Time Series Analysis

Lecture 4

Mixed Autoregressive Moving Average (ARMA) Models

Autoregressive Integrated Moving Average (ARIMA) Models

Seasonal ARIMA (SARIMA) Models

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Mathematical Formulation and Properties of ARMA Models

Mathematical Formulation of ARMA Models (2)

Using AR and MA operators defined in the last lecture, the ARMA(p,q) model can be expressed concisely as

$$\phi(B)x_t = \theta(B)w_t \tag{4.1.3}$$

As the process is stationary, its mean is constant over time:

$$\boxed{\mu = \alpha + \phi_1\mu + \cdots + \phi_p\mu} \tag{4.1.4}$$

or

$$\mu = \frac{\alpha}{1 - \phi_1 - \cdots - \phi_p} \tag{4.1.5}$$

Properties of ARMA Models: The Case of ARMA(1,1)

To study the properties of **ARMA(p,q)** model, let's first consider the **ARMA(1,1)** model:

$$x_t = \phi x_{t-1} + \omega_t + \theta\omega_{t-1} \tag{4.1.6}$$

where $\{\omega_t\} \sim WN(0, \sigma^2)$ and $\phi + \theta \neq 0$.

Rewriting the model using the backward shift operator:

$$\phi(B)x_t = \theta(B)\omega_t \tag{4.1.7}$$

where $\phi(B)$ and $\theta(B)$ are the linear filters

$$\phi(B) = 1 - \phi B \text{ and } \theta(B) = 1 + \theta(B) \tag{4.1.8}$$

where the zeros of $\phi(z)$ lie outside the unit circle for the model to be stationary. In other words, in the simple ARMA(1,1) model, A stationary solution exists if and only if $\phi \neq 1 or - 1$.

Mean, Variance, and Autocovariance of ARMA(1,1) Model

Let's write the process using the infinite MA representation:

$$x_t = \sum_{j=0}^{\infty} \psi_j \omega_{t-j} \tag{4.1.9}$$

It follows immediately that $E(x_t) = 0$.

The autocovariance function is

$$\gamma(h) = cov(x_{t+h}, x_t) = \sigma_\omega^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \qquad h \geq 0 \tag{4.1.10}$$

Autocorrelation Function of the ARMA(1,1) Model

ACF of ARMA(1,1):

The autocovariance function satisfies

$$\gamma(h) - \phi\gamma(h-1) = 0, \quad h = 2, 3, \ldots \tag{4.1.11}$$

With a few lines of algebra, we can arrive at the following form:

$$\gamma(h) = \frac{\gamma(1)}{\phi}\phi_h = \sigma_\omega^2 \frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\phi^{h-1} \tag{4.1.12}$$

As the autocorrelation function at time lag h is the ratio of the autocovariance function at time lag h divided by the variance:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} \tag{4.1.13}$$

$$\rho(h) = \frac{(1+\theta\phi)(\phi+\theta)}{1-\phi^2}\phi^{h-1} \qquad h \geq 1 \tag{4.1.14}$$

Autocorrelation Function of the ARMA(1,1) Model (2)

Recall that the ACF of an AR(1) model takes the form

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h, \qquad h \geq 0 \tag{4.1.15}$$

Note that the general form of the autocorrelation functions between the two models are not that different from each other. For this reason, it is very hard to distinguish an ARMA(1,1) and an AR(1) model based only on the sample ACF.

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