Problem 6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and E' have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points.

Proof. E' is closed. By definition E' is closed if and only if E'', the limit points of E', are conained within E'. So we need to show that every limit point of E' is a limit point of E.

Let x be a limit point of E'. Every neighbourhood U_x around x then contains some $y \in E'$. Since U_x is open, we can find some neighbourhood U_y of y such that $U_y \subseteq U_x$. Because y is a limit point of E, we also know the U_y contains points of E. Hence U_x contains points of E. This holds for any neighbourhood U_x, and thus $x \in E'$, as desired. ☐

Proof. E and $\overline{E}$ have the same limit points. We prove both inclusions.

Since $E \subseteq \overline{E}$, every neighbourhood U_x of a limit point $x \in E$ will intersect at least E. Hence $E' \subseteq \overline{E}'$.

For the other inclusion, let x be a limit point of $\overline{E}$ and U_x be any neighbourhood of x. Then U_x intersects $\overline{E}$ at some $y \neq x$. Now either $y \in E$ or $y \in E'$, since $\overline{E} = E \cup E'$. In the first case, we deduce that U_x intersects E. In the second case, we know that y is a limit point of E, so any neighbourhood U_y around y intersects E. We can choose U_y to be contained in U_x and *not* contain x by making it sufficiently small. (Since U_x is open and $y \neq x$.) Hence U_x contains U_y and therefore some point of E which is not x. In both cases, U_x intersects E. This holds for any U_x, so x must be a limit point of E.

Hence the proof is complete. ☐

Proof. E and E' do not always have the same limit points. Let $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$ is the limit points of E, but E' clearly doesn't have any limit points itself. ☐

Problem 9. Let E° denote the set of all interior points in a set E.

(a) Prove that E° is always open

Proof. For every $x \in E^\circ$, choose some neighbourhood U_x. We claim that $\bigcup_{x \in E^\circ} U_x = U = E^\circ$. Indeed, clearly U contains every point of E° so $E^\circ \subseteq U$. At the same time, every U_x is contained in E°, so $U \subseteq E^\circ$. Now U is a union of open sets, so it is open. Hence $E^\circ = U$ is open. ☐

(b) Prove that E is open if and only if $E^\circ = E$.

Proof. ($\Rightarrow$): If $E = E^\circ$, then part (a) shows that E is open.

($\Leftarrow$): If E is open, then for every $x \in E$ and U_x a neighbourhood of x, $U_x \subseteq E$. Hence $x \in E^\circ$ and $E = E^\circ$. ☐

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Proof. Consider G°. If $x \in G^\circ$, then there is a neighbourhood $U_x \subseteq G \subseteq E$. Hence $x \in E^\circ$ and $G^\circ \subseteq E^\circ$. But G is open, so part (b) shows that $G = G^\circ$, and thus $G \subseteq E^\circ$. $\square$

(d) Prove that the complement of E° is the closure of the complement of E.

Proof. We want to show that $(E^\circ)^c = \overline{E^c}$ by proving both inclusions.

First, $(E^\circ)^c \subseteq \overline{E^c}$. For any $x \in (E^\circ)^c$, either x is in E^c or $E \setminus E^\circ$. In the first case, clearly $x \in \overline{E^c}$. In the second case, we know that x is not in the interior of E, so any open neighbourhood U_x contains some point y of E^c. We can see that $y \neq x$ since $x \in E$ while $y \in E^c$. Therefore x is a limit point of E^c, hence $x \in \overline{E^c}$.

For the other inclusion, since $E^\circ \subseteq E$, we have $E^c \subseteq (E^\circ)^c$. Hence $\overline{E^c} \subseteq \overline{(E^\circ)^c} = (E^\circ)^c$. The last equality comes from the fact that, since E° is open, $(E^\circ)^c$ must be closed. This completes the proof. $\square$

(e) Do E and $\overline{E}$ always have the same interiors?

No. Let $E = \mathbb{Q} \subseteq \mathbb{R}$. Then $E^\circ = \varnothing$, while $\overline{E}^\circ = \mathbb{R}^\circ = \mathbb{R}$.

(f) Do E and E° always have the same closures?

No. Let $E = \mathbb{Q} \subseteq \mathbb{R}$. Then $E^\circ = \varnothing$. Hence $\overline{E} = \mathbb{R} \neq \overline{E^\circ} = \varnothing$.

Problem 10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. We first prove the $d(p,q)$ is a metric. By definition $d(p,q) = 0$ if and only if $p = q$. Clearly $d(p,q) = d(q,p)$. Finally, we can prove $d(p,q) + d(q,r) \geq d(p,r)$ with some casework.

If the LHS is 0, then it must be that $d(p,q) = d(q,r) = 0$, implying that $p = q = r = 0$. Hence the RHS is also 0 and inequality holds.

If the LHS is 1, then one term is 1 while the other is zero. Without loss of generality assume $d(p,q) = 1$ and $d(q,r) = 0$. Then $p \neq q$ and $q = r$ implies $p \neq r$. Thus $d(p,r) = 1$ and inequality holds.

If the LHS is 2, then inequality always holds since $2 > 1 \geq d(p,r)$.

We conclude that $d(p,q)$ is indeed a metric. $\square$

Proposition. *The open and closed sets in this metric are $\mathcal{P}(X)$, i.e. the indiscrete toplogy.*

Proof. For any p, $N_r(p) = \{q : d(p,q) < r\} = \{p\}$. If $r < 1$, then only $d(p,p) = 0 < 1$, so $N_r(p) = \{p\}$. So every point is an open set. Then we can construct every subset of X by taking suitable unions of the points. Hence all subsets are open. At the same time, every subset is a complement of another, so every subset is complement to a open set. Hence every subset is also closed. □

Proposition. *A subset A of X is compact if and only if A is finite.*

Proof. ($\Rightarrow$): Let $\mathcal{C}$ be a cover of A. If A is finite, then for each $x \in A$ choose some $U_x \in \mathcal{C}$ containing x. Then $\bigcup_{x\in A} U_x$ is a finite union coving A.

($\Leftarrow$): If A is compact then the cover $\mathcal{C}$ made of $\{x\}$ for every $x \in A$ has size equal to A. The only subcover of $\mathcal{C}$ is $\mathcal{C}$ itself, so $\mathcal{C}$ must be finite. Hence A is finite. □