1. $E(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,z)\, dx\, dz$, $E(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z \cdot f(x,z)\, dx\, dz$

$E(x+z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+z) f(x,z)\, dx\, dz$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,z)\, dx\, dz + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z f(x,z)\, dx\, dz$

$= E(x) + E(z)$

$V(x+z) = E[(x+z)^2] - (E(x+z))^2$

$= E(x^2 + 2xz + z^2) - (E(x)^2 + E(z)^2 + 2E(x)E(z))$

$= E(x^2) + E(z^2) - E(x)^2 - E(z)^2$

$= V(x) + V(z)$

2. μ에 대해서 미분

$\frac{d \ln p(x|\mu,\sigma^2)}{d\mu} = \frac{1}{\sigma^2}\sum(x_n - \mu)$

$= \frac{1}{\sigma^2}\sum x_n - \frac{1}{\sigma^2}\sum \mu = 0$

$\frac{1}{\sigma^2}\sum x_n = \frac{1}{\sigma^2} N\mu$ $\mu = \frac{1}{N}\sum_{n=1}^{N} x_n$

σ^2으로 미분 $\sigma^2 = \alpha$로 치환

$\frac{d \ln(x|\mu,\sigma^2)}{d\alpha} = \frac{1}{2\alpha^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2}\times\frac{1}{\alpha} = 0$

$= \frac{1}{\alpha^2}\sum(x_n-\mu)^2 = \frac{N}{\alpha}$

$= \sum(x_n-\mu)^2 = \alpha N$

$\therefore \alpha = \sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu)^2$

3. $KL(p\|q) = -\int p(x)\left(\ln\frac{q(x)}{p(x)}\right)dx = -\int p(x)\ln q(x)dx + \int p(x)\ln p(x)\,dx$

$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $\quad \ln p(x) = \ln\frac{1}{\sqrt{2\pi\sigma^2}} - \ln\frac{(x-\mu)^2}{2\sigma^2} =$

$q(x) = \frac{1}{\sqrt{2\pi s^2}}\exp\left(-\frac{(x-m)^2}{2s^2}\right)$ $\quad \ln q(x) = \ln\frac{1}{\sqrt{2\pi s}} - \ln\frac{(x-m)^2}{2s^2}$

$p(x)\ln p(x) = -\frac{1}{2}(1+\ln 2\pi\sigma^2)$

$p(x)\ln q(x) = \frac{1}{2}\ln(2\pi s^2) - \int p(x)\left(-\frac{(x-m)^2}{2s^2}\right)dx$

$\left[\begin{array}{l} \int N(x|\mu,\sigma^2)=1 \\ \int x^2 N(x|\mu,\sigma^2)=\mu^2+\sigma^2 \\ \int x N(x|\mu,\sigma^2)=\mu \end{array}\right.$

$= \frac{1}{2}\ln(2\pi s^2) + \frac{\int p(x)x^2dx - \int p(x)2xm\,dx + \int p(x)m^2dx}{2s^2}$

$= \frac{1}{2}\ln(2\pi s^2) + \frac{\sigma^2+\mu^2-2\mu m+m^2}{2s^2} = \frac{1}{2}\ln(2\pi s^2) + \frac{\sigma^2+(\mu-m)^2}{2s^2}$

$\therefore KL(p\|q) = \frac{1}{2}\ln(2\pi s^2) + \frac{\sigma^2+(\mu-m)^2}{2s^2} - \frac{1}{2}(1+\ln 2\pi\sigma^2)$

$= \ln\frac{s}{\sigma} + \frac{\sigma^2+(\mu-m)^2}{2s^2} - \frac{1}{2}$

4. Bernoulli $\Rightarrow p(x|\theta) = \theta^x(1-\theta)^x$, $\mu=\theta$, $\sigma^2=\theta(1-\theta)$

$H(x) = -\sum_{x\in\{0,1\}} p(x|\theta)\ln p(x|\theta)$

$= -p(x=0|\theta)\ln p(x=0|\theta) - p(x=1|\theta)\ln p(x=1|\theta)$

$= -(1-\theta)\ln(1-\theta) - \theta\ln\theta$ $\quad \left[\begin{array}{l} p(x=0|\theta)=1-\theta \\ p(x=1|\theta)=\theta \end{array}\right.$

$\therefore \theta=\mu \Rightarrow -\mu\ln\mu - (1-\mu)\ln(1-\mu)$