Problem 3.4. Let $(V, \langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space. Assume V is *separable* – that is, assume there exists a countable set E which is dense in V. Construct an orthonomral Schauder basis for V, using the following process:

(a) Fill in the details of the following procedure: Pick $v_1 \in E \setminus \{0\}$. Given vectors $v_1, \ldots, v_n \in V$ which are pairwise orthogonal and nonzero, pick an arbitrary $u \in E$. If $u \in \operatorname{span}\{v_1, \cdots, v_n\}$, discard it; otherwise, take v_{n+1} to the part of u that lies in $\operatorname{span}\{v_1, \cdots, v_n\}^\perp$. This gives you a sequence $(v_n)_{n=1}^\infty$ of orthogonal vectors (briefly say why).

(b) Put $W = \operatorname{span}\{v_j\}_{j=1}^\infty$ and prove that $\overline{W} = V$.

(c) Using Bessel's inequality in the next section, Exercise 2.7 from Chapter 3, and the fact that $W^\perp = \{0\}$ to prove that $(v_j)_{j=1}^\infty$ is a Schauder basis for V.

Proof. We proceed with each part separately:

(a) The steps are already laid out, so we'll just give some quick justification. Since every new v_{n+1} is chosen to be in $\operatorname{span}\{v_1, \ldots, v_n\}^\perp$, it is guaranteed to be orthogonal to $\{v_1, \ldots, v_n\}$. Applying transfinite induction, the entire sequence $(v_n)_{n=1}^\infty$ will be pairwise orthogonal.

(b) Note that $W = \operatorname{span}\{v_j\}_{j=1}^\infty = \operatorname{span} E$. If this wasn't the case, then there would exist some $u \in E$ such that some part of u lies in $(\operatorname{span}\{v_j\}_{j=1}^\infty)^\perp$. But this is impossible by the construction of W, therefore we must have $W = E$. Since E is dense in V, then so must be W, as desired.

(c) Aside: the author gave up on this one, although he has the right idea? Essentially, if $(v_n)_{n=1}^\infty$ is a (Hamel) basis for W, then it is a Schauder basis for $\overline{W}$, or something like this, but the author has no idea how to write this intuition out formally.

$\square$

Problem 3.5. Let $(V, \langle\cdot,\cdot\rangle)$ be a real or complex inner product space, and let $U, W \leq V$.

(a) Show that if $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

(b) Prove that $W^\perp$ is always a *closed* subspace of V.

(c) Show that if $V = W \oplus W^\perp$, then $(W^\perp)^\perp = W$.

(d) Prove by way of example that the equality $(W^\perp)^\perp = W$ can fail.

(e) Show that if V is a Hilbert space, then in general we have $(W^\perp)^\perp = \overline{W}$, where $\overline{W}$ denotes the closure of W in V.

Proof. We proceed with each part separately.

(a) Suppose $w \in W^\perp$, then for all $v \in W$, we have $\langle w, v\rangle = 0$, but $U \subseteq W$, so this implies for all $v \in U$, we have $\langle w, v\rangle = 0$, which exactly means $w \in U^\perp$.

(b) We show that $W^\perp$ contains its closure, i.e. let $v \in \overline{W}$ and let $(v_n)_{n=1}^\infty$ be a sequence in $W^\perp$ such that $v_n \to v \in V$. By definition, we have $\langle v_n, w\rangle = 0$ for all $w \in W$, so $\lim_{n\to\infty}\langle v_n, w\rangle = \langle \lim_{n\to\infty} v_n, w\rangle = \langle v, w\rangle = 0$, as desired.

(c) We show both inclusions.

First $(W^\perp)^\perp \subseteq W$: Suppose $v \in (W^\perp)^\perp$. Since we know that $V = W \oplus W^\perp$, write $v = w + w^\perp$. By definition, for all $w' \in W^\perp$, we have $\langle v, w'\rangle = 0$. Thus:

$$0 = \langle w + w^\perp, w'\rangle = \langle w, w'\rangle + \langle w^\perp, w'\rangle = \langle w^\perp, w'\rangle.$$

This must hold for all w', so choose $w' = w^\perp$ to conclude that $\langle w^\perp, w^\perp\rangle = 0 \Rightarrow w^\perp = 0$. Thus $v = w \in W$.

For the other side, suppose $w \in W$. We want to show that, for all $w^\perp \in W^\perp$, $\langle v, w'\rangle = 0$. But this is clearly true, so $w \in (W^\perp)^\perp$.

(d) Consider the space $\ell^2(\mathbb{N};\mathbb{R})$. Let W be the subspace of finitely supported sequences such that $a_1 = 0$. i.e.

$$W = \{a_n \in \mathbb{R}^\mathbb{N} \mid \sum_{n=1}^\infty |a_n|^2 < \infty, a_1 = 0, a_n \text{ finitely supported}\}.$$

Then $W^\perp$ are the sequences such that $a_n = 0$ for all $n \neq 1$. Then

$$(W^\perp)^\perp = \{a_n \in \mathbb{R}^\mathbb{N} \mid \sum_{n=1}^\infty |a_n|^2 < \infty, a_1 = 0\},$$

and we don't necessarily need to be finitely supported anymore.

(e) We show both inclusions.

First $\overline{W} \subseteq (W^\perp)^\perp$: Suppose $w \in W$, then trivially $\langle w, w^\perp\rangle = 0$ for all $w^\perp in W^\perp$, so $w \in (W^\perp)^\perp$. From part (b), $(W^\perp)^\perp$ is closed. $\overline{W}$ is the smallest closed set containing W, so it must be that $\overline{W} \subseteq (W^\perp)^\perp$.

On the other hand, we show the contrapositive. Suppose $v \notin \overline{W}$. Then there exists some $w \perp\in \overline{W}^\perp$, $w \neq 0$ and $w \in \overline{W}$ such that $v = w^perp + w$. Since $W \subseteq \overline{W}$, part (a) shows $w^\perp \in \overline{W}^\perp \subseteq W^\perp$. But then $\langle v, w^\perp\rangle = \langle w^\perp + w, w^\perp\rangle = \langle w^\perp, w^\perp\rangle \neq 0$. Thus

$v \notin (W^{\perp})^{\perp}$, as desired.

$\square$

Problem 1.1. (The Basel Problem)

(a) Compute the Fourier coefficients of the function $f : [-\pi, \pi] \to \mathbb{C}$ given by $f(x) = x$.

(b) Using Parseval's identity, together with part (a), prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Proof. We proceed with each part separately:

(a) We compute, for $n \neq 0$:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \frac{i(-1)^n}{n}$$

(we leave out the work to keep things simple) and $c_0 = 0$, since x is odd.

(b) So then

$$\begin{aligned} \|f(x)\|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3} \\ &= \sum_{n=-\infty}^{\infty} \left| \frac{i(-1)^n}{n} \right|^2 = \sum_{n=1}^{\infty} \frac{2}{n^2}. \end{aligned}$$

Thus we conclude:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

$\square$