Problem 3.1. Prove that the integral defining $\Gamma(x)$ converges for all $x > 0$.

Proof. Split the integral up between $(0, 1)$ and $(1, \infty)$ and deal with the convergence of each with different strategies.

On the unit interval, we have $\int_0^1 t^{x-1}dt = \frac{t^x}{x}\Big|_0^1 = \frac{1}{x}$. But $t^{x-1}e^{-t} < t^{x-1}$ on $t \in (0, 1)$, thus $\int_0^1 t^{x-1}e^t dt < \frac{1}{x}$ converges.

On the $(1, \infty)$ ray, consider $t^{x-1}e^{-t} \le e^{-t/2}$. This inequality is not outright true, but

$$t^{x-1}e^{-t} \le e^{-t/2} \iff \frac{t^{x-1}}{e^{t/2}} \le 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^{x-1}}{e^{t/2}} = 0.$$

So for each t, there is some N such that the inequality holds for all $x \ge N$. Thus we may split the integral again,

$$\begin{aligned}
\int_1^\infty t^{x-1}e^{-t}dt &= \int_1^N t^{x-1}e^{-t}dt + \int_N^\infty t^{x-1}e^{-t}dt \\
&\le X + \int_N^\infty e^{-t/2}dt \\
&= X - 2e^{-t/2}\Big|_N^\infty \\
&= X + 2e^{-N/2} \\
&< \infty,
\end{aligned}$$

where X is just some finite value. Thus $\Gamma(x)$ converges on both $(0, 1)$ and $(1, \infty)$, so it is a convergent integral, as desired. $\square$

Problem 3.2. Using Hölder's inequality, show that the Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is log-convex.

Proof. We want to show that $\log(\Gamma(\lambda a+(1-\lambda)b)) \le \lambda \log(\Gamma(a))+(1-\lambda) \log(\Gamma(b))$. Indeed, we use Hölder's inequality in the form of Exercise 2.3:

$$\begin{aligned}
\log(\Gamma(\lambda a + (1 - \lambda)b)) &= \log\left(\int_0^\infty t^{\lambda a+(1-\lambda)b-1}e^{-t}dt\right) \\
&= \log\left(\int_0^\infty t^{(a-1)\lambda}e^{-t\lambda}t^{(b-1)(1-\lambda)}e^{-t(1-\lambda)}dt\right) \\
&= \log\left(\int_0^\infty (t^{a-1}e^{-t})^\lambda(t^{b-1}e^{-t})^{1-\lambda}dt\right)
\end{aligned}$$

$$\begin{aligned}
&\le \log\left[\left(\int_0^\infty t^{a-1}e^{-t}dt\right)^\lambda \left(\int_0^\infty t^{b-1}e^{-t}dt\right)^{1-\lambda}\right] \\
&= \lambda \log\left(\int_0^\infty t^{a-1}e^{-t}dt\right) + (1-\lambda)\log\left(\int_0^\infty t^{b-1}e^{-t}dt\right) \\
&= \lambda \log(\Gamma(a)) + (1-\lambda)\log(\Gamma(b)),
\end{aligned}$$

$\square$

Problem 3.3. Let $B : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

This is called the *beta function.*

(a) Prove that $B(1,y) = \frac{1}{y}$, for $y \in (0,\infty)$.

(b) Prove that for each fixed $y \in (0,\infty)$, the function $x \mapsto B(x,y)$ is log-convex on $(0,\infty)$.

(c) Using an integration-by-parts on the identity

$$B(x+1,y) = \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}dt,$$

prove that $B(x+1,y) = \frac{x}{x+y}B(x,y)$.

(d) Argue that for each $y > 0$, the function $f_y : (0,\infty) \to \mathbb{R}$ defined by

$$f_y(x) = \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y)$$

satisfies the hypotheses of the Bohr-Mollerup Theorem. Conclude that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0.$$

(e) Using the conclusion of part (d), and the substitution $t = \sin^2\theta$ in the definition of the beta function, show that $B(\frac{1}{2},\frac{1}{2}) = (\Gamma(\frac{1}{2}))^2 = \pi$.

(f) Using the conclusion of part (e), and the substitution $t = s^2$ in the definition of the

Gamma function, conclude that

$$\int_{-\infty}^{\infty} e^{-s^2} dx = \sqrt{\pi}.$$

Proof. We proceed with each part.

(a) We have

$$B(1,y) = \int_0^1 (1-t)^{y-1} dt = -\frac{(1-t)^y}{y}\bigg|_0^1 = -\frac{0}{y} + \frac{1}{y} = \frac{1}{y}.$$

(b) We want to show that $\log(B(\lambda a + (1-\lambda)b, y)) \leq \lambda \log(B(a,y)) + (1-\lambda)\log(B(b,y))$. Indeed, we have again by Hölder's inequality,

$$\begin{aligned}
B(\lambda a + (1-\lambda)b, y) &= \int_0^1 t^{\lambda a + (1-\lambda)b - 1}(1-t)^{y-1} dt \\
&= \int_0^1 t^{(a-1)\lambda}(1-t)^{(y-1)\lambda} t^{(b-1)(1-\lambda)}(1-t)^{(y-1)(1-\lambda)} dt \\
&= \int_0^1 (t^{a-1}(1-t)^{y-1})^{\lambda}(t^{b-1}(1-t)^{y-1})^{1-\lambda} dt \\
&\leq \left(\int_0^1 t^{a-1}(1-t)^{y-1} dt\right)^{\lambda} \left(\int_0^1 t^{b-1}(1-t)^{y-1}\right)^{1-\lambda} \\
&= B(a,y)^{\lambda} B(b,y)^{1-\lambda}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\log(B(\lambda a + (1-\lambda)b, y)) &\leq \log(B(a,y)^{\lambda} B(b,y)^{1-\lambda}) \\
&= \lambda \log(B(a,y)) + (1-\lambda)\log(B(b,y)),
\end{aligned}$$

as desired.

(c) Integrating by parts with $u = \left(\frac{t}{1-t}\right)^x$ and $dv = (1-t)^{x+y-1}$, we compute first that $du = \left(\frac{t}{1-t}\right)^{x-1} \frac{x dt}{(1-t)^2}$ and $v = -\frac{(1-t)^{x+y}}{x+y}$. Then,

$$\begin{aligned}
B(x+1,y) &= \int_0^1 \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1} dt \\
&= uv|_0^1 - \int_0^1 v du
\end{aligned}$$

$$\begin{aligned}
&= -\left(\frac{t}{1-t}\right)^x \frac{(1-t)^{x+y}}{x+y}\Bigg|_0^1 + \int_0^1 \frac{(1-t)^{x+y}}{x+y}\left(\frac{t}{1-t}\right)^{x-1}\frac{xdt}{(1-t)^2} \\
&= -\frac{t^x(1-t)^y}{x+y}\Bigg|_0^1 + \frac{x}{x+y}\int_0^1 t^{x-1}(1-t)^{y-1}dt \\
&= 0 - 0 + \frac{x}{x+y}B(x,y) \\
&= \frac{x}{x+y}B(x,y),
\end{aligned}$$

as desired.

(d) We prove properties Γ(1-3).

(Γ1): We have

$$\begin{aligned}
f_y(x+1) &= \frac{\Gamma(x+1+y)}{\Gamma(y)}B(x+1,y) \\
&= \frac{(x+y)\Gamma(x+y)}{\Gamma(y)}\frac{x}{x+y}B(x,y) \\
&= x\frac{\Gamma(x+y)}{\Gamma(y)}B(x,y) = xf_y(x).
\end{aligned}$$

(Γ2): We have

$$f_y(1) = \frac{\Gamma(1+y)}{\Gamma(y)}B(1,y) = \frac{y\Gamma(y)}{\Gamma(y)}\frac{1}{y} = 1.$$

(Γ3): Since the product of log-convex functions are convex, it suffices to show that for fixed y, $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $B(x,y)$ are log-convex. The latter was proven in part (b). The former is a translation and scaling of a log-convex function, which is clearly log-convex. Thus $f_y(x)$ must also be log-convex.

So $f_y(x)$ satisfies the hypotheses of the Bohr-Mollerup Theorem. Therefore, it must be true that

$$f_y(x) = \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y) = \Gamma(x).$$

We conclude that
$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0.$$

(e) With the substitution $t = \sin^2\theta$, we have $dt = 2\cos\theta\sin\theta d\theta$. Thus
$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = 2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta.$$
Setting $x = y = 1/2$, we have
$$\begin{aligned} B\left(\frac{1}{2},\frac{1}{2}\right) &= 2\int_0^{\pi/2}(\sin\theta)^{2(1/2)-1}(\cos\theta)^{2(1/2)-1}d\theta \\ &= \int_0^{\pi/2} d\theta = 2\pi/2 = \pi. \end{aligned}$$
From part (d), $B(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = (\Gamma(\frac{1}{2}))^2$. Thus $B(\frac{1}{2},\frac{1}{2}) = (\Gamma(\frac{1}{2}))^2 = \pi$, as desired.

(f) With the substitution $t = s^2$, we have $dt = 2sds$. Fixing $x = 1/2$, we have
$$\Gamma(1/2) = \int_0^\infty t^{1/2-1}e^{-t}dt = \int_0^\infty s^{2(1/2-1)}e^{-s^2}2sds = 2\int_0^\infty e^{-s^2}ds.$$

Thus, remarkably, we conclude with part (d) that
$$\int_{-\infty}^\infty e^{-s^2}dx = 2\int_0^\infty e^{-x^2}dx = \Gamma(1/2) = \sqrt{\pi}.$$

$\square$