Problem 2.1. This exercise gives you some practice working with some technical aspects of convolutions and approximate identities. To minimize the monotony of the write-up, challenge yourself to find shortcuts to make your solution as efficient as possible.

For each $n \in \mathbb{N}$, define $\phi_n : \mathbb{R} \to \mathbb{R}$ by $\phi_n = n1_{I_n}$, where $I_n = \left(-\frac{1}{2n}, \frac{1}{2n}\right)$. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|1_{(-2,2)}(x)$. Write down an (n-dependent) formula for the function $g_n = \phi_n * f$. You should write your formula explicitly enough so that is does not contain any integrals. Use basic calculus considerations to identify intervals where $\phi_n * f$ is concave up, concave down, increasing, decreasing, differentiable, etc.

Proof. Think about the convolution visually as we slide ϕ_n across f. With this in mind, the behavior of the convolution reduces to casework.

(a) When $x \leq -2 - \frac{1}{2n}$ or $2 + \frac{1}{2n} \leq x$, the ϕ_n block is disjoint from $(-2, 2)$, so the integral is zero.

(b) When $x \in (-2 - \frac{1}{2n}, -2 + \frac{1}{2n})$, the interval where the integral is non-zero is $(-2, 2) \cap (x - \frac{1}{2n}, x + \frac{1}{2n}) = (-2, x + \frac{1}{2n})$, thus the integral is:

$$\begin{aligned}\int_{-2}^{x+\frac{1}{2n}} -ny\,dy &= -\frac{ny^2}{2}\bigg|_{-2}^{x+\frac{1}{2n}} \\ &= \frac{4n}{2} - \frac{n(x+\frac{1}{2n})^2}{2}.\end{aligned}$$

Note that we may replace $|y|$ with $-y$ because $y < 0$ throughout.

(c) When $x \in (-2 + \frac{1}{2n}, -\frac{1}{2n})$, the interval where the integral is non-zero is $(x - \frac{1}{2n}, x + \frac{1}{2n})$, thus the integral is:

$$\begin{aligned}\int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} -ny\,dy &= -\frac{ny^2}{2}\bigg|_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} \\ &= \frac{n(x-\frac{1}{2n})^2}{2} - \frac{n(x+\frac{1}{2n})^2}{2} \\ &= \frac{n}{2}(x - \frac{1}{2n} + x + \frac{1}{2n})(x - \frac{1}{2n} - (x + \frac{1}{2n})) \\ &= \frac{n}{2}(2x)(-2/2n) = -x\end{aligned}$$

Note that we may replace $|y|$ with $-y$ because $y < 0$ throughout.

(d) When $x \in (-\frac{1}{2n}, \frac{1}{2n})$, things get a bit more complicated because we can no longer get rid of the absolute values. The relevant interval we want is still $((x - \frac{1}{2n}, x + \frac{1}{2n}))$, but we need to split between $x < 0$ and $x > 0$:

$$\begin{aligned}\int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} n|y|dy &= \int_{x-\frac{1}{2n}}^{0} -nydy + \int_{0}^{x+\frac{1}{2n}} nydy \\ &= \frac{ny^2}{2}\Big|_0^{x-\frac{1}{2n}} + \frac{ny^2}{2}\Big|_0^{x+\frac{1}{2n}} \\ &= \frac{n(x-\frac{1}{2n})^2}{2} + \frac{n(x+\frac{1}{2n})^2}{2} \\ &= nx^2 + \frac{1}{4n}\end{aligned}$$

(e) When $x \in (\frac{1}{2n}, 2-\frac{1}{2n})$, we can abuse symmetry and reuse the result from case (c) to deduce that the integral must be:

$$\int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} nydy = x$$

(f) When $x \in (2-\frac{1}{2n}, 2+\frac{1}{2n})$, we can abuse symmetry and reuse the result from case (b) to deduce that the integral must be:

$$\int_{x-\frac{1}{2n}}^{2} nydy = \frac{4n}{2} - \frac{n(x+\frac{1}{2n})^2}{2}.$$

Thus, in total, our function is

$$g(x) = \begin{cases} 0 & x \le -2-\frac{1}{2n} \vee 2+\frac{1}{2n} \le x \\ -\frac{n}{2}\left(x+\frac{1}{2n}\right)^2 + 2n & x \in [-2-\frac{1}{2n}, -2+\frac{1}{2n}] \\ -x & x \in [-2+\frac{1}{2n}, -\frac{1}{2n}] \\ nx^2 + \frac{1}{4n} & x \in [-\frac{1}{2n}, \frac{1}{2n}] \\ x & x \in [\frac{1}{2n}, 2-\frac{1}{2n},] \\ -\frac{n}{2}\left(x-\frac{1}{2n}\right)^2 + 2n & x \in [2-\frac{1}{2n}, 2+\frac{1}{2n}]. \end{cases}$$

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