Problem 1.6. Let X be a normed vector space. Let U be an open subset of X, and let $f : U \to \mathbb{R}$ be a function. Recall that we say that f has a *local maximum* at $a \in U$ if there exists a neighborhood V of a such that $f(x) \leq f(a)$ for all $x \in V$. Prove that if a is a local maximum of f at which f is differentiable, then $f'(a) = 0$.

Proof. Choose a set of basis vectors $\{e_i\}_{i \in I}$ appropriately such that $x + e_i \in V$ for all $i \in I$. Then, we know that for each $i \in I$,

$$\lim_{h \to 0^+} \frac{\|f(x + he_i) - f(x) - f'(x)he_i\|}{\|he_i\|} = \lim_{h \to 0^-} \frac{\|f(x + he_i) - f(x) - f'(x)he_i\|}{\|he_i\|} = 0.$$

Because $f(x) \leq f(x + he_i)$, the first limit imposes $D_i f(x) \geq 0$, bit the second limit imposes $D_i f(x) \leq 0$. Thus we must have $D_i f(x) = 0$ for all $i \in I$. Therefore, $f'(x) = 0$, as desired. $\square$

Problem 2.1. Differentiation past the integral.

(a) Let Y be a metric space; suppose $f : [a, b] \times Y \to \mathbb{R}$ is continuous. Show that the function $F : Y \to \mathbb{R}$ defined by $F(y) = \int_a^b f(x, y)dx$ is continuous.

(b) Assume $f : [a, b] \to \mathbb{R}$ is continuous; assume $\partial_2 f(x, y)$ also exists and is continuous on $[a, b] \times (c, d)$. Define $F : (c, d) \to \mathbb{R}$ by $F(y) = \int_a^b f(x, y)dx$. Prove that $F \in C^1((c, d); \mathbb{R})$, with $F'(y) = \int_a^b \partial_2 f(x, y)$.

Proof. We proceed with each part separately.

(a) Let $\varepsilon > 0$. Since f is continuous, there exists an δ such that for all $\|z - y\| < \delta$, we have $|f(z) - f(y)| < \varepsilon/(b - a)$. Then we have

$$\begin{aligned} |F(z) - F(y)| &= \left| \int_a^b [f(x, z) - f(x, y)]dx \right| \\ &\leq \int_a^b |f(x, z) - f(x, y)|dx \\ &< \int_a^b \frac{\varepsilon}{b - a} dx = \varepsilon, \end{aligned}$$

as desired.

(b) We show that $F'(y)$ exists and is equal to $\int_a^b \partial_2 f(x, y)dx$. Indeed,

$$\lim_{h \to 0} \frac{F(y + h) - F(y)}{h} = \lim_{h \to 0} \int_a^b \frac{f(x, y + h) - f(x, y)}{h} dx$$

$$= \int_a^b \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} dx$$
$$= \int_a^b \partial_2 f(x,y) dx,$$

where swapping the order of the limit and integration relies on the fact that continuous functions on compact intervals are also uniformly continuous. By part (a), since $\partial_2 f(x,y)$ is continuous, so is $\int_a^b \partial_2 f(x,y)dx$, so $F(y) \in C^1$, as desired.

□

Problem 2.2. Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ bia the following rule when $y \geq 0$:

$$f(x,y) = \begin{cases} x, & 0 \leq x \leq \sqrt{y} \\ -x + 2\sqrt{y}, & \sqrt{y} \leq x \leq 2\sqrt{y} \\ 0, & x \geq 2\sqrt{y} \text{ or } x \leq 0. \end{cases}$$

Extend $f(x,y)$ to all of $\mathbb{R}^2$ by letting it be odd in the second argument: $f(x,y) = -f(x,-y)$ for $y < 0$.

(a) Show that f is continuous everywhere, and $(\partial_2 f)(x,0) = 0$ for all x.

(b) Define $F(y) = \int_{-1}^1 f(x,y)dx$. Prove that $F(y) = y$ for $|y| < \frac{1}{4}$. Conclude that $F'(0) \neq \int_{-1}^1 \partial_2 f(x,0)dx$.

(c) Compare part (b) of the present exercise with part (b) of Exercise 2.1. Why are these two not in contradiction with one another?

Proof. We proceed with each part separately.

(a) Within each region, f is continuous, so we just need to show that it is continuous at the boundaries $x = \sqrt{y}$ and $x = 2\sqrt{y}$. Indeed, we have if $x = \sqrt{y}$, then $x = -x + 2x = -x + 2\sqrt{y}$; if $x = 2\sqrt{y}$, then $-x + 2\sqrt{y} = -2\sqrt{y} + 2\sqrt{y} = 0$. Thus f has no discontinuities at the boundaries, so it is continuous everywhere.

   Furthermore, consider $\partial_2 f(x,0) = \lim_{h\to 0}(f(x,h) - f(x,0))/h = \lim_{h\to 0} f(x,h)/h$. For any x, we may choose h small enough that $2\sqrt{h} \leq x$, so the limit $\partial_2 f(x,0)$ will always be 0.

(b) This is fairly painful, but if we draw out the graph its easier to visualize. We have for

$y > 0$:

$$\begin{aligned}F(y) &= \int_{-1}^{1} f(x,y)dx \\ &= \int_{0}^{\sqrt{y}} x dx + \int_{\sqrt{y}}^{2\sqrt{y}} (-x + 2\sqrt{y})dx \\ &= \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} + \left(\frac{-x^2}{2}\right)_{\sqrt{y}}^{2\sqrt{y}} + 2y \\ &= \frac{y}{2} + \frac{-3y}{2} + 2y = y.\end{aligned}$$

Since f is odd, for $y < 0$, we have $F(y) = F(-|y|) = -F(|y|) = -|y| = y$ as well, as desired.

(c) The reason why this doesn't contradict part (b) of Exercise 2.1 is because $\partial_2 f(x,y)$ is not continuous.

$\square$