Exercises 7, 11, 13, 14, 16, 30, 31 (expect (e)), pp. 256-260.

Problem 7. Let R be a commutative ring with 1. Prove that the principal ideal generated by x in the polynomial ring $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

Proof. **TODO** $\square$

Problem 11. Assume R is commutative. Prove that if P is a prime ideal of R and P contains no zero divisors then R is an integral domain.

Proof. **TODO** $\square$

Problem 13. Let $\varphi : R \to S$ be a homomorphism of commutative rings.

(a) Prove that if P is a prime ideal of S then either $\varphi^{-1}(P) = R$ or $\varphi^{-1}(P)$ is a prime ideal of R. Apply this to the special case when R is a subring of S then $P \cap R$ is either R or a prime ideal of R.

(b) Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R. Give an example to show that this need not be the case if φ is not surjective.

Proof. **TODO** $\square$

Problem 14. **TODO**

Proof. **TODO** $\square$

Problem 16. Let $x^2 - 16$ be an element of the polynomial ring $E = \mathbb{Z}[x]$ and use the bar notation to denote passage to the quotient ring $\mathbb{Z}[x]/(x^3 - 2x + 1)$. Let $p(x) = 2x^7 - 7x^5 + 4x^3 - 9x + 1$ and let $q(x) = (x-1)^4$.

(a) Express each of the following elements of $\overline{E}$ in the form $\overline{f(x)}$ for some polynomial $f(x)$ of degree ≤ 2: $\overline{p(x)}, \overline{q(x)}, \overline{p(x) + q(x)}$, and $\overline{p(x)q(x)}$.

(b) Prove that $\overline{E}$ is not an integral domain.

(c) Prove that $\overline{x}$ is a unit in $\overline{E}$.

Proof. **TODO** $\square$

Problem 30. Let I be an ideal of the commutative ring R and define

$$\text{rad } I = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$$

called the *nilradical* of I. Prove that rad I is an ideal containing I and that (rad $I)/I$ is the nilradical of the quotient ring R/I, i.e. (rad $I/I) = \mathfrak{N}(R/I)$.

Proof. **TODO** □

Problem 31. An ideal I of the commutative ring R is called a *radical ideal* if rad $I = I$.

(a) Prove that every prime ideal of R is a radical ideal.

(b) Let $n > 1$ be an integer. Prove that 0 is a radical ideal in $\mathbb{Z}/n\mathbb{Z}$ if and only if n is a product of distinct primes to the first power (i.e. n is square free). Deduce that (n) is a radical ideal of $\mathbb{Z}$ if and only if n is a product of distinct primes in $\mathbb{Z}$.

Proof. **TODO** □

Exercises 1, 2, 5 pp. 267-269.

Problem 1. An element $e \in R$ is called an *idempotent* if $e^2 = e$. Assume that e is an idempotent in R and $er = re$ for all $r \in R$. Prove that Re and $R(1-e)$ and two-sided ideals of R and that $R \cong Re \times R(1-e)$. Show that e and $1-e$ are identities for the subrings Re and $R(1-e)$ respectively.

Proof. **TODO** □

Problem 2. Let R be a finite Boolean ring with identity $1 \neq 0$. Prove that $R \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$.

Proof. **TODO** □

Problem 5. Let $n_1, n_2, \cdots, n_k$ be integers which are relatively prime in pairs: $(n_i, n_j = 1$ for all $i \neq j$.

(a) Show that the Chinese Remainder Theorem implies that for any $a_1, \cdots, a_k \in \mathbb{Z}$ there is a solution $x \in \mathbb{Z}$ to the simultaneous congruences

$$x \equiv a_1 \text{ mod } n_1, \quad x \equiv a_2 \text{ mod } n_2, \quad \cdots, \quad x \equiv a_k \text{ mod } n_k$$

and that the solution x is unique mod $n = n_1 n_2 \cdots n_k$.

(b) Let $n_i' = n/n_i$ and t_i be the inverse of n_i' mod n_i. Prove that the solution x in (a) is given by

$$x = a_1 t_1 n_1' + a_2 t_2 n_2' + \cdots + a_k t_k n_k' \bmod n.$$

(c) Solve the simultaneous system of congruences

$$x \equiv 1 \bmod 8, \quad x \equiv 2 \bmod 25, \quad x \equiv 3 \bmod 81$$

and

$$y \equiv 5 \bmod 8, \quad y \equiv 12 \bmod 25, \quad y \equiv 47 \bmod 81.$$

Proof. **TODO** □

More to be added...? **TODO**