Chocolate 4

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Problem 0. [0]

Chocolate Problem: 1 chocolate bar

Reminder: If you solve a chocolate problem (which you can do in groups of size up to 3), please e-mail David with the solution — do not submit it on Gradescope. Also, feel free to list preferences or dietary restrictions for/against particular types of chocolate.

In discussion section, you saw how to generalize the idea of Binary Search to trees: each tree has a node v such that when you query that node and the answer points to one of the subtrees from that node, there are at most $n/2$ nodes in that subtree. Thus, repeating this querying at most $\log_2(n)$ times, you can find any node in a tree from the answers to such queries.

Here, we want to generalize the idea further, from trees to arbitrary undirected graphs. In non-tree graphs, it does not make sense to talk about "the subgraph containing the node". For instance, if your graph is a cycle, then no matter which node you query, you can go around the cycle in either direction to get to any other node. So we will clarify the answer as follows: when a node v is queried, and the correct answer is t, the answer will reveal an edge out of v that lies on a *shortest* path from v to t. If there are multiple shortest paths from v to t (with different edges out of v), then any of them could be returned.

You now play the following game: both you and your opponent know the undirected graph G. Your opponent picks a node $t \in G$, without telling you what it is. In each round, you get to point to a vertex v. If $v = t$, then the game is over. Otherwise, your opponent reveals an edge e incident on v that lies on a shortest path from v to t. The game repeats until you've found t. Your goal is to get there with few queries. Prove the following:

Lemma 1. *For every graph $G = (V, E)$, there exists a vertex v you can query such that never mind which edge e incident on v your opponent reveals, the set of remaining vertices that are consistent with this answer shrinks by at least a factor 2. That is, if S is the set of all nodes t such that e is on a shortest path from v to t, then $|S| \leq |V|/2$.*

To round out the answer, show how to use the lemma to guarantee that you can find the node t in at most $\log_2 n$ queries — this part is basically trivial once you have proved the lemma.