Homework 1

Hanting Zhang

Problem 1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. We prove the contrapositive of both statements.

If r is rational and $r+x$ is rational, then we may express both as fractions $r = a/b$ and $r + x = c/d$. Hence $x = (r+x) - x = c/d - a/b = (cb - ad)/bd$, which is clearly rational. Thus completes the first proof.

If r is rational and rx is rational, then again write $r = a/b$ and $rx = c/d$. So then $x = rx/r = (ca)/(bd)$, which is again rational, as desired. $\square$

Problem 2. Prove that there is no rational number whose square is 12.

Proof. Assume for the sake of contradiction that there exists some $q \in \mathbb{Q}$ such that $q^2 = 12$. Then $4 \mid q^2 \implies 2 \mid q$. Hence write $q = 2p$, and substitute to simplify $4p^2 = 12 \implies p^2 = 3$. Now p is rational so we can write it as a reduced fraction a/b.

Then $a^2/b^2 = 3$, implying $a^2 = 3b^2$. This is an equation over the integers, so 3 must divide a. Write $a = a'/3$. Substitiuting, we have another integer equation $3a'^2 = b^2$. By the same logic, 3 divides b. But now we conclude that 3 is a common factor of a and b, contradicting our assumption that a/b is reduced!

Hence such a q cannot exist. $\square$