Problem 4.2. Verify that the sequence (23) is Cauchy in $L^p_{\mathcal{R}}(\mathbb{R})$.

$$f_n = \frac{1_{(1/n,1]}}{x^{1/2p}}.$$

Proof. Without loss of generality for $n > m > N$, N to be fixed later, we have

$$\begin{aligned}
\|f_n - f_m\|_{L_p}^p &= \int_{-\infty}^{\infty} \left| \frac{1_{(1/n,1]} - 1_{(1/m,1]}}{x^{1/2p}} \right|^p dx \\
&= \int_{-\infty}^{\infty} \left| \frac{1_{(1/n,1/m]}}{x^{1/2}} \right| dx \\
&= \int_{1/n}^{1/m} \frac{1}{|x|^{1/2}} dx \\
&= 2\sqrt{1/m} - 2\sqrt{1/n}.
\end{aligned}$$

Next, we may bound

$$2\sqrt{1/m} - 2\sqrt{1/n} = \frac{2(\sqrt{n} - \sqrt{m})}{\sqrt{nm}} \le \frac{2\sqrt{n}}{\sqrt{nm}} = \frac{2}{\sqrt{m}} < \frac{2}{\sqrt{N}}.$$

Thus choose $N > 4/\varepsilon^2$, and we conclude $\|f_n - f_m\|_{L_p}^p < \varepsilon$, which suffices. □

Problem 5.1. Suppose $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$ is T-periodic, and assume that $1 \le p < \infty$. Show that given $\varepsilon > 0$ there exists a *continuous* T-periodic function g such that $\|g\|_u \le 4\|f\|_u$ and

$$\int_0^T |f(x) - g(x)|^p dx < \varepsilon.$$

Proof. We can follow the constructions of Corollary 5.2 and Lemma 5.3, while taking extra care to maintain the periodicity of each construction. Indeed, since $f \in \mathcal{R}_{\text{loc}}(\mathbb{R})$, we have $f|_{[0,T]} \in \mathcal{R}([0,T])$. Thus there is some step function

$$\ell|_{[0,T]} = \sum_{j=1}^{n} m_j 1_{[p_{j-1},p_j)}$$

such that $\|f|_{[0,T]} - \ell|_{[0,T]}\|_{L^1([0,T])} < \varepsilon$ and $\|\ell|_{[0,T]}\|_u \le 2\|f|_{[0,T]}\|_u$. Since f is T-periodic, we can "copy-paste" this construction across every T interval and obtain an T-periodic extension ℓ of $\ell|_{[0,T]}$. We also have

$$\|\ell\|_u = \max_n \|\ell|_{[nT,(n+1)T]}\|_u \le \max_n 2\|f|_{[nT,(n+1)T]}\| = 2\|f\|_u.$$

Next, from Corollary 5.2 we may construct from $\ell|_{[0,T]}$ a continuous function $g|_{[0,T]} = \sum_{j=1}^n c_j g_j$. Then it is easy to see that

$$g = \sum_n g|_{[nT,(n+1)T]}$$

is a T-periodic function. We also have $\|\ell - g\|_{L^1([0,T])} < \varepsilon$ and

$$\|g\|_u = \max_n \|g\|_{[nT,(n+1)T]}\|_u \leq \max_n 2\|\ell|_{[nT,(n+1)T]}\| = 2\|\ell\|_u.$$

(Actually, there is a bit of "leakage" across the boundary points at $0, T, 2T, \ldots$, so the equations are not exactly true – but these leakages don't effect the uniform norm.)

Together, we conclude that there is some T-periodic function g such that $\|g\|_u \leq 4\|f\|_u$ and $\|f - g\|_{L^1([0,T])} < \varepsilon$ (approx.).

To extend this to all $1 \leq p < \infty$, note that $f \in \mathcal{R}([0,T])$ implies that f in bounded. Thus g is bounded by construction. Thus we can bound $|f(x) - g(x)|^{p-1} \leq L$ on $[0,T]$, and therefore

$$\begin{aligned}\int_0^T |f(x) - g(x)|^p dx &= \int_0^T |f(x) - g(x)||f(x) - g(x)|^{p-1} dx \\ &\leq TL \int_0^T |f(x) - g(x)| dx < \varepsilon.\end{aligned}$$

□

Problem 1.1. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the following:

(a) ϕ is compactly supported and continuously differentiable.

(b) f is compactly supported and is Riemann integrable on an interval containing its support.

Prove that the function $g : \mathbb{R} \to \mathbb{R}$ defined $g = \phi * f$ is continuously differentiable on $\mathbb{R}$, with derivative $g' = \phi' * f$. What can you say about the case where ϕ and f are k and ℓ times continuously differentiable, respectively (and still both be compactly supported)?

Proof. We have

$$\begin{aligned}g'(x) &= \lim_{h \to 0} \left(\frac{\phi * f(x+h) - \phi * f(x)}{h} \right) \\ &= \lim_{h \to 0} \frac{1}{h} \left(\int_{-\infty}^{\infty} \phi(x+h-y) f(y) dy - \int_{-\infty}^{\infty} \phi(x-y) f(y) dy \right)\end{aligned}$$

$$= \lim_{h\to\infty} \int_{-\infty}^{\infty} \left(\frac{\phi(x-y+h)-\phi(x-y)}{h} \right) f(y)dy.$$

Now we want to show that $g'(x) - \phi' * f < \varepsilon$, so consider

$$\begin{aligned} g'(x) - \phi' * f &= \lim_{h\to\infty} \int_{-\infty}^{\infty} \left(\frac{\phi(x-y+h)-\phi(x-y)}{h} - \phi'(x-y) \right) f(y)dy \\ &= \lim_{h\to\infty} \int_{-\infty}^{\infty} E_h(y) f(y)dy, \end{aligned}$$

where $E_h(y) = \frac{\phi(x-y+h)-\phi(x-y)}{h} - \phi'(x-y)$. Now we can see that pointwise as $h \to 0$,

$$\frac{\phi(x-y+h)-\phi(x-y)}{h} - \phi'(x-y) \to 0.$$

However, this is insufficient to move change the order of the limit/integration. But note that

$$\begin{aligned} E_h(y) &= \frac{\phi(x-y+h)-\phi(x-y)}{h} - \phi'(x-y) \\ &= \frac{1}{h} \left(\int_{x-y}^{x-y+h} \phi'(t)dt - \int_{x-y}^{x-y+h} \phi'(x-y)dt \right) \\ &= \frac{1}{h} \left(\int_{x-y}^{x-y+h} \phi'(t) - \phi'(x-y)dt \right). \end{aligned}$$

Since ϕ is compactly supported and continuously differentiable, this implies that ϕ' is uniformly continuous. Thus we see that in fact $E_h(y) \to 0$ uniformly as $h \to 0$. Furthermore, consider the fact that f has compact support, so we can limit the bounds of integration to some $[-L, L]$, and thus we may move the limit in:

$$\begin{aligned} g'(x) - \phi' * f &= \lim_{h\to\infty} \int_{-\infty}^{\infty} E_h(y) f(y)dy \\ &= \lim_{h\to\infty} \int_{-L}^{L} E_h(y) f(y)dy \\ &= \int_{-L}^{L} \lim_{h\to\infty} E_h(y) f(y)dy \\ &= \int_{-L}^{L} 0 f(y)dy = 0. \end{aligned}$$

Thus $g'(x) = \phi' * f$, as desired. $\square$