out that the normalization constant $2\sigma_1$ cancels in the final results. So in a practical calculation the eigenfunction $f_0(z_*;\phi)$ may be determined with an arbitrary normalization.

As a side note, let us remark that the argument given above will apply also to other singular points $z'_* \neq z_*$ as long as the eigenvalue $\lambda_k(z)$ of the operator $\hat{L} + 3(\ln g + 1)$ is nondegenerate when it vanishes at $z = z'_*$. If the relevant eigenvalue becomes degenerate, the singularity structure will not be of the form $\sqrt{z - z'_*}$ but rather $(z - z'_*)^s$ with some other power $0 < s < 1$.

Now we are ready to obtain the asymptotic form of the distribution $\rho(\mathcal{V};\phi)$ for $\mathcal{V} \to \infty$. We deform the integration contour in the inverse Laplace transform (8.53) such that it passes near the real axis around $z = z_*$. Then we use Eq. (8.85) for $g(z;\phi)$ and obtain the leading asymptotic

$$\begin{aligned} \rho(\mathcal{V};\phi) &\approx \frac{1}{2\pi \mathrm{i}} \sigma(\phi) \left[\int_{-\infty}^{z_*} - \int_{z_*}^{-\infty} \right] \sqrt{z - z_*}\, e^{z\mathcal{V}} dz \\ &= \frac{1}{2\sqrt{\pi}} \sigma(\phi) \mathcal{V}^{-3/2} e^{z_* \mathcal{V}}. \end{aligned} \tag{8.87}$$

The subdominant terms come from the higher-order terms in the expansion in Eq. (8.85) and are of the order $\mathcal{V}^{-1}$ times the leading term shown in Eq. (8.87).

Finally, we show that z_* must be real-valued and that there are no other singularities z'_* with $\operatorname{Re} z'_* = \operatorname{Re} z_*$. This is so because the integral

$$g_1(z_*;\phi) = -\int_0^\infty \rho(\mathcal{V};\phi) e^{-z_* \mathcal{V}} \mathcal{V} d\mathcal{V} = \infty \tag{8.88}$$

will definitely diverge for purely real z_* if it diverges for a nonreal value $z'_* = z_* + \mathrm{i}A$. If, on the other hand, the integral (8.88) diverges for a real z_*, it will *converge* for any nonreal $z'_* = z_* + \mathrm{i}A$ with real $A \neq 0$ because the function $\rho(\mathcal{V};\phi)$ has the large-$\mathcal{V}$ asymptotic of the form (8.87) and the oscillations of $\exp(\mathrm{i}A\mathcal{V})$ will make the integral (8.88) convergent.

8.3.3 FPRV distribution of a field Q

Consider a fluctuating field Q such that the Fokker–Planck operator $\hat{L}$ is of the form (8.51). We are interested in the portion $\mathcal{V}_{Q_R}$ of the total reheated volume $\mathcal{V}$ where the field Q has a value within a given interval $[Q_R, Q_R + dQ]$. We denote by $\rho(\mathcal{V}, \mathcal{V}_{Q_R}; \phi_0, Q_0)$ the joint probability distribution of the volumes $\mathcal{V}$ and $\mathcal{V}_{Q_R}$ for initial H-regions with initial values $\phi = \phi_0$ and $Q = Q_0$. The generating function $\tilde{g}(z, q; \phi, Q)$ corresponding to that distribution is defined by

$$\tilde{g}(z, q; \phi, Q) \equiv \int e^{-z\mathcal{V} - q\mathcal{V}_Q} \rho(\mathcal{V}, \mathcal{V}_Q; \phi, Q) d\mathcal{V} d\mathcal{V}_Q. \tag{8.89}$$

Since this generating function satisfies the same multiplicative property (8.37) as the generating function $g(z; \phi, Q)$, we may repeat the derivation of Eq. (8.50) without modifications for the function $\tilde{g}(z, q; \phi, Q)$. Hence, $g(z, q; \phi, Q)$ is the solution of the same equation as $g(z; \phi, Q)$. The only difference is the boundary conditions at reheating, which are given not by Eq. (8.52) but by

$$\tilde{g}(z, q; \phi_*, Q) = \exp\left[- \left(z + q\delta_{QQ_R} \right) H^{-3}(\phi_*, Q) \right], \tag{8.90}$$

where (with a slight abuse of notation) δ_{QQ_R} is the indicator function of the interval $[Q_R, Q_R + dQ]$, *i.e.*

$$\delta_{QQ_R} \equiv \theta(Q - Q_R)\theta(Q_R + dQ - Q). \tag{8.91}$$

We employ this "finite" version of the δ-function only because we cannot use a standard Dirac δ-function under the exponential. This slight technical inconvenience will disappear shortly.

The solution for the function $\tilde{g}(z, q; \phi, Q)$ may be obtained in principle and will provide complete information about the distribution of possible values of the volume $\mathcal{V}_Q$ together with the total reheating volume $\mathcal{V}$ to the future of an initial H-region. In the context of the RV cutoff, one is interested in the event when $\mathcal{V}$ is finite and very large. Then one expects that $\mathcal{V}_Q$ also becomes typically very large while the ratio $\mathcal{V}_Q / \mathcal{V}$ remains roughly constant. In other words, one expects that the distribution of $\mathcal{V}_Q$ is sharply peaked around a mean value $\langle \mathcal{V}_Q \rangle$, and that the limit $\langle \mathcal{V}_Q \rangle / \mathcal{V}$ is well-defined at $\mathcal{V} \to \infty$. The value of that limit is the only information we need for calculations in the RV cutoff. Therefore, we do not need to compute the entire distribution $\rho(\mathcal{V}, \mathcal{V}_Q; \phi, Q)$ but only the mean value $\langle \mathcal{V}_Q |_{\mathcal{V}} \rangle$ at fixed $\mathcal{V}$.

Let us therefore define the generating function of the mean value $\langle \mathcal{V}_Q |_{\mathcal{V}} \rangle$ as follows,

$$h(z; \phi, Q) \equiv \left\langle \mathcal{V}_{Q_R} e^{-z\mathcal{V}} \right\rangle_{\mathcal{V} < \infty} = -\frac{\partial \tilde{g}}{\partial q}(z, q = 0; \phi, Q). \tag{8.92}$$

(The dependence on the fixed value of Q_R is kept implicit in the function $h(z; \phi, Q)$ in order to make the notation less cumbersome.) The differential equation and the boundary conditions for $h(z; \phi, Q)$ follow straightforwardly by taking the derivative ∂_q at $q = 0$ of Eqs. (8.50) and (8.90). It is clear from the definition of $\tilde{g}$ that $\tilde{g}(z, q = 0; \phi, Q) = g(z; \phi, Q)$. Hence we obtain

$$\hat{L}h + 3\left(\ln g(z;\phi,Q) + 1\right)h = 0, \tag{8.93}$$

$$h(z;\phi_*,Q) = \frac{e^{-zH^{-3}(\phi_*,Q)}}{H^3(\phi_*,Q)}\delta_{QQ_R}, \tag{8.94}$$

$$h(z;\phi_{\text{Pl}},Q) = 0. \tag{8.95}$$

Note that it is the generating function g, not $\tilde{g}$, that appears as a coefficient in Eq. (8.93).

Since the "finite" δ-function δ_{QQ_R} now enters only linearly rather than under an exponential, we may replace δ_{QQ_R} by the ordinary Dirac δ-function $\delta(Q - Q_R)$. To maintain consistency, we need to divide h by dQ, which corresponds to computing the *probability density* of the reheated volume with $Q = Q_R$. This probability density is precisely the goal of the present calculation.

The RV-regularized probability density for values of Q is defined as the limit

$$\begin{aligned} p(Q_R) &= \lim_{\mathcal{V}\to\infty} \frac{\left\langle \mathcal{V}_{Q_R} \middle|_{\mathcal{V}} \right\rangle_{\mathcal{V}<\infty}}{\mathcal{V}\,\rho(\mathcal{V};\phi,Q)} \\ &= \lim_{\mathcal{V}\to\infty} \frac{\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{z\mathcal{V}} h(z;\phi,Q)dz}{\mathcal{V}\int e^{z\mathcal{V}} g(z;\phi,Q)dz}. \end{aligned} \tag{8.96}$$

To compute this limit, we need to consider the asymptotic behavior of $\left\langle \mathcal{V}_{Q_R} \middle|_{\mathcal{V}} \right\rangle_{\mathcal{V}<\infty}$ at $\mathcal{V} \to \infty$. This behavior is determined by the leading singularity of the function $h(z;\phi,Q)$ in the complex z plane. The arguments of Sec. 8.3.2 apply also to $h(z;\phi,Q)$ and show that h cannot have a ϕ- or Q-dependent singularity in the z plane.

Moreover, $h(z;\phi,Q)$ has precisely the same singular points, in particular $z = z_*$, as the basic generating function $g(z;\phi,Q)$ of the reheating volume. Indeed, the function $h(z;\phi,Q)$ can be expressed through the Green's function $G(z;\phi,Q,\phi',Q')$ of the operator $\hat{L} + 3(\ln g + 1)$, similarly to the function $g_1(z;\phi)$ considered in Sec. 8.3.2. For $z \neq z_*$, this operator is invertible on the space of functions $f(\phi,Q)$ satisfying zero boundary conditions. Hence, $h(z;\phi,Q)$ is nonsingular at $z \neq z_*$ and becomes singular precisely at $z = z_*$.

Let us now obtain an explicit form of $h(z;\phi,Q)$ near the singular point $z = z_*$. We assume again the eigenfunction decomposition of the Green's function (with the same caveats as in Sec. 8.3.2),

$$G(z;\phi,Q,\phi',Q') = \sum_{n=0}^{\infty} \frac{1}{\lambda_n(z)} f_n(z;\phi,Q) f_n^*(z;\phi',Q'), \tag{8.97}$$

where f_n are appropriately normalized eigenfunctions of the z-dependent operator $\hat{L} + 3(\ln g + 1)$ with eigenvalues $\lambda_n(z)$. The eigenfunctions f_n must satisfy zero boundary conditions at reheating and Planck boundaries. Similarly to the way we derived Eq. (8.74), we obtain the explicit solution

$$h(z;\phi,Q)=\sum_{n=0}^{\infty}\frac{f_n(z;\phi,Q)}{\lambda_n(z)}\left[\frac{\partial f_n}{\partial\phi}\frac{D_{\phi\phi}e^{-zH^{-3}}}{H^4}\right]_{\phi_*,Q_R}. \tag{8.98}$$

The value of $h(z;\phi,Q)$ for $z\approx z_*$ is dominated by the contribution of the large factor $1/\lambda_0(z)\propto(z-z_*)^{-1/2}$, so the leading term is

$$h(z;\phi,Q)\approx\frac{f_0(z_*;\phi,Q)}{\lambda_0(z)}\left[\frac{\partial f_0}{\partial\phi}\frac{D_{\phi\phi}e^{-z_*H^{-3}}}{H^4}\right]_{\phi_*,Q_R}. \tag{8.99}$$

The asymptotic behavior of the mean value $\langle\mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle_{\mathcal{V}<\infty}$ is determined by the singularity of $h(z;\phi,Q)$ at $z=z_*$. As before, we may deform the integration contour to pass near the real axis around $z=z_*$. We can then express the large-$\mathcal{V}$ asymptotic of $\langle\mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle_{\mathcal{V}<\infty}$ as follows,

$$\begin{aligned}\langle\mathcal{V}_{Q_R}|_{\mathcal{V}}\rangle&=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}e^{z\mathcal{V}}h(z;\phi,Q)dz\\&\approx\frac{f_0(z_*;\phi,Q)}{\sqrt{\pi}\sigma_0\sqrt{\mathcal{V}}}e^{z_*\mathcal{V}}\left[\frac{\partial f_0}{\partial\phi}\frac{D_{\phi\phi}e^{-z_*H^{-3}}}{H^4}\right]_{\phi_*,Q_R},\end{aligned} \tag{8.100}$$

where σ_0 is the constant defined by Eq. (8.82).

We now complete the analytic evaluation of the limit (8.96). Since the denominator of Eq. (8.96) has the large-$\mathcal{V}$ asymptotics of the form

$$\mathcal{V}\int_0^{\infty}g(z;\phi,Q)e^{z\mathcal{V}}d\mathcal{V}\propto f_0(z_*;\phi,Q)\mathcal{V}^{-\frac{1}{2}}e^{z_*\mathcal{V}}, \tag{8.101}$$

where f_0 is the same eigenfunction, the dependence on ϕ and Q identically cancels in the limit (8.96). Hence, that limit is independent of the initial values ϕ and Q but is a function only of Q_R, on which $h(z;\phi,Q)$ implicitly depends. Using this fact,, we can significantly simplify the rest of the calculation. It is not necessary to compute the denominator of Eq. (8.96) explicitly. The distribution of the values of Q at $\phi=\phi_*$ is simply proportional to the Q_R-dependent part of Eq. (8.100); the denominator of Eq. (8.96) serves merely to normalize that distribution. Hence, the RV cutoff yields

$$p(Q_R)=\text{const}\left[\frac{\partial f_0(z_*;\phi,Q)}{\partial\phi}\frac{D_{\phi\phi}e^{-z_*H^{-3}}}{H^4}\right]_{\phi=\phi_*,Q=Q_R}, \tag{8.102}$$

where the normalization constant needs to be chosen such that $\int p(Q_R)dQ_R=1$. This is the final analytic formula for the RV cutoff applied to the distribution of Q at reheating. The the value z_*, and the corresponding solution $g(z_*;\phi,Q)$ of Eq. (8.22), and the eigenfunction $f_0(z_*;\phi,Q)$ need to be obtained numerically unless an analytic solution is possible.

Let us comment on the presence of the factor $D_{\phi\phi}$ in the formula (8.102). The "diffusion" coefficient $D_{\phi\phi}$ is evaluated at the reheating boundary and is thus small since the fluctuation amplitude at reheating is (in slow-roll inflationary models)

$$\frac{\delta\phi}{\phi} \sim \frac{H^2}{\dot{\phi}} = \frac{\sqrt{8\pi^2 D_{\phi\phi} H}}{v_\phi} \sim 10^{-5}. \tag{8.103}$$

Nevertheless it is not possible to set $D_{\phi\phi} = 0$ directly in Eq. (8.102). This is so because the existence of the Green's function of the Fokker–Planck operator such as $\hat{L}$ depends on the fact that $\hat{L}$ is a second-order differential operator of elliptic type. If one sets $D_{\phi\phi} = 0$ near the reheating boundary, the operator $\hat{L}$ becomes first-order in ϕ at that boundary. Then one needs to use a different formula than Eq. (8.60) for reducing an equation with inhomogeneous boundary conditions to an inhomogeneous equation with zero boundary conditions. Accordingly, one cannot use formulas such as Eq. (8.64) for the solutions. Alternative ways of solving the relevant equations in that case will be used in Sec. 8.3.4.

8.3.4 Calculations for an inflationary model

In this section we perform explicit calculations of RV cutoff for a model of slow-roll inflation driven by a scalar field with a potential shown in Fig. 6.1. The kinetic coefficients $D(\phi)$ and $v(\phi)$ are such that $D(\phi) = D_0$, $v(\phi) = 0$, and $H(\phi) = H_0$ in the flat region $\phi_1 < \phi < \phi_2$, where the constants D_0 and H_0 are

$$H_0 = \sqrt{\frac{8\pi G}{3} V_0}, \quad D_0 = \frac{H_0^3}{8\pi^2}. \tag{8.104}$$

In the slow-roll regions $\phi_1 < \phi < \phi_*^{(1)}$ and $\phi_2 < \phi < \phi_*^{(2)}$, the coefficient $D(\phi)$ is set equal to zero, while $v(\phi) \neq 0$ and $H(\phi)$ is not constant any more. The number of e-foldings in the two slow-roll "shoulders" can be computed by the standard formula,

$$N_j = \int_{\phi_j}^{\phi_*^{(j)}} \frac{H}{v} d\phi = -4\pi G \int_{\phi_j}^{\phi_*^{(j)}} \frac{H}{H'} d\phi, \quad j = 1, 2. \tag{8.105}$$

The first step of the calculation is to determine the singular point $z = z_*$ of solutions $g(z;\phi)$ of Eq. (8.44). We expect z_* to be real and negative. The boundary conditions for $g(z;\phi)$ are

$$g(z;\phi_*^{(1,2)}) = \exp\left[-zH^{-3}(\phi_*^{(1,2)})\right]. \tag{8.106}$$

In each of the two deterministic regions, $\phi_*^{(1)} < \phi < \phi_1$ and $\phi_2 < \phi < \phi_*^{(2)}$, Eq. (8.44) becomes

$$\frac{v}{H}\partial_\phi g + 3g \ln g = 0, \tag{8.107}$$

with the general solution

$$g(z;\phi) = \exp\left[C\exp\left(-3\int^{\phi}\frac{H}{v}d\phi\right)\right], \tag{8.108}$$

where C is an integration constant. Since the equation is first-order within the deterministic regions, the solutions are fixed by the boundary condition (8.106) in the respective region,

$$g(z;\phi) = \exp\left[-zH^{-3}(\phi_*^{(1,2)})\exp\left(-3\int_{\phi_*^{(1,2)}}^{\phi}\frac{H}{v}d\phi\right)\right]. \tag{8.109}$$

We may therefore compute the values of $g(z;\phi)$ at the boundaries $\phi_{1,2}$ of the self-reproduction region as

$$g(z;\phi_{1,2}) = \exp\left[-zH^{-3}(\phi_*^{(1,2)})\exp\left(3N_{1,2}\right)\right]. \tag{8.110}$$

Now we need to solve Eq. (8.44) with these boundary conditions in the region $\phi_1 < \phi < \phi_2$. The equation has then the form

$$\frac{D_0}{H_0}\partial_\phi\partial_\phi g + 3g\ln g = 0. \tag{8.111}$$

Exact solutions of Eq. (8.111) were studied in [188], to which the reader is referred for further details. It is easy to show that Eq. (8.111) is formally equivalent to a one-dimensional motion of a particle with coordinate $g(\phi)$ in a potential $U(g)$,

$$U(g) = \frac{6\pi^2}{H_0^2}g^2\left(2\ln g - 1\right), \tag{8.112}$$

while ϕ plays the role of time. A solution $g(z;\phi)$ with boundary conditions (8.110) corresponds to a trajectory that starts at the given value $g(z;\phi_1)$ with the initial velocity chosen such that the motion takes precisely the specified time interval $\phi_2 - \phi_1$ and reaches $g(z;\phi_2)$. For $z < 0$ the boundary conditions specify $g(z;\phi_{1,2}) > 1$, *i.e.* the trajectory begins and ends to the right of the minimum of the potential (see Fig. 8.1). Since the system is conservative, there is a constant of motion E (the "energy") such that $E = U(g_0)$ at the highest point of the trajectory g_0 where the "kinetic energy" vanishes. The solution $g(z;\phi)$ can be written implicitly as one of the two alternative formulas,

$$\pm\int_{g(z;\phi)}^{g(z;\phi_{1,2})}\frac{dg}{\sqrt{2E(z) - 2U(g)}} = \phi - \phi_{1,2}, \tag{8.113}$$

valid in appropriate intervals $\phi_1 < \phi < \phi_0$ and $\phi_0 < \phi < \phi_2$ respectively, where ϕ_0 is the value of ϕ corresponding to the turning point $g_0 = g(z;\phi_0)$. The value $E = E(z)$ in Eq. (8.113) must be chosen such that the total "time" is $\phi_2 - \phi_1$,

$$\left[\int_{g_0}^{g(z;\phi_1)} + \int_{g_0}^{g(z;\phi_2)}\right]\frac{dg}{\sqrt{2E - 2U(g)}} = \phi_2 - \phi_1. \tag{8.114}$$

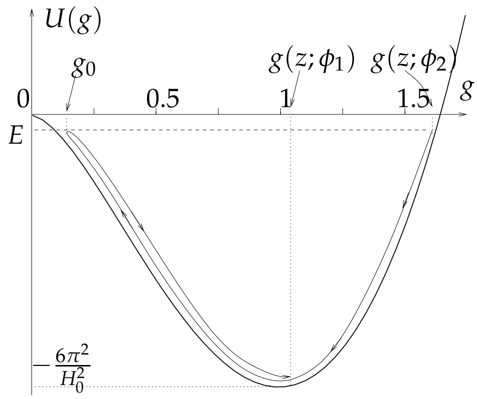

Figure 8.1: Potential $U(g)$ given by Eq. (8.112) can be used to interpret solutions $g(z;\phi)$ as mechanical motion in "time" ϕ at constant total energy E. The potential vanishes at $g = 0$ and $g = e^{1/2}$ and has a minimum at $g = 1$. The value g_0 is the turning point where $U(g_0) = E$. A trajectory corresponding to $z < 0$ will begin and end with $g > 1$, *i.e.* to the right of the minimum of the potential. Solutions cease to exist when $z < z_*$; the solution $g(z_*;\phi)$, shown by the thin line with arrows, corresponds to a trajectory that starts at rest (as demonstrated in the text). The energy of this trajectory is $E \approx 0$, and so the value $g(z_*;\phi_2)$ is close to $e^{1/2}$ while $g(z_*;\phi_1)$ is close to 1.

This condition together with $E = U(g_0)$ implicitly determine the values $E = E(z)$ and $g_0 = g_0(z)$.

The singularity $z = z_*$ of the solution $g(z;\phi)$ is found by using the condition $\partial g/\partial z \to \infty$. Differentiating Eq. (8.113) with respect to z and substituting Eq. (8.110) for $g(z;\phi_{1,2})$, we obtain the condition

$$\begin{aligned} &-\frac{\partial g}{\partial z}\frac{1}{\sqrt{2E(z)-2U(g)}} + \frac{e^{3N_{1,2}}H^{-3}(\phi_*^{(1,2)})}{\sqrt{2E(z)-2U(g(z;\phi_{1,2}))}} \\ &-E'(z)\int_{g(z;\phi)}^{g(z;\phi_{1,2})}\frac{dg}{\left[2E(z)-2U(g)\right]^{3/2}} = 0. \end{aligned} \tag{8.115}$$

It follows that $\partial g/\partial z \to \infty$ when

$$E(z) = U(g(z;\phi_{1,2})). \tag{8.116}$$

This condition is interpreted in the language of Fig. 8.1 as follows. As the value of z becomes more negative, the initial and the final values of g given by Eq. (8.110) both grow. The last available trajectory starts from rest at $\phi = \phi_2$ and at the value of g such that $U(g) = E$.

To obtain a specific result, let us assume that $\phi_2 - \phi_1$ is sufficiently large to provide self-reproduction ($\phi_2 - \phi_1 \gg H_0$) and that the number of e-foldings in channel 1 is smaller than that in channel 2,

$$H^{-3}(\phi_*^{(1)})\exp(3N_1) \ll H^{-3}(\phi_*^{(2)})\exp(3N_2). \tag{8.117}$$

Then the value $g(z;\phi_2)$ will grow faster than $g(z;\phi_1)$ as z becomes more negative. It follows that $g(z;\phi_2)$ will reach the singular point first. Since the "time" $\phi_2 - \phi_1$ is large, the constant E will be close to 0 so that the trajectory spends a long "time" near $g = 0$. Then the value $g(z_*;\phi_2)$ will be close to $e^{1/2}$. Hence the value of z_* is approximately

$$z_* \approx -\frac{1}{2}H^3(\phi_*^{(2)})\exp(-3N_2). \tag{8.118}$$

For this value of z_*, the starting point of the trajectory will be

$$g(z_*;\phi_1) \approx \exp\left[\frac{1}{2}\exp(3N_1 - 3N_2)\right] \approx 1. \tag{8.119}$$

Hence, the solution $g(z_*;\phi)$ at the singular point $z = z_*$ can be visualized as the thin line in Fig. 8.1, starting approximately at $g(z_*;\phi_1) = 1$ and finishing at $g(z_*;\phi_2) \approx e^{1/2}$.

An approximate expression for $g(z_*;\phi)$ can be obtained by setting $E \approx 0$ in Eq. (8.113); then the integral can be evaluated analytically. In the range $\phi_0 < \phi < \phi_2$ we obtain

$$\phi_2 - \phi \approx \int_{g(z_*;\phi)}^{g(z_*;\phi_2)}\frac{dg}{\sqrt{-2U(g)}} \approx \frac{H_0}{\sqrt{12\pi^2}}\left.\sqrt{1-2\ln g}\right|_{\phi_2}^{\phi}, \tag{8.120}$$

so the solution is

$$g(z_*;\phi) \approx \exp\left[\frac{1}{2} - \frac{6\pi^2}{H_0^2}(\phi_2 - \phi)^2\right], \quad \phi_0 < \phi < \phi_2. \tag{8.121}$$

In the range $\phi_1 < \phi < \phi_0$ we obtain within the same approximation

$$g(z_*;\phi) \approx \exp\left[\frac{1}{2} - \frac{6\pi^2}{H_0^2}\left(\phi - \phi_1 + \frac{H_0}{\sqrt{12\pi^2}}\right)^2\right]. \tag{8.122}$$

These approximations are valid for ϕ within the indicated ranges and away from the turning point ϕ_0. The value of ϕ_0 can be estimated by requiring that the value of $g(z_*;\phi_0)$ obtained from Eq. (8.121) be equal to that obtained from Eq. (8.122). This yields

$$\phi_0 \approx \frac{1}{2}\left[\phi_2 + \phi_1 - \frac{H_0}{\sqrt{12\pi^2}}\right] \approx \frac{\phi_2 + \phi_1}{2}. \tag{8.123}$$

We note that the value $g(z_*;\phi_0)$ can be obtained somewhat more precisely by approximating the solution $g(z_*;\phi)$ in a narrow interval near $\phi = \phi_0$ by a function of the form $\exp\left[A + B(\phi - \phi_0)^2\right]$ and matching both the values and the derivatives of $g(z_*;\phi)$ to the approximations (8.121) and (8.122) at some intermediate points straddling $\phi = \phi_0$. In this way, a uniform analytic approximation for $g(z_*;\phi)$ can be obtained. However, the accuracy of the approximations (8.121) and (8.122) is sufficient for the present purposes.

Having obtained adequate analytic approximations for z_* and $g(z_*,\phi)$, we can now proceed to the calculation of the mean volumes $\left\langle \mathcal{V}_{1,2}|_{\mathcal{V}}\right\rangle$ of regions reheated through channels 1 and 2 respectively, conditioned on the event that the total volume of all reheated regions is $\mathcal{V}$. We use the formalism developed in Sec. 8.3.3, where the variable Q now takes only the discrete values 1 and 2, so instead let us denote that value by j. The relevant generating function $h_j(z;\phi)$ is defined by

$$h_j(z;\phi) \equiv \left\langle \mathcal{V}_j e^{-z\mathcal{V}}\right\rangle_{\mathcal{V}<\infty}, \quad j = 1,2, \tag{8.124}$$

and is found as the solution of Eq. (8.93), which now takes the form

$$\left[\frac{D(\phi)}{H(\phi)}\partial_\phi\partial_\phi + \frac{v(\phi)}{H(\phi)}\partial_\phi + 3(\ln g(z;\phi) + 1)\right] h_j(z;\phi) = 0, \tag{8.125}$$

with boundary conditions imposed at the reheating boundaries,

$$h_1(z;\phi_*^{(1)}) = H^{-3}(\phi_*^{(1)})e^{-zH^{-3}(\phi_*^{(1)})}, \quad h_1(z;\phi_*^{(2)}) = 0; \tag{8.126}$$

$$h_2(z;\phi_*^{(1)}) = 0, \quad h_2(z;\phi_*^{(2)}) = H^{-3}(\phi_*^{(2)})e^{-zH^{-3}(\phi_*^{(2)})}. \tag{8.127}$$

In the present toy model the diffusion coefficient is set to zero at reheating, so the formalism developed in Sec. 8.3.3 needs to be modified. We will first solve Eq. (8.125) analytically in the no-diffusion intervals of ϕ and obtain the boundary conditions for h at the boundaries of the self-reproduction regime $[\phi_1, \phi_2]$ where $D(\phi) \neq 0$. Then the methods of Sec. 8.3.3 will be applicable to the boundary value problem for the interval $[\phi_1, \phi_2]$.

Implementing this idea in the first no-diffusion region $\phi_*^{(1)} < \phi < \phi_1$, we use the solution (8.109) for $g(z;\phi)$ and reduce Eq. (8.125) to

$$\partial_\phi h_j + \frac{3H}{v}\left[1 - zH^{-3}(\phi_*^{(1)}) \exp\left(-3\int_{\phi_*^{(1)}}^{\phi} \frac{H}{v} d\phi\right)\right] h_j = 0. \tag{8.128}$$

This equation is easily integrated together with the boundary conditions (8.126)–(8.127) and yields the values of h_j at ϕ_1,

$$h_j(z;\phi_1) = \delta_{j1} H^{-3}(\phi_*^{(1)}) \exp\left[3N_1 - zH^{-3}(\phi_*^{(1)}) e^{3N_1}\right]. \tag{8.129}$$

Similarly we can determine the values $h_j(z;\phi_2)$. Since the value $z = z_*$ is important for the present calculation, we now find the values of h_j at $z = z_*$ using the assumption $N_2 \gg N_1$ and the estimate (8.118),

$$h_j(z_*;\phi_i) \approx \delta_{ij} \frac{\exp\left[3N_i + \frac{1}{2}\delta_{i2}\right]}{H^3(\phi_*^{(i)})}, \quad i, j = 1, 2. \tag{8.130}$$

We have thus reduced the problem of determining $h_j(z;\phi)$ to the boundary-value problem for the interval $[\phi_1, \phi_2]$ where the methods of Sec. 8.3.3 apply but the boundary conditions are given by Eq. (8.130).

The next step, according to Sec. 8.3.3, is to compute the eigenfunction $f_0(z_*;\phi)$ of the operator

$$\hat{\hat{L}} \equiv \frac{D_0}{H_0} \partial_\phi \partial_\phi + 3(\ln g(z_*;\phi) + 1) \tag{8.131}$$

such that

$$\hat{\hat{L}} f_0 = 0; \quad f_0(z_*;\phi_{1,2}) = 0. \tag{8.132}$$

As we have shown, this eigenfunction with eigenvalue 0 exists precisely at $z = z_*$. Once this eigenfunction is computed, the ratio of the RV-regulated mean volumes in channels 1 and 2 will be expressed through the derivatives of f_0 at the endpoints and through the modified boundary conditions (8.130) as follows,

$$\frac{P(2)}{P(1)} = \frac{h_2(z_*;\phi_2)}{h_1(z_*;\phi_1)} \frac{\left|\partial_\phi f_0(z_*;\phi_2)\right|}{\partial_\phi f_0(z_*;\phi_1)}. \tag{8.133}$$

The absolute value is taken to compensate for the negative sign of the derivative $\partial_\phi f_0$ at the right boundary point (assuming that $f_0 \geq 0$ everywhere). Since $h_{1,2}(z_*;\phi_{1,2})$ are already known, it remains to derive an estimate for $f_0(z_*;\phi)$.

The eigenvalue equation $\hat{\hat{L}} f_0 = 0$ formally resembles a one-dimensional Schrödinger equation with the coordinate ϕ and the "potential"

$$\tilde{V}(\phi) \equiv -\frac{12\pi^2}{H_0^2} (\ln g(z_*;\phi) + 1). \tag{8.134}$$

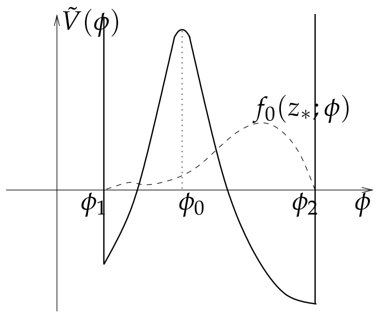

Figure 8.2: Sketch of the eigenfunction $f_0(z_*;\phi)$ (dashed line) interpreted as the wavefunction of a stationary state with zero energy in the potential $\tilde{V}(\phi)$ (solid line). Due to exponential suppression by the potential barrier, the amplitude of f_0 in the right region is exponentially larger than that in the left region.

The eigenfunction $f_0(z_*;\phi)$ is then interpreted as the "wavefunction" of a stationary state with zero energy and zero boundary conditions at $\phi = \phi_{1,2}$. According to Eqs. (8.121) and (8.122), the function $\tilde{V}(\phi)$ has a maximum at $\phi \approx \phi_0$ (see Fig. 8.2), while its values at the endpoints are

$$\tilde{V}(\phi_1) \approx -\frac{12\pi^2}{H_0^2}, \quad \tilde{V}(\phi_2) \approx -\frac{18\pi^2}{H_0^2}. \tag{8.135}$$

Using the terminology of quantum mechanics, there is a potential barrier separating two classically allowed regions near $\phi = \phi_1$ and $\phi = \phi_2$. Since the "potential well" at $\phi = \phi_2$ is deeper, the ground state is approximately the ground state of that one well, with an exponentially small amplitude of being near $\phi = \phi_1$. The shape of the eigenfunction is sketched in Fig. 8.2. The exponential suppression of the amplitude near $\phi = \phi_1$ can be found using the WKB approximation, which yields

$$\frac{\left|\partial_\phi f_0(z_*;\phi_2)\right|}{\partial_\phi f_0(z_*;\phi_1)} = A_{21} \exp\left[\int_{\tilde{\phi}_1}^{\tilde{\phi}_2} \sqrt{\tilde{V}(\phi)}d\phi\right], \tag{8.136}$$

where $\tilde{\phi}_{1,2}$ are the turning points such that $\tilde{V}(\tilde{\phi}_{1,2}) = 0$. The pre-exponential factor A_{21} is of order 1 and can, in principle, be obtained from a more detailed matching of the WKB-approximated solution across the barrier to the solutions in the "classically allowed" regions, or by determining the solution $f_0(z_*;\phi)$ numerically. However, we will omit this calculation since the main result will consist of an exponentially large factor. That factor can be estimated using Eqs. (8.121), (8.122), and (8.134) as

$$\int_{\tilde{\phi}_1}^{\tilde{\phi}_2} \sqrt{\tilde{V}(\phi)}d\phi \approx 2\int_{\phi_1}^{\phi_0} \sqrt{-\frac{12\pi^2}{H_0^2}\ln g}\, d\phi \approx \frac{3\pi^2}{\sqrt{2}H_0^2}\left(\phi_2 - \phi_1\right)^2. \tag{8.137}$$

Hence, the ratio (8.133) is simplified to

$$\frac{P(2)}{P(1)} = A_{21} \frac{H^{-3}(\phi_*^{(2)})}{H^{-3}(\phi_*^{(1)})} \frac{e^{3N_2+\frac{1}{2}}}{e^{3N_1}} \exp\left[\frac{3\pi^2}{\sqrt{2}H_0^2}(\phi_2-\phi_1)^2\right]. \tag{8.138}$$

This is the main result quoted above in Eq. (8.34).

9 The RV measure for the landscape

The existing measure proposals that apply directly to landscape scenarios are the "holographic" measure [107, 66] (see also the recent proposal [242]), the "comoving horizon cutoff" [222, 232, 224], the "stationary measure" [53, 101], the measure on transitions [239], and the "pseudo-comoving" measure [115, 238, 241, 237]. In the absence of a unique definition of the measure, one judges a cutoff prescription viable if its predictions are not obviously pathological. Possible pathologies include the dependence on choice of spacetime coordinates [179, 235], the "youngness paradox" [187, 94], and the "Boltzmann brain" problem [247, 248, 249, 115, 251, 252, 223, 253]. Various measures have been used for predicting cosmological parameters, most notably the cosmological constant, in the landscape scenarios (see, *e.g.*, [114, 101, 108, 109, 265, 266, 267]).

In this chapter I extend the RV measure, originally formulated in the context of random-walk inflation, to landscape scenarios. The basic idea of the RV proposal is to select multiverses that are very large but (by rare chance) have a finite total number of observers. In the context of a string landscape scenario (or a "recycling universe"), this can happen if sufficiently many anti-de Sitter or Minkowski bubbles nucleate everywhere, collide, and merge. In such a case, there will be a finite time after which no de Sitter regions remain and no further nucleations can occur. Hence, there will be a finite time after which no more observers are created anywhere. By this construction, one obtains a subensemble of multiverses having a fixed, finite total number N_{obs} of observers. These finite multiverses with very large N_{obs} are regarded as controlled approximations to the actual infinite multiverse. The observer-weighted statistical distribution of any quantity within a finite multiverse can be obtained by ordinary counting, since the total number of observers within any such multiverse is finite. The limit of that statistical distribution as $N_{\mathrm{obs}} \to \infty$ is the final result of the RV prescription.

It was shown in Chapters 7 and 8 that the RV measure is gauge-invariant, independent of the initial conditions, and free of the youngness paradox in the context of random-walk inflation. Presently I investigate whether the same features persist in an application of the RV measure to landscape scenarios. In particular, it is important to obtain RV predictions with respect to the "Boltzmann brain" problem that has been widely discussed.

In principle, the RV prescription can be extended to landscape models in different ways, depending on the precise choice of the ensemble of finite multiverses. The ensemble of multiverses with a fixed total number of observers N_{obs} (where one counts both the ordinary observers and the "Boltzmann brains") appears to be the natural choice. However, it is difficult to compute the number of observers directly and unambiguously. Instead of the total number of observers, I propose to fix the total number $n_{\rm tot}$ of bubbles nucleated to the future of an initial bubble.

The total number of ordinary observers in bubbles of a given kind is proportional to the volume of the reheating surface in those bubbles. It is known that the square bubble approximation [223], which neglects the effects of bubble wall geometry, is adequate for the purposes of volume counting. Then the evolution of the landscape is well described by the approximate model called "inflation in a box" [189, 75, 246]. In that approximation, one keeps track only of the number of new bubbles nucleated in previously existing bubbles, and each new bubble is assumed to be instantaneously nucleated exactly of Hubble size in comoving coordinates. Motivated by this approximation, I study a simplified definition of the RV measure for a landscape scenario (see Sec. 9.2 for details): One requires the total number of bubbles of all types, $n_{\rm tot}$, to be finite and evaluates the statistical distribution of bubble types (or other cosmological observables) in the limit $n_{\rm tot} \to \infty$. In principle, this limit can be calculated if the bubble nucleation rates are known. Since this is a first attempt to perform this technically challenging calculation, I concentrate only on bubble abundances and on the relative abundance of Boltzmann brains. I neglect the increased number of observers due to additional slow-roll inflation within bubbles; this effect was considered in [246] and requires additional complications in the formalism.

A landscape scenario may be specified by enumerating the available N types of vacua by a label j ($j = 1, ..., N$) and by giving the Hubble rates H_j within bubbles of type j. One can, in principle, compute the nucleation rate $\Gamma_{j\to k}$ describing the probability (per unit four-volume of spacetime) of creating a bubble of type k within bubbles of type j.[1] It is convenient to work with the dimensionless rates,

$$\kappa_{j\to k} \equiv \frac{4\pi}{3}\Gamma_{j\to k}H_j^{-4}. \tag{9.1}$$

The rate $\kappa_{j\to k}$ equals the probability of having a bubble of type k within a 3-volume of one horizon in a bubble of type j, during a single Hubble time.[2] Explicit expressions for $\kappa_{j\to k}$ are available in some landscape scenarios. In what follows, I assume that $\kappa_{j\to k}$ are known.

In Sec. 9.3 I apply the RV measure proposal to a toy model of the landscape with four vacua (the FABI model of [222]). In this model, the vacua labeled F and I are de Sitter (dS) and the vacua labeled A and B are anti-de Sitter (AdS) states. One assumes that only the transitions $F \to I$, $I \to F$, $F \to A$, and $I \to B$ are allowed, with known nucleation rates κ_{FI}, κ_{IF}, etc., per unit Hubble 4-volume. I show in Eq. (9.88) that the RV-regulated bubble abundances depend

[1] For some recent work concerning the determination of the bubble nucleation rates, see [268, 264, 269, 270, 271].

[2] The notation $\kappa_{j\to k}$, chosen here and in [254] for its visual clarity, corresponds to κ_{kj} of [222] and to Γ_{jk} of [246].

on the value of the dimensionless number

$$\eta \equiv \left(\frac{\kappa_{IB}}{\kappa_{FA}}\right)^{\nu+1} \frac{\kappa_{FI}}{\kappa_{IF}}, \quad \nu \equiv e^3. \tag{9.2}$$

Here the constant ν, introduced for convenience, is simply the number of statistically independent Hubble regions after one e-folding. Barring fine-tuned cases, one expects that the value of η is either much larger than 1 or much smaller than 1, since the nucleation rates may differ by exponentially many orders of magnitude. By relabeling the vacua ($F \leftrightarrow I$ and $A \leftrightarrow B$) if necessary, we may assume that $\eta \ll e^{-6}$. Then the bubble abundances are approximately described by the ratios

$$p(I) : p(F) : p(A) : p(B) \approx \frac{1}{\nu^2} : \frac{1}{\nu} : 1 : \left[\frac{\eta}{\nu^{\nu-1}}\right]^{\frac{1}{\nu+1}}. \tag{9.3}$$

This result can be interpreted as follows. Each of the I bubbles produces ν bubbles of type F, and each of the F bubbles produces ν bubbles of type A. The abundance of B bubbles is negligible compared with other bubbles. Heuristically, the chain of transitions $I \to F \to A$ can be interpreted as the "dominant" chain in the landscape. The fine-tuned case, $e^{-6} < \eta < e^{6}$, is considered separately, and the result is given by Eq. (9.79).

I then consider the abundance of Boltzmann brains (BBs) in the FABI landscape (Sec. 9.3.2). The total number of BBs is proportional to the total number of Hubble regions (H-regions, or 4-volumes of order H^{-4}) in de Sitter bubbles. The coefficient of proportionality is the tiny nucleation rate Γ^{BB} of Boltzmann brains, which is of order $\exp(-10^{50})$ or smaller. In comparison, ordinary observers occur at a rate of at least 1 per horizon volume. It turns out that (after applying the RV cutoff) the total number of H-regions of dS types F or I is approximately equal to the total number of nucleated bubbles of the same type. Hence, the BBs are extremely rare compared with ordinary observers.

In Sec. 9.4 I extend the same calculations to a general landscape with an arbitrary number of vacua. The RV prescription predicts a definite ratio $p(j)/p(k)$ between the number of bubbles of types j and k. I derive a formula for the ratio $p(j)/p(k)$ that involves all the parameters of the landscape. With the help of mathematical results derived in Sections 9.4.5 and 9.4.6, it is possible to show in full generality that the RV measure gives well-defined results that are independent of the initial conditions. Nevertheless, actually performing the required calculations for an arbitrary landscape remains a daunting task. To obtain explicit expressions in a semi-realistic setup, I calculate the bubble abundances for a landscape that contains a single high-energy vacuum and a large number of low-energy vacua. The result is an approximate formula [Eq. (9.149)] for the ratio $p(j)/p(k)$ expressed directly through the nucleation rates of the landscape.

I also demonstrate in Sec. 9.4.3 that the abundance of Boltzmann brains is negligible compared with the abundance of ordinary observers in the same bubble type.

9.1 Regulating the number of terminal bubbles

The "landscape" scenarios where transitions between metastable vacuum states occur via bubble nucleation, promise to explain the values of presently observed cosmological parameters, such as the effective cosmological constant Λ. For this reason it is important to be able to apply the RV measure to landscape-type scenarios and, in particular, to compute the relative abundances of different bubble types.

In the terminology of [222], "terminal bubbles" are those with nonpositive value of Λ. No further transitions are possible from such bubbles because bubbles with $\Lambda < 0$ rapidly collapse while bubbles with $\Lambda = 0$ do not support tunneling instantons. The RV measure, as presently formulated, can be used directly for comparing the abundances of *terminal* bubbles. (Extending the RV prescription to non-terminal bubble types is certainly possible but is delegated to a future publication.)

It was shown in [223] that the bubble volume calculations may use the simplifying "square bubble" approximation, which neglects the effects of bubble wall geometry. In this approximation, the evolution of the landscape is well described by the "inflation in a box" model [75], defined as follows. All the vacuum states are labeled by $j = 1, ..., N$ of which the terminal states are $j = 1, ..., N_T$. During a time step δt, an initial H-region of type j expands into $n_j \equiv e^{3H_j\delta t}$ independent daughter H-regions of type j. Each of the daughter H-regions then has probability κ_{jk} of changing into an H-region of type k (for convenience we define $\kappa_{jj} \equiv 1 - \sum_{k\neq j} \kappa_{jk}$). The process is repeated *ad infinitum* for each resulting H-region, except for H-regions of terminal types. A newly created H-region of terminal type will admit no further transitions and will not expand (or, perhaps, will expand only by a fixed amount of slow-roll inflation occurring immediately after nucleation). This imitates the behavior of anti-de Sitter or Minkowski vacua that do not admit further bubble nucleations.

To implement the RV cutoff in this model, let us consider the probability $p_j(n, n'; k)$ of producing a finite total number n of terminal H-regions of which n' are of type j, starting from one initial H-region of (nonterminal) type k. A generating function for this distribution can be defined by

$$g_j(z, q; k) \equiv \sum_{n,n'=0}^{\infty} z^n q^{n'} p_j(n, n'; k) \equiv \left\langle z^n q^{n'} \right\rangle_{n<\infty}. \tag{9.4}$$

One can show that this generating function satisfies the following system of nonlinear algebraic equations,

$$g_j^{1/n_k}(z, q; k) = \sum_{i=1}^{N_T} \kappa_{ki} z q^{\delta_{ij}} + \sum_{i=N_T+1}^{N} \kappa_{ki} g_j(z, q; i), \tag{9.5}$$

I will merely sketch the derivation of Eq. (9.5) here.[3] The generating function g_j satisfies a multiplicative property analogous to Eq. (8.37). This property applies

[3]This equation is similar to the equations for generating functions used in the theory of branching

to the n_k independent daughter H-regions created by expansion from an H-region of type k. Therefore, $g_j(z,q;k)$, which is the expectation value of $z^n q^{n'}$ in an initial H-region of type k, is equal to the product of n_k expectation values of $z^n q^{n'}$ in the n_k daughter H-regions (which may be of different types). The latter expectation value is given by the right-hand side of Eq. (9.5). This yields Eq. (9.5) after raising both sides to the power $1/n_k$.

If the generating functions g_j are known, the distribution $p_j(n,n';k)$ can be recovered by computing derivatives of $g_j(z,q;k)$ at $z=0$ and $q=0$. Further, the mean fraction of H-regions of type j at fixed total number n of terminal H-regions is found as

$$p(j|n) \equiv \frac{\langle n'|_n \rangle}{n} = \frac{\partial_z^n \partial_q g_j(z=0,q=1;k)}{n\, \partial_z^n g_j(z=0,q=1;k)}. \tag{9.6}$$

Then the RV cutoff *defines* the probability of terminal type j, among all the possible *terminal* types, through the limit

$$p(j) \equiv \lim_{n\to\infty} p(j|n) = \lim_{n\to\infty} \frac{\partial_z^n \partial_q g_j(z=0,q=1;k)}{n\, \partial_z^n g_j(z=0,q=1;k)}, \tag{9.7}$$

similarly to Eq. (7.4). Again one expects that the limit exists and is independent of the initial bubble type, as long as the initial bubble is not of terminal type.

As a specific example requiring fewer calculations, let us consider a toy model with only three bubble types. There is one de Sitter ($\Lambda > 0$) vacuum labeled $j=3$ that can decay into two possible anti-de Sitter terminal bubbles labeled $j=1$ and $j=2$. The growth rate n_3 and the nucleation probabilities κ_{31} and κ_{32} are assumed known. To mimic interesting features of the landscape, let us also assume that there is a period of slow-roll inflation inside the bubbles 1 and 2, generating respectively N_1 and N_2 additional e-foldings of inflationary expansion after nucleation. Hence, the model is determined by the parameters $n_3, \kappa_{31}, \kappa_{32}, N_1$, and N_2. For convenience we define $\kappa_{33} \equiv 1 - \kappa_{31} - \kappa_{32}$.

We now perform the calculations for the RV cutoff in this simple model. There are only two generating functions, $g_1(z,q;k)$ and $g_2(z,q;k)$, that need to be considered. The only meaningful initial value is $k=3$ (*i.e.*, the initial bubble is of de Sitter type) since the two other bubble types do not lead to eternal inflation. Hence, we will suppress the argument k in $g_j(z,q;k)$. We also need to modify Eq. (9.5) to take into account the additional expansion inside the terminal bubbles. Let us denote the volume expansion factors by

$$Z_1 \equiv e^{3N_1}, \quad Z_2 \equiv e^{3N_2}. \tag{9.8}$$

Then the functions g_1 and g_2 are solutions of

$$g_1^{1/n_3} = \kappa_{31} z^{Z_1} q^{Z_1} + \kappa_{32} z^{Z_2} + \kappa_{33} g_1, \tag{9.9}$$

$$g_2^{1/n_3} = \kappa_{31} z^{Z_1} + \kappa_{32} z^{Z_2} q^{Z_2} + \kappa_{33} g_2. \tag{9.10}$$

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processes (see *e.g.* the book [272] for a mathematically rigorous presentation). A more detailed derivation will be given in Sec. 9.4.4.

An explicit solution of these equations is impossible for a general n_3 (barring the special cases $n_3 = 2, 3, 4$). Nevertheless, sufficient information about the limit (9.7) can be obtained by the following method. Introduce the auxiliary function $F(x)$ as the solution $F > 0$ of the algebraic equation

$$F^{1/n_3} = x + \kappa_{33} F, \tag{9.11}$$

choosing the branch connected to the value $F(0) = 0$. (It is straightforward to see that Eq. (9.11), has at most two positive solutions, and that $F(x)$ is always the smaller solution of the two.) The generating functions g_1 and g_2 are then expressed through $F(x)$ as

$$g_1(z, q) = F(\kappa_{31} z^{Z_1} q^{Z_1} + \kappa_{32} z^{Z_2}), \tag{9.12}$$

$$g_2(z, q) = F(\kappa_{31} z^{Z_1} + \kappa_{32} z^{Z_2} q^{Z_2}). \tag{9.13}$$

The limit (9.7) involves derivatives of these functions of very high order with respect to z. We note that the function $F(x)$ is analytic; thus the functions g_1 and g_2 are also analytic in z.

To evaluate the high-order derivatives, we need an elementary result from complex analysis. The asymptotic growth of high-order derivatives of an analytic function $f(z)$ is determined by the location of the singularities of $f(z)$ in the complex z plane. The required result can be derived by the following elementary argument. Consider the derivative $d^n f / dz^n$ at $z = 0$, and assume that $f(z)$ admits an expansion around the singularity z_* nearest to $z = 0$, such as

$$f(z) = c_0 + c_1 (z - z_*)^s + ..., \tag{9.14}$$

where $s \neq 0, 1, 2, ...$ is the power of the leading-order singularity, and the omitted terms are either higher powers of $z - z_*$ or singularities at points z'_* located further away from $z = 0$. The singularity structure (9.14) yields the large-n asymptotics

$$\left.\frac{d^n f}{dz^n}\right|_{z=0} \approx c_1 (-z_*)^s \frac{\Gamma(n-s)}{\Gamma(-s)} z_*^{-n} + ... \tag{9.15}$$

This formula enables one to evaluate large-n limits such as Eq. (9.7). It can be seen from this formula that any other singular point z'_* located further away from $z = 0$ gives a contribution that is smaller by the factor $|z'_*/z_*|^{s-n}$. The contribution of a subdominant singularity of the form $(z - z_*)^{s'}$, *i.e.* at the same point $z = z_*$ but with a higher power $s' > s$, is suppressed, in comparison with the term in Eq. (9.15), by the factor

$$\frac{\Gamma(n-s')}{\Gamma(n-s)} \frac{\Gamma(-s)}{\Gamma(-s')} \approx \frac{1}{(n-1)^{s'-s}} \frac{\Gamma(-s)}{\Gamma(-s')}. \tag{9.16}$$

It is clear that the terms omitted from Eq. (9.15) indeed give subleading contributions at large n, and so Eq. (9.15) is indeed the leading asymptotic term at $n \to \infty$.

To proceed, we need to determine the location of the singularities of $F(x)$. Since $F(x)$ is obtained as an intersection of a curve F^{1/n_3} and a straight line $x + \kappa_{33}F$, there will be a value $x = x_*$ where the straight line is tangent to the curve. At this value of x the function $F(x)$ has a singularity of the type

$$F(x) = F(x_*) + F_1\sqrt{x - x_*} + O(x - x_*), \tag{9.17}$$

where F_1 is a constant that can be easily determined; we omit further details that will not be required below. The value of x_* is found from the condition that dF/dx diverge at $x = x_*$. The value of dF/dx at $x \neq x_*$ is found as the derivative of the inverse function, or by taking the derivative of Eq. (9.11),

$$\frac{dF}{dx} = \frac{1}{\frac{1}{n_3}F^{\frac{1}{n_3}-1} - \Gamma_{33}}. \tag{9.18}$$

This expression diverges at the values

$$F(x_*) = (n_3\kappa_{33})^{-\frac{n_3}{n_3-1}}, \tag{9.19}$$

$$x_* = (n_3 - 1)\,\kappa_{33}F(x_*) = \kappa_{33}\frac{n_3 - 1}{n_3^{n_3/(n_3-1)}}. \tag{9.20}$$

Note that $\kappa_{33} \approx 1$ and $n_3 \gg 1$, hence x_* is a constant of order 1.

Rather than compute the limit (9.41) directly, we will perform an easier computation of the *ratio* of the mean number of bubbles of types 1 and 2 at fixed total number n of terminal bubbles,

$$\frac{\left\langle n'_{(1)} \Big|_n \right\rangle}{\left\langle n'_{(2)} \Big|_n \right\rangle} = \frac{\partial_z^n \partial_q g_1(z = 0, q = 1)}{\partial_z^n \partial_q g_2(z = 0, q = 1)}. \tag{9.21}$$

The derivatives $\partial_q g_1$ and $\partial_q g_2$ can be evaluated directly through Eqs. (9.12)–(9.13). For instance, we compute $\partial_q g_1$ as

$$\left.\frac{\partial g_1(z,q)}{\partial q}\right|_{q=1} = F'(\kappa_{31}z^{Z_1} + \kappa_{32}z^{Z_2})\kappa_{31}Z_1 z^{Z_1}. \tag{9.22}$$

It is clear that the functions $\partial_q g_1$ and $\partial_q g_2$ have a singularity at $z = z_*$ corresponding to the singularity $x = x_*$ of the function $F(x)$, where z_* is found from the condition

$$\kappa_{31}z_*^{Z_1} + \kappa_{32}z_*^{Z_2} = x_*. \tag{9.23}$$

Let us analyze this equation in order to estimate z_*. If the nucleation rates κ_{31} and κ_{32} differ by many orders of magnitude, we may expect that one of the terms in Eq. (9.23), say $\kappa_{31}z_*^{Z_1}$, dominates. Then the value z_* is well approximated by

$$z_* \approx \left(\frac{x_*}{\kappa_{31}}\right)^{1/Z_1}. \tag{9.24}$$

This approximation is justified if the first term in Eq. (9.23) indeed dominates, which is the case if

$$\left(\frac{\kappa_{32}}{x_*}\right)^{1/Z_2} \ll \left(\frac{\kappa_{31}}{x_*}\right)^{1/Z_1}. \tag{9.25}$$

If the reversed inequality holds, we can relabel the bubble types 1 and 2 and still use Eq. (9.24). If neither Eq. (9.25) nor the reversed inequality hold, the approximation (9.24) for z_* can be used only as an order-of-magnitude estimate. To be specific, let us assume that the condition (9.25) holds.

We can now compute the ratio (9.21) asymptotically for large n using Eqs. (9.17) and (9.15). The singularities of the functions g_1 and g_2 are directly due to the singularity of the function F. Then the dominant singularity structure of the function (9.22) is found as

$$\left.\frac{\partial g_1(z,q)}{\partial q}\right|_{q=1} \approx \left.\frac{\kappa_{31} Z_1 z^{Z_1}}{2\sqrt{\frac{\partial}{\partial z}\left(\kappa_{31} z^{Z_1} + \kappa_{32} z^{Z_2}\right)}}\right|_{z=z_*} \frac{F_1}{\sqrt{z - z_*}}. \tag{9.26}$$

This fits Eq. (9.14), where $f(z) \equiv \partial_q g_1(z, q = 1)$ and $s = -1/2$. After canceling the common n-dependent factors, we obtain

$$\frac{p(1)}{p(2)} = \lim_{n\to\infty} \frac{\left\langle n'_{(1)}\Big|_n \right\rangle}{\left\langle n'_{(2)}\Big|_n \right\rangle} = \left.\frac{\kappa_{31} Z_1 z^{Z_1}}{\kappa_{32} Z_2 z^{Z_2}}\right|_{z=z_*}. \tag{9.27}$$

This is the final probability ratio found by applying the RV cutoff to a toy model landscape containing two terminal vacua. Substituting the approximation (9.24), which assumes the condition (9.25), we can simplify Eq. (9.27) to a more suggestive form

$$\frac{p(1)}{p(2)} \approx \frac{\kappa_{31} Z_1}{\kappa_{32} Z_2} \left(\frac{\kappa_{31}}{x_*}\right)^{-1+Z_2/Z_1}. \tag{9.28}$$

We can interpret this as the ratio of nucleation probabilities κ_{31}/κ_{32} times the ratio of volume expansion factors, Z_1/Z_2, times a certain "correction" factor. As we have seen, the correction factor is actually a complicated function of all the parameters of the landscape. The correction factor takes the simple form

$$\left(\frac{\kappa_{31}}{x_*}\right)^{-1+Z_2/Z_1} \tag{9.29}$$

only if the condition (9.25) holds.

We note that the result (9.28) is similar to but does not exactly coincide with the results obtained in previously studied volume-based measures. For comparison, the volume-based measures proposed in [222] and [53] both yield

$$\frac{p(1)}{p(2)} = \frac{\kappa_{31} Z_1}{\kappa_{32} Z_2}, \tag{9.30}$$

which is readily interpreted as the ratio of nucleation probabilities enhanced by the ratio of volume factors. The "holographic" measure [107], which is not a volume-based measure, gives the ratio

$$\frac{p(1)}{p(2)} = \frac{\kappa_{31}}{\kappa_{32}} \tag{9.31}$$

that does not depend on the number of e-foldings after nucleation. While the discrepancy between volume-based and worldline-based measures is to be expected, the correction factor that distinguishes Eq. (9.28) from Eq. (9.30) is model-dependent and may be either negligible or significant depending on the particular model.

As a specific example, consider bubbles that nucleate with equal probability, $\kappa_{31} = \kappa_{32} \ll 1$, but have very different expansion factors, $Z_1 \gg Z_2$. Then the condition (9.25) holds and the "correction" factor is given by Eq. (9.29), which is an exponentially large quantity of order κ_{31}^{-1}. Qualitatively this means that the RV measure rewards bubbles with a larger slow-roll expansion factor even more than previous volume-based measures.

On the other hand, if $Z_1 = Z_2$ but $\kappa_{31} \gg \kappa_{32}$, the "correction" factor disappears and we recover the result found in the other measures.

To conclude, we note that the result (9.27) does not depend on the durations of *time* spent during slow-roll inflation inside the terminal bubbles, but only on the number of e-foldings gained. This confirms that the RV measure does not suffer from the youngness paradox.

So far we were able to apply of the RV measure to the comparison of the abundances of terminal vacua. In the next section, the RV measure can be extended to arbitrary observables in landscape models.

9.2 Regulating the total number of bubbles

The RV measure prescription as formulated in the previous section applies only to the calculation of abundances of terminal bubbles. We will now extend the RV measure to computing arbitrary statistics on any landscape.

We first note that RV measure prescription can be applied, strictly speaking, only to landscapes that contain *some* terminal bubble types (*i.e.* vacua from which no further tunneling is possible). However, this limitation is quite benign, for two reasons. First, a landscape without any Minkowski or AdS states is not expected to be realized in any realistic string theory scenario without an exceptional amount of fine-tuning. Second, the previously proposed volume-based and the world-line based measure prescriptions agree for a landscape without terminal bubble types [224, 107]. One may therefore consider the measure problem as solved in such landscapes and turn one's attention to more realistic landscapes where terminal bubble types are present.

Let us take an initial bubble of a nonterminal type j and consider the statistical ensemble $E_n(j)$ of all possible evolutions of the initial bubble such that the total number of nucleated bubbles of all types is finite and equals n (not counting the initial bubble). The total number of nucleated bubbles in a multiverse can be finite only if terminal bubbles nucleate everywhere and merge globally to the future of the initial bubble. This can happen by rare chance; however, it is important the total probability of all events in the ensemble $E_n(j)$ is always nonzero for any given n, so that the ensembles $E_n(j)$ are well-defined and nonempty.

The ensemble $E_n(j)$ may be described in the language of "transition trees" used in [107]. The ensemble consists of all trees that have a total number $n+1$ of bubbles, including the initial bubble of type j. The trees in $E_n(j)$ are finite because all the "outer" leaves are bubbles of terminal types. The motivation for considering the ensemble $E_n(j)$ is that a finite but very large tree (with n large) is a controlled approximation to infinite trees that typically occur. Hence, we are motivated to consider $E_n(j)$ with n finite but very large.

Note that the ensemble $E_n(j)$ differs from the ensemble defined in [246]; in $E_n(j)$ the total number of bubbles of *all* types is equal to n, rather than the total number of *terminal* bubbles as in [246]. Thus, the current proposal, which appears more natural, is an extension of that of [246]. Future work will show whether this technical difference is significant; presently I will investigate the consequences of the current proposal.

Once the ensemble $E_n(j)$ is defined, one may consider the statistical distribution of some cosmological observable within the multiverses belonging to the set $E_n(j)$. For instance, one can count the number of bubbles of some type k, or the number of observers within bubbles of type k, or the number of observations of some physical process, etc. In a very large multiverse belonging to $E_n(j)$ with $n \gg 1$, one may expect that the statistics of observations will be independent of the initial bubble type j. Indeed, this will be one of the results of this chapter. Hence, let us suppress the argument j and write simply E_n.

Each multiverse belonging to E_n has a naturally defined probability weight, which is simply equal to the probability of realizing that multiverse. This probability weight needs to be taken into account when computing the statistical distributions of observables. The sum of all probability weights of multiverses within E_n is equal to the total probability of E_n, which is exponentially small for large n but always nonzero. Since the multiverses from the ensemble E_n are by construction finite, *i.e.* each multiverse supports only a finite total number of possible observers, we are assured that any statistics we desire to compute on E_n will be well-defined.

We can now consider the probability distribution $p(Q|E_n)$ of some interesting observable Q within the ensemble E_n and take the limit $n \to \infty$. One expects that the probability distribution $p(Q|E_n)$ will have a well-defined limit for large n,

$$p(Q) \equiv \lim_{n\to\infty} p(Q|E_n). \tag{9.32}$$

It was shown in previous work on the RV prescription [246, 255] that the distribution $p(Q)$ is well-defined for a simplest toy landscape as well as in the case of random-walk eternal inflation. In this work I extend these results to a general landscape scenario. Below (Sec. 9.4.5) I will prove rigorously that the limit (9.32) indeed exists and is independent of the chosen initial bubble type j as long as the initial bubble is not of terminal type and as long as the landscape is irreducible (every vacuum can be reached from every other non-terminal vacuum by a chain of nucleations). Thus, the distribution $p(Q)$ is unique and well-defined. This distribution is the final result of applying the RV prescription to the observable Q.

In practice, it is necessary to compute the distribution $p(Q|E_n)$ asymptotically in the limit of large n. A direct numerical calculation of probabilities in that limit by enumerating all possible evolution trees is extremely difficult because of the exponential growth of the number of possible evolutions. Instead, I derive explicit formulas for the distribution $p(Q)$ in a generic landscape by evaluating the limit (9.32) analytically. These formulas are the main result of the present chapter.

9.3 A toy landscape

I begin by applying the RV measure to a toy landscape with very few vacua. Using this simple example, I develop the computational techniques needed for the practical evaluation of the limit such as Eq. (9.32). In Sec. 9.4 the same techniques will be extended to a more general landscape with an arbitrary number of vacua.

In Sec. 9.1 we considered the simplest possible nontrivial landscape: a single dS and two AdS (terminal) vacua. The next least complicated example that can be treated analytically is a toy landscape having two nonterminal and two terminal vacua. This toy landscape was called the "FABI" model in [222] and consists of the vacua labeled F, I, A, B with the transition diagram $A \leftarrow F \leftrightarrow I \rightarrow B$. In other words, one assumes that the vacuum F ("false" vacuum) can nucleate only bubbles of types A and I, the vacuum I ("intermediate" vacuum) can nucleate bubbles of types F and B, while A and B are terminal vacua that do not have further nucleations.

To describe the finitely produced probability in this landscape, I use the discrete picture called the "eternal inflation in a box" [75, 246], which is closely related to the "square bubble" approximation [223]. In this picture, one considers the evolution of discrete, causally disjoint homogeneous H-regions in discrete time. All possible vacuum types are labeled by $j = 1, ..., N$. During one time step of order $\delta t = H_j^{-1}$, where H_j is the local Hubble rate in a given H-region of type j, the evolution consists of expanding the H-region into

$$e^{3H_j \delta t} = e^3 \equiv \nu \tag{9.33}$$

daughter H-regions of type j. Each of the daughter H-regions has then the probability $\kappa_{j \to k}$ of changing immediately into an H-region of type $k \neq j$; this imitates a nucleation of a horizon-size bubble of type k. If no transition has taken

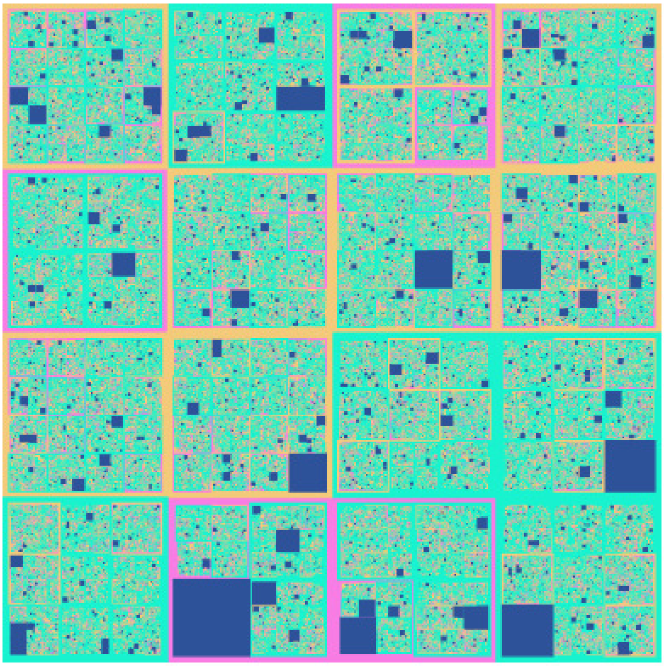

Figure 9.1: An example simulation of "eternal inflation in a box" in two spatial dimensions. Bubbles (H-regions) are represented in comoving coordinates by squares. Dark shades indicate bubbles of terminal types. Other shades correspond to nested bubbles of various nonterminal ("recyclable") types. For the purposes of visual illustration, nucleation rates were chosen of order one, and colored lines were drawn at bubble boundaries.

place, the daughter H-region retains its type j. For convenience, we denote by

$$\kappa_{j\to j} \equiv 1 - \sum_{k\neq j} \kappa_{j\to k} \tag{9.34}$$

the probability of no transitions during one Hubble time. The process of expansion and transition is continued *ad infinitum*, independently for each resulting H-region. The H-regions of terminal types will admit no further transitions and will not expand further (except, perhaps, by a fixed amount due to slow-roll inflation occurring immediately after nucleation). An example simulation is shown in Fig. 9.1.

In the remainder of this section I perform explicit calculations of the RV cutoff in the FABI model.

9.3.1 Bubble abundances

The first task is to compute the relative abundance of bubbles of different types. Consider the probability $p(n_{\text{tot}}, n_F, n_I, n_A, n_B; k)$ of having a finite total number n_{tot} of bubbles of which n_j are of type j (where $j = F, I, A, B$), if one starts from a single initial H-region of type k ($k = F, I$). By construction, $p \neq 0$ only for $n_{\text{tot}} = n_F + n_I + n_A + n_B$. A generating function for this probability distribution can be defined by

$$
\begin{aligned}
g(z, \{q_j\}; k) &\equiv \sum_{n\geq 0, n_j \geq 0} z^n p(n, \{n_j\}; k) \prod_{j=F,I,A,B} q_j^{n_j} \\
&\equiv \langle z^n q_F^{n_F} q_I^{n_I} q_A^{n_A} q_B^{n_B} \rangle_{n<\infty;k},
\end{aligned} \tag{9.35}
$$

where the notation $\langle ... \rangle_{n<\infty;k}$ stands for a statistical average restricted to events with a finite total number n of bubbles nucleated to the future of an initial bubble of type k.

The generating function g plays a crucial role in the entire calculation. Since we will be only interested in the initial bubbles of types F and I, let us denote

$$
F(z, \{q_j\}) \equiv g(z, \{q_j\}; F), \quad I(z, \{q_j\}) \equiv g(z, \{q_j\}; I). \tag{9.36}
$$

The generating functions F and I satisfy the following system of nonlinear algebraic equations [246],

$$
F^{\frac{1}{\nu}} = zq_A\kappa_{FA} + zq_I\kappa_{FI}I + \kappa_{FF}F, \tag{9.37}
$$

$$
I^{\frac{1}{\nu}} = zq_B\kappa_{IB} + zq_F\kappa_{IF}F + \kappa_{II}I. \tag{9.38}
$$

Here we denoted for brevity $\kappa_{FF} \equiv 1 - \kappa_{FA} - \kappa_{FI}$ and $\kappa_{II} \equiv 1 - \kappa_{IB} - \kappa_{IF}$; within our assumptions, $\kappa_{FF} \approx 1$ and $\kappa_{II} \approx 1$. In Eqs. (9.37)–(9.38) the generating variable z multiplies only the terms that correspond to changing the type of the H-region (which imitates the nucleation of new bubbles) but not the terms $\kappa_{FF}F$ and $\kappa_{II}I$ that correspond to the eventuality of not changing the type of the H-region during one Hubble time.

The nonlinear equations (9.37)–(9.38) may have several real-valued solutions, as well as complex-valued solutions that are certainly not of physical interest. In particular, for $z = 1$ and $q_j = 1$ there exists the "trivial" solution $F = I = 1$ as well as a nontrivial solution with $F \ll 1$ and $I \ll 1$. Similarly, for z near 0 there exists the solution that approaches $F(0) = I(0) = 0$,

$$
F = (zq_A\kappa_{FA})^\nu + O(z^{2\nu-1}), \; I \approx (zq_B\kappa_{IB})^\nu + O(z^{2\nu-1}), \tag{9.39}
$$

as well as the solution

$$
F \approx \kappa_{FF}^{\frac{\nu}{1-\nu}} \approx 1, \quad I \approx \kappa_{II}^{\frac{\nu}{1-\nu}} \approx 1 \tag{9.40}
$$

and solutions where $F \approx 1$ and $I \approx 0$ and vice versa. It is important to determine the solution branch $F(z), I(z)$ that has the physical significance as the actual generating function of the finitely produced distribution of H-regions.

The functions $F(z), I(z)$ are solutions of algebraic equations and thus are continuous functions of z that are analytic everywhere in complex z plane except for branch cuts. It is easy to see that the solution $F = I = 1$ at $z = 1, q_j = 1$ is continuously connected with the solution (9.40) at $z \approx 0$, while the solution (9.39) is continued to a solution with $F(z) \ll 1$ and $I(z) \ll 1$ for all $0 < z < 1$. The values $F(1)$ and $I(1)$ are the probabilities of the events that the evolution of an initial bubble of type F or I ends globally. These probabilities are extremely small and of order κ_{FA}^{ν} and κ_{IB}^{ν} respectively. This is easy to interpret because, for instance, κ_{FA}^{ν} is the probability of nucleating ν terminal regions at once after one Hubble time within an H-region of type F. Hence, the generating functions for the finitely produced distribution of bubbles are given by the solution branch having $F \ll 1$ and $I \ll 1$ rather than by the solution $F = I = 1$ at $z = 1$. More precisely, the physically meaningful solution $F(z), I(z)$ is selected by the asymptotic behavior (9.39). This argument removes the ambiguity inherent in solving Eqs. (9.37)–(9.38). Below we refer to the branch of solutions $F(z), I(z)$ connected to the nontrivial solution (9.39) near $z = 0$ as the "main branch."

Once the main branch of the generating functions $g(z, \{q_j\}; k)$ are known, one can express the mean number of bubbles of type i ($i = F, I, A, B$) at a fixed total number n_{tot} as

$$p(i|n_{\mathrm{tot}}) \equiv \frac{\langle n_i \rangle_{n_{\mathrm{tot}}}}{n_{\mathrm{tot}}} = \frac{\partial_z^{n_{\mathrm{tot}}} \partial_{q_i} g(z = 0, \{q_j = 1\}; k)}{n_{\mathrm{tot}} \partial_z^{n_{\mathrm{tot}}} g(z = 0, \{q_j = 1\}; k)}. \tag{9.41}$$

One expects that the limit of this ratio at $n_{\mathrm{tot}} \to \infty$ will be independent of the initial bubble type k since the ensemble $E_{n_{\mathrm{tot}}}$ will consist of H-regions having a very long evolution, so that the initial conditions are forgotten. Below I will show explicitly that this is indeed the case.

It is more convenient to compute the ratios of the number of bubbles of types i and i' at fixed n_{tot},

$$\frac{p(i|n_{\mathrm{tot}})}{p(i'|n_{\mathrm{tot}})} = \left. \frac{\partial_z^{n_{\mathrm{tot}}} \partial_{q_i} g}{\partial_z^{n_{\mathrm{tot}}} \partial_{q_{i'}} g} \right|_{z=0, \{q_j = 1\}}. \tag{9.42}$$

Then one only needs to compute derivatives $\partial_{q_i} g \equiv g_{,q_i}$ evaluated at $q_j = 1$. These derivatives satisfy a system of *linear* equations that can be easily derived from Eqs. (9.37)–(9.38). For instance, the derivatives $F_{,q_A}$ and $I_{,q_A}$ satisfy

$$\frac{1}{\nu} F^{\frac{1}{\nu}-1} F_{,q_A} = z\kappa_{FA} + z\kappa_{FI} I_{,q_A} + \kappa_{FF} F_{,q_A}, \tag{9.43}$$

$$\frac{1}{\nu} I^{\frac{1}{\nu}-1} I_{,q_A} = z\kappa_{IF} F_{,q_A} + \kappa_{II} I_{,q_A}. \tag{9.44}$$

Rewriting these equations in a matrix form, we obtain

$$\begin{pmatrix} \frac{1}{\nu} F^{\frac{1}{\nu}-1} - \kappa_{FF} & -z\kappa_{FI} \\ -z\kappa_{IF} & \frac{1}{\nu} I^{\frac{1}{\nu}-1} - \kappa_{II} \end{pmatrix} \begin{bmatrix} F_{,q_A} \\ I_{,q_A} \end{bmatrix} = \begin{bmatrix} z\kappa_{FA} \\ 0 \end{bmatrix}. \tag{9.45}$$

The coefficients of the z-dependent matrix

$$\hat{M}(z) \equiv \begin{pmatrix} \frac{1}{\nu}F^{\frac{1}{\nu}-1} - \kappa_{FF} & -z\kappa_{FI} \\ -z\kappa_{IF} & \frac{1}{\nu}I^{\frac{1}{\nu}-1} - \kappa_{II} \end{pmatrix} \tag{9.46}$$

are the main branch of solutions of the nonlinear equations (9.37)–(9.38) at $q_j = 1$ but at arbitrary z.

One can verify using Eq. (9.39) that the matrix $\hat{M}(z)$ is invertible near $z = 0$. Hence the solution of Eq. (9.45) can be written, at least within some range of z where $\hat{M}(z)$ remains invertible, as

$$\begin{bmatrix} F_{,q_A} \\ I_{,q_A} \end{bmatrix} = \hat{M}^{-1}(z) \begin{bmatrix} z\kappa_{FA} \\ 0 \end{bmatrix}. \tag{9.47}$$

Similarly, the derivatives $F_{,q_F}$ and $I_{,q_F}$ satisfy the equations

$$\hat{M}(z) \begin{bmatrix} F_{,q_F} \\ I_{,q_F} \end{bmatrix} = \begin{bmatrix} 0 \\ z\kappa_{IF}F \end{bmatrix}, \tag{9.48}$$

whose solution is

$$\begin{bmatrix} F_{,q_F} \\ I_{,q_F} \end{bmatrix} = \hat{M}^{-1}(z) \begin{bmatrix} 0 \\ z\kappa_{IF}F \end{bmatrix}. \tag{9.49}$$

Other generating functions can be expressed in the same manner.

One could in principle obtain a numerical solution for $F_{,q_A}(z)$, $I_{,q_A}(z)$, and all the other generating functions at any given value of z. However, the numerical solution is not particularly useful at this point because the next step in the calculation is the evaluation of Eq. (9.42) in the limit of very large n_{tot}, for instance,

$$\frac{p(A|n_{\text{tot}})}{p(F|n_{\text{tot}})} = \left. \frac{\partial_z^{n_{\text{tot}}} F_{,q_A}}{\partial_z^{n_{\text{tot}}} F_{,q_F}} \right|_{z=0}. \tag{9.50}$$

It is generally not feasible to compute the n-th derivative of a numerically obtained function in the limit $n \to \infty$, because the unavoidable round-off errors are amplified by a fixed factor with each successive numerical differentiation. Therefore, we need a way to evaluate the ratio (9.42) in the limit $n_{\text{tot}} \to \infty$ without using numerics. Indeed we will be able to compute the ratio $p(A)/p(F)$. We will also show that, for instance,

$$\frac{p(A)}{p(F)} = \lim_{n\to\infty} \left. \frac{\partial_z^n F_{,q_A}}{\partial_z^n F_{,q_F}} \right|_{z=0} = \lim_{n\to\infty} \left. \frac{\partial_z^n I_{,q_A}}{\partial_z^n I_{,q_F}} \right|_{z=0}; \tag{9.51}$$

in other words, that the final RV-regulated ratio $p(A)/p(F)$ is independent of whether the initial bubble is of type F or of type I.

To proceed, we use the fact that $F_{,q_A}(z)$, $I_{,q_A}(z)$, $F_{,q_F}(z)$, etc. are analytic functions of the parameter z. The asymptotic growth of high-order derivatives of an analytic function $f(z)$ is determined by the location of the singularities of $f(z)$ in the complex z plane, as shown in Eq. (9.15).

We now need to determine the location of the singularities of $F_{,q_A}(z)$, $I_{,q_A}(z)$, etc., as functions of z. This task is much simplified once we observe that all these quantities are expressed through the inverse matrix $\hat{M}^{-1}(z)$, and hence it remains to analyze the singularities of $\hat{M}^{-1}(z)$. That matrix can be singular at some value $z = z_*$ either because some of the coefficients of $\hat{M}(z)$ are singular, or because the matrix $\hat{M}(z)$ is degenerate (noninvertible) at $z = z_*$. The coefficients of $\hat{M}(z)$ depend on solutions $F(z), I(z)$ of algebraic equations (9.37)–(9.38) at $q_j = 1$ and thus cannot be divergent as functions of z. These coefficients can be singular only in that some derivative in z diverges. The derivatives $F_{,z}$ and $I_{,z}$ satisfy the equations

$$\hat{M}(z) \begin{bmatrix} F_{,z} \\ I_{,z} \end{bmatrix} = \begin{bmatrix} \kappa_{FA} + \kappa_{FI} I \\ \kappa_{IB} + \kappa_{IF} F \end{bmatrix}. \tag{9.52}$$

Therefore, $F_{,z}$ and $I_{,z}$ diverge only for those z for which the matrix $\hat{M}(z)$ is degenerate. For these z, all the derivatives $F_{,z}$, $I_{,z}$, $F_{,q_A}$, etc. will be divergent at the same time since they are all proportional to the inverse matrix $\hat{M}^{-1}(z)$.

We note that the matrix $\hat{M}(z)$ is the Jacobian of the nonlinear system (9.37)–(9.38). As long as $\hat{M}(z)$ is nondegenerate, all the different branches of the solutions $F(z), I(z)$ do not meet and remain smooth functions of z. As we noted above, the "main" branch $F(z), I(z)$ is the one connected to the nontrivial solution (9.39) at $z \approx 0$. The matrix $\hat{M}(z)$ is nondegenerate near $z = 0$; therefore, the solutions $F(z), I(z)$ remain smooth functions of z for all z such that $\hat{M}(z)$ is nondenegerate. We conclude that the only possible singularities of $F(z), I(z)$ are those values $z = z_*$ where $\det \hat{M}(z) = 0$.

Below it will be shown that the behavior of $\det \hat{M}(z)$ near $z = z_*$ is

$$\det \hat{M}(z) \approx c_1 \sqrt{z_* - z}, \tag{9.53}$$

i.e. of the form (9.14) with $c_0 = 0$ and $s = \frac{1}{2}$. It now follows from Eqs. (9.47) and (9.49) that the derivatives such as $F_{,q_A}(z)$, $I_{,q_A}(z)$, $F_{,q_F}(z)$, etc., all diverge at $z = z_*$ with the same asymptotic behavior, namely proportional to $(z_* - z)^{-1/2}$. We may express the inverse matrix as

$$\hat{M}^{-1}(z) = \frac{1}{\det \hat{M}(z)} \hat{\hat{M}}(z), \tag{9.54}$$

$$\hat{\hat{M}}(z) \equiv \begin{pmatrix} \frac{1}{\nu} I^{\frac{1}{\nu}-1} - \kappa_{II} & z\kappa_{FI} \\ z\kappa_{IF} & \frac{1}{\nu} F^{\frac{1}{\nu}-1} - \kappa_{FF} \end{pmatrix}, \tag{9.55}$$

where we introduced the algebraic complement matrix $\hat{\hat{M}} \equiv \hat{M}^{-1} \det \hat{M}$, which remains well-defined at $z = z_*$. It then follows from Eq. (9.47) that the asymptotic behavior of the functions $F_{,q_A}(z)$ and $I_{,q_A}(z)$ near $z = z_*$ is given by

$$\begin{aligned} \begin{bmatrix} F_{,q_A} \\ I_{,q_A} \end{bmatrix} &\approx \frac{1}{c_1 \sqrt{z_* - z}} \hat{\hat{M}}(z_*) \begin{bmatrix} z_* \kappa_{FA} \\ 0 \end{bmatrix} \\ &= \frac{1}{c_1 \sqrt{z_* - z}} \begin{bmatrix} \left(\frac{1}{\nu} I^{\frac{1}{\nu}-1} - \kappa_{II} \right) z_* \kappa_{FA} \\ z_*^2 \kappa_{IF} \kappa_{FA} \end{bmatrix}. \end{aligned} \tag{9.56}$$

Similarly, using Eq. (9.49) we find near $z = z_*$

$$\begin{bmatrix} F_{,q_F} \\ I_{,q_F} \end{bmatrix} \approx \frac{1}{c_1\sqrt{z_* - z}} \hat{\hat{M}}(z_*) \begin{bmatrix} 0 \\ z_*\kappa_{IF}F \end{bmatrix}$$
$$= \frac{1}{c_1\sqrt{z_* - z}} \begin{bmatrix} z_*^2\kappa_{FI}\kappa_{IF}F \\ \left(\frac{1}{\nu}F^{\frac{1}{\nu}-1} - \kappa_{FF}\right) z_*\kappa_{IF}F \end{bmatrix}. \tag{9.57}$$

Using Eq. (9.15) and noticing that all n-dependent factors are the same in $\partial_z^n F_{,q_A}$ and other such derivatives, we conclude that

$$\lim_{n\to\infty} \left.\frac{\partial_z^n F_{,q_A}}{\partial_z^n F_{,q_F}}\right|_{z=0} = F^{\frac{1}{\nu}} \frac{\left(\frac{1}{\nu}I_*^{\frac{1}{\nu}-1} - \kappa_{II}\right)\kappa_{FA}}{z_*\kappa_{FI}\kappa_{IF}F_*}, \tag{9.58}$$

$$\lim_{n\to\infty} \left.\frac{\partial_z^n I_{,q_A}}{\partial_z^n I_{,q_F}}\right|_{z=0} = \frac{z_*\kappa_{FA}}{\left(\frac{1}{\nu}F_*^{\frac{1}{\nu}-1} - \kappa_{FF}\right)F_*}. \tag{9.59}$$

It is important that the two limits above are equal; this is so because the condition $\det \hat{M}(z_*) = 0$ yields

$$\det \hat{M} = \left[\frac{F_*^{\frac{1}{\nu}-1}}{\nu} - \kappa_{FF}\right]\left[\frac{I_*^{\frac{1}{\nu}-1}}{\nu} - \kappa_{II}\right] - z_*^2\kappa_{FI}\kappa_{IF} = 0, \tag{9.60}$$

where we denoted $F_* \equiv F(z_*)$, $I_* \equiv I(z_*)$ for brevity. It follows that the ratio of A-bubbles to F-bubbles is independent of whether the initial bubble is of type F or of type I. In other words, the RV-regulated ratio of A-bubbles to F-bubbles is

$$\frac{p(A)}{p(F)} = \frac{z_*\kappa_{FA}}{\left(\frac{1}{\nu}F_*^{\frac{1}{\nu}-1} - \kappa_{FF}\right)F_*} \tag{9.61}$$

independently of the initial bubble type. Below (Sec. 9.4.5) the independence of initial conditions will be rigorously proved for a general landscape using mathematical techniques of the theory of nonnegative matrices. Presently we have shown this independence using explicit formulas available for the toy landscape under consideration.

In a similar way, we find the RV-regulated ratio of A-bubbles to B-bubbles,

$$\frac{p(A)}{p(B)} = \frac{z_*\kappa_{FA}\kappa_{IF}}{\left(\frac{1}{\nu}F_*^{\frac{1}{\nu}-1} - \kappa_{FF}\right)\kappa_{IB}}, \tag{9.62}$$

and the ratio of F-bubbles to I-bubbles,

$$\frac{p(F)}{p(I)} = \frac{\left(\frac{1}{\nu}F_*^{\frac{1}{\nu}-1} - \kappa_{FF}\right)F_*}{z_*\kappa_{FI}I_*}. \tag{9.63}$$

It remains to compute z_*, F_*, I_* and to justify Eq. (9.53). We will do this using the explicit form of the matrix $\hat{M}(z)$. To determine z_*, F_*, and I_*, we need to solve Eq. (9.60) simultaneously with the equations

$$F_*^{\frac{1}{\nu}} = z_* \kappa_{FA} + z_* \kappa_{FI} I_* + \kappa_{FF} F_*, \tag{9.64}$$

$$I_*^{\frac{1}{\nu}} = z_* \kappa_{IB} + z_* \kappa_{IF} F_* + \kappa_{II} I_*, \tag{9.65}$$

the latter two being Eqs. (9.37)–(9.38) after setting $q_j = 1$. We are interested in the solution z_* closest to $z = 0$. By definition (9.35), the generating functions are nonsingular for $|z| \leq 1$; hence, the only possible values of z_* are in the domain $|z| > 1$ in the complex plane.

Moreover, we can show that the solutions $F(z), I(z)$ are growing functions of z for real $z < z_*$. Initially at $z \approx 0$ these functions have the form (9.39) and hence are positive. The determinant $\det \hat{M}$ shown in Eq. (9.60) is also positive for $z < z_*$. One can then find from Eq. (9.55) that the algebraic complement matrix $\hat{\hat{M}}$ and hence the inverse matrix $\hat{M}^{-1}$ has all positive elements for those z. It is then evident from Eq. (9.52) that $\partial_z F > 0$ and $\partial_z I > 0$ as long as the values of F and I are themselves positive. Therefore, the solutions $F(z), I(z)$ are positive and growing functions of z as long as $\det \hat{M}(z) > 0$.

We may thus visualize the behavior of $\det \hat{M}(z)$ as z grows. Both $F(z)$ and $I(z)$ will grow with z, so that the terms in square brackets diminish while the second term, $z_*^2 \kappa_{FI} \kappa_{IF}$, grows. Eventually the product of the square brackets in Eq. (9.60) will be balanced by the second term, and the determinant will vanish. We need to determine the smallest value $z = z_*$ for which $\det \hat{M}(z) = 0$.

Since the physically significant branch of the solution involves always very small values $F(z)$ and $I(z)$ for $|z| \leq 1$, while other (unphysical) branches have either F or I approximately equal to 1, it is reasonable to assume that $F(z_*) \ll 1$ and $I(z_*) \ll 1$ also at $z = z_*$ (the self-consistency of this assumption will be confirmed by later calculations). Then the terms $\kappa_{FF} F_* \approx F_*$ and $\kappa_{II} I_* \approx I_*$ can be disregarded in comparison with $F_*^{1/\nu}$ and $I_*^{1/\nu}$ in Eqs. (9.64)–(9.65). In this approximation, we can simplify Eqs. (9.64)–(9.65) to

$$F_*^{\frac{1}{\nu}} = z_* \kappa_{FA} + z_* \kappa_{FI} I_*, \tag{9.66}$$

$$I_*^{\frac{1}{\nu}} = z_* \kappa_{IB} + z_* \kappa_{IF} F_*, \tag{9.67}$$

while Eq. (9.60) becomes

$$(\kappa_{FA} + \kappa_{FI} I_*)(\kappa_{IB} + \kappa_{IF} F_*) = \nu^2 \kappa_{FI} \kappa_{IF} F_* I_*. \tag{9.68}$$

The ratios (9.61)–(9.63) are also simplified and can be written more concisely as

$$\begin{aligned} p(A) : p(I) : p(F) : p(B) &= \kappa_{FA} : (\kappa_{FI} I_*) \\ &: \frac{\kappa_{FA} + \kappa_{FI} I_*}{\nu} : \kappa_{IB} \frac{\kappa_{FA} + \kappa_{FI} I_*}{\nu \kappa_{IF} F_*}. \end{aligned} \tag{9.69}$$

To determine z_*, we will obtain an explicit approximation for the main branch $F(z), I(z)$ for all z. For small enough z, the solutions of Eqs. (9.64)–(9.65) are well approximated by Eq. (9.39),

$$F(z) \approx (z\kappa_{FA})^{\nu}, \quad I(z) \approx (z\kappa_{IB})^{\nu}. \tag{9.70}$$

These solutions are obtained under the assumption that the terms $z\kappa_{FA}$ and $z\kappa_{IB}$ are numerically small ($z\kappa_{FA} \ll 1$, $z\kappa_{IB} \ll 1$) and yet dominant in Eqs. (9.64)–(9.65). These terms only remain dominant as long as

$$\kappa_{FA} \gg \kappa_{FI} I(z), \quad \kappa_{IB} \gg \kappa_{IF} F(z). \tag{9.71}$$

Substituting Eq. (9.39) for $F(z)$ and $I(z)$ into Eq. (9.71), we obtain the conditions

$$z \ll \kappa_{IB}^{-1} \left(\frac{\kappa_{FA}}{\kappa_{FI}} \right)^{\frac{1}{\nu}}, \quad z \ll \kappa_{FA}^{-1} \left(\frac{\kappa_{IB}}{\kappa_{IF}} \right)^{\frac{1}{\nu}}. \tag{9.72}$$

We need to check whether $\det \hat{M}(z)$ could vanish already for some z_* within the range (9.72). Using Eq. (9.68) in the regime (9.71), we find

$$z_* \approx \left[(\kappa_{FA}\kappa_{IB})^{\nu-1} \nu^2 \kappa_{FI}\kappa_{IF} \right]^{-\frac{1}{2\nu}}. \tag{9.73}$$

This value is within the range (9.72) only if the following simultaneous inequalities hold,

$$e^{-6} \equiv \frac{1}{\nu^2} \ll \left(\frac{\kappa_{IB}}{\kappa_{FA}} \right)^{\nu+1} \frac{\kappa_{FI}}{\kappa_{IF}} \ll \nu^2 \equiv e^6. \tag{9.74}$$

Let us denote by η the quantity in Eq. (9.74),

$$\eta \equiv \left(\frac{\kappa_{IB}}{\kappa_{FA}} \right)^{\nu+1} \frac{\kappa_{FI}}{\kappa_{IF}}, \tag{9.75}$$

then Eq. (9.74) becomes simply $|\ln \eta| < 6$. One would expect that η is generically either very large or very small, so the inequalities (9.74) can hold only in a fine-tuned landscape. Additionally, we need to require

$$z_* \ll \min \left(\frac{1}{\kappa_{FA}}, \frac{1}{\kappa_{IB}} \right), \tag{9.76}$$

which entails

$$\frac{\kappa_{IB}}{\kappa_{IF}} \ll \nu\sqrt{\eta}, \quad \frac{\kappa_{FA}}{\kappa_{FI}} \ll \frac{\nu}{\sqrt{\eta}}. \tag{9.77}$$

However, this requirement is weaker than Eq. (9.74) since typically $\kappa_{FA} \ll \kappa_{FI}$ and $\kappa_{IB} \ll \kappa_{IF}$, so

$$z_* \ll \kappa_{IB}^{-1} \left(\frac{\kappa_{FA}}{\kappa_{FI}} \right)^{\frac{1}{\nu}} \ll \kappa_{IB}^{-1}; \; z \ll \kappa_{FA}^{-1} \left(\frac{\kappa_{IB}}{\kappa_{IF}} \right)^{\frac{1}{\nu}} \ll \kappa_{FA}^{-1}. \tag{9.78}$$

Let us assume, for the moment, that Eqs. (9.74) and (9.77) hold. Then we use Eq. (9.69) to obtain

$$p(A) : p(I) : p(F) : p(B) \approx 1 : \frac{\sqrt{\eta}}{\nu} : \frac{1}{\nu} : \sqrt{\eta}. \tag{9.79}$$

Due to the fine-tuning assumption (9.74), these ratios are all within the interval $\left[\nu^{-2}, \nu^{2}\right] = \left[e^{-6}, e^{6}\right]$.

Having considered the fine-tuned case, let us now turn to the generic case. Generically one would expect that the quantity η is either extremely large or extremely small, and in any case outside the logarithmically narrow range $\left[e^{-6}, e^{6}\right]$. In that case, one of the inequalities (9.74) does not hold; generically, either $\eta \ll \nu^{-2}$ or $\eta \gg \nu^{2}$. It follows that $\det \hat{M}(z) \neq 0$ for all z within the range (9.72). To reach the value z_* at which $\det \hat{M}(z_*) = 0$, we need to increase z further, until one of the terms $z\kappa_{FI}I$ or $z\kappa_{IF}F$ in Eqs. (9.64)–(9.65) becomes dominant and the solution (9.70) becomes invalid.

It is impossible that *both* the terms $z\kappa_{FI}I$ and $z\kappa_{IF}F$ are dominant in Eqs. (9.64)–(9.65) at $z = z_*$ because then Eq. (9.68) would yield a contradiction,

$$\kappa_{FI} I_* \kappa_{IF} F_* = \nu^2 \kappa_{FI} \kappa_{IF} F_* I_*. \tag{9.80}$$

Hence, the determinant $\det \hat{M}(z)$ first vanishes at the value $z = z_*$ such that only *one* of those terms is dominant in its respective equation. Without loss of generality, we may relabel the vacua ($F \leftrightarrow I$, $A \leftrightarrow B$) such that the term $z\kappa_{IF}F$ becomes dominant in Eq. (9.65) while the term $z\kappa_{FA}$ is still dominant in Eq. (9.64). This is equivalent to assuming $\eta \ll \nu^{-2}$. In the range of z for which this is the case,

$$z\kappa_{FI} I(z) \ll z\kappa_{FA} \ll 1; \quad z\kappa_{IB} \ll z\kappa_{IF} F(z) \ll 1, \tag{9.81}$$

the approximate solution of Eqs. (9.64)–(9.65) can be written as

$$F(z) \approx \left(z\kappa_{FA}\right)^{\nu}; \; I(z) \approx \left(z\kappa_{IF} F\right)^{\nu} = \kappa_{FA}^{\nu^2} \kappa_{IF}^{\nu} z^{(\nu+1)\nu}. \tag{9.82}$$

With these values of $F(z)$ and $I(z)$, the consistency requirement (9.81) yields the following range of z,

$$\frac{\kappa_{IB}}{\kappa_{IF}\kappa_{FA}^{\nu}} \ll z^{\nu} \ll \left[\kappa_{FA}^{\nu^2 - 1} \kappa_{FI} \kappa_{IF}^{\nu}\right]^{-\frac{1}{\nu+1}}. \tag{9.83}$$

This range is nonempty if

$$\eta \equiv \left(\frac{\kappa_{IB}}{\kappa_{FA}}\right)^{\nu+1} \frac{\kappa_{FI}}{\kappa_{IF}} \ll 1, \tag{9.84}$$

which is indeed one of the two possible ways that the fine-tuning (9.74) can fail. Having assumed that the above condition holds, we need to determine the value of z_* and check that it belongs to the range (9.81). Using Eq. (9.68) in the regime (9.81), we obtain

$$z_* \approx \left[\nu^2 \kappa_{IF}^{\nu} \kappa_{FI} \kappa_{FA}^{\nu^2-1}\right]^{-\frac{1}{\nu(\nu+1)}}. \tag{9.85}$$

Substituting this value into the inequalities (9.83), we find the condition

$$\eta \ll \frac{1}{\nu^2} \ll 1, \tag{9.86}$$

which holds identically under the current assumption, $\eta \ll \nu^{-2}$. (The relabeling $F \leftrightarrow I$, $A \leftrightarrow B$ is necessary if the opposite case, $\eta \gg \nu^2$, holds.) Therefore, z_* is within the regime (9.81), and our approximations are self-consistent, yielding

$$F_* \approx (z_* \kappa_{FA})^{\nu} = \frac{\kappa_{FA}}{\left[\nu^2 \kappa_{IF}^{\nu} \kappa_{FI}\right]^{\frac{1}{\nu+1}}}, \quad I_* \approx \frac{\kappa_{FA}}{\nu^2 \kappa_{FI}}. \tag{9.87}$$

The ratios (9.69) become

$$p(A) : p(I) : p(F) : p(B) \approx 1 : \frac{1}{\nu^2} : \frac{1}{\nu} : \left[\frac{\eta}{\nu^{\nu-1}}\right]^{\frac{1}{\nu+1}}. \tag{9.88}$$

This is the result of applying the RV prescription to a generic $FABI$ toy landscape in the second regime.

It remains to justify the statement of Eq. (9.53). Below in Sec. 9.4.5 I will demonstrate that the property (9.53) holds for a general landscape. Here only a simple argument is presented to illustrate this property for Eqs. (9.66)–(9.67). Using those equations, we can express I_* through F_* and derive a closed algebraic equation for F_*,

$$F_* = z^{\nu} \left[\kappa_{FA} + \kappa_{FI} z^{\nu} \left(\kappa_{IB} + \kappa_{IF} F_*\right)^{\nu}\right]^{\nu} \equiv f(z; F_*). \tag{9.89}$$

The solution $F(z)$ is given by the intersection of the line $y = F$ and the curve $y = f(z;F)$ in the $y - F$ plane. The function $f(z;F)$ is convex in F for $F > 0, z > 0$; hence, there will be a value $z = z_*$ for which the curve $f(z_*;F)$ is tangent to the line $y = F$, *i.e.* $f_{,F}(z_*;F_*) = 1$. This value of z_* will then implicitly determine F_*. For values $F \approx F_*$, $z \approx z_*$ the dependence of $F(z)$ on z will exhibit the singularity behavior of the type $\sqrt{z_* - z}$. To see this formally, we may expand

$$\begin{aligned} F = f(z;F) &\approx F_* + f_{,F}\left(F - F_*\right) + \frac{1}{2} f_{,FF}\left(F - F_*\right)^2 \\ &+ f_{,z}\left(z - z_*\right). \end{aligned} \tag{9.90}$$

Since $f_{,F}(z_*;F_*) = 1$, we obtain

$$F \approx F_* + \left.\frac{2 f_{,z}}{f_{,FF}}\right|_{F_*, z_*} \sqrt{z_* - z}. \tag{9.91}$$

This shows explicitly the singularity structure of the form (9.14). The determinant of the matrix $\hat{M}$ is a smooth function of $F(z)$ and $I(z)$ near $z = z_*$. Expressing $\det \hat{M}$ as a function only of F, we obtain for $z \approx z_*$ the required formula (9.53),

$$\det \hat{M}(z) \approx (F - F_*) \left. \frac{d}{dF} \right|_{F_*} \det \hat{M} \propto \sqrt{z_* - z}. \tag{9.92}$$

9.3.2 "Boltzmann brains"

Let us now use the same techniques to compute the relative abundance of "Boltzmann brain" observers to ordinary observers.

We need to introduce an appropriate set of generating functions. "Boltzmann brains" can be created with a fixed probability per unit 4-volume, unlike ordinary observers who can appear only within a narrow interval of time after creation of a given bubble. Let us therefore compare the total number of bubbles n_j with the total number of 4-volumes H_j^{-4} in bubbles of type j.

To be specific, let us fix $j = F$ (no Boltzmann brains can be expected in a terminal vacuum). Consider the probability $p(n_{\text{tot}}, n_F, N_F; k)$ of having a finite total number n_{tot} of bubbles of which n_F are of type F and N_F is the total number of H-regions of type F, if one starts from a single initial H-region of type k ($k = F, I$). The number of "Boltzmann brains" in bubbles of type F is proportional to N_F with a small proportionality constant, κ_F^{BB}, which we will include at the end of the calculation. A generating function g for the probability distribution p can be defined by

$$\begin{aligned} g(z, q, r; k) &\equiv \sum_{n, n_F, N_F \geq 0} z^n q^{n_F} r^{N_F} p(n, n_F, N_F; k) \\ &\equiv \left\langle z^n q^{n_F} r^{N_F} \right\rangle_{n<\infty; k}. \end{aligned} \tag{9.93}$$

For the initial bubble types $k = F$ and $k = I$, let us denote for brevity

$$F(z, q, r) \equiv g(z, q, r; F), \quad I(z, q, r) \equiv g(z, q, r; I). \tag{9.94}$$

The generating functions F and I satisfy the following system of equations,

$$F^{\frac{1}{\nu}} = z\kappa_{FA} + z\kappa_{FI} I + r\kappa_{FF} F, \tag{9.95}$$

$$I^{\frac{1}{\nu}} = z\kappa_{IB} + zrq\kappa_{IF} F + \kappa_{II} I. \tag{9.96}$$

These equations differ from the analogous Eqs. (9.37)–(9.38) in that the generating parameter q appears when a *new bubble* of type F is created, while the parameter r appears every time a new H-region of type F is created, which can happen via Hubble expansion of old F-bubbles as well as through nucleation of new F-bubbles.

We would like to compare the mean number of H-regions of type F with the mean number of new bubbles of type F, so we compute (*e.g.* starting with I-bubbles)

$$\frac{\langle N_F \rangle_{n;I}}{\langle n_F \rangle_{n;I}} = \left. \frac{\partial_z^n \partial_r I}{\partial_z^n \partial_q I} \right|_{z=0,r=1,q=1} \tag{9.97}$$

and take the limit $n \to \infty$. Taking the limit as $n \to \infty$ of the ratio $\partial_z^n F_{,r} / \partial_z^n F_{,q}$ will yield the same result, but the limit of Eq. (9.97) is more straightforwardly analyzed.

As before, we first obtain the equations for the derivatives $F_{,q}$, $I_{,q}$, $F_{,r}$, $I_{,r}$ at $q = r = 1$ from Eqs. (9.95)–(9.96). The derivatives $F_{,q}$ and $I_{,q}$ satisfy the same equations as before, namely Eq. (9.48), while $F_{,r}$ and $I_{,r}$ satisfy

$$\hat{M}(z) \left[\begin{array}{c} F_{,r} \\ I_{,r} \end{array} \right] = \left[\begin{array}{c} \kappa_{FF} F \\ z\kappa_{IF} F \end{array} \right], \tag{9.98}$$

whose solution is

$$\left[\begin{array}{c} F_{,r} \\ I_{,r} \end{array} \right] = \hat{M}^{-1}(z) \left[\begin{array}{c} \kappa_{FF} F \\ z\kappa_{IF} F \end{array} \right]. \tag{9.99}$$

Using the same arguments as in the previous section, we evaluate the limit of Eq. (9.97),

$$\lim_{n\to\infty} \left. \frac{\partial_z^n \partial_r I}{\partial_z^n \partial_q I} \right|_{z=0,r=1,q=1} = \nu F_*^{1-\frac{1}{\nu}} \kappa_{FF} + 1. \tag{9.100}$$

To analyze this simple result, we do not actually need to use the complicated decision procedure of Sec. 9.3. Since $\kappa_{FF} \approx 1$ and $F_* < 1$ in any case,[4] the value (9.100) is bounded from above by $\nu + 1$, which is not a large number. Hence, the mean total number of H-regions of type F is at most $\nu + 1$ times larger than the mean total number of bubbles of type F. (Both numbers are finite in finite multiverses, and the relationship persists in the limit $n_{\text{tot}} \to \infty$.)

Analogous results are obtained for regions of type I. We simply need to replace F with I and κ_{FF} by κ_{II} in Eq. (9.100).

The "Boltzmann brains" are created at a very small rate κ_j^{BB} per H-region of type j, while ordinary observers are created at a much larger rate per reheated 3-volume H_j^{-3} in bubbles of the same type. We conclude that the RV-regulated abundance of "Boltzmann brains" is always negligible compared with the abundance of ordinary observers in bubbles of the same type.

9.4 A general landscape

In the previous section we performed computations in a simple toy model of the landscape. Let us now consider a general landscape containing N vacua, labeled $j = 1, ..., N$. The dimensionless transition rates $\kappa_{j\to k}$ between vacua j and k are considered known. The main task is to compute the RV-regulated ratio

[4] Typically $F_* \ll 1$ but we can do with a weaker bound $F_* < 1$ here.

of abundances of bubbles of kinds j and k. We will also compare the abundances of ordinary observers with that of Boltzmann brains.

9.4.1 Bubble abundances

It is convenient to denote by T the set of terminal bubble types and to relabel the vacua such that $j = 1, ..., N_r$ are the "recyclable" (nonterminal) vacua. Thus, $T = \{N_r + 1, ..., N\}$. We start by considering the probability $p(n, \{n_j\}; k)$ of having n_j bubbles of type j, with total $n = \sum_j n_j$ bubbles of all types, to the future of an initial bubble of type $k \notin T$. The generating function $g(z, \{q_j\}; k)$ for this probability distribution can be defined by a straightforward generalization of Eq. (9.35),

$$g(z, \{q_l\}; k) \equiv \sum_{n,\{n_l\}\geq 0} p(n, \{n_l\}; k) z^n \prod_j q_j^{n_j}. \tag{9.101}$$

Here z is the generating parameter for n, and q_j are the generating parameters for n_j. The generating function can be written symbolically as the average

$$g(z, \{q_l\}; k) \equiv \langle z^n q_1^{n_1} ... q_N^{n_N} \rangle_{n<\infty;k}, \tag{9.102}$$

where the subscript $(n < \infty)$ indicates that only the events with a finite total number of bubbles contribute to the statistical average.

For terminal bubble types $k \in T$, the definition (9.101) yields $g(z, \{q_j\}; k) = 1$ since there are no further bubbles to the future of terminal bubbles, thus $n = n_j = 0$ with probability 1.

The generating function $g(z, \{q_j\}; k)$, $k = 1, ..., N_r$ satisfies the system of N_r nonlinear equations

$$\begin{aligned} g^{\frac{1}{\nu}}(z, \{q_j\}; k) = &\sum_{i\notin T, i\neq k} z q_i \kappa_{k\to i} g(z, \{q_j\}; i) \\ &+ \sum_{i\in T} z q_i \kappa_{k\to i} + \kappa_{k\to k} g(z, \{q_j\}; k), \end{aligned} \tag{9.103}$$

where $\nu \equiv e^3$ as before, and the quantities $\kappa_{k\to k}$ defined by Eq. (9.34). A derivation of Eqs. (9.103) will be given below in Sec. 9.4.4.

The solution of Eqs. (9.103) can be visualized as N_r analytic functions $g(z, \{q_j\}; k)$, $k = 1, ..., N_r$, of the free parameters z and q_j ($j = 1, ..., N$). Arguments similar to those of Sec. 9.3 show that the physically significant solution of Eqs. (9.103) is the "main branch" that has the following asymptotic form at $z \to 0$,

$$g(z; k) = z^\nu \left[\sum_{i\in T} q_i \kappa_{k\to i} \right]^\nu + O(z^{2\nu-1}) \tag{9.104}$$

(see also the argument at the end of Sec. 9.4.4).

Once the generating function $g(z, \{q_j\}; k)$ is determined, the RV measure gives the ratio of the mean number of bubbles of type j to that of type k as

$$\frac{p(j)}{p(k)} = \lim_{n\to\infty} \left.\frac{\partial_z^n \partial_{q_j} g(z, \{q_i\}; i')}{\partial_z^n \partial_{q_k} g(z, \{q_i\}; i')}\right|_{z=0,\{q_i\}=1}, \tag{9.105}$$

where, for clarity, we wrote explicitly the type i' of the initial bubble. We will now use the methods developed in Sec. 9.3.1 to reduce Eq. (9.105) to an expression that does not contain limits and so can be analyzed more easily. It will then become evident that the limit (9.105) is independent of i'.

For a fixed bubble type j, we first consider the derivatives $\partial_{q_j} g(z, \{q_i\}; k)$, $k = 1, ..., N_r$, evaluated at $\{q_i\} = 1$. Let us denote these N_r derivatives by $h_j(z; k)$,

$$h_j(z; k) \equiv \left.\frac{\partial}{\partial q_j}\right|_{q_i=1} g(z, \{q_i\}; k), \quad k = 1, ..., N_r. \tag{9.106}$$

These quantities are conveniently represented by an N_r-dimensional vector, which we will denote by $|h_j(z)\rangle$ using the Dirac notation (although no connection to quantum mechanics is present here). This vector satisfies an inhomogeneous linear equation that follows straightforwardly by taking the derivative ∂_{q_j} at $\{q_i\} = 1$ of Eqs. (9.103). That equation can be written in the matrix form as follows,

$$\sum_{i=1}^{N} M_{ki}(z) h_j(z; i) = z\kappa_{k\to j} g(z; j), \quad k \neq j, \tag{9.107}$$

$$\sum_{i=1}^{N} M_{ki}(z) g_{,q_j}(z; i) = 0, \quad k = j, \tag{9.108}$$

where we denoted by $g(z; j) \equiv g(z; \{q_i\} = 1; j)$ the solution of Eqs. (9.103) at $\{q_i\} = 1$, written as

$$g^{\frac{1}{\nu}}(z; k) = z \sum_{i\notin T, i\neq k} \kappa_{k\to i} g(z; i) + z \sum_{i\in T} \kappa_{k\to i} + \kappa_{k\to k} g(z; k), \tag{9.109}$$

while the matrix $M_{ki}(z)$ is defined by

$$\hat{M}(z) \equiv M_{ki}(z) = \begin{cases} \frac{1}{\nu} g^{\frac{1}{\nu}-1}(z; k) - \kappa_{k\to k}, & k = i; \\ -z\kappa_{k\to i}, & k \neq i. \end{cases} \tag{9.110}$$

For convenience we rewrite Eqs. (9.107)–(9.108) in a more concise form,

$$\hat{M}(z) |h_j(z)\rangle = |Q_j(z)\rangle, \tag{9.111}$$

where $|Q_j\rangle$ is the vector with the components $|Q_j\rangle_i$, $i = 1, ..., N_r$ given by the right-hand sides of Eqs. (9.107)–(9.108),

$$|Q_j(z)\rangle_i \equiv z\kappa_{i\to j}g(z;j)\left[1-\delta_{ij}\right]. \tag{9.112}$$

The solution of Eq. (9.111) is found symbolically as

$$|h_j(z)\rangle = \hat{M}^{-1}(z)\,|Q_j(z)\rangle, \tag{9.113}$$

provided that the inverse matrix $\hat{M}^{-1}(z)$ exists. Of course, it is impractical to obtain the inverse matrix explicitly; but we will never need to do that.

The next task is to determine z for which the matrix $\hat{M}(z)$ remains nondegenerate, $\det \hat{M}(z) \neq 0$. It will be shown in Sec. 9.4.5 that there exists an eigenvalue $\lambda_0(z)$ of $\hat{M}(z)$ such that $\lambda_0(z_*) = 0$ at some real $z_* > 0$, while $\lambda_0(z) > 0$ for $z < z_*$. Moreover, the (left and right) eigenvectors corresponding to the eigenvalue $\lambda_0(z)$ are always nondegenerate and can be chosen with all positive components. At the same time, all the other eigenvalues of $\hat{M}(z)$ remain nonzero for all $z \leq z_*$. It will also be shown (see Sec. 9.4.6) that the following asymptotic expansion holds near $z = z_*$,

$$\lambda_0(z) = c_1\sqrt{z_* - z} + O(z_* - z), \quad c_1 > 0. \tag{9.114}$$

One can see from the small-z asymptotics (9.104) and from Eq. (9.110) that, for small enough z, $\hat{M}(z)$ contains very large positive numbers on the diagonal and very small negative numbers off the diagonal. Hence $\det \hat{M}(z) > 0$ for all sufficiently small z. Since the eigenvalues of $\hat{M}(z)$ remain nonzero except for $\lambda_0(z)$ that first vanishes at $z = z_*$, we conclude that $\det \hat{M}(z)$ remains positive for all $0 < z < z_*$, and that $z = z_*$ is the smallest positive value of z for which $\det \hat{M}(z) = 0$.

Now we restrict our attention to $0 < z < z_*$, for which $\det \hat{M}(z) > 0$ and $\hat{M}^{-1}(z)$ exists. We need to analyze the behavior of $\hat{M}^{-1}$ near $z \approx z_*$. We treat $\hat{M}(z)$ as an operator in N_r-dimensional real space V. The matrix $\hat{M}$ is not symmetric and may not be diagonalizable. Instead of diagonalizing $\hat{M}(z)$, we split the space V into the 1-dimensional subspace corresponding to the smallest eigenvalue $\lambda_0(z)$, and into the $(N_r - 1)$-dimensional complement subspace V_1. We know that the eigenvalue $\lambda_0(z)$ is nondegenerate. Thus, we may symbolically write

$$\hat{M}(z) = \lambda_0(z)\left|v^0(z)\right\rangle\left\langle u^0(z)\right| + \hat{M}_1(z), \tag{9.115}$$

where $|v^0\rangle$ and $\langle u^0|$ are the right and the left eigenvectors corresponding to $\lambda_0(z)$, normalized such that

$$\left\langle u^0 | v^0\right\rangle = 1, \tag{9.116}$$

and it is implied that $\hat{M}_1(z)$ vanishes on $|v^0\rangle$ and $\langle u^0|$ but is nonsingular on the complement space V_1. In other words, we have

$$\hat{M}\left|v^0\right\rangle = \lambda_0\left|v_0\right\rangle, \quad \left\langle u^0\right|\hat{M} = \lambda_0\left\langle u^0\right|, \tag{9.117}$$

$$\hat{M}_1\left|v^0\right\rangle = 0, \quad \left\langle u^0\right|\hat{M}_1 = 0. \tag{9.118}$$

There exists a matrix $\hat{M}^{-1}_{1(V_1)}$ that acts as the inverse to $\hat{M}_1$ when restricted to the subspace V_1 and again vanishes on $\left|v^0\right\rangle$ and $\left\langle u^0\right|$. Using that matrix, we may write the inverse matrix $\hat{M}^{-1}$ explicitly as

$$\hat{M}^{-1}(z) = \frac{1}{\lambda_0(z)} \left|v^0(z)\right\rangle \left\langle u^0(z)\right| + \hat{M}^{-1}_{1(V_1)}(z). \tag{9.119}$$

Hence, the solution (9.113) can be written as

$$\left|h_j(z)\right\rangle = \frac{\langle u^0|Q_j\rangle}{\lambda_0(z)} \left|v^0\right\rangle + \hat{M}^{-1}_{1(V_1)} \left|Q_j\right\rangle. \tag{9.120}$$

Now it is clear from Eq. (9.114) that $\left|h_j(z)\right\rangle$ diverges at $z = z_*$ as $\propto (z_* - z)^{-1/2}$. It also follows that $\left|h_j(z)\right\rangle$ does not diverge at any smaller real z.

We can also show that $\lambda_0(z)$ cannot vanish at some *complex* value of z that is closer to $z = 0$ than $z = z_*$. If $\lambda_0(z'_*) = 0$ with a complex-valued z'_*, then a derivative of the generating function, such as $\partial_z g(z; j)$, would diverge at $z = z'_*$. Using the definition (9.101) of g and substituting $q_i = 1$, we find that the following sum diverges,

$$\left.\frac{\partial g(z;j)}{\partial z}\right|_{z=z'_*} = \sum_{n\geq 1} \left(z'_*\right)^{n-1} np(n;j) = \infty. \tag{9.121}$$

The sum of absolute values is not smaller than the above, and hence also diverges:

$$\left.\frac{\partial g(z;j)}{\partial z}\right|_{z=|z'_*|} = \sum_{n\geq 1} \left|z'_*\right|^{n-1} np(n;j) = \infty. \tag{9.122}$$

So $\partial_z g$ has also a singularity at a *real* value $z = |z'_*|$. As we have shown, $z = z_*$ is the smallest such real-valued singularity point; hence $|z'_*| \geq z_*$. It follows that z_* is equal to the radius of convergence of the series (9.121), which is a Taylor series for the function $\partial_z g$. There remains the possibility that a singularity z'_* is located directly on the circle of convergence, so that $|z'_*| = z_*$ and $z'_* = z_* e^{\mathrm{i}\phi}$ with $0 < \phi < 2\pi$. This possibility can be excluded using the following argument. The function $\partial_z g$ can have only finitely many singularities on the circle $|z| = z_*$; infinitely many singularities on the circle would indicate an accumulation point which would be an essential singularity, *i.e.* not a branch point. However, by construction $g(z; j)$ is an algebraic function of z that cannot have singularities other than branch points. Since $g(z; j)$ is regular at $z = z'_*$, all branch points of g must be of the form $\left(z - z'_*\right)^s$ with $s > 0$. Using the explicit formula (9.15) for the n-th derivative of an analytic function with a branch cut singularity of the form $\left(z - z'_*\right)^s$, we find that the coefficients $np(n; j)$ of the Taylor series (9.121) decay at large n asymptotically as

$$\begin{aligned} np(n;j) &= \frac{1}{n!}\partial_z^n \partial_z g \propto \frac{\Gamma(n+1-s)}{\Gamma(n+1)} \left(z'_*\right)^{-n} \left(1 + O(n^{-1})\right) \\ &\propto n^{-s} \left(z'_*\right)^{-n} \left(1 + O(n^{-1})\right). \end{aligned} \tag{9.123}$$

The function g has a finite number of singularities $z'_* = z_* e^{i\phi}$ on the circle $|z| = z_*$, and each singularity gives a contribution of the form (9.123). Hence, we may estimate (for sufficiently large n, say for $n \geq n_0$)

$$np(n;j) = c_0 n^{-s} z_*^{-n} \left(1 + O(n^{-1})\right), \quad c_0 > 0. \tag{9.124}$$

Then the partial sum for $n \geq n_0$ of the series (9.121) is estimated by

$$\begin{aligned}\sum_{n \geq n_0} \left(z'_*\right)^{n-1} np(n;j) &\approx \sum_{n \geq n_0} \left(z'_*\right)^{n-1} c_0 n^{-s} z_*^{-n} \\ &= \frac{c_0}{z_*} \sum_{n \geq n_0} n^{-s} e^{in\phi}.\end{aligned} \tag{9.125}$$

The latter series converges for $s > 0$ and $0 < \phi < 2\pi$. (The neglected terms of order n^{-1-s} build an absolutely convergent series and hence introduce an arbitrarily small error into the estimate.) Therefore, the series (9.121) also converges at $z'_* \neq z_*$, contradicting the assumption that other singularities exist on the circle $|z| = z_*$.

We conclude from these arguments that the singularity of $g(z;j)$ nearest to $z = 0$ in the complex z plane is indeed at a real value $z = z_*$, and all other singular points z'_* satisfy the strict inequality $|z'_*| > z_*$. The same statement about the locations of singularities holds for the generating functions $g(z, \{q_i\}; j)$ and hence for their derivatives such as $\left|h_j(z)\right\rangle$.

To compute the final expression (9.105), we need to evaluate the n-th derivative ∂_z^n of the functions $h_j(z;i)$ at $z = 0$ and for very large n. To this end, we use the formula (9.15), which requires to know the location $z = z_*$ of the singularity of $h_j(z;i)$ nearest to $z = 0$. We have just found that this singularity is at a real value $z = z_*$ and has the form

$$h_j(z;i) \approx \frac{1}{c_1\sqrt{z_* - z}} \left\langle u^0 | Q_j \right\rangle v_i^0, \quad z \approx z_*, \tag{9.126}$$

where v_i^0 is the i-th component of the eigenvector $|v_0\rangle$. The value of the proportionality constant c_1 is not required for computing the ratios (9.105). Using Eqs. (9.15), (9.112), and (9.126), we evaluate the limit (9.105) as

$$\frac{p(j)}{p(k)} = \left.\frac{\left\langle u^0 | Q_j \right\rangle v_{i'}^0}{\left\langle u^0 | Q_k \right\rangle v_{i'}^0}\right|_{z=z_*} = \frac{\sum_{i \neq j} u_i^0(z_*) \kappa_{i \to j}}{\sum_{i \neq k} u_i^0(z_*) \kappa_{i \to k}} \frac{g(z_*;j)}{g(z_*;k)}. \tag{9.127}$$

Since the component $v_{i'}^0$ cancels, we find that the limit (9.105) is indeed independent of the initial bubble type i'. We also note that the normalization of the eigenvector $\left\langle u^0 \right|$ is irrelevant for the ratio.

The formula (9.127) can be simplified further for *nonterminal* types j, k if we use the relationship $\left\langle u^0(z_*) \right| \hat{M}(z_*) = 0$ together with the explicit definition (9.110) of $\hat{M}$:

$$0 = \sum_{i \neq j} u_i^0(z_*) M_{ij}(z_*) + u_j^0(z_*) M_{jj}(z_*)$$
$$= -\sum_{i \neq j} u_i^0(z_*) z_* \kappa_{i \to j} + u_j^0(z_*) \left[\frac{1}{\nu} g^{\frac{1}{\nu}-1}(z_*; j) - \kappa_{j \to j} \right]. \tag{9.128}$$

The last term in the square brackets in Eq. (9.128) can be neglected since $\kappa_{j \to j} \approx 1$ while $g(z_*; j) \ll 1$. Hence, Eq. (9.127) is simplified to

$$\frac{p(j)}{p(k)} \approx \frac{u_j^0(z_*)}{u_k^0(z_*)} \left[\frac{g(z_*; j)}{g(z_*; k)} \right]^{1/\nu}. \tag{9.129}$$

This is the main formula for the RV-regulated relative abundances of bubbles of arbitrary (nonterminal) types j and k. For a terminal type $k \in T$, one needs to use Eq. (9.127) together with $g(z; k) \equiv 1$.

We note that the expression (9.129) depends on the components u_i^0 of the left eigenvector of the matrix $\hat{M}(z_*)$ with eigenvalue $\lambda_0(z_*) = 0$. Although we have been able to evaluate the limit (9.105) analytically and obtained Eq. (9.129), the task of computing the values of z_*, $g(z_*; j)$, and $u_i^0(z_*)$ remains quite difficult. The outline of the required computations is as follows: One first needs to determine $g(z; j)$ ($j = 1, ..., N_r$) as the "main branch" of the solution of Eqs. (9.109) with the small-z asymptotic given by Eq. (9.104). The functions $g(z; j)$ determine the matrix elements $M_{ij}(z)$ using Eq. (9.110). Then one needs to find the smallest value $z = z_* > 0$ such that the determinant of the matrix $\hat{M}(z)$ vanishes. Finally, one needs to compute a left eigenvector $\langle u^0(z_*) |$ that corresponds to the eigenvalue $\lambda_0(z_*) = 0$ of the matrix $\hat{M}(z_*)$. The mathematical construction shown below guarantees that z_* exists and that $\langle u^0(z_*) |$ is nondegenerate and has all positive components; the results are independent of the normalization of $\langle u^0 |$. However, a brute-force numerical computation of these quantities appears to be impossible due to the huge number N_r of the simultaneous equations (9.109) and to the wide range of numerical values of the coefficients $\kappa_{i \to j}$ in a typical landscape. Even the numerical value of z_* is likely to be too large to be represented efficiently in computers. In the next section we will consider an example landscape where an analytic approximation can be found.

9.4.2 Example landscape

We begin with some qualitative considerations regarding the behavior of the functions $g(z; k)$.

To determine $g(z_*; k)$, we need to follow the main branch $g(z; k)$ as the value of z is increased from $z = 0$ until $\det \hat{M}(z)$ vanishes, which will determine the value $z = z_*$. We note that the asymptotic form (9.104) is valid in some range near $z = 0$. Since $g(z; k) \ll 1$ at those z, the matrix $\hat{M}(z)$ is dominated by large positive diagonal terms $g^{\frac{1}{\nu}-1}(z; k)\delta_{kj}$. As z increases to z_*, all the functions $g(z; j)$ also increase while $\det \hat{M}(z)$ decreases monotonically to zero (this statement is proved rigorously in Sec. 9.4.5). One can visualize the changes in the matrix elements of $\hat{M}(z)$ if one notes that the positive diagonal terms decrease with z while the negative off-diagonal terms ($M_{jk} = -z\kappa_{j\to k}$, $j \neq k$) grow in magnitude. Eventually $\det \hat{M}(z)$ vanishes at $z = z_*$ such that the off-diagonal terms become sufficiently large at least in some rows and columns of the matrix $\hat{M}(z)$.

Let us determine an upper bound on z such that Eq. (9.104) remains a good approximation for the generating function $g(z;j) \equiv g(z, \{q_i = 1\}; j)$. This will be the case if the term $z\sum_{i\in T}\kappa_{k\to i}$ in the right hand side of Eq. (9.109) dominates over all other terms, for every k. Denoting for brevity

$$\kappa_{k\to T} \equiv \sum_{i\in T} \kappa_{k\to i} \tag{9.130}$$

the total transition probability from k to all terminal vacua, we have therefore the conditions (for every k)

$$z\kappa_{k\to T} \gg g(z;k), \quad z\kappa_{k\to T} \gg z \sum_{i\notin T, i\neq k} \kappa_{k\to i} g(z;i). \tag{9.131}$$

These conditions are satisfied, consistently with Eq. (9.104), if z is bounded (for every k) simultaneously by

$$z \ll \kappa_{k\to T}^{-1}, \quad z \ll \left[\sum_{i\notin T, i\neq k} \frac{\kappa_{k\to i}}{\kappa_{k\to T}} \kappa_{i\to T}^{\nu}\right]^{-\frac{1}{\nu}}. \tag{9.132}$$

Since typically the rate of transitions to terminal vacua is much smaller than the rate of transitions to dS vacua, one can expect that this regime will include $z = 1$. Nevertheless, one expects that $\det \hat{M}(z)$ remains nonzero for these z. The detailed consideration of the FABI model in Sec. 9.3.1 showed that the value of z_* lies outside the regime (9.132) unless the landscape parameters are fine-tuned. In a general landscape, one expects the analogous fine-tuning to be much stronger or even impossible to satisfy. In other words, one expects that the functions $g(z_*;k)$ at least for some k will violate the conditions (9.131), although these conditions might be satisfied for other k.

Now we would like to estimate the value of z for which the matrix $\hat{M}(z)$ first becomes degenerate. We can use a general theorem for estimating the eigenvalues of a matrix (Gershgorin's theorem, see [225], chapter 7). For each of the diagonal elements M_{kk}, $k = 1, ..., N_r$ one needs to draw a circle in the complex λ plane, centered at the diagonal element M_{kk} with radius

$$\rho_k \equiv \sum_{i\neq k} |M_{ki}| \,. \tag{9.133}$$

Thus one obtains N_r circles

$$|\lambda - M_{kk}| \leq \rho_k, \quad k = 1, ..., N_r, \tag{9.134}$$

called the Gershgorin circles. The Gershgorin theorem says that all the eigenvalues of a matrix M_{jk} are located within the union of these circles. An elementary proof is as follows. An eigenvector v_i with eigenvalue λ can be normalized such that its component of largest absolute value is equal to 1. Let $v_1 = 1$ be this component (renumbering the components if necessary), then it is easy to show that λ is within a circle with center M_{11} and radius ρ_1. Namely, we start from the eigenvalue equation,

$$\sum_{i \neq 1} M_{1i} v_i + M_{11} v_1 = \lambda v_1, \tag{9.135}$$

and use the properties $v_1 = 1$ and $|v_i| \leq 1$ for $i \neq 1$:

$$|\lambda - M_{11}| = \left| \sum_{i \neq 1} M_{1i} v_i \right| \leq \sum_{i \neq 1} |M_{1i}| = \rho_1. \tag{9.136}$$

Applying the theorem to the matrix $\hat{M}(z)$ at small z such that Eq. (9.104) is a good approximation for the generating function $g(z; j)$, we find that the Gershgorin circles are centered at large positive values

$$M_{kk} = \frac{1}{\nu} g^{\frac{1}{\nu} - 1}(z; k) - \kappa_{k \to k} \approx \frac{1}{\nu} \frac{1}{\left(z \kappa_{k \to T}\right)^{\nu - 1}} \gg 1, \tag{9.137}$$

while the radii of the circles are small,

$$\rho_k = z \sum_{i \neq k, i \notin T} \kappa_{k \to i} \ll 1. \tag{9.138}$$

Hence, for small z none of the circles can contain $\lambda = 0$, so the matrix $\hat{M}(z)$ is nondegenerate. As z increases, the radii ρ_k increase while the centers M_{kk} move monotonically towards zero. The circle closest to $\lambda = 0$ is the one with the center closest to zero and the largest radius. This circle is labeled by k with largest rates $\kappa_{k \to T}$ and $\kappa_{k \to i}$. This value of k corresponds to the high-energy vacua in the landscape, for which tunneling to any other vacua is much easier than for low-energy vacua. Therefore the eigenvalue $\lambda(z_*) = 0$ will be located inside the Gershgorin circle(s) centered at $M_{kk}(z_*)$ for k corresponding to high-energy vacua. The Gershgorin circle to which the eigenvalue belongs indicates the largest component of the eigenvector. Hence, one expects that the eigenvector $\langle u^0(z_*) |$ has the largest values of its components u_k^0 corresponding to the high-energy vacua k. The Gershgorin theorem requires that $\lambda_0(z_*) = 0$ be inside the circle centered at $M_{kk}(z_*)$ with radius $\rho_k(z_*)$, *i.e.*

$$M_{kk}(z_*) = \frac{1}{\nu} g^{\frac{1}{\nu} - 1}(z_*; k) - \kappa_{k \to k} \leq z_* \sum_{i \neq k, i \notin T} \kappa_{k \to i}. \tag{9.139}$$

Disregarding the terms $\kappa_{k \to k}$, which are small in comparison with the remaining terms, and using Eq. (9.109) for $g(z_*; k)$ we obtain the following condition,

$$z_*^{\nu} \geq \frac{1}{\nu} \frac{\left[\kappa_{k \to T} + \sum_{i \neq k, i \notin T} \kappa_{k \to i} g(z_*; i) \right]^{1 - \nu}}{\sum_{i \neq k, i \notin T} \kappa_{k \to i}}. \tag{9.140}$$

Proceeding further requires at least an estimate of $g(z_*; i)$ for all i, which is so far not available.

While we are as yet unable to obtain an explicit estimate of z_* and $\langle u^0|$ for an arbitrary landscape, let us consider a workable example that is more realistic than the FABI model. In this example there is a single high-energy vacuum, labeled $k = 1$, and a large number of dS low-energy vacua, $k = 2, ..., N_r$, as well as a number of terminal vacua, $k = N_r + 1, ..., N$. We assume that the downward tunneling rates $\kappa_{1\to i}$ for $i \neq 1$ are much larger than the rates $\kappa_{i\to 1}$ or $\kappa_{i\to j}$ for $i, j \neq 1$. Then $g(z; 1) \gg g(z; i)$ for $i \neq 1$, and so the first Gershgorin circle is the one closest to $\lambda = 0$. Hence we may expect that the eigenvalue $\lambda_0(z)$ always belongs to that circle. Moreover, it is likely that z_* is within the regime (9.131) for the low-energy vacua ($k \neq 1$) but not for the high-energy vacuum $k = 1$. We will now proceed with the calculation and later check that these assumptions are self-consistent.

If the regime (9.131) holds for every low-energy vacuum $k \neq 1$, we have

$$g(z; k) \approx z^\nu \kappa^\nu_{k\to T}, \quad k = 2, ..., N_r. \tag{9.141}$$

For the high-energy vacuum $k = 1$ we use Eq. (9.109) directly to find

$$\begin{aligned} g(z; 1) &= \left[\kappa_{1\to T} + z \sum_{i\notin T, i\neq 1} \kappa_{1\to i} g(z; i) + \kappa_{1\to 1} g(z; 1) \right]^\nu \\ &\approx z^{\nu^2+\nu} \left[\sum_{i=2}^{N_r} \kappa_{1\to i} \kappa^\nu_{i\to T} \right]^\nu, \end{aligned} \tag{9.142}$$

where we have neglected the terms with $\kappa_{1\to T}$ and $\kappa_{1\to 1}$. The condition that the regime (9.131) holds for $k \neq 1$ is

$$\kappa_{i\to T} \gg z \sum_{j\neq i, j\notin T} \kappa_{i\to j} g(z; j), \quad i = 2, ..., N_r. \tag{9.143}$$

In the matrix $\hat{M}(z)$ the off-diagonal elements M_{ij} for $i \neq 1$, $j \neq 1$ are much smaller than all other elements. Hence, we can approximate $\hat{M}(z)$ by a matrix that has only its first row, first column, and the diagonal elements,

$$\begin{aligned} \hat{M}(z) &\approx \begin{pmatrix} M_{11} & -z\kappa_{1\to 2} & -z\kappa_{1\to 3} & \cdots & -z\kappa_{1\to N_r} \\ -z\kappa_{2\to 1} & M_{22} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ -z\kappa_{N_r\to 1} & 0 & 0 & \cdots & M_{N_r N_r} \end{pmatrix}, \\ M_{kk} &\approx \frac{1}{\nu} g^{\frac{1}{\nu}-1}(z; k). \end{aligned} \tag{9.144}$$

This approximate matrix is much easier to analyze; in particular, its determinant and the eigenvectors can be computed in closed form. The determinant of this matrix is

$$\det \hat{M}(z) \approx M_{11} ... M_{N_r N_r} \left[1 - \sum_{i=2}^{N_r} \frac{z^2 \kappa_{1\to i} \kappa_{i\to 1}}{M_{11}(z) M_{ii}(z)} \right]. \tag{9.145}$$

Therefore, the condition $\det \hat{M}(z_*) = 0$ can be written as

$$M_{11}(z_*) \approx z_*^2 \sum_{i=2}^{N_r} \frac{\kappa_{1\to i}\kappa_{i\to 1}}{M_{ii}(z_*)}. \tag{9.146}$$

Using Eqs. (9.141), (9.142), and (9.144), we transform this condition to

$$z_*^{-\nu-\nu^2} \approx \nu^2 \left[\sum_{i=2}^{N_r} \kappa_{1\to i}\kappa_{i\to T}^{\nu} \right]^{\nu-1} \sum_{i=2}^{N_r} \kappa_{1\to i}\kappa_{i\to 1}\kappa_{i\to T}^{\nu-1}. \tag{9.147}$$

However, one does not need this value apart from checking explicitly that the assumptions (9.143) hold.

The left eigenvector $\langle u^0(z_*)|$ corresponding to the eigenvalue 0 of the matrix $\hat{M}(z_*)$ is found approximately as

$$u_1^0 = 1, \quad u_k^0 \approx \frac{z_*\kappa_{1\to k}}{M_{kk}(z_*)}, \tag{9.148}$$

where the normalization $u_1^0 = 1$ was chosen arbitrarily for convenience. A perturbative improvement of this approximation along the lines of [114, 265, 266, 267] may be possible, but the precision obtained from the present approximation is sufficient for our purposes.

The bubble abundance ratios (9.129) for nonterminal types are then expressed as follows,

$$\frac{p(j)}{p(k)} \approx \frac{\kappa_{1\to j}}{\kappa_{1\to k}} \left(\frac{\kappa_{j\to T}}{\kappa_{k\to T}} \right)^{\nu}, \quad j,k = 2, ..., N_r, \tag{9.149}$$

$$\frac{p(1)}{p(k)} \approx \frac{1}{\nu} \frac{\sum_{i=2}^{N_r} \kappa_{1\to i}\kappa_{i\to T}^{\nu}}{\kappa_{1\to k}\kappa_{k\to T}^{\nu}}, \quad k = 2, ..., N_r. \tag{9.150}$$

These equations are the main result of this section, yielding RV-regulated bubble abundances for a landscape with a large number of low-energy vacua. The formula (9.150) agrees with that obtained in Sec. 9.3.1 for the ratio $p(F)/p(I) \approx 1/\nu$ in the FABI landscape, which may be considered a special case of the present model with $N_r = 2$.

9.4.3 "Boltzmann brains"

We now investigate the abundance of "Boltzmann brains" in a general landscape, relative to the abundance of ordinary observers.

We first need to derive the equations for the suitable generating functions, analogously to Eqs. (9.95)–(9.96). Let us introduce the generating function

$$g(z, \{q_i\}, \{r_i\}; j) \equiv \left\langle z^{n_{\text{tot}}} \prod_i q_i^{n_i} r_i^{N_i} \right\rangle_{n_{\text{tot}}<\infty; j}, \tag{9.151}$$

where n_i is the total number of bubbles of type i and N_i is the total number of H-regions of type i. The number of BBs in bubbles of type i is proportional to N_i with a proportionality constant κ_i^{BB}, which is the (extremely small) nucleation rate of a BB per Hubble 4-volume H_i^{-4}.

The generating function $g(z, \{q_i\}, \{r_i\}; j)$, which we will denote for brevity by $g(..., j)$, satisfies a system of equations analogous to Eq. (9.103),

$$g^{\frac{1}{\nu}}(..., j) = \sum_{k \neq j} \kappa_{j \to k} z q_k r_k g(..., k) + \kappa_{j \to j} r_j g(..., j). \tag{9.152}$$

Let us compute the RV-regulated ratio of the number of H-regions of type j to the number of bubbles of the same type j. This ratio is given by

$$\lim_{n \to \infty} \frac{\langle N_j \rangle_n}{\langle n_j \rangle_n} = \lim_{n \to \infty} \left. \frac{\partial_z^n \partial_{r_j} g(..., i')}{\partial_z^n \partial_{q_j} g(..., i')} \right|_{z=0, r_i=1, q_i=1}, \tag{9.153}$$

where i' is the initial bubble type. To evaluate the limit, we use the methods developed in Sec. 9.4.1. The derivatives $\partial_{q_j} g \equiv |h_j(z)\rangle$, as defined in Eq. (9.106), satisfy Eq. (9.111). The vector of derivatives

$$\left. \frac{\partial}{\partial r_j} \right|_{r_i=1, q_i=1} g(..., k) \equiv |\rho_j(z)\rangle_k \tag{9.154}$$

satisfies the linear equations that follow from Eq. (9.152),

$$\hat{M}(z) |\rho_j(z)\rangle = |\beta_j(z)\rangle, \tag{9.155}$$

where $|\beta_j\rangle$ is the vector with the components $|\beta_j\rangle_i$, $i = 1, ..., N_r$ defined by

$$|\beta_j(z)\rangle_i \equiv z \kappa_{i \to j} g(z; j) \left[1 - \delta_{ij}\right] + \kappa_{j \to j} g(z; j) \delta_{ij}. \tag{9.156}$$

By the same considerations that lead to Eq. (9.127), we now obtain

$$\lim_{n \to \infty} \left. \frac{\partial_z^n \partial_{r_j} g(..., i')}{\partial_z^n \partial_{q_j} g(..., i')} \right|_{z=0, r_i=1, q_i=1} = \left. \frac{\langle u^0 | \beta_j \rangle}{\langle u^0 | Q_j \rangle} \right|_{z=z_*}, \tag{9.157}$$

where, as before, $\langle u^0(z_*)|$ is the unique eigenvector of the matrix $\hat{M}(z_*)$ with eigenvalue 0. (The normalization of $\langle u^0|$ is again irrelevant.) Using the definitions (9.112) and (9.156) and the relationship (9.128), we compute

$$\begin{aligned} \left. \frac{\langle u^0 | \beta_j \rangle}{\langle u^0 | Q_j \rangle} \right|_{z=z_*} &= \frac{u_j^0(z_*) \kappa_{j \to j} + \sum_{i \neq j} u_i^0(z_*) z_* \kappa_{i \to j}}{\sum_{i \neq j} u_i^0(z_*) z_* \kappa_{i \to j}} \\ &= \frac{\frac{1}{\nu} g^{\frac{1}{\nu}-1}(z_*; j)}{\frac{1}{\nu} g^{\frac{1}{\nu}-1}(z_*; j) - \kappa_{j \to j}}. \end{aligned} \tag{9.158}$$

The last ratio is always very close to 1 since typically $g(z_*;j) \ll 1$ while $\kappa_{j\to j} \approx 1$. In particular, $g(z_*;j) \ll 1$ for vacua j with low-energy Hubble scale, since for those vacua we may approximate $g(z_*;j)$ by Eq. (9.104),

$$g(z_*;j) \approx \left[z_* \sum_{i\in T} \kappa_{j\to i} \right]^{\nu} \ll 1. \tag{9.159}$$

Therefore, the RV-regulated ratio N_j/n_j of the total number of H-regions of type j to the total number of bubbles of type j is never large.

Using this result, we can now estimate the RV-regulated ratio of BBs to ordinary observers. The number of ordinary observers per one H-region of type j is not precisely known but is presumably at least of order 1 or larger, as long as bubbles of type j are compatible with life. On the other hand, the number of Boltzmann brains per H-region, *i.e.* within a four-volume H_j^{-4} of spacetime, is negligibly small. It follows that the abundance of BBs in the RV measure is always negligible relative to the abundance of ordinary observers in the same bubble type.

9.4.4 Derivation of Eq. (9.103)

To derive Eq. (9.103), we need to consider the expansion of a single initial H-region during one Hubble time. Within the "inflation in a box" model, an initial H-region of type j is split after one Hubble time $\delta t = H_j^{-1}$ into $\nu \equiv e^3$ statistically independent "daughter" H-regions.[5] Each of the daughter H-regions may change its vacuum type from j to $k \neq j$ ($k = 1, ..., N$) with probability $\kappa_{j\to k}$. The quantity $\kappa_{j\to j}$ was defined for convenience by Eq. (9.34) to be the probability of *not* changing the bubble type j during one Hubble time.

The generating function g is defined by Eq. (9.101),

$$g(z, \{q_i\}; j) \equiv \left\langle z^{n_{\text{tot}}} q_1^{n_1} ... q_N^{n_N} \right\rangle_{n_{\text{tot}}<\infty; j} \tag{9.160}$$

where n_i is the number of bubbles of type i, while the notation $\langle ... \rangle_{n_{\text{tot}}<\infty;j}$ stands for a probabilistic average evaluated for the initial H-region of type j on the sub-ensemble of finite total number of bubbles n_{tot}. Note that the initial bubble of type j is *not* counted in n_i or n_{tot}. Our goal is to obtain a relationship between $g(z, \{q_i\}; j)$ and the generating functions $g(z, \{q_i\}; k)$ with $k \neq j$.

To this end, we equate two expressions for the average $\left\langle z^{n_{\text{tot}}} \prod_i q_i^{n_i} \right\rangle_{n_{\text{tot}}<\infty; j}$. The first expression is the left-hand side of Eq. (9.160). The second expression is found by considering the ν daughter H-regions evolved out of the initial H-region and by using the fact that the same average for a daughter region of type k is equal to $g(z, \{q_i\}; k)$. However, two details need to be accounted for: First, the generating functions $g(z, \{q_i\}; k)$ evaluated for the daughter regions

[5] To avoid considering a non-integer number ν of daughter regions, we may temporarily assume that ν is an integer parameter. At the end of the derivation, we will set $\nu \equiv e^3 \approx 20.1$ in the final equations.

do not count the daughter bubbles themselves. Second, the type k of each of the daughter regions is a random quantity. Before deriving a general relationship, let us illustrate the procedure using an example.

It is possible that, say, only two of the daughter regions change their type to k while all other daughter regions retain the initial bubble type j. Denote temporarily by $p_{kkj...j}$ the probability of this event; binomial combinatorics yields

$$p_{kkj...j} = \frac{\nu!}{2!(\nu-2)!}\kappa_{j\to k}^2 \kappa_{j\to j}^{\nu-2}. \tag{9.161}$$

Then the average of $z^{n_{\mathrm{tot}}} \prod_i q_i^{n_i}$ receives a contribution

$$p_{kkj...j} z^2 q_k^2 \left[g(z,\{q_i\};k)\right]^2 \left[g(z,\{q_i\};j)\right]^{2-\nu} \tag{9.162}$$

from this event. The factor z^2 describes two additional bubbles that contribute to n_{tot}; the factor q_k^2 accounts for two additional bubbles of type k; no factors of q_j appear since no additional bubbles of type j are generated. Finally, the powers of g account for all the bubbles generated in the daughter H-regions, but these generating functions do not count the daughter H-regions themselves. Those daughter H-regions are explicitly counted by the extra factors $z^2 q_k^2$.

To compute the average, we need to add the contributions from all the possible events of this kind. Since all of the daughter H-regions are independent and statistically equivalent, the average $\left\langle z^{n_{\mathrm{tot}}} \prod_i q_i^{n_i} \right\rangle_{n_{\mathrm{tot}}<\infty;j}$ splits into the product of ν averages, each evaluated over a single daughter region.

The average over a single daughter region of type j has contribution from transitions to other types $k \neq j$ and a contribution from the event of no transition. Let us first consider a *terminal* type k. With probability $\kappa_{j\to k}$ a given daughter region becomes a vacuum of type k. Thereafter, no more bubbles will be nucleated inside it; the average of $\left\langle z^{n_{\mathrm{tot}}} \prod_i q_i^{n_i} \right\rangle_{n_{\mathrm{tot}}<\infty;j}$ over that daughter region is simply zq_k, meaning that there is a total of one bubbles and only one bubble of type k. Hence, the contribution of that event to the statistical average is $\kappa_{j\to k} z q_k$.

Now let us consider a nonterminal type $k \neq j$, $k \notin T$. The corresponding contribution to the average is $\kappa_{j\to k} z q_k g(z,\{q_i\};k)$. Since we have defined $g(z,\{q_i\};k) \equiv 1$ when k is a terminal bubble type, we may write the contribution as $\kappa_{j\to k} z q_k g(z,\{q_i\};k)$ for both terminal and nonterminal types $k \neq j$.

Finally, we consider the case of $k = j$ (the daughter region retains the original bubble type). Since no new bubbles were nucleated, the contribution to the average is simply $\kappa_{j\to j} g(z,\{q_i\};j)$ without any factors of z or q_k.

Putting these ingredients together, we obtain an equation for $g(z,\{q_i\};j)$,

$$\begin{aligned} &g(z,\{q_i\};j) \\ &= \left[\sum_{k\neq j} \kappa_{j\to k} z q_k g(z,\{q_i\};k) + \kappa_{j\to j} g(z,\{q_i\};j)\right]^{\nu}. \end{aligned} \tag{9.163}$$

This is equivalent to Eq. (9.103). One can also verify that the binomial expansion of Eq. (9.163) indeed yields all the terms such as the one given in Eq. (9.162).

We can now analyze the behavior of $g(z, \{q_i\}\,; j)$ in the limit $z \to 0$. By definition, the generating function $g(z, \{q_i\}\,; j)$ is a power series in z whose coefficient at z^n is equal to the probability of the event that a multiverse has exactly n bubbles to the future of the initial bubble j. It is clear that this power series starts with the term z^ν, corresponding to the probability that the initial bubble j expands exactly into ν terminal bubbles, signaling the global end of the multiverse. The probability of having fewer than ν bubbles in the entire multiverse is equal to zero.[6] The next term of the binomial expansion is of order $z^{2\nu-1}$ since it is the product of $z^{\nu-1}$ and g itself. Therefore, the small-z behavior of the generating function $g(z, \{q_i\}\,; j)$ must be given by Eq. (9.104). This condition, together with analyticity in z, selects the unique physically relevant solution of Eqs. (9.103).

9.4.5 Eigenvalues of $\hat{M}(z)$

According to the definition (9.110), the matrix $\hat{M}(z)$ has positive elements on the diagonal and nonpositive elements off the diagonal. Such a matrix can be rewritten in the form

$$\hat{M}(z) = \mu\hat{1} - \hat{A}(z), \tag{9.164}$$

where a constant $\mu > 0$ is introduced, the notation $\hat{1}$ stands for an identity matrix, and $\hat{A}(z)$ is a suitable nonnegative matrix, *i.e.* a matrix with all nonnegative elements. For instance, we may choose μ as the largest of the diagonal elements of $\hat{M}$. The theory of nonnegative matrices gives powerful results for the eigenvalues of matrices such as $\hat{M}$ and $\hat{A}$ (see *e.g.* the book [225], chapter 9). For the present case, the most important are the properties of the algebraically smallest eigenvalue of the matrix $\hat{M}$.

It will be convenient to drop temporarily the argument z since all the results of matrix theory will hold for every fixed z. By the Perron-Frobenius theorem (see [225], chapter 9), under the condition of irreducibility[7] a nonnegative ma-

[6] The property that there are exactly ν daughter bubbles is, of course, an artifact of the "inflation in a box" approximation. In the actual multiverse, one has bubbles of spherical shape that can intersect in complicated ways, so a given bubble may end in one, two, or any other number of terminal bubbles. However, we are using the box approximation to obtain results in the limit of very large total number of bubbles, so we disregard the imprecision in the description of multiverses with a very small total number of bubbles. Also, the value of ν may be considered a variable parameter of the "box" model; the final results will not be overly sensitive to the value of ν.

[7] The irreducibility condition means, in the context of landscape models, that any two recyclable vacua in the landscape can be connected by a chain of transitions with nonzero nucleation rates. This condition has been discussed in [222, 242]. Note that a nonzero transition probability from vacuum i to vacuum j (where both i and j are recyclable) entails also a nonzero transition probability for the backward transition. If some subset of vacua form a "disconnected island" in the landscape, such that transitions to and from the "island" are forbidden, one can regard the "island" as a separate irreducible landscape and apply the same techniques to that landscape. Hence, it is sufficient to consider only irreducible landscapes; we will from now on always assume that the landscape is irreducible.

trix $\hat{A}$ has a unique nondegenerate, real eigenvalue $\alpha_0 > 0$ such that all the eigenvalues of $\hat{A}$ (which may be complex-valued) are located within the circle $|\lambda| \leq \alpha_0$ in the complex λ plane. This "dominant" eigenvalue α_0 has a corresponding (right) eigenvector $\left|v^0\right\rangle$ that can be chosen with all strictly positive components $v_i^0 > 0$. The same property holds for the relevant left eigenvector $\left\langle u^0\right|$ (the matrix need not be symmetric, so the left and the right eigenvectors do not, in general, coincide). Therefore it is possible to choose the eigenvectors $\left\langle u^0\right|$ and $\left|v^0\right\rangle$ such that the normalization $\left\langle u^0|v^0\right\rangle = 1$ holds. This normalization will be convenient for further calculations, and so we assume that such eigenvectors have been chosen.

It follows that $\left|v^0\right\rangle$ and $\left\langle u^0\right|$ are also the right and left eigenvectors of the matrix $\hat{M}$ with the eigenvalue

$$\lambda_0 \equiv \mu - \alpha_0, \tag{9.165}$$

while all the other eigenvalues of $\hat{M}$ are located within the circle $|\mu - \lambda| \leq \alpha_0$ in the complex λ plane. Since $\alpha_0 > 0$, all the other eigenvalues of $\hat{M}$ are strictly to the right (in the complex plane) of the real eigenvalue λ_0. In other words, λ_0 is the eigenvalue of $\hat{M}$ with the algebraically smallest real part.

Restoring now the argument z of the matrix $\hat{M}$, we find that $\hat{M}$ always has a real, nondegenerate eigenvalue $\lambda_0(z)$, which is at the same time the eigenvalue with the algebraically smallest real part among all the eigenvalues of $\hat{M}(z)$. We know that $\det \hat{M}(z) > 0$ for sufficiently small z; hence $\lambda_0(z) > 0$ for those z. Moreover, $\det \hat{M}(z)$ will remain positive as long as $\lambda_0(z) > 0$, since no other eigenvalue can become negative unless $\lambda_0(z)$ first becomes negative. We will now show that $\det \hat{M}(z)$ cannot remain positive for all real $z > 0$. It will then follow by continuity of $\lambda_0(z)$ that there will be a value z_* such that $\lambda_0(z) > 0$ for all $0 < z < z_*$ but $\lambda_0(z_*) = 0$.

We will use the property that the inverse matrix $\hat{M}^{-1}(z)$ has all positive elements as long as $\lambda_0(z) > 0$ (equivalently if $\alpha_0 < \mu$). The derivation of this property is simple:

$$\hat{M}^{-1} = \left(\mu\hat{1} - \hat{A}\right)^{-1} = \mu^{-1}\hat{1} + \mu^{-2}\hat{A} + \mu^{-3}\hat{A}^2 + ..., \tag{9.166}$$

which yields explicitly a matrix with all nonnegative elements. [The matrix-valued series in Eq. (9.166) converges because all the eigenvalues of $\hat{A}$ are strictly smaller than μ by absolute value.] Moreover, the irreducibility condition means that some chain of transitions will connect every pair of recyclable vacua; this is equivalent to saying that for any vacua i, j there exists some integer s such that $\hat{A}^s$ has a nonzero matrix element $(\hat{A}^s)_{ij}$. Hence, every matrix element of $\hat{M}^{-1}$ is strictly positive as long as $\lambda_0(z) > 0$.

Further, we can deduce that $g(z; j)$ is a strictly increasing, real-valued function of z for those z for which $\lambda_0(z) > 0$. To show this, we consider the vector $|\partial_z g\rangle$ whose components are the N_r derivatives $\partial_z g(z; j)$, $j = 1, ..., N_r$. It follows from Eq. (9.103) that the vector $|\partial_z g\rangle$ satisfies the inhomogeneous equation

$$\hat{M}(z) \left|\partial_z g\right\rangle = \left|\zeta\right\rangle, \tag{9.167}$$

where we denoted by $|\zeta\rangle$ the vector with the components

$$\zeta_k(z) \equiv \sum_{i\neq k} \kappa_{k\to i} g(z;i). \tag{9.168}$$

The solution of Eq. (9.167) is

$$|\partial_z g\rangle = \hat{M}^{-1}(z)\, |\zeta(z)\rangle\,. \tag{9.169}$$

Since all the matrix elements of $\hat{M}^{-1}(z)$ are positive and all the components of $|\zeta\rangle$ are nonnegative as long as $g(z;i) > 0$, it follows that all the components of $|\partial_z g\rangle$ are strictly positive. Equation (9.104) shows that $g(z;i) > 0$ for sufficiently small $z > 0$, and it follows that $g(z;i)$ will remain positive for all $z > 0$ such that $\lambda_0(z) > 0$. Therefore, $g(z;i)$, $i = 1, ..., N_r$ are strictly increasing functions of z for all those z.

Nevertheless, the functions $g(z;i)$ are bounded from above. To see this, consider the relationship

$$\left\langle u^0 \right| \hat{M} = \lambda_0 \left\langle u^0 \right|, \tag{9.170}$$

written in components as

$$u_j^0(z) \left[\frac{1}{\nu} g^{\frac{1}{\nu}-1}(z;j) - \kappa_{j\to j}\right] - \sum_{i\neq j} u_i^0(z) z \kappa_{i\to j} = \lambda_0 u_j^0. \tag{9.171}$$

Since all the components u_i^0 are strictly positive (as long as $\lambda_0(z) > 0$), it follows that

$$\frac{1}{\nu} g^{\frac{1}{\nu}-1}(z;j) - \kappa_{j\to j} > 0 \tag{9.172}$$

and hence

$$g(z;j) < \left(\nu\kappa_{j\to j}\right)^{-\frac{\nu}{\nu-1}} < \frac{1}{\nu^{\frac{\nu}{\nu-1}}} \approx \frac{1}{\nu}. \tag{9.173}$$

Furthermore, we can show that $\lambda_0(z)$ monotonically decreases as z grows. This follows from the perturbation theory formula for nondegenerate eigenvalues, which allows us to express $d\lambda_0/dz$ as a matrix product with normalized left and right eigenvectors,

$$\frac{d\lambda_0(z)}{dz} = \left\langle u^0(z) \right| \frac{d\hat{M}}{dz} \left| v^0(z) \right\rangle. \tag{9.174}$$

As we have just shown, $g(z;i)$ grows with growing z, so $d\hat{M}/dz$ is a matrix with all nonpositive elements. Since the vectors $\left\langle u^0 \right|$ and $\left| v^0 \right\rangle$ have strictly positive components while at least *some* matrix elements of the nonpositive matrix $d\hat{M}/dz$ are strictly negative, we obtain the strict inequality

$$\frac{d\lambda_0(z)}{dz} = \left\langle u^0(z) \right| \frac{d\hat{M}}{dz} \left| v^0(z) \right\rangle < 0. \tag{9.175}$$

Similarly, we can show that $\det \hat{M}(z)$ monotonically decreases with z:

$$\frac{d}{dz} \det \hat{M}(z) = \big(\det \hat{M}(z)\big) \mathrm{Tr}\big(\hat{M}^{-1} \frac{d\hat{M}}{dz}\big) < 0 \tag{9.176}$$

since it was already found that the matrix $\hat{M}^{-1}(z)$ has all positive elements while $d\hat{M}/dz$ has all nonpositive elements. However, the monotonic decrease alone of $\lambda_0(z)$ and of $\det \hat{M}(z)$ is not yet sufficient to establish that the matrix $\hat{M}(z)$ actually becomes singular at some finite z.

The results derived so far — the monotonic behavior of $g(z;j)$ and $\lambda_0(z)$, the positivity of the matrix elements of $\hat{M}^{-1}$, the bounds on g — hold for all z for which $\lambda_0(z) > 0$. Now we will show that $\lambda_0(z)$ cannot remain positive for all real $z > 0$. We can rewrite Eq. (9.109) as

$$z \sum_{i \neq k} \kappa_{k\to i} g(z;i) = g^{\frac{1}{\nu}}(z;k) - \kappa_{k\to k} g(z;k). \tag{9.177}$$

Using the property $g(z;j) > 0$ and the bound (9.173), we obtain (for every k) an upper bound on z,

$$z = \frac{g^{\frac{1}{\nu}}(z;k) - \kappa_{k\to k} g(z;k)}{\sum_{i \notin T, i \neq k} \kappa_{k\to i} g(z;i) + \kappa_{k\to T}} < \frac{g^{\frac{1}{\nu}}(z;k)}{\kappa_{k\to T}} < \frac{\nu^{1-\nu}}{\kappa_{k\to T}}. \tag{9.178}$$

In other words, no real-valued solutions of Eq. (9.109) exist for larger z. Let us then show that an upper bound on z contradicts the assumption that $\lambda_0(z) > 0$ for all z. We know that there exists a real-valued solution branch $g(z;j)$ near $z = 0$ such that $0 < g(z;j) < \infty$ and $0 < \partial_z g(z;j) < \infty$ for all those $z > 0$ for which this solution branch remains real-valued. Hence, $g(z;j)$ can be viewed as a solution of a differential equation $\partial_z g(z;j) = |\partial_z g\rangle$ with continuous coefficients and everywhere positive right-hand side. The solution of such differential equations, if bounded, will exist for all $z > 0$. Indeed, if the solution $g(z;j)$ existed only up to some $z = z_1$, we would have, by assumption, $\lambda_0(z_1) > 0$ and hence a finite value $g(z_1;j) > 0$ and a finite derivative $\partial_z g(z_1;j) > 0$. So the solution $g(z;j)$ could then be continued further to some $z > z_1$. Therefore, the real-valued solution branch $g(z;j)$ must exist for all $z > 0$. This is incompatible with the bound (9.178).

We conclude that there exists a value $z_* > 0$ such that $\lambda_0(z_*) = 0$ but $\lambda_0(z) > 0$ for all $0 < z < z_*$. Within the range $0 < z < z_*$ the functions $g(z;j)$ grow monotonically but remain bounded by Eq. (9.173), while $\lambda_0(z)$ and $\det \hat{M}(z)$ both decrease monotonically to zero.

9.4.6 The root of $\lambda_0(z)$

It remains to establish that $\lambda_0(z)$ indeed has the form (9.114) near $z = z_*$. We again restrict our attention to the interval $0 < z < z_*$ where $\lambda_0(z) > 0$. For these z we expand $\lambda_0(z)$ in Taylor series and express the value $\lambda_0(z_*) \equiv 0$ as

$$0 = \lambda_0(z_*) = \lambda_0(z) + \frac{d\lambda_0(z)}{dz}(z_* - z) + O[(z_* - z)^2], \tag{9.179}$$

hence

$$\frac{d\lambda_0(z)}{dz} = -\frac{\lambda_0(z)}{z_* - z} + O(z_* - z). \tag{9.180}$$

We then use Eq. (9.174) to express $d\lambda_0/dz$ in another way,

$$\begin{aligned}\frac{d\lambda_0(z)}{dz} &= \left\langle u^0(z)\right| \frac{d}{dz}\hat{M}(z) \left|v^0(z)\right\rangle \\ &= \left\langle u^0(z)\right| \left[\sum_i \frac{\partial \hat{M}}{\partial g(z;i)}\partial_z g(z;i) + \partial_z \hat{M}(z)\right] \left|v^0(z)\right\rangle,\end{aligned} \tag{9.181}$$

where in the second line we interpreted $\hat{M}(z)$ as a function of N_r variables $g(z;i)$, $i = 1, ..., N_r$, and explicitly of z, in order to express d/dz through $\partial/\partial g$ and $\partial/\partial z$. Using Eq. (9.169), we then find

$$\frac{\lambda_0(z)}{z - z_*} \approx \left\langle u^0(z)\right| \left[\sum_{i,k} \frac{\partial \hat{M}}{\partial g(z;i)} \hat{M}^{-1}_{ik} \zeta_k + \frac{\partial \hat{M}(z)}{\partial z}\right] \left|v^0(z)\right\rangle. \tag{9.182}$$

Since we are only interested in the qualitative behavior of $\lambda_0(z)$ at $z = z_*$, we do not need to keep track of the complicated coefficients in Eq. (9.182). Near $z = z_*$ we have, by Eq. (9.119),

$$\hat{M}^{-1}(z) \approx \frac{1}{\lambda_0(z)} \left|v^0\right\rangle \left\langle u^0\right| + O(1), \tag{9.183}$$

so the dominant singular terms in Eq. (9.182) near $z = z_*$ are

$$-\frac{\lambda_0(z)}{z_* - z} \approx \frac{C_1}{\lambda_0(z)} + O(1). \tag{9.184}$$

Therefore we obtain

$$\lambda_0(z) = c_1\sqrt{z_* - z} + O(z_* - z). \tag{9.185}$$

The positivity of $\lambda_0(z)$ for $z < z_*$ entails $c_1 > 0$. This concludes the derivation of Eq. (9.114).

10 Conclusion

In this book I have explored the problem of extracting observational consequences from inflationary models with spatial variation of the physical constants in the universe. The approach was based on the version of the anthropic principle proposed in [5]. The main issue in this line of work is to construct a well-motivated prescription for a measure on a multiverse.

Models of inflation are generically future-eternal in that inflation never ends globally (Chapter 3). Much of the work is performed in the context of inflation driven by a hypothetical scalar field ϕ (the inflaton), as is currently observationally favored. A second much researched cosmological scenario is the so called string-theoretic landscape where the vacuum state of fields is metastable. Transitions between vacua can proceed via bubble nucleation.

The stochastic description of inflation based on Fokker–Planck equations for the spatial probability distribution of the inflaton field ϕ is reviewed in Chapter 4. I explain the main properties of the stochastic approach and also give an introduction to the stochastic description of the string-theoretic landscape using the master equation. These mathematical tools allow one to obtain a concrete stochastic process representing eternal inflation and to study its properties quantitatively.

I give a broad overview of the currently present measure proposals in Chapter 6. In all the measure proposals, the infinite volume of the multiverse is cut off in a controlled manner. However, the measures differ in the specific implementation of the volume cutoff and in the way the volume approaches infinity when the cutoff is removed. The existing measure prescriptions can be categorized into volume-based and worldline-based. I characterize these prescriptions from a physical standpoint and explain their main features. The volume-based measures compute the observation of a randomly situated observer in a global infinite multiverse, and are suitable for predicting "cosmological" conditions, *i.e.* external conditions found on first observation. The worldline-based measures are suitable for sending a "message to the future" containing information about the expected cosmological conditions in the future given the knowledge of the present conditions. Since the message can be opened only by observers near the message box, the worldline selects the correct statistical ensemble of observers. In this way I show that the two classes of measures look for answers to very different physical questions.

I proceed to present a new multiverse measure proposal — the RV measure (Chapter 7). The RV measure is based on the construction of an ensemble of finite multiverses, which can be visualized as a cutoff in probability space, rather than a geometric cutoff in the real spacetime. This is an attractive feature of the RV proposal since no arbitrarily chosen geometric constructions need to be used.

In chapters 8 and 9 I develop a mathematical framework for the technically difficult analysis of the probability distributions of observables in the RV cutoff. There are two main contexts in which the RV cutoff is applied: the slow-roll inflation driven by a scalar field, and the landscape of string theory. In both contexts I show by rigorous methods that the RV cutoff is well-defined and independent of the choice of initial conditions at the beginning of inflation. For slow-roll inflation I derive formulas that are suitable for direct numerical calculations. For the landscape, I derive an analytic expression for relative bubble abundances of low-energy vacua. I also perform explicit analytic calculations for a toy model of slow-roll inflation and for several toy models of the landscape.

These calculations confirm that the RV measure has the desirable properties expected of a volume-based measure: coordinate invariance, independence of initial conditions, and the absence of the youngness paradox. A calculation for a general landscape also shows that the RV measure does not suffer from the "Boltzmann brain" problem. Thus the RV measure is a promising solution to the long-standing problem of obtaining probabilities in models of eternal inflation. Ultimately, the viability of the RV measure proposal will depend on its performance in various example cases. In the calculations performed so far, it is found that RV measure yields results that do not always coincide with the results of any other measure proposal. Hence, the RV measure is not equivalent to any of the earlier proposals.

An extension of the RV measure to landscape scenarios can be achieved in several ways. In Sec. 9.1 I considered the set of all possible future evolutions of a single nonterminal bubble and defined the ensemble E_N of evolutions yielding a finite total number N of *terminal* bubbles. In Sec. 9.2 I used the ensemble of evolutions with a finite total number N of daughter bubbles (of *all* types). One could also consider the ensemble E'_N of evolutions yielding a finite total number N of *observers* in bubbles of all types. After computing the distribution of some desired quantity by counting the observations made within the finite set of N bubbles (or observers), the cutoff parameter N can be increased to infinity. It remains to be seen whether the limit distributions are different for differently defined ensembles, such as E_N and E'_N, and if so, which definition is more suitable. Future work will show whether some extension of the RV measure can provide a satisfactory answer to the problem of predictions in eternal inflation.

To conclude, the present work demonstrates that the RV measure has attractive features and may be considered a viable candidate for the solution of the measure problem in multiverse cosmology. More work is needed to investigate the dependence of the predictions on the precise details of the definition of the ensemble E_N. I have developed an extensive mathematical framework for the calculations in the RV prescription and obtained first results for specific landscapes. However, a more powerful approximation scheme is desirable so that the predictions of the RV measure can be more easily obtained for landscapes of general type. Ultimately, the viability of the RV measure is to be judged by its predictions for cosmological observables in realistic landscapes. These issues are topics for future research.

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Index