Exercises 6, 12, pp. 52-53.

6. (a) If H is a subgroup of G, then for any $h, h' \in H$, we have $h^{-1}h'h \in H$. Hence $h^{-1}Hh = H$, and $h \in N_G(H)$. Therefore $H \leq N_G(H)$.

   If H is not a subgroup of G, then multiplication fails so we have no reason to expect $h^{-1}h'h \in H$. For example, let

$$H = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right\}.$$

   Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 0 & 3 \end{pmatrix} \notin H.$$

   Hence $H \not\leq N_G(H)$.

   (b) If $H \leq C_G(H)$, then for any $h, h' \in H$, we have $h^{-1}h'h = h' \Rightarrow h'h = hh'$. Hence H is abelian, as desired.

12. Too much work for now.

Exercises 16, 17, pp. 65-66.

16. (a) Since G is finite there can only be a finite amount of subgroups. In particular, there are only a finite amount of subgroups $\{H_i\}_{i=1}^n$ containing H. Then any chain $H \leq H_{i_1} \leq H_{i_2} \leq \cdots \leq H_{i_k} \leq G$ is finite, and we may prescribe H_{i_k} as the maximal subgroup containing H.

   (b) Suppose $\langle r \rangle \leq K$. Then $|\langle r \rangle| \leq |K|$ while $|K| \mid |G|$. But $\langle r \rangle$ has order n and G has order $2n$. Hence $|K|$ can only be n, in which case $H = K$, or $2n$, in which case $K = G$. This is exactly the definition of H being maximal, as desired.

   (c) The order of x^p is n/p, so $|\langle x^p \rangle| = n/p$. If K contains $\langle x^p \rangle$, then $n/p \leq |K|$ while $|K| \mid n \Rightarrow a|K| = n$ for some a.

17. (a) The chain $\mathcal{C}$ is a set of subgroups $\{H_i\}_{i \in \mathcal{I}}$ on a total order $\mathcal{I}$ such that $H_i \leq H_j$ for all $i \leq j$.

   We first show that if $x, y \in \bigcup_{i \in \mathcal{I}} H_i = H$, then

$$xy \in bigcup_{i \in \mathcal{I}} H_i = H.$$

   Since $x \in H$, we have $x \in H_i$ for some $i \in I$. Similiarly $y \in H_j$ for some $j \in \mathcal{I}$. Furthermore, I is a total order so either $i \leq j$ or $i \geq j$. Without loss of generality assume that $i \leq j$, since we could just swap the labels if instead $j \leq i$. Then $H_i \leq H_j$, so $x \in H_i \leq H_j$ and $y \in H_j$ imply $xy \in H_j \leq H$.

The other subgroup axioms are straightforward: $e \in H$ since every H_i is a subgroup. For any $x \in H$, $\exists i, x \in H_i \Rightarrow x^{-1} \in H_i \leq H$.

Hence H is a subgroup of G.

(b) Assume for the sake of contradiction that H is *not* a proper subgroup, i.e. $H = G$. Then each g_i must lie in some H_{α_i}. There are only finite g_i, therefore we can compute the finite maximum $\max(\alpha_i) = \alpha_j$ for some fixed j. Then H_{α_j} is both in $\mathcal{C}$ and contains each g_i. Then $\langle g_1, \ldots, g_n \rangle \subset H_{\alpha_j}$. But $\langle g_1, \ldots, g_n \rangle = G$! So H_{α_j} is not proper, contradicting our assuptions about $\mathcal{C}$.

(c) Part (b) shows that for any chain $\mathcal{C}$, the union of all subgroups in the chain H is an upper bound on $\mathcal{C}$ that is proper. In other words, $H \in \mathcal{S}$, and hence we may apply Zorn's lemma to deduce that $\mathcal{S}$ contains at least one maximal element. This concludes the proof.

Exercises 1, 18, 24, 40, 41 pp. 85-89.

Problem 1. Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$. If $E \trianglelefteq H$, then $\varphi^{-1}(E) \trianglelefteq G$.

Problem 18. bruh

Problem 24. Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

Problem 40. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$.

Proof. ($\Rightarrow$): If $x^{-1}y^{-1}xy \in N$, then $(x^{-1}y^{-1}xy)N = N \Rightarrow xyN = Nyx$. But N is normal, so we can swap the left and right cosets. Thus $xNyN = xyN = Nyx = yxN = yNxN$, as desired.

($\Leftarrow$): If $xNyN = yNxN$, then we can just run the argument backwards:

$$xyN = yxN \Rightarrow x^{-1}y^{-1}xyN = N \Rightarrow x^{-1}y^{-1}xy \in N.$$

$\square$

Problem 41. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy | x, y \in G \rangle$ is a normal subgroup of G and G/N is abelian.

Proof. N is a normal subgroup of G: Let $\varphi_g(n) = g^{-1}ng$. Note that conjugation by g is a homomorphism. Let $n \in N$, which will have the form $a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_n^{\epsilon_n}$, where each $a_i = x^{-1}y^{-1}xy$

for some $x, y \in G$ and $\epsilon_i = \pm 1$. Now we want to show that $g^{-1}ng = \varphi_g(n) \in N$ for any $g \in G$. Since φ_g is a homomorphism, we have

$$\varphi_g(n) = \varphi_g(a_1)^{\epsilon_1}\varphi_g(a_2)^{\epsilon_2}\ldots\varphi_g(a_n)^{\epsilon_n}.$$

Because N is a subgroup, it suffices now to prove that each $\varphi_g(a_i) \in N$. We have

$$\varphi_g(a_i) = \varphi_g(x^{-1}y^{-1}xy) = \varphi_g(x^{-1}y^{-1}xy) = \varphi_g(x)^{-1}\varphi_g(y)^{-1}\varphi_g(x)\varphi_g(y).$$

The LHS is of the form $x'^{-1}y'^{-1}x'y'$ for $x' = \varphi_g(x)$ and $y' = \varphi_g(y)$, so it must be in N. Hence $\varphi_g(a_i) \in N$. By extension, $\varphi_g(n) \in N$. Therefore N is normal. □

Proof. N is abelian: By Exercise 40 we have that $\overline{x}$ and $\overline{y}$ commute in G/N is and only if $x^{-1}y^{-1}xy \in N$. But by definition, □

Exercise 4, pp. 111.

4.