Math 410 Homework 2

Due Date: **something**

Exercises 8, 9, 12, 26, 36, pp. 21-23.

8. (a) We show that G satifies that group axioms under multiplcation.

   Identity: Clearly $1 \in G$. And 1 is the identity since for any $g \in G \subset \mathbb{C}$, $g * 1 = 1 * g = g$.

   Associativity: Note that $G \subset \mathbb{C}$. Since $\mathbb{C}$ is associative under normal $*$, and G carries the same (restricted) $*$, G must be associative as well.

   Closure: Let $g, h \in G$. Then by definition there exist some $n, m \in \mathbb{Z}^+$ such that $g^n = h^m = 1$. We want to find N such that $(gh)^N = 1$. Let $N = nm$, directly calucating $(gh)^N$ yields $(gh)^{nm} = g^{nm} * h^{nm} = (g^n)^m * (h^m)^n = 1^m * 1^n = 1$. Therefore $gh \in G$.

   Inverse: Let $g \in G$, with $g^n = 1$ for some $n \in \mathbb{Z}^+$. Let $h = \overline{g}$, the complex conjugate of g. Then

   (b)

9.

1. (a) We show that G satisfies the group axioms under addtion.

   Identity: Clearly $0 = 0 + 0\sqrt{2} \in G$. And 0 is the identity since for any $g \in G \subset \mathbb{R}$, $g + 0 = 0 + g = g$.

   Associativity: Note that $G \subset \mathbb{R}$. Since $\mathbb{R}$ is associative under normal $+$, and G carries the same (restricted) $+$, G must be associative as well.

   Closure: Let $g, h \in G$. Then by definition there exist some $p, q, r, s \in \mathbb{Q}$ such that $p + q\sqrt{2} = g$ and $r + s\sqrt{2} = h$. We want to find $x, y \in \mathbb{Q}$ such that $x + y\sqrt{2} = g + h$. Clearly we want $x = p + r$ and $y = q + s$, so that $g + h = (p + r) + (q + s)\sqrt{2} = x + y\sqrt{2}$.

   Inverse: Let $g \in G$, with $a + b\sqrt{2} = g$ for some $a, b, \in \mathbb{Q}$. Then $-a + -b\sqrt{2} \in G$ is the inverse of g, since $a + b\sqrt{2} - a - b\sqrt{2} = 0$. Hence G has inverses.

   (b) Let g be a non-zero element of G such that $a + b\sqrt{2} = g$ for some $a, b \in \mathbb{Q}$ (where a and B are not both 0). Then note that $1/g = 1/(a + b\sqrt{2})$ is in G, since

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

   Letting $x = \frac{a}{a^2 - 2b^2}$ and $y = \frac{-b}{a^2 - 2b^2}$, we have $1/g = x + y\sqrt{2}$. Both x and y are rational, since they are made up of rational expressions. Hence $1/g$ (in $\mathbb{R}$) is the inverse og g in G.

Note. This makes G a *field*. In fact it is the field $\mathbb{Q}[\sqrt{2}]$, the result of adjoining $\sqrt{2}$ to $\mathbb{Q}$.

12. 1

Exercises 3, 9, pp. 27-28. Exercises 2, 4, 13, 16, 20, pp. 32-34. Exercises 17, 18, pp. 40. Exercises 18, 19, pp. 45.