Midterm 2

Hanting Zhang

Problem 1. Determine the convergence or divergence in (a)-(c).

(a)
$$\sum_{k=1}^{\infty}(-1)^k\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}$$

(b)
$$\sum_{k=1}^{\infty}\frac{k!}{(k+2)!}$$

(c)
$$\sum_{k=1}^{\infty}\frac{2^k}{k^k}$$

Proof. We do each separately:

(a) **Converges.** We can simplify the terms with the difference of squares by multiplying the top and bottom by $\sqrt{k+1}+\sqrt{k}$:

$$\sum_{k=1}^{\infty}(-1)^k\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}\cdot\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}}=\sum_{k=1}^{\infty}\frac{(-1)^k}{\sqrt{k}(\sqrt{k+1}+\sqrt{k})}.$$

The absolute magnitudes of the terms $1/\sqrt{k}(\sqrt{k+1}+\sqrt{k})$ are monotonically decreasing, since $\sqrt{k}(\sqrt{k+1}+\sqrt{k})$ is monotonically increasing. Furthermore, $1/\sqrt{k}(\sqrt{k+1}+\sqrt{k})\to 0$ as $k\to\infty$. Hence we can apply the alternating series test to conclude that

$$\sum_{k=1}^{\infty}(-1)^k\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}$$

converges.

(b) **Converges.** We can bound each term:

$$\frac{k!}{(k+2)!}=\frac{1}{(k+1)(k+2)}\leq\frac{1}{k^2}.$$

Now we know $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ converges, so the comparsion test gives the convergence of $\sum_{k=1}^{\infty}\frac{2^k}{k^k}$.

(c) **Converges.** For $k \geq 3$,

$$\frac{2^k}{k^k} = \left(\frac{2}{k}\right)^k \leq \left(\frac{2}{3}\right)^k.$$

Then

$$\sum_{k=1}^{\infty} \frac{2^k}{k^k} = \frac{2}{1} + \frac{2^2}{2^2} + \sum_{k=3}^{\infty} \frac{2^k}{k^k} \leq 2 + 1 + \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k.$$

The RHS is finite amount plus a geometric series with $r < 1$, therefore it converges. Thus $\sum_{k=1}^{\infty} \frac{2^k}{k^k}$ converges.

□

Problem 2. Consider the power series $\sum a_n z^n$ and assume that the coefficients a_n are integers, infinitely many of which are not zero. Prove that the radius of convergence $R \leq 1$.

Proof. Recall the definition of R given in Theorem 3.39:

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{|a_n|}, \qquad R = \frac{1}{\alpha}.$$

Only finitely many of the a_n can be zero, so there is some $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n| \geq 1$. Thus we have $\sqrt[n]{|a_n|} \geq 1 \Rightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$. This exactly means that $R = 1/\alpha \leq 1$, as desired. □

Problem 3. Consider a function $f : M \to \mathbb{R}$. It's graph is the set,

$$G(f) = \{(x, y) \in M \times \mathbb{R} \mid y = f(x)\}.$$

(a) Prove that if f is continuous then $G(f)$ is closed as a subset of $M \times \mathbb{R}$.

(b) Prove that if f is continuous and M is compact then $G(f)$ is compact.

(c) Prove that if $G(f)$ is compact then f is continuous.

Proof. We do each part separately:

(a) If f is continuous, then any convergent sequence $x_n \to x$ has a convergent image under f, i.e. $f(x_n) \to f(x)$. Thus $(x_n, f(x_n))$ converges to $(x, f(x))$ in the product space. Thus any subseqeuence of $G(f)$ converges in $G(f)$, which implies that $G(f)$ is closed.

(b) Define $F : M \to M \times \mathbb{R}$ by mapping $x \mapsto (x, f(x))$, i.e. F is the pair (id_M, f). Both id_M and f are continuous, so F is continuous as well. The image of F is $G(f)$; thus we may invoke Theorem 4.14 to give that $F(M) = G(f)$ is compact.

(c) We prove the contrapositive. Suppose f is not continuous. Then there exist points $x \in M$ and $\varepsilon > 0$ such that there is a sequence $x_n \to x$ with $|f(x_n) - f(x)| \geq \varepsilon$. Now consider the sequence $\{(x_n, f(x_n))\} \subseteq G(f)$ obtained by mapping $\{x_n\}$ under f. Because $G(f)$ is compact, we have a convergent subsequence $\{(x_{n_i}, f(x_{n_i}))\}$ with limit (x, y). We know that $y \neq f(x)$ by construction and that $G(f)$ must contain a unique pair of the form $(x, *)$ (namely $(x, f(x))$). Thus $(x, y) \notin G(f)$, implying that $G(f)$ is not compact, as desired.

□

Problem 4. Let $I = [0, 1]$ and let $F : I \to I$ be continuous. Prove that F has at least one fixed point. Quoting a fixed points theorem is not acceptable.

Proof. Extend the codomain of F to $\mathbb{R}$ and consider the map $f(x) = F(x) - x$. We have the bounds $0 \leq F(0) - 0 = F(0) \leq 1$ and $-1 \leq F(1) - 1 \leq 0$. Thus the interval $[f(0), f(1)]$ contains the point 0. The continuity of $F(x)$ implies the continuity of $f(x)$; the intermediate value theorem guarentees the existence of $x_0 \in [0, 1]$ such that $f(x_0) = 0$. Thus $F(x_0) = x_0$ and x_0 is a fixed point. □

Problem 5. Let X and Y be metric spaces and $F : X \to Y$ be a continuous mapping onto Y. If D is a dense subset of X, prove that $F(D)$ is a dense subset of Y.

Proof. Let $y \in F(X)$. If $y \in F(D)$, then there is nothing to prove. Otherwise, if $y \in F(X) \setminus F(D)$, then it suffices to show that y is a limit point of $F(D)$. Let $x \in F^{-1}(y)$. Since D is dense in X, x is a limit point of X. Thus there is some sequence $x_n \to x$. The continuity of F implies that $\lim_{x_n \to x} F(x_n) = y$, so $F(x_n) = y_n \to y$. For each n, $F(x_n) \in F(D)$, so y is a limit point of $F(D)$.

Now y was arbitrary; we have $F(D)$ is dense in $F(X)$. □

EXTRA CREDIT (10 POINTS):
Problem 6. Prove the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}.$$

Proof. We begin by bounding the partial sums $\sum_{n=1}^{N} \sin n$.

Consider $S_N = \sum_{n=1}^{N} \cos n + i \sin n$. Then $\mathrm{Im}(S_N) = \sum_{n=1}^{\infty} \sin n$ is the quantity we need.

Now $|\mathrm{Im}(z)| \leq |z|$ for any $z \in \mathbb{C}$, so we have

$$|\mathrm{Im}(S_N)| \leq |S_N| \leq \left|\sum_{n=1}^{N} e^{in}\right| = \left|e^i \frac{1 - e^{iN}}{1 - e^i}\right|,$$

where the last equality follows from the formula for a geometric series. We can bound $|1 - e^{iN}| \leq 2$, so $|\mathrm{Im}(S_N)| \leq 2/|1 - e^i| < \infty$.

We can apply Theorem 3.42 with $a_n = \sin n$ and $b_n = 1/n$. We have the three needed conditions: (a) S_N is bounded, (b) $1/n$ is decreasing, and (c) $\lim_{n\to\infty} 1/n = 0$. Thus $\sum_{n=1}^{\infty} \sin n/n$ converges. $\square$