Running list of notes and calculations through the project.

$$\Delta v \ = \ v_e \ln\left(\frac{m_i}{m_f}\right)$$

Start with Newton's Law of Gravitation expressed in its form as a second order differential equation.

$$\overrightarrow{\ddot{r}} \ = \ \frac{-\mu}{|\vec{r}\,|^2} \ \hat{r}$$

Now decompose into cartesian coordinates.

$$\begin{aligned} x \ &= \ r \ \cos\phi \ \sin\theta \\ y \ &= \ r \ \sin\phi \ \sin\theta \\ z \ &= \ r \ \cos\theta \end{aligned}$$

In our case, the 'r' is actually the force vector decomposition.

$$\begin{aligned} F_x \ &= \ F \ \cos\phi \ \sin\theta \\ F_y \ &= \ F \ \sin\phi \ \sin\theta \\ F_z \ &= \ F \ \cos\theta \end{aligned}$$

First, x-coordinates…

$$\ddot{x} \ = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \cos\phi \ \sin\theta = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \frac{x}{(x^2+y^2)^{1/2}} \sin\theta$$

$$\ddot{x} \ = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \frac{x}{(x^2+y^2)^{1/2}} \frac{(x^2+y^2)^{1/2}}{(x^2+y^2+z^2)^{1/2}}$$

$$\ddot{x} \ = \ \frac{-\mu x}{(x^2+y^2+z^2)^{3/2}}$$

Now y-coordinates…

$$\ddot{y} \ = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \sin\phi \ \sin\theta = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \frac{y}{(x^2+y^2)^{1/2}} \sin\theta$$

$$\ddot{y} \ = \ \frac{-\mu}{(x^2+y^2+z^2)} \ \frac{y}{(x^2+y^2)^{1/2}} \frac{(x^2+y^2)^{1/2}}{(x^2+y^2+z^2)^{1/2}}$$

$$\ddot{y} = \frac{-\mu y}{(x^2 + y^2 + z^2)^{3/2}}$$

Finally, z-coordinates…

$$\ddot{z} = \frac{-\mu}{(x^2 + y^2 + z^2)} \cos\theta = \frac{-\mu}{(x^2 + y^2 + z^2)} \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\ddot{z} = \frac{-\mu z}{(x^2 + y^2 + z^2)^{3/2}}$$

Hohmann Transfer and Trajectory Convergence

Now that we have a functional simulation code, the next step is to determine numerical solutions to this boundary value problem. The initial conditions we're solving for are the velocity components that take us from Earth to Mars within the constraints. Then, we can determine the magnitude of the velocity, which is the Δv that will ultimately represent one point on the pork chop plot.

We will use ephemeris data to build $\vec{r}_{Earth}(t)$ and $\vec{r}_{Mars}(t)$, and then pick an initial and final time, $t_1,\ t_2$, and then constrain the problem as follows:

$$\overrightarrow{r_i} = \vec{r}_{Earth}(t_1)$$

$$\overrightarrow{r_f} = \vec{r}_{Mars}(t_2)$$

$$\Delta t = t_2 - t_1$$

And of course, the trajectory must obey the laws of physics at all times. Specifically, the trajectory must obey Newton's Law of Gravitation (shown again, below):[1]

$$\overrightarrow{\ddot{r}} = \frac{-\mu}{|\,\vec{r}\,|^2}\ \hat{r}$$

So what does this mean for our computer simulation? For any given sample point (on the eventual pork chop plot), we already know $\overrightarrow{r_i},\ \overrightarrow{r_f},\ \Delta t$, so it makes the most sense to simply

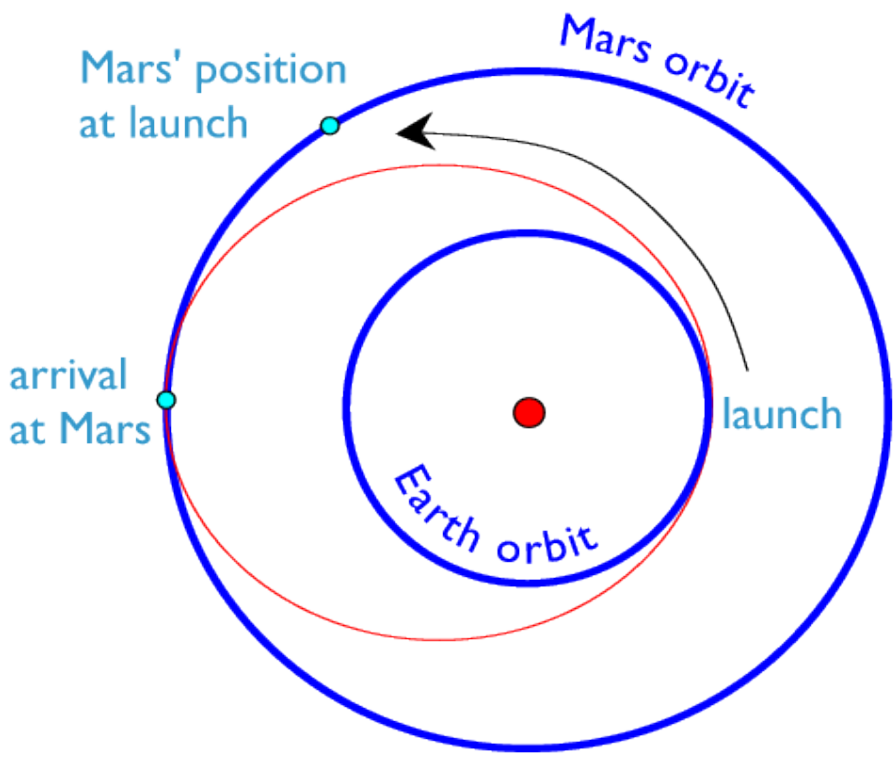
[1] There's a small caveat here. If we allow the spacecraft to make course corrections (ie: fire its rockets) at times during the orbit, then this is not necessarily the case. However, we are sticking to a simple instantaneous perfect impulse burn to jump out of the Earth orbit, and then a purely ballistic trajectory to Mars.

start the simulation at point $\overrightarrow{r_i}$ with some initial velocity vector, $(\dot{x}, \dot{y}, \dot{z})$, and then simulate forward for time Δt. Then we ask how far away the final position is from its target by defining the following "displacement vector":

$$\vec{r}_{error} \equiv \sqrt{\overrightarrow{r_f}^2 - \vec{r}^2_{Mars}(t_2)}$$

Ultimately, we should. be able to build a 3d (or 2d) contour plot with variables $\dot{x}, \dot{y}, \dot{z}$ (or variables $\dot{x}, \dot{y}$) that contribute to $|\vec{r}_{error}|$. Obviously the correct trajectory has $|\vec{r}_{error}| = 0$, so we will try to find the minima of this system. Depending on how long this code takes to converge, we may specify to accept something like $|\vec{r}_{error}| \leq 500$ km, or some other sensical convergence value.

First, let us build a proof of concept using a model we may solve analytically: the basic Hohmann Transfer. Let us use circular orbit approximations for both Earth and Mars.



Hohmann transfer trajectory

Perhaps later we will make our own plot, but for now this one will suffice. In this example, $\overrightarrow{r_i} \;=\; \vec{r}_{Earth}(t_1)$ is located at the "launch" text. $\overrightarrow{r_f} \;=\; \vec{r}_{Mars}(t_2)$ is located at "arrival at Mars." The trajectory is the (top half of) the red line. For clarity, this diagram also shows Mars' position at launch, but that will not factor into the math in any way.

Ultimately, we define $\overrightarrow{r_i} \;=\; |\vec{r}_{Earth}|\;\hat{x}$ and $\overrightarrow{r_f} \;=\; -|\vec{r}_{Mars}|\;\hat{x}$. Below, we plug in numbers using the approximation that earth and Mars both have perfectly circular orbits. (That is, we look up semi-major axis, a, and assert that $r_{circ} = a$)

	Semi-Major Axis (orbital "radius" for approximation)		Velocity
Earth:	1.0000 AU	1.496 $\times 10^{11}$ m	$29{,}790$ m/s
Mars:	1.5237 AU	2.279 $\times 10^{11}$ m	$24{,}130$ m/s

With our position constraints introduced, we now only need the time constraint, Δt.

For time, we simply use Kepler's Third Law. The period for an orbit is below:

$$T \;=\; 2\pi\sqrt{\frac{a^3}{GM}}$$

Since we are taking a symmetrical slice of half the orbit, we can simply cut this number in half, for the time from periapsis to apoapsis. For the transfer orbit, we define the semi-major axis from the Earth an Mars circular orbit radii, $a = (r_{Earth} + r_{Mars})/2$.

$$\Delta t \;=\; \pi\sqrt{\frac{a^3}{GM}} \qquad\qquad a = 1.888 \times 10^{11}\text{ m}$$

$$\Delta t = 22{,}368{,}649 \text{ sec} \approx 258 \text{ days}$$

And so all of our constraints are fulfilled. For the computer, we would now begin to iterate through possible initial conditions (possible initial velocities), but in this we may also solve for the velocity analytically, to check our answer.

To analytically solve for the velocity required, we may use the vis-viva equation.

$$v^2 \;=\; GM\left(\frac{2}{r} - \frac{1}{a}\right) \qquad\qquad |v_i| = 32{,}735 \text{ m/s}$$

Note that G, M, and the semi-major axis, a, are constant, so the only variable is r. Plug in the value $r = r_{Earth}$, and you've calculated $|v_i|$ Furthermore, we know that In this toy example, $\overrightarrow{v_i} \;=\; |v_i|\hat{y}$, so we now know the magnitude and direction of the initial velocity.

Optimization Algorithm
So we did all that and we calculated 10,000 trajectories and got a nice constraint using a 100x100 grid sample method. Thing is, we don't want to have to run an 8 hour simulation for every single location on our (also probably 100x100) pork chop plot.

So we need an optimization algorithm. Using an algorithm to follow the gradient into the minima will allow us to find the minima in maybe 10-20 simulation runs instead of 10,000, which is helpful for two reasons (well, one reason that has two consequences): 1. It takes *way* less time so we can actually accomplish this task in human timescales. 2. It takes *way* less time so we can run more accurate orbit simulations. The aforementioned 8 hour sim only used N=40,000 sample points, which corresponds to a point every 15 minutes or so. But we've already learned that for 0.1% accuracy we need N=1,000,000, which gets us a sample point every 30 *seconds*. Each of these simulations takes about 20 times longer, but if we're using 1,000 times fewer simulations, we're still coming out on top.

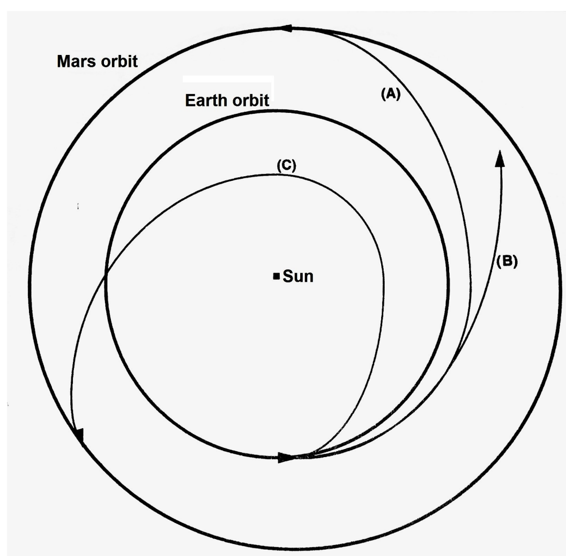
So the next step is finding an algorithm that doesn't suck.

Scipy has a suite of algorithms that can be used from the scipy.optimize.minimize function. We got this up and running fairly easily, and recreated our Hohmann transfer result in less than five minutes (as opposed to eight hours). As we watched the simulation converge to within kilometers, then meters, then centimeters of Mars, we realized that (believe it or not) Mars actually has a radius of 3390 km, so we can set the "good enough" criterion to a few thousand kilometers. I think a good benchmark is within a sphere of Low Martian Orbit radius. What does that mean?

The Mars Reconnaissance Orbiter has a periapsis of 255 km above the Martian surface, so let's call that LMO. So our "good enough" criterion should be…

$$3390 + 255 \text{ km} = 3645 \text{ km}$$

Making a good guess
A lot of computation time can be saved by making intelligent guesses for what the optimized trajectory actually looks like. Obviously if I knew that I wouldn't have to simulate it, but we can ballpark it pretty well by thinking about the problem.[2] The most (energetically) efficient way to get to Mars is the Hohmann transfer we discussed in the previous problem. So *any* trajectory will need *at least* that much delta-v to get started. That's a good first constraint, since it rules out a great deal of possible through-the-sun trajectories, which the computer hates



[2] In fact, I think we could drastically improve computational time if we dug deeper into the analytical and physical side of things (instead of just brute forcing Newton's Law of Gravitation), but this is a computational project and math is hard.

anyway.[3]

If we limit our thinking to the xy-plane for a second, any trajectory that gets to Mars *faster* than the Hohmann transfer needs to have a stronger prograde velocity at the periapsis, which will allow for closest approach at some point closer than the apoapsis. I've included a figure of this stolen from Wikipedia.

Look at trajectories A and B.[4] Trajectory A is the nominal Hohmann transfer, which is energetically favorable. But if we're in a hurry, we might consider trajectory B, which leaves at the exact same time but arrives much sooner since it intersects Mars earlier in it's orbit with a higher velocity. The problem with B is that you spend more velocity *leaving* Earth *and arriving* at Mars, since the velocities of the spacecraft and Mars are poorly matched. So as we decrease transfer time, we increase minimum velocity. Presumably this argument applies in three dimensions as well, but I'm not really sure what that would look like.

These two rules help us with initial guesses for a wide variety of trajectories, but obviously not all. If Mars is lagging in phase behind Earth by, say, 20 degrees, the optimal burn is probably radially out, possibly with some retrograde component. Not exactly what we've defined above. But if we stick to reasonable trajectories that spacecraft might actually be able to accomplish, these rules should work.

$$\Delta v = \sqrt{C_3 + \frac{2GM_e}{r_{LEO}}} - \sqrt{\frac{GM_e}{r_{LEO}}}$$

[3] Things tend to break the speed of light, since we haven't taught the computer how relativity works, and don't intend to do so in the near future.

[4] We're not counting C since it's a two burn trajectory. See how the ellipse has a kink near the sun? It's a retrograde burn at Earth and then a prograde burn closer to the sun, which takes advantage of the Oberth effect. Cool, but beyond the scope of this project.