Classical Mechanics Special Relativity for Starters

C. Freek. J. Pols

Abstract

Abstract: This book provides an introduction for freshman students into the world of classical mechanics and special relativity theory. Much of physics is build on the basic ideas from classical mechanics. Hence an early introduction to the topic can be beneficial for new students. However, at the start of studying physics, lots of the required math is not available yet. That means that all kind of concepts that are very useful can not be invoked in the lectures and teaching. That does not have to be a disadvantage. It can also be used to help the students by introducing some math and coupling it directly to the physics, making more clear why mathematics should be studied and what its 'practical use' is. With this book, we aim to introduce new students directly at the start of their studies into the world of physics, more specifically the world of Newton, Galilei and many others who laid the foundation of physics. We aim to help students getting a good understanding of the theory, i.e. the framework of physics. What is 'work' and why do we use it? Why is kinetic energy $\frac{1}{2}mv^2$ and not $\frac{1}{3}mv^2$ or $\frac{1}{2}mv^3$? Both 3's are fundamentally wrong, but what is behind it?

1 Introduction

1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and pressumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version, though good use has been made by open educational resources.

1.1.1 Special features

In this book you will find some 'special' features. Some of these are emphasized by their own style:

Exercise 1:
Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3

Intermezzos
Intermezzos contain background information on the topic, of the people working on the concepts.

Experiments
We include some basic experiments that can be done at home.

Examples
We provide various examples showcasing, e.g., calculations.

Python
We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'.

New concepts, such as *Free body diagram*, are introduced with a hoover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

1.1.2 Feedback

Do you see a mistake, do you have suggestions for exercises, are parts missing or abundant. Tell us! You can use the feedback button at the top right button. You will need a (free) GitHub account to report an issue!

1.2 Authors

- Prof.Dr. Robert F. Mudde, Department of Chemical Engineering, Delft University of Technology
- Prof.Dr. Bernd Rieger, Department of Imaging Physics, Delft University of Technology
- Dr. Freek Pols, Science & Engineering Education, Delft University of Technology

Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files.

1.2.1 About the authors

Robert Mudde is Distinguished Professor of Science Education at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

Bernd Rieger is Antoni van Leeuwenhoek Professor in the Department of Imaging Physics at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

Freek Pols is an assistant professor in the Science & Engineering Education group at the faculty of Applied Sciences of Delft University of Technology in The Netherlands.

1.3 Open Educational Resource

This website is a Jupyter Book. MarkDown source files are available for download using the button on the top right.

The bar on the right of this page opens Hypothesis, a service that allows you to add comments to pieces of text in this book.

1.3.1 Software and license

This website is a Jupyter Book. Markdown source files are available for download using the button on the top right.

1.3.2 Images, videos, apps

The cover image is inspired by the work of 3blue1brown developer Grant Sanderson.

All vecor images have been made by Hanna den Hertog, and are available in vector format through the repository. For reuse, adapting and redistrution, adhere to the CC-BY-licences.

We embedded several clips from 3blue1brown in accord with their licences requirements.

The embedded vpython apps are made freely available from trinket.

Some videos from NASA are included, where we adhere to their regulations.

At various places we use pictures which are in the public domain. We comply to the regulations with regard to references.

1.3.3 How to cite this book

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*,

2 Special Relativity

2.1 Cracks in Classical Mechanics

As the years progressed, Classical Mechanics developed further and further. So, in the first half of the nineteenths century it felt like classical mechanics was an all encompassing theory and that physics would become a discipline of working out problems based on a well-established, complete theory. But that wasn't going to be the case at all. Around 1850-1860 several cracks in the theory started to become visible. And they were fundamental!

2.1.1 Rutherford & the atom

Atomic theory The idea that matter is made of atoms is old. However, the notion of atoms as we have now is relatively young.

In the ancient Greek world, it was as early as the fifth century B.C. that Leucippus and later one of his pupils Democritus proposed that the world (matter), is made up of tiny, indivisible particles. These particles were called atoms, derived from the Greek word 'atomos', which means uncuttable. These particles would float in a vacuum, that was called *void* by Democritus. We currently have a view that is remarkably close, but at the same time quite different from these first ideas.

In ancient India even earlier (records go back to the eighth' century B.C.) philosophers like Uddalaka Aruni talk about matter being made up of tiny particles. They did not use terms like atoms, but instead referred to the 'building blocks' of matter as 'kana' which means particles. In the Islamic world, atomic theories were developed in e.g. the Asharite school by Al Ghazali (1058-1111). In his thinking, atoms are the only material things that live forever. Everything else, any event or interaction is due to God's intervention.

Although these early thoughts point at atoms as the underlying elements of matter and as such resemble our current understanding of matter, they also differ quite a bit. The early ideas are based on philosophy and the notion that matter is either a continuum that can always be cut in smaller parts that still maintain all characteristics or that at some point a further splitting in smaller pieces is no longer possible with at least losing some of the characteristics.

In more recent history, the notion of atoms as elementary building blocks is guided by experiments. The English physicist and chemist John Dalton (1766-1844) did ground braking work. He noticed that water, when decomposed, always resulted in the same elements: hydrogen and oxygen. Moreover, the relative weights of these two was always the same. Furthermore, he came to the conclusion that there is a unique atom for each element. More chemists noted that many substances were made of the elements in very specific ratios. In our modern view we would say: water is formed in a 1 to 2 ratio of oxygen and hydrogen, never in 1 to 2.1 or any other non-integer number.

Although the evidence from chemistry was clear, the notion of atoms as the building blocks remained controversial. The laws of definite proportion (e.g. water is decomposed in a fixed ratio in hydrogen and oxygen) were generally accepted, but the hypothesis that everything was made of atoms was not. As a consequence, the work of Ludwig Boltzmann (1844-1906) on statistical thermodynamics that is entirely based on an atomic (or molecular) view was not accepted during Boltzmann's life.

In the second half of the nineteenth century William Thomson (1824-1907) -later Lord Kelvin- proposed the so-called vortex theory of the atom. Based on the discoveries by chemists of only a few different atoms that made up the rest of matter, Thomson proposed that atoms are stable vortices, not in an ordinary fluid like water, but in the omni-present luminiferous aether.

Stable vortices have the shape of rings with no beginning or end. In air they are easily made and made visible with smoke and are indeed surprisingly stable. According to the vortex theory, atoms are vortices in aether. The simplest one is a single ring, which was hydrogen. More complicated forms, called knots represented the other elements.

Figure 3: Lord Kelvin working on the vortex theory of the atom

At the end of the nineteenth century, in 1897, Joseph John Thomson discovered the electron. It allowed him to further refine the scientific model of the atom and ended the vortex theory. In Thomson's view, an atom has internal structure: the electrons are moving in it. As electrons have a negative charge and atoms are neutral, there must be a balancing positive charge in an atom as well. Thomson had no idea what that would be. He figured that the positive charge was everywhere in the atom (that he thought of as being a sphere), with the electrons moving inside that sphere as tiny particles. From this picture, the Thomson model got its name: *the plum pudding model*, although it is a bit misleading as the idea was that the positively charged sphere was more a liquid in which the electrons 'float' than a solid.

Figure 6: Thomson refined the idea of the atomic model: electrons moving within the atom, though the atom is still neutral in charge.

The model did not hold very long as we will see in the next paragraph. Nevertheless, it marks the start of physicist becoming really interested in an atom theory.

Rutherford's scattering experiment The plum pudding model was abandoned in 1911. That year Ernest Rutherford (1871-1937), a former student of Joseph Thomson, performed a ground-breaking experiment. Rutherford had been working on the newly discovered radio-activity of certain elements. He discovered that there were two types of radiation that were different from X-rays. He called them 'alpha' and 'beta' rays. Later he proved that 'alpha' rays consist of He-nuclei. Rutherford, in cooperation with Frederick Soddy, was the first one to prove Marie Curie's conjecture that radioactivity was an atomic phenomenon, which could lead to changes in the atom itself, from one element to another. This idea thus countered the prior idea that an atom was seen as the ultimate, indestructible form of matter: atoms could not change from one form (element) to another.

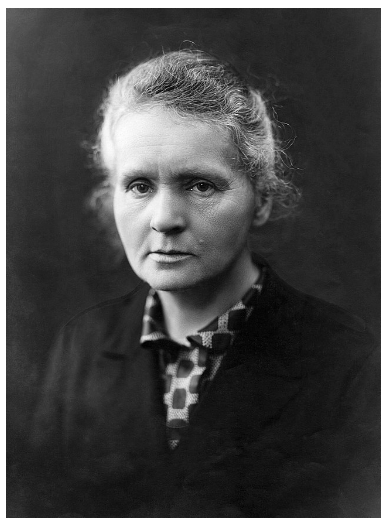


Figure 7: Marie Curie (1867-1934). From Wikimedia Commons, public domain.

Rutherford, in cooperation with Hans Geiger (one of the inventors of what we now call the Geiger counter) and Ernest Masden, built an apparatus that could count the alpha particles. Moreover, he showed that the alpha particles were He-nuclei with a positive charge of $2e4$. In 1917, he showed that Nitrogen could become Oxygen by bombarding it with the alpha particles. This was the first time that someone succeeded in artificially changing one element into another.

Scattering at a gold As mentioned, Rutherford is responsible for overthrowing the plum pudding model and replacing it by our modern view: an atom is made of a tiny, positively charged nucleus with the electrons orbiting around it.

The start of this paradigm-shift was formed by Rutherford's observation that some of the alpha-particles were deflected by a thin metal sheet in front of his alpha-counter. This puzzled him as the plum pudding model could not explain this: when using that model the particles were either colliding or passing straight. Rutherford, Geiger and Marsden thus set up an experiment in which they led the alpha particles scatter at a very thin gold foil to investigate further.

In the experiment, the source would emit α-particles through a small diaphragm onto the gold foil. The diaphragm made sure that all α-particles were traveling on the same line. After moving through the gold foil, the particles could be detected by looking via a microscope at the tiny light flashes an α-particle caused when hitting the detection screen. The microscope and detection screen could be placed under an angle with the original trajectory of the α-particles. By measuring at all possible angles, the scattering of the α-particles by the gold foil could be completely mapped and quantified.

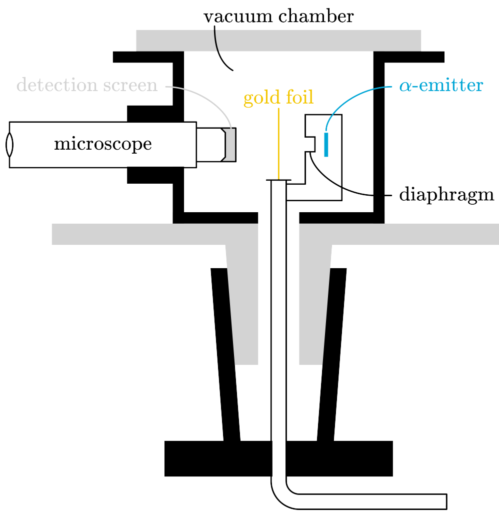


Figure 8: Experimental setup of α-scattering at a gold foil.

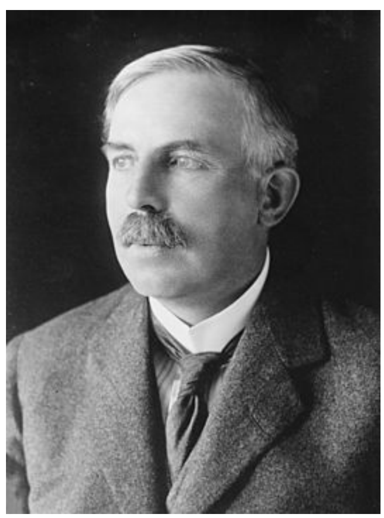


Figure 9: Ernest Rutherford (1871-1937). From Wikimedia Commons, public domain.

The story goes, that Rutherford's students would, together with Geiger, do the measurements as an assignment of their studies. The principle is simple: set the microscope under a know angle and, for a given period in time, count the number of hits. Repeat this for the next angle of the microscope. Obviously, the first measurements were all done on the side of the foil opposite to the α-emitter. One was expecting small deviations from the undisturbed trajectory.

When the experiments were basically done, so goes the story, still a student was left over that also needed an assignment. One of Rutherford's assistant suggested that this student would then measure with the microscope at the same side of the foil as the α-emitter. They did not expect anything to see, but they needed an assignment for this student. Whether the story of the student assignments is true or not, fact is that the team found also hits on the detector for angles of about 180°. That is, some α-particles seemed to bounce back from the foil!

There is no way that the plum pudding model could explain this. The argumentation to show that, goes as follows.

- The size of the atoms of gold is known: they are on the order of $r_0 \approx 10^{-10}$m.
  This value can be found from the density of gold, the mass of a gold atom and the mass and volume of the gold foil (or any other amount of gold). By treating the atoms as small spheres that are stacked back to back, the size of the atom is easily found.
- An α-particle has a charge of 2e and is deflected by a gold atom due to the charge of the gold atom. As gold has number 79 in the periodic table, we know that the charge of a gold atom is $+79$e in the 'plum pudding' and -79e of all electrons floating in the pudding.
  However, an $alpha$-particle is much heavier than an electron. Hence in the Coulomb interaction between the α -particle and an electron, the acceleration (of deflection) of the 'heavy' α-particle is negligible: the electrons are pushed out of the way (or actually attracted); they don't influence the trajectory of the α-particle.
  It is the positive charge of the pudding itself, that does the deflection. The atom (i.e. the pudding) can not move out of the way as it is part of the gold foil which is many orders of magnitude heavier than the incoming particle.

Rutherford knew the theory of Maxwell for Electro-Magnetism and could estimate The force an α-particle would feel from the atom. He deduced that the force on the α-particle is always smaller than:

$$F_c \leq \frac{q_\alpha Q_g}{4\pi\epsilon_0}\frac{1}{r_0^2} \tag{1}$$

with Q the charge of the atom (i.e. $+79$e), q_α the charge ($+2$e) of the α-particle, ϵ_0 the permittivity of vacuum ($\frac{1}{4\pi\epsilon} = 9 \cdot 10^9 Nm^2/C^2$) and r_0 the radius of a gold atom.

The deflection of the particle is biggest if the Coulomb force is perpendicular to the trajectory. So, we take that for our estimate. The true effect, when passing through the atom, will be smaller.

- It is easiest to compute the change of momentum. The particle comes in with a know momentum p. If the change in momentum

Δp is much smaller than p itself, the deflection will be small.

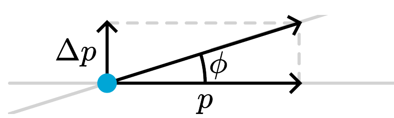


Figure 10: Relation of angle of deflection and change in momentum.

$$\tan\phi = \frac{\Delta p}{p} \Rightarrow \phi \approx \frac{\Delta p}{p} \text{ if } \phi \ll 1 \tag{2}$$

- The momentum change is due to the force F_c working for a time period

Δt on the particle:

$$dP = Fdt - \rightarrow \Delta p \approx F_c \Delta t \tag{3}$$

The time the particle is in the atom is estimated as follows. The particle has a velocity v_0 and it has to travel a distance $2r_0$ through the atom, thus $\Delta t \approx \frac{2r_0}{v_0}$. We assume that the change in momentum is small, so that we can still use v_0 as an estimate of the velocity with which the α-particle travels.

- If we put everything together, we find:

$$\frac{\Delta p}{p} \ll \frac{q_\alpha Q_g}{4\pi\epsilon_0}\frac{1}{r_0^2}\cdot\frac{2r_0}{v_0} = \frac{q_\alpha Q_g}{2\pi\epsilon_0}\frac{1}{r_0 v_0} \ll 1 \tag{4}$$

We have used the know charge of a gold atom (79e) and that of the α-particle, the radius of the gold atom and the incoming velocity of the α-particle, $v_0 \approx 1.6 \cdot 10^7$m/s.

With this estimate and the fact that Rutherford's gold foil was about 400 atoms thick, there is no way that we can explain α-particles bouncing back.

Rutherford and his colleagues, had no other option than to conclude that the positive charge of the gold atom must be confined to a much smaller part of space. After all, the only parameter in our estimate that is not measured is r_0. That was estimated based on the plum pudding model.

They redid the calculation, but now with r_0 as a free parameter to be backed out of the calculation. They changed the requirement of small scattering angles (i.e. small deviation from the original path) to the experimental finding that scattering angles of about 180° were possible. That gave that r_0 is on the order of 10^{-14}m.

Conclusion: the atom has a nucleus that is much smaller than the size of the atom that contains all positive charge. The electrons must orbit this nucleus as a mini-solar system. These electrons 'define' the size of the atom.

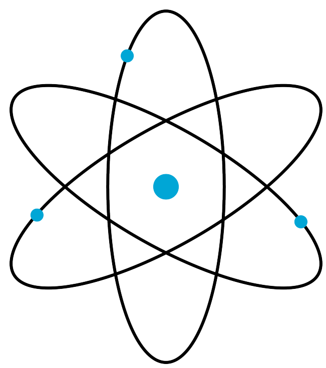


Figure 11: Rutherford's model of an atom.

This new model would spark a whole new set of questions, setting up one of the biggest changes in physics: Quantum Theory.

Collapse of matter? An immediate consequence of this new view on atoms and matter came from the analogy with Newton's work on the solar system and the Kepler Laws. In the case of the sun and planets, the interaction force is gravity: $\vec{F}_g = -G\frac{m_1 m_2}{r_{12}^2}\hat{r_{12}}$. When dealing with a nucleus with its orbiting electrons the interaction force is the Coulomb force: $\vec{F}_C = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r_{12}}$.

As both Gravity and Coulombs forces are central, conservative forces and are inversely proportional to the square of distance from the two interacting particles the motion of a 'tiny' planet around the 'massive' sun is mathematically completely analogous to that of a 'tiny' electron around its 'massive' nucleus.

Thus an electron orbits the nucleus in an ellipse. Consequently, it is in a permanent state of acceleration. However, from Maxwell's theory of Electro-Magnetism it is well known (already in the times of Rutherford as the theory of Maxwell dates back to around 1860) that accelerating charged particles radiate energy in the form of electro-magnetic waves. This means that the electron constantly looses energy and thus moves to an elliptical orbit closer to the nucleus until, eventually, its orbit collapses onto the nucleus. This process would go very fast and matter in its present form could not exist. Now we know, that the idea of an atom being a miniature solar system is wrong. But out of questions and dilemma's as these grew very quickly quantum mechanics opening a whole new world and a completely different picture of things at the small scales. A world with new rules and new consequences, where our intuition based on daily life and large scale structures composed of many, many atoms fails.

Scattering Theory The work of Rutherford and co-workers forms the start of a new branch of physics: nuclear physics. By using radiation in the form of X-rays (i.e. high energy photons) and electrons or protons, physicists are able to probe the internal properties of molecules, atom, nuclei and even elementary particles (or at least, what we once thought were elementary particles).

The idea is to send high energy particles towards the object of investigation and look at the scattering that is a consequence of the interaction between the object and the incoming particles. The internal structure of the object dictates the scattering. Thus, by measuring the scattering features and back tracing the underlying physical interaction can be found.

It is done with facilities of a very large scale to research particles at the smallest scales. For instance, in CERN researchers accelerate particles (protons, electrons, etc) to velocities almost the speed of light. Then, they let these particles collide, that is undergo interactions involving enormous amounts of energy, and measure the fragments and all kind of exotic particles that result from these collisions.

Figure 12: Circular Accelerator of CERN depicted in its environment. ESO/José Francisco, licensed under CC-BY 4.0.

The principles used in scattering can be illustrated by revisiting Rutherford's experiment.

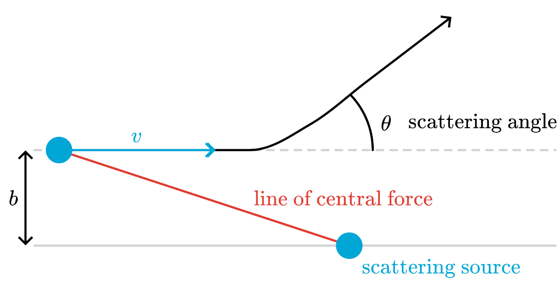


Figure 13: Scattering of an incoming particle at a fixed source.

Consider fig.(7.7): a particle of mass m and velocity v is moving towards a fixed second particle. The latter is fixed in the origin and act like a force-source. The force is central, i.e. works along the direction of the red line in fig.(7.7). In the drawing the forces is repelling and the incoming particle will deviate from its straight line. Eventually it will continue moving over a straight line, when the influence of the force is no longer felt. The angle of the new direction with the incoming one, is θ, the scattering angle. We are looking for the relation between b, the distance at which the incoming particle would have passed by the origin if there was no force and the scattering angle θ.

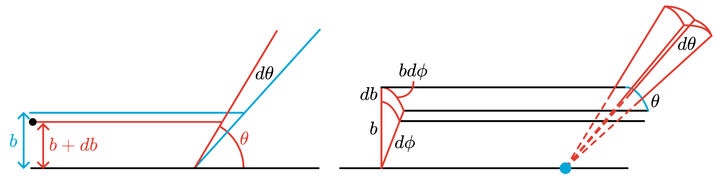


Figure 14: left: scattering in 2D, right: scattering in 3D.

In fig.(7.8) scattering in a 2D world and in the 3D world is schematically depicted. In the 3-dimensional world the scattering takes place in the solid angle $d\theta$. Like the 2d equivalent, where the scattering angle can go from 0 to 2π (that is the full circle), in 3d it goes from 0 to 4π reflecting that it is now a full sphere.

2.1.2 Kinetic theory of gases

Already in the 18$^{\text{th}}$ century, work was done on what we call the kinetic theory of gases. The Swiss scientist Daniel Bernoulli proposed that gases were a large collection of molecules, i.e tiny particles moving in all directions. According to Bernoulli, their collision with walls was felt macroscopically as pressure and their averaged kinetic energy was in essence the temperature of the gas.

Figure 17: Two famous scientists working on the physics of gases.

It took a while before these ideas were accepted, partly because the law on conservation of energy was not fully developed. Moreover, people had difficulty accepting that at a molecular level collisions could be perfectly elastic.

With the further development of Thermodynamics, the kinetic theory of gases also refined. In 1856, August Krönig came up with a simple kinetic model for gases in which he only considered the possibility of translational motion of the molecules. In essence, he treated gas molecules as point particles. A year later, Rudolf Clausius incorporated the possibility of rotation and vibrations. Two years after this, James Clerk Maxwell continued along this line. He found the velocity distribution of the molecules and established a firm connection between temperature and the average kinetic energy of a molecule. However, he also noted that the theoretical predictions were not in line with experiments. What was the problem?

Specific heat of gases For ideal gases, we have the ideal gas law: $pV = nRT$ with n the number of moles of the gas in question. Or written in terms of number of molecules, N, it reads as: $pV = NkT$, k being the Boltzmann constant.

The ideal gas law helps in understanding how gases behave under changing conditions. For instance, if we compress a given amount of gas, we may expect that the pressure goes up. But we also see that this depends on whether or not the temperature changes. And in principle the temperature will change.

If we would do a compression experiment with a fixed number of molecules, N, and we would compress the gas such that no heat can escape from the container, then the changes in temperature, volume and pressure are such that $pV^\gamma = const$. This is called adiabatic compression. The quantity γ is the ratio of the specific heat at constant pressure over the specific heat at constant volume. Both these quantities are easily measured in experiments and, hence, γ can be found for many gases.

The kinetic theory predicts γ for various classes of gases. For instance, for monatomic gasses as Helium, it is $5/3 \approx 1.667$; for diatomic gases, such as Oxygen or Hydrogen, it should be $9/7 \approx 1.286$. And so on. Moreover, γ does, according to the kinetic theory, not depend on temperature.

In the table below, the ratio of the specific heats c_p/c_V is listed for a number of gasses.

Gas	c_p/c_v	kin.gas.th.
He	1.663	1.667
Ne	1.667	1.667
Kr	1.656	1.667
Br_2	1.28	1.286
Cl_2	1.34	1.286
H_2	**1.405**	1.286
N_2	**1.40**	1.286
O_2	**1.395**	1.286

we see, for the noble gases it is quite ok (at $T = 293$K !), but not so for the diatomic gases.

For really high temperatures (˜2000K) for both O_2 and H_2, γ it is close to 1.286. But if we go to low temperature, their respective γ's increase and H_2 reaches a value of 1.66! Hence, Maxwell concluded, that the laws of classical mechanics could not be the final answer.

As we have seen when discussing Rutherford's experiment, in the early twentieth century more cracks became visible. These led scientists to Quantum Mechanics.

2.1.3 The problem with Maxwell's equations

In the early 1860s Maxwell extended Ampères law, in combination with Gauss and Faraday laws. This led to four coupled differential equations describing the generation of electro-magnetic fields from charges and currents. They are now just known as *the Maxwell equations*. They read in modern (differential) notation as follows for the electric $\vec{E}(\vec{x},t)$ and magnetic $\vec{B}(\vec{x},t)$ field in free space

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}
\end{aligned} \tag{5}$$

With $\rho(\vec{x})$ the charge density distribution and $\vec{J}(\vec{x},t)$ the electric current density, and constants ϵ_0 the vacuum permittivity and μ_0 the vacuum magnetic permeability.

You will learn all about Maxwell's equations in classes on *Electromagnetism*. The mathematics of these equations is quite difficult as each equation is $3D + t$ and the equations are coupled.

In vacuum ($\rho = 0$ and $\vec{J} = 0$) we can simplify these equation. Furthermore, we could look at 1-dimensional cases, that is the electric field has only a component E_y which is only depending on time t and the x-coordinate. This leads us to the *wave equation*

$$\frac{\partial^2 E_y}{\partial x^2} - \epsilon_0 \mu_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \tag{6}$$

This equation describes the propagation of the electric field through vacuum (you can of course derive the same equation for the magnetic field). In the wave equation a second derivative in space is coupled to a second derivative in time. The solution to this differential equation is $E_y(x,t) \propto \cos(kx - \omega t)$, with the *wave number* k related to the wave length $k = 2\pi/\lambda$ and the angular frequency ω to the frequency ν according to $\omega = 2\pi\nu$. Like for all waves, the frequency, wave length and velocity of the wave are coupled: $\nu\lambda = c$ with c the speed of the wave, i.e. in this case the speed of light.

Light is identified as an electro-magnetic wave and from the wave equation we see that the wave velocity is given by

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} \equiv c = 2.998 \cdot 10^8 \text{ m/s} \approx 300,000 \text{ km/s} \tag{7}$$

If the Maxwell equations are laws of physics all inertial observers must be able to write down the equation in the same form. Therefore for an observer S', traveling at constant velocity $V\hat{x}$ with respect to S, we would write down the wave equation for a field that propagates only along the x-direction with amplitude in the z-direction (without loss of generality) $\vec{E} = (0, E_y(x,t), 0)$ as

$$\frac{\partial^2 E'_{y'}}{\partial x'^2} - \epsilon_0\mu_0 \frac{\partial^2 E'_{y'}}{\partial t'^2} = 0 \; (*) \tag{8}$$

This has exactly the same form as for S which is good if it should represent a physical law. However, for S' the speed of the wave is also given by $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$. As the speed is covered by universal constants ϵ_0, μ_0, the speed is the same of S and S' or in other words $c = c'$! This is not what should happen! From the Galileo Transformation we know that we should find the same form, but with $c' = c - V$ the relative velocity of the two observers.

If we apply the coordinate transformation from $S \to S'$ according to the Galilean Transformation, the wave equation reads thus as

$$\frac{\partial^2 E_z}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t'^2} - \frac{V^2}{c^2}\frac{\partial^2 E_z}{\partial x'^2} + \frac{2V}{c^2}\frac{\partial^2 E_z}{\partial x' \partial t'} = 0 \tag{9}$$

Now we still need to find a transformation $E_z \to E'_{z'}$ (and $c' \to c$) trying to retrieve the general form of the wave equation, but there is no such transformation. Therefore the wave equation of electromagnetic waves is not Galilei invariant at all! This was a serious issue at the time.

Hypothesis of the aether As light is a wave, people naturally thought there must be a medium to transport the wave, *something* must be oscillating. Vacuum was considered nothing, not something. A water wave, needs water as medium to transport the wave; the water oscillates. Or take sound waves, they need gas, liquid or a solid to oscillate. What could be the medium that light, electromagnetic waves, use to oscillate? This medium must be all around us, in the space between the sun and earth, just everywhere. To save the Galilei invariance of Maxwell's equations this also needs to be a very special kind of medium that behaves differently than other media. This strange hypothetical medium was termed *aether*. The search for the properties of the aether lead to the Michelson-Morley experiment - which showed that there was no aether at all! Lorentz and Fitzgerald found an ad hoc way to save the day by postulating an adapted version of the Galilei transformation and a contraction of length. Later more about that, and how Einstein showed that all of this ad hoc business is not needed, things follow directly from his second axiom.

2.1.4 The Michelson-Morley experiment

The Michelson-Morley experiment was performed in between 1880-1890 to investigate properties of the hypothetical aether. The experiment returned a null-result, i.e. there was no sign of the existence of the aether - and to this day there is none.

The idea is to check the speed of light for two observers S and S'. One is moving with respect to the other with the highest possible speed, the orbit speed of the earth around the sun ~ 30 km/s. Of course, that is still only 10^{-4} compared to 300,000 km/s of the speed of light but the effects could be measured spectroscopically by interference of light.

The experiment essentially consists of a Michelson interferometer. Light is send to a 50/50 beam splitter such that half of the light is reflected up towards arm L_1 and half is transmitted to arm L_2. The mirrors at the end of each arm reflect the light back. On the way back again half of the light is transmitted and reflected at the beamsplitter, such that half of the light from both arms is now traveling downwards towards the image plane/camera. At the image plane the light from both arms forms an interference pattern, depending on the path length difference induced by the difference of $L_1 - L_2$.

The whole setup is mounted for stability on a heavy table that is floating in liquid mercury, to reduce vibrations coupling to the setup. If now one arm is parallel to the earth's orbit with $V = 30$km/s, while the other is perpendicular to it, there will be some difference between the length of the two paths traveled: $\Delta\lambda_1$. If we rotate

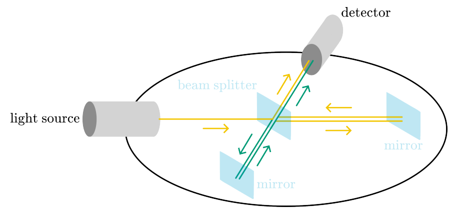
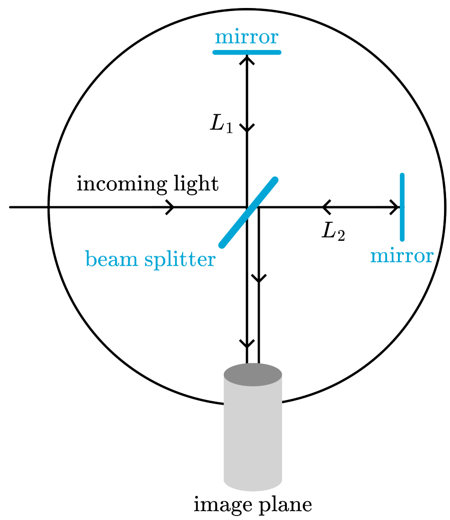


Figure 18: Michelson & Morley setup



the setup by 90 degree (easily done in the mercury bath), then the roles of L_1 and L_2 are exchanged, leading to another phase shift $\Delta\lambda_2$. Therefore after rotation the fringes of the interference pattern on the detector should shift as

$$\Delta\phi = 2\pi\frac{\Delta\lambda_1 - \Delta\lambda_2}{\lambda} = 2\pi\frac{(L_1 + L_2)}{\lambda}\frac{V^2}{c^2} \tag{10}$$

If we fill in the numbers $\lambda = 550$ nm, $L_1 + L_2 = 11$ m and $V^2/c^2 = 10^{-8}$ this results in an expected $\Delta\phi = 0.4\pi$. However, Michelson and Morely found only $\Delta\phi \leq 0.01\pi$. The experiment to find the aether failed.

Physics was in serious trouble until 1905.

NB: Back in the days, white light was used for the actual measurement and monochromatic coherent light of e.g. a sodium lamp for alignment. As white light produces a colored interference pattern which is much easier to observe visually. Otherwise temperature changes or vibrations, resulted in constant fringe drift. Today monochromatic laser light can be used in combination with environmental temperature control better than to 0.1 C and sensitive CCD cameras. Today experiments have confirmed the null-result of Michelson and Morley but to much better precision. The anisotropy in the speed of light is $c_\perp/c_{||} \leq 10^{-17}$.

Although the proposed hypothetical medium aether (or ether) does not exist, as proven a long time ago, the terminology did not drop from everyday language.

2.1.5 Einstein's axioms

In 1905 Einstein formulated his special theory of relativity with the article *Zur Elektrodynamik bewegter Körper*, Annalen der Physik, **17**:891-921, 1905. He choose the Maxwell equations and the Michelson Morely experiment as a starting point for this argument to arrive at

1. The laws of physics are the same in all inertial frames of reference.
2. The velocity of light in vacuum is the same in all inertial frames.

That does not sound like a lot or world changing, but it certainly was. You can directly see that the second axioms violates Galilean addition of velocities, but that is what was found experimentally by Michelson and Morley!

If you think these two axioms stubbornly through and take their consequences seriously, things get confusing, surprising and almost impossible to believe. Nevertheless, we will do this. Why? Because nature is this way, whether we like it or not.

Extra reading with a historic perceptive. In a 200 page book Wolfgang Pauli - *Theory of Relativity*, Dover (the original German version is available online *Relativitätstheorie*) - summarizes all that was known about special relativity as a request made by this PhD advisor Arnold Sommerfeld. It is worth a read, although the notation is a bit outdated.

Extra reading Hoe de ether verdween uit de natuurkunde. This article by Jos Engelen in the *Nederandse Tijdschrijft voor Natuurkunde* explains the Michelsen-Morley experiment, places it into historic perspective and then adds the work of Lorentz, Poincaré and Einstein leading to the Lorentz transformation.

Exercise 1: Assume the Michelson-Morley experiment uses arms of length $L = 11$ m and an aether wind speed v due to the motion of the earth around the sun.
Distance earth-sun: $150 \cdot 10^6$km.
Calculate the expected time difference Δt between light traveling parallel and perpendicular to the aether wind.

The sun itself is orbiting the center of our Milky Way at an even higher speed: 250 km/s. What would this mean for the expected time difference in the Michelson and Morley experiment?

Note: in the experiment of 1887, Michelson and Morley had used multiple mirrors in their set up for each of the two light beam paths to make the traveling length of the light, from the splitter to the mirror and the edge of the table and back, much longer than only the radius of the table and back. This is how they achieved a path length of 11m. The table itself was of course not of a diameter of 22m.

Solution to Exercise 1: Assume the Michelson-Morley experiment uses arms of length
The orbit of the earth around the sun is almost circular. Thus, we can estimate the velocity of the earth as $V = \frac{2\pi R}{T}$ with $R = 150 \cdot 10^6$ km and $T = 1$ year $= 31.6\ 10^6$ s. This gives $V = 30$ km/s.

We compute the traveling time from light leaving the beam splitter, reflecting at the mirror on the side of the table and reaching the beam splitter again. The rest of the path is identical for both light beams and does not lead to a time difference.

Time for light parallel to V:

- one part - tail wind from aether and velocity (according to Classical Mechanics with Galilei Transformation) $c + V$.
- Other part: head wind with velocity $c - V$. Thus traveling time:

$$t_{//} = \frac{L}{c-V} + \frac{L}{c+V} = \frac{2L}{c}\frac{1}{1-\frac{V^2}{c^2}} \tag{11}$$

Time to travel perpendicular to V:

$$t_{\perp} = \frac{L}{\sqrt{c^2 - V^2}} + \frac{L}{\sqrt{c^2 - V^2}} = \frac{2L}{c} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{12}$$

Putting in the numbers, we find $\Delta t = 3.67 \cdot 10^{-16} s$

This time difference may be way to small to measure. And indeed, no 'stop-watch' experiment will work. But Michelson & Morley used interferometry, i.e. interference of light. So, relevant is the difference in phase of the two light beams. This can be assessed by turning the time difference into a length: $\Delta s = c\Delta t = 1.1 \cdot 10^{-7}$m. Compare this to the wave length of the (yellow) light used by Michelson and Morley: $\lambda \approx 500nm = 5 \cdot 10^{-7}m$. Conclusion: the expected time difference is well in reach of interferometry.

2.1.6 Three body Problem

Now that we have reduced a two-particle system to a single particle problem, the question arises: can we repeat this 'trick' and turn a three-body problem in a two body problem, that in its turn can be reduced to a single particle problem?

The answer is: no. There is no general strategy to reduce a three body problem two a two body-one.

The three body problem is an old one. Already Newton himself worked on it. Its importance stems e.g. from navigation on see. It would be of great help if the position of the moon could be predicted in advance with great accuracy. Then sailors in the 17$^{\text{th}}$, 18$^{\text{th}}$ and 19$^{\text{th}}$ could have found much better their position at full sea. But no one succeeded in providing a closed solution in basic functions.

The king of Sweden, Oscar II, announced, as celebration of his 60$^{\text{th}}$ birthday, a competition with the price awarded to the one that came up with a general solution. But it took a different course. The price went to the French mathematician and physicist Henri Poincaré.

Figure 19: Click here for the Wikipedia page of Poincaré.

He showed that it was impossible to find such a solution as he reached the conclusion that the three body problem showed chaotic features. It led Poincaré to develop a whole new field: dynamic systems and what we call now *deterministic chaos*.
The work of Poincaré was the trigger of yet another 'revolution' in our understanding of the universe.

It doesn't mean that there are no known solutions of specific cases of the three body problem. On the contrary, in the movie below 20 solutions are given. Notice that they all have a high degree of symmetry.

Figure 20: Click here to see some exact solutions of the three body problem (By Perosello - Uploaded by Author, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=133294338).

Alpha Centauri A, Alpha Centauri B and Alpha Centauri C The three body problem can also be studied by numerical means. As the equations of motion are easily set up and put into a computer code, this allows us to investigate for instance the three stars of the Alpha Centauri system: Alpha Centauri A, Alpha Centauri B and Alpha Centauri C. This system is a little over 4 million light years away from us: these stars are our closest (star) neighbors. Although they form a three body system, it is stable due to the much smaller mass op Alpha Centauri C compared to the other two. Alpha Centauri A and Alpha Centauri B are of similar mass, that is 1.1 and 0.9 the mass of our sun, respectively. Alpha Centauri C, on the other hand has a mass of only 0.12 of that of the sun.

Gaurav Deshmukh has written a nice python-based web-page on this system. Below we show some examples of the simulations, that you can do yourself with the code given by Deshmukh.

First, we ignore Alpha Centauri C and used that A and B have about the same mass. The two stars start rotating around each other in ellipsoidal orbits, as we already know from the two body problem.

Figure 21: Alpha Centauri A and B circling each other.

Then, we add third small one object (not Centauri C, but one with a much smaller mass): $m_A = 1.1$, $m_B = 0.907$ (both actual relative masses), $m_C = 0.001$.

m_C tries to orbit its closest star, but at some point comes under the influence of the second star and gets 'tossed around'.

Figure 22: Alpha Centauri A and B circling each other with a third object.

If we let the simulations run for a much longer time, we see that at some point the conditions for our small star are such that it is 'shot' into space and disappears for ever.

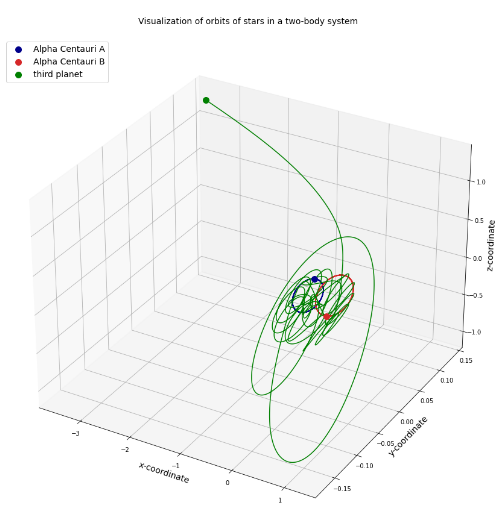


Figure 23: Alpha Centauri A and B circling each other with a third object. The third 'planet' is finally escaping into space.

Note: this is a chaotic system and computations need great care.

2.1.7 Exercises, examples & solutions

2.2 Special Relativity - Lorentz Transformation

As we discussed, in the second half of the nineteenth century it became clear that there was something wrong in classical mechanics. However, people would not easily give up the ideas of classical mechanics. We saw that the luminiferous aether was introduced as a cure and as a medium in which Electromagnetic waves could travel and oscillate. Moreover, Lorentz and Fitzgerald managed to find a coordinate transformation that made the wave equation of Maxwell invariant. Fitzgerald came even up with length contraction: if the arm moving parallel to the aether of the interferometer of Michelson and Morley would contract according to $L_n = L\sqrt{1-\frac{V^2}{c^2}}$ then, the M&M experiment should result in no time difference for the two paths, in agreement with the experimental findings. However, there was no fundamental reasoning, no physics under pinning the transformation and the length contraction. It worked, but had an ad hoc character. Very unsatisfying for physicists!

And as we have mentioned, it took the work of a single man to change this and underpin the Lorentz Transformation, making Classical Mechanics a valid limit of Relativity Theory, only applicable at velocities small compared to the speed of light and to small distances compared to those of interest in cosmology.

Figure 24: Albert Einstein (1879-1955).

Lorentz Transformation

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\
x' &= \gamma\left(x - \frac{V}{c}ct\right) \\
y' &= y \\
z' &= z
\end{aligned} \tag{13}
$$

with

$$
\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{14}
$$

But there is more! Einstein also changed our view on the universe and on time itself. In the world of Newton and Galilei, people could not even think about relativity of time. Of course time was the same for everyone. There was only one time, one master clock - the same for all of us. It is hard coded in the Galilei Transformation:

Galilei Transformation

$$
\begin{aligned}
t' &= t \\
x' &= x - Vt \\
y' &= y \\
z' &= z
\end{aligned} \tag{15}
$$

Lorentz Transformation

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\
x' &= \gamma\left(x - \frac{V}{c}ct\right) \\
y' &= y \\
z' &= z
\end{aligned} \tag{16}
$$

Now, with the Lorentz Transformation, that is no longer true: different observers may have different time. We will see that this has very peculiar consequences, some of which are very counterintuitive. However, they have been tested over and over again. And so far: they firmly hold. And there is no other way then to accept that the world and our universe is different from what we thought and from what we experience in our daily lives.

Do note, that the Galilei Transform is a limit of the Lorenz Transformation. If we let $c \to \infty$, we see that $\gamma \to 1$ and $\frac{V}{c} \to 0$. And this gives us: $t' = t$ and $x' = x - Vt$, that is the Galilei Transformation! Now, this should not come as a surprise (even if it for a moment did). After all, Classical Mechanics does an outstanding job in many, many physics problems and the agreement with experiments is excellent.

2.2.1 The Lorentz Transformation

The way we wrote down the Lorentz transformation is a bit particular in a sense that we combine time t with the speed of light c into the "time" axis ct which now has unit length. We can do this as c is constant for all observers independent of their frame of reference. The speed of light is said to be a **Lorentz invariant**. In this notation the transform between S and S' (moving with velocity V away) is *easy to remember*!

S and S' We will discuss most of the consequences for two observers S and S', traveling with a constant velocity $\vec{V}$ with respect to each other. They have taken their x, resp. x' axis parallel to $\vec{V}$. Hence, we only need to talk about V, knowing that this is the only component of the relative velocity between the two observers and that it is along the x, x' axis.

Furthermore, their y and y' coordinates are taken in the same direction. This also holds for the z-component. Finally, when S and S' pass each other (they are then both at the same point), they put their clocks to zero: $t = 0$ and $t' = 0$.

Note: S is sitting in her origin $\mathcal{O}$ (with coordinates, according to S (x,y,z)=(0,0,0)) and stays there. Similarly for S' who is sitting in $\mathcal{O}'$ (with coordinates, according to S' (x',y',z')=(0,0,0)).

The standard sketch is given in the figure below.

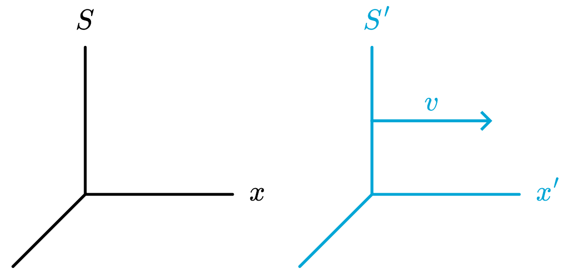


Figure 25: S and S': relative velocity parallel to the x and x' axes.

N.B. It is crucial to be very precise in your notation when it comes to coordinates and quantities. For instance: S might talk about the x-component of the velocity of an object and denote this by v_x. S', on the other hand can also talk about that component, but will not call it the x-component: in the world of S' x "does not exist", only x' does. So it is better to write $v'_{x'}$ for the x'-component of the velocity of the object according to S'. It may look cumbersome, and to a certain extend it is, but it actually does make sense. S' would say that this component is $\frac{dx'}{dt'}$ both space and time having a prime. Hence, naturally S' would talk about $\vec{r}' = x'\hat{x}' + y'\hat{y}' + z'\hat{z}'$ or $\vec{v}' = v'_{x'}\hat{x}' + v'_{y'}\hat{y}' + v'_{z'}\hat{z}'$

Lorentz Transformation and its Inverse The Lorentz Transformation, like the Galilei Transformation is a communication protocol for S and S'. It allows them to interpret information that they get from each other in their own 'world', i.e. coordinate system.

For instance, if S sees an object moving with v_x, S' can 'translate' this information via the Lorentz Transform into $v'_{x'}$ and $v'_{y'}$ or so if applicable. Of course, S also needs such a translation scheme when receiving information from S'. That is: S needs the inverse transformation.

Luckily, the inverse is very easy to reconstruct from the Lorentz Transform itself. LT from S to S' is

$$\begin{array}{rcl} ct' &=& \gamma\left(ct - \frac{V}{c}x\right) \\ x' &=& \gamma\left(x - \frac{V}{c}ct\right) \\ y' &=& y \\ z' &=& z \end{array} \tag{17}$$

The inverse is found by invoking 'relativity', after all it is called Relativity Theory. If S sees $'$ moving at a constant velocity V, then - because motion is relative- S' will say that S moves with $-V$. And thus, if S' writes down the Lorentz Transformation, she uses $-V$.

The inverse is therefore given by

$$\begin{aligned} ct &= \gamma(ct' + \frac{V}{c}x') \\ x &= \gamma(x' + \frac{V}{c}ct') \\ y &= y' \\ z &= z' \end{aligned} \tag{18}$$

with the *Lorentz factor* $\gamma(V) \equiv \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \geq 1$. Note that as γ is quadratic in V, both S and S' use the same value! That is why we don't talk about γ': it is equal to γ.

The structure of the formulas is very symmetric and therefore needs little remembering.

From the Lorentz transformation it is clear that time is not universal anymore ($ct' \neq ct$ in general). This is a large step from Newton and Galileo. Now the time coordinated is mixed somehow with the space coordinated depending on the speed V.

Lorentz factor The Lorentz factor (or γ-factor)

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \geq 1 \tag{19}$$

is a dimensionless constant depending on the ratio of the velocity V to the speed of light c. Sometimes this ratio V/c is abbreviated further as $\beta \equiv \frac{V}{c} \leq 1$. For the ratio we know that it is smaller than 1 as c is a limit velocity. From that it follows that the γ-factor is always equal to or larger than one, $\gamma \geq 1$.

In many exercises the speed V is given already as fraction of c, e.g. $V = 0.8c$. Analytically only for very few speeds a nice γ-factor is computed. These are for instance

$$\begin{aligned} V = \tfrac{3}{5}c \quad &\Leftrightarrow \quad \gamma = \tfrac{5}{4} \\ V = \tfrac{4}{5}c \quad &\Leftrightarrow \quad \gamma = \tfrac{5}{3} \\ V = \tfrac{12}{13}c \quad &\Leftrightarrow \quad \gamma = \tfrac{13}{5} \end{aligned} \tag{20}$$

Note that this list goes on for ever: there is a simple rule to find the triplets. Think about it yourself. Hint: the first one uses (3,4,5), the third one (5,12,13). What is special about them? $5^2 - 4^2 = 5 + 4 = 3^2$ and $13^2 - 12^2 = 13 + 12 = 5^2$. Do you see the pattern? Can you derive the general rule? What is the next one? How about (7,24,25)?

In the limit In the **limit of low speeds** with respect to the speed of light $\frac{V}{c} \ll 1 \Rightarrow \gamma = 1$. Practically, this happens for about $V < 0.1c \sim 30.000$ km/s. In this limit the Lorentz transformation also reduces to the Galileo transformation.

$$\begin{aligned} ct' &= ct \\ x' &= x - Vt \\ y' &= y \\ z' &= z \end{aligned} \tag{21}$$

In the **limit of infinity speed of light** ($c \to \infty$) the γ-factor is again one: $\gamma = 1$ and the ratio $V/c \to 0$. Also here the LT reduces to the GT. The case of infinite speed of light represents the case that GT is generally valid, i.e. $c' = c + V$.

It is always important to verify that an extension of the known theory reduces to the known theory that has proofed itself for most circumstances.

Historical context Lorentz did not derive the transformation that now has his name, based on Einstein's axioms. Lorentz, however, saw that Maxwell's equations were not GT invariant, therefore he tried to find a transformation

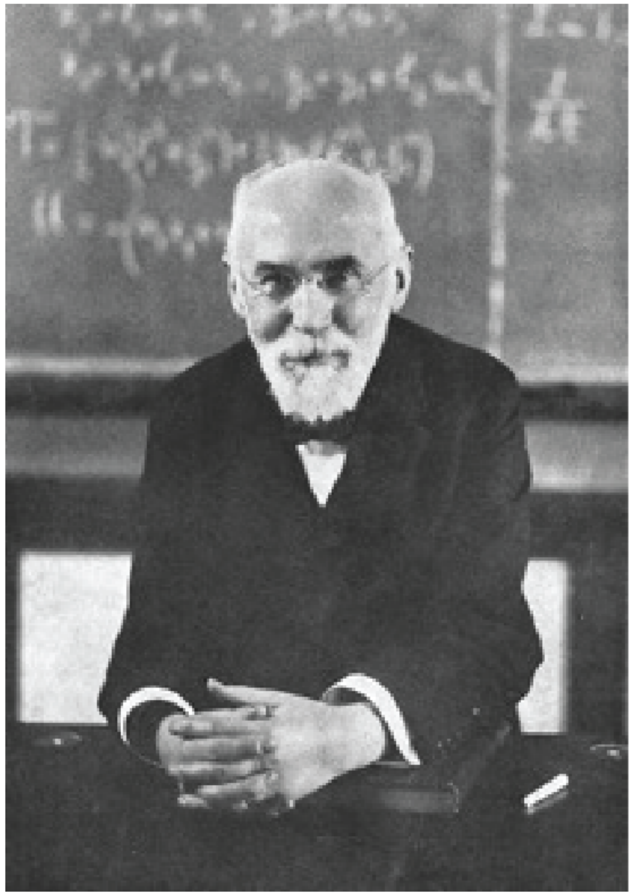


Figure 26: Hendrik Lorentz (1853-1928).

under which they were invariant. He did so (with a bit of help from Poincaré afterwards). FitzGerald did also derive the transformation, but too did not understand its implications.

Before Einstein's idea spread, Lorentz thought about the transformation as a fix to Galileo Transformation. Later he understood, of course. Unfortunately, Fitzgerald did not live long enough to see the first publication of Einstein on Relativity in 1905.

The electro-magnetic wave equation can be transformed from S to S'. And indeed, if you would do that, you would find that the wave equation maintains its form with the same c, not a new c'. Lorentz had found this, but it was Einstein who underpinned and generalized the use of the Lorentz Transformation to all mechanics, replacing the Galilean Transformation.

Exercise 1: Close your book, laptop. Shut down your screen, put aside your mobile, tablet. Put away you notes and put an empty clean sheet in front of you. All you have is that sheet of paper, one pen and your brain.

- Write down the Lorentz Transformation and its inverse.
- Repeat so you don't forget it (for the rest of your life: no one may call him/herself a physicist if he/she does not know the LT by heart ;-)).

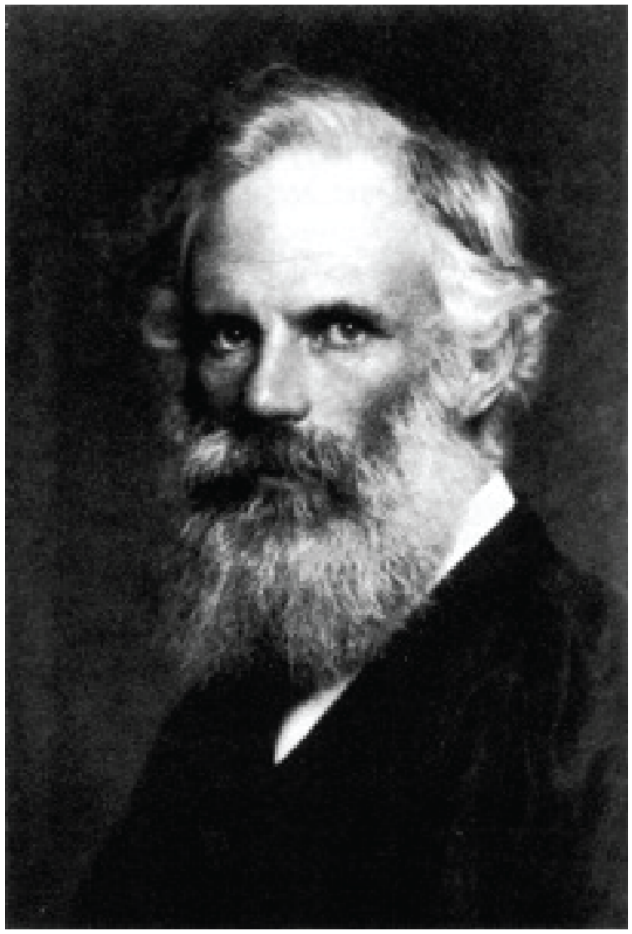


Figure 27: George Fitzgerald (1851-1901).

2.2.2 Length contraction & Time dilatation

First Implications As we have seen, we need to use the Lorentz Transformation instead of the Galilei one when two observers, S and S', want to exchange information. What does change if we do so? Let's first do some examples and see some of the consequences and the 'strange' conclusions we need to draw.

Note: we will frequently use high velocities and large distances. It is convenient not to write these in units like m and m/s. The numbers in front of them become so large that keeping an overview becomes cumbersome. Therefore, we will change to a different unit for distance: the light second. That is per definition the distance a photon of light ray travels in one second:

$$1 \text{ lightsecond} = 1ls = c \cdot 1sec = 3.0 \cdot 10^8 m \tag{22}$$

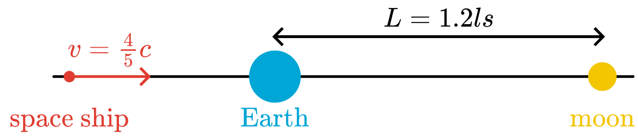
For instance, it takes a photon about 8.3 minutes to travel from the sun to the earth. Thus, the distance from the sun to the earth is $8.3lmin = 500ls$. That is equivalent to $150 \cdot 10^6 km$.

Worked Example A space ship is flying at a velocity 0.8c passed earth in the direction of the moon. The moon is at a distance of 1.2 ls (that is some $3.6 \cdot 10^8$m) from the earth. The clocks on earth and in the space ship are set to zero when the space ship passes the earth.

At time $t = 1.7ls$ observer S of the earth observes that a big comet strikes the moon surface.

When does S', who is on the space ship, see this happening?

Solution First we make a sketch.



Next, we need to carefully clarify what we mean by observe, know, see. This is very important as observations are made by someone at a certain time, while being at a certain position. Since now both time and place information gets into the transformation, being sloppy might lead to very strange and wrong conclusions.

Thus, we will from now on, specify **Events**. An event is a physical phenomenon happening at a certain place at a certain time. For instance, you catching a frisbee at 12:45 (i.e. t_f) on the campus (at location x_f, y_f, z_f). This will be denoted as:

$$\text{frisbee caught: } E_f = (ct_f, x_f, y_f, z_f) = (.., .., .., ..) \tag{23}$$

That is, four coordinates are specified (in m or ls or ...). Note: this is information as used by S: the coordinates do not carry a prime.

So, back to our example: we have our first event:

$$\text{S observes 'comet hits moon' } E_1 = (ct_1, x_1, y_1, z_1) = (1.7, 0, 0, 0) \tag{24}$$

What does this mean? Observer S, who is sitting in $\mathcal{O} = (0, 0, 0)$ literally sees that the comet hits the moon. He does so at $ct_1 = 1.7s$. In terms of physics: a photon hits his eye at ct_1. The observer has zero-size, that is everything he observes is done at (0,0,0).

Now, we need to realize, that the actual impact of the comet took place earlier. By how much? Well, a photon that was generated at this moment of impact due to the impact will have to travel 1.2 s to reach S. That requires $1.2ls$, as photons travel with the speed of light.

Thus, S concludes that the actual impact -which is event E_2- took place at $ct_2 = 0.5ls$ and he writes down:

$$\text{comet hits moon } E_2 = (ct_2, x_2, y_2, z_2) = (0.5, 1.2, 0, 0) \tag{25}$$

Again notice that we have updated this event not only by using the actual time, but also the actual place, i.e. at x_2.

S passes this information on to S'. She has to translate it to her own coordinates and uses for that the Lorentz transform.

First, she needs to calculate the γ-factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{16}{25}}} = \frac{5}{3} \tag{26}$$

Now she computes her coordinates for the same event:

$$\begin{aligned}
ct_2' &= \gamma\left(ct_2 - \frac{V}{c}x_2\right) = \frac{5}{3}\left(0.5 - \frac{4}{5}1.2\right) = -0.767ls \\
x_2' &= \gamma\left(x_2 - \frac{V}{c}ct_2\right) = \frac{5}{3}\left(1.2 - \frac{4}{5}0.5\right) = 1.333ls \\
y_2' &= y_2 = 0 \\
z_2' &= z_2 = 0
\end{aligned} \tag{27}$$

We will not further deal with the y and z coordinates as they are trivial.

But, we might get our first surprise here. According to S the impact of the comet happens at $t = 0.5s$ That is at a positive time. Then, Space Ship has passes the earth and is on its way to the moon. Actually, at $t = 0.5s$ the location of Space Ship is, according to S: $x_{SS}(t) = Vt = \frac{V}{c}ct \rightarrow x_{SS}(0.5s) = \frac{4}{5}0.5 = 0.4ls$. Space ship is already at 1/3 of the distance to the moon.

So far nothing strange.

But now we consider S'. She says: the impact of the comet was at $t' = -0.767$. This means that according to her, the impact took place when she was still approaching the earth. After all, negative times mean that Space Ship is approaching the earth (and is to the left of it in our sketch), while positive times mean that Space Ship has passed the earth and is moving away - thus is at the right side of earth in our sketch.

And this is so according to both S and S'. They may use different times, but they have set their clocks to zero when earth and Space ship were in 'the same position'.

Ok, let's be puzzled for a while: how can S' at the same time be both at the left side and at the right side of the earth? That doesn't make any sense!!!! What is **wrong** with this new theory? The answer is: **nothing!**

It is us, mixing stuff up. Who said that it is 'at the same time'?!? Nobody (with perhaps for a moment us as the exception). S and S' agree upon the event: a comet hits the moon. This physical phenomena is not disputed at all. It happened. They don't agree that it took place at the same time according to their clocks.

But this is not all: according to S at the moment of the impact Space Ship was at a distance of $1.2 - 0.4 = 0.8ls$ from the moon. But S' just calculated that she was $1.33ls$ from the moon. One can not be at two different distance form the moon at the same time!

Ok, let's push this somewhat further and see if we can get a contradiction.

We do know, from S that the event took place at $ct_2 = 0.5ls$. Then, definitely S' has passed earth. S has reconstructed this event from observation Event E_1 S' got the information of event E_2 from S and backed out the coordinates of the event in her coordinate system. From these data, S' can easily predict when she will see the impact. That is obviously later then the time of the event: the photons have to travel to her. How can we compute when S' literally sees the event?

That is remarkably easy: we know that according to S' the event tokes place at $ct'_2, x'_2) = (-0.767ls, 1.333ls)$. At that moment and that place a photon was generated that moves in her direction. Since the velocity of each photon is always c, we can easily find the time when S' sees the photon, i.e. detect it at location $x' = 0$. The photon has to travel a distance $1.33ls$ at a speed of $1c$. That will take 1.33s. The photon started traveling at time $ct_2 = -0.767$. It's trajectory according to S' is $x'_p(t') = x'_p(0) - c(t' - t'_2)$.

Thus, the photon gets measure at event E_3: $x'_3 = 0 \rightarrow ct'_3 = x'_2 + ct'_2 = 0.567ls$. Thus we have our third event:

$$\text{Space Ship observes impacting comet: } E_3 = (ct'_3, x'_3) = (0.567, 0) \tag{28}$$

And as we by now kind of expected: indeed, then is Space Ship on the right side of the earth. What does S say about this event? He receives the coordinates of $E3$ from S' and plugs them in, in the inverse LT:

$$\begin{aligned} ct_3 &= \gamma\left(ct'_3 + \frac{V}{c}x'_3\right) = \frac{5}{3}\left(0.567 + \frac{4}{5}0\right) = 0.945ls \\ x_3 &= \gamma\left(x'_3 + \frac{V}{c}ct'_3\right) = \frac{5}{3}\left(0 + \frac{4}{5}0.567\right) = 0.756ls \end{aligned} \tag{29}$$

Now does this make any sense? It does! If we concentrate on S only and what he observes and knows:

- E_1 - S observes -comet hits moon: (ct_1,x_1) = (1.2, 0) ls
- E_2 - the comet actually hits the moon: (ct_2,x_2) = (0.5, 1.2) ls

-
- E_3 - S' observes that the comet hits the moon: (ct_3,x_3) = (0.945, 756) ls

Obviously, if the actual impact is at positive t, then S' will see it before S does as for positive time t S' is closer the moon than S. And this is all reflected in the events. Moreover, if you would compute the events as S will model things, you will find event E_3 just based on event E_2 and the motion of Space Ship according to S (and when it will encounter a photon that was generated at the actual impact of the comet on the moon). Do the calculation yourself and see, that nothing strange happens.

We can draw the position of earth, moon and space ship in space-time plot. It is customary to use as horizontal axis the x or x' coordinate and as the vertical one ct or ct'. S will see the earth and moon standing still and thus draw a vertical line in the space-time diagram for each of them: they do not change position, but their time is changing, i.e. the clock ticks. S would draw for Space Ship a straight line moving from left bottom to upper right as the space ship moves in the positive direction.

Similarly, S' will draw a vertical line for Space Ship itself, as in the frame of reference of S' the space ship, obviously, does not move. The earth and moon move to the left, thus their trajectories are straight line from the bottom right to the upper left in the (x', ct')-diagram.

At some moment in time-space the comet impacts the moon and a photon is moving in the negative x-direction towards the earth. Somewhat later, this photon is received by earth. In the (x, ct)-diagram this is a straight line from lower right to upper left.

In the animation below the whole scenery is shown from the perspective of S on the left side and from S' on the right side. The diagrams are made such, that the event "Space Ship passes earth" is simultaneous in both diagrams, i.e. it happens for both observers at their time equal to 0. All other events happen at different times according to the clocks of the observers.

An animation is given above.

- the three squares represent the position of earth, moon and Space Ship according to

S at $ct = -1$ls. In the diagram for S, these three are, of course, on a horizontal line as they are at the same time according to S.

However, S' sees that differently: there are absolutely not at the same time!!!

Earth, moon and Space Ship do travel in the space-time diagrams. Their trajectories are shown by dashed lines. Their space-time location is represented by the (moving) dots. The diagrams are made such, that indeed both observers pass each other at

$ct = ct' = 0$ and $x = x' = 0$. The dots represent, where according to S (left diagram) and S' (right diagram) earth, moon and Space Ship are at a certain time on the clock of that observer. Note that both position and time have really different values if you compare the diagrams of S and S'.

In both diagrams, at some point in time the comet impacts the moon and a photon starts traveling in the negative

x and x'-direction. The photon is shown by the blue dot. Again nothing happens at the same time. But the order of events is the same: first the photon is emitted and only after that it is observed. That should of course hold!

Notice that the photon is emitted at

ct=0.5ls according to S and observed at ct=1,7ls. So for S, the photon traveled for 1.2ls (and covered a distance of 1.2ls: of course, photons travel with velocity c). However, for S' this is quite different: the photon is emitted at ct'=-23/30, that is much earlier than S reports. Moreover, it is only registered by S on ct'=85/30ls. It traveled for 3.6 seconds on the clock of S'!!

Puzzled by this all? Confused? Hard to believe?

Welcome the 'Magical World of Relativity'. And don't worry: you will get used to it. Moreover, we will develop a mathematical framework that helps us and prevents our failing intuition to take the wrong path.

Conclusions:

- we need to be careful with interpreting distances and times, things are not what they seem at first glance.
- within the framework of one observer nothing funny happens.
- we better work with well defined events: they represent physical phenomena happening. Both observers will agree upon these and on the logic, e.g. first the impact than the observation of a photon - not the other way around!

Time & Space Here we have a look at the consequences of axioms 1 & 2. We know how two observers S and S' (moving away with V) transform their respective coordinates into each other, via the Lorentz transformation.

LT
Lorentz Transformation

$$
\begin{aligned}
ct' &= \gamma\left(ct - \frac{V}{c}x\right)\\
x' &= \gamma\left(x - \frac{V}{c}ct\right)\\
y' &= y\\
z' &= z
\end{aligned}
\tag{30}
$$

with

$$
\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}
\tag{31}
$$

We will look at the consequences for time and space coordinates.

Relativity of simultaneity From the Lorentz transformation it is clear that time is not universal anymore ($ct' \neq ct$ in general). This is a large step from Newton and Galileo. Now the time coordinate is mixed somehow with the space coordinates depending on the speed V.

Let us consider 2 events in the reference frame of S;

- event A with coordinates (ct_1,x_1)
- and event B with ($ct_2,x_2)$.

If the two events in S are simultaneous, i.e. $t_1 = t_2 \to ct_1 - ct_2 = 0$, then in S' they are in general not! Simultaneity is relative!

$$
\begin{array}{rcl}
ct_1' &=& \gamma\left(ct_1 - \frac{V}{c}x_1\right)\\
ct_2' &=& \gamma\left(ct_2 - \frac{V}{c}x_2\right)\\
\Rightarrow ct_1' - ct_2' &=& \gamma(ct_1 - ct_2) + \gamma\frac{V}{c}(x_1 - x_2)
\end{array}
\tag{32}
$$

Even though the first term $(ct_1 - ct_2) = 0$ the second term $(x_1 - x_2)$ is never zero unless $x_1 = x_2$, and $ct_1' - ct_2' \neq 0$ in general.

In words: The events A and B that are simultaneous for S, are never simultaneous for S', unless the events are happening at the same place.

Relativität der Gleichzeitigkeit as Einstein called it, is the first very counterintuitive consequence by simple application of the Lorentz transformation. Our brains are not trained and build to cope with this aspect of nature. There is just no evolutionary advantage to it as all relevant speeds are much smaller than the speed of light.

Time dilation We investigate how time intervals between a stationary and a moving observers are transformed. We can expect that these time intervals are not the same.

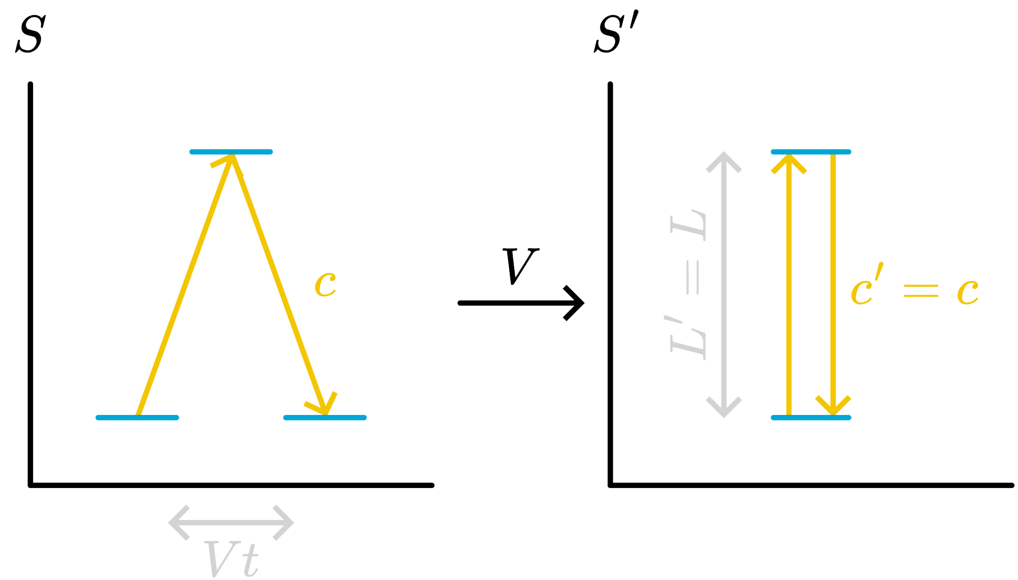


Figure 28: Clock stationary according to S' but moving for S.

If you consider the sketch above, we see how time intervals are counted for a moving observer and for an observer in the rest frame. A light ray is traveling between 2 mirrors. This up and down traveling of the light is a counter for the time. If you have never thought how time is measured, think a bit how a clock actually does that. Today, the second is defined as a (very large) number of tiny energy transitions (vibrations) of the Caesium-133 atom (see e.g. Atomic Clock).

Consider the time light travels for the observer S who sees the clock moving with velocity V. The clock counts one unit of time, t if the light has gone from the bottom mirror to the top one and back to the bottom mirror. Thus from bottom to top takes $t/2$. This means that the length of the light path from bottom mirror to top mirror is equal to $ct/2$ as light travels with velocity c. In that same period of time, the top mirror has moved a distance $Vt/2$, as the clock and thus the mirrors move with velocity V with respect to observer S. Now, we can relate the length of the light path from the bottom to the top mirror to the size of the clock, L and the displacement of the mirror, $Vt/2$: $L^2 + \frac{V^2}{4}t^2 = \frac{c^2}{4}t^2$ where we used Pythagoras, see figure below.

We can solve this for the time t that the stationary observer S puts to the moving clock

$$t = \frac{2L/c}{\sqrt{1-\frac{V^2}{c^2}}} = \gamma\frac{2L}{c} > \frac{2L}{c} \tag{33}$$

We see directly that the time the stationary observer S records is larger than the moving observer S' itself which is just $2L/c$ (the time in his rest frame)! The time interval gets longer/dilated by the γ-factor.

$$\Delta T = \gamma \Delta T_0 \tag{34}$$

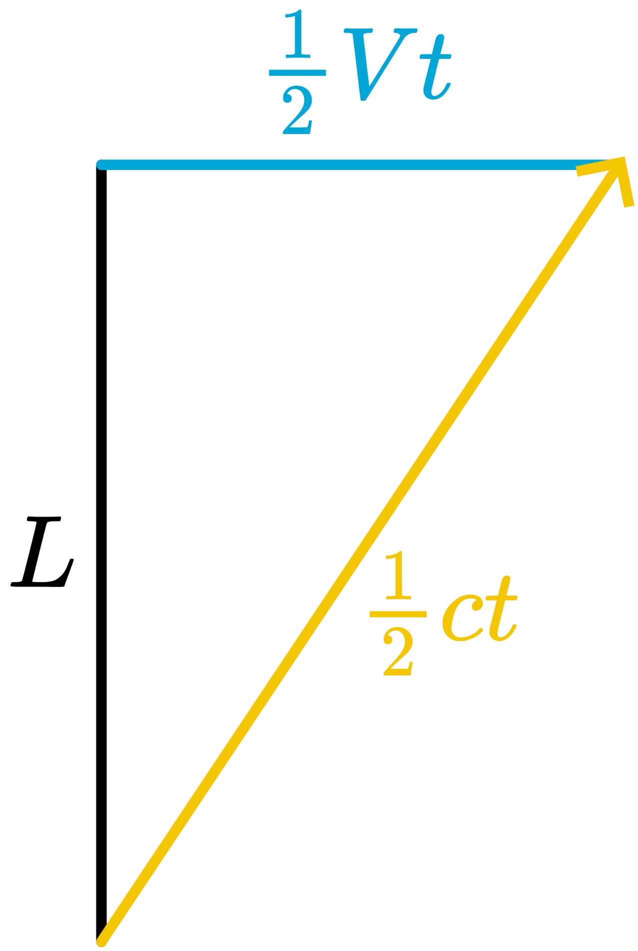


Figure 29: Light path in a moving clock.

with $\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} > 1$ and T_0 the ***proper time*** or *eigen time* in the rest frame.

Note: a time interval is also the counting of your heart. That means the moving observer ages more slowly compared to the observer at rest. See the examples below for some experimental evidence of the time dilation.

** Conclusion: moving clocks run slower, time gets stretched**

Length contraction The length of moving objects becomes smaller/contracted for the observer at rest. To explain this effect, we consider a moving rod with velocity V and with length L_0 in the rest frame.

Now that we have seen that time intervals are no longer universal, we need to think about:

"what is it, measuring the length of an object?"

Normally, we measure the length of an object by seeing how many times a measuring stick fits in the object. We obviously do this in the frame of reference in which the object doesn't move. There we don't need to worry about the moment we start at the left side of the object and arrive with our measuring stick on the right side. But if we would do so in a frame of reference in which the object is moving, that wouldn't work of course. By the time we would reach the right side of the object, it would no longer be at its starting position when we began our measurement and the number of times our ruler fits in the object is now influenced by the motion of the right side of the object.

To measure the length of a moving object, we thus need a different strategy. What we could do, is having a very long ruler fixed in our system. The object is moving passed it. If we have two observers, one concentrating on the left side of the object and the other on the right side, we could ask them to measure the position of the left and right side of the object along the ruler **at the same time**. Then the difference of the left and right side on the ruler will give us the length of the object.

Thus: the length is measured from the difference of two events in space-time of the front and the back of the rod. We will call the events $E_L : (ct_1, x_1)$ and $E_R : (ct_2, x_2)$. As we measure size, we require: $t_1 = t_2$, that is the measurements are done simultaneously in S. According to S, the length of the rod is $L = x_2 - x_1$, nothing special here.

Next, we transform the events E_L and E_R to S':

$$ \begin{split}

x'_1 &= \gamma \left (ct_1 - \frac{V}{c} x_1 \right) \ x'_2 &= \gamma \left (ct_2 - \frac{V}{c} x_1 \right) \

\end{split}$$

For S' the difference between x'_2 and x'_1 is of course the length of the rod. It doesn't matter for S' whether or not the coordinates the left and right side of the rod are measured at the same time. The rod is not moving in the frame of S'. Thus S' gets as length of the rod:

$$L_0 = x'_2 - x'_1 \tag{35}$$

with L_0 the ** proper length** of the rod, i.e. the length according to an observer moving with the rod.

Now we invoke the Lorentz transformation for the two events E_L and E_R to find the relation between the coordinates used by the two observers:

$$L_0 = x'_2 - x'_1 = \gamma(x_2 - x_1) - \gamma\frac{V}{c}(ct_2 - ct_1) \tag{36}$$

As we measure x_1, x_2 at the same time in S, we have $ct_2 = ct_1$.

$$L_0 = \gamma(x_2 - x_1) = \gamma L \Leftrightarrow L = \frac{L_0}{\gamma} \tag{37}$$

The length of the moving object observed by the stationary observer is not the same as the length in the rest frame. The length observed by the stationary observer S gets smaller/contracted by $\gamma > 1$ compared to the length in the rest frame of S': $L < L_0$.

** Conclusion: moving rods are shorter, space shrinks**

Paradox: twins and barns

Barn & Ladder There are many variants of the following paradox. The word *paradox* already implies that there is only an apparent contradiction, not a real one. Here we will formulate the paradox with a ladder & barn and resolve it, but you can also think about it as a train & tunnel, or tank & trench etc. The resolution is always the same.

As an example we consider a ladder of rest length $L_l = 26$ m and a barn of rest length $L_b = 10$ m. Obviously, the ladder does not fit in the barn, isn't it?

Now consider that the ladder is moving with velocity $V = \frac{12}{13}c$ ($\gamma = \frac{13}{5}$) towards the barn.

- For an observer in the barn, the length of the ladder is contracted to $L_l/\gamma = 26 \cdot \frac{5}{13} = 10$ m exactly fitting in the barn which in her rest frame is 10 m.
- For an observer moving with the ladder, the barn gets contracted to $L_b/\gamma = 10 \cdot \frac{5}{13} = 50/13 \sim 4$ m, being much too small to fit in the ladder. The ladder in his rest frame is 26 m.

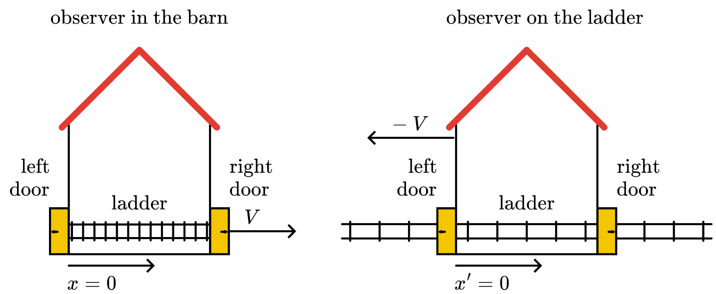


Figure 30: Ladder & Barn: perspective from two observers.

We have applied the Lorentz transformation or length contraction (time dilation) and the concept of relativity correctly, but something seems wrong! The physical outcome must be the same for both observers, but one observer claims the ladder perfectly fits into the barn, the other say it does not! That is: the observer in the barn can close the left and right door when the ladder is just inside the barn. Of course, the doors need to be open again very quickly as the ladder is moving with high velocity to the right. But that doesn't take away the fact that doors were closed and the ladder was inside the barn. How does the other observer cope with this?

You can have the same paradox not with length contraction, but time dilation, then it is called the *twin paradox*. We discuss the twin paradox later in the framework of Minkowski-diagrams.

Solution

```
The key to the resolution of the paradox is always the relativity of simultaneity. In this instance of
```

Here we analyze the situation in detail using the Lorentz transformation. Later you can analyze it again qualitatively using a Minkowski-diagram which is quite insightful.

Our above "analysis" was a bit short: using length contraction. It is also a bit 'dangerous' as length contraction assumes simultaneous events in one frame.

We will consider how both observers would actually *measure* things in their respective frames of reference and in which order these happen. It turns out that both points of view are correct, but with a twist. We define 4 events to analyze the situation.

1. Event 1: right end ladder at left door barn
2. Event 2: right end ladder at right door barn
3. Event 3: left end ladder at left door barn
4. Event 4: left end ladder at right door barn (not really needed)

The four events are sketched in the figure below

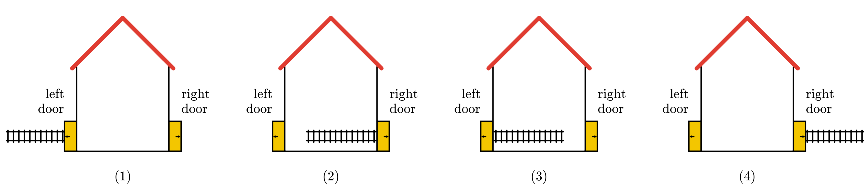


Figure 31: Four events of the ladder & barn paradox

Note: the size of the ladder in the sketch above is of course open for debate between the two observers :-).

Observer Barn (B) will conclude that the ladder fits inside the barn and actually is inside the barn if Event 3 is earlier then Event 2, according to the clock of observer B. If, however, Event 3 is later than Event 2, the ladder does not fit. Similarly, observer Ladder (L) will draw the same conclusions, but based on the clock of observer L.

Let's analyze these events. We will denote the coordinates of observer B as (ct, x) and those of observer L as (ct', x'). Both observers agree that they will call the position of the left door the origin, that is $x_{LD} = x'_{LD} = 0$. Moreover, they agree that at the moment the right end of the ladder is at the left door, they will set their clocks to 0. Remember: according to observe B, the length of the ladder is $L_{0L}/\gamma = 10$ m, which happens to be the size of the barn according to B. We anticipate that B will conclude that the ladder fits.

Next, we need to give the events their space-time coordinates, e.g. in the frame of Be and transform these coordinates according to the LT to the frame of L. This is done below, where we used: L_{0B} = proper length of barn, i.e. in the rest frame of the barn and L_{0L} = proper length of ladder, that is in the rest frame of the ladder. Note: $V/c = 12/13 \Rightarrow \gamma = 13/5$

Event	Barn (ct,x)	Ladder (ct',x')
1	(0, 0)	(0, 0)
2	(\frac{c}{V}L_{0B}, L_{0B})	(\frac{c}{V}\frac{L_{0B}}{\gamma}, 0)
3	(\frac{c}{V}L_{0B}, 0)	(\gamma \frac{c}{V}\frac{L_{0B}}{\gamma}, -L_{0L})

As we see, according to B, the left and right end of the ladder are exactly at the same moment at the left and right door of the barn, respectively (time coordinate of events 2 & 3 $ct_2 = ct_3 = \frac{c}{V}L_{0B}$). Consequently, observer B measures that the ladder (just) fits into the barn as anticipated by us. So B can close both doors and have the ladder inside the barn.

However, if we look at events 2 & 3 according to L, we see that L measures that the right end of the ladder is much earlier at the right door (event 2 $ct'_2 = \frac{c}{V}\frac{L_{0B}}{\gamma}$), than the left end is at the left door (event 3 $ct'_3 > ct'_2$). So, according to L, when the ladder hits the right end of the barn, the left part of the ladder is still left from the left door, thus outside the barn. The ladder does not fit. Of course, L sees that B closes the doors of the barn, but contrary to what B says: 'I closed the doors simultaneously and the ladder was in my barn", L will respond: "that may be true for you, but I clearly observed that you first shut the right door, while the left was still open. Then you quickly opened the right door to let the ladder pass and only after a while, when the left side of the ladder was just inside your bar, you closed the left door. The ladder was never inside the barn with both doors closed!"

The paradox is, that both observers are right. Again we see demonstrated that simultaneous for one does not necessarily mean simultaneous for another. Very counter intuitive and yet: very true.

As you see both observers do not agree where the ladder is when the left door is closed. Where for the barn observer both doors closes at the same time, this does not happen for the ladder observer.

Worked Example This problem became known through John Bell.

??? "Why you absolutely need to know John Bell"

John Bell became famous by the inequalities that have his name attached. Bell's theorem from 1964 started to end (post mortem) the twist between Einstein and Bohr about quantum mechanics in favor for Bohr. In 1935 Einstein, Polodsky and Rosen came up with a paradox, named EPR paradox after their names, that seemed to show that quantum mechanics cannot be "complete" (.i.e *the real thing* describing reality). Bell's inequalities allowed to experimentally test who was right, and Einstein was fundamentally wrong. In 2022 the Nobel Prize in Physics was awarded to Clauser, Aspect and Zeilinger for their efforts to experimentally show that the Bell's inequalities are violated (and Bohr was right). In Delft Roland Hanson performed a *loophole-free Bell test* in 2015 which was big news.

Why is this so important? It touches the heart of what is reality, is it deterministic and/or local now that quantum mechanics turned out *to be* the real thing? How we see reality now boils down to how we interpret quantum mechanics - and that is difficult to comprehend. The Copenhagen interpretation is so frustrating as the wave function collapses at measurement, however, the many-world interpretation that avoids the collapse is also not very appealing as it needs an infinite number of universes. This remains one of the important open ends in physics.

In this thought experiment we have two space ships B and C initially at rest and space ship A as observer. B and C are connected by a tight but fragile string between them. A simultaneously signals B and C to accelerate equally, and B and C will have the same velocity at every time from the start.

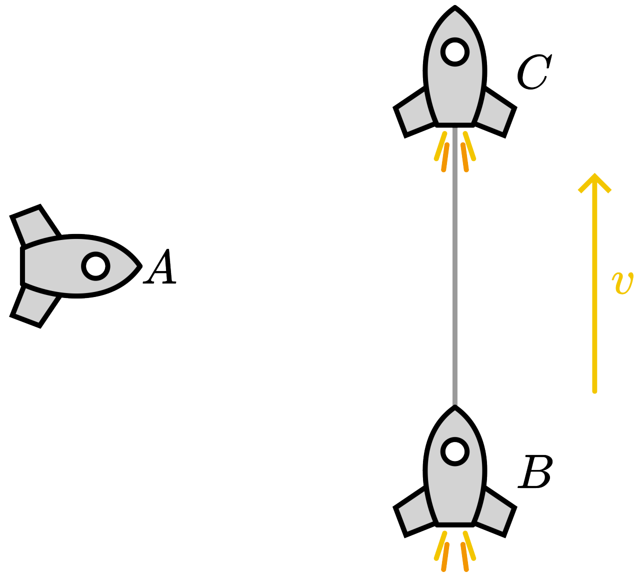


Figure 32: Bell's paradox: accelerating space ships and a thin wire.

Question: Will the string between B and C break eventually?

??? "Answer"

```
Yes.
```

Explanation:

One might think that the whole assembly of the two ships B and C and string undergo length contraction together, thus the string would not break, but that is incorrect.

- As seen from A's rest frame, B and C will have at every moment the same velocity, and so remain displaced one from the other by a fixed distance. The tying will not be long enough anymore due to length contraction and therefore break.
- The distance between B and C in the rest frame of B or C *increases* however as the acceleration from neither of them is simultaneous (if you work this out the relativity of simultaneity is the issue)! The thread breaks also in their frame.

If you got this wrong, do not worry, most people do (that is trained physicists).

If you think about this example for a bit, it becomes clear that relativistic acceleration is very troublesome for the structural integrity of extended objects! Another problem for our hopes of space travel to far away places.

2.2.3 Exercises, examples & solutions

Muon production in the upper atmosphere
Muons are elementary particles of the lepton family, the heavier brother of the electron. Muons decay via $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ (or $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. NB: You need the neutrinos to conserve lepton number) with a mean lifetime of $\tau = 2.2\ \mu$s. Muons are generated in the upper atmosphere (20 km) when a high energetic cosmic ray hits a nuclei as decay products. The speed of the muons is about $v = 0.99c$. If you compute velocity times lifetime $\tau v < 1$ km, then we conclude that nearly no muons should be detectable on the ground (assuming no other process interferes in the muons path). But we do? How is this possible?

Solution
You can solve this by considering the time dilation for an earth observer, as the lifetime is with respect to the rest frame! The lifetime for an earth observer is therefor stretched to $\gamma\tau \sim 16\ \mu$s. Therefore muons only need to travel about 4 lifetimes, and a decent fraction $(1/16)$ can still be measured on the earth surface. You can also reason via length contraction of the path the muons travel 20 km/γ.

Special relativistic correction to GPS timing
GPS uses satellites orbiting the earth at a lower altitude to determine the position. If you receive the signals from 4 or more satellites, you can compute your position by triangulation, e.g. measurement of time difference of the received signals. To this end you need a very precise timing of the signals. The satellites velocity is "slow" with $v = 4 \cdot 10^3$ m/s, and thus $\gamma \sim 10^{-5} \ll 1$. But the error in time measurement accumulates and due to time dilation even this small γ-factor will increase within 1 hour to a time error of 10^{-7} s or a position error of about 100 m. This would not be useful for navigation in a city and would required a recalibration of the system every few minutes. Later we see that a **general relativistic effect** is even more prominent!

Relativistic correction to wavelength of electrons in a TEM
In a standard Transmission Electron Microscope the electrons are accelerated via electric potential differences of up to 300 kV. Assuming that particles have a wavelength via the idea of de Broglie $E = mc^2 = pc = h\frac{c}{\lambda} \Rightarrow \lambda = \frac{h}{p}$ we can use electrons as waves to image and magnify as with a normal light microscope. The smallest detail you can image with waves imaging in the far-field is given by the diffraction or Abbe resolution limit to $d \sim \frac{\lambda}{2}$. For microscopy with visible light ($\lambda \sim$ 500 nm) this limit is a hard restriction. For electrons of low speeds we can use $\lambda = \frac{h}{mv}$, but for 300 kV acceleration the speed would be already larger than c! Later in the course you learn how to compute the **relativistic momentum**, filling in the numbers and the rest mass of the electron of 511 keV we obtain $\lambda \sim 2$ pm. About 10% *smaller* than from classical considerations. The diffraction limit to resolution is not an issue practically for the electrons as the distances between atoms in a solid are typically > 10 pm.

Examples

Exercise 1: During their quest to find planets at other stars than our sun, ESA researcher discover a planet that shows striking similarities with earth. This planet orbits a star 40 lightyears from us. They start planning an expedition with astronauts. ESA requires that the astronauts upon arrival at the planet have aged no more than 30 years.

In this exercise, we ignore possible effects of acceleration. A lightyear is the distance traveled by a photon in one year.

1. What is the required velocity of the space ship (with respect to the reference frame of the earth) to ensure a journey of 30 years (ignore the time spent on the other planet)?

2. What is according to the astronauts the distance the have to travel? Does that agree with the journey time of 30 years?
3. To inform Mission Control on earth the astronauts send yearly (according to their clock) a report to earth. Of course, the report is coded in the form of a light pulse. What is the period between receiving two consecutive reports according to Mission Control?

Exercise 2: An observer S' is traveling in a fast train. According to S', the train has a length $2L'$. The train is speeding with V over a track that is along the x-axis. At $t' = 0$ S' passes the origin of the frame of reference of S, who is stationary with respect to the track. At the moment of passing, S sets her clock to $t = 0$.

S' is in the middle of the train. He send at $t' = 0$ two light pulses out. One in the direction of the front of the train, where this pulse reflects on a mirror and is traveling back to S'. The other pulse is send to the back of the train and reflects there back to S'. S' claims that both pulses are received back at the same time.

1. Define the events that define this problem and give the coordinates as S' would do.
2. Translate the events to the frame of S.
3. Does S also see the two pulses reach $'$ at the same time?
4. Draw a (ct, x) diagram in which the the trajectories of S', front and back mirror as well as the two pulses are shown. Note: the ct-axis is the vertical axis in such a graph. Can you graphically understand whether or not the two pulses arrive at S' at the same time according to S.

Exercise 3: Observer S' is traveling with velocity V with respect to observer S. Both observers have aligned their x, x' axis and set their clocks to zero when their origins coincide.

According to S', an object is approaching S' at a velocity -V. At $ct' = 0$, the object is a distance L' from S'. At some moment in time it will collide with S'.

1. The initial time and position of the object at $ct' = 0$ is marked as Event 1 by S'. Provide the coordinates of E1 according to S' and according to S.
2. Determine the event "object collides with S'" (event E2) according to S' and according to S.
3. Can you understand the values of x_1 and x_2?

Exercise 4: Observer S' is traveling with velocity V/c=4/5 with respect to observer S. Both observers have aligned their x, x' axis and set their clocks to zero when their origins coincide.

According to S, an object is moving at a velocity -V/c = -4/5. At $ct = 0$, the object is in the origin of S. At some moment in time, ct, it is located somewhere on the negative x-axis.

Do the exercise twice: first for observers in the world of Einstein and Lorentz, second time for the world of Newton and Galilei.

1. Define two events: one (E1) where the object is at $ct = 0$ and the other (E2) where it is at ct. Transform both objects to S'.

2. For an object moving at constant velocity, the velocity can be found from two locations at two separate moments in time. Find the velocity of the object according to S' and show that its magnitude is smaller than the speed of light in the world of Lorentz and Einstein. To people living in the world of Newton and Galilei, this is a surprising result. They would have found a velocity magnitude larger than c.

Exercises

Solution to Exercise 1: During their quest to find planets at other stars than our sun, ESA researcher discover a planet that shows striking similarities with earth. This planet orbits a star 40 lightyears from us. They start planning an expedition with astronauts. ESA requires that the astronauts upon arrival at the planet have aged no more than 30 years.

1. Denote Mission control by S and the space ship by S'. According to S, the distance to the planet is $L = 40ly$. Thus the traveling time will be $\delta t_e = \frac{L}{V}$, with V the velocity of the space ship according to S. S' time dilation $\to \delta t_e = \gamma \delta t_0$
   Requirement: $\delta t_0 = 30ly \to \frac{L}{V} = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\delta t_0 \Rightarrow \frac{V}{c} = \frac{4}{5}$
2. Length contraction: $L' = \frac{L}{\gamma} \to L' = \frac{40}{5/3} = 24ly$
   According to the astronauts, the planet is approaching them with a velocity $-V \Rightarrow \frac{V}{c} = -\frac{4}{5}$.
   So they have to wait $\delta t'_w = \frac{L'}{\frac{4}{5}c} = 30y$
3. in S' a light pulse every year. Define event = n$^{\text{th}}$ pulse $(ct'_n, x') = (n, 0)$. The (n+1) pulse $(ct'_{n+1}, x'_{n+1}) = (n+1, 0)$ Transform to S via inverse LT

$$
\begin{aligned}
n^{th}pulse: &\begin{cases} ct_n = \gamma\left(ct'_n + \dfrac{V}{c}x'_n\right) = \gamma ct'_n \\ x_n = \gamma\left(x'_n + \dfrac{V}{c}ct'_n\right) = \gamma V t'_n \end{cases} \\
(n+1)^{th}pulse: &\begin{cases} ct_{n+1} = \gamma\left(ct'_{n+1} + \dfrac{V}{c}x'_{n+1}\right) = \gamma ct'_{n+1} \\ x_{n+1} = \gamma\left(x'_{n+1} + \dfrac{V}{c}ct'_{n+1}\right) = \gamma V t'_{n+1} \end{cases}
\end{aligned} \tag{38}
$$

The n$^{\text{th}}$ arrives at earth after traveling the distance x_n with the speed of light. Hence, the moment of receiving is:

$$t_{n,e} = t_n + \frac{x_n}{c} = \gamma n\left(+\frac{V}{c}\right) \tag{39}$$

Similarly for the (n+1)$^{\text{th}}$:

$$t_{n+1,e} = t_n + 1 + \frac{x_n + 1}{c} = \gamma(n+1)\left(+\frac{V}{c}\right) \tag{40}$$

So, we conclude that the time between receiving two consecutive pulses by Mission Control is:

$$\delta t_e = t_{n+1,e} - t_{n,e} = \gamma\left(+\frac{V}{c}\right) = 3year \tag{41}$$

Is that possible?

The astronauts send 30 reports while on their way to the planet as their journey to the planet takes 30 years. According to S this journey takes $\frac{L}{V} = 50year$. The last pulse is send 50 years after S' has left earth. This pulse need to travel 40ly and that takes 40 years. Thus it is received after 90 years. In those 90 years, 30 pulses have been received, hence Mission Control receives a pulse every 90/30 = 3 years.

This is a great example, that you need to be careful with quick answers based on time dilation. That would give $\gamma \cdot 1year = \frac{5}{3}year$ in between two pulses. But than we have forgotten that these pulses are not send from the same location.

Solution to Exercise 2: An observer

1. Events:

E0 - pulses send: $(ct'_0, x'_0) = (0, 0)$
E1R - forward traveling pulse hits front mirror: $(ct'_{1R}, x'_{1R}) = (L', L')$
E1L - backward traveling pulse hits back mirror: $(ct'_{1L}, x'_{1L}) = (L', -L')$
E2 - pulses send: $(ct'_2, x'_2) = (2L', 0)$

2. LT the events to S

E0: $(ct_0, x_0) = (0, 0)$
E1R: $(ct_{1R}, x_{1R}) = (\gamma(L' + \frac{V}{c}L', \gamma(L' + \frac{V}{c}L') = \gamma\left(1 + \frac{V}{c}\right)L'$
E1L: $(ct_{1L}, x_{1L}) = (\gamma(L' + \frac{V}{c} - L', \gamma(-L' + \frac{V}{c}L') = \gamma\left(1 - \frac{V}{c}\right)L'$
E2: $(ct_2, x_2) = (\gamma 2L', \gamma 2\frac{V}{c}L')$

3. right pulse: first part of the traveling time is longer as the right mirror moves away, but on the way back S' approaches the pulse. The left pulse does exactly the opposite: first going to a mirror that is approaching and then moving after S' that is 'running away'.
4. This becomes evident in the (ct, x) diagram.

Solution to Exercise 3: Observer

1. E1:

$$(ct'_1, x'_1) = (0, L') \Rightarrow \left\{\begin{matrix} ct_1 = \gamma\left(ct'_1 + \dfrac{V}{c}x'_1\right) = \gamma\dfrac{V}{c}L' \\ x_1 = \gamma\left(x'_1 + \dfrac{V}{c}ct'_1\right) = \gamma L' \end{matrix}\right\} \Leftrightarrow (ct_1, x_1) = (\gamma\frac{V}{c}L', \gamma L') \tag{42}$$

2. trajectory object according to S' $\rightarrow$ linear motion with velocity $-V$: $x'(ct') = L' - \frac{V}{c}ct'$

collision with $S' \Rightarrow x'(ct'_2) = 0 \rightarrow ct'_2 = \frac{L'}{V/c}$

Thus, E2: $(ct'_2, x'_2) = (\frac{L'}{V/c}, 0)$

according to observer S:

$$
\begin{aligned}
ct_2 &= \gamma\left(ct'_2 + \frac{V}{c}x'_2\right) = \gamma\frac{L'}{V/c} \\
x_2 &= \gamma\left(cx'_2 + \frac{V}{c}ct'_2\right) = \gamma L'
\end{aligned} \tag{43}
$$

3. So, according to S the object hasn't moved! In retrospect, this is logical: S' sees S moving at velocity $-V$ and also sees the object moving at $-V$. Thus in S the object has zero velocity.

Note: we will come back to the transformation of velocities. That is more subtle than it know may look.

Solution to Exercise 4: Observer
Special Relativity with LT

1. E1: $(ct_1, x_1) = (0, 0)$ en $(ct_2, x_2) = (ct, -\frac{V}{c}ct)$

LT naar S' with $\frac{V}{c} = \frac{4}{5}$ and $\gamma = \frac{5}{3}$:

$$
\begin{aligned}
&(ct'_1, x'_1) = (0, 0) \\
&(ct'_2, x'_2) = \left(\gamma\left(ct - \frac{V}{c}\frac{-V}{c}ct\right), \gamma\left(-\frac{V}{c}ct - \frac{V}{c}ct\right)\right) = \left(\gamma\left(1 + \frac{V^2}{c^2}\right)ct, -2\gamma\frac{V}{c}ct\right)
\end{aligned} \tag{44}
$$

2. velocity According to S: $\frac{v}{c} = \frac{x_2 - x_1}{ct_2 - ct_1} = \frac{-\frac{V}{c}ct}{ct} = -\frac{V}{c}$

According to S':

$$
\frac{v'}{c} = \frac{x'_2 - x'_1}{ct'_2 - ct'_1} = \frac{-2\gamma\frac{V}{c}cdt}{\gamma\left(1 + \frac{V^2}{c^2}\right)ct} = -\frac{4/5}{1 + 16/25} = -\frac{40}{41} \tag{45}
$$

Thus the magnitude of the velocity according to S' is less than c.

Newtonian mechanics with GT

1. E1: $(ct_1, x_1) = (0, 0)$ en $(ct_2, x_2) = (ct, -\frac{V}{c}ct)$

GT:

$$
\begin{aligned}
ct' &= ct \\
x' &= x - \frac{V}{c}ct
\end{aligned} \tag{46}
$$

GT naar S' with $\frac{V}{c} = \frac{4}{5}$:

$$
\begin{aligned}
(ct'_1, x'_1) &= (0, 0) \\
(ct'_2, x'_2) &= \left(ct, -\frac{V}{c}ct - \frac{V}{c}ct\right) = \left(ct, -2\frac{V}{c}ct\right)
\end{aligned}
\tag{47}
$$

2. velocity According to S: $\frac{v}{c} = \frac{x_2 - x_1}{ct_2 - ct_1} = \frac{-\frac{V}{c}ct}{ct} = -\frac{V}{c}$ as before.

According to S':

$$
\frac{v'}{c} = \frac{x'_2 - x'_1}{ct'_2 - ct'_1} = \frac{-2\frac{V}{c}ct}{ct} = -2\frac{V}{c} = -\frac{8}{5} \tag{48}
$$

Thus the magnitude of the velocity according to S' is higher than c.

We will come back to this peculiar result in the world of Einstein and Lorentz.

Answers

2.3 Velocity Transformation & Doppler shift

Imagine we have two space ships moving each with a speed of $\frac{3}{4}c$ as shown below. What is the speed that either the red or yellow space ships sees for the other space ship speed?

We should, first of all realize, that the information regarding the velocity of the two space ships is given by an observer S who is neither in the red nor the yellow ship. We need to transform this information to an observer in the red or in the yellow ship.

Figure 34: Two space ships approaching each other.

For the GT we have derived the velocity transformation to be

$$v'_{x'} = v_x - V \tag{49}$$

So, let's translate our velocity information from the observer S to someone in the red ship. The relative velocity between S and the red ship is $V = \frac{3}{4}c$. Thus according to the observer in the red ship, S_R, her velocity is $V'_R = V_R - V_R = 0$, obviously.

However, she will denote the velocity of the yellow ship as $V'_y = V_y - V_R = (-3/4 - 3/4)c = \frac{3}{2}c > c$. In the world of Galilei and Newton, this is no problem at all: velocities can be as big as you can imagine. However, in reality, this is not true. We have to use Special relativity if the velocities start to approach c. It is not possible for any object to move faster than the speed of light, as we will see later.

In the above, we have only looked at the velocity component in the x-direction. We have in addition found $v'_{y'} = v_y, v'_{z'} = v_z$.

As our universe does not follow Galilei and Newton, we need to derive the transformation/addition formula for velocities with the LT. So let's do it.

2.3.1 Velocity Transformation

Let us start from the definition of velocity (assuming we deal with constant velocities, so we don't need to worry about differentiation and integration). We will denote velocities by u to avoid confusion with V, the relative velocity between the two observers.

Observer S' will write down:

$$u'_{x'} = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{\Delta x'}{\Delta t'} \quad \text{and} \quad u'_{y'} = \frac{y'_2 - y'_1}{t'_2 - t'_1} = \frac{\Delta y'}{\Delta t'} \quad (*) \tag{50}$$

We have left out the z'-component as that will be completely analogous to the y'-coordinate.

Observer S will use similar definitions. How do these observers translate velocity information that they get from each other?

We need to use the LT to transform (ct', x', y') to (ct, x, y):

$$\begin{aligned} x'_2 - x'_1 &= \gamma\left(x_2 - \tfrac{V}{c}ct_2\right) - \gamma\left(x_1 - \tfrac{V}{c}ct_1\right) \\ &= \gamma(x_2 - x_1) - \gamma\tfrac{V}{c}(ct_2 - ct_1) \\ y'_2 - y'_1 &= y_2 - y_1 \end{aligned} \tag{51}$$

and

$$\begin{aligned} ct_2' - ct_1' &= \gamma\left(ct_2 - \tfrac{V}{c}x_2\right) - \gamma\left(ct_1 - \tfrac{V}{c}x_1\right) \\ &= \gamma(ct_2 - ct_1) - \gamma\tfrac{V}{c}(x_2 - x_1) \end{aligned} \tag{52}$$

From the last line it is clear that also the y, z components of the velocity $\vec{u}$ will be influenced by the transformation although the relative motion between the two observers is only along the x-direction. Substituting the expressions for the space and time difference into $v'_{x'}$, we obtain

$$\begin{aligned} u'_{x'} &= \frac{\gamma\Delta x - \gamma\frac{V}{c}\Delta ct}{\gamma\Delta ct - \gamma\frac{V}{c}\Delta x} = \frac{\frac{\Delta x}{\Delta t} - V}{1 - \frac{V}{c^2}\frac{\Delta x}{\Delta t}} \\ &= \frac{u_x - V}{1 - \frac{V u_x}{c^2}} \end{aligned} \tag{53}$$

For the transverse components y, z, we obtain due to the change of the time interval

$$u'_{y'} = \frac{\Delta y}{\gamma\Delta ct - \gamma\frac{V}{c}\Delta x} = \frac{u_y}{\gamma\left(1 - \frac{V u_x}{c^2}\right)} \tag{54}$$

In the limit of $u_x, V \ll c$ both formulas reduce to the Galileo transformation as required. For $u_x \to c$ and $V \to -c$ the combined velocity will stay smaller than c. Check yourself that this is true.

The formula for the velocity transformation/addition are not so easy to remember. Later you will see how to derive them from the transformation properties of the 4-velocity, which is easy to remember.

For our example of the two approaching space ships, $u_x = -\frac{3}{4}c, V = \frac{3}{4}c$ we find for the speed of the yellow approaching the red ship

$$u''_{x''} = \frac{-\frac{3}{4} - \frac{3}{4}}{1 + \frac{3}{4}\frac{3}{4}}c = -\frac{24}{25}c < c \tag{55}$$

This is again smaller (in absolute sense) than c. For the other ship we find of course the same, but with different sign.

2.3.2 Doppler effect

The Doppler effect is well-known from waves. You know it from daily life. If a car is passing you at high speed, the frequency of the sound you hear changes from approaching to driving away from you. The received frequency f_{obs} by you is higher than the emitted frequency f_{src} while the car is approaching, and smaller when it drives off.

Here we investigate the effect of a moving source that is emitting light with f_{src} (electro-magnetic waves). This is one of the few cases where the relativistic case is easier than the classical effect. In the latter it matters if the source is moving or the medium. For EM-waves, however, there is no medium (aether) as we have seen which simplifies things.

Figure 35: Effect on sound waves due to motion.

For the case of an observer with speed v_{obs} and speed of sound in the medium u and moving source v_{src} (e.g. car) the classical formula of the frequency shift is

$$f_{obs} = f_{src}\frac{u \pm v_{obs}}{u \mp v_{src}} \tag{56}$$

where for the stationary observer and medium, we have $+/-$ and for the moving observer and stationary source $-/+$.

The origin of the observed frequency shift of a moving source is visible in the picture. In the direction of motion, more wave maxima arrive per unit time, as the car is moving closer between two wave maxima.

For the relativistic effect we consider a moving source with velocity $\vec{u}$ moving with observer S' relative to S. The frequency of the source is $f_0 = \frac{1}{T_0}$ in the rest frame, with T_0 the period of the oscillation.

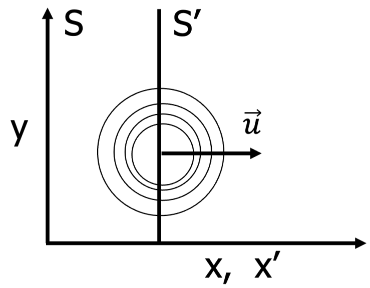
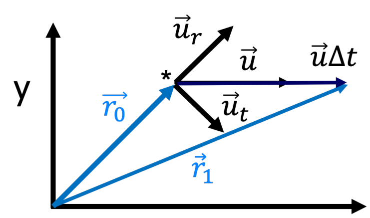


Figure 36: Observer S' and source both moving with respect to S.

We now consider the situation for S as shown in the figure below. The position of the source $\vec{r}_0$ is indicated with the star $*$.



We do know that according to S', the proper frequency is f_0 and the proper period $T_0 = 1/f_0$. Thus if a maximum is send at t'_0 the next one will be at $t'_0 + \frac{1}{f_0}$.

S will denote the first maximum with time $t_1 = t_0$, but will have to take time dilation into account for the second one: $t_2 = t_0 + \frac{\gamma}{f_0}$. Note that these two time instants are the moments, according to S when the two maxima are send, not when they are received by S.

During this time interval $\frac{\gamma}{f_0}$ the source moves from $\vec{r}_0$ to $\vec{r}_1 = \vec{r}_0 + \vec{u}\frac{\gamma}{f_0}$. Thus, the distance that the second maximum has to travel is different from that of the first one, just like in the classical Doppler case.

We consider the 2 consecutive wave maxima that are emitted in S' and received in S:

- 1st maximum in S' at t'_0, that is received in S at $t_1 = t_0 + \frac{r_0}{c}$. The additional time $\frac{r_0}{c}$ is needed for the light to travel from $\vec{r}_0$ to the observer at the origin of S.
- 2nd maximum in S' at $t'_0 + \frac{1}{f_0}$, is received in S at $t_2 = (t_0 + \frac{\gamma}{f_0}) + \frac{r_1}{c}$.

To move further we split the velocity of the source into a radial component (in line to the observer in S) and a tangential part perpendicular $\vec{u} = \vec{u}_r + \vec{u}_t = u_r\hat{r} + u_t\hat{t}$. If the distance $r_0 \gg \vec{u}\frac{\gamma}{f_0}$ then the distance $r_1 = r_0 + u_r\frac{\gamma}{f_0}$.

Note that we could drop the vector notation here from the exact relation above. Classically only the radial component is relevant as only the distance matters.

With this simplification on the distances we can compute t_2

$$t_2 = (t_0 + \frac{\gamma}{f_0}) + \frac{r_1}{c} \approx t_0 + \frac{\gamma}{f_0} + \frac{r_0 + u_r\frac{\gamma}{f_0}}{c} \tag{57}$$

For the frequency f in S we now subtract the two arrival times

$$\frac{1}{f} = t_2 - t_1 = \frac{\gamma}{f_0} + \frac{u_r}{c}\frac{\gamma}{f_0} \tag{58}$$

Rewriting this into a ratio of the emitted and received frequency, we obtain for the relativistic Doppler effect

$$\frac{f_0}{f} = \gamma\left(1 + \frac{u_r}{c}\right) = \frac{1 + \frac{u_r}{c}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{59}$$

Please observe that the γ factor is of $\gamma(u)$ that means even for only tangential movement $(u_r = 0)$ there is a Doppler shit.

Cosmic background radiation The most well-known frequency shift is the red-shift from the expanding universe.

The astronomer Edwin Hubble first found in the 1920s that the universe does not only consists out of our own galaxy, the milky way, but there must be (many) other galaxies, which were called *nebula* at that time. Second he could show that all further away galaxies move away from us by measuring the Doppler shift of well-known emission lines of stars and their distance from periodic intensity variation of Cepheid Variable stars. It turned out that the distance of the galaxies d was roughly linearly proportional to the red-shift which is again linearly related to the the radial velocity v as we derived. This is known now as Hubble's law $v = H_0 d$ with the Hubble constant $(H_0 \sim 70\ km/s/Mpc)$. Further away galaxies move faster away, but why? And why is no galaxy approaching us?

At end of the 1920s Georges Lemaître applied Einstein general theory of relativity to cosmology and found that the universe must be expanding, while it started in a "primeval atom", now known as the *Big Bang*. He could explain the red-shift relation from the expanding universe hypothesis.

The Big Bang theory was highly debated early on, in particular by Einstein, but is now fully accepted. The strongest experimental evidence was the discovery of the *cosmic background radiation* in 1965 (by accident).

The whole cosmos is nearly uniformly filled by a background radiation of about 2.7 K (wavelength in the μm range) with small inhomogeneities as shown in the picture by the Planck satellite around 2013.

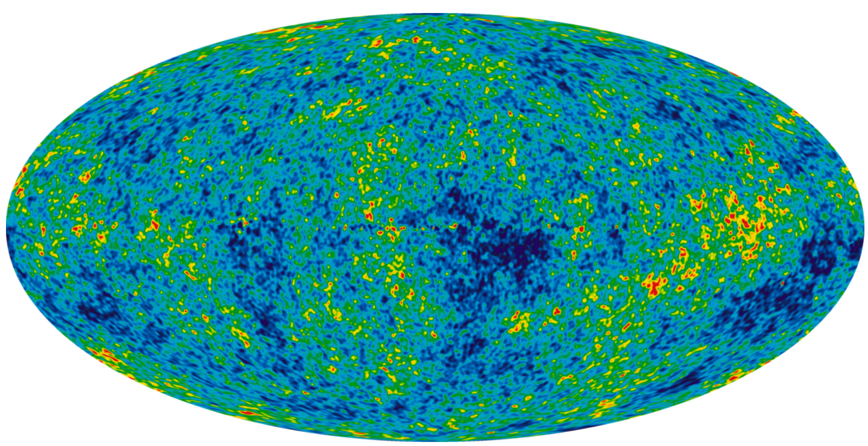


Figure 37: Background radiation in the universe as observed from earth. By NASA / WMAP Science Team - http://map.gsfc.nasa.gov/media/121238/ilc_9yr_moll4096.png, Public Domain.

This radiation is the red shifted radiation from around 380,000 years after the Big Bang when the universe became transparent. At that time the temperature had dropped (due to the adiabatic expansion) to around 3000 K, at which protons and electrons can form stable hydrogen atoms $p + e^- \rightarrow H$. This event is called *recombination*. At higher temperatures photons are scattered from the free electrons (and protons) constantly, effectively the photons have a very short mean free path and the universe is opaque. At the recombination temperature all of a sudden the photon could travel without strong scattering, the universe was transparent. The 3K cosmic background radiation that we measure today is the red-shifted version of this 3000 K light. It gives us a glimpse of how the universe looked at that time. Apart from the background radiation there were no other light sources in the universe as stars had not formed yet, the Dark Ages of the universe began.

The red-shift here is actually caused by the expansion of the universe itself (the universe expands causing the photons' wavelength to expand). NB: Time in cosmology is often given in units of red-shift (e.g. the red-shift for recombination is $z = 1089$).

Wavelength temperature relation

How can we relate the wavelength of electro-magnetic radiation to temperature?

Matter emits electro-magnetic radiation depending on its temperature. This relation is given by Planck's law with which quantum mechanics started in 1900 as he considered *black body radiation*. The emitted spectral density per solid angle depends on the thermal energy kT and is given by

$$u(\lambda, T) = \frac{2hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \tag{60}$$

Here for the first time h, Planck's constant, was introduced to quantize energy packages hf of oscillation.

Phenomenologically, the relation between the peak of the spectrum and the temperature was found by Wien already earlier to be $\lambda_{peak} = \frac{b}{T}$ with b Wien's constant $b \sim 2900\ \mu m \cdot K$.

NB: If you buy a light bulb for a lamp, then a temperature is indicated on the packaging, e.g. 2700 K for "warm white", 4000 K for "cool white" to describe the light color. Quantum physics at your local super market.

2.3.3 Examples, exercises and solutions

21cm hydrogen line
21 cm line of hydrogen in radio astronomy.
The proton and electron in the hydrogen atom both have a magnetic dipole moment related to their spin. The total quantum mechanical wave function can have 2 states for the spins, parallel or anti parallel, where anti parallel is energetically favorable. The transition between these two states is forbidden to first order (you will learn more about this in the three courses on *Quantum mechanics* in the second and third year). By Fermi's golden rule of quantum mechanics that means the probability that it happens per second is very small, here $10^{-15}s^{-1}$ or that the lifetime of the excited state is very long $\sim 10^7$ years. As space is vast and there is much hydrogen, however, this still happens a lot such that we can observe it.

Due to the very long life time, the emission line is very sharp, i.e. it has a small natural spectral broadening. This can be seen from Heisenberg's uncertainty principle $\Delta E \Delta t \sim \hbar$. If Δt is very large, then ΔE is small and the spectral line is very sharp. Therefore shifts to this line must come from Doppler shifts which can be used to measure speeds accurately.

Examples

Exercise 1: Observers S' is moving at $V/c = 3/5$ with respect to S. Both observers have their x and x' axis aligned. If they are at the same position ($x = x' = 0$), they set their clocks to zero.

S' observes an object traveling at $4/5$ of the speed of light in the negative x'-direction.

Calculate the velocity according to S.

Exercise 2: Same situation as in ex.(14.1), but now S' observes that the object is moving in the y'-direction with velocity $\frac{4}{5}c$.

Show that the magnitude of the velocity of the object according to S is smaller than c.

Exercise 3: In order to send information via electro-magnetic waves, people use amplitude modulation (AM) and frequency modulation (FM). AM means that the amplitude of the wave that is send out varies: the variations can be detected by the receiver and 'decoded' to the message. FM, on the other hand, means that the frequency of the wave is changing. This can also be detected and decoded to the message.

Captain Kirk on board of the starship USS Enterprise is traveling at a speed of $\frac{V}{c} = \frac{40}{41}
with respect to earth. He uses FM and sends his monthly report to mission control using a center frequency of 10GHz. What is the frequency that Mission Control needs to look for in case:

1. Enterprise is moving straight towards earth;
2. Enterprise moves radially away from earth;
3. Enterprise moves tangentially to earth.

Exercise 4: In the year 2525 a young Applied Physics student (who doesn't take his study to seriously) is caught ignoring a red traffic light and gets a fine. Trying to be a smarty, he refuses to pay and calls for a hearing in court.

The judge asks the student why he doesn't want to pay: ignoring a red traffic light is dangerous and a fine is in place.

The student argues, that he wasn't ignoring a red light: the light was clearly green.
The judge asks: "which light: the bottom one, the middle one or the top one?"

The students replies: the top one of course. I was riding my fat bike at a lovely high speed and noticed that only the top light of the traffic light was on. And it was definitely green."

The judge has heard enough. She adjourns the session and retreats to her office. There, she picks up her notebook and calculates what the velocity of the student was. Then she calculates the fine for speeding, using the formula "fine = 5Euro * (speed (in km/h) - 40km/h)".

She returns to the court room and the session is continued by her ruling. The student is acquitted of running a red light but is fined for speeding.

What is the amount of the fine?

Exercises

Solution to Exercise 1: Observers
According to S' the object has velocity $u'_{x'}/c = -4/5$. Observer S uses the velocity transformation for the x-component of velocities:

$$\frac{u_x}{c} = \frac{\frac{u'_{x'}}{c} + \frac{V}{c}}{1 + \frac{V}{c}\frac{u'_{x'}}{c}} = \frac{\frac{-4}{5} + \frac{3}{5}}{1 - \frac{3}{5}\cdot\frac{4}{5}} = -\frac{5}{13} \tag{61}$$

Solution to Exercise 2: Same situation as in ex.(14.1), but now
According to S' the object has velocity $u'_{x'} = 0$ and $u'_{y'}/c = 4/5$. Observer S uses the velocity transformation for the x and y-component of velocities:

$$\frac{u_x}{c} = \frac{\frac{u'_{x'}}{c} + \frac{V}{c}}{1 + \frac{V}{c}\frac{u'_{x'}}{c}} = \frac{3}{5} \tag{62}$$

$$\frac{u_y}{c} = \frac{\frac{u'_{y'}}{c}}{\gamma(V)\left(1 + \frac{V}{c}\frac{u'_{x'}}{c}\right)} = \frac{\frac{4}{5}}{\frac{5}{4}} = \frac{16}{25} \tag{63}$$

Thus the magnitude of $\vec{u}$ is:

$$\frac{u}{c} = \sqrt{\frac{u_x^2}{c^2} + \frac{u_y^2}{c^2}} = \sqrt{\frac{481}{625}} \approx \frac{22}{25}1 \tag{64}$$

Solution to Exercise 3: In order to send information via electro-magnetic waves, people use amplitude modulation (AM) and frequency modulation (FM). AM means that the amplitude of the wave that is send out varies: the variations can be detected by the receiver and 'decoded' to the message. FM, on the other hand, means that the frequency of the wave is changing. This can also be detected and decoded to the message.
Doppler shift

$$\frac{f_0}{f} = \frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{65}$$

In this case: $u/c = 40/41 \rightarrow \gamma = \frac{41}{9}$

1. $u_r/c = -40/41 \rightarrow \frac{f_0}{f} = \frac{1}{41}\frac{41}{9} \rightarrow f = 9f_0 = 90GHz$
2. $u_r/c = 40/41 \rightarrow \frac{f_0}{f} = \frac{81}{41}\frac{41}{9} \rightarrow f = \frac{1}{9}f_0 = 1.11GHz$
3. $u_r/c = 0 \rightarrow \frac{f_0}{f} = \frac{41}{9} \rightarrow f = \frac{9}{41}f_0 = 2.20GHz$

Solution to Exercise 4: In the year 2525 a young Applied Physics student (who doesn't take his study to seriously) is caught ignoring a red traffic light and gets a fine. Trying to be a smarty, he refuses to pay and calls for a hearing in court.
Obviously, the student tries to claim that due to his high speed, the red color of the traffic light was green to him. As he is approaching the light source, with a velocity V/c, he may also take the point of view of an observer in a frame in which he is not moving, but the traffic light is approaching with V/c,

The wave length of red light is 630nm and of green light 530nm. Or in terms of the corresponding frequencies: $f_r = \frac{c}{\lambda_r} = 4.76 \cdot 10^{14}$Hz and $f_g = 5.66 \cdot 10^{14}$Hz. In the rest frame of the traffic light, the frequency is thus: $f_0 = f_r$, whereas in the frame of the student it is $f = f_g$.

If we plug this into the Doppler shift formula, we get:

$$\begin{gathered} \frac{f_0}{f} = \frac{f_r}{f_g} = 0.82 = \frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}} = \sqrt{\frac{1+V/c}{1-V/c}} \Rightarrow \\ \frac{1+V/c}{1-V/c} = 0.68 \rightarrow \frac{V}{c} = 0.2 \end{gathered} \tag{66}$$

Thus the biker claims to have a speed of 20% of the speed of light, that is 2.16 10^8 km/h and accordingly gets a fine of 1.08 billion Euro.

Answers

2.4 Spacetime and 4-vectors

2.4.1 Space time

In 3D space we define a point/coordinate by its components (x, y, z) where all components have the same unit. We can do this also in 4D space time by an event (ct, x, y, z) as ct has unit length (it should be called *time space* by this ordering, but what ever). The same unit for all components is needed if we want to do geometry with the coordinates.

If we want to measure distances Δs between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) we do this in 3D Euclidean space as $\Delta s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$. These distances are Galileo invariant, observer S and S' moving with $\vec{V}$ measure the same distance $\Delta s^2 = \Delta s^{'2}$. Note, that we take these two pints at the sam time t: $t_1 = t_2$. Or rephrased: we perform the measurement in the rest frame of the object we measure. That makes sense: measuring the length of an object that is moving requires that we measure the left and right side at the same time. Otherwise, the motion of the object will interfere with our measurements of the length.

The above statement is easily shown by invoking the Galilei Transformation:

$$
\begin{aligned}
x' &= x - Vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned} \tag{67}
$$

We transform the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) at the same time t, we get:

$$
\begin{aligned}
&x_1' = x_1 - Vt, x_2' = x_2 - Vt \Rightarrow x_2' - x_1' = x_2 - x_1 \\
&y_1' = y_1, y_2' = y_2 \Rightarrow y_2' - y_1' = y_2 - y_1 \\
&z_1' = z_1, z_2' = z_2 \Rightarrow z_2' - z_1' = z_2 - z_1 \\
&t' = t
\end{aligned} \tag{68}
$$

If we want to measure distances in space time and require that the distance is now Lorentz invariant, we cannot measure distance the same way! If we measure in S the positions at the same time, that will in general be at different times according to S'. Time is relative!

To do geometry, measure angles etc. we need an inner product and the inner product provides a distance measure (a metric) by the norm. For 3D you know that for two vectors $\vec{r}_1$ and $\vec{r}_2$: $\Delta s^2 = ||\vec{r}_1 - \vec{r}_2||^2 = (\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2) = \Delta x^2 + \Delta y^2 + \Delta z^2$. Clearly the inner product in 4D space time cannot be defined in the same way.

We want that two relativistic observers measure the same distance (e.g. between two events), that is, it must be Lorentz invariant. We start by noting that the speed of light is constant for both observers. A light wave traveling in S and S' must therefore obey

$$
c^2t^2 - x^2 - y^2 - z^2 = 0 = c^2t^{'2} - x^{'2} - y^{'2} - z^{'2} \tag{69}
$$

Given this observation it is needed (and natural) to define the distance in space time as

$$
ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{70}
$$

!!! important "Warning" Notice directly that the distance Δs^2 can be negative! (And we are OK with that).

It is straightforward to show that the above distance ds^2 is indeed a Lorentz Invariant, i.e. $ds'^2 = ds^2$. Suppose we have two events: $E_1 : (ct, x, y, z)$ and $E_2 : (c(t + dt), x + dx, y + dy, z + dz)$. We can transform these to S' via the standard Lorentz Transformation:

$$\begin{aligned} ct_2' &= \gamma\left(c(t+dt) - \tfrac{V}{c}(x+dx)\right) \Rightarrow \\ cdt' &= \gamma\left(cdt - \tfrac{V}{c}dx\right) \\ x_2' &= \gamma\left((x+dx) - \tfrac{V}{c}c(t+dt)\right) \Rightarrow \\ cdx' &= \gamma\left(dx - \tfrac{V}{c}cdt\right) \\ y_2' &= y_2 \Rightarrow \\ dy' &= dy \\ z_2' &= z_2 \Rightarrow \\ dz' &= dz \end{aligned} \tag{71}$$

Clearly, we do only have to concentrate on the cdt and dx terms:

$$\begin{aligned} cdt'^2 - dx'^2 &= \gamma^2\left(cdt - \tfrac{V}{c}dx\right)^2 - \gamma^2\left(dx - \tfrac{V}{c}cdt\right)^2 \\ &= \gamma^2\left(c^2dt^2 - 2\tfrac{V}{c}cdtdx + \tfrac{V^2}{c^2}dx^2 - dx^2 + 2\tfrac{V}{c}dxcdt - \tfrac{V^2}{c^2}c^2dt^2\right) \\ &= \underbrace{\gamma^2\left(1 - \frac{V^2}{c^2}\right)}_{=1}\left(c^2dt^2 - dx^2\right) \\ &= c^2dt^2 - dx^2 \end{aligned} \tag{72}$$

Note that if we had used a $+$ sign, that is $ds^2 \equiv c^2dt^2 + dx^2$, we would **not** have arrived at a Lorentz Invariant.

Example 15.1: Pythagoras gets mixed up
We are used to all kind of 'obvious' results that hold in our Galilei/Newtonian world. For instance, for a triangle with a perpendicular angle we can apply Pythagoras theorem:

$$a^2 + b^2 = c^2 \tag{73}$$

Example: for a triangle with sides (3,4,5) this would give the figure below.

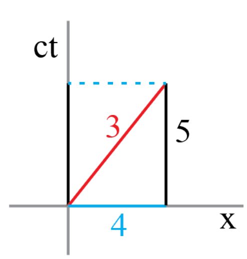
How does this work in our Lorentz/Einstein world?

Consider the following: according to S, a particle is moving with velocity $\frac{V}{c} = \frac{4}{5}$ over the x-axis. The particle is at $ct = 0$ at $x = 0$. Obviously, $5ls$ later it is at position $x = 4$. So, we can define two events:

$$\begin{aligned} E1 :&(ct_1, x_1) = (0, 0) \\ E2 :&(ct_2, x_2) = (5, 4) \end{aligned} \tag{74}$$

Can we draw this? Sure, now we need an (ct, x) diagram. It is a convention to draw the ct-axis vertically.

The figure is going to look like this.



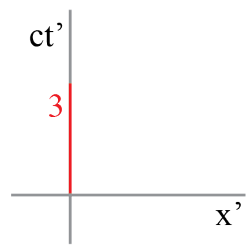
Much to our surprise, the hypotenusa is shorter than each of the other two sides!

Why does this make sense? In the world of Special Relativity, we can find answers by looking at a different frame of reference. What will observer S', who is traveling with the particle, say about this?

We have to translate the two events of E_1 and E_2 to the frame of S':

$$\begin{split} E1: (ct'_1, x'_1) &= (0,0) \ E2: (ct'_2, x'_2) &= \left (\gamma \left (ct_2 - \frac{V}{c} x_2 \right), \gamma \left (x_2 - \frac{V}{c} ct_2 \right) \right) \ &= \left (\frac{5}{3} \left (5 - \frac{4}{5} 4 \right), \frac{5}{3} \left (4 - \frac{4}{5} 5 \right) \right) \ &= (3, 0)

\end{split}$$



Of course, as we knew, the length of the interval stays the same: $\Delta s^2 = \Delta s'^2 = 3^2$.

2.4.2 4-vector

The idea of having to work with a 'position' vector with 4 components with an inproduct as we discussed above, is generalized to vectors, i.e. quantities with a direction and a magnitude.

We define a 4-vector $\vec{A} = A^\mu = (A^0, A^1, A^2, A^3)$ to be a vector that transforms between two observers S and S' moving with V along the x-direction by the LT

$$\begin{array}{lcl} A^{0'} & = & \gamma\left(A^0 - \frac{V}{c}A^1\right) \\ A^{1'} & = & \gamma\left(A^1 - \frac{V}{c}A^0\right) \\ A^{2'} & = & A^2 \\ A^{3'} & = & A^3 \end{array} \tag{75}$$

Other tuples of 4 values are not 4-vectors. The requirement that the 4-vector must transform via the LT is essential. We will use this later for the 4-velocity and 4-momentum.

Inner product & conventions Like the distance also the inner product can be defined between two *4-vectors*. We use a capital letter for a 4-vector

$$\vec{A} = A^\mu = (A^0, A^k) = (A^0, A^1, A^2, A^3) = (A^0, \vec{a}) \tag{76}$$

This notation is just to a make clear distinction with 3-vectors that only have spatial coordinates. With a Greek index μ, A^μ we indicate all 4 components of the vector, while with a Latin index k, A^k we only indicate the spatial components. We also start counting at 0 for the first component, which is 'time'.

The inner product between two 4-vectors $\vec{A}, \vec{B}$ is now defined according to the rule we already saw before

$$\vec{A} \cdot \vec{B} \equiv A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 \tag{77}$$

This is not a "choice" for the inner product, but follows strictly from the requirement that distance or length should not change under LT. A space with this inner product is called *Minkowski space* or the space has a *Minkowski metric* after Hermann Minkowski.

Notice that time component $(+)$ is treated differently than the spatial components $(-)$ in the inner product. Sometimes the inner product is also called *pseudo Euclidean* as there are -1 and $+1$ present in the inner product (instead of only $+1$ for Euclidean space).

Lorentz invariants As is clear by the above construction the inner product of two 4-vectors must be LT invariant, that is for observers $S : \vec{A}, \vec{B}$ and $S' : \vec{A}', \vec{B}'$ it holds

$$\vec{A} \cdot \vec{B} = \vec{A}' \cdot \vec{B}' \tag{78}$$

This property can be a *very* powerful tool (OK, we constructed it that way). If we know the value of the inner product in one frame of reference, it will be the same in all other inertial frames of reference! We will use that later often. It is also clear that the distance interval ds^2 is a Lorentz invariant.

??? "Inner product LT invariant: the hard way"

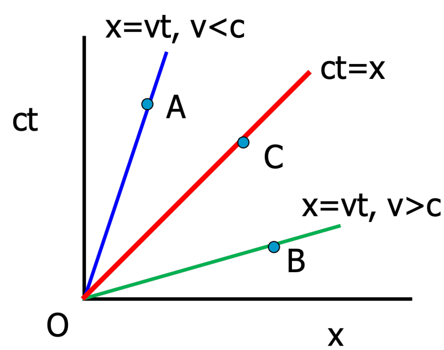
If you do not believe that the inner product is LT invariant you can write it out of course (with $\beta \equiv \frac{V}{c}$, a short hand notation that is frequently used).

We compute $\vec{A}' \cdot \vec{B}'$. We will concentrate on only $A^0B^0 - A^1B^1$, as with the standard Lorentz Transformation the A^2 and A^3 component are trivial.

$$\begin{array}{rcl} \vec{A}' \cdot \vec{B}' & = & \gamma(A^0 - \beta A^1) \cdot \gamma(B^0 - \beta B^1) - \gamma(A^1 - \beta A^0) \cdot \gamma(B1 - \beta B^0) \\ & = & \gamma^2 \left(A^0B^0 - \beta A^1B^0 - \beta A^0B^1 + \beta^2 A^1B^1\right) \\ & - & \gamma^2 \left(A^1B^1 - \beta A^0B^1 - \beta A^1B^0 + \beta^2 A^0B^0\right) \\ & = & \gamma^2(1 - \beta^2)(A^0B^0 - A^1B^1) \\ & = & A^0B^0 - A^1B^1 \\ & = & \vec{A} \cdot \vec{B} \end{array} \tag{79}$$

2.4.3 The light cone

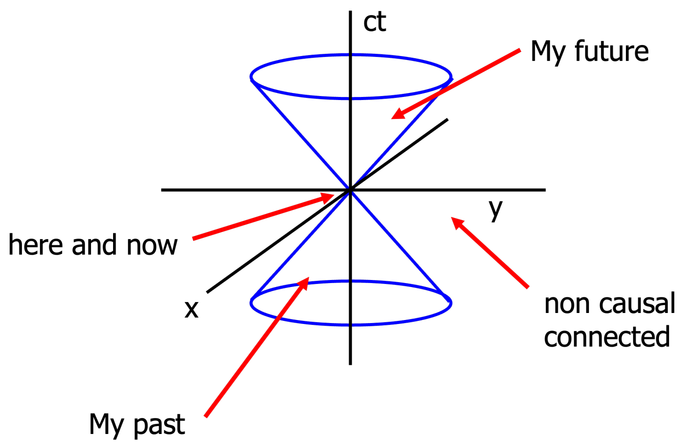
Let us consider an event in space time $\vec{X} = X^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$. For sake of simplicity we only consider one space like component here. In the sketch we have the space axis (x or x^1) to the right and the time axis (ct or x^0) up. We consider 3 events A, B, C (points in space time) and their connection to the origin O



- OA: The point A can be reached from O with velocity $v < c$, therefore it is called *causally connected* or *time like*. For the distance $OA : \Delta s^2$, we see from projection of the coordinates A onto the time and space axis $|x_A - 0| < (ct - 0) \Rightarrow \Delta s^2 > 0$. Because the time component is larger than the space component, it is called *time like*. The distance is positive.
- OB: The point B can be reached from O only with velocity $v > c$, therefore it is called *non-causally connected* or *space like*. For the distance $OB : \Delta s^2$, we see from projection of the coordinates B onto the time and space axis $|x_B - 0| > (ct - 0) \Rightarrow \Delta s^2 < 0$. Because the space component is larger than the time component, it is called *space like*. The distance squared is negative.
- OC: The point C can be reached from O only with velocity $v = c$, therefore it is called *light like* or *null*. For the distance $OB : \Delta s^2$, we see from projection of the coordinates C onto the time and space axis

$|x_C - 0| = (ct - 0) \Rightarrow \Delta s^2 = 0$. Because the space component is equal to the time component, it is called *light like*. The distance is zero. Therefore it is also called *null*.

Here you visually can observe that the sign of the distance using the Minkowski inner product classifies parts of space time.

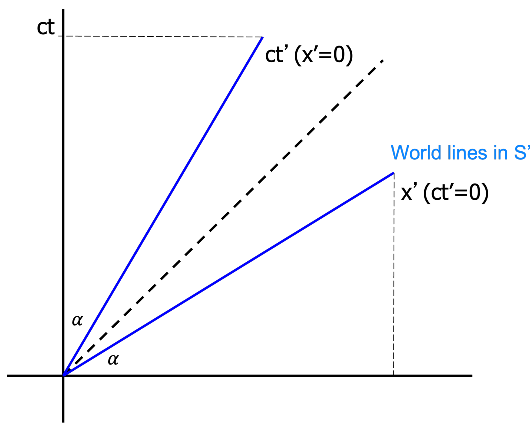


This is even more evident if you look at the light cone in the sketch. The cone mantel is generated by revolving the line $x = ct$, a light line. Here only a 2D cone is shown (ct, x, y), but of course this should be a 3D cone (ct, x, y, z). The inside of the cone at negative times is the *past* that could have influenced me at *now*. My *now* can influence my *future* (inside the cone to positive times). All the rest, outside the cone is not causally connected to me.

2.4.4 Minkowski-diagram

Now we can have a look at world lines of an observer S' with respect to S traveling with V along the $x-$axis in a graphical manner. The world line of an object is the path that an object travels in the 4-dimensional spacetime.

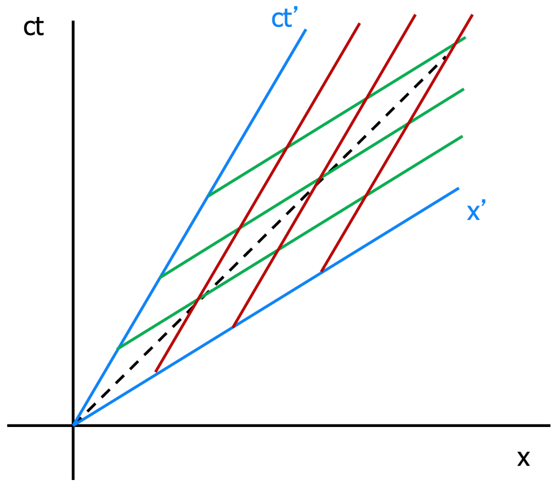
We plot the coordinate system of S' (blue) in the coordinate system of S (black).



- The time line of S' in S is given by the fact that $x' = 0$. From the LT we have $x' = \gamma(x - \frac{V}{c}ct) = 0 \Rightarrow x = \frac{V}{c}ct$. The angle α of the ct'-line with the ct axis is given by $\tan\alpha = \frac{V}{c}$.
- The space line of S' in S is given by the fact that $ct' = 0$. From the LT we have $ct' = \gamma(ct - \frac{V}{c}x) = 0 \Rightarrow ct = \frac{V}{c}xt$. The angle α of the x'-line with the x axis is given by $\tan\alpha = \frac{V}{c}$.

Both lines of S' make the same angle α with the coordinates axis of S. They lie symmetric around the light line $x = ct$ (diagonal with $\alpha = 45$ deg). The higher the speed V the higher the angle and the closer the lines lie to the light line. See the animation below, where the (ct', x') axis are plotted in the (ct, x) diagram of S for different values of V/c.

To further investigate how this plot can help us, let us consider lines of equal time in S. These are just the lines perpendicular to the ct-axis, and parallel to the x-axis, as you expect. And of course, lines parallel to ct, perpendicular to x are lines of constant space coordinate.



For the frame of reference S' that is only a bit different.

- Lines of constant time in S' are parallel to x'
- Lines of constant space coordinate in S' are parallel to ct'

With this information in hand, we can investigate how events are transferred from S to S'. We can graphically do a LT without the explicit computation.

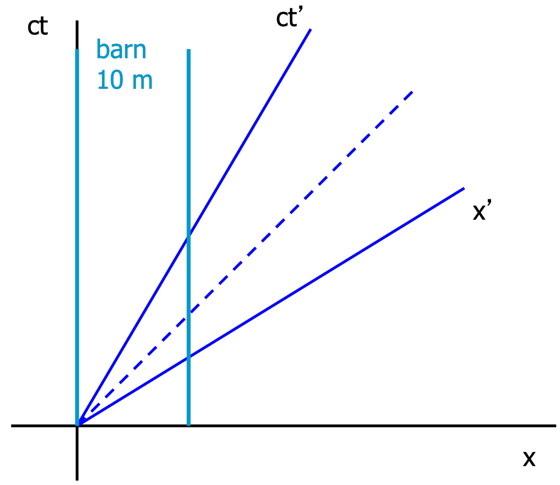
In the animation below, we see the effect of different values of V/c on the lines of constant ct' and x' as seen by S. For clarity, these are only drawn for $V/c \geq 0$

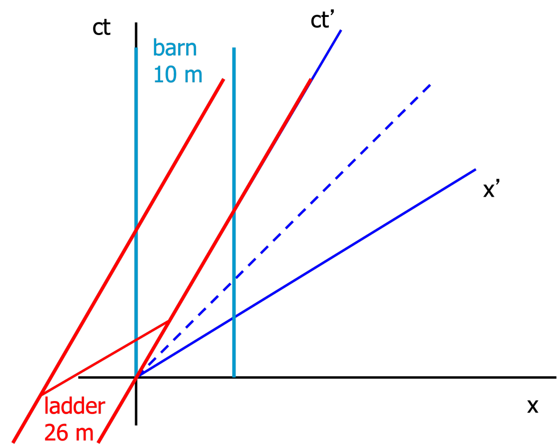
The ladder & barn revisited We will now take a look back at the ladder and barn paradox. We had a barn of 10 m wide and a ladder of 26 m long (both measured in their rest frame). The ladder was moving towards the barn with high velocity. We start by drawing the barn S (black) and ladder S' (blue) coordinate systems in the Minkowski diagram. Now we add the barn world line into the diagram (light blue) with 2 lines of constant space coordinate (parallel to ct) in S.

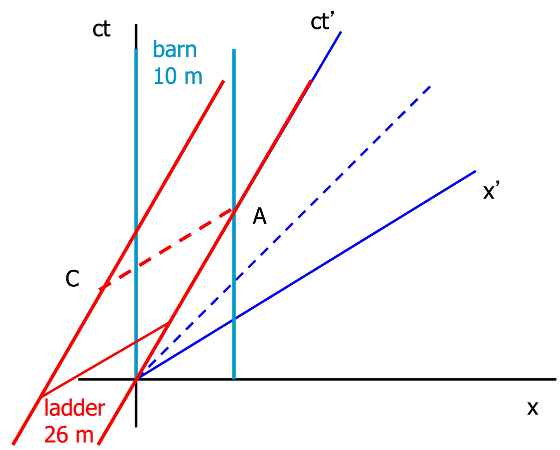
Now we can add the ladder to S'. It has rest length of 26 m and in the (x', ct') plane it is a world line of constant space coordinate, therefore parallel to ct'. The ladder itself is a line of constant time in ct' and therefore parallel to x'.

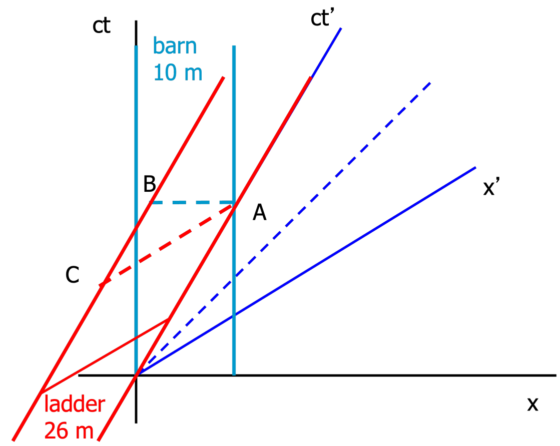
As the ladder moves (we move it parallel to x' between the world lines) it will eventually enter the barn and hit the right door of the barn (dashed red line). This event is indicated by the space time point A. For S' the other end of the ladder is then still outside the barn at space time point C. According to S' the ladder does not fit into the barn.

When the ladder hits the right door for S at space time point A, he makes a measurement of the ladder. To this end we draw a line of constant time (dashed light blue, parallel to x) until it intersects the world line of the ladder at space time point B. Observer S measures that the ladder fits into the barn.


ct
barn
10 m
ct'
x'
x


ct
barn
10 m
ct'
x'
ladder
26 m
x


ct
barn
10 m
ct'
x'
A
C
ladder
26 m
x



From this diagram it is obvious that the events B and C are not the same, therefore it is not strange that S and S' disagree about the outcome of the measurement. Both are right! But they would not be able to agree that both doors shut at the same time, to capture the ladder.

The twin paradox Let there be two twins, Alice and Bob. Bob leaves earth in a space ship with relativistic speed $\vec{v}$, while Alice remains back home on earth. At some time Bob turns around, with $-\vec{v}$ and comes back to Alice. Based on time dilation Alice will argue that Bob is younger than she due to $\Delta T = \gamma \Delta T_0$. For the γ-factor it does not matter if Bob is moving away or approaching as it is quadratic in the velocity. For each year she ages, her brother only ages $1/\gamma$ years. Bob can argue that due to the principle of relativity, he is at rest and his twin sister is moving away and then coming back, therefore she will be younger than he - and we have a paradox.

This paradox has two issues:

1. The principle of relativity is not applicable as Bob must *turn around*. This requires acceleration of his frame and breaks the symmetry of the problem.
2. Bob will be younger than Alice, due to the relativity of simultaneity changing around the turning point. We can see this by looking at the Minkowski-diagram below. Just before Bob is turning around, his line of simultaneity is x', but just after turning around his line of simultaneity is x''. On the time line of Alice, Bob lines of simultaneity first is at point A, but then makes a jump around the turning point to B. Bob will be younger than Alice, by the length of this jump on her time line from A to B.

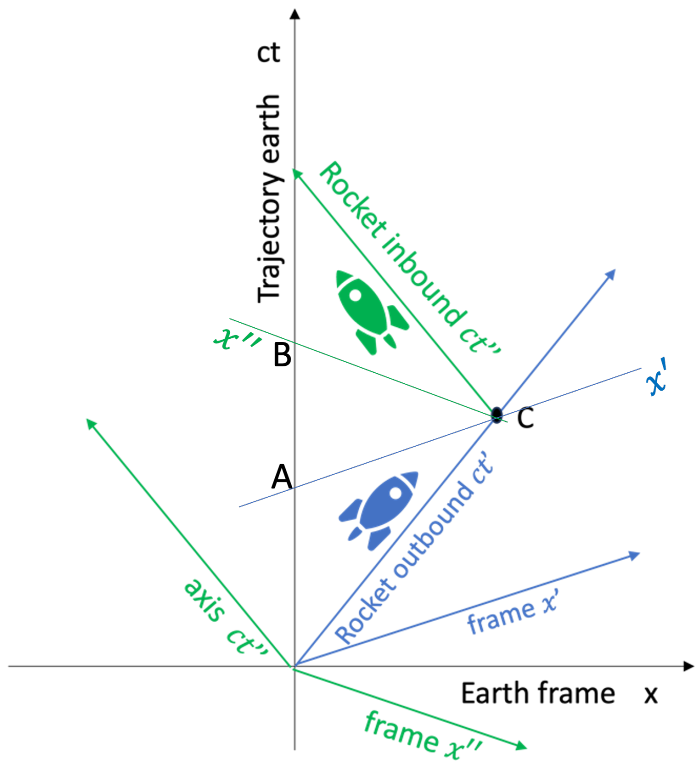
Extra: We symmetries the problem. Both Alice and Bob move in space ships away from each other at the same but opposite speed, then turn around and meet again. Who is older now?

??? "Answer"

```
They are the same age. You can now reason with symmetry even though both are accelerated. You can also
```

Worked example: the rabbit and the turtle We consider the relativistic race between the well-known rabbit (R) with speed v_R and his buddy turtle (T) with speed $v_T < v_R$. Both turtle and rabbit are point particles. To give turtle a chance, it does not need to run as far as rabbit ($L_T < L_R$). The distances are chosen such that an observer at rest (the audience) records that R and T finish at the same time.

1. Draw a Minkowski-diagram of the situation described above.
2. Indicate the following events in space time.
   - R finishes in his frame (A)
   - T finishes in his frame (B)



- In the frame of R, when he finishes, the event where T is then (C)
- In the frame of T, when he finishes, the event where R is then (D)

3. Who has won according to R and who according to T. Do they agree?

Solutions:

We start by drawing the audience frame with (ct, x) and an equal time line for the finish of R and T. From that we draw the coordinate system of R as (ct_R, x_R) and of T as (ct_T, x_T). As $v_T < v_R$, the coordinate system (ct_R, x_R) is closer to the light line. The length L_R and L_T follow as the intersections of ct_R and ct_T with the line of equal time for the audience.

These intersections are also directly the events A and B.

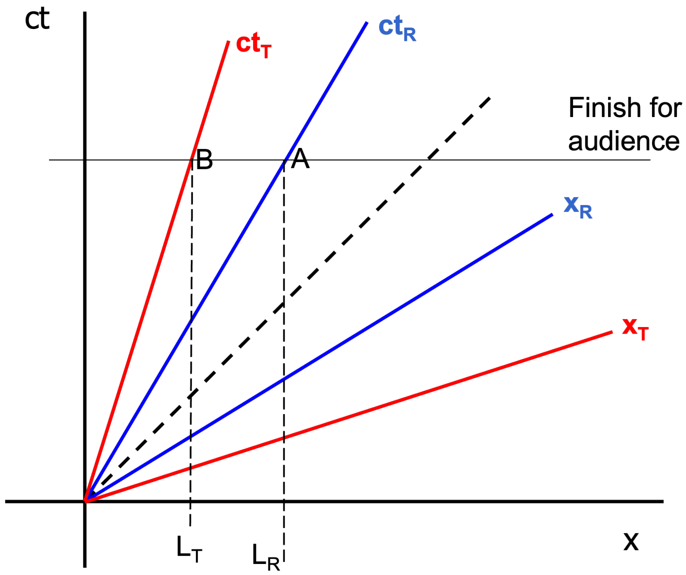
For the events C and D, we first draw from A a line of constant time for R (parallel to x_R) and then look at the intersection with the world line of T and mark it with C. The same for the event D. We draw a line parallel to x_T of constant time for T through B to see where R is when T finishes and mark it with D.

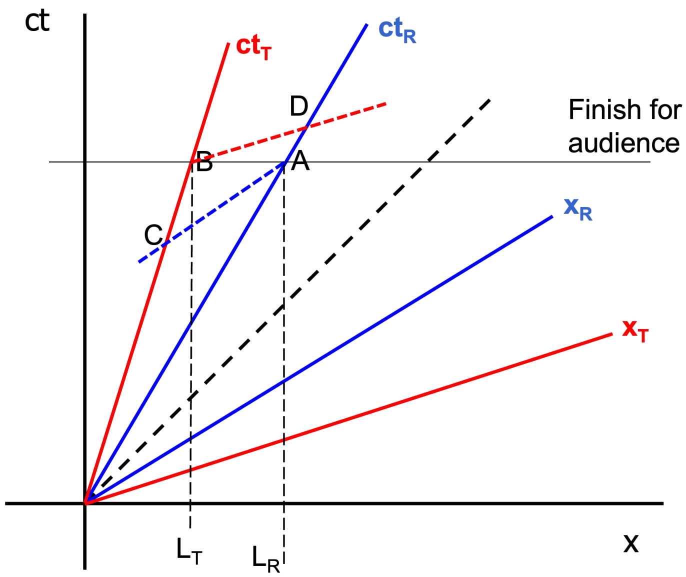
Both R and T agree that R has won, but the audience does of course not agree.

Worked example: moving particle Consider a standard situations: S' moving at $V/c = 3/5$ with respect to S. CLocks are synchronized at $ct' = ct = 0$ when $x' = x = 0$.

According to S, a particle is moving with $U/c = 4/5$ over the x-axis. S describes the trajectory of the particle as $x_p(ct) = \frac{U}{c}ct$. In the animation below a Minkowski diagram is plotted as S would do. The motion of S' is made visible by the moving blue dot. Similarly, the pink dot shows the motion of the particle. The grey grid is giving coordinates according to S. The black dashed lines show the ct and x coordinate of the particle as S uses.

The green dashed lines is the grid of S' translated to the world of S. The pink dashed lines show the corresponding coordinates of the particle in the world of S': they intersect the ct' and x' axes at the position and time as S'


ct
ct_T
ct_R
Finish for audience
B
A
x_R
x_T
L_T
L_R
x


ct
ct_T
ct_R
D
Finish for audience
B
A
C
x_R
x_T
L_T
L_R
x

would use. Notice that the clock of S' is indeed slow. Of course the x' coordinate of the particle stays relatively small: S' is 'chasing' the particle.

Lines of invariant distance We have seen that the length interval ds^2 is a Lorentz invariant. Therefore we can use it to also indicate corresponding time and space units in a Minkowski diagram for two moving observers. If we fix ds^2 then the equation $ds^2 = c^2dt^2 - dx^2$ describes a hyperbola in (ct, x) of the Minkowski diagram.

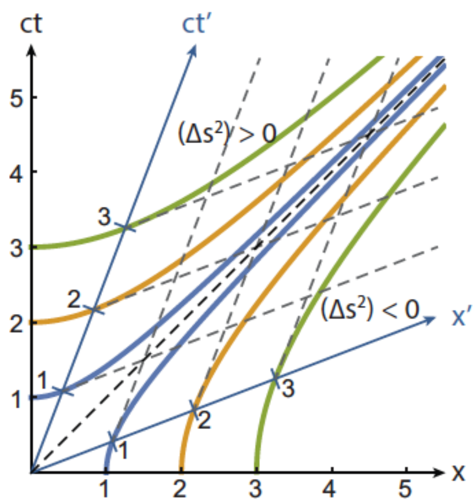


Figure 38: Image from T. Idema, *Mechanics and Relativity*.

For $ds^2 < 0$ we find the corresponding space units (the interval is space-like), and for $ds^2 > 0$ the corresponding time units (the interval is time-like. All hyperbola have the light line $ds^2 = 0$ as asymptotes.

Example 15.3: Circles are not circular??
We define a circle as the set of points (in a plane) that have the same distance to some given point M. We can easily extend this to three dimensions: that the circle becomes the surface of a sphere. If we stick to Eucledian spaces, we can do this for any dimension: a spherical surface in n-dimensional space, is the collection of points with the same distance to a given point M. Now the point has to be represented by n coordinates. But our measure of distance follows the same inner-product as we use in 2 and three dimensions:

let $\{M_i\}$ with $i = 1..n$ be a point in n-dimensional space. Then all points $\{X_i\}$ with $i = 1..n$ that obey the rule

$$\sum_{i=1}^{n}(X_i - M_i)(X_i - M_i) = R^2 \tag{80}$$

form a spherical surface with distance R to M. The above rule is actually the inner product of $\vec{X} - \vec{M}$: $(\vec{X} - \vec{M}) \cdot (\vec{X} - \vec{M}) = R^2$

Without loss of generality, we can chose the origin at M. That simplifies notation: $\vec{X} \cdot \vec{X} = R^2$ is now the surface of a sphere of radius R with center $\mathcal{O}$.

What if we leave our Eucledian space and go to the Minkowski space of special relativity? We still would define e circle as a set of point with the same distance to a given point. But now, our measure of distance is different. Let's again take the origin as the central point. Then, we are looking for the set of point $\{X^\mu\}_i$ such that $\vec{X} \cdot \vec{X} = R^2$. This means:

$$X^0X^0 - X^1X^1 - X^2X^2 - X^3X^3 = R^2 \tag{81}$$

Or, if we only consider ct, x:

$$c^2t^2 - x^2 = R^2 \tag{82}$$

These are the 'circles' in Minkowski ct, x-space. Of course, we would have the tendency to call them hyperbola, as they have the mathematical expression of a hyperbola. But in fact, the interpretation in Minkowsi space would be that of circles, that is the collection of points with the same distance to the origin.

Note, that $R = 0$ now does not reduce the set to a single point, but instead refers to the light lines. Second note: we do not have a problem here with negative distances. Thus if we take R to be a pure imaginary number, we will still get hyperbola, but just rotated by 90°.

2.4.5 LT as a rotation

This part is optional, but insightful.

You can think of the LT as a rotation of the 4 coordinates of Minkowski space time. Obviously it is not a "normal" rotation with a rotation matrix $R \in SO(n)$ as we encountered in change to polar coordinates.

The LT in matrix notation reads as follows with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = V/c$.

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{83}$$

The matrix transfers the space time coordinates between two observers moving with V. From this it is clear that transferring between more than two observers $S \rightarrow S' \rightarrow S^{''} \rightarrow \dots$ can be done easily by multiplying the respective Lorentz transformation matrices into one overall LT. This must be possible, of course, as the LT is a linear transformation in space time (ct, x).

From the matrix notation it is also clear that for rotations around "different axis", speeds in x, y, z direction, the order of change of frame matters as matrix multiplication does not commute.

In 3D normal space, distance is persevered under rotation with $R \in SO(n)$, in Minkowski space distance is preserved under Lorentz transformation which too is a rotation.

You can see the rotation clearer if we introduce the quantity *rapidity* α, which is defined as $\tanh\alpha \equiv \frac{V}{c}$ (a relativistic generalization of the modulus of the velocity. It goes from 0 for $v = 0$ to ∞ for $v = c$). We will not use the rapidity except here, however, it is used for relativistic velocity decompositions. With $\tanh\alpha = \frac{V}{c}$ we can write the Lorentz transformation as (using $\gamma = \frac{1}{\sqrt{1-\tanh^2\alpha}} = \cosh\alpha$ and $\gamma\beta = \frac{\tanh\alpha}{\sqrt{1-\tanh^2\alpha}} = \sinh\alpha$)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{84}$$

Notice the similarity to the rotation with sine and cosine.

With that LT is a rotation in hyperbolic space with "angle" α (where α is the rapidity), we identify the matrix as $L(\alpha)$. That the hyperbolic functions appear should not be a surprise as they are equivalent to the sine and cosine for the circle, $(ct^2 + x^2 = 1)$, for the hyperbola $(ct^2 - x^2 = 1)$. Notice the relation to the inner products for standard and Minkowski space.

Minkowski made the sketch below to show that the Lorentz transformation is a rotation over a hyperbola not a circle as we were used to. The asymptotes of the hyperbola are given by the light lines.

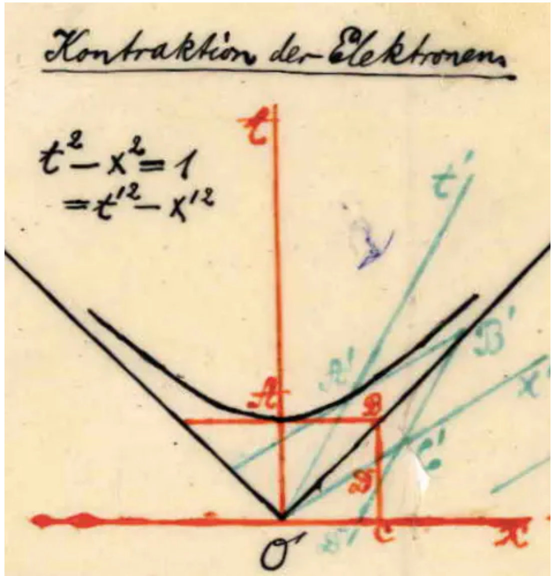


Figure 39: Drawing by Minkowski

The addition of velocities that we derived earlier is easy with this notation with rotations and rapidity $L(\alpha_1)L(\alpha_2) = L(\alpha_1 + \alpha_2)$. In terms of speeds this reads

$$\beta = \tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 + \tanh\alpha_1 \tanh\alpha_2} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \tag{85}$$

The addition of velocities is brought back to hyperbolic identities".

2.4.6 Exercises, examples & solutions

Exercise 1: Consider the following pairs of events and determine whether they are time like, space like or light like connected.

a. E1: $(ct_1, x_1) = (1, 0)$ and E2: $(ct_2, x_2) = (0, 1)$
b. E3: $(ct_3, x_3) = (1, 3)$ and E4: $(ct_4, x_4) = (-2, 1)$
c. E5: $(ct_5, x_5) = (1, 2)$ and E6: $(ct_6, x_6) = (3, 4)$

S' travels at $V/c = 12/13$ in the positive x-direction with respect to S. Their clocks are synchronized when their origins coincide.

d. Answer the same questions, but now from the perspective of S'.

Exercise 2: In the frame of S a laser is placed at $(x_1, y_1, z_1) = (4, 0, 0)$. A receiver is located at $(x_2, y_2, z_2) = (0, 3, 0)$. At $ct = 0$ the laser fires a light puls towards the receiver.

Find the events "pulse send" and "pulse received" and determine the distance between the two events.

Secondly, an observer S' moves with $V/c = 4/5$ with respect to S. The velocity points in the positive x-direction. Both observers have their x resp. x' axis aligned and their clocks synchronized: $ct' = ct = 0$ when $x' = x = 0$.

Find the events for S' and show that the same distance is found between the two events.

Exercise 3: Observer S' moves at a constant velocity of $V/c = 12/13$ with respect to S. They have aligned their axes and synchronized their clocks in the usually way.

Consider the two events $E1 : (ct_1, x_1) = (3, 3)$ and $E2 : (ct_2, x_2) = (4, 5)$

a. Compute the distance between the two events, Δs^2, according to S.
b. Compute the event coordinates according to S'.
c. Compute $\Delta s'^2$ and convince yourself that this is of course equal to Δs^2.

Exercise 4: Observer S' moves at a constant velocity of $V/c = 3/5$ with respect to S. They have aligned their axes and synchronized their clocks in the usually way. In the world of S' the following events happen:

E0: $(ct'_0, x'_0) = (0, 0)$ preparation is made to send a signal;
E1: $(ct'_1, x'_1) = (1, 0)$ a light signal is send over the positive x' axis;
E2: $(ct'_2, x'_2) = (2, 1)$ the signal is received;
E3: $(ct'_3, x'_3) = (3, 1)$ the signal is processed and a second one is emitted in the negative x' direction;
E4: $(ct'_4, x'_4) = (4, 0)$ the signal is received;
E5: $(ct'_5, x'_5) = (5, 0)$ the signal is processed.

Find the corresponding (ct, x) coordinates according to S. Draw the events in two diagrams. The first one has both ct and ct' as the vertical axis and x and x' as the horizontal axis. The second on is a Minkowski diagram from the perspective of S.

Exercise 5: A Space Ship, with S' on board, is moving at $V/c = 3/5$ with respect to Mission Control (where S is located) on earth. Both S and S' have aligned their axes and synchronized their clocks in the usual way.

Mission control receives at $t = 1.7ls$ images from the impact of a meteorite on the moon. The distance from Mission Control to the moon is 1.2ls (according to S). This event E1. Event E2 is the impact itself (that is where and when of the impact), Event 3 is the receiving of images of the impact by S'. Of course, images travel in space at the speed of light.

a. Lorentz transform the events to S'. b. Find the position of S' at the time of the three events according to S. This provides additional events. c. Make a (ct, x) diagram in which you plot all the above events. Draw the the world line of S' in the diagram. d. Do the same but now for S'. e. Make a Minkowski diagram (from the perspective of S) and draw the grid-lines of S' for the events E1 and E2.

Exercises ##s Answers

Solution to Exercise 1: Consider the following pairs of events and determine whether they are time like, space like or light like connected.
a. E1: $(ct_1, x_1) = (1, 0)$ and E2: $(ct_2, x_2) = (0, 1)$

$$\rightarrow \Delta s_{12}^2 = (1-0)^2 - (0-1)^2 = 0 \text{ light like} \tag{86}$$

b. E3: $(ct_3, x_3) = (1, 3)$ and E4: $(ct_4, x_4) = (-2, 1)$

$$\rightarrow \Delta s_{34}^2 = (1+2)^2 - (3-1)^2 = 5 \text{ time like} \tag{87}$$

c. E5: $(ct_5, x_5) = (1, 2)$ and E6: $(ct_6, x_6) = (3, 4)$

$$\rightarrow \Delta s_{56}^2 = (1-3)^2 - (2-4)^2 = 0 \text{ light like} \tag{88}$$

d. Transform to S': $V/c = 12/13 \rightarrow \gamma = 13/5$

$$\begin{aligned} ct' &= \gamma \left(ct - \frac{V}{c} x \right) \\ x &= \gamma \left(x - \frac{V}{c} ct \right) \end{aligned} \tag{89}$$

E1: $(ct_1', x_1') = (13/5, -12/5)$ and E2: $(ct_2, x_2) = (-12/5, 13/5)$

$$\rightarrow \Delta s_{12}^{'2} = (13/5 + 12/5)^2 - (-12/5 - 13/5)^2 = 0 \text{ light like} \tag{90}$$

E3: $(ct_3', x_3') = (-23/5, 27/5)$ and E4: $(ct_4, x_4) = (-38/5, 37/5)$

$$\begin{aligned} \rightarrow \Delta s_{34}^{'2} &= (-23/5 + 38/5)^2 - (27/5 - 37/5)^2 \\ &= 225/25 - 100/25 = 5 \text{ time like} \end{aligned} \tag{91}$$

E5: $(ct_5', x_5') = (-11/5, 14/5)$ and E6: $(ct_6', x_6') = (-9/5, 16/5)$

$$\rightarrow \Delta s_{56}^{'2} = (-11/5 + 9/5)^2 - (14/5 - 16/5)^2 = 0 \text{ light like} \tag{92}$$

Of course, for all cases we find $\Delta s'^2 = \Delta s^2$: distance defined according to our Minkowski inproduct is a Lorentz invariant, i.e. the same for all inertial observers.

Solution to Exercise 2: In the frame of
For S:

$$E1 : (ct_1, x_1, y_1, z_1) = (0, 4, 0, 0) \tag{93}$$

$$E2 : (ct_2, x_2, y_2, z_2) = (5, 0, 3, 0) \tag{94}$$

$$\delta s_{12}^2 = (0-5)^2 - (4-0)^2 - (0-3)^2 - (0-0)^2 = 0 \tag{95}$$

light-like of course, as we deal with a light pulse.

For S': LT with $V/c = 4/5 \to \gamma = 5/3$

$$\begin{aligned} ct' &= \frac{5}{3}\left(ct - \frac{4}{5}x\right) \\ x' &= \frac{5}{3}\left(x - \frac{4}{5}ct\right) \\ y' &= y \\ z' &= z \end{aligned} \tag{96}$$

Thus:

$$E1 : (ct'_1, x'_1, y'_1, z'_1) = (-16/3, 20/3, 0, 0) \tag{97}$$

$$E2 : (ct'_2, x'_2, y'_2, z'_2) = (25/3, -20/3, 3, 0) \tag{98}$$

$$\begin{aligned} \delta s'^2_{12} &= (-16/3 - 25/3)^2 - (20/3 + 20/3)^2 - (0-3)^2 - (0-0)^2 \\ &= \frac{41^2}{9} - \frac{40^2}{9} - \frac{81}{9} = 0 \end{aligned} \tag{99}$$

Solution to Exercise 3: Observer

We start with writing down the LT. As $V/c = 12/13$ we have $\gamma = 13/5$. Thus, for this case LT reads as:

$$\begin{aligned} ct' &= \frac{13}{5}\left(ct - \frac{12}{13}x\right) \\ x' &= \frac{13}{5}\left(x - \frac{12}{13}ct\right) \end{aligned} \tag{100}$$

a.

$$\begin{aligned} \Delta s^2 &\equiv (ct_2 - ct_1)^2 - (x_2 - x_1)^2 \\ &= (4-3)^2 - (5-3)^2 \\ &= -3 \end{aligned} \tag{101}$$

b. event E1:

$$\begin{aligned} ct'_1 &= \frac{13}{5}\left(3 - \frac{12}{13}3\right) = \frac{3}{5} \\ x'_1 &= \frac{13}{5}\left(3 - \frac{12}{13}3\right) = \frac{3}{5} \end{aligned} \tag{102}$$

event E2:

$$\begin{aligned} ct'_2 &= \frac{13}{5}\left(4 - \frac{12}{13}5\right) = -\frac{8}{5} \\ x'_2 &= \frac{13}{5}\left(5 - \frac{12}{13}4\right) = \frac{17}{5} \end{aligned} \tag{103}$$

c.

$$\begin{aligned} \Delta s'^2 &\equiv (ct'_2 - ct'_1)^2 - (x'_2 - x'_1)^2 \\ &= (-\frac{8}{5} - \frac{3}{5})^2 - (\frac{17}{5} - \frac{3}{5})^2 \\ &= \frac{121}{25} - \frac{196}{25} = -3 \end{aligned} \tag{104}$$

Solution to Exercise 4: Observer
Lorentz Transformation

$$
\begin{aligned}
ct &= \gamma\left(ct' + \frac{V}{c}x'\right) \\
x &= \gamma\left(x' + \frac{V}{c}ct'\right) \\
\text{with } \frac{V}{c} &= \frac{3}{5} \text{ and } \gamma = \frac{5}{4}
\end{aligned}
\tag{105}
$$

This gives:

E0: $(ct_0, x_0) = (0, 0)$
E1: $(ct_1, x_1) = (5/4, 3/4)$
E2: $(ct_2, x_2) = (13/4, 11/4)$
E3: $(ct_3, x_3) = (9/2, 7/2)$
E4: $(ct_4, x_4) = (5, 3)$
E5: $(ct_5, x_5) = (25/4, 15/4)$

This gives the two required plots.

Solution to Exercise 5: A Space Ship, with

2.5 4-Momentum & E=mc2̂

2.5.1 Proper time

We have seen that in Special Relativity events are described by four coordinates: (ct, x, y, z). Moreover, distance is measured via a inner product $A^\mu \cdot B^\mu = A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3$. That opens the question: what about other quantities that we use in mechanics.

If position is $X^\mu = (ct, x, y, z)$ then what is velocity? Is $v^\mu \equiv \frac{dX^\mu}{dt}$ a good choice? It is what we are used to: velocity is change in position over time. However, we need to be careful. We require that our quantities are four-vectors, i.e. they transform according to the Lorentz Transformation. And the length, i.e. the inner product with itself, is the same for all inertial observes.

However, in our our first choice of the definition, we take the derivative with respect to time. But time is not the same for different observers!

We do know that the distance ds^2 is LT invariant, as is c^2, therefore we can combine both into another invariant - of time

$$d\tau^2 \equiv \frac{ds^2}{c^2} \tag{106}$$

If we spell out ds^2 we can write

$$d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \tag{107}$$

$d\tau$ is called *proper time* or *Eigenzeit* because for the rest frame S' we have $(dx' = dy' = dz' = 0)$ and thus

$$d\tau^2 = dt'^2 \tag{108}$$

We associate to a moving particle the 3-velocity $\vec{u} = (u_x, u_y, u_z) = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$. This is the velocity that we normally use: it is distance as measured in our frame of reference over time as we see on our clocks. We can relate the proper time $d\tau$ to the frame/coordinate time dt:

\$\$\begin{split} d\tau2̂ &= dt2̂-\frac{1}{c2̂}(dx2̂+dy2̂+dz2̂)\

&=dt2̂\left [1-\frac{1}{c2̂} \left (\left (\frac{dx}{dt} \right)2̂+ \left (\frac{dy}{dt} \right)2̂+ \left (\frac{dz}{dt} \right)2̂ \right) \right] \end{split}\$\$

Here we use the magnitude of the 3-velocity u. In other words

$$\frac{d\tau^2}{dt^2} = 1 - \frac{u^2}{c^2} \Rightarrow dt = \gamma(u)d\tau \tag{109}$$

The proper time interval relates to the frame time via the γ-factor for the velocity u.

2.5.2 4-velocity

Now we can tackle the 4-velocity. In order to make any sense we must define a velocity whose length is an invariant. Furthermore, velocity must be something like displacement over time interval. For the displacement the obvious choice is: dX^μ, i.e. a particle has moved from X^μ to $X^\mu + dX^\mu$. The displacement dX^μ transforms, of course, via the Lorentz Transformation. Moreover, its length is a Lorentz Invariant. In order to arrive at an adequate velocity, we must thus divide the displacement by a time interval that is also a Lorentz Invariant. Luckily, we have just seen that proper time is a Lorentz Invariant.

Therefore the 4-velocity $\vec{U}$ is

$$U^\mu \equiv \frac{dX^\mu}{d\tau} \tag{110}$$

where the derivative of the 4-position vector is taken with respect to the proper time τ. We obtain the relation to the 3-velocity $\vec{u}$ just from filling in $d\tau = dt/\gamma(u)$

$$U^\mu = \gamma(u)\left(\frac{dct}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (\gamma(u)c, \gamma(u)\vec{u}) \tag{111}$$

4-velocity transfers between frames moving with speed V as given by the Lorentz transformation as $\vec{U}$ is a 4-vector.

Be careful with 4-vector interpretation We compute the inner product of $\vec{U}$ with itself $U^2 = \gamma^2(u)(c^2 - u^2)$. That is a LT invariant of course. Therefore we can choose the frame such that $u = 0$, or in other words $U^2 = c^2$. The 4-velocity length is constant! That is not intuitive at all. Even stranger as the vector has constant length, it follows that the 4-velocity is always perpendicular to the 4-acceleration.

$$\frac{d}{d\tau}U^2 = 2\vec{U}\cdot\frac{d}{d\tau}\vec{U} = 0 \tag{112}$$

The counter intuitive stuff happens of course due to the pseudo-Euclidean metric.

Revisit 3-velocity transformation Earlier we transformed the velocity u of a particle in S to S' which was moving with V. This was quite complicated and the formula is difficult to remember. However, there is no need to remember the formula, you can always derive it from the transformation of the 4-velocity.

For the 4-velocity $\vec{U} = (\gamma(u)c, \gamma(u)\vec{u})$ we can write down the LT of a 4-vector between S and S'.

$$\begin{array}{rcl} \gamma(u')c & = & \gamma(V)\left(\gamma(u)c - \frac{V}{c}\gamma(u)u_x\right) \\ \gamma(u')u'_x & = & \gamma(V)\left(\gamma(u)u_x - \frac{V}{c}\gamma(u)c\right) \\ \gamma(u')u'_y & = & \gamma(u)u_y \\ \gamma(u')u'_z & = & \gamma(u)u_z \end{array} \tag{113}$$

If we now divide the second of these equations by the first we obtain

$$\frac{u'_x}{c} = \frac{\frac{u_x}{c} - \frac{V}{c}}{1 - \frac{Vu_x}{c^2}} \tag{114}$$

and if we divide the third of these equations by the first we obtain

$$\frac{u'_y}{c} = \frac{\frac{u_y}{c}}{\gamma(V)\left(1 - \frac{Vu_x}{c^2}\right)} \tag{115}$$

Just what we have derived before, but now in a way that you can always do this on the spot if you know the definition of the 4-velocity and the LT of a 4-vector.

2.5.3 4-momentum

If we postulate that the mass m is LT invariant we can define the 4-momentum simply by

$$\vec{P} = m\vec{U} = (m\gamma(u)c, m\gamma(u)\vec{u}) \equiv (P^0, \vec{p}) \tag{116}$$

with the 3-momentum $\vec{p} = m\gamma(u)\vec{u} = m\frac{d\vec{x}}{d\tau}$.

!!! warning "mass is a LT invariant" The mass m *does not* change as a function of velocity $\vec{u}$. You still sometimes see $\tilde{m} \equiv \gamma(u)m$ and with this $\vec{P} = (\tilde{m}c, \tilde{m}\vec{u})$. That is not practical as it mixes kinetic energy with inertial mass.

Conservation of 4-momentum For collisions now the total 4-momentum is conserved (per component)

$$\sum_{i,before} \vec{P}_i = \sum_{j,after} \vec{P}_j \tag{117}$$

If the total momentum is conserved than this must hold for the components $(m\gamma(u)c, \vec{p})$.

Note, that we did not write "mass is conserved". We postulate that it is a LT invariant, that is: it is the same for all inertial observers. But that does not imply that for collisions the mass should equal before and after the collision.

2.5.4 E=mc^2

The most famous equation in physics.

We will derive it by looking at N2 in its relativistic form.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\gamma(u)\vec{u}) = m\frac{d\vec{u}}{d\tau} \tag{118}$$

Kinetic energy was defined as work done on a mass. We again start from that and fill in N2 and take it step by step

$$\begin{aligned} \Delta E_{kin} &= \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 \vec{F}\cdot\vec{u}dt \\ &= \int_1^2 \frac{d}{dt}(m\gamma(u)\vec{u})\cdot\vec{u}dt \\ &= m\int_0^{\tilde{u}} \vec{u}\cdot d\gamma(u)\vec{u} \end{aligned} \tag{119}$$

This integration is more difficult than what we had before as the $\gamma(u)$ factor appears additional in the differential (for small velocities we have $\gamma(u) = 1$ and we just get $\frac{1}{2}mu^2$ as before). Now we apply integration by parts

$$\begin{aligned} \Delta E_{kin} &= m[\vec{u}\cdot\gamma(u)\vec{u}]_0^{\tilde{u}} - m\int_0^{\tilde{u}} \gamma(u)\vec{u}\cdot d\vec{u} \\ &= m\gamma(\tilde{u})\tilde{u}^2 - m\int_0^{\tilde{u}} \frac{\vec{u}\cdot d\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \\ &= m\gamma(\tilde{u})\tilde{u}^2 - m\int_0^{\tilde{u}} \frac{\frac{1}{2}du^2}{\sqrt{1-\frac{u^2}{c^2}}} \\ &= m\gamma(\tilde{u})\tilde{u}^2 - mc^2\left[\sqrt{1-\frac{u^2}{c^2}}\right]_0^{\tilde{u}} \\ &= m\gamma(\tilde{u})\tilde{u}^2 - mc^2\left(-\sqrt{1-\frac{\tilde{u}^2}{c^2}}+1\right) \\ &= m\gamma(\tilde{u})\tilde{u}^2 + \frac{mc^2}{\gamma(\tilde{u})} - mc^2 \\ &= -mc^2 + mc^2\gamma(\tilde{u})\left(\frac{\tilde{u}^2}{c^2}+1-\frac{\tilde{u}^2}{c^2}\right) \\ &= mc^2(\gamma(\tilde{u})-1) \end{aligned} \tag{120}$$

??? "Integration by parts"

Easy to remember integration by parts formula, from the product rule

$$\begin{array}{rcl}(fg)' & = & f'g+fg' \\ \Rightarrow \int (fg)' & = & \int f'g + \int fg' \\ \int f'g & = & [fg] - \int fg'\end{array} \tag{121}$$

In the derivation of the kinetic energy we used $f' = d\gamma(u)\vec{u}$ and $g = \vec{u}$.

If we now inspect the limiting cases for the velocity

$$\Delta E_{kin} = mc^2(\gamma(u) - 1) \tag{122}$$

- particle at rest: $u = 0 \Rightarrow \gamma(u) = 1 \Rightarrow \Delta E_{kin} = 0$
- small velocity $\frac{u}{c} \ll 1 \Rightarrow \gamma(u) = 1 + \frac{1}{2}\frac{u^2}{c^2} + \mathcal{O}(\frac{u^4}{c^4}) \Rightarrow \Delta E_{kin} = \frac{1}{2}mu^2$

The limiting cases work out. Very reassuring.

We can add a constant (LT invariant) to the kinetic energy $E = E_{kin} + mc^2 = m\gamma(u)c^2$. Adding constants to the energy/potential is always allowed as only the change of it is physically relevant (or the relative energies). The reason for *this* constant will be apparent below as this allows to include the energy in 4-momentum nicely.

We obtain

$$E = m\gamma(u)c^2 \tag{123}$$

or in the rest frame $(u = 0 \Rightarrow \gamma(u) = 1)$

$$E = mc^2 \tag{124}$$

With this energy $E = m\gamma(u)c^2$ we can define the 4-momentum as follows (we had $\vec{P} = (m\gamma(u)c, \vec{p})$)

$$\vec{P} = \left(\frac{E}{c}, \vec{p}\right) \tag{125}$$

??? “4-momentum with a different energy?”

With a different energy (addition of another constant to E_{kin} than what we did above) the length of the 4-momentum would not be LT invariant and $\vec{P}$ not a 4-vector. If we would have used $E = mc^2(\gamma - 1)$ then P^2 would not be LT invariant. You see this by computing $P^2 = \frac{E_{kin}^2}{c^2} - p^2c^2 = m^2c^2(2 - 2\gamma)$.

And we have finally derived *the* most famous equation in physics. We will use, however, $E = m\gamma(u)c^2$ most of the time as we are not always in the rest frame. The equation says essentially that mass is the same as energy. They are different manifestations of the same thing. A particle has energy in itself at rest without being in any potential.

NB: As gravitation acts on mass, it should also act on energy if they are the same! This is indeed the case, also photons, massless particles, feel gravity. More about that in Einstein’s theory of general relativity.

Mass in units of energy The mass of an electron $m_e = 9.13 \cdot 10^{-31}$ kg is often given as 512 keV, [kilo electron Volts]. Mass of all elementary particles is given actually in units of eV.

One electron volt is

$$1eV = 1.6 \cdot 10^{-19}C \cdot 1V = 1.6 \cdot 10^{-19}J \tag{126}$$

The conversion to mass via $E = mc^2$

$$m_ec^2 = 8.2 \cdot 10^{-14}J = \frac{8.2 \cdot 10^{-14}}{1.6 \cdot 10^{-19}} = 512keV \tag{127}$$

The fame The origin of the fame is probably twofold.

- Firstly, mass is no longer conversed as was a central pillar in Newton's mechanics. It can be converted. This was shocking for *physicists only*.
- Secondly, when mass is actually converted into energy e.g. in a nuclear fission bomb or inside the sun with nuclear fusion, the effect is immense. The drop of the two nuclear bombs (little boy and fat man) on Hiroshima and Nagasaki made the equation inglorious world-known; life changing for *all people*.
- Einstein's rock star status helped certainly quite a bit.

2.5.5 Energy-momentum relation

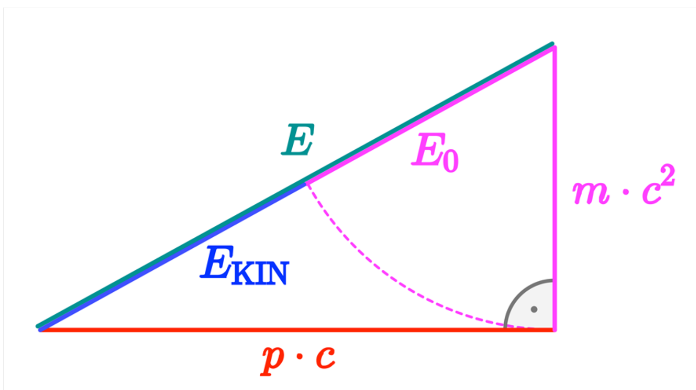
The 4-momentum is, of course, a 4-vector and therefore P^2 is LT invariant. Let us have a look at the outcome with $\vec{P} = (\frac{E}{c}, \vec{p})$

$$\begin{aligned} P^2 = \frac{E^2}{c^2} - p^2 &= m^2\gamma^2(u)c^2 - m^2\gamma^2(u)u^2 \\ &= m^2\gamma^2(u)c^2\left(1 - \frac{u^2}{c^2}\right) = m^2c^2 \\ \Rightarrow E^2 - p^2c^2 &= m^2c^4 \end{aligned} \tag{128}$$

Indeed, we find that P^2 is LT invariant as m and c are LT invariants. Rearranging the equation, we obtain

$$E^2 = (mc^2)^2 + (pc)^2 \tag{129}$$

This converts back to $E = mc^2$ in the rest frame.



Einstein triangle

You can visualize the energy momentum relation with the Einstein triangle shown here, as the relation has the form of $c^2 = a^2 + b^2$. With the kinetic energy as $E_{kin} = mc^2(\gamma(u) - 1)$. $E = E_0 + E_{kin} \equiv mc^2 + E_{kin}$.

LT invariance of P² Above we found a very useful, but bit hidden relation in the derivation

$$P^2 = m^2c^2 \tag{130}$$

This is of course LT invariant, as m and c are LT invariants (and the momentum is a 4-vector), but more importantly we can use this for computations of relativistic collisions. By the conservation of 4-momentum we can of course compute all collisions by equating the 4 components of the momentum before and after the collision. It is often, however, mathematically easier to write down the conservation of momentum and then square it. Because you can write down $P^2 = m^2c^2$ directly, this saves often computations.

2.5.6 Photons

For photons we have the energy given by $E = \hbar\omega$ and the momentum as $p = \frac{\hbar\omega}{c}$. The 4-momentum of a photon is

$$\vec{P} = P^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\right)\left(\frac{h\nu}{c}, \frac{h\nu}{c}\right) \tag{131}$$

It is directly clear that for photons the LT invariant $P^2 = 0$.

We could substitute the photon 4-momentum into the energy-momentum relation, we find

$$E^2 = (pc)^2 + (mc^2)^2 \Rightarrow m = 0 \tag{132}$$

This seems to confirm that photons do not have mass. But we need to be careful: photons do not have a 4-momentum of the form $P^\mu = (m\gamma c, m\gamma u)$. They can't: (1) their velocity is always c, which would lead to ∞ for their $\gamma(c)$, (2) with a mass $m = 0$ we multiply γc by zero. Together, this would gives us $0 \times \infty$ which is not defined in a unique way.

Thus: photons do not have mass. Do not get confused with $E = mc^2$.

Rest frame of a photon? Does a photon have a rest frame? It travels with the speed of light c (obviously) in all frames.

The answers is no and we give three good arguments.

- A rest frame implies that in this frame the object is at rest. But for a photon, traveling at c, which is LT invariant, there is no frame at which it is at rest, but only frames with $v = c$.
- The proper time of a photon is $d\tau^2 = dt^2 - \frac{1}{c^2}d\vec{x}^2$ but this is always equal to 0! A photon does not experience the passage of time, therefore it is reasonable to state that do not have a rest frame.
- In the hypothetical rest frame for a photon there would be no electro-magnetic radiation/interaction possible. In this frame e.g. the interaction between electrons would be zero.

Doppler revisited In chapter 14 we discussed the Doppler effect from a relativistic point of view. With the concept of 4-momentum it is easy to derive the Doppler shift of photons as observed in different frames of reference. We take the usual LT between S' and S. In S' a photon is moving along the x'-direction. It has frequency f'. Its 4-momentum is

$$P'^\mu_{photon} = \left(\frac{hf'}{c}, \pm\frac{hf'}{c}\right) \tag{133}$$

The $\pm$-sign indicates the direction of the photon: $+$ for moving in the positive x'-direction, - for moving in the negative x'-direction.

Using the Lorentz Transformation, we can easily transform the 4-momentum to the frame of S:

$$\begin{aligned} \frac{hf}{c} &= \gamma\left(\frac{hf'}{c} + \frac{V}{c}\frac{\pm hf'}{c}\right) = \gamma\left(1 \pm \frac{V}{c}\right)\frac{hf'}{c} \Rightarrow \\ \frac{f}{f'} &= \frac{1 \pm V}{\sqrt{1 - V^2}} \end{aligned} \tag{134}$$

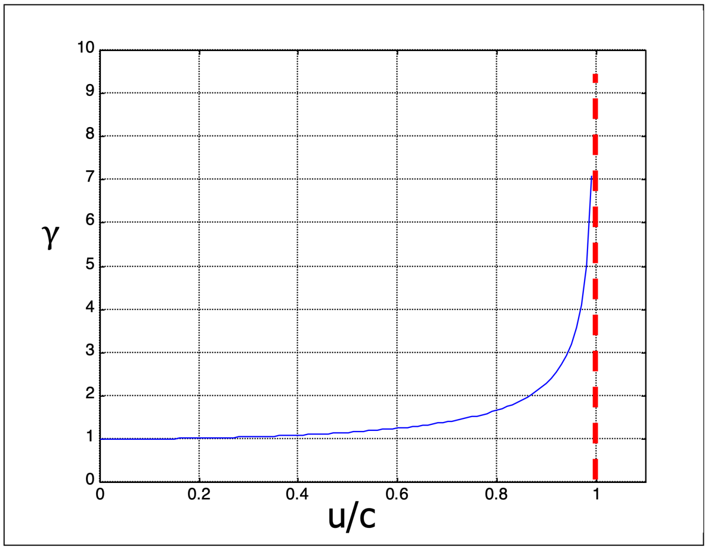
Note that we didn't use the transformation of P'^1_{photon} as this will give the same result.

2.5.7 Speed of light as limiting velocity

The γ factor increases strongly if the speed approaches the speed of light $u/c \rightarrow 1$ as can be seen in this plot

```
import numpy as np
import matplotlib.pyplot as plt
```

```
Hier code voor de plot
```



For a massive particle this has strong consequences. In the limit $u \rightarrow c$ the factor goes towards infinity. If we consider that the kinetic energy is $E = m(\gamma(u) - 1)c^2$, the amount of work done to increase the speed increases with γ. Therefore no massive particle can move with the speed of light (or faster) as this would require an infinite amount of energy for the acceleration.

NB: c is the speed of light in vacuum. In matter the speed of light v is smaller than c, characterized by the *refractive index* n as $n = c/v$. This leads e.g. to refraction by Snell's law at an interface. In matter the speed of massive particles can be larger than the speed of light there. This happens e.g. in a nuclear reactor when electrons move faster than the speed of light in water $(0.75c)$. As water is a dielectric, the light waves generated from the response to the moving charge lag behind and a phenomena similar to a sonic boom is created. This phenomena is termed Cherenkov radiation. If you have the opportunity to see it in a nuclear reactor, we highly recommend to take it. The color is a very intense deep blue.

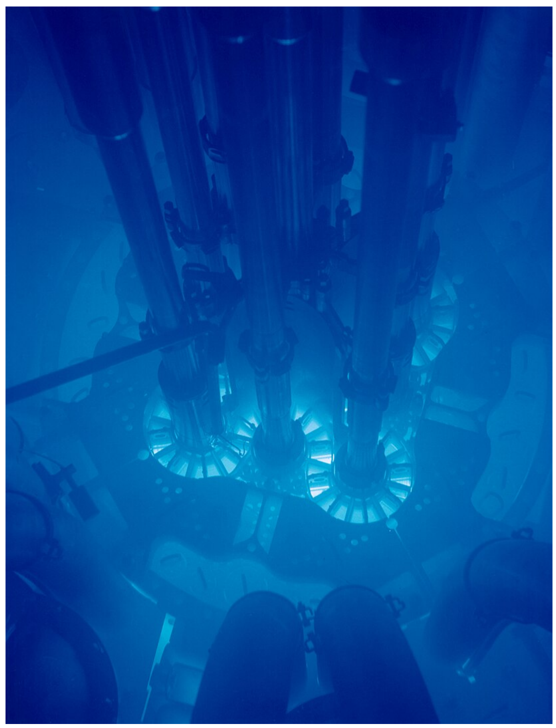


Figure 42: Cherenkov radiation glowing in the core of the Advanced Test Reactor at Idaho National Laboratory (Wikipedia Commons, CC BY-SA 2.0)

2.5.8 Exercises, examples & solutions

Exercise 1: Observer S and S' are connected via a Lorentz Transform of the form

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\ x' &= \gamma\left(x - \frac{V}{c}ct\right) \end{aligned} \tag{135}$$

with $V/c = 12/13$.

S' observes a particle of mass m traveling in the positive x'-direction with velocity $U'/c = 40/41$.

Find, using the 4-velocity, the velocity of m according to S.

Exercise 2: Observer S and S' are connected via a Lorentz Transform of the form

$$\begin{aligned} ct' &= \gamma\left(ct - \frac{V}{c}x\right) \\ x' &= \gamma\left(x - \frac{V}{c}ct\right) \end{aligned} \tag{136}$$

with $V/c = 12/13$.

S' observes a particle of mass m traveling in the positive y'-direction with velocity $U'/c = 40/41$.

Find, using the 4-velocity, the velocity of m according to S.

Exercise 3: According to S' a photon is emitted at $t' = 0$ from position $L_0 = 1ls$. It has a frequency f_0. S' is traveling at $V/C = 3/5$ in the positive x-direction with respect to S. They have synchronized their clocks when their origins coincide. Determine the time of detection of the photon by S' and by S. Find the frequency that S measures.

Exercise 4: In this exercise, the photon is emitted to S' a photon over the y'-axis. It has again a frequency f_0. S' is traveling at $V/C = 3/5$ in the positive x-direction with respect to S. They have synchronized their clocks when their origins coincide.

Find the frequency that S measures and the angle the traveling photon makes with the x-axis.

Exercises

Solution to Exercise 1: Observer
According to S'

$$\begin{aligned} U'_0 &= \gamma(U')c = \frac{41}{9}c \\ U'_1 &= \gamma(U')U' = \frac{40}{9}c \end{aligned} \tag{137}$$

LT to S using $\gamma(V) = \frac{13}{5}$:

$$\begin{aligned} U_0 &= \gamma(V)\left(U'_0 + \frac{V}{c}U'_1\right) = \frac{13}{5}\left(\frac{41}{9}c + \frac{12}{13}\frac{40}{9}c\right) = \frac{1013}{45}c \\ U_1 &= \gamma(V)\left(U'_1 + \frac{V}{c}U'_0\right) == \frac{13}{5}\left(\frac{40}{9}c + \frac{12}{13}\frac{41}{9}c\right) = \frac{1012}{45}c \end{aligned} \tag{138}$$

We find u_x by taking the ratio $\frac{U_1}{U_0} = \frac{\gamma(U)u}{\gamma(U)c}$:

$$\begin{aligned} u_x &= \frac{1012}{1013}c1 \\ u_y &= u_z = 0 \end{aligned} \tag{139}$$

Solution to Exercise 2: Observer
According to S'

$$\begin{aligned} U'_0 &= \gamma(U')c = \frac{41}{9}c \\ U'_1 &= 0 \\ U'_2 &= \gamma(U')U' = \frac{40}{9}c \end{aligned} \tag{140}$$

LT naar S using $\gamma(V) = \frac{13}{5}$:

$$\begin{aligned} U_0 &= \gamma(V)\left(U'_0 + \frac{V}{c}U'_1\right) = \frac{13}{5}\left(\frac{41}{9}c + 0\right) = \frac{533}{45}c \\ U_1 &= \gamma(V)\left(U'_1 + \frac{V}{c}U'_0\right) == \frac{13}{5}\left(0 + \frac{12}{13}\frac{41}{9}c\right) = \frac{492}{45}c \\ U_2 &= U'_2 = \frac{40}{9}c \end{aligned} \tag{141}$$

We find u_x by taking the ratio $\frac{U_1}{U_0} = \frac{\gamma(U)u_x}{\gamma(U)c}$:

$$u_x = \frac{492}{533}c \tag{142}$$

Similarly:

$$u_y = \frac{U_2}{U_0} = \frac{\gamma(U)u_y}{\gamma(U)c} = \frac{40}{533}c \tag{143}$$

The magnitude of the volocity accoring to $S4 is

$$u = \sqrt{u_x^2 + u_y^2} = \sqrt{\frac{243664}{284089}}c \approx 0.93c1 \tag{144}$$

Solution to Exercise 3: According to
According to S' the photon is send at $E_1 : (ct'_1, x'_1) = (0,1)ls$. Thus, it is received at $E_2 : (ct'_2, x'_2) = (1,0)$. Hence, for S event E_1 has coordinates:

$$\begin{aligned} ct_1 &= \frac{5}{4}\left(0+\frac{3}{5}1\right) = \frac{3}{4}ls \\ x_1 &= \frac{5}{4}\left(1+\frac{3}{5}0\right) = \frac{5}{4}ls \end{aligned} \tag{145}$$

and thus, S receives this photon at $(ct_3, x_3) = (2,0)ls$.

For S' the 4-Momentum of the photon is: $\left(\frac{hf_0}{c}, -\frac{hf_0}{c}\right)$. If we transform this to the frame of S, we get:

$$\frac{hf}{c} = \frac{5}{4}\left(\frac{hf_0}{c} + \frac{3}{5}\cdot -\frac{hf_0}{c}\right) = \frac{1}{2}\frac{hf_0}{c} \Rightarrow f = \frac{1}{2}f_0 \tag{146}$$

Solution to Exercise 4: In this exercise, the photon is emitted to
In this case for S' the 4-momentum of the photon is:

$$P'^{\mu} = \left(\frac{hf_0}{c}, 0, \pm\frac{hf_0}{c}, 0\right) \tag{147}$$

If we translate this to the world of S, we need to realize that momentum is a vector and that the spatial parts, i.e. P^1, P^2, P^3 form a 3-vector. In this case, there is no z-component and we can write the x and y-components as the length of the vector times a $\cos$ and a $\sin$, respectively:

$$\begin{aligned} \frac{hf}{c} &= \frac{5}{4}\left(\frac{hf_0}{c}+\frac{3}{5}0\right) = \frac{5}{4}\frac{hf_0}{c} \\ \frac{hf}{c}\cos\alpha &= \frac{5}{4}\left(0+\frac{3}{5}\frac{hf_0}{c}\right) = \frac{3}{4}\frac{hf_0}{c} \\ \frac{hf}{c}\sin\alpha &= \pm\frac{hf_0}{c} \end{aligned} \tag{148}$$

Thus, from the time-like component we conclude: $f = \frac{5}{4}f_0$. This should be in agreement with the spatial components. Let's check:

$$\begin{aligned} \frac{h^2f^2}{c^2} &= \frac{h^2f^2}{c^2}\cos^2\alpha + \frac{h^2f^2}{c^2}\sin^2\alpha \\ &= \frac{3^2}{4^2}\frac{h^2f_0^2}{c^2} + \frac{h^2f_0^2}{c^2} \\ &= \frac{5^2}{4^2}\frac{h^2f_0^2}{c^2} \end{aligned} \tag{149}$$

Indeed, the two spatial components are in agreement with the time-like one.

Finally, we have that according to S, the photon travels at an angle $\tan\alpha = \pm\frac{4}{3} \rightarrow \alpha = \pm 53.13°$ with the x-axis.

Answers

2.6 Relativistic dynamics and collisions

2.6.1 4-force

In the previous chapter we have seen that 4-momentum is defined by taking the derivative of the 4-velocity with respect to proper time: $P^\mu \equiv \frac{dU^\mu}{d\tau}$. This way, 4-momentum became a 4-vector that transforms according to the Lorentz Transformation.

In Special Relativity, we deal with inertial observers. The particles they encounter can, however, accelerate under the influence of forces. As momentum is now a 4-vector, we need to define a 4-force. Following Newton, momentum changes due to a force: $\frac{d\vec{p}}{dt} = \vec{F}$. In chapter 2 we discussed Newton's second Law in the form $\vec{F} = m\vec{a}$. We saw that the acceleration did not provide any problems: we had rulers and clocks. Hence, we could measure the acceleration using know and measurable concepts like position, distance and time.

The connection between force and acceleration is of a different nature: it is a physical law, i.e. a formulation that reflects how we think nature works at its principle level. It is a hypothesis; something we need to check over and over. A rule that holds until we find inconsistencies: contradictions between theory and experiment. It takes only one experiment to overthrow a theory.

We postulate, that force is a 4-vector. Moreover, we require that in the limit of $v/c \ll 1$, we recover Newton's second Law from the spatial components of our new 4-vector force law. After all, for low velocities, Classical Mechanics of Newton and Galilei works like a charm. This indicates that we need to differentiate 4-momentum with respect to time. But, if we require force to be a 4-vector, we need to differentiate with respect to proper time. Thus, we postulate:

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma(u)\frac{d}{dt}\left(m\gamma(u)c, m\gamma(u)\vec{u}\right) \tag{150}$$

with $E = m\gamma(u)c^2$ we can rewrite this to

$$\vec{F} = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, \frac{d}{dt}m\gamma(u)\vec{u}\right) = \gamma(u)\left(\frac{1}{c}\frac{dE}{dt}, \vec{f}\right) \tag{151}$$

with the 3-force $\vec{f} = \frac{d}{dt}(m\gamma(u)\vec{u})$

Work and Impulse How about our ideas of force performing work by that force acting over a distance or providing momentum by a force working during a time interval? These ideas and concepts still apply, but they take a relativistic form. Let's see how that works.

First, the natural extension of the definition of work is now:

$$dW = F^\mu dX^\mu \tag{152}$$

If we repeat what we did in chapter 4, we will replace dX^μ by $U^\mu \equiv \frac{dX^\mu}{d\tau}$ and instead of F^μ we write $\frac{dP^\mu}{d\tau}$:

$$\begin{aligned}
dW &= F^\mu dX^\mu \\
&= \frac{dP^\mu}{d\tau}U^\mu d\tau \\
&= m\frac{dU^\mu}{d\tau}U^\mu d\tau \\
&= mU^\mu dU^\mu \\
&= \frac{1}{2}m\, d\left(U^\mu U^\mu\right)
\end{aligned} \tag{153}$$

However, $U^\mu U^\mu = \gamma^2 c^2 - \gamma^2 \vec{u} \cdot \vec{u} = c^2$. That is, it is a constant (for all inertial observers the same). Thus $dU^\mu U^\mu = 0$. And we must conclude that

$$dW = F^\mu dX^\mu = 0 \tag{154}$$

Surprisingly, 4-force does perform zero work, always?! It is, on second thought, less surprising. Let's see how it works out in terms of 4-momentum:

$$\begin{aligned}
0 = dW &= F^\mu dX^\mu \\
&= \frac{dP^\mu}{d\tau} dX^\mu \\
&= \gamma \frac{dP^0}{dt} cdt - \gamma \frac{dP^1}{dt} u_x dt - \gamma \frac{dP^2}{dt} u_y dt - \gamma \frac{dP^3}{dt} u_z dt \\
&= \gamma \frac{dE/c}{dt} c - \gamma \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt}
\end{aligned} \tag{155}$$

Thus we can divide γ out of this equation and write $cE/c = E$:

$$0 = \frac{dE}{dt} - \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt} \Rightarrow \frac{dE}{dt} = \vec{u} \cdot \frac{d\gamma m \vec{u}}{dt} \tag{156}$$

But this is the relativistic equivalent of

$$\mathcal{P} \equiv \frac{dE}{dt} = \vec{f} \cdot \vec{u} \tag{157}$$

In words: the inner product of 3-force and 3-velocity is the power $\mathcal{P}$.

2.6.2 Collisions

We will now concentrate on collisions. From our earlier discussions, for collisions we assume that we can look 'over' the collision, that is: we applie conservation of momentum and -for elastic collisions- kinetic energy. The latter implies: no non-conservative forces that dissipate mechanical energy and the potential energy prior and after the collision is the same.

We do that also for our relativistic collisions. But, we don't require that it only holds for perfectly elastic collisions. Instead, we apply it to cases where there is no possibility to turn some of the energy involved into heat. So, we focus on collisions of elementary particles that do not convert part to their energy to heat.

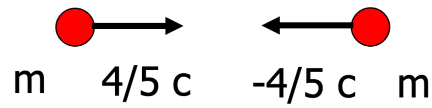
The 4-momentum is conserved. For $\vec{P} = (\frac{E}{c}, \vec{p})$ we have

$$\sum_{i,before} \vec{P}_i = \sum_{j,after} \vec{P}_j \tag{158}$$

and the energy-momentum relation from the LT invariance of $\vec{P} \cdot \vec{P}$

$$E^2 = (mc^2)^2 + (pc)^2 \tag{159}$$

With $E = m\gamma(u)c^2$ and $\vec{p} = m\gamma(u)\vec{u}$.



Example: head on collision Two elementary particles collide head on, see the figure below.

Both particles have mass m, after the collision there is only one particle with unknown mass M. What is the mass M and the velocity v of that one particle after the collision/fusion?

We consider the conservation of 4-momentum, in 1D:

$$\begin{aligned} P^{\mu}_{before} &= (m\gamma(u)c, m\gamma(u)u) + (m\gamma(-u)c, -m\gamma(-u)u) \\ &= (2m\gamma(u)c, 0) \\ P^{\mu}_{after} &= (M\gamma(v)c, M\gamma(v)v) \end{aligned} \tag{160}$$

with $\gamma(u) = \gamma(-u)$. The 4-momentum is conserved per component, from the space component we see $0 = M\gamma(v)v \Rightarrow v = 0$. With $\gamma(u) = 5/3$ and $\gamma(v) = 1$ we find for the time-component $2m\frac{5}{3} = M$.

This leads to $M = \frac{10}{3}m > 2m$. Thus, the energy prior to the collision was composed of energy associated with the masses themselves and with kinetic energy. After the collision, there is no kinetic energy but their is mass-energy and there is more of this than prior to the collision.

Example: decay of a photon into an electron and positron We discuss if a photon (of sufficient energy $E > 1024$ keV) can decay into an electron e^- and positron e^+.

If we place us in the Center of Mass (CM) frame of the electron e^- and positron e^+ after the decay, then the total spatial momentum is $\vec{p} = 0$. The momentum before the decay of the photon is $\vec{p} = \frac{hf}{c} > 0$ therefore the decay cannot happen in free space. Momentum must be transferred to an additional different particle.

$$\left(\frac{E_e}{c}, \vec{p}\right) + \left(\frac{E_p}{c}, -\vec{p}\right) \neq \left(\frac{hf}{c}, \frac{hf}{c}\right) \tag{161}$$

Example: Electron-positron annihilation We consider an electron and positron annihilation, resulting in two photons (after the collision). Remember that the decay cannot happen into one photon as shown above (Remember: equations are invariant under time reversal).

In the CM of the e^-e^+ system we have for the 4-momentum before

$$P^{\mu}_{before} = (m_e\gamma(u)c, m_e\gamma(u)u, 0, 0) + (m_e\gamma(-u)c, -m_e\gamma(-u)u, 0, 0) \tag{162}$$

After we have two photons, with different frequencies f, f' and traveling in different directions $\hat{s}, \hat{s}'$

$$P^{\mu}_{after} = \left(\frac{hf}{c}, \frac{hf}{c}\hat{s}\right) + \left(\frac{hf'}{c}, \frac{hf'}{c}\hat{s}'\right) \tag{163}$$

From the conservation of 4-momentum we have

$$\begin{aligned} 2m_e\gamma(u)c &= \tfrac{hf}{c} + \tfrac{hf'}{c} \\ 0 &= \tfrac{hf}{c}\hat{s} + \tfrac{hf'}{c}\hat{s}' \end{aligned} \tag{164}$$

From the second equation we see

$$\frac{hf}{c}\hat{s} = -\frac{hf'}{c}\hat{s}' \Rightarrow \hat{s} = -\hat{s}', \quad f = f' \tag{165}$$

The two photons are emitted in opposite directions (in the CM system) with the same frequency.

Filling this into the first equation $hf = m_e\gamma(u)c^2 \approx m_e c^2 = 512$ keV. The speed in the CM frame is typically $u \ll c \Rightarrow \gamma(u) = 1$.

NB: please observe that analysis in the CM frame is often a good idea.

Example: Compton scattering Compton scattering describes the (elastic) scattering of an incoming photon by a (bound) charged particle, typical an electron.

In the rest frame of the electron, we have for the 4 different 4-momenta:

$$\begin{aligned} P_{e,b} &= (m_e c, 0, 0, 0) \\ P_{\gamma,b} &= (E/c, E/c, 0, 0) \\ P_{e,a} &= (\tfrac{E_e'}{c}, m_e\gamma(u)u\cos\phi, -m_e\gamma(u)u\sin\phi, 0) \\ P_{\gamma,a} &= (\tfrac{E'}{c}, \tfrac{E'}{c}\cos\theta, \tfrac{E'}{c}\sin\theta, 0) \end{aligned} \tag{166}$$

We have

$$P_{e,b} + P_{\gamma,b} = P_{e,a} + P_{\gamma,a} \tag{167}$$

Now we make use of the LT invariance of $\vec{P}^2$

$$(P_{e,b} + P_{\gamma,b} - P_{\gamma,a})^2 = P_{e,a}^2 \tag{168}$$

$$P_{e,b}^2 + P_{\gamma,b}^2 + P_{\gamma,a}^2 + 2P_{e,b}P_{\gamma,b} - 2P_{e,b}P_{\gamma,a} - 2P_{\gamma,b}P_{\gamma,a} = P_{e,a}^2 \tag{169}$$

where we know $P_{e,b}^2 = P_{e,a}^2 = m_e^2c^2$ (totally elastic collison) and $P_\gamma^2 = 0$ directly as shown before. Evaluating the cross terms gives

$$m_e^2c^2 + 0 + 0 + 2m_eE' - 2m_eE - 2\frac{EE'}{c^2}(1-\cos\theta) = m_e^2c^2 \tag{170}$$

We isolate the energy after the collision E'

$$E' = \frac{Em_ec^2}{m_ec^2 + E(1-\cos\theta)} \tag{171}$$

With $E = hc/\lambda$ we obtain

$$\frac{\lambda'}{hc} = \frac{m_ec^2 + \frac{hc}{\lambda}(1-\cos\theta)}{\frac{hc}{\lambda}m_ec^2} \tag{172}$$

Now we only multiply both sides by hc and on the right we divide out, to obtain

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\theta) \tag{173}$$

Alternatively, we could try and solve the collision by directly using conservation of momentum. This is much more work than the P2̂ trick. The calculation goes as follows.

In the rest frame of the electron

$$P^\mu_{before} = \left(\frac{hf}{c}, \frac{hf}{c}, 0, 0\right) + (m_e c, 0, 0, 0) \tag{174}$$

After the scattering

$$\begin{aligned} P^\mu_{after} = &\left(\frac{hf'}{c}, \frac{hf'}{c}\cos\theta, \frac{hf'}{c}\sin\theta, 0\right) + \\ &+ (m_e\gamma(u)c,\, m_e\gamma(u)u\cos\phi,\, -m_e\gamma(u)u\sin\phi,\, 0) \end{aligned} \tag{175}$$

We have 3 equations, but 4 unknowns (f', u, ϕ, θ). Therefore the outgoing frequency f' is not uniquely determined, but dependent on the scattering angle θ. We can eliminate 2 (here u, ϕ) of the 4 unknowns, to remain with a relation for the other two.

For the spatial momentum we have

$$\begin{array}{rcl} \frac{hf}{c} & = & \frac{hf'}{c}\cos\theta + m_e\gamma(u)u\cos\phi \\ 0 & = & \frac{hf'}{c}\sin\theta - m_e\gamma(u)u\sin\phi \end{array} \tag{176}$$

We rewrite the equations slightly, before squaring them and then adding them to eliminate ϕ

$$\begin{array}{rcl} \frac{hf}{c} - \frac{hf'}{c}\cos\theta & = & m_e\gamma(u)u\cos\phi \\ \frac{hf'}{c}\sin\theta & = & m_e\gamma(u)u\sin\phi \end{array} \tag{177}$$

We indeed eliminate ϕ to

$$\frac{h^2f^2}{c^2} - 2\frac{hfhf'}{c^2}\cos\theta + \frac{h^2f'^2}{c^2} = m_e^2\gamma^2(u)u^2 \quad (*) \tag{178}$$

The right hand side of the equation is the space component squared of the momentum after: $p_{e'}^2 = m_e^2\gamma^2(u)u^2$, but this can be related to the energy via the momentum-energy relation for the moment after $(p_{e'}c)^2 = E_{e'}^2 - (m_e c^2)^2$. We will use this to eliminate the unknown speed u.

The energies can be related via the 0-component of the 4-momentum

$$\begin{array}{rcl} \frac{hf}{c} + m_e c & = & \frac{hf'}{c} + \frac{E_e'}{c} \\ \Rightarrow E_e'^2 & = & (hf - hf' + m_e c^2)^2 \end{array} \tag{179}$$

Substituting the energy $E_e'^2$ into the momentum-energy relation and replacing the right hand side of equation $(*)$ after multiplying by c^2 to

$$h^2f^2 - 2hfhf'\cos\theta + h^2f'^2 = (hf - hf' + m_e c^2)^2 - (m_e c^2)^2 \tag{180}$$

Indeed we have removed the speed u and angle ϕ. We cannot do more, but remain with a relation for the frequency f' after scattering as function of angle θ. To this end we evaluate the square in the equation, cancel a few terms and rearrange to

$$\begin{aligned} 2hfm_ec^2 - 2hf'm_ec^2 &= 2h^2ff'(1-\cos\theta) \\ \frac{c}{f'} - \frac{c}{f} &= \frac{h}{m_ec}(1-\cos\theta) \end{aligned} \tag{181}$$

Finally, by replacing the frequency with the wavelength $f\lambda = f'\lambda' = c$

$$\lambda' - \lambda = \frac{h}{m_ec}(1-\cos\theta) \tag{182}$$

This is, of course, the same result as we derived earlier.

As $\cos\theta < 1$ we find $\lambda' > \lambda$, which makes sense as the photon can only loose energy to the electron in the initial rest frame of the electron. After the scattering the electron can pick up some speed.

To analyze the outcome we check for

- $\theta = 0$ (no scattering): $\Rightarrow \lambda' = \lambda$ which makes sense
- $\theta = \pi$: backwards scattering, maximal $\Delta\lambda = \frac{2h}{m_ec}$ largest energy transfer

2.6.3 Exercises, examples & solutions

Momentum of an accelerated electron

Momentum of an accelerated electron: compute the momentum and speed of an electron after acceleration in a potential of $V = 300$ kV.

From $E^2 = (mc^2)^2 + (pc)^2$ we have $p = \frac{1}{c}\sqrt{E^2 - (mc^2)^2}$ and using $E = mc^2 + E_{kin}$ we have

$$p = \frac{1}{c}\sqrt{2mc^2 E_{kin} + E_{kin}^2} \tag{183}$$

With $E_{kin} = 300$ keV and $m_e = 511$ keV. The speed can be computed from rearranging $E_{kin} = mc^2(\gamma - 1)$ to $\frac{v}{c} = \sqrt{1 - \frac{(mc^2)^2}{(E_{kin}+mc^2)^2}} = \sqrt{1 - \frac{511^2}{811^2}} = 0.77$. Please observe how practical it is to use the units eV!

Decay of a neutral kaon

Decay of a neutral kaon into three pions. $K^0 \rightarrow \pi^- + \pi^+ + \pi^0$. Show that the three pions trajectories are in one plane.

In the rest frame of the kaon we have $\vec{p}_K = 0$ before the decay. By conservation of momentum we have after the decay $\vec{p}_{\pi^-} + \vec{p}_{\pi^+} + \vec{p}_{\pi^0} = 0$. A necessary and sufficient condition for three vectors $\vec{p}_1, \vec{p}_2, \vec{p}_3$ to lie in one plane is that $\vec{p}_1 \cdot (\vec{p}_2 \times \vec{p}_3) = 0$ (Remember that this expression gives the volume of the parallelepiped spanned by the three vectors). From the conservation of momentum we have $\vec{p}_1 = -\vec{p}_2 - \vec{p}_3$. Now we can compute $(-\vec{p}_2 - \vec{p}_3) \cdot (\vec{p}_2 \times \vec{p}_3) = -\vec{p}_2 \cdot (\vec{p}_2 \times \vec{p}_3) - \vec{p}_3 \cdot (\vec{p}_2 \times \vec{p}_3) = 0$. The two terms are each zero individually as the term in the bracket is perpendicular to $\vec{p}_2$ and $\vec{p}_3$ respectively.

If the trajectories in the rest frame of the kaon are in one plane, then they are also in one plane in all other frames. A coordinate transformation only shifts or rotates, which transfers a plane into a plane, but does not e.g. shear or bend a plane.

Worked Examples

Exercise 1: A particle of mass M disintegrates into two fragments. In the rest frame of M, fragment m_1 has a mass of $\frac{1}{4}M$ and a velocity $u_1/c = 3/5$.

Find the mass and velocity of the other fragment.

Exercise 2: A particle of mass m is initially at rest (in frame S). A photon of frequency f is traveling over the x-axis and will be absorbed by the particle. Another photon is emitted. This photon is also traveling over the x- axis but in the opposite direction as incoming photon.

The incoming photon energy equals mc^2, the emitted photon $\frac{1}{4}mc^2$. Find the velocity and mass of the particle after the process of absorbing and emitting the photons.

Exercise 3: An elementary particle of mass M moves in the frame of observer S with a velocity $v/c = 12/13$. The particle is unstable and decays into a particle of mass m and a photon. The particle m has velocity $u/c = 4/5$. Both M and m move along the x-axis in the positive direction.

1. Find the mass m in terms of M.
2. What is the frequency of the photon?

Exercise 4: A particle of mass m moves with velocity $v_1/c = 1/2$ in the positive direction over the x-axis. At the same time, an identical particle is moving with the same velocity in the positive y-direction over the y-axis. At some point in time the two particles collide and as a result a new particle of mass M is formed.

Find the mass and velocity-vector of the new particle.

Exercise 5: A particle of mass $\frac{3}{5}m$ is moving at velocity $v_1/c = 4/5$ over the $x - axis$. From the other side a particle with mass $\frac{4}{5}m$ is approaching with velocity $v_2/c = 3/5$. The two particles will collide. After the collision, they maintained each their original mass. The collision is perfectly elastic.

1. Find the velocities of both masses in the world of Galilei and Newton.
2. The same but now in the world of Lorentz and Einstein.

Exercises

Solution to Exercise 1: A particle of mass
Prior to the disintegration, particle M has 4-momentum:

$$P_i^{\mu} = (Mc, 0) \tag{184}$$

After the disintegration, we have two particles with 4-momentum:

$$P_{1,a}^{\mu} = \left(\frac{1}{4}M\frac{5}{4}c, \frac{1}{4}M\frac{5}{4}\frac{3}{5}c\right) \tag{185}$$

and

$$P_{2,a}^{\mu} = (m_2\gamma_2 c, m_2\gamma_2 u_2) \tag{186}$$

From conservation of momentum we get:

$$\begin{aligned} 1 &= \frac{5}{16} + \frac{m_2}{M}\gamma_2 \rightarrow \frac{m_2}{M}\gamma_2 = \frac{11}{16} \\ 0 &= \frac{3}{16} + \frac{m_2}{M}\gamma_2\frac{u_2}{c} \rightarrow \frac{m_2}{M}\gamma_2\frac{u_2}{c} = -\frac{3}{16} \end{aligned} \tag{187}$$

Take the ratio of the last two equations:

$$\frac{u_2}{c} = -\frac{3}{11} \tag{188}$$

and from this we find

$$\frac{m_2}{M} = \frac{4\sqrt{7}}{16} \tag{189}$$

Thus, we see that the mass after the disintegration is $\frac{1}{4}M + \frac{4\sqrt{7}}{16}M \approx 0.911M$.

Solution to Exercise 2: A particle of mass
Before the absorption of the photon the 4-momentum is:

$$P_i^\mu = \left(\frac{hf}{c}, \frac{hf}{c}\right) + (mc, 0) = (2mc, mc) \tag{190}$$

After emitting the photon, the particle has mass M and velocity u. The emitted photon has as frequency $\tilde{f}$ and 4-momentum $\left(\frac{h\tilde{f}}{c}, -\frac{h\tilde{f}}{c}\right) = (\frac{1}{4}mc, -\frac{1}{4}mc)$. The total momentum after the process is:

$$P_f^\mu = \left(\frac{1}{4}mc + M\gamma c, -\frac{1}{4}mc + M\gamma u\right) \tag{191}$$

Since 4-momentum is conserved, we find:

$$\begin{aligned} 2mc &= \frac{1}{4}mc + M\gamma c \\ mc &= -\frac{1}{4}mc + M\gamma u \end{aligned} \tag{192}$$

We rearrange the two above equations:

$$\begin{aligned} M\gamma c &= \frac{7}{4}mc \\ M\gamma u &= \frac{5}{4}mc \end{aligned} \tag{193}$$

If we divide the second equation by the first, we have:

$$\frac{u}{c} = \frac{5}{7} \tag{194}$$

The mass of the particle is:

$$M = \frac{7}{4\gamma}m = \frac{1}{2}\sqrt{6}\,m \tag{195}$$

Solution to Exercise 3: An elementary particle of mass
Initially, the 4-Momentum is

$$P_i^\mu = (M\gamma(v)c, M\gamma(v)v) \tag{196}$$

with

$$\frac{v}{c} = \frac{12}{13} \to \gamma(v) = \frac{13}{5} \tag{197}$$

After the decay, we have

$$P_f^\mu = \left(m\gamma(u)c + \frac{hf}{c}, m\gamma(u)u + \frac{hf}{c}\hat{f}\right) \tag{198}$$

with $\hat{f}$ a unit vector pointing in the $\pm x$-direction. We know $\frac{u}{c} = \frac{4}{5} \rightarrow \gamma(u) = \frac{5}{3}$. Conservation of 4-momentum now leads to::

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc \pm \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{199}$$

We still need to find out which direction the photon travels: parallel to m or in the opposite direction. According to the above conservation of 4-momentum both seem possible. We require that in the above $f0$.

First we inspect the negative sign of $\pm$:

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc - \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{200}$$

If we solve for f, we get $f0$, which conflicts our requirement. That leaves us with the $+$sign:

$$\begin{aligned} \frac{5}{3}mc + \frac{hf}{c} &= \frac{13}{5}Mc \\ \frac{4}{3}mc + \frac{hf}{c} &= \frac{12}{5}Mc \end{aligned} \tag{201}$$

Solving for m gives: $m = \frac{3}{5}M$. If we plug this back in, we find for the photon $hf = \frac{8}{5}Mc^2$.

Solution to Exercise 4: A particle of mass
The total 4-momentum before the collision is

$$P_i^\mu = \left(2m\gamma c, \frac{1}{2}m\gamma c, \frac{1}{2}m\gamma c\right) \text{ with } \gamma = \frac{2}{3}\sqrt{3} \tag{202}$$

After the collision, we have only one particle with 4-momentum

$$P_f^\mu = (M\gamma_f c, M\gamma_f u_x, M\gamma_f u_y) \text{ with } \gamma_f = \frac{1}{\sqrt{1 - \frac{u_x^2 + u_y^2}{c^2}}} \tag{203}$$

In this process, 4-momentum is conserved.

From P^1 and P^2 we get

$$\begin{aligned} \frac{1}{2}m\gamma c &= M\gamma_f u_x \\ \frac{1}{2}m\gamma c &= M\gamma_f u_y \end{aligned} \tag{204}$$

hence, $u_x = u_y$. The new particle moves over the line $x = y$.

If we combine P^0 with P^1, we find:

$$\begin{aligned}2m\gamma c &= M\gamma_f c\\ \frac{1}{2}m\gamma c &= M\gamma_f u_x\end{aligned} \tag{205}$$

This gives $\frac{u_x}{c} = \frac{1}{4}$. Thus, the new particle moves with velocity $\vec{u} = \frac{1}{4}c\hat{x} + \frac{1}{4}c\hat{y}$. We find its mass by calculating $\gamma_f = \frac{1}{\sqrt{1-2\frac{1}{16}}} = 2\sqrt{\frac{2}{7}}$ and using this in the P^0 equation:

$$2m\gamma c = M\gamma_f c \rightarrow M = \sqrt{\frac{14}{3}}m \tag{206}$$

Solution to Exercise 5: A particle of mass
a. In classical mechanics, we use -for this type of collision- conservation of momentum and of kinetic energy. This gives us:

$$\begin{aligned}p:&\quad \frac{3}{5}m\frac{4}{5}c - \frac{4}{5}m\frac{3}{5}c = \frac{3}{5}mu + \frac{4}{5}mU \rightarrow U = -\frac{3}{4}u\\ E_{kin}:&\quad \frac{1}{2}\frac{3}{5}m\left(\frac{4}{5}c\right)^2 + \frac{1}{2}\frac{4}{5}m\left(\frac{3}{5}c\right)^2 = \frac{1}{2}\frac{3}{5}mu^2 + \frac{1}{2}\frac{4}{5}mU^2\end{aligned} \tag{207}$$

This set has as solution (not surprising): $u = -\frac{4}{5}c, U = \frac{3}{5}c$.

b. Now we use 4-momentum conservation:

$$P_i^\mu = \left(\frac{3}{5}m\frac{5}{3}c, \frac{3}{5}m\frac{5}{3}\frac{4}{5}c\right) + \left(\frac{4}{5}m\frac{5}{4}c, -\frac{4}{5}m\frac{5}{4}\frac{3}{5}c\right) = \left(2mc, \frac{1}{5}mc\right) \tag{208}$$

Note: the spatial part of momentum is thus non-zero, in contrast to the classical case.

After the collision we have:

$$P_f^\mu = \left(\frac{3}{5}m\gamma_1 c, \frac{3}{5}m\gamma_1 u\right) + \left(\frac{4}{5}m\gamma_2 c, -\frac{4}{5}m\gamma_2 U\right) \tag{209}$$

with

$$\gamma_1 = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \text{ and } \gamma_2 = \frac{1}{\sqrt{1-\frac{U^2}{c^2}}} \tag{210}$$

Next, we use conservation of 4-momentum: $P_i^\mu = P_f^\mu$. This is, however, hard to do analytical! Thus we use either a graphical or numerical method. If you do this, you will find:

$$u = -0.7355c \quad \text{and} \quad U = +0.8050c \tag{211}$$

Answers

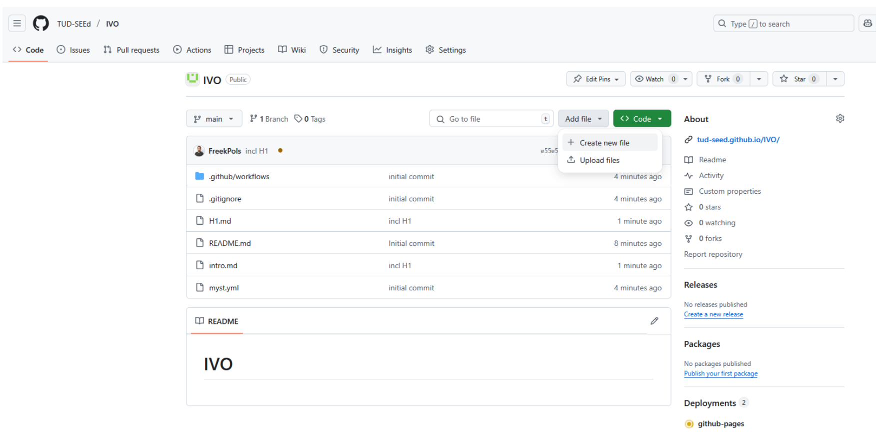
3 For developers

3.1 How to

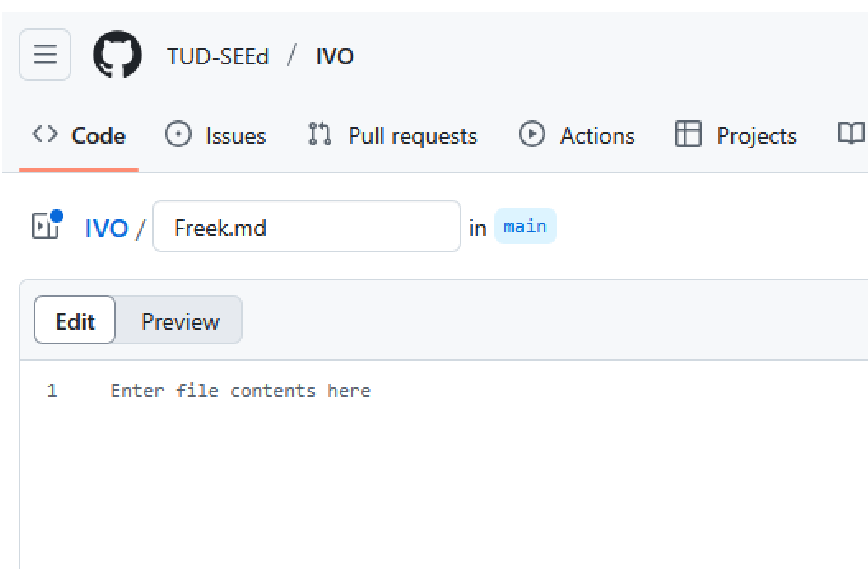
3.1.1 Introductie

Welkom bij Jupyter Book!

- Ga naar de website van Github en maak een account aan als je dat nog niet hebt.
- Geef je accountnaam door aan Freek, hij voegt jou toe aan het boek.
- Als je toegang hebt, kun je aan de slag met een eigen hoofdstuk maken of een bestaand hoofdstuk editen. De repo waar je toegang toe krijgt (voor dit specifieke boek) vind je hier.
- Ga naar de folder *content* en klik op *Add file* en *Create a new file*, zie hieronder.



- Geef je file een naam met als extensie *.md* bijv. *Freek.md*



- In die file kun je jouw inhoud stoppen / ontwikkelen.
- Maak een hoofdstuk titel (# Mijn eerste titel) en een section titel (## Mijn eerste sectie).

- Druk op de groene *Commit changes* knop om je aanpassingen door te zetten naar de repo. Je kunt de commit een passende titel geven (of niet).
- Let op! Het mechanica boek is gewijzigd tov de template, dusdanig dat je het hoofdstuk ook in de **Table of Contenst** moet zetten. Deze staat in de hoofdmap, in het bestand myst.yml

Wat er nu gebeurt is dat het boek opnieuw gemaakt wordt en via GitHub pages gepubliceerd. Na ongeveer 2 minuten kun je dus het resultaat op de website zien!

- Bekijk eens op de site van Jupyter Book naar wat je allemaal kunt toevoegen en pas dat aan in je eigen gemaakte hoofdstuk: klik daartoe op je gemaakte hoofdstuk en dan op het pennetje aan de rechterkant (*edit this file*)
- Je kunt natuurlijk ook de features bekijken in het volgende hoofdstuk.
- Succes!

Note
Goed om te weten... dit boek is gemaakt in MyST de meest recente versie van Jupyter Books.

3.1.2 Feedback / issue report / vragen

Rechtsboven op de page staat een knop met FEEDBACK. Wanneer je daar op klikt kom je op de issues pagina van de github van dit boek. Je kunt een nieuwe issue aanmaken (groene knop, *New issue*). Daarmee kom je bij een formulier die vraagt om een titel, en een beschrijving van het probleem. Je kunt verder iemand aanwijzen (*assignees*) om het probleem te koppelen aan iemand die het waarschijnlijk kan oplossen. Daarnaast is er de mogelijkheid om een label er aan te hangen (bijv. bug / invalid / help wanted).

Wanneer je de issue hebt gerapporteerd (Create) belandt deze in de to-do list en wordt het issue opgepakt wanneer daar tijd voor is.

Wil je tekeningen bij een specifiek onderwerp, tag dan *Hanna*. Beschrijf wat je voor tekening wilt, als dat onvoldoende helder is vanuit de vraagstelling zelf.

3.1.3 Opzetten van een lokale server

Wanneer je lokaal werkt en een push doet naar github, zal het boek opnieuw gebouwd worden en online te zien zijn. Een andere mogelijkheid is lokaal werken en je output (bijna) live te volgen. Wanneer je een document opslaat, wordt dit gedetecteerd en wordt alleen de pagina die je hebt gewijzigd opnieuw gemaakt.

Om direct te output van de wijzigingen te zien (lokaal), ga je via de terminal (anaconda prompt of de mac terminal) naar de folder waar het myst.yml bestand van dit boek staat. Typ in de terminal `myst start` (de eerste keer dat je het boek bouwt moet dit `myst init` zijn). Op dat moment worden de boeken geconverteerd naar een website, welke lokaal te zien is. Het adres wordt gegeven in de terminal, veelal is dat: `http://localhost:3000`. Via een webbrowser kan dit adres gekopieerd worden. Wanneer je een bestand opslaat, wordt deze binnen ~5 s zichtbaar.

3.1.4 Werken met GIT

Werken met Git heeft het voordeel dat je goed kunt samenwerken. Via de repository worden de bestanden gesynchroniseerd. Om hier goed gebruik van te maken is de volgende workflow handig:

Bij starten van nieuwe edits doe je een git pull, zie Figure 43.

Ben je klaar, dan commit & push je de wijzigingen naar de repository. Vergeet niet een samenvatting van de wijzigingen toe te voegen! Tussendoor kun je ook een push doen, om bijv. het resultaat online te bekijken.

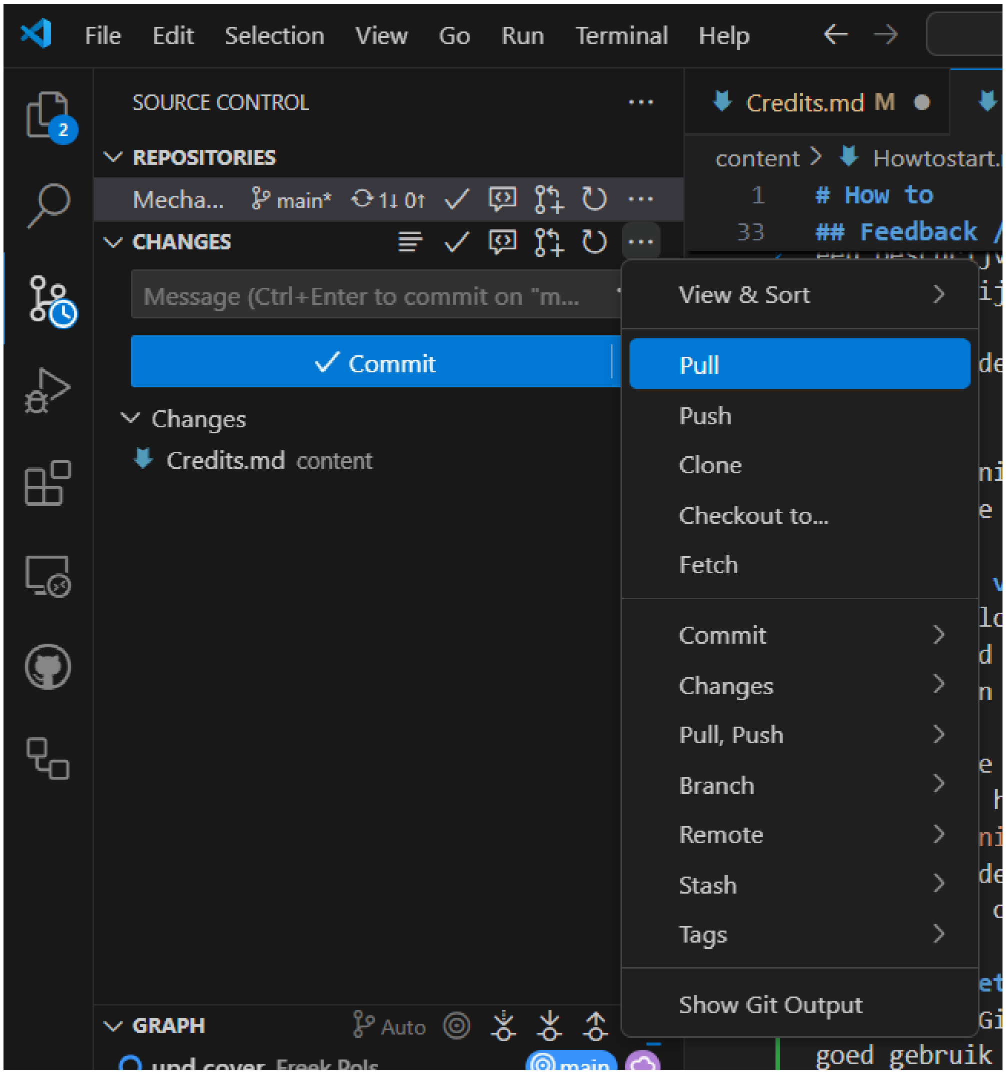


Figure 43: Bij de start doe je een pull.

3.1.5 Admonitions

Er zijn diverse admonitions mogelijk: danger / tips / exercises.

Het is ook mogelijk om eigen admonitions te maken. Voor nu zijn er: intermezzo en experiment.

Ik ben een intermezzo
Hier dan tekst.

Wil je een experiment doen?
Altijd.

Ik ben een example
Genoeg voorbeelden

Is er behoefte aan meer admonition types, laat het weten via een issue!

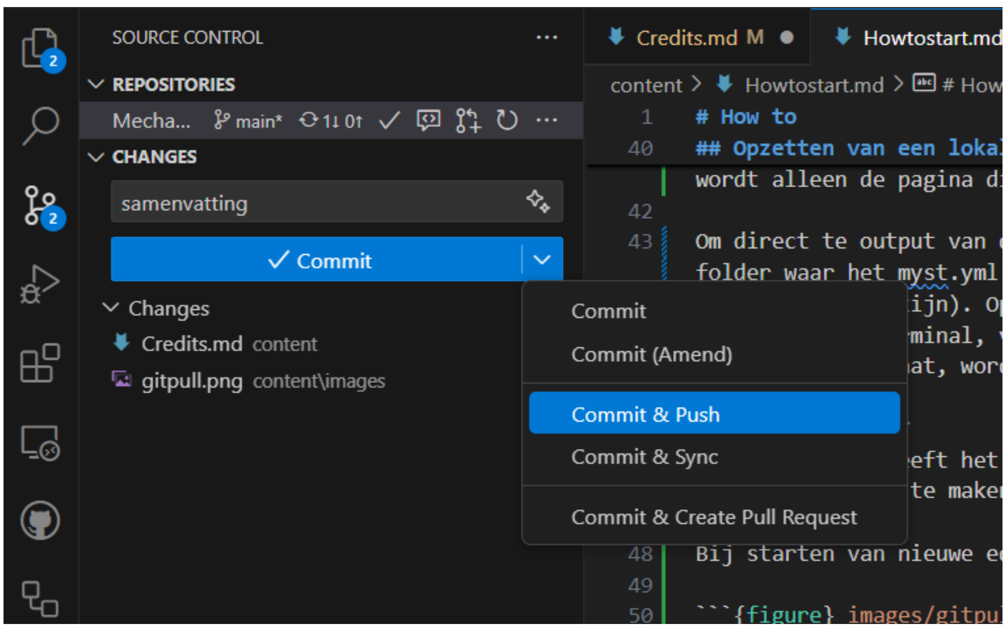


Figure 44: Aan het eind doe je een gitpush, de wijzigingen worden doorgestuurd naar de repository.

3.2 Markdown (Cheatsheet)

Markdown is een eenvoudige opmaaktaal: platte tekst die *opgemaakt* wordt met kleine stukjes 'code'. Die tekst is vervolgens snel te exporteren naar allerlei andere formats zoals pdf, word, html etc.

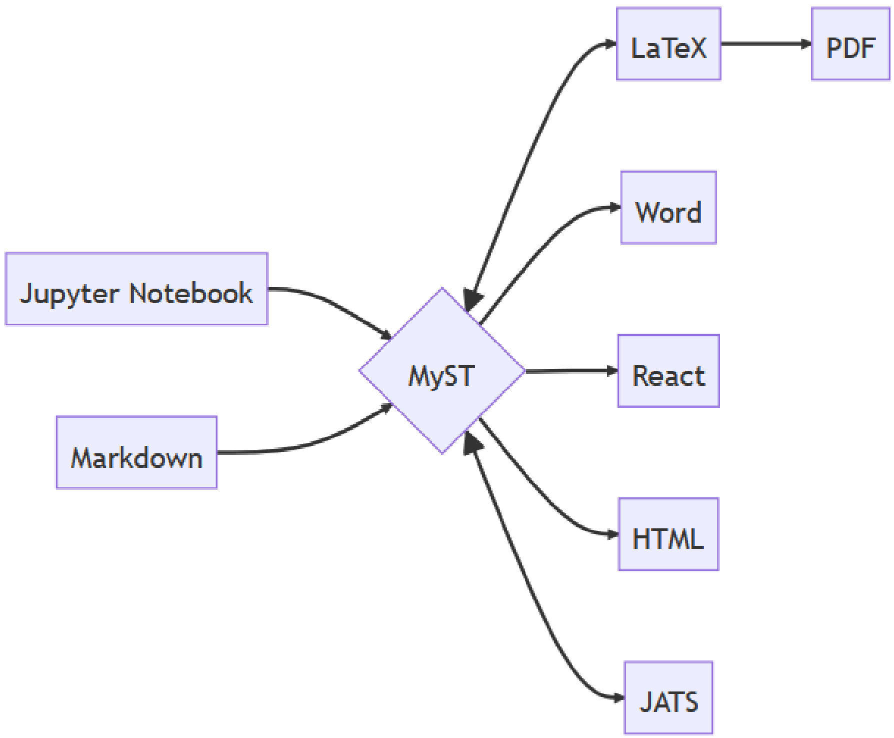


Figure 45: *
Een Jupyter Book gemaakt met MyST vraagt een collectie van markdown en jupyter notebooks die vervolgens geëxporteerd kunnen worden naar pdf, html maar ook word.

3.2.1 Structuur

We kunnen hier onderscheid maken in twee structuren: die van de inhoud van de boek (een collectie van verschillende documenten), en de (interne)structuur van de hoofdstukken.

Table of Contents De software waar we gebruik van maken bouwt zelf een inhoudsopgave (Table of contents, ookwel ToC). Dat gaat op alfabetische volgorde. Maar je kunt ook zelf de ToC specificeren. Dit kan wel het beste door offline te werken (myst init –toc), zie de documentatie van MyST.

Hoofdstukken Om onderscheid te maken tussen hoofdstuk, sectie en subsectie (en verder) wordt er gewerkt met aantal #, zie hieronder.

```
# H1 hoofdstuk
## H1.1 sectie
### H1.1.3 subsectie
```

Tip
Nummer je hoofdstukken en sectie niet! Dit gebeurt automatisch.

Een nieuwe regel krijg je door of een harde enter en een witregel, of door een \ achter de zin en een enter of door twee spaties achter de zin.

Nieuwe regel

3.2.2 Basic opmaak

Markdown is een opmaaktaal waarbij de formatting van de tekst gedaan wordt met kleine stukjes code (net als bij HTML).

Element	Syntax	Voorbeeld
Bold	`**dik gedrukte tekst**`	**Bold**
Italic	`*italics*`	*Italics*
Emphasis	`***emphasis***`	***emphasis***
in line Formule	`$F = m \cdot a$`	$F = m \cdot a$
Super en subscript	`H{sub}‘2‘O, and 4{sup}‘th‘ of July`	H_2O, and 4th of July
Footnote	`- A footnote reference[m̂yref] \ [m̂yref]: This is an auto-numbered footnote definition.`	- A footnote reference[m̂yref] \ [m̂yref]: This is an auto-numbered footnote definition.

Lijsten optie 1

Lijsten optie 2

Afvinklijsten

3.2.3 Formules

Voor de betavakken zijn wiskundige vergelijkingen essentieel. Ook in JB's kun je vergelijkingen opnemen. Wat in LaTeX kan, kan in JB ook, bijv:

$$F_{res} = m \cdot a \tag{212}$$

Waarbij gelabelde vergelijkingen, zoals (212) naar verwezen kan worden.

`$$ Vergelijking $$`

Maar je kunt ook inline vergelijkingen opnemen zoals deze: $s = v_{gem}t$. Daarbij gebruik je een enkele dollar teken voor en na je `$ Vergelijking $`

Naam	Script	Symbolen
wortel	`\sqrt{4}`	$\sqrt{4}$
macht	`{2x}`	2x
breuk	`\frac{2}{3}`	$\frac{2}{3}$
subscript	`_{gem}`	gem
superscript	`{N}`	N
vermenigvuldig	`\cdot`	$\cdot$

wat voorbeelden:

Naam	Script	Output
Afgeleide	`\frac{\Delta f}{\Delta t}`	$\frac{\Delta f}{\Delta t}$
Integraal	`\int_aˆb dx`	$\int_a^b dx$
sinus	`sin(x)`	$sin(x)$

: https://en.wikibooks.org/wiki/LaTeX/Mathematics

3.2.4 Admonitions

Je kunt speciale blokken toevoegen die gehighlight worden in de tekst. Zie bijvoorbeeld onderstaande waarschuwing.

> **Warning**
> Hier een waarschuwing

Daar zijn verschillende varianten van zoals:

- tip
- admonition
- warning
- note
- objective
- see also ...

> **De gouden...**
> Exercises zijn een speciaal soort admonition.

> **Exercise 1: Opdracht 1**
> Maak de som $4 + 2$

> **Solution to Exercise 1: Opdracht 1**
> 6

Opdrachten

3.2.5 Figuren

Een site / boek kan natuurlijk niet zonder figuren. Er zijn grofweg twee manieren om een figuur te maken

Snelle figuur, zonder opmaak mogelijkheden

— Snelle figuur — `![](link naar figuur)` —

Betere manier met meer controle:

Hier hebben we gebruik gemaakt van figuren die op het internet staan, maar je kunt ook figuren zelf toevoegen aan een folder (bijv. genaamd *Figuren*), waarbij je dan een relatief pad op geeft.

3.2.6 Tabellen

Tabellen worden gemaakt met scheidingsteken |

Of via ...

Methode 2 heeft als voordeel de mogelijkheid tot refereren.

3.2.7 Tabbladen

```
::::{tab -set}
:::{tab -item} Tab 1
Hier tekst in tab 1
:::

:::{tab -item} Tab 2
Hier tekst in tab 2
:::
::::
```

3.2.8 YouTube

Voor het embedden van YouTube filmpjes op de site heb je de embed YT link nodig. De code wordt dan:

> **YT in pdf**
> De embedded YT filmpjes worden niet opgenomen in de pdf. Een oplossing zou bijv. een qr code opnemen kunnen zijn.

3.2.9 Referenties

3.2.10 Replacing

To find and replace all HTML anchor tags like:
parsec
with Markdown-style links like:
parsec
You can use regular expressions in Visual Studio Code's Find and Replace:
FIND

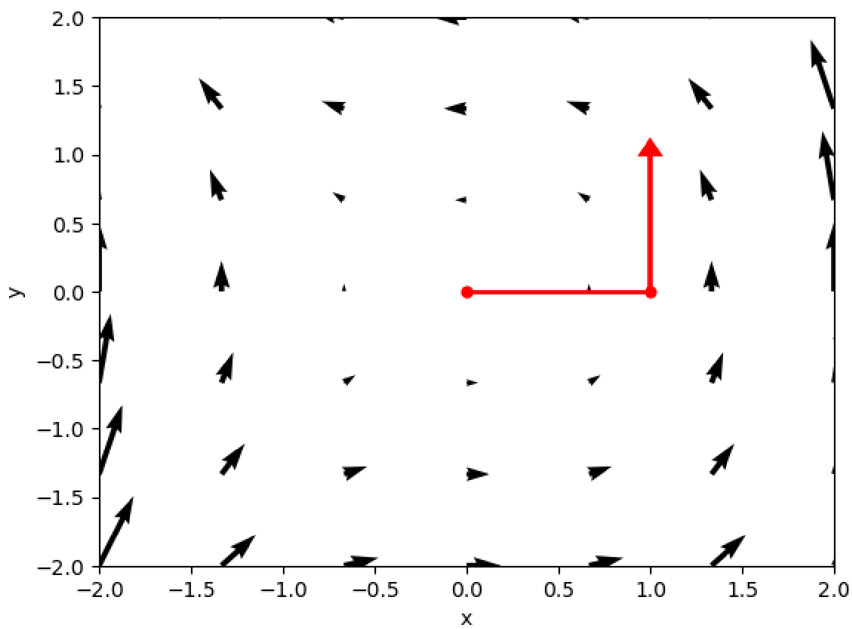
```
<a\s+href="([^"]+)">([^<]+)<\/a>
```

REPLACE

```
[$2]($1)
```

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from IPython.display import HTML

# Simulatieparameters
dt = 0.05
t_max = 10
t_values = np.arange(0, t_max, dt)

# Fysische parameters
vx = 1.0
Fy = 1.0
m = 1.0
ay = Fy / m

# Posities berekenen
x = vx * t_values
y = np.zeros_like(t_values)

x_burn_start = 2.0
x_burn_end = 4.0
i_start = np.argmax(x >= x_burn_start)
i_end = np.argmax(x >= x_burn_end)

for i in range(i_start, i_end+1):
    t_burn = t_values[i] - t_values[i_start]
    y[i] = 0.5 * ay * t_burn**2

vy_final = ay * (t_values[i_end] - t_values[i_start])
y0 = y[i_end]
t0 = t_values[i_end]
for i in range(i_end, len(t_values)):
    y[i] = y0 + vy_final * (t_values[i] - t0)

# Plot
fig, ax = plt.subplots(figsize=(8, 4))
ax.set_xlim(0, np.max(x)+1)
ax.set_ylim(0, np.max(y)+1)
ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_title(" Raket met stuwfase tussen x=2 en x=4")

# Raket (emoji als tekst)
rocket = ax.text(0, 0, '', fontsize=14)

# Trail
trail, = ax.plot([], [], 'r -', lw=1)

# Tijd
time_text = ax.text(0.98, 0.95, '', transform=ax.transAxes,
                    ha='right', va='top', fontsize=12)

# Init
```

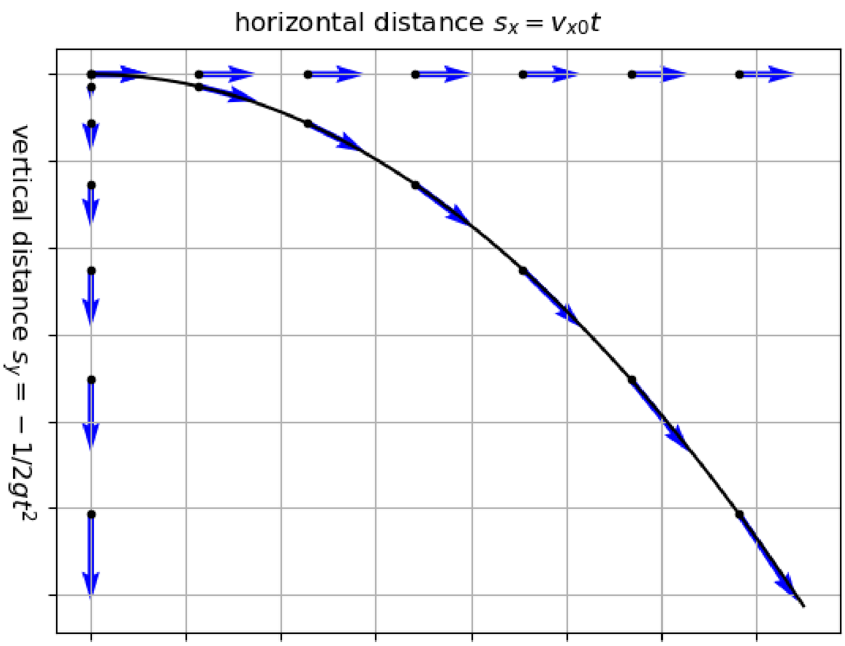
```
def init():
    rocket.set_position((0, 0))
    trail.set_data([], [])
    time_text.set_text('')
    return rocket, trail, time_text

# Update
def update(frame):
    rocket.set_position((x[frame], y[frame]))
    trail.set_data(x[:frame+1], y[:frame+1])
    time_text.set_text(f"t = {t_values[frame]:.2f} s")
    return rocket, trail, time_text

# Animatie
ani = FuncAnimation(fig, update, frames=len(t_values),
                    init_func=init, interval=dt*1000, blit=True)

plt.close()
HTML(ani.to_jshtml())

import numpy as np
import matplotlib.pyplot as plt

x = np.linspace( -2, 2, 7)
y = np.linspace( -2, 2, 7)
X, Y = np.meshgrid(x, y)
U = - Y
V = X**2

path_x = [0, 1]
path_y = [0, 0]


plt.figure()
plt.arrow(1, 0, 0, 1, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.plot(path_x, path_y, color='red', linewidth=2, marker='o', markersize=5)

plt.quiver(X, Y, U, V, color='k')
plt.xlim( -2, 2)
plt.ylim( -2, 2)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('images/force_field.png', dpi=300)
plt.show()
```



1. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} * 10 * 2^2 = 20\text{J}$
2. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 10 \cdot 2^2 = 20\text{J}$
3. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}(10)(2)^2 = 20\text{J}$
4. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}.(10).(2)^2 = 20\text{J}$
5. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}(10\text{kg})(2\text{m/s})^2 = 20\text{J}$
6. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}.(10\text{kg}).(2\text{m/s})^2 = 20\text{J}$
7. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 10 \bullet 2^2 = 20\text{J}$

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0, 5, 100)
v_x = 30
a = 9.81

s_x = v_x * t
s_y = -0.5 * a * t**2


N = 15

plt.figure()
plt.plot(s_x, s_y, 'k -' )
plt.plot(s_x[::N], s_y[::N], 'k.' )
plt.plot(s_x[::N], s_y[::N]*0, 'k.' )
plt.plot(s_x[::N]*0, s_y[::N], 'k.' )

plt.quiver(s_x[::N], s_y[::N]*0, v_x, 0, color='blue', scale=400)
plt.quiver(s_x[::N]*0, s_y[::N], 0, -a*t[::N], color='blue', scale=400)
```

```
plt.quiver(s_x[::N], s_y[::N], v_x, -a*t[::N], color='blue', scale=400)


plt.gca().set_xticklabels([])
plt.gca().set_yticklabels([])
plt.grid(visible=True)
plt.text(30, 10, 'horizontal distance $s_x=v_{x0}t$', fontsize=12, color='black')
plt.text( -20, -100, 'vertical distance $s_y= -1/2gt^2$', fontsize=12, color='black',rotation= -90)
plt.savefig('../images/parmotionv.png', dpi=300)
plt.show()
```



Measure restitution coefficient
Use a pingpong ball and the app phyphox using the acoustic chronometer to determine the coefficient of restitution of the pingpongball. How does this coefficient changes with different surfaces?

```
import numpy as np
import matplotlib.pyplot as plt

N = 1.5
x = np.linspace( -N, N, 15)
y = np.linspace( -N, N, 15)
X, Y = np.meshgrid(x, y)
U = Y
V = -X


plt.figure(figsize=(4, 4))
```

```
plt.plot([0,1], [0,0], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([1,1], [0,1], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([1,0], [1,1], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([0,0], [1,0], color='red', linewidth=2, marker='o', markersize=5)

plt.arrow(1, 0, 0, .5, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(0, 1, 0, -.5, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(0, 0, 0.5, 0, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(1, 1, -0.5, 0, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)

#plt.plot(path_y, path_z, color='red', linewidth=2, marker='o', markersize=5)


plt.quiver(X, Y, U, V, color='k')
plt.xlim( -.5*N, N)
plt.ylim( -.5*N, N)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('../images/StokesTheoremExample.png', dpi=300)
plt.show()
```