Exercises 3, 8, 20, 23, 25, pp. 40-41;

3. ($\Rightarrow$): If H is abelian, then for any $a, b \in G$, $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$, where we use all our hypotheses. But φ is injective, so $ab = ba$. Hence G is abelian.

   ($\Leftarrow$): If G is abelian, then do the same argument with φ^{-1}. For any $a, b \in H$, $\varphi^{-1}(ab) = \varphi^{-1}(a)\varphi^{-1}(b) = \varphi^{-1}(b)\varphi^{-1}(a) = \varphi^{-1}(ba)$. Now φ^{-1} is injective, so G is abelian.

8. The orders of S_n and S_m are $n!$ and $m!$, respectively. Since sizes are non-equal so there cannot exist a bijection between S_n and S_m.

20. We prove the group axioms for $\mathrm{Aut}(G)$.

   (a) *Identity*: Let $\mathrm{id}_G : G \to G$ be the identity. Clearly for any $\varphi \in \mathrm{Aut}(G)$, $\varphi \circ \mathrm{id}_G = \mathrm{id}_G \circ \varphi = \varphi$. Hence id_G is the identity.

   (b) *Associativity*: Note that function composition is associative, so multiplication in $\mathrm{Aut}(G)$ is by definition associative.

   (c) *Closure*: We need to prove that for any $\varphi, \phi \in \mathrm{Aut}(G)$, $\varphi \circ \phi \in \mathrm{Aut}(G)$. Indeed, if φ and ϕ are isomorphisms, then for any $g, h \in G$, $\varphi(\phi(gh)) = \varphi(\phi(g)\phi(b)) = \varphi(\phi(g))\varphi(\phi(h))$. Hence $\varphi \circ \phi$ is a homomorphism. Furthermore, the composition of two bijections is a bijection, so $\varphi \circ \phi$ is a isomorphism as well.

   (d) *Inverses*: If $\varphi \in \mathrm{Aut}(G)$, then the function inverse $\varphi^{-1} G \to G$ is also the inverse of φ in $\mathrm{Aut}(G)$.

23.

25.

Exercises 4, 5, 6, 20, 21, pp. 44-45.

4.

5.

6.

20.

21.

Exercises 8, 10, 15 ,17, pp. 48-49.

8.

10.

15.

17.

Exercises 5, 10, 12, pp. 52-53.

5.

10.

12.