Problem 1. Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

Proof. **Counterexample**: Let $f(x) = 0$ if $x \neq 0$ and $f(0) = 1$. At every point $x \neq 0$, the limits $\lim_{h\to 0} f(x \pm h)$ are well defined and equal to zero. Hence $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ for all $x \neq 0$. If $x = 0$, then we must have $f(x \pm h) = 0$, and thus $\lim_{h\to 0} f(x \pm h) = 0$.

Then our hypothesis $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ is satisfied for every $x \in \mathbb{R}$, but clearly f is not continuous at 0. $\square$

Problem 2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subseteq \overline{f(E)}$$

for every set $E \subseteq X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof. We want to show that $y \in f(\overline{E}) \Rightarrow y \in \overline{f(E)}$. Indeed, let $y \in f(\overline{E})$. There is $x \in \overline{E}$ such that $f(x) = y$. If x is in E, then $f(x) \in f(E) \subseteq \overline{f(E)}$. Otherwise, x must be a limit point of E. In this case, there must be a sequence $\{x_n\}$ of E that converges to x. By the continuity of f, we have $f(x_n) \to f(x)$. Thus $f(x)$ is a limit point of $f(E)$, and $f(x) \in \overline{f(E)}$.

Hence we conclude that $f(\overline{E}) \subseteq \overline{f(E)}$.

For an example where the inclusion is strict, take $X = \mathbb{Q}$, $Y = \mathbb{R}$, and $E = [0,1] \cap \mathbb{Q} \subseteq \mathbb{Q}$. If f is the embedding $x \mapsto x$, then $\overline{E} = E$, so $f(\overline{E}) = [0,1] \cap \mathbb{Q}$. Meanwhile, $\overline{f(E)} = \overline{[0,1] \cap \mathbb{Q}} = [0,1]$ (since after the map we are now in $\mathbb{R}$), which strictly includes $f(\overline{E})$. $\square$

Problem 3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof. We prove that the complement of $Z(f)$ is open. Let $x \notin Z(f)$ and $f(x) \neq 0$. Choose $0 < \epsilon < f(x)$. The continuity of f allows us to find an $\delta > 0$ such that for any $y \in X$, $d_X(y,x) < \delta$ implies $d_Y(f(y), f(x)) < \epsilon$.

Since $\epsilon < f(x)$, $d_Y(f(y), f(x)) < \epsilon$ implies that $f(y) \neq 0$. Note that $d_X(y,x) < \delta$ is equivalent to $y \in B_x(\delta)$. Hence $y \notin Z(f)$ for every $y \in B_x(\delta)$. Putting everything together, we have

$$x \in B_x(\delta) \subseteq X \setminus Z(f).$$

This holds for all $x \notin Z(f)$, so $X \setminus Z(f)$ is open. Thus $Z(f)$ is closed. $\square$

Problem 4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

Proof. *$f(E)$ is dense.* Let $y \in f(X)$. If $y \in f(E)$, then there is nothing to prove. Otherwise, if $y \in f(X) \setminus f(E)$, then we want to show that y is a limit point of $f(E)$. Let $x \in f^{-1}(y)$. Since E is dense in X, there is some sequence $x_n \to x$. The continuity of f implies that $\lim_{x_n \to x} f(x_n) = y$, so $f(x_n) = y_n \to y$. For each n, $f(x_n) \in f(E)$, so we can conclude that y is a limit point of $f(E)$.

Since y was arbitrary by assumption, we have $f(E)$ is dense in $f(X)$. $\square$

Proof. *$g(p) = f(p)$ for all $p \in X$.* Suppose $p \in X$. If $p \in E$, then there is nothing to prove. Otherwise, assume that $p \in X \setminus E$. By the density of E, there is at least one sequence $x_n \to p$. For any of these sequences, the continuity of f and g guarentee the equalities

$$\lim_{x_n \to p} f(x_n) = f(p) \text{ and } \lim_{x_n \to p} g(x_n) = g(p).$$

But since $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$, we also know that $\lim_{x_n \to p} f(x_n) = \lim_{x_n \to p} g(x_n)$. Thus $f(p) = g(p)$, as desired. $\square$