Geostatistical modelling of spatially structured zero-inflation

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Geostatistical methods for prevalence mapping

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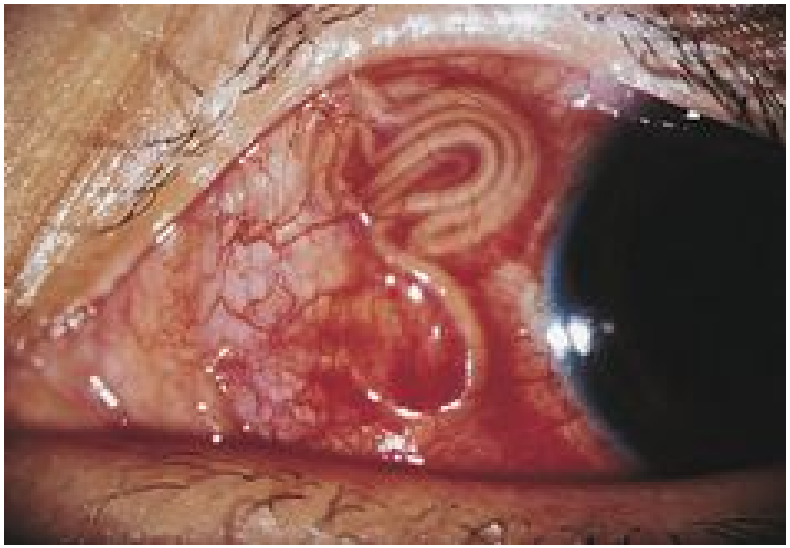
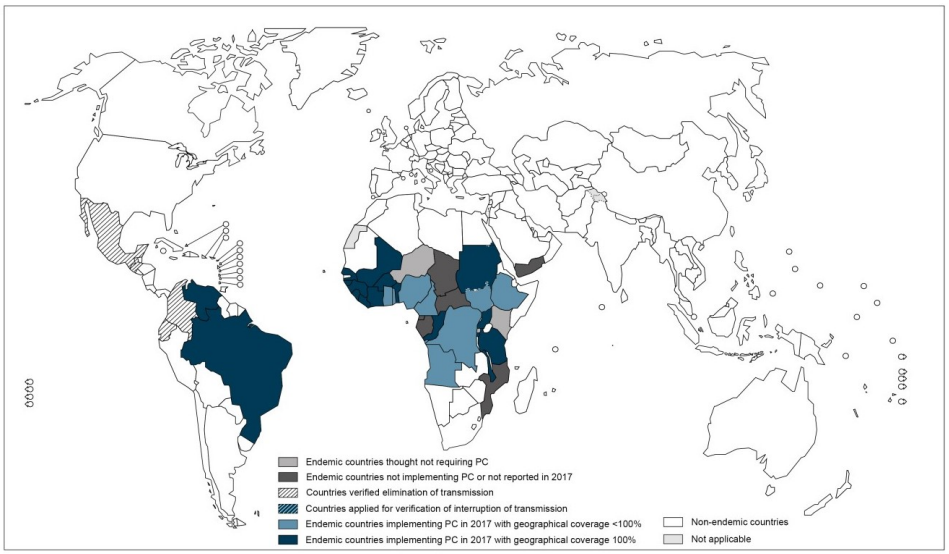
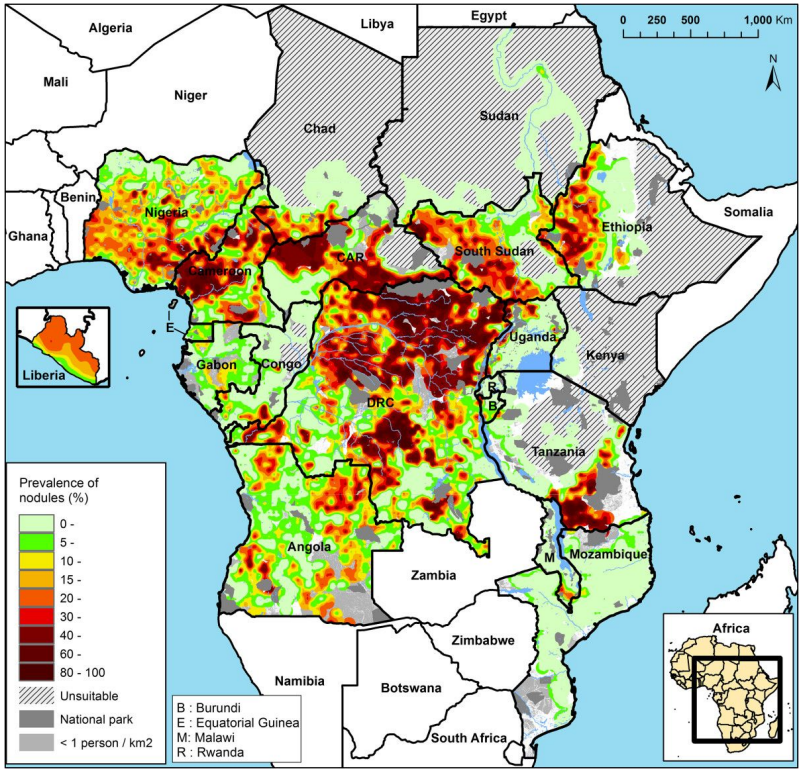
  4. $\log\{p(x_i)/[1 - p(x_i)]\} = d(x_i)^\top \beta + S(x_i) + Z_i$.

Onchocerciasis: the disease

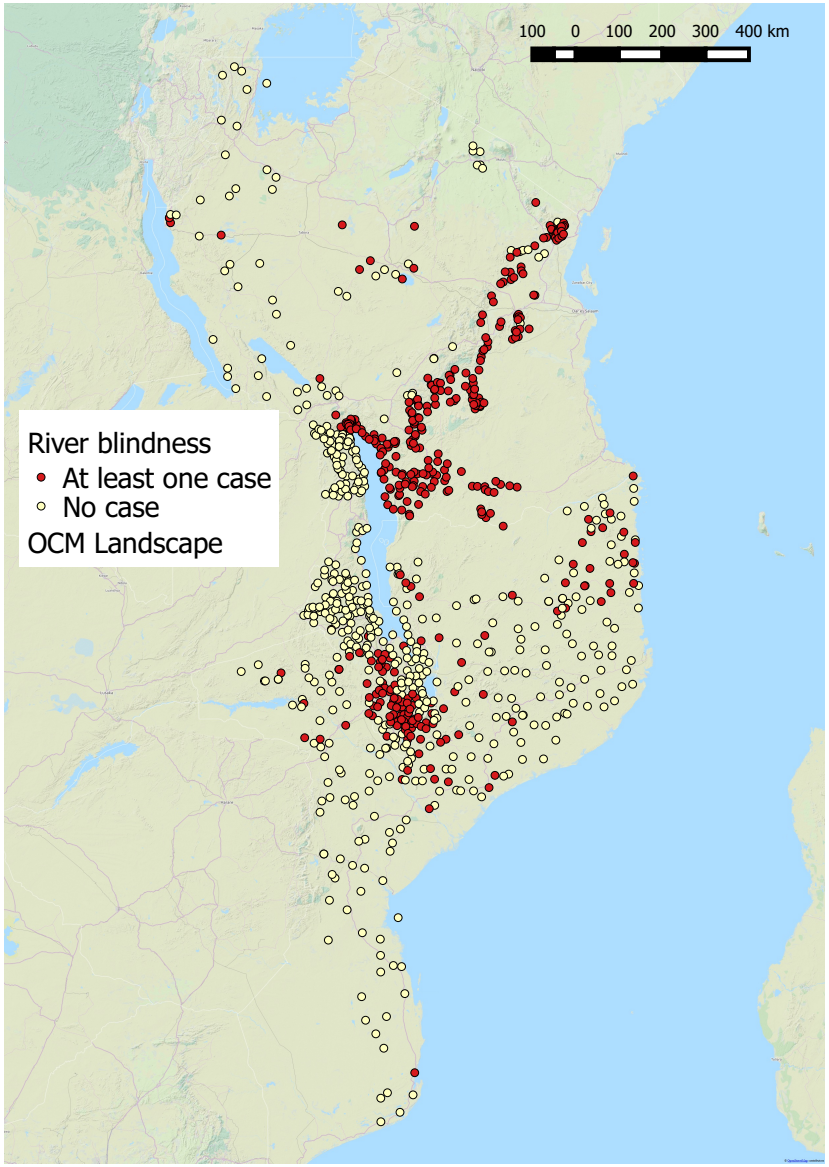




The REMO project



A closer look



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- How should we model $\pi(x)$?

What happens at the border?

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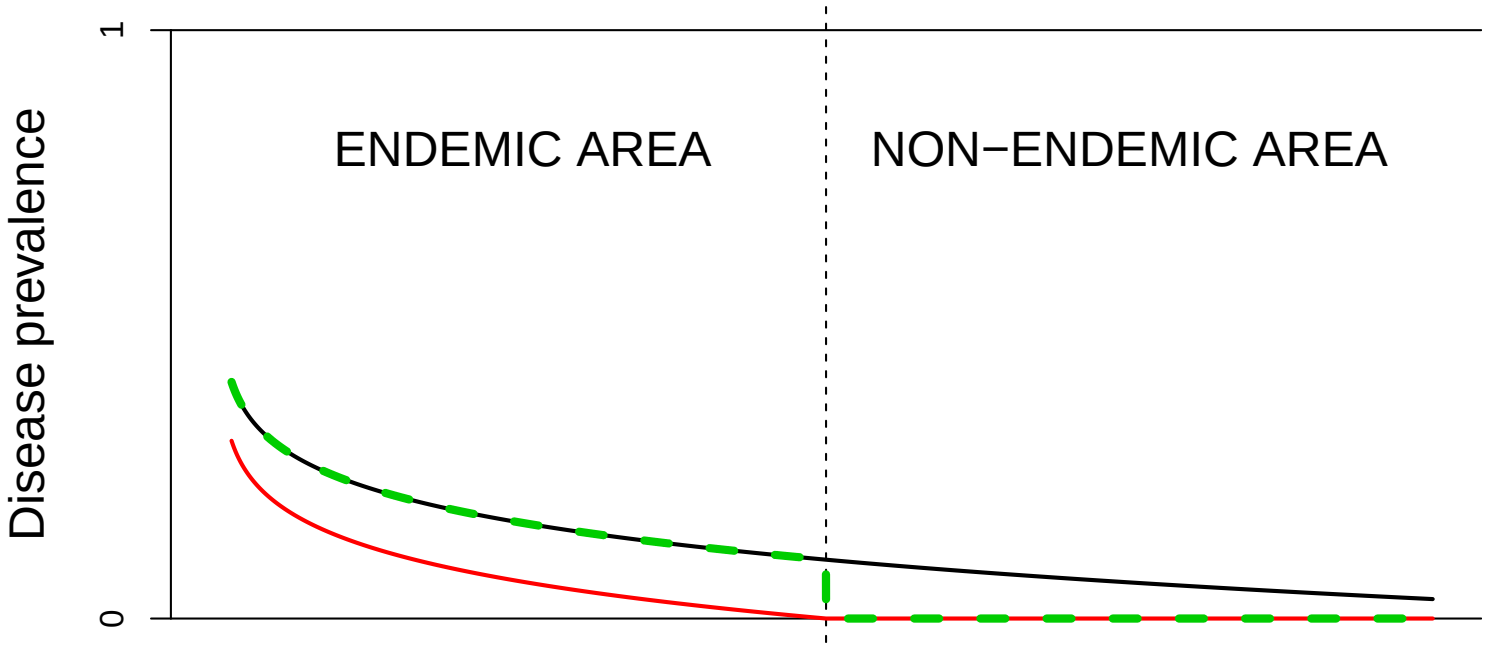
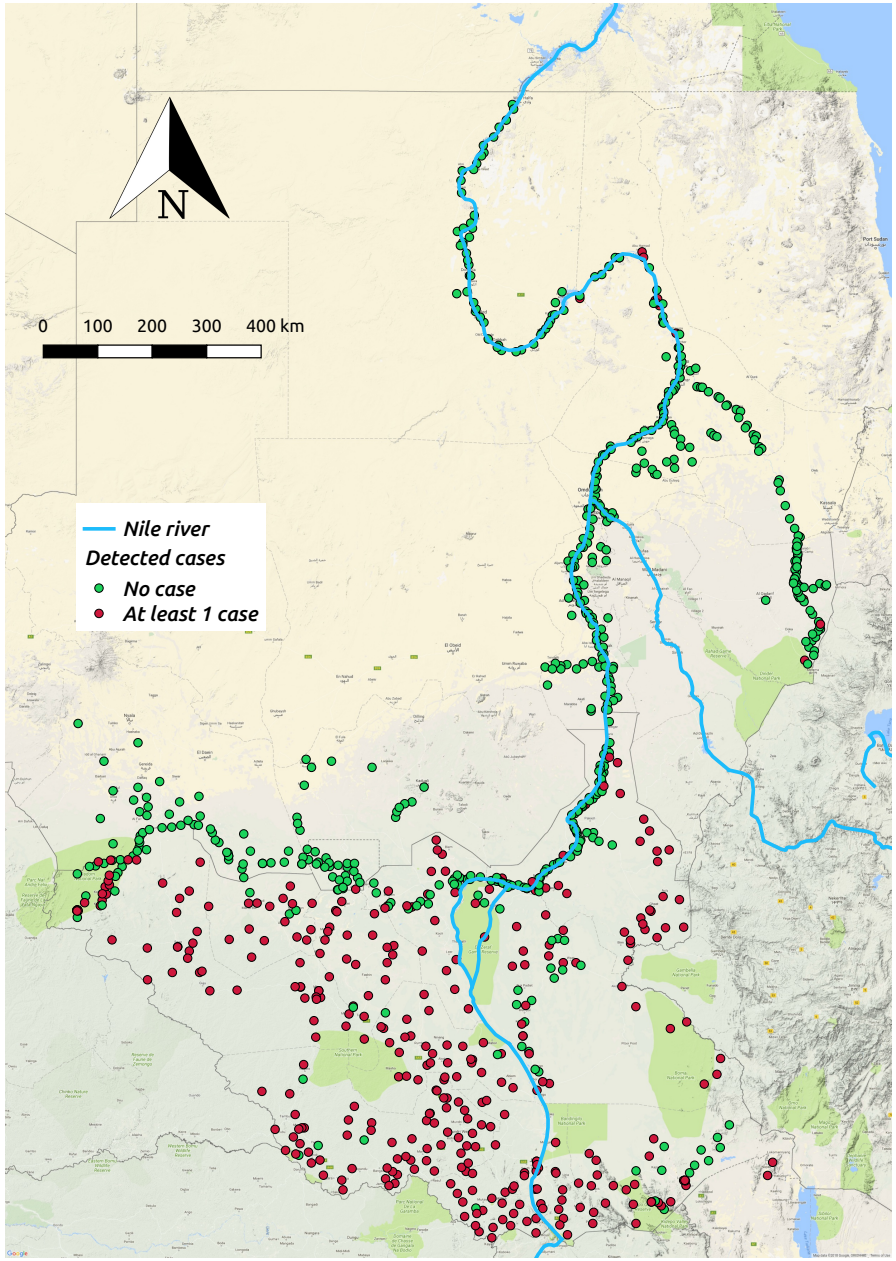


Onchocerciasis in Sudan and South Sudan



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$$\begin{aligned} \log\left\{\frac{\pi(x_i)}{1-\pi(x_i)}\right\} &= \gamma + T(x_i) \\ \mathrm{cov}\{T(x), T(x')\} &= \nu^2 \exp\{-\|x - x'\|/\delta\} \end{aligned}$$

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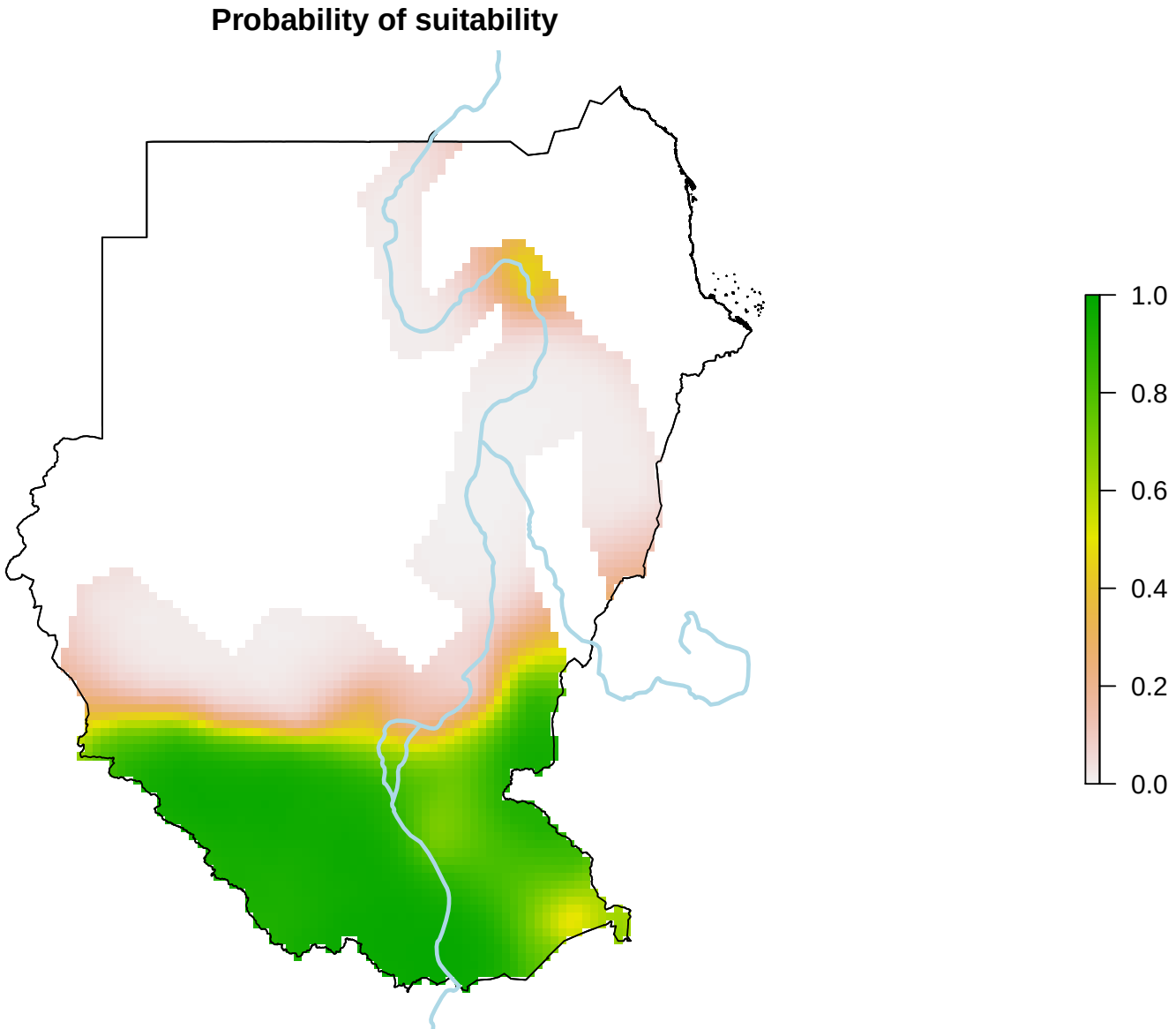
- Model for $Y_{2,i}$

$$\begin{aligned} \log\left\{\frac{p(x_i)}{1-p(x_i)}\right\} &= \beta + S(x_i) + Z_i, \\ \text{cov}\{S(x), S(x')\} &= \sigma^2 \exp\{-\|x - x'\|/\phi\} \end{aligned}$$

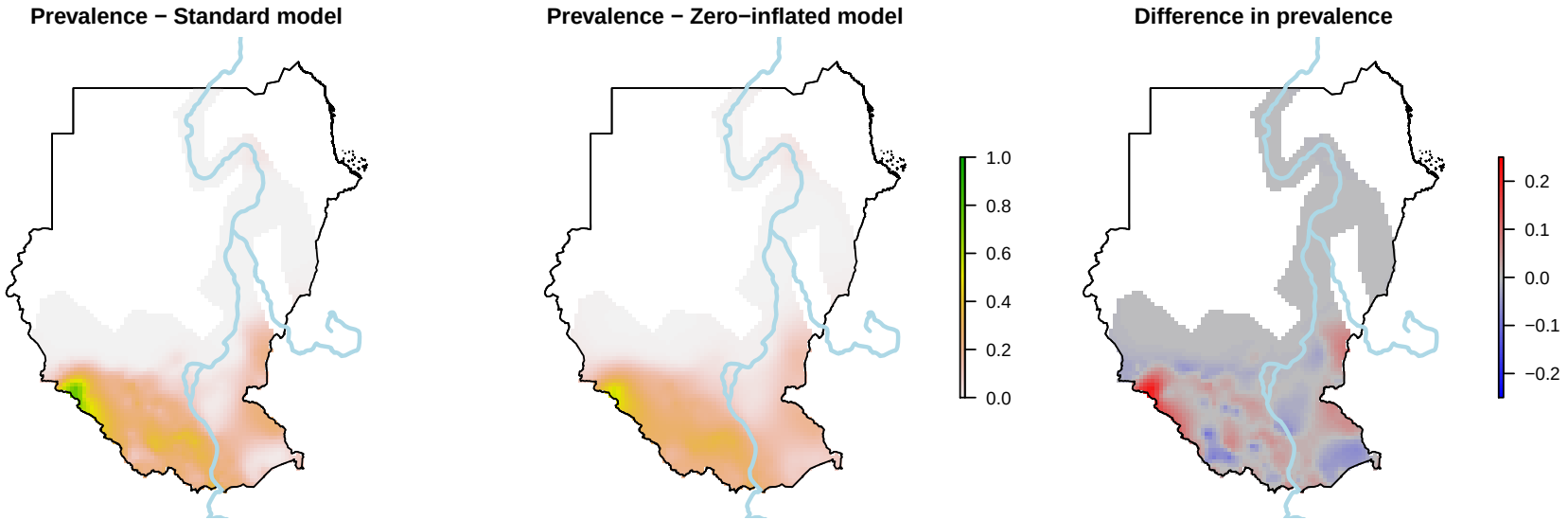
Monte Carlo maximum likelihood estimation

	Standard model		Zero-inflated model	
Parameter	Estimate	95% CI	Estimate	95% CI
β	-4.851	(-11.862, 2.160)	-1.819	(-2.618, -1.019)
$\log(\sigma^2)$	3.140	(1.413, 4.867)	-0.310	(-1.093, 0.474)
$\log(\phi)$	7.392	(5.647, 9.136)	5.769	(4.911, 6.627)
$\log(\tau^2/\sigma^2)$	-3.295	(-5.051, -1.539)	-0.812	(-1.600, -0.025)
γ			-1.561	(-9.526, 6.404)
$\log(\nu^2)$			3.217	(1.337, 5.097)
$\log(\delta)$			7.565	(5.697, 9.434)

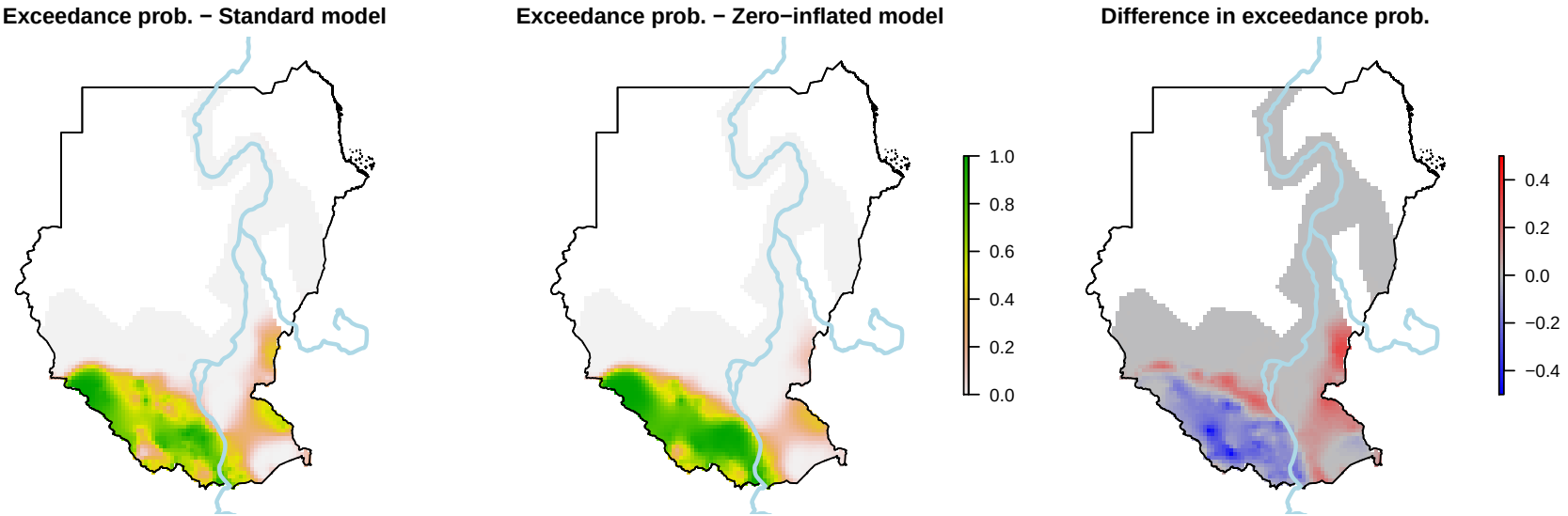
Mapping of $\pi(x)$



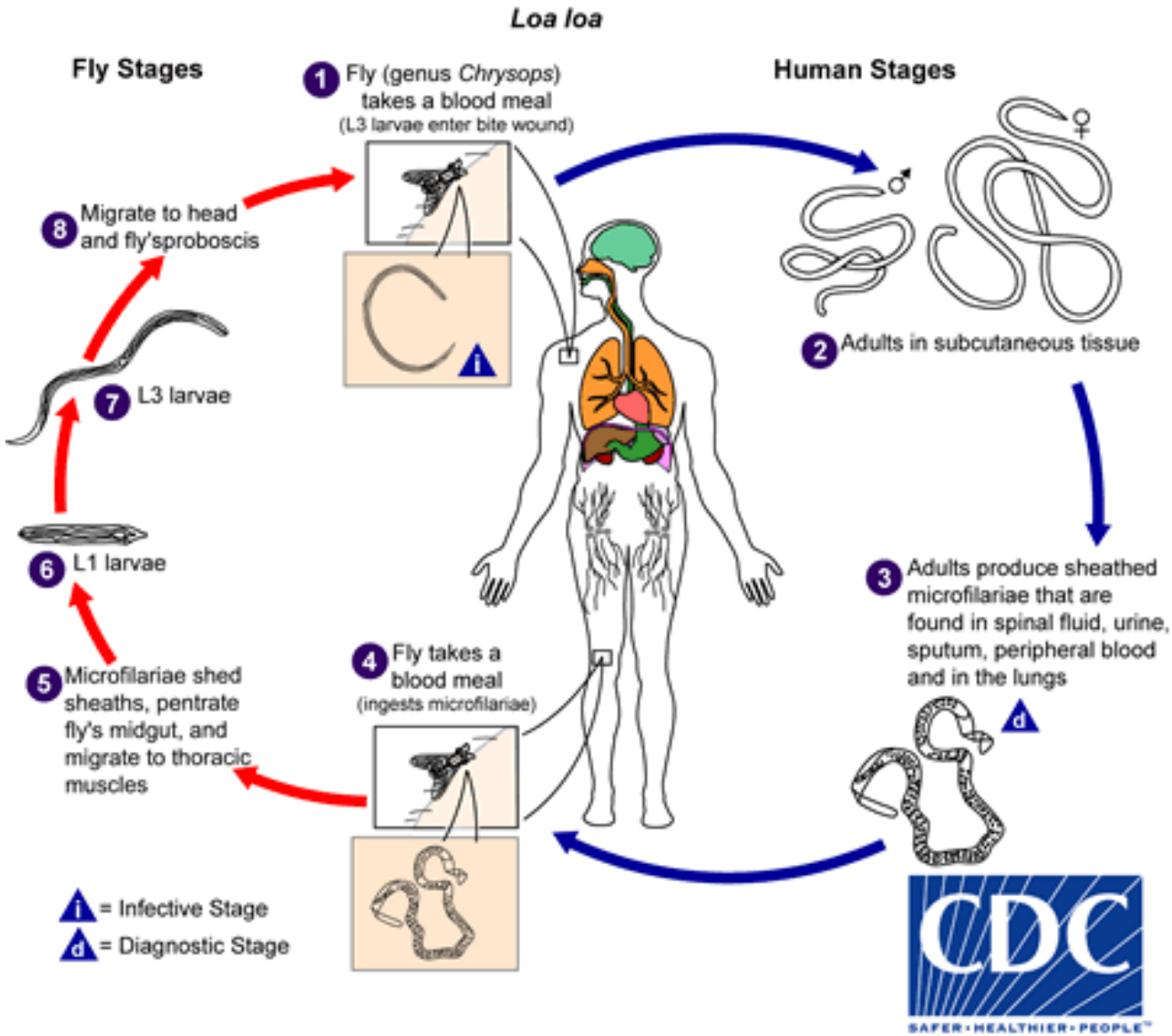
Mapping of oncho prevalence: $\pi(x)p(x)$



Mapping of exceedance probabilities: $P[\pi(x)p(x) > 0.2|y]$



Loa loa: the disease



The data

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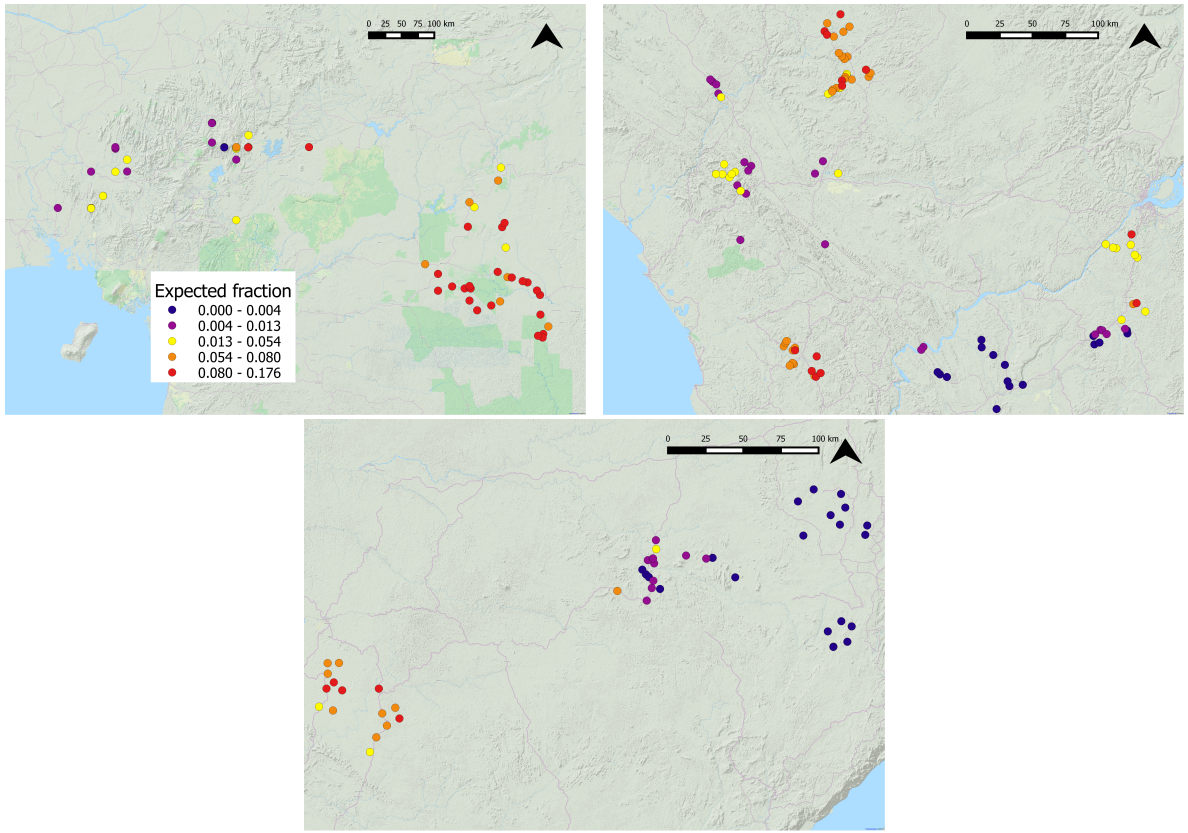
- **Outcome:** $Y_j(x_i)$, number of micro-filariae eggs per ml in a blood samples



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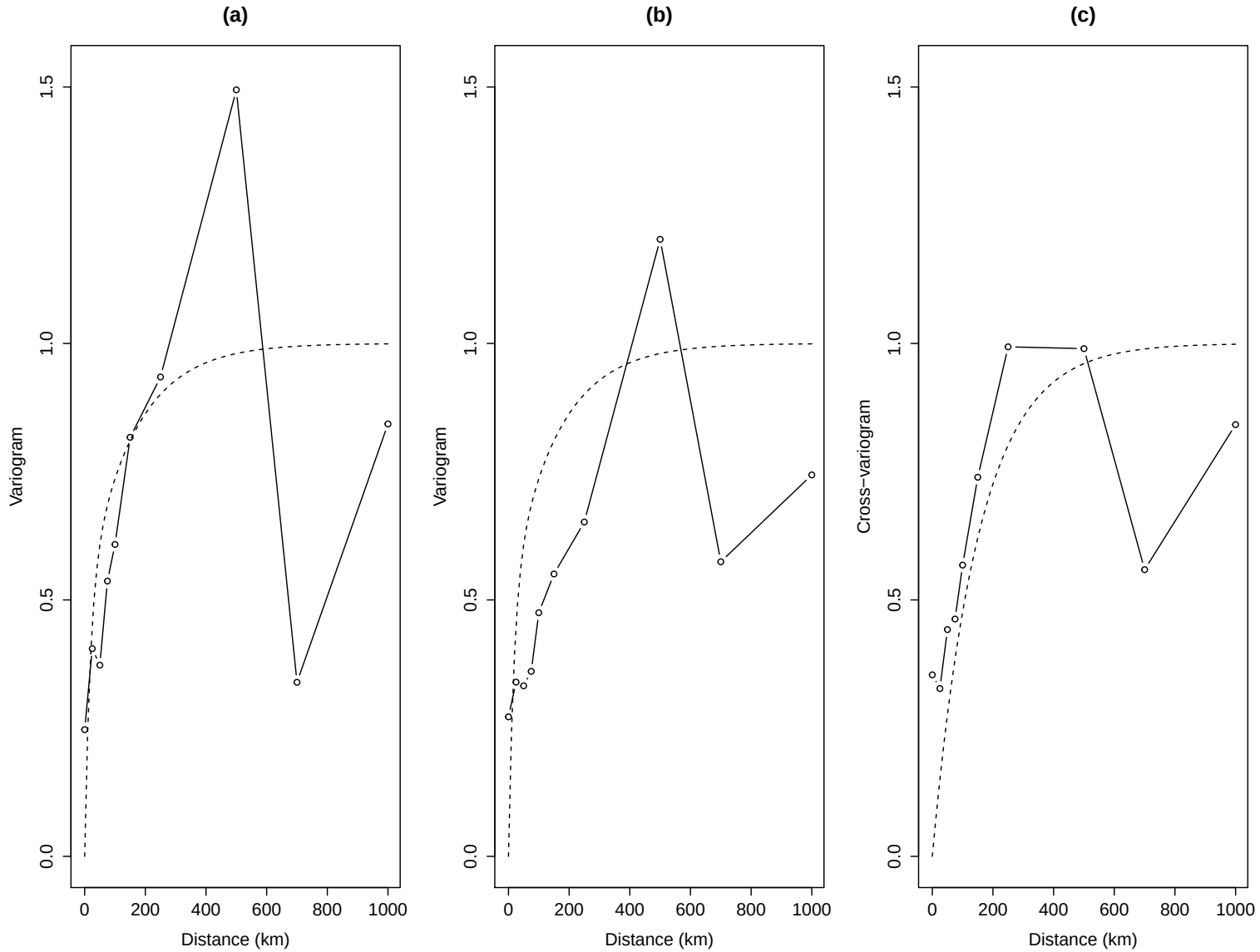
- The standardized cross-variogram:

$$\begin{aligned} \gamma_{12}(u) &= \frac{1}{2}E[(S_1(x) + T(x) - S_2(x') - T(x'))^2] \\ &= 1 - \exp\{-u/\phi_T\}. \end{aligned}$$

Monte Carlo maximum likelihood estimation

	Estimate	95% CI
μ_1	-2.187	(-2.230, -2.144)
μ_2	8.258	(8.190, 8.327)
σ_1^2	0.874	(0.663, 1.152)
σ_2^2	0.146	(0.111, 0.193)
ϕ_S	17.982	(13.012, 24.850)
ϕ_T	154.520	(72.402, 329.774)
κ	0.552	(0.537, 0.568)

Variogram check



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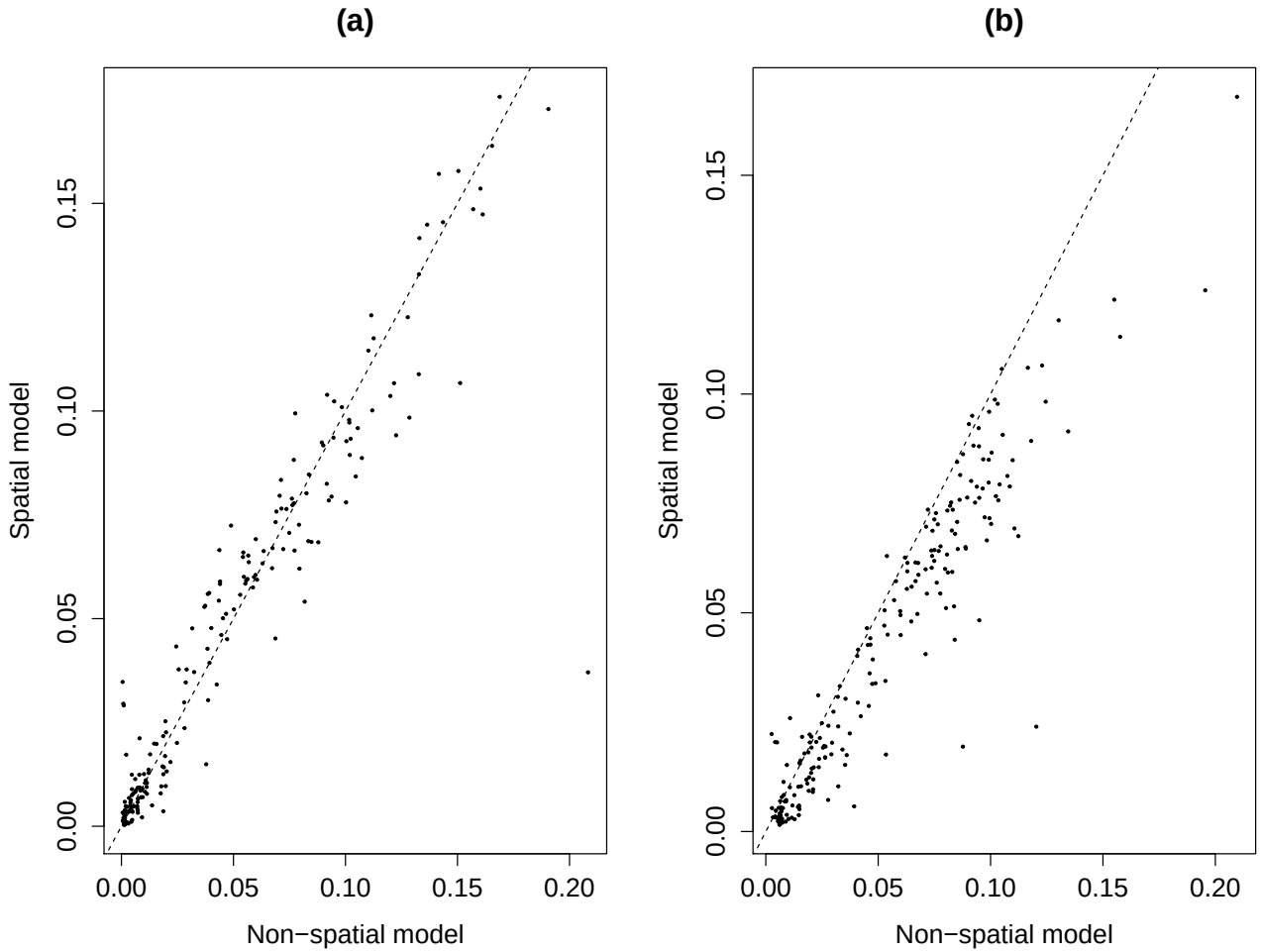

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Model	RMSE	PIL90	PIL95	PIL99	CP90	CP95	CP99
Spatial	0.013	0.033	0.040	0.053	0.895	0.947	0.989
Nonspatial	0.015	0.041	0.049	0.066	0.901	0.949	0.988

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p(x_i) &= 2\frac{\exp\{\alpha + S(x_i)\}}{1 + \exp\{\alpha + S(x_i)\}} - 1
\end{aligned}
$$

References

1. Diggle PJ, Giorgi E, (2019). Model-based Geostatistics for Global Public Health: Methods and Applications. Chapman & Hall/CRC.
2. Giorgi, E., Schl′uter, D. K., Diggle, P. J. (2017). Bivariate geostatistical modelling of the rela- tionship between Loa loa prevalence and intensity of infection. Environmetrics. doi:10.1002/env.2447
3. Diggle PJ, Giorgi E (2016). Model-based geostatistics for prevalence mapping in low-resource settings (with discussion). Journal of the American Statistical Association. 111:1096-1120