Problem 15. Show that Theorem 2.36 and its Corolary become false (in $\mathbb{R}$, for example) if the word "compact" is replaced by "closed" or "bounded."

1. Closed: If we instead consider a collection of closed subsets, then the theorem is false because we can take advantage of the fact that the subsets can be unbounded. For example, take $C_i = [i, \infty) \subseteq \mathbb{R}$ for $i \in \mathbb{N}$. Then the intersection every finite subset $\{C_i\}_{i \in A}$ for $A \subseteq \mathbb{N}$ is just $C_{\max A}$, which clearly is non-empty. However, the total intersection $\bigcap_{i=0}^{\infty} C_i$ is empty.
2. Bounded: If we instead only assume that our subsets are bounded, then notice that $\varnothing$ is bounded. Therefore any collection that contains the empty set will have an empty intersection. So the theorem fails in this case as well.

Problem 16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

First, let us prove a few useful lemmas:

Lemma. *For any $p \in \mathbb{R}$, the set $U = \{x \in \mathbb{Q} : x^2 < p\}$ is open in our metric.*

Proof. We want to show that there exists some $r > 0$ such that $B_r(x) \subseteq U$. This is equivalent to showing that for every $-r < \epsilon < r$, $(x + \epsilon) \in U \Rightarrow (x + \epsilon)^2 < p$. Solving for ϵ in this quadratic in $\mathbb{R}$ reveals $|x + \epsilon| < \sqrt{p}$. Taking the positive branch of the absolute value (since $r > 0$), we have $\epsilon < \sqrt{p} - x \Rightarrow r \leq \sqrt{p} - x$. We now return to $\mathbb{Q}$ by choosing a suitable value of r that is rational. Hence $B_r(x) \subseteq U$ and U is open. $\square$

Lemma. *For any $p \in \mathbb{R}$, the set $U = \{x \in \mathbb{Q} : x^2 > p\}$ is open in our metric.*

Proof. I will drop the proof since it is extremely similar to the one above. (Just replace $<$ with $>$ in most places.) $\square$

Corollary. *If p is prime, then $U = \{x \in \mathbb{Q} : x^2 < p\}$ and $V = \{x \in \mathbb{Q} : x^2 > p\}$ is also closed.*

Proof. For U, we need to prove that $U^c = \{x \in \mathbb{Q} : x^2 >= p\}$ is open. Note that because p is prime, there is no $x \in \mathbb{Q}$ such that $x^2 = p$, so we can drop the $=$ conditioning without losing any points, hence $U^c = \{x \in \mathbb{Q} : x^2 > p\}$, which we proved is open.

Similarly, V is open since V^c can be written as $\{x \in \mathbb{Q} : x^2 < p\}$. $\square$

Now we can do the exercise quite smoothly!

Proof. 1. E is closed: Note that $E = \{p : 2 < p^2\} \cap \{p : p^2 < 3\}$. Applying our lemmas, then, E is a intersection of closed sets, so it is also closed.

2. E is bounded: E is bounded by $M = 9$. If $p >= 3$, then $p^2 >= 9$ and $p \notin E$. Hence $U \subseteq B_9(0)$.

3. E is not compact: Let $\mathcal{C}$ be the collection of open sets $B_q(0)$ for each $q \in E$. Clearly $\bigcup_{q\in E} B_q(0) = E$, but there is no finite subcover because any finite subcollection $B_{q_1}(0), \cdots B_{q_n}(0)$ can be bounded by their maximum $B_{\max q_i}(0)$. But $\sqrt{2}$ and $\sqrt{3}$ are irrational so we will always be missing points between $\max q_i$ and $sqrt2$, for example.

4. E is open: Apply our lemmas because $E = \{p : 2 < p^2\} \cap \{p : p^2 < 3\}$ is an intersection of open sets.

□