Problem 6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and E' have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points.

Proof. *E' is closed.* By definition E' is closed if and only if E'', the limit points of E', are conained within E'. So we need to show that every limit point of E' is a limit point of E.

Let x be a limit point of E'. Every neighbourhood U_x around x then contains some $y \in E'$. Since U_x is open, we can find some neighbourhood U_y of y such that $U_y \subseteq U_x$. Because y is a limit point of E, we also know the U_y contains points of E. Hence U_x contains points of E. This holds for any neighbourhood U_x, and thus $x \in E'$, as desired. $\square$

Proof. *E and $\overline{E}$ have the same limit points.* We prove both inclusions.

Since $E \subseteq \overline{E}$, every neighbourhood U_x of a limit point $x \in E$ will intersect at least E. Hence $E' \subseteq \overline{E}'$.

For the other inclusion, let x be a limit point of $\overline{E}$ and U_x be any neighbourhood of x. Then $\square$

E and E' do not always have the same limit points. Let $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$ is the limit points of E, but E' clearly doesn't have any limit points itself.

Problem 9. Let E° denote the set of all interior points in a set E.

(a) Prove that E° is always open

Proof. For every $x \in E^\circ$, choose some neighbourhood U_x. We claim that $\bigcup_{x \in E^\circ} U_x = U = E^\circ$. Indeed, clearly U contains every point of E° so $E^\circ \subseteq U$. At the same time, every U_x is contained in E°, so $U \subseteq E^\circ$. Now U is a union of open sets, so it is open. Hence $E^\circ = U$ is open. $\square$

(b) Prove that E is open if and only if $E^\circ = E$.

Proof. ($\Rightarrow$): If $E = E^\circ$, then part (a) shows that E is open.

($\Leftarrow$): If E is open, then for every $x \in E$ and U_x a neighbourhood of x, $U_x \subseteq E$. Hence $x \in E^\circ$ and $E = E^\circ$. $\square$

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Proof. Consider G°. If $x \in G^\circ$, then there is a neighbourhood $U_x \subseteq G \subseteq E$. Hence $x \in E^\circ$ and $G^\circ \subseteq E^\circ$. But G is open, so part (b) shows that $G = G^\circ$, and thus $G \subseteq E^\circ$. $\square$

(d) Prove that the complement of E° is the closure of the complement of E.

Proof. We want to show that $(E^\circ)^c = \overline{(E^c)}$ by proving both inclusions.

First, $(E^\circ)^c \subseteq \overline{(E^c)}$. Let $x \in (E^\circ)^c$ □

(e) Do E and $\overline{E}$ always have the same interiors?

No. Let $E = \mathbb{Q} \subseteq \mathbb{R}$. Then $E^\circ = \varnothing$, while $\overline{E}^\circ = \mathbb{R}^\circ = \mathbb{R}$.

(f) Do E and E° always have the same closures?

No. Let $E = \mathbb{Q} \subseteq \mathbb{R}$. Then $E^\circ = \varnothing$. Hence $\overline{E} = \mathbb{R} \neq \overline{E^\circ} = \varnothing$.

Problem 10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. We first prove the $d(p,q)$ is a metric. By definition $d(p,q) = 0$ if and only if $p = q$. Clearly $d(p,q) = d(q,p)$. Finally, we can prove $d(p,q) + d(q,r) \geq d(p,r)$ with some casework.

If the LHS is 0, then it must be that $d(p,q) = d(q,r) = 0$, implying that $p = q = r = 0$. Hence the RHS is also 0 and inequality holds.

If the LHS is 1, then one term is 1 while the other is zero. Without loss of generality assume $d(p,q) = 1$ and $d(q,r) = 0$. Then $p \neq q$ and $q = r$ implies $p \neq r$. Thus $d(p,r) = 1$ and inequality holds.

If the LHS is 2, then inequality always holds since $2 > 1 \geq d(p,r)$.

We conclude that $d(p,q)$ is indeed a metric. □

Proposition. *The open and closed sets in this metric are $\mathcal{P}(X)$, i.e. the indiscrete toplogy.*

Proof. For any p, $N_r(p) = \{q : d(p,q) < r\} = \{p\}$. If $r < 1$, then only $d(p,p) = 0 < 1$, so $N_r(p) = \{p\}$. So every point is an open set. Then we can construct every subset of X by taking suitable unions of the points. Hence all subsets are open. At the same time, every subset is a complement of another, so every subset is complement to a open set. Hence every subset is also closed. □

Proposition. *A subset A of X is compact if and only if A is finite.*

Proof. ($\Rightarrow$): Let $\mathcal{C}$ be a cover of A. If A is finite, then for each $x \in A$ choose some $U_x \in \mathcal{C}$ containing x. Then $\bigcup_{x \in A} U_x$ is a finite union coving A.

($\Leftarrow$): If A is compact then the cover $\mathcal{C}$ made of $\{x\}$ for every $x \in A$ has size equal to A. The only subcover of $\mathcal{C}$ is $\mathcal{C}$ itself, so $\mathcal{C}$ must be finite. Hence A is finite. □