Institut für Betriebssysteme und Rechnerverbund
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Masterarbeit

Computational Aspects of MaxMin Triangulations

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for my daughter—in the hope that she will understand sometime

Erklärung

Ich versichere, die vorliegende Arbeit selbstständig und nur unter Benutzung der angegebenen Hilfsmittel angefertigt zu haben. Bei den Experimenten sind keine unbeteiligten Dreiecke zu Schaden gekommen.

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Braunschweig, den 1. August 2013

Abstract

This thesis studies the MaxMin Length Triangulation (MMLT) problem which has recently be proven to be NP-hard, which implies that there is no polynomial time algorithm for finding an optimal MMLT solution unless P=NP. We verify that instances with randomly distributed points can be solved in polynomial time on average, however.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit dem MaxMin Length Triangulation (MMLT) Problem, von dem kürzlich bewiesen wurde, dass es NP-schwer ist. Folglich gib es keinen Algorithmus, der immer eine optimale MMLT Lösung in polynomieller Laufzeit findet, es sei denn P=NP. Wir überprüfen, dass Instanzen mit zufällig verteilten Punkten dennoch im Durchschnitt in polynomieller Zeit gelöst werden können.

Contents

1. Introduction

Triangulations, that is subdividing the plane (or a polygon) into triangles, are a popular topic in computational geometry — not only because of their connections to other problems but also due to their practical applications. They can be helpful as a preprocessing in other algorithms or as a tool in geometric proofs. One popular example is the artgallery problem [22]. Another area where triangulations are widely used is mesh generation and approximation of complex geometric structures. [2]

For different use cases the objectives for a triangulation vary. For example, the widely known Delaunay Triangulation [3, Section 9.2] tends to avoid “skinny” triangles and is therefore useful for meshes. One can imagine many different properties to be optimized: Edge lengths, triangle area, inner angles, and degree of a triangulation vertex are some of them. Many have already been looked into, but for some of them no application is known by now—so they remain theoretical problems. In chapter 3 we will have a brief overview of different kinds of triangulations.

This thesis will focus on the MaxMin Length Triangulation (MMLT) problem introduced in chapter 4. Stated an open problem in 1991 [19], it has been proven to be NP-complete in 2012 [21]. This implies that the worst case running time for an algorithm solving MMLT is more than polynomial unless P≠NP. However our assumption is that the worst case instances (for example those constructed in [21]) are rare+ and that random instances can be solved in polynomial time on average.

In chapter 4 we examine some properties of the MMLT problem and develop an algorithmic idea. The key observation, which we were not able to prove, is that the shortest line segment having endpoints in a given point set, and which is not crossed by a longer segment is likely to be short. Our experimental results in chapter 6 confirm this assumption and lead to a sub-linear bound for the index of such a line segment in a set sorted by length.

We implemented the algorithm presented in chapter 4 to compare its running time to natural approaches such as formulating MMLT as a standardized optimization problem and passing it to a solver. The components of our program are described in chapter 5. For the geometric part of the problem, we made use of the well established Computational Geometry Algorithms Library (CGAL) [6].

To ease approaching the combinatorial aspects of the MMLT problem and to agree on necessary definitions, we cover some of the basics in chapter 2. This allows us to come up with a solely combinatorial definition of Triangulation problems in chapter 3, which we will make extensive use of later within this thesis. In fact, abstracting geometric aspects of Triangulation problems grants more flexibility, e.g. when transferring approaches from the plane to higher dimensions. However, this generalization comes to the prize of higher computational complexity.

2. Integer Programming

This chapter focuses on some preliminaries around integer programming—which is a subarea of (mathematical) optimization. In that context we introduce three common optimization problems which can be solved using integer programming: SAT, Vertex Cover, and Independent Set. They are discussed later within this thesis (chapters 3 and 4) in association with the MaxMin Length Triangulation (MMLT) problem.

An integer program is basically a problem description as a system of (in-)equations involving only variables with integer values. Additionally, there can be an objective function for modeling optimization problems. Throughout this thesis, we assume that every integer program has only linear (in-)equations. This is referred to as a *linear integer program.*

Definition 2.1 ((Linear) Integer Program (IP))
An IP is a system of integer variables $x \in \mathbb{Z}^n (n \in \mathbb{N})$ with a set of constraints on them and optionally an objective function. We consider only the case where the constraints are linear with respect to x.

A natural standardized way of formulating (linear) IPs is the so called *canonical form.* It combines the coefficients of the variable vector $x \in \mathbb{Z}^n (n \in \mathbb{N})$ for all $m \in \mathbb{N}$ (in-)equations in a matrix $A \in \mathbb{Z}^{m \times n}$, all constant terms are summed up to a vector $b \in \mathbb{Z}^m$, and the objective function is represented as a multiplication of the variables x and a constant target vector $c \in \mathbb{Z}^n$. For simplicity, we slightly modify the canonical form within this thesis, e.g. by allowing sums.

Problem 2.2 (IP in canonical form [41])

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n \end{aligned} \qquad \text{or} \qquad \begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \leq 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

($\leq$ and $\geq$ here denote the row-wise comparison of two vectors)

Even though the structure of IPs looks very simple, it is a well known fact that solving them is not easy. Nevertheless they are useful for various optimization problems—especially because there are many practical applications involving only integers. A common strategy to simplify the problem is leaving out the integrity restriction for retrieving bounds of the optimal solution.

Theorem 2.3
Solving IPs is NP-hard.

Proof:
Even the special case where there is no objective function, only binary variables, and only equality constraints is NP-complete [32]. Therefore the more general problem is NP-hard.

2.1. SAT

The (boolean) satisfiability problem (SAT) asks for a variable assignment which fulfills a logical term. It was the first problem to be proven NP-complete [12] and has since been a common choice for NP-hardness proofs. Many times, even more restricted such as the 3-SAT problem which allows only for three variables in each "sub-term" (clause).

Problem 2.4 (3-SAT)

Given: Set of boolean literals X, a formula in conjunctive normal form consisting of clauses C each involving exactly three literals (or their negations)

Sought: An assignment for X which lets all clauses in C evaluate to "true"

Shortly after the concept of NP-completeness was introduced in [12], Richard Karp applied the idea to several other well-known problems [32]. One of them was the 3-SAT problem which has more structure than the general boolean satisfiability problem and is therefore easier to embed into constructive NP-hardness or NP-completeness proofs.

Theorem 2.5 (NP-completeness of 3-SAT)
Problem 2.4 is NP-complete. [32, Satisfiability with at most 3 Literals per Clause]

As mentioned earlier, the 3-SAT problem can be formulated as an IP. This easily confirms the NP-hardness of solving IPs. Additionally it allows for applying IP solvers (such as CPLEX, see also section 5.2.4) to 3-SAT problems. Usually however, SAT solvers are used, which take advantage of the problem structure better and therefore, have less overhead.

Problem 2.6 (IP formulation of 3-SAT)

$$\begin{aligned} &(\text{minimize} && 0^T x) \\ &\text{subject to} && \forall c \in C : \sum_{x_i \in c} x_i + \sum_{\neg x_i \in c} (1 - x_i) = 1 \\ & && \forall x_i \in X : \qquad\qquad\qquad x_i \in \{0, 1\} \end{aligned}$$

($x_i \in c$ (or $\neg x_i \in c$) herein denotes that x_i (or $\neg x_i$) is part of the clause c)

For geometric and graph problems, an even more restricted class of 3-SAT problem has been established, called the *Planar 3-SAT*. It asks for the connection graph of all clauses and their variables to be embeddable in the plane.

Problem 2.7 (Planar 3-SAT)
An instance of the 3-SAT problem with literals X and clauses C which can be represented by a planar graph $G = (V, E)$ such that

$$\begin{aligned} V &= \{v : v \in X \cup C\} \\ E &= \{\{x, c\} : x \in X,\ c \in C,\ (x \in c) \vee (\neg x \in c)\} \end{aligned}$$

Figure 2.1 shows an example of a 3-SAT instance with a planar variable-clause-connection graph. There is a vertex for every clause (mint colored squares) and every variable (red circles), and edges in between if a variable (or its negation) is part of a clause.

It was shown that even this restricted class of satisfiability problems is NP-complete. This way problems can be proven to be NP-hard in the plane without having to deal with intersections. This result can then be used for higher dimensions (usually, problems become harder to solve if they are elevated to higher dimensions).

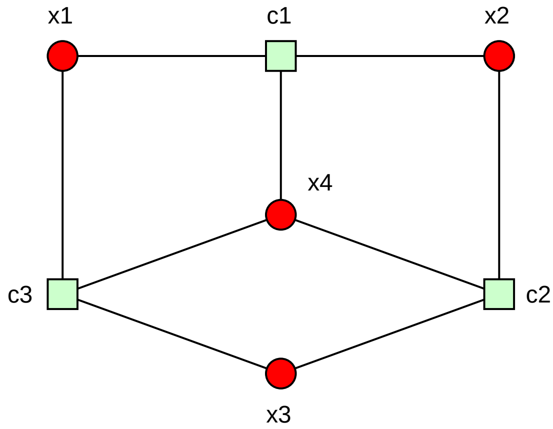


Figure 2.1.: Example of a Planar 3SAT instance which represents the term $\underbrace{(x_1 \lor x_2 \lor x_4)}_{c_1} \land \underbrace{(\neg x_2 \lor x_3 \lor \neg x_4)}_{c_2} \land \underbrace{(x_1 \lor \neg x_3 \lor \neg x_4)}_{c_3}$

Theorem 2.8 (NP-completeness of Planar 3-SAT)
Problem 2.7 is still NP-complete. [34]

2.2. Vertex Cover

Another problem in the collection of [32] is *Vertex Cover*, which aims to cover (all) edges of a graph by their incident vertices. It is closely related to other graph problems such as Edge Cover, Clique, Independent Set (see section 2.3), Coloring, and the more general Set Cover.

Definition 2.9 (Vertex Cover)
Given an undirected graph $G = (V, E)$, a Vertex Cover $V_{\text{cover}} \subseteq V$ for G is a vertex set such that every edge $e \in E$ is incident to at least one vertex $v \in V_{\text{cover}}$:

$$\forall e \in E : \exists v \in V_{\text{cover}} : v \in e$$

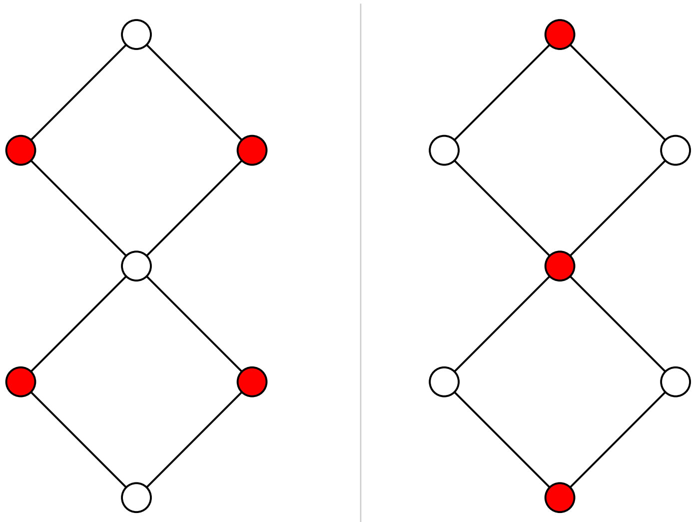
A trivial Vertex Cover of any graph $G = (V, E)$ would be the complete vertex set V. In general, such a Vertex Cover is not useful but one asks for the covering vertex set to be minimal (roughly speaking without unnecessary vertices).

Definition 2.10 (Minimal Vertex Cover)
A Vertex Cover V_{cover} for an undirected graph $G = (V, E)$ is minimal if there is no vertex $v \in V_{\text{cover}}$ such that $V_{\text{cover}} \setminus \{v\}$ remains a Vertex Cover.

Still, *any* Minimal Vertex Cover is not always wanted as it needs not be unique. Hence, there can be a much smaller covering vertex set. An extreme example is a star-shaped graph where one vertex can cover all edges but the set of all other vertices would also be minimal. Therefore we distinguish between a Minimal Vertex Cover and a *Minimum* Vertex Cover; i.e. a smallest possible one. Figure 2.2 shows the difference between Minimal and Minimum Vertex Cover by example.

Definition 2.11 (Minimum Vertex Cover)
A Vertex Cover V_{cover} for an undirected graph $G = (V, E)$ is minimum if there is no other Vertex Cover V'_{cover} for G which has fewer vertices:

$$\forall\ V'_{\text{cover}} \text{ Vertex Cover} : |V_{\text{cover}}| \leq |V'_{\text{cover}}|$$



(a) Minimal (b) Minimum

Figure 2.2.: Example of Minimal vs. Minimum Vertex Cover, covering vertices in red

The IP formulation of Minimum Vertex Cover can be directly deduced from the definition: We minimize the number of covering vertices such that for every edge at least one vertex is selected. The optimal vertex set V_{cover} can then be retrieved from the IP solution vector by just collecting all selected vertices: $V_{\text{cover}} = \{v \in V : x_v = 1\}$

Problem 2.12 (IP Formulation of Minimum Vertex Cover)

$$\begin{aligned} \text{minimize} \quad & \sum_{v \in V} x_v \\ \text{subject to } \forall \{v, w\} \in E : \quad & x_v + x_w \geq 1 \\ \forall v \in V : \quad & x_v \in \{0, 1\} \end{aligned}$$

What follows is that the Minimum Vertex Cover is also NP-complete because it is a subset of the Set Cover problem [32] and clearly lies in NP as it can be (polynomially) reduced to solving IPs.

Theorem 2.13 (NP-completeness of Minimum Vertex Cover)
Minimum Vertex Cover is NP-complete. [32]

We now introduce the class of *well-covered* graphs which we will need in chapter 3. They combine all graphs for which there is no difference between Minimal and Minimum Vertex Cover. Minimal Vertex Covers can be calculated with significantly less effort (i.e. in polynomial time) than Minimum Vertex Covers (e.g. by just iteratively removing vertices from the set until it would destroy the covering property). This leads to the Minimum Vertex Cover being solvable in polynomial time for *well-covered* graphs.

A subset of *well-covered* graphs are connected bipartite graphs $G = (A \cup B, E \subseteq A \times B)$ with both vertex sets having the same size: $|A| = |B|$. Either one of A and B is a Minimal and a Minimum Vertex Cover for G. Clearly this graph class is not empty which already implies that the class of *well-covered* graph is not empty either.

Definition 2.14 (*Well-covered* Graph)
An undirected graph $G = (V, E)$ is *well-covered* if and only if every Minimal Vertex Cover for G is also a Minimum Vertex Cover for G. [42]

As a logical consequence of definition 2.14, every two vertex sets that are a Minimal Vertex Cover for a *well-covered* graph G have the same size. If one of them had more vertices than the other one, it could not have been a Minimum Vertex Cover for G, which contradicts G being *well-covered*.

Theorem 2.15
In a *well-covered* graph, all Minimal Vertex Covers have the same cardinality. [42]

2.3. Independent Set

Another prominent graph problem is the *Independent Set*, which asks for a vertex set that is "independent", meaning that no two vertices are adjacent. It can be seen as the counter part of Vertex Cover (see section 2.2) because it requires *at most* one vertex for every edge, while Vertex Cover demands *at least* one vertex for every edge. We consider this connection in theorem 2.19.

Definition 2.16 (Independent Set)
Given an undirected graph $G = (V, E)$, an independent set $V_{\text{IS}} \subseteq V$ is a vertex set such that no two vertices $v, w \in V_{\text{IS}}$ are incident to the same edge $\{v, w\} \in E$:

$$\forall v \in V_{\text{IS}} : \forall \{v, w\} \in E : w \notin V_{\text{IS}}$$

Again we can find a trivial Independent Set for every graph in constant time: the empty set. However, similar to Vertex Cover one usually asks for a "large" Independent Set as defined below:

Definition 2.17 (Maximal/Maximum Independent Set)
For an undirected graph $G = (V, E)$, an independent set $V_{\text{IS}} \subseteq V$ is maximal if there is no vertex $v \in V \setminus V_{\text{IS}}$ such that $V_{\text{IS}} \cup \{v\}$ remains an independent set. It is maximum if there is no independent set V_{IS}' with larger cardinality.

The IP for Maximum Independent Set maximizes the number of selected vertices, such that for every edge only one of the incident vertices is selected. Analogous to the IP formulation of Minimum Vertex Cover, the optimal V_{IS} solution can directly be retrieved from the IP solution: $IP = \{v \in V : x_v = 1\}$

Problem 2.18 (IP Formulation of Maximum Independent Set)

$$\begin{aligned}
\text{maximize} \quad & \sum_{v \in V} x_v \\
\text{subject to } \forall \{v, w\} \in E : \quad & x_v + x_w \leq 1 \\
\forall v \in V : \quad & x_v \in \{0, 1\}
\end{aligned}$$

We already mentioned the close connection between Vertex Cover and Independent Set earlier. Note that in fig. 2.2 the Minimal Vertex Cover is a Maximum Independent Set and the Minimum Vertex Cover a Maximal Independent Set. From the IP formulations it can also been seen that the problems behave the opposite way:

Theorem 2.19 (Independent Set and Vertex Cover)
For an undirected graph $G = (V, E)$, $V_{\text{IS}} \subseteq V$ is a Maximum Independent Set if and only if $V_{\text{cover}} = V \setminus V_{\text{IS}}$ is a Minimum Vertex Cover.

Proof:
Let V_{cover} be a Vertex Cover for G.

$$\begin{aligned}
\forall e \in E : \exists v \in V_{\text{cover}} : v \in e \iff & \forall \{v, w\} \in E : v \in V_{\text{cover}} \vee w \in V_{\text{cover}} \\
\iff & \forall \{v, w\} \in E : \neg(v \notin V_{\text{cover}} \wedge w \notin V_{\text{cover}}) \\
\iff & \forall \{v, w\} \in E : \neg(v \in (V \setminus V_{\text{cover}}) \wedge w \in (V \setminus V_{\text{cover}})) \\
\iff & \forall v \in (V \setminus V_{\text{cover}}) : \forall \{v, w\} \in E : w \notin (V \setminus V_{\text{cover}}) \\
\iff & (V \setminus V_{\text{cover}}) \text{ independent set}
\end{aligned}$$

Assume V_{cover} is a Minimum Vertex Cover for G and the independent set $V_{\text{IS}} = V \setminus V_{\text{cover}}$ is not maximum. Then there is an independent $V'_{\text{IS}} \subseteq V$ with $|V_{\text{IS}}| < |V'_{\text{IS}}|$. But then for the Vertex Cover $V'_{\text{cover}} = V \setminus V'_{\text{IS}}$ the following holds: $|V'_{\text{cover}}| < |V_{\text{cover}}|$—which is a contradiction to V_{cover} being minimum. The same argumentation applies in the other direction.

Now that the link between both problems is proven, we can apply properties of the Vertex Cover problem to Independent Sets. Firstly, it follows that the Maximum Independent Set problem is NP-complete as well because every solution can be transformed into a Minimum Vertex Cover and vice versa in polynomial time.

Theorem 2.20 (NP-Completeness of Maximum Independent Set)
Theorems 2.13 and 2.19 imply that Maximum Independent Set is NP-complete.

Additionally, we can adapt the concept of *well-covered* graphs to Independent Sets. The following theorem implies that for *well-covered* graphs, the complexity of finding a Maximum Independent Set is reduced to polynomial time.

Theorem 2.21 (Independent Set in *Well-covered* Graphs)
For a *well-covered* graph $G = (V, E)$, every maximal independent set has the same size and is therefore maximum.

Proof:
Theorem 2.21 follows directly from definition 2.14 and theorems 2.15 and 2.19.

Next we show a straightforward approach for finding Maximal Independent Sets in any undirected graph. It is similar to a method for Minimum Vertex Cover mentioned earlier. The idea consists in iteratively adding vertices to the set (as long as they do not violate the Independent Set property), until all vertices have been considered. Such an approach is called "greedy" because it constantly extends the solution until this is no longer possible—without removing part of the solution at any time.

Algorithm 2.1 : Greedy Algorithm for Independent Set

Input : Undirected graph $G = (V, E)$
Output : Maximal Independent Set $V_{\mathrm{IS}} \subseteq V$ for G

Set $V_{\mathrm{IS}} = \emptyset$
foreach $v \in V$ **do**
  if $\forall \{v, w\} \in E : (w \notin V_{\mathrm{IS}})$ **then**
    Set $V_{\mathrm{IS}} = V_{\mathrm{IS}} \cup \{v\}$
return V_{IS}

This simple algorithm performs surprisingly well—as be seen in the following theorem. The Independent Set property can not be ensured without taking all edges into account which implies that the running time of algorithm 2.1 is even asymptotically optimal.

Theorem 2.22 (Correctness and Complexity of Algorithm 2.1)
Algorithm 2.1 always finds a Maximal Independent Set in $O(|E|)$ time.

Proof:
Because the vertices are processed sequentially, for every edge $\{v, w\} \in E$ at most one of v and w is added to V_{IS}. Therefore V_{IS} is an independent set. Additionally every vertex $v \in V$ is processed and if there is no $w \in V_{\text{IS}}$ with $\{v, w\} \in E$ then $v \in V_{\text{IS}}$. So V_{IS} is maximal.

The for-loop runs $|V|$ times but the if-statement is only executed twice for every edge $e \in E$. Every other statement runs in $O(1)$ time. Thus algorithm 2.1 needs $O(|E|)$ time.

Finally, we can take advantage of *well-covered* graphs when running algorithm 2.1 on them. It implicitly returns a Maximum Independent Set for *well-covered* graphs without even comparing it to other Maximal Independent Sets.

Theorem 2.23 (Algorithm 2.1 in *well-covered* Graphs)
For a *well-covered* graph algorithm 2.1 always finds a Maximum Independent Set in $O(|E|)$ time.

Proof:
Theorem 2.23 follows directly from theorems 2.21 and 2.22.

3. Triangulations

In the following chapter, we introduce triangulations along with some of their different variants and mention related work that has been done in this field. Many triangulations are usually seen as a part of computational geometry—including the most popular one, the Delaunay Triangulation [3, Section 9.2]. However, the underlying structure that they share can also be interpreted as a combinatorial problem separated from geometric aspects. Thereby, we encounter some of the basics from chapter 2 again.

To begin with, we borrow a definition from graph theory: the *Complete Graph*. It is a simple (undirected) graph with the maximum number of edges. Often it is denoted as K_n where n is the number of vertices. We modify this definition slightly to let the graph be induced by a given vertex set:

Definition 3.1 (Complete Graph)
Given a vertex set V, the *Complete Graph* $K_V = (V, E)$ for V contains all possibles undirected edges between each pair of vertices in V:

$$E = \{e = \{v, w\} : v, w \in V \land v \neq w\}$$

Next, we develop the term of *conflicts*. This concept tries to abstract geometric properties of mutually exclusive objects (such as intersecting line segments). That way we can represent (some) geometric restrictions in combinatorial problems.

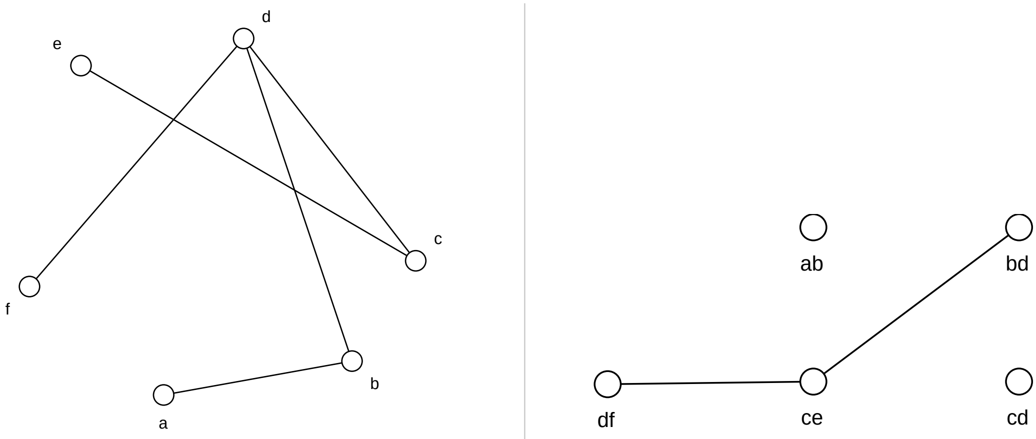
Definition 3.2 (Conflicts)
For a set of objects O, *conflicts* X are a set of unordered object pairs:

$$X \subseteq \{\{o_i, o_j\} : o_i, o_j \in O \land o_i \neq o_j\}$$

As we defined it, conflicts of objects are indistinguishable from edges of a simple graph having the objects as vertices. Below, we call such a graph the *Conflict Graph*:

Definition 3.3 (Conflict Graph)
The *Conflict Graph* $G_{\mathrm{conf}}(O, X) = (O, X)$ for a set of objects O and a set of conflicts X is an undirected graph with O as vertices and X as edges.

Refer to fig. 3.1 for an example of the Conflict Graph with line segments being the objects and their conflicts representing all pairwise intersections.



(a) Line Segments (b) Conflict Graph

Figure 3.1.: Example of the Conflict Graph for a given set of line segments and the conflicts being all intersections

Now we can combine the previous definitions to characterize triangulations. Note that we start off with a combinatorial definition without any geometric components. In the course of this chapter, we show how to apply this formulation to geometric problems.

Definition 3.4 (Triangulation)
Given the Complete Graph $K_V = (V, E)$ for a vertex set V and a set of conflicts X for E such that the Conflict Graph $G_{\mathrm{conf}}(E, X)$ is *well-covered*. A *Triangulation* $T(V, X) \subseteq E$ of V with respect to X is a maximum set of non-conflicting edges:

$$e_i \in T(V, X) \iff e_i \in E \land \forall e_j \in T(V, X) : \{e_i, e_j\} \notin X$$

Bringing the Maximum Independent Set from section 2.3 back to mind, we can see that a Triangulation is essentially the same problem. Apart from Triangulations operating on the Conflict Graph, both problems maximize the number of "independent" vertices.

Theorem 3.5 (Equality of Triangulation and Maximum Independent Set)
Every Triangulation $T(V, X)$ of a vertex set V with respect to conflicts X is a Maximum Independent Set for the Conflict Graph $G_{\text{conf}}(E, X)$ and vice versa. Herein E are the edges of the Complete Graph K_V.

Proof:
Theorem 3.5 follows directly from definition 2.17 and theorem 2.21.

Now we can make use of algorithm 2.1 from chapter 2 to gain a first bound on the time complexity of finding *any* Triangulation (for a given vertex set and conflicts). Later, we give more accurate time bounds for special classes of Triangulations.

Theorem 3.6 (Time Complexity of Triangulations)
From theorems 2.23 and 3.5 follows that finding a Triangulation $T(V, X)$ of a vertex set V with respect to conflicts X takes $O(|X|)$ time.

One important property of geometric objects has not been considered for our definition of Triangulations yet: *forbidden objects*. Besides conflicting objects from which only one can be part of the same Triangulation, there may also be objects which are not wanted at all as we consider the Complete Graph. This can be the case when a Triangulation is required to lie completely within a certain boundary or when one object overlaps multiple others.

Definition 3.7 (Triangulation with Forbidden Edges)
A *Triangulation with Forbidden Edges* $T(V, X, F)$ is a Triangulation of the vertex set V with respect to conflicts X which does not contain any of the edges in F:

$$\forall e \in F : e \notin T(V, X, F)$$

Even though definition 3.7 models geometric triangulation problems better, it is not solvable in polynomial time anymore (assuming P≠NP). Many geometric triangulations are solvable in polynomial time however (as we see later) so this approach is clearly not the best one to solve them.

Theorem 3.8 (NP-completeness of Triangulation with Forbidden Edges)
The decision problem whether a Triangulation with Forbidden Edges $T(V, X, F)$ exists is NP-complete. [35, triangulation existence problem]

There is another approach to ensure that boundary is part of a Triangulation which we call *constraints*. Basically, we require a Triangulation to contain certain objects, e.g. the boundary. This leaves the problem that objects outside the boundary can be part of the Triangulation, but those can be removed from the Triangulation in polynomial time (because triangulations have polynomial size).

Definition 3.9 (Constrained Triangulation)
A *Constrained Triangulation* $T(V, X, C)$ is a Triangulation of the vertex set V with respect to conflicts X which contains the edge constrains C:

$$\forall e \in C : e \in T(V, X, C)$$

In contrast to the Triangulation with Forbidden Edges, Constrained Triangulations can be found in polynomial time. Therefore we slightly modify algorithm 2.1 to begin with the constraints as initial Independent Set and then proceed as before. This assumes that the constraints themselves are all valid and not conflicting. Even without that guarantee, an additional check before running the algorithm would require at most quadratic time with respect to the constraints—so the whole running time is still polynomial.

Theorem 3.10 (Time Complexity of Constrained Triangulations)
Using algorithm 2.1 a Constrained Triangulation $T(V, X, C)$ of the vertex set V with respect to conflicts X and constrains C can be calculated in $O(|C|^2 + |X|)$ time. In case C is guaranteed to have the Independent Set property, running time reduces to $O(|X|)$.

3.1. Point Set Triangulations

In this section we present the first geometric triangulation problem and draw the connection to our previous definition. The difference between a (topological) Triangulation as we have defined it earlier and a Point Set Triangulation which we get to in this section can be seen as the distinction between a graph and its embedding.

To make things clear, we start by defining some geometric terms which we use in the following. The equivalent of edges in a topological triangulation are line segments for a geometric triangulation in the plane. Since line segments are not restricted to the plane, we do not focus on two dimensions here. However according to some definitions, triangulations for higher dimensions contain also geometric objects of higher dimension (e.g. tetrahedra in three dimensions). For simplicity, we consider those objects be implicitly defined by their bounding line segments.

Definition 3.11 (Line Segments)
A *line segment* $s = (p, q)$ is determined by its endpoints $p, q \in P$ with P being a point set (of arbitrary dimension). For compatibility with other definitions, s is directed from p to q, i.e. $(p, q) \neq (q, p)$ and contains all points m "between" p and q:

$$m \in s \iff \exists a \in [0, 1] : m = p + a \cdot (q - p)$$

Clearly, the counterpart of conflicting edges are intersecting line segments. Nevertheless, depending on the definition, intersection of two line segments with the same endpoint is either empty or the common endpoint. This is why we assume intersection of any geometric objects to be the set intersection of all (usually infinitely many) points contained in the objects and introduce the already widely used term *crossing*:

Definition 3.12 (Crossing)
Two line segments $s_i = (p_i, q_i)$ and $s_j = (p_j, q_j)$ with different slope are *crossing*, if their intersection is not empty and not an endpoint, i.e.

$$s_i, s_j \textit{ crossing} \iff (p = s_i \cap s_j) \wedge (|s_i \cap s_j| = 1) \wedge (p \notin \{p_i, q_i, p_j, q_j\})$$

Two line segments s_i and s_j are *non-crossing* if they are not *crossing*. A set S of line segments is *crossing* if at least two segments $s_i, s_j \in S$ are *crossing*. It is *non-crossing* if each pair $s_i, s_j \in S$ is *non-crossing*.

We do not require points to be in general position which is why we have to deal with degeneracies in the following. General position often preempts interesting instances—e.g. those which are heavily symmetric. Additionally, many man-made structures aim for collinearity, so extra effort has to be expended to make real world instances fit the general position requirements. The following definition identifies all unwanted line segments in case of collinear points. Refer to fig. 3.2 for examples of such degeneracies.

Definition 3.13 (Overlapping Line Segments)
Given a point set P and a line segment $s = (p, q)$ with $p, q \in P$. s is *overlapping* in P if and only if there is a point $p' \in P$ which lies in its interior:

$$s \textit{ overlapping} \iff \exists p' \in P : (p' \in s) \land (p' \notin \{p, q\})$$

s is *non-overlapping* if it is not *overlapping*.

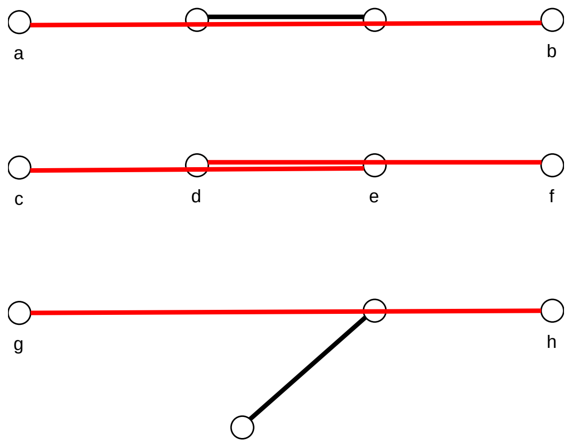


Figure 3.2.: Examples of overlapping line segments ($\{a, b\}, \{c, e\}, \{d, f\}, \{g, h\}$)

Earlier in this section, we already mentioned the connection between topological and geometric triangulations—and here comes the formal definition. Note that we distinguish between points (which have a geometry, i.e. coordinates) and vertices (which are solely topological objects).

Definition 3.14 (Topological Representation)
A vertex set $V(P)$ *represents* a point set P if there is exactly one vertex $v_p \in V(P)$ for each point $p \in P$ and v_p can be identified by p and vice versa.

After having all basic terms at hand, we now proceed with *Point Set Triangulations*. They are the geometric equivalent of topological triangulations for point sets. To emphasize that Point Set Triangulations without further restrictions can be calculated in polynomial time (compare theorems 3.6 and 3.16), we avoid a definition based on Triangulations with Forbidden Edges (see definition 3.7)—despite the fact that overlapping line segments are forbidden edges. Note that in this case that finding a Triangulation involving overlapping line segments takes polynomial time. Afterwards they can be replaced by the line segments that they include in polynomial time (because there are polynomially many line segments).

Definition 3.15 (Point Set Triangulation)
The Triangulation $T(P)$ of a point set P is a triangulation $T(P) = T(V, X)$ where the vertex set $V = V(P)$ represents P, the conflicts X are all *crossing* line segments, and which contains no *overlapping* line segments:

$$\begin{aligned}(p, q), (p', q') \textit{ crossing} &\iff \big\{\{v_p, v_q\}, \{v_{p'}, v_{q'}\}\big\} \in X \\ (p, q) \textit{ overlapping} &\implies \{v_p, v_q\} \notin T(P)\end{aligned}$$

For convenience we define $s = (p, q) \equiv e = \{v_p, v_q\}$ such that $s \in T(P) \iff e \in T(P)$. See also [3, Section 9.1] for a slightly different yet equivalent definition of Point Set Triangulations.

The asymptotic time complexity of finding a Triangulation can be improved for the special case of Point Set Triangulations. For example by making use of sweep line algorithms which exploit that Triangulation properties can be ensured locally, the running time drops to $O(n \log n)$ as in [23].

Theorem 3.16 (Time Complexity of Planar Point Set Triangulation)
A planar point set $P \subseteq \mathbb{R}^2$ can be triangulated in $O(n \log n)$ time with $n = |P|$. [3, Theorem 9.12]

In definition 3.9 we already showed a way to force certain objects (e.g. boundaries) to be part of a Triangulation. The same concept can be transferred to Point Set Triangulations as in the following definition:

Definition 3.17 (Constrained Point Set Triangulation)
A *Constrained Point Set Triangulation* $T(P, C)$ of a point set P with line segment constraints C is a Point Set Triangulation $T(P, C) = T(P)$ such that $C \subseteq T(P, C)$.

The same time bound holds in the presence of constraints. Sweep line algorithms can be modified for taking constraints into account. Note that constraints can destroy desired properties of a Point Set Triangulation (such as large inner angles). We come back to some of those additional requirements later in this chapter.

Theorem 3.18 (Constrained Point Set Triangulation)
A *Constrained Point Set Triangulation* $T(P, C)$ for a planar point set $P \subseteq \mathbb{R}^2$ and line segment constraints $C \subseteq P^2$ can be calculated in $O(n \log n)$ time. [7]

3.2. Intersection Graph

When we defined the Point Set Triangulation (definition 3.15), we assumed that the set of conflicts (i.e. all pairs of *crossing* line segments) is already given, which might not be they case. This section fills the gap by introducing the *Intersection Graph*. For consistency with other definitions, we kept the name even though it actually contains only *crossing* line segments (according to our previous definition of intersection).

Definition 3.19 (Intersection Graph)
For a set of line segments S the *Intersection Graph* $G_{\text{cross}}(S) = (V_S, X)$ consists of a vertex $v_s \in V_S$ for every line segment $s \in S$ and an edge $\{v_{s_i}, v_{s_j}\} \in X$ for every pair of *crossing* segments $s_i, s_j \in S$. It is the geometric equivalent of the Conflict Graph (see definition 3.3).

Generating the Intersection Graph for line segments in the plane can be done efficiently using an output sensitive sweep line algorithm. By output sensitive we refer to the (asymptotic) running time depending on the output size, i.e. how many line segments intersect.

Theorem 3.20 (Time Complexity of Intersection Graph)
Given a set of line segments S in the plane, the Intersection Graph $G_{\text{cross}}(S) = (V_S, X)$ can be calculated in $O(m \log m + i \log m)$ time where $m = |V_S| = |S|$ and $i = |X|$ is the number of intersections in S. [3, Lemma 2.3]

The referenced method works well for a small number of intersections but fails to do so when there are significantly more intersections than line segments. Let us first state how many intersections there are when all possible connections in a point set are considered:

Theorem 3.21 (Complexity of Point Set Intersections)
For a planar point set P with n points, the set of all line segments S with endpoints in P has $\Theta(n^4)$ intersection points. [38] Thus calculating the Intersection Graph for S takes $\Omega(n^4)$ time.

This bound on the number of intersections shows that the sweep line approach is not a good choice in our case—especially because it does not reach the lower time bound of $\Omega(n^4)$ (even though we have not yet shown that this is possible at all).

Theorem 3.22 (Non-Optimality of Sweep Algorithm for Point Sets)
The sweep algorithm presented in [3, Section 2.1] with the time complexity of theorem 3.20 is not optimal for finding all *crossing* line segments with endpoints in a given planar point set P of size $n = |P|$ as it takes $O(n^4 \log n)$ time.

Now let us consider the straightforward approach of simply checking all pairs of line segments if they are *crossing*. Algorithm 3.1 realizes this idea with a small optimization: To avoid checking each pair of line segments twice, we take the length $|s|$ of a line segment s into account. The same can be done by using a sorted list instead of a set for the line segments (which even works for line segments with the same length).

Algorithm 3.1 : Naive Intersection Algorithm

Input : Set of line segments S
Output : Intersection Graph $G_{\text{cross}}(S) = (V_S, X)$

Set $V_S = \{v_s : s \in S\}$
Set $X = \emptyset$
foreach $s \in S$ **do**
    foreach $s_\times \in S$ *with* $|s| \leq |s_\times|$ **do**
        if s *and* $s_\times$ *are crossing* **then**
            Add $\{v_s, v_{s_\times}\}$ to X
return $G_{\text{cross}}(S) = (V_S, X)$

We can directly deduce the asymptotic running time from the algorithm and its correctness is implied by the fact that every possible combination of potentially *crossing* line segments is considered.

Theorem 3.23 (Time Complexity and Correctness of Algorithm 3.1)
Algorithm 3.1 finds all *crossing* line segments in $O(m^2)$ time with $m = |S|$.

Again, the most simple approach is indeed asymptotically optimal for our application as for algorithm 2.1 in chapter 2. Additionally algorithm 3.1 does not assume that the set of line segments is planar.

Theorem 3.24 (Optimality of Algorithm 3.1 for Point Set)
Algorithm 3.1 is asymptotically optimal for finding all *crossing* line segments with endpoints in a given planar point set P as it takes $O(n^4)$ time for $n = |P|$.

3.3. Polygon Triangulations

One case where boundary needs to be taken into account is when triangulating the interior of a polygon. We can reduce this case to the already defined Constrained Point Set Triangulation of the previous section 3.1:

Definition 3.25 (Polygon Triangulation)
A Triangulation $T(P)$ of a polygon P bounded by line segments are all boundary and interior edges of P in a Constrained Point Set Triangulation of the polygon vertices with the polygon boundary as constraints.

We can also reverse the reduction such that a Point Set Triangulation can be constructed by finding a Polygon Triangulation first:

Theorem 3.26 (Generalization of Point Set Triangulation)
Every triangulation of a point set P is a triangulation $T(\text{conv}(P) \cup P)$ of the polygon bounded by the convex hull $\text{conv}(P)$ of P and containing all inner points of P.

Proof:
First recall that the Triangulation $T(\text{conv}(P) \cup P)$ is a Constrained Point Set Triangulation $T_c(P, C)$ for the point set P with the constraints $C = \text{conv}(P)$ being all line segments on the convex hull of P. Now we only need to show that the choice of C does not exclude any Point Set Triangulation by proving that every line segment $s \in C$ has to be part of every Point Set Triangulation for P:

Assume that there is a Point Set Triangulation T' for P which does not contain a line segment $s \in C$. Because T' is a maximal set of *non-crossing* line segments by definition, there has to be a line segment $s' \in T'$ such that s and s' are *crossing*. This implies that the endpoints of s' have to be on opposite sides of s and since s is part of the convex hull, one of the endpoints of s' has to lie outside of the convex hull. This contradicts the definition of the convex hull.

3.4. Edge Flipping

Besides the already mentioned sweep algorithms for Triangulations of planar point sets, there is another common approach called *Edge Flipping*, which starts with any Triangulation and iteratively replaces an edge by another one until the Triangulation has a certain desired property (e.g. large inner angles).

Definition 3.27 (Edge Flip)
Given a triangulation $T(V, X)$ for a vertex set V with respect to a set of conflicts X, (e, f) with $e \in T(V, X)$ and $f \notin T(V, X)$ is an *edge flip* iff $T(V, X) \setminus \{e\} \cup \{f\}$ is a triangulation for V with respect to X.

An edge flip is not possible with any pair of edges. For planar point sets, both edges need to be diagonals of a convex quadrangle. More generally, they can only be conflicting edges:

Theorem 3.28 (Edge Flips are Conflicts)
Given a Triangulation $T(V, X)$ for a vertex set V with respect to a set of conflicts X, every edge flip (e, f) is a conflict: $\{e, f\} \in X$.

Proof:
Assume $\{e, f\} \notin X$. Further assume that

$$\neg\exists e' \in T(V, X) \setminus \{e\} : \{e', f\} \in X.$$

Then f can be added to $T(V, X)$ (without removing e) and therefore $T(V, X)$ is no triangulation—which is a contradiction. Now let $e' \in T(V, X) \setminus \{e\}$ such that $\{e', f\} \in X$. Then $e' \in T(V, X) \setminus \{e\} \cup f$ —which contradicts that $T(V, X) \setminus \{e\} \cup f$ is a triangulation. Therefore every edge flip (e, f) is a conflict.

Continuously applying edge flips to a Triangulation can be seen as traversing a path within the graph of all Triangulations. Such a graph with vertices for every Triangulation and edges for every possible edge flip is called the *Flip Graph*. An example can be seen in fig. 3.3.

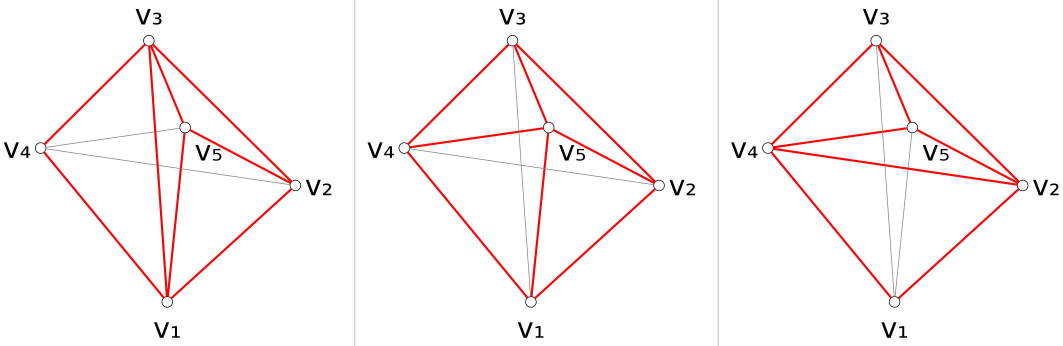
Definition 3.29 (Flip Graph)
The *Flip Graph* $G_{\text{flip}}(V, X) = (V_T, E_{\text{flip}})$ for a vertex set V and edge conflicts X contains a vertex $v \in V_T$ for every triangulation of V with respect to X and edges $e \in E_{\text{flip}}$ for every possible edge flip.

Changing any Triangulation into any other Triangulation using edge flips is only possible if the Flip Graph is connected; i.e. there is a path between every two vertices. This is the case for two dimensions but is not yet completely explored for higher dimensions.

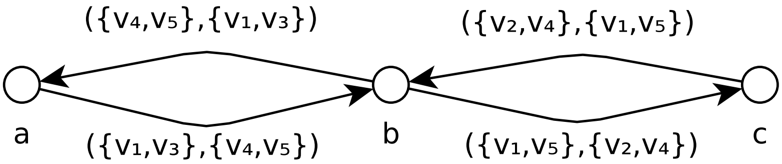
Theorem 3.30 (Connectivity of the Flip Graph)
The Flip Graph $G_{\text{flip}}(V, X)$ is connected in two dimensions [50, Behauptung 4] and has a diameter of at most $6n - 30$ for $n = |V|$. [5] Therefore every Triangulation $T(V, X)$ of a vertex set V with respect to edge conflicts X can be transformed into any other Triangulation of V with respect to X in $O(n)$ time.

For three dimensions, it is still an open problem whether or not the Flip Graph is connected. [16]

Another issue of edge flipping algorithms are Triangulations which have additional non-locally restricted objectives. If for every two edges in a potential edge flip it can be determined which one improves the Triangulation with respect to the objective, the



(a) (b) (c)



(d) Flip Graph

Figure 3.3.: Example triangulations and their flip graph

Flip Graph can be searched for a Triangulation for which all edge flips would worsen the solution with respect to the objective. This is possible for optimizing the inner angles of triangles. In chapter 4 we show that it is not possible for the MaxMin Length Triangulation (MMLT). Therefore the whole Flip Graph has to be considered, which has exponential size.

3.5. Related Work

In the following, we briefly mention some results for different kinds of Triangulations. These problems have kept researchers busy for over 100 years [27] and several books have been written which cover the topic, e.g. [15].

The most famous kind is probably the Delaunay Triangulation [3, Section 9.2]. It forces every circumcircle of a triangle to be empty of other points and therefore maximizes the minimum angle [3, Theorem 9.9]. There is an edge flipping algorithm which calculates it in $O(n \log n)$ expected time using $O(n)$ space [3, Theorem 9.12].

The counterpart of a Delaunay Triangulation, minimizing the maximum angle, takes $O(n^2 \log n)$ time and $O(n)$ space [4]. The same approach can also produce triangulations which maximize the minimum height of a triangle. Finally, the same reference shows also that minimizing the maximum slope and minimizing the maximum eccentricity can both be done in $O(n^3)$ time and $O(n^2)$ space.

Optimizing the area of triangles has been studied in [49] resulting in $O(n^2 \log n)$ time and $O(n^2)$ space for minimizing or maximizing the area.

There are several results for optimal edge length Triangulations. One of the first publications [19] shows that minimizing the maximum edge length can be done in $O(n^2)$ time. Minimizing the edge length sum (also known as the Minimum Weight Triangulation) was proven NP-hard [40]. Maximizing the minimum edge length was stated as an open problem [19] but 20 years later it has been shown that it is NP-complete [21]. It remains NP-hard for polygons with holes and interior points [8] but can be solved in $O(n^3)$ time for simple polygons and even in linear time for convex polygons [28].

4. MaxMin Length Triangulation

Subsequently, we get on to the main problem of this thesis: the MaxMin Length Triangulation (MMLT). This chapter covers its complexity, mentions different potential approaches, and finally presents an algorithm for solving it. Let us begin with a formal definition of the problem by reducing it to the Point Set Triangulation (definition 3.15):

Problem 4.1 (MMLT)

Given: Set of points P, length function $|s|$ for each line segment s with endpoints in P (e.g. Euclidean distance of the endpoints)

Sought: Point Set Triangulation $T_{\text{opt}} = T(P)$ of P which maximizes $\min\limits_{s \in T_{\text{opt}}} |s|$

A first observation that can be made is that an optimal solution for MMLT needs not be unique, i.e. there can be multiple solutions T_{opt} with the same value for the shortest segment in T_{opt}. See also fig. 4.1 for an example.

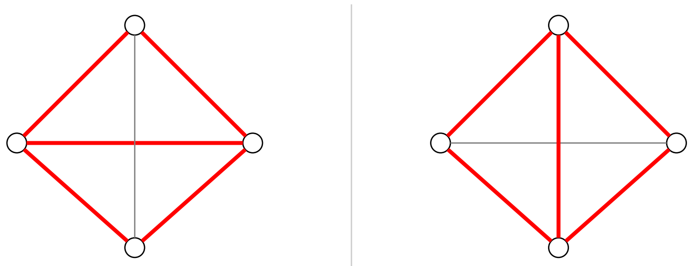


Figure 4.1.: Example of different optimal MMLT solutions for the same point set using Euclidean distance of the endpoints as length for each line segment

4.1. Complexity

As already mentioned in section 3.5, the MMLT problem was stated as an open problem by Edelsbrunner and Tan in 1991 [19] and has recently been proven to be NP-hard by Fekete [21]. In the following, we briefly sketch the proof.

Firstly, we adopt the definition of the Covering by Disjoint Segments (CDS) problem which has been part of [21] and which we will need afterwards:

Problem 4.2 (CDS)

Given: Set of line segments S, set of target points $T \subseteq \{s_i \cap s_j : s_i, s_j \in S\}$

Sought: Set of non-intersecting line segments $S_{\text{opt}} \subseteq S$ such that T is covered:

$$\forall p \in T : \exists s \in S_{\text{opt}} : p \in s$$

Theorem 4.3 (NP-completeness of CDS)
Problem 4.2 (CDS) is NP-complete.

Proof:
CDS can be reduced to Planar 3-SAT (problem 2.7) and vice-versa. Therefore we construct a CDS instance from the planar variable-clause-connection graph of a Planar 3-SAT instance. Each variable vertex gets replaced by a cycle of target points connected by line segments which alternating represent assigning either true or false to the variable. Additionally, the target points contain all clause vertices which are the intersection point of three long segments intersecting the respective assignment line segment of the variables taking part in the clause. See fig. 4.2 for an example of such a construction. Note that this construction can be done in polynomial time and the CDS can be solved if and only if there is an assignment to the Planar 3-SAT variables which lets all clauses evaluate to “true”. For further details of the proof refer to [21].

Now we can proceed with the MMLT problem:

Theorem 4.4 (NP-hardness of MMLT)
Problem 4.1 (MMLT) is NP-hard—even for the case where P is planar.

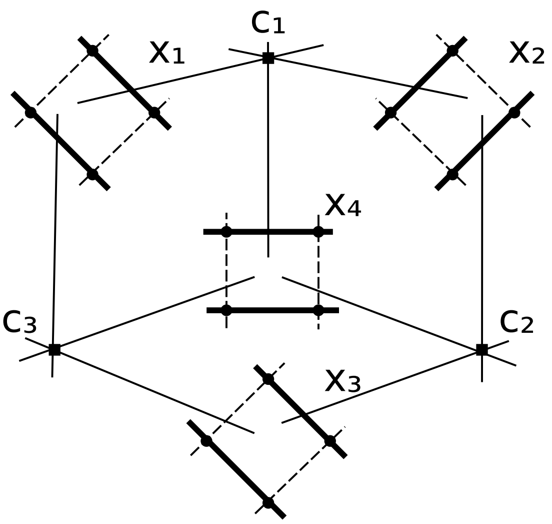


Figure 4.2.: Example of a CDS construction for the Planar 3-SAT instance from fig. 2.1, bold line segments represent assignment of variables to “true”, dashed line segments represent “false”, and long line segments connected clauses with the respective variable assignments

Proof:
We summarize the reduction from CDS to MMLT of [21]. Let us continue from the construction of the proof for theorem 4.3. We assume that three points are collinear only if they correspond to the endpoints of a line segment s and a target point covered by s (this can be achieved by perturbation). Furthermore, let δ be the minimum distance of any point to a line segment it is not part of and assume that every line segment has at least length δ (can be achieved by appropriate scaling). We now replace every target point of the CDS instance by a tuple of points with distance ε such that $\varepsilon \ll \delta$.

The shortest line segment in an optimal MMLT solution for the constructed instance has length ε if and only if the initial CDS has a solution. This is due to the fact that the only line segments of length ε are those which replaced the target points and every other line segment is longer. From the definition of MMLT it follows, that every ε-segment is part of the solution if and only if there is no *crossing* line segment in the solution.

For the ε-segments which replaced the clause vertices, only the long line segments connecting a clause with its variables are *crossing*. This implies that the ε-segment for a clause it part of an optimal MMLT solution if and only if the clause can satisfied in the 3-SAT problem (or the target point at that position can be covered in the CDS problem).

In a variable cycle, ε-segments are only picked if neither all line segment representing an assignment to “true” nor all of those representing “false” can be part of the MMLT solution. This is the case when two clauses of the 3-SAT problem require the variable to have conflicting assignments.

Finally, let us point out that the above transformation of a CDS instance into a MMLT instance can be done in polynomial time such that we reduce CDS to MMLT—which implies MMLT being NP-hard.

4.2. Geometric Approaches

In chapter 3 we mentioned two efficient methods for finding Triangulations: sweep algorithms and edge flipping. Here we shortly explain why these approaches can not directly be applied to the MMLT problem.

Sweep algorithms compute solutions for geometric problems (usually in the plane) gradually and rely on the assumption that every part of the solution “behind” the current position of the sweep line is fixed and needs not to be changed anymore. For the MMLT problem, the last line segment reached by the sweep line may change for every line segment it crosses whether it is part of the solution, which may then force itself an update of other line segments. Therefore no part of the solution can be fixed until all line segments are considered which does not comply with the idea of sweep algorithms.

For the edge flipping strategy (traversing a path in the Flip Graph) we show an example in fig. 4.3 which requires an edge flip to worsen the MMLT before the optimal solution can be reached. This essentially requires that every edge flip of the whole Flip Graph has to be considered—which leads to exponential running time.

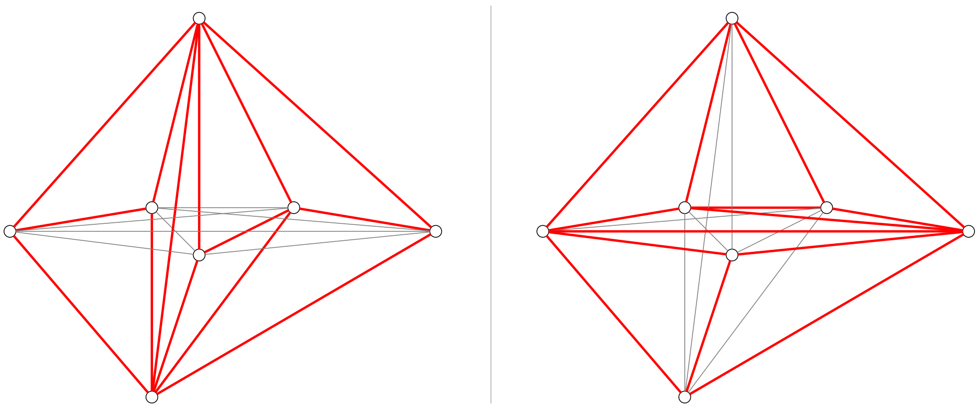


Figure 4.3.: Example of necessary locally non–optimal flips: The shortest edge needs to be flipped in before the optimal solution can be achieved.

4.3. Combinatorial Approach

In this section, we demonstrate an alternative way of finding optimal MMLT solutions based on the underlying combinatorial problem. We come up with another problem, the MaxMin Edge Length Index Triangulation (MELT), which is similar to MMLT yet solely combinatorial.

First of all, we define a new strict total order on the potential edges of a triangulation to circumvent the fact that two different line segments can have the same length, which implies that two optimal MMLT solutions may have a different shortest line segment.

Definition 4.5 (Edge Length Order)
Given a set of edges E representing a set of line segments S and two edges $e_{s_i}, e_{s_j} \in E$ representing two line segments $s_i, s_j \in S$, respectively. Let $|s|$ for $s \in S$ be the segment length and $s_i < s_j$ the lexicographical order of $s_i, s_j \in S$. The *Edge Length Order* is defined as

$$e_{s_i} < e_{s_j} \iff |s_i| < |s_j| \vee ((|s_i| = |s_j|) \wedge (s_i < s_j)).$$

Furthermore, we define a short cut for the position that an edge has in a set sorted by the Edge Length Order, which can be seen as a higher level length function.

Definition 4.6 (Edge Length Index)
Given a set of edges E representing a set of line segments S and an edge $e \in E$. The *Edge Length Index* $\mathrm{idx}(e)$ is the index of e in E sorted by Edge Length Order.

As a side effect of the Edge Length Order, we can now address every edge in a set uniquely by its Edge Length Index, which we state in the following theorem.

Theorem 4.7 (Uniqueness of Edge Length Index)
For a set of edges E representing line segments S, there can be no two different edges $e, e' \in E$ with the same Edge Length Index.

Proof:
For two segments $s, s' \in S$ the lexicographical order is unique—i.e. either $s < s'$ or $s > s'$ but not both. The same holds for the Edge Length Order.

Using the Edge Length Index for the Triangulation edges instead of the line segment length, we can now formulate the combinatorial equivalent of the MMLT problem:

Problem 4.8 (MELT)

Given: Set of points P

Sought: Point Set Triangulation $T_{\text{opt}} = T(P)$ of P which maximizes the minimum Edge Length Index $\min\limits_{e \in T_{\text{opt}}} \text{idx}(e)$.

Even though, there can still be arbitrary many optimal MELT solutions (as for MMLT) at least the "shortest" edge (meaning the one with minimum Edge Length Index) is the same in all of them. Because neither the MMLT nor the MELT problem ask for the whole vector of edges to be maximum, there can still be different combinations of edges with higher Edge Length Index.

Theorem 4.9 (Uniqueness of Optimal MELT Solutions)
From Theorem 4.7 follows directly that every optimal solution for problem 4.8 (MELT) has the same edge with minimum Edge Length Index.

Even though the link between MMLT and MELT may be obvious, the proof is yet to come and is covered by the following theorem.

Theorem 4.10 (Equality of MELT and MMLT)
Every optimal MELT solution is an optimal MMLT solution.

Proof:
Let T_{opt} be an optimal MELT solution for a point set P and assume that there is a MMLT solution T with $\min\limits_{s \in T} |s| < \min\limits_{s \in T_{\text{opt}}} |s|$. By definitions 4.5 and 4.6 $\text{idx}(\arg\min\limits_{s \in T} |s|) < \text{idx}(\arg\min\limits_{s \in T_{\text{opt}}} |s|)$—which contradicts T_{opt} being optimal.

Determining the complexity of MELT is straightforward as a reduction from MMLT can easily be done considering theorem 4.10. Compare also section 4.1 for further details of the complexity of the MMLT problem.

Theorem 4.11 (NP-hardness of MELT)
From theorem 4.10 follows that MELT is NP-hard.

Despite the fact that MMLT instances can be solved through the corresponding MELT problem, there is one important difference between both problems that we will exploit in the following: Making use of the Edge Length Index, MELT instances contain only integers. This allows us to transform the problem into an Integer Program (IP) (see also chapter 2).

Problem 4.12 (IP Formulation of MELT)

$$
\begin{aligned}
E &: \textit{non-overlapping} \text{ line segments with endpoints in } P \\
X &: \text{pairs of } \textit{crossing} \text{ line segments} \\
T &: \text{MELT solution}
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & \min_{e \in T} x_e \cdot \mathrm{idx}(e) \\
\text{subject to } \forall \{e_i, e_j\} \in X : & \quad x_{e_i} + x_{e_j} \leq 1 \\
\forall e_i \in E : & \quad x_{e_i} + \sum_{\{e_i, e_j\} \in X} x_{e_j} \geq 1 \\
\forall e \in E : & \quad x_e \in \{0, 1\}
\end{aligned}
$$

4.4. Separators

The IP for MELT contains n^2 variables and $O(n^4)$ restrictions for n points—even though most of them are irrelevant for the optimal solution. Therefore we identify two groups of interesting edges: *Short Edges*, which are potential candidates for the smallest Edge Length Index in an optimal MELT solution, and *Separators*, which may avoid certain Short Edges.

Definition 4.13 (Short Edges)
Short Edges within a set of edges E are all edges with an Edge Length Index smaller than a certain threshold:

$$E_{\text{short}}(E, \ell) := \{e \in E : \text{idx}(e) < \ell\}$$

Definition 4.14 (Separators)
Given a set of edges E and edge conflicts $X \subseteq E^2$. The set of *Separators* $E_{\text{sep}}(E, X, e)$ for an edge $e \in E$ are all edges that improve the MELT solution, i.e. all e_{sep} with $\{e, e_{\text{sep}}\} \in X$ which have a higher Edge Length Index:

$$E_{\text{sep}}(E, X, e) := \{e_{\text{sep}} \in E : \text{idx}(e) < \text{idx}(e_{\text{sep}}) \wedge \{e, e_{\text{sep}}\} \in X\}$$

The preceding definitions 4.13 and 4.14 allow us to restrict the edges that may influence an optimal MELT solution T_{opt} such that we can formulate an upper bound on the minimum Edge Length Index in T_{opt}.

Theorem 4.15 (Upper Bound for MELT)
Given the optimal MELT solution T_{opt} for a point set P, let E be all *non-overlapping* line segments contained in the Complete Graph $K_{V(P)}$ where $V(P)$ represents P, and let X be all conflicts in the Intersection Graph of P. Every edge $e \in E$ without Separators is an upper bound for T_{opt}:

$$\forall e \in E,\ E_{\text{sep}}(E, X, e) = \emptyset : \min_{e_{\min} \in T_{\text{opt}}} \text{idx}(e_{\min}) \leq \text{idx}(e)$$

which is equivalent to

$$\forall e \in E : \neg \exists e_{\text{sep}} \in E : \text{idx}(e) < \text{idx}(e_{\text{sep}}) \wedge\ \{e, e_{\text{sep}}\} \in X \implies \min_{e_{\min} \in T_{\text{opt}}} \text{idx}(e_{\min}) \leq \text{idx}(e)$$

Proof:
Assume

$$\exists e \in E,\ E_{\text{sep}}(E, X, e) = \emptyset : \min_{e_{\min} \in T_{\text{opt}}} \text{idx}(e_{\min}) > \text{idx}(e)$$

This implies $e \notin T_{\text{opt}}$ and

$$\forall e' \in E : \text{idx}(e') < \text{idx}(e) \implies e' \notin T_{\text{opt}}.$$

With $E_{\text{sep}}(E, X, e) = \emptyset$ it follows that $\forall \{e, e_\times\} \in X : e_\times \notin T_{\text{opt}}$. This means that for T_{opt} to be a Triangulation e has to be in T_{opt}—which is a contradiction.

From all the edges without Separators, the one with smallest Edge Length Index is clearly the one which restricts an optimal MELT solution in theorem 4.15 the most. Therefore we give it a name such that we can refer to it later:

Definition 4.16 (Shortest Non-separable Edge)
Given a set of edges E and edge conflicts $X \subseteq E^2$. The *Shortest Non-separable Edge* e_{nose} is the edge with the smallest Edge Length Index from all edges which has no Separators:

$$e_{\text{nose}} := \operatorname*{arg\,min}_{e \in E : E_{\text{sep}}(E,X,e) = \emptyset} \operatorname{idx}(e)$$

Figure 4.4 shows an example where the upper bound for an optimal MELT solution in theorem 4.15 is not tight for e_{nose}. However, we assume that considering e_{nose} is still of great help for reducing the problem size significantly—though we have only experimental evidence by now.

Theorem 4.17 (Index of e_{nose})
The Edge Length Index of the Shortest Non-separable Edge e_{nose} for a random point set P of size $n = |P|$ is at most $\operatorname{idx}(e_{\text{nose}}) \in O(n)$ on average.

Proof:
The proof is still unknown to us, yet we have experimental evidence for this to be true (see section 6.1).

The logical consequence of theorem 4.17 is that we only need to consider part of the edges. Our new task is to find Separators for Short Edges where possible, which is reflected in the Non-Conflicting Separators (NOCS) problem.

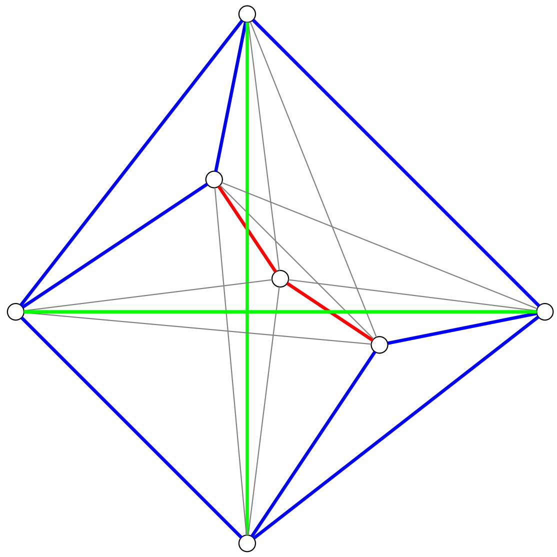


Figure 4.4.: Example where the upper bound from theorem 4.15 is not tight. One of the Short Segments (red) is part of an optimal MELT solution because their Separators (green) cross. Thus the shortest segment in the solution is shorter than the shortest one of all segments without Separators (blue).

Problem 4.18 (NOCS)

Given: Set of Short Edges E and their Separators E_{sep}, edge conflicts $X \subseteq (E \cup E_{\text{sep}})^2$

Sought: Set of non-conflicting edges:

$$E_{\text{opt}} \subseteq E \cup \bigcup_{e \in E} E_{\text{sep}}(e)$$

which contains for every edge $e \in E$ either e itself or at least one Separator $e_{\text{sep}} \in E_{\text{sep}}(e)$ and maximizes the smallest Edge Length Index $\min\limits_{e \in E_{\text{opt}}} \text{idx}(e)$

An essential observation at this point is that MELT and NOCS pursue the same objective and lead to the same result—only that the NOCS solution does not contain edges which do not influence the minimum Edge Length Index. Therefore those "irrelevant" edges have to be added to any optimal NOCS solution to retrieve an optimal MELT solution.

Theorem 4.19 (Equality of MELT and NOCS)
Given a point set P along with all *non-overlapping* edges E of the Complete Graph $K_{V(P)}$, and all conflicts X from the Intersection Graph for E. Let e_{nose} be the shortest non-separable edge in E. An optimal MELT solution T_{opt} for P and an optimal NOCS solution S_{opt} for $E_{\text{short}}(E, \text{idx}(e_{\text{nose}}))$ have the same value:

$$\min_{e \in T_{\text{opt}}} \text{idx}(e) = \min_{e \in S_{\text{opt}}} \text{idx}(e)$$

A direct consequence of the Shortest Non-separable Edge e_{nose} not being restricted in general (so it can be the edge with highest Edge Length Index in the worst case) is that both problems MELT and NOCS have the same complexity:

Theorem 4.20 (NP-hardness of NOCS)
For $E_{\text{short}}(E, \text{idx}(e_{\text{nose}})) \cup e_{\text{nose}} = E$, NOCS and MELT are the same problem. Therefore NOCS is also NP-hard.

Additionally, the IP formulation for NOCS is also the same as for MELT—besides the fact that we may have less variables and restrictions depending on the Shortest Non-separable Edge e_{nose}.

Problem 4.21 (IP Formulation of NOCS)

$$\begin{aligned} E &: \text{Short Edges} \\ E_{\text{sep}} &: \text{Separators} \\ X &: \text{edge conflicts} \\ S &: \text{NOCS solution} \end{aligned}$$

$$\begin{array}{lll} \text{maximize} & \min_{e \in S} x_e \cdot \text{idx}(e) & \\ \text{subject to} & \forall \{e_i, e_j\} \in X : & x_{e_i} + x_{e_j} \leq 1 \\ & \forall e_i \in E : x_{e_i} + \sum_{e_j \in E_{\text{sep}}(e_i)} x_{e_j} \geq 1 & \\ & \forall e \in E \cup \bigcup_{e \in E} E_{\text{sep}}(e) : & x_e \in \{0, 1\} \end{array}$$

Assuming that theorem 4.17 is correct, the IP for NOCS is significantly smaller than for MELT as it only has $O(n)$ variables and $O(n^3)$ restrictions in case all the Short Edges conflict with every other edge—and we can assume that this does not happen for random point sets.

The presence of Separators which are not necessary (i.e. which only conflict with Short Edges already conflicting with other Separators) in a NOCS solution is not defined. This is what we intend to change by introducing *complete* NOCS solutions:

Definition 4.22 (Completion of NOCS)
Given an optimal NOCS solution T_{opt} for a set of Short Edges E, their Separators E_{sep}, and conflicts X, the *Completion* of T_{opt} is

$$T := T_{\text{opt}} \cup \left\{ e_{\text{sep}} \in \bigcup_{e \in E} E_{\text{sep}}(e) : \neg \exists e \in T : \{e, e_{\text{sep}}\} \in X \right\}.$$

If $T = T_{\text{opt}}$, T_{opt} is *complete*.

It is important to realize that Completion of an optimal NOCS solution does not destroy its optimality:

Theorem 4.23 (Optimality of NOCS Completion)
Given an optimal NOCS solution T_{opt} the Completion T of T_{opt} is also an optimal NOCS solution.

After we defined that "unnecessary" Separators may as well be in an optimal NOCS solution if it is complete, we can exploit the newly acquired structure to recycle the Independent Set problem from chapter 2:

Theorem 4.24 (Connection of NOCS and Independent Set)
Every complete optimal NOCS solution T_{opt} for a set of Short Edges E, their Separators E_{sep}, and conflicts $X \subseteq (E \cup E_{\text{sep}})^2$ is a Maximal Independent Set (definition 2.17) for the simple graph $G = (E \cup E_{\text{sep}}, X)$.

4.5. Algorithms

At last, we combine earlier concepts and observations to algorithms for solving MMLT instances. Chapter 5 describes our implementation and chapter 6 shows the results of running it on random point sets. Note that (according to the common assumption that P≠NP) polynomial time algorithms for NP-hard problems—and therefore also for MMLT—are unlikely to exist.

Transforming MMLT instances into MELT instances requires only for sorting the line segments by length and then by lexicographical order to determine the Edge Length Index for every edge. An algorithm for doing so is trivial and therefore we omit it here.

By making use of theorem 4.19, we can give a straightforward algorithm for solving MELT instances though reduction to NOCS.

Algorithm 4.1 : MELT Algorithm

Input : Point Set P

Output : Optimal MELT Solution T_{opt} for P

Let S be all *non-overlapping* line segments with endpoints in P

Let E be edges representing all line segments in S sorted by Edge Length Order

Find the Shortest Non-Separable Edge $e_{\text{nose}} \in E$ and compute the partial Intersection Graph $G_{\text{cross}}(S') = (E', X)$ for all line segments $S' \subseteq S$ represented by edges E' with $E' = E_{\text{short}}(E, \text{idx}(e_{\text{nose}}))$

Find an optimal NOCS solution E_{opt} for the Short Edges E' with Separators $E_{\text{sep}}(E, X, e)$ for every $e \in E'$ and conflicts X

Compute a Constrained Point Set Triangulation T_{opt} for P with the constraints E_{opt}

return T_{opt}

Constructing the partial Intersection Graph and finding the Shortest Non-separable Edge at the same time can be done using a slightly modified version of algorithm 3.1 keeping the running time of $O(n^4)$ for n points.

Solving the NOCS instance can be done by directly using the IP formulation. In our implementation however, we run a binary search on the Edge Length Index starting with the interval 0 to $\text{idx}(e_{\text{nose}})$, and solve the corresponding decision problem for NOCS. This gives us the chance to abort the algorithm at any time gaining upper and lower bound for the optimal NOCS solution—and, if wanted, a "good" MELT solution instead of an optimal one.

5. Implementation

In this chapter, we will briefly describe the program components roughly split into the geometry and the optimization part. To let the program run close to the hardware layer (which usually leads to fast execution times), the code was written in C++. First attempts to use Python instead (for the sake of clarity and better readability) stumbled over the non-readiness of the Computational Geometry Algorithms Library (CGAL) bindings and the lack of good alternatives. For the technical documentation, please refer to appendix A.

5.1. Geometry

The geometry part consists of number, point and segment types, data structures for triangulation, convex hull, bounding box, and segment intersection. For most of it we made extensive use of CGAL, which will be introduced in section 5.1.1.

5.1.1. CGAL

CGAL [6] is an Open Source library (mainly) for computational geometry written in C++. It includes most of the common algorithms in the field and also offers efficient data structures. Through the use of C++ templates it is flexible and extendable: for example it is common to adjust the underlying number types to the application.

5.1.2. Kernel

A kernel in CGAL is something like a computational geometry operating system: it holds the basic type definitions like numbers, points, lines, and line segments. Basic operations such as intersection, angle calculations, and comparisons are also part of it.

In our application we use the built-in `Exact_predicates_inexact_constructions_kernel`[1] [20], which uses double as a number type and is not capable of constructing new geometric objects from existing ones accurately. Both properties lead to faster execution time yet do not produce wrong results in our case.

On top of the CGAL kernel there are two modifications. One is for printing points and segments without the need to use streams, the other one to output them to SVG (see also sections 5.3.2 and 5.3.6). Additionally, segments are indexed by length and have the information whether they overlap with other segments attached to them.

[1] Thanks to Michael Hemmer for making me aware that I should use it!

5.1.3. Triangulation

CGAL brings along a constrained triangulation already [10], which triangulates a point set with respect to a given mandatory set of (non-crossing) edges. For this application the class was extended to be drawable to SVG and to find the shortest edge which is part of the triangulation.

5.1.4. Convex Hull

This class directly calls the `convex_hull_2` function of CGAL [11] which itself defaults to the algorithm of Akl & Toussaint [1]. The only extension is that the class serves as container which holds the output points and contains a function to find the shortest segment within.

For the algorithm this class is not necessary as its bound is worse than the one of the SAT solution. It is left in the implementation however, as it is a measure of quality for our algorithm.

5.1.5. Intersection Algorithm

To find all pairs of intersecting segments we use algorithm 3.1. For the intersection check itself we make use of the CGAL function `do_intersect` [17]. In contrast to the `intersection` function [30], it does not actually compute the intersection and therefore performs much better.[2]

5.1.6. Intersection Graph

The intersection graph (as defined in definition 3.19) stores for every segment the indices of all intersecting segments. This graph data structure (adjacency list) performs well for few edges (in this case intersections) per vertex (in this case line segment)—which we assume here. It may however in future versions of the implementation be replaced by a sparse adjacency matrix from the section 5.3.1 library.

5.2. Optimization

The optimization part itself consists mainly of data structures for SAT problems and solutions, an interface for SAT solvers and the solvers themselves (currently only CPLEX).

Only segment indices and intersections are passed to the SAT problem. This is because geometry does not influence the problem, only topology. Also, it is easier that way to keep track of which segments take part in the restrictions.

5.2.1. SAT problem

This class serves two purposes: to grant an interface to the relevant data for the SAT (i.e. segments and intersections), and to set the short segment range.

[2]Thanks again to Michael Hemmer!

5.2.2. SAT solution

The SAT solution class mainly just stores the segment indices derived from solving the SAT problem—which can be none if no feasible solution is found. Additionally it offers methods for drawing short segments and separators and for finding the shortest segment of the solution.

5.2.3. SAT solver

To unify the way solving the SAT problem is done, there are three interfaces: the base SAT solver and two derived interfaces for decision and optimization problems. They all share methods for adding forbidden segments, intersection and separation restrictions and for running the actual solving. Furthermore, there is a method for binding short segments to the objective function for optimization problems.

5.2.4. CPLEX

IBM ILOG CPLEX [29] is a commercial optimization suite written in C. It contains a standalone tool for solving optimization problems and also includes libraries for being used in other programs or even other programming languages. According to [39] CPLEX is one of the two fastest Integer Program (IP) solvers.

In our application we use the Concert API for C++ [9] to access CPLEX. It allows for adding variables and restrictions to a model, extracting them to more efficient data structures and then running several solving algorithms on it.

5.3. Remaining Components

Besides the geometry and optimization parts, our implementation contains the following components: a controller class for the whole algorithm which combines all the other components and utility classes for reading input files, debug output and assertions, test case generation, and drawing certain states of the algorithm to SVG.

5.3.1. Boost

Boost [14] is a collection of free (as in freedom) C++ libraries containing tools for various tasks. It makes extensive use of the C++ pre-processor (mostly templates) and aims to extend the C++ standard library (STL).

In our implementation, we use the Spirit [26] library for parsing the JSON input files (see section 5.3.3), and the Program Options library [43] for parsing command line arguments and configuration files. Later versions of the implementation may also use the Boost Graph Library [48] for the intersection graph, and the Geometry library [24] for creating SVG images (see section 5.3.6).

5.3.2. Qt

Qt [46] is a framework for cross-platform application development written in C++. It features some enhancements to the C++ standard library including an own string class `QString` [45] which supports a different formatting syntax than the `std::string`, offers a UI description format with integration into a even processing framework, and also comes with a simplified built system qmake which wraps platform dependent tools such as GNU make.

We make use of `QString` for our logger class (section 5.3.4), generate SVG images through the `QPainter` class, and build our applications using qmake (which also allows for integration in the C++ IDE Qt Creator).

5.3.3. JSON Parser

JSON Spirit [51] uses the Boost Spirit library [26] for parsing the input point files in JSON (JavaScript Object Notation) [31] format. As of writing, the built-in JSON support of Qt unfortunately has a bug [37] such that it is not compatible with the C++ standard library (and therefore also neither with CGAL nor with Boost). After parsing the input file, all points are stored in a sorted set to allow for fast lookup.

5.3.4. Logger

Our logger class supports different levels of verbosity (debug, info, error, print) and adds the current timestamp to each of them. Additionally, there are shortcut methods for output of measured times, and the current status of the algorithm. Debug output is only included in the programs if they are compiled in debug mode. For convenience, all methods accept `QString`.

5.3.5. Point Generator

For testing and analyzing the algorithm, we generated different instances of point sets and stored them for repeated runs. We hereby rely on the `Random_points_in_square_2` class [47] in combination with the `Creator_uniform_2` class [13]—both part of CGAL (section 5.1.1).

5.3.6. SVG Painter

Mainly for debugging purposes and to visualize different steps of the algorithm, we included the possibility to draw certain data structures to SVG using the `QPainter` class [44].

6. Results

This chapter covers the results from running our implementation (chapter 5) for the algorithm presented in chapter 4 on a uniform distribution of random points in the unit square. All experiments were run on an Intel® Core™ 2 Duo CPU E6850 with 2 GB RAM. For everything but the IP-solving with CPLEX only one core is used. We compiled with the GCC 4.7.2 and following options: `-frounding-math -std=c++11 -O3`. The libraries we compiled against are Qt 5.0.0, CGAL 4.0.2, Boost 1.49, and JSON Spirit 4.06.

6.1. Segment Lengths

Our first experiments were made to verify theorem 4.17 by computing the Edge Length Index of the Shortest Non-separable Edge $\text{idx}(e_{\text{nose}})$ for various numbers of input points. Figures 6.1 and 6.2 show the trend of e_{nose} in comparison with the shortest line segment, and with the minimum Edge Length Index of the MaxMin Edge Length Index Triangulation (MELT) solution, respectively. For each data point 100 instances were run, though some of them have been aborted after 30 minutes (refer to tables B.1 and B.2).

Even though the specific progression of the Shortest Non-separable Edge index $\text{idx}(e_{\text{nose}})$ is not clear from fig. 6.2, it can be assumed that it is sub-linear. From fig. 6.1, we can also see that the shortest line segment length significantly drops below the length of e_{nose}, which results in more segments being in between for a uniform distribution.

6.2. Execution Time

Furthermore, we made an analysis of our algorithm's running time with respect to the number of input points. Again we let 100 instances run for each data point. The resulting times are real time (in contrast to user or system time)—i.e. execution time of non-related background processes is not excluded. This decision was made because it is usually impossible to guarantee that no other (system) tasks are running, so real time is more meaningful (yet less accurate).

Figure 6.3 shows which part of our algorithm takes what amount of time in comparison to the full execution execution time. As can be seen, generating the Complete Graph and sorting the line segments by Edge Length Order takes polynomial time (red bars). The high variance in the time for Integer Program (IP) solving (blue) is due to heuristics being used in CPLEX to reduce the exponential worst case running time. All remaining parts of the algorithm, including constructing of the Intersection Graph, and computing the Constrained Point Set Triangulation take significantly less time (green). By now, we can not explain the gap for 470 to 490 input points. For the complete data refer to table B.5.

For a comparison in terms of running time of our algorithm versus solving the complete MELT instance as an IP (involving the construction of a complete Intersection Graph) see fig. 6.4. For the data points, we use the median of all running times combined with the median absolute deviation. As fig. 6.5 shows, we have to deal with heavy outliers starting at a problem size of 80 input points, which is why we chose to use median instead of average.

6.3. Aborted instances

As you will have noticed in fig. 6.4, there are only few data points, and also in appendix B there are some cases where not all 100 instances are part of the table. This is because we aborted every instance after exceeding a running time of 30 minutes. See fig. 6.6 for an impression on how many instances could be completed within a certain time (here 23 minutes and 20 seconds).

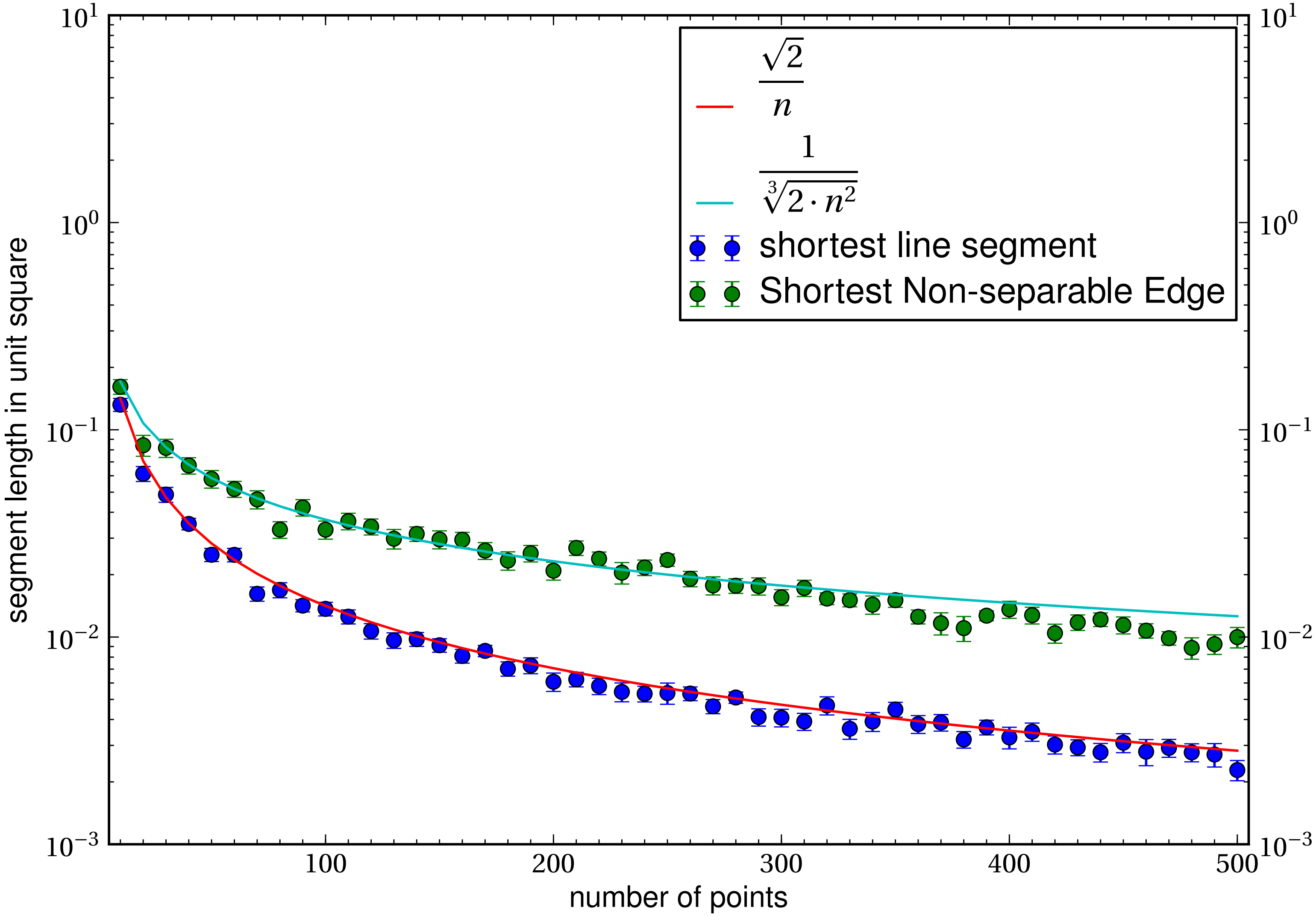


Figure 6.1.: Comparison of segment lengths

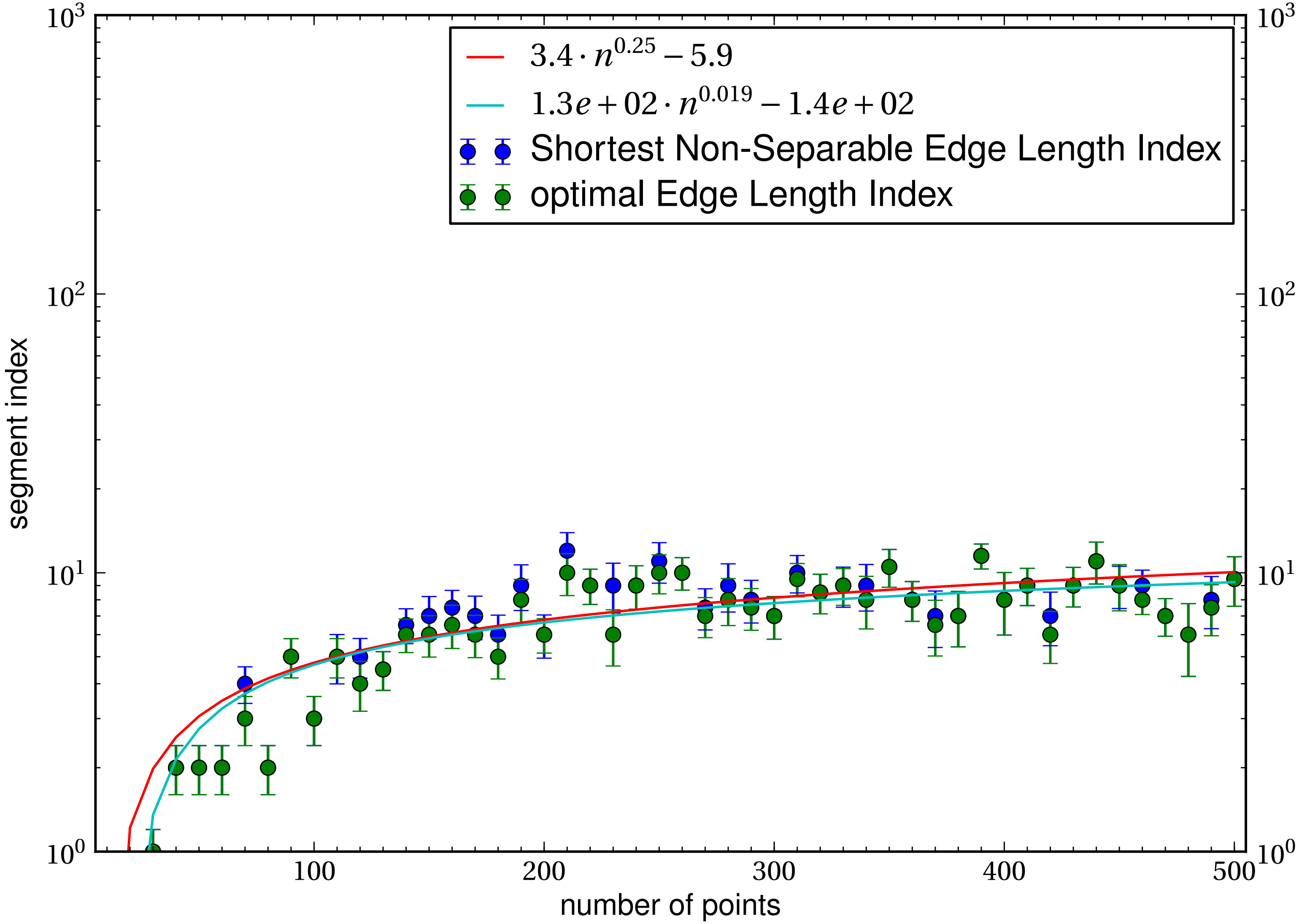


Figure 6.2.: Comparison of segment indices

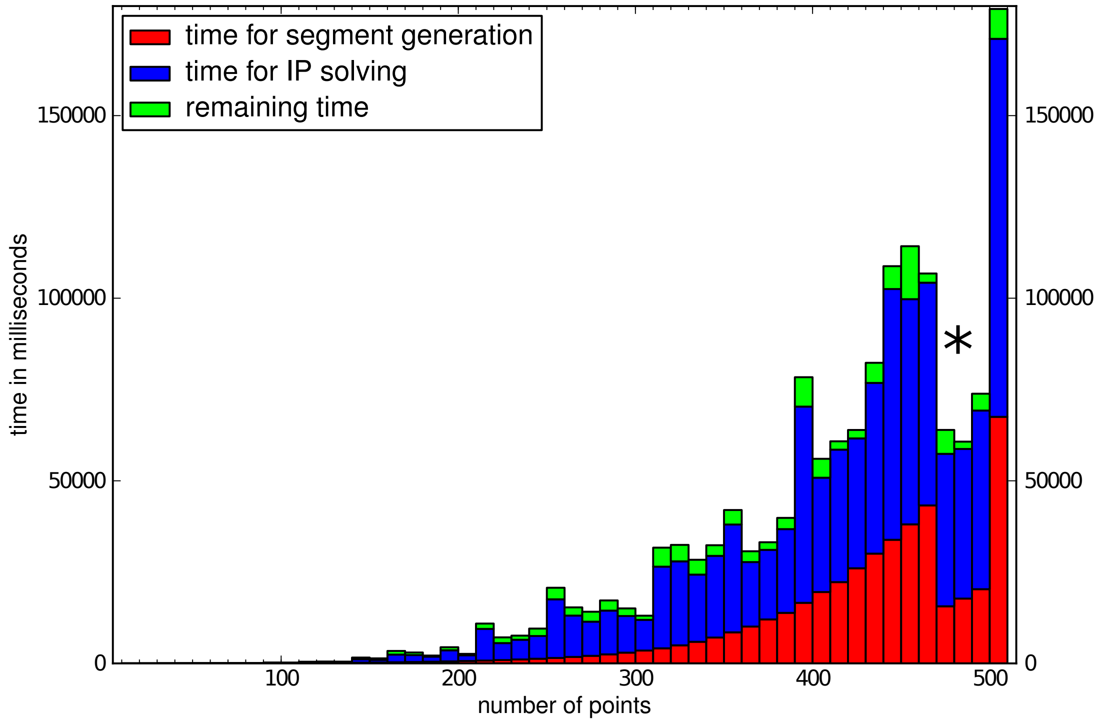
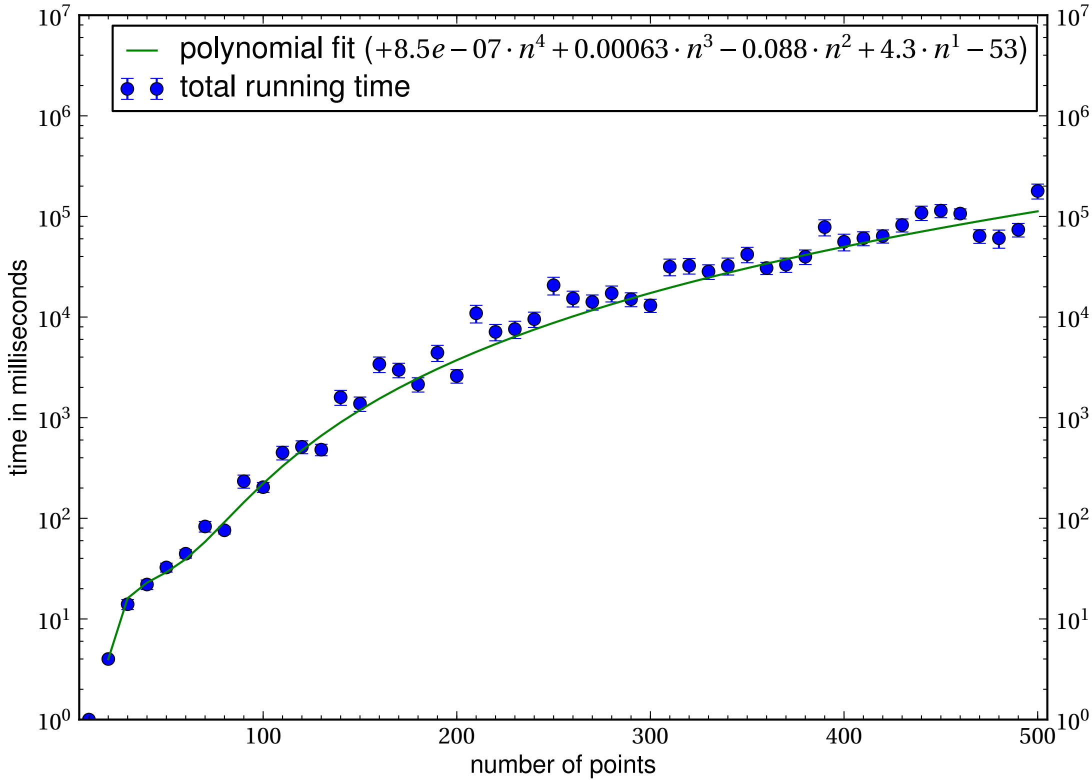
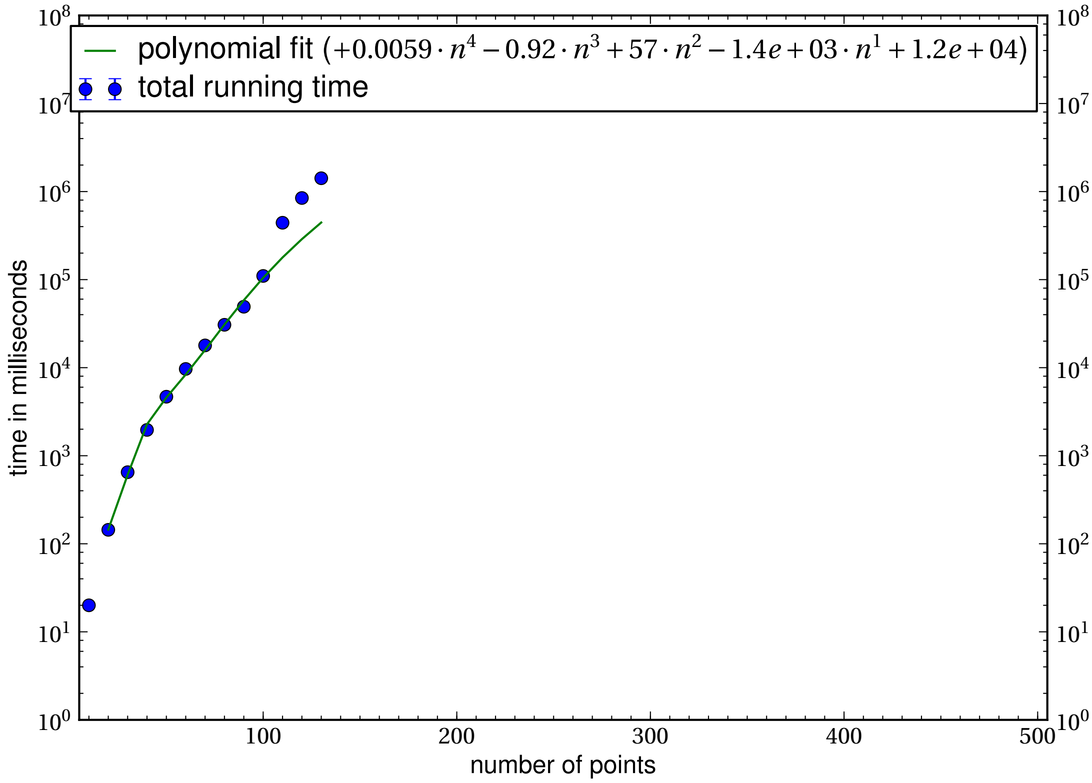


Figure 6.3.: Composition of execution times

* is where the magic happens: we currently can not explain this gap

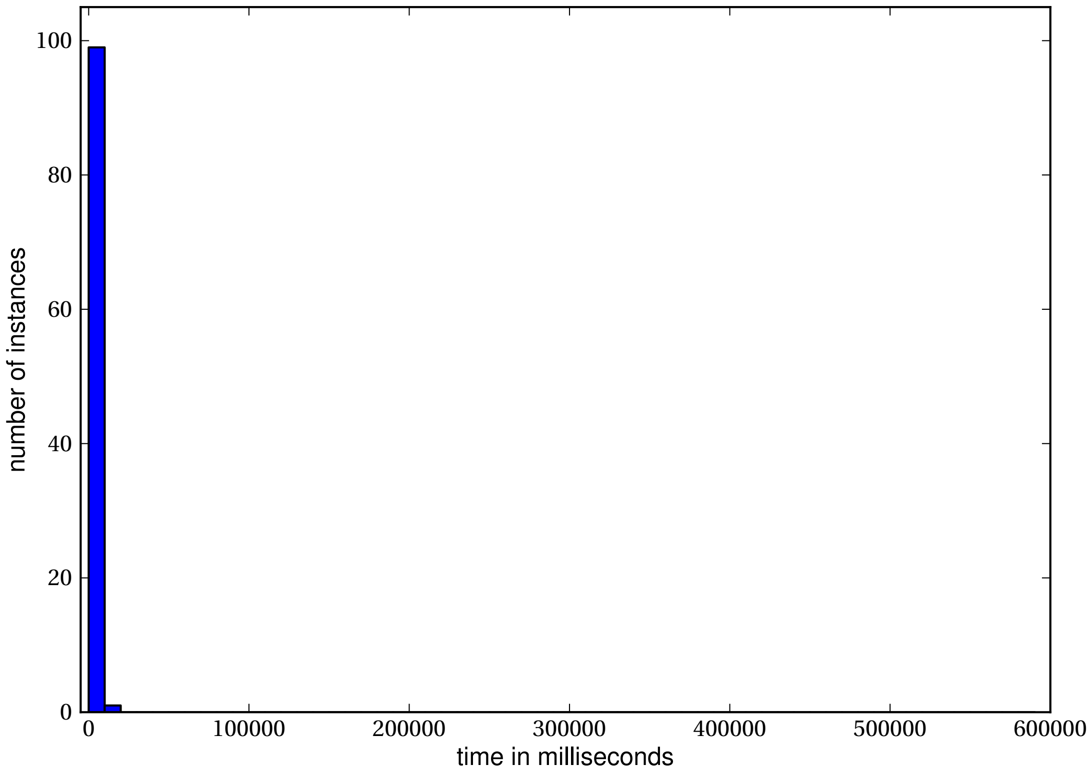


(a) Our Algorithm

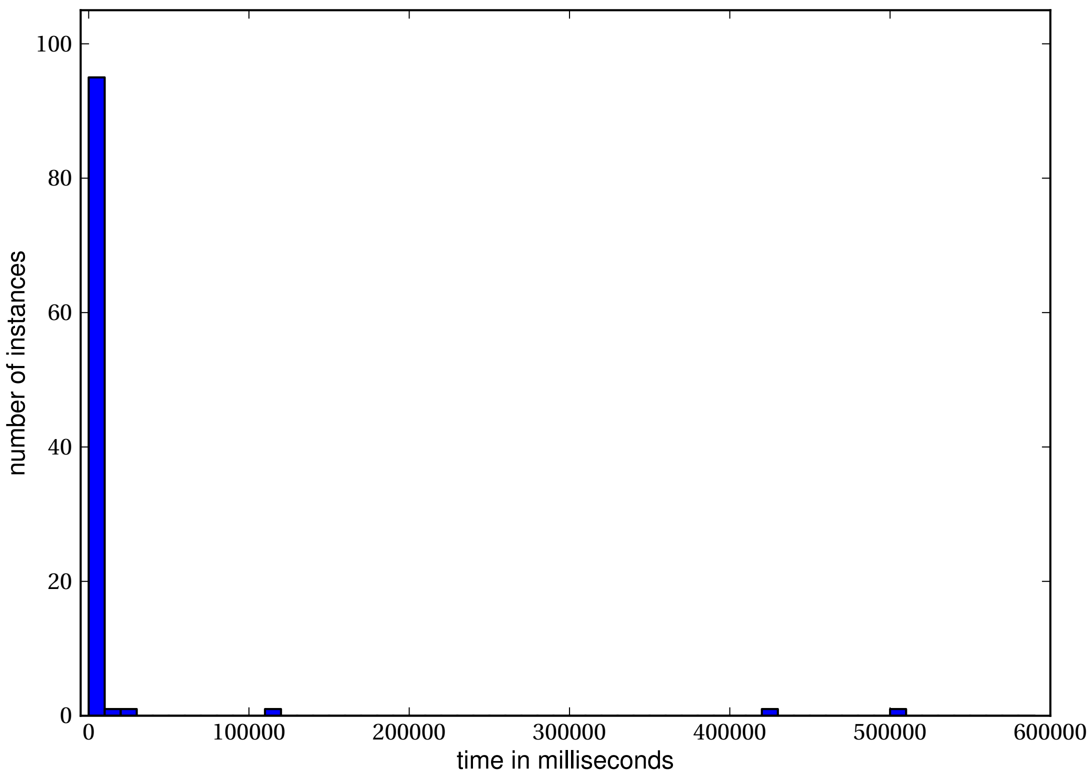


(b) Complete Intersection Graph and IP

Figure 6.4.: Total execution time

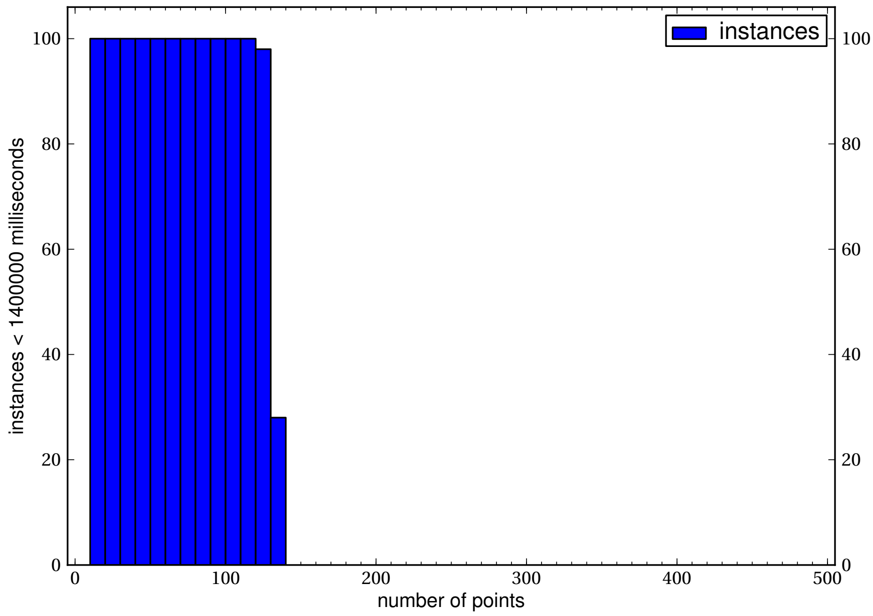


(a) 70 points

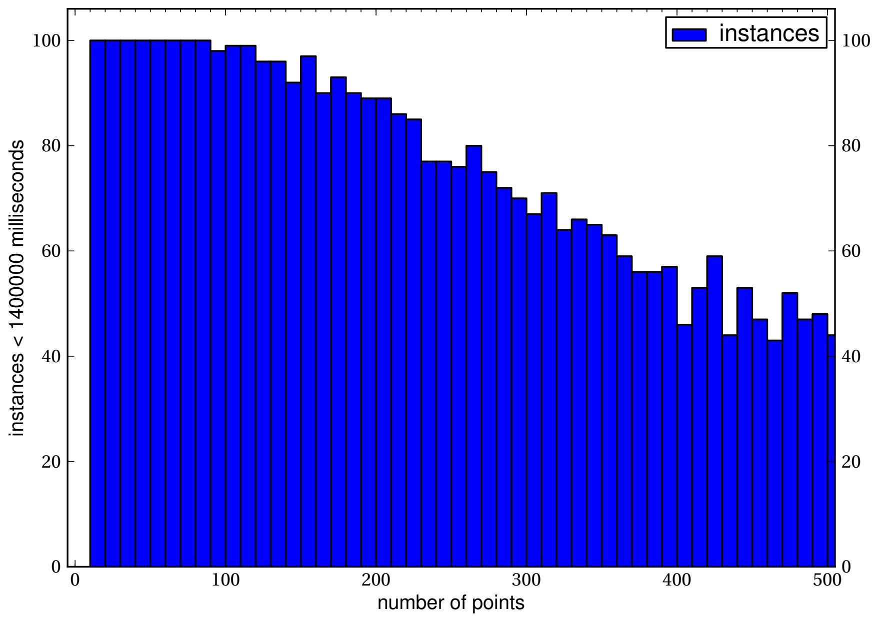


(b) 80 points

Figure 6.5.: Histogram of execution times



(a) Complete SAT



(b) Improved method

Figure 6.6.: Instances with running time $<$ 23:20min

7. Conclusion

The claim of this thesis was that even though the MaxMin Length Triangulation (MMLT) problem was proven to be NP-hard, instances with randomly distributed points can still be solved in polynomial time. Therefore we explored some properties of the problem and consequently presented an algorithm. Its implementation was run on uniformly distributed random points in the unit square, and we compared its running time to solving the complete Integer Program (IP) representing the MMLT problem. The results show that indeed we can achieve a polynomial running time on average, and that our approach is significantly better than simply working with the IP.

As a side effect of analyzing our algorithm, we found out that given randomly distributed points P, the shortest line segment $s \in S$ of all line segments S with endpoints in P that does not cross any longer line segment $s_\times \in S$ ($|s| < |s_\times|$) has a sub-linear index in S sorted by length with respect to the number of points in P. We verified this assumption in our experiments but the proof is still open, and we could not find any publications on that topic.

There are several motivations for future work on the MMLT problem: We have not shown that our algorithm is the best approach by giving a lower bound for the running time on MMLT instances with randomly distributed points, such that there may be other algorithms with better asymptotic running time. Additionally, our implementation can be improved by using heuristics such as applying a greedy algorithm for Weighted Independent Set on the Intersection Graph with the line segment lengths as weights. It is also possible that the use of SAT solvers instead of IP solvers leads to better results as the SAT problem is more restricted.

Another way of improving our results would be to consider different distributions for random point sets or other underlying geometries. Common choices for random point sets are grids, clusters, and Gaussian distribution. Instead of selecting all points from the unit square one may as well consider rectangles, circles or geometric objects of higher dimensions. Besides, the resulting data may benefit from using accurate number types instead of double precision floats. In general, it might also be of interest to run our algorithm on real world instances—even though we could not find any applications for the the MMLT problem.

Furthermore, our algorithm may exploit certain structures (which we do not know of yet) in the input or the Intersection Graph to avoid solving the IP at all. For instances with $\mathrm{idx}(e_{\mathrm{nose}}) = 0$ we do so already. Strategies like blue rule and red rule for Minimum Spanning Trees may have analogies for the MMLT problem.

Finally, some of our constructions and results can be applied to Polygon Triangulations. The MMLT problem is also NP-hard for general polygons such that adapting our algorithm may lead to new insights. In general only few things have to be changed to forbid line segments crossing the boundary, and forcing the boundary to be part of the solution.

A. Documentation

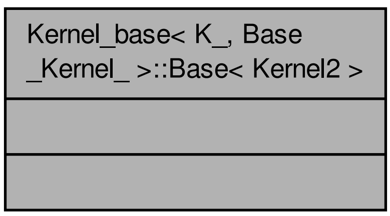
The following is the technical documentation of the program, including class structure and interfaces. It was generated using Doxygen [18]. For an overview of the program components see chapter 5. The source code itself is hosted at: `https://bitbucket.org/winniehell/mmlt`

Contents

1 Class Documentation

1.1 Kernel_base< K_, Base_Kernel_ >::Base< Kernel2 > Struct Template Reference

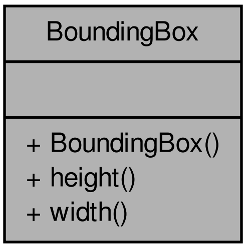
Collaboration diagram for Kernel_base< K_, Base_Kernel_ >::Base< Kernel2 >:



1.2 BoundingBox Class Reference

```
#include <bounding_box.h>
```

Collaboration diagram for BoundingBox:



Public Member Functions

- BoundingBox (const PointSet &points)
- Number height () const
- Number width () const

1.2.1 Constructor & Destructor Documentation

1.2.1.1 BoundingBox::BoundingBox (const PointSet & *points*) [inline]

1.2.2 Member Function Documentation

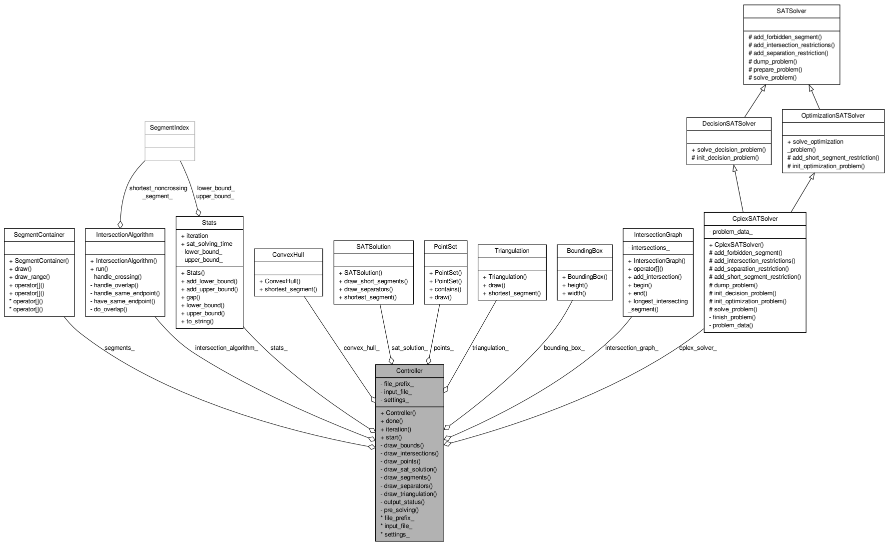
1.2.2.1 Number BoundingBox::height ( ) const [inline]

1.2.2.2 Number BoundingBox::width ( ) const [inline]

1.3 Controller Class Reference

```
#include <controller.h>
```

Collaboration diagram for Controller:



Public Member Functions

- Controller (const QString &file_prefix, QFile &input_file, const QSettings &settings)
- void done ()
- bool iteration ()
- bool start ()

Private Member Functions

- void draw_bounds () const
- void draw_intersections () const
- void draw_points (SVGPainter &painter) const
- void draw_sat_solution () const
- void draw_segments (SVGPainter &painter) const
- void draw_separators () const
- void draw_triangulation () const
- void output_status () const
- void pre_solving ()

Private Attributes

independent members

- CplexSATSolver cplex_solver_
- IntersectionAlgorithm intersection_algorithm_
- SATSolution sat_solution_
- Stats stats_

input parameters

- const QString & file_prefix_
- QFile & input_file_
- const QSettings & settings_

dependent on input parameter

- const PointSet points_

dependent on input points

- const BoundingBox bounding_box_
- const ConvexHull convex_hull_
- SegmentContainer segments_
- Triangulation triangulation_

dependent on segments

- IntersectionGraph intersection_graph_

1.3.1 Constructor & Destructor Documentation

1.3.1.1 Controller::Controller (const QString & *file_prefix*, QFile & *input_file*, const QSettings & *settings*)

1.3.2 Member Function Documentation

1.3.2.1 void Controller::done ( )

called after the algorithm finished

1.3.2.2 void Controller::draw_bounds ( ) const [private]

1.3.2.3 void Controller::draw_intersections ( ) const [private]

1.3.2.4 void Controller::draw_points (SVGPainter & *painter*) const [private]

1.3.2.5 void Controller::draw_sat_solution ( ) const [private]

1.3.2.6 void Controller::draw_segments (SVGPainter & *painter*) const [private]

1.3.2.7 void Controller::draw_separators ( ) const [private]

1.3.2.8 void Controller::draw_triangulation ( ) const [private]

1.3.2.9 bool Controller::iteration ( )

run next iteration

Returns

true if next iteration should be triggered

1.3.2.10 void Controller::output_status ( ) const [private]

dumps the current algorithm status

1.3.2.11 void Controller::pre_solving ( ) [private]

does some pre-processing

1.3.2.12 bool Controller::start ( )

start the algorithm

Returns

true if iteration should be triggered

1.3.3 Member Data Documentation

1.3.3.1 const BoundingBox Controller::bounding_box_ [private]

1.3.3.2 const ConvexHull Controller::convex_hull_ [private]

1.3.3.3 CplexSATSolver Controller::cplex_solver_ [private]

1.3.3.4 const QString& Controller::file_prefix_ [private]

1.3.3.5 QFile& Controller::input_file_ [private]

1.3.3.6 IntersectionAlgorithm Controller::intersection_algorithm_ [private]

1.3.3.7 IntersectionGraph Controller::intersection_graph_ [private]

1.3.3.8 const PointSet Controller::points_ [private]

1.3.3.9 SATSolution Controller::sat_solution_ [private]

1.3.3.10 SegmentContainer Controller::segments_ [private]

1.3.3.11 const QSettings& Controller::settings_ [private]

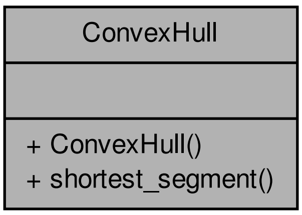
1.3.3.12 Stats Controller::stats_ [private]

1.3.3.13 Triangulation Controller::triangulation_ [private]

1.4 ConvexHull Class Reference

#include <convex_hull.h>

Collaboration diagram for ConvexHull:



Public Member Functions

- ConvexHull (const PointSet &points)
- const SegmentIndex & shortest_segment (const SegmentContainer &segments) const

1.4.1 Constructor & Destructor Documentation

1.4.1.1 ConvexHull::ConvexHull (const PointSet & *points*)

compute convex hull of given point set

1.4.2 Member Function Documentation

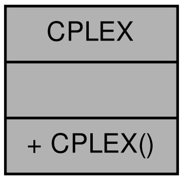
1.4.2.1 const SegmentIndex & ConvexHull::shortest_segment (const SegmentContainer & *segments*) const

find the convex hull segment with minimum length

1.5 CPLEX Class Reference

`#include <concert.h>`

Collaboration diagram for CPLEX:



Public Member Functions

- CPLEX ()

1.5.1 Detailed Description

ugly CPLEX code is not our fault helper class for CPLEX concert API

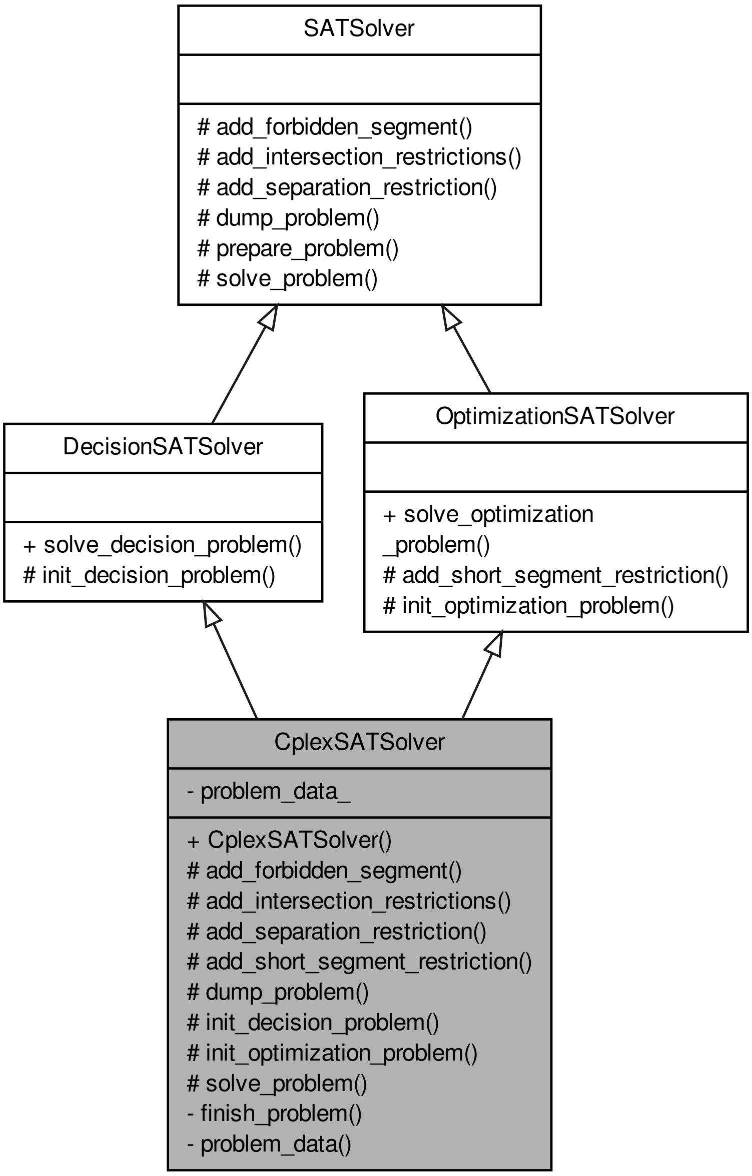
1.5.2 Constructor & Destructor Documentation

1.5.2.1 CPLEX::CPLEX ( ) [inline]

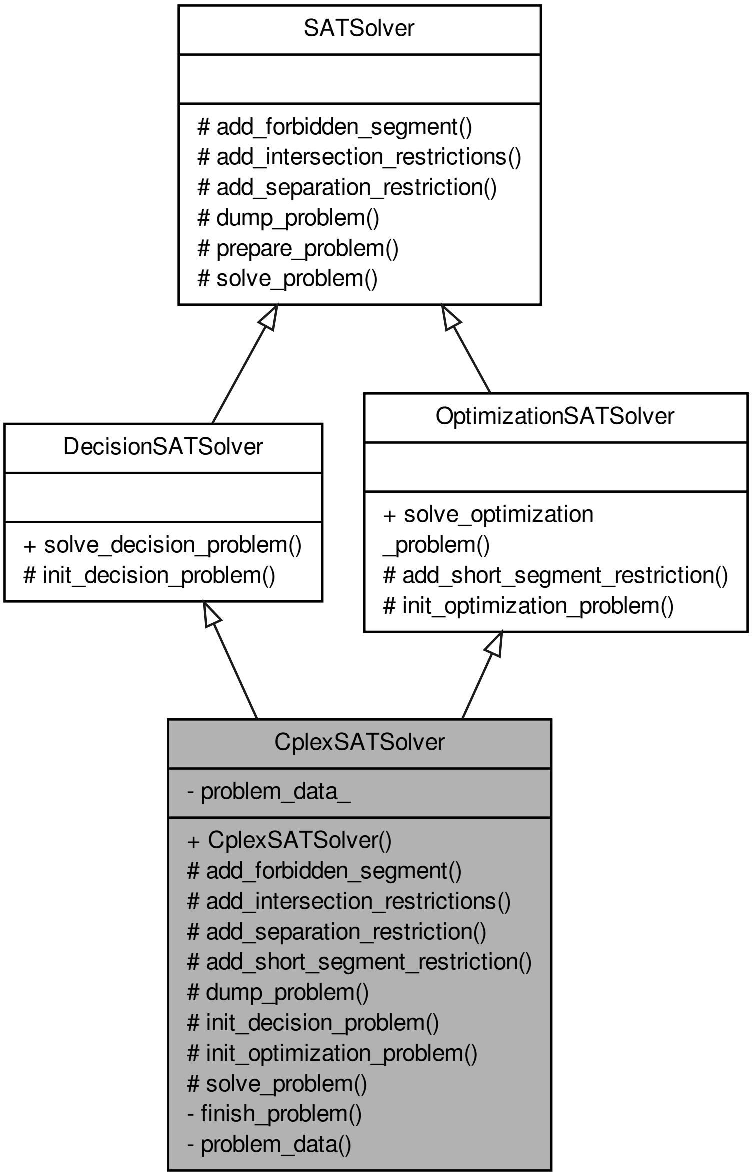
1.6 CplexSATSolver Class Reference

`#include <cplex_sat_solver.h>`

Inheritance diagram for CplexSATSolver:



Collaboration diagram for CplexSATSolver:



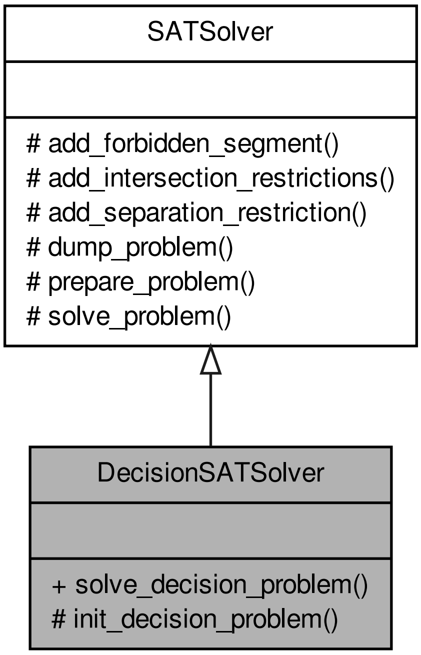
Classes

- struct ProblemData

Public Member Functions

- CplexSATSolver ()

Collaboration diagram for DecisionSATSolver:



Public Member Functions

- void solve_decision_problem (const QSettings &settings, const QString &file_prefix, const SATProblem &problem, SATSolution &solution)

Protected Member Functions

- virtual void init_decision_problem (const SATProblem *problem)=0

1.6.1 Detailed Description

interface for decision SAT solvers

1.6.2 Member Function Documentation

1.6.2.1 virtual void DecisionSATSolver::init_decision_problem (const SATProblem * *problem*) [protected], [pure virtual]

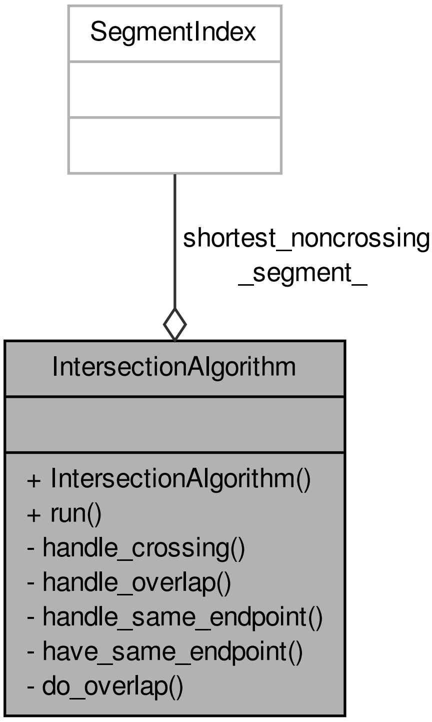
Implemented in CplexSATSolver.

1.6.2.2 void DecisionSATSolver::solve_decision_problem (const QSettings & *settings*, const QString & *file_prefix*, const SATProblem & *problem*, SATSolution & *solution*)

1.7 IntersectionAlgorithm Class Reference

`#include <intersection_algorithm.h>`

Collaboration diagram for IntersectionAlgorithm:



Public Member Functions

- IntersectionAlgorithm ()
- void run (IntersectionGraph &igraph, SegmentContainer &segments)

Public Attributes

- SegmentIndex shortest_noncrossing_segment_

Private Member Functions

- void handle_crossing (IntersectionGraph &igraph, const Segment &s1, const Segment &s2)
- void handle_overlap (IntersectionGraph &igraph, const Segment &s1, const Segment &s2)
- void handle_same_endpoint (const Segment &s1, const Segment &s2) const
- bool have_same_endpoint (const Segment &s1, const Segment &s2) const
- bool do_overlap (Segment &s1, Segment &s2) const

1.7.1 Constructor & Destructor Documentation

1.7.1.1 IntersectionAlgorithm::IntersectionAlgorithm ( )

1.7.2 Member Function Documentation

1.7.2.1 bool IntersectionAlgorithm::do_overlap (Segment & *s1*, Segment & *s2*) const [private]

checks if two segments overlap

Returns

the outer segment

1.7.2.2 void IntersectionAlgorithm::handle_crossing (IntersectionGraph & *igraph*, const Segment & *s1*, const Segment & *s2*) [private]

segments cross

1.7.2.3 void IntersectionAlgorithm::handle_overlap (IntersectionGraph & *igraph*, const Segment & *s1*, const Segment & *s2*) [private]

segments intersect but do not cross

1.7.2.4 void IntersectionAlgorithm::handle_same_endpoint (const Segment & *s1*, const Segment & *s2*) const [private]

segments have the same end point

1.7.2.5 bool IntersectionAlgorithm::have_same_endpoint (const Segment & *s1*, const Segment & *s2*) const [private]

checks if two segments share an endpoint

Returns

the endpoint

1.7.2.6 void IntersectionAlgorithm::run (IntersectionGraph & *igraph*, SegmentContainer & *segments*)

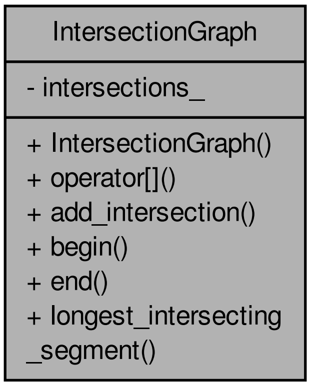
1.7.3 Member Data Documentation

1.7.3.1 SegmentIndex IntersectionAlgorithm::shortest_noncrossing_segment_

1.8 IntersectionGraph Class Reference

`#include <intersection_graph.h>`

Collaboration diagram for IntersectionGraph:



Public Member Functions

- IntersectionGraph (const SegmentIndex &size)
- const Intersections & operator[] (const SegmentIndex &index) const
- void add_intersection (const Segment &s1, const Segment &s2)
- IntersectionsVector::const_iterator begin () const
- IntersectionsVector::const_iterator end () const
- const SegmentIndex & longest_intersecting_segment (const SegmentIndex &index) const

Private Attributes

- IntersectionsVector intersections_

1.8.1 Constructor & Destructor Documentation

1.8.1.1 IntersectionGraph::IntersectionGraph (const SegmentIndex & *size*)

default constructor

1.8.2 Member Function Documentation

1.8.2.1 void IntersectionGraph::add_intersection (const Segment & *s1,* const Segment & *s2*)

add two intersecting segments to the graph

1.8.2.2 IntersectionsVector::const_iterator IntersectionGraph::begin ( ) const [inline]

1.8.2.3 IntersectionsVector::const_iterator IntersectionGraph::end ( ) const [inline]

1.8.2.4 const SegmentIndex & IntersectionGraph::longest_intersecting_segment (const SegmentIndex & *index*) const

1.8.2.5 const Intersections& IntersectionGraph::operator[] (const SegmentIndex & *index*) const [inline]

Returns

all intersecting segments for a segment

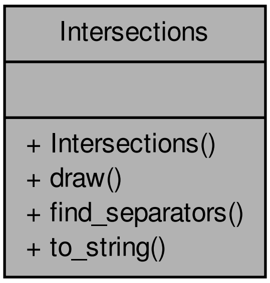
1.8.3 Member Data Documentation

1.8.3.1 IntersectionsVector IntersectionGraph::intersections_ [private]

1.9 Intersections Class Reference

`#include <intersections.h>`

Collaboration diagram for Intersections:



Public Member Functions

- Intersections ()
- void draw (QPainter &painter, const SegmentContainer &segments) const
- void find_separators (const SegmentIndex &segment_index, const SegmentContainer &segments, std::vector< SegmentIndex > &separators) const
- QString to_string (const SegmentContainer &segments) const

1.9.1 Detailed Description

sorted set of intersecting segments

1.9.2 Constructor & Destructor Documentation

1.9.2.1 Intersections::Intersections ( ) [inline]

default constructor

1.9.3 Member Function Documentation

1.9.3.1 void Intersections::draw (QPainter & *painter,* const SegmentContainer & *segments*) const

draws intersections using QPainter

1.9.3.2 void Intersections::find_separators (const SegmentIndex & *segment_index*, const SegmentContainer & *segments*, std::vector< SegmentIndex > & *separators*) const

finds all separators for a given segment and stores them in the passed container

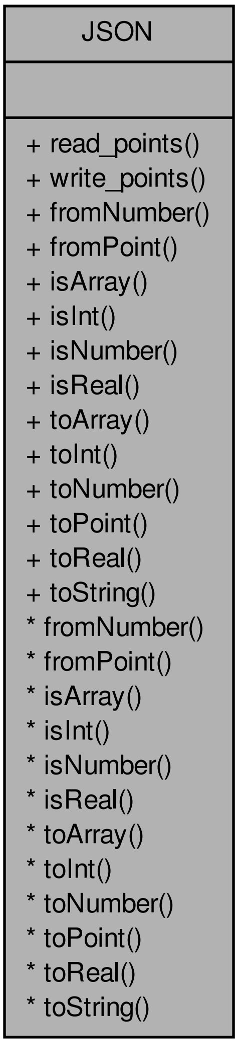
1.9.3.3 QString Intersections::to_string (const SegmentContainer & *segments*) const

output intersections to QString

1.10 JSON Class Reference

```
#include <json.h>
```

Collaboration diagram for JSON:



Static Public Member Functions

- template<typename OutputIterator >
  static bool read_points (QFile &file, OutputIterator output)
- template<typename Container >
  static bool write_points (const std::string &file_name, Container points)

helper functions

- static JSONValue fromNumber (const Number &value)
- static JSONArray fromPoint (const Point &point)
- static bool isArray (const JSONValue &value)
- static bool isInt (const JSONValue &value)
- static bool isNumber (const JSONValue &value)
- static bool isReal (const JSONValue &value)
- static const JSONArray & toArray (const JSONValue &value)
- static int toInt (const JSONValue &value)
- static Number toNumber (const JSONValue &value)
- static Point toPoint (const JSONValue &value)
- static double toReal (const JSONValue &value)
- static const std::string & toString (const JSONValue &value)

1.10.1 Member Function Documentation

1.10.1.1 JSON::JSONValue JSON::fromNumber (const Number & *value*) [static]

1.10.1.2 JSON::JSONArray JSON::fromPoint (const Point & *point*) [static]

1.10.1.3 bool JSON::isArray (const JSONValue & *value*) [static]

1.10.1.4 bool JSON::isInt (const JSONValue & *value*) [static]

1.10.1.5 bool JSON::isNumber (const JSONValue & *value*) [static]

1.10.1.6 bool JSON::isReal (const JSONValue & *value*) [static]

1.10.1.7 template<typename OutputIterator > static bool JSON::read_points (QFile & *file,* OutputIterator *output*) [inline], [static]

1.10.1.8 const JSON::JSONArray & JSON::toArray (const JSONValue & *value*) [static]

1.10.1.9 int JSON::toInt (const JSONValue & *value*) [static]

1.10.1.10 Number JSON::toNumber (const JSONValue & *value*) [static]

1.10.1.11 Point JSON::toPoint (const JSONValue & *value*) [static]

1.10.1.12 double JSON::toReal (const JSONValue & *value*) [static]

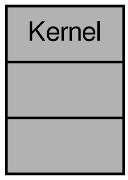
1.10.1.13 const std::string & JSON::toString (const JSONValue & *value*) [static]

1.10.1.14 template<typename Container > static bool JSON::write_points (const std::string & *file_name,* Container *points*) [inline], [static]

1.11 Kernel Struct Reference

```
#include <kernel.h>
```

Collaboration diagram for Kernel:



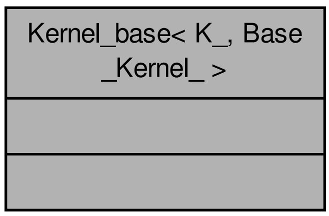
1.11.1 Detailed Description

customized kernel

1.12 Kernel_base< K_, Base_Kernel_ > Class Template Reference

```
#include <kernel.h>
```

Collaboration diagram for Kernel_base< K_, Base_Kernel_ >:



Classes

- struct Base

1.12.1 Detailed Description

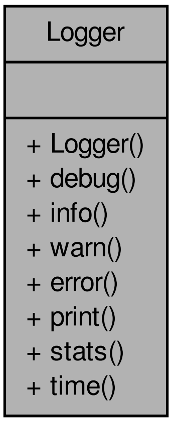
template< typename K_, typename Base_Kernel_>class Kernel_base< K_, Base_Kernel_ >

kernel base with customized PointC2 and SegmentC2

1.13 Logger Class Reference

```
#include <logger.h>
```

Collaboration diagram for Logger:



Public Member Functions

- Logger ()
- void debug (const QString &message) const
- void info (const QString &message) const
- void warn (const QString &message) const
- void error (const QString &message) const
- void print (const QString &message) const
- void stats (const Stats &stats) const
- void time (const QString &identifier, int milliseconds) const

1.13.1 Constructor & Destructor Documentation

1.13.1.1 Logger::Logger ( )

1.13.2 Member Function Documentation

1.13.2.1 void Logger::debug (const QString & *message*) const

1.13.2.2 void Logger::error (const QString & *message*) const

1.13.2.3 void Logger::info (const QString & *message*) const

1.13.2.4 void Logger::print (const QString & *message*) const

1.13.2.5 void Logger::stats (const Stats & *stats*) const

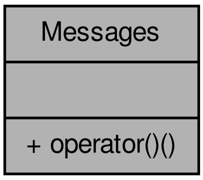
1.13.2.6 void Logger::time (const QString & *identifier,* int *milliseconds*) const

1.13.2.7 void Logger::warn (const QString & *message*) const

1.14 Messages Class Reference

```
#include <logger.h>
```

Collaboration diagram for Messages:



Public Member Functions

- QString operator() (const char ∗text)

1.14.1 Detailed Description

helper class for string literals

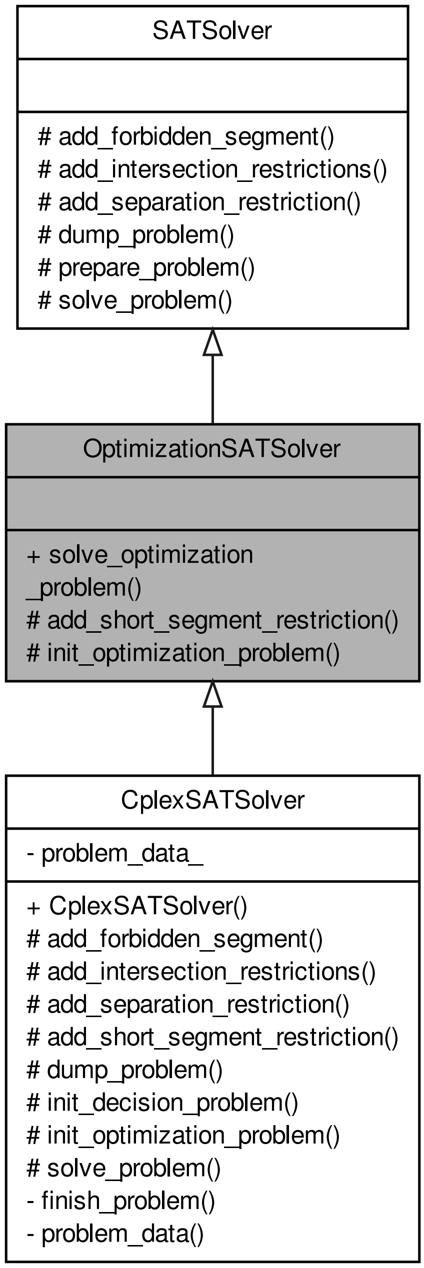
1.14.2 Member Function Documentation

1.14.2.1 QString Messages::operator() (const char ∗ *text*) [inline]

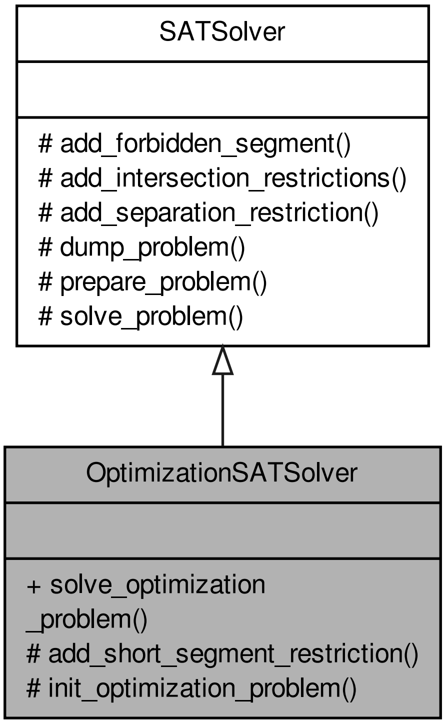
1.15 OptimizationSATSolver Class Reference

```
#include <sat_solver.h>
```

Inheritance diagram for OptimizationSATSolver:



Collaboration diagram for OptimizationSATSolver:



Public Member Functions

- void solve_optimization_problem (const QSettings &settings, const QString &file_prefix, const SATProblem &problem, SATSolution &solution)

Protected Member Functions

- virtual void add_short_segment_restriction (const SATProblem ∗problem, const SegmentIndex &index)=0
- virtual void init_optimization_problem (const SATProblem ∗problem)=0

1.15.1 Detailed Description

interface for optimization SAT solvers

1.15.2 Member Function Documentation

1.15.2.1 virtual void OptimizationSATSolver::add_short_segment_restriction (const SATProblem ∗ *problem,* const SegmentIndex & *index*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.15.2.2 virtual void OptimizationSATSolver::init_optimization_problem (const SATProblem * problem) [protected], [pure virtual]

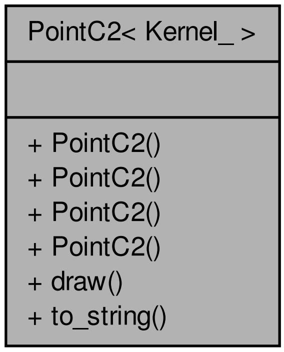
Implemented in CplexSATSolver.

1.15.2.3 void OptimizationSATSolver::solve_optimization_problem (const QSettings & settings, const QString & file_prefix, const SATProblem & problem, SATSolution & solution)

1.16 PointC2< Kernel_ > Class Template Reference

`#include <point.h>`

Collaboration diagram for PointC2< Kernel_ >:



Public Member Functions

- PointC2 ()
- PointC2 (const CGAL::Origin &origin)
- PointC2 (const FT &x, const FT &y)
- PointC2 (const FT &hx, const FT &hy, const FT &hw)
- void draw (QPainter &painter) const
- QString to_string () const

Private Types

- typedef Kernel_::FT FT
- typedef CGAL::PointC2< Kernel_ > PointBase

1.16.1 Detailed Description

template<class Kernel_>class PointC2< Kernel_ >

customized point type

1.16.2 Member Typedef Documentation

1.16.2.1 template<class Kernel_ > typedef Kernel_::FT **PointC2**< Kernel_ >::**FT** `[private]`

1.16.2.2 template<class Kernel_ > typedef CGAL::PointC2<Kernel_> **PointC2**< Kernel_ >::**PointBase** `[private]`

1.16.3 Constructor & Destructor Documentation

1.16.3.1 template<class Kernel_ > **PointC2**< Kernel_ >::**PointC2** ( ) `[inline]`

empty constructor

1.16.3.2 template<class Kernel_ > **PointC2**< Kernel_ >::**PointC2** (const CGAL::Origin & *origin*) `[inline]`

origin constructor

1.16.3.3 template<class Kernel_ > **PointC2**< Kernel_ >::**PointC2** (const **FT** & *x*, const **FT** & *y*) `[inline]`

Cartesian constructor

1.16.3.4 template<class Kernel_ > **PointC2**< Kernel_ >::**PointC2** (const **FT** & *hx*, const **FT** & *hy*, const **FT** & *hw*) `[inline]`

homogeneous constructor

1.16.4 Member Function Documentation

1.16.4.1 template<class Kernel_ > void **PointC2**< Kernel_ >::draw (QPainter & *painter*) const

draw segment using given QPainter

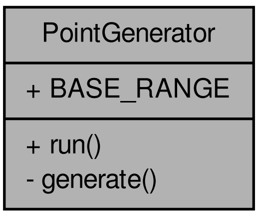
1.16.4.2 template<class Kernel_ > QString **PointC2**< Kernel_ >::to_string ( ) const

dump point to QString

1.17 PointGenerator Class Reference

`#include <point_generator.h>`

Collaboration diagram for PointGenerator:



Static Public Member Functions

- static void run (const QSettings &settings)

Static Public Attributes

- static const double BASE_RANGE = 100.0

Static Private Member Functions

- template<typename GeneratorType >
  static void generate (const QString &base_name, std::size_t num_points, std::size_t num_iterations)

1.17.1 Member Function Documentation

1.17.1.1 template<typename GeneratorType > static void PointGenerator::generate (const QString & *base_name,* std::size_t *num_points,* std::size_t *num_iterations*) [inline], [static], [private]

1.17.1.2 static void PointGenerator::run (const QSettings & *settings*) [inline], [static]

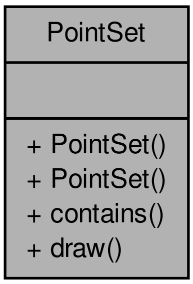
1.17.2 Member Data Documentation

1.17.2.1 const double PointGenerator::BASE_RANGE = 100.0 [static]

1.18 PointSet Class Reference

`#include <point_set.h>`

Collaboration diagram for PointSet:



Public Member Functions

- PointSet ()
- PointSet (QFile &input_file)
- bool contains (const Point &point) const
- void draw (QPainter &painter) const

1.18.1 Detailed Description

(sorted) set of points

1.18.2 Constructor & Destructor Documentation

1.18.2.1 PointSet::PointSet ( )

empty set

1.18.2.2 PointSet::PointSet (QFile & *input_file*)

read points from file

1.18.3 Member Function Documentation

1.18.3.1 bool PointSet::contains (const Point & *point*) const `[inline]`

shortcut for STL count()

Returns

true if point is in set

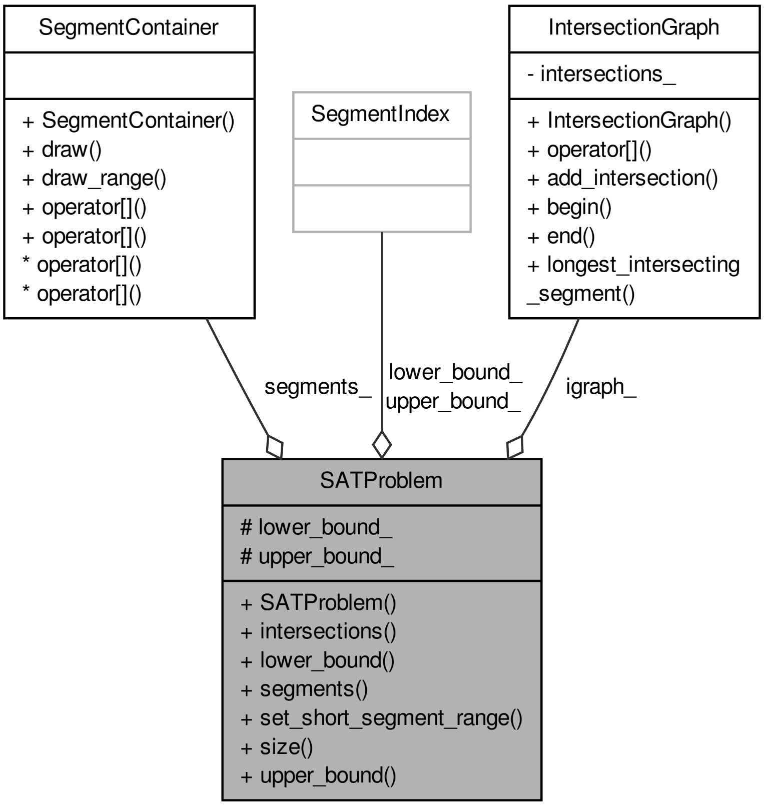
1.18.3.2 void PointSet::draw (QPainter & *painter*) const

output point set using QPainter

1.19 SATProblem Class Reference

`#include <sat_problem.h>`

Collaboration diagram for SATProblem:



Public Member Functions

- SATProblem (const IntersectionGraph &igraph, const SegmentContainer &segments)
- const Intersections & intersections (const SegmentIndex &index) const
- const SegmentIndex & lower_bound () const
- const SegmentContainer & segments () const
- void set_short_segment_range (const SegmentIndex &lower_bound, const SegmentIndex &upper_bound)
- SegmentIndex size () const
- const SegmentIndex & upper_bound () const

Protected Attributes

- const IntersectionGraph & igraph_
- const SegmentContainer & segments_
- SegmentIndex lower_bound_
- SegmentIndex upper_bound_

1.19.1 Constructor & Destructor Documentation

1.19.1.1 SATProblem::SATProblem (const IntersectionGraph & *igraph*, const SegmentContainer & *segments*)

default constructor

1.19.2 Member Function Documentation

1.19.2.1 const Intersections& SATProblem::intersections (const SegmentIndex & *index*) const [inline]

1.19.2.2 const SegmentIndex& SATProblem::lower_bound ( ) const [inline]

1.19.2.3 const SegmentContainer& SATProblem::segments ( ) const [inline]

1.19.2.4 void SATProblem::set_short_segment_range (const SegmentIndex & *lower_bound,* const SegmentIndex & *upper_bound*)

set range of segments to consider

1.19.2.5 SegmentIndex SATProblem::size ( ) const [inline]

1.19.2.6 const SegmentIndex& SATProblem::upper_bound ( ) const [inline]

1.19.3 Member Data Documentation

1.19.3.1 const IntersectionGraph& SATProblem::igraph_ [protected]

1.19.3.2 SegmentIndex SATProblem::lower_bound_ [protected]

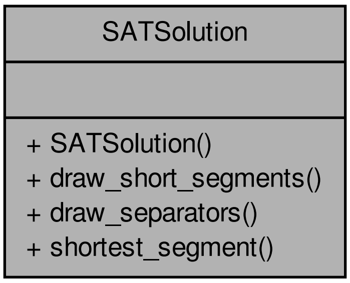
1.19.3.3 const SegmentContainer& SATProblem::segments_ [protected]

1.19.3.4 SegmentIndex SATProblem::upper_bound_ [protected]

1.20 SATSolution Class Reference

`#include <sat_solution.h>`

Collaboration diagram for SATSolution:



Public Member Functions

- SATSolution ()
- void draw_short_segments (QPainter &painter, const SegmentIndex &num_short_segments, const SegmentContainer &segments) const
- void draw_separators (QPainter &painter, const SegmentIndex &num_short_segments, const SegmentContainer &segments) const
- const SegmentIndex & shortest_segment () const

1.20.1 Constructor & Destructor Documentation

1.20.1.1 SATSolution::SATSolution ( ) [inline]

1.20.2 Member Function Documentation

1.20.2.1 void SATSolution::draw_separators (QPainter & *painter*, const SegmentIndex & *num_short_segments*, const SegmentContainer & *segments*) const

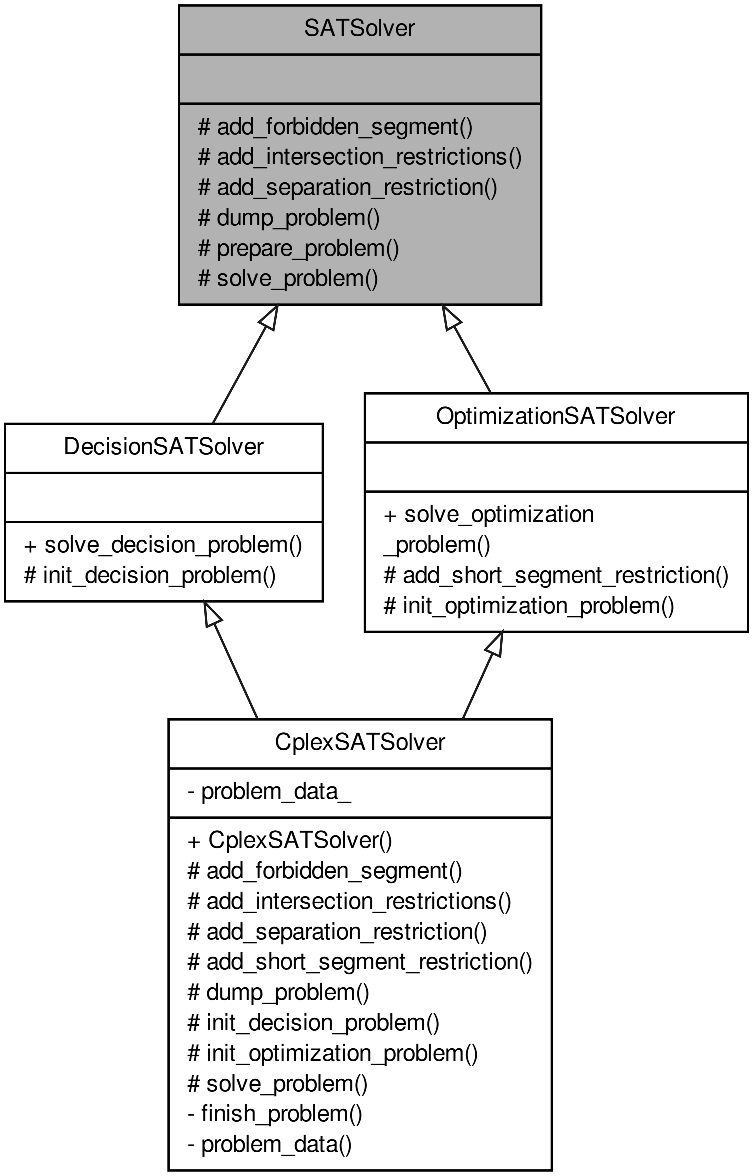
1.20.2.2 void SATSolution::draw_short_segments (QPainter & *painter*, const SegmentIndex & *num_short_segments*, const SegmentContainer & *segments*) const

1.20.2.3 const SegmentIndex & SATSolution::shortest_segment ( ) const

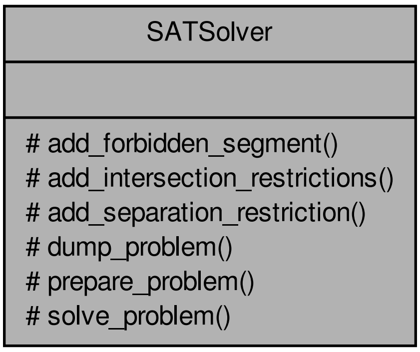
1.21 SATSolver Class Reference

```
#include <sat_solver.h>
```

Inheritance diagram for SATSolver:



Collaboration diagram for SATSolver:



Protected Member Functions

- virtual void add_forbidden_segment (const SATProblem ∗problem, const SegmentIndex &index)=0
- virtual void add_intersection_restrictions (const SATProblem ∗problem, const SegmentIndex &index, const Intersections &igroup)=0
- virtual void add_separation_restriction (const SATProblem ∗problem, const SegmentIndex &index, const std::vector< SegmentIndex > &separators)=0
- virtual void dump_problem (const QString &file_prefix, const SATProblem ∗problem)=0
- void prepare_problem (const SATProblem &problem)
- virtual void solve_problem (const SATProblem ∗problem, SATSolution &solution)=0

1.21.1 Detailed Description

interface for SAT solvers

1.21.2 Member Function Documentation

1.21.2.1 virtual void SATSolver::add_forbidden_segment (const SATProblem ∗ *problem,* const SegmentIndex & *index*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.21.2.2 virtual void SATSolver::add_intersection_restrictions (const SATProblem ∗ *problem,* const SegmentIndex & *index,* const Intersections & *igroup*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.21.2.3 virtual void SATSolver::add_separation_restriction (const SATProblem ∗ *problem,* const SegmentIndex & *index,* const std::vector< SegmentIndex > & *separators*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.21.2.4 virtual void SATSolver::dump_problem (const QString & *file_prefix,* const SATProblem ∗ *problem*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.21.2.5 void SATSolver::prepare_problem (const SATProblem & *problem*) [protected]

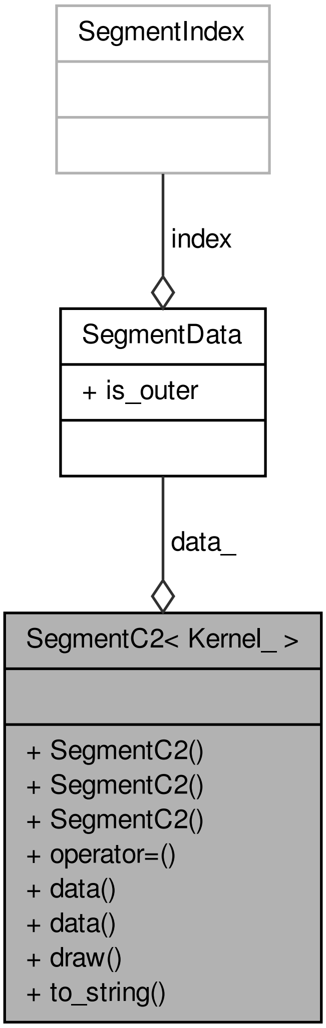
1.21.2.6 virtual void SATSolver::solve_problem (const SATProblem * *problem*, SATSolution & *solution*) [protected], [pure virtual]

Implemented in CplexSATSolver.

1.22 SegmentC2< Kernel_ > Class Template Reference

`#include <segment.h>`

Collaboration diagram for SegmentC2< Kernel_ >:



Public Member Functions

- SegmentC2 ()
- SegmentC2 (const Point_2 &source, const Point_2 &target)

- SegmentC2 (const SegmentC2 &other)
- SegmentC2 & operator= (const SegmentC2 &other)
- SegmentData & data ()
- const SegmentData & data () const
- void draw (QPainter &painter) const
- QString to_string () const

Private Attributes

- SegmentData data_

1.22.1 Detailed Description

template<class Kernel_>class SegmentC2< Kernel_ >

customized segment type

1.22.2 Constructor & Destructor Documentation

1.22.2.1 template<class Kernel_ > SegmentC2< Kernel_ >::SegmentC2 ( ) [inline]

empty constructor

1.22.2.2 template<class Kernel_ > SegmentC2< Kernel_ >::SegmentC2 (const Point_2 & *source*, const Point_2 & *target*) [inline]

base constructor

1.22.2.3 template<class Kernel_ > SegmentC2< Kernel_ >::SegmentC2 (const SegmentC2< Kernel_ > & *other*) [inline]

copy constructor

1.22.3 Member Function Documentation

1.22.3.1 template<class Kernel_ > SegmentData& SegmentC2< Kernel_ >::data ( ) [inline]

getter for attached data

1.22.3.2 template<class Kernel_ > const SegmentData& SegmentC2< Kernel_ >::data ( ) const [inline]

constant getter for attached data

1.22.3.3 template<class Kernel_ > void SegmentC2< Kernel_ >::draw (QPainter & *painter*) const

draw segment using given QPainter

1.22.3.4 template<class Kernel_ > SegmentC2& SegmentC2< Kernel_ >::operator= (const SegmentC2< Kernel_ > & *other*) [inline]

assignment operator

1.22.3.5 template<class Kernel_ > QString SegmentC2< Kernel_ >::to_string ( ) const

dump segment to QString

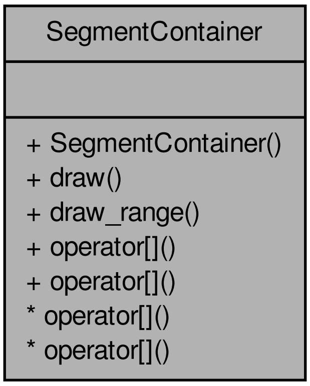
1.22.4 Member Data Documentation

1.22.4.1 template<class Kernel_ > SegmentData SegmentC2< Kernel_ >::data_ [private]

1.23 SegmentContainer Class Reference

```
#include <segment_container.h>
```

Collaboration diagram for SegmentContainer:



Public Member Functions

- SegmentContainer (const PointSet &points)
- void draw (QPainter &painter) const
- void draw_range (QPainter &painter, const SegmentIndex &lower_bound, const SegmentIndex &upper_-bound) const

access i-th shortest segment

these operators assume that the segment set is not changed after construction

- Segment & operator[] (const SegmentIndex &index)
- const Segment & operator[] (const SegmentIndex &index) const

1.23.1 Detailed Description

container of segments sorted by length

1.23.2 Constructor & Destructor Documentation

1.23.2.1 SegmentContainer::SegmentContainer (const PointSet & *points*)

construct segments for all point pairs from set

1.23.3 Member Function Documentation

1.23.3.1 void SegmentContainer::draw (QPainter & *painter*) const

draws all segments

1.23.3.2 void SegmentContainer::draw_range (QPainter & *painter,* const SegmentIndex & *lower_bound,* const SegmentIndex & *upper_bound*) const

draw a range of segments

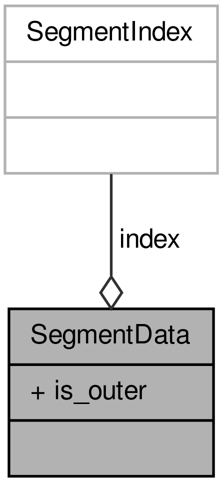
1.23.3.3 Segment & SegmentContainer::operator[] (const SegmentIndex & *index*)

1.23.3.4 const Segment & SegmentContainer::operator[] (const SegmentIndex & *index*) const

1.24 SegmentData Struct Reference

`#include <segment.h>`

Collaboration diagram for SegmentData:



Public Attributes

- SegmentIndex index
- bool is_outer

1.24.1 Detailed Description

data attached to a segment

1.24.2 Member Data Documentation

1.24.2.1 SegmentIndex SegmentData::index

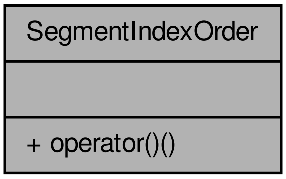
1.24.2.2 bool SegmentData::is_outer

true if the segment includes another

1.25 SegmentIndexOrder Struct Reference

`#include <orders.h>`

Collaboration diagram for SegmentIndexOrder:



Public Member Functions

- CGAL::Comparison_result operator() (const Segment &s, const Segment &t) const

1.25.1 Detailed Description

CGAL order for Segment by index

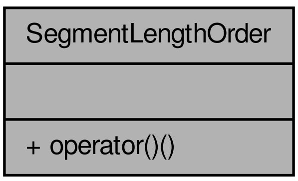
1.25.2 Member Function Documentation

1.25.2.1 CGAL::Comparison_result SegmentIndexOrder::operator() (const Segment & *s*, const Segment & *t*) const

1.26 SegmentLengthOrder Struct Reference

`#include <orders.h>`

Collaboration diagram for SegmentLengthOrder:



Public Member Functions

- CGAL::Comparison_result operator() (const Segment &s, const Segment &t) const

1.26.1 Detailed Description

CGAL order for Segment by length

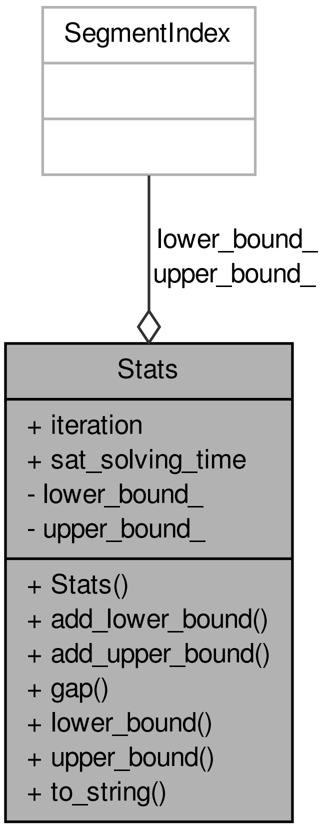
1.26.2 Member Function Documentation

1.26.2.1 CGAL::Comparison_result SegmentLengthOrder::operator() (const Segment & *s*, const Segment & *t*) const

1.27 Stats Class Reference

`#include <stats.h>`

Collaboration diagram for Stats:



Public Member Functions

- Stats ()
- void add_lower_bound (const SegmentIndex &bound)

- void add_upper_bound (const SegmentIndex &bound)
- SegmentIndex gap () const
- const SegmentIndex & lower_bound () const
- const SegmentIndex & upper_bound () const
- QString to_string () const

Public Attributes

- size_t iteration
- quint64 sat_solving_time

Private Attributes

- SegmentIndex lower_bound_
- SegmentIndex upper_bound_

1.27.1 Constructor & Destructor Documentation

1.27.1.1 Stats::Stats ( ) [inline]

1.27.2 Member Function Documentation

1.27.2.1 void Stats::add_lower_bound (const SegmentIndex & *bound*) [inline]

1.27.2.2 void Stats::add_upper_bound (const SegmentIndex & *bound*) [inline]

1.27.2.3 SegmentIndex Stats::gap ( ) const [inline]

1.27.2.4 const SegmentIndex& Stats::lower_bound ( ) const [inline]

1.27.2.5 QString Stats::to_string ( ) const [inline]

1.27.2.6 const SegmentIndex& Stats::upper_bound ( ) const [inline]

1.27.3 Member Data Documentation

1.27.3.1 size_t Stats::iteration

1.27.3.2 SegmentIndex Stats::lower_bound_ [private]

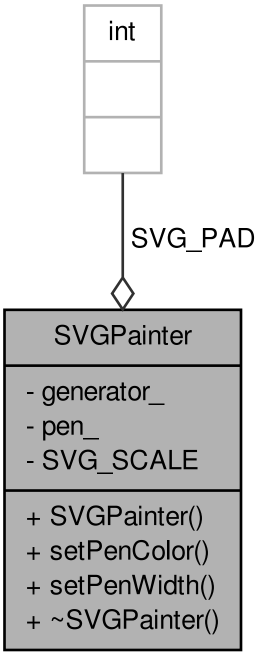
1.27.3.3 quint64 Stats::sat_solving_time

1.27.3.4 SegmentIndex Stats::upper_bound_ [private]

1.28 SVGPainter Class Reference

```
#include <svg_painter.h>
```

Collaboration diagram for SVGPainter:



Public Member Functions

- SVGPainter (const QString &file_prefix, const QString &file_name, const BoundingBox &bbox)
- void setPenColor (const QColor &color)
- void setPenWidth (int width)
- ∼SVGPainter ()

Private Attributes

- QSvgGenerator generator_
- QPen pen_

Static Private Attributes

- static const int SVG_PADDING = 10
- static const double SVG_SCALE = 4.0

1.28.1 Constructor & Destructor Documentation

1.28.1.1 SVGPainter::SVGPainter (const QString & *file_prefix*, const QString & *file_name*, const BoundingBox & *bbox*)

1.28.1.2 SVGPainter::∼SVGPainter ( )

1.28.2 Member Function Documentation

1.28.2.1 void SVGPainter::setPenColor (const QColor & *color*)

1.28.2.2 void SVGPainter::setPenWidth (int *width*)

1.28.3 Member Data Documentation

1.28.3.1 QSvgGenerator SVGPainter::generator_ [private]

1.28.3.2 QPen SVGPainter::pen_ [private]

1.28.3.3 const int SVGPainter::SVG_PADDING = 10 [static], [private]

padding for SVG images

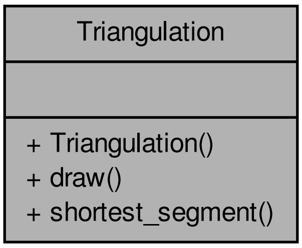
1.28.3.4 const double SVGPainter::SVG_SCALE = 4.0 [static], [private]

scale for SVG images

1.29 Triangulation Class Reference

`#include <triangulation.h>`

Collaboration diagram for Triangulation:



Public Member Functions

- Triangulation (const PointSet &points)
- void draw (QPainter &painter) const
- const SegmentIndex & shortest_segment (const SegmentContainer &segments) const

1.29.1 Constructor & Destructor Documentation

1.29.1.1 Triangulation::Triangulation (const PointSet & *points*)

default constructor

1.29.2 Member Function Documentation

1.29.2.1 void Triangulation::draw (QPainter & *painter*) const

draw triangulation segments using given QPainter

1.29.2.2 const SegmentIndex & Triangulation::shortest_segment (const SegmentContainer & *segments*) const

B. Result Data

The following tables contain the data plotted in chapter 6.

	Shortest Non-Separable Edge Length Index		optimal Edge Length Index		instances
	median	**err**	**median**	**err**	
number of points					
10	0.0	0.000000	0.0	0.000000	100
20	0.0	0.000000	0.0	0.000000	100
30	1.0	0.200000	1.0	0.200000	100
40	2.0	0.400000	2.0	0.400000	100
50	2.0	0.400000	2.0	0.400000	100
60	2.0	0.400000	2.0	0.400000	100
70	4.0	0.600000	3.0	0.600000	100
80	2.0	0.400000	2.0	0.400000	100
90	5.0	0.804030	5.0	0.804030	99
100	3.0	0.600000	3.0	0.600000	100
110	5.0	1.005038	5.0	0.804030	99
120	5.0	0.812277	4.0	0.812277	97
130	4.5	0.714435	4.5	0.714435	96
140	6.5	0.928279	6.0	0.825137	94
150	7.0	1.218415	6.0	1.015346	97
160	7.5	1.146829	6.5	1.146829	92
170	7.0	1.244342	6.0	1.036952	93
180	6.0	1.048285	5.0	0.838628	91
190	9.0	1.677256	8.0	1.467599	91
200	6.0	1.059998	6.0	0.847998	89
210	12.0	1.929803	10.0	1.715380	87
220	9.0	1.301583	9.0	1.301583	85
230	9.0	1.823369	6.0	1.367527	77
240	9.0	1.595448	9.0	1.595448	77
250	11.0	1.823369	10.0	1.595448	77
260	10.0	1.333333	10.0	1.333333	81

	Shortest Non-Separable Edge Length Index		optimal Edge Length Index		instances
	median	**err**	**median**	**err**	
number of points					
270	7.5	1.261787	7.0	1.147079	76
280	9.0	1.767767	8.0	1.532065	72
290	8.0	1.394972	7.5	1.278724	74
300	7.0	1.221694	7.0	1.221694	67
310	10.0	1.532065	9.5	1.296362	72
320	8.5	1.375000	8.5	1.375000	64
330	9.0	1.477098	9.0	1.354006	66
340	9.0	1.710372	8.0	1.710372	67
350	10.5	1.625000	10.5	1.625000	64
360	8.0	1.301889	8.0	1.301889	59
370	7.0	1.603567	6.5	1.469937	56
380	7.0	1.575677	7.0	1.575677	58
390	11.5	1.181758	11.5	1.181758	58
400	8.0	2.020726	8.0	2.020726	48
410	9.0	1.373606	9.0	1.373606	53
420	7.0	1.524002	6.0	1.270001	62
430	9.0	1.458650	9.0	1.458650	47
440	11.0	1.887760	11.0	1.887760	55
450	9.0	1.555635	9.0	1.697056	50
460	9.0	1.219989	8.0	0.914991	43
470	7.0	1.078720	7.0	1.078720	55
480	6.0	1.750380	6.0	1.750380	47
490	8.0	1.697056	7.5	1.555635	50
500	9.5	1.916745	9.5	1.916745	46

Table B.1.: Edge Length Index

	shortest line segment		Shortest Non-separable Edge		instances
	median	**err**	**median**	**err**	
number of points					
10	0.132175	0.009518	0.161383	0.013075	100
20	0.061388	0.005080	0.084226	0.009750	100
30	0.048673	0.004021	0.081751	0.008222	100
40	0.035132	0.002244	0.067223	0.006086	100
50	0.024943	0.001819	0.057941	0.005639	100
60	0.024920	0.001863	0.051782	0.004601	100
70	0.016162	0.001276	0.046139	0.004624	100
80	0.016871	0.001409	0.032974	0.003039	100
90	0.014177	0.000955	0.042176	0.003836	99
100	0.013673	0.001032	0.032963	0.003303	100
110	0.012539	0.000975	0.036238	0.003316	99
120	0.010689	0.000916	0.034097	0.003012	97
130	0.009648	0.000830	0.029811	0.003224	96
140	0.009766	0.000751	0.031462	0.002516	94
150	0.009127	0.000681	0.029580	0.002950	97
160	0.008100	0.000617	0.029461	0.002537	92
170	0.008574	0.000536	0.026119	0.002434	93
180	0.007030	0.000557	0.023387	0.002379	91
190	0.007293	0.000642	0.025347	0.002313	91
200	0.006084	0.000619	0.020896	0.002083	89
210	0.006249	0.000503	0.026915	0.002161	87
220	0.005801	0.000527	0.023859	0.001853	85
230	0.005436	0.000567	0.020446	0.002426	77
240	0.005324	0.000463	0.021653	0.001859	77
250	0.005367	0.000629	0.023539	0.001594	77
260	0.005336	0.000406	0.019110	0.001627	81
270	0.004629	0.000364	0.017739	0.001778	76

	shortest line segment		Shortest Non-separable Edge		instances
	median	**err**	**median**	**err**	
number of points					
280	0.005107	0.000320	0.017681	0.001439	72
290	0.004110	0.000393	0.017616	0.001677	74
300	0.004083	0.000399	0.015546	0.001381	67
310	0.003912	0.000370	0.017238	0.001552	72
320	0.004676	0.000468	0.015333	0.000997	64
330	0.003608	0.000397	0.015066	0.001078	66
340	0.003911	0.000409	0.014327	0.001447	67
350	0.004471	0.000369	0.015060	0.001151	64
360	0.003801	0.000376	0.012530	0.000966	59
370	0.003871	0.000355	0.011661	0.001430	56
380	0.003203	0.000294	0.011038	0.001522	58
390	0.003669	0.000299	0.012678	0.000702	58
400	0.003278	0.000392	0.013592	0.001286	48
410	0.003492	0.000353	0.012744	0.001185	53
420	0.003028	0.000302	0.010433	0.001097	62
430	0.002933	0.000259	0.011771	0.001014	47
440	0.002779	0.000286	0.012152	0.000917	55
450	0.003091	0.000324	0.011425	0.001075	50
460	0.002797	0.000403	0.010733	0.000856	43
470	0.002917	0.000289	0.009871	0.000735	55
480	0.002776	0.000277	0.008862	0.001049	47
490	0.002710	0.000351	0.009240	0.001000	50
500	0.002280	0.000255	0.009988	0.001130	46

Table B.2.: Line segment length

	time for IP solving		total running time		time for segment generation		instances
	median	**err**	**median**	**err**	**median**	**err**	
number of points							
10	0.0	0.000000	1.0	0.000000	0.0	0.000000	100
20	0.0	0.000000	4.0	0.200000	1.0	0.000000	100
30	5.0	1.000000	14.0	1.600000	4.0	0.000000	100
40	7.0	1.400000	22.0	2.400000	7.0	0.000000	100
50	9.0	1.800000	32.5	3.300000	12.0	0.000000	100
60	13.0	2.600000	44.5	4.300000	19.0	0.000000	100
70	23.5	4.700000	83.0	10.000000	28.0	0.000000	100
80	20.0	4.000000	76.0	6.200000	39.0	0.000000	100
90	81.0	16.281613	234.0	34.774308	54.0	0.201008	99
100	65.0	13.000000	204.0	23.100000	74.0	0.200000	100
110	172.0	34.573301	450.0	69.347609	97.0	0.402015	99
120	258.0	52.391862	514.0	75.135616	125.0	0.406138	97
130	161.5	32.966049	481.5	62.359926	159.0	0.408248	96
140	931.5	169.256227	1597.0	270.954361	203.0	0.618853	94
150	720.0	146.209848	1379.0	224.797641	251.0	0.609208	97
160	2093.5	398.679560	3407.5	599.166169	311.0	0.834058	92
170	1903.0	347.586208	2984.0	488.611639	377.0	1.036952	93
180	1390.0	291.423185	2147.0	348.869194	456.0	1.257942	91
190	3042.0	637.776495	4426.0	808.017952	551.0	1.467599	91
200	1576.0	334.111332	2605.0	401.739197	657.0	1.695997	89
210	8672.0	1859.471980	10905.0	2165.452898	777.0	2.144225	87
220	4635.0	1005.472672	7127.0	1315.900157	920.0	3.470887	85
230	5389.0	1228.267093	7600.0	1479.436204	1085.0	4.786344	77
240	6224.0	1212.084691	9534.0	1653.795886	1271.0	7.065556	77
250	16057.0	3426.110771	20711.0	4131.070897	1492.0	12.079821	77
260	11315.0	2248.888889	15339.0	2735.555556	1772.0	16.000000	81

	time for IP solving		total running time		time for segment generation		instances
	median	err	median	err	median	err	
number of points							
270	9351.0	1824.543331	14146.0	2442.015779	2071.5	19.729753	76
280	11973.5	2443.878887	17242.0	3103.138109	2477.0	30.052038	72
290	10032.0	1969.467496	15060.0	2380.867889	2938.0	35.688025	74
300	8401.0	1540.068016	13059.0	1947.625282	3519.0	52.044183	67
310	22352.5	4701.435137	31695.0	5958.670994	4132.0	55.036478	72
320	22990.5	4828.625000	32445.0	5711.500000	4949.5	53.125000	64
330	18390.5	3765.368709	28373.0	4726.713254	5916.5	86.287135	66
340	22363.0	5464.150568	32356.0	6194.479507	7078.0	97.246878	67
350	29520.0	6378.375000	41994.5	7310.625000	8504.5	91.500000	64
360	17658.0	3468.492966	30706.0	4232.962252	10085.0	111.441708	59
370	19042.0	5089.188568	33177.0	5380.236061	12042.0	103.430101	56
380	22932.5	6022.369543	39843.0	6585.280221	13835.0	126.448095	58
390	53730.0	12351.470913	78368.0	14289.553862	16584.0	137.215222	58
400	31308.5	9037.985451	56022.5	10531.734935	19533.0	154.585535	48
410	36274.0	8444.652750	60817.0	9633.371071	22254.0	118.679527	53
420	35559.5	9032.122032	63916.5	9586.604587	26051.5	146.177146	62
430	46750.0	10841.853088	82310.0	12262.578105	30052.0	196.917739	47
440	68665.0	15874.170602	108771.0	17664.036397	33845.0	228.418913	55
450	61658.0	14333.337297	114239.0	16987.816154	38078.0	237.870721	50
460	60985.0	11096.100975	106764.0	12283.759840	43260.0	259.247570	43
470	41744.0	8316.659823	63928.0	9935.818213	15636.0	107.871978	55
480	40974.0	11953.344323	60723.0	12425.071706	17746.0	125.443893	47
490	48911.5	10191.530037	73846.5	11342.134192	20321.0	168.432835	50
500	103511.0	27683.554246	179198.5	30294.308963	67510.0	426.844463	46

Table B.3.: Composition of running times

	total running time		instances
	median	err	
number of points			
10	1.0	0.000000	100
20	4.0	0.200000	100
30	14.0	1.600000	100
40	22.0	2.400000	100
50	32.5	3.300000	100
60	44.5	4.300000	100
70	83.0	10.000000	100
80	76.0	6.200000	100
90	234.0	34.774308	99
100	204.0	23.100000	100
110	450.0	69.347609	99
120	514.0	75.135616	97
130	481.5	62.359926	96
140	1597.0	270.954361	94
150	1379.0	224.797641	97
160	3407.5	599.166169	92
170	2984.0	488.611639	93
180	2147.0	348.869194	91
190	4426.0	808.017952	91
200	2605.0	401.739197	89
210	10905.0	2165.452898	87
220	7127.0	1315.900157	85
230	7600.0	1479.436204	77
240	9534.0	1653.795886	77
250	20711.0	4131.070897	77
260	15339.0	2735.555556	81
270	14146.0	2442.015779	76

	total running time		instances
	median	**err**	
number of points			
280	17242.0	3103.138109	72
290	15060.0	2380.867889	74
300	13059.0	1947.625282	67
310	31695.0	5958.670994	72
320	32445.0	5711.500000	64
330	28373.0	4726.713254	66
340	32356.0	6194.479507	67
350	41994.5	7310.625000	64
360	30706.0	4232.962252	59
370	33177.0	5380.236061	56
380	39843.0	6585.280221	58
390	78368.0	14289.553862	58
400	56022.5	10531.734935	48
410	60817.0	9633.371071	53
420	63916.5	9586.604587	62
430	82310.0	12262.578105	47
440	108771.0	17664.036397	55
450	114239.0	16987.816154	50
460	106764.0	12283.759840	43
470	63928.0	9935.818213	55
480	60723.0	12425.071706	47
490	73846.5	11342.134192	50
500	179198.5	30294.308963	46

Table B.4.: Total running times for our algorithm

	total running time		instances
	median	**err**	
number of points			
10	20.0	0.200000	100
20	144.0	0.600000	100
30	650.0	2.400000	100
40	1963.0	6.300000	100
50	4678.0	10.900000	100
60	9668.5	29.000000	100
70	17897.0	50.800000	100
80	30678.5	47.000000	100
90	49213.0	113.900000	100
100	110061.0	998.600000	100
110	442401.0	3036.800000	100
120	845285.0	4047.890305	99
130	1417066.0	5876.938265	96

Table B.5.: Total running times for the complete IP

Glossary

Acronyms

CDS . . . . Covering by Disjoint Segments

CGAL . . . . Computational Geometry Algorithms Library

IP . . . . Integer Program

MELT . . . . MaxMin Edge Length Index Triangulation

MMLT . . . . MaxMin Length Triangulation

NOCS . . . . Non-Conflicting Separators

Properties

crossing . . . . see definition 3.12

non-crossing . . . . see definition 3.12

non-overlapping . . . . see definition 3.13

overlapping . . . . see definition 3.13

well-covered . . . . see definition 2.14

Sets

E_{flip} . . . . set of edge flips (see definition 3.27)

E_{sep} . . . . set of separators (see definition 4.14)

E_{short} . . . . set of short edges (see definition 4.13)

G_{conf} . . . . conflict graph (see definition 3.3)

G_{cross} . . . . intersection graph (see definition 3.19)

G_{flip} . . . . flip graph (see definition 3.29)

S_{opt} . . . . an optimal set of segments

T_{opt} . . . . an optimal triangulation

V_{IS} . . . . independent set (see definition 2.16)

V_{cover} . . . . vertex cover (see definition 2.9)

Other Symbols

e_{nose} . . . . shortest non-separable edge

e_{sep} . . . . separating edge

$s_{\times}$ . . . . crossing segment

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With the MMLT algorithm at hand, Greg, who loves triangles [36], finally made his way to world domination.