# Math 410 Homework 2

Due Date: **something**

Exercises 8, 9, 12, 26, 36, pp. 21-23.

8. (a) Since $G$ is a subset of $\mathbb{C}$, it suffices to prove that $G$ is a subgroup of $\mathbb{C}$. For any $g, h \in G$, we need to show that $gh^{-1} \in G$. By definiton there exist some $n, m \in \mathbb{Z}^+$ such that $g^n = h^m = 1$.

   We want to find $N$ such that $(gh^{-1})^N = 1$. Let $N = nm$ and notice that $(gh^{-1})^{nm} = g^{nm} * h^{-nm} = (g^n)^m * (h^m)^{-n} = 1^m * 1^{-n} = 1$. Hence $gh^{-1} \in G$, and $G$ is a subgroup of $\mathbb{C}$, which makes it a group in general.

   (b)

9.

1. (a) Again we prove that $G$ is a group by proving it is a subgroup of $\mathbb{R}$.

   For any $g, h \in G$, there are some $a, b, c, d \in \mathbb{Q}$ with $g = a + b\sqrt{2}$ and $h = c\sqrt{d}$. then $g - h = (a - c) + (b - d)\sqrt{2}$. Clearly $a - c$ and $b - d$ are rational, so $g - h \in G$, as desired.

   (b) Let $g$ be a non-zero element of $G$ such that $a + b\sqrt{2} = g$ for some $a, b \in \mathbb{Q}$ (where $a$ and $B$ are not both 0). Then note that $1/g = 1/(a + b\sqrt{2})$ is in $G$, since

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

   Letting $x = \frac{a}{a^2 - 2b^2}$ and $y = \frac{-b}{a^2 - 2b^2}$, we have $1/g = x + y\sqrt{2}$. Both $x$ and $y$ are rational, since they are made up of rational expressions. Hence $1/g$ (in $\mathbb{R}$) is the inverse og $g$ in $G$.

Note. This makes $G$ a *field*. In fact it is the field $\mathbb{Q}[\sqrt{2}]$, the result of adjoining $\sqrt{2}$ to $\mathbb{Q}$.

12. We can just calculate the orders:

$$|\overline{1}| = 0$$
$$\overline{-1}^2 = 1, |\overline{-1}| = 2$$

Exercises 3, 9, pp. 27-28.

We will have to use the fact that $D_{2n} = \{1, r, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}$. This is not easy to prove formally because we don't have a formal definition of a presentation yet, so for now I will take it to be true.

3. Since $x$ is not a rotation, it must be of the form $sr^i$ for $0 \leq i < n$. Then $(sr^i)^2 = sr^isr^i = ssr^{-i}r^i = 1 * 1 = 1$. Thus $sr^i$ has order 2.

   We can combine $s$ and $sr$ to get $ssr = r$. Hence $s$ and $sr$ generate $r$, which together will generate $D_{2n}$.

9. Note that the orientation of the tetrahedron is determined the orientation of a single edge. Call the edge $ab$. Then there are 4 choices for $a$ and 3 choices for $b$, giving a total of 12 orientations, or 12 possible rigid motions we can do.

Exercises 2, 4, 13, 16, 20, pp. 32-34.

2. Write everything in cycle notation. We have $\sigma = (1,13,5,10)(3,15,8)(4,14,11,7,12,9)$ and $\tau = (1,14)(2,9,15,13,4)(3,10)(5,12,7)(8,11)$.

   Then quick calculations show that:

$$\begin{aligned}
\sigma^2 &= (1,5)(5,10)(3,15,8)(4,11,12)(14,7,9) \\
\sigma\tau &= (1,11,3)(2,4)(5,9,8,7,10,15)(13,14) \\
\tau\sigma &= (1,4)(2,9)(3,13,12,15,11,5)(8,10,14) \\
\tau^2\sigma = \tau(\tau\sigma) &= (1,2,15,8,3,4,14,11,12,13,7,5,10)
\end{aligned}$$

4. (a) $S_3$ has 6 elements: $\{(),(12),(23),(13),(123),(132)\}$. Direct calculation shows that these elements have orders $0,2,2,2,3,3$.

   (b) $S_4$ has 24 elements:

      i. 1 identity: $()$; order $= 0$;

      ii. 6 transpositions: $(12),(23),(34),(14),(13),(24)$; order $= 2$;

      iii. 3 disjoint products of transpositions: $(12)(34),(23)(14),(13)(24)$; order $= 2$;

      iv. 8 3-cycles: $(123),(132),(124),(142),(134),(143),(234),(243)$; order $= 3$;

      v. 6 4-cycles: $(1234),(1243),(1324),(1342),(1423),(1432)$; order $= 4$.

13. ($\Rightarrow$): If $\sigma$ is a product of commuting 2-cycles. Let $\sigma = (a_1b_1)(a_2b_2)\dots(a_kb_k)$. Then

$$\sigma^2 = (a_1b_1)(a_2b_2)\dots(a_kb_k)(a_1b_1)(a_2b_2)\dots(a_kb_k) = (a_1b_1)^2\dots(a_kb_k)^2,$$

   since all the factors commute. But squaring a 2-cycle makes the term vanish, so $\sigma^2 = 1$ and $\sigma$ has order 2.

   ($\Leftarrow$): Suppose $\sigma$ has order 2. Decompose $\sigma$ as the product of disjoint cycles $c_1,\dots c_k$, with lengths $\ell_1,\dots,\ell_k$. Since $\sigma^2 = 1$ and disjoint cycles do not affect each other, we must have $c_1^2 = c_2^2 = \cdots = c_k^2 = 1$. Then $\ell_1 = \cdots = \ell_k = 2$ and everything is a 2-cycle, as desired.

16. We make a combinatorial argument. To form an $m$-cycle we must choose $m$ objects out of $n$, where order matters. This gives $n(n-1)\dots(n-m+1)$ possible choices. However, for each cycle, we have $m$ different representations created by shifting the cycle over $m$ times. Hence we have overcounted by a factor of $m$. Thus the final answer is

$$\frac{n(n-1)\dots(n-m+1)}{m}.$$

20. We know that $S_3 = \{(),(12),(23),(13),(123),(132)\}$ and that $(12)(23) = (123)$. So $(12)$ and $(23)$ can generate $()$, $(123)$, and $(132)$. Then $(132)(12) = (23)$. Hence $a = (12)$ and $b = (23)$ generate $S_3$. The relations are at least $a^2 = b^2 = 1$. This isn't enough though, because it tells us nothing about how $ab = (123)$ behaves. Thus we need another relation $ab^3 = 1$ to constrain $(123)$. This gives

$$S_3 = \langle a, b : a^2 = b^2 = 1, (ab)^3 = 1\rangle.$$

Exercises 17, 18, pp. 40.

17. Let $i : G \to G$ be the map $g \mapsto g^{-1}$. Then $i(gh) = (gh)^{-1} = h^{-1}g^{-1}$. Clearly $i(gh) = i(g)i(h)$ if and only if $h^{-1}$ commutes with $g^{-1}$ for all $g,h \in G$, i.e. $G$ is abelian.

18. Let $s : G \to G$ be the map $g \mapsto g^2$. Then $s(gh) = (gh)^2 = ghgh$. If $G$ is abelian, then $ghgh = gghh = g^2h^2 = s(g)s(h)$. Thus $s$ is a homomorphism. Conversely, if $s(gh) = s(g)s(h)$, then we can take $ghgh = gghh$ and multiply by $g^{-1}$ on the left and $h^{-1}$ on the right to cancel. The result is $gh = hg$. This holds for any $g,h \in G$, therefore $G$ is abelian.

Exercises 18, 19, pp. 45.

18. Let $H$ be a left action on $A$. The relation $a \sim b$ defined by $a = hb$ for some $h \in H$ defines a equivalence relation:

    *Reflexive:* Let $h = 1$. Then clearly $a = 1a$ for all $a \in A$. Hence $a \sim a$ for all $a \in A$.

    *Symmetric:* Let $a \sim b$ with $a = hb$. Then $h^{-1}a = b$. Thus $b \sim a$.

    *Transitive:* Let $a \sim b$ and $b \sim c$ with $a = h_1 b$ and $b = h_2 c$. Then substitute $b$ to see that $a = (h_1 h_2)c$. Hence $a \sim c$.

19. Define $f : H \to \mathcal{O}$ to be the map $h \mapsto hx$. If $f(x) = f(y)$, then $hx = hy$. Cancelling implies $x = y$; thus $f$ is injective. Furthermore, for any $y \in \mathcal{O}$, we know by definition that $y \sim x$. Hence there is some $h \in H$ such that $y = hx$. Thus $f(h) = y$, and $f$ is surjective. The two combine show that $f$ is bijective. We conclude that $|H| = |\mathcal{O}|$.

    But this is true for every orbit $\mathcal{O}$, so all the orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$ have size $|H|$. Since the orbits partition $G$, we have
    $$|G = \sum_{i=1}^{k} |\mathcal{O}_i| = \sum_{i=1}^{k} |H| = k|H|.$$
    (We may do the sum since $G$ is finite.) Therefore we have *Lagrange's Theorem*: $|H|$ divides $|G|$ for any subgroup $H$ of a finite group $G$.