Chapter 5, #1,2 3 and 26

Problem 1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

Proof. Without loss of generality assume that $x \geq y$ and $x = y + \delta$. Then we may rewrite the given equation as

$$\frac{|f(y+\delta) - f(y)|}{\delta} \leq \delta.$$

Then for any y, as $\delta \to 0$, we have $\lim_{\delta \to 0} |(f(y+\delta) - f(y))/\delta| \leq 0$. Thus $f'(y)$ is defined and equal to zero. Theorem 5.11 gives $f'(x) = 0$ implies f is constant. □

Problem 2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing on (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof. For any $x, y \in (a, b)$, $x > y$, apply MVT to see that $f(y) - f(x) = (y - x)f'(c)$ for some $x < c < y$. Both $y - x > 0$ and $f'(c) > 0$, so the RHS is positive; hence $f(y) > f(x)$. This shows that f is strictly increasing.

We deduce that f is injective, and therefore it is bijective on its image, $(f(a), f(b))$. Thus we may construct an inverse g. For any $y \in (f(a), f(b))$, consider the limit

$$\lim_{s \to y} \frac{g(s) - g(y)}{s - y} \quad (f(a) < y < f(b)).$$

Since g is the inverse of f, there is a unique mapping $f(x) = y$ and $f(t) = s$ such that

$$\lim_{s \to y} \frac{g(s) - g(y)}{s - y} = \lim_{t \to x} \frac{t - x}{f(t) - f(x)} = \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x} \right)^{-1} = \frac{1}{f'(x)}.$$

Thus g is differentiable, and with $y = f(x)$, we deduce that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

□

Problem 3. Suppose that g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$).

Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is injective if ε is small enough. (A set of admissable values of ε can be determined which depends only on M.)

Proof. Let $\varepsilon < 1/M$. Then $f'(x) = 1 + \varepsilon g'(x)$. Now $|\varepsilon g'(x)| < (1/M)M = 1$, so we have $f'(x) > 0$. Thus f is strictly increasing. The reals form a total order so this implies that f is injective. $\square$

Problem 26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Following the hint given by the textbook, let $M_0 = \sup |f(x)|$ and $M_1 = \sup |f'(x)|$ for $x \in [a, b]$. For any x, we have $|f(x)| \leq M_1(b - a) \leq A(b - a)M_0$. We deduce that $\sup |f(x)| = M_0 \leq A(b - a)M_0$. Hence $M_0 = 0$ if $A(b - a) < 1$. So choose $A < 1/(b - a)$; then $f = 0$. $\square$