Exercises 6, 13, 14, 21, 25, 26, pp. 230-233.

Problem 6. Are the following subrings of the ring of all functions from the closed interval $[0, 1]$ to $\mathbb{R}$.

(a) the set of all functions $f(x)$ such that $f(q) = 0$ for all $q \in \mathbb{Q} \cap [0, 1]$:

Yes.

(b) the set of all polynomial functions:

Yes.

(c) the set of all functions which have only finite numer of zeros, together with the zero function:

Yes.

(d) the set of all functions which have an infinite number of zeros:

Yes.

(e) the set of all functions f such that $lim_{x\to 1^-} f(x) = 0$:

Yes.

(f) the set of all rational linear combinations of the fuctions $\sin nx$ and $\cos nx$, where $m, n \in \{0, 1, 2, \dots\}$:

Yes.

And we're done.

Problem 13. An element x in R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$.

(a) Show that if $n = a^k b$ for some integers a and b then $\overline{ab}$ is a nilpotent element of $\mathbb{Z}/n\mathbb{Z}$.

(b) If $a \in \mathbb{Z}$ is an integer, show that the element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if every prome divisor of a. In particular, determine the nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$ explicitly.

(c) Let R be the ring of functions from a nonempty set X to a field F. Prove that R contains no nonzero nilpotent elements.

Proof. TODO $\square$

Problem 14. Let x be a nilpotent element of the commutative ring R.

(a) Prove that x is either zero or a zero divisor.

(b) Prove that rx is nilpotent for all $r \in R$.

(c) Prove that $1 + x$ is a unit in R.

(d) Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. TODO □

Problem 21. Let X be any nonempty set.

(a) Prove that $\mathcal{P}(X)$ is a ring under the addition and multiplication given.

(b) Prove that this ring is commutative, has an identity and is a Boolean ring.

Proof. TODO □

Problem 25. Let I be the ring of integral Hamilton Quaterions and define

$$N : I \to \mathbb{Z} \text{ by } N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$$

(the map N is called the *norm*).

(a) Prove that $N(\alpha) = \alpha\overline{\alpha}$ for all $\alpha \in I$, where if $\alpha = a+bi+cj+dk$ then $\overline{\alpha} = a-bi-cj = dk$.

(b) Prove that $N(\alpha\beta) = N(\alpha)(\beta)$ for all $\alpha, \beta \in I$.

(c) Prove that an element of I is a unit if and only if it has norm $+1$. Show that $I^\times$ is isomprphic to the quaterion group of order 8.

Proof. TODO □

Problem 26. Let K be a field and $\nu : K^\times \to \mathbb{Z}$ a discrete valuation on K. Let R be the valuation ring of ν.

(a) Prove that R is a subring of K which contains the identity.

(b) Prove that for each nonzero element $x \in K$ either x or x^{-1} is in R.

(c) Prove that an element x is a unit of R if and only if $\nu(0) = 0$.

Proof. TODO □

Exercises 3, 4, 10, 11, pp. 238-239.
Problem 3. Let $R[[x]]$ be the *formal power series* of R in x. Define addition and multiplication as the textbook does.

(a) Prove that $R[[x]]$ is a commutative ring with 1.

(b) Show that $1 - x$ is a unit in $R[[x]]$ with inverse $1 + x + x^2 + \cdots$.

(c) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

Proof. TODO □

Problem 4. Prove that if R is an integral domain then the ring of formal power series $R[[x]]$ is also an integral domain.

Proof. TODO □

Problem 10. Consider the following elements of the integral group ring $\mathbb{Z}S_3$:

$$\alpha = 3(1,2) - 5(2,3) + 14(1,2,3) \text{ and } \beta = 6(1) + 2(2,3) - 7(1,3,2)$$

(where (1) is the identity of S_3). Compute the following elements:

(a) $\alpha + \beta$, **(b)** $2\alpha - 3\beta$, **(c)** $\alpha\beta$, **(d)** $\beta\alpha$, **(e)** a^2.

Proof. TODO □

Problem 11. Repeat the precedeing exercise under the assumption that the coefficients of α and β are in $\mathbb{Z}/3\mathbb{Z}$.

Proof. TODO □

Exercises 15, 17, 18, 19, 24, 26, pp. 247-251.

Problem 15. Prove that the map $\mathcal{P}(X) \to R$ defined by $A \mapsto \chi_A$ is a ring homomorphism.

Proof. TODO □

Problem 17. Let R and S be nonzero rings with identity and denote their respective identities by 1_R and 1_S. Let $\varphi : R \to S$ be a nonzero homomorphism of rings.

(a) Prove that if $\varphi(1_R) \neq 1_S$, then $\varphi(1_R)$ is a zero divisor in S. Deduce that if S is an integral domain then every ring homomorphism from R to S sends the identity of R to the identity of S.

(b) Prove that if $\varphi(1_R) = 1_S$ then $\varphi(u)$ is a unit in S and that $\varphi(u^{-1}) = \varphi(u)^{-1}$ for each unit $u \in R$.

Proof. TODO □

Problem 18. Let R be a ring.

(a) If I and J are ideals of R prove that their intersection $I \cap J$ is also an ideal of R.

(b) Prove that the intersection of an arbitrary nonempty collection of ideals is again an ideal of R.

Proof. TODO $\square$

Problem 19. Prove that if $I_1 \subseteq I_2 \subseteq \dots$ are ideals of R then $\bigcup_{n=1}^{\infty} I_n$ is an ideal of R.

Proof. TODO $\square$

Problem 24. Let $\varphi : R \to S$ be a ring homomorphism.

(a) Prove that if J is an ideal of S then $\varphi^{-1}(J)$ is an ideal of R. Apply this to the special case when R is a subring of S and φ is the inclusion homomorphism to deduce that if J is an ideal of S then $J \cap R$ is an ideal of R.

(b) Prove that if φ is surjective and I is an ideal of R then $\varphi(I)$ is an ideal of S. Give an example where this fails if φ is not surjective.

Proof. TODO $\square$

Problem 26. Let R be a ring. For any $n \in \mathbb{Z}$ and $r \in R$, define $nr = r + \dots + r$ (n times).

(a) Prove that the map $\mathbb{Z} \to R$ defined by $k \mapsto k1_R$ is a ring homomorphism whose kernel is $n\mathbb{Z}$, where n is the characteristic of R.

(b) Determine the characteristics of the rings $\mathbb{Q}$, $\mathbb{Z}[x]$, and $\mathbb{Z}/n\mathbb{Z}[x]$.

(c) Prove that if p is a prime and if R is a commutative ring of characteristic p, then $(a+b)^p = a^p + b^p$ for all $a, b \in R$.

Proof. TODO $\square$