Problem 1.1. Prove Proposition 1.5. In the right-most expression in (52), interpret $\inf(\varnothing) = \infty$ when necessary, or equivalently, the the infimum over $C \geq 0$ belonging to $\overline{\mathbb{R}}$. Caution: You are not guaranteed the existence of an $x \in X$ such that $\|T_x\|_Y = \|T\|_{X \to Y}\|x\|_X$.

Proof. Let's deal with each equality, starting with the middle (2) = (3). We have

$$\{\|Tx\|_Y : \|x\|_X = 1\} \subseteq \{\|Tx\|_Y : \|x\|_X \leq 1\} \Rightarrow \sup_{\|x\|=1} \|Tx\|_Y \leq \sup_{\|x\|\leq 1} \|Tx\|_Y.$$

But also, by linearity, $\|Tx_1\| \leq \|Tx_2\|$ for all $\|x_1\| \leq \|x_2\|$, and for all x_1 such that $\|x_2\| \leq 1$, there is some x_2 such that $\|x_1\| \leq \|x_2\| = 1$. Thus also $\sup_{\|x\|\leq 1} \|Tx\|_Y \leq \sup_{\|x\|=1} \|Tx\|_Y$, and we have $\sup_{\|x\|=1} \|Tx\|_Y = \sup_{\|x\|\leq 1} \|Tx\|_Y$.

Now (3) = (4). We have by definition of sup / inf:

$$\sup_{\|x\|=1} \|Tx\|_Y = \inf\{C : \|Tx\|_Y \leq C, \|x\| = 1\}.$$

Consider the map ψ such that $x \mapsto x/\|x\|_X$. For all $\|x\| = 1$, the union preimages $\psi^{-1}(x)$ is the entire space X. Furthermore, if $\|Tx\|_Y \leq C$ for $\|x\| = 1$, then for every point w in the preimage, it holds that $\|Tw\| \leq C\|w\|$, by linearity. Thus we may write:

$$\begin{aligned}\sup_{\|x\|=1} \|Tx\|_Y &= \inf\{C : \|Tx\|_Y \leq C, \|x\| = 1\} \\ &= \inf\{C : \|Tw\|_Y \leq C\|w\|, w \in X\},\end{aligned}$$

as desired. Finally, the right side. We have by definition of sup / inf:

$$\sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf\{C : \frac{\|Tx\|_Y}{\|x\|_X} \leq C, x \neq 0\}.$$

A little rearranging gives (note we can add in $x = 0$ since it doesn't affect the answer):

$$\begin{aligned}\sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} &= \inf\{C : \frac{\|Tx\|_Y}{\|x\|_X} \leq C, x \neq 0\} \\ &= \inf\{C : \|Tx\|_Y \leq C\|x\|_X, x \in X\},\end{aligned}$$

as desired. $\square$

Problem 1.2. Prove Proposition 1.9. For convenience, a "checklist" is provided below.

(a) Start with a Cauchy sequence $(T_n)_{n=1}^{\infty}$ in $(\mathcal{B}(X,Y), \|\cdot\|_{X\to Y})$.

(b) Find a candidate $T : X \to Y$ for the limit. (Use the completeness of $(Y, \|\cdot\|_Y)$.)

(c) Prove that T is linear and continuous.

(d) Prove that $\lim_{n\to\infty} \|T_n - T\|_{X\to Y} = 0$, and finish the argument.

Proof. We proceed with the steps given: Let $(T_n)_{n=1}^{\infty}$ be a Cauchy in $(\mathcal{B}(X,Y), \|\cdot\|_{X\to Y})$ and $\varepsilon > 0$. Then there exists N such that for all $n, m > N$, we have $\|T_n - T_m\|_{X\to Y} < \varepsilon$. This implies for all $x \in X$, we have $\|T_n(x) - T_m(x)\|_Y < \varepsilon$. Thus $(T_n(x))_{n=1}^{\infty}$ is shown to also be Cauchy; and knowing that $(Y, \|\cdot\|_Y)$ is complete, we must have $T_n(x) \to T_x$ for all $x \in X$. Then define $T : X \to Y$ to be $x \mapsto T_x$. Indeed, T is linear and continuous. Let $x_1, x_2 \in X$ and $k \in F$. We have

$$\begin{aligned}
T(x_1 + x_2) &= \lim_{n\to\infty} (T_n(x_1 + x_2)) \\
&= \lim_{n\to\infty} (T_n(x_1) + T_n(x_2)) = \lim_{n\to\infty} T_n(x_1) + \lim_{n\to\infty} T_n(x_2) \\
&= T(x_1) + T(x_2)
\end{aligned}$$

and

$$\begin{aligned}
T(kx_1) &= \lim_{n\to\infty} (T_n(kx_1)) \\
&= \lim_{n\to\infty} (kT_n(x_1)) = k \lim_{n\to\infty} T_n(x_1) \\
&= kT(x_1).
\end{aligned}$$

For continuity, it suffices to show that T is bounded. We know that all the T_ns are bounded uniformly by some K. Then for all $x \in X$, we abuse the limit to conlude:

$$\|T(x)\| = \left\| \lim_{n\to\infty} T_n(x) \right\| \leq K\|x\|.$$

Thus $T \in \mathcal{B}(X,Y)$. Finally, we must check that actually $T_n \to T$ in the $\|\cdot\|_{X\to Y}$ norm. Since $T_n(x) \to T$, we have $\|T_n(x) - T\| < \varepsilon$. Then $\|T_n - T\| = \sup_{\|x\|=1} \|T_n(x) - T(x)\| < \varepsilon$, as desired. $\square$

Problem 1.3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be finite-dimensional normed F-vector spaced, with $F = \mathbb{R}$ or $\mathbb{C}$. Give an explanation (as concise as possible) for why any linear bijection $T : X \to Y$ is automatically a normed vector space isomorphism.

Proof. Construct the space $(Y, \|\cdot\|_Y)$ by $\|y\|_Y = \|\psi^{-1}y\|_X$. Then X and Y are isomorphic as normed vector spaces by construction. Recall that all norms on finite dimensional F-vector spaces are equivalent. Thus we have the isomorphism

$$(X, \|\cdot\|_X) \xrightarrow{\sim} (Y, \|\cdot\|_Y) \xrightarrow{\sim} (Y, \|\cdot\|),$$

where the second map is id_Y, but converts the topology. $\square$