Problem 1.1. Prove that if G is Frechet differentiable at x_0, then its Frechet derivative at x_0 is unique.

Proof. Let T_1 and T_2 be two Frechet derivatives of G at x_0. We want to show that

$$\lim_{z\to 0}\frac{\|G(x_0+z)-G(z_0)-T_1 z\|}{\|z\|}=\lim_{z\to 0}\frac{\|G(x_0+z)-G(z_0)-T_2 z\|}{\|z\|}.$$

Indeed, move everything to the left to get

$$\lim_{z\to 0}\frac{\|G(x_0+z)-G(z_0)-T_1 z-(G(x_0+z)-G(z_0)-T_2 z)\|}{\|z\|}=\lim_{z\to 0}\frac{\|T_2 z-T_1 z\|}{\|z\|}=0.$$

This implies $\|T_2-T_1\|(u)=0$ for all unit vectors u. But of course, we can simply extend this linearly and conclude that $\|T_2-T_1\|=0$, i.e. $T_1=T_2$, as desired. □

Problem 1.2. Explain what is wrong with the following argument, letting G, G_1, G_2 be as in Example 1.14: "Since G_2 is a linear transformation, it is its own derivative, $G_2' \equiv G_2$ Therefore $G_2'(G_1(f)) = G_2(G_1(f)) = \int_a^x f(t)^2 dt$." (Hint: The short answer to this question is: The equality at the end is nonsense. But be more specific as to why.)

Proof. The claim $G_2' \equiv G$ is nonsense. The first is a map from $X \to \mathcal{B}(X;Y)$. The second is a map $X \to Y$. It is only true that, for every $x \in X$, $G_2'(x) = G$. □

Problem 1.3. Compute the Frechet derivative of the function

$$G : (C([0,\pi]), \|\cdot\|_u) \to (C^1([0,\pi]), \|\cdot\|_{C^1}), \quad G(f)(x)=\int_0^x \sin(f(t)^2)dt.$$

Proof. We compute a candidate with the chain rule, $G_1=\sin(f(t)^2)$ and $G_2=\int_0^x f(t)dt$. Then,

$$\begin{aligned}[G_1'(f)z](x) &= 2f(x)\cos(f(x)^2)z(x)\\ G_2'(f) &= G_2,\end{aligned}$$

since G_2 is linear. Therefore,

$$\begin{aligned}G'(f)z &= G_2'(G_1(f))\circ G_1'(f) = G_2(2f\cos(f^2)z)\\ &= \int_0^x 2f(t)\cos(f(t)^2)z(t)dt.\end{aligned}$$

□

Problem 1.4. Let X and Y be real normed spaces, and let U and V be open subsets of X and Y, respectively. Assume there exists a bijection $G : U \to V$ such that G is differentiable at every point of U and G^{-1} is a differentiable at every point of V. Then for every $x \in U$, $G'(x)$ is invertible, with inverse

$$G'(x)^{-1} = (G^{-1})'(G(x)).$$

In particular, $G'(x)$ is a vector space isomorphism (which must be an isomorphism of normed vector spaces if X and Y is known to be finite-dimensional.)

Proof. Just apply the chain rule to $\mathrm{id}_X(x) = (G^{-1} \circ G)(x)$. We have,

$$\begin{aligned}
(G^{-1})'(G(x)) \circ G'(x) &= \mathrm{id}_X \\
\Rightarrow (G^{-1})'(G(x)) \circ G'(x) \circ G'(x)^{-1} &= \mathrm{id}_X \circ G'(x)^{-1} \\
\Rightarrow (G^{-1})'(G(x)) &= G'(x)^{-1}.
\end{aligned}$$

□

Problem 1.5. Let X and Y be real normed vector spaces; let E be a connected open subset of X. Assume $f : E \to Y$ is differentiable on E and that $f'(x)$ is the zero element of $\mathcal{B}(X;Y)$ for all $x \in E$. Prove that f is constant on E. (Hint: Recall that a set E is connected if and only if E has no proper subsets that are both open and closed in E. It will be useful to consider functions of the form $g_z(t) = f(a + tz)$ for part of your argument.)

Proof. **TODO** □