Problem 1.1. Show that replacing 'rectangles' with 'squares' in the definition of a set of measure zero leads to an equivalent definition. (Skipped.)

Problem 1.2. The following sets have measure zero:

(a) Any finite subset of $\mathbb{R}^n$ has measure zero.

(b) Any finite or countable union of sets of measure zero itself has measure zero.

(c) Any subset of a measure-zero set has measure zero.

(d) $\mathbb{R}^{n-1} \times \{0\}$ has measure zero in $\mathbb{R}^n$.

(e) If $Z \subseteq \mathbb{R}^n$ has measure zero and $\varphi : Z \to \mathbb{R}^n$ is Lipschtiz, then $\varphi(Z)$ has measure zero.

Proof. We proceed with each part.

(a) Let $\{s_1, \cdots, s_k\} \subseteq \mathbb{R}^n$ be a finite set. Then simply cover the set with k cubes of volume ε/k each.

(b) Let $\{S_i\}_{i=1}^\infty$ be a finite or countable collection of sets, where it is understood that we stop enumerating at k if the collection is finite. If each set has measure zero, then for any ε, the set S_i can be covered with a collection of rectangles with volume less than $\varepsilon/2^i$. Then the total volume of the union of all the collections covers $\bigcup_{i=1}^\infty S_i$ is no more than $\sum_{i=1}^\infty \varepsilon/2^i = \varepsilon$, as desired.

(c) If $Z \subseteq \mathbb{R}^n$ has measure zero, then for any ε, $\{R_i\}$ is a cover with total volume less than ε. But thus $\{R_i\}$ trivially covers any subset of Z with volume less than ε also, and thus any subset of Z has measure zero.

(d) Note that we can break the hyperplane into *tiles*:

$$\mathbb{R}^{n-1} \times \{0\} = \bigcup_{v \in \mathbb{N}^{n-1}} \left(\bigotimes_{i=1}^{n-1} [v_i, v_i + 1] \times \{0\} \right).$$

For any ε, each tile at $v \in \mathbb{N}^{n-1}$ can be covered with $\bigotimes_{i=1}^{n-1} [v_i, v_i + 1] \times [-\varepsilon/2, -\varepsilon/2]$. Thus each tile is a set of measure zero. Since $\mathbb{N}^{n-1}$ is a countable set, by part (2), $\mathbb{R}^{n-1} \times \{0\}$ is a countable union of measure zero sets, and therefore has measure zero itself.

(e) Let Z be a measure zero set. For any ε, let $\{R_i\}$ be a cover of Z with total volume less than ε/L^n. Since φ is Lipschtiz with L, we have $|\varphi(R_i)| \le L^n |R_i|$. Since $\{R_i\}$ covers Z, we know that $\{\varphi(R_i)\}$ covers $\varphi(R_i)$ with volume at most $L^n(\varepsilon/L^n) = \varepsilon$, as desired.

$\square$

Problem 1.3. For real numbers $A < B$, prove that the interval $[A, B]$ is *not* a set of measure zero, using the following outline. First, argue by contradiction, assuming that there is some 'bad' open covering $\mathcal{B} = \{(a_i, b_i)\}_{i=1}^{\infty}$ of $[A, B]$ such that

$$\sum_{i=1}^{\infty}(b_i - a_i) < B - A$$

(where the sum is understood to be truncated at some finite index if $\mathcal{B}$ is a finite collection.) Then, justify the following steps.

(a) Show that, without loss of generality, you may assume that $\mathcal{B}$ is finite.

(b) Define N to be the smallest possible number of intervals that make up a putative 'bad' covering $\mathcal{B}$. Argue that N cannot be equal to 1.

(c) Let $\mathcal{B}$ be a 'bad' covering consisting of N intervals. Show that without loss of generality, you may assume that $A \in (a_1, b_1)$ and that $a_2 < b_1$.

(d) Use (c) to fashion a 'bad' covering $\mathcal{B}'$ with $N - 1$ elements, and explain why this completes the proof.

Proof. We proceed with each part.

(a) Since $\mathcal{B}$ is a cover of a compact set, it has some finite subcover. Thus without loss of generality we may always assume that $\mathcal{B}$ is finite.

(b) If $N = 1$, then we could only have one interval (a_1, b_1). We have to have $a_1 < A$ and $b_1 > B$, but then $b_1 - a_1 > B - A$, which contradicts our original assumption on $\mathcal{B}$. Thus $N > 1$.

(c) Without loss of generality, order the intervals by a_i in increasing order.

Then we must have $a_1 < A$. If it is the case that $b_1 < A$ as well, then (a_1, b_1) doesn't cover $[A, B]$ at all, and we can safely remove it from $\mathcal{B}$ for a smaller cover. But we assumed that N is the smallest sized covers that exist, so this cannot happen. Thus $b_1 > A$ and $A \in (a_1, b_1)$.

Furthermore, if $a_2 > b_1$, then all $a_i > b_1$, for $i \geq 2$. This implies that $b_1 \notin (a_i, b_i)$ for all $i \geq 2$ and $b_1 \notin (a_1, b_1)$, which is impossible since $\mathcal{B}$ is a cover, and thus must contain b_1. Hence $a_2 < a_1$.

(d) With (c), construct another 'bad' covering (a_1, b_2) with all (a_i, b_i) for $i > 2$. This is clearly still a cover since (c) guarantees that $(a_1, b_2) = (a_1, b_1) \cup (a_2, b_2)$. This new cover $\mathcal{B}'$ has $N - 1$ elements. But this contradicts the minimality of N. Thus any such 'bad' cover cannot exist, as desired.

□

Problem 1.4. If f is the function in Example 1.3 of Chapter 9, show that $\mathrm{osc}_0(f) = 1$.

Proof. Along the curve $x_2 = x_1^2$, the value of f is constant $1/2$. Along the curve $x_2 = -x_1^2$, the value of f is constant $-1/2$. Thus

$$\limsup_{(a,b)\to(0,0)} f(a,b) - \liminf_{(a,b)\to(0,0)} f(a,b) = \frac{1}{2} + \frac{1}{2} = 1,$$

as desired. □

Problem 1.5. Let f and g be two locally Riemann integrable functions defined on all of $\mathbb{R}$. Show that $\int_a^b f(x)dx = \int_a^b g(x)dx$ *for all* compact intervals $[a,b]$ if and only if the set $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ has measure zero.

Proof. Note $\int_a^b f(x)dx = \int_a^b g(x)dx$ if and only if $\left|\int_a^b (f(x) - g(x))dx\right| < \varepsilon$ for all $\varepsilon > 0$. Thus it suffices to work with this instead.

($\Rightarrow$): **TODO**

($\Leftarrow$): If $Z = \{x \in \mathbb{R} : f(x) \neq g(x)\}$ is a set of measure zero, then for any $\varepsilon > 0$, let $\{R_i\}$ be a cover with total length less than ε/M, where $M = \sup_Z |f(x) - g(x)|$. Then we have

$$\begin{aligned}
\left|\int_a^b (f(x) - g(x))dx\right| &= \left|\int_Z (f(x) - g(x))dx\right| \\
&\leq \int_Z |f(x) - g(x)|dx \\
&\leq |Z| \sup_Z |f(x) - g(x)| \\
&= M(\varepsilon/M) = \varepsilon,
\end{aligned}$$

as desired. □