Homework 12

Hanting Zhang

Chapter 7; # 4 and 8 (pg. 175)

Problem 4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what interval does it converge uniformly? On what interval does it fail to converge uniformly? Is f continuous whenever it converges? If f bounded?

We first need a small result:

Proposition. *Let I be a bounded interval. If $f : I \to \mathbb{R}$ is uniformly continuous, then f is bounded.*

Proof. Set $\varepsilon = 1$. Since f is uniformly continuous, we may find δ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < 1$. Then partition I with points $x_1 < x_2 < \cdots < x_n$ with $x_{i+1} - x_i < \delta$ (which we may do since I is bounded). Then for any points x such that $x_i < x < x_{i+1}$, we have $|f(x)| \leq |f(x_1)| + \sum_{j=1}^{i-1} |f(x_i) - f(x_{i+1})| + |f(x_i) - f(x)| \leq |f(x_1)| + i \leq |f(x_1)| + n$. Thus f is bounded by $M = |f(x_1)| + n$, as desired. $\square$

Proof. The series converges for all real x except for $x = 0$ and $x = -\frac{1}{n^2}$ for $n > 0$. For $x = 0$, we have $1 + 1 + \ldots$, which diverges. For $x = -\frac{1}{n^2}$, the nth term of the series is undefined. For all other x, the series has the same growth rate as $\sum \frac{1}{n^2}$, so it converges.

The first reaction is that all intervals not containing $X = \{0, -1, -\frac{1}{4}, \ldots\}$ should be correct. However, the problem with this is that if our interval has a limit point in X, then the neighbourhoods around such a limit point will not be bounded, and hence fail the Weierstrass M-test. The way to amend these limit points is to simply take closed intervals instead; hence we claim that f converges uniformly for any interval of the form $[a, b]$ disjoint from X. (This includes intervals of the form $[a, \infty)$ and $(-\infty, b]$.) Indeed, then each term of f is bounded by either the value at a or b on the boundary points (whichever is greater for each n). These values are $\sim 1/n^2$, so the sum is $f \sim \sum 1/n^2$, which converges.

From what we just talked about, f will fail to converge uniformly on any interval that has a limit point in X. Explicitly, suppose $a = -\frac{1}{n^2} \in X$ is a limit point of a considered interval I. Then the nth term of f is unbounded as $x \to a$. Thus f itself cannot be uniformly continuous and the series will not converge uniformly.

Uniform convergence show that the limit f is coninuous on any of the intervals it converges uniformly on. But the union of all intervals of the form $[a, b]$ disjoint from X is just $\mathbb{R} - X$. Hence f is continuous whenever it is defined.

Since f diverges around all the points of X, f is clearly not bounded. $\square$

Problem 8. if

$$I(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \le x \le b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof. The first part follows immediately from Theorem 7.10 in the text; let $f_n = c_n I(x - x_n)$, then $|f_n| \le |c_n|$ and $\sum |c_n|$ converges, so $\sum f_n$ converges, as desired.

Furthermore, $f(x)$ is pointwise continuous at x when each $f_n(x)$ is continuous at x; hence $f(x)$ is at least continuous for all $x \neq x_n$. □