Problem 1. True or False: If true prove it, if false counterexample it.

(a) Let $\{F_n\}$ be a countable collection of closed subsets of $\mathbb{R}$ such that for any finite sub-collection
$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing.$$
Then
$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(b) Add the condition that each F_n is bounded and repeat (2a).

(c) Repeat (1a) where closed and bounded $F_n \subseteq X$, and arbitrary metric space.

Proof.

(a) This claim is **false**. Consider the subsets $F_n = [n, \infty)$. Then for any finite sub-collection $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$, let $n = \max_k(n_k)$. We can compute the intersection to be:
$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} = F_n \neq \varnothing.$$
However, since for any $x \in \mathbb{R}$ we may find some $n \geq x$, there is always some F_n such that $x \notin F_n$. Hence
$$\bigcap_{n=1}^{\infty} F_n = \varnothing.$$
This disproves the claim.

(b) This claim is **true**. If F_n are both closed and bounded subsets of $\mathbb{R}$, then the Heine-Borel Theorem guarantees that F_n is compact. Now apply Theorem 2.36 from the textbook to conclude that
$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(c) This claim is **false**. Let $X = \mathbb{Q}$ with the relative topology inherited from $\mathbb{R}$. Then consider the subsets $F_n = \overline{B_{1/n}(\sqrt{2})}$ as the closed balls centered at $\sqrt{2}$ with radius $1/n$, where p_n is tne nth prime. In particular, since $\sqrt{2} \pm 1/n$ are irrational, the boundary points of F_n don't exist in $\mathbb{Q}$, and hence we can drop them without changing anything: $F_n = B_{1/n}(\sqrt{2})$.

Now we check that finite intersections are nonempty. Indeed, if $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$ are a finite sub-collection, then their intersection is just the ball of minimum radius $r =$

$\min_k(1/n_k)$. This radius is clearly greater than 0, so we know that $F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing$.

However, if we consider $\bigcap_{n=1}^{\infty} F_n$, then for any $x \neq \sqrt{2}$, we can find a k such that $k > 1/|x - \sqrt{2}|$. This implies $1/n < |x - \sqrt{2}|$. Hence by definition $x \notin F_k$, so $x \notin \bigcap_{n=1}^{\infty} F_n$. So all $x \neq \sqrt{2}$ are not in our intersection. But also $\sqrt{2}$ is not in $\mathbb{Q}$! Hence in $\mathbb{Q}$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is empty. This disproves the claim.

□

Problem 2. Show that every compact metric space is complete.

Proof. Let X be a compact metric space. We must show that every Cauchy sequence $\{x_n\}$ converges. Since X is compact, there is a convengent subsequence $x_{n_k} \to x \in X$. We claim that in fact $x_n \to x$.

Indeed, since $x_{n_k} \to x$, there is N_1 such that $n_k \geq N_1$ implies $d(x_{n_k}, x) < \varepsilon/2$. Furthermore, given that $\{x_n\}$ is Cauchy, choose N_2 such that $n, m \geq N_2$ implies $d(x_n, x_m) < \varepsilon/2$.

Set $N = \max(N_1, N_2)$ and $n_k \geq N$. Then

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon.$$

Thus $x_n \to x \in X$, and X is complete. □

Problem 3. The following "Theorem" is not true. Find an error in the "proof" and construct a counterexample.

Theorem: (Bogus) Let $f : X \to Y$ be a continuous mapping from a metric space X to a metric space Y. Let $E \subseteq X$ be a closed subset and assume the diameter, $\operatorname{diam}(E) < 1$. Then $f(E)$ is bounded.

Proof: (Junk) Since $\operatorname{diam}(E) < 1$, E can be contained in a ball

$$B_2(x_0) = \{x \in X \mid d(x, x_0) < 2\}.$$

Therefore E is bounded. Since E is assumed to be closed, E is therefore compact. Since F is continuous, $f(E)$ is therefore compact and therefore bounded.

Proof. The error is in this step: "Since E is assumed to be closed, E is therefore compact." Because X is any arbitrary metric space the equivalence between closed and bounded iff compact does not hold. Indeed, let $f : (0,1) \to \mathbb{R}$ with $x \mapsto 1/x$. The subspace topology gives that $(0, 1/2]$ is closed and bounded. But $f((0, 1/2]) = (2, \infty)$ is clearly not bounded. □

Problem 4. Let $I = [0,1]$ and let $f : I \to I$ be continuous. Prove that f has at least one fixed point.

Proof. Extend the codomain of f to $\mathbb{R}$ and consider the map $g(x) = f(x) - x$. We have the bounds $0 \le f(0) - 0 = f(0) \le 1$ and $-1 \le f(1) - 1 \le 0$. Thus the interval $[g(0), g(1)]$ contains the point 0. The continuity of $f(x)$ implies the continuity of $g(x)$; the application of the intermediate value theorem guarentees the existence of $x_0 \in [0,1]$ such that $g(x_0) = 0$. Thus $f(x_0) = x_0$ and x_0 is a fixed point. □

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ and suppose

$$|f(x) - f(y)| \le |x - y|^{1+\alpha}$$

for all real x and some fixed real $\alpha > 0$. Prove that f is a constant function.

Proof. Without loss of generality assume that $x \ge y$ and $x = y + \delta$. Then we may rewrite the given equation as

$$\frac{|f(y+\delta) - f(y)|}{\delta} \le \delta^\alpha.$$

Note that $\alpha > 0$ gives the important limit $\lim_{\delta\to 0} \delta^\alpha = 0$. Then for any y, we have $\lim_{\delta\to 0} |(f(y+\delta) - f(y))/\delta| \le 0$. Thus $f'(y)$ is defined and equal to zero. Theorem 5.11 gives that if $f'(x) = 0$, then f must be constant. □

Problem 6. Let $g : \mathbb{R} \to \mathbb{R}$ and suppose that $g'(x)$ exists for all x. Also assume that there is a constant $M > 0$ such that $|g'(x)| \le M$ for all $x \in \mathbb{R}$. Define $f(x) = x + \delta g(x)$ where δ is a fixed real number.

(a) Show f is 1-to-1 if $|\delta|$ is sufficiently small. Find an estimate δ must satisfy.

(b) Assuming δ satisfies the condition in (6a), find an expression for $\frac{d}{dx} f^{-1}(x)$.

Proof.

(a) Let $\delta < 1/M$. Then $f'(x) = 1 + \delta g'(x)$. Now $|\delta g'(x)| < (1/M)M = 1$, so we have $f'(x) > 0$. Thus f is strictly increasing. The reals form a total order so this implies that f is injective. Thus δ is about as small as $1/M$.

(b) By definition $f(f^{-1}(x)) = x$. Applying the chain rule, we see that

$$f'(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1.$$

Hence
$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

□

Problem 7. Define
$$\int_a^\infty f(x)d\alpha(x) = \lim_{N\to\infty}\int_a^N f(x)d\alpha(x)$$
provided the limit exists. Let $f(x) = 1/x^2$ and $\alpha(x) = \lfloor x/2 \rfloor$.

Show the above limit exists and compute $\int_{\frac{1}{2}}^\infty f(x)d\alpha(x)$.

Proof. Fix some N. Now note that
$$\alpha(x) = \sum_{a/2\le n<N/2} I(x-2n)$$
on the interval $[a, N)$. Hence we may rewrite
$$\int_a^N f(x)d\alpha(x) = \sum_{a/2\le n<N/2} f(2n) = \sum_{a/2\le n<N/2}\frac{1}{4x^2}.$$
This series is less than $\sum 1/x^2$, so it converges as $N \to \infty$. If $a = 1/2$, then we have
$$\int_{\frac{1}{2}}^\infty f(x)d\alpha(x) = \sum_{n\ge 1}\frac{1}{4x^2} = \frac{6}{4\pi^2} = \frac{3}{2\pi^2}.$$

□

Problem 8. Let $f \in C^1([0, 2\pi])$ and define
$$a_n = \int_0^{2\pi} f(x)\cos nx dx.$$
Prove that $a_n \to 0$ as $n \to \infty$.

Proof. Since f is differentiable we may use integration by parts to find that
$$a_n = \int_0^{2\pi} f(x)\cos nx dx$$

$$
\begin{aligned}
&= \left(f(2\pi)\frac{\sin 2\pi n}{n} - f(0)\frac{\sin 0n}{n} \right) - \int_0^{2\pi} f'(x)\frac{\sin nx}{n}dx \\
&= -\int_0^{2\pi} f'(x)\frac{\sin nx}{n}dx \\
&= -\frac{1}{n}\int_0^{2\pi} f'(x)\sin nx dx.
\end{aligned}
$$

The domain of f' is compact, so f' must be bounded. Since $\sin nx$ is also bounded by some $M > 0$, we conclude that $f'(x)\sin nx$ is bounded. Therefore

$$
|a_n| = \frac{1}{n}\left|\int_0^{2\pi} f'(x)\sin nx dx\right| \leq \frac{1}{n}\left|\int_0^{2\pi} M dx\right| \leq \frac{2\pi M}{n}.
$$

Now it is clear that $a_n \to 0$ as $n \to \infty$. □

Problem 9. Define BUC $= \{f : \mathbb{R} \to \mathbb{R} \mid f$ is bounded and uniformly continuous on $\mathbb{R}\}$ and $d(f, g) = \sup_{\mathbb{R}} |f(t) - g(t)|$. For $\delta \in (0, 1)$ and $f \in$ BUC define

$$
f_\delta(t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} f(s)ds = \frac{1}{2\delta}\int_0^{2\delta} f(t - \delta + \tau)d\tau.
$$

Show

(a) $f_\delta \in$ BUC,

(b) $f_\delta \in C^1$,

(c) the collection $\{f_\delta \mid 0 < \delta < 1\}$ is dense in BUC, i.e. for each $\varepsilon > 0$ there is a $\delta \in (0, 1)$ such that $d(f, f_\delta) < \varepsilon$.

Proof.

(a) By assumption f is bounded by some $M \geq 0$. Then

$$
f_\delta(t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} f(s)ds \leq \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} M ds = M.
$$

Hence f_δ is also bounded by M.

Now let $\varepsilon > 0$. Then choose $\gamma < \frac{\varepsilon\delta}{M}$. Then for all $|t_1 - t_2| < \gamma$, we have

$$
|f_\delta(t_1) - f_\delta(t_2)| = \frac{1}{2\delta}\left|\int_{t_1-\delta}^{t_1+\delta} f(s)ds - \int_{t_2-\delta}^{t_2+\delta} f(s)ds\right|
$$

$$
\begin{aligned}
&= \frac{1}{2\delta}\left|\int_{t_1-\delta}^{t_2-\delta} f(s)ds - \int_{t_1+\delta}^{t_2+\delta} f(s)ds\right| \\
&\leq \frac{1}{2\delta}\left(\left|\int_{t_1-\delta}^{t_2-\delta} f(s)ds\right| + \left|\int_{t_1+\delta}^{t_2+\delta} f(s)ds\right|\right) \\
&\leq \frac{1}{2\delta}M\left|t1-t2\right| + \frac{1}{2\delta}M\left|t1-t2\right| \\
&= \frac{M|t_1-t_2|}{\delta} \\
&< \varepsilon.
\end{aligned}
$$

(Note that the second equality can be seen by drawing out the integrals geometrically.) This proves that f_δ is uniformly continous. Hence $f_\delta \in \text{BUC}$.

(b) Let $F(x) = \int f(s)ds$. By the fundamental theorem of calculus we have

$$f_\delta(t) = \frac{1}{2\delta}(F(t+\delta) - F(t-\delta)).$$

Since $F \in C^1$, we have $f_\delta \in C_1$ as well.

(c) Set $\varepsilon > 0$. Since f is uniformly continuous we have some $\gamma > 0$ such that $|t_1 - t_2| < \gamma$ implies $|f(t_1) - f(t_2)| < \varepsilon$. Now simjply choose $\delta = \gamma$. Then the integral of $f(t)$ between $t \pm \gamma$ is bounded above and below by $f(t) \pm \varepsilon$, which implies

$$\frac{1}{2\gamma}(f(t)-\varepsilon)2\gamma \leq \frac{1}{2\gamma}\int_{t-\gamma}^{t+\gamma} f(s)ds \leq \frac{1}{2\gamma}(f(t)+\varepsilon)2\gamma.$$

Thus $f(t) - \varepsilon \leq f_\gamma(t) \leq f(t) + \varepsilon$ for all t implies $d(f, f_\gamma) \leq \varepsilon$, as desired.

$\square$