Time series analysis

Dr Emanuele Giorgi
Lancaster University
e.giorgi@lancaster.ac.uk

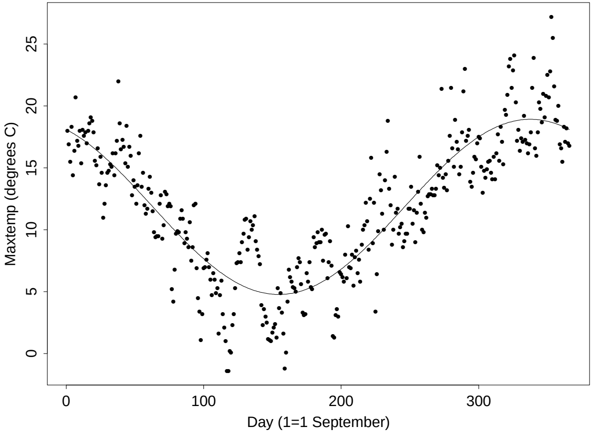
Toowoomba 21-25 October 2019

Overview

- Autocorrelation.
- Linear and harmonic regression models, detecting residual autocorrelation.
- Auto-regressive models for discrete time series models.
- Gaussian process models for continuous time series models.

A meteorological time series

- maximum daily temperatures (degrees C) at Bailrigg (Lancaster) field-station, September 1995 to August 1996
- note that an unusually cold Christmas 1995 was followed by a mild period in January-February



Bailrigg temperature data: points for discussion

- what are the main features of the data?
- how did I fit the smooth curve to the data?
- what features are and are not explained by the fitted curve?

A harmonic regression model

$$\begin{aligned} Y(t) &= \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual} \\ &= \mu + \beta_1 \cos(2\pi t/p) + \beta_2 \sin(2\pi t/p) + \text{residual} \end{aligned}$$

- $\mu =$ overall mean value (of time series $Y(t)$)
- $p =$ period
- $\alpha =$ amplitude
- $\phi =$ phase

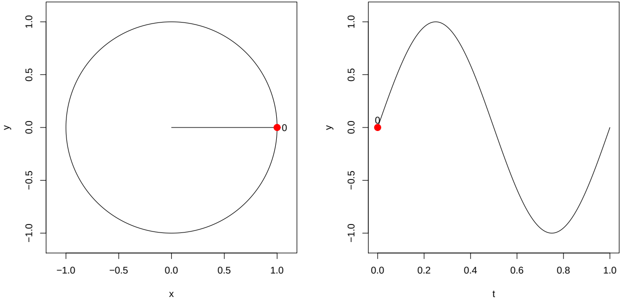
Usually, the period is known, but the mean, amplitude and phase are not

Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

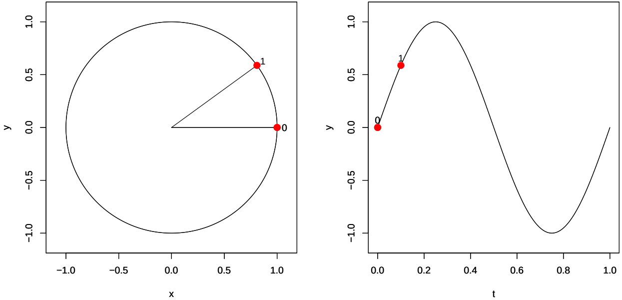


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

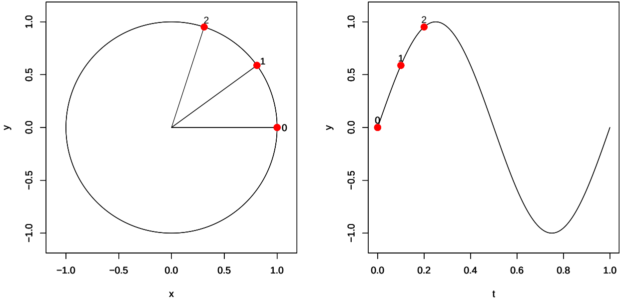


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

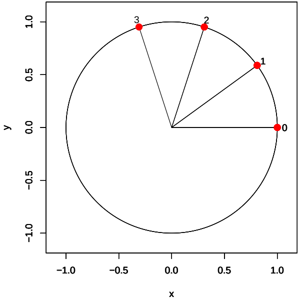


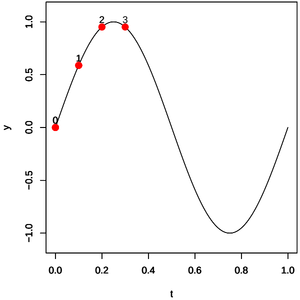
Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle



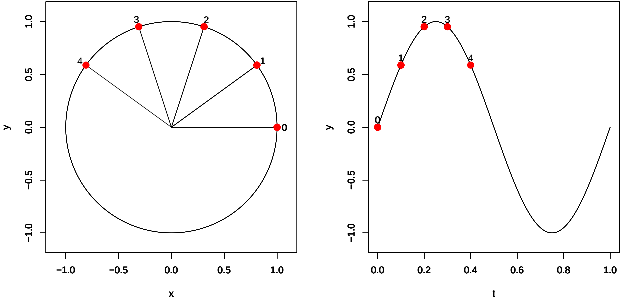


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

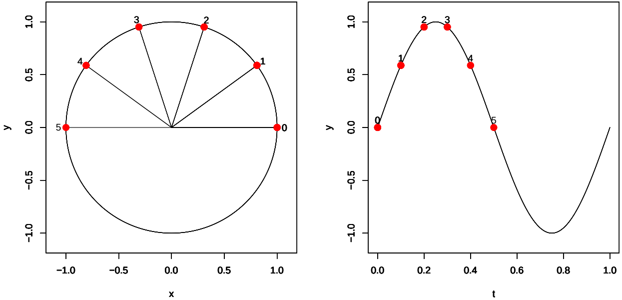


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

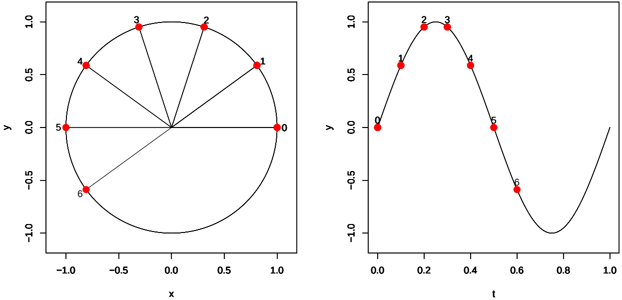


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

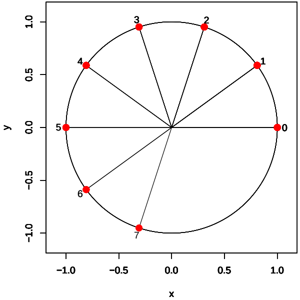


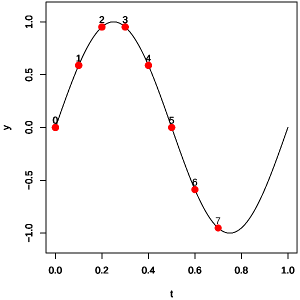
Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle



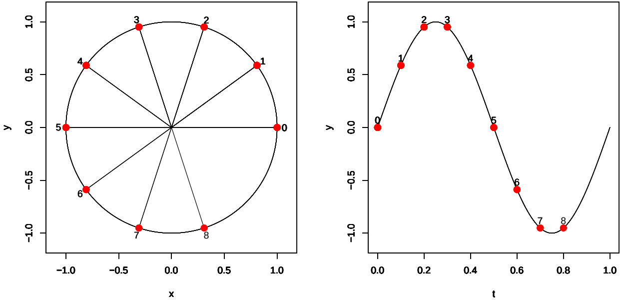


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle

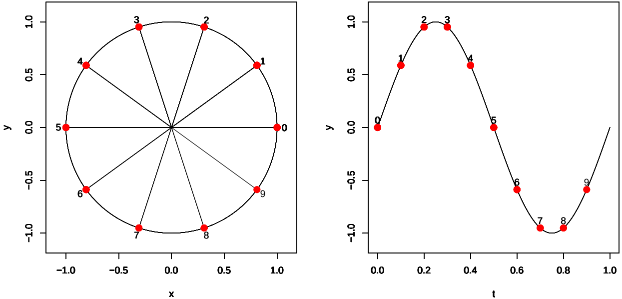


Why does the model work?

Use the first form of the model,

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

Now imagine tracking the vertical displacement of a particle moving at constant speed around a circle



Lifting, stretching, shifting

$$Y(t) = \mu + \alpha \cos(2\pi t/p + \phi) + \text{residual}$$

- Lifting

  μ: adjust to match observed and modelled average value

- Stretching

  α: adjust to match observed and modelled range

- Shifting

  ϕ: adjust to match times of observed and modelled peak

Fitting the model

Use the second form of the model,

$$Y(t) = \mu + \beta_1 \cos(2\pi t/p) + \beta_2 \sin(2\pi t/p) + \text{residual}$$

Note that the following quantities are known, i.e. they can be calculated without having to estimate anything

- $x_1(t) = \cos(2\pi t/p)$
- $x_2(t) = \sin(2\pi t/p)$

Re-write the model as a linear model,

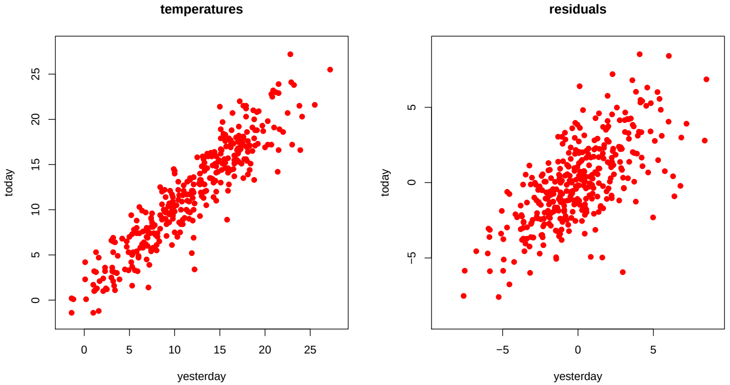
$$Y = \mu + \beta_1 x_1 + \beta_2 x_2$$

After fitting, amplitude and phase can be recovered using

$$\alpha = \sqrt{\beta_1^2 + \beta_2^2} \qquad \phi = \tan^{-1}(\beta_2/\beta_1)$$

Autocorrelation

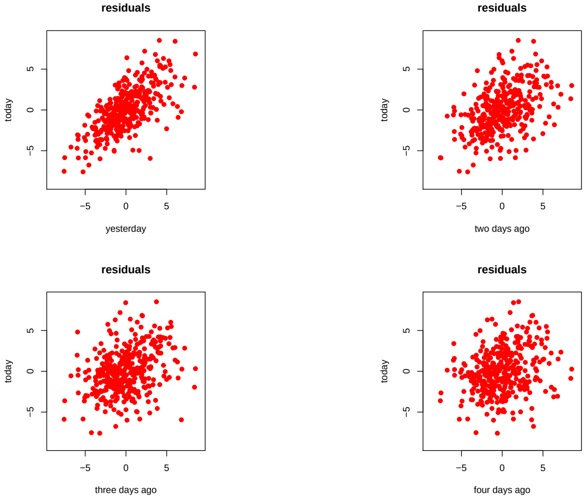
- relationship between today's and yesterday's temperature?
- relationship between today's and yesterday's residual?



- how and why are the two relationships different?

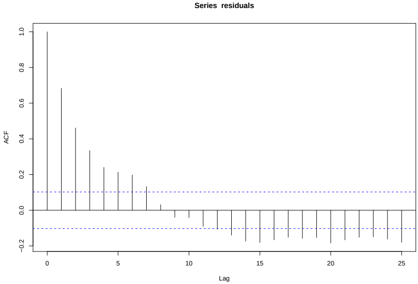
Autocorrelation (2)

How does the relationship between residuals today and k days ago change as k increases?



Autocorrelation (3)

- lag-k autocorrelation is the correlation between pairs of values from the same time series k time-units apart
- correlogram is a plot of lag-k autocorrelation against k



- dashed lines at $\pm 2/\sqrt{n}$ are pointwise 95% limits for uncorrelated residuals
- but overall pattern is more important than individual numerical values

Exercise

Imagine that you have data on daily maximum temperatures for several years, up to today.

1. How would make a forecast of:
   - tomorrow's temperature?
   - the temperature one month from now?
2. In what ways are your two answers different, and why?

Time series models and random effects

$$Y(t) = \alpha + \beta_1 \cos(2\pi t/p) + \beta_2 \sin(2\pi t/p) + \text{residual}(t)$$

1. Model consists of a time-varying mean, also called the trend,

$$\mu(t) = \alpha + \beta_1 \cos(2\pi t/p) + \beta_2 \sin(2\pi t/p)$$

and stochastic variation, residual(t), about the trend.

2. Decompose the residual into two terms:

$$\text{residual}(t) = S(t) + Z_t$$

- $\mathrm{Cov}\{S(t), S(t-u)\} = \sigma^2 \exp(-u/\phi)$ (random effect)
- Z_t uncorrelated $\mathrm{N}(0, \tau^2)$ (noise/measurement error)

Modelling autocorrelation structure

Time: is a continuous variable in nature, but typically treated as discrete in text-books on time series analysis.

Sampling or aggregation?

$$S(t) : t \geq 0 \quad \rightarrow Y_t = S(t) : t = 1, 2, ... \qquad \text{sampling}$$

$$\rightarrow Y_t = \int_t^{t+1} S(u) du : t = 1, 2, ... \qquad \text{aggregation}$$

Interpretation of the model parameters differs under the two scenarios – we will give an example later

Stationarity and the ARMA class of discrete-time processes

$$Y_t : t = 0, 1, ...$$

- To keep things simple, assume:
  - any non-zero mean has been removed, so $\mathrm{E}[Y_t] = 0$
  - Y_t is Normally distributed (this is less important)
- Stationarity: $\mathrm{Var}(Y_t) = \sigma^2 \qquad \mathrm{Corr}\{Y_t, Y_{t-u}) = \rho(u)$
- The $\mathrm{ARMA}(p, q)$ model

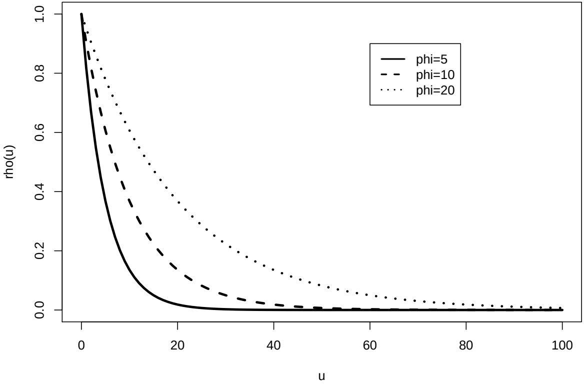
$$Y_t = \sum_{i=1}^{p} \alpha_i Y_{t-i} + \sum_{j=0}^{q} Z_{t-q}$$

where $Z_t \sim \mathrm{N}(0, \tau^2)$ are mutually independent (white noise)

Example: an exponentially correlated process

Notation: $S(t) \sim \mathrm{SGP}(\mu, \sigma^2, \rho(u))$

One of many possible models for the correlation structure is
$\rho(u) = \exp(-|u|/\phi) : \phi > 0$



An exponentially correlated process (continued)

$$S(t) \sim \mathrm{SGP}(\mu, \sigma^2, \rho(u)) \qquad \rho(u) = \exp(-|u|/\phi)$$

Discrete-time counterparts?

- Sampled version

$$Y_t = S(t) : t = 1, 2, ...$$

equivalent to AR(1)

$$Y_t = \alpha Y_{t-1} + Z_t \qquad \alpha = \exp(-1/\phi)$$

- Aggregated version

$$Y_t = \int_t^{t+1} S(u) du$$

equivalent to ARMA(1,1)

$$Y_t = \alpha Y_{t-1} + Z_t + \beta Z_{t-1} \qquad \alpha = \exp(-1/\phi)$$

Example: modelling reported plague cases in Madagascar from 2000 to 2008