Problem 1. If $x \in \mathbb{R}^k$, $k \geq 2$, show that there exists $y \in \mathbb{R}^k$ such that $x \cdot y = 0$.

Proof. One possible solution is simply $y = 0$; hence trivially $x \cdot y = 0$. But that's not very interesting so I'll give a nontrivial example.

If x is trivial then y can be anything, so assume that x is nontrivial. Then there is at least one index i where $x_i \neq 0$. Since $n \geq 2$, pick some $j \neq i$. Set $y_i = -x_j$ and $y_j = x_i$, and $y_k = 0$ for all $k \neq i$ and $k \neq j$. Hence this defines y as nontrivial. Then

$$x \cdot y = \sum_{k=1}^{n} x_k y_k = x_i y_i + x_j + y_j = -x_i x_j + x_j x_i = 0.$$

□

Problem 2. True or False: If true prove it, if false counterexample it.

(a) Let $\{F_n\}$ be a countable collection of closed subsets of $\mathbb{R}$ such that for any finite sub-collection

$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing.$$

Then

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(b) Add the condition that each F_n is bounded and repeat (2a).

(c) Repeat (2a) where closed and bounded $F_n \subseteq X$, and arbitrary metric space.

Proof. Part (a): This claim is **false**. Consider the subsets $F_n = [n, \infty)$. Then for any finite sub-collection $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$, let $n = \max_k n_k$. We can compute the intersection to be:

$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} = F_n \neq \varnothing.$$

However, since for any $x \in \mathbb{R}$ we may find some $n \geq x$, there is always some F_n such that $x \notin F_n$. Hence

$$\bigcap_{n=1}^{\infty} F_n = \varnothing.$$

This disproves the claim. □

Proof. Part (b): This claim is **true**. If F_n are both closed and bounded subsets of $\mathbb{R}$, then the Heine-Borel Theorem guarantees that F_n is compact. Now apply Theorem 2.36 fromthe textbook to conclude that

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

∎

Proof. Part (c): This claim is **false**. Let $X = \mathbb{Q}$ with the relative topology inherited from $\mathbb{R}$. Then consider the subsets $F_n = \overline{B_{1/n}(\sqrt{2})}$ as the closed balls centered at $\sqrt{2}$ with radius $1/n$, where p_n is tne nth prime. In particular, since $\sqrt{2} \pm 1/n$ are irrational, the boundary points of F_n don't exist in $\mathbb{Q}$, and hence we can drop them without changing anything: $F_n = B_{1/n}(\sqrt{2})$.

Now we check that finite intersections are nonempty. Indeed, if $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$ are a finite sub-collection, then we their intersection is just the ball of minimum radius $r = \min_k(1/n_k)$. This radius is clearly greater than 0, so we know that $F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing$.

However, if we consider $\bigcap_{n=1}^{\infty} F_n$, then for any $x \neq \sqrt{2}$, we can find a k such that $k > 1/|x - \sqrt{2}|$. This implies $1/n < |x - \sqrt{2}|$. Hence by definition $x \notin F_k$, so $x \notin \bigcap_{n=1}^{\infty} F_n$. But $\sqrt{2}$ is not in $\mathbb{Q}$! Hence in $\mathbb{Q}$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is empty. This disproves the claim. ∎

Problem 3. Consider the metric space $\mathbb{Q}$ of all rationals on the real line with the Euclidean metric. Prove that if $K \neq \varnothing$ is a compact subset of $\mathbb{Q}$ then K cannot contain an open subset of $\mathbb{Q}$. *Hint: Consider the relative topology.*

Proof. bruh ∎

Problem 4. Let F and K be nonempty closed subsets of the metric space X with $K \cap F = \varnothing$. Show that if K is compact there is a positive distance from F to K, i.e.

$$\inf\{d(x, y) \mid x \in F, y \in K\} = \delta > 0.$$

Is is still true if K is only assumed to be closed? If not find a counterexample.

Problem 5. A *base* for a topological space X is a collection $\{V_\alpha | \alpha \in A\}$ of open subsets of X such that for every open subset of $G \subseteq X$, one has $G = \bigcup_{\alpha \in \mathcal{B}} V_\alpha$ where $\mathcal{B} \subseteq \mathcal{A}$.

Prove that every compact metric space X has a countable base.

Proof. For each $q > 0 \in \mathbb{Q}$, consider the collection of subsets $\mathcal{C}_q = \{B_q(x) \mid x \in X\}$. For each q, this clearly defines an open cover of X, so we may construct a finite subcover $\mathcal{D}_q = \{B_{q_1}(x_1), B_{q_2}(x_2), \ldots, B_{q_n}(x_n)\}$. In articular, we have a *countable* amount *finite* covers, so their union is countable. Given this, we claim that

$$\bigcup_{q \in \mathbb{Q}} \mathcal{D}_q$$

forms a countable base of X. ∎