**Problem 1.1.** (Modified Single-Variable Inverse Function Theorem) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function with $f'(x) > 0$ for all $x \in (a, b)$.

(a) Prove that $f$ is injective, and argue that its image must be an open interval $(c, d)$ (with $c$ and/or $d$ possibly infinite).

(b) By part (a), there exists a function $g : (c, d) \to (a, b)$ such that $g(f(x)) = x$ for all $x \in (a, b)$. Prove that $g$ is continuous.

(c) Prove that $g$ is differentiable, and the $g'(f(x)) = \frac{1}{f'(x)}$, for all $x \in (a, b)$. (Hint: Pick $y \in (c, d)$, and let $(y_n)_{n=1}^{\infty}$ be a sequence in $(c, d)$ that converges to $y$. Write the difference quotient $\frac{g(y_n)-g(y)}{y_n-y}$ in terms of $f$ and a sequence $(x_n)_{n=1}^{\infty}$ in $(a, b)$.)

**Problem 1.2.** Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2 = (x + y)(2x^2 - 2xy + 2y^2 - 3x + 3y).$$

(a) Find the four points in $\mathbb{R}^2$ where the gradient of $f$ is zero. Use the Second Derivative Test to show that $f$ has exactly one local maximum and one local minimum in $\mathbb{R}^2$.

(b) Let $S$ denote the level set $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ of $f$ at the value 0. Let $S_1$ denote the subset of $S$ consisting of those points $(x, y)$ of $S$ at which $\partial_1 f(x, y) = 0$. Determine $S_1$ completely (it consists of four points).

(c) Failure of the hypotheses of the Implicit Function Theorem at a given point $(a, b)$ doesn't guarantee that one cannot 'solve for one variable in terms of the other' near $(a, b)$. However, it turns out that in this particular example, one cannot solve for $x$ as a function of $y$ near any of the four points of $S_1$. Give a heuristic argument for this statement, making reference to Figure 1(A).

(d) Pick one of the points $(a, b)$ in $S_1$ and demonstrate rigorously that one cannot solve for $x$ as a function of $y$ near $(a, b)$. That is for any neighborhood $U$ of $(a, b)$, show that there exist $y \in \mathbb{R}$ such that $(x_1, y), (x_2, y) \in U$, $f(x_1, y) = f(x_2, y) = 0$, and $x_1 \neq x_2$.

**Problem 1.3.** Let $X$ be a real normed vector space and let $U$ be a open subset of $X$. Assume that $f : U \to \mathbb{R}$ is continuous and let $a$ be a point of $U$.

(a) Show that if $f$ achieves a local minimum at $a$, then $f$ is not injective on any neighborhood of $a$. Hint: Reduce this to a problem about a function of a single real variable, and use the Intermediate Value Theorem. Not that $f$ is not assumed to be differentiable anywhere. (For future reference, note also that the same conclusion holds if $f$ has a local maximum at $a$ instead of a local minimum.)