Time Series Analysis
Lecture 2

Regression With Time Series, an Intro to Exploratory Time Series Data Analysis, and Time Series Smoothing

datascience@berkeley

Autoregressive Models Part 1b Mathematical Formulation and Properties

Autoregressive Models

The $AR(1)$ model

$$x_t = \phi x_{t-1} + \omega_t$$

is **stationary** with mean

$$E\left(x_t\right) = \sum_{j=0}^{\infty} \phi^j \overset{0}{E\left(w_{t-j}\right)} = 0$$

due to the assumption that w_t is a *white noise series with mean zero and a constant variance.*

The variance and autocorrelation functions are

$$\rho_k = \phi^k (k \geq 0 and \left|\phi\right| < 1)$$

$$\gamma_k = \frac{\phi^k \omega_w^2}{(1 - \phi^2)}$$

Autoregressive Models: Second-Order Properties

Second-order properties of an AR(1) model

AR(1) process is given by

$$x_t = \alpha x_{t-1} + w_t$$

where $\{w_t\}$ is a white noise series with mean zero and variance σ^2. It can be shown that the second-order properties follow as

$$\left.\begin{aligned} \mu_x &= 0 \\ \gamma_k &= \alpha^k \sigma^2 / (1 - \alpha^2) \end{aligned}\right\}$$

$Bx_t = x_{t-1}$

Using $\mathbf{B}$, a stable AR(1) process ($|\alpha| < 1$) can be written as

$$\left.\begin{aligned} (1 - \alpha \mathbf{B}) x_t &= w_t \\ \Rightarrow x_t &= (1 - \alpha \mathbf{B})^{-1} w_t \\ &= w_t + \alpha w_{t-1} + \alpha^2 w_{t-2} + \ldots = \textstyle\sum_{i=0}^{\infty} \alpha^i w_{t-i} \end{aligned}\right\}$$

Autoregressive Models: Second-Order Properties

Hence, the mean is given by

$$E\left(x_t\right) = E\left(\sum_{i=0}^{\infty} \alpha^i w_{t-i}\right) = \sum_{i=0}^{\infty} \alpha^i E\left(w_{t-i}\right) = 0$$

and the autocovariance follows as

$$\begin{aligned}
\gamma_k = \operatorname{Cov}\left(x_t, x_{t+k}\right) &= \operatorname{Cov}\left(\sum_{i=0}^{\infty} \alpha^i w_{t-i}, \sum_{j=0}^{\infty} \alpha^j w_{t+k-j}\right) \\
&= \sum_{j=k+i} \alpha^i \alpha^j \operatorname{Cov}\left(w_{t-i}, w_{t+k-j}\right) \\
&= \alpha^k \sigma^2 \sum_{i=0}^{\infty} \alpha^{2i} = \alpha^k \sigma^2 / (1 - \alpha^2)
\end{aligned}$$

Autoregressive Models: General Case

- As we will see when studying MA models, we need to specify the invertability condition because this condition ensures the existence of the autoregressive representation of a MA model. As for stationarity condition, MA model is always stationary regardless of its parameter values.

- In contract to MA models, an AR model is always invertible, but restrictions on parameters need to be imposed in order for the process to be stationary.

Berkeley SCHOOL OF INFORMATION