Problem 3.1. Prove that
$$\lim_{L\to\infty}\sum_{\ell=0}^{2L+1}\sum_{j+k=\ell}x_{j,k}=-\frac{2}{3}.$$

Proof. We may rearrange the finite sum $\sum_{\ell=0}^{2L+1}\sum_{j+k=\ell}x_{j,k}$ into $\sum_{n=0}^{L}\sum_{m=n}^{2L+1-n}x_{n,m}$. Formally checking that this is a bijection is quite painful, but the geometric argument is that the second sum simply counts the entries column by column instead of along the diagonals.

Now, the definition of $x_{n,m}$ gives:

$$\begin{aligned}\sum_{n=0}^{L}\sum_{m=n}^{2L+1-n}x_{n,m}&=\sum_{n=0}^{L}\left(-1+\sum_{m=n+1}^{2L+1-n}x_{n,m}\right)\\ \text{(since } m>n \text{ we have } x_{n,m}=2^{n-m}\text{)}\quad&=\sum_{n=0}^{L}\left(-1+\sum_{m=n+1}^{2L+1-n}2^{n-m}\right)\\ \text{(reindex with } m'=m-n\text{)}\quad&=\sum_{n=0}^{L}\left(-1+\sum_{m'=1}^{2L+1-2n}\frac{1}{2^{m'}}\right)\\ &=\sum_{n=0}^{L}\left(-1+1-\frac{1}{2^{2L+1-2n}}\right)\\ &=-\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^{L-n}}=-\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^{n}}\end{aligned}$$

Now as $L\to\infty$, we have $-\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^n}\to-\frac{1}{2}\left(\frac{1}{1-1/4}\right)=-2/3$, as desired. ☐

Problem 3.2. Let $(x_{n,m})_{n,m=0}^{\infty}$ be a double sequence of complex numbers. Let $\phi:\mathbb{N}_0\to\mathbb{N}_0\times\mathbb{N}_0$ be a bijection, $\phi(n)=(j(n),k(n))$. Prove that if $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_{n,m}|$ converges, then so does the series $\sum_{n=0}^{\infty}x_{j(n),k(n)}$, and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_{n,m}|=\sum_{n=0}^{\infty}x_{j(n),k(n)}.$$

Lemma 1. *Let $\phi:\mathbb{N}_0\to\mathbb{N}_0\times\mathbb{N}_0$ be a bijection, $\phi(n)=(j(n),k(n))$. For any L, there exists some N such that $[0,L]\times[0,L]\subseteq\phi([0,N])$. i.e. ϕ will always "fill up" the $L\times L$ square.*

Proof. Proof by contradiction. Suppose this wasn't the case for all N. Then ϕ would not be bijective, since there exists some $(x,y)\in[0,L]\times[0,L]$ with no preimage. ☐

Proof. Let $\varepsilon > 0$. Define

$$y_m = \sum_{n=0}^{\infty} |x_{m,n}|, \quad z_n = \sum_{m=0}^{\infty} |x_{m,n}|, \quad A = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m}.$$

By the same argument as Lemma 3.3, there exists some M_0 and N_0 such that, for any $M > M_0$ and $N > N_0$,

$$\sum_{m=M+1}^{\infty} y_m < \frac{\varepsilon}{2}, \quad \sum_{n=N+1}^{\infty} x_n < \frac{\varepsilon}{2}.$$

To show that $\sum_{n=0}^{\infty} x_{j(n),k(n)}$ converges, it suffices to show that there exists some P such that

$$\left| A - \sum_{n=0}^{P} x_{j(n),k(n)} \right| < 2\varepsilon.$$

Indeed, choose $L = \max(M, N)$. Then by our above lemma, there is some P such $[0, L] \times [0, L] \subseteq \phi([0, P])$. Let $S = [0, P] \setminus \phi^{-1}([0, L] \times [0, L])$ be the points in the interval that don't get mapped into the square. Also note that $S \subseteq [L + 1, \infty) \times [0, \infty) \cup [0, \infty) \times [L + 1, \infty)$. Thus we have,

$$\begin{aligned}
\left| \sum_{n=0}^{P} x_{j(n),k(n)} - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| &\leq \sum_{n\in S} |x_{j(n),k(n)}| \\
&\leq \sum_{(i,j)\in[L+1,\infty)\times[0,\infty)} |x_{j,k}| + \sum_{(i,j)\in[0,\infty)\times[L+1,\infty)} |x_{j,k}| \\
&\leq \sum_{j=L+1}^{\infty} y_j + \sum_{k=L+1}^{\infty} x_k < \varepsilon.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\left| A - \sum_{n=0}^{P} x_{j(n),k(n)} \right| &\leq \left| A - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| + \left| \sum_{n=0}^{P} x_{j(n),k(n)} - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| \\
&< \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}$$

as desired. Thus our proof is complete. $\square$

Problem 4.1. Provide the missing details of the claims in Remark 4.6. That is,

(a) Prove that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, and

(b) Explain why f cannot be represented as a power series of the form $\sum_{n=0}^{\infty} c_n x^n$.

Lemma 2. *Let $f(x) = e^{-1/x^2}$ if $n \neq 0$ and $f(0) = 0$. Then $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$ if $n \neq 0$ and $f^{(n)}(0) = 0$, where P_n is polynomial in $1/x$.*

Proof. We proceed by induction on n. Clearly $n = 0$ holds as $P_0 = 1$, which is polynomial, $f^{(0)}(x) = P_0 e^{-1/x^2}$, and $f^{(0)}(0) = 0$.

Now assume that for some k, P_k is polynomial in $1/x$ and $f^{(k)}(x) = P_k(1/x)e^{-1/x^2}$. Then we have

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx} P_k(1/x)e^{-1/x^2} \\
&= P_k(1/x)\frac{d}{dx}e^{-1/x^2} + e^{-1/x^2}\frac{d}{dx}P_k(1/x) \\
&= P_k(1/x) \cdot \frac{2}{x^3} \cdot e^{-1/x^2} + e^{-1/x^2} \cdot \frac{-1}{x^2} \cdot P_k'(1/x) \\
&= \left(2P_k(1/x)\frac{1}{x^3} - P_k'(1/x)\frac{1}{x^2}\right) e^{-1/x^2}.
\end{aligned}$$

Setting $w = 1/x$, we see that $P_{k+1}(w) = 2P_k(w)w^3 - P_k'(w)w^2$ is indeed polynomial in $w = 1/x$. Furthermore, $f^{(k+1)}$ continuous at 0 since $-1/x^2 \to -\infty$ as $x \to 0$. The exponential skrinks faster than $1/x^m$ for all m and we end up with $\lim_{x\to 0} f^{(0)}(x) = 0$. Thus our claim holds for $k + 1$, and our induction is complete. □

Now we get with the actual problem...

Proof. We proceed with each part.

(a) By Lemma 2, just forget the $P_n(1/x)$ part and see that we've also shown $f^{(n)}(0) = 0$.

(b) From the notes, functions must be the same if their power series are the same. However, the power series for e^{-1/x^2} is 0, but it is not the same as the zero function. Therefore e^{-1/x^2} is not representable as a power series.

□