Dealing with over-dispersion using generalized linear mixed models

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Pre-requisites

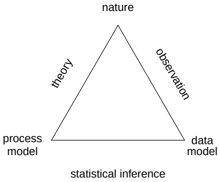
- Good knowledge of generalized linear models
- Basic knowledge of R
- Notions of probability calculus (e.g. conditional distribution and expectation).
- Basic mastering of mathematical equations

Learning outcomes of the workshop

You should be able:

- to understand the limitations of generalized linear models;
- to test for the presence of spatial correlation using variogram-based techniques;
- to formulate a suitable geostatistical model for data-analysis;
- to understand and correctly interpret the results from a geostatistical analysis;
- to fit generalized linear geostatistical models and carry out spatial prediction using `PrevMap` in R.

Science and statistics



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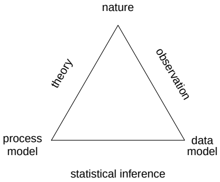

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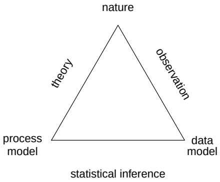

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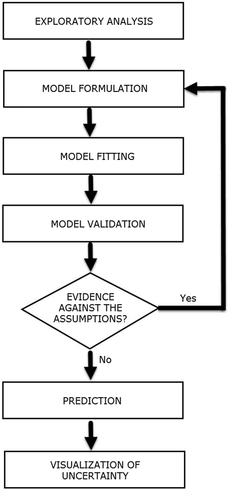
$$[Y, S] = [S][Y|S]$$

Statistical analysis



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- Under the assumptions of classical GLMs, we can ignore $[S]$.

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2. *Random effects models.* $S = d^\top \beta + Z$, where Z is a stochastic process.

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Are the Y_i mutually independent?

- **Examples:** 1) $Y_i|Z_i \sim \text{Poisson}(e^{d_i^\top \beta + Z_i})$ and $Z_i \sim N(-\tau^2/2, \tau^2)$ i.i.d.; $E[Y_i] =$ .... and $Var[Y_i] =$ .... (Hint: Use the law of total expectation and variance)
  2) $Y_i|Z_i \sim \text{Poisson}(e^{d_i^\top \beta + Z_i})$, $e^{Z_i} \sim \text{Gamma}(k, k)$ i.i.d.; show that Y_i is a Negative Binomial distribution.

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  1. Obtain $\hat{\theta}$ (MLE).
  2. Obtain $\hat{\theta}_0$, the MLE constrained by fixing p values of β to 0.
  3. Compute the log-likelihood ratio

$$D = 2(logL(\hat{\theta}) - logL(\hat{\theta}_0)) \sim \chi^2_p$$

  4. P-value: $P(D > D_{obs}|H_0)$

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- Y_i="number of positively tested individuals for river-blindness out of n_i.
- d_i="elevation of the i-th village"
- **Question:** How should we formulate and estimate a model to understand the association between d_i and the probability of being positive for river-blindness, p_i?