

Chapter 5, #1, 2, 3 and 26

Problem 1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is constant.

Proof. Without loss of generality assume that $x \geq y$ and $x = y + \delta$. Then we may rewrite the given equation as

$$\frac{|f(y + \delta) - f(y)|}{\delta} \leq \delta.$$

Then for any y , as $\delta \rightarrow 0$, we have $\lim_{\delta \rightarrow 0} |(f(y + \delta) - f(y))/\delta| \leq 0$. Thus $f'(y)$ is defined and equal to zero. Theorem 5.11 gives $f'(x) = 0$ implies f is constant. \square

Problem 2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing on (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof. For any $x, y \in (a, b)$, $x > y$, apply MVT to see that $f(y) - f(x) = (y - x)f'(c)$ for some $x < c < y$. Both $y - x > 0$ and $f'(c) > 0$, so the RHS is positive; hence $f(y) > f(x)$. This shows that f is strictly increasing.

We \square

Problem 3. Suppose that g is a real function on \mathbb{R}^1 , with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is injective if ε is small enough. (A set of admissible values of ε can be determined which depends only on M .)

Problem 26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.