Problem 3.2. Assume $f \in C^2(W;\mathbb{R})$ for some open subset W of $\mathbb{R}^n$.

(a) Explicitly write out all the terms of the second order Taylor polynomial $P_2(z)$ if $n = 1$, $n = 2$, or $n = 3$, by evaluating all the multi-index powers and factorials. (This is quick for $n = 1$ and $n = 2$ but takes a bit of writing for $n = 3$.)

(b) For arbitrary n, write down a formula for $P_2(z)$ in terms of the Jacobian and Hessian matrices.

(c) (The Second Derivative Test.) A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be *positive-definite* if $x^T Ax > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. Prove that if $Jf(x_0) = 0$ and $Hf(x_0)$ is a positive definite, then f has a local minimum at x_0. (Hint: The function $z \mapsto H(x_0)z\cdot z$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$; its restriction to the (compact!) st $S^1 = \{x \in \mathbb{R}^n : |x| = 1\}$ is of course continuous.) Similarly, show that if $Jf(x_0) = 0$ and $Hf(x_0)$ is *negative-definite* (you can guess the definition), then f has a local maximum at x_0.

(d) Show that a 2 symmetric matrix $A = (a_{i,j})_{i,j=1}^2$ is positive definite if and only if $\det A$ and $a_{1,1}$ are positive. Use this information to write down another version of the Second Derivative Test for a function of 2 variables.

Proof. We proceed with each part separately.

(a) Recall the formulas for $P_2(z)$. Suppose $n = 1$. If $|\alpha| = 0$, then we have $\alpha = (0)$. If $|\alpha| = 1$, then we have $\alpha = (1)$. If $|\alpha| = 2$, then we have $\alpha = (2)$. Thus,

$$\begin{aligned} P_2(z) &= \sum_{k=0}^{2}\sum_{|\alpha|=k}\frac{\partial^\alpha f(a)}{\alpha!}z^\alpha \\ &= f(a) + \frac{df(a)}{dz}z + \frac{1}{2}\frac{d^2f(a)}{dz^2}z^2. \end{aligned}$$

Suppose $n = 2$. If $|\alpha| = 0$, then we have $\alpha = (0,0)$. If $|\alpha| = 1$, then we have $\alpha \in \{(1,0),(0,1)\}$. If $|\alpha| = 2$, then we have $\alpha \in \{(2,0),(1,1),(0,2)\}$. Thus,

$$\begin{aligned} P_2(z) &= \sum_{k=0}^{2}\sum_{|\alpha|=k}\frac{\partial^\alpha f(a)}{\alpha!}z^\alpha \\ &= f(a) + \partial_1 f(a)z + \partial_2 f(a)z + \frac{1}{2}\partial_1^2 f(a)z^2 + \frac{1}{2}\partial_2^2 f(a)z^2 + \partial_1\partial_2 f(a)z^2. \end{aligned}$$

Suppose $n = 3$. If $|\alpha| = 0$, then we have $\alpha = (0,0,0)$. If $|\alpha| = 1$, then we have $\alpha \in \{(1,0,0),(0,1,0),(0,0,1)\}$. If $|\alpha| = 2$, then we have

$$\alpha \in \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}.$$

Thus,

$$\begin{aligned}P_2(z) = f(a) &+ \partial_1 f(a)z + \partial_2 f(a)z + \partial_1 f(a)z \\ &+ \frac{1}{2}\partial_1^2 f(a)z^2 + \frac{1}{2}\partial_2^2 f(a)z^2 + \frac{1}{2}\partial_3^2 f(a)z^2 \\ &+ \partial_1\partial_2 f(a)z^2 + \partial_2\partial_3 f(a)z^2 + \partial_1\partial_3 f(a)z^2.\end{aligned}$$

(b) Reading off the terms, we can see that

$$P_2(z) = f(a) + \nabla f(a)^T z + \frac{1}{2}z^T Hf(a)z.$$

(Note how the 1/2 factor perfectly handles the duplicated partials in the off diagonals.)

(c) **TODO**

(d) We want to show that $z^T Az > 0$ for all $z \in \mathbb{R}^2$. Let $a_{1,1} = a$, $a_{1,2} = a_{2,1} = b$, and $a_{2,2} = c$ and let $z = (x, y)$. Expanding this out, we have

$$z^T Az = ax^2 + 2bxy + cy^2.$$

($\Rightarrow$): If $z^T Az > 0$, then setting $z = (1, 0)$ implies $a > 0$. Next consider $z = (x, 1)$. Then we know that $ax^2 + 2bx + c > 0$ for all x. This means that the polynomial has no zeros. Thus its discriminant $4b^2 - 4ac$ must be negative, i.e. $ac - b^2 = \det A > 0$. ($\Leftarrow$): Do casework on $y = 0$ and do almost the opposite argument. For all $z = (x, 0)$, $a_{1,1} > 0$ implies $ax^2 = z^T Az > 0$.

If $y \neq 0$, then consider $t = x/y$ and $q(t) = at^2 + 2bt + c = \frac{1}{y^2}z^T Az$. Since $\det A > 0$, we know that $b^2 - ac < 0$, so $q(t)$ has no roots. Since $a > 0$, it must be that $q(t) > 0$ for all t. This implies $z^T Az > 0$ for all (x, y), $y \neq 0$.

Thus $z^T Az$ holds for all $z \in \mathbb{R}^2$, as desired.

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