Homework MIDTERM

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Problem 1. If $x \in \mathbb{R}^k$, $k \geq 2$, show that there exists $y \in \mathbb{R}^k$ such that $x \cdot y = 0$.

Proof. One possible solution is simply $y = 0$; hence trivially $x \cdot y = 0$. But that's not very interesting so I'll give a nontrivial example.

If x is trivial then y can be anything, so assume that x is nontrivial. Then there is at least one index i where $x_i \neq 0$. Since $n \geq 2$, pick some $j \neq i$. Set $y_i = -x_j$ and $y_j = x_i$, and $y_k = 0$ for all $k \neq i$ and $k \neq j$. Hence this defines y as nontrivial. Then

$$x \cdot y = \sum_{k=1}^{n} x_k y_k = x_i y_i + x_j + y_j = -x_i x_j + x_j x_i = 0.$$

□

Problem 2. True or False: If true prove it, if false counterexample it.

(a) Let $\{F_n\}$ be a countable collection of closed subset of $\mathbb{R}$ such that for any finite subcollection

$$F_{n_1} \cap F_{n_2} \cap \cdots \cap F_{n_k} \neq \varnothing.$$

Then

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(b) Add the condition that each F_n is bounded and repost (2a).

(c) Repeat (2a) where closed and bounded $F_n \subseteq X$, and arbitrary metric space.

Proof. Part (a): □

Problem 3. Consider the metric space $\mathbb{Q}$ of all rationals on the real line with the Euclidean metric. Prove that if $K \neq \varnothing$ is a compact subset of $\mathbb{Q}$ then K cannot contain an open subset of $\mathbb{Q}$. *Hint: Consider the relative topology.*

. Let F and K be nonempty closed subsets of the metric space X with $K \cap F = \varnothing$. Show that if K is compact there is a positive distance from F to K, i.e.

$$\inf\{d(x,y) | x \in F, y \in K\} = \delta > 0.$$

Is is still true if K is only assumed to be closed? If not find a counterexample.

Problem 4. A *base* for a topological space X is a collection $\{V_\alpha | \alpha \in A\}$ of open subsets of X such that for every open subset of $G \subseteq X$, one has $G = \bigcup_{\alpha \in \mathcal{B}} V_\alpha$ where $\mathcal{B} \subseteq \mathcal{A}$.

Prove that every compact metric space X has a countable base.