Problem 3.1. Prove Proposition 3.3 for complex inner product spaces, using the following strategy. Choose $w_v \in W$ such that (36) holds. Fix $z \in W$ and consider the two functions

$$f_z : \mathbb{R} \to [0, \infty), \quad f_z(\alpha) = \|v - w_v + \alpha z\|^2,$$
$$g_z : \mathbb{R} \to [0, \infty), \quad g_z(\beta) = \|v - w_v + i\beta z\|^2.$$

Argue that $f_z'(0) = g_z'(0) = 0$. Then, use this to show that $\langle v - w_v, z \rangle = 0$. Conclude that $v - w_v \in W^\perp$.

Problem 3.2. Prove Proposition 3,7, using the following outline.

(a) Given $v \in V$, let $(w_n)_{n=1}^\infty$ be a sequence in W such that $\|w_n - v\| \to \delta(v)$ as $n \to \infty$, where $\delta(v) = \inf_{w \in W} \|v - w\|$. (In fewer than 10 words, cite a reason why such a sequence exists.) Prove that

$$\|w_n - w_m\|^2 = 2\|w_n - v\|^2 + 2\|w_m - v\|^2 - 4 \left\| \frac{w_n + w_m}{2} - v \right\|^2,$$

by applying the parallelogram law to $w_n - w_m = (w_n - v) - (w_m - v)$.

(b) Use the identity from part (a), together with the definition of $\delta(v)$, to prove that $(w_n)_{n=1}^\infty$ is a Cauchy sequence.

(c) Let w_v denote the element of W to which the sequence $(w_n)_{n=1}^\infty$ converges. (The fact that w_v exists is guaranteed by the completeness of W, together with part (b).) Give a short argument for why $\|v - w_v\| = \delta(v)$.

Problem 3.3. Prove the second half of Corollary 3.8. That is, prove that if $(V, \langle \cdot, \cdot, \rangle)$ is a real or complex Hilbert space and W is a subspace of V, which is not complete, then $V \neq W \oplus W^\perp$.