Problem 25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0, f(b) > 0, f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

Essentially, draw the tangent line at $f(x_n)$ and look at its intersection with the x-axis. That is the value of x_{n+1}.

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi.$$

Proof. We first show that $x_n > \xi$ for all n by induction. The base case $n = 1$ is assumed by $x_1 \in (\xi, \beta)$. Now let $x_n > \xi$. By IVT we have $f(x_n) - f(x) = f'(z)(x_n - x)$ for some $z \in (x, x_n)$. The function $f'(x)$ is increasing since $f''(x) \geq 0$, so

$$f'(z) \leq f'(x_n) \Rightarrow \frac{f(x_n) - f(x)}{x_n - x} \leq f'(x_0).$$

Rearranging gives

$$f(x_n) + f'(x_0)(x - x_n) \leq f(x).$$

In words, this just means that the tangent line at $f(x_n)$ is less than $f(x)$, since $f(x)$ is convex. In particular, when the LHS is zero, then $x = x_{n+1}$. Thus $0 = f(\xi) < f(x_{n+1}) \Rightarrow \xi < x_{n+1}$, which completes the induction as desired. (Note we drop the equality on the $\leq$ since $f(x_{n+1})$ cannot be equal to zero.)

As $x_n > \xi$ for all n, we know that $f(x_n) > 0$. Thus $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$, as desired.

To see that $\lim_{n\to\infty} x_n = \xi$, note that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)}.$$

Thus we deduce that

$$\lim_{n\to\infty} \frac{f(x_n)}{f'(x_n)} \Longrightarrow \lim_{n\to\infty} f(x_n) = 0.$$

The continuity of f means that $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = 0$. Thus we finally have $\lim_{n\to\infty} x_n = \xi$. $\square$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. By Taylor's theorem, there is a point $t \in (\xi, x_n)$ such that

$$0 = f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(x_n)}{2}(\xi - x_n)^2.$$

Dividing by $f'(x_n)$ gives

$$0 = \frac{f(x_n)}{f'(x_n)} + (\xi - x_n) + \frac{f''(x_n)}{2f'(x)}(\xi - x_n)^2,$$

thus by the definition of x_{n+1},

$$\xi - x_{n+1} + \frac{f''(x_n)}{2f'(x)}(\xi - x_n)^2 = 0.$$

Hence

$$x_{n+1} - \xi = \frac{f''(x_n)}{2f'(x)}(\xi - x_n)^2.$$

□

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chapter 3.)

Proof. Given our result from part (c) and given that $f'(x) \geq \delta > 0$ and $0 \leq f''(x) \leq M$, we have

$$0 \leq s_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \leq \frac{M}{2\delta}(x_n - \xi)^2 = A(x_n - \xi)^2.$$

Now we proceed by induction. We can verify that the base case $n = 1$ holds:

$$0 \leq x_2 - \xi \leq A(x_1 2 - \xi)^2 = \frac{1}{A}[A(x_1 - \xi)]^2 = \frac{1}{A}[A(x_1 - \xi)]^{2^1}.$$

Assume for $k \leq n$ that the result holds for k. Then applying part (c) shows that

$$0 \leq x_{k+2} - \xi \leq A(x_{k+1} - \xi)^2 \leq A\left[\frac{1}{A}[A(x_1 - \xi)]^{2^n}\right]^2 = \frac{1}{A}[A(x_1 - \xi)]^{2^{n+1}}.$$

Hence the induction is complete and the result holds for all n.

To compare this with problems 16 and 18, problem 18 is a special case of this exercise with $f(x) = x^p - \alpha$. Then problem 16 is an further specialization to $p = 2$. □

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

Proof. Any x_0 is a fixed point of $g(x)$ if and only if $f(x_0) = 0$, which happens if and only if $x_0 = \xi$. Thus ξ is the unique fixed point of $g(x)$.

Now with the quotient rule, we have

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} - \frac{f(x)f''(x)}{[f'(x)]^2},$$

which implies that $g'(x) \to 0$ as $x \to \xi$. □

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

We compute $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$, neither of which are bounded above or below. So we expect Newton's formula to fail.

Indeed,

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

Thus for any x_1, we have the sequence $x_n = (-2)^{n-1}x_1$, which clearly blows up unless $x_1 = 0$.

I hope this is the correct formulation:
Problem 26. Suppose $\mathbf{r} : [a, b] \to \mathbb{R}^k$ is differentiable such that $\mathbf{r}(a) = 0$ and there is a real number A such that

$$|\mathbf{r}'(x)| \leq A|\mathbf{r}(x)|$$

on $[a, b]$. Then $\mathbf{r}(x) = 0$ for all $x \in [a, b]$.

Proof. This proof is largely the same as the one-dimensional version. Let $M_0 = \sup |\mathbf{r}(x)|$ and $M_1 = \sup |\mathbf{r}'(x)|$ for $x \in [a, b]$. Here $|\mathbf{r}(\cdot)|$ is the vector norm on $\mathbb{R}^k$. For any x, we have $|f(x)| \leq M_1(b-a) \leq A(b-a)M_0$. We deduce that $\sup |\mathbf{r}(x)| = M_0 \leq A(b-a)M_0$. Hence $M_0 = 0$ if $A(b-a) < 1$. So choose $A < 1/(b-a)$; then $\mathbf{r} = \mathbf{0}$. $\square$