Problem 1.4. Fix $A > 1$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t + At^2 \sin(1/t) & t \neq 0 \\ 0 & t = 0. \end{cases}$$

The graph of f near t is pictured in Figure 1(B).

(a) Show that f is differentiable everywhere, and compute its derivative (including at zero).

(b) Show that f is not injective on any interval around zero. Hint: Use Exercise 1.3(b), and keep in mind that it's not necessary to exhibit an extremum in order to rigorously prove it exists.

(c) Discuss how (a) and (b) relate to the (single-variable) Inverse Function Theorem (restricting attention to a neighborhood of 0). Which if the hypotheses of the Theorem are satisfied and which are not? In particular, this example demonstrates that one of the hypotheses cannot be removed. Which one?

Proof. We proceed with each part separately:

(a) For $t \neq 0$, the derivative is

$$2A \sin\left(\frac{1}{x}\right) x - A \cos\left(\frac{1}{x}\right) + 1.$$

For the derivative at 0, we have to compute it directly,

$$\begin{aligned} f'(0) &= \lim_{t \to 0} \frac{f(t) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{t + At^2 \sin(1/t) - 0}{t} \\ &= \lim_{t \to 0} 1 + At \sin(1/t) \\ &= 1 + A, \end{aligned}$$

where in the last part, we use the fact that $\lim_{t \to 0} t \sin(1/t) = 1$.

(b) Consider the sequence of points $t_n = 1/\pi n$. We have

$$\begin{aligned} f(t_n) &= 2A \sin(\pi n) \frac{1}{\pi n} - A \cos(\pi n) + 1 \\ &= 0 - A(-1)^n + 1 = A(-1)^{n+1} + 1. \end{aligned}$$

Since $A > 1$, this implies that as $t_n \to 0$, $f'(t_n)$ constantly flips between the positive and negative. Thus for any neighborhood U around 0, we can find sufficiently large n such that $t_n, t_{n+1}, t_{n+2} \in U$, and the sign of $f'(t_n), f'(t_{n+1}), f'(t_{n+2})$ is positive/negative/positive. Since $f'(t)$ is continuous, that means there exist points $M \in [t_n, t_{n+1}]$ and $m \in [t_{n+1}, t_{n+2}]$, such that $f'(M) = f'(m) = 0$, and M and m are the local maxima and minima of f on U, respectively. By Exercise 1.3(b), this shows that f is never injective on U, as desired.

(c) The hypotheses of $f'(x) > 0$ (and the other trivial ones) have been satisfied. The one thing that is missing is $f \in C^1(U; \mathbb{R})$. Subtly, $f(t)$ is not C^1 at $t = 0$, since $f'(t)$ is not continuous at $t = 0$.

□

Problem 3.1. Consider the system of equations

$$\begin{cases} x + y + \sin(xy) = z \\ \sin(x^2 + y) = 2z \end{cases} \qquad x, y, z \in \mathbb{R}.$$

(a) Show that there exists a neighborhood V of 0 in $\mathbb{R}$, a neighborhood W of $(0,0)$ in $\mathbb{R}^2$, and a function $g : V \to W$, $g(z) = (g_1(z), g_2(z))$, such that $(x, y, z) = (g_1(z), g_2(z), z)$ solves the above system for every $z \in V$. Compute $Jg(0)$.

(b) Repeat part (a), with the following change: Instead of writing x and y as a function of z near $(0, 0, 0)$, write z and z as a function of y.

Proof. **TODO** □