Homework Midterm 2

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Problem 1. Determine the convergence or divergence in (a)-(c).

(a)
$$\sum_{k=1}^{\infty}(-1)^k \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}$$

(b)
$$\sum_{k=1}^{\infty} \frac{k!}{(k+2)!}$$

(c)
$$\sum_{k=1}^{\infty} \frac{2^k}{k^k}$$

Proof. We do each separately:

(a) **TODO**

(b) **TODO**

(c) **TODO**

□

Problem 2. Consider the power series $\sum a_n z^n$ and assume that the coefficients a_n are integers, infinitely many of which are not zero. Prove that the radius of convergence $R \leq 1$.

Proof. **TODO:** radius of convergence □

Problem 3. Consider a function $f : M \to \mathbb{R}$. It's graph is the set,

$$G(f) = \{(x, y) \in M \times \mathbb{R} \mid y = f(x)\}.$$

(a) Prove that if f is continuous then $G(f)$ is closed as a subset of $M \times \mathbb{R}$.

(b) Prove that if f is continuous and M is compact then $G(f)$ is compact.

(c) Prove that if $G(f)$ is compact then f is continuous.

Proof. We do each part separately:

(a) **TODO:** $\Rightarrow$

(b) **TODO:** $\Rightarrow$

(c) **TODO:** $\Rightarrow$

$\square$

Problem 4. Let $I = [0, 1]$ and let $F : I \to I$ be continuous. Prove that F has at least one fixed point. Quoting a fixed points theorem is not acceptable.

Proof. **TODO:** fixed point, no thm $\square$

Problem 5. Let X and Y be metric spaces and $F : X \to Y$ be a continuous mapping onto Y. If D is a dense subset of X, prove that $F(D)$ is a dense subset of Y.

Proof. **TODO:** dense $\Rightarrow$ dense $\square$

EXTRA CREDIT (10 POINTS):

Problem 6. Prove the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}.$$

Proof. **TODO** $\square$