Time Series Analysis

Lecture 4

Mixed Autoregressive Moving Average (ARMA) Models

Autoregressive Integrated Moving Average (ARIMA) Models

Seasonal ARIMA (SARIMA) Models

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Seasonal ARIMA
Introduction and Mathematical Formulation

Seasonal ARIMA: Introduction

- So far we have been side-stepping the issue of seasonal effects.
- The dependence on the past often tends to occur most strongly at multiples of some underlying seasonal lag *s*.
- ARIMA models can be extended to include seasonal effects.
- A seasonal ARIMA (SARIMA) model uses differencing at a lag equal to the number of season(s) to remove additive seasonal effects.
- A seasonal ARIMA model is formed by including <u>additional seasonal terms</u> in the ARIMA models, as written below.

$$\text{ARIMA} \quad \underbrace{(p, d, q)}_{\uparrow \; \left(\substack{\text{Non-seasonal part} \\ \text{of the model}}\right)} \quad \underbrace{(P, D, Q)_m}_{\uparrow \; \left(\substack{\text{Seasonal part} \\ \text{of the model}}\right)}$$

s

where m is the number of periods per season.

Seasonal ARIMA: Mathematical Formulation

- Uppercase notation is for the seasonal parts of the model.
- Lowercase notation for the nonseasonal parts of the model.
- The seasonal part of the model consists of terms that are very similar to the nonseasonal components of the model, but they involve backshifts of the seasonal period.
- The general-form seasonal ARIMA model can be written as below.

The seasonal ARIMA$(p, d, q)(P, D, Q)_s$ model can be most succinctly expressed using the backward shift operator

$$\boxed{\Theta_P(\mathrm{B}^s)\theta_p(\mathrm{B})(1-\mathrm{B}^s)^D(1-\mathrm{B})^d x_t = \Phi_Q(\mathrm{B}^s)\phi_q(\mathrm{B})w_t}$$

where Θ_P, θ_p, Φ_Q, and ϕ_q are polynomials of orders P, p, Q, and q

Seasonal ARIMA: Explanation of the Formula

The seasonal ARIMA$(p, d, q)(P, D, Q)_s$ model can be most succinctly expressed using the backward shift operator

$$\Theta_P(\mathrm{B}^s)\theta_p(\mathrm{B})(1-\mathrm{B}^s)^D(1-\mathrm{B})^d x_t = \Phi_Q(\mathrm{B}^s)\phi_q(\mathrm{B})w_t$$

where Θ_P, θ_p, Φ_Q, and ϕ_q are polynomials of orders P, p, Q, and q.

For example, an ARIMA(1,1,1)(1,1,1)$_4$ model (without a constant) is for quarterly data (i.e., m=4) and can be written as:

$$\underbrace{(1-\phi_1 B)}_{\text{Non-seasonal AR(1)}}\ \underbrace{(1-\Phi_1 B^4)}_{\boxed{\text{Seasonal AR(1)}}}\ \underbrace{(1-B)}_{\text{Non-seasonal difference}}\ \underbrace{(1-B^4)}_{\boxed{\text{Seasonal difference}}}y_t = \underbrace{(1+\theta_1 B)}_{\text{Non-seasonal MA(1)}}\ \underbrace{(1+\Theta_1 B^4)}_{\boxed{\text{Seasonal MA(1)}}}e_t.$$

Seasonal ARIMA: Examples

(a) A simple AR model with a seasonal period of 12 units, denoted as ARIMA(0, 0, 0)(1, 0, 0)$_{12}$, is $x_t = \alpha x_{t-12} + w_t$. Such a model would be appropriate for monthly data when only the value in the month of the previous year influences the current monthly value. The model is stationary when $|\alpha^{-1/12}| > 1$.

(b) It is common to find series with stochastic trends that nevertheless have seasonal influences. The model in (a) above could be extended to $x_t = x_{t-1} + \alpha x_{t-12} - \alpha x_{t-13} + w_t$. Rearranging and factorising gives $(1 - \alpha \mathbf{B}^{12})\boxed{(1 - \mathbf{B})}x_t = w_t$ or $\Theta_1(\mathbf{B}^{12})(1 - \mathbf{B})x_t = w_t$, which, on comparing with Equation (7.3), is ARIMA(0, 1, 0)(1, 0, 0)$_{12}$. Note that this model could also be written $\nabla x_t = \alpha \nabla x_{t-12} + w_t$, which emphasises that the *change* at time t depends on the change at the same time (i.e., month) of the previous year. The model is non-stationary since the polynomial on the left-hand side contains the term $(1 - \mathbf{B})$, which implies that there exists a unit root $B = 1$.

Seasonal ARIMA: Examples (2)

(c) A simple quarterly seasonal moving average model is $x_t = (1 - \beta B^4) w_t = w_t - \beta w_{t-4}$. This is stationary and only suitable for data without a trend. If the data also contain a stochastic trend, the model could be extended to include first-order differences, $x_t = x_{t-1} + w_t - \beta w_{t-4}$, which is an ARIMA$(0, 1, 0)(0, 0, 1)_4$ process. Alternatively, if the seasonal terms contain a stochastic trend, differencing can be applied at the seasonal period to give $x_t = x_{t-4} + w_t - \beta w_{t-4}$, which is ARIMA$(0, 0, 0)(0, 1, 1)_4$.

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