Problem 3.1. Prove that

$$\lim_{L\to\infty}\sum_{\ell=0}^{2L+1}\sum_{j+k=\ell} x_{j,k} = -\frac{2}{3}.$$

Proof. We may rearrange the finite sum $\sum_{\ell=0}^{2L+1}\sum_{j+k=\ell} x_{j,k}$ into $\sum_{n=0}^{L}\sum_{m=n}^{2L+1-n} x_{n,m}$. Formally checking that this is a bijection is quite painful, but the geometric argument is that the second sum simply counts the entries column by column instead of along the diagonals.

Now, the definition of $x_{n,m}$ gives:

$$\begin{aligned}
\sum_{n=0}^{L}\sum_{m=n}^{2L+1-n} x_{n,m} &= \sum_{n=0}^{L}\left(-1+\sum_{m=n+1}^{2L+1-n} x_{n,m}\right)\\
(\text{since } m>n \text{ we have } x_{n,m}=2^{n-m}) \quad &= \sum_{n=0}^{L}\left(-1+\sum_{m=n+1}^{2L+1-n} 2^{n-m}\right)\\
(\text{reindex with } m'=m-n) \quad &= \sum_{n=0}^{L}\left(-1+\sum_{m'=1}^{2L+1-2n} \frac{1}{2^{m'}}\right)\\
&= \sum_{n=0}^{L}\left(-1+1-\frac{1}{2^{2L+1-2n}}\right)\\
&= -\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^{L-n}} = -\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^{n}}
\end{aligned}$$

Now as $L\to\infty$, we have $-\frac{1}{2}\sum_{n=0}^{L}\frac{1}{4^n} \to -\frac{1}{2}\left(\frac{1}{1-1/4}\right) = -2/3$, as desired. □

Problem 3.2. Let $(x_{n,m})_{n,m=0}^{\infty}$ be a double sequence of complex numbers. Let $\phi : \mathbb{N}_0 \to \mathbb{N}_0\times\mathbb{N}_0$ be a bijection, $\phi(n) = (j(n), k(n))$. Prove that if $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_{n,m}|$ converges, then so does the series $\sum_{n=0}^{\infty} x_{j(n),k(n)}$, and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|x_{n,m}| = \sum_{n=0}^{\infty} x_{j(n),k(n)}.$$

Lemma. *Let $\phi : \mathbb{N}_0 \to \mathbb{N}_0\times\mathbb{N}_0$ be a bijection, $\phi(n) = (j(n), k(n))$. For any L, there exists some N such that $[0, L]\times[0, L] \subseteq \phi([0, N])$. i.e. ϕ will always "fill up" the $L\times L$ square.*

Proof. Proof by contradiction. Suppose this wasn't the case for all N. Then ϕ would not be bijective, since there exists some $(x, y) \in [0, L]\times[0, L]$ with no preimage. □

Proof. Let $\varepsilon > 0$. Define

$$y_m = \sum_{n=0}^{\infty} |x_{m,n}|, \quad z_n = \sum_{m=0}^{\infty} |x_{m,n}|, \quad A = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} x_{n,m}.$$

By the same argument as Lemma 3.3, there exists some M_0 and N_0 such that, for any $M > M_0$ and $N > N_0$,

$$\sum_{m=M+1}^{\infty} y_m < \frac{\varepsilon}{2}, \quad \sum_{n=N+1}^{\infty} x_n < \frac{\varepsilon}{2}.$$

To show that $\sum_{n=0}^{\infty} x_{j(n),k(n)}$ converges, it suffices to show that there exists some P such that

$$\left| A - \sum_{n=0}^{P} x_{j(n),k(n)} \right| < 2\varepsilon.$$

Indeed, choose $L = \max(M, N)$. Then by our above lemma, there is some P such $[0, L] \times [0, L] \subseteq \phi([0, P])$. Let $S = [0, P] \setminus \phi^{-1}([0, L] \times [0, L])$ be the points in the interval that don't get mapped into the square. Also note that $S \subseteq [L + 1, \infty) \times [0, \infty) \cup [0, \infty) \times [L + 1, \infty)$. Thus we have,

$$\begin{aligned}
\left| \sum_{n=0}^{P} x_{j(n),k(n)} - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| &\leq \sum_{n\in S} |x_{j(n),k(n)}| \\
&\leq \sum_{(i,j)\in[L+1,\infty)\times[0,\infty)} |x_{j,k}| + \sum_{(i,j)\in[0,\infty)\times[L+1,\infty)} |x_{j,k}| \\
&\leq \sum_{j=L+1}^{\infty} y_j + \sum_{k=L+1}^{\infty} x_k < \varepsilon.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\left| A - \sum_{n=0}^{P} x_{j(n),k(n)} \right| &\leq \left| A - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| + \left| \sum_{n=0}^{P} x_{j(n),k(n)} - \sum_{j=0}^{L}\sum_{k=0}^{L} x_{j,k} \right| \\
&< \varepsilon + \varepsilon = 2\varepsilon
\end{aligned}$$

as desired. Thus our proof is complete. $\square$