Chapter 7; # 4 and 8 (pg. 175)

Problem 4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what interval does it converge uniformly? On what interval does it fail to converge uniformly? Is f continuous whenever it converges? If f bounded?

Proof. The series converges for all real x except for $x = 0$ and $x = -\frac{1}{n^2}$ for $n > 0$. For $x = 0$, we have $1 + 1 + \ldots$, which diverges. For $x = -\frac{1}{n^2}$, the nth term of the series is undefined. For all other x, the series has the same growth rate as $\sum \frac{1}{n^2}$, so it converges.

The first reaction is that all intervals not containing $X = \{0, -1, -\frac{1}{4}, \ldots\}$ should be correct. However, the problem with this is that if our interval has a limit point in X, then the neighbourhoods around such a limit point will not be bounded, and hence fail the Weierstrass M-test. The way to amend these limit points is to simply take closed intervals instead; hence we claim that f converges uniformly for any interval of the form $[a, b]$ disjoint from X. (This includes intervals of the form $[a, \infty)$ and $(-\infty, b]$.) Indeed, **TODO**

From what we just talked about, f will fail to converge uniformly on any interval that has a limit point in X. Explicitly, suppose $a = -\frac{1}{n^2} \in X$ is a limit point of a considered interval I. Let $\varepsilon = 1$ and $\delta > 0$, then

$$\begin{aligned}
|f(x) - f(x+\delta)| &= \left| \frac{1}{1+n^2x} - \frac{1}{1+n^2(x+\delta)} \right| \\
&= \left| \frac{n^2\delta}{(1+n^2x)(1+n^2(x+\delta))} \right| \\
&\geq \frac{n^2\delta}{(1+n^2|x|)(1+n^2(|x|+\delta))}
\end{aligned}$$

Uniform convergence show that the limit f is coninuous on any of the intervals it converges uniformly on. But the union of all intervals of the form $[a, b]$ disjoint from X is just $\mathbb{R} - X$. Hence f is continuous whenever it is defined.

Since f diverges around all the points of X, f is clearly not bounded. □

Problem 8. if

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof. The first part follows immediately from Theorem 7.10 in the text; let $f_n = c_n I(x - x_n)$, then $|f_n| \leq |c_n|$ and $\sum |c_n|$ converges, so $\sum f_n$ converges, as desired.

Furthermore, $f(x)$ is pointwise continuous at x when each $f_n(x)$ is continuous at x; hence $f(x)$ is at least continuous for all $x \neq x_n$. $\square$