Homework 9

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Exercises 3, 7, 8, 10 pp. 277-279.

Problem 3. Let R be a Euclidean Domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero (if such a element exists) is a unit.

Problem 7. Find a generator for the ideal $(85, 1+13i)$ in $\mathbb{Z}[i]$, i.e., a greatest common divisor of 85 and $1 = 13i$, by the Euclidean Algorithm. Do the same for the ideal $(47-13i, 53+56i)$.

Problem 8. Let $F = \mathbb{Q}(\sqrt{D})$ be a quadratic field with associated quadratic integer ring $\mathcal{O}$ and field norm N as in Section 7.1.

(a) Suppose D is $-1, -2, -3, -7$ or -11. Prove that $\mathcal{O}$ is a Euclidean Domain with respect to N. [Modify the proof for $\mathbb{Z}[i]$ $(D = -1)$ in the text.]

(b) Suppose that $D = -43, -67$ or -163. Prove that $\mathcal{O}$ is not a Euclidean Domain with respect to any norm. [Apply the same proof as for $D = -19$ in the text.]

Problem 10. Prove that the quotient ring $\mathbb{Z}[i]/I$ is finite for any nonzero ideal I of $\mathbb{Z}[i]$.

Exercises 1, 3, 4, 5, 6 pp. 282-283.

Problem 1. Prove that in a Principal Ideal Domain two ideals (a) and (b) are comaximal if and only if a greatest common divisor of a and b is 1 (in which case a and b are said to be *coprime* or *relatively prime*.)

Problem 3. Prove that a quotient of a P.I.D. by a prime ideal is once again a P.I.D..

Problem 4. Let R be an integral domain. Prove that if the following two conditions hold then R is a P.I.D.:

(i) any two nonzero elements a and b in R have a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$, and

(ii) if $a_1, a_2, a_3, \ldots$ are nonzero elements of R such that $a_{i+1} \mid a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

Problem 5. Let R be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Define the ideals $I_2 = (2, 1+\sqrt{-5})$, $I_3 = (3, 2+\sqrt{-5})$, and $I_3' = (3, 2-\sqrt{-5})$.

(a) Prove that I_2, I_3, and I_3' are nonprincipal ideals in R.

(b) Prove that the product of two nonprincipal ideals can be principal by showing that I_2^2 is the principal ideal generated by 2, i.e., $I_2^2 = (2)$.

(c) Prove similarly that $I_2 I_3 = (1-\sqrt{-5})$ and $I_2 I_3' = (1+\sqrt{-5})$ are principal. Conclude that the principal ideal (6) is the product of 4 ideals: $(6) = I_2^2 I_3 I_3'$.

Problem 6. Let R be an integral domain and suppose that every *prime* ideal in R is principal. This exercise proves that every ideal of R is principal, i.e., R is a P.I.D.

(a) Assume that the set of ideals of R that are not principal is nonempty and prove that this set has a maximal element under inclusion (which, by hypothesis, is not prime). [Use Zorn's Lemma.]

(b) Let I be an ideal which is maximal with respect to being nonprincipal, and let $a, b \in R$ with $ab \in I$ but $a \notin I$ and $b \notin I$. Let $I_a = (I, a)$ be the ideal generated by I and a, let $I_b = (I, b)$ be the ideal generated by I and b, and define $J = \{r \in R \mid rI_a \subseteq J\}$. Prove that $I_a = (\alpha)$ and $J = (\beta)$ are principal ideals in R with $I \subset I_b$ and $I_aJ = (\alpha\beta) \subseteq I$.

(c) If $x \in I$ show that $x = s\alpha$ for some $s \in J$. Deduce that $I = I_aJ$ is principal, a contradiction, and conclude that R is a P.I.D.

Exercises 6, 8 pp. 282-283.

TODO