Problem 2.1. Suppose $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are sequences of complex numbers , and the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_1 and R_2, respectively. Show that the radius of convergence R of the Cauchy product of these two series satisfies $R \geq \min\{R_1, R_2\}$. Give an example of two series where strict inequalities folds, $R > \min\{R_1, R_2\}$.

Proof. By proposition 1.2, the series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converges absolutely for values z such that $|z| < R_1$ and $|z| < R_2$, respectively. By Merten's Theorem, the Cauchy product at any point z such that $|z| < \min\{R_1, R_2\}$ converges to the product of each series at z. Thus, since we converge on the interval $(0, \min\{R_1, R_2\})$, the radius of convergence for the Cauchy product must at least be $R \geq \min\{R_1, R_2\}$, as desired. □

An example of the strict inequality is as follows: Let $f(z) = (1+z)^{1/2}$ and $g(z) = (1+z)^{-1/2}$. We can construct the Maclaurin series of $f(z)$ and $g(z)$, which each have radii of convergence 1. This follows because $f(z)$ blows up at $z = -1$ and $g(z)$ blows up at $x = 1$. However, the Cauchy product of the series' of $f(z)$ and $g(z)$ is simply $(1 + z)^{1/2}(1 + z)^{-1/2} = 1 = 1 + 0z^1 + 0z^2 + \cdots$; this has radius of convergence $\infty > 1$, as desired.

Problem 2.2. Prove that the Cauchy product of two absolutely convergent series is itself absolutely convergent.

Proof. Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series, i.e. $\sum |a_n| < A$ and $\sum |b_n| < B$ for some constants A, B.

Then we have:

$$\begin{aligned}
\sum_{n=0}^{N} |c_n| &= \sum_{n=0}^{N} \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \\
&\leq \sum_{n=0}^{N} \sum_{k=0}^{n} |a_k||b_{n-k}| \\
&= |a_0||b_0| + |a_0||b_1| + |a_1||b_0| + |a_0||b_2| + |a_1||b_1| + |a_2||b_0| + \cdots \\
&< \sum_{n=0}^{N} |a_n| \sum_{k=0}^{N-n} |b_k| \\
&< \sum_{n=0}^{N} |a_n| B \\
&< AB
\end{aligned}$$

This is independent of N, thus as $N \to \infty$, $\sum |c_n|$ is bounded by AB, which proves the absolute convergence of $\sum c_n$. □