Problem 1.1. Suppose $F = \mathbb{R}$ or $\mathbb{C}$.

(a) Show that F^X is finite-dimensional if and only if X is a finite set.

(b) The space of 'sequences in F that are eventually zero' is an infinite-dimensional vector space. Give a more precise definition for this space; then give an example of a Hamel basis for it.

Proof. We proceed with each part.

(a) F^X is finite-dimensional if and only if there exists a basis of finite size. Note that all bases have the same cardinality (this may depend on AoC, I think). But X itself is trivially a basis of F^X. Thus this occurs if and only if X is finite.

(b) Say a function $f : A \to F$ has *finite support* if the set $\operatorname{supp}(f) = \{x \in A \mid f(x) \neq 0\}$ is finite. Sequences in F are just functions $\mathbb{N} \to F$. They are eventually zero means that they must have finite support. Thus we must show that the set of finitely supported functions, $\{f : \mathbb{N} \to F \mid \operatorname{supp}(f) \text{ is finite}\}$, is a vector space.

Indeed, let's just check the axioms. Let $fg : \mathbb{N} \to F$, then it is easy to see that $\operatorname{supp}(f + g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$, which is finite. Thus addition is closed. Let $k \in F$, it is also easy to see that $\operatorname{supp}(kf) \subseteq \operatorname{supp}(f)$, which is finite. Thus scalar multiplication is closed. We don't really need to check all the other commutative/associative/distributive axioms as they are tedious yet obvious. Thus $\{f : \mathbb{N} \to F \mid \operatorname{supp}(f) \text{ is finite}\}$ is indeed a vector space, as desired.

$\square$

Problem 2.1. Let $F = \mathbb{R}$ or $\mathbb{C}$. Prove that $(BC(X; F), \|\cdot\|_u)$ is a Banach space., using the Uniform Limit Theorem.

Problem 2.2. Let $F = \mathbb{R}$ or $\mathbb{C}$. Use the results of this section, together with the completeness of F^n under the Euclidean norm $\|\cdot\|$, to prove that any finite-dimensional normed F-vector space $(V, \|\cdot\|_V)$ is complete.

Proof. Any finite-dimensional F-vector space V with dimension n is isomorphic (as a topological vector space) to F^n. Thus any norm on V is equivalent to some norm on F^n. Since F^n is complete, V is complete. $\square$

Problem 2.3. In this problem, we show that the L^2 norm on $C([0, 1])$ is strictly stronger than the L^1 norm. (Actually, we show a little more.)

(a) Assume $1 \le p < q < +\infty$. Prove that for any $f \in C([a,b])$, we have

$$\|f\|_{L^p([a,b])} \le (b-a)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q([a,b])}.$$

(b) Prove that the continuous functions $f_n(x) = n^2(1/n - x)1_{[-,1/n]}(x)$ have constant L^1 norm, but their L^2 norm tends to $+\infty$ as $n \to \infty$. Conclude that the L^2 norm is strictly stronger than the L^1 norm on $C([0,1])$.

Proof. We proceed with each part.

(a) Apply Hölder's inequality on functions $|f|^p$ and 1, with conjugates q/p and $q/(q-p)$. Note that since $p < q$, we have $q/p, q/(q-p) > 1$ so these conjugates are valid. Now,

$$\int_a^b |f|^p dx \le \left(\int_a^b (|f|^p)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} \left(\int_a^b 1^{\frac{q}{q-p}} dx\right)^{\frac{q-p}{q}} = (b-a)^{\frac{q-p}{q}} \left(\int_a^b |f|^q dx\right)^{\frac{p}{q}}.$$

Taking the pth root on both sides, we have

$$\begin{aligned} \|f\|_{L^p([a,b])} = \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} &\le (b-a)^{\frac{q-p}{pq}} \left(\int_a^b |f|^q dx\right)^{\frac{1}{q}} \\ &= (b-a)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b |f|^q dx\right)^{\frac{1}{q}} \\ &= (b-a)^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q([a,b])}, \end{aligned}$$

as desired.

(b) In the L^1 norm,

$$\begin{aligned} \|f_n\|_1 &= \int_0^{1/n} n^2 \left(\frac{1}{n} - x\right) dx = \int_0^{1/n} (n - n^2 x) dx \\ &= 1 - n^2 \frac{(1/n)^2}{2} = \frac{1}{2}. \end{aligned}$$

So indeed the functions have constant L^1 norm.

In the L^2 norm,

$$\|f_n\|_2 = \left(\int_0^{1/n} (n - n^2 x)^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\frac{n(nx-1)^3}{3} \Big|_0^{1/n} \right)^{\frac{1}{2}}$$
$$= \left(\frac{n}{3} \right)^{\frac{1}{2}}.$$

Thus clearly $\|f_n\|_2 \to \infty$ as $n \to \infty$. This shows that L^1 and L^2 are not equivalent, and thus L^2 must be stronger than L^1.

□

Problem 2.4. What is the relationship between $\|\cdot\|_{L^1([0,1])}$ and $\|\cdot\|_u$ on $C([0,1])$? Justify your answer.

Proof. $\|\cdot\|_u$ is *strictly stronger* than $\|\cdot\|_1$. We simply have,

$$\|f\|_1 = \int_0^1 |f|dx \leq \sup f \times 1 = \|f\|_u.$$

And there are many functions where $\sup f = \infty$ but $\|f\|_1 < \infty$, so the two norms are definitely not equivalent. Thus $\|\cdot\|_u$ is strictly stronger than $\|\cdot\|_1$. □