

Problem 1.6. Let X be a normed vector space. Let U be an open subset of X , and let $f : U \rightarrow \mathbb{R}$ be a function. Recall that we say that f has a *local maximum* at $a \in U$ if there exists a neighborhood V of a such that $f(x) \leq f(a)$ for all $x \in V$. Prove that if a is a local maximum of f at which f is differentiable, then $f'(a) = 0$.

Proof. Choose a set of basis vectors $\{e_i\}_{i \in I}$ appropriately such that $x + e_i \in V$ for all $i \in I$. Then, we know that for each $i \in I$,

$$\lim_{h \rightarrow 0^+} \frac{\|f(x + he_i) - f(x) - f'(x)he_i\|}{\|he_i\|} = \lim_{h \rightarrow 0^-} \frac{\|f(x + he_i) - f(x) - f'(x)he_i\|}{\|he_i\|} = 0.$$

Because $f(x) \leq f(x + he_i)$, the first limit imposes $D_i f(x) \geq 0$, but the second limit imposes $D_i f(x) \leq 0$. Thus we must have $D_i f(x) = 0$ for all $i \in I$. Therefore, $f'(x) = 0$, as desired. \square

Problem 2.1. Differentiation past the integral.

- (a) Let Y be a metric space; suppose $f : [a, b] \times Y \rightarrow \mathbb{R}$ is continuous. Show that the function $F : Y \rightarrow \mathbb{R}$ defined by $F(y) = \int_a^b f(x, y) dx$ is continuous.
- (b) Assume $f : [a, b] \rightarrow \mathbb{R}$ is continuous; assume $\partial_2 f(x, y)$ also exists and is continuous on $[a, b] \times (c, d)$. Define $F : (c, d) \rightarrow \mathbb{R}$ by $F(y) = \int_a^b f(x, y) dx$. Prove that $F \in C^1((c, d); \mathbb{R})$, with $F'(y) = \int_a^b \partial_2 f(x, y) dx$.

Proof. We proceed with each part separately.

- (a) Let $\varepsilon > 0$. Since f is continuous, there exists an δ such that for all $\|z - y\| < \delta$, we have $|f(z) - f(y)| < \varepsilon/(b - a)$. Then we have

$$\begin{aligned} |F(z) - F(y)| &= \left| \int_a^b [f(x, z) - f(x, y)] dx \right| \\ &\leq \int_a^b |f(x, z) - f(x, y)| dx \\ &< \int_a^b \frac{\varepsilon}{b - a} dx = \varepsilon, \end{aligned}$$

as desired.

- (b) We show that $F'(y)$ exists and is equal to $\int_a^b \partial_2 f(x, y) dx$. Indeed,

$$\lim_{h \rightarrow 0} \frac{F(y + h) - F(y)}{h} = \lim_{h \rightarrow 0} \int_a^b \frac{f(x, y + h) - f(x, y)}{h} dx$$

$$\begin{aligned}
&= \int_a^b \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} dx \\
&= \int_a^b \partial_2 f(x, y) dx,
\end{aligned}$$

where swapping the order of the limit and integration relies on the fact that continuous functions on compact intervals are also uniformly continuous. By part (a), since $\partial_2 f(x, y)$ is continuous, so is $\int_a^b \partial_2 f(x, y) dx$, so $F(y) \in C^1$, as desired.

□

Problem 2.2. Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ via the following rule when $y \geq 0$:

$$f(x, y) = \begin{cases} x, & 0 \leq x \leq \sqrt{y} \\ -x + 2\sqrt{y}, & \sqrt{y} \leq x \leq 2\sqrt{y} \\ 0, & x \geq 2\sqrt{y} \text{ or } x \leq 0. \end{cases}$$

Extend $f(x, y)$ to all of \mathbb{R}^2 by letting it be odd in the second argument: $f(x, y) = -f(x, -y)$ for $y < 0$.

- Show that f is continuous everywhere, and $(\partial_2 f)(x, 0) = 0$ for all x .
- Define $F(y) = \int_{-1}^1 f(x, y) dx$. Prove that $F(y) = y$ for $|y| < \frac{1}{4}$. Conclude that $F'(0) \neq \int_{-1}^1 \partial_2 f(x, 0) dx$.
- Compare part (b) of the present exercise with part (b) of Exercise 2.1. Why are these two not in contradiction with one another?

Proof. We proceed with each part separately.

- Within each region, f is continuous, so we just need to show that it is continuous at the boundaries $x = \sqrt{y}$ and $x = 2\sqrt{y}$. Indeed, we have if $x = \sqrt{y}$, then $x = -x + 2x = -x + 2\sqrt{y}$; if $x = 2\sqrt{y}$, then $-x + 2\sqrt{y} = -2\sqrt{y} + 2\sqrt{y} = 0$. Thus f has no discontinuities at the boundaries, so it is continuous everywhere.

Furthermore, consider $\partial_2 f(x, 0) = \lim_{h \rightarrow 0} (f(x, h) - f(x, 0))/h = \lim_{h \rightarrow 0} f(x, h)/h$. For any x , we may choose h small enough that $2\sqrt{h} \leq x$, so the limit $\partial_2 f(x, 0)$ will always be 0.

- This is fairly painful, but if we draw out the graph its easier to visualize. We have for

$y > 0$:

$$\begin{aligned} F(y) &= \int_{-1}^1 f(x, y) dx \\ &= \int_0^{\sqrt{y}} x dx + \int_{\sqrt{y}}^{2\sqrt{y}} (-x + 2\sqrt{y}) dx \\ &= \left(\frac{x^2}{2} \right)_0^{\sqrt{y}} + \left(-\frac{x^2}{2} \right)_{\sqrt{y}}^{2\sqrt{y}} + 2y \\ &= \frac{y}{2} + \frac{-3y}{2} + 2y = y. \end{aligned}$$

Since f is odd, for $y < 0$, we have $F(y) = F(-|y|) = -F(|y|) = -|y| = y$ as well, as desired.

- (c) The reason why this doesn't contradict part (b) of Exercise 2.1 is because $\partial_2 f(x, y)$ is not continuous.

□