M425a Midterm 2 Due Monday, March 28

1. Determine convergence or divergence in (a)-(c).

(a) $\sum_{1}^{\infty}(-1)^k \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}$ (b) $\sum_{1}^{\infty} \frac{k!}{(k+2)!}$

(c) $\sum_{1}^{\infty} \frac{2^k}{k^k}$

2. Consider the power series $\sum a_n z^n$ and assume the coefficients a_n are integers, infinitely many of which are not zero. Prove that the radius of convergence $R \leq 1$.

3. Consider a function $f : M \to \mathbb{R}$. It's graph is the set,

$$G(f) = \{(x, y) \in M \times \mathbb{R} : \ y = f(x)\}.$$

   (a) Prove that if f is continuous then $G(f)$ is closed as a subset of $M \times \mathbb{R}$.

   (b) Prove that if f is continuous and M is compact then $G(f)$ is compact.

   (c) Prove that if $G(f)$ is compact then f is continuous.

4. Let $I = [0, 1]$ and let $F : I \to I$ be continuous. Prove that F has at least one fixed point, i.e. a point $x \in I$ such that $F(x) = x$. Quoting a Fixed Point Theorem is not acceptable.

5. Let X and Y be metric spaces and $F : X \to Y$ be a continuous mapping onto Y. If D is a dense subset of X prove $F(D)$ is dense in Y.

EXTRA CREDIT (10 Points):

6. Prove convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}.$$