Problem 5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof. Set $a^* = \limsup_{n\to\infty} a_n$, $b^* = \limsup_{n\to\infty} b_n$, and $c^* = \limsup_{n\to\infty}(a_n + b_n)$.

We do not consider the case $a^* = \infty, b^* = -\infty$. If at least one of $a^* = \infty$ or $b^* = \infty$ (or both), then we necessarily have $c^* \leq \infty$.

Otherwise, if $a^* = -\infty$, then $a_n \to -\infty$. Then any subsequence of $\{a_n + b_n\}$ also tends to $-\infty$. Thus $c^* = -\infty \leq -\infty + b^*$. The same argument can be done in the case where $b^* = -\infty$.

Hence we may assume that $a^*, b^* \in \mathbb{R}$. By Theorem 3.17 in the textbook, we have for any $\epsilon > 0$, there are $N_s, N_t \in \mathbb{N}$ such that $n \geq \max(N_s, N_t)$ implies $a_n < a^* + \epsilon/2$ and $b_n < b^* + \epsilon/2$. Thus $a_n + b_n \leq a^* + b^* + \epsilon$, and all subsequential limits of $\{a_n + b_n\}$ are bounded by $a^* + b^* + \epsilon$. By definition $\sup E = c^*$, where E is the set of subsequential limits of $\{a_n + b_n\}$, so we must have $c^* \leq a^* + b^* + \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that

$$c^* \leq a^* + b^* \Rightarrow \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

□

Problem 7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof. The idea to this proof is that we must somehow linearize $\frac{\sqrt{a_n}}{n}$ by bounding it with $p_n a_n + q_n$. Then all that is required for

$$\sum \frac{\sqrt{a_n}}{n}$$

to converge is for

$$\sum p_n a_n + q_n = \sum p_n a_n + \sum q_n$$

to converge.

Indeed, pick $p_n = 1$ and $q_n = \frac{1}{n^2}$. Since $a_n, n \geq 0$, we have

$$\begin{aligned} 0 &\leq a_n^2 + \frac{a_n}{n^2} + \frac{1}{n^4} \\ \Rightarrow \frac{a_n}{n^2} &\leq a_n^2 + 2\frac{a_n}{n^2} + \frac{1}{n^4} = \left(a_n + \frac{1}{n^2}\right) \\ \Rightarrow \frac{\sqrt{a_n}}{n} &\leq a_n + \frac{1}{n^2}. \end{aligned}$$

Then both $\sum p_n a_n = \sum a_n$ and $\sum q_n = \sum n^{-2}$ clearly converge. Hence $\sum \sqrt{a_n}/n$ converges. □

Problem 8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof. Since $\{b_n\}$ is bounded and montonic, it converges to some $b \in \mathbb{R}$. If b_n is increasing, set $c_n = b - b_n$, otherwise $c_n = b_n - b$. This new sequence c_n is decreasing and converges to 0 by construction, therefore we may apply Theorem 3.42 from the textbook to obtain the convergence of $\sum a_n c_n$. Whether we have $c_n = b - b_n$ or $c_n = b_n - b$, the sum $\sum a_n c_n$ differs from $\sum a_n b_n$ by some constant of $\pm \sum a_n b$. Hence $\sum a_n b_n$ converges. □