Exercises 6, 12, pp. 52-53.

6. (a) If H is a subgroup of G, then for any $h, h' \in H$, we have $h^{-1}h'h \in H$. Hence $h^{-1}Hh = H$, and $h \in N_G(H)$. Therefore $H \leq N_G(H)$.

   If H is not a subgroup of G, then multiplication fails so we have no reason to expect $h^{-1}h'h \in H$. For example, let

$$H = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right\}.$$

   Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 0 & 3 \end{pmatrix} \notin H.$$

   Hence $H \not\leq N_G(H)$.

   (b) ($\Rightarrow$): If H is abelian, then clearly $h'h = hh'$ for all $h, h' \in H$. Then $H \leq C_G(H)$.

   ($\Leftarrow$): If $H \leq C_G(H)$, then for any $h, h' \in H$, we have $h^{-1}h'h = h' \Rightarrow h'h = hh'$. Hence H is abelian, as desired.

12. We do each part:

   (a) Define

$$p(x_1, \ldots, x_4) = 12x_1^5x_2^7x_3 - 18x_2^3x_3 + 11x_1^6x_2x_3^3x_4^{23}.$$

   Direct calculation shows that:

$$\begin{aligned} \sigma \cdot p &= (1234) \cdot p \\ &= 12x_2^5x_3^7x_4 - 18x_3^3x_4 + 11x_1^{23}x_2^6x_3x_4^3 \end{aligned}$$

$$\begin{aligned} \tau \cdot (\sigma \cdot p) &= (123) \cdot 12x_2^5x_3^7x_4 - 18x_3^3x_4 + 11x_1^{23}x_2^6x_3x_4^3 \\ &= 12x_1^7x_3^5x_4 - 18x_1^3x_4 + 11x_1x_2^{23}x_3^6x_4^3 \end{aligned}$$

$$\begin{aligned} (\tau \circ \sigma) \cdot p &= (1342) \cdot p \\ &= 12x_1^7x_3^5x_4 - 18x_1^3x_4 + 11x_1x_2^{23}x_3^6x_4^3 \end{aligned}$$

$$\begin{aligned} (\sigma \circ \tau) \cdot p &= (1324) \cdot p \\ &= 12x_2x_3^5x_4^7 - 18x_4^3 + 11x_1^{23}x_2^3x_3^6x_4. \end{aligned}$$

(b) This definition gives a left group action of S_4 on R. If $\sigma, \tau \in S_4$ and $p \in R$, then

$$\begin{aligned}\tau \cdot (\sigma \cdot p) &= \tau \cdot (p(x_{\sigma(1)}, \ldots, x_{\sigma(4)})) \\ &= p(x_{\tau(\sigma(1))}, \ldots, x_{\tau(\sigma(4))}) \\ &= p(x_{(\tau\circ\sigma)(1)}, \ldots, x_{(\tau\circ\sigma)(1)}) \\ &= (\tau \circ \sigma) \cdot p.\end{aligned}$$

Hence composition in S_4 is compatible with its action on R. Clearly $e \cdot p = p$. Thus we have satisfied the axioms for a group action, as desired.

(c) The permutations that stabilize x_4 are the ones that fix 4. The subset of S_4 that does this is: $\{e, (12), (23), (13), (123), (231)\}$. Looking at these permutations in cycle notation, clearly they are isomorphic to S_3. (They are the image of the embedding $\iota : S_3 \hookrightarrow S_4$.)

(d) An element σ stabilizes $x_1 + x_2$ satisfy $x_1 + x_2 = x_{\sigma(1)} + x_{\sigma(2)}$. Hence we must have either $(\sigma(1), \sigma(2)) = (1, 2)$ or $(\sigma(1), \sigma(2)) = (2, 1)$. In the first case, σ fixes 1 and 2, so the possible values are $\{e, (34)\}$. In the second case, σ must permute (12), so the possible values are $\{3, (12), (12)(34)\}$. Letting $x = (12), y = (34)$, we see that the stabilizer of $x_1 + x_2$ is $\{e, x, y, xy\}$ with $x^2 = y^2 = e$. This is clearly the (abelian) group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(e) If σ stabilizes $x_1x_2 + x_3x_4$, then

$$x_{\sigma(1)}x_{\sigma(2)} + x_{\sigma(3)}x_{\sigma(4)} = x_1x_2 + x_3x_4.$$

Then there are two cases:

$$\begin{aligned}&(\sigma(1), \sigma(2)) \in \{(1, 2), (2, 1)\} \wedge (\sigma(3), \sigma(4)) \in \{(3, 4), (4, 3)\}, \\ &(\sigma(1), \sigma(2)) \in \{(3, 4), (4, 3)\} \wedge (\sigma(3), \sigma(4)) \in \{(1, 2), (2, 1)\}\end{aligned}$$

The first case has solutions $\sigma \in \{e, (12), (34), (12)(34)\}$. And the second case has solutions $\sigma \in \{(13)(24), (1324), (1423), (14)(23)\}$.
To see that these two sets combine to form D_8, map $(12) \mapsto s$ and $r \mapsto (1324)$. Then we have:

$$\begin{gathered}e \mapsto e,\ \ (1324) \mapsto r,\ \ (12)(34) \mapsto r^2,\ \ (1423) \mapsto r^3, \\ (12) \mapsto r,\ \ (13)(24) \mapsto sr,\ \ (34) \mapsto sr^2,\ \ (14)(23) \mapsto sr^3.\end{gathered}$$

It can be checked that this is an isomorphism. Hence the stabilizer of $x_1x_2 + x_3x_4$ is indeed isomorphic to D_8.

(f) Again, for the map to stablize, we must have either $x_1 + x_2 = x_{\sigma(1)} + x_{\sigma(2)}$ or

$x_1 + x_2 = x_{\sigma(3)} + x_{\sigma(4)}$. The same equations follow for $x_3 + x_4$. So there are two cases:

$$(\sigma(1), \sigma(2)) \in \{(1,2), (2,1)\} \wedge (\sigma(3), \sigma(4)) \in \{(3,4), (4,3)\},$$
$$(\sigma(1), \sigma(2)) \in \{(3,4), (4,3)\} \wedge (\sigma(3), \sigma(4)) \in \{(1,2), (2,1)\}$$

As we've seen in part (e), this subset is isomorphic to D_8.

Exercises 16, 17, pp. 65-66.

16. (a) Since G is finite there can only be a finite amount of subgroups. In particular, there are only a finite amount of subgroups $\{H_i\}_{i=1}^n$ containing H. Then any chain $H \leq H_{i_1} \leq H_{i_2} \leq \cdots \leq H_{i_k} \leq G$ is finite, and we may prescribe H_{i_k} as the maximal subgroup containing H.

    (b) Suppose $\langle r \rangle \leq K$. Then $|\langle r \rangle| \leq |K|$ while $|K| \mid |G|$. But $\langle r \rangle$ has order n and G has order $2n$. Hence $|K|$ can only be n, in which case $H = K$, or $2n$, in which case $K = G$. This is exactly the definition of H being maximal, as desired.

    (c) The order of x^p is n/p, so $|\langle x^p \rangle| = n/p$. If K contains $\langle x^p \rangle$, then $n/p \leq |K| \Rightarrow n/|K| \leq p$ while $|K| \mid n \Rightarrow n/|K| = a$ for some $a \in \mathbb{Z}$. But the only possible factors of p are 1 and p, and $|K| \neq n$, so we must have $a = p$. Because their orders are equal and one is a subset of the other, $K = \langle x^p \rangle$. Hence $\langle x^p \rangle$ is maximal.

17. (a) The chain $\mathcal{C}$ is a set of subgroups $\{H_i\}_{i \in \mathcal{I}}$ on a total order $\mathcal{I}$ such that $H_i \leq H_j$ for all $i \leq j$.

    We first show that if $x, y \in \bigcup_{i \in \mathcal{I}} H_i = H$, then

    $$xy \in \bigcup_{i \in \mathcal{I}} H_i = H.$$

    Since $x \in H$, we have $x \in H_i$ for some $i \in I$. Similiarly $y \in H_j$ for some $j \in \mathcal{I}$. Furthermore, I is a total order so either $i \leq j$ or $i \geq j$. Without loss of generality assume that $i \leq j$, since we could just swap the labels if instead $j \leq i$. Then $H_i \leq H_j$, so $x \in H_i \leq H_j$ and $y \in H_j$ imply $xy \in H_j \leq H$.

    The other subgroup axioms are straightforward: $e \in H$ since every H_i is a subgroup. For any $x \in H$, $\exists i, x \in H_i \Rightarrow x^{-1} \in H_i \leq H$.

    Hence H is a subgroup of G.

    (b) Assume for the sake of contradiction that H is *not* a proper subgroup, i.e. $H = G$. Then each g_i must lie in some H_{α_i}. There are only finite g_i, therefore we can compute the finite maximum $\max(\alpha_i) = \alpha_j$ for some fixed j. Then H_{α_j} is both in

$\mathcal{C}$ and contains each g_i. Then $\langle g_1, \ldots, g_n \rangle \subset H_{\alpha_j}$. But $\langle g_1, \ldots, g_n \rangle = G$! So H_{α_j} is not proper, contradicting our assuptions about $\mathcal{C}$.

(c) Part (b) shows that for any chain $\mathcal{C}$, the union of all subgroups in the chain H is an upper bound on $\mathcal{C}$ that is proper. In other words, $H \in \mathcal{S}$, and hence we may apply Zorn's lemma to deduce that $\mathcal{S}$ contains at least one maximal element. This concludes the proof.

Exercises 1, 18, 24, 40, 41 pp. 85-89.

Problem 1. Let $\varphi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\varphi^{-1}(E) \leq G$. If $E \trianglelefteq H$, then $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Proof. Part 1: We show that $\varphi^{-1}(E)$ is a subgroup with the subgroup property. Suppose $g, h \in \varphi^{-1}(E)$. Then by definition $\varphi(g), \varphi(h) \in E$, and since E is a subgroup, we have in particular $\varphi(h)^{-1} = \varphi(h^{-1}) \in E$. Thus $\varphi(g)\varphi(h^{-1}) = \varphi(gh^{-1}) \in E$. By definiton this means $gh^{-1} \in \varphi^{-1}(E)$, which proves that $\varphi^{-1}(E)$ is indeed a subgroup.

Part 2: Now suppose that $E \trianglelefteq H$. To show that $\varphi^{-1}(E) \trianglelefteq G$, we have to prove $gng^{-1} \in \varphi^{-1}(E)$ for all $n \in \varphi^{-1}(E)$ and $g \in G$. By definition, $\varphi(n) \in E$, and also the normality of E implies $\varphi(g)\varphi(n)\varphi(g)^{-1} \in H$. Applying the properties of homomorphisms, we have can deduce $\varphi(gng^{-1}) \in H$, and so $gng^{-1} \in \varphi^{-1}(E)$. Hence $\varphi^{-1}(E)$ is normal in G.

Part 3: Immediately from part 2, since $\{e\} \trianglelefteq H$ and by definition $\ker \varphi = \varphi^{-1}(e)$, we have $\ker \varphi \trianglelefteq G$. □

Problem 18. Let G be the quasidihedral group of order 16:

$$G = \langle \sigma\tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$$

and let $\overline{G} = G/\langle \sigma^4 \rangle$ be the quotient of G by the subgroup generated by σ^4 (this subgroup is the center of G, hence is normal).

Proof. We do each part in this proof:

(a) The subgroup $\langle \sigma^4 \rangle$ has order 2, so Lagrange's theorem implies that $|\overline{G}| = |G|/|\langle \sigma^2 \rangle| = 16/2 = 8$.

(b) We have $\overline{G} = \{\overline{\tau}^a \overline{\sigma}^b \mid a = 0, 1,\ \ b = 0, 1, 2, 3\}$. This gives 8 elements which cannot be further reduced with the rules $\overline{\tau}^2 = \overline{\sigma}^4 = 1$ and $\sigma\tau = \tau\sigma^3$. Hence these must *exactly* be the elements of $\overline{G}$.

(c) Let $x = \overline{\tau}$ and $y = \overline{\sigma}$. The orders can be computed pretty easily:

$$\begin{aligned}
&x^0y^0 = 1 \Rightarrow |x^0y^0| = 1\\
&x^1y^0 = 1 \Rightarrow |x^1y^0| = |x| = 2\\
&x^0y^1 = 1 \Rightarrow |x^0y^1| = |y| = 4\\
&x^1y^1 = 1 \Rightarrow xyxy = xxy^3y = 1 \Rightarrow |x^1y^1| = 2\\
&x^0y^2 = 1 \Rightarrow |x^0y^2| = 2\\
&x^1y^2 = 1 \Rightarrow |x^1y^2| = 2\\
&x^0y^3 = 1 \Rightarrow |x^0y^3| = |y^{-1}| = 4\\
&x^1y^3 = 1 \Rightarrow |x^1y^3| = |xy^{-1}| = 2
\end{aligned}$$

(d) Again let $x = \overline{\tau}$ and $y = \overline{\sigma}$. Then

$$\begin{gathered}
yx = xy^3\\
xy^{-2}x = xy^2x = xyyx = xyxy^3 = xxy^3y^3 = y^2\\
x^{-1}y^{-1}xy = xy^3xy = xy^2yxy = xy^2xy^3y = xy^2x = y^2
\end{gathered}$$

(e) Consider the map φ such that $x \mapsto s$ and $y \mapsto r$. Then, looking at (c), clearly x and y interact in the same way s and r do hence $\overline{G} \cong D_8$.

□

Problem 24. Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

Proof. Suppose $N \trianglelefteq G$ and $H \leq G$. Let $n \in H \cap N$ and $h \in H$. So $h \in G$ and $n \in N$, for which we duduce that $hnh^{-1} \in N$. Also $n \in H$, and so $hnh^{-1} \in H$. Hence $hnh^{-1} \in N \cap H$. Hence $N \cap H \trianglelefteq H$. □

Problem 40. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that $\overline{x}$ and $\overline{y}$ commute in $\overline{G}$ if and only if $x^{-1}y^{-1}xy \in N$.

Proof. ($\Rightarrow$): If $x^{-1}y^{-1}xy \in N$, then $(x^{-1}y^{-1}xy)N = N \Rightarrow xyN = Nyx$. But N is normal, so we can swap the left and right cosets. Thus $xNyN = xyN = Nyx = yxN = yNxN$, as desired.

($\Leftarrow$): If $xNyN = yNxN$, then we can just run the argument backwards:

$$xyN = yxN \Rightarrow x^{-1}y^{-1}xyN = N \Rightarrow x^{-1}y^{-1}xy \in N.$$

□

Problem 41. Let G be a group. Prove that $N = \langle x^{-1}y^{-1}xy \mid x, y \in G\rangle$ is a normal subgroup of G and G/N is abelian.

Proof. N is a normal subgroup of G: Let $\varphi_g(n) = g^{-1}ng$. Note that conjugation by g is a homomorphism. Let $n \in N$, which will have the form $a_1^{\epsilon_1}a_2^{\epsilon_2}\dots a_n^{\epsilon_n}$, where each $a_i = x^{-1}y^{-1}xy$ for some $x, y \in G$ and $\epsilon_i = \pm 1$. Now we want to show that $g^{-1}ng = \varphi_g(n) \in N$ for any $g \in G$. Since φ_g is a homomorphism, we have

$$\varphi_g(n) = \varphi_g(a_1)^{\epsilon_1}\varphi_g(a_2)^{\epsilon_2}\dots\varphi_g(a_n)^{\epsilon_n}.$$

Because N is a subgroup, it suffices now to prove that each $\varphi_g(a_i) \in N$. We have

$$\varphi_g(a_i) = \varphi_g(x^{-1}y^{-1}xy) = \varphi_g(x^{-1}y^{-1}xy) = \varphi_g(x)^{-1}\varphi_g(y)^{-1}\varphi_g(x)\varphi_g(y).$$

The LHS is of the form $x'^{-1}y'^{-1}x'y'$ for $x' = \varphi_g(x)$ and $y' = \varphi_g(y)$, so it must be in N. Hence $\varphi_g(a_i) \in N$. By extension, $\varphi_g(n) \in N$. Therefore N is normal. □

Proof. N is abelian: By Exercise 40 we have that $\overline{x}$ and $\overline{y}$ commute in G/N is and only if $x^{-1}y^{-1}xy \in N$. But this implies that $\overline{x^{-1}y^{-1}xy} = 1$ in G/N. Rearranging gives $\overline{xy} = \overline{yx}$, as desired. □

Exercise 4, pp. 111.

Problem 4. Prove that $S_n = \langle (12), (123\dots n)\rangle$ for all $n \geq 2$.

Proof. It suffices to show that every transposition can be generated from $x = (12)$ and $y = (123\dots n)$. Indeed, direct calculation shows that we can obtain transpositions of the form $(i, i+1)$ by conjugating $y^{i-1}xy^{1-i}$:

$$\begin{aligned}
&i \mapsto 1 \mapsto 2 \mapsto i+1 \\
&i+1 \mapsto 2 \mapsto 1 \mapsto i \\
&j \mapsto j-i+1 \notin \{1,2\} \mapsto j \quad \forall j \neq i, i+1
\end{aligned}$$

Next, transpositions of the form $1i$ can be generated recursively using $(1, i+1) = (1i)(i, i+1)(1i)$, starting with the base case (12). Finally, general transpositions of the form (ij) can be computed with $(1i)(1j)(1i)$, which clearly maps $i \mapsto j$ and $j \mapsto i$. And with all the transpositions, we're done. □