Chapter 6, # 9, 10 & 13

**Problem 10.** Let $p$ and $q$ be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathcal{R}(\alpha), g \in (\alpha), f \geq 0, g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If $f$ and $g$ are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality.* When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

*Proof.* We proceed with each part:

(a) We use Jensen's inequality. (I know this is slightly illegal since we haven't defined what $\log(x)$ is, but I couldn't figure out any other way to do it.) If $a = 0$ or $b = 0$ then there's nothing to prove, so assume $a, b > 0$. By Jensen's inequality,

$$\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) \geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) = \log u + \log v = \log(uv).$$

Since $\log x$ is monotonically increasing, we conclude that

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}.$$

(b) From part (a) we know that

$$f(x)g(x) \le \frac{f(x)^p}{p} + \frac{g(x)^q}{q}.$$

Integrating both sides gives

$$\begin{aligned}\int_a^b f(x)g(x)dx &\le \int_a^b \frac{f(x)^p}{p} + \frac{g(x)^q}{q}dx \\ &= \frac{1}{p}\int_a^b f(x)^p dx + \frac{1}{q}\int_a^b g(x)^q dx \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

(c) Normalize $f(x)$ and $g(x)$ by taking

$$\frac{|f(x)|}{\left(\int_a^b |f(x)|^p\right)^{1/p}} \quad \text{and} \quad \frac{|g(x)|}{\left(\int_a^b |g(x)|^q\right)^{1/q}},$$

so we can apply part (b). We have,

$$\int_a^b \frac{|f(x)|}{\left(\int_a^b |f(x)|^p\right)^{1/p}} \frac{|g(x)|}{\left(\int_a^b |g(x)|^q\right)^{1/q}} \le 1,$$

which implies

$$\left|\int_a^b fg d\alpha\right| \le \int_a^b |f||g| d\alpha \le \left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q d\alpha\right\}^{1/q},$$

as desired.

(d) There is no need for any other assumptions. We can just push the limits around since everything is continuous:

$$\begin{aligned}
\left|\int_a^\infty fg d\alpha\right| &= \left|\lim_{b\to\infty}\int_a^b fg d\alpha\right| \\
&\le \lim_{b\to\infty}\left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \lim_{b\to\infty}\left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\lim_{b\to\infty}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \left\{\lim_{b\to\infty}\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\lim_{b\to\infty}\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \left\{\int_a^\infty |f|^p d\alpha\right\}^{1/p}\left\{\int_a^\infty |g|^q d\alpha\right\}^{1/q},
\end{aligned}$$

as desired.

□

**Problem 13.** Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and $c$ is a constant.

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2)dt$ converge?

*Proof.* We proceed with each part:

(a) Assume that $x > 0$ throughout. We begin with the hint from the book. Make the substitution $u = t^2$ with $dt = du/2\sqrt{u}$ to obtain

$$f(x) = \int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} du.$$

We integrate by parts with $f(x) = \sin x$ and $G(x) = 1/2\sqrt{x}$. So $F(x) = -\cos x$ and

$g(x) = -1/4x^{3/2}$. Hence

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Now we try to bound $f(x)$ above and below by $1/x$ and $-1/x$, respectively. To bound $f(x) < 1/x$, note that we can simplify the last integral term with the inequality by replacing $\cos u$ with 1:

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du < \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du = \frac{1}{2(x+1)} - \frac{1}{2x}.$$

Thus

$$\begin{aligned} f(x) &< \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1-\cos(x^2)}{2x} - \frac{1-\cos((x+1)^2)}{2(x+1)} \\ &\leq \frac{1-\cos(x^2)}{2x} \\ &\leq \frac{1}{x}. \end{aligned}$$

On the other hand, we can also replace $\cos u$ with $-1$, which gives the opposite effect:

$$\begin{aligned} f(x) &> \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2(x+1)} - \frac{1}{2x} \\ &= \frac{1-\cos((x+1)^2)}{2(x+1)} - \frac{1-\cos(x^2)}{2x} \\ &\geq \frac{1-\cos((x+1)^2)}{2(x+1)} \\ &\geq -\frac{1}{x}. \end{aligned}$$

Thus $|f(x)| < 1/x$.

(b) We have

$$\begin{aligned}
2xf(x) &= \cos(x^2) - \frac{2x\cos((x+1)^2)}{2(x+1)} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \\
&= \cos(x^2) - \frac{2(x+1)\cos((x+1)^2) - 2\cos((x+1)^2)}{2(x+1)} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \\
&= \cos(x^2) - \cos((x+1)^2) + \frac{\cos((x+1)^2)}{x+1} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du.
\end{aligned}$$

Thus we may identify

$$r(x) = \frac{\cos((x+1)^2)}{x+1} - \frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du.$$

Now we cannot bound the last intgral again by $\mathcal{O}(1/x)$ again because of the factor of $x$ in the front. Hence the messy (but somewhat natural) thing to do is simply integrate by parts again. Let $du = \sin x dx$ and $v = \frac{1}{x^{3/2}}$, so that $u = \cos x$ and $dv = -\frac{3}{2x^{5/2}}dv$. Thus

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du = \frac{\sin((x+1)^2)}{(x+1)^3} - \frac{\sin(x^2)}{x^3} + \int_{x^2}^{(x+1)^2} \frac{3\sin u}{2u^{5/2}}du.$$

Now we may use the same technique as in part (a) to bound

$$-\frac{3}{2x^3} < \int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du < \frac{3}{2x^3}.$$

Thus we can bound $r(x)$ loosely with

$$\begin{aligned}
|r(x)| &= \left|\frac{\cos((x+1)^2)}{x+1}\right| + \left|\frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du\right| \\
&< \frac{1}{x} + \frac{3}{2x^2} \\
&< \frac{2}{x}.
\end{aligned}$$

(c) We claim that the lower and upper limits of $xf(x)$ are $\pm 1$. Indeed, since $r(x) \to 0$ as

$x \to \infty$ we can not worry about it. So consider the behavior of

$$\begin{aligned}\frac{\cos(x^2) - \cos((x+1)^2)}{2} &= -\sin\left(\frac{x^2 + (x+1)^2}{2}\right)\sin\left(\frac{x^2 - (x+1)^2}{2}\right) \\ &= \sin\left(x^2 + x + \frac{1}{2}\right)\sin\left(x + \frac{1}{2}\right).\end{aligned}$$

Intuitively, we need both arguments inside the sines to be close to $2\pi(n + 1/2)$. This will *probably (?)* never happen exactly due to the transendental nature of $\pi$. But because we are looking at only the upper and lower limits, we have "an epsilon of room" to work with. In particular, intuitively, if $x + 1/2 = 2\pi(n + 1/2)$ for some $n$ and $n$ is large enough, the neighbourhood around

$$2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} \pm \varepsilon$$

will map to some interval with length proportional to $\varepsilon n$ under $x \mapsto x^2$. To be precise, if we have

$$x^- = 2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} - \varepsilon \quad \text{and} \quad x^+ = 2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} + \varepsilon,$$

then (I will just skip all the calculation...)

$$\begin{aligned}(x^+)^2 + x^+ + 1 - (x^-)^2 - x^- - 1 &= (x^+)^2 - (x^-)^2 + 2\varepsilon \\ &= 2\varepsilon(\pi(4n+2) + 1) + 2\varepsilon \\ &\geq 2\pi\varepsilon(4n+2)\end{aligned}$$

So for any $\varepsilon > 0$, we may choose $n > \frac{2-\varepsilon}{8\varepsilon}$ so that $2\pi\varepsilon(4n+2) > 2\pi$. Thus there exist $a, b$ in the interval such that $\sin(a^2 + a + 1) = 1$ and $\sin(b^2 + b + 1) = -1$, where $|x - a| < \varepsilon$ and $|x - b| < \varepsilon$. So we have $af(a) > 1 - \varepsilon$ and $bf(b) < -1 + \varepsilon$. (I may have lost some factors in there somewhere.) This holds for any $\varepsilon > 0$, so the upper and lower limits of $xf(x)$ are $\pm 1$.

(d) The integral does converge. For any integer $N$ we have

$$\begin{aligned}\int_0^{N+1} \sin(t^2)dt &= \sum_{n=0}^{N} f(n)\\ &= f(0) + \sum_{n=1}^{N} \frac{1}{2n}\big(\cos(n^2) - \cos((n+1)^2) + r(n)\big)\\ &= f(0) + \sum_{n=1}^{N} \frac{r(n)}{2n} + \frac{1}{2}\sum_{n=1}^{N} \frac{\cos(n^2)}{n} - \frac{1}{2}\sum_{n=2}^{N} \frac{\cos(n^2)}{n-1}\\ &= f(0) + \sum_{n=1}^{N} \frac{r(n)}{2n} + \frac{\cos 1}{2} - \frac{\cos((N+1)^2)}{2} + \sum_{n=2}^{N} \frac{\cos(n^2)}{n(n-1)}\end{aligned}$$

Since $|r(n)| < 2/n$ and $|\cos(n^2)| \le 1$, both sums are comparable to $\sum_{n=0}^{N} 1/n^2$, we conclude that they converge in the limit $n \to \infty$. Hence

$$\int_0^{\infty} \sin(t^2)dt$$

converges in the limit as well.

$\square$