Problem 2.1. This exercise is about the formulas (28) and (29) in the definition of trigonometric polynomials.

(a) Show that two formulas (28) and (29) are equivalent ways to define the space of complex valued trigonometric functions. That is, given a trigonometric polynomial of the form (28), find the numbers a_n, b_n so that the same polynomial can be written in the form (29). Then go the other direction – give a formula for the c_n's in terms of the a_n's and b_n's.

(b) Using part (a), formulate a condition on the coefficients c_n so that (28) defined a real-valued function. Your condition should be as general as possible.

Proof. We proceed with each part separately.

(a) We have $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$, so

$$\begin{aligned}
p(\theta) &= \sum_{n=-N}^{N} c_n e^{in\theta} = \sum_{n=-N}^{N} c_n \cos(n\theta) + c_n i \sin(n\theta) \\
&= c_0 + \sum_{n=1}^{N} c_n \cos(n\theta) + c_{-n}\cos(-n\theta) + c_n i \sin(n\theta) + c_{-n} i \sin(-n\theta) \\
&= c_0 + \sum_{n=1}^{N} (c_n + c_{-n})\cos(n\theta) + (c_n - c_{-n}) i \sin(n\theta).
\end{aligned}$$

Thus $a_0 = c_0$, $a_n = c_n + c_{-n}$, and $b_n = i(c_n - c_{-n})$.

On the other hand, we have $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$$\begin{aligned}
p(\theta) &= a_0 + \sum_{n=1}^{N} a_n \cos(n\theta) + b_n \sin(n\theta) \\
&= a_0 + \sum_{n=1}^{N} \frac{a_n}{2}(e^{in\theta} + e^{-in\theta}) + \frac{b_n}{2i}(e^{in\theta} - e^{-in\theta}) \\
&= a_0 + \sum_{n=1}^{N} \frac{a_n - ib_n}{2} e^{in\theta} + \frac{a_n + ib_n}{2} e^{-in\theta} \\
&= a_0 + \sum_{n=1}^{N} \frac{a_n - ib_n}{2} e^{in\theta} + \sum_{n=-1}^{-N} \frac{a_{-n} + ib_{-n}}{2} e^{in\theta}.
\end{aligned}$$

Thus we have $c_0 = a_0$, $c_n = \frac{1}{2}(a_n - ib_n)$ for $1 \le n \le N$, and $c_n = \frac{1}{2}(a_{-n} + ib_{-n})$ for

$-N \leq n \leq -1$.

(b) From part (a), for $p(\theta)$ to be real, we need a_n and b_n to be real, thus $c_n + c_{-n}$ and $i(c_n - c_{-n})$ must both be real. The first condition implies that $\text{Im}(c_n) = -\text{Im}(c_{-n})$; the second implies that $\text{Re}(c_n) = \text{Re}(c_{-n})$. Thus we need exactly $c_n = \overline{c_{-n}}$ for $0 \leq n \leq N$.

$\square$

Problem 2.2. For $F = \mathbb{R}$ or $\mathbb{C}$, prove that the space $P(S^1; F)$ defined in Section 2.5 is an F-algebra that separates points and vanishes at no point of S^1. If $F = \mathbb{C}$, also show that $P(S^1; F)$ is self-adjoint.

Proof. The sums of polynomials are clearly polynomials. The products of polynomials are clearly polynomials. The scalar product of polynomials are clearly polynomials. Thus $P(S^1; F)$ is an F-algebra. It also separates points with the element z and vanishes nowhere with the element 1.

Suppose $F = \mathbb{C}$ and $p(z) \in P(S^1; \mathbb{C})$. Note that for $z \in S^1$, we have $1/z = \overline{z}/\|z\| = \overline{z}$. Thus

$$\overline{p(z)} = \overline{\sum_{n=-N}^{N} c_n z^n} = \sum_{n=-N}^{N} \overline{c_n z^n} = \sum_{n=-N}^{N} \overline{c_n} \frac{1}{z^n} = \sum_{n=-N}^{N} \overline{c_{-n}} z^n \in P(S^1; \mathbb{C}).$$

$\square$

Problem 2.3. Let $P^+(S^1; \mathbb{C})$ be the space of trigonometric polynomials of the form

$$p(z) = \sum_{n=0}^{N} c_n z^n, \quad z \in S^1, N \in \mathbb{N}, c_n \in \mathbb{C}.$$

Prove that $P^+(S^1; \mathbb{C})$ is a complex algebra that separates points and vanishes at no point of S^1, yet $p^+(S^1; \mathbb{C})$ is not dense in $(C(S^1; \mathbb{C}), \|\cdot\|_u)$. Why does this not contradict the (complex) Stone-Weierstrass Theorem?

Proof. The same argument from Exercise 2.2 applies; $P^+(S^1; \mathbb{C})$ is an complex algebra. However, it is not self-adjoint: For all $f(z) = z^n$, $n \geq 0$, we have $\int_{-\pi}^{\pi} f(e^{i\theta}) e^{i\theta} d\theta = \int_{-\pi}^{\pi} e^{i(n+1)\theta} d\theta = 0$. Thus for any $p \in P^+(S^1; \mathbb{C})$,

$$\int_{-\pi}^{\pi} p(e^{i\theta}) e^{i\theta} d\theta = \sum_{n=0}^{N} \int_{-\pi}^{\pi} c_n (e^{i\theta})^n e^{i\theta} d\theta = 0.$$

But if $n = -1$, then $\int_{-\pi}^{\pi} e^{-i\theta} e^{i\theta} d\theta = 2\pi \neq 0$, so $\overline{z} \notin P^+(S^1; \mathbb{C})$, i.e. it is not self-adjoint. $\square$