Problem 1. If $x \in \mathbb{R}^k$, $k \geq 2$, show that there exists $y \in \mathbb{R}^k$ such that $x \cdot y = 0$.

Proof. One possible solution is simply $y = 0$; hence trivially $x \cdot y = 0$. But that's not very interesting so I'll give a nontrivial example.

If x is trivial then y can be anything, so assume that x is nontrivial. Then there is at least one index i where $x_i \neq 0$. Since $n \geq 2$, pick some $j \neq i$. Set $y_i = -x_j$ and $y_j = x_i$, and $y_k = 0$ for all $k \neq i$ and $k \neq j$. Hence this defines y as nontrivial. Then

$$x \cdot y = \sum_{k=1}^{n} x_k y_k = x_i y_i + x_j y_j = -x_i x_j + x_j x_i = 0.$$

□

Problem 2. True or False: If true prove it, if false counterexample it.

(a) Let $\{F_n\}$ be a countable collection of closed subsets of $\mathbb{R}$ such that for any finite sub-collection

$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing.$$

Then

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(b) Add the condition that each F_n is bounded and repeat (2a).

(c) Repeat (2a) where closed and bounded $F_n \subseteq X$, and arbitrary metric space.

Proof. Part (a): This claim is **false**. Consider the subsets $F_n = [n, \infty)$. Then for any finite sub-collection $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$, let $n = \max_k(n_k)$. We can compute the intersection to be:

$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} = F_n \neq \varnothing.$$

However, since for any $x \in \mathbb{R}$ we may find some $n \geq x$, there is always some F_n such that $x \notin F_n$. Hence

$$\bigcap_{n=1}^{\infty} F_n = \varnothing.$$

This disproves the claim. □

Proof. Part (b): This claim is **true**. If F_n are both closed and bounded subsets of $\mathbb{R}$, then the Heine-Borel Theorem guarantees that F_n is compact. Now apply Theorem 2.36 from the textbook to conclude that

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

□

Proof. Part (c): This claim is **false**. Let $X = \mathbb{Q}$ with the relative topology inherited from $\mathbb{R}$. Then consider the subsets $F_n = \overline{B_{1/n}(\sqrt{2})}$ as the closed balls centered at $\sqrt{2}$ with radius $1/n$, where p_n is tne nth prime. In particular, since $\sqrt{2} \pm 1/n$ are irrational, the boundary points of F_n don't exist in $\mathbb{Q}$, and hence we can drop them without changing anything: $F_n = B_{1/n}(\sqrt{2})$.

Now we check that finite intersections are nonempty. Indeed, if $F_{n_1}, F_{n_2}, \ldots, F_{n_k}$ are a finite sub-collection, then their intersection is just the ball of minimum radius $r = \min_k(1/n_k)$. This radius is clearly greater than 0, so we know that $F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing$.

However, if we consider $\bigcap_{n=1}^{\infty} F_n$, then for any $x \neq \sqrt{2}$, we can find a k such that $k > 1/|x - \sqrt{2}|$. This implies $1/n < |x - \sqrt{2}|$. Hence by definition $x \notin F_k$, so $x \notin \bigcap_{n=1}^{\infty} F_n$. So all $x \neq \sqrt{2}$ are not in our intersection. But also $\sqrt{2}$ is not in $\mathbb{Q}$! Hence in $\mathbb{Q}$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is empty. This disproves the claim. $\square$

Problem 3. Consider the metric space $\mathbb{Q}$ of all rationals on the real line with the Euclidean metric. Prove that if $K \neq \varnothing$ is a compact subset of $\mathbb{Q}$ then K cannot contain an open subset of $\mathbb{Q}$. *Hint: Consider the relative topology.*

Proof. By Theorem 2.33 from the textbook, a subset $K \subseteq \mathbb{Q}$ is compact in $\mathbb{Q}$ if and only if $K \subseteq \mathbb{R}$ is compact. But $\mathbb{Q}$ has empty interior in $\mathbb{R}$, so K must also have empty interior in $\mathbb{R}$. Thus any open subset of K with respect to $\mathbb{R}$ must be $\varnothing$. Furthermore, by the relative topology, open sets U of $\mathbb{Q}$ must be equal equal to $V \cap \mathbb{Q}$ for some open set V of $\mathbb{R}$. We conclude that the only open subset of K in $\mathbb{Q}$ is $\varnothing$. $\square$

Problem 4. Let F and K be nonempty closed subsets of the metric space X with $K \cap F = \varnothing$. Show that if K is compact there is a positive distance from F to K, i.e.

$$\inf\{d(x,y) \mid x \in F, y \in K\} = \delta > 0.$$

Is is still true if K is only assumed to be closed? If not find a counterexample.

Proof. Assume for the sake of contradiction that $\inf\{d(x,y) \mid x \in K, y \in F\} = 0$. Then there is a squence of pairs $a_n \in K$ and $b_n \in F$ such that $\lim_{n\to\infty} |a_n - b_n| = 0$. Since K is compact, by Theorem 2.37 in the textbook, there is a limit point a of $\{a_n\}_{n=1}^{\infty}$. This implies that $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence $\lim_{n\to\infty} a_{m_n} = a$.

Now for any $\epsilon > 0$, choose large enough N such that $|a_N - b_N| < \epsilon/2$ and $|a - a_N| < \epsilon/2$ (with a_N is the subsequence). The triangle inequality then shows:

$$|a - b_N| < |a - a_N| + |a_N - b_N| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus for any $n \in \mathbb{N}$, b_n will have some neighbourhood which contains $a \in K$. Hence a is a limit point of $\{b_n\}_{n=1}^{\infty}$. But F is closed implies $a \in F$, contradicting our assumption that $F \cap K = \varnothing$. And we're done. $\square$

If K is not closed, then the claim does not hold in general. For example, take $X = \mathbb{Q}$. Then the relative open sets $A = [0, \sqrt{2}] \cap \mathbb{Q}$ and $B = [\sqrt{2}, 2] \cap \mathbb{Q}$ are closed, disjoint in $\mathbb{Q}$, and yet $d(A, B) = 0$ since we can find subsequences that converge on both sides of $\sqrt{2}$.

Problem 5. A *base* for a topological space X is a collection $\{V_\alpha | \alpha \in A\}$ of open subsets of X such that for every open subset of $G \subseteq X$, one has $G = \bigcup_{\alpha \in \mathcal{B}} V_\alpha$ where $\mathcal{B} \subseteq \mathcal{A}$.

Prove that every compact metric space X has a countable base.

First we prove a lemma that makes things slightly easier:

Lemma. *Let X be a topological space. Let $\{V_\alpha \mid a \in \mathcal{A}\}$ be a collection of open sets and let $U \subseteq X$ be open. If for any $x \in U$, there exists some V_{α_x} such that $x \in V_{\alpha_x} \subseteq U$, then there exists be $\mathcal{B} \subseteq \mathcal{A}$ such that $U = \bigcup_{\alpha \in B} V_\alpha$.*

Proof. Consider $\mathcal{B} = \{\alpha_x \mid x \in U\}$. Then clearly $\mathcal{B} \subseteq \mathcal{A}$. Furthermore, the assumption $V_{\alpha_x} \subseteq U$ implies that $\bigcup_{\alpha_x \in \mathcal{B}} V_{\alpha_x} \subseteq U$. For the other inclusion, every $x \in U$ is in V_{α_x}, so $x \in \bigcup_{\alpha_x \in \mathcal{B}} V_{\alpha_x}$. Hence we also have $U \subseteq \bigcup_{\alpha_x \in \mathcal{B}} V_{\alpha_x}$. Thus we have the equality $U = \bigcup_{\alpha_x \in \mathcal{B}} V_{\alpha_x}$, as desired. □

Now we proceed to the actual proof:

Proof. For each $q > 0 \in \mathbb{Q}$, consider the collection of subsets $\mathcal{C}_q = \{B_q(x) \mid x \in X\}$. For each q, this clearly defines an open cover of X, so we may construct a finite subcover $\mathcal{D}_q = \{B_{q_1}(x_1), B_{q_2}(x_2), \ldots, B_{q_n}(x_n)\}$. In particular, we have a *countable* amount *finite* covers, so their union is countable. Given this, we claim that

$$\bigcup_{q \in \mathbb{Q}} \mathcal{D}_q$$

forms a countable base of X.

Indeed, let $U \subseteq X$ be a open set. Applying the lemma, we want to show that for any $x \in U$, there is some $V_x \in \bigcup_{q \in \mathbb{Q}} \mathcal{D}_q$ such that $x \in V_x \subseteq U$. Because U is open, there is some neighbourhood $B_r(x)$ contained in U. Now choose rational $q < r/2$. Since $\mathcal{D}_q$ covers X, there is some $V_x = B_q(y) \in \mathcal{D}_q$ such that $x \in V_x$. By choosing $q < r/2$, we are also guaranteed that $V_x \subseteq B_r(x)$. (This can be seen through the inequality $\forall z \in V_x, |z - x| \leq |z - y| + |y - x| < r/2 + r/2 = r$.)

Hence $x \in V_x \subseteq U$, which is exactly what we need. Thus we can write U as a union of elements of $\bigcup_{q \in \mathbb{Q}} \mathcal{D}_q$, and we conclude that $\bigcup_{q \in \mathbb{Q}} \mathcal{D}_q$ is indeed a countable base of X. □