Homework 11

Hanting Zhang

Chapter 6, # 9, 10 & 13

Problem 9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.

Proposition. *Let $f(x)$ and $g(x)$ be continuously differentiable functions defined on $[a, \infty)$. If $\lim_{x\to\infty} f(x)g(x)$ exists and $\int_a^\infty f(x)g'(x)dx$ converges, then $\int_a^\infty f'(x)g(x)dx$ converges.*

Proof. Let $b > a$. Applying the normal integration by parts gives

$$\int_a^b f'(x)g(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x)dx.$$

In the limit, our hypotheses guarantee that the RHS is well defined. Thus the LHS also exists and is well defined, as desired. □

So now let $f(x) = \sin x$ and $g(x) = \frac{1}{1+x}$. Then $f'(x) = \cos x$ and $g'(x) = -\frac{1}{(1+x)^2}$. Since

$$\left|\frac{\sin x}{(1+x)^2}\right| \le \frac{1}{x^2},$$

we know that $\int_0^\infty f(x)g'(x)dx$ converges absolutely; and $\lim_{x\to\infty} f(x)g(x) = 0$. So indeed,

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

To see that $\int_0^\infty \frac{\cos x}{1+x}dx$ does not converge absolutely, we need to show that

$$\int_0^\infty \frac{|\cos x|}{1+x}dx$$

diverges. Indeed, Consider breaking up the integral onto each interval $[\pi n, \pi(n+1)$, so that we may compare it to the harmonic series:

$$\int_0^\infty \frac{|\cos x|}{1+x}dx = \sum_{n=0}^\infty \left(\int_{\pi n}^{\pi(n+1)} \frac{|\cos x|}{1+x}dx\right)$$

$$
\begin{aligned}
&\geq \sum_{n=0}^{\infty}\left(\int_{\pi n}^{\pi(n+1)} \frac{|\cos x|}{1+\pi(n+1)} dx\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{2}{1+\pi(n+1)}\right) \\
&\sim \sum_{n=0}^{\infty} \frac{1}{n}.
\end{aligned}
$$

And we're done!

Problem 10. Let p and q be positive real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p}+\frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathcal{R}(\alpha), g \in (\alpha), f \geq 0, g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left|\int_a^b fg d\alpha\right| \leq \left\{\int_a^b |f|^p d\alpha\right\}^{1/p}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q}.$$

This is *Hölder's inequality.* When $p=q=2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

Proof. We proceed with each part:

(a) We use Jensen's inequality. (I know this is slightly illegal since we haven't defined what $\log(x)$ is, but I couldn't figure out any other way to do it.) If $a = 0$ or $b = 0$ then there's nothing to prove, so assume $a, b > 0$. By Jensen's inequality,

$$\log\left(\frac{u^p}{p} + \frac{v^q}{q}\right) \geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) = \log u + \log v = \log(uv).$$

Since $\log x$ is monotonically increasing, we conclude that

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

(b) From part (a) we know that

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}.$$

Integrating both sides gives

$$\begin{aligned}\int_a^b f(x)g(x)dx &\leq \int_a^b \frac{f(x)^p}{p} + \frac{g(x)^q}{q}dx \\ &= \frac{1}{p}\int_a^b f(x)^p dx + \frac{1}{q}\int_a^b g(x)^q dx \\ &= \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

(c) Normalize $f(x)$ and $g(x)$ by taking

$$\frac{|f(x)|}{\left(\int_a^b |f(x)|^p\right)^{1/p}} \quad \text{and} \quad \frac{|g(x)|}{\left(\int_a^b |g(x)|^q\right)^{1/q}},$$

so we can apply part (b). We have,

$$\int_a^b \frac{|f(x)|}{\left(\int_a^b |f(x)|^p\right)^{1/p}} \frac{|g(x)|}{\left(\int_a^b |g(x)|^q\right)^{1/q}} \leq 1,$$

which implies

$$\left|\int_a^b fg d\alpha\right| \leq \int_a^b |f||g| d\alpha \leq \left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q d\alpha\right\}^{1/q},$$

as desired.

(d) There is no need for any other assumptions. We can just push the limits around since everything is continuous:

$$\begin{aligned}
\left|\int_a^\infty fg d\alpha\right| &= \left|\lim_{b\to\infty}\int_a^b fg d\alpha\right| \\
&\leq \lim_{b\to\infty}\left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \lim_{b\to\infty}\left\{\int_a^b |f|^p d\alpha\right\}^{1/p} \lim_{b\to\infty}\left\{\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \left\{\lim_{b\to\infty}\int_a^b |f|^p d\alpha\right\}^{1/p} \left\{\lim_{b\to\infty}\int_a^b |g|^q d\alpha\right\}^{1/q} \\
&= \left\{\int_a^\infty |f|^p d\alpha\right\}^{1/p} \left\{\int_a^\infty |g|^q d\alpha\right\}^{1/q},
\end{aligned}$$

as desired.

□

Problem 13. Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2) dt$ converge?

Proof. We proceed with each part:

(a) Assume that $x > 0$ throughout. We begin with the hint from the book. Make the substitution $u = t^2$ with $dt = du/2\sqrt{u}$ to obtain

$$f(x) = \int_{x^2}^{(x+1)^2} \frac{\sin u}{2\sqrt{u}} du.$$

We integrate by parts with $f(x) = \sin x$ and $G(x) = 1/2\sqrt{x}$. So $F(x) = -\cos x$ and $g(x) = -1/4x^{3/2}$. Hence

$$f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Now we try to bound $f(x)$ above and below by $1/x$ and $-1/x$, respectively. To bound $f(x) < 1/x$, note that we can simplify the last integral term with the inequality by replacing $\cos u$ with 1:

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du < \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du = \frac{1}{2(x+1)} - \frac{1}{2x}.$$

Thus

$$\begin{aligned} f(x) &< \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1-\cos(x^2)}{2x} - \frac{1-\cos((x+1)^2)}{2(x+1)} \\ &\leq \frac{1-\cos(x^2)}{2x} \\ &\leq \frac{1}{x}. \end{aligned}$$

On the other hand, we can also replace $\cos u$ with -1, which gives the opposite effect:

$$\begin{aligned} f(x) &> \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} + \frac{1}{2(x+1)} - \frac{1}{2x} \\ &= \frac{1-\cos((x+1)^2)}{2(x+1)} - \frac{1-\cos(x^2)}{2x} \\ &\geq \frac{1-\cos((x+1)^2)}{2(x+1)} \\ &\geq -\frac{1}{x}. \end{aligned}$$

Thus $|f(x)| < 1/x$.

(b) We have

$$\begin{aligned}
2xf(x) &= \cos(x^2) - \frac{2x\cos((x+1)^2)}{2(x+1)} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \\
&= \cos(x^2) - \frac{2(x+1)\cos((x+1)^2) - 2\cos((x+1)^2)}{2(x+1)} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du \\
&= \cos(x^2) - \cos((x+1)^2) + \frac{\cos((x+1)^2)}{x+1} - 2x\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}}du.
\end{aligned}$$

Thus we may identify

$$r(x) = \frac{\cos((x+1)^2)}{x+1} - \frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du.$$

Now we cannot bound the last intgral again by $\mathcal{O}(1/x)$ again because of the factor of x in the front. Hence the messy (but somewhat natural) thing to do is simply integrate by parts again. Let $du = \sin x dx$ and $v = \frac{1}{x^{3/2}}$, so that $u = \cos x$ and $dv = -\frac{3}{2x^{5/2}}dv$. Thus

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du = \frac{\sin((x+1)^2)}{(x+1)^3} - \frac{\sin(x^2)}{x^3} + \int_{x^2}^{(x+1)^2} \frac{3\sin u}{2u^{5/2}}du.$$

Now we may use the same technique as in part (a) to bound

$$-\frac{3}{2x^3} < \int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du < \frac{3}{2x^3}.$$

Thus we can bound $r(x)$ loosely with

$$\begin{aligned}
|r(x)| &= \left|\frac{\cos((x+1)^2)}{x+1}\right| + \left|\frac{x}{2}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}}du\right| \\
&< \frac{1}{x} + \frac{3}{2x^2} \\
&< \frac{2}{x}.
\end{aligned}$$

(c) We claim that the lower and upper limits of $xf(x)$ are ± 1. Indeed, since $r(x) \to 0$ as

$x \to \infty$ we can not worry about it. So consider the behavior of

$$\begin{aligned}\frac{\cos(x^2) - \cos((x+1)^2)}{2} &= -\sin\left(\frac{x^2 + (x+1)^2}{2}\right)\sin\left(\frac{x^2 - (x+1)^2}{2}\right) \\ &= \sin\left(x^2 + x + \frac{1}{2}\right)\sin\left(x + \frac{1}{2}\right).\end{aligned}$$

Intuitively, we need both arguments inside the sines to be close to $2\pi(n+1/2)$. This will *probably (?)* never happen exactly due to the transendental nature of π. But because we are looking at only the upper and lower limits, we have "an epsilon of room" to work with. In particular, intuitively, if $x + 1/2 = 2\pi(n+1/2)$ for some n and n is large enough, the neighbourhood around

$$2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} \pm \varepsilon$$

will map to some interval with length proportional to εn under $x \mapsto x^2$. To be precise, if we have

$$x^- = 2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} - \varepsilon \quad \text{and} \quad x^+ = 2\pi\left(n + \frac{1}{2}\right) - \frac{1}{2} + \varepsilon,$$

then (I will just skip all the calculation...)

$$\begin{aligned}(x^+)^2 + x^+ + 1 - (x^-)^2 - x^- - 1 &= (x^+)^2 - (x^-)^2 + 2\varepsilon \\ &= 2\varepsilon(\pi(4n+2) + 1) + 2\varepsilon \\ &\geq 2\pi\varepsilon(4n+2)\end{aligned}$$

So for any $\varepsilon > 0$, we may choose $n > \frac{2-\varepsilon}{8\varepsilon}$ so that $2\pi\varepsilon(4n+2) > 2\pi$. Thus there exist a, b in the interval such that $\sin(a^2 + a + 1) = 1$ and $\sin(b^2 + b + 1) = -1$, where $|x - a| < \varepsilon$ and $|x - b| < \varepsilon$. So we have $af(a) > 1 - \varepsilon$ and $bf(b) < -1 + \varepsilon$. (I may have lost some factors in there somewhere.) This holds for any $\varepsilon > 0$, so the upper and lower limits of $xf(x)$ are ± 1.

(d) The integral does converge. For any integer N we have

$$\begin{aligned}\int_0^{N+1} \sin(t^2)dt &= \sum_{n=0}^{N} f(n) \\ &= f(0) + \sum_{n=1}^{N} \frac{1}{2n}\left(\cos(n^2) - \cos((n+1)^2) + r(n)\right)\end{aligned}$$

$$= f(0) + \sum_{n=1}^{N} \frac{r(n)}{2n} + \frac{1}{2}\sum_{n=1}^{N} \frac{\cos(n^2)}{n} - \frac{1}{2}\sum_{n=2}^{N} \frac{\cos(n^2)}{n-1}$$
$$= f(0) + \sum_{n=1}^{N} \frac{r(n)}{2n} + \frac{\cos 1}{2} - \frac{\cos((N+1)^2)}{2} + \sum_{n=2}^{N} \frac{\cos(n^2)}{n(n-1)}$$

Since $|r(n)| < 2/n$ and $|\cos(n^2)| \leq 1$, both sums are comparable to $\sum_{n=0}^{N} 1/n^2$, we conclude that they converge in the limit $n \to \infty$. Hence

$$\int_0^\infty \sin(t^2) dt$$

converges in the limit as well.

$\square$