Math 3108 - Linear Algebra | **Course Schedule** | **Summer 201~~6~~7**

Monday	Tuesday	Wednesday	Thursday	Friday
6/5 Sec 1.1-1.2	6/6 Sec 1.2-1.3	6/7 Sec 1.4-1.5	6/8 Sec 1.5,1.7	6/9 Sec 1.8-1.9
6/12 Sec 1.9-2.1	6/13 Sec 2.2-2.3	6/14 Sec 2.5	6/15 Review	6/16 Exam 1
6/19 Sec 3.1-3.2	6/20 Sec 4.1	6/21 Sec 4.2	6/22 Sec 4.3	6/23 Sec 4.4
6/26 Sec 4.5	6/27 Sec 4.6	6/28 Sec 4.7	6/29 Sec 5.1	6/30 Sec 5.2
7/3 Sec 5.3	7/4 Holiday	7/5 Sec 5.4	7/6 Review	7/7 Exam 2
7/10 Sec 5.5	7/11 Sec 5.7	7/12 Sec 6.1	7/13 Sec 6.2	7/14 Sec 6.3
7/17 Sec 6.4	7/18 Sec 6.5	7/19 Sec 6.7	7/20 Sec 6.8	7/21 Sec 7.1
7/24 Sec 7.2	7/25 Sec 7.4	7/26 Review		7/28 Final Exam 8:00-10:00 AM

Moving to CSF G5E

mymathlab

Calc III — 3-d views

hour exams (3)

6/16 Exam 1
7/7 Exam 2
7/28 Exam 3 / Final

500 points
100 HW
300 exams
100 comprehensive final

2-week free MML

Final on Friday 8:00 am

ask Roe for help

daily homework

Tommy's here!

read ahead in the book
read again after class

homeworks through exam 1 are available

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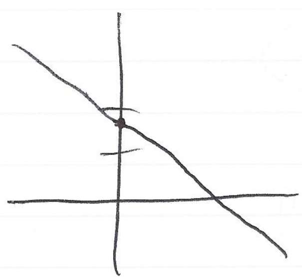
linear equations — variables to first power
no funny functions

to solve
- solve & substitute

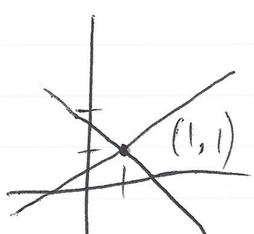
or
- pivot

$2x + 3y = 5$
multiple
(infinite)

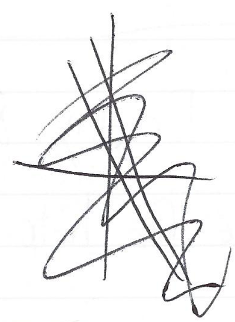


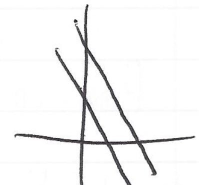
$2x + 3y = 5$
$x - 4y = \cancel{?} -3$

exactly one solution



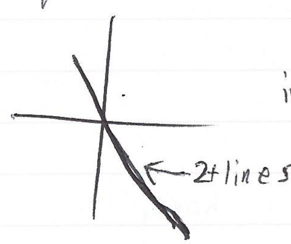
possibility





no solutions

possibility



infinite # of solutions

a square system may have 0, 1, or ∞ solutions, like non-square systems

plane — ~~equat~~ 3 unknowns, 1 equation

$5x + 7y + 2z = 7$

2 planes might
- be parallel and distinct
- be identical
- intersect on a line

3 planes might
- never intersect
- intersect in a line
- intersect in one point

$$\begin{aligned} 5x - y + 2z &= 7 \\ -2x + 6y + 9z &= 0 \\ -7x + 5y - 3z &= -7 \end{aligned} \longrightarrow \begin{pmatrix} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{pmatrix} \xrightarrow{2R2+R1} \begin{pmatrix} 1 & 11 & 20 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{pmatrix}$$

3 operations
- scale equation by nonzero const
- add one equation to another
- swap two equations

$$\xrightarrow{2R1 + R2} \begin{pmatrix} 0 & 28 & 49 & 14 \end{pmatrix}$$

$$\xrightarrow{7R1 + R3} \begin{pmatrix} 0 & 82 & 137 & 4 \end{pmatrix}$$

$$\tfrac{1}{28} R2 \rightarrow \begin{pmatrix} \cancel{1} & \cancel{\tfrac{7}{4}} & \cancel{\tfrac{1}{2}} & \\ 0 & 1 & \tfrac{7}{4} & \tfrac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 11 & 20 & 7 \\ 0 & 1 & \tfrac{7}{4} & \tfrac{1}{2} \\ 0 & 0 & \left(137 - \tfrac{82 \cdot 7}{4}\right) & 1 \end{pmatrix}$$

there is exactly one solution

$$\left(137 - \frac{82 \cdot 7}{4}\right) z = 1$$

solve for z
plug in
solve for y
plug in
solve for X

rref

$$\begin{pmatrix} 1 & 0 & 0 & 7 + \tfrac{40}{13} - \tfrac{110}{13} \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \tfrac{10}{13} \end{pmatrix}$$

$\tfrac{40}{13}$

there will be arithmetic on the exam

reduced row echelon form

1. the first nonzero number in each row is 1
2. each column containing one of those 1's contains only that 1 (all other entri are zer

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notes from homework

to find the value of h such that

$$\begin{bmatrix} \cdots & h & \cdots \\ \vdots & \ddots & \\ & & \ddots \end{bmatrix}$$

is inconsistent,

find the value of h such that

~~$hx_n = b_k$, where b_k~~

$0x_n = b_k$, where $b_k \neq 0$

(it's possible that no such value exists)

$m \times n \Rightarrow m$ rows, n columns

solution set — the set of solutions to a system

2 fundamental questions

whether "the" solution exists

whether there is only one solution

1.2 homework

the reduced echelon form of a matrix is unique
(every matrix has exactly one equivalent matrix in reduced echelon form)

~~a~~ a parametric description of a solution set to a system can only be found if there is at least one solution to the system

matrices A and B are <u>row equivalent</u> ~~(or just equivalent?)~~
— row operations can make A and B identical

systems A and B are <u>equivalent</u> iff they have equivalent solution sets and thus their matrices are row equivalent

system A is <u>consistent</u> — A has at least one solution

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book notes

1.1

coefficient matrix — matrix of coefficients
— a matrix of the essential information of a linear system — aligns ~~coeffient~~ coefficients of each v in columns

augmented matrix — coefficients with additional ~~columns~~ column of RHS's of ~~the~~ the equations (which are constants)

row operations
1. replacement
2. interchange
3. scaling

all row operations are reversible

1.2

a rectangular matrix is in echelon form iff
1. All nonzero rows are above rows with zeros
2. each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below leading entries are zero
~~4. the leading entry in each nonzero r~~

a rectangular matrix is in reduced row echelon form iff, additionally,
4. the leading entry in each nonzero row is 1
5. each leading 1 is the only nonzero entry in its colu

pivot position — a leading 1 in reduced echelon for
pivot column — column that contains a pivot position

basic variables — pivot variables
free variable — non-basic variable

numerical notes
- computer programs choose the largest leading values as pivots
- a flop is one operation

parametric representations of systems: — (consistent systems only)
treat one variable as parameter, (multiple solutions ~~on~~ systems only)
~~let~~
(we usually choose the free variables as parameters)

1.3

column vector — vector — matrix with one column

$$\begin{bmatrix}4\\7\end{bmatrix} \neq \begin{bmatrix}7\\4\end{bmatrix}$$

$$\begin{bmatrix}3\\-1\end{bmatrix} \neq \begin{bmatrix}3 & -1\end{bmatrix}$$

$$\begin{bmatrix}3\\-1\end{bmatrix} = (3, -1)$$

let u and v be vectors in R^n

$c(u+v) = cu + cv$

$\cancel{cu+}(c+d)u = cu + du$

linear combination (y) let v_1 — v_p be vectors in R^n.

$y = c_1v_1 + c_2v_2 + c_3v_3 + \ldots + c_pv_p$

1.2 h/w notes

forward phase — reducing a matrix to echelon form

backward phase — reducing a matrix to reduced echelon for

general solution — an explicit description of all solutions to the system

every row has a pivot position $\Rightarrow$ the sytem is consistent

the system is consistent

there are no free variables $\Rightarrow$ there is exactly one solution to the system

Roe 6/6/2017

1.2

$$\begin{pmatrix} & & & \\ 0 & 0 & 0 & 7 \end{pmatrix} \rightarrow 0x+0y+0z=7 \rightarrow \text{no solutions}$$

consistent system — a system with at least one solutio

leading entry — the first nonzero entry in a row of a matrix

A matrix is in <u>row echelon form</u> if the leading entry of each row has only zero below it and any row of all zeros is at the bottom.

A matrix is in <u>Reduced Row Echelon Form</u> if the leading entries are all 1's and there are only zeroes in the remaining entries of the column.

Positions that have leading entrier in a row echelon form matrix are called <u>pivot positions.</u>
Pivot positions are unique.

A system is <u>inconsistent</u> iff there is a pivot position in the last column of the augmented matrix
(If we don't have a pivot position in the last column of the augmented matrix, then the system is consistent)

system of m equations and n unknowns —

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{pmatrix} \underline{1} & 0 & 2 & 2 \\ 0 & \underline{1} & -1 & 3 \\ 0 & 0 & 0 & \underline{1} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

ref
not rref

pivot positions are underlined

$$\begin{array}{cccc} & & \text{free variable} & \\ & & \downarrow & \end{array}$$
$$\begin{pmatrix} \underline{1} & 0 & 3 & -1 \\ 0 & \underline{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rref

$x_1 + 3x_3 = -1$
$x_2 - x_3 = 3$

$x_2 = 3 + x_3$
$x_1 = -1 - 3x_3$

$x_3 = t$
$x_2 = 3 + t$
$x_1 = -1 - 3t$

x_2 is free

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

ref
not rref

with multiple free varibles, use $t, s, \ldots$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \to x_1 + x_2 = 4 \\ \to x_3 = -1 \\ \to x_4 = 1 \\ \end{array}$$

rref

$x_1 = -t + 4$
$x_2 = t$
$x_3 = -1$
$x_4 = 1$

1.3

A vector in $\mathbb{R}^n$ is an $n \times 1$ matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Def. addition of vectors $\bar{u} + \bar{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$

scalar multiplication $c\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix}$

properties
1) $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
2) $(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w})$
3) $\bar{u} + \bar{0} = \bar{u}$
4) $\bar{u} + -\bar{u} = \bar{0}$
5) $c(\bar{u} + \bar{v}) = c\bar{u} + c\bar{v}$
6) $(c+d)\bar{u} = c\bar{u} + d\bar{u}$
7) $c(d\bar{u}) = (cd)\bar{u}$
8) $1\bar{u} = \bar{u}$

$x_1 = 1$
$x_2 = 3$
$2x_1 + x_2 = 5$
$x_1 - x_2 = -3$
$x_1 + x_2 = 4$

is consistent

5 lines in a plane that all go through $(1, 1)$

(augmented matrix of system) $\to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Linear Combination

if $\bar{u}_1 \ldots \bar{u}_m$ are vectors in $\mathbb{R}^n$
and $c_1 \ldots c_n$ are scalars
then $\bar{y} = c_1\bar{u}_1 + c_2\bar{u}_2 + \ldots + c_m\bar{u}_m$
is a linear combination
of the vectors $\bar{u}_1 \ldots \bar{u}_m$

$$\bar{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

any vector in R^3 is a linear combination of $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

is $\bar{y} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ a linear combination of $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} \right\}$?

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h/w 1.2.29

undetermined system — a system of linear equations with fewer equations than unknowns

An undetermined system either has infinitely many solutions or no solutions

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book 1.2 & 1.3

general solution

$$\underset{\text{rref}}{\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \Longrightarrow \begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases}$$

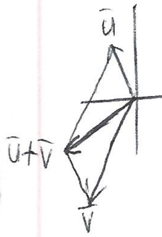
↑ general solution

parametric descriptions of solution sets don't have to use the free variable as the parameter, but we usually make the free variable the parameter

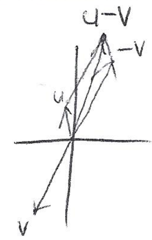
A if a system is inconsistent then it has no parametric representation

A system is consistent iff the augmented column is not a pivot column.

If a system is consistent, then either
- it has exactly one solution, or
- it has an infinite number of solutions



parallelogram rule



zero vector — $\bar{0}$

Span$\{v_1, v_2, v_3, \ldots, v_p\}$
— the set of all linear combinations of $v_1, v_2, v_3, \ldots v_p$

1.4

matrix equation — an equation of the form

$$A\bar{x} = \bar{b}$$

If A is $m \times n$ then $\bar{x} \in R^n$ and $\bar{b} \in R^m$

product of matrix A and vector $\bar{x}$
— $A\bar{x}$
— linear combination of the columns of A using the corresponding entries in $\bar{x}$ as weights

we can now view a system of linear equations as
- a matrix equation
- a vector equation
- a system of linear equations

$A\bar{x} = \bar{b}$ has a solution $\Leftrightarrow$ $\bar{b}$ is a linear combination of the columns of A $\Leftrightarrow$ $Ax = b$ is consistent

important + equivalent statements

Let A be a non-augmented coefficient matrix.
then the columns of A span R^m
$\Leftrightarrow$ every b in R^m is a linear combination of the columns of A
$\Leftrightarrow$ $A\vec{x} = \bar{b}$ has a solution for all $\bar{b} \in R^m$
$\Leftrightarrow$ A has a pivot position in every row

to determine if the columns of A span R^m, bring A to row echelon form. If there are any rows of all zeroes, then A does not span R^m, else A spans R^m.

$\cancel{A\bar{u}}$

$A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$
$A(c\bar{u}) = c(A\bar{u})$

1.4

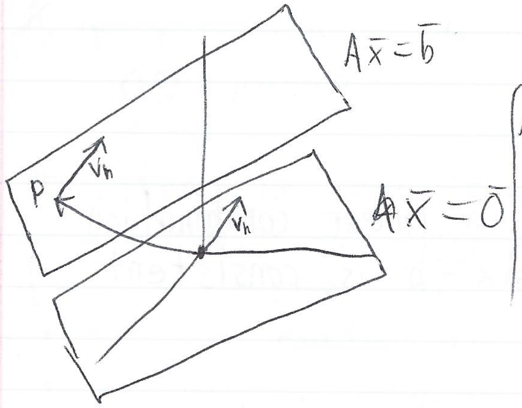
a system is <u>homogeneous</u>
$\Leftrightarrow$ it can be written in the form $A\bar{x} = \bar{0}$

trivial solution — $\bar{x} = \bar{0}$

any equation $A\bar{x} = \bar{0}$ has a trivial solution

a nontrivial solution can have some zeroes, but cannot be all zeroes

theorem 6
<u>suppose $A\bar{x} = \bar{b}$ is consistent</u>
let p be a solution to $A\bar{x} = \bar{b}$
let v_h be any solution to $A\bar{x} = \bar{0}$
then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form $w = p + v_h$



<u>proof</u>
$A\bar{w} = A(p + v_h) = Ap + Av_h = \bar{b} + \bar{0} = \bar{b}$
so $A\bar{w} = \bar{b}$

6/7/2017

Roe

A vector $\bar{y}$ in $\mathbb{R}^n$ is ~~the~~ a <u>linear combination</u> of the set of vectors $\bar{u}_1, \bar{u}_2, \dots \bar{u}_m$ if there are scalars $c_1, c_2, \dots, c_m$ such that $\bar{y} = c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_m\bar{u}_m$

Span$\{\bar{u}_1, \dots, \bar{u}_m\}$ is the set of all vectors that are linear combinations of $\bar{u}_1, \dots, \bar{u}_m$

is $\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$ in Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix} \right\}$?

$$\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \end{pmatrix} + c_3 \dots$$

$$= \begin{pmatrix} c_1 \\ 2c_1 \\ 0 \\ 3c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_2 \\ 2c_2 \\ -2c_2 \end{pmatrix} + \begin{pmatrix} c_3 \\ c_3 \\ 3c_3 \\ -c_3 \end{pmatrix}$$

$-2 = c_1 - c_2 + c_3$

$3 = 2c_1 + c_2 + c_3$

$1 = 2c_2 + 3c_3$

$0 = 3c_1 - 2c_2 - c_3$

$$\begin{pmatrix} 1 & -1 & 1 & -2 \\ 2 & 1 & 1 & 3 \\ 0 & 2 & 3 & 1 \\ 3 & -2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 11 & -11 \\ 0 & 0 & 11 & -11 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since the system has a sol $\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$ is in the

what is the linear combination?

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$c_1 = 1$

$c_2 = 2$

$c_3 = -1$

is every vector in $\mathbb{R}^4$ in the span?

$$\begin{pmatrix} 1 & -1 & 1 & a \\ 2 & 1 & 1 & b \\ 0 & 2 & 3 & c \\ 3 & -2 & 1 & d \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & * \\ 0 & 1 & -4 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & c-2(d-3a)-(b-2a)-3(d-3a) \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 & * \\ 0 & 1 & -4 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & c-b+2a-5(d-3a) \end{pmatrix}$$

$c-b+2a-5(d-3a) \neq 0$

so not every vector in $\mathbb{R}^4$ is in the span

Roe

"1.4"

If A is an $m\times n$ matrix and $\bar{x}\in\mathbb{R}^n$ then

$$A\bar{x} = x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2}\end{pmatrix} + \dots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn}\end{pmatrix} = x_1 a_1 + x_2 a_2 + \dots$$

represents a linear combination of the columns of A with the entries of $\bar{x}$ as coeffieients

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

$$\Leftrightarrow \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1}\end{pmatrix}x_1 + \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2}\end{pmatrix}x_2 + \dots + \begin{pmatrix} a_{n1} \\ a_{n2} \\ a_{mn}\end{pmatrix}x_n = b$$

$$\Leftrightarrow A\bar{x} = \bar{b}$$

$$A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$$
$$A(c\bar{u}) = c(A\bar{u})$$

"1.5"

a linear system of equations is <u>homogeneous</u> if the RHS equals zero, e.i. $A\bar{x}=\bar{0}$

homogeneous systems have to be consistent, with $\bar{x}=\bar{0}$

either $\bar{x}=\bar{0}$ is the only solution or there are infinitely many solutions

If $\bar{u}$ is a soln. to $A\bar{x}=\bar{b}$ and $\bar{v}$ is soln. to $A\bar{x}=\bar{0}$ then $A(\bar{u}+\bar{v}) = A\bar{u}+A\bar{v} = \bar{b}+\bar{0} = \bar{b}$, so $\bar{u}+\bar{v}$ is a solution to the non-homogeneous system $A\bar{x}=\bar{b}$

h/w 1.4 & 1.5 notes

if a coefficient matrix A is $m \times n$ and $n < m$, A cannot span $\mathbb{R}^m$

parametric vector form

$$\bar{x} = (\text{free variable})\begin{bmatrix} \text{vector} \end{bmatrix} + \dots + \begin{bmatrix} \text{vector} \end{bmatrix}$$

$$\cancel{pvf\left(\begin{bmatrix} 1 & -2 & 0 & -4 & 4 & 8 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = pvf\left(\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)}$$

$$\bar{x} = x_2\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6\begin{bmatrix} -12 \\ 0 \\ 0 \\ -7 \\ -6 \\ 1 \end{bmatrix}$$

is the parametric vector form of

$$\begin{bmatrix} 1 & -2 & 0 & -4 & 4 & 8 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\bar{x} = \bar{0}$$

which reduces to

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 11$$
$$x_2 - x_3 = -1$$

can be described in parametric vector form as

$$\bar{x} = \begin{bmatrix} 11 \\ -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$pvf(x_1 + 7x_2 - 5x_3 = -4)$ is

$$\bar{x} = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} -7 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

book 1.4 & 1.5

implicit description of solution set — system of equations
explicit description of solution set — solved ~~x~~ for pivot varia
parametric vector form — explicit description of solution set using vectors

writing a solution set in parametric vector form
1. Row reduce the augmented matrix to reduced row echelon form
2. Express each basic variable in terms of any free variables
3. Write a typical vector solution $\bar{x}$ whose entries depend on free variables
4. Decompose $\bar{x}$ into a linear combination of vectors with numeric entries using the free variables as parameters

book 1.7

an indexed set of vectors $\{v_1, \ldots, v_p\}$ in R^n is said to be <u>linearly independent</u> if the vector equation
$x_1\bar{v}_1 + x_2\bar{v}_2 + \ldots + x_p\bar{v}_p = \bar{0}$ (1)
has only the trivial solution

$\{v_1, v_2, \ldots, v_p\}$ is <u>linearly dependent</u> if there exist weights $c_1, \ldots, c_p$, not all zero, such that
$c_1\bar{v}_1 + c_2\bar{v}_2 + \ldots + c_p\bar{v}_p = \bar{0}$ (2)

equation (2) is called a <u>linear dependence relation</u> among $v_1, \ldots,$
to "find a linear dependence relation," find numbers for $c_1 \ldots c_p$ that satisfy the equation (not a solution description)

a linear dependence relation among the columns of A correspo a nontrivial solution to $A\bar{x} = \bar{0}$

a set containing only one vector is linearly ~~in~~dependent
if and only if the one vector is the zero vector

a set containing only two vectors $\bar{a}$ and $\bar{b}$ is linearly ~~in~~dependent
if and only if ~~the two vectors are multiples of each other~~
either $\bar{a} = c\bar{b}$ or $\bar{b} = c\bar{a}$ for some $c \in \mathbb{R}$

$S = \{v_1, \ldots, v_p\}$ is linearly dependent if and only if
at least one of the vectors in S is a linear combinatio
of the others, ~~in which case every vector in S is a linear combination of the others~~

Tanner observes: $\bar{0} \in S \Rightarrow S$ is linearly dependent
"Theorem 9"

if S is a set of vectors in $\mathbb{R}^m$, $|S| > m \Rightarrow S$ is linearly dependent

if a set of vectors $S = \{v_1, \ldots, v_n\}$ is linearly dependen
there may be ~~vecto~~ a vector v_k such that v_k is
not a linear combination of $S - \{v_k\}$

Roe
6/8/2017

example

$$\begin{aligned} x_1 + 2x_2 + x_3 + 3x_4 &= 1 \\ x_1 + 2x_2 + 2x_3 + 5x_4 &= 3 \\ -x_1 - 2x_2 - x_3 - 3x_4 + x_5 &= 2 \\ -x_1 - 2x_2 - 3x_3 - 7x_4 &= -5 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 0 & 1 \\ 1 & 2 & 2 & 5 & 0 & 3 \\ -1 & -2 & -1 & -3 & 1 & 2 \\ -1 & -2 & -3 & -7 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_2 \cancel{+} -x_4 - 1 \\ x_2 \\ -2x_4 + 2 \\ x_4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix}$$

$$\bar{x} = t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 3 \end{pmatrix}$$

"parametric solution" where t and s are paramete

Soln set to the homogeneous system is

$$\bar{v}_h = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

soln set to homogeneous system $= \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}\right\}$

skipping 1.6

1.7

Def a set of vectors $\bar{u}_1, \dots, \bar{u}_m$ in $\mathbb{R}^n$ is said to be <u>linearly independent</u> iff the only soln to

$$c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_m\bar{u}_m = \bar{0} \text{ is } c_1 = c_2 = \dots = c_m = 0$$

equivalently, $[\bar{u}_1\ \bar{u}_2 \dots \bar{u}_m]\bar{x} = \bar{0}$ has only the trivial solution

- otherwise the vectors are <u>linearly dependent</u>

If $\bar{u}_1, \ldots, \bar{u}_m$ are linearly dependent then there must be $c_1, c_2, \ldots, c_m$, not all zero, so that $c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_m\bar{u}_m = \bar{0}$, and we can always write at least one of the vectors as a linear combination of the other vectors, that is, $\exists\, i \in \mathbb{Z}^+, i \le m \wedge \bar{u}_i = c_1\bar{u}_1 + c_2\bar{u}_2 + \ldots + c_{i-1}\bar{u}_{i-1} + c_{i+1}\bar{u}_{i+1} + \ldots + c$
for some $c_1, c_2, \ldots, c_m \in \mathbb{R}$ (which may all be zero)

is $\left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} \right\}$ linearly independent?

~~Yes~~, No since the matrix would ~~have to~~ produce at least one free variable

$\left(\begin{array}{cccc|c} 1 & -3 & 3 & -2 & 0 \\ -3 & 7 & -1 & 2 & 0 \\ 0 & 1 & -4 & 3 & 0 \end{array}\right)$ There are at most 3 pivot positions (1 per row) so there must be at least one free variable

is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 10 \\ -3 \end{pmatrix} \right\}$ LI?

No If LD, then one vector ~~must be~~ is a linear combination of the others.
They are LI, because of term 3

is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 10 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 8 \\ -2 \end{pmatrix} \right\}$ LI?

you'll need to perform ~~f~~ row reductions is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \\ -5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 1 \\ 3 \end{pmatrix} \right\}$ LI?

since $0\bar{v} = \bar{0}$ for any $\bar{v} \in \mathbb{R}^n$, and the set contains $\bar{0}$ of $\mathbb{R}^4$, the set is LD.

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ -1 & 4 & 0 & 1 & 0 \\ 3 & -5 & 0 & 3 & 0 \end{array}\right) \\ \uparrow \\ \text{free} \end{array}$$

Notes from hw 1.5 & 1.7

A is linearly independent $\Leftrightarrow$ A has a pivot position in every column
$\Leftrightarrow$ A has no free variables
else A is linearly dependent

If $\bar{x}$ and $\bar{y}$ are linearly independent, and if $\{\bar{x}, \bar{y}, \bar{z}\}$ is linearly dependent, then $\bar{z} \in \text{Span}\{\bar{x}, \bar{y}\}$ and $\bar{z}$ is a linear combination of $\bar{x}$ and $\bar{y}$

If A is 6×2, $A = (a_1\ a_2)$, and a_2 is not a multiple of a_1, then A could have an echelon form of

$$\begin{pmatrix} \square & * \\ 0 & \square \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \square \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

1.8 matrix transformations

if A is $m \times n$, domain of $A\bar{x}$ is R^n
co-domain of $A\bar{x}$ is R^m
range of $A\bar{x}$ is the set of all linear combinations of the columns of A

Roe 6/9/2017

Vector Fields

vector field — a vector valued function
$\bar{F}(\bar{x}, \bar{y}) = \langle f(\bar{x}, \bar{y}), g(\bar{x}, \bar{y}) \rangle$

If A is an $m \times n$ matrix and $\bar{x} \in R^n$ then $Ax \in R^m$.

transformation — a function whose inputs and outputs are vectors

if T is a transformation from R^n to R^m we denote this $T: R^n \to R^m$

the <u>range</u> of T is the set of all vectors $\bar{y} \in R^m$ such that ($\exists\, \bar{x} \in R^n$ such that $\bar{y} = T(\bar{x})$)

<u>Def.</u> A transformation $T: R^n \to R^m$ is said to be a <u>linear transformation</u> if for every $\bar{u}$ and $\bar{v}$ in R^n
(1) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$
(2) $T(c\bar{u}) = c(T(\bar{u}))$

$f(x) = ax + b$ is <u>not</u> a linear transformation theorem $T: R^n \to R^m$ is linear $\Rightarrow T(\bar{0}) = \bar{0}$

proof

$\bar{0} = \bar{u} + (-1)\bar{u}$ for any $\bar{u} \in R^n$

by properties of linear transformations $T(\bar{u} + (-1)\bar{u}) = T(\bar{u}) + T((-1)\bar{u})$

$= T(\bar{u}) + (-1)T(\bar{u})$

$= \bar{0}$

$T(\bar{0}) = \bar{0} \not\Rightarrow T$ is linear

but $T(\bar{0}) \neq \bar{0} \Rightarrow T$ is not linear

proof that $T: R^n \to R^m$ $T(\bar{x}) = A\bar{x}$ is linear

$T(\bar{x} + \bar{y}) = A(\bar{x} + \bar{y}) = A\bar{x} + A\bar{y} = T(\bar{x}) + T(\bar{y})$

$T(c\bar{x}) = A(c\bar{x}) = c(A\bar{x}) = c(T(\bar{x}))$

proof that $T(\bar{x}) = 3\bar{x}$ is linear $T: R^n \to R^n$

$T(\bar{x} + \bar{y}) = 3(\bar{x} + \bar{y}) = 3\bar{x} + 3\bar{y} = T(\bar{x}) + T(\bar{y})$

$T(c\bar{x}) = 3(c\bar{x}) = c(3\bar{x}) = cT(\bar{x})$

$T(\bar{x}) = c\bar{x}$ is called a dialation (or, if $|c| < 1$, a contract

proof that $\cancel{T(\bar{x})}$ $T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$ $T: R^n \to R^{n-1}$ is linear

$T(\bar{x} + \bar{y}) = T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_{n-1}+y_{n-1} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = T(\bar{x}) + T(\bar{y})$

$\vdots$

projection

<u>lemma</u> $T: R^n \to R^m$ is a linear transformation and $\bar{x}_1, \bar{x}_2, \ldots \bar{x}_k$ are vectors in R^n, $c_1 \ldots c_k$ are scalars, then $T(c_1\bar{x}_1 + \ldots + c_k\bar{x}_k) = c_1T(\bar{x}_1) + c_2T(\bar{x}_2) \ldots c_kT(\bar{x}_k)$

If $T: R^n \to R^m$ is a linear transformation and we know
$T\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, T\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots T\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$
then we know the ~~values~~ value of T at every vector in R^n

ex $T: R^3 \to R^2$ is linear

$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$ $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\\-4\end{pmatrix}$

find $T\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$$T\begin{pmatrix}3\\-1\\2\end{pmatrix} = 3\begin{pmatrix}2\\3\end{pmatrix} + (-1)\begin{pmatrix}-1\\1\end{pmatrix} + (2)\begin{pmatrix}5\\-4\end{pmatrix} = \begin{pmatrix}6\\9\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}10\\-8\end{pmatrix}$$
$$= \begin{pmatrix}17\\0\end{pmatrix}$$

it turns out $T(\bar{x}) = \begin{pmatrix}2 & -1 & 5\\3 & 1 & -4\end{pmatrix}\bar{x}$

any linear transformation can be written as a matrix multiplication

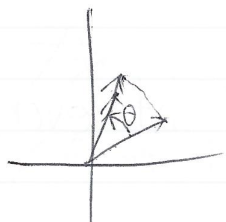
constructive proof:

$A = \left[T(e_1)\ T(e_2)\ \ldots\ T(e_3)\right]$

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$T(\bar{x}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 1 & 0 \end{pmatrix} \bar{x}$$

$T: R^2 \to R^2$ $T(\bar{x}) = \bar{x}$ rotated about origin by angle θ

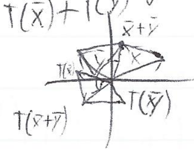


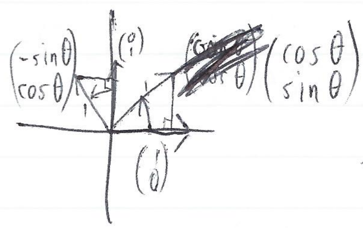
is T linear

$T(\bar{0}) = \bar{0}$ ✓

$T(c\bar{x}) = cT(\bar{x})$ ✓

$T(\bar{x}+\bar{y}) = T(\bar{x}) + T(\bar{y})$ ✓





$$T(\bar{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \bar{x}$$

<u>rotation</u>

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homework notes (1.8 & 1.9)

Tanner observes

$T: R^n \to R^m$ is ~~1~~ one-to-one iff ~~it~~s corresponding matrix is linearly independent.

if $\{v_1, \ldots, v_k\}$ is linearly dependent, then $\{T(v_1), T(v_2), \ldots, T(v_k)\}$ is also linearly dependent

<u>not inversely</u>

let $T: R^n \to R^m$ be a linear transformation

$T\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \iff$ the first column of T's matrix is $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$T\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \iff$ the second column of T's matrix is $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$

$\vdots$

$T\left(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} \iff$ the nth column of T's matrix is $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}$

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book 1.9 notes

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

let $T: R^n \to R^m$ be a linear transformation
then T is one-to-one iff $T(\bar{x}) = \bar{0}$ has only the trivial solution,
T is onto iff its matrix has a pivot position in every row, that is, if the columns of its matrix span R^m

book

2.1

warnings about matrix products

1) in general $AB \neq BA$

2) $AB = AC \not\Rightarrow B = C$

3) $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

diagonal entries of $A = [a_{ij}]$ — $a_{11}\ a_{22}\ a_{33} \dots$

main diagonal ~~of a square matrix~~ — the diagonal entries

diagonal matrix — a square matrix whose nondiagonal entries are all zero

zero matrix — an $m \times n$ matrix whose entries are all zero

A equals B iff they have the same size and their corresponding columns are equal

A is row equivalent to B iff there is a sequence of row operations ~~to b that~~ on A that result in a matrix to B

the sum $A+B$ is a matrix whose columns are the sum of the corresponding columns in A and B
(A, B, and the resulting matrix must be $m \times n$)

scalar multiple rA — a matrix whose columns are r times the corresponding columns in A

$A + B = B + A$

$(A+B)+C = A + (B+C)$

$A + 0 = A$

$r(A+B) = rA + rB$

$(r+s)A = rA + sA$

$r(sA) = (rs)A$

important

if A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $b_1, b_2, \ldots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, Ab_2, \ldots, Ab_p$, that is

$$AB = A\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$

Each column of AB is a linear combination of the columns of A using ~~the columns~~ weights from the corresponding column of B

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

the ith row of AB is the ith row of A times B

$$\text{row}_i(AB) = \text{row}_i(A) \cdot B$$

row vectors

$$A(BC) = (AB)C$$
$$A(B+C) = AB + AC$$
$$(B+C)A = BA + CA$$
$r(AB) = (rA)B = A(rB)$ for any scalar r
$$I_m A = A = A I_n$$

given an $m \times n$ matrix A, the <u>transpose</u> of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A

if A is n×n (square)

$A^k = \underbrace{AA\ldots A}_{k\ A\text{'s}}$

$A^0 = I_n$

$(A^T)^T$

$(A+B)^T = A^T + B^T$

for any scalar r, $(rA)^T = rA^T$

$(AB)^T = B^T A^T$

$(AB)^T \neq A^T B^T$

note from h/w

$x \mapsto Ax$ is one-to-one iff A has no free variables $\Leftrightarrow$ A is linearly independent

A and B commute with each other $\Leftrightarrow AB = BA$

~~An n×n matrix is invertible iff~~

An $n \times n$ matrix A is <u>invertible</u> $\Leftrightarrow \exists$ an $n \times n$ matrix C such that $CA = I$ and $AC = I$ (where $I = I_n$)

C is called the <u>inverse</u> of A

C is uniquely determined by A

proof that inverse matrix is unique

$AC = I$

$AB = I$

$B = BI = B(AC) = (BA)C = IC = C$

a matrix that is not invertible is called a <u>singular</u> matrix

an invertible matrix is called a <u>nonsingular</u> matrix theorem 4

$AA^{-1} = I \quad A^{-1}A = I$

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad-bc \neq 0$,

then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

theorem 5 if A is invertible and $A\bar{x} = \bar{b}$ then $\bar{x} = A^{-1}\bar{b}$

proof

~~$A\bar{x} = \bar{b} = I\bar{b} = (AA^{-1})\bar{b} = A(A^{-1}\bar{b})$~~

$\bar{x} = A^{-1}\bar{b} \quad A\bar{x} = A(A^{-1}\bar{b}) \quad A\bar{x} = (AA^{-1})b \quad A\bar{x} = I\bar{b} \quad A\bar{x} = \bar{b}$

to prove that $\bar{x} = A^{-1}\bar{b}$ is unique, let $A\bar{u} = \bar{b}$

then $A^{-1}(A\bar{u}) = A^{-1}\bar{b} \quad (A^{-1}A)\bar{u} = A^{-1}\bar{b} \quad I\bar{u} = A^{-1}\bar{b} \quad \bar{u} = A^{-1}\bar{b} = \bar{x}$

note from h/w

the standard matrix of a shear transformation has the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

horizontal vertical

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possible echelon forms of the standard matrix of $T: \mathbb{R}^4 \to \mathbb{R}^3$ if T is onto

$$\begin{pmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & \square & * \end{pmatrix} \quad \begin{pmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & \square \end{pmatrix} \quad \begin{pmatrix} \square & * & * & * \\ 0 & 0 & \square & * \\ 0 & 0 & 0 & \square \end{pmatrix} \quad \begin{pmatrix} 0 & \square & * & * \\ 0 & 0 & \square & * \\ 0 & 0 & 0 & \square \end{pmatrix}$$

2.2

theorem 6

a. $(A^{-1})^{-1} = A$

b. $(AB)^{-1} = B^{-1}A^{-1}$

c. $(A^T)^{-1} = (A^{-1})^T$

proof

~~a. A~~

b. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= A(I)A^{-1} = AA^{-1} = I$

elementary matrix — a matrix that can be formed from the identity matrix with a single row operation (where adding a constant times row i to row j $(i \neq j)$ is considered a single row operation)
"elimination matrix" — Gilbert Strang

if an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written EA, where the $m \times m$ matrix E is created by perfor the same row operation on I_m

~~note from h/w: if A and B are symetric, $(AB)^T = BA$~~

$$\begin{pmatrix} A & I \end{pmatrix} \text{ is row equivalent to } \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

~~Famer observes $AB = (BA)^T$~~

~~multiplying by matrix performs row operation~~

$$AB = \begin{bmatrix} col_1(A) & col_2(A) & \cdots & col_n(A) \end{bmatrix} \begin{bmatrix} row_1(B) \\ row_2(B) \\ \vdots \\ row_n(B) \end{bmatrix}$$

$$= col_1(A)row_1(B) + col_2(A)row_2(B) + \ldots + col_n(A)row_n(B)$$

if $AB = I$ then A and B are both invertible with $A = B^{-1}$ and $B = A^{-1}$

$A = LU$

how to find LU factorization

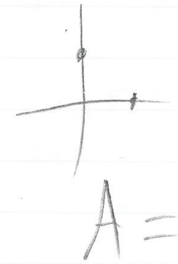
1. row reduce $[A|I]$, without performing row swaps
2. ~~it~~ you get $[U|L^{-1}]$. Negate all non-diagonal entries of L^{-1} to get L

Monday 6/12/2017

sign attendance

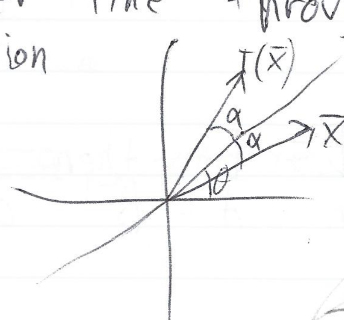
if $T: R^n \rightarrow R^n$

$$A = \begin{bmatrix} T(e_1) & T(e_2) \dots T(e_n) \end{bmatrix}$$



$$A = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos \end{pmatrix}$$

exercise: show reflection over line through origin is linear transformation and find matrix A



note: onto and one-to-

<u>def</u> a transformation $T: R^n \rightarrow R^m$ is said to be <u>onto</u> R^m iff for every vector $\bar{y} \in R^m$ there is at least one vector $\bar{x} \in R^n$ such that $\bar{y} = T(\bar{x})$

~~If~~ T is a linear transformation that is onto, $\Leftrightarrow$ $T(\bar{x}) = A\bar{x}$, ~~$\Leftrightarrow$~~ $A\bar{x} = \bar{y}$ has ~~a solution for~~ at least one solut for every $\bar{y} \in R^m$ $\overset{?}{\Leftrightarrow}$ ~~and~~ A has a pivot position in every row $\Leftrightarrow$ the columns of A span R^m

~~This happens when each row of A has a pivot position~~

<u>Def</u> a transformation $T: R^n \rightarrow R^m$ is <u>one-to-one</u> iff for every $\bar{b}$ in R^m there is <u>at most one</u> vector $\bar{x}$ in R^n such that $T(\bar{x}) = \bar{b}$

$\Leftrightarrow T(\bar{x}_1) = T(\bar{x}_2) \Rightarrow x_1 = x_2$

~~If additionally, T is linear~~

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, then T is one-to-one iff $T(\bar{x}) = \bar{0}$ has exactly one solution

suppose T is not 1-to-1, $\Rightarrow$ for some $\bar{x}_1 \neq \bar{x}_2$, $T(\bar{x}_1) = T(\bar{x}_2)$
$\Rightarrow T(\bar{x}_1) - T(\bar{x}_2) = \bar{0} \Rightarrow T$ is linear $\Rightarrow T(x_1 - x_2) = \bar{0}$
$\Rightarrow T(\bar{0}) = \bar{0}$ so $\bar{x}_1 - \bar{x}_2 = 0$ and $\bar{x}_1 = \bar{x}_2$, contradiction

$T(\bar{x}) = A\bar{x}$ is one-to-one iff $\bar{x} = \bar{0}$ has only one solution $\Leftrightarrow$ there are no free variables in A

2.1 w/ Roe

matrix addition $A+B$
- must be same size
- add corresponding entries

scalar multiplication cA
- multiply every entry by scalar

if A is $m \times n$ and B is $n \times p$ then AB is a $m \times p$ matrix where $AB = [A b_1 \; A b_2 \cdots A b_p]$

note: BA is not defined

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$AB = \left(A\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \; A\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \; A\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} \right)$$

$$AB = \left(\begin{pmatrix} 5 \\ -1 \end{pmatrix} \begin{pmatrix} 15 \\ 9 \end{pmatrix} \begin{pmatrix} 8 \\ -4 \end{pmatrix} \right)$$

$$AB = \begin{pmatrix} 5 & 15 & 8 \\ -1 & 9 & -4 \end{pmatrix}$$

linear trans

$T: \mathbb{R}^m \to \mathbb{R}^n$ $\quad S\circ T(\bar{x}+\bar{y}) = S(T(\bar{x}+\bar{y})) = S(T(\bar{x}) + T(\bar{y})) = S(T(\bar{x})) + S(T(\bar{y}))$

$S: \mathbb{R}^n \to \mathbb{R}^p$ $\quad S\circ T(c\bar{x}) = S(T(c\bar{x})) = S(cT(\bar{x})) = c\,S(T(\bar{x}))$

$S\circ T(\bar{x})$

$S(\bar{x}) = B\bar{x}$

$T(\bar{x}) = A\bar{x}$

$S\circ T(\bar{x}) = BA\bar{x} = C\bar{x} = x_1\bar{c}_1 + x_2\bar{c}_2 + \ldots + x_p\bar{c}_p$

$S\circ T(\bar{e}_i) = \bar{c}_i$

~~$S\circ T(\bar{x}) =$~~

$S(\bar{x}) = A\bar{x} \quad T(\bar{x}) = B\bar{x}$

$S\circ T(\bar{x}) = S(B\bar{x}) = S(x_1\bar{b}_1 + x_2\bar{b}_2 + \ldots + x_p\bar{b}_p) = x_1S(\bar{b}_1) + x_2S(\bar{b}_2) + \ldots + x_pS(\bar{b}_p) = x_1A\bar{b}_1 + x_2A\bar{b}_2 + \ldots + x_pA\bar{b}_p$

$S\circ T(\bar{e}_i) = A\bar{b}_i \longleftarrow$ ith col of AB

Roe 2.1

Sample Exam

Thm. If A is $m\times n$, B is $n\times p$, then ~~the~~ the (i,j) entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$

~~ij entry of AB is in the jth column of of AB~~

jth col of AB is $A\bar{b}_j$ where $\bar{b}_j$ is the jth col of B

$$A\bar{b}_j = b_{1j}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{pmatrix} + b_{2j}\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + b_{ij}\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} + \ldots + b_{nj}\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{pmatrix}$$

ex

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \end{pmatrix}$$

<u>Def</u> an $n \times n$ matrix where the ij entry equals 1 if $i=j$ and 0 if $i \neq j$ is called an <u>identity matrix</u> and is denoted I or I_n

If a is $m \times n$ and I is $n \times n$ then $AI = A$

If a is $m \times n$ and I is $m \times m$ then $IA = A$

<u>Def</u> the transpose of an $m \times n$ matrix A is the $n \times m$ matrix obtained by interchanging the rows and columns of A denoted A^T

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 4 & -1 \end{pmatrix}$$

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(cA)^T = c(A)^T$

$(AB)^T = B^T A^T$

proof:

ij entry of $B^T A^T = (i^{th}$ row of $B^T)(j^{th}$ col of $A^T)$

$= (i^{th}$ col of $B)(j^{th}$ row of $A)$

$=$ ~~ij entry of (AB)^T~~

$= (ji$ entry of $AB)$

$= ij$ entry of $(AB)^T$

<u>Def</u> a matrix A is <u>invertible</u> iff there exists a matrix B so that $AB = BA = I$

for A to be invertible, A must be a square matrix, that is, # rows = # columns if A is invertible, we denote the inverse by A^{-1}

Given an $n \times n$ matrix A, to find A^{-1},
we need to find B so that $AB = I$,

$$\sum_{k=1}^{n} a_{ik} b_{kj} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \longleftarrow n^2 \text{ linear equations in the variable } b_{ij}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} 2b_{11} + 3b_{21} + 0b_{12} + 0b_{22} \\ \cancel{2b_{12} + 1b_{22} + 0b} \\ 3b_{11} + 5b_{21} + 0b_{12} + 0b_{22} \\ 0b_{11} + 0b_{21} + 2b_{12} + 5b_{22} \\ 0b_{11} + 0b_{21} + 3b_{12} + 5b_{22} = \end{cases}$$

Def ~~if I~~ if we do a single row operation to I, we get an <u>elementary matrix</u> (E)

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thm if A and B are invertible
then AB is invertible,
and $(AB)^{-1} = B^{-1}A^{-1}$

prf
$$(AB)(B^{-1}A^{-1}) = I \qquad (B^{-1}A^{-1})(AB) = I$$
$$A(BB^{-1})A^{-1} = I \qquad B^{-1}B = I$$
$$AA^{-1} = I \qquad I = I$$
$$I = I$$

if $A_1, A_2, \ldots, A_k$ are $n \times n$ invertible matrices,
then $A_1 A_2 \cdots A_k$ is invertible
and $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$

multiplying A on the left by an E matrix performs the row operation on A

EA

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2a+d & 2b+e & 2c\cdot \\ g & h & i \end{pmatrix}$$

$$A \to E_1 A \to E_2 E_1 A \to \dots \to I \Rightarrow (E_1 E_2 \cdots E_k) \text{ is invertible}$$

Ex $A = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & -4 & 0 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & -10 & -2 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{10}R_2} \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{1}{10} \end{pmatrix}$$

$$\xrightarrow{-3R_2+R_1} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{3}{10} \\ 0 & 1 & \frac{2}{10} & -\frac{1}{10} \end{array}\right)$$

sample Exam

we won't get to 2.5 before the exam

Book

$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

h/w 2.2.33

$$\begin{pmatrix} k & 0 & 0 & \cdots & 0 \\ k & k & 0 & & \vdots \\ k & k & k & & \vdots \\ \vdots & & & \ddots & \\ k & \cdots & & & k \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{k} & 0 & 0 & \cdots & 0 \\ -\frac{1}{k} & \frac{1}{k} & 0 & & \vdots \\ 0 & -\frac{1}{k} & \frac{1}{k} & \ddots & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & -\frac{1}{k} & \frac{1}{k} \end{pmatrix}$$

Book 2.2

Theorem 7

an $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, in which case the same sequence of row operations reduces I_n to A^{-1}

$(A^{-1})^{-1} = A$

$AD = I \Leftrightarrow (Ad_1 \; Ad_2 \cdots Ad_p) = I$

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

---

study plan 2.3.20

if E and F are $n \times n$

$EF = I \Longrightarrow FE = I$

$FE = I \Longrightarrow EF = I$

thus

F and E commute

book 2.9

a mapping ~~T is said to be onto~~
$T: R^n \to R^m$ is said to be onto R^m if each $\bar{b}$ in R^m is the image of at least one $\bar{x} \in R^n$

a mapping $T: R^n \to R^m$ is said to be one-to-one if each $\bar{b}$ in R^m is the image of at most one $\bar{x} \in R^n$

$$(A^{-1})^T = (A^T)^{-1}$$

if A is an invertible $n \times n$ matrix, $\forall \bar{b} \in R^n$, $A\bar{x} = \bar{b}$ has the unique solution $\bar{x} = A^{-1}b$

if $\bar{x} = \bar{0}$ is a solution to $A\bar{x} = \bar{b}$, then $\bar{b} = \bar{0}$ and the system is homogeneous

$T(\bar{x}) = A\bar{x} \Rightarrow T$ is a linear transformation

2.3 (substitute ~~lecturer~~ lecturer) "Sahan"

$(A|I) \to (I|A^{-1})$

$(AC = I \wedge DA = I) \iff A$ is invertible

$\Downarrow$

A is square

IVT: let A be an $n \times n$ matrix. Then all of the following are equivalent.

(a) A is invertible

$\Downarrow$ by definition of invertible

by definition of invertible

(k) $\exists$ $n \times n$ matrix D such that $AD = I$

(j) $\exists$ $n \times n$ matrix C such that $CA = I$

~~$A\bar{x} = \bar{0}$; since A is invertible, A^{-1} exists, so $A^{-1}A\bar{x} = A^{-1}\bar{0}$, $\bar{x} = A^{-1}\bar{0} = \bar{0}$~~

$\Downarrow$ since A is invertible, A^{-1} exists, so $A\bar{x} = \bar{0} \Rightarrow A^{-1}A\bar{x} = A^{-1}\bar{0} \Rightarrow \bar{x} = A^{-1}\bar{0} \Rightarrow \bar{x} = \bar{0}$, so $\bar{x} = \bar{0}$ is the only solution

$A\bar{x} = \bar{b}$ $\Downarrow$ $\bar{x} = A^{-1}\bar{b}$

(d) $A\bar{x} = \bar{0}$ has only the trivial solution

$A\bar{x} = \bar{0}$ has only the trivial solution $\iff$ A has no free variables $\iff$ A has a pivot position in every column

~~(e)~~ (c) A has n pivot positions

$\Downarrow$ since A is $n \times n$ and has n p.p.'s, the p.p.'s are on the main diagonal; so $\operatorname{rref}(A) = I$

(b) A is row equivalent to the $n \times n$ identity matrix

(g) the equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in R^n

sample exam w/ subsititute

1. $x_1 + 2x_3 = 5$
$3x_1 + x_2 + 5x_3 = 18$
$2x_1 + 1x_2 + 3x_3 = 13$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 & 5 \\ 3 & 1 & 5 & 18 \\ 2 & 1 & 3 & 13 \end{pmatrix}$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \end{pmatrix} \xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + 2x_3 = 5$ $\quad x_1 = 5 - 2x_3$
$x_2 - x_3 = 3$ $\quad x_2 = 3 + x_3$
let $t = x_3$

$$\bar{x} = t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$$

substitute writes

$$\bar{x} = \begin{pmatrix} 5-2x_3 \\ 3+x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} x_3$$

$f \Rightarrow LD$

4. $$A = \begin{pmatrix} -4 & -3 & -7 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \\ -5 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) linearly dependent since $col_3(A) = col_1(A) + col_2(A)$

(b) $T(\bar{x}) = A\bar{x}$ $\quad T: R^j \rightarrow R^k$ $\quad j, k?$

$j = 3$, since A has 3 cols, ~~and~~ so the multiplication is only defined if $\bar{x} \in R^3$

$k = 4$, since $A\bar{x}$ is a linear combination of vectors in R^4

(c) No, since it has has at least one row without a pivot position

(d) No, since A has free variables

THEOREM 8 **The Invertible Matrix Theorem**

Let A be a square $\underline{n \times n}$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

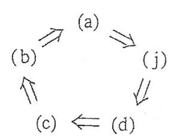
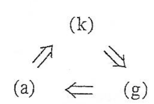


FIGURE 1

First, we need some notation. If the truth of statement (a) always implies that state ment (j) is true, we say that (a) *implies* (j) and write (a) $\Rightarrow$ (j). The proof will establis the "circle" of implications shown in Figure 1. If any one of these five statements i true, then so are the others. Finally, the proof will link the remaining statements of th theorem to the statements in this circle.

PROOF If statement (a) is true, then A^{-1} works for C in (j), so (a) $\Rightarrow$ (j). Next, (j) $\Rightarrow$ (d by Exercise 23 in Section 2.1. (Turn back and read the exercise.) Also, (d) $\Rightarrow$ (c) by Exercise 23 in Section 2.2. If A is square and has n pivot positions, then the pivot must lie on the main diagonal, in which case the reduced echelon form of A is I_n. Thus (c) $\Rightarrow$ (b). Also, (b) $\Rightarrow$ (a) by Theorem 7 in Section 2.2. This completes the circle in Figure 1.



Next, (a) $\Rightarrow$ (k) because A^{-1} works for D. Also, (k) $\Rightarrow$ (g) by Exercise 24 in Section 2.1, and (g) $\Rightarrow$ (a) by Exercise 24 in Section 2.2. So (k) and (g) are linked to the circle. Further, (g), (h), and (i) are equivalent for any matrix, by Theorem 4 in Section 1.4 and Theorem 12(a) in Section 1.9. Thus, (h) and (i) are linked through (g) to the circle.

$$(g) \Leftrightarrow (h) \Leftrightarrow (i)$$

$$(d) \Leftrightarrow (e) \Leftrightarrow (f)$$

$$(a) \Leftrightarrow (l)$$

Since (d) is linked to the circle, so are (e) and (f), because (d), (e), and (f) are all equivalent for *any* matrix A. (See Section 1.7 and Theorem 12(b) in Section 1.9.) Finally, (a) $\Rightarrow$ (l) by Theorem 6(c) in Section 2.2, and (l) $\Rightarrow$ (a) by the same theorem with A and A^T interchanged. This completes the proof.

Because of Theorem 5 in Section 2.2, statement (g) in Theorem 8 could also be written as "The equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for each $\mathbf{b}$ in $\mathbb{R}^n$." This statement certainly implies (b) and hence implies that A is invertible.

The next fact follows from Theorem 8 and Exercise 8 in Section 2.2.

Let A and B be square matrices. If $AB = I$, then A and B are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

preimage $\downarrow$ image $\downarrow$

$$T(\bar{x}) = \bar{y}$$

Roe ~~16~~ 6/15/2017 Review for Exam 1

T or F

if $\bar{w}$ is a linear combination of $\bar{u}$ and $\bar{v}$ in R^n then $\bar{v}$ is a linear combination of $\bar{u}$ and $\bar{w}$

$\bar{w} = c_1\bar{u} + c_2\bar{v}$ ~~$c_2\bar{v} = \bar{w} -$~~

$\bar{v} = \frac{\bar{w}}{c_2} - c_1/c_2\,\bar{u}$ $\cancel{c_2}$

F, if $c_2 = 0$, then it doesn't work

$$T(\bar{x}) = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\bar{x}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\bar{x}$$

- solve linear system $\Rightarrow$ puf
- dependent/independent
- one-to-one/onto
- span
- shear, dilation, projection, rotation, reflection

$$x_1 + 3x_2 = k$$
$$4x_1 + hx_2 = 8$$

$$\begin{pmatrix} 1 & 3 & k \\ 4 & h & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{pmatrix}$$

inconsistent $\Leftrightarrow$ $h=12 \quad 8-4k \neq 0$
$k \neq 2$

unique solution $\Leftrightarrow$ $h \neq 12$ k anything

infinitely many solutions $\Leftrightarrow$ $h=12 \quad k=2$

4.

$$\begin{pmatrix} -4 & -3 & -7 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \\ 5 & 4 & 9 \end{pmatrix} \xrightarrow[-5R_3+R_4]{4R_3+R_1} \begin{pmatrix} 0 & -3 & -3 \\ 0 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 4 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

free $\Rightarrow$ linearly dependent

$$AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{pmatrix}$$

show $c_1 A\bar{b}_1 + c_2 A\bar{b}_2 + \dots + c_n A\bar{b}_n = \bar{0}$
where at least one constant $c \neq 0$

$$A(c_1\bar{b}_1 + c_2\bar{b}_2 + \dots + c_n\bar{b}_n) = \bar{0}$$

T onto $\Leftrightarrow$ every $\bar{b}$ in R^m has $T(\bar{x}) = \bar{b}$ for some $\bar{x} \in R^n$
$T: R^n \to R^m$

Book 3.1

determinant — $n \times n$ only $\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$

(you choose i)

example

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\det A = 1 \cdot \det\begin{pmatrix} 4 & -1 \\ -2 & 0 \end{pmatrix} - 5 \cdot \det\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}$$

$$= 1 \cdot (0-2) - 5(0-0) + 0(-4-0)$$

$$= -2$$

Roe

Test distribution

8 A | 9 B | 19 C | 4 D

2 100% s (I'm one of them)

chapter 3 — determinants

~~determinant~~

if A is an $n \times n$ matrix, the determinant of A, denoted $\det A$, sometimes $|A|$, is a number which is zero if A is not invertible and nonzero if A is invertible

$\det A = \pm$ the product of the diagonal entries of the row echelon form of A if the scalar multiply operation is not used. Each row swap introduces a negative.

If $T: R^2 \to R^2$ is linear, then $T(\bar{x}) = A\bar{x}$.
If S is a square in R^2,
~~$\text{Area}(S) = (\det A) \cdot \text{Area}(T(S))$~~

derivatives and integrals are linear transformations

ex $A = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix} \to \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

$\det A = (2)(1) = 2$

<u>alternate def</u> det of 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

is $(a)(d)-(c)(b)$

show it

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ 0 & -\frac{cb}{a}+d \end{pmatrix}$$

$$\det = (a)\left(-\frac{cb}{a}+d\right) = -cb + da = da - cb$$

for $n\times n$ $\det A = a_{i1}(-1)^{i+1}\det(A_{i1}) + a_{i2}(-1)^{i+2}\det(A_{i2}) + \dots + a_{in}(-1)^{i+n}\det(A_{in})$

where A_{ij} are obtained by eliminating the ith row and jth column of A

— cofactor expansion about the ith row

ex

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \\ 2 & 1 & 4 \end{pmatrix} = \cancel{4(-1)^{2+1}\det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + 0** + (-1)(-1)^{2+3}\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$$

zero ⇒ use cofactor with this row

$$\cancel{= (-4)(4-3\cdot2) + 0 + 1(1\cdot1-2\cdot2)}$$

$$\cancel{= (-4)(}$$

$$= 4(-1)^{2+1}\det\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + 0** + (-1)(-1)^{2+3}\det\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$= -23$$

$$\det\begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 9 \\ 0 & 8 & 3 \end{pmatrix} = 3\det\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 8 & 3 \end{pmatrix}$$

use column cofactor

$$\det A = a_{1j}(-1)^{j+1}\det A_{1j} + a_{2j}(-1)^{j+2}\det A_{2j} + \dots + a_{nj}(-1)^{j+n}\det A_{nj}$$

for 3X3 only

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + chd - (ceg + bdi + ahf)$$

Cramer's rule — let $A\vec{x}=\vec{b}$ be a square system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$x_k = \frac{\det(\tilde{A}_k)}{\det(A)},$$ where $\tilde{A}_k$ is obtained by replacing the kth column of A by $\vec{b}$

not a good algorithm

note from h/w

~~multiplying a~~
scaling a row multiplies the determinant by the same scalar $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \frac{1}{k}\begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix}$

(i, j) – cofactor — the number $C_{ij} = (-1)^{i+j}\det(A_{ij})$

interchanging any 2 rows negates the determinant

$\cancel{\det(A+B) = \det(A) + \det(B)}$

$$\det(A^{-1}) = (\det(A))^{-1} = \frac{1}{\det(A)}$$

$$\det(A^k) = (\det(A))^k$$

$$\det AB = \det A \det B$$

$$\det A^T = \det A$$

$$\det(rA) = r^n \det(A)$$

book chapter 3 notes

$$A^{-1} = \frac{1}{\det A} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{transposed cofactors}}$$

adjugate of A — $\underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{cofactors transposed}}$

aka. "classical adjoint"

if A is 2×2, the area of the parallelogram determined by the columns of A is $|\det A|$

if A is 3×3, ~~the area of~~ the volume of the parallelepiped determined by the columns of A is $|\det A|$

let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation determined by the columns of A.
If S is a parallelogram in $\mathbb{R}^2$, then
$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation determined by the columns of A. If S is a parallelepiped in then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

notice that shear transformations do not change the area of parallelograms, since their ~~de~~ matrices determinants are always 1 let a_1, a_2 be nonzero vectors. Then for any scalar c, the ~~areas of +~~ area of the parallelogram determined by a_1 and a_2 ~~is~~ equals the area of the parallelogram determined by a_1 and $a_2 + ca_1$

Chapter 4 w/ Roe

4.1

Vector Spaces

<u>Def</u> a <u>vector space</u> is a set V of 'things' called vec and another set S of 'things' called scalars ~~vectors~~ together with two operations $+$ and $\cdot$ where $+$ we call vector addition, and $\cdot$ we call scalar multiplication that satisfy these 10 properties

the three with arrows must be checked, the rest come naturally for subsets of vector spaces

(1) for every $\bar{u}$ and $\bar{v}$ in V, $\bar{u}+\bar{v}$ is in V

(2) $\bar{u}+\bar{v} = \bar{v}+\bar{u}$

(3) $(\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$

(4) there is a thing in V, which we denote by $\bar{0}$, such that $\bar{0}+\bar{u}=\bar{u}$

(5) for any $\bar{u}$ in V there is a corresponding thing $-\bar{u}$ in V such that $\bar{u}+-\bar{u}=\bar{0}$

(6) for every $\bar{u}$ in V and every c in S, $c\cdot\bar{u}$ is in V

(7) $(c+d)\cdot\bar{u} = c\cdot\bar{u} + d\cdot\bar{u}$

not same + signs

(8) $c\cdot(\bar{u}+\bar{v}) = c\cdot\bar{u} + c\cdot\bar{v}$

same + signs

(9) $cd\cdot\bar{u} = c\cdot(d\cdot\bar{u})$

(10) $1\cdot\bar{u} = \bar{u}$

vector spaces

ex "2"

$V = M_{mn}$ — set of mxn matrices

V is a vector space

you could call an mxn matrix a "vector"

---

ex

V = infinite sequences of real numbers

$\bar{u} = u_1, u_2, u_3, \dots, u_n, \dots$

$\bar{v} = v_1, v_2, v_3, \dots, v_n, \dots$

$\bar{u} + \bar{v} = u_1 + v_1,\ u_2 + v_2, \dots, u_n + v_n, \dots$

$c\bar{u} = cu_1, cu_2, cu_3, \dots, cu_n, \dots$

$\bar{0} = 0, 0, \dots$

$-\bar{u} = -u_1, -u_2, \dots$

---

ex

V = set of all real valued functions defined on a set D
a subset of R

we need to define +, just add $f(x) + g(x)$

$\bar{0} = f$ for which $f(x) = 0$ for all x

$-f$ is the function whose value is $-(f(x))$ for all $x \in R$

$c \cdot f$ is the function whose value at x is ~~$c(f(x))$~~ $c(f(x))$

Thm $-1 \cdot \bar{u} = -\bar{u}$

If V is a vector space and W is a subset of V we can ask if W is a vector space if we use the same $+$ and $\cdot$

ex

Is the set of nxn invertible matrices a vector space?

T says

No, $\cancel{\bar{0}}$ $-A + A = \bar{0}$

$\bar{0}$ is not invertible

T was correct

$$\begin{array}{rl} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{inv} \\ + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \text{inv} \\ \hline \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{not inv} \end{array}$$

ex

$$W = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in R \right\}$$

subset of V for "ex2"

(1) $\begin{pmatrix} a & b \\ c & a \end{pmatrix} + \begin{pmatrix} e & f \\ g & e \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & a+e \end{pmatrix}$

ex

W = set of real valued functions defined on R that satisfy the differential equation $3y' + 4y = 0$

$\bar{0} = f : f(x) = 0$

$c f'$

$\cancel{4(f+g)'}$ $3(f+g)' + 4(f+g) \cancel{=0}$

$= 3f' + 3g' + 4f + 4g$

$= 3f' + 4f + 3g' + 4g$

$= 0 + 0 = 0$

If V is a vector space and W is a subset of V is also a vector space (when we use the same $+$ and $\cdot$) then W is called a <u>subspace</u> of V

ex
If V is any vector space and ~~W is~~ $W = \{\bar{0}\}$ ← the zero vector from V then W is a subspace of V

---

h/w notes

P_6 — the set of all 6th degree polynomials

book 4.1 note
definition: a subspace of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H
b. H is closed under vector addition

c. H is closed under multiplication by scalars

zero subspace — $\{\bar{0}\}$

vector — an element of a vector space

a vector space is a subspace of itself

since $R^2 \not\subseteq R^3$, R^2 is not a subspace of R^3

Book 4.1

a <u>subspace</u> of a vector space V is a subset H of V that has three properties

a. the zero vector of V is in H

b. H is closed under vector addition.
   That is, for each $\bar{u}$ and $\bar{v}$ in H, $\bar{u}+\bar{v}$ is in H.

c. H is closed under scalar multiplication.
   That is, for each $\bar{u}$ in H and each scalar c, the vector $c\bar{u}$ is in H

4.2

the <u>null space</u> of an $m \times n$ matrix A, written Nul A, is the set of all solutions to the homogeneous equation $A\bar{x} = \bar{0}$

$$\text{Nul } A = \{\bar{x} : \bar{x} \text{ is in } R^n \text{ and } A\bar{x} = \bar{0}\}$$

Roe 6/21/2017

If A is an $m \times n$ matrix
then the set of all solns to the homogeneous equation $A\bar{x} = \bar{0}$ is a ~~subset~~ subspace of R^n

4. is $\bar{0}$ in this set? $A\bar{0} = \bar{0}$ ~~so $\bar{0}$ is~~

1. $\bar{u}$ and $\bar{v}$ are in this set. Is $\bar{u}+\bar{v}$ a soln? $A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$
$= \bar{0} + \bar{0} = \bar{0}$.

6. If $\bar{u}$ is a solution and c is a scalar is $c\bar{u}$ a solution? $A(c\bar{u}) = c(A\bar{u}) = c\bar{0} = \bar{0}$

Is the set of all solutions to the ~~homogeneous~~ non-homogeneous system $A\bar{x} = \bar{b}$ a subspace of $\mathbb{R}^n$?
Is $\bar{0}$ a solution? $A\bar{0} = \bar{0} \neq \bar{b}$ $\bar{b} \neq 0$ since the system is not homogeneous
No!

Def The subspace of $\mathbb{R}^n$ that is the set of all solutions to the homogeneous system $A\bar{x} = \bar{0}$ (A $m \times n$) is called the null space of A, denoted Nul A

ex find Nul A

$$A = \begin{pmatrix} 1 & -6 & 4 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & -6 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} \boxed{1} & -6 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ & f & & f & \end{array}\right) \to \begin{array}{l} x_1 - 6x_2 = 0 \\ x_3 = 0 \end{array} \quad \bar{x} = x_2 \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 ($$

$$\text{Nul } A = \text{Span}\left\{ \begin{pmatrix} 6 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

If $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ are vectors in R^m

Is $V = \text{Span}\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ a subspace of R^m?

<u>Ck</u> Is $\bar{0}$ in V? yes $\bar{0} = 0 \cdot u_1$

If $\bar{u}$ and $\bar{v}$ are ~~are~~ in V, is $\bar{u}+\bar{v}$ in V?

$$\left.\begin{array}{l} \bar{u} \in V \Rightarrow \exists\, c_1 \ldots c_k \ni \bar{u} = c_1\bar{u}_1 + c_2\bar{u}_2 + \ldots + c_k\bar{u}_k \\ \bar{v} \in V \Rightarrow \exists\, d_1 \ldots d_k \ni \bar{v} = d_1\bar{u}_1 + d_2\bar{u}_2 + \ldots + d_k\bar{u}_k \end{array}\right\}$$

$$\Rightarrow \bar{u} + \bar{v} = (c_1\bar{u}_1 + c_2\bar{u}_2 + \ldots + c_k\bar{u}_k) + (d_1\bar{u}_1 + d_2\bar{u}_2 + \ldots + d_k\bar{u}_k)$$

$$= (c_1 + d_1)\bar{u}_1 + (c_2 + d_2)\bar{u}_2 + \ldots + (c_k + d_k)\bar{u}_k$$

✓

If $\bar{v}$ is in V is $c\bar{v}$ in V?

$$\bar{v} = d_1\bar{u}_1 + d_2\bar{u}_2 + \ldots + d_k\bar{u}_k$$

$$c\bar{v} = c(d_1\bar{u}_1 + d_2\bar{u}_2 + \ldots + d_k\bar{u}_k)$$

$$= c\,d_1\bar{u}_1 + c\,d_2\bar{u}_2 + \ldots + c\,d_k\bar{u}_k$$

✓

Thm If $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ are vectors in R^m

then $\text{Span}\{u_1, u_2, \ldots, u_k\}$ is a subspace of R^m

$$\text{Span}\{1, x, x^2, \dots, x^n\} = P_n$$

If A is an $m \times n$ matrix, $A = (a_1 \ a_2 \ \cdots \ a_n)$
the subspace of R^m

$\text{Span}\{a_1, a_2, \dots, a_m\}$

is called the <u>column space of A</u>,
denoted Col A

$$\text{Col } A = \{c_1 a_1 + c_2 a_2 + \dots + c_n a_n : c_1, c_2 \dots c_n \in R\}$$
$$= \{\bar{b} : \bar{b} = A\bar{x} \text{ for some } \bar{x} \text{ in } R^n\}$$

if $m \neq n$ $\text{Col } A \cap \text{Nul } A = \phi$

if $m = n$ $\bar{0} \in (\text{Col } A \cap \text{Nul } A)$

If V and W are vector spaces and $T: V \to W$
~~and~~ T is a linear transformation if
(1) $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$ for all $\bar{u}$ and $\bar{v}$ in V
(2) $T(c \cdot \bar{u}) = c \cdot T(\bar{u})$ for all scalars c and $\bar{u}$ in V

Def If $T: V \to W$ is a linear transformation,
the set of all vector $\bar{x}$ in V
such that $T(\bar{x}) = \bar{0}$ is called the kernal of T

If $T: R^n \to R^m$ and $T(\bar{x}) = A\bar{x}$ then
kernal of T = Nul A

$\{\bar{w} \in W \ni \bar{w} = T(\bar{v})$ for some $\bar{v} \in V\}$ is
called the Range of T

If $T: R^n \to R^m$ and $T(\bar{x}) = A\bar{x}$
Range of T = Col A

ex
let V be vector space of all continuous fns defined on R

let $T(f) = \int_0^x f(t)\,dt = F(x) - F(0)$

~~T is a tra~~

$T: V \to V$

$$T(f+g) = \int_0^x (f(t) + g(t))\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt$$
$$= F(x) - F(0) + G(x) - G(0)$$
$$= T(f) + T(g)$$

other side

$$T(cf) = \int_0^x c f(\cancel{t})\, dt = c \int_0^x f(t)\,dt = c\,T(f)$$

find kernel and range ...

---

ex $T: P_3 \to R^2$

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$$

...

h/w notes

a subspace is a vector space
a null space is a subspace of R^n

kernel of T = $\{$ all u in domain of T : T(u)

Roe 6/22/2017

4.2 cont

V = set of continuous functions

$T: V \to V \quad T(f) = \int_0^x f(t)\,dt$

kernel of $T = \{\bar{0}\}$ identically zero function

range of T = Not V, consider $|x|$

$|x| \in V$, but ~~$T(|x|)$ is not continuous~~ there is no f such that $T(f) = |x|$

$T: P_3 \to R^2 \quad T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}$

is T a linear transformation

$$T(p(x)+q(x)) = \begin{pmatrix} p(0)+q(0) \\ p(1)+q(1) \end{pmatrix} = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \end{pmatrix}$$

$$= T(p(x)) + T(q(x))$$

$$T(c\,p(x)) = \begin{pmatrix} c\,p(0) \\ c\,p(1) \end{pmatrix} = c\begin{pmatrix} p(0) \\ p(1) \end{pmatrix} = c\,T(p(x))$$

so T is linear

---

what is the kernel of T

$$T(c_0 + c_1x^1 + c_2x^2 \cancel{+ \dots +} + c_3x^3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} c_0 + 0 + 0 + 0 \\ c_0 + c_1 + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$c_0 = 0 \quad c_1 = -c_2 - c_3$

kernel of $T = \{0 + (-c_2 - c_3)x + c_2x^2 + c_3x^3\}$

$$= \text{Span}\left\{ -x + x^2,\ -x + x^3 \right\}$$

T is not 1-1

Range of T

since T is a linear transformation, Range of T is a subspace of R^2

$T(c_0 + c_1x + c_2x^2 + c_3x^3) = \begin{pmatrix} c_0 \\ c_0+c_1+\dots \end{pmatrix}$

range $= R^2$ T is onto

V is LI $\Leftrightarrow$ the only solution to $c_1\bar{v}_1 + c_2 v_2 + \ldots + c_n v$
is $c_1 = c_2 = \ldots = c_n = 0$

is $\{\sin(x), \cos(x)\}$ LI?

0 fn
for a

is $c_1 = c_2 = 0$ the only solution to $c_1 \sin(x) + c_2 \cos(x) =$

yes ~~no,~~ ~~since~~
so $\{\sin(x), \cos(x)\}$ are LI

is $\{\sin^2(x), \cos^2(x), \sin(2x), \cos(2x)\}$ LI

~~$\sin^2(x) + \cos^2(x)$~~

$\sin(2x) = 2\sin x \cos x$

$(1)\sin^2(x) + (-1)\cos^2(x) + 0\sin(2x) + (1)\underbrace{\cos(2x)}_{\cos^2(x) - \sin^2(x)} = 0$
for all x

so the set is LD

<u>Def</u> if H is a vector space and
$B = \{v_1 \ldots v_k\}$ are vectors in H then

B is a <u>basis</u> for H iff
(1) $H = \text{Span}\{B\}$
(2) B is linearly independent

example

$B = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ • spans R^3

• is linearly independent

so B is a basis for R^3

If A is 3×3 and A is row eq. to I
do the columns of A span R^3

(1) since there is a pivot in every row, the columns of A span R^3

(2) since there is a pivot in every column, the columns of A are linearly independent

Thm if H is a vector space and $B = \{\bar{V}_1, \ldots, \bar{V}_k\}$ is a basis for H, then there is one and only one set of scalars $c_1, \ldots, c_k$ such that $\bar{b} = c_1\bar{V}_1 + c_2\bar{V}_2 + \cdots + c_k\bar{V}_k$

- there is one because B spans H
- suppose there is a second set $\{d_1, \ldots, d_k\}$ such that $\bar{b} = d_1\bar{V}_1 + d_2\bar{V}_2 + \ldots + d_k\bar{V}_k$

$\bar{0} = \bar{b} - \bar{b} = (c_1\bar{V}_1 + c_2\bar{V}_2 + \ldots + c_k\bar{V}_k) - (d_1\bar{V}_1 + d_2\bar{V}_2 + \ldots + d_k\bar{V}_k)$

$= (c_1 - d_1)\bar{V}_1 + (c_2 - d_2)\bar{V}_2 + \ldots + (c_k - d_k)\bar{V}_k$

$c_1 - d_1 = 0 \quad c_2 - d_2 = 0 \quad \ldots \quad c_k - d_k = 0$ since $\bar{V}$'s are linearly independent

$\left\{ \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \right\}$ $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ is called the <u>standard basis</u> for R^n

If A is $m \times n$ with $m \neq n$, could the columns of A be a basis for R^m

either $m > n$ or $m < n$

$m > n$

$A = \left(\Big| \; \cdots \; \Big| \; \Big| \right)$

there cannot be a pivot in every ~~column~~ row, so the columns of A cannot span R^m, so the columns of A are not a basis for R^m

$m < n$

$A = \left(| \;\; | \;\; \cdots \;\; | \;\; | \right)$

there must be a free variable, so the columns of A are not Linearly independent

so the columns of A are not a basis for R^m

Thm Any set than spans R^m has a subset that is a basis for R^m

Roe

Sec 4.4

If V is a vector space and
$B = \{ \bar{b}_1, \dots, \bar{b}_k \}$ is a basis for V
then for every vector $\bar{v}$ in V there is exactly one set of scalars $c_1, \dots, c_k$ so that $\bar{v} = c_1 b_1 + c_2 b_2 + \dots + c_k b_k$

We can define a transformation $T: V \to R^k$

$$T(\bar{v}) = \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}}_{\text{scalars for basis } B}$$

Is T a linear transformation?

$$T(\bar{v}+\bar{w}) \stackrel{?}{=} T(\bar{v}) + T(\bar{w})$$
$$T(c\bar{v}) \stackrel{?}{=} cT(\bar{v})$$

$$\bar{v} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + \dots + c_k \bar{b}_k$$
$$\bar{w} = d_1 \bar{b}_1 + d_2 \bar{b}_2 + \dots + d_k \bar{b}_k$$
$$\bar{v} + \bar{w} = (c_1 + d_1)\bar{b}_1 + (c_2 + d_2)\bar{b}_2 + \dots + (c_k + d_k)\bar{b}_k$$

$$T(\bar{v}+\bar{w}) = \begin{pmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_k+d_k \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix} = T(\bar{v}) + T(\bar{w})$$

$$c\bar{v} = cc_1 \bar{b}_1 + cc_2 \bar{b}_2 + \dots + cc_k \bar{b}_k$$

$$T(c\bar{v}) = \begin{pmatrix} cc_1 \\ cc_2 \\ \vdots \\ cc_k \end{pmatrix} = c\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$ T is linear

Is T one-to-one?

T is one to one if the only vector that maps to $\bar{0}$ is $\bar{0}$

$$T(\bar{v}) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \bar{v} = 0\bar{b}_1 + 0\bar{b}_2 + \dots + 0\bar{b}_k = \bar{0}$$ yes

Is T onto R^k? for any $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ in R^k is there at least one $\bar{v}$ in such that $T(\bar{v}) = \begin{pmatrix} a_1 \\ a_k \end{pmatrix}$ $\quad \bar{v} = a_1\bar{b}_1 + a_2\bar{b}_2 + \dots + a_k\bar{b}_k$ yes change of basis is a one-to-one, ~~linear~~ onto linear transformation

isomorphism — a one-to-one, onto, linear transformation from V to W; V and W are said to be <u>isomorphic</u>

let $B = \{\bar{b}_1, \bar{b}_2, \dots \bar{b}_k\}$ a basis for V

$T: V \to R^k \quad T(c_1\bar{b}_1 + c_2 b_2 + \dots + c_k \bar{b}_k)$

we call this isomorphism a <u>coordinate transformation</u>

denoted $[\bar{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$

$c_1, c_2, \dots, c_k$ are called the coordinates of $\bar{v}$ with respect to the basis B

$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \left[\begin{pmatrix} a \\ b \end{pmatrix} \right]_B = \begin{pmatrix} a \\ b \end{pmatrix}$

$C = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 2 \\ 0 & \boxed{7} \end{pmatrix}$ so C is a basis for R^2

$\left[\begin{pmatrix} 5 \\ 3 \end{pmatrix}_C \right] = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longleftrightarrow c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 5 \\ 3 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 5 \\ 0 & -7 & -12 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5 - \frac{24}{7} \\ 0 & 1 & \frac{12}{7} \end{pmatrix} \begin{matrix} \to c_1 = \frac{11}{7} \\ \to c_2 = \frac{12}{7} \end{matrix}$$

$$\left[\begin{pmatrix} 5 \\ 3 \end{pmatrix}_C \right] = \begin{pmatrix} \frac{11}{7} \\ \frac{12}{7} \end{pmatrix}$$

if $V = \mathbb{R}^k$ and $B = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_k\}$ is a basis for $\mathbb{R}^k$

then $[\bar{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ where $\bar{v} = c_1\bar{b}_1 + c_2\bar{b}_2 + \dots + c_k\bar{b}_k$

$$= \begin{bmatrix} b_1 & b_2 & \dots & b_k \end{bmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

so $[\bar{v}]_B = P_B^{-1}\bar{v}$

otation $P_B = \begin{pmatrix} b_1 & b_2 & \dots & b_k \end{pmatrix}$ — $k \times k$ matrix whose columns are the basis vectors

ex

~~$B = \{1+$~~

$B = \{1 - t^2,\ t - t^2,\ 2 - 2t + t^2\}$

vectors in $\mathbb{P}_2$ — polynomials of degree 2

are there c_1, c_2, c_3 such that $c_1(1-t^2) + c_2(t - t^2) + c_3(2 - 2t + t^2) = a + bt + ct^2$ for all $t \in \mathbb{R}$

$c_1 + 2c_3 = a$

$c_2 - 2c_3 = b$

$-c_1 - c_2 + c_3 = c$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & * \\ 0 & 1 & -2 & * \\ -1 & -1 & 1 & * \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

pivot in every row $\Rightarrow$ spans $\mathbb{P}_2$

pivot in every column $\Rightarrow$ LI

$\Rightarrow$ B is a basis for $\mathbb{P}_2$

question $[1 + 4t + 7t^2]_B = ?$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ -1 & -1 & 1 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & -1 & 3 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & 28 \\ 0 & 0 & 1 & 12 \end{array}\right) \Rightarrow -23(1-t^2) + 28(t - t^2) + 12(2 - 2t + t^2) = 1 + 4t + 7t^2$$

ans $\begin{pmatrix} -23 \\ 28 \\ 12 \end{pmatrix}$

thm

if $T: V \to W$ is a linear transformation from the vector space V to the vector space W and $\{\bar{v}_1 \cdots \bar{v}_k\}$ is a linearly dependent set of vectors in V, then $\{T(\bar{v}_1), T(\bar{v}_2), \dots, T(\bar{v}_k)\}$ is linearly dependent vectors in W

not inversely!

h/w notes

a plane in R^3 (through origin) can be isomorphic to R^2

$x \to [x]_B$ is called the coordinate mapping

let $B = \{\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots \bar{b}_n\}$ be a basis for the vector space V.

the B-coordinate vectors of $b_1, \dots, b_n$ (vectors in V) are $e_1, \dots, e_n$

if $\{\bar{v}_1, \dots, \bar{v}_5\}$ is a linearly dependent spanning set ~~for V~~ of the vector space V, then for every $\bar{w}$ in V, there is more than one linear combination of $\bar{v}_1, \dots, \bar{v}_5$ that equals $\bar{w}$

Book 4.5

Thm if a vector space V has a basis of n vectors, then every ~~vector~~ basis of V must consist of exactly n vectors

the dimension of V, denoted dim V, is the number of vectors in a basis for V

thm let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded to a basis for H.
also $\dim H \leq \dim V$

Roe 6/26/2017

4.4

thm If V is a vector space and $B = \{\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_k\}$ is a basis for V and if $\{\bar{v}_1, \dots, \bar{v}_p\}$ are vectors in V with $p > k$ then $\bar{v}_1, \dots, \bar{v}_p$ are linearly dependent

because we can have only k pivots but we have p columns in $(v_1 \dots v_p)$, and $p > k$

~~prf~~

~~There are $c_1, c_2, \dots, c_p$~~

Any two bases for a vector space must have the same number of vectors

prf suppose not, suppose $\exists$ bases ~~B_1, B~~ B_1 with k vectors and B_2 with p vectors with $k \neq p$
either $k > p$ or $p > k$
if $k > p$ & B_2 is a basis, then B_1 is linearly dependent

if $p > k$ & B_1 is a basis then B_2 is LD def if V is a vector space and $\{b_1, \ldots, b_k\}$ is a basis for V then the dimension of V is k and is denoted $\dim V$

ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^3$

so $\dim \mathbb{R}^3 = 3$

ex $\{1, t, t^2\}$ is a basis for P_2

so $\dim P_2 = 3$

$\{t, t^2-1, t^2+1, 1+t+t^2\}$ is LD since $\dim P_2 = 3$ and all vectors in the set are in P_2

does $\{1+t+t^2, 1-t^2\}$ span P_2?

~~no, we agree that $\{1, t, t^2\}$ is a basis for P_2~~

~~if $\{1+$~~

No, in order to span P_2, ~~they would have~~ there would have to be at least $\dim P_2 = 3$ vectors in the set

"Def"

the vector space $\{\bar{0}\}$ has no basis

- if $B = \{\bar{0}\}$, $3\bar{0} = \bar{0}$ so B is LD
- if $B = \{b_1\}$ where $\bar{b}_1 \neq \bar{0}$, then $\text{Span } B \neq \{\bar{0}\}$

If V is a vector space having nonzero vectors and there is no finite set of vectors than span V then $\dim V = \infty$

P = set of all polynomials

suppose $\{p_1(t), p_2(t), \ldots, p_k(t)\}$ did span P

let n_i = degree$(p_i(t))$

let $N = \max(n_1, n_2, \ldots, n_k)$

So if $p(t)$ is a polynomial with degree $> N$,

$$\underbrace{p(t)}_{\deg > N} \neq \underbrace{c_1 p_1(t) + c_2 p_2(t) + \cdots + c_k p_k(t)}_{\deg \leq N}$$

If $M_{m,n}$ is the set of all $m \times n$ matrices with real entries (m and n are fixed)

$\dim M_{m,n} = m \cdot n$

consider $m=2$ $n=3$

$$B = \left\{ \begin{pmatrix} 1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1 \end{pmatrix} \right\}$$

suppose V is a vector space with $\dim V = n$

H is a subspace of V.

How does $\dim H$ compare to $\dim V$

$\dim H \leq \dim V$

if A is an $m \times n$ matrix $? \leq \dim \text{Nul } A \leq ?$

$\text{Nul } A = \{\bar{x} : A\bar{x} = \bar{0}\}$

$\bar{x} \in R^n$

~~$\dim \text{Nul } A = n$~~ $0 \leq \dim \text{Nul } A \leq n$

if A is invertible, $\text{Nul } A = \bar{0}$

~~$? \leq \dim \text{Col } A \leq ?$~~ $0 \leq \dim \text{Col } A \leq m$

~~n m~~

if $A = \begin{pmatrix} 0&0&0\\0&0&0 \end{pmatrix}$

$\dim \text{Col } A = 0$

$m \times n$

if A row reduces to a matrix having k pivots
then $\dim \text{Nul } A = n-k = \#$ free variables
and $\dim \text{Col } A = k$

Book

Spanning Set Theorem
~~a. If~~ let $S = \{\bar{v}_1, \ldots, \bar{v}_p\}$ be a set
let $H = \text{Span}\{\bar{v}_1, \ldots, \bar{v}_p\}$

a. If one of the vectors in S, say $\bar{v}_k$, is a linear combination of the others, then the set formed from S by removing $\bar{v}_k$ still spans H

b. If $H \neq \{\bar{0}\}$, some subset of S is a basis for H

the pivot columns of a matrix A form a basis for Col A

warning: only the pivot cols of A itself form a basis, not the pivot cols of A's echelon form

---

$\bar{x} = c_1\bar{b}_1 + c_2\bar{b}_2 + \ldots + c_n\bar{b}_n$

$[\bar{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ — the coordinates of $\bar{x}$ relative to the basis B

$P_B = \begin{pmatrix} b_1 & b_2 & \ldots & b_n \end{pmatrix}$ ← matrix

$\bar{x} = P_B [\bar{x}]_B$

$B = \{ b_1 \dots b_n \}$

$\bar{x} \mapsto [\bar{x}]_B$ is a one-to-one linear transformation from V onto R^n

If V is spanned by a finite set, then V is said to be <u>finite-dimensional</u>, and the <u>dimension</u> of V, written dim V, is the number of vectors in a basis for V

$\dim \{\bar{0}\} = 0$

If V is not spanned by a finite set, then V is said to be infinite-dimensional

let H be a subspace of V.
any linearly independent set in H can be expanded to a basis for V, and
$\dim H \leq \dim V$

let V be a p-dimensional vector space, $p \geq 1$.
Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V

~~t~~the dimension of Nul A is the number of free variables in A

the dimension of Col A is the number of pivot positions in A

Tanner Observes

The number of pivot positions in A^T is the same as the number of pivot positions in A

proof:

The number of pivot positions in A is the dimension of the columnspace and row space of A. $\text{col } A^T = \text{row } A$ and $\text{row } A^T = \text{col } A$ so $\dim \text{col } A^T = \dim \text{row } A$ and $\dim \text{row } A^T = \dim \text{col}$ and thus the number of pivot positions in A^T is the number of pivot positions in A

4.7 Roe

suppose V is a vector space

$B = \{\bar{b}_1, \ldots, \bar{b}_k\}$ and

$D = \{\bar{d}_1, \ldots, \bar{d}_k\}$ are bases for V

~~How are~~

If $\bar{x}$ is a vector in V, how are $[\bar{x}]_B$ and $[\bar{x}]_D$ related?

$\cancel{B}$ $[\bar{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$

$\Leftrightarrow c_1 \bar{b}_1 + c_2 \bar{b}_2 + \ldots + c$

$= \bar{x}$

recall:

If V is $\mathbb{R}^k$

$\bar{x} = P_B[\bar{x}]_B$

$\bar{x} = P_D[\bar{x}]_D$

where both P_B and P_D are invertible $k \times k$ matrices

$\Leftrightarrow P_B^{-1}\bar{x} = [\bar{x}]_B \Leftrightarrow P_B^{-1}P_D[\bar{x}]_D = [\bar{x}]_B$

$\Rightarrow$ $[\bar{x}]_B$ is a linear transformation of $[\bar{x}]_D$

$\Leftrightarrow \exists P \ni [\bar{x}]_D = P[\bar{x}]_B$

$\Rightarrow$

$[b_1]_D = P[b_1]_B = P\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =$ 1st column of P

$[b_i]_D = P[b_i]_B = P\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith entry $=$ ith column of P

change of coordinates matrix

notation $\underset{D \leftarrow B}{P}$

changing from basis B to basis D

$$\underset{D\leftarrow B}{P} = \underset{B\leftarrow D}{P}^{-1} \qquad \underset{D\leftarrow B}{P} = \begin{pmatrix} [b_1]_D & [b_2]_D & \dots & [b_k]_D \end{pmatrix}$$

$$[\bar{X}]_D = \underset{D\leftarrow B}{P} [\bar{X}]_B$$

ex find $\underset{D\leftarrow B}{P}$ where $B = \left\{ \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix} \right\}$

$[\bar{x}]_D = \underset{D\leftarrow B}{P} [\bar{x}]_B$ $\qquad D = \left\{ \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\}$.

$$\underset{D\leftarrow B}{P} = \left(\left[\begin{pmatrix} 7 \\ 5 \end{pmatrix}\right]_D \left[\begin{pmatrix} -3 \\ -1 \end{pmatrix}\right]_D \right)$$

$$c_1 \begin{pmatrix} 1 \\ -5 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & -8 & 40 & -16 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ 0 & 1 & -5 & 2 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right)$$

$c_1 = -3, \ 1$

$c_2 = -5, \ 2$

$$\underset{D\leftarrow B}{P} = \begin{pmatrix} -3 & 1 \\ -5 & 2 \end{pmatrix}$$

check

make up $\bar{X}$

find $[\bar{X}]_B$

find $[\bar{X}]_D$

find $\underset{D\leftarrow B}{P} [\bar{X}]_B$, it should be $[\bar{X}]_D$

It can be shown that $B = \{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t\}$ are LI vectors in the vector space of real-valued functions

Let V be the vector space spanned by B.
then B is a basis for V

$D = \{1, \cos t, \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}$

$\cos(2t) = 2\cos^2 t - 1$

all vectors in D are linear combinations of vectors in B

$$\left[5\cos^3 t - 6\cos^4 t + 7\cos^5 t\right]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -6 \\ 7 \end{pmatrix}$$

$$\left[\quad \right]_D$$

$$\cancel{\underset{D \leftarrow B}{P} = \left([1]_D \; [\cos t]_D \; [\cos(2t)]_D \; [\cos(3t)]_D \; [\cos(4t)]_D \right)}$$

$$\underset{D \leftarrow B}{P} = \left([1]_D \; [\cos t]_D \; [\cos^2 t]_D \; [\cos^3 t]_D \; [\cos^4 t]_D \; [\cos^5 t]_D \right)$$

$$\underset{B \leftarrow D}{P} = \begin{pmatrix} 1 & 0 & -1 & \\ 0 & 1 & 0 & \\ 0 & 0 & 2 & \cdots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{pmatrix}$$

$$\underset{D \leftarrow B}{P} = \underset{B \leftarrow D}{P}^{-1}$$

$$[\bar{X}]_D = \underset{B \leftarrow D}{P}^{-1} [\bar{X}]_B$$

Book

the column space of an $m \times n$ matrix is a subspace of R^m

$\text{Col } A = \{\bar{b} : \bar{b} = A\bar{x} \text{ for some } \bar{x} \in R^n\}$

$\text{Col } A = R^m \iff A\bar{x} = \bar{b}$ has a solution for all $\bar{b} \in R^m$
$\iff$ A has a pivot position in every row

an indexed set of vectors $\{\bar{v}_1 \ldots \bar{v}_p\}$ is LI
$\iff \exists j \ni$ ~~v_j~~ $\bar{v}_j$ is a linear combination of $\bar{v}_1, \ldots, \bar{v}_{j-1}$

an indexed set of vectors $B = \{b_1, b_2, \ldots, b_p\}$ is a <u>basis</u> for a vector space V
$\iff \begin{pmatrix} B \text{ is LI} \\ \wedge\ H = \text{Span}\{b_1, \ldots, b_p\} \end{pmatrix}$

the standard basis for P_n is $\{1, t, \ldots, t^n\}$
$\dim P_n$ is $n+1$

Spanning Set Theorem

Roe 6/29/2017

skipping last 2 sections of chapter 4

is it ever true that for matrix A and vector $\bar{x}$

$$A\bar{x} = c\bar{x}$$

- A would have to be square
- ex
  - $I\bar{x} = (1)\bar{x}$ eigen
  - $A\bar{0} = c\bar{0}$ ← not eigen
  - $(cI)\bar{x} = c\bar{x}$ eigen
  - $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \underset{\text{eigenvalue}}{2} \cdot \underset{\text{eigenvector}}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

def If A is an $m \times n$ matrix and $\bar{x}$ is a non-zero vector such that $A\bar{x} = c\bar{x}$ then $\bar{x}$ is called an eigenvector of A with eigenvalue c

eigenvalues are denoted with λ

if $\bar{x}$ is an eigenvector of A with eigenvalue λ, must $c\bar{x}$ be an eigenvector of A with λ?

$$A(c\bar{x}) = \lambda(c\bar{x})?$$
$$\Updownarrow$$
$$c(A\bar{x}) = c(\lambda\bar{x})$$
$\Updownarrow$ if $c \neq 0$, and it's not since $\bar{0}$ is not eigenvector
$$A\bar{x} = \lambda\bar{x}$$

If $\bar{x}$ and $\bar{y}$ are eigenvectors of A with eigenvalue λ is $\bar{x}+\bar{y}$ an eigenvector of A with eigenvalue λ?

$$A(\bar{x}+\bar{y}) = A\bar{x} + A\bar{y} \overset{?}{=} \lambda(\bar{x}+\bar{y}) = \lambda\bar{x}+\lambda\bar{y}$$

given $A\bar{x} = \lambda\bar{x}$ $A\bar{y} = \lambda\bar{y}$

substituting

$$A\bar{x} + A\bar{y} = \cancel{\lambda\bar{x} + \lambda\bar{y}}\ A\bar{x} + A\bar{y}$$

identity confirmed

So, if A is an $n \times n$ matrix and λ is an eigenvalue of A, if W = the set of all eigenvectors of A with eigenvalue λ is W a subspace of $\mathbb{R}^n$?

No, since $\bar{0} \notin W$

W is called an eigenspace for λ

The set of all vectors that satisfy $A\bar{x} = \lambda\bar{x}$ is not a subspace of $\mathbb{R}^n$ and is an eigenspace

is 3 an eigenvalue of $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$

$$A\bar{x} = 3\bar{x}$$

$$A\bar{x} - 3\bar{x} = \bar{0} \qquad \text{WRONG: } \cancel{(A-3)\bar{x} = 0}$$

$$A\bar{x} - 3(I\bar{x}) = \bar{0}$$

$$A\bar{x} - 3I\bar{x} = \bar{0}$$

$$(A-3I)\bar{x} = \bar{0}$$

$$\begin{pmatrix} 3-3 & -2 \\ 1 & 0-3 \end{pmatrix}\bar{x} = \bar{0} \qquad \left(\begin{array}{cc|c} 0 & -2 & 0 \\ 1 & -3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

Only solution $\bar{x} = \bar{0}$, so 3 is **not** an eigenvalue for matrix if λ is an eigenvalue for A then there has to be a nonzero solution to $A\bar{x} = \lambda\bar{x}$

$$A\bar{x} = \lambda\bar{x}$$
$$\downarrow$$
$$A\bar{x} - (\lambda I)\bar{x} = \bar{0}$$
$$\downarrow$$
$$(A - \lambda I)\bar{x} = \bar{0}$$

example

if $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{pmatrix}$$
$$\downarrow$$

$$\left(\begin{array}{cc|c} 3-\lambda & -2 & 0 \\ 1 & -\lambda & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & -2-(3-\lambda)(-\lambda) & 0 \\ 1 & -\lambda & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 0 & -2+\lambda(3-\lambda) & 0 \\ 1 & -\lambda & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & -2+3\lambda-\lambda^2 & 0 \\ 1 & -\lambda & 0 \end{array}\right)$$

$\to$ in order to have a nonzero solution,

$$-2+3\lambda-\lambda^2 = 0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = \frac{3 \pm \sqrt{9-4(1)(2)}}{2(1)} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm \sqrt{1}}{2}$$
$$= \frac{3 \pm 1}{2}$$
$$\lambda = 2 \quad \lambda = 1$$

If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of matrix A with associated eigenvectors $\bar{x}_1, x_2, \dots, x_k$ then $\bar{x}_1 \dots \bar{x}_k$ are linearly independent

<u>not conversely</u>

<u>prf</u> assume $\bar{x}_1 \dots \bar{x}_k$ are LD,

then $\exists\, c_1 \dots c_k$ not all zero, such that $c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_k\bar{x}_k = \bar{0}$

let p be the smallest number such that $\bar{x}_p$ is a linear combination of $\bar{x}_1, \dots, \bar{x}_{p-1}$.

Then $\bar{x}_1, \dots, \bar{x}_{p-1}$ are LI

$$\bar{x}_p = c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_{p-1}\bar{x}_{p-1}$$

multiplying both sides by A

$$A\bar{x}_p = \lambda_p\bar{x}_p = A(c_1\bar{x}_1 + c_2\bar{x}_2 + \dots + c_{p-1}\bar{x}_{p-1}) = c_1\lambda_1\bar{x}_1 + c_2\lambda_2\bar{x}_2 + \dots + c_{p-1}\lambda_{p-1}\bar{x}_{p-1}$$

equation subtraction

$$\lambda_p\bar{x}_p = c_1\lambda_p\bar{x}_1 + c_2\lambda_p x_2 + \dots \cancel{+ c_{p-1}\bar{x}_{p-1}\lambda_p} + c_{p-1}\lambda_p\bar{x}_{p-1}$$

$$\bar{0} = c_1(\lambda_1 - \lambda_p)x_1 + c_2(\lambda_2 - \lambda_p)x_2 + \dots + c_{p-1}(\lambda_p - \lambda_{p-1})\bar{x}_{p-1}$$

$\bar{x} = \bar{0}$

contradiction

h/w note

every non-invertible square matrix has eigenvalue $\lambda=0$

an $n \times n$ matrix A is not invertible
$\iff$ A has an eigenvalue $\lambda = 0$

If v_1, v_2 are LI, they may correspond to the same eigenvalue

The eigenvalues of a triangular matrix are on its main diagonal

the eigenspace of A is the null space of $A - \lambda I$

if $A^2 = \bar{0}$ then the only eigenvalue of A is $\lambda = 0$

prf

$$0\bar{x} = A^2\bar{x} = A(A\bar{x}) = A(\lambda\bar{x}) = \lambda A\bar{x} = \lambda^2\bar{x}$$
$$= \bar{0}$$

since $\bar{x} \neq 0$, $\lambda = 0$

diagonalization h/w notes

- $A = PDP^{-1}$, where
  - D is a diagonal matrix consisting of the eigenvalues of A $\begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \\ 0 & \cdots & \end{pmatrix}$
  - the columns of P are the eigenvectors corresponding to the eigenvalues. Scaling ~~the~~ any column(s) of P by a nonzero constant ~~results in~~ results in another P that still works (P^{-1} must be changed)

$$D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad P = \begin{pmatrix} \begin{pmatrix} \text{evect corresponding to } \lambda_1 \end{pmatrix} & \begin{pmatrix} \text{evect cor. to } \lambda_2 \end{pmatrix} & \cdots & \begin{pmatrix} \text{evect cor. to } \lambda_n \end{pmatrix} \end{pmatrix}$$

Roe 7/3/2017

new meaning of P

$$AB = \begin{pmatrix} A\bar{b}_1 & A\bar{b}_2 & \cdots & A\bar{b}_p \end{pmatrix}$$

$$AB = \begin{pmatrix} \text{row}_1(A)B \\ \text{row}_2(A)B \\ \vdots \\ \text{row}_m(A)B \end{pmatrix}$$

Diagonal matrices are nice

$$\begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & d_{nn} \end{pmatrix} \begin{pmatrix} - \bar{r}_1 - \\ - \bar{r}_2 - \\ \vdots \\ - \bar{r}_n - \end{pmatrix} = \begin{pmatrix} - d_{11}\bar{r}_1 - \\ - d_{22}\bar{r}_2 - \\ \vdots \\ - d_{nn}\bar{r}_n - \end{pmatrix}$$

$$\begin{pmatrix} | & | & & | \\ \bar{c}_1 & \bar{c}_2 & \cdots & \bar{c}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & d_{nn} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ d_{11}\bar{c}_1 & d_{22}\bar{c}_2 & \cdots & d_{nn}\bar{c}_n \\ | & | & & | \end{pmatrix}$$

$$\begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & \vdots \\ 0 & \cdots & & d_{nn} \end{pmatrix}^k = \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & & \vdots \\ \vdots & & & \\ 0 & \cdots & & d_{nn}^k \end{pmatrix}$$

~~sup~~ A and D are similar $\Leftrightarrow \exists$ invertible P such that..

$$P^{-1}AP = D \Leftrightarrow A = PDP^{-1}$$

$$A^k = (PDP^{-1})^k$$
$$= (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}DP^{-1})$$
$$= PD^kP^{-1}$$

if A is ~~m~~ $n \times n$ and $\{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\}$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A

you are gauranteed $\{\bar{b}_1 \ldots \bar{b}_n\}$ exists if A has n distinct eigenvalues

since $I = P^{-1}P = (P^{-1}\bar{b}_1\ P^{-1}\bar{b}_2 \ldots$
$= (e_1\ e_2 \ldots$

let $P = (\bar{b}_1\ \bar{b}_2 \ldots \bar{b}_n)$

so P is $n \times n$ & invertible

$$[\lambda_1 e_1\ \lambda_2 e_2 \ldots \lambda_n e_n] = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & & \lambda_n \end{pmatrix}$$

$$P^{-1}AP = P^{-1}(A[\bar{b}_1\ \bar{b}_2 \cdots \bar{b}_n])$$
$$= P^{-1}(A\bar{b}_1\ A\bar{b}_2 \cdots A\bar{b}_n)$$
$$= P^{-1}(\lambda_1\bar{b}_1\ \lambda_2\bar{b}_2 \ldots \lambda_n\bar{b}_n)$$
$$= [\lambda_1 P^{-1}\bar{b}_1\ \lambda_2 P^{-1}\bar{b}_2 \ldots \lambda_n P^{-1}\bar{b}_n]$$

ex

$$A=\begin{pmatrix}4&0&1\\-2&1&0\\-2&0&1\end{pmatrix}$$

$\lambda=1: \bar{x}=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\lambda=2: \bar{x}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$

$\lambda=3: \bar{x}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$

$$P^{-1}AP=\begin{pmatrix}0&-1&-1\\1&2&1\\0&2&1\end{pmatrix}$$

ex

$$A=\begin{pmatrix}4&2&3\\-1&1&-3\\2&4&9\end{pmatrix}$$

$\lambda=3 \Rightarrow \begin{pmatrix}-2\\1\\0\end{pmatrix}\begin{pmatrix}-3\\0\\1\end{pmatrix}$

$\lambda=8 \Rightarrow \begin{pmatrix}1\\-1\\2\end{pmatrix}$

$$\cancel{P=\begin{pmatrix}-2&-3\\1&0\\0&1\end{pmatrix}}$$

$$P=\begin{pmatrix}-2&-3&1\\1&0&-1\\0&1&2\end{pmatrix}$$

$$\begin{pmatrix}3&&\\&3&\\&&8\end{pmatrix}$$

or

$$P=\begin{pmatrix}-2&1&-3\\1&-1&0\\0&2&1\end{pmatrix}$$

$$\begin{pmatrix}3&&\\&8&\\&&3\end{pmatrix}$$

suppose P is invertible

so $P^{-1}AP$ is a diagonal matrix

$P^{-1}AP=D$

$AP=PD \neq \cancel{PD}$

$$=\left(P\begin{pmatrix}d_{11}\\0\\\vdots\\0\end{pmatrix}\ P\begin{pmatrix}0\\d_{22}\\\vdots\\0\end{pmatrix}\ \cdots\ P\begin{pmatrix}0\\0\\\vdots\\0\\p_{nn}\end{pmatrix}\right)$$

$$=\left(d_{11}\bar{p}_1\ \ d_{22}\bar{p}_2\ \cdots\ d_{nn}\bar{p}_n\right)$$

$$AP=\left(A\bar{p}_1\ \ A\bar{p}_2\ \cdots\ A\bar{p}_n\right)$$

so

$\forall_{i\in\mathbb{Z}\ 0<i\leq n}\ A\bar{p}_i = d_{ii}\bar{p}_1$

so P is the eigenvectors corresponding to d entries as λ's

ex

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ just one evect

<u>def</u> A is <u>diagonalizable</u> iff there is an invertible matrix P such that $P^{-1}AP = D$

diagonal matrix

A is diagonalizable $\Leftrightarrow$ A has n distinct eigenvectors

A has n distinct eigenvalues $\Rightarrow$ A has n distinct eigenvectors

if A does not have n distinct eigenvectors, then A may or may not be diagonalizable

the <u>geometric multiplicity</u> of an eigenvalue of A is the dimension of the corresponding eigenspace

geom mult $\lambda \leq$ algebraic mult λ

A is diagonalizable $\Leftrightarrow \forall \lambda$ of A, geom mult $\lambda =$ alg mult λ

Jordan form

$$\begin{pmatrix} \lambda_1 & 1 & \cdots & & 0 \\ \vdots & \lambda_2 & \ddots & & \\ & & \lambda_i & 1 & \\ & & & \lambda_j & \\ 0 & & & & \lambda_n \end{pmatrix}$$

if $\lambda_1 = \lambda_2$ with geom mult 1

if $\lambda_i = \lambda_j$ with geom mult 1

$\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\theta$

$(\text{Row } A)^\perp = \text{Nul } A \qquad (\text{Col } A)^\perp = \text{Nul } A^T$

Roe 7/19/2017

$$\beta_0 + \beta_1 x_1 + \beta_2 x_1^2 = y_1$$
$$\beta_0 + \beta_1 x_2 + \beta_2 x_2^2 = y_2$$
$$\vdots$$
$$\beta_0 + \beta_1 x_n + \beta_2 x_n^2 = y_n$$

$$A\bar{\beta} = \bar{y} \qquad A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ & \vdots & \\ & \vdots & \\ 1 & x_n & x_n^2 \end{pmatrix}$$

$$A^T A \bar{\beta} = A^T \bar{y}$$

ellipse

$$r = \beta + e(r\cos\theta)$$

radius

eccentricity

$$\beta + e(r_1\cos\theta_1) = r_1$$
$$\beta + e(r_2\cos\theta_2) = r_2$$

$$\beta + e(r_5\cos(\theta_5)) = r_5$$

$$\begin{pmatrix} 1 & r_1\cos(\theta_1) \\ 1 & r_2\cos(\theta_2) \\ & \\ 1 & r_5\cos(\theta_5) \end{pmatrix} \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix}$$

$$y = ce^{ax}$$

vars

we need to ln

$$\ln(y) = \ln(ce^{ax})$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \tilde{c} \\ a \end{pmatrix} = \begin{pmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_n \end{pmatrix}$$

$$\ln y = \ln c + ax\ln e$$
$$= \ln c + ax$$

still not linear, but

$$\ln y = \tilde{c} + ax$$ linear

6.7 Roe

def if V is a vector space
an <u>inner product</u> on V
is a function whose input
are pairs of vectors
from V and whose
outputs are real numbers
that satisfy the conditions
for dot products on R^n.
denoted $<\bar{u}, \bar{v}>$

properties

1. $<\bar{u}, \bar{v}> = <\bar{v}, \bar{u}>$
2. $<\bar{u}+\bar{v}, \bar{w}> = <\bar{u}, \bar{w}> + <\bar{v}, \bar{w}>$
3. $<c\bar{u}, \bar{v}> = c<\bar{u}, \bar{v}>$
4. $<\bar{u}, \bar{u}> \geq 0$
   and $<\bar{u}, \bar{u}> = 0 \Longleftrightarrow \bar{u} = \bar{0}$

ex

$V = R^3$, is $<\bar{u}, \bar{v}> = \bar{u}^T\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\bar{v}\right)$
an inner product

$$<\bar{u}, \bar{v}> = (u_1, u_2, u_3)\begin{pmatrix} 2v_1 \\ 3v_2 \\ v_3 \end{pmatrix} = 2u_1v_1 + 3u_2v_2 + u_3v_3$$

(1) $<\bar{u}, \bar{v}> = 2u_1v_1 + 3u_2v_2 + u_3v_3 = v_1(2u_1) + v_2(3u_2) + v_3(u_3)$
$= \bar{v}^T\left(\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \end{pmatrix}\bar{u}\right) = <\bar{v}, \bar{u}>$

(2) $<\bar{u}+\bar{v}, \bar{w}> = (\bar{u}+\bar{v})^T\left(\begin{pmatrix} 2 & & \\ & 3 & \\ & & 1 \end{pmatrix}\bar{w}\right) = \ldots$

(3)
(4)

what about
$<\bar{u}, \bar{v}> = u^T\left(\begin{pmatrix} 2 & & \\ & 3 & \\ & & -1 \end{pmatrix}\bar{v}\right)$ $<(0 0 1), (0 0 1)> = -1$ X

Book 4.3

$B = \{b_1 \dots b_p\}$ is a basis for H
$\Longleftrightarrow$
(i) B is linearly independent, and
(ii) the subspace spanned by B coincides with H, that is, $H = \text{Span}\{\bar{b}_1, \dots, \bar{b}_p\}$

Roe 6/27/2017

<u>Def</u> If A is an $m \times n$ matrix, the <u>row space of A</u>, denoted Row A, is the subspace of $\mathbb{R}^n$ spanned by the rows of A, viewed as vectors

~~Row A is a su~~

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 1 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix} \sim A' = \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{pmatrix}$$

Row A is a subspace of $\mathbb{R}^5$
dim Row $A \leq 4$

Row A' = Row A

<u>prf</u> $\cancel{r_2}$ $d_1(r_2) + d_2(r_1 + 2r_2) + d_3$ ...

$= c_1 r_1 + c_2 r_2 + c_3 r_3 + c_4 r_4$

$d_1 \dots d_4$ can be found by solving, in terms of $c_1 \cdots c_4$
no div by zero is possible

$$A'' = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$\sim$

~~eliminating row~~
- rows 1–3 are LI since each has a pivot position.
- since Row A'' = Row A,
$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{pmatrix} \right\}$ is a basis for Row A $\Rightarrow$ dim Row A = 3

ex from last page

~~Col A is~~

find Basis for Col A

eliminating free variables

$$\text{Col } A = \text{Span}\left\{\begin{pmatrix}-2\\1\\3\\1\end{pmatrix}, \begin{pmatrix}-5\\3\\11\\7\end{pmatrix}, \begin{pmatrix}0\\1\\7\\5\end{pmatrix}\right\}$$

so $\text{Dim Col } A = 3$

(since # pivot cols = # pivot rows,
for any matrix A,
Dim Col A = Dim Row A

$$\text{Nul } A = \text{Span}\left\{\begin{pmatrix}-1\\2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\-3\\0\\5\\1\end{pmatrix}\right\}$$

$\dim \text{Nul } A = 2$

(since Dim Nul A = num free variables,
Dim Nul A = # cols − (Dim Col A = Dim Row A)

<u>def</u> the <u>rank</u> of A = Dim Col A = Dim Row A
= # of pivots

same ex

$$A^T = \begin{pmatrix}-2 & 1 & 3 & 1\\ -5 & 3 & 11 & 7\\ 8 & -5 & -19 & -13\\ 0 & 1 & 7 & 5\\ -17 & 5 & 1 & -3\end{pmatrix}$$

for any matrix A

$\text{Col } A^T = \text{Row } A$

$\text{Row } A^T = \text{Col } A$

$\text{Nul } A^T$ is a subspace of Row A (Nul A is a subspa… of Co…

Q: A is 5×6 and all solutions to $A\bar{x}=\bar{0}$ are multiples of one vector.

4.6.19 A: since all solns are multiples of 1 vector, $\bar{x} = t\begin{pmatrix} * \\ * \\ * \end{pmatrix}$, so the ~~m~~ matrix has 1 free variable, so there is a pivot in all 5 rows, so there is a solution to $A\bar{x}=\bar{b}$ has a solution for every $\bar{b}$

4.6.20 Q $\underset{A}{6 \times 8}$ ~~6 pivots~~ 2 free variables

Book 4.7

change-of-coordinates matrix from B to C — $\underset{C \leftarrow B}{P}$
— the ~~matrix~~ unique $n \times n$ matrix such that $[\bar{x}]_C = \underset{C \leftarrow B}{P} [\bar{x}]_B$

$$\underset{C \leftarrow B}{P} = \begin{pmatrix} [b_1]_C & \cdots & [b_k]_C \end{pmatrix}$$

$$\left(c_1 \ c_2 \mid b_1 \ b_2 \right) \sim \left(I \mid \underset{C \leftarrow B}{P} \right)$$

Book 4.3

Spanning Set Theorem
let $S = \{V_1 \ldots V_p\}$ let $H = \text{Span}\, S$

a) if one of the vectors of S, say V_k, is a linear combination of the ~~others,~~ remaining vectors in S, then the set formed by removing V_k still spans S

b) if $H \neq \{\bar{0}\}$ some subset of S is a basis for H

theorem 6
the pivot columns of a matrix form a basis for the matrix

Roe 6/30/2017

5.1

how many distinct eigenvalues can a matrix have

if a is square and $\lambda_1 \ldots \lambda_k$ are distinct eigenvalues with $x_1 \ldots x_k$ eigenvectors then the eigenvectors are LI

since each eval goes with 1 evect, at most n

5.2

given an $n \times n$ matrix A and we want to find the eigenstuff of A we need to find the non-zero vector solutions to

$$A\bar{x} = \lambda\bar{x} \underset{\text{equation}}{=} (A-\lambda I)\bar{x} = \bar{0}$$

$$\Downarrow$$

nonzero solns $\iff$ $A-\lambda I$ has free variables

$\iff$ $\det(A-\lambda I) \cancel{\neq} = 0$

n^{th} degree polynomial equation (has at most n solutions)

$\det(A-\lambda I)=0$ is called the characteristic equation for A find eigenvalues/eigenvectors

$$A = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 4-\lambda & 2 & 3 \\ -1 & 1-\lambda & -3 \\ 2 & 4 & 9-\lambda \end{pmatrix} = 0$$

$$= 2(-1)^{1+2}\det\begin{pmatrix} -1 & -3 \\ 2 & 9-\lambda \end{pmatrix} + (1-\lambda)(-1)^{2+2}\det\begin{pmatrix} 4-\lambda & 3 \\ 2 & 9-\lambda \end{pmatrix}$$
$$+ 4(-1)^{3+2}\det\begin{pmatrix} 4-\lambda & 3 \\ -1 & -3 \end{pmatrix}$$

$$= -2\left((-1)(9-\lambda) - (-3)(2)\right) + (1-\lambda)\left((4-\lambda)(9-\lambda) - 3\cdot 2\right)$$
$$+ (-4)\left((4-\lambda)(-3) - (3)(-1)\right)$$

$$= (-2)(\lambda - 9 + 6) + (1-\lambda)(36 - 9\lambda - 4\lambda + \lambda^2 - 6)$$
$$- 4(3\lambda - 12 + 3)$$

$$= (-2)(\lambda - 3) + (1-\lambda)(\lambda^2 - 13\lambda + 30)$$
$$- 4(3\lambda - 9)$$

$$= (-2)\lambda + 6 + (\lambda^2 - 13\lambda + 30$$

$\vdots$

$$\lambda^3 - 14\lambda^2 + 57\lambda - 72 = 0$$

<u>Rational Root Theorem</u>

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 18, \pm 24, \pm 36, \pm 72$

$\lambda = 3$

$$\lambda^2 - 11\lambda + 24 = (\lambda - 3)(\lambda - 8)$$

$$\lambda - 3 \,\overline{)\, \lambda^3 - 14\lambda^2 + 57\lambda - 72}$$
$$\underline{\lambda^3 - 3\lambda^2}$$
$$-11\lambda^2 + 57\lambda$$
$$\underline{-11\lambda^2 + 33\lambda}$$
$$24\lambda - 72$$
$$\underline{24\lambda - 72}$$
$$0$$

algebraic multiplicity 2

$$(\lambda - 3)^2(\lambda - 8) = 0$$

$\lambda = 3 \qquad \lambda = 8$

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_1 + 2x_2 + 3x_3 =$$

f f

$$\bar{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} t + ($$

the dimension of an eigenspace is at most the algebraic multiplicity of the eigenvalue def if A and B are $n \times n$ matrices and there is an invertible matrix P such that $A = P^{-1}BP$ then A and B are said to be similar

Thm if A and B are similar, then they have the same set of eigenvalues (in general not the same eigenvectors)

prf show that

$$\det(A - \lambda I) = \det(B - \lambda I)$$

the determinant of a product is the product of the determinants

If $\bar{x}$ is an eigenvector of A with eigenvalue λ then $P\bar{x}$ is an eigenvector of B with eigenv

prf $A\bar{x} = \lambda \bar{x} \Leftrightarrow P^{-1}BP\bar{x} = \lambda\bar{x} \Leftrightarrow BP\bar{x} = P\lambda\bar{x}$
$\Leftrightarrow B(P\bar{x}) = \lambda(P\bar{x})$

H/w notes

$\det A^T = \det A$

row operations usually change the eigenvalues of a matrix

$(\det A)(\det B) = \det AB$

Book

thm the pivot columns of a matrix A form a basis for Col A

Unique Representation Theorem
if B is a basis for V then for each $\bar{x}$ in V there exists a unique set of scalars $c_1 \dots c_n$ such

$c_1\bar{b}_1 + c_2\bar{b}_2 + \dots + c_n\bar{b}_n = \bar{X}$

namely $[\bar{x}]_B$

P_B

$\bar{x} = P_B[\bar{x}]_B$

P_B ← change of coordinates matrix from B to the standard bas

since the columns of P_B form a basis for $\mathbb{R}^n$, P_B is invertible

$[\bar{x}]_B = P_B^{-1}\bar{X}$

thm $\bar{X} \to [\bar{x}]_B$ is a one-to-one, onto, linear transforma

isomorphism - a one-to-one, onto, linear transformation

thm if B is a basis for V having n vectors, any set of vectors in V with more than n vectors must be linearly dependent

basis theorem
if V is a p-dimensional vector space, $p \geq 1$, then any linearly independent set of p vectors is automatically a basis for V

dim Nul A is the number of free variables in A
dim Col A is the number of pivot columns in A

rank A + dim Nul A = n

eigenspace for A corresponding to eigenvalue λ
— the set of all eigenvectors corresponding to eigenvalue λ

thm if $\bar{v}_1 \dots \bar{v}_r$ are eigen vectors corresponding to unique eigenvalues then $\{\bar{v}_1 \dots \bar{v}_r\}$ is a linearly independent set

---

book notes

matrix for T relative to B and D — not commuta

— $\left[[T(\bar{b}_1)]_D \; [T(\bar{b}_2)]_D \; \cdots \; [T(\bar{b}_n)]_D \right]$

satisfies

$$[T(\bar{x})]_D = M[\bar{x}]_B$$

matrix for T relative to B — B-matrix for T ~
— satisfies $[T(\bar{x})]_B = [T]_B [\bar{x}]_B$

$[T]_B$

7/5/2017

Roe

~~matrix for T relative to B a~~

5.4

recall from 1.9
if $T: R^n \to R^m$ is a linear transformation, then there is an $m \times n$ matrix A such that
$T(\bar{x}) = A\bar{x}$

$T(\bar{e}_1) = A\bar{e}_1$ ~~so $T(e_1$~~

so $A = \left(T(\bar{e}_1)\ T(\bar{e}_2) \ldots T(\bar{e}_n) \right)$

suppose V is a vector space $\mathcal{V} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n)$
and W ~~is a basis~~ is a vector space $\mathcal{W} = (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_n$
$T: V \to W$ is a linear transformation
is there a matrix for this?
So that $[T(\bar{x})]_{\mathcal{W}} = \underset{m \times n}{M} [\bar{x}]_{\mathcal{V}}$

~~does $M = [[T(v_1)]_M\ T(v_2) \ldots$~~

does
$M = \left[[T(\bar{v}_1)]_{\mathcal{W}}\ [T(\bar{v}_2)]_{\mathcal{W}} \ldots [T($

$\bar{x} = c_1 v_1 + c_2 v_2 + \ldots + c_n \bar{v}_n \Rightarrow [\bar{x}]_{\mathcal{V}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

$$M \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = c_1 [T(\bar{v}_1)]_{\mathcal{W}} + c_2 [T(\bar{v}_2)]_{\mathcal{W}} + \ldots + c_n [T(\bar{v}_n)]_{\mathcal{W}}$$
$$= [T(c_1 \bar{v}_1 + c_2 \bar{v}_2 + \ldots + c_n \bar{v}_n)]_{\mathcal{W}}$$
$$= [T(\bar{x})]_{\mathcal{W}}$$

M is called the matrix of T with respect to $\mathcal{V}$ and $\mathcal{W}$

ex

$T: P_2 \to P_4 \qquad T(p) = p(t) + t^2 p(t)$

is T linear

$$T(p+q) = (p(t)+q(t)) + t^2(p(t) + q($$
$$= p(t) + q(t) + t^2 p(t) + t^2(q(t))$$
$$= T(p) + T(q)$$

$$T(cp) = cp(t) + t^2(cp(t))$$
$$= c(p(t) + t^2 p(t))$$
$$= cT(p)$$

basis for $P_2 = \{1, t, t^2\} = \mathcal{B}$

basis for $P_4 = \{1, t, t^2, t^3, t^4\} = \mathcal{C}$

what is the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$

$$M = \left[\cancel{[T(b_1)]_{\mathcal{C}}\ [T(b_2)]_{\mathcal{C}}}\right.$$
$$= \left[\, [T(1)]_{\mathcal{C}}\ [T(t)]_{\mathcal{C}}\ [T(t^2)]_{\mathcal{C}} \,\right]$$

$T(1) = 1 + t^2(1) \qquad [T(1)]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
$\quad = 1 + t^2$

$[T(t)]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$T(t) = t + t^3$

$T(t^2) = t^2 + t^4 \qquad [T(t)]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$T(2+3t-4t^2)$
$= 2+3t-4t^2$
$+ t^2(2+3t-4$
$= 2-3t-2t^2+3t$
-4

$[T(2+3t-4t^2)]_{\mathcal{C}}$
$= M[2+3t-$
$= M\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = ($

ex 2

suppose T is a linear transformation from $R^n \to R^m$

$T(\bar{x}) = A\bar{x}$

if the eigenvectors $\{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\}$ of A form a basis for R^n

$$M = \left[[T(\bar{b}_1)]_B \ [T(\bar{b}_2)]_B \ \dots \ [T(\bar{b}_n)]_B \right]$$

$$= \left[[A\bar{b}_1]_B \ [A\bar{b}_2]_B \dots [A\bar{b}_n]_B \right]$$

since $\bar{b}$'s are eigenvectors of A

$$= \left[[\lambda_1 \bar{b}_1]_B \ [\lambda_2 \bar{b}_2]_B \ \dots \ [\lambda_n \bar{b}_n]_B \right]$$

$$= \left[\lambda_1 [\bar{b}_1]_B \ \lambda_2 [\bar{b}_2]_B \ \dots \ \lambda_n [\bar{b}_n]_B \right]$$

$$= \left[\begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{pmatrix} \right]$$

$$= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

M is diagonal

$$P = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_n \end{bmatrix}$$

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = M$$

$$A = PMP^{-1}$$

$$A\bar{x} = PMP^{-1}\bar{x}$$

$$A\bar{x} = PM[\bar{x}]_B$$

$$\bar{x} = \begin{bmatrix} \bar{b}_1 & \dots & \bar{b}_n \end{bmatrix}[\bar{x}]_B$$

$$\bar{x} = P_B[\bar{x}]_B$$

$$P_B^{-1}\bar{x} = [\bar{x}]_B$$

ch4 supplementary

3. $\bar{w} = t\bar{u}_1 + s\bar{u}_2$

$\cancel{w_1}$

$$\bar{w} = t\begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$$

$\cancel{w_1 = -2}$

$$\bar{w} = t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} +$$

Roe review for exam2 7/6/2017

sample Exan

#4

$$A = \cancel{\begin{pmatrix} 1 & 3 & 3 & -4 \\ 0 & 4 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{pmatrix}} \sim \begin{pmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{C \leftarrow B}{P} = \left[[\bar{b}_1]_C \ [\bar{b}_2]_C \ \ldots \ [\bar{b}_n]_C \right]$$

check T/f via Google

7/10/2017

Exam 2 (me)

just 1 100% 2 98%'s

67.5 <u>median</u>

5.5

Recall $\cancel{A} = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}$

$A\bar{x} = \lambda\bar{x}$

$\det(A - \lambda I) = 0$

$\det\begin{pmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{pmatrix} = 0$

$\lambda = \frac{\sqrt{3} \pm i}{2}$

$\bar{x} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

for $\lambda = \frac{\sqrt{3} - i}{2}$

$\left(\begin{array}{cc|c} \frac{1}{2}i & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}i & 0 \end{array}\right) \xrightarrow{iR_1 + R_2} \left(\begin{array}{cc|c} \frac{1}{2}i & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\frac{1}{2}i x_1 - \frac{1}{2}x_2 = 0$

$x_1 =$

<u>def</u> <u>complex conjugate</u> of a complex number $a + bi$ is $a - bi$

notation: $\overline{a + bi} = a - bi$

$\overline{2 - 3i} = 2 + 3i$

if $\vec{x}$ or A are vectors in $\mathbb{C}^n$ or $\mathbb{C}^{n^2}$

$\mathbb{C}^n$ = vectors with n complex entries

$\mathbb{C}^{n^2}$ = $n \times n$ matrices with complex entries

then the complex conjugate of $\vec{x}$ or A

denoted $\overline{\vec{x}}$ or $\bar{A}$

is $\bar{x}$ or $\bar{A}$ whose entries are complex conjugates of those in $\vec{x}$ or A

$$\overline{A\vec{x}} = \bar{A}\,\bar{\vec{x}} = A\overline{\vec{x}}$$

$$\overline{(\lambda\vec{x})} = \bar{\lambda}\,\overline{\vec{x}}$$

so $A\bar{x} = \lambda\bar{x} \Leftrightarrow A\bar{\bar{x}} = \bar{\lambda}\bar{\bar{x}}$

ex

cubics always have at least one root

$$A = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$$

$$\det\begin{pmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 5(-2) = 0$$

$$3 - 3\lambda - \lambda + \lambda^2 + 10 = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 2\sqrt{-9}}{2}$$

$$= \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$P = \begin{pmatrix} -5 & -5 \\ -1+3i & -1-3i \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2-3i & 0 \\ 0 & 2+3i \end{pmatrix}$$

$$\cancel{P^{-1} = \frac{1}{15}\begin{pmatrix} -1-3i & 5 \\ 1-3i & -5 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} \frac{1}{15}+\frac{1}{5}i & \frac{1}{3} \\ \frac{1}{5}i-\frac{1}{15} & \frac{1}{3} \end{pmatrix}}$$

A is similar to a scalar times a rotation matrix

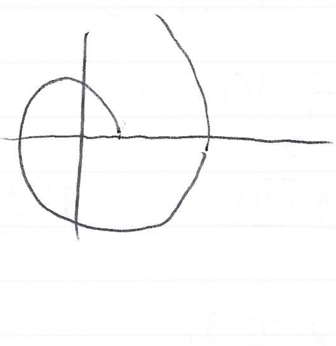
If A is a 2x2 matrix with $\lambda = a - bi$
then a is similar to $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$P = \begin{pmatrix} \bar{c}_1 & \bar{c}_2 \end{pmatrix}$ where $\bar{c}_1 + \bar{c}_2 i$ is the eigenvector associated with λ_1

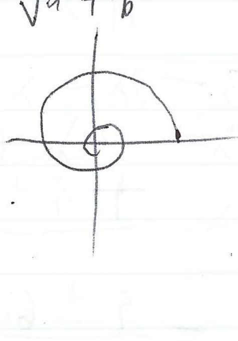
$$P^{-1}AP = R = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$\sqrt{a^2+b^2} = 1$

$\sqrt{a^2+b^2} > 1$



$\sqrt{a^2+b^2} < 1$



h/w & book note

Thm 9 let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and associated eigenvector $\bar{v}$ in C^2. Then $A = PCP^{-1}$ where $P = [\mathrm{Re}\,\bar{v} \;\; \mathrm{Im}\,\bar{v}]$ and $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

minus

($\bar{v}$ can be <u>any</u> eigen vector of A, as long as $\lambda = a - bi$ is the corresponding eigen~~vector~~ value)

For any complex $c \in C$

$c \cdot \bar{c} \in R$

and it follows that

$\bar{\vec{x}}^T \vec{x} \in R$

$a + bi = c + di$

$\Longleftrightarrow$

$a = c \wedge b = d$

5.5 supplement notes

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ has $\lambda = a+bi$ and $\lambda = a-bi$

a rotation matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ ~~has~~
rotates by φ,

$$\varphi = k\pi + \tan^{-1}\left(\frac{b}{a}\right)$$

and scales by r

$$r = \sqrt{a^2 + b^2}$$

5.~~6~~7 Roe

suppose $x_1(t), x_2(t), \ldots, x_n(t)$ are differentiable functions of t

consider the system

$x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t)$

$x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t)$

$\vdots$

$x_n'(t) = a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t)$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

~~If A is diagonalizable~~

if the matrix is diagonal, then we can solve

$\bar{y}(t) = P^{-1}x(t)$

~~$P^{-1}AP = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$~~

ex

$$\bar{x}' = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix} \bar{x}$$

$x_1'(t) = 2x_1(t) + 3x_2(t)$

$x_2'(t) = 2x_1(t) - x_2(t)$

$$\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$(2-\lambda)(-1-\lambda) - (3)(2) = 0$

$-2 - 2\lambda + \lambda + \lambda^2 - 6 = 0$

$\lambda^2 - \lambda - 8 = 0$

$$\lambda = \frac{1 \pm \sqrt{1-4(1)(-8)}}{2} = \frac{1 \pm \sqrt{1+32}}{2}$$

$$= \frac{1 \pm \sqrt{33}}{2}$$

$\lambda = \frac{1+\sqrt{33}}{2}$ $\quad \lambda = \frac{1-\sqrt{33}}{2}$

$$\begin{pmatrix} -3 \\ \frac{3}{2} - \frac{\sqrt{33}}{2} \end{pmatrix} \quad \begin{pmatrix} -3 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} \end{pmatrix}$$

$$P = \begin{pmatrix} -3 & -3 \\ \frac{3}{2} + \frac{\sqrt{33}}{2} & \frac{3}{2} - \frac{\sqrt{33}}{2} \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{1-\sqrt{33}}{2} & 0 \\ 0 & \frac{1+\sqrt{33}}{2} \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} -3 & -3 \\ \frac{3+\sqrt{33}}{2} & \frac{3-\sqrt{33}}{2} \end{pmatrix} \begin{pmatrix} e^{\frac{1-\sqrt{33}}{2}t} & 0 \\ 0 & e^{\frac{1+\sqrt{33}}{2}t} \end{pmatrix} P^{-1}$$

$\bar{x}' = A\bar{x} \quad \bar{x}(0) = \bar{x}_0$

If $\bar{x}_0 = \bar{v}_k$ — k^{th} eigenvector of A

$$\bar{x}(t) = P\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{pmatrix} P^{-1}\bar{v}_k = P\begin{pmatrix} 0 \\ \vdots \\ e^{\lambda_k t} \\ \vdots \\ 0 \end{pmatrix} = e^{\lambda_k t}\bar{v}_k$$

$$P^{-1}\bar{v}_k = \bar{e}_k$$

If $\bar{x}_0$ is not an eigenvector

$$\bar{x}(t) = c_1 e^{\lambda_1 t}\bar{v}_1 + c_2 e^{\lambda_2 t}\bar{v}_2 + \dots + c_n e^{\lambda_n t}\bar{v}_n$$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at+bit} = e^{at}e^{(bt)i} = e^{at}(\cos(bt) + i\sin(bt))$$

$\lambda = a+bi \quad \lambda = a-bi$

$$\bar{x}(t) = c_1 e^{\lambda t}\begin{pmatrix} w_1 + z_1 i \\ w_2 + z_2 i \end{pmatrix} + c_2 \overset{\text{complex conjugate}}{e^{\bar{\lambda} t}}\begin{pmatrix} w_1 - z_1 i \\ w_2 - z_2 i \end{pmatrix}$$

$$\Downarrow$$

$$\bar{x}(t) = (c_1 + c_2)e^{at}\left(\cos(bt)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \sin(bt)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$$

~~$\bar{x}(t) = c_1 e$~~

~~$\bar{x}(t) = c_1 ($~~ If $\bar{x}' = A\bar{x}$, then

$$\bar{x}(t) = c_1\bar{v}_1 e^{\lambda_1 t} + \dots + c_n \bar{v}_n e^{\lambda_n t}$$

where λ_i is the eigenvalue corresponding to eigenvector v_i
and $\bar{c} = P^{-1}\bar{x}(0)$ where P is $(\bar{v}_1 \dots \bar{v}_n)$

the eigenvalues of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$
are $a+bi$ and $a-bi$

A ~~m~~ 2x2 matrix A can be factored
$A = PCP^{-1}$ where $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $P = \begin{pmatrix} \operatorname{Re}\bar{v} & \operatorname{Im}\bar{v} \end{pmatrix}$
($\lambda = a \pm bi$ $\bar{v}$ ~~is either~~ the eigenvector corresponding to $\lambda = a - bi$)

h/w notes

how to find "the general real solution"

1. use

$$\bar{X} = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} \cancel{+ \cdots + c_n \bar{v}_n e^{\lambda_n t}}$$

to solve the system of differential ~~so~~ equations ($\bar{v}_k$ and λ_k may be imaginary)

2. ~~pick any term~~
pick either the first term or the second term of the solution, remove the c, multiply t through the complex λ, and use $e^{ri} = \cos(r) + i\sin(r)$.

3. Distribute the $(\cos(-bt) + i\sin(-bt))$ through the vector $\bar{v}$

4. Separate the real and imaginary parts of $\bar{v}$ so that the system looks like

$$\cancel{\bar{x} = c\,re(\bar{v})e^{at} + i\,c\,im(\bar{v})e^{at}}$$

$$\cancel{\bar{x} = Re(v)\,e^{at} + i\,Im(v)\,e^{at}}$$

$$\bar{x} = \bar{u}e^{at} + i\bar{v}e^{at}$$

5. Write that as

"general real solution"

$$\bar{x} = c_1 \bar{u} e^{at} + c_2 \bar{v} e^{at}$$

note that these are completely different c's than those at the top of the page

6.1 Roe

def If $\bar{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ and $\bar{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ are vectors in $\mathbb{R}^n$, then the inner product (or dot product) of $\bar{u}$ and $\bar{v}$, denoted $\bar{u} \cdot \bar{v}$ is $\bar{u}^T \bar{v}$

$$(u_1 \cdots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

properties

(i) $\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

(ii) $(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$

(iii) $(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v})$ ← 3 types of multiplicati

(iv) $\bar{u} \cdot \bar{u} \geq 0$ and $\bar{u} \cdot \bar{u} = 0 \iff \bar{u} = \bar{0}$

def the length of a vector, denoted $\|\bar{u}\|$, is $\sqrt{\bar{u} \cdot \bar{u}}$

the length of $\bar{u}$ is 0 iff $\bar{u} = \bar{0}$

Thm $\|c\bar{u}\| = |c| \|\bar{u}\|$

prf $\|c\bar{u}\|^2 = (c\bar{u}) \cdot (c\bar{u}) = c(\bar{u} \cdot (c\bar{u})) = c^2(\bar{u} \cdot \bar{u})$

$\|c\bar{u}\| = \sqrt{c^2(\bar{u} \cdot \bar{u})} = \sqrt{c^2}\sqrt{\bar{u} \cdot \bar{u}}$

def The distance from $\bar{u}$ to $\bar{v}$, denoted $\text{dist}(\bar{u}, \bar{v})$, is $\|\bar{u} - \bar{v}\|$

$\text{dist}(\bar{u}, \bar{v}) = \text{dist}(\bar{v}, \bar{u})$

prf

$$\|\bar{u} - \bar{v}\| \stackrel{?}{=} \|\bar{v} - \bar{u}\|$$
$$= \|(-1)(\bar{u} - \bar{v})\|$$
$$= |-1| \, \|\bar{u} - \bar{v}\|$$
$$= \|\bar{u} - \bar{v}\|$$

def Two vectors $\bar{u}$ and $\bar{v}$ in R^n are orthogonal iff $\bar{u} \cdot \bar{v} = 0$

ex

~~2 vectors~~ iff ~~in R~~ e_i and e_j are orthogonal iff $i \neq j$

$\bar{0}$ is orthogonal to every vector

suppose $\text{dist}(\bar{u}, \bar{v}) = \text{dist}(\bar{u}, -\bar{v})$

$$\|\bar{u} - \bar{v}\| = \|\bar{u} + \bar{v}\|$$
$$\sqrt{(\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v})} = \sqrt{(\bar{u} + \bar{v}) \cdot (\bar{u} + \bar{v})}$$
$$(\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) = (\bar{u} + \bar{v}) \cdot (\bar{u} + \bar{v})$$
$$(\bar{u} - \bar{v}) \cdot \bar{u} + (\bar{u} - \bar{v}) \cdot (-\bar{v}) = (\bar{u} + \bar{v}) \cdot \bar{u} + (\bar{u} -$$
$$\bar{u} \cdot \bar{u} + \bar{u} \cdot (-\bar{v}) + \bar{u} \cdot (-\bar{v}) + \bar{v} \cdot \bar{v}$$
$$= \bar{u} \cdot \bar{u} + \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{v} + \bar{v} \cdot \bar{v}$$
$$-2(\bar{u} \cdot \bar{v}) = 2(\bar{u} \cdot \bar{v})$$
$$0 = 4(\bar{u} \cdot \bar{v})$$
$$\bar{u} \cdot \bar{v} = 0$$

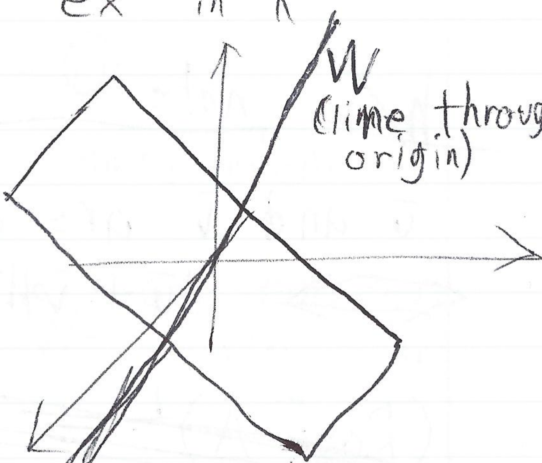
$\bar{u}$ and $\bar{v}$ are orthogonal

thm $\bar{u}$ and $\bar{v}$ are orthogonal iff
$\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$

$(\Rightarrow)$ $\bar{u}\cdot\bar{v}=0 \Rightarrow \|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + 2\underbrace{(\bar{u}\cdot\bar{v})}_{0} + \|\bar{v}\|^2$

$(\Leftarrow)$

def if W is a subspace of R^n let $W^\perp$ ("w perp") be the set of all vectors in R^n that are orthogonal to every vector in W



is $W^\perp$ always a subspace?

(1) does $W^\perp$ contain $\bar{0}$?
$\bar{0}$ is orthogonal to every vector ✓

(2) if $\bar{u}\in W^\perp$, is $c\bar{u}\in W^\perp$?
$\bar{v}\in W \Rightarrow \bar{u}\cdot\bar{v}=0$
$(c\bar{u})\cdot\bar{v} = c(\bar{u}\cdot\bar{v}) = c0 = 0$ ✓

(3) if $\bar{u}$ and $\bar{v}\in W^\perp$ is $\bar{u}+\bar{v}\in W^\perp$
$\forall\, \bar{w}\in W$ $\bar{u}\cdot\bar{w}=0$ $\bar{v}\cdot\bar{w}=0$
$(\bar{u}+\bar{v})\cdot\bar{w} = \bar{u}\cdot\bar{w} + \bar{v}\cdot\bar{w} = 0+0=0$ ✓

$(W^\perp)^\perp = W$

~~If A is an m×n~~ ~~Row A~~

If A is an $m\times n$ matrix

Then Row A is a subspace of R^n

$(\text{Row } A)^{\perp}$ = all $\bar{x} \in R^n$ so that $\bar{x}\cdot\bar{y}=0$ where $\bar{y} \in$ Row

$$\bar{x}\cdot\bar{y} = \bar{x}\cdot(c_1\bar{r}_1 + c_2\bar{r}_2 + \cdots + c_m\bar{r}_m)$$
$$= c_1\,\bar{x}\cdot\bar{r}_1 + c_2\,\bar{x}\cdot\bar{r}_2 + \cdots + c_m\,\bar{x}\cdot\bar{r}_m$$

$$\begin{aligned} \bar{x}\cdot\bar{r}_1 &= 0 \\ \bar{x}\cdot\bar{r}_2 &= 0 \\ &\vdots \\ \bar{x}\cdot\bar{r}_m &= 0 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} -\bar{r}_1- \\ -\bar{r}_2- \\ \vdots \\ -\bar{r}_m- \end{pmatrix} \begin{pmatrix} | \\ \bar{x} \\ | \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

so $(\text{Row } A)^{\perp} = \text{Nul } A$

and $(\text{Col } A)^{\perp} = \text{Nul } A^T$ (since Col A = Row A^T)

h/w note

$\bar{u}$ and $\bar{v}$ are orthogonal

$\Longleftrightarrow \quad \|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$

~~$(\text{Row } A)^{\perp} = \text{Nul } A$~~

book

$(\text{Row } A)^{\perp} = \text{Nul } A$

$(\text{Col } A)^{\perp} = \text{Nul } A^T$

$\text{dist}(\bar{u}, \bar{v}) = \|\bar{u}-\bar{v}\|$

1. a vector $\bar{x}$ is in $W^{\perp}$ $\Longleftrightarrow$ $\bar{x}$ is orthogonal to every vector in W
2. $W^{\perp}$ is a subspace of R^n

def a set of vectors $\{\bar{u}_1 \dots \bar{u}_n\}$ in $\mathbb{R}^n$ is an <u>orthogonal set</u> if

$\bar{u}_i \cdot \bar{u}_j = 0$ whenever $i \neq j$

Thm if $\{\bar{u}_1 \dots \bar{u}_k\}$ is an orthogonal set on <u>non-zero</u> vectors in $\mathbb{R}^n$, then the set is linearly independent

pf

$$c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_n\bar{u}_n = \bar{0}$$
$$\bar{u}_i \cdot (c_1\bar{u}_1 + c_2\bar{u}_2 + \dots + c_n\bar{u}_n) = \bar{u}_1 \cdot \bar{0}$$
$$c_1\bar{u}_i\cdot\bar{u}_1 + c_2\bar{u}_i\cdot\bar{u}_2 + \underset{c_i\bar{u}_i\cdot\bar{u}_i}{\dots} + c_n\bar{u}_i\cdot\bar{u}_n$$

~~0 + 0~~

if all vectors are orthogonal

$$0 + 0 + \dots + c_i\bar{u}_i\cdot\bar{u}_i + \dots + 0 = 0$$

so $c_i = 0 \Rightarrow \bar{u}_1 \dots \bar{u}_n$ are linearly independent

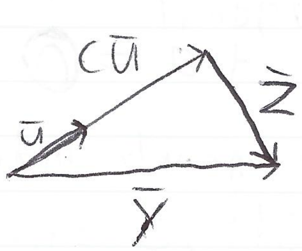
If $\{\bar{u}_1, \dots, \bar{u}_n\}$ are an orthogonal set of nonzero vectors then it is a basis for $W = \text{Span}\{\bar{u}_1, \dots, \bar{u}_n\}$
We call this an <u>orthogonal basis</u>

Thm If $\{\bar{u}_1, \dots, \bar{u}_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and $\bar{y}$ is a vector in W, then there are $c_1, \dots, c_k$
so that $\bar{y} = c_1\bar{u}_1 + \dots + c_k\bar{u}_k$
where $c_i = \dfrac{\bar{u}_i \cdot \bar{y}}{\bar{u}_i \cdot \bar{u}_i}$

$$\bar{y} = c_1\bar{u}_1 + \cdots + c_k\bar{u}_k$$
$$\bar{u}_i \cdot \bar{y} = \bar{u}_i \cdot (c_1\bar{u}_1 + \cdots + c_k\bar{u}_k)$$
$$\bar{u}_i \cdot \bar{y} = c_1\,\bar{u}_i \cdot \bar{u}_1 + \cdots + c_k\,\bar{u}_i \cdot \bar{u}_k$$
$$\bar{u}_i \cdot \bar{y} = 0 + \cdots + c_i\,\bar{u}_i \cdot \bar{u}_i + \cdots + 0$$
$$\frac{\bar{u}_i \cdot \bar{y}}{\bar{u}_i \cdot \bar{u}_i} = c_i$$

---

let $\bar{u}$ be a nonzero vector in $\mathbb{R}^n$ and $\bar{y}$ be a vector in $\mathbb{R}^n$.
Can we find a vector $\bar{z}$ and scalar c so that
$\bar{y} = c\bar{u} + \bar{z}$ where $\bar{u} \cdot \bar{z} = 0$



$$\bar{y} = \left(\frac{\bar{u}\cdot\bar{y}}{\bar{u}\cdot\bar{u}}\right)\bar{u} + \bar{z}$$
$$\Rightarrow \bar{z} = \bar{y} - \left(\frac{\bar{u}\cdot\bar{y}}{\bar{u}\cdot\bar{u}}\right)\bar{u}$$

if $L = \text{Span}\{\bar{u}\}$
$\left(\frac{\bar{u}\cdot\bar{y}}{\bar{u}\cdot\bar{u}}\right)\bar{u}$ is the <u>projection of $\bar{y}$ onto L</u>

$$\hat{y} = \left(\frac{\bar{u}\cdot\bar{y}}{\bar{u}\cdot\bar{u}}\right)\bar{u} = \text{proj}_L\,\bar{y}$$

ex $\left\{ \begin{pmatrix}3\\-2\\1\\3\end{pmatrix}, \begin{pmatrix}-1\\3\\-3\\4\end{pmatrix}, \begin{pmatrix}3\\8\\7\\0\end{pmatrix} \right\} = B$

(all pairs dot to zero)

find, if possible, c_1, c_2, c_3 so that

$$\begin{pmatrix}5\\-28\\-5\\1\end{pmatrix} = c_1\begin{pmatrix}3\\-2\\1\\3\end{pmatrix} + c_2\begin{pmatrix}-1\\3\\-3\\4\end{pmatrix} + c_3$$

$$c_1 = \frac{\begin{pmatrix}5\\-28\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\\3\end{pmatrix}}{\begin{pmatrix}3\\-2\\1\\3\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\1\\3\end{pmatrix}} = 3$$

$$c_2 = \frac{\begin{pmatrix}5\\-28\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-3\\4\end{pmatrix}}{\begin{pmatrix}-1\\3\\-3\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\3\\-3\\4\end{pmatrix}} = -2$$

$$c_3 = \frac{\begin{pmatrix}5\\-28\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}3\\8\\7\\0\end{pmatrix}}{\begin{pmatrix}3\\8\\7\\0\end{pmatrix}\cdot\begin{pmatrix}3\\8\\7\\0\end{pmatrix}} = -2$$

but we don't know that $\bar{y} \in \text{Span}\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$

$$3\begin{pmatrix}3\\-2\\1\\3\end{pmatrix} + (-2)\begin{pmatrix}-1\\3\\-3\\4\end{pmatrix} + (-2)\begin{pmatrix}3\\8\\7\\0\end{pmatrix} = \begin{pmatrix}5\\-28\\-5\\1\end{pmatrix}$$

now we do

def A set of vectors $\bar{u}_1, \dots, \bar{u}_k$ in R^n that are both orthogonal set and $\|\bar{u}_i\| = 1$ for all i is called an <u>orthonormal set</u>

If $\{\bar{u}_1, \dots, \bar{u}_k\}$ is an orthonormal set in R^n

ex $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is orthonormal

Let $U = [\bar{u}_1 \cdots \bar{u}_k]$ so U is an $n \times k$ matrix

$$U^T U = \begin{pmatrix} -\bar{u}_1^T- \\ \vdots \\ -\bar{u}_n^T- \end{pmatrix} \begin{pmatrix} | & & | \\ \bar{u}_1 & \cdots & \bar{u}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

not square

square

but $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow UU^T, \text{ not } U^T U$$

(wrong)

Let U be an $m \times n$ matrix with orthonormal columns and $\bar{x}, \bar{y}$ be vectors in R^n

$$\begin{aligned} (U\bar{x}) \cdot (U\bar{y}) = (U\bar{x})^T (U\bar{y}) &= \bar{x}^T U^T U \bar{y} \\ &= \bar{x}^T I \bar{y} \\ &= \bar{x}^T \bar{y} \\ &= \bar{x} \cdot \bar{y} \end{aligned}$$

so $(U\bar{x}) \cdot (U\bar{y}) = 0 \Leftrightarrow \bar{x} \cdot \bar{y} = 0$

and
$$\begin{aligned} \|U\bar{x}\| &= \sqrt{(U\bar{x}) \cdot (U\bar{x})} \\ &= \sqrt{\bar{x} \cdot \bar{x}} \\ &= \|\bar{x}\| \end{aligned}$$

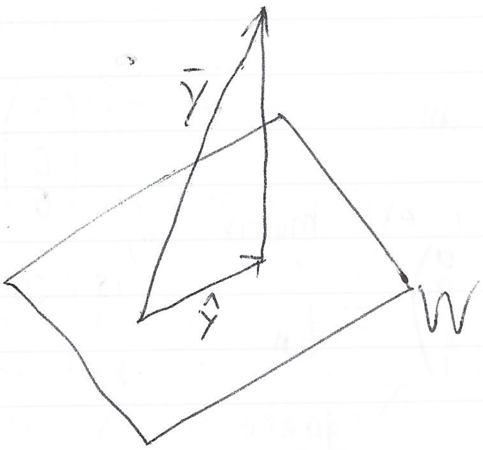
def an orthonormal matrix is a <u>square</u> matrix U where $U^{-1} = U^T$ $\quad U^T U = I \quad UU^T = I$

6.3 Roe

If $\bar{y}$ is in W and $\{\bar{u}_1 \dots \bar{u}_k\}$ is an orthogonal basis for W, then

~~let W be a subspace of R^n spanned by $\bar{u}_1 \dots \bar{u}_k$, where $\{\bar{u}_1 \dots \bar{u}_k\}$ is an orthogonal basis for W~~

$$\text{proj}_W \bar{y} = \hat{y} = \left(\frac{\bar{u}_1 \cdot \bar{y}}{\bar{u}_1 \cdot \bar{u}_1}\right)\bar{u}_1 + \left(\frac{\bar{u}_2 \cdot \bar{y}}{\bar{u}_2 \cdot \bar{u}_2}\right)\bar{u}_2 + \dots + \left(\frac{\bar{u}_k \cdot \bar{y}}{\bar{u}_k \cdot \bar{u}_k}\right)\bar{u}_k =$$



$\text{dist}(\bar{y}, \hat{y}) \le \text{dist}(\bar{y},$

for all $\bar{x}$ in W

$\text{proj}_W \bar{y}$ is the vector in W that is closest to $\bar{y}$

suppose A is an $m \times n$ matrix

for $A\bar{x} = \bar{y}$ to have a solution, $\bar{y}$ must be in Col A

$\bar{y} \notin \text{Col } A \Rightarrow A\bar{x} = \bar{y}$ is inconsistent.

But $\text{proj}_{\text{Col } A} \bar{y}$ is in Col A.

$\bar{y} = \text{proj}_{\text{Col } A} \bar{y} \Rightarrow A\bar{x} = \bar{y}$ has a solution.

If $\bar{y}$ is not in Col A, $\|\bar{y} - A\bar{x}\| \neq 0$

but if $\hat{x}$ is a soln to $A\bar{x} = \hat{y}$

then $\|\bar{y} - A\hat{x}\|$ is as small as possible.

prf

let $\bar{x}$ be in W, $\bar{x} \neq \hat{y}$

$\text{dist}(\bar{y}, \bar{x}) = \|\bar{y} - \bar{x}\|$

$\bar{y} - \bar{x} = \bar{y} - \hat{y} + \hat{y} - \bar{x}$

$\bar{y} - \hat{y}$ is in $W^\perp$

$\hat{y} - \bar{x}$ is in W

so $\bar{y} - \hat{y}$ is orthogonal to $\hat{y} - \bar{x}$

so, by pythagorean theorem,

$\|\bar{y} - \hat{y}\|^2 + \|\hat{y} - \bar{x}\|^2 = \|\bar{y} -$

since $\hat{y} \neq \bar{x}$, $\|\hat{y} - \bar{x}\|^2 > 0$

so

$\|\bar{y} - \hat{y}\|^2 < \|\bar{y} - \bar{x}\|^2$

so, since lengths are positive

$\|\bar{y} - \hat{y}\| < \|\bar{y} - \bar{x}\|$

let $\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ and $W = \text{Span}\{\bar{u}_1, \ldots, \bar{u}_k\}$.

Then
$$\text{proj}_W \bar{y} = \left(\frac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\right)\bar{u}_1 + \cdots + \left(\frac{\bar{y}\cdot\bar{u}_k}{\bar{u}_k\cdot\bar{u}_k}\right)\bar{u}_k$$

$$= (\bar{y}\cdot\bar{u}_1)\bar{u}_1 + \cdots + (\bar{y}\cdot\bar{u}_k)\bar{u}_k$$

$$= \begin{bmatrix}\bar{u}_1 & \cdots & \bar{u}_k\end{bmatrix}\begin{bmatrix}\bar{y}\cdot\bar{u}_1 \\ \vdots \\ \bar{y}\cdot\bar{u}_k\end{bmatrix}$$

orthonormal matrix

$$= \begin{bmatrix}\bar{u}_1 & \cdots & \bar{u}_k\end{bmatrix}\begin{bmatrix}\bar{u}_1^T\bar{y} \\ \vdots \\ \bar{u}_k^T\bar{y}\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}\bar{u}_1 & \cdots & \bar{u}_k\end{bmatrix}\begin{bmatrix}- & \bar{u}_1^T & - \\ & \vdots & \\ - & \bar{u}_k^T & -\end{bmatrix}}_{(\text{If } UU^T = I_k)}\begin{pmatrix}| \\ \bar{y} \\ |\end{pmatrix}$$

$$= I_k\begin{pmatrix}| \\ \bar{y} \\ |\end{pmatrix}$$

Book

Best Approximation Theorem

let W be a subspace of R^n and let $\bar{y}$ be a vector in R^n and let $\hat{y}$ be the orthogonal projection of $\bar{y}$ onto W. Then $\hat{y}$ is the closest point to $\bar{y}$ in W, that is,

$\|\bar{y}-\hat{y}\| < \|\bar{y}-\bar{v}\|$ for all $\bar{v}$ in W distinct from $\hat{y}$

$\bar{y} \in W \Leftrightarrow \text{proj}_W \bar{y} = \bar{y}$

if $\{\bar{u}_1 \ldots \bar{u}_p\}$ is an orthogonal basis for W, then
$$\hat{y} = \left(\frac{\bar{y}\cdot\bar{u}_1}{\bar{u}_1\cdot\bar{u}_1}\right)\bar{u}_1 + \left(\frac{\bar{y}\cdot\bar{u}_2}{\bar{u}_2\cdot\bar{u}_2}\right)\bar{u}_2 + \ldots + \left(\frac{\bar{y}\cdot\bar{u}_p}{\bar{u}_p\cdot\bar{u}_p}\right)\bar{u}_p$$
and $\bar{y}-\hat{y}$ is orthogonal to all of $\bar{u}_1 \cdots \bar{u}_p$ and is thus in $W^\perp$

Theorem 10

if $\{\bar{u}_1, \ldots, \bar{u}_p\}$ is an orthonormal basis for a subspace W of R^n, then
$$\text{proj}_W \bar{y} = (\bar{y}\cdot\bar{u}_1)\bar{u}_1 + (\bar{y}\cdot\bar{u}_2)\bar{u}_2 + \ldots + (\bar{y}\cdot\bar{u}_p)\bar{u}_p$$
if $U = [\bar{u}_1\ \bar{u}_2 \cdots \bar{u}_p]$ then
$$\text{proj}_W \bar{y} = UU^T\bar{y} \text{ for all } \bar{y} \in R^n$$

let U be an $m \times n$ matrix with orthonormal columns, and let $\bar{x}$ and $\bar{y}$ be in R^n

then

$$\|U\bar{x}\| = \|\bar{x}\|$$

$$(U\bar{x}) \cdot (U\bar{y}) = \bar{x} \cdot \bar{y}$$

$$(U\bar{x}) \cdot (U\bar{y}) = 0 \Leftrightarrow \bar{x} \cdot \bar{y} = 0$$

an $m \times n$ matrix U has orthonormal columns

$\Leftrightarrow U^T U = I$

A nonzero vector cannot correspond to two different eigenvalues of A

prf

$$\left.\begin{array}{l} A\bar{v} = \lambda_1 \bar{v} \\ A\bar{v} = \lambda_2 \bar{v} \end{array}\right\} \Rightarrow \lambda_1 \bar{v} = \lambda_2 \bar{v} \Leftrightarrow \left\{\begin{array}{l} \lambda_1 v_1 = \lambda_2 v_1 \\ \lambda_1 v_2 = \lambda_2 v_2 \\ \vdots \\ \lambda_1 v_n = \lambda_2 v_n \end{array}\right.$$

$\bar{v} \neq \bar{0}$, so $\exists i$ such that $v_i \neq 0$

$\lambda_1 v_i = \lambda_2 v_i$

since $v_i \neq 0$

$\lambda_1 = \lambda_2$

Gram-Schmidt process

6.4

Given a basis $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_k\}$ for V how do we find an orthogonal basis for V

① Let $\bar{u}_1 = \bar{b}_1$

② $\text{proj}_{\text{Span}\{\bar{u}_1\}} \bar{b}_2 + \bar{z}_1 = \bar{b}_2$

$$\bar{b}_2 = \left(\frac{\bar{b}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\right)\bar{u}_1 + \bar{z}_1$$

let $\bar{u}_2 = \bar{b}_2 - \text{proj}_{\{\bar{u}_1\}} \bar{b}_2$ so $\{\bar{u}_1, \bar{u}_2\}$ is an orthogonal set

③ $\bar{b}_3 = \text{proj}_{\{\bar{u}_1, \bar{u}_2\}} \bar{b}_3 + \bar{z}_2$

let $\bar{u}_3 = \bar{b}_3 - \text{proj}_{\{\bar{u}_1, \bar{u}_2\}} \bar{b}_3$ so $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is an orthogonal set

Find an orthogonal basis for Col A

$$A = \begin{pmatrix} 3 & -5 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & -2 & 1 & 4 \\ -1 & 5 & 3 & -4 & -2 & 1 \\ 3 & -7 & -1 & 4 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} D & 0 & & & 0 & \\ & D & & & 0 & \\ & & & & D & \\ & & & & 0 & \end{pmatrix}$$

$$\text{Basis Col } A = \left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} \right\}$$

$$\bar{u}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

$$\bar{u}_2 = \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - \text{proj}_{\left\{\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}\right\}} \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - \left(\frac{-15 + 1 - 5 - 21}{9 + 1 + 1 + 9} \right) \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - \left(\frac{-40}{20} \right) \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} + 2 \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} -5 - 10 \\ 1 + 2 \\ 5 + 10 \\ -7 - 14 \end{pmatrix} = \begin{pmatrix} -15 \\ 3 \\ \\ \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 3 \\ 3 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix}$$

to generate an orthonormal base,
divide each vector by its length
~~(which~~
(all vectors will be the same length)

then, let $U = [\bar{u}_1 \cdots \bar{u}_n]$

$U^T U = I$

~~if $A = QR$~~

suppose A is $m \times n$ and has LI columns

Q is orthonormal
R is ~~lower~~ triangular with non-zero entries along diagonal

then to solve $A\bar{x} = \bar{b}$

$$Q R \bar{x} = \bar{b}$$
$$Q^T (Q R \bar{x}) = Q^T b$$
$$(Q^T Q) R \bar{x} = Q^T b$$
$$R\bar{x} = Q^T \bar{b} \qquad \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & r_{mn} \end{pmatrix}$$
$$\cancel{\begin{pmatrix} r_{11} & \cdots & & 0 \\ r_{21} & r_{22} & & \vdots \\ \vdots & & \ddots & \\ r_{n1} & \cdots & & r_{nn} \end{pmatrix}} \bar{x} = Q^T \bar{b}$$

$\Downarrow$

find x_1, plug into equation 2, find x_2,
plug into equation 3 . . .

$A = (\bar{a}_1 \ \bar{a}_2 \ldots \bar{a}_n)$ a_i's are LI

$\bar{u}_1 = \dfrac{\bar{a}_1}{\|a_1\|} - \bar{a}_1$

$\bar{u}_2 = \bar{a}_2$ ———

let Q be orthonormal matrix whose columns are ~~&~~ a basis for Col A (obtain via gram schmidt)

$Q^T A = Q^T(QR)$

$Q^T A = R$

~~&~~

$\|A\bar{x}\| \cancel{=} \det A \, \|\bar{x}\|$ does not equal, in general

If an $m \times n$ matrix A has linearly independen columns, then ~~∃ ortho~~ Q

$A = QR$

where Q is an orthonormal martrix whose column space is the same as that of A and R is upper triangular with strictly positive entries along the main diagonal

Q is $m \times n$, like A

R is $n \times n$, square

A and B are similar $\Leftrightarrow \exists P, \; A = PBP^{-1}$

$\Rightarrow$ A and B have the same ~~eigen vectors~~ eigenvalues

$\not\Rightarrow$ A and B have same eigenvectors

does not imply

A and B have same eigenvalues

$\not\Rightarrow$ A and B are similar

B-matrix for T

$[T(\bar{x})]_B =$

If A is an $m \times n$ matrix and $\bar{b}$ is in $\mathbb{R}^m$, a least squares solution of $A\bar{x} = \bar{b}$ is an $\cancel{\bar{x}}$ $\hat{x}$ in $\mathbb{R}^n$ such that
$$\|\bar{b} - A\hat{x}\| \leq \|\bar{b} - A\bar{x}\| \quad \text{for all } \bar{x} \in \mathbb{R}^n$$

~~If $\hat{b}$ = proj_b Col A~~

If $\hat{b} = \text{proj}_{\text{Col} A}\, \bar{b}$ then $\|\bar{b} - \hat{b}\| \leq \|\bar{b} - \bar{y}\|$ for all $\bar{y}$ in

To find a least squares solution, we could
(1) use gram-schmidt to find an orthogonal basis $\{\bar{u}_1 \dots \bar{u}_k\}$ for Col A
(2) Use the orthogonal basis to find
$$\text{proj}_{\text{Col} A}\, \bar{b} = \frac{\bar{b} \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\bar{u}_1 + \dots + \frac{\bar{b} \cdot \bar{u}_k}{\bar{u}_k \cdot \bar{u}_k}\bar{u}_k$$
(3) solve $A\hat{x} = \hat{b}$ for $\hat{x}$

recall $\bar{b} = \hat{b} + \bar{z}$ $\quad \bar{z} \in (\text{Col} A)^\perp$

so $\bar{b} - \hat{b} = \bar{z}$ so $\bar{b} - \hat{b}$ is orthogonal to every col in A

so
$$A^T(\bar{b} - \hat{b}) = A^T(\bar{b} - A\hat{x}) = \begin{pmatrix} - a_1 - \\ \vdots \\ - a_n - \end{pmatrix}(\bar{b} - \hat{b}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \bar{0}$$

so
$$A^T(\bar{b} - A\hat{x}) = \bar{0} \Leftrightarrow A^T\bar{b} - A^TA\hat{x} = \bar{0} \Leftrightarrow A^T\bar{b} = A^TA\hat{x}$$
$$\Leftrightarrow \boxed{(A^T\bar{b}) = (A^TA)\hat{x}}$$

ex

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \bar{x} = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

inconsistent

$$\begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \bar{x}$$

$$\cancel{\begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} \hat{x}} =$$

$$\hat{x} = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

let's try it!

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} = \text{proj}_{\text{Col } A} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}$$

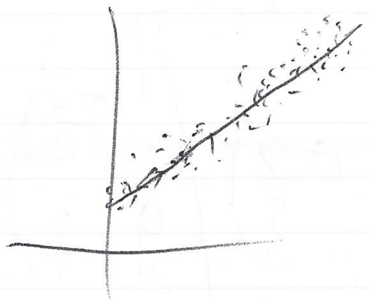
$\|\bar{b} - \hat{b}\|$ is as small as possible

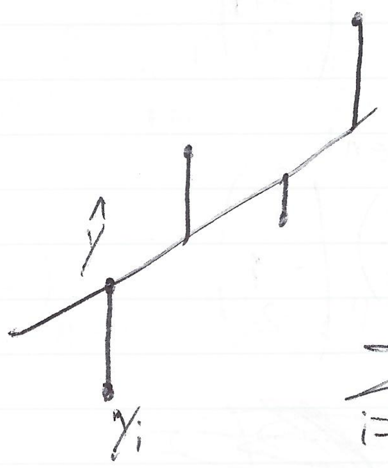
least-squares error $\rightarrow$ $$\left\| \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ -1 \\ 1 \end{pmatrix} \right\| = \sqrt{20}$$

$$= \sqrt{(b_1 - \hat{b}_1)^2 + \cdots + (b_m - \hat{b}_m)^2}$$

least squares

best fit of data   linear regression





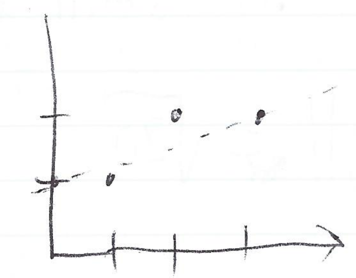
$$\sum_{i=1}^{n} (y_i - \hat{y})^2 = \min$$

$$\hat{y}_i = \beta_0 + \beta_1 x_i$$

$$\beta_0 + \beta_1 \cancel{X_0} = \cancel{Y_0}$$
$$\beta_0 + \beta_1 \cancel{X_1} = y_1$$
$$\beta_0 + \beta_1 x_2 = y_2$$
$$\vdots$$
$$\beta_0 + \beta_1 x_n = y_n$$

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} (\beta_0 \ \beta_1) = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

ex



$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} (\beta_0 \ \beta_1) = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 11 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} (\beta_0 \ \beta$$

$$\left(\begin{array}{cc|c} 4 & 6 & 6 \\ 6 & 14 & 11 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{9}{10} \\ 0 & 1 & \frac{2}{5} \end{array}\right)$$

$$y = \frac{9}{10} + \frac{2}{5}$$

h/w note

$A\hat{x} = \hat{b} \quad A = QR$

$\Downarrow$

$\hat{x} = R^{-1} Q^T \bar{b}$ ← not $\hat{b}$

Roe
7/19/2017

$\bar{V}$ = continuous real-valued functions defined on $[0, 3]$

$$\langle f, g \rangle = \int_0^3 f(t) g(t)\, dt$$

1 $\langle f, g \rangle = \langle g, f \rangle$

2 $\langle f+g, h \rangle = \int_0^3 (f(t) + g(t))\, h(t)\, dt$

$$= \int_0^3 f(t)h(t)\,dt + \int_0^3 g(t)h(t)\,dt$$

$$= \langle f, h \rangle + \langle g, h \rangle$$

3 $\langle cf, g \rangle = \int_0^3 cf(t) g(t)\, dt$

$$= c\int_0^3 f(t) g(t)\, dt$$

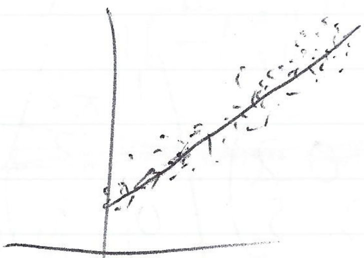
$$= c\langle f, g \rangle$$

so this is an inner product

4 $\langle f, f \rangle = \int_0^3 (f(t))^2\, dt \geq 0$

$\int_0^3 (f(t))^2\, dt = 0 \Longleftrightarrow f(t) = 0$ on $[0, 3]$

best fit of data linear regression



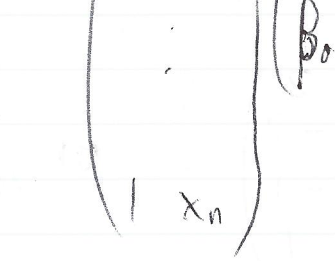
$$\sum_{i=1}^{n} (y_i - \hat{y})^2 = \min$$

$$\hat{y}_i = \beta_0 + \beta_1 x_i$$

$$\beta_0 + \beta_1 \cancel{x_0} = \cancel{x_0}$$
$$\beta_0 + \beta_1 \cancel{x_1}$$
$$\beta_0 + \beta_1 x_2 \cong \cancel{y_1}\ \cancel{y_2}$$
$$\vdots$$
$$\beta_0 + \beta_1 x_n = y_n$$

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 & \beta_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

ex



$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta_0 & \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 11 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} (\beta_0\ \beta$$

$$\left(\begin{array}{cc|c} 4 & 6 & 6 \\ 6 & 14 & 11 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{9}{10} \\ 0 & 1 & \frac{2}{5} \end{array}\right)$$

$$y = \frac{9}{10} + \frac{2}{5}$$

h/w note

$A\hat{x} = \hat{b} \qquad A = QR$

$\Downarrow$

$\hat{x} = R^{-1} Q^T \bar{b}$ ← not $\hat{b}$

Roe
7/19/2017

$\bar{V}$ = continuous real-valued functions defined on $[0, 3]$

$$\langle f, g \rangle = \int_0^3 f(t)g(t)\,dt$$

1 $\langle f, g \rangle = \langle g, f \rangle$

2 $\langle f+g, h \rangle = \int_0^3 (f(t)+g(t))\,h(t)\,dt$

$$= \int_0^3 f(t)h(t)\,dt + \int_0^3 g(t)h(t)\,dt$$

$$= \langle f, h \rangle + \langle g, h \rangle$$

3 $\langle cf, g \rangle = \int_0^3 cf(t)g(t)\,dt$

$$= c\int_0^3 f(t)g(t)\,dt$$

$$= c\langle f, g \rangle$$

so this is an inner product

4 $\langle f, f \rangle = \int_0^3 (f(t))^2\,dt \geq 0$

$\int_0^3 (f(t))^2\,dt = 0 \iff f(t) = 0$ on $[0, 3]$

V, a vector space with an inner product $<x, y>$

def $\|\bar{v}\| = \sqrt{<\bar{v}, \bar{v}>}$

ex's $\|f\| = \sqrt{\int_0^3 (f(t))^2 dt}$ | $\|u\| = \sqrt{2u_1^2 + 3u_2^2 + u_3^2}$

def

V vector sp, $<\bar{u}, \bar{v}>$

$\bar{u}$ is orthogonal to $\bar{v}$ iff $<\bar{u}, \bar{v}> = 0$.

~~cauchy-schwatz~~

Cauchy-Schwarz inequaity

$$|<\bar{u}, \bar{v}>| \leq \|\bar{u}\|\|\bar{v}\|$$

for any inner product

Triangle Inequality

$\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$

~~Ex,~~

$C[a, b]$ is the set of all real-valued functions that are continuous on $[a, b]$

Roe 7/20/2017

ex

$V = P_n$

$$\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \dots + p(t_n)q(t_n)$$

where $t_0, \dots, t_n$ are real numbers

$$\langle p, q \rangle = p(t_0)q(t_0) + \dots + p(t_n)q(t_n)$$
$$= q(t_0)p(t_0) + \dots + q(t_n)p(t_n)$$
$$= \langle q, p \rangle$$

$$\langle p+q, r \rangle = (p(t_0)+q(t_0))r(t_0) + \dots + (p(t_n)+q(t_n))r(t_n)$$
$$= p(t_0)r(t_0) + q(t_0)r(t_0) + \dots + p(t_n)r(t_n) + q(t_n)r(t_n)$$

$$\langle cp, q \rangle = cp(t_0)q(t_0) + \dots + cp(t_n)q(t_n)$$
$$= c(p(t_0)q(t_0) + \dots + p(t_n)q(t_n))$$
$$= c\langle p, q \rangle$$

$$\langle p, p \rangle = p(t_0)p(t_0) + \dots + p(t_n)p(t_n)$$
$$= p(t_0)^2 + \dots + p(t_n)^2$$

so $\langle p, p \rangle \geq 0$

and $\langle \bar{0}, \bar{0} \rangle = 0$

but what about $\langle p, p \rangle = 0$

$\Rightarrow p(t_0) = 0 \wedge p(t_1) = 0 \wedge \dots \wedge p(t_n) =$

$\cancel{\Rightarrow p = \bar{0}}$

$\Rightarrow$ if $p(t) \neq \bar{0}$, $p(t) = (t-t_0)(t-t_1)\cdots(t-t_n)$

$= t^{n+1} \dots$

$\Rightarrow p \notin P_n$

so, since $p \in P_n$, $p(t) = \bar{0}$

$\text{dist}(\bar{u}, \bar{v}) = \|\bar{v} - \bar{u}\| = \sqrt{\langle \bar{v}-\bar{u}, \bar{v}-\bar{u} \rangle}$

$\text{dist}(\bar{u}, \bar{v})$ is small

$\Rightarrow$ at $t_0 \ldots t_n$, $u(t_0)$ is close to $v(t_0)$, $u(t_1)$ is close to $v(t_1)$, ..., $u(t_n)$ is close to $v(t_n)$

$\text{dist}(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| = \sqrt{((u-v)(t_0))^2 + ((u-v)(t_1))^2 + \ldots + ((u-v)(t_n))^2}$

$\text{proj}_{\{\bar{u}_1\}} \bar{u}_2 = \left(\dfrac{\langle \bar{u}_1, \bar{u}_2 \rangle}{\langle \bar{u}_1, \bar{u}_1 \rangle} \right) \bar{u}_1$ ...

is $\beta = \{1, t, t^2, t^3, t^4\}$ an orthogonal basis with respect to $\langle \bar{u}, \bar{v} \rangle = u(-2)v(-2) + u(-1)v(-1) + u(0)v(0) + u(1)v(1) + u(2)v(2)$

$\langle 1, t \rangle = (1)(-2) + (1)(-1) + (1)(0) + (1)(1) + (1)(2)$
$= -2 - 1 + 1 + 2 = 0$

$\langle 1, t^2 \rangle = (1)(-2)^2 + (1)(-1)^2 + (1)(0)^2 + (1)(1)^2 + (1)$
$= 4 + 1 + 0 + 1 + 4$
$= 10 \neq 0$

so β is **not** an orthogonal set with respect to the inner product

| next page

find an orthogonal basis with respect to the inner product

use Gram Schmidt

we know 1 and t are orthogonal

$$\cancel{\text{proj}_{\{1,t\}} t^2 = \bar{u}_3 = \left(\frac{\langle 1, t^2\rangle}{\langle 1, 1\rangle}\right)1 + \left(\frac{\langle t, t^2\rangle}{\langle t, t\rangle}\right)t}$$

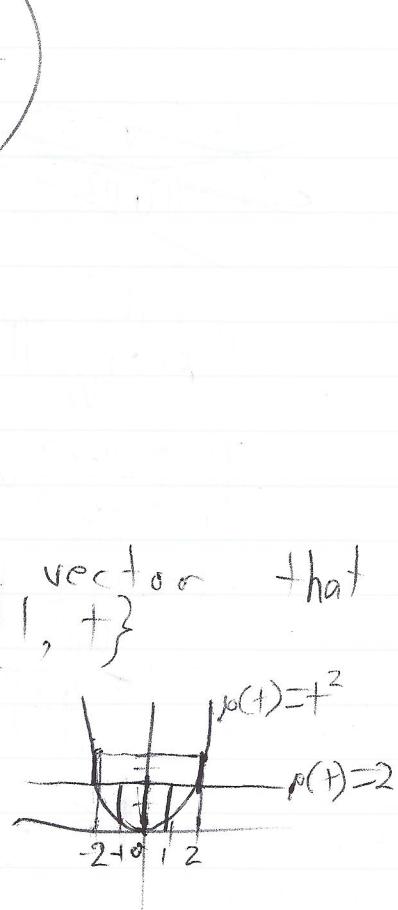
$$\bar{u}_3 = t^2 - \left(\left(\frac{\langle 1, t^2\rangle}{\langle 1, 1\rangle}\right)1 + \left(\frac{\langle t, t^2\rangle}{\langle t, t\rangle}\right)t\right)$$

$$= t^2 - \left(\left(\frac{10}{5}\right)1 + (0)t\right)$$

$$= t^2 - \frac{10}{5}$$

$$= t^2 - 2$$

notice $p(t)=2$ is the vector that is closest to t^2 in Span$\{1, t\}$



$$\bar{u}_4 = t^3 - \left(\left(\frac{\langle 1, t^3\rangle}{\langle 1, 1\rangle}\right) 1 + \left(\frac{\langle 1, t^3\rangle}{\langle t, t\rangle}\right) + \right.$$

$$\left. + \left(\frac{\langle t^3, t^2-2\rangle}{\langle t^2-2, t^2-2\rangle}\right)(t^2-2)\right)$$

If V is a vector space with inner product $\langle *, * \rangle$

then $|\langle \bar{u}, \bar{v}\rangle| \leq \|\bar{u}\| \|\bar{v}\|$

proof

let $W = \text{Span}\{\bar{u}\}$

$$\text{proj}_W \bar{v} = \frac{\langle \bar{v}, \bar{u}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u} = \frac{\langle \bar{v}, \bar{u}\rangle}{\|\bar{u}\|^2}\bar{u}$$

we know $\bar{v} - \frac{\langle \bar{v}, \bar{u}\rangle}{\|\bar{u}\|^2}\bar{u}$ is in $W^\perp$

$$\cancel{\left(\frac{\langle \bar{v}, \bar{u}\rangle}{\|\bar{u}\|^2}\bar{u}\right)}$$

$$\|\text{proj}_W \bar{v}\| \leq \|\bar{v}\|$$

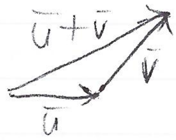
$$\left\| \frac{\langle \bar{u}, \bar{v}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u}\right\| \leq \|\bar{v}\|$$

$$\left|\frac{\langle \bar{u}, \bar{v}\rangle}{\|\bar{u}\|^2}\right| \|\bar{u}\| \leq \|\bar{v}\|$$

$$\left|\frac{\langle \bar{u}, \bar{v}\rangle}{\|\bar{u}\|}\right| \leq \|\bar{v}\|$$

$|\langle \bar{u}, \bar{v}\rangle| \leq \|\bar{u}\|\|\bar{v}\|$ cauchy-Schwartz inequ

triangle inequality
$\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$



these are true for any inner product

prf
$\|\bar{u}+\bar{v}\|^2 = \langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle =$
$\langle \bar{u}, \bar{u}+\bar{v} \rangle + \langle \bar{v}, \bar{u}+\bar{v} \rangle = \langle \bar{u}, \bar{u} \rangle + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle + \langle \bar{v}, \bar{v} \rangle$

$= \|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2$

by cauchy schwartz $\leq \|\bar{u}\|^2 + 2\|\bar{u}\|\|\bar{v}\| + \|\bar{v}\|^2$

$= (\|\bar{u}\| + \|\bar{v}\|)^2$

so

$\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$

ex
$V =$ continuous functions on $[0, 2\pi]$
$\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)\,dt$

are $\sin(t)$ and $\cos(t)$ orthogonal?

$\langle \sin(t), \cos(t) \rangle = \int_0^{2\pi} \sin(t)\cos(t)\,dt$

$u = \sin(t)$
$du = \cos(t)dt$ $\quad dv = \cos(t)\,dt$
$\int \cos(t)dt = \sin(t) + c$
$v = \sin(t)$

$= 0$ apparently

yes, $\sin(t)$ and $\cos(t)$ are orthogonal

$\| \cos(t) \| = \sqrt{<co}$

if m and n are positive <u>integers</u> are $\cos(mt)$ and $\cos(nt)$ orthogonal

$$\langle \cos(mt), \cos(mt) \rangle$$

$$= \int_0^{2\pi} \cos(mt)\cos(nt)\, dt$$

$$= \int_0^{2\pi} \frac{1}{2}\left(\cos((m+n)t) + \sin((m+n)t)\right) dt$$

$$= \frac{1}{2}\left[\frac{\sin((m+n)t)}{(m+n)} + \frac{\cos((m+n)t)}{(m+n)}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(0 - \frac{1}{m+n}\right) - \left(0 - \frac{1}{m+n}\right)\right]$$

$$= \frac{1}{2}[0]$$

$$= 0$$

so they are orthogonal

---

forier transformations ←
projecting a function onto

$$\{ \cancel{\cos}\ 1, \cos(t), \cos(2t), \cos(3t) \ldots$$

$$\sin(t), \sin(2t), \sin(3t), \ldots \}$$

Book

weighted Least-Squares solutions

normal equation:

$$(WA)^T(WA\hat{x}) = (WA)^T(W\bar{y})$$

(as opposed to $A^T A\hat{x} = A^T\bar{y}$)

W is the <u>weight matrix</u>, with weights $w_1 \dots w_n$ along the diagonal

$$\begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}$$

each weight w_i is assciated with a data point $(x_i, y$

to ~~fi~~ make a polynomial fit for data points, use $\langle p(t), q(t) \rangle = p(t_0)q(t_0) + \dots + p(t$

and use gram-schmidt to build degree

$< f(t), g(t) >$

~~the orthogonal projection of $\frac{a_0}{2} + a_1 \sin(t)$~~

Roe 7/21/2017

$C[0, 2\pi]$ — v.s. of continuous functions on $[0, 2\pi]$

$$< f(t), g(t) > = \int_0^{2\pi} f(t) g(t)\, dt$$

If m and n are positive integers with $m \neq n$

$< \cos(mt), \cos(nt) > = 0$

so $\cos(mt)$ is orthogonal to $\cos(nt)$

$< \sin(mt), \sin(nt) > = 0$

$< \sin(mt), \cos(nt) > = 0$

$< 1, \cos(nt) > = < 1, \sin(nt) > = 0$

$\|u\| = \sqrt{\langle u, u \rangle}$

$$\|1\| = \left(\int_0^{2\pi} (1)^2 \, dt \right)^{\frac{1}{2}} = \sqrt{2\pi}$$

$$\| \cos(nt) \| = \left(\int_0^{2\pi} \cos(nt)\, dt \right)^{\frac{1}{2}} = \sqrt{\pi}$$

$$\| \sin(nt) \| = \sqrt{\pi}$$

$$B = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\cos(2t)}{\sqrt{\pi}}, \ldots, \frac{\cos(nt)}{\sqrt{\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \frac{\sin(2t)}{\sqrt{\pi}}, \ldots, \frac{\sin(nt)}{\sqrt{\pi}} \right\}$$

B is an orthonormal basis for a $2n+1$ dimensional su

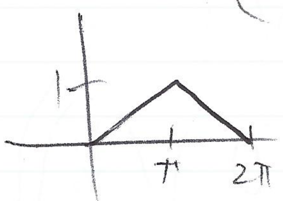
$$\text{proj}_{\text{Span } B} f(t) = \underbrace{\left(\frac{\left\langle \frac{1}{\sqrt{2\pi}}, f(t) \right\rangle}{\left\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \right\rangle} \right)}_{1} \frac{1}{\sqrt{2\pi}} + \Big(\langle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt + $$

$$\frac{1}{\pi} \cos(t) \int_0^{2\pi} f(t) \cos(t)\,dt$$

$$+$$

$$\text{let } f(t) = \begin{cases} \frac{t}{\pi} & \text{if } 0 \le t \le \pi \\ 2 - \frac{t}{\pi} & \text{if } \pi < t \le 2\pi \end{cases}$$



Use ~~fourier~~ approximation for stuff like ~~musical tones~~ sound wave functions

$A\bar{x} = \bar{y}$ is not consistent

$\Longrightarrow$ $\bar{y}$ is not in Col A

But $A\hat{x} = \text{proj}_{\text{Col}A}\bar{y}$ is consistent.

$\hat{x}$ can be found with $A^TA\hat{x} = A^T\bar{y}$

$\|\bar{y} - A\hat{x}\|$ is min in R^n with usual inner product

$$\|\bar{y} - A\hat{x}\| = \cancel{((\bar{y} - \hat{y}_1)^2 + (\bar{y}_2}$$
$$((y_1 - \hat{y}_1)^2 + (y_2 - \hat{y}_2)^2 + \cdots + (y_n -$$

weighted data

minimize

$$w_1^2(y_1 - \hat{y}_1)^2 + w_2^2(y_2 - \hat{y}_2)^2 + \cdots + w_n^2(y_n -$$

$$= (y_1\ y_2 \cdots y_n)\begin{pmatrix} w_1^2 & & & 0 \\ & w_2^2 & & \\ & & \ddots & \\ 0 & & & w_n^2 \end{pmatrix}\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix} =$$

$$= \left(\begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right)^T \left(\begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \ddots & \\ 0 & & & w_n \end{pmatrix}\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{pmatrix}\right)$$

$$\boxed{(WA)^T(WA)\hat{x} = (WA)^T W\bar{y}}$$

Final @ 8:00 am
Friday

$\beta_0 + \beta_1 x = y$

ex

$$W = \begin{pmatrix} 2 & & & & & & \\ & 2 & & & & 0 & \\ & & 1 & & & & \\ & & & 2 & & & \\ & & & & 2 & & \\ & 0 & & & & 2 & \\ & & & & & & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_7 \end{pmatrix}$$

$$W \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = W \begin{pmatrix} 3.1 \\ 3.9 \\ 4.2 \\ 5.9 \\ 7.2 \\ 8.1 \\ 9.05 \end{pmatrix} \qquad WA = \begin{pmatrix} 2 & 2x_1 \\ 2 & 2x_2 \\ 1 & x_3 \\ 2 & 2x_4 \\ 2 & 2x_5 \\ 2 & 2x_6 \\ 2 & 2x_7 \end{pmatrix}$$

$$(WA)^T \left(W \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \right) \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = (WA)^T W \vec{y}$$

$$\bar{\beta} \approx \begin{pmatrix} 3.107 \\ 2.043 \end{pmatrix}$$

without W, you would get
$$\bar{\beta} = \begin{pmatrix} 2.186 \\ 2.529 \end{pmatrix}$$

$y = 2x + 3$ w/ noise and pt 3 messed

Book 7.1

Spectral Theorem

An $n \times n$ symmetric matrix has these properties
a. A has n real eigenvectors, counting multiplicity
b. the dimension of each eigenspace is the algebraic multiplicity of the corresponding eigenvalue λ in the characteristic polynomial
c.

Roe 7.1

symmetric matrices

<u>def</u> an $n \times n$ matrix A is <u>symmetric</u> if $a_{ij} = a_{ji}$ for all $1 \le i, j \le n$

$(A^T = A)$

if $\vec{x}$ is a possibly complex vector and we let $q = (\overline{\vec{x}})^T A \vec{x}$ ~~then (A is real)~~ (A is real and symmetric)

then q is a possibly complex number

$$\overline{q} = \overline{(\overline{\vec{x}})^T A \vec{x}} = \overline{(\overline{\vec{x}})^T}\;\overline{A}\;\overline{\vec{x}}$$

since this is just a scala

$$= \vec{x}^T \overline{A}\, \overline{\vec{x}} = \vec{x}^T A \overline{\vec{x}} = \left((\vec{x})^T A \overline{\vec{x}}\right)^T$$

$$= \overline{\vec{x}}^T A^T \vec{x}$$

$$= q$$

since $\overline{q} = q$, q is real

suppose $\vec{x}$ is an eigenvector of A with eigenvalue λ

$\Longrightarrow A\vec{x} = \lambda \vec{x}$ A is symmetric & real

$$\underbrace{(\overline{\vec{x}})^T (A\vec{x})}_{\text{real (shown above)}} = (\overline{\vec{x}})^T (\lambda \vec{x})$$

$$= \lambda \underbrace{(\overline{\vec{x}})^T \vec{x}}_{\text{real (shown above)}} \Longrightarrow \lambda \text{ is real}$$

$\Longrightarrow$ real symmetric matrices only have real ei real symmetric matrices only have real eigenvalues

A real symmetric $n \times n$ with distinct λ_1, λ_2

$$(\lambda_1 \bar{x}_1) \cdot \bar{x}_2 = (A\bar{x}_1) \cdot \bar{x}_2 = (A\bar{x}_1)^T \bar{x}_2 = \bar{x}_1^T A^T \bar{x}_2$$

$$= \bar{x}_1 \cdot (A\bar{x}_2) = \bar{x}_1 \cdot (\lambda_2 \bar{x}_2) = \lambda_2 \bar{x}_1 \cdot \bar{x}_2$$

so $\lambda_1 \bar{x}_1 \cdot \bar{x}_2 = \lambda_2 \bar{x}_1 \cdot \bar{x}_2$ (λ_1, λ_2 distinct)

$\Rightarrow$ ~~$\lambda_1 = \lambda_2$ but~~ $\bar{x}_1 \cdot \bar{x}_2 = 0$

$\Rightarrow$ $\bar{x}_1$ and $\bar{x}_2$ are orthogonal

Thm A real symmetric $n \times n \Rightarrow$ dim λ eigenspace = algebraic multiplicity of λ

A real, symmetric $\Rightarrow A$ is diagonalizable

If algebraic multiplicity of λ is more than one then we can use Gram-Schm to get an orthogonal basis for the λ eigenspace

So we can make a basis for $\mathbb{R}^n$ of orthogonal eigenvectors of A

ex

$$A = \begin{pmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{pmatrix}$$

~~$\lambda = 1, 3$~~

$\lambda = -3, 15$

$\underline{\lambda = -3}$

$$(A - (-3I))\bar{x} = \bar{0}$$

$$\begin{pmatrix} 8 & 8 & -4 \\ 8 & 8 & -4 \\ -4 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\to 2x_1 + 2x_2 - x_3 = 0$

$\to$ ~~$2x_1$~~ $x_3 = 2x_1 + 2x_2$

$\to$

$$\bar{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} x_2$$

$$\bar{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \qquad \bar{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \operatorname{proj}_{\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \left(\frac{4}{5}\right) \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{5} \\ 1 \\ \frac{2}{5} \end{pmatrix}$$

ortogonal basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -\frac{4}{5} \\ 1 \\ \frac{2}{5} \end{pmatrix} \right\}$

orthogonal matrix — should be called orthonormal matrix

$\lambda = 15$

$$(A - 15I)\vec{x} = \vec{0}$$

$$\begin{pmatrix} -10 & 8 & -4 \\ 8 & -10 & -4 \\ -4 & -4 & -16 \end{pmatrix} x = \vec{0}$$

$$\to \begin{pmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ -2 & -2 & -8 \end{pmatrix} \to \begin{pmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ 1 & 1 & 4 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 9 & 18 \\ 0 & -9 & -18 \\ 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} x_1 + 2x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 = x_3 \end{matrix} \quad \vec{x} = \begin{pmatrix} -2x_3 \\ -2x_3 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -\frac{4}{5} \\ 1 \\ \frac{2}{5} \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

~~trav~~ ~~ortog~~ gram-schmidt

$U^T A U = ($

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \right.$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & -\frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & -\frac{2}{3} \\ \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \end{pmatrix}$$

$$U^T A U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Rightarrow U U^T A U U^T = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^T$$

$$A = U \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} U^T$$

$$= \lambda_1 U U^T + \lambda_2 U U^T + \cdots + \lambda_n U U^T$$

~~U $m \times n$~~

U $n \times m$ w/ orthonormal columns

$\Rightarrow U^T U = I$

but UU^T not identity matrix

however

$(UU^T)\bar{x} = \text{proj}_{\text{Col } U} \vec{x}$

$$\Rightarrow A\bar{x} = \lambda_1 UU^T \bar{x} + \lambda_2 UU^T \bar{x} + \cdots + \lambda_n UU^T x$$

$$= \lambda_1 \text{proj}_{\bar{u}_1} \bar{x} + \lambda_2 \text{proj}_{\bar{u}_2} \bar{x} + \cdots + \lambda_n \text{proj}_{\bar{u}_n}$$

Roe 7/25/2017

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\bar{x}^T A \bar{x} = (x \ y \ z) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= (x \ y \ z) \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+iz \end{pmatrix}$$

$$= ax^2 + bxy + cxz +$$

$$\bar{x}^T A \bar{x} = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots$$

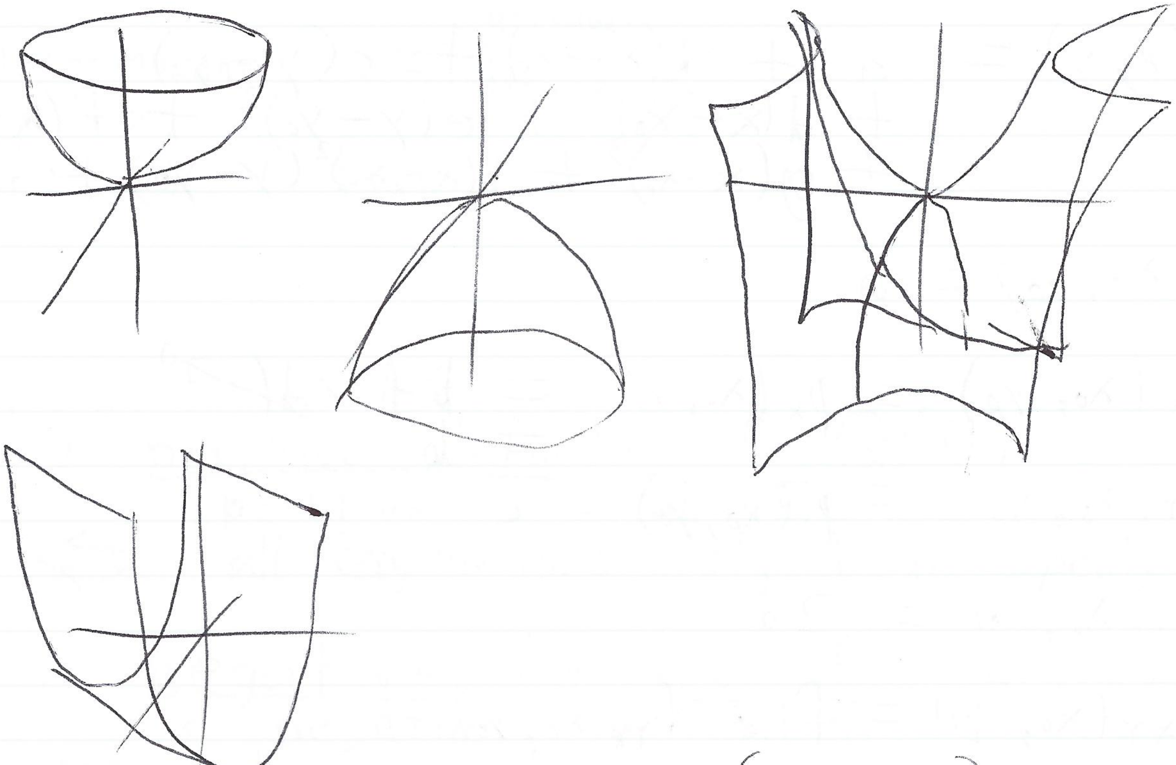
called a <u>quadratic form</u>

ex

$$\bar{X}^T \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -5 & 1 \\ -1 & -5 & -8 & 7 \\ 3 & 1 & 7 & 6 \end{pmatrix} \bar{X} = 1x_1^2 + 4x_2^2 + (-8)x_3^2 + (6)x_4^2 + (2)(2)x_1x_2 + (-1)(2)x_1x_3 + (3)(2)x_1x_4 + (-5)(2)x_2x_3 + (1)(2)x_2x_4 + (7)(2)x_3x_4$$

$f(x, y) = ax^2 + by^2 + cxy$

~~4~~ possibilitier



There are orthonormal eigenvectors $\{\bar{u}_1, \bar{u}_2\}$ of A so that if

$$P = \begin{pmatrix} \bar{u}_1 & \bar{u}_2 \end{pmatrix} \quad \text{the} \quad P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$y = P^{-1}\bar{x} = P^T\bar{x}$

$x^T A x =$

$\lambda_1 y_1^2 + \lambda_2 y_2^2$

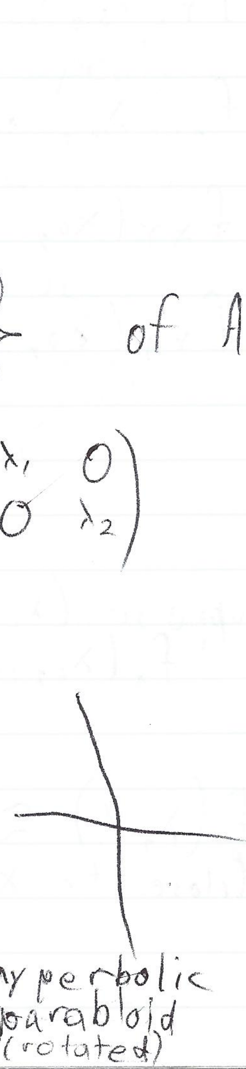
ex

$$\begin{array}{cc} a & \frac{b}{2} \end{array}$$

$$2x^2 + 6xy - 6y^2 = \bar{x}^T \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} \bar{x}$$

$\lambda = -7 \quad \lambda = 3$

$2x^2 + 6xy - 6y^2 = -7y_1^2 + 3y_2^2 \Rightarrow$ hyperbolic paraboloid (rotated)



$$f(x,y) = x\sin(xy)$$

Taylor Polynomials

$$\begin{aligned} f(x,y) = {} & a + b(x-x_0) + c(y-y_0) \\ & + d(x-x_0)^2 + e(y-y_0)^2 + f(x-x_0)(y \\ & + g(x-x_0)^3 + h(x-x_0)^2(y-y_0) + \cdots \end{aligned}$$

$$f(x_0,y_0) = a$$

$$f_x(x_0,y_0) = p_x(x_0,y_0) = b + 2\cancel{d(}\to 0$$
$$= b$$

$$f_y(x_0,y_0) = p_x(x_0,y_0) = c$$

$$f_{xx}(x_0,y_0) = 2d$$

$$f_{xy}(x_0,y_0) = f = f_{yx}(x_0,y_0)$$

$$f_{yy}(x_0,y_0) = 2e$$

suppose (x_0,y_0) is a critical point, that is, $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$

$$\underset{(\text{close to } x_0\ y_0)}{f(x,y)} \approx f(x_0,y_0) + \underbrace{\begin{pmatrix} x-x_0 & y-y_0 \end{pmatrix}\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}_{\text{determined by eigenvalues}}$$

det = product of eigenvalues

both eigenvalues positive
$\Rightarrow$ local min
both negative
$\Rightarrow$ local max

Determinant $+ \Rightarrow$ eigenvalues same sign
$- \Rightarrow$ different sign

Calc 3 $\Rightarrow$
$D(x, y) = f_{xx}($

$Q(\bar{x}) = \bar{x}^T A \bar{x}$
is positive definite if $Q(\bar{x})$ is positive for all $\bar{x}$
$\Leftrightarrow$ all eigenvalues of A are positive

is negative definite if $Q(\bar{x})$ is negative for all $\bar{x}$
$\Leftrightarrow$ all eigenvalues are negative

is indefinite if $Q(\bar{x}) > 0$ for some $\bar{x}$ and $Q(\bar{x}) < 0$ for other $\bar{x}$

H/W note

$P =$ orthonormal eigenvectors of ~~diago~~ symmetric A

$$a x_1^2 + b x_1 x_2 + c x_2^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$y = P^{-1}\bar{x}$$

$$\cancel{P^T} \quad \bar{x} = P\bar{y}$$

$$(P\bar{y})^T A P\bar{y} = \bar{y}^T P^T A P\bar{y} = \bar{y}^T P^{-1} A P\bar{y} = \bar{y}^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \bar{y}$$

$m = \min\{\bar{x}^T A \bar{x} : \|\bar{x}\| = 1\}$ $\quad M = \max\{\bar{x}^T A \bar{x} : \|\bar{x}\| = 1\}$

then m is the ~~le~~ smallest eigenvalue of A
and M is the largest eigenvalue of A

$\in V$ $\quad$ $\in W$

Kernel — $\{\bar{x} : T(\bar{x}) = \bar{0}\}$

Range — $\{\bar{y} : \exists\, \bar{x} \ni T(\bar{x}) = \bar{y}\}$

$T: V \to W$

if $T(\bar{x}) = A\bar{x}$, then
kernel of T = Nul A
and Range of T = Col A

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [\bar{b}_1]_C & \cdots & [\bar{b}_n]_C \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{Re}\,\bar{v} & \mathrm{Im}\,\bar{v} \end{bmatrix} \qquad T(\bar{p}) = p'(x)$$

real ~~coe~~ imaginary
coefficients

$[T(\bar{v})]$

matrix for T relative to
B and C

source $\qquad$ dest

$B = \{1 \;\; t \;\; t^2 \;\; t^3\}$

$C = \{t^2 \;\; t \;\; 1\}$

ex find basis for $T: B \to C$

$T(\bar{v}) = \bar{v}'(t)$

$\cancel{A} =$

$$\left([T(\bar{b}_1)]_C \;\; [T(\bar{b}_2)]_C \;\; [T(\bar{b}_3)]_C \;\; [T(\bar{b}_4)] \right.$$

$1' = 0 \quad \frac{d}{dt}(t) = 1 \quad \frac{d}{dt}(t^2) = 2t$

$\frac{d}{dt}(t^3) = 3t^2$

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$T(2 + 3t - t^2 + 4t^3)$

$= 3 - 2t + 12t^2$

8:00am

Friday

this room

$$[T(2 + 3t - t^2 + 4t^3)]_C$$

$$\cancel{= \begin{pmatrix} 3 \\ -2 \\ 12 \end{pmatrix}} = \begin{pmatrix} 12 \\ -2 \\ 3 \end{pmatrix}$$