Problem 2.1. Let V be a real vector space, and let E be a convex subset of V. Then $f : E \to \mathbb{R}$ is a convex function if and only if for every $a, b \in E$ and $\lambda \in [0, 1]$, we have

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

Proof. We want to show that for every pair of points $(a, f(a)), (b, f(b)) \in \text{epi}(f)$ and every $\lambda \in [0, 1]$, we have $\lambda(a, f(a)) + (1 - \lambda)(b, f(b)) \in \text{epi}(f)$.

Indeed, we have

$$\lambda(a, f(a)) + (1 - \lambda)(b, f(b)) = (\lambda a + (1 - \lambda)b, \lambda f(a) + (1 - \lambda)f(b)).$$

Since E is convex, $\lambda a + (1 - \lambda)b = c$ for some point $c \in E$. Then by assumption,

$$y = \lambda f(a) + (1 - \lambda)f(b) \geq f(\lambda a + (1 - \lambda b)) = f(c).$$

Thus we satisfy the conditions that show

$$(\lambda a + (1 - \lambda)b, \lambda f(a) + (1 - \lambda)f(b)) \in \text{epi}(f).$$

□

Problem 2.2. Given an example of a convex function $f : \mathbb{R} \to \mathbb{R}$ which is convex, but whose square in not convex.

Proof. The function $x^2 - 1$ works. This function is a parabola and obivously convex. Its square is $(x+1)^2(x-1)^2$. This is not convex since letting $a = (-1, 0), b = (1, 0)$ and $\lambda = 1/2$, the point $(0, 0)$ is not in the epigraph of the square. Essentially the "squaring" makes a W shape that destroys convexity. □

Problem 2.3. (Hölder's inequality.) Assume $f, g \in \mathcal{R}_{\text{loc}}(\mathbb{R})$, and let p and q be positive real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and assume that the integrals $\int_{-\infty}^{\infty} |f(x)|^p dx$ and $\int_{-\infty}^{\infty} |g(x)|^q dx$ converge. Then $\int_{-\infty}^{\infty} f(x)g(x)dx$ converges absolutely, and

$$\int_{-\infty}^{\infty} |f(x)||g(x)|dx \leq \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(x)|^q dx\right)^{\frac{1}{q}}.$$

Lemma 1. *Suppose f and g are nonnegative functions satisfying $\int_{-\infty}^{\infty} f(x)^p dx = \int_{-\infty}^{\infty} g(x)^q dx = 1$. Then in this special case, Hölder's inequality holds.*

Proof. By Young's inequality, we have for all x, $f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}$. Thus, noting that $f = |f|$ and $g = |g|$,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)g(x)dx &\leq \frac{1}{p}\int_{-\infty}^{\infty} f(x)^p dx + \frac{1}{q}\int_{-\infty}^{\infty} g(x)^q dx \\
&= \frac{1}{p} + \frac{1}{q} = 1 = (1)^{\frac{1}{p}}(1)^{\frac{1}{q}} \\
&= \left(\int_{-\infty}^{\infty} f(x)^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} g(x)^q dx\right)^{\frac{1}{q}},
\end{aligned}$$

we've shown Hölder's inequality as desired. □

Proof. Now for Hölder's inequality in the general case of arbitrary functions $f, g \in \mathcal{R}_{\text{loc}}(\mathbb{R})$. Denote $\|f(x)\| = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}}$. If $\|f\| = 0$ or $\|g\| = 0$, then by Exercise 1.6, the LHS of the inequality is 0, and thus trivial. Otherwise, define $F = |f|/\|f\|$ and $G = |g|/\|g\|$. We have f' and g' are nonnegative functions which satisfy

$$\int_{-\infty}^{\infty} F(x)^p dx = \frac{1}{\|f\|^p}\int_{-\infty}^{\infty} |f(x)|^p dx = \frac{\|f\|^p}{\|f\|^p} = 1.$$

and

$$\int_{-\infty}^{\infty} G(x)^q dx = \frac{1}{\|g\|^q}\int_{-\infty}^{\infty} |g(x)|^q dx = \frac{\|g\|^q}{\|g\|^q} = 1.$$

Thus we may apply Lemma 1 and see that,

$$\int_{-\infty}^{\infty} F(x)G(x)dx \leq \left(\int_{-\infty}^{\infty} F(x)^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} G(x)^q dx\right)^{\frac{1}{q}},$$

which implies

$$\frac{1}{\|f\|\|g\|}\int_{-\infty}^{\infty} |f(x)|^p|g(x)|^q dx \leq \frac{1}{\|f\|^{p(1/p)}\|g\|^{q(1/q)}} \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |g(x)|^q dx\right)^{\frac{1}{q}}.$$

Clearing $\|f\|\|g\|$ on boths sides gives Hölder's inequality, as desired. □

Problem 2.4. Tweak Hölder's inequality slightly to prove that for $\lambda \in [0, 1]$, we have

$$\int_{-\infty}^{\infty} |f(x)|^\lambda |g(x)|^{1-\lambda} dx \leq \left(\int_{-\infty}^{\infty} |f(x)| dx \right)^\lambda \left(\int_{-\infty}^{\infty} |g(x)| \right)^{1-\lambda},$$

whenever $\lambda \in [0, 1]$ and $f, g \in \mathcal{R}_{\text{loc}}(\mathbb{R})$.

Proof. This trivially follows from Exercise 2.3 with setting $p = 1/\lambda$, $q = 1/(1-\lambda)$, $f \leftarrow f^{1/p}$, and $g \leftarrow g^{1/q}$. There is a slight hiccup with the case $\lambda = 0, 1$, but in these special cases, the inequality reduces to a trivial one. $\square$

Problem 2.5. This Exercise establishes some properties of log-convex functions.

(a) Let $f : (\alpha, \beta) \to (0, \infty)$ (with $-\infty \leq \alpha \leq \beta \leq \infty$) be a function. Then f is log-convex if and only if for any $a, b \in (\alpha, \beta)$, and $\lambda \in [0, 1]$, we have

$$f(\lambda a + (1 - \lambda)b) \leq f(a)^\lambda f(b)^{1-\lambda}.$$

(b) Tweak Young's inequality to prove that for $A, B \geq 0$ and $\lambda \in [0, 1]$, one has

$$A^\lambda B^{1-\lambda} \leq \lambda A + (1 - \lambda)B.$$

Using this inequality and part (a), conclude that every log-convex function $f : (\alpha, \beta) \to (0, \infty)$ (with $-\infty \leq \alpha \leq \beta \leq \infty$) is convex.

(c) Prove that the product of log-convex functions is log-convex.

Proof. We proceed with each part.

(a) We have f is log-convex iff $\log \circ f$ is convex. This occurs iff

$$\log(f(\lambda a + (1 - \lambda)b)) \leq \lambda \log(f(a)) + (1 - \lambda) \log(f(b)),$$

iff (use log rules)

$$\log(f(\lambda a + (1 - \lambda)b)) \leq \log(f(a)^\lambda f(b)^{1-\lambda}),$$

iff (monoticity of log)

$$f(\lambda a + (1 - \lambda)b) \leq f(a)^\lambda f(b)^{1-\lambda}.$$

(b) Just modify the proof of Young's inequality:

$$A^\lambda B^{1-\lambda} = \exp(\log A^\lambda B^{1-\lambda}) = \exp(\lambda \log A + (1 - \lambda) \log B) \leq \lambda A + (1 - \lambda)B.$$

Thus with part (a), log-convex functions satisfy

$$f(\lambda a + (1-\lambda)b) \leq f(a)^\lambda f(b)^{1-\lambda} \leq \lambda f(a) + (1-\lambda) f(b).$$

i.e. they are convex, as desired.

(c) Let f and g be log-convex functions. We want to show that $\log \circ fg$ is convex. Indeed, for all $a, b \in (\alpha, \beta)$, $\lambda \in [0,1]$, and $c = \lambda a + (1-\lambda)b$),

$$\begin{aligned} \log((f \cdot g)(c)) &= \log(f(c)) + \log(g(c)) \\ &\leq \lambda \log(f(a)) + (1-\lambda)\log(f(b)) + \log(g(a)) + (1-\lambda)\log(g(b)) \\ &= \lambda \log((f \cdot g)(a)) + (1-\lambda)\log((f \cdot g)(b)). \end{aligned}$$

Thus the condition for convexity holds, as desired.

$\square$