Problem 1.1. In the proof of Weierstrass's Polynomial Approximation Theorem above, we added the additional simplifying hypotheses that $[a, b] = [0, 1]$ and $f(0) = f(1) = 0$. Finish the proof of the original statement, with this special case in hand.

Proof. Let $t : [0, 1] \to [a, b]$ be a bijection defined by $\lambda \mapsto \lambda a + (1 - \lambda)b$. Let $T : C([0, 1]) \to C([a, b])$ extend t via $T(f)(x) = f(t^{-1}(x))$. Note that T is a bijection, thus for any $f \in C([a, b])$, we have $T^{-1}(f) \in C([0, 1])$, so there exists a sequence of polynomials $P_n \to f$. Then

$$T(P_n)(x) = P_n(t^{-1}(x)) \to f(t^{-1}(x)) = T(f)(x),$$

as desired. $\square$

Problem 1.2. Assume $f \in C([a, b])$.

(a) Assume that $\int_a^b f(x)x^n dx = 0$ for all $n \in \mathbb{N}$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(b) Now assume that instead that $\int_a^b f(x)x^n dx = 0$ for all $n \geq N$, $n \in \mathbb{N}$. Can you still conclude that $f \equiv 0$ on $[a, b]$? Prove your answer is correct.

Proof. We proceed with each part separately.

(a) Suppose $P_n \to f$ by the Weierstrass Approximation Theorem. We have

$$\begin{aligned}
\int_a^b f(x)f(x)dx &= \int_a^b (\lim_{n\to\infty} P_n)f(x)dx \\
\text{(because } P_n \to f \text{ uniformly)} \quad &= \lim_{n\to\infty} \int_a^b P_n f(x)dx \\
&= \lim_{n\to\infty} \int_a^b \left(\sum_{k=0}^{\infty} a_k^{(n)} x^k \right) f(x)dx \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k^{(n)} \int_a^b x^k f(x)dx \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k^{(n)} 0 = \lim_{n\to\infty} 0 = 0.
\end{aligned}$$

But this means $\int_a^b f^2(x)dx = 0$ and $f^2(x) \geq 0$ for all x, thus this is only possible if $f^2(x) = 0$. This of course implies $|f(x)| = 0$, and $f(x) = 0$ for all x.

(b) Yes. Apply part (a) to $g(x) = x^N f(x)$ to conclude that $g = 0$. Thus $f = 0$.

$\square$

Problem 1.3. Suppose that $f \in C([1,\infty))$ and $\lim_{x\to+\infty} f(x) = a \in \mathbb{R}$. Prove that for any $\varepsilon > 0$, there is a polynomial p such that $\sup_{x\in[1,\infty)} |p(1/x) - f(x)| < \varepsilon$.

Proof. Let $g : [0,1] \to \mathbb{R}$ and define $g(x) = f(1/x)$ when $x \in (0,1]$ and $g(0) = a$. Then $f(x) = g(1/x)$. By the Weierstrass Approximation Theorem, there exists some polynomial p such that $\|p - g\|_u < \varepsilon$. Now, then

$$\sup_{x\in[1,\infty)} |p(1/x) - f(x)| = \sup_{x\in[1,\infty)} |p(1/x) - g(1/x)| = \sup_{y\in[0,1]} |p(y) - g(y)| < \varepsilon,$$

as desired. $\square$

Problem 1.4. Suppose that $f \in C^1([a,b])$, that is, f is continuously differentiable on the interval $[a,b]$. Recall the "C^1 norm," defined by

$$\|f\|_{C^1([a,b])} = \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)|.$$

Prove that for any $\varepsilon > 0$, there exists a polynomial p such that $\|f - p\|_{C^1([a,b])} < \varepsilon$.

Proof. By the Weierstrass Approximation Theorem, there is some polynomial q such that $\|q - f'\|_u < \frac{\varepsilon}{2(b-a)}$. Then let $p(x) = \int_a^x q$ and

$$\begin{aligned}
\|f - p\|_{C^1([a,b])} &= \sup_{x\in[a,b]} |f(x) - p(x)| + \sup_{x\in[a,b]} |f'(x) - q(x)| \\
&< \sup_{x\in[a,b]} \left| \int_a^x f'(y) - q(y) dy \right| + \frac{\varepsilon}{2(b-a)} \\
&< \sup_{x\in[a,b]} \int_a^x \left| \frac{\varepsilon}{2(b-a)} \right| dy + \frac{\varepsilon}{2(b-a)} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)} < \varepsilon,
\end{aligned}$$

as desired. $\square$