**Problem 2.5.** For $p \in [1, \infty)$, define $\ell^p(\mathbb{N}; \mathbb{C})$ to be the set of all complex-valued sequences $(x_j)_{j=1}^{\infty}$ such that

$$\|(x_j)_{j=1}^{\infty}\|_{\ell^p} := \left[\sum_{j=1}^{\infty} |x_j|^p\right]^{\frac{1}{p}} < +\infty\|.$$

Define addition of sequences and multiplication by (complex) scalars componentwise in each case, i.e.,

$$(x_j)_{j=1}^{\infty} + (y_j)_{j=1}^{\infty} = (x_j + y_j)_{j=1}^{\infty}; \qquad c(x_j)_{j=1}^{\infty} = (cx_j)_{j=1}^{\infty}.$$

Prove that $(\ell^p(\mathbb{N}; \mathbb{C}), \|\cdot\|_{\ell^p})$ is a normed vector space, using the following outline.

(a) Adapt the proof of Hölder's inequality (Theorem 2.13 and Exercise 2.3) to prove that for complex-valued sequences $(x_j)_{j=1}^{\infty} \in \ell^p(\mathbb{N}; \mathbb{C}), (y_j)_{j=1}^{\infty} \in \ell^q(\mathbb{N}; \mathbb{C})$, where $p$ and $q$ are Hölder conjugates, we have

$$\left|\sum_{j=1}^{\infty} x_j y_j\right| \leq \|(x_j)_{j=1}^{\infty}\|_{\ell^p} \|(y_j)_{j=1}^{\infty}\|_{\ell^q}.$$

(b) Mimic the proof of Minkowski's inequality (Theorem 2.14) to prove that $\|\cdot\|_{\ell^p}$ is a norm on $\ell^p(\mathbb{N}; \mathbb{C})$, for $p \in [1, \infty)$.

**Lemma 1.** *Define $\mathcal{F} : \ell^p(\mathbb{N}; \mathbb{C}) \to \mathcal{R}_{loc}(\mathbb{R})$ by*

$$\mathcal{F}((x_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} x_j 1_{[j-1,j]}.$$

*Then we have the nice property that,*

$$\|(x_j)_{j=1}^{\infty}\|_{\ell^p} = \|\mathcal{F}((x_j)_{j=1}^{\infty})\|_{L^p}$$

*Proof.* Indeed, by definition

$$\begin{aligned}
\|\mathcal{F}((x_j)_{j=1}^{\infty})\|_{L^p} &= \left(\int_{-\infty}^{\infty} \left|\sum_{j=1}^{\infty} x_j 1_{[j-1,j]}\right|^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_{-\infty}^{\infty} \sum_{j=1}^{\infty} |x_j|^p 1_{[j-1,j]} dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= \left( \sum_{j=1}^{\infty} \int_{j-1}^{j} |x_j|^p dt \right)^{\frac{1}{p}} \\
&= \left( \sum_{j=1}^{\infty} (j-(j-1))|x_j|^p \right)^{\frac{1}{p}} \\
&= \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{1}{p}} = \|(x_j)_{j=1}^{\infty}\|_{\ell^p}.
\end{aligned}
$$

Note $\mathcal{F}$ is well defined since $\mathcal{F}((x_j)_{j=1}^{\infty}) < +\infty$. $\square$

*Proof.* We proceed with the lemma, which makes things much easier.

(a) Note that

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\mathcal{F}((x_j)_{j=1}^{\infty})||\mathcal{F}((y_j)_{j=1}^{\infty})| dt &= \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} x_j y_j 1_{[j-1,j]} dt \\
&= \sum_{j=1}^{\infty} \int_{j-1}^{j} x_j y_j dt \\
&= \sum_{j=1}^{\infty} x_j y_j.
\end{aligned}
$$

By Hölder's inequality on $L^p$, we have

$$
\begin{aligned}
\sum_{j=1}^{\infty} x_j y_j &= \int_{-\infty}^{\infty} |\mathcal{F}((x_j)_{j=1}^{\infty})||\mathcal{F}((y_j)_{j=1}^{\infty})| dt \\
&\leq \|\mathcal{F}((x_j)_{j=1}^{\infty})\|_{L^p} \|\mathcal{F}((y_j)_{j=1}^{\infty})\|_{L^q} \\
&= \|(x_j)_{j=1}^{\infty}\|_{\ell^p} \|(y_j)_{j=1}^{\infty}\|_{\ell^q},
\end{aligned}
$$

as desired.

(b) Note again that

$$
\begin{aligned}
\|\mathcal{F}((x_j)_{j=1}^{\infty}) + \mathcal{F}((y_j)_{j=1}^{\infty})\|_{L^p} &= \left( \int_{-\infty}^{\infty} |\mathcal{F}((x_j)_{j=1}^{\infty}) + \mathcal{F}((y_j)_{j=1}^{\infty})|^p dt \right)^{\frac{1}{p}} \\
&= \left( \int_{-\infty}^{\infty} \left| (x_j + y_j) 1_{[j-1,j]} \right|^p dt \right)^{\frac{1}{p}}
\end{aligned}
$$

$$= \|\mathcal{F}((x_j + y_j)_{j=1}^\infty)\|_{L^p}.$$

By Minkowski's inequality on $L^p$, we have

$$\begin{aligned} \|(x_j + y_j)_{j=1}^\infty\|_{\ell^p} &= \|\mathcal{F}((x_j)_{j=1}^\infty) + \mathcal{F}((y_j)_{j=1}^\infty)\|_{L^p} \\ &\leq \|\mathcal{F}((x_j)_{j=1}^\infty)\|_{L^p} + \|\mathcal{F}((y_j)_{j=1}^\infty)\|_{L^p} \\ &= \|(x_j)_{j=1}^\infty\|_{\ell^p} + \|(y_j)_{j=1}^\infty\|_{\ell^p}, \end{aligned}$$

as desired.

□

**Problem 2.6.** Prove that $(\ell^p(\mathbb{N};\mathbb{C}), \|\cdot\|_{\ell^p})$ is complete for all $p \in [1, \infty)$.

*Proof.* Let $x^{(n)} = (x_j^{(n)})_{j=1}^\infty \in \ell^p(\mathbb{N};\mathbb{C})$ be a sequence for each $n \in \mathbb{N}$, so that $(x^{(n)})_{n=1}^\infty$ is a sequence of sequences. For this problem, assume that $(x^{(n)})_{n=1}^\infty$ is Cauchy. We want to show that it converges to some $x = (x_j)_{j=1}^\infty$.

Indeed, being Cauchy implies that for any $\varepsilon > 0$, there is some $N$ such that for any $n, m > N$, we have $\|x^{(n)} - x^{(m)}\|_{\ell^p} < \varepsilon$. Thus for all $j$,

$$|x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j=1}^\infty |x_j^{(n)} - x_j^{(m)}|^p = \|x^{(n)} - x^{(m)}\|_{\ell^p}^p < \varepsilon^p,$$

and we may conclude that each componentwise sequence $(x_j^{(n)})_{n=1}^\infty$ is Cauchy in $\mathbb{C}$. Since $\mathbb{C}$ is complete, all of these converge to some $x_j$. Thus we can define the sequence $x = (x_j)_{j=1}^\infty$.

Now fix some $k \in \mathbb{N}$. We have

$$\sum_{j=1}^k |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j=1}^\infty |x_j^{(n)} - x_j^{(m)}|^p = \|x^{(n)} - x^{(m)}\|_{\ell^p}^p < \varepsilon^p.$$

Taking $m \to \infty$, we have $\sum_{j=1}^k |x_j^{(n)} - x_j|^p < \varepsilon^p$. Taking $k \to \infty$ (note the importance of the order we take these limits!), we have $\sum_{j=1}^\infty |x_j^{(n)} - x_j|^p = \|x^{(n)} - x\|_{\ell^p}^p < \varepsilon^p$. Minkowski's inequality says that, for all $n > N$,

$$\|x\|_{\ell^p} \leq \|x - x^{(n)}\|_{\ell^p} + \|x^{(n)}\|_{\ell^p} = \|x^{(n)} - x\|_{\ell^p} + \|x^{(n)}\|_{\ell^p} < \varepsilon + \|x^{(n)}\|_{\ell^p}.$$

Thus we know that $\|x\|_{\ell^p}$ is bounded and in $\ell^p(\mathbb{N};\mathbb{C})$. Furthermore, $\|x^{(n)} - x\|_{\ell^p}^p < \varepsilon^p$ implies $\lim_{n\to\infty} x^{(n)} = x$. This shows that $\ell^p(\mathbb{N};\mathbb{C})$ is complete, as desired. □

**Problem 2.7.** Let $(V, \|\cdot\|)$ be a normed vector space. We say that a series $\sum_{n=1}^{\infty} v_n$ in $V$ *converges* in $V$ if there exists $w \in V$ such that $\| \sum_{n=1}^{N} v_n - w\| \to 0$ as $N \to \infty$, that is, if the sequence $(s_N)_{N=1}^{\infty}$ of partial sums $s_N = \sum_{n=1}^{N} v_n$ converges in $V$ under the norm $\|\cdot\|$. We say that the series $\sum_{n=1}^{\infty} v_n$ converges *absolutely* if the series of numbers $\sum_{n=1}^{\infty} \|v_n\|$ converges in $\mathbb{R}$.

Prove that $(V, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series in $V$ converges in $V$.

*Proof.* $(\Rightarrow)$: Suppose we have a Banach space $(V, \|\cdot\|)$ and an absolutely convergent series $\sum_{n=1}^{\infty} v_n$. Fix any $\varepsilon > 0$. Because $\sum_{n=1}^{\infty} \|v_n\|$ converges, there exists some $N$ such that for all $i, j > N$, assuming $i > j$ without loss of generality, $\sum_{n=i}^{j} \|v_n\| < \varepsilon$. Thus

$$\|s_i - s_j\| = \left\| \sum_{n=i+1}^{j} v_n \right\| \leq \sum_{n=i}^{j} \|v_n\| < \varepsilon.$$

We conclude that $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in $V$. Thus it must converge, as desired.

$(\Leftarrow)$: Suppose that if a sequence $\sum_{n=1}^{\infty} v_n$ converges absolutely, then it converges in $V$. Let $(s_n)_{n=1}^{\infty}$ be a Cauchy sequence in $V$. For any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists some $N$ such that for any $n_k, n_{k+1} > N$, we have $\|s_{n_k} - s_{n_{k+1}}\| < \varepsilon/2^k$. Choose the $n_k$ in a proper manner to define $a_k = s_{n_k} - s_{n_{k+1}}$, so as to form a subsequence $(a_{n_k})_{k=1}^{\infty}$. Now we have,

$$\sum_{k=1}^{\infty} \|a_k\| < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

which converges absolutely. By assumption then $\sum_{k=1}^{\infty} a_k$ converges. Thus the subsequence $\sum_{i=1}^{k} a_i = s_{n_k}$ converges. We know that if a Cauchy sequence has a subsequential limit, then the entire sequence does too, so $(s_n)_{n=1}^{\infty}$ also converges, as desired. $\square$