Problem 1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Proof. Suppose $\{s_n\}$ converges to s. Then for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $n \geq N$ implies $|s - s_n| \leq \epsilon$. We claim that $\{|s_n|\}$ converges to $|s|$. Indeed, if $\epsilon > 0$, then choose the same $N \in \mathbb{N}$ we did for $\{s_n\}$. We have for any $n \geq N$:

$$||s| - |s_n|| \leq |s - s_n| \leq \epsilon.$$

(We proved the first inequality in homework 2!) Hence $\{|s_n|\}$ converges. □

Problem 2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

We need a few lemmas (I continue to use this version sorry):

Lemma. $\lim_{n\to\infty}(\sqrt{n+c} - \sqrt{n}) = 0$. *For any* $c \in \mathbb{R}$.

Proof. Indeed, suppose $\epsilon > 0$. Then choose $N \geq \frac{c^2}{4\epsilon^2} - c$. (This value will make sense after the the calucations.)

Notice that if $\delta = |\sqrt{n+c} - \sqrt{n} - 0| = \sqrt{n+c} - \sqrt{n}$, then we can use difference of squares to see that $\delta(\sqrt{n+c} + \sqrt{n}) = n + c - n = c$. Since $\sqrt{n+c} + \sqrt{n}$ will not be zero as $n \to \infty$, we may write $\delta = \frac{c}{\sqrt{n+c}+\sqrt{n}}$. Now for all $n \geq N$, we have:

$$\begin{aligned} n \geq \frac{c^2}{4\epsilon^2} - c \Rightarrow n + c &\geq \frac{c^2}{4\epsilon^2} \\ &\Rightarrow \frac{1}{n+c} \leq \frac{4\epsilon^2}{c^2} \\ &\Rightarrow \frac{1}{\sqrt{n+c}} \leq \frac{2\epsilon}{c} \\ &\Rightarrow \frac{c}{2\sqrt{n+c}} \leq \epsilon \\ &\Rightarrow \delta = \frac{c}{\sqrt{n+c}+\sqrt{n}} \leq \epsilon \end{aligned}$$

Thus $\lim_{n\to\infty}(\sqrt{n+c} - \sqrt{n}) = 0$. □

Lemma. *If* $\lim_{n\to\infty} f(n) = L$ *and* $\lim_{n\to\infty} g(n) = \infty$, *then*

$$\lim_{n\to\infty} f(g(n)) = L.$$

Proof. Set $\epsilon > 0$. Then there is some N_f such that $n \geq N_f$ implies $|f(n) - L| \leq \epsilon$ by assumption. Futhermore, there is some N_g such that $n \geq N_g$ implies $g(n) \geq N_f$. Thus

$n \geq N_g$ implies $|f(g(n)) - L| \leq \epsilon$. This shows that

$$\lim_{n\to\infty} f(g(n)) = L.$$

$\square$

Now we can finally do the real problem!

Proof. We can calculate a slightly different equation: $\sqrt{n^2 + n + \frac{1}{4}} - n$. We can complete the square inside the radical to see that

$$\sqrt{\left(n + \frac{1}{4}\right)^2} - n = n + \frac{1}{2} - n = \frac{1}{2}.$$

Thus we may rewrite:

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2+n} - n) &= \lim_{n\to\infty}\left(\sqrt{n^2+n} - \sqrt{n^2+n+\frac{1}{4}} + \sqrt{n^2+n+\frac{1}{4}} - n\right) \\ &= \lim_{n\to\infty}\left(\sqrt{n^2+n} - \sqrt{n^2+n+\frac{1}{4}} + \frac{1}{2}\right)\end{aligned}$$

We claim that the difference has a limit of 0. Let $g(n) = n^2 + n$ and note that $g(n) \to \infty$, so we may apply our lemma to obtain:

$$\lim_{m\to\infty}\left(\sqrt{m} - \sqrt{m + \frac{1}{4}} + \frac{1}{2}\right).$$

But we know by our first lemma that this converges to $\frac{1}{2}$, because we have proved that $\lim_{m\to\infty}\left(\sqrt{m} - \sqrt{m+\frac{1}{4}}\right) = 0$. Thus $\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$. $\square$

Problem 3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

Proof. By the complete upper bound property of the reals, it suffices to prove that $\{s_n\}$ bounded above by 2 and monotonically increasing.

Indeed, we prove that $s_n < 2$ for all $n = 1, 2, 3, \ldots$ by induction. Clearly $s_1 = \sqrt{2} < 2$, so the base case is true. Now assume the induction hypothesis that $s_k < 2$ for some k. Then $\sqrt{s_k} < \sqrt{2}$. Thus

$$s_{k+1} = \sqrt{2 + \sqrt{s_k}} < \sqrt{2 + \sqrt{2}} < \sqrt{2+2} = 2,$$

which completes the induction.

Similarly, we prove that $\{s_n\}$ is monotonically increasing by induction.

Clearly, $\sqrt{2} < \sqrt{2 + \sqrt{s_1}} = s_2$. Hence the base case is true.

Now assume the induction hypothesis that $s_{k-1} < s_k$ for some k. Now since $\sqrt{x}$ is a monotonically increasing function, we have

$$\begin{aligned} s_{k-1} < s_k &\Rightarrow \sqrt{s_{k-1}} < \sqrt{s_k} \\ &\Rightarrow 2 + \sqrt{s_{k-1}} < 2 + \sqrt{s_k} \\ &\Rightarrow \sqrt{2 + \sqrt{s_{k-1}}} < \sqrt{2 + \sqrt{s_k}} \\ &\Rightarrow s_k < s_{k+1}, \end{aligned}$$

which completes the induction.

Combining the two results shows that $\{s_n\}$ converges. □

Problem 4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof. First we prove by induction that for any $m \geq 1$,

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m}, \qquad s_{2m+1} = 1 - \frac{1}{2^m}$$

Clearly the base case is true, since $s_2 = 0 = 1/2 - 1/2^1$ and $s_3 = 1/2 = 1 - 1/2^1$.

Now assume for the sake of induction that

$$s_{2k} = \frac{1}{2} - \frac{1}{2^k}, \qquad s_{2k+1} = 1 - \frac{1}{2^k}.$$

Then

$$s_{2k+2} = \frac{s_{2k+1}}{2} = \frac{1}{2}\left(1 - \frac{1}{2^k}\right) = \frac{1}{2} - \frac{1}{2^{k+1}}.$$

Furthermore we can use this new value of s_{2k+2} to compute $s_{2(k+1)+1}$:

$$s_{2(k+1)+1} = \frac{1}{2} + s_{2k+2} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.$$

Hence the induction is complete.

Now we can compute the upper and lower limits by using theorem 3.17 in the textbook. Define E, s^*, and s_* as in Definition 3.16.

First, $\limsup_{n\to\infty}(s_n) = 1$. Consider the subsequence of only odd indices. Then that subsequence clearly converges to 1, so $1 \in E$. Futhermore, if $x > 1$, then clearly for any $n \in \mathbb{N}$ we have $s_n < x$ since 1 bounds s_n. Hence theorem 3.17 tells us that $s^* = 1$.

Second, $\liminf_{n\to\infty}(s_n) = \frac{1}{2}$. Consider the subsequence of only even indices. Then that subsequence clearly converges to $\frac{1}{2}$, so $\frac{1}{2} \in E$. Also, if $x < \frac{1}{2}$, then take N such that $2^{N/2} > 1/(x - 1/2)$. Then for any $n \geq N$, if n is odd we clearly have $s_n > \frac{1}{2}$, and if n is even we have $s_n = \frac{1}{2} - \frac{1}{2^{n/2}}$. We also have $\frac{1}{2} - x > \frac{1}{2^{n/2}} < x$ by assumption, so:

$$x > \frac{1}{2} - \frac{1}{2^{n/2}},$$

which proves $s_* = \frac{1}{2}$. $\square$