Exercises 3, 8, 20, 23, 25, pp. 40-41;

3. ($\Rightarrow$): If H is abelian, then for any $a, b \in G$, $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$, where we use all our hypotheses. But φ is injective, so $ab = ba$. Hence G is abelian.

   ($\Leftarrow$): If G is abelian, then do the same argument with φ^{-1}. For any $a, b \in H$, $\varphi^{-1}(ab) = \varphi^{-1}(a)\varphi^{-1}(b) = \varphi^{-1}(b)\varphi^{-1}(a) = \varphi^{-1}(ba)$. Now φ^{-1} is injective, so G is abelian.

8. The orders of S_n and S_m are $n!$ and $m!$, respectively. Since sizes are non-equal so there cannot exist a bijection between S_n and S_m.

20. We prove the group axioms for $\mathrm{Aut}(G)$.

    (a) *Identity*: Let $\mathrm{id}_G : G \to G$ be the identity. Clearly for any $\varphi \in \mathrm{Aut}(G)$, $\varphi \circ \mathrm{id}_G = \mathrm{id}_G \circ \varphi = \varphi$. Hence id_G is the identity.

    (b) *Associativity*: Note that function composition is associative, so multiplication in $\mathrm{Aut}(G)$ is by definition associative.

    (c) *Closure*: We need to prove that for any $\varphi, \phi \in \mathrm{Aut}(G)$, $\varphi \circ \phi \in \mathrm{Aut}(G)$. Indeed, if φ and ϕ are isomorphisms, then for any $g, h \in G$, $\varphi(\phi(gh)) = \varphi(\phi(g)\phi(b)) = \varphi(\phi(g))\varphi(\phi(h))$. Hence $\varphi \circ \phi$ is a homomorphism. Furthermore, the composition of two bijections is a bijection, so $\varphi \circ \phi$ is a isomorphism as well.

    (d) *Inverses*: If $\varphi \in \mathrm{Aut}(G)$, then the function inverse $\varphi^{-1} G \to G$ is also the inverse of φ in $\mathrm{Aut}(G)$.

23. To show that every element g can be written as $x^{-1}\sigma(x)$, it is equivalent to prove that the map $x \mapsto x^{-1}\sigma(x)$ is surjective. Since G is finite, this is equivalent to showing that $x \mapsto x^{-1}\sigma(x)$ is injective via a cardinality argument.

    Let $x, y \in G$ such that $x^{-1}\sigma(x) = y^{-1}\sigma(y)$.

25.

Exercises 4, 5, 6, 20, 21, pp. 44-45.

4. (a) We proceed by using the subgroup formula. Let H be the kernel of the action of G on A. Suppose $x, y \in H$, i.e. both x and y fix all elements of A. Then for any $a \in A$, we have $(xy^{-1}) \cdot a) = x \cdot (y^{-1} \cdot a)$. We know that $y \cdot a = a$, so $a = y^{-1} \cdot a$. Hence we can simplify $(xy^{-1}) \cdot a = a$ and conclude that $xy^{-1} \in H$. Hence H is a subgroup.

   (b) Let $G_a = \{g \in G : ga = a\}$. We use the subgroup formula again. Suppose $x, y \in G_a$. Again, $(xy^{-1}) \cdot a = x \cdot (y^{-1} \cdot a) = x \cdot a = a$. Hence $xy^{-1} \in G_a$ and we have a subgroup.

5. The kernel K of the group action of G on A is defined as $\{g \in G : \forall a \in A, ga = a\}$. If $g \in K$, then the permutation σ_g associated with g is the identity permutation. But we have $\varphi : G \to S_A$ defined by $\varphi(g) = \sigma_g$, so $\sigma_g = 1 \Rightarrow g \in \ker\varphi$. Hence $K \subseteq \ker\varphi$.

   For the other inclusion, consider $g \in \ker\varphi$. Then σ_g is the identity permutation, so clearly $ga = a$ for all $a \in A$. Then $g \in K$. Hence the two subgroups K and $\ker\varphi$ are equal.

6. For a faithful action of G on A, the corresponding permutation representation is injective. Hence $\ker\varphi$ is the trivial subgroup. By exercise 5, the kernel of the action is therefore also trivial.

20. Imagine the group of such rigid motions G acting on the vertices of the tetrahedron. Then we have an action from G on $\{$four vertices$\}$. This is equivalent to a homomorphism $\varphi : G \to S_4$. Furthermore, the only action that fixes all the vertices is clearly the trivial action. Hence the action is faithful, and G embedds into S_4, i.e. G is isomorphic to its image (which is some subgroup) in S_4 under φ.

21. Let the rigid motions of the cube G act on the four pairs of opposite vertices that make up the cube. This is a valid action because opposite vertices remain opposite after rigid motions. The associated permutation representation is a map $\varphi : G \to S_4$. Furthermore, the only action that does not permute any vertices is the identity action. Then G acts faithfully; hence φ is injective.

    Now we prove that φ is surjective. Let σ be some permutation of opposite pairs of vertices.

Exercises 8, 10, 15 ,17, pp. 48-49.

8.

10.

15.

17.