Problem 5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

We first prove a useful lemma.

Lemma. *Let $\{a_n\}$ be a sequence. Then*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_k \mid k \ge n\}).$$

Proof. Define $A_n = \sup\{a_k \mid k \ge n\}$ and $\lim_{n\to\infty} A_n = L$. □

Proof. Let $A_n = \sup\{a_k \mid k \ge n\}$ and $B_n = \sup\{b_k \mid k \ge n\}$. Then for any $k \ge n$, we have $a_k + b_k \le A_n + B_n$, so the property of the supremum says that $\sup\{a_k + b_k \mid k \ge n\} \le A_n + B_n$. This holds for all $n \in \mathbb{N}$, so we take the limit to obtain

$$\begin{aligned}\limsup_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty} \sup\{a_k + b_k \mid k \ge n\} \\ &\le \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n \\ &= \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,\end{aligned}$$

as desired. □

Problem 7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \ge 0$.

Proof. The idea to this proof is that we must somehow linearize $\frac{\sqrt{a_n}}{n}$ by bounding it with $p_n a_n + q_n$. Then all that is required for

$$\sum \frac{\sqrt{a_n}}{n}$$

to converge is for

$$\sum p_n a_n + q_n = \sum p_n a_n + \sum q_n$$

to converge.

Indeed, pick $p_n = 1$ and $q_n = \frac{1}{n^2}$. Since $a_n, n \geq 0$, we have

$$
\begin{aligned}
0 &\leq a_n^2 + \frac{a_n}{n^2} + \frac{1}{n^4} \\
\Rightarrow \frac{a_n}{n^2} &\leq a_n^2 + 2\frac{a_n}{n^2} + \frac{1}{n^4} = \left(a_n + \frac{1}{n^2}\right) \\
\Rightarrow \frac{\sqrt{a_n}}{n} &\leq a_n + \frac{1}{n^2}.
\end{aligned}
$$

Then both $\sum p_n a_n = \sum a_n$ and $\sum q_n = \sum n^{-2}$ clearly converge. Hence $\sum \sqrt{a_n}/n$ converges. □

Problem 8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Since $\{b_n\}$ is bounded and montonic, it converges to some $b \in \mathbb{R}$. If b_n is increasing, set $c_n = b - b_n$, otherwise $c_n = b_n - b$. This new sequence c_n is decreasing and converges to 0 by construction, therefore we may apply Theorem 3.42 from the textbook to obtain the convergence of $\sum a_n c_n$. Whether we have $c_n = b - b_n$ or $c_n = b_n - b$, the sum $\sum a_n c_n$ differs from $\sum a_n b_n$ by some constant of $\pm \sum a_n b$. Hence $\sum a_n b_n$ converges. □