Chapter 7; # 1, 2, 3 (pg. 175)

Problem 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Suppose $f_n \to f$ uniformly on some set E such that each $|f_n(x)| \leq M_n$ is bounded. Then there is some N such that $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in E$. Thus we have $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| \leq 2 + M_N$ for all $x \in E$.

Set $M = \max(M_1, M_2, \ldots, M_{N-1}, 2 + M_N)$. Then f_n is uniformly bounded by M, as desired. $\square$

Problem 2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof. Let $\varepsilon > 0$. Since $f_n \to f$ and $g_n \to g$ uniformly, we have N_f and N_g such that $n \geq N_f$ implies $|f_n - f| < \varepsilon/2$ and $n \geq N_g$ implies $|g_n - g| < \varepsilon/2$. Then set $N = \max(N_f, N_g)$, so that for $n \geq N$ we have

$$|(f_n + g_n| - (f + g) < |f_n - f| + |g_n - g| = \varepsilon.$$

Thus $f_n + g_n \to f + g$ uniformly.

Now let $\varepsilon > 0$ again, but assume that f_n and g_n are bounded. Suppose $|f| \leq M$ and $|g| \leq M'$. There are integers N_f and N_g such that $n \geq N_f, N_g$ implies $|f_n - f| < 1$ and $|g_n - g| < 1$. Thus for all $n \geq N = \max(N_f, N_g$, we have $|f_n| < 1 + M$ and $|g_n| < 1 + M'$.

Now choose L_f and L_g such that $n \geq L_f, L_g$ implies $|f_n - f| < \varepsilon/2(1 + M')$ and $|g_n - g| < \varepsilon/2(1 + M)$. For any $n \geq \max(L_f, L_g, N)$, we have

$$\begin{aligned}
|f_n g_n - fg| &\leq |f_n g_n - f_n g| + |f_n g - fg| \\
&\leq |f_n||g_n - g| + |f_n - f||g| \\
&\leq \frac{(1 + M)\varepsilon}{2(1 + M)} + \frac{\varepsilon M'}{2(1 + M')} \\
&\leq \varepsilon.
\end{aligned}$$

Hence $f_n g_n \to fg$ uniformly, and we're done. $\square$

Problem 3. Construct sequences $\{f_n\}$ and $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E.

Proof. Let $f_n(x) = 1 - 1/n$ and $g_n(x) = 1/x$ on $E = (0, 1)$. The intuition is that although $1/x$ is not uniformly continuous on $(0, 1)$, the *constant sequence* $g_n(x) = 1/x$ is trivially a uniformly convergent sequence. Thus we have $f_n \to 1$ and $g_n \to 1/x$.

However, $f_n g_n = (1 - \frac{1}{n})\frac{1}{x}$ does not converge uniformly to $1/x$ since the pointwise sequences blow up as $x \to 0$. $\square$