Homework 5

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Exercises 9, 10, pp. 116-117.

Problem 9. Assume G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ be the distinct orbits of H on A.

(a) Prove that G permutes the sets $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ in the sense that for each $g \in G$ and each $i \in \{1, \dots, r\}$ there is a j such that $g\mathcal{O}_i = \mathcal{O}_j$, where $g\mathcal{O} = \{g \cdot a \mid a \in \mathcal{O}\}$. Prove that G is transitive on $\{\mathcal{O}_1, \dots \mathcal{O}_r\}$. Deduce that all orbits of H on A have the same cardinality.

(b) Prove that if $a \in \mathcal{O}_1$ then $|\mathcal{O}_1| = |H : H \cap G_a|$ and prove that $r = |G : HG_a|$.

Proof. Part (a): $\square$

Problem 10. Let H and K be subgroups of the group G. For each $x \in G$ define the HK *double coset* of x in G to be the set

$$HxK = \{hxk \mid h \in H, k \in K\}.$$

Exercises 8, 14, pp. 122-123.

Problem 8. Prove that if H has finite index n then there is a normal subgroup $K \trianglelefteq G$ with $K \leq H$ and $|G : K| \leq n!$.

Proof. Let G act on the left cosets of H with permutation representation $\pi_{G/H} : G \to S_{|G/H|}$. Then let $K = \ker(\pi_{G/H})$, which is clearly a subset of K and also a normal subgroup of G. By the first isomorphism theorem, $G/\ker(\pi_{G/H}) = G/K \cong \operatorname{im}(\pi_{G/H})$. Comparing the cardinality of both sides, we have

$$|G : K| = |G/K| = |\operatorname{im}(\pi_{G/H})| \leq |S_{|G/H|}| = n!.$$

$\square$

Problem 14. Let G be a finite group of composite order n with the property that G has a subgroup of order k dividing n for each positive integer k dividing n. Prove that G is not simple.

Proof. Choose k such that $n/k = p$, where p is the minimum prime dividing n. Then by Corollary 5, $\square$