Math 425a - Midterm 1 - Spring 2022 Due February 14

You may consult books or notes but do not consult other individuals.

1. If $x \in \mathbb{R}^k,\ k \geq 2$, show there exists $y \in \mathbb{R}^k$ such that $x \cdot y = 0$.

2. True or False: If true prove it, if false counterexample it.

   (a) Let $\{F_n\}$ be a countable collection of closed subsets of $\mathbb{R}$ such that for any finite sub- collection

   $$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \emptyset$$

   Then

   $$\bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

   (b) Add the condition that each F_n is bounded and repeat (2a).

   (c) Repeat (2a) where closed and bounded $F_n \subset X$, an arbitrary metric space.

3. Consider the metric space $\mathbb{Q}$ of all rationals on the real line with the euclidean metric. Prove that if $K \neq \emptyset$ is a compact subset of $\mathbb{Q}$ then K cannot contain an open subset of $\mathbb{Q}$. *Hint*: Consider the relative topology.

4. Let F and K be nonempty closed subsets of the metric space X with $K \cap F = \emptyset$. Show that if K is compact there is a positive distance from F to K, i.e.

   $$\inf\{d(x,y) \mid x \in F,\ y \in K\} = \delta > 0.$$

   Is it still true if K is only assumed to be closed? If not, find a counterexample.

5. A *base* for a topological space X is a collection $\{V_\alpha | \alpha \in \mathcal{A}\}$ of open subsets of X such that for every open subset of $G \subset X$, one has $G = \cup_{\alpha \in \mathcal{B}} V_\alpha$ where $\mathcal{B} \subset \mathcal{A}$.

   Prove that every compact metric space X has a countable base.