Problem 1.1. Prove that the integral $\int_1^\infty \cos(x)dx$ does not converge.

Proof. Simply integrate:

$$\int_1^\infty \cos(x)dx = \lim_{b\to\infty}\int_1^b \cos(x)dx = \lim_{b\to\infty}(-\sin(b)+\sin(1))$$

which has no limit $b\to\infty$ since $\sin(b)$ oscillates forever. $\square$

Problem 1.2. Consider the two integrals

$$\int_\pi^\infty \frac{\cos x}{x}dx \quad \text{and} \quad \int_\pi^\infty \frac{\sin x}{x^2}dx$$

(a) Prove that one of these two integrals converges absolutely, but the other does not.

(b) Prove that both integrals converge, to the same value.

Proof. We proceed part by part.

(a) We claim that the first does not converge absolutely while the second does.

Indeed, consider the union of closed intervals of radius $\pi/2$ centered along $\pi\mathbb{N}$. On each interval, the value of $|\cos(x)| \geq 1/2$, thus

$$\begin{aligned}\int_\pi^\infty \frac{|\cos x|}{x}dx &\geq \sum_{n=1}^\infty \int_{\pi n-\pi/2}^{\pi n+\pi/2} \frac{|\cos x|}{x}dx \\ &\geq \sum_{n=1}^\infty \int_{\pi n-\pi/2}^{\pi n+\pi/2} \frac{1}{2x}dx \\ &\geq \sum_{n=1}^\infty \frac{\pi}{2\pi n+\pi} = \infty,\end{aligned}$$

where the final sum diverges due to the diverge of the harmonic series.

Meanwhile, we have

$$\int_\pi^\infty \frac{|\sin x|}{x^2}dx \leq \int_\pi^\infty \frac{1}{x^2}dx,$$

and this obviously converges since the exponent is > 1.

(b) However, it is the case that both integrals converge to the same value. Indeed, we can show this indirectly via integration by parts:

$$\int_{\pi}^{\infty} \frac{\cos x}{x} dx = \frac{\sin x}{x}\bigg|_{\pi}^{\infty} + \int_{\pi}^{\infty} \frac{\sin x}{x^2} dx = \int_{\pi}^{\infty} \frac{\sin x}{x^2} dx$$

Thus since the sin integral converges, so does the cos integral, and they must have the same value.

□

Problem 1.3. Prove that the integral $\int_1^\infty \cos(x^2)dx$ does not converge absolutely.

Proof. Substitute $x^2 \mapsto u$ to get $\int_1^\infty \frac{\cos u}{2u} du$. By Exercise 1.1, we know that this does not converge absolutely. □

Problem 1.4. Find a function $f \in \mathcal{R}_{\text{loc}}((0,1])$ such that $\int_0^1 f(x)dx$ converges, but not absolutely.

Proof. Let $I_k = (2^k, 2^{k+1}]$. Note that $\bigcup_{i=0}^\infty I_k = (0,1]$. Now define

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{2^{k+1}}{k} 1_{I_k}.$$

Thus

$$\int_0^1 f(x)dx = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{2^{k+1}}{k} 1_{I_k} dx = \sum_{k=0}^{\infty} (-1)^k \int_0^1 \frac{2^{k+1}}{k} 1_{I_k} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k}$$

which converges. However, similarly, we have

$$\int_0^1 |f(x)|dx = \int_0^1 \sum_{k=0}^{\infty} \frac{2^{k+1}}{k} 1_{I_k} dx = \sum_{k=0}^{\infty} \int_0^1 \frac{2^{k+1}}{k} 1_{I_k} = \sum_{k=0}^{\infty} \frac{1}{k},$$

which diverges. □