Homework 10

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Chapter 6, Problems 1, 2, 4 and 9

Problem 1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in R(\alpha)$ and that $\int f d\alpha = 0$.

Proof. Let $\varepsilon > 0$. Because α is continuous, we may find $\delta > 0$ such that $|x - x_0| < \delta$ implies $|\alpha(x) - \alpha(x_0)| < \varepsilon$. Consider any partition $a = t_0 < t_1 < \cdots < t_n = b$ with $n \geq 2$ such that $t_i - t_{i-1} < \delta/2$ for all i. Then there exists an index i such that $t_{i-1} < x_0 < t_i$. For any choice of $t_1^*, \ldots, t_n^*$, we have

$$\begin{aligned}\left|\sum_{j=1}^{n} f(t_j^*)(\alpha(t_j) - \alpha(t_{j-1}))\right| &\leq |f(t_i^*)||\alpha(t_{i+1}) - \alpha(t_i)| \\ &\leq |\alpha(t_{i+1}) - \alpha(t_i)| \\ &\leq \varepsilon.\end{aligned}$$

(Actually there is some ickiness when $t_i = x_0$, but this has no real effect the computation.) By definition of the Riemann-Stieltjes integral, this means that $f \in \mathcal{R}(\alpha)$ and $\int f d\alpha = 0$. □

Problem 2. Suppose $f \geq 0$, f is continuous on $[a, b]$ and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. We prove the contrapositive. Suppose there is some x_0 such that $f(x_0) > 0$; we want to show that $\int_a^b f(x)dx > 0$.

Set $\varepsilon = \frac{f(x_0)}{2} > 0$. The continuity of f gives us some $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{f(x_0)}{2} = \varepsilon$ if $|x - x_0| < \delta$. Let $\ell = \min(\delta, \max(x_0 - a, b - x_0)$ and note that $\ell > 0$. Let I be the interval $[x_0 - \ell, x_0]$ if it is contained in $[a, b]$ and $[x_0, x_0 + \ell]$ otherwise. (These are all just technical details to fit the right interval in.) In any case, we have $f(x) \geq \frac{f(x_0)}{2} = \varepsilon$ for all $x \in I$. Now consider the function

$$f'(x) = \begin{cases} \varepsilon & x \in I \\ 0 & x \in [a, b] \setminus I \end{cases}$$

which is continuous on $[a, b]$ except at two points, so it is Riemann integrable. At the same time we have $f(x) \geq f'(x)$ for all $x \in [a, b]$, so

$$\int_a^b f(x)dx \geq \int_a^b f'(x)dx = \int_{x \in I} \varepsilon dx = \ell\varepsilon > 0.$$

This completes the proof. Note here, compared to Exercise 1, the continuity of f made a key difference. □

Problem 4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Let $a < b$ and let $a = t_0 < t_1 < \cdots < t_n = b$ be any partition of $[a, b]$. Since both the rationals and irrationals are dense in $\mathbb{R}$, we can also find $t_1^*, \ldots, t_n^*$ such that they are all either rational or irrational. With these two choices we will have

$$\left|\sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})\right| = b - a \quad \text{or} \quad \left|\sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})\right| = 0.$$

This holds for any $a < b$ and any partition, so the upper Riemann sum is $b - a$ while the lower Riemann sum is 0. These do not match so $f(x)$ is not Riemann integrable. □

Problem 9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.

Proposition. *Let $f(x)$ and $g(x)$ be continuously differentiable functions defined on $[a, \infty)$. If $\lim_{x\to\infty} f(x)g(x)$ exists and $\int_a^\infty f(x)g'(x)dx$ converges, then $\int_a^\infty f'(x)g(x)dx$ converges.*

Proof. Let $b > a$. Applying the normal integration by parts gives

$$\int_a^b f'(x)g(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x)dx.$$

In the limit, our hypotheses guarantee that the RHS is well defined. Thus the LHS also exists and is well defined, as desired. □

So now let $f(x) = \sin x$ and $g(x) = \frac{1}{1+x}$. Then $f'(x) = \cos x$ and $g'(x) = -\frac{1}{(1+x)^2}$. Since

$$\left|\frac{\sin x}{(1+x)^2}\right| \leq \frac{1}{x^2},$$

we know that $\int_0^\infty f(x)g'(x)dx$ converges absolutely; and $\lim_{x\to\infty} f(x)g(x) = 0$. So indeed,

$$\int_0^\infty \frac{\cos x}{1+x} dx = \int_0^\infty \frac{\sin x}{(1+x)^2} dx.$$