Classical Mechanics & Special Relativity for Starters

C. Freek. J. Pols

Abstract

Abstract: This book provides an introduction for freshman students into the world of classical mechanics and special relativity theory. Much of physics is build on the basic ideas from classical mechanics. Hence an early introduction to the topic can be beneficial for new students. However, at the start of studying physics, lots of the required math is not available yet. That means that all kind of concepts that are very useful can not be invoked in the lectures and teaching. That does not have to be a disadvantage. It can also be used to help the students by introducing some math and coupling it directly to the physics, making more clear why mathematics should be studied and what its 'practical use' is. With this book, we aim to introduce new students directly at the start of their studies into the world of physics, more specifically the world of Newton, Galilei and many others who laid the foundation of physics. We aim to help students getting a good understanding of the theory, i.e. the framework of physics. What is 'work' and why do we use it? Why is kinetic energy $\frac{1}{2}mv^2$ and not $\frac{1}{3}mv^2$ or $\frac{1}{2}mv^3$? Both 3's are fundamentally wrong, but what is behind it?

1 Introduction

1.1 About this book

Classical mechanics is the starting point of physics. Over the centuries, via Newton's three fundamental laws formulated around 1687, we have built a solid framework describing the material world around us. On these pages, you will find a text book with animations, demos and exercises for studying introductory classical mechanics. Moreover, we will consider the first steps of Einstein's Special Theory of Relativity published 1905.

This material is made to support first year students from the BSc Applied Physics at Delft University of Technology during their course *Classical Mechanics and Relativity Theory*, MechaRela for short. But, of course, anybody interested in Classical Mechanics and Special Relativity is more than welcome to use this book.

With this e-book our aim is to provide learning material that is:

- self-contained
- easy to modify and thus improve over the years
- interactive, provide additional demos and exercises next to the lectures

This book is based on Mudde & Rieger 2025.

That book was already beyond introductory level and pressumed a solid basis in both calculus and basic mechanics. All texts in this book were revised, additional examples and exercises were included, picture and drawings have been updated and interactive materials have been included. Hence, this book should be considered a stand-alone new version, though good use has been made by open educational resources.

1.1.1 Special features

In this book you will find some 'special' features. Some of these are emphasized by their own style:

Exercise 1:
Each chapter includes a variety of exercises tailored to the material. We distinguish between exercises embedded within the instructional text and those presented on separate pages. The in-text exercises should be completed while reading, as they offer immediate feedback on whether the concepts and mathematics are understood. The separate exercise sets are intended for practice after reading the text and attending the lectures.

To indicate the level of difficulty, each exercise is marked with 1, 2, or 3

Intermezzos
Intermezzos contain background information on the topic, of the people working on the concepts.

Experiments
We include some basic experiments that can be done at home.

Examples
We provide various examples showcasing, e.g., calculations.

Python
We include in-browser python code, as well as downloadable .py files which can be executed locally. If there is an in-browser, press the ON-button to 'enable compute'.

New concepts, such as *Free body diagram*, are introduced with a hoover-over. If you move your mouse over the italicized part of the text, you will get a short explanation.

1.1.2 Feedback

Do you see a mistake, do you have suggestions for exercises, are parts missing or abundant. Tell us! You can use the feedback button at the top right button. You will need a (free) GitHub account to report an issue!

1.2 Authors

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Special thanks to Hanna den Hertog for (re)making most of the drawings, Luuk Fröling for his technical support and Dion Hoeksema for converting the .js scripts to .py files.

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1.3.3 How to cite this book

R.F. Mudde, B. Rieger, C.F.J. Pols, *Classical Mechanics & Special Relativity for Beginners*,

2 Mechanics

2.1 The language of Physics

Physics is the science that seeks to understand the fundamental workings of the universe: from the motion of everyday objects to the structure of atoms and galaxies. To do this, physicists have developed a precise and powerful language: one that combines mathematics, both colloquial and technical language, and visual representations. This language allows us not only to describe how the physical world behaves, but also to predict how it will behave under new conditions.

In this chapter, we introduce the foundational elements of this language, covering how to express physical ideas using equations, graphs, diagrams, and words. You'll also get a first taste of how physics uses numerical simulations as an essential complement to analytical problem solving.

This language is more than just a set of tools—it is how physicists *think*. Mastering it is the first step in becoming fluent in physics.

2.1.1 Representations

Physics problems and concepts can be represented in multiple ways, each offering a different perspective and set of insights. The ability to translate between these representations is one of the most important skills you will develop as a physics student. In this section, we examine three key forms of representation: equations, graphs and drawings, and verbal descriptions using the context of a base jumper, see Figure 1.

Figure 1: A base jumper is used as context to get familiair with representation, picture from https://commons.wikimedia.org/wiki/File:04SHANG4963.jpg

Verbal descriptions Words are indispensable in physics. Language is used to describe a phenomenon, explain concepts, pose problems and interpret results. A good verbal description makes clear:

- What is happening in a physical scenario;
- What assumptions are being made (e.g., frictionless surface, constant mass);
- What is known and what needs to be found.

Base jumper: Verbal description
Let us consider a base jumper jumping from a 300 m high building. We take that the jumper drops from that height with zero initial velocity. We will assume that the stunt is performed safely and in compliance with all regulations/laws. Finally, we will assume that the problem is 1-dimensional: the jumper drops vertically down and experiences only gravity, buoyancy and air-friction.

We know (probably from experience) that the jumper will accelerate. Picking up speed increases the drag force acting on the jumper, slowing the *acceleration* (meaning it still accelerates!). The speed keeps increasing until the jumper reaches its terminal velocity, that is the velocity at which the drag (+ buoyancy) exactly balance gravity and the sum of forces on the jumper is zero. The jumper no longer accelerates.

Can we find out what the terminal velocity of this jumper will be and how long it takes to reach that velocity?

Visual representations Visual representations help us interpret physical behavior at a glance. Graphs, motion diagrams, free-body diagrams, and vector sketches are all ways to make abstract ideas more concrete.

- **Graphs** (e.g., position vs. time, velocity vs. time) reveal trends and allow for estimation of slopes and areas, which have physical meanings like velocity and displacement.
- **Drawings** help illustrate the situation: what objects are involved, how they are moving, and what forces act on them.

Base jumper: Free body diagram
The situation is sketched in Figure **??** using a Free body diagram. Note that all details of the jumper are lost in the sketch.

- m = mass of jumper in kg;
- v = velocity of jumper in m/s;
- F_g = gravitational force in N;
- F_f = drag force by the air in N;
- F_b = buoyancy in N: like in water also in air there is an upward force, equal to the weight of the displaced air.

Equations Equations are the compact, symbolic expressions of physical relationships. They tell us how quantities like velocity, acceleration, force, and energy are connected.

Base jumper: equations
The forces acting on the jumper are already shown in Figure **??**. Balancing of forces tells us that the jumper might reach a velocity such that the drag force and buoyancy exactly balance gravity and the jumper no longer accelerates:

$$F_g = F_f + F_b \tag{1}$$

We can specify each of the force:

$$\begin{aligned} F_g &= -mg = \rho_p V_p g \\ F_f &= \frac{1}{2}\rho_{air} C_D A v^2 \\ F_b &= \rho_{air} V_p g \end{aligned} \tag{2}$$

with g the acceleration of gravity, ρ_p the density of the jumper ($\approx 10^3 \text{kg/m}^3$), V_p the volume of the jumper, ρ_{air} the density of air ($\approx 1.2 \text{kg/m}^3$), C_D the so-called drag coefficient, A the frontal area of the jumper as seen by the air flowing past the jumper.

A physicist is able to switch between these representations, carefully considering with representations suits best for the given situation. We will practice these when solving problems.

Danger
Note that in the example above we neglected directions. In our equation we should have been using vector notation, which we will cover in one of the next sections in this chapter.

2.1.2 How to solve a physics problem?

One of the most common mistakes made by 'novices' when studying problems in physics is trying to jump as quickly as possible to the solution of a given problem or exercise by picking an equation and slotting in the numbers. For simple questions, this may work. But when stuff gets more complicated, it is almost a certain route to frustration.

There is, however, a structured way of problem solving, that is used by virtually all scientists and engineers. Later this will be second nature to you, and you apply this way of working automatically. It is called IDEA, an acronym that stands for:

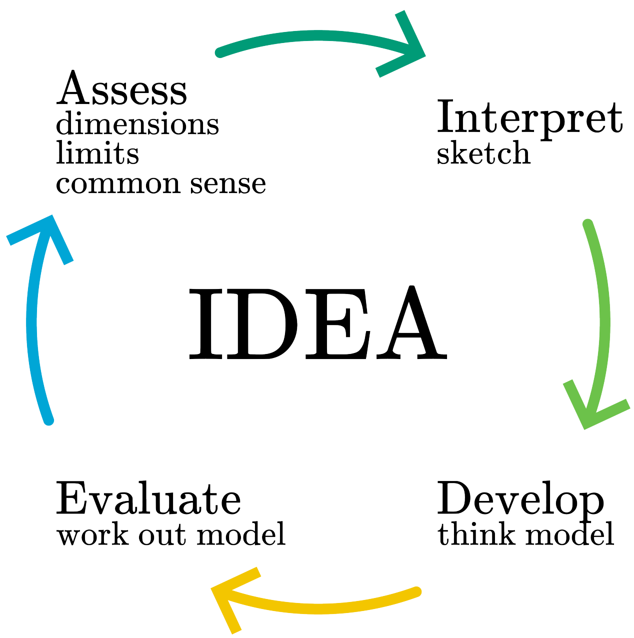


Figure 3: IDEA

- **Interpret** - First think about the problem. What does it mean? Usually, making a sketch helps. Actually *always start with a sketch*;
- **Develop** - Build a model, from coarse to fine, that is, first think in the governing phenomena and then subsequently put in more details. Work towards writing down the equation of motion and boundary conditions;
- **Evaluate** - Solve your model, i.e. the equation of motion;

- **Assess** - Check whether your answer makes any sense (e.g. units OK? What order of magnitude did we expect?).

We will practice this and we will see that it actually is a very relaxed way of working and thinking. We strongly recommend to apply this strategy for your homework and exams (even though it seems strange in the beginning).

The first two steps (Interpret and Develop) typically take up most of the time spend on a problem.

Good Practice
It is a good habit to make your mathematical steps small: one-by-one. Don't make big jumps or multiple steps in one step. If you make a mistake, it will be very hard to trace it back.

Next: check constantly the dimensional correctness of your equations: that is easy to do and you will find the majorities of your mistakes.

Finally, use letters to denote quantities, including π. The reason for this is:

- letters have meaning and you can easily put dimensions to them;
- letters are more compact;
- your expressions usually become easier to read and characteristic features of the problem at hand can be recognized.

powers of ten
In physics, powers of ten are used to express very large or very small quantities compactly and clearly, from the size of atoms (~10^{-10} m) to the distance between stars (~10^{16} m). This notation helps compare scales, estimate orders of magnitude, and maintain clarity in calculations involving extreme values.

We use prefixes to denote these powers of ten in front of the standard units, e.g. km for 1000 meters, ms for milli seconds, GB for gigabyte that is 1 billion bytes. Here is a list of prefixes.

Prefix	Symbol	Math	Prefix	Symbol	Math
Yocto	y	10^{-24}	Base	-	10^{0}
Zepto	z	10^{-21}	Deca	da	10^{1}
Atto	a	10^{-18}	Hecto	h	10^{2}
Femto	f	10^{-15}	Kilo	k	10^{3}
Pico	p	10^{-12}	Mega	M	10^{6}
Nano	n	10^{-9}	Giga	G	10^{9}
Micro	µ	10^{-6}	Tera	T	10^{12}
Milli	m	10^{-3}	Peta	P	10^{15}
Centi	c	10^{-2}	Exa	E	10^{18}
Deci	d	10^{-1}	Zetta	Z	10^{21}
Base	-	10^{0}	Yotta	Y	10^{24}

On quantities and units
Each quantity has a unit. As there are only so many letters in the alphabet (even when including the Greek alphabet), letters are used for multiple quantities. How can we distinguish then meters from mass, both denoted with the letter m? Quantities are expressed in italics (m) and units are not (m).

We make extensively use of the International System of Units (SI) to ensure consistency and precision in measurements across all scientific disciplines. The seven base SI units are:

- Meter (m) – length
- Kilogram (kg) – mass
- Second (s) – time
- Ampere (A) – electric current
- Kelvin (K) – temperature
- Mole (mol) – amount of substance
- Candela (cd) – luminous intensity

All other quantities can be derived from these using dimension analysis:

$$\begin{aligned} &W = F \cdot s = ma \cdot s = m\frac{\Delta v}{\Delta t} \cdot s \\ &[\mathrm{J}] = [\mathrm{N}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot [\mathrm{m/s^2}] \cdot [\mathrm{m}] = [\mathrm{kg}] \cdot \frac{[\mathrm{m/s}]}{[\mathrm{s}]} \cdot [\mathrm{m}] = [\frac{\mathrm{kg\ m^2}}{\mathrm{s^2}}] \end{aligned} \tag{3}$$

Tip
For a more elaborate description of quantities, units and dimension analysis, see the manual of the first year physics lab course.

Example

2.1.3 Calculus

Most of the undergraduate theory in physics is presented in the language of Calculus. We do a lot of differentiating and integrating, and for good reasons. The basic concepts and laws of physics can be cast in mathematical expressions, providing us the rigor and precision that is needed in our field. Moreover, once we have solved a certain problem using calculus, we obtain a very rich solution, usually in terms of functions. We can quickly recognize and classify the core features, that help us understanding the problem and its solution much deeper.

Given the example of the base jumper, we would like to know how the jumper's position as a function of time. We can answer this question by applying Newton's second law (though it is covered in secondary school, the next chapter explains in detail Newton's laws of motion):

$$\sum F = F_g - F_f = ma = m\frac{dv}{dt} \tag{4}$$

$$m\frac{dv}{dt} = mg - \frac{1}{2}\rho_{air} C_D A v^2 \tag{5}$$

Clearly this is some kind of differential equation: the change in velocity depends on the velocity itself. Before we even try to solve this problem ($v(t) = ...$), we have to dig deeper in the precise notation, otherwise we will get lost in directions and sign conventions.

Differentiation Many physical phenomena are described by differential equations. That may be because a system evolves in time or because it changes from location to location. In our treatment of the principles of classical mechanics, we will use differentiation with respect to time a lot. The reason is obviously found in Newton's 2^{nd} law: $F = ma$.

The acceleration a is the derivative of the velocity with respect to time; velocity in itself is the derivative of position with respect to time. Or when we use the equivalent formulation with momentum: $\frac{dp}{dt} = F$. So, the change of momentum in time is due to forces. Again, we use differentiation, but now of momentum.

There are three common ways to denote differentiation. The first one is by 'spelling it out':

$$v = \frac{dx}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{6}$$

- Advantage: it is crystal clear what we are doing.
- Disadvantage: it is a rather lengthy way of writing.

Newton introduced a different flavor: he used a dot above the quantity to indicate differentiation with respect to time. So,

$$v = \dot{x}, \text{ or } a = \dot{v} = \ddot{x} \tag{7}$$

- Advantage: compact notation, keeping equations compact.
- Disadvantage: a dot is easily overlooked or disappears in the writing.

Finally, in math often the prime is used: $\frac{df}{dx} = f'(x)$ or $\frac{d^2f}{dx^2} = f''(x)$. Similar advantage and disadvantage as with the dot notation.

Important

$$v = \frac{dx}{dt} = \dot{x} = x' \tag{8}$$

$$a = \frac{dv}{dt} = \dot{v} = \frac{d^2x}{dt^2} = \ddot{x} \tag{9}$$

It is just a matter of notation.

2.1.4 Definition of velocity, acceleration and momentum

In mechanics, we deal with forces on particles. We try to describe what happens to the particles, that is, we are interested in the position of the particles, their velocity and acceleration. We need a formal definition, to make sure that we all know what we are talking about.

1-dimensional case

In one dimensional problems, we only have one coordinate to take into account to describe the position of the particle. Let's call that x. In general, x will change with time as particles can move. Thus, we write $x(t)$ showing that the position, in principle, is a function of time t. How fast a particle changes its position is, of course, given by its velocity. This quantity describes how far an object has traveled in a given time interval: $v = \frac{\Delta x}{\Delta t}$. However, this definition gives actually the average velocity in hte time interval Δt. The (momentary) velocity is defined as:

Velocity

$$\text{definition velocity: } v \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = \frac{dx}{dt} \tag{10}$$

Similarly, we define the acceleration as the change of the velocity over a time interval Δt: $a = \frac{\Delta v}{\Delta t}$. Again, this is actually the average acceleration and we need the momentary one:

Acceleration

$$\text{definition acceleration: } a \equiv \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{(t+\Delta t) - t} = \frac{dv}{dt} \tag{11}$$

Consequently,

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} \tag{12}$$

Now that we have a formal definition of velocity, we can also define momentum: momentum is mass times velocity, in math:

Momentum

$$\text{definition momentum: } p \equiv mv = m\frac{dx}{dt} \tag{13}$$

In the above, we have not worried about how we measure position or time. The latter is straight forward: we use a clock to account for the time intervals. To find the position, we need a ruler and a starting point from where we measure the position. This is a complicated way of saying the we need a coordinate system with an origin. But once we have chosen one, we can measure positions and using a clock measure changes with time.

Figure 4: Calculating velocity requires both position and time, both easily measured e.g. using a stopmotion where one determines the position of the car per frame given a constant time interval.

Vectors - more dimensional case Position, velocity, momentum, force: they are all *vectors*. In physics we will use vectors a lot. It is important to use a proper notation to indicate that you are working with a vector. That can be done in various ways, all of which you will probably use at some point in time. We will use the position of a particle located at point P as an example.

Tip
See the linear algebra book on vectors.

Position vector We write the position**vector** of the particle as $\vec{r}$. This vector is a 'thing', it exists in space independent of the coordinate system we use. All we need is an origin that defines the starting point of the vector and the point P, where the vector ends.

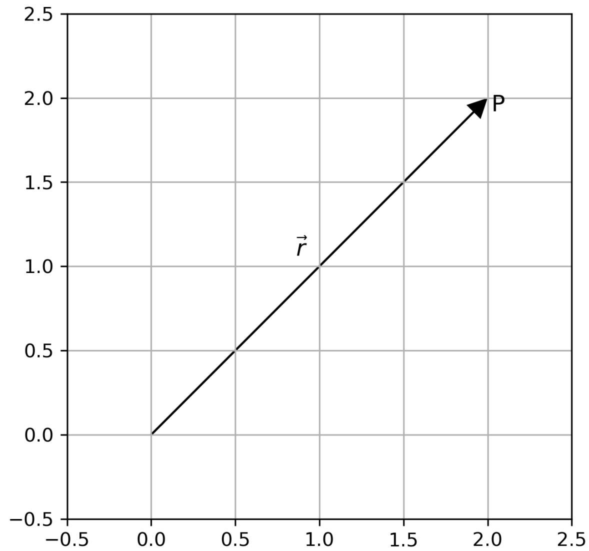


Figure 5: Some physical quantities (velocity, force etc) can be represented as a vector. The have in common the direction, magnitude and point of application.

A coordinate system allows us to write a representation of the vector in terms of its coordinates. For instance, we could use the familiar Cartesian Coordinate system {x,y,x} and represent $\vec{r}$ as a column.

$$\vec{r} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{14}$$

Alternatively, we could use unit vectors in the x, y and z-direction. These vectors have unit length and point in the x, y or z-direction, respectively. They are denoted in varies ways, depending on taste. Here are 3 examples:

$$\begin{aligned} &\hat{x},\ \ \hat{i},\ \ \vec{e}_x \\ &\hat{y},\ \ \hat{j},\ \ \vec{e}_y \\ &\hat{z},\ \ \hat{k},\ \ \vec{e}_z \end{aligned} \tag{15}$$

With this notation, we can write the position vector $\vec{r}$ as follows:

$$\begin{aligned} \vec{r} &= x\hat{x}\ + y\hat{y}\ + z\hat{z} \\ \vec{r} &= x\hat{i}\ \ + y\hat{j}\ + z\hat{k} \\ \vec{r} &= x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \end{aligned} \tag{16}$$

Note that these representations are completely equivalent: the difference is in how the unit vectors are named. Also note, that these three representations are all given in terms of vectors. That is important to realize: in contrast to the column notation, now all is written at a single line. But keep in mind: $\hat{x}$ and $\hat{y}$ are perpendicular **vectors**.

Other textbooks
Note that other textsbooks may use bold symbols to represent vectors

$$\vec{F} = m\vec{a} \tag{17}$$

is the same as

$$\mathbf{F} = m\mathbf{a} \tag{18}$$

Differentiating a vector We often have to differentiate physical quantities: velocity is the derivative of position with respect to time; acceleration is the derivative of velocity with respect to time. But you will also come across differentiation with respect to position.
As an example, let's take velocity. Like in the 1-dimensional case, we can ask ourselves: how does the position of an object change over time? That leads us naturally to the definition of velocity: a change of position divided by a time interval:

Velocity vector

$$\text{definition velocity: } \vec{v} \equiv \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{19}$$

What does it mean? Differentiating is looking at the change of your 'function' when you go from x to $x + dx$:

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{20}$$

In 3 dimensions we will have that we go from point P, represented by $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ to 'the next point' $\vec{r} + d\vec{r}$. The small vector $d\vec{r}$ is a small step forward in all three directions, that is a bit dx in the x-direction, a bit dy in the y-direction and a bit dz in the z-direction.
Consequently, we can write $\vec{r} + d\vec{r}$ as

$$\begin{aligned} \vec{r} + d\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} + dx\hat{dx} + dy\hat{y} + dz\hat{z} \\ &= (x+dx)\hat{x} + (y+dy)\hat{y} + (z+dz)\hat{z} \end{aligned} \tag{21}$$

Now, we can take a look at each component of the position and define the velocity component as, e.g., in the x-direction

$$v_x = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt} \tag{22}$$

Applying this to the 3-dimensional vector, we get

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= \frac{d}{dt}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) \\ &= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} \\ &= v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \end{aligned} \tag{23}$$

Note that in the above, we have used that according to the product rule:

$$\frac{d}{dt}(x\hat{x}) = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} = \frac{dx}{dt}\hat{x} \tag{24}$$

since $\frac{d\hat{x}}{dt} = 0$ (the unit vectors in a Cartesian system are constant). This may sound trivial: how could they change; they have always length 1 and they always point in the same direction. Trivial as this may be, we will come across unit vectors that are not constant as their direction may change. But we will worry about those examples later.

Now that the velocity of an object is defined, we can introduce its momentum:

Momentum Vector

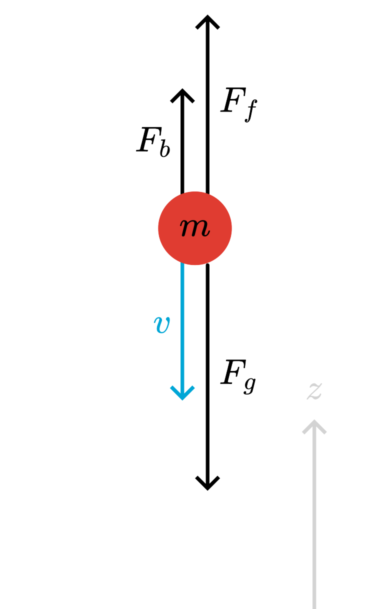
$$\text{definition momentum: } \vec{p} \equiv m\vec{v} = m\frac{d\vec{r}}{dt} \tag{25}$$

We can use the same reasoning and notation for acceleration:

Acceleration Vector

$$\text{definition acceleration: } \vec{a} \equiv \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{26}$$

The base jumper Given the above explanation, we can now reconsider our description of the base jumper.



We see a z-coordinate pointing upward, where the velocity. As gravitational force is in the direction of the ground, we can state

$$\vec{F}_g = -mg\hat{z} \tag{27}$$

Buoyancy is clearly along the z-direction, hence

$$\vec{F}_b = \rho_{air} V g\hat{z} \tag{28}$$

The drag force is a little more complicated as the direction of the drag force is always against the direction of the velocity $-\vec{v}$. However, in the formula for drag we have v^2. To solve this, we can write

$$\vec{F_f} = -\frac{1}{2}\rho_{air}C_D A|v|\vec{v} \tag{29}$$

Note that $\hat{z}$ is missing in (29) on purpose. That would be a simplification that is valid in the given situation, but not in general.

2.1.5 Numerical computation and simulation

In simple cases we can come to an analytical solution. In the case of the base jumper, an analytical solution exists, though it is not trivial and requires some advanced operations as separation of variables and partial fractions:

$$v(t) = \sqrt{\frac{mg}{k}}\tanh(\sqrt{\frac{kg}{m}}t) \tag{30}$$

with

$$k = \frac{1}{2}\rho_{air}C_D A \tag{31}$$

In that case there is nothing to add or gain from a numerical analysis. Nevertheless, it is instructive to see how we could have dealt with this problem using numerical techniques. One way of solving the problem is, to write a computer code (e.g. in python) that computes from time instant to time instant the force on the jumper, and from that updates the velocity and subsequently the position.

```
some initial conditions
t = 0
x = x_0
v = 0
dt = 0.1

for i is 1 to N:
    compute F: formula
    compute new v: v[i+1] = v[i] - F[i]/m*dt
    compute new x: x[i+1] = x[i] + v[i]*dt
    compute new t: t[i+1] = t[i] + dt
```

You might already have some experience with numerical simulations. (Figure 6) presents a script for the software Coach, which you might have encountered in secondary school.

The base jumper Let us go back to the context of the base jumper and write some code.

First we take: $k = \frac{1}{2}\rho_{air}C_D A$ which eases writing. The force balance then becomes:

$$m\vec{a} = -m\vec{g} - k|v|\vec{v} \tag{32}$$

We rewrite this to a proper differential equation for v into a finite difference equation. That is, we go back to how we came to the differential equation:

$$m\lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t} = \vec{F}_{net} \tag{33}$$

with $\vec{F}_{net} = -m\vec{g} - k|v|\vec{v}$

'Stop condition is set 'Computations are based on Euler x := x + flow_1*Δt1 v := v + flow_2*Δt1 t1 := t1 + Δt1 flow_1 := v Fz := m*g Fw := 6*d*d*v*v f := Fz - Fw a := f/m flow_2 := a	t1 := 0 's Δt1 := 0.01 's x := 0 'm v := 0 'm/s m := 75 'kg g := 9.81 'm/s^2 d := 2.5 'm flow_1 := v 'm/s Fz := m*g 'N Fw := 6*d*d*v*v 'N f := Fz - Fw 'N a := f/m 'm/s^2 flow_2 := a 'm/s^2

Figure 6: An example of a numerical simulation made in Coach. At the left the iterative calculation proces, at the right the initial conditions.

On a computer, we can not literally take the limit of Δt to zero, but we can make Δt very small. If we do that, we can rewrite the difference equation (thus not taken the limit):

$$\vec{v}(t+\Delta t) = \vec{v}(t) + \frac{\vec{F}}{m}\Delta t \tag{34}$$

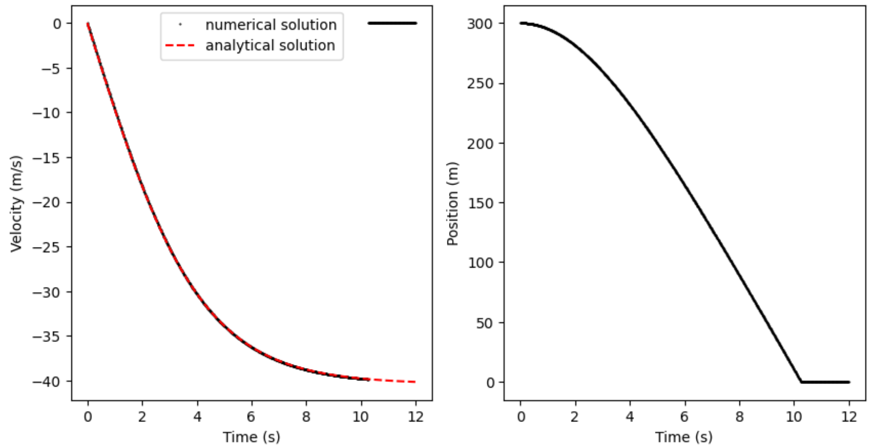
This expression forms the heart of our numerical approach. We will compute v at discrete moments in time: $t_i = 0, \Delta t, 2\Delta t, 3\Delta t, \dots$. We will call these values v_i. Note that the force can be calculated at time t_i once we have v_i.

$$\begin{aligned} F_i &= mg - k|v_i|v_i \\ v_{i+1} &= v_i + \frac{F_i}{m}\Delta t \end{aligned} \tag{35}$$

Similarly, we can keep track of the position:

$$\frac{dx}{dt} = v \Rightarrow x_{i+1} = x_i + v_i\Delta t \tag{36}$$

With the above rules, we can write an iterative code:



Important to note is the sign-convention which we adhere to. Rather than using v^2 we make use of $|v|v$ which takes into account that drag is always against the direction of movement. Note as well the similarity between the analytical solution and the numerical solution.

To come back to our initial problem:
It roughly takes $10\mathrm{s}$ to get close to terminal velocity (note that without friction the velocity would be $98\mathrm{m/s}$). The building is not high enough to reach this velocity (safely).

2.1.6 Exercises

Exercises

Solutions

2.2 Newton's Laws

Now we turn to one of the most profound breakthroughs in the history of science: the laws of motion formulated by Isaac Newton. These laws provide a systematic framework for understanding how and why objects move, and form the backbone of classical mechanics. Using these three laws we can predict the motion of a falling apple, a car accelerating down the road, or a satellite orbiting Earth (though some adjustments are required in this context to make use of e.g. GPS!). More than just equations, they express deep principles about the nature of force, mass, and interaction.

In this chapter, you will begin to develop the core physicist's skill: building a simplified model of the real world, applying physical principles, and using mathematical tools to reach meaningful conclusions.

2.2.1 Newton's Three Laws

Much of physics, in particular Classical Mechanics, rests on three laws that carry Newton's name:

Newton's first law (N1)
If no force acts on an object, the object moves with constant velocity.

Newton's second law (N2)
If a (net) force acts on an object, the momentum of the object will change according to:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{37}$$

Newton's third law (N3)
If object 1 exerts a force $\vec{F}_{12}$ on object 2, then object 2 exerts a force $\vec{F}_{21}$ equal in magnitude and opposite in direction on object 1:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{38}$$

N1 has, in fact, been formulated by Galileo Galilei. Newton has, in his N2, build upon it: N1 is included in N2, after all:
if $\vec{F} = 0$, then $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} =$ constant $\rightarrow \vec{v} =$ constant, provided m is a constant.

Most people know N2 as

$$\vec{F} = m\vec{a} \tag{39}$$

For particles of constant mass, the two are equivalent:
if $m =$ constant, then

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} \tag{40}$$

Nevertheless, in many cases using the momentum representation is beneficial. The reason is that momentum is one of the key quantities in physics. This is due to the underlying conservation law, that we will derive in a minute. Moreover, using momentum allows for a new interpretation of force: force is that quantity that - provided it is allowed to act for some time interval on an object - changes the momentum of that object. This can be formally written as:

$$d\vec{p} = \vec{F}dt \leftrightarrow \Delta\vec{p} = \int \vec{F}dt \tag{41}$$

The latter quantity $\vec{I} \equiv \int \vec{F}dt$ is called the impulse.

Note
Momentum is in Dutch **impuls** in Dutch; the English **impulse** is in Dutch **stoot**.

In Newton's Laws, velocity, acceleration and momentum are key quantities. We repeat here their formal definition.

Definition

$$\begin{aligned} \text{velocity: } \vec{v} &\equiv \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \\ \text{acceleration: } \vec{a} &\equiv \lim_{\Delta t\to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \\ \text{momentum: } \vec{p} &\equiv m\vec{v} = m\frac{d\vec{r}}{dt} \end{aligned} \tag{42}$$

Intermezzo: Isaac Newton
At the end of the year of Galilei's death, Isaac Newton was born in Woolsthorpe-by-Colsterworth in England. He is regarded as the founder of classical mechanics and with that he can be seen as the father of physics.

In 1661, he started studying at Trinity College, Cambridge. In 1665, the university temporarily closed due to an outbreak of the plague. Newton returned to his home and started working on some of his breakthroughs in calculus, optics and gravitation. Newton's list of discoveries is unsurpassed. He invented calculus (at about the same time and independent of Leibniz). Newton is known for 'the binomium of Newton', the the cooling law of Newton. He proposed that light is made of particles. And he formulated his law of gravity. Finally, he postulated his three laws that started classical mechanics and worked on several ideas towards energy and work. Much of our concepts in physics are based on the early ideas and their subsequent development in classical mechanics. The laws and rules apply to virtually all daily life physical phenomena and only do they require adaptation when we go the very small scale or extreme velocities and cosmology. In what follows, we will follow his footsteps, but in a modern way that we owe to many physicist and mathematicians that over the years shaped the theory of classical mechanics in a much more comprehensive form. We do not only stand on shoulders of giants, we stand on a platform carried by many.

Interesting to know is that his mentioning of *standing of shoulders* can be interpreted as a sneer towards Hooke, with he was in a fight with over several things. Hooke was a rather short man...

Important
In Newtons mechanics time does not have a preferential direction. That means, in the equations derived based on the three laws of Newton, we can replace t by $-t$ and the motion will have different sign, but that's it. The path/orbit will be the same, but traversed in opposite direction. Also in special relativity this stays the same.

However, in daily life we experience a clear distinction between past, present and future. This difference is not present in this lecture at all. Only by the second of law thermodynamics the time axis obtains a direction, more about this in classes on Statistical Mechanics.

2.2.2 Newton's laws applied

Force addition, subtraction and decomposition Newton's laws describe how forces affect motion, and applying them often requires combining multiple forces acting on an object, see Figure 8. This is done through vector addition, subtraction, and decomposition—allowing us to find the net force and analyze its components in different directions, see this chapter in the book on linear algebra for a full elaboration on vector addition and subtraction.

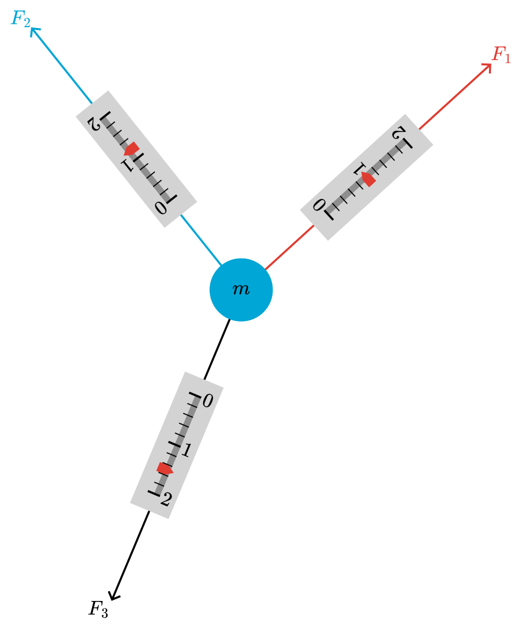


Figure 8: Three forces acting on a particle. In which direction will it accelerate?

Three forces acting on a particle
Consider three forces acting on a particle:

$\vec{F}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{F}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{F}_3 = \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}$

What is the net force acting on the particle and in which direction will the particle accelerate?

Incline
The box in Figure **??** is at rest. Calculate the frictional force acting on the box.

Acceleration due to gravity In most cases the forces acting on an object are not constant. However, there is a classical case that is treated in physics (already at secondary school level) where only one, constant force acts and other forces are neglected. Hence, according to Newton's second law, the acceleration is constant.

When we first consider only the motion in the z-direction, we can derive:

$$a = \frac{F}{m} = \text{const.} \tag{43}$$

Hence, for the velocity:

$$v(t) = \int_{t_0}^{t_e} a dt = a(t_e - t_0) + v_0 \tag{44}$$

assuming $t_0 = 0$ and $t_e = t \Rightarrow v(t) = v_0 + at$ the position is described by

$$s(t) = \int_0^t v(t) dt = \int_0^t at + v_0 dt = \frac{1}{2}at^2 + v_0 t + s_0 \tag{45}$$

Rearranging:

$$s(t) = \frac{1}{2}at^2 + v_0t + s_0 \tag{46}$$

2D-motion We only considered motion in the vertical direction, however, objects tend to move in three dimension. We consider now the two-dimensional situation, starting with an object which is horizontally thrown from a height.

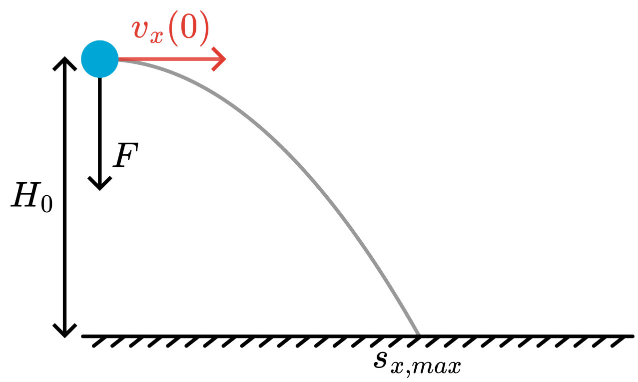


Figure 10: A sketch of the situation where an object is thrown horizontally and the horizontal distance should be calculated.

In the situation given in Figure 10 the object is thrown with a horizontal velocity of v_{x0}. As no forces in the horizontal direction act on the object (N1), its horizontal motion can be described by

$$s_x(t) = v_{x0}t \tag{47}$$

In the vertical direction only the gravitational force acts (N2), hence the motion can be described by (46). Taking the y-direction upward, a starting height $y(0) = H_0$ and $v_y(0) = 0$ it becomes:

$$s_y(t) = H_0 - \frac{1}{2}gt^2 \tag{48}$$

The total horizontal traveled distance of the object before hitting the ground then becomes:

$$s_{x,max} = v_x\sqrt{\frac{2H_0}{g}} \tag{49}$$

This motion is visualized in Figure 11. The trajectory is shown with s_x on the horizontal axis and s_y on the vertical axis. At regular time intervals Δt, velocity vectors are drawn to illustrate the motion. Note that the horizontal and vertical components of velocity, v_x and v_y, vary independently throughout the trajectory. Moreover, $\vec{v}(t)$ is the tangent of $s(t)$.

> **Danger**
> Understand that the case above is specific in physics: in most realistic contexts multiple forces are acting upon the object. Hence the equation of motion does not become $s(t) = s_0 + v_0t + 1/2at^2$

Frictional forces There are two main types of frictional force:

- **Static friction** prevents an object from starting to move. It adjusts in magnitude up to a maximum value, depending on how much force is trying to move the object. This maximum is given by

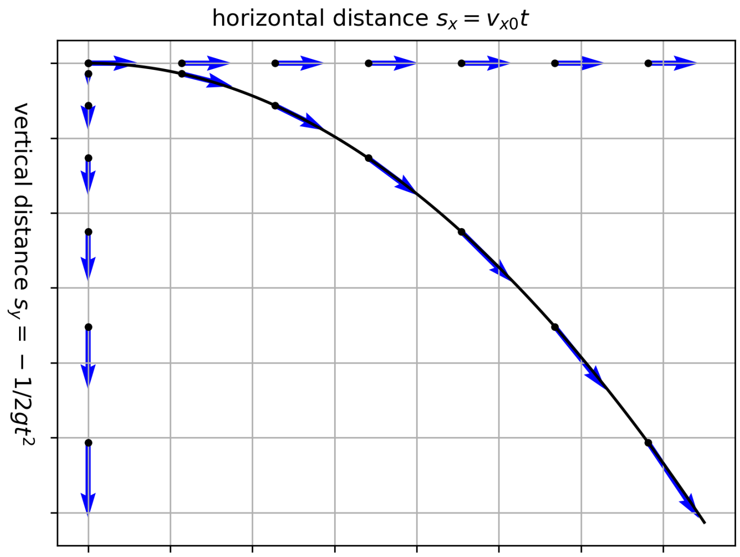


Figure 11: The parabolic motion is visualized with blue velocity vectors v, v_x and v_y shown at various points along the trajectory.

Figure 12: *
The trajectory of a ball shot under an angle visualized.

$$F_{static,max} = \mu_s N \tag{50}$$

where μ_s is the coefficient of static friction and N is the normal force. If the applied force exceeds this maximum, the object begins to slide.

- **Kinetic (dynamic) friction** opposes motion once the object is sliding. Its magnitude is generally constant and given by

$$F_{kinetic} = \mu_k N \tag{51}$$

where μ_k is the coefficient of kinetic friction. This force does not depend on the velocity of the object, only on the normal force and surface characteristics.

Friction always acts opposite to the direction of intended or actual motion and is essential in both preventing and controlling movement.

Material Pair	Static Friction (μ_s)	Kinetic Friction (μ_k)
Rubber on dry concrete	1.0	0.8
Steel on steel (dry)	0.74	0.57
Wood on wood (dry)	0.5	0.3
Aluminum on steel	0.61	0.47
Ice on ice	0.1	0.03
Glass on glass	0.94	0.4
Copper on steel	0.53	0.36
Teflon on Teflon	0.04	0.04
Rubber on wet concrete	0.6	0.5
Leather on wood	0.56	0.4

Values are approximate and can vary depending on surface conditions.

2.2.3 Conservation of Momentum

From Newton's 2$^{\text{nd}}$ and 3$^{\text{rd}}$ law we can easily derive the law of conservation of momentum.
Assume there are only two point-particle (i.e. particles with no size but with mass), that exert a force on each other. No other forces are present. From N2 we have:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{21} \frac{d\vec{p}_2}{dt} = \vec{F}_{12} \tag{52}$$

From N3 we know:

$$\vec{F}_{21} = -\vec{F}_{12} \tag{53}$$

And, thus by adding the two momentum equations we get:

$$\left.\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} = -\vec{F}_{21} \end{aligned}\right\} \Rightarrow \tag{54}$$

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0 \rightarrow \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0 \tag{55}$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = const \;\; \text{i.e. does } \textit{not} \text{ depend on time} \tag{56}$$

Note the importance of the last conclusion: **if objects interact via a mutual force then the total momentum of the objects can not change.** No matter what the interaction is. It is easily extended to more interacting particles. The crux is that particles interact with one another via forces that obey N3. Thus for three interacting point particles we would have (with $\vec{F}_{ij}$ the force from particle i felt by particle j):

$$\left.\begin{aligned} \tfrac{d\vec{p}_1}{dt} &= \vec{F}_{21} + \vec{F}_{31} \\ \tfrac{d\vec{p}_2}{dt} &= \vec{F}_{12} + \vec{F}_{32} = -\vec{F}_{21} + \vec{F}_{32} \\ \tfrac{d\vec{p}_3}{dt} &= \vec{F}_{13} + \vec{F}_{23} = -\vec{F}_{31} - \vec{F}_{32} \end{aligned}\right\} \tag{57}$$

Sum these three equations:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \frac{d\vec{p}_3}{dt} = 0 \to \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2 + \vec{p}_3\right) = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \text{const. i.e. does \textit{not} depend on time} \quad (58)$$

For a system of N particles, extension is straight forward.

Momentum example The above theoretical concept is simple in its ideas:

- a particle changes its momentum whenever a force acts on it;
- momentum is conserved;
- action $=$ - reaction.

But it is incredible powerful and so generic, that finding when and how to use it is much less straight forward. The beauty of physics is its relatively small set of fundamental laws. The difficulty of physics is these laws can be applied to almost anything. The trick is how to do that, how to start and get the machinery running. That can be very hard. Luckily there is a recipe to master it: it is called practice.

2.2.4 Forces & Inertia

Newton's laws introduce the concept of force. Forces have distinct features:

- forces are vectors, that is, they have magnitude and direction;
- forces change the motion of an object:
  - they change the velocity, i.e. they accelerate the object

$$\vec{a} = \frac{\vec{F}}{m} \leftrightarrow d\vec{v} = \vec{a}dt = \frac{\vec{F}dt}{m} \quad (59)$$

- or, equally true, they change the momentum of an object

$$\frac{d\vec{p}}{dt} = \vec{F} \leftrightarrow d\vec{p} = \vec{F}dt \quad (60)$$

Many physicists like the second bullet: forces change the momentum of an object, but for that they need time to act.

Momentum is a more fundamental concept in physics than acceleration. That is another reason why physicists prefer the second way of looking at forces.

Connecting physics and calculus
Let's look at a particle of mass m, that has initially (say at $t = 0$) a velocity v_0. For $t > 0$ the particle is subject to a force that is of the form $F = -bv$. This is a kind of frictional force: the faster the particle goes, the larger the opposing force will be.

We would like to know how the position of the particle is as a function of time.

We can answer this question by applying Newton 2:

$$m\frac{dv}{dt} = F \Rightarrow m\frac{dv}{dt} + bv = 0 \quad (61)$$

Clearly, we have to solve a differential equation which states that if you take the derivative of v you should get something like $-v$ back. From calculus we know, that exponential function have the feature that when we differentiate them, we get them back. So, we will try $v(t) = Ae^{-\mu t}$ with A and μ to be determined constants.

We substitute our trial v:

$$m \cdot A \cdot -\mu e^{-\mu t} + b \cdot Ae^{-\mu t} = 0 \tag{62}$$

This should hold for all t. Luckily, we can scratch out the term $e^{-\mu t}$, leaving us with:

$$-mA\mu + Ab = 0 \tag{63}$$

We see, that also our unknown constant A drops out. And, thus, we find

$$\mu = \frac{b}{m} \tag{64}$$

Next we need to find A: for that we need an initial condition, which we have: at $t = 0$ is $v = v_0$. So, we know:

$$v(t) = Ae^{-\frac{b}{m}t} \text{ and } v(0) = v_0 \tag{65}$$

From the above we see: $A = v_0$ and our final solution is:

$$v(t) = v_0 e^{-\frac{b}{m}t} \tag{66}$$

From the solution for v, we easily find the position of m as a function of time. Let's assume that the particle was in the origin at $t = 0$, thus $x(0) = 0$. So, we find for the position

$$\frac{dx}{dt} \equiv v = v_0 e^{-\frac{b}{m}t} \Rightarrow x = v_0 \cdot \left(-\frac{m}{b} e^{-\frac{b}{m}t}\right) + B \tag{67}$$

We find B with the initial condition and get as final solution:

$$x(t) = \frac{mv_0}{b}\left(1 - e^{-\frac{b}{m}t}\right) \tag{68}$$

If we inspect and assess our solution, we see: the particle slows down (as is to be expected with a frictional force acting on it) and eventually comes to a stand still. At that moment, the force has also decreased to zero, so the particle will stay put.

Inertia Inertia is denoted by the letter m for mass. And mass is that property of an object that characterizes its resistance to changing its velocity. Actually, we should have written something like m_i, with subscript i denoting inertia.

Why? There is another property of objects, also called mass, that is part of Newton's Gravitational Law.

Two bodies of mass m_1 and m_2 that are separated by a distance r_{12} attract each other via the so-called gravitational force ($\hat{r}_{12}$ is a unit vector along the line connecting m_1 and m_2):

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{69}$$

Here, we should have used a different symbol, rather than m. Something like m_g, as it is by no means obvious that the two 'masses' m_i and m_g refer to the same property. If you find that confusing, think about inertia and electric forces. Two particles with each an electric charge, q_1 and q_2, respectively exert a force on each other known as the Coulomb force:

$$\vec{F}_{C,12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \tag{70}$$

We denote the property associated with electric forces by q and call it charge. We have no problem writing

$$\vec{F} = m\vec{a}\vec{F}_C = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} \tag{71}$$

We do not confuse q by m or vice versa. They are really different quantities: q tells us that the particle has a property we call 'charge' and that it will respond to other charges, either being attracted to, or repelled from. How fast it will respond to this force of another charged particle depends on m. If m is big, the particle will only get a small acceleration; the strength of the force does not depend on m at all. So far, so good. But what about m_g? That property of a particle that makes it being attracted to another particle with this same property, that we could have called 'gravitational charge'. It is clearly different from 'electrical charge'. But would it have been logical that it was also different from the property inertial mass, m_i?

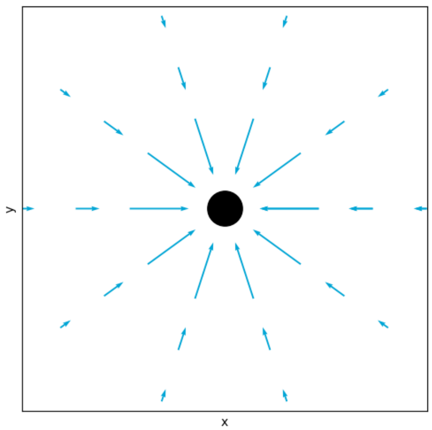
$$\begin{aligned}\vec{F} &= m_i\vec{a} \\ \vec{F}_g &= -G\frac{m_g M_g}{r^2}\hat{r}\end{aligned} \tag{72}$$

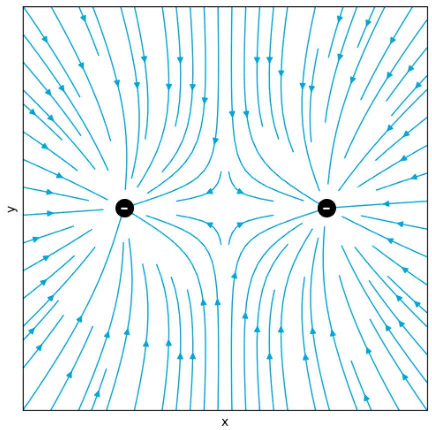
As far as we can tell (via experiments) m_i and m_g are the same. Actually, it was Einstein who postulated that the two are referring to the same property of an object: there is no difference.

Force field We have seen, forces like gravity and electrostatics act between objects. When you push a car, the force is applied locally, through direct contact. In contrast, gravitational and electrostatic forces act over a distance — they are present throughout space, though they still depend on the positions of the objects involved.

One powerful way to describe how a force acts at different locations in space is through the concept of a **force field**. A force field assigns a force vector (indicating both direction and magnitude) to every point in space, telling you what force an object would experience if placed there.

For example, the graph below at the left shows a gravitational field, described by $\vec{F}_g = G\frac{mM}{r^2}\hat{r}$. Any object entering this field is attracted toward the central mass with a force that depends on its distance from that mass's center.





Measuring mass or force So far we did not address how to measure force. Neither did we discuss how to measure mass. This is less trivial than it looks at first side. Obviously, force and mass are coupled via N2: $F = ma$.

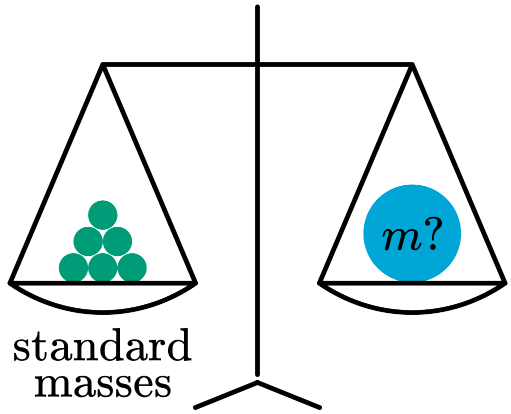


Figure 13: Can force be measured using a balance?

The acceleration can be measured when we have a ruler and a clock, i.e. once we have established how to measure distance and how to measure time intervals, we can measure position as a function of time and from that velocity and acceleration.

But how to find mass? We could agree upon a unit mass, an object that represents by definition 1kg. In fact we did. But that is only step one. The next question is: how do we compare an unknown mass to our standard. A first reaction might be: put them on a balance and see how many standard kilograms you need (including fractions of it) to balance the unknown mass. Sounds like a good idea, but is it? Unfortunately, the answer is not a 'yes'.

As on second thought: the balance compares the pull of gravity. Hence, it 'measures' gravitational mass, rather than inertia. Luckily, Newton's laws help. Suppose we let two objects, our standard mass and the unknown one, interact under their mutual interaction force. Every other force is excluded. Then, on account on N2 we have

$$\left\{ \begin{array}{l} m_1 a_1 = F_{21} \\ m_2 a_2 = F_{12} = -F_{21} \end{array} \right. \tag{73}$$

where we used N3 for the last equality. Clearly, if we take the ratio of these two equations we get:

$$\frac{m_1}{m_2} = \left| \frac{a_2}{a_1} \right| \tag{74}$$

irrespective of the strength or nature of the forces involved. We can measure acceleration and thus with this rule express the unknown mass in terms of our standard.

Note

We will not use this method to measure mass. We came to the conclusion that we can't find any difference in the gravitational mass and the inertial mass. Hence, we can use scales and balances for all practical purposes. But the above shows, that we can safely work with inertial mass: we have the means to measure it and compare it to our standard kilogram.

Now that we know how to determine mass, we also have solved the problem of measuring force. We just measure the mass and the acceleration of an object and from N2 we can find the force. This allows us to develop 'force measuring equipment' that we can calibrate using the method discussed above.

Intermezzo: kilogram, unit of mass
In 1795 it was decided that 1 gram is the mass of 1 cm^3 of water at its melting point. Later on, the kilogram became the unit for mass. In 1799, the *kilogramme des Archives* was made, being from then on the prototype of the unit of mass. It has a mass equal to that of 1 liter of water at 4°C (when water has its maximum density).

In recent years, it became clear that using such a standard kilogram does not allow for high precision: the mass of the standard kilogram was, measured over a long time, changing. Not by much (on the order of 50 micrograms), but sufficient to hamper high precision measurements and setting of other standards. In modern physics, the kilogram is now defined in terms of Planck's constant. As Planck's constant has been set (in 2019) at exactly $h = 6.62607015 \cdot 10^{-34}$ kgm^{2}s^{-1}, the kilogram is now defined via h, the meter and second.

Eötvös experiment on mass The question whether inertial mass and gravitational mass are the same has put experimentalists to work. It is by no means an easy question. Gravity is a very weak force. Moreover, determining that two properties are identical via an experiment is virtually impossible due to experimental uncertainty. Experimentalist can only tell the outcome is 'identical' within a margin. Newton already tried to establish experimentally that the two forms of mass are the same. However, in his days the inaccuracy of experiments was rather large. Dutch scientist Simon Stevin concluded in 1585 that the difference must be less than 5%. He used his famous 'drop masses from the church' experiments for this (they were primarily done to show that every mass falls with the same acceleration).

A couple of years later, Galilei used both fall experiments and pendula to improve this to: less than 2%. In 1686, Newton using pendula managed to bring it down to less than 1‰ .

An important step forward was set by the Hungarian physicist, Loránd Eötvös (1848-1918). We will here briefly introduce the experiment. For a full analysis, we need knowledge about angular momentum and centrifugal forces that we do not deal with in this book.

The experiment The essence of the Eötvös experiment is finding a set up in which both gravity (sensitive to the gravitational mass) and some inertial force (sensitive to the inertial mass) are present. Obviously, gravitational forces between two objects out of our daily life are extremely small. This will be very difficult to detect and thus introduce a large error if the experiment relies on measuring them. Eötvös came up with a different idea. He connected two different objects with different masses, m_1 and m_2, via a (almost) massless rod. Then, he attached a thin wire to the rod and let it hang down.

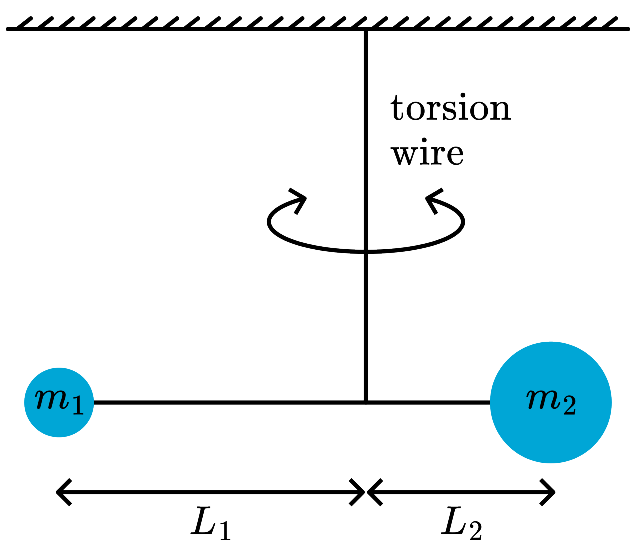


Figure 15: Torsion balance used by Eötvös.

This is a sensitive device: any mismatch in forces or torques will have the setup either tilt or rotate a bit. Eötvös attached a tiny mirror to one of the arms of the rod. If you shine a light beam on the mirror and let it reflect and be projected on a wall, then the smallest deviation in position will be amplified to create a large motion of the light spot on the wall.

In Eötvös experiment two forces are acting on each of the masses: gravity, proportional to m_g, but also the centrifugal force $F_c = m_i R\omega^2$, the centrifugal force stemming from the fact that the experiment is done in a frame of reference rotating with the earth. This force is proportional to the inertial mass. The experiment is designed such that if the rod does not show any rotation around the vertical axis, then the gravitational mass and inertial mass must be equal. It can be done with great precision and Eötvös observed no measurable rotation of the rod. From this he could conclude that the ratio of the gravitational over inertial mass differed less from 1 than $5 \cdot 10^{-8}$. Currently, experimentalist have brought this down to $1 \cdot 10^{-15}$.

Note

The question is not if m_i/m_g is different from 1. If that was the case but the ratio would always be the same, then we would just rescale m_g, that is redefine the value of the gravitational const G to make m_g equal to m_i. No, the question is whether these two properties are separate things, like mass and charge. We can have two objects with the same inertial mass but give them very different charges. In analogy: if m_i and m_g are fundamentally different quantities then we could do the same but now with inertial and gravitational mass.

Tip

Want to know more about this experiment? Watch this videoclip.

2.2.5 Examples, exercises and solutions

Here are some examples and exercises that deals with forces. Make sure you practice IDEA.

Exercise 1: Force on a particle
Consider a point particle of mass m, moving at a velocity v_0 along the x-axis. At $t = 0$ a constant force acts on the particle in the negative x-direction. The force lasts for a small time interval Δt.

What is the strength of the force, if it brings the particle exactly to a zero-velocity? Start by making a drawing.

Exercise 2:
A ball is shot from a 10m high hill with a velocity of 10m/s under an angle of 30°, see Figure **??**.

1. How long is the ball in the air?
2. How far does it travel in the horizontal direction?
3. With what velocity does the ball hit the ground?

Exercise 3:
A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$, a constant force acts on the particle. This is a 1-dimensional problem.

- Derive the acceleration of the particle as a function of time.
- Derive the velocity of the particle as a function of time.
- Derive the position of the particle as a function of time.

Exercise 4:
A particle of mass m moves along the x-axis. At time $t = 0$ it is at the origin with velocity v_0. For $t > 0$ the particle is subject to a force $F_0 \sin(2\pi f_0 t)$. This is a 1-dimensional problem.

- Calculate the acceleration of the particle as a function of time.
- Calculate the velocity of the particle as a function of time.
- Calculate the position of the particle as a function of time.

Exercise 5:
A particle follows a straight path with a constant velocity. At $t = 0$ the particle is at point A with coordinate $(0, y_A)$, while at $t = t_1$ it is at B with coordinate $(x_B, 0)$. The coordinates are given in a Cartesian system. The problem is 2-dimensional.

1. Make a sketch.
2. Find the position of the particle at arbitrary time $0 < t < t_1$.
3. Derive the velocity of the particle from position as function of time.

Represent vectors in a Cartesian coordinate system using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Exercise 6:
In Classical Mechanics we often use a coordinate system to describe motion of object. In this exercise, you will look at two Cartesian coordinate systems. System S has coordinates (x, y) and corresponding unit vectors $\hat{x}$ and $\hat{y}$.
The second system, S', uses (x', y') and corresponding unit vectors. The x'-axis makes an angle of $30°$ with the x-axis (measured counter-clockwise).

1. Make a sketch.
2. Determine the relations between $\hat{x}'$ and $\hat{x}, \hat{y}$ as well as between $\hat{y}'$ and $\hat{x}, \hat{y}$
   An object has, according to S, a velocity of $\vec{v} = 3\hat{x} + 5\hat{y}$.
3. Determine the velocity according to S'.

Exercise 7:
According to your observations, a particle is located at position (1,0) (you use a Cartesian coordinate system). The particle has no velocity and no forces are acting on it.
Another observer, S', uses a Cartesian coordinate system described by (x', y'). You notice that her unit vectors rotate at a constant speed compared to your unit vectors:

$$\hat{x}' = \cos(2\pi f t)\hat{x} + \sin(2\pi f t)\hat{y} \tag{75}$$

$$\hat{y}' = -\sin(2\pi f t)\hat{x} + \cos(2\pi f t)\hat{y} \tag{76}$$

1. Find the position of the particle according to the other observer, S'.
2. Calculate the velocity of the particle according to S'.

Exercise 8:
A particle of mass m moves at a constant velocity v_0 over a friction less table. The direction it is moving in, is at $45°$ with the positive x-axis. At some point in time, the particle experiences a force $\vec{F} = -b\vec{v}$ with $b > 0$. We call this time $t = 0$ and take the position of the particle at that time as our origin.

1. Make a sketch.
2. Determine whether this problem needs to be analyzed as a 1D or a 2D problem.
3. Set up N2 in the form $m\frac{d\vec{v}}{dt} = \vec{F}$
4. Solve N2 and find the velocity of the particle as a function of time.
5. What happens to the particle for large t?

Exercise 9: Parabolic trajectory with maximum area [1]
A ball is thrown at speed v from zero height on level ground. We want to find the angle θ at which it should be thrown so that the area under the trajectory is maximized.

1. Sketch of the trajectory of the ball.
2. Use dimensional analysis to relate the area to the initial speed v and the gravitational acceleration g.
3. Write down the x and y coordinates of the ball as a function of time.
4. Find the total time the ball is in the air.
5. The area under the trajectory is given by $A = \int y \mathrm{d}x$. Make a variable transformation to express this integral as an integration over time.
6. Evaluate the integral. Your answer should be a function of the initial speed v and angle θ.
7. From your answer at (f), find the angle that maximizes the area, and the value of that maximum area. Check that your answer is consistent with your answer at (b).

Exercise 10: Two attracting particles [2]
Two particles on a line are mutually attracted by a force $F = -ar$, where a is a constant and r the distance of separation. At time $t = 0$, particle A of mass m is located at the origin, and particle B of mass $m/4$ is located at $r = 5.0 \text{ cm}$.

1. If the particles are at rest at $t = 0$, at what value of r do they collide?
2. What is the relative velocity of the two particles at the moment the collision occurs?

Exercises set 1

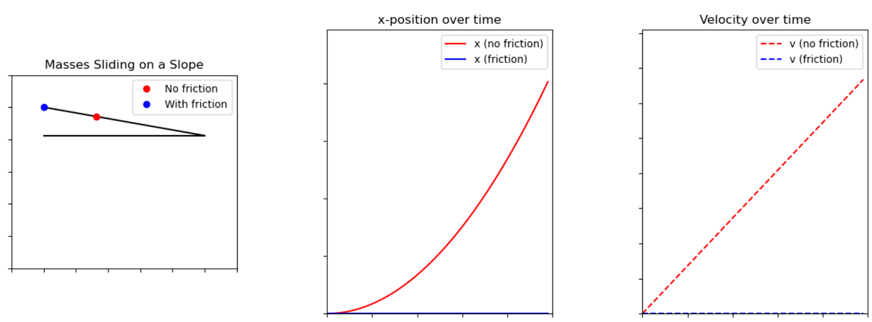
Answers set 1

Warning
Onderstaande app nog niet correct qua waarden!

Exercises set 2

```
interactive(children=(IntSlider(value=10, description='F_girl (N)', max=25, min=5), IntSlider(value=2, 
```

```
<function __main__.update_plot(Fgirl=10, Delta=2)>
```



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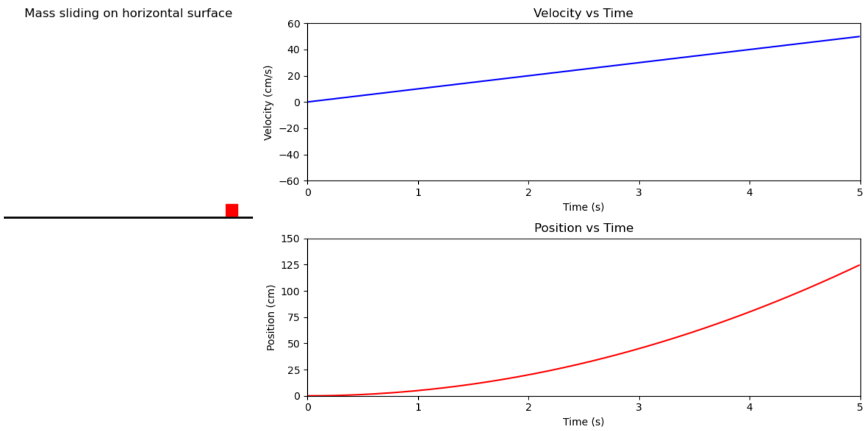
Animation size has reached 20979181 bytes, exceeding the limit of 20971520.0. If you're sure you want a
```



Answers set 2

2.3 Work & Energy

2.3.1 Work

Work and energy are two important concepts. Work is the transfer of energy that occurs when a force is applied to an object and causes displacement in the direction of that force, calculated as 'force times path'. However, we need a formal definition:

if a point particle moves from $\vec{r}$ to $\vec{r} + d\vec{r}$ and during this interval a force $\vec{F}$ acts on the particle, then this force has performed an amount of work equal to:

$$dW = \vec{F} \cdot d\vec{r} \tag{77}$$

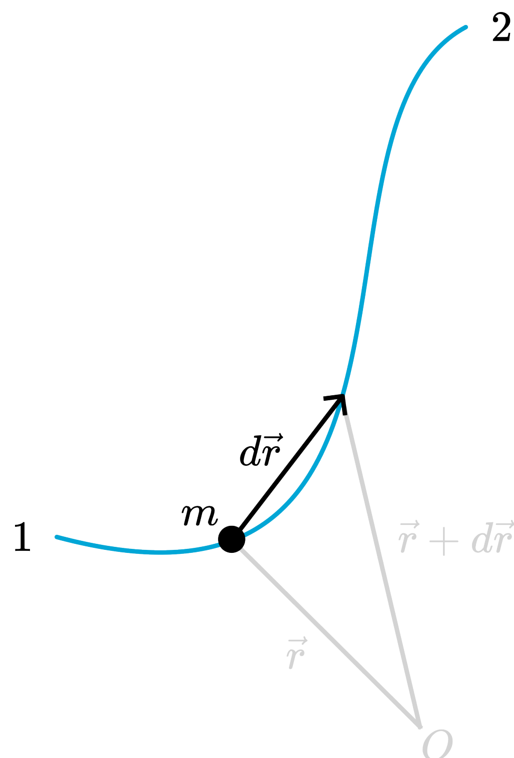


Figure 17: Path of a particle.

Note that this an *inner product* between two vectors, resulting in a *scalar* . In other words, work is a number, not a vector. It has no direction. That is one of the advantages over force.

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz \tag{78}$$

Work done on m by F during motion from 1 to 2 over a prescribed trajectory:

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} \tag{79}$$

Keep in mind: in general the work depends on the starting point 1, the end point 2 and on the trajectory. Different trajectories from 1 to 2 may lead to different amounts of work.

Tip
See also the chapter in the linear algebra book on the inner product

2.3.2 Kinetic Energy

Kinetic energy is defined and derived using the definition of work and Newton's 2^{nd} Law.

The following holds: if work is done on a particle, then its kinetic energy must change. And vice versa: if the kinetic energy of an object changes, then work must have been done on that particle. The following derivation shows this.

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_1^2 \vec{F} \cdot \vec{v} dt = \int_1^2 m \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_1^2 \vec{v} \cdot d\vec{v} = m[\frac{1}{2}\vec{v}^2]_1^2 = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2 \quad (80)$$

It is from the above that we indicate $\frac{1}{2}m\vec{v}^2$ as kinetic energy. It is important to realize that the concept of kinetic energy does not bring anything that is not contained in N2 to the table. But it does give a new perspective: kinetic energy can only be gained or lost if a force performs work on the particle. And vice versa: if a force performs work on a particle, the particle will change its kinetic energy.

Obviously, if more than one force acts, the net work done on the particle determines the change in kinetic energy. It is perfectly possible that force 1 adds an amount W to the particle, whereas at the same time force 2 will take out an amount $-W$. This is the case for a particle that moves under the influence of two forces that cancel each other: $\vec{F}_1 = -\vec{F}_2$. From Newton 2, we immediately infer that if the two forces cancel each other, then the particle will move with a constant velocity. Hence, its kinetic energy stays constant. This is completely in line with the fact that in this case the net work done on the particle is zero:

$$W_1 + W_2 = \int_1^2 \vec{F}_1 \cdot d\vec{r} + \int_1^2 \vec{F}_2 \cdot d\vec{r} = \int_1^2 \vec{F}_1 \cdot d\vec{r} - \int_1^2 \vec{F}_1 \cdot d\vec{r} = 0 \quad (81)$$

2.3.3 Worked Examples

Reminder of path/line integral from Analysis
As long as the path can be split along coordinate axis the separation above is a good recipe. If that is not the case, then we need to turn back to the way how things have been introduced in the Analysis class. We need to make a 1D parameterization of the path.

Line integral of a vector valued function $\vec{F}(x, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$ over a curve $\mathcal{C}$ is given as

$$\int_{\mathcal{C}} \vec{F}(x, y, z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(\tau)) \cdot \frac{d\vec{r}(\tau)}{d\tau} d\tau \quad (82)$$

We integrate in the definition of the work from point 1 to 2 over an implicitly given path. To compute this actually, you need to parameterize the path by $\vec{r}(\tau) = (x(\tau), y(\tau), z(\tau))$. The integration variable τ tells you where you are on the path, $\tau \in [a, b] \in \mathbb{R}$. The derivative of $\vec{r}$ with respect to τ gives the tangent vector to the curve, the "speed" of walking along the curve. In the analysis class you used $\vec{v}(\tau) \equiv \frac{d\vec{r}(\tau)}{d\tau}$ for the speed. The value of the line integral is independent of the chosen parameterization. However, it changes sign when reversing the integration boundaries.

Example 4.3
Now we integrate from $(0, 0) \to (1, 1)$ but along the diagonal. A parameterization of this path is $\vec{r}(\tau) =$

$(0,0)+(1,1)\tau = (\tau,\tau), \tau \in [0,1]$. The derivative is $\frac{d\vec{r}(\tau)}{d\tau} = (1,1)$. Therefore we can write the work of $\vec{F}(x,y) = -y\hat{x} + x^2\hat{y}$ along the diagonal as

$$\int_0^1 \vec{F}(\tau,\tau)\cdot(1,1)\,d\tau = \int_0^1 (-\tau,\tau^2)\cdot(1,1)\,d\tau = \int_0^1 -\tau+\tau^2\,d\tau = -\frac{1}{6} \tag{83}$$

Integration of the same force $\vec{F}(x,y) = -y\hat{x} + x^2\hat{y}$ from $(0,0)\to(1,1)$ but along a normal parabola. A parameterization of the path is $\vec{r}(\tau) = (0,0) + (\tau,\tau^2), \tau \in [0,1]$ and the derivative is $\frac{d\vec{r}}{d\tau} = (1,2\tau)$. The work then is

$$\int_0^1 \vec{F}(\tau,\tau^2)\cdot(1,2\tau)\,d\tau = \int_0^1 (-\tau^2,\tau^2)\cdot(1,2\tau)\,d\tau = \int_0^1 -\tau^2+2\tau^3\,d\tau = \frac{1}{6} \tag{84}$$

2.3.4 Gravitational potential energy

Let's consider an object close to the surface of any planet, where the acceleration due to gravity can be described by $F_g = mg$. Raising the object to a height H requires us to do work:

$$W = \int_0^H F dx = \int_0^H -mg dx = -mgH \tag{85}$$

Note that there is a minus sign, we have done work against the gravitational force. As energy is a conservative quantity, the object has the 'gained' some energy. We call this potential energy, more particular in this case gravitational potential energy.

When the object is released from that height H, this gravitational potential energy is converted to kinetic energy. The gravitational force does work on the object:

$$W == \int_H^0 F dx = \int_H^0 mg dx = mgH = \Delta E_{kin} \tag{86}$$

From this, it follow that the object will reach a velocity of $v = \sqrt{2gH}$.

2.3.5 Conservative force

Work done on m by F during motion from 1 to 2 over a prescribed trajectory, is defined as:

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} \tag{87}$$

In general, the amount of work depends on the path followed. That is, the work done when going from $\vec{r}_1$ to $\vec{r}_2$ over the red path in the figure below, will be different when going from $\vec{r}_1$ to $\vec{r}_2$ over the blue path. Work depends on the specific trajectory followed.

However, there is a certain class of forces for which the path does not matter, only the start and end point do. These forces are called conservative forces. As a consequence, the work done by a conservative force over a closed path, i.e start and end are the same, is always zero. No matter which closed path is taken.

$$\text{conservative force} \Leftrightarrow \oint \vec{F}\cdot d\vec{r} = 0 \text{ for } \textbf{ALL} \text{ closed paths} \tag{88}$$

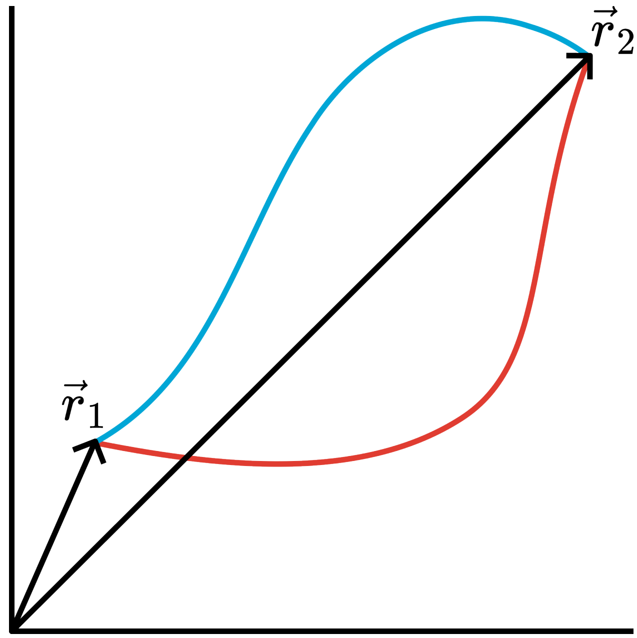


Figure 18: Two different paths.

Figure 19: Sir George Stokes (1819-1903). From Wikimedia Commons, public domain.

Stokes' Theorem It was George Stokes who proved an important theorem, that we will use to turn the concept of conservative forces into a new and important concept.

His theorem reads as:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{89}$$

In words: the integral of the force over a closed path equals the surface integral of the curl of that force. The surface being 'cut out' by the close path. The term $\vec{\nabla} \times \vec{F}$ is called the curl of F: it is a vector. The meaning of it and some words on the theorem are given below.

Intermezzo: intuitive proof of Stokes' Theorem
Consider a closed curve in the xy-plane. We would like to calculate the work done when going around this curve. In other words: what is $\oint \vec{F} \cdot d\vec{r}$ if we move along this curve?

We can visualize what we need to do: we cut the curve in small part; compute $\vec{F} \cdot d\vec{r}$ for each part (i.e. the red, green, blue, etc. in Figure **??** and sum these to get the total along the curve. If we make the parts infinitesimally small, we go from a (Riemann) sum to an integral.

We are going to compute much more: take a look at Figure **??**. We have put a grid in the xy-plane over a closed curve Γ. Hence, the interior of our curve is fool op squares. We are not only computing the parts along the curve, but also along the sides of alle curves. This will sound like way too much work, but we will see that it actually is a very good idea.

See Figure **??**: we calculate $\oint \vec{F} \cdot d\vec{r}$ counter clockwise for the green square. Than we have at least the green part of our $\oint \vec{F} \cdot d\vec{r}$ done in the right direction. Hence, we compute $\int \vec{F} \cdot d\vec{r}$ along the right side of the green square. We do that from bottom to top as we go counter clockwise along the green square. Let's call that $\int_g \vec{F} \cdot d\vec{r}$.
Then we move to the blue square and repeat in counter clockwise direction our calculation. But this means that we compute along the left side of blue the square from top to bottom. We will call this $\int_b \vec{F} \cdot d\vec{r}$.

Note that we will add all contributions. Thus we get $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r}$. But these two cancel each other as they are exactly the same but done in opposite directions. Thus if we use that $\int_1^2 f dx = -\int_2^1 f dx$ for any integration, it becomes obvious that $\int_g \vec{F} \cdot d\vec{r} + \int_b \vec{F} \cdot d\vec{r} = 0$.

Note that this will happen for all side of the squares that are in the interior of our curve. Thus, the integral over all squares is exactly the integral along the curve Γ.

It seems, we do a lot of work for nothing. But there is another way of looking at the path-integrals along the squares. If we make the square small enough, the calculation along one square can be approximated:

$$\begin{aligned}
\oint_{square} \vec{F} \cdot d\vec{r} &\approx F_x(x,y)dx + F_y(x+dx,y)dy - F_x(x,y+dy)dx - F_y(x,y)dy \\
&\approx \frac{F_x(x,y) - F_x(x,y+dy)}{dy} dxdy + \frac{F_y(x+dx,y) - F_y(x,y)}{dx} dxdy \\
&\approx \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy
\end{aligned} \tag{90}$$

The results get more accurate the smaller we make the square.

If we now sum up all squares and make these squares infinitesimally small, the sum becomes an integral, but now an integral over the surface enclosed by the curve:

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dxdy \tag{91}$$

The right hand side of the above equation is an surface integral of the 'vector' $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. Obviously, we did not provide a rigorous proof, but only an intuitive one. For a mathematical proof, see your calculus classes.

Moreover, we only worked in the xy-plane. If we would extend our reasoning to a closed curve in 3 dimensions, we would get Stokes theorem, which reads as:

$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{92}$$

Here, $d\vec{\sigma}$ is a small element out of the surface. Note that it is a vector: it has size and directions. The latter is perpendicular to the surface element itself. Furthermore, we have the vector $\vec{\nabla} \times \vec{F}$. This is the cross-product of the nabla-operator and our vector field $\vec{F}$. The nabla operator is a strange kind of vector. Its components are: partial differentiation. In a Cartesian coordinate system this can be written as:

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \tag{93}$$

or if you prefer a column notation:

$$\vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{94}$$

The curl of F can be found from e.g.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z} \tag{95}$$

Note of warning: do be careful with the nabla-operator. It is not a standard vector. For instance, ordinary vectors have the property $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. This does not hold for the nabla-operator.

Second note of warning: the representation of the nabla-operator does change quite a bit when using other coordinate systems like cylindrical or spherical. For instance, in cylindrical coordinates it is ***not*** equal to $\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}$. This can be easily seen as both r, z have units length, i.e. meters, but ϕ has no units.

5.1

Suppose we need to calculate the integral of the vectorfield $\vec{F}(x, y) = y\hat{x} - x\hat{y}$ over the closed curve formed by a square from $(0, 0)$ to $(1, 0)$, $(1, 1)$, $(0, 1)$ and back to $(0, 0)$.

We go counter clockwise.

$$\begin{aligned} \oint \vec{F} \cdot d\vec{r} = & \int_{x=0}^{1} F_x(x, y=0)dx + \int_{y=0}^{1} F_y(x=1, y)dy + \\ & + \int_{x=1}^{0} F_x(x, y=1)dx + \int_{y=1}^{0} F_y(x=0, y)dy \\ & = \int_0^1 0dx + \int_0^1 -1dy + \int_1^0 1dx + \int_1^0 -0dx \\ & = 0 - [y]_0^1 + [x]_1^0 - 0 \\ & = -2 \end{aligned} \tag{96}$$

Now we try this using Stokes' Theorem:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} \tag{97}$$

We first calculate $\vec{\nabla} \times \vec{F}$:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y} \right) \hat{z} = -2\hat{z} \tag{98}$$

Thus, in this example $\vec{\nabla} \times \vec{F}$ has only a z-component.

An elementary surface element of the square is: $d\vec{\sigma} = dxdy\hat{z}$. This also has only a z-component. Note that it points in the positive z-direction. This is a consequence of the counter clockwise direction that we use to go along the square.

According to Stokes Theorem, we this find:

$$\oint \vec{F} \cdot d\vec{r} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_{x=0}^{1} \int_{y=0}^{1} (-2) dxdy = -2 \tag{99}$$

Indeed, we find the same outcome.

Conservative force and $\vec{\nabla} \times \vec{F}$ For a conservative force the integral over the closed path is zero for any closed path. Consequently, $\vec{\nabla} \times \vec{F} = 0$ everywhere. How do we know this? Suppose $\vec{\nabla} \times \vec{F} \neq 0$ at some point in space. Then, since we deal with continuous differentiable vector fields, in the close vicinity of this point, it must also be non-zero. Without loss of generality, we can assume that in that region $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$. Next, we draw a closed curve around this point, in this region. We now calculate the $\oint \vec{F} \cdot d\vec{r}$ along this curve. That is, we invoke Stokes Theorem. But we know that $\vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} > 0$ on the surface formed by the closed curve. Consequently, the outcome of the surface integral is non-zero. But that is a contradiction as we started with a conservative force and thus the integral should have been zero.
The only way out, is that $\vec{\nabla} \times \vec{F} = 0$ everywhere.

Thus we have:

$$\text{conservative force } \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere} \tag{100}$$

2.3.6 Potential Energy

A direct consequence of the above is:

if $\vec{\nabla} \times \vec{F} = 0$ everywhere, a function $V(\vec{r})$ exists such that $\vec{F} = -\vec{\nabla} V$

$$\text{conservative force } \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \text{ everywhere} \Updownarrow \vec{F} = -\vec{\nabla} V \Leftrightarrow V(\vec{r}) = -\int_{ref} \vec{F} \cdot d\vec{r} \tag{101}$$

where in the last integral, the lower limit is taken from some, self picked, reference point. The upper limit is the position $\vec{r}$.

This function V is called the potential energy or the potential for short. It has a direct connection to work and kinetic energy.

$$E_{kin,2} - E_{kin,1} = W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = V(\vec{r}_2) - V(\vec{r}_1) \tag{102}$$

or rewritten:

$$E_{kin,1} + V(\vec{r}_1) = E_{kin,2} + V(\vec{r}_2) \tag{103}$$

In words: **for a conservative force, the sum of kinetic and potential energy stays constant**.

Energy versus Newton's Second Law We, starting from Newton's Laws, arrived at an energy formulation for physical problems.
Question: can we also go back? That is: suppose we would start with formulating the energy rule for a physical problem, can we then back out the equation of motion?
Answer: yes, we can!

It goes as follows. Take a system that can be completely described by its kinetic plus potential energy. Then: take the time-derivative and simplify, we will do it for a 1-dimensional case first.

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(x) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(x)\right] = \frac{dE_0}{dt} = 0 \Rightarrow \\
&mv\dot{v} + \frac{dV}{dx}\underbrace{\frac{dx}{dt}}_{=v} = 0 \Rightarrow \\
&v\left(m\dot{v} + \frac{dV}{dx}\right) = 0
\end{aligned} \tag{104}$$

The last equation must hold for all times and all circumstances. Thus, the term in brackets must be zero.

$$m\dot{v} + \frac{dV}{dx} = 0 \Rightarrow m\ddot{x} = -\frac{dV}{dx} = F \tag{105}$$

And we have recovered Newton's second law.

In 3 dimensions it is the same procedure. What is a bit more complicated, is using the chain rule. In the above 1-d case we used $\frac{dV}{dt} = \frac{dV(x(t))}{dt} = \frac{dV}{dx}\frac{dx(t)}{dt}$. In 3-d this becomes:

$$\frac{dV}{dt} = \frac{dV(\vec{r}(t))}{dt} = \frac{dV}{d\vec{r}} \cdot \frac{d\vec{r}(t)}{dt} = \vec{\nabla}V \cdot \vec{v} \tag{106}$$

Thus, if we repeat the derivation, we find:

$$\begin{aligned}
&\frac{1}{2}mv^2 + V(\vec{r}) = E_0 \Rightarrow \\
&\frac{d}{dt}\left[\frac{1}{2}mv^2 + V(\vec{r})\right] = 0 \Rightarrow \\
&m\vec{v} \cdot \dot{\vec{v}} + \vec{\nabla}V \cdot \vec{v} = 0 \Rightarrow \\
&v\left(m\vec{a} + \vec{\nabla}V\right) = 0 \Rightarrow \\
&m\vec{a} = -\vec{\nabla}V = \vec{F}
\end{aligned} \tag{107}$$

And we have recovered the 3-dimensional form of Newton's second Law. This is a great result. It allows us to pick what we like: formulate a problem in terms of forces and momentum, i.e. Newton's second law, or reason from energy considerations. It doesn't matter: they are equivalent. It is a matter of taste, a matter of what do you see first, understand best, find easiest to start with. Up to you!

2.3.7 Stable/Unstable Equilibrium

A particle (or system) is in equilibrium when the sum of forces acting on it is zero. Then, it will keep the same velocity, and we can easily find an inertial system in which the particle is at rest, at an equilibrium position. The equilibrium position (or more general state) can also be found directly from the potential energy.

Potential energy and (conservative) forces are coupled via:

$$\vec{F} = -\vec{\nabla} V \tag{108}$$

The equilibrium positions $\left(\sum_i \vec{F}_i = 0\right)$ can be found by finding the extremes of the potential energy:

$$\text{equilibrium position } \Leftrightarrow \vec{\nabla} V = 0 \tag{109}$$

Once we find the equilibrium points, we can also quickly address their nature: is it a stable or unstable solution? That follows directly from inspecting the characteristics of the potential energy around the equilibrium points.

For a stable equilibrium, we require that a small push or a slight displacement will result in a force pushing back such that the equilibrium position is restored (apart from the inertia of the object that might cause an overshoot or oscillation).

However, an unstable equilibrium is one for which the slightest push or displacement will result in motion away from the equilibrium position.

The second derivative of the potential can be investigated to find the type of extremum. For 1D functions that is easy, for scalar valued functions of more variables that is a bit more complicated. Here we only look at the 1D case $V(x) : \mathbb{R} \rightarrow \mathbb{R}$

$$\text{equilibrium: } \vec{\nabla} V = 0 \begin{cases} \text{stable:} & \frac{d^2V}{dx^2} > 0 \\ \text{unstable:} & \frac{d^2V}{dx^2} < 0 \end{cases} \tag{110}$$

Luckily, the definition of potential energy is such that these rules are easy to visualize in 1D and remember, see fig.(?.?).

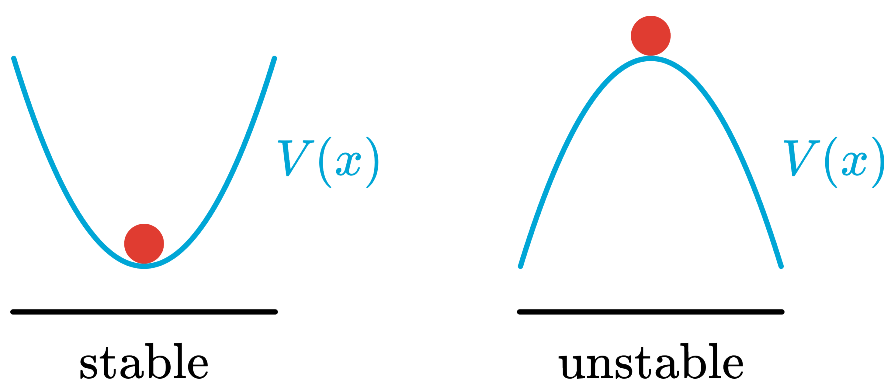


Figure 22: Stable and unstable position of a particle in a potential.

A valley is stable; a hill top is unstable.
NB: Now the choice of the minus sign in the definition of the potential is clear . Otherwise a hill would be stable, but that does not feel natural at all.

It is also easy to visualize what will happen if we distort that particle from the equilibrium state:

- The valley, i.e., the stable system, will make the particle move back to the lowest point. Due to inertia, it will not stop but will continue to move. As the lowest position is one of zero force, the particle will 'climb' toward the other end of the valley and start an oscillatory motion.
- The top, i.e., the unstable point, will make the particle move away from the stable point. The force acting on the particle is now pushing it outwards, down the slope of the hill.

Taylor Series Expansion of the Potential The Taylor expansion or Taylor series is a mathematical approximation of a function in the vicinity of a specific point. It uses an infinite series of polynomial terms with coefficients given by value of the derivative of the function at that specific point: the more terms you use, the better the approximation. If you use all terms, then it is exact. Mathematically, it reads for a 1D scalar function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + ... \tag{111}$$

For our purpose here, it suffices to stop after the second derivative term:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{112}$$

A way of understanding why the Taylor series actually works is the following.
Imagine you have to explain to someone how a function looks around some point x_0, but you are not allowed to draw it. One way of passing on information about $f(x)$ is to start by giving the value of $f(x)$ at the point x_0:

$$f(x) \approx f(x_0) \tag{113}$$

Next, you give how the tangent at x_0 is; you pass on the first derivative at x_0. The other person can now see a bit better how the function changes when moving away from x_0:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{114}$$

Then, you tell that the function is not a straight line but curved, and you give the second derivative. So now the other one can see how it deviates from a straight line:

$$f(x) \approx f(x_0) + \frac{1}{1!}f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 \tag{115}$$

Note that the prefactor is placed back. But the function is not necessarily a parabola; it will start deviating more and more as we move away from x_0. Hence we need to correct that by invoking the third derivative that tells us how fast this deviation is. And this process can continue on and on.

Important to note: if we stay close enough to x_0 the terms with the lowest order terms will always prevail as higher powers of $(x - x_0)$ tend to zero faster than a lower powers (for instance: $0.5^4 << 0.5^2$).

This 3Blue1Brown clip explains the 1D Taylor series nicely.

Figure 23: *
A 3blue1brown clip on Taylor series.

For scalar valued functions as our potentials $V(\vec{r}) : \mathbb{R}^3 \to \mathbb{R}$ the extension of the Taylor series is not too difficult. If we expand the function around a point

$$\begin{aligned} V(\vec{r}) &\approx V(\vec{r}_0) + \vec{\nabla}V(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) \\ &\quad + \frac{1}{2}(\vec{r} - \vec{r}_0) \cdot (\partial^2 V)(\vec{r}_0) \cdot (\vec{r} - \vec{r}_0) + \mathcal{O}(r^3) \end{aligned} \tag{116}$$

The second derivative of the potential indicated by $\partial^2 V$ is the Hessian matrix.

Conceptually the extrema of the function are again the hills and valleys. The classification of the extrema has next to hills and valleys also saddle points etc. In this course we will not bother about these more dimensional cases, but only stick to simple ones.

2.3.8 Examples, exercises and solutions

Exercises

Answers

Exercise set 2

2.4 Angular Momentum, Torque & Central Forces

2.4.1 Torque & Angular Momentum

From experience we know that if we want to unscrew a bottle, lift a heavy object on one side, try to unscrew a screw, we better use 'leverage'.

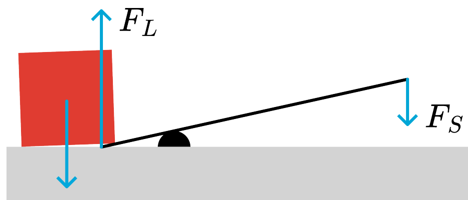


Figure 24: Lifting is easier using leverage.

With a relatively small force F_S, we can lift the side of a heavy object. The key concept to use here is torque, which in words is loosely formulated: apply the force using a long arm and the force seems to be magnified. The torque is then force multiplied by arm: $\Gamma = Force \text{ times } arm$

This is, of course, too sloppy for physicists. We need strict, formal definitions. So, we put the above into a mathematical definition.

torque

$$\vec{\Gamma} \equiv \vec{r} \times \vec{F} \tag{117}$$

That is: torque (or krachtmoment in Dutch) is the outer product of 'arm' as a vector(!) and the force. We notice a few peculiarities.

1. like force, torque is a vector. That is: it has a magnitude and a direction. In principle: three components.
2. its direction is perpendicular to the force vector $\vec{F}$ *and* perpendicular to the arm $\vec{r}$.
3. the arm is not a number: it is a vector!

We further know from experience that we can balance torques, like we can balance forces. Rephrased: the net effect of more than one force is found by adding all the forces (as vectors!) and using the net force in Newtons second law: $m\vec{a} = \sum \vec{F}_i = \vec{F}_{net}$. From Newtons first law, we immediately infer: if $\sum \vec{F}_i = \vec{F}_{net} = 0$ then the object moves at constant velocity. We can move with the object at this speed and conclude that it from this perspective has zero velocity: it doesn't move, i.e. it is in equilibrium.

The same hols for torque: we can work with the sum of all torques that act on an object: $\sum \vec{\Gamma}_i = \vec{\Gamma}_{net}$. And if this sum is zero, the object is in equilibrium.

However, there is a catch: using torques requires that we are much more explicit and precise about the choice of our origin. Why? The reason is in the 'arm'. That is only defined if we provide an origin.

The seesaw and torque Let's consider a simple example (simple in the sense that we are all familiar with it): the seesaw.

It is obvious that the adult -seesawing with the child- should sit much closer to the pivot point than the child. That is: we assume that the mass of the adult is greater than that of the child.

Let's turn this picture into one that captures the essence and includes the necessary physical quantities, and then draw a free-body diagram.

What did we **draw**?

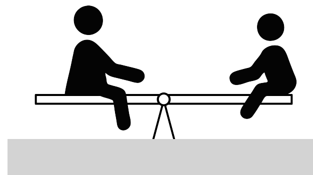


Figure 25: An adult (left) and a child (right) on a seesaw.

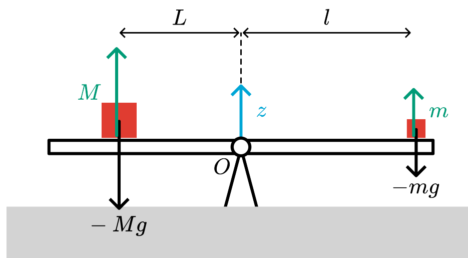


Figure 26: Free-body diagram of the seesaw and the masses.

1. The force of gravity acting on the two masses M and m. That is obvious: without forces nothing will happen and there is nothing to be analyzed.
2. The 'reaction forces' from the seesaw on both masses. Why? If the seesaw is in equilibrium, then each of the masses is in equilibrium and the sum of forces on each mass must be zero.
3. The distance of each of the masses to the pivot point. Why? Leverage! The heavy M must be closer to the pivot point the get equilibrium.
4. An origin O. Why? We need a point to measure the 'arm', 'leverage', from.
5. The z-coordinate. Why? We deal with forces in the vertical direction. Hence a coordinate, a direction that we all use, is handy.

Analysis
Time for a first analysis: what keeps this seesaw in equilibrium?

1. The sum of forces on each of the masses is zero. As gravity pulls them down, the seesaw must exert a force of the same magnitude but in the opposite direction. These are the green forces.
2. With this idea we have the masses in equilibrium, but not necessarily the seesaw. Why? We did not consider forces on the seesaw. Which are these: (a) the reaction force (i.e. the N3 pair) of the green force from the seesaw on mass M. We did not draw that! Similarly, for the mass m.
3. Now that we focus on the seesaw itself: this is in equilibrium (that is given), but there are two forces acting on it in the negative z-direction as we found in (2). Even if we consider the mass of the seesaw: that will not help, gravity will pull it downwards. What did we forget? The force at the pivot point, of course! The pivot will exert an upward force, preventing the seesaw from falling down. For simplicity, we assume that the seesaw has zero mass. Thus, there are three forces acting on it: $-Mg$, $-mg$, F_p with $F_p - Mg - mg = 0$.

Let's redraw, now concentrating on the forces on the seesaw.

Analysis part 2
We know that the seesaw is in equilibrium, thus

$$F_p - Mg - mg = 0 \tag{118}$$

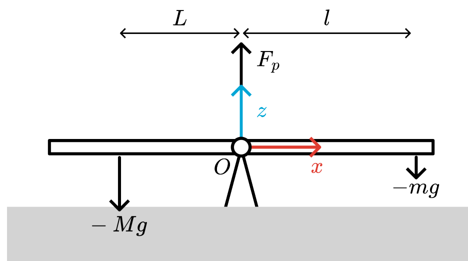


Figure 27: Free-body diagram of the seesaw.

This guarantees that the seesaw does not change its velocity, and as it does not move at some time t_0, it doesn't move for all $t > t_0$.

But this doesn't guarantee that the seesaw doesn't rotate around the pivot point. For that we need that the 'leverages' on the left and right side 'perform' the same.
Making this precise: the torques exerted on the seesaw must also equate to zero.
We have 3 forces, thus 3 torques: $-Mg$ with arm L, $-mg$ with arm l and F_p with arm zero.

Now we need to be even more precise: torque is a vector and it is made as an outer product of the vector 'arm' and the force.
We have already drawn an x-coordinate in the figure, that will allow us to write the 'arm' as a vector. After all, we need to evaluate the outer product $\vec{r} \times \vec{F}$. We do that for the three forces, starting on the left:

$$\vec{\Gamma}_1 = -L\hat{x} \times (-Mg)\hat{z} = MLg\,\hat{x} \times \hat{z} = MLg\,(-\hat{y}) = -MLg\,\hat{y} \tag{119}$$

We have used here, that the outer product of $\hat{x}$ with $\hat{z}$ is equal to $-\hat{y}$ with $\hat{y}$ the unit vector in the y-direction pointing into the screen.

Similarly for the force coming from the small mass m on the right side:

$$\vec{\Gamma}_2 = l\hat{x} \times (-mg)\hat{z} = -mlg\,\hat{x} \times \hat{z} = mlg\,\hat{y} \tag{120}$$

Finally, the torque from the force exerted by the pivot point:

$$\vec{\Gamma}_3 = 0\hat{x} \times F_p\hat{z} = 0 \tag{121}$$

Next, we evaluate the total torque:

$$\vec{\Gamma}_1 + \vec{\Gamma}_2 + \vec{\Gamma}_3 = (mlg - MLg)\hat{y} \tag{122}$$

In order for the seesaw not to start rotating, we must have that the torque is zero and thus:

$$\sum \vec{\Gamma}_i = 0 \Rightarrow mlg = MLg \rightarrow \frac{m}{M} = \frac{L}{l} \tag{123}$$

A result we expected: the greater mass should be closer to the pivot point.

Different origin So far, so good. But what if we had chosen the origin not at the pivot point, but somewhere to the left? Then all 'arm' will change length. And all torques will be different. Wouldn't that make $\sum \vec{\Gamma}_i \neq 0$?
No, it wouldn't! Let's just do it and recalculate. In the figure below, we have moved the origin to the left end of the seesaw. The distance from the heavy mass to the origin is Λ (green arrow).

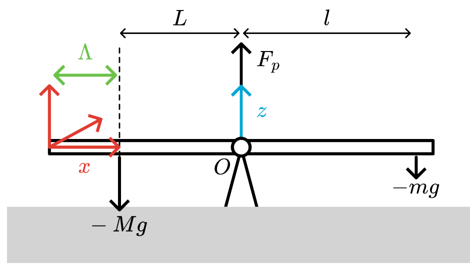


Figure 28: Free-body diagram with the origin located at the seesaw's end.

We still have that the sum of forces is zero. But what about the sum of torques? Obviously, the choice of the origin can not affect the seesaw: it is still in balance, regardless of our choice of the origin. Let's see if that is correct:

$$\sum \vec{\Gamma}_i = \Lambda\hat{x} \times -Mg\hat{z} + (\Lambda + L) \times F_p\hat{z} + (\Lambda + L + l)\,\hat{x} \times -mg\hat{z} \tag{124}$$

We have drawn the three unit vectors $\hat{x}, \hat{y}, \hat{z}$ in the figure. We will use again: $\hat{x} \times \hat{z} = -\hat{y}$. This simplifies the torque equation above to:

$$\sum \vec{\Gamma}_i = [Mg\Lambda - (\Lambda + L)F_p + mg(\Lambda + L + l)]\,\hat{y} \tag{125}$$

This is clearly more complicated than the expression we had with the first choice of the origin. Moreover, the force from the pivot point shows up in our expression.

Luckily, we have equilibrium. Hence: $F_p - Mg - mg = 0 \Rightarrow F_p = Mg + mg$. We substitute this into our torque equation:

$$\begin{aligned}\sum \vec{\Gamma}_i &= [Mg\Lambda - (\Lambda + L)(Mg + mg)) + mg(\Lambda + L + l)]\,\hat{y} \\ &= [Mg(\Lambda - (\Lambda + L)) + mg(-(\Lambda + L) + \Lambda + L + l)]\,\hat{y} \\ &= [-MgL + mgl]\,\hat{y}\end{aligned} \tag{126}$$

Which is exactly the same expression as we found before. So, indeed, the choice of the origin doesn't matter.

Conclusion

For equilibrium we need that the sum of torques is zero:

$$\sum_i \vec{\Gamma}_i = 0 \tag{127}$$

2.4.2 Angular Momentum

From our seesaw example we learn: the seesaw can only be in equilibrium if the sum of torques is zero. What if this sum is non-zero? That is, a net torque operates on the seesaw.

We know that the seesaw will rotate and in order to balance it, we have to shift at least one of the masses.

In which direction will it rotate?

Before answering: first we need to think about **direction of rotation**. Does it have direction and if so: how do we make clear what that is?

Again the seesaw will give guidance. Suppose we remove the smaller mass all together. Then, it is obvious: the 'heavy' left side will rotate to the ground and the light right side upwards. From the point of view we are standing: the seesaw will rotate counter clockwise.

We will use the corkscrew rule or right hand rule to give that a direction: rotate a corkscrew clockwise and the screw will move into the cork away from you; rotate a corkscrew counter clockwise and it will move out of the cork, towards you. Of course, in stead of a corkscrew you can think of a screwdriver or a water tap: closing is rotating 'clock wise, opening the tap is counter clockwise.

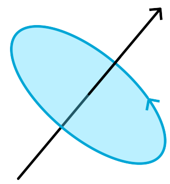


Figure 29: The rotation is given by the black arrow. You can find it by using the corkscrew rule: rotating a corkscrew as the blue arrow indicates gives that the screw moves forward like the black arrow.

With this, we can define the direction of rotation better than via clock or counter clock wise. In our seesaw example, we will say: if the seesaw rotates clockwise, its rotation is described by a vector that points in the positive y-direction as given in the figure, that is pointing into the paper (or screen).

Now, we can couple this to the direction of the torque. We saw -taking the origin at the pivot point- two torques acting on the seesaw. The large mass has its torque pointing in the negative y-direction: it points out of the screen/paper. And this torque tries to rotate the seesaw counter clockwise. On the other hand, the small mass has a torque pointing in the positive y-direction which is in line with it trying to rotate the seesaw clockwise.
Which of the two is 'strongest' determines the direction of rotation: if $MgL > mgl$ then the net torque is in the minus-y direction. That is, the torque of the larger mass is more negative than the smaller one is positive: $-MgL + mgl < 0$ and the net torque points towards us.

The quantity that goes with this, is the angular momentum. It is defined as

angular momentum

$$\vec{l} \equiv \vec{r} \times \vec{p} \tag{128}$$

Note that it is a cross product as well. Hence it is a vector itself. Further note that $\vec{r} \times \vec{p} \neq \vec{p} \times \vec{r}$. The order matters! First $\vec{r}$ then $\vec{p}$. If you do it the other way around, you unwillingly have introduced a minus sign that should not be there.
Furthermore, note that since $\vec{l} \equiv \vec{r} \times \vec{p}$, $\vec{l}$ is perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$.

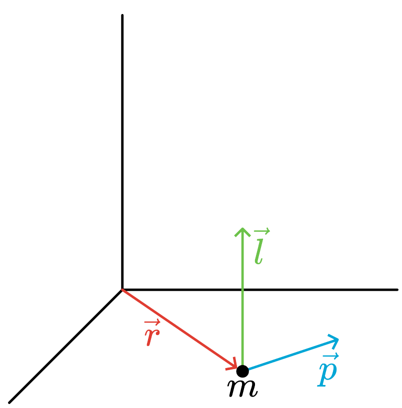


Figure 30: Angular momentum of a particle at a certain position with momentum.

Torque & Analogy to N2 Angular momentum obeys a variation of Newton's second law that ties it directly to torque.

$$\vec{l} = \vec{r} \times \vec{p} \Rightarrow \tag{129}$$

$$\frac{d\vec{l}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} = \underbrace{\underbrace{\frac{d\vec{r}}{dt}}_{=\vec{v}} \times \vec{p}}_{=0 \text{ since } \vec{v}//\vec{p}} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{\text{N2: } =\vec{F}} = \vec{r} \times \vec{F} \tag{130}$$

Thus, we find a general law for the angular momentum:

N2 for angular momentum

$$\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \tag{131}$$

Again, note that the right hand side is a cross product, so the order does matter.

With the torque denoted by $\vec{\Gamma}$, we have

$$\vec{\Gamma} \equiv \vec{r} \times \vec{F} \tag{132}$$

then we can write down an equation similar to N2 ($\dot{\vec{p}} = \vec{F}$) but now for angular motion

$$\dot{\vec{l}} = \vec{\Gamma} \tag{133}$$

where the force is replaced by the torque and the linear momentum by the angular momentum.

NB: Note that the torque and angular moment change if we choose a different origin as this changes the value of $\vec{r}$.

Intermezzo: cross product
Here is some recap for the cross product. See also the Lin. Alg. book page. A cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{c} = \vec{a} \times \vec{b} \equiv \|a\|\|b\| \sin\theta \, \hat{n} \tag{134}$$

Here θ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector normal to the plane spanned by $\vec{a}, \vec{b}$ with direction given by the *right-hand rule*.

From the definition it is clear that $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram spanned by $\vec{a}, \vec{b}$.

The cross product is bilinear, anti commutative ($\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$) and distributive over addition.

The formula is for computation in an orthonormal basis is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{135}$$

The formula can be derived from the cross product for orthonormal basis vectors, e.g. $\hat{x}, \hat{y}, \hat{z}$

$$\begin{aligned} \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y} \end{aligned} \tag{136}$$

Notice the cyclic structure of the equations.

It is a common mistake to identify angular momentum with rotational motion. That is not correct. A particle that travels in a straight line will, in general, also have a non-zero angular momentum, see Figure 33. Here we look at a free particle: there are no forces working on it. So it travels in a straight line, with constant momentum.

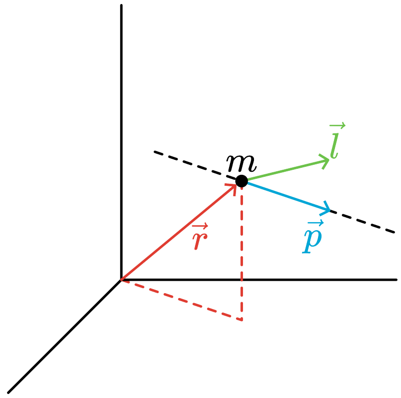


Figure 33: Angular momentum of a free particle.

However, the particle position does change over time. So: is its angular momentum constant or not?
That is easy to find out. We could take 'N2' for angular momentum:

$$\frac{d\vec{l}}{dt} = \vec{r} \times \underbrace{\vec{F}}_{\substack{=0 \\ \text{free particle}}} = 0 \Rightarrow \vec{l} = \text{const} \tag{137}$$

Clearly, the angular momentum of a free particle is constant. Moreover, the momentum of a free particle is also constant. But what about the position vector: isn't that changing over time and eventually becomes very, very long? Why does that not change $\vec{r} \times \vec{p}$?
Take a look at Figure 34. We have chosen the xy-plane such that both $\vec{r}$ and $\vec{p}$ are in it. Furthermore, we have taken it such that $\vec{p}$ is parallel to the x-axis.

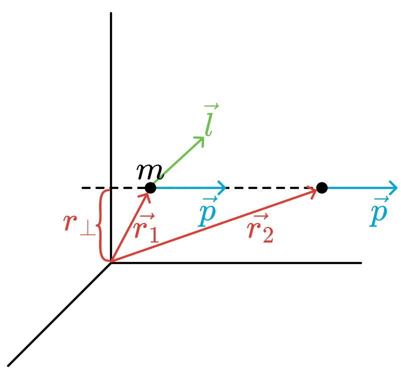


Figure 34: Angular momentum of a free particle is constant.

At some point in time, the particle is at position $\vec{r}_1$. Its angular momentum is perpendicular to the xy-plane and has magnitude $||\vec{r}_1 \times \vec{p}|| = r_\perp p$. Later in time it is at position $\vec{r}_2$. Still, its angular momentum is perpendicular to the xy-plane and has magnitude $||\vec{r}_2 \times \vec{p}|| = r_\perp p$, indeed identical to the earlier value. This shows that indeed the angular momentum of a free particle is constant.

2.4.3 Examples

Example: Throwing a basketball
As seen in class: one person throws a basketball to another via a bounce on the ground, the basketball starts to spin after hitting the ground although initially it did not.

When the ball hits the ground a friction force is acting on the ball. This force will apply a torque on the ball. The friction is directed opposite to the direction of motion. The arm $\vec{r}$ from the center of the ball to where the force is acting, is downwards. Using the right-hand rule we find that the torque is pointing in the plane of the screen, and thus the rotation is clockwise (forwards spin).

The forwards momentum of the ball is reduced by the action of the force. The upwards components is just flipped by the bounce on the ground. Therefore the outgoing ball is bouncing up at a steeper angle than it is was incoming.

Conservation of angular momentum & spinning wheel
As seen in class, we have a student sitting on a chair that can rotate (swivel chair). The student is holding a bicycle wheel in horizontal position.

Once the student starts to spin the wheel while sitting on the chair, the student will start to rotate in the opposite direction (with smaller angular velocity, later on we will see why their speeds are different). There is no external force on the student + wheel. Consequently, the total angular momentum mst stay constant. But the student exerts an angular momentum on the wheel, causing it to rotate. But at the same time, due to action = - reaction, the wheel exerts also a torque on the student. But in the opposite direction. Thus, to compensate the angular momentum pointing up (counter clockwise rotation), an angular momentum pointing down (clockwise rotation) of the same magnitude must occur, keeping the total angular momentum of student + wheel constant.

2.4.4 Exercises

Exercise 1: A point particle (mass m) is initially located at position $P = (x_0, H, 0)$. At $t = 0$, it is released from rest and falls in a force field of constant acceleration $\vec{a} = (0, -a, 0)$ that acts on the mass.

Analyze what happens to the angular momentum of m.

Exercise 2: The same question, but now the particle has an initial velocity $\vec{v} = (v_0, 0, 0)$.

Exercise 3: 73
Similar situation: can you find an example of a falling object for which the angular momentum stays constant? Ignore friction with the air.

Solution to Exercise 1: A point particle (mass
The particle falls under a force that points in the negaive y-direction. As a consequence, it will start moving vertically downwards:

$$\begin{aligned} &\text{x:} && m\frac{dv_x}{dt} = 0 \rightarrow v_x = C_1 = 0 \\ &\text{y:} && m\frac{dv_y}{dt} = -ma \rightarrow v_y = -at + C_2 = -at \end{aligned} \tag{138}$$

Thus, we find for the momentum om the particle: $\vec{p} = (0, -mat)$.

The position of m follows from:

$$
\begin{aligned}
\text{x:} \qquad & \frac{dx}{dt} = v_x = 0 \rightarrow x(t) = C_3 = x_0 \\
\text{y:} \qquad & \frac{dy}{dt} = v_y = -at \rightarrow y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2
\end{aligned}
\tag{139}
$$

We can now compute the angular momentum:

$$
\begin{aligned}
\vec{l} &= \vec{r} \times \vec{p} \\
&= \left(x_0\hat{x} + (H - \frac{1}{2}at^2)\hat{y} \right) \times (-mat\hat{y}) \\
&= -mx_0at \underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + x_0(H - \frac{1}{2}at^2) \underbrace{\hat{y} \times \hat{y}}_{=0} \\
&= -mx_0at\,\hat{z}
\end{aligned}
\tag{140}
$$

Thus, the angular momentum is pointing in the negative z-direction and increases linearly with time in magnitude.

We could have tried to find this via the variation of N2 for angular momentum. Now, we need to compute the torque on m and solve $\frac{d\vec{l}}{dt} = \vec{\Gamma}$. This goes as follows:

$$
\begin{aligned}
\vec{\Gamma} &= \vec{r} \times \vec{F} \\
&= (x\hat{x} + y\hat{y}) \times -ma\hat{y} \\
&= -ma\,x\hat{z}
\end{aligned}
\tag{141}
$$

And thus:

$$
\frac{d\vec{l}}{dt} = -ma\,x\hat{z} \tag{142}
$$

To get any further, we need information about $x(t)$. From the x-component of N2 we know (see above): $x(t) = x_0$. If we plug this in, we get:

$$
\frac{d\vec{l}}{dt} = -ma\,x_0\hat{z} \rightarrow \vec{l} = -mx_0at + C_5 = -mx_0at \tag{143}
$$

where we have used: $t = 0 \rightarrow \vec{p} = 0 \rightarrow \vec{l} = 0 \Rightarrow C_5 = 0$

Solution to Exercise 2: The same question, but now the particle has an initial velocity
We can follow the same procedure as in exercise (6.1). But now, the outcome of the x-component of N2 changes.

$$
\begin{aligned}
\text{x:} \qquad & m\frac{dv_x}{dt} = 0 \rightarrow v_x = C_1 = v_0 \\
\text{y:} \qquad & m\frac{dv_y}{dt} = -ma \rightarrow v_y = -at + C_2 = -at
\end{aligned} \tag{144}
$$

Thus, we find for the momentum om the particle: $\vec{p} = (mv_0, -mat)$.

The position of m follows from:

$$
\begin{aligned}
\text{x:} \qquad & \frac{dx}{dt} = v_x = v_0 \rightarrow x(t) = v_0 t + C_3 = x_0 + v_0 t \\
\text{y:} \qquad & \frac{dy}{dt} = v_y = -at \rightarrow y(t) = -\frac{1}{2}at^2 + C_4 = H - \frac{1}{2}at^2
\end{aligned} \tag{145}
$$

We can now compute the angular momentum:

$$
\begin{aligned}
\vec{l} &= \vec{r} \times \vec{p} \\
&= \left((x_0 + v_0 t)\hat{x} + (H - \frac{1}{2}at^2)\hat{y} \right) \times (mv_0\hat{x} - mat\hat{y}) \\
&= -m(x_0 + v_0 t)at \underbrace{\hat{x} \times \hat{y}}_{=\hat{z}} + (H - \frac{1}{2}at^2)mv_0 \underbrace{\hat{y} \times \hat{x}}_{=-\hat{z}} \\
&= -m\left(Hv_0 + x_0 at + \frac{1}{2}v_0 at^2 \right)\hat{z}
\end{aligned} \tag{146}
$$

Thus, the angular momentum still points in the negative z-direction but increases quadratically with time in magnitude.

Solution to Exercise 3: 73
We can take the situation of Note **??**, but shift our origin such that at $t = 0 \rightarrow x = 0$. Now the particle will fall along the y-axis. It has its momentum also in the y-direction and consequently $\vec{l} = \vec{r} \times \vec{p} = 0$ and stays zero!

2.4.5 Central Forces

We have looked at a specific class of forces: the conservative ones. Here we will inspect a second class, that is very useful to identify: the central forces.

A force is called a central force if:

$$
\vec{F} = |\vec{F}(\vec{r})|\hat{r} \tag{147}
$$

In words: a force is central of it points always into the direction of the origin or exactly in the opposite direction. The reason to identify this class is in the consequences it has for the angular momentum.

Suppose, a particle of mass m is subject to a central force. Then we can immediately infer that its angular momentum is a constant:

$$\text{if } \vec{F} = |\vec{F}(\vec{r})|\hat{r} \text{ then } \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = |\vec{F}(\vec{r})|\vec{r} \times \hat{r} = 0 \tag{148}$$

where we have use that $\vec{r}$ and $\hat{r}$ are always parallel so their cross-product is zero.

The above is rather trivial, but has a very important consequence: a particle that moves under the influence of a central force, moves with a constant angular momentum (vector!) and must move in a plane. It can not get out of that plane. Thus its motion is at maximum a 2-dimensional problem. We can always use a coordinate system, such that the motion of the particle is confined to only two of the three coordinates, e.g. we can choose our x, y plane such that the particle moves in it and thus always has $z(t) = 0$.

Why is this so? Why does the fact that the angular momentum vector is a constant immediately imply that the particle motion is in a plane? The argumentation goes as follows.
Imagine a particle that moves under the influence of a central force. At some point in time it will have position $\vec{r}_0$ and momentum $\vec{p}_0$. Neither of them is zero. We will assume that $\vec{r}_0$ and $\vec{p}_0$ are not parallel (in general they will not be). Thus they define a plane. Due to the cross-product $\vec{l}_0 = \vec{r}_0 \times \vec{p}_0$ is perpendicular to this plane.
A little time later, say Δt later, both position and momentum will have changed. Since the force is central, the force is also in the plane defined by the initial position and momentum. Thus the change of momentum is in that plane as well: $\vec{p}(t + \Delta t) = \vec{p}(t) + \vec{F}\Delta t$. The right hand side is completely in our plane. And thus, the new momentum is also in the plane. But that means that the velocity is also in the same plane. An thus the new position $\vec{r}(t + \Delta t) = \vec{r}(t) + \frac{\vec{p}}{m}\Delta t$ must be in the same plane as well. We can repeat this argument for the next time and thus see, that both momentum and position can not get out of the plane. This is, of course, fully in agreement with the fact that $\vec{l} = const$ for a central force.

2.4.6 Central forces: conservative or not?

We can further restrict our class of central forces:

$$\text{if } \vec{F}(\vec{r}) = f(r)\hat{r} \text{ then F is central and conservative} \tag{149}$$

In the above, $|\vec{F}(\vec{r})| = f(r)$, that is: *the magnitude of the force only depends on the distance from the origin not on the direction*. **Rephrased**: *the force is spherically symmetric*. If that is the case, the force is automatically conservative and a potential does exist.

Both the concept of central forces and potential energy play a pivotal role in understanding the motion of celestial bodies, like our earth revolving the sun. The planetary motion is an example of using the concept of central forces. It is, however, also an example in its own right. Using his new theory, Newton was able to prove that the motion of the earth around the sun is an ellipsoidal one. It helped changing the way we viewed the world from geo-centric to helio-centric.

Keppler's Laws Before we embark at the problem of the earth moving under the influence of the sun's gravity, we will go back in time a little bit.

Intermezzo: Tycho Brahe & Johannes Kepler
We find ourselves back in the Late Renaissance, that is around 1550-1600 AD. In Europe, the first signs of the scientific revolution can be found. Copernicus proposed his heliocentric view of the solar system. Galilei used his telescope to study the planets and found further evidence for the heliocentric idea. In Denmark, Tycho Brahe (1546-1601) made astronomical observations with data of unprecedented precision. He did so without the telescope as the first records of telescopes date back to around 1608 AD.

Brahe initially studied law, but developed a keen interest in astronomy. He is heavily influenced by the solar eclipse of August 21$^{\text{st}}$ in 1560. The eclipse had been predicted via the theory of celestial motion at that time. However, the prediction was off by a day. This led Brahe to the conclusion that in order to advance celestial science, many more and much better observations were needed. He devoted much of his time in achieving this. One of his best assistants was his younger sister, Sophie.

On November 11th 1572, Brahe observed a bright, new star in the constellation Cassiopeia (it consists of five bright stars forming a M or W). That was another event that made him decide to spend his days (or rather nights) gathering astronomical data. The general believe in those days was still that everything beyond the Moon was eternal, never changing. So, this new star, that all in a sudden appeared, must be closer to the earth than the Moon itself. Brahe set out to measure its daily parallax against the five stars of Cassiopeia. But he didn't observe any parallax. Consequently, the new star's position had to be farther out than the Moon and the other planets that did show daily parallax. Moreover, Brahe kept measuring for months and still found no parallax. That meant that this new star was even further out than the known planets that show no daily parallax but did so for periods of month. Brahe reached the conclusion that this new 'thing' thus could not be yet another planet, but that it was a star. Another nail to the coffin of the Aristotle view. Brahe wrote a small book about it, called De Nova Stella (published in 1573). He uses the term 'nova' for a new star. We see this back in our name for the phenomenon observed by Brahe: we call it a supernova. By now it is known that this new star, this supernova is some 8,000 light years away from us. Brahe was upset by those who denied the new findings. In his introduction of De Nova Stella he writes (given here in our modern words): "Oh, coarse characters. Oh, blind spectators of heaven". The work and the booklet made his name in Europe as a leading scientist and astronomer.

In the winter of 1577-1578 a comet, known as the "Great Comet" appeared in the skies. It was observed by many all over the globe (from the Aztecs in the America's via European researchers to the Arabic world, India all the way to Japan). Brahe made thousands of recordings, some simultaneously done in Denmark (close to Copenhagen) and Prague. That way, Brahe could establish that the comet was much beyond the Moon.

At the end of his life, Brahe moved to Prague to become the official imperial astronomer under the protection of Rudolf II, the Holy Roman Emperor. In the later part of his life, Brahe had Johannes Kepler as his assistant.

Kepler was 6 years old when the Great Comet appeared in the sky. He recorded in his writings that his mother had taken him to a high place to look at it. At the age of nine, he witnessed a lunar eclipse in which the Moon is in the Earth shadow, darkening it and turning quite red. As a child he suffered from smallpox making his vision weak and limited ability to use his hands. This made it difficult for him to make astronomical observations. It pushed him to mathematics. But there he was confronted with the Ptolemaic and the Copernican view on planetary motion. Kepler became a math professor at the Protestant Stiftsschüle in Graz. He wrote his ideas about the universe, following the thoughts of Copernicus in a book, that was read by Tycho Brahe. This brought him into contact with Brahe. In 1600 Kepler and his family moved to Prague as a consequence of political and religious oppression. He was appointed as assistant to Brahe and worked with Brahe on a new star catalogue and planetary tables. Brahe died unexpectedly on October 24th 1601. Two days later, Kepler was appointed as his successor.

Kepler worked on a heliocentric version of the universe and in the period 1609-1619 published his first two laws. With these, he changed from trying circular orbits to other closed ones, to arrive at an elliptical one for Mars. That one was in very good agreement with the Brahe data, much better than had been achieved before. Kepler realized that the other planets might also be in elliptical orbits. In comparison with Copernicus he stated: the planetary orbits are not circles with epi-circles. Instead it are ellipses. Secondly, The sun is not at the center of the orbit, but in one of the focal points of the ellipse. Thirdly, the speed of a planet is not a constant.

Kepler's work was not immediately recognized. On the contrary, Galilei completely ignored it and many criticized Kepler for introducing physics into astronomy.

Kepler has formulated three laws that describe features of the orbits of the planets around the sun.

1. The orbit of a planet is an ellipse with the Sun at one of the two focal points.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time (Law of Equal Areas).

3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

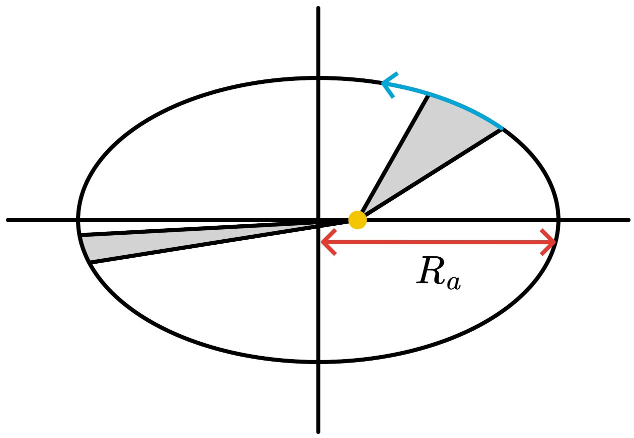


Figure 39: Kepler's 2nd Law of Equal Area.

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = \text{const.} \tag{150}$$

Warning
Ugly app that needs to be updated!

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C:\Users\fpols\AppData\Local\Temp\ipykernel_10644\2187262885.py:54: MatplotlibDeprecationWarning: Setti
  planet.set_data(px, py)
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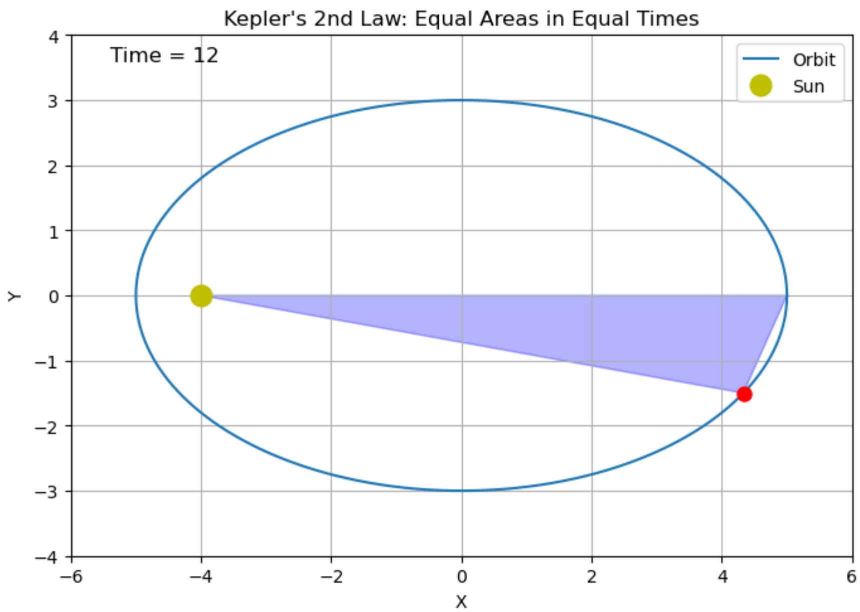


It is important to realize, that Kepler came to his laws by -what we would now call- curve fitting. That is, he was looking for a generic description of the orbits of planets that would match the Brahe data. He abandoned the Copernicus idea of circles with epi-circles with the sun in the center of the orbit. Instead he arrived at ellipses with the sun out of the center, in one of the focal points of the ellipse.

But, there was no scientific theory backing this up. It is purely 'data-fitting'. Nevertheless, it is a major step forward in the thinking about our universe and solar system. It radically changed from the idea that the universe is 'eternal', that is for ever the same and build up of circles and spheres: the mathematical objects with highest symmetry showing how perfect the creation of the universe is.

Kepler had formulated his laws by 1619 AD. It would take another 60 years before Isaac Newton showed that these laws are actually imbedded in his first principle approach: all that is needed is Newton's second law and his Gravitational Law.

2.4.7 Newton's theory and Kepler's Laws

The planets move under the influence of the gravitational force between them and the sun. We start with inspecting and classifying the force of gravity. Newton had formulated the Law of gravity: two objects of mass m_1 and m_2, respectively, exert a force on each other that is inversely proportional to the square of the distance between the two masses and is always attractive. In a mathematical equation, we can make this more precise:

$$\vec{F}_g = -G\frac{m_1 m_2}{r_{12}^2}\hat{r}_{12} \tag{151}$$

In the figure below, the situation is sketched. We have chosen the origin somewhere and denote te position of the sun and the planet by $\vec{r}_1$ and $\vec{r}_2$. Gravity works along the vector $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$. The corresponding unit vector is defined as $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$.

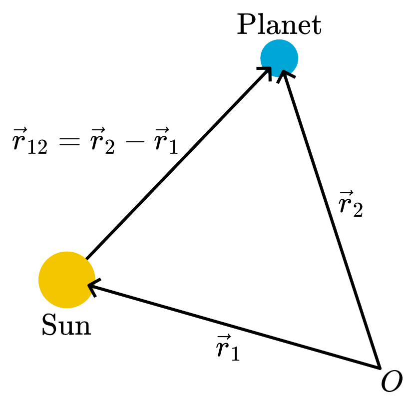


Figure 40: The sun and a planet.

Newton realized that he could make a very good approximation. Given that the mass of the sun is much bigger than that of a planet, the acceleration of the sun due to the gravitational force of the planet on the sun is much less than the acceleration of the planet due to the sun's gravity. For this, we only need Newton's 3rd law:

$$F_{g,sun\ on\ planet} = -F_{g,planet\ on\ sun} \tag{152}$$

Hence

$$m_{sun}a_{sun} = -m_{planet}a_{planet} \rightarrow a_{sun} = \frac{m_{planet}}{m_{sun}}a_{planet} \ll a_{planet} \tag{153}$$

Newton concluded, that for all practical purposes, he could treat the sun as not moving. Next, he took the origin at the position of the sun. And from here on, we can ignore the sun and pretend that the planet feels a force given by

$$\vec{F}(\vec{r}) = -G\frac{mM}{r^2}\hat{r} \tag{154}$$

with M the mass of the sun and m that of the planet. r is now the distance from the planet to the origin and $\hat{r}$ the unit vector pointing from the origin to the planet.

First observation: The force is central!

First conclusion: Then the angular momentum of the planet is conserved (is a constant during the motion of the planet) and the motion is in a plane, i.e. we deal with a 2-dimensional problem!

Second Observation: The force is of the form $\vec{F}(\vec{r}) = f(r)\hat{r}$

Second conclusion: Thus, we do know that a potential energy can be associated with it. It is a conservative force. This also implies that the mechanical energy of the planet, that is the sum of kinetic en potential energy, is a constant over time. In other words, there is no frictional force and the motion can continue forever. This seems to be inline with our observation of the universe: the time scales are so large that friction must be small.

Constant Angular Momentum: Equal Area Law The first clue towards the Kepler Laws comes from angular momentum. Let's consider the earth-sun system (ignoring all other planets in our solar system). As we saw above, gravity with the sun pinned in the origin, is a central force and thus

$$\frac{d\vec{l}}{dt} = \vec{r} \times \left(-G\frac{mM}{r^2}\frac{\vec{r}}{r}\right) = 0 \tag{155}$$

Thus, $\vec{l} =$ const. both in length and in direction. From the latter, we can infer that the motion of the earth around the sun is in a plane. Hence, we deal with a 2-dimensional problem.

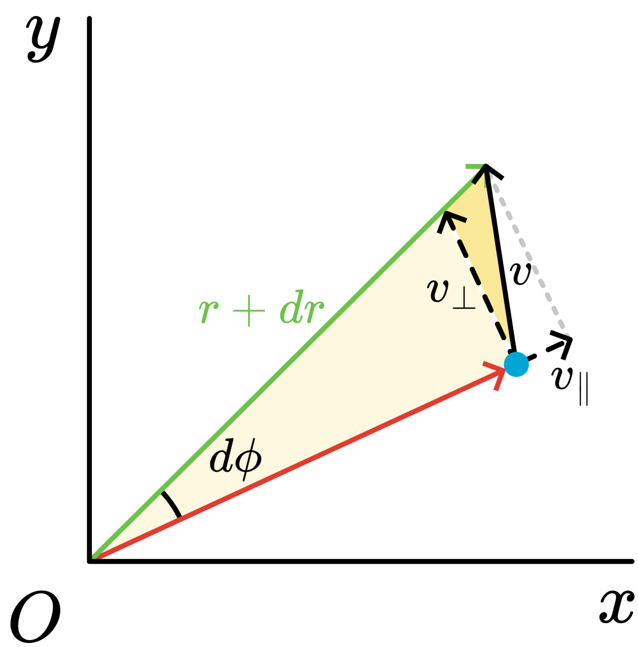


Figure 41: A free body diagram to help determine the area.

At some point in time, the earth is at location $\vec{r}$ (see red arrow in Figure 41). It has velocity $\vec{v}$, given by the black arrow. In a small time interval dt, the earth will move a little and arrive at $\vec{r} + d\vec{r}$ (the green arrow). As the time interval is very short, we can treat the velocity as a constant and thus write: $d\vec{r} = \vec{v}dt$.

Instead of concentrating on the motion of the earth, we focus on the area traced out by the earth orbit in the interval dt. That is the yellow area in the figure. This area is composed of two parts: the light yellow part and a smaller, bright yellow part. The light yellow part has an area $A_1 = \frac{1}{2}$height x base. If we make dt very small, the height is almost equal to r and the base becomes $v_\perp dt$ and thus $A_1 \approx \frac{1}{2}rv_\perp dt$. For the smaller yellow triangle we have: $A_2 = \frac{1}{2}dr$ x base $\approx \frac{1}{2}(v_{//}dt) \cdot (v_\perp dt) = \frac{1}{2}v_{//}v_\perp dt^2$.

The total area traced out by the earth orbit during dt is thus in good approximation:

$$dA = A_1 + A_2 = \frac{1}{2}\left(rv_\perp + v_{//}v_\perp dt\right) dt \tag{156}$$

We divide both sides by dt and take the limit $dt \to 0$:

$$\frac{dA}{dt} = (\frac{1}{2}rv_\perp + \frac{1}{2}v_\perp v_{/}/dt) \to \frac{1}{2}rv_\perp \tag{157}$$

In stead of $v_\perp$ we can also write $\frac{p_\perp}{m}$:

$$\frac{dA}{dt} = \frac{1}{2m}rp_\perp \tag{158}$$

But $rp_\perp$ is the magnitude of $\vec{r} \times \vec{p}$. And that is the magnitude of the angular momentum: $l = ||\vec{r} \times \vec{p}|| = rp_\perp$!!! We know l is constant, thus we have found:

$$\frac{dA}{dt} = \frac{1}{2m}rp_\perp = \frac{l}{2m} = \text{constant} \tag{159}$$

We can easily integrate this equation:

$$\frac{dA}{dt} = \frac{l}{2m} \to A(t) = \frac{l}{2m}t + C \tag{160}$$

We can set the constant C to zero at some point in time t_0 and start counting the increase of the swept area. But we immediately infer that if we check the swept area between t and $t + \Delta t$, this will be $\Delta A = \frac{l}{2m}\Delta t$ regardless of where the earth is in its orbit. In words: in equal time intervals, the earth sweeps an area that is the same for any position of the earth. We have established the Equal Area Law!

Newton's theory and Kepler's Laws - part 2 We have:

- The sun is replaced by a force field originating at the origin. This force field is a central force.
  1. Thus, the angular momentum is conserved.
  2. The orbit is in a plane: we deal with a 2-dimensional problem.
- The force is conserved: a potential exists.

Based on these, we will derive Kepler's laws only using Newtonian Mechanics. This is easiest in polar coordinates (r, ϕ). However, in this course we do not deal with these coordinates. We will thus give a coarse overview of the steps that should be taken.

The first thing we notice, is that the constant angular momentum provides a constraint on the relation between $\vec{r}$ and $\vec{p}$. This constraint can be used to rewrite the kinetic energy $E_{kin} = \frac{1}{2}mv^2$ into $E_{kin} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$.

What does this mean? The coordinate r is the distance from the sun to the earth. Its time derivative ($\dot{r} = \frac{dr}{dt} = v_r$) is the velocity of the earth away from the sun. This is called the radial component of the velocity. Figure 42 illustrates this.

It is important to realize that $\dot{r}$ tells us if we are moving such that we are getting closer to the sun or further away. But it does not tells us how we move 'around' the sun. For that we need the information of the component of the velocity perpendicular toe $\vec{r}$ (the other grey vector in the figure).

So, it seems that we are working with incomplete information. And in a sense we do. But it will turn out to be very useful to understand the physics of the earth's orbit.

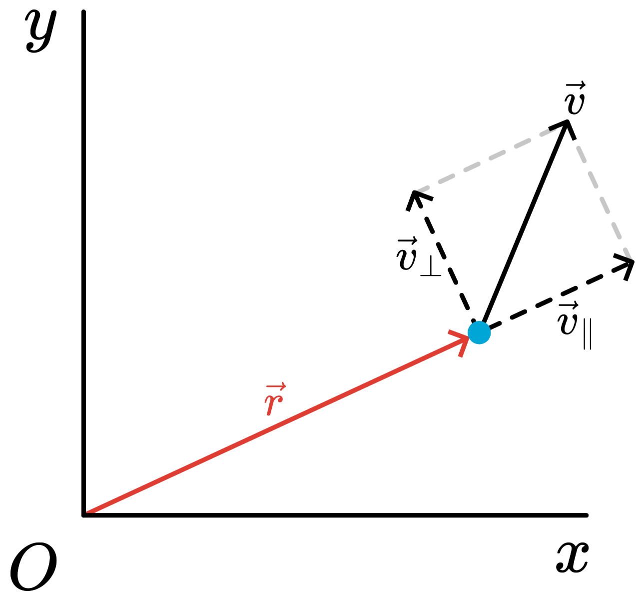


Figure 42: The coordinate r is the distance from the sun to the earth. Its time derivative ($\dot{r} = \frac{dr}{dt} = v_r$) is the velocity of the earth away from the sun.

We already saw that in this case gravity is a conservative force. The potential energy is found by solving $V(r) = -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r}$. We can plug in $\vec{F}_g = -G\frac{mM}{r^2}\hat{r}$. Thus only the radial coordinate is of importance in the inner product in the integral. Furthermore, we will use as reference boundary: ∞. Thus, the potential energy is:

$$\begin{aligned} V(r) &= -\int_{r_{ref}}^{r} \vec{F}_g \cdot d\vec{r} \\ &= GmM\int_{\infty}^{r} \frac{dr}{r^2} \\ &= -G\frac{mM}{r} \end{aligned} \tag{161}$$

Thus, energy conservation can be written as:

$$\frac{1}{2}m(v_x^2 + v_y^2) - -G\frac{mM}{r} = E_0 = const \tag{162}$$

As expected: we have an equation with two unknowns $(x(t), y(t))$. Once we solved the problem, we will thus have the coordinates of the planet's trajectory as a function of time. However, we will not do that. Reason: it is complicated and we don't need it! What we need is to find what kind of figure the trajectory is.

Our first step is to bring the number of unknowns in the energy equation down from two to one. For that, we use $E_{kin} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2}$.

$$\frac{1}{2}\dot{r}^2 + \frac{l^2}{2mr^2} - -G\frac{mM}{r} = E_0 = const \tag{163}$$

Now we have an equation with only one unknown $r(t)$.

We can interpret this in a different way: the second term, with the angular momentum, originates from kinetic energy, but now looks like a potential energy. And that is exactly what we are going to do: treat it as a potential energy.

Hence, we can first inspect the global features of our energy equation. Notice that the gravity potential energy is an increasing function of the distance from the planet to the sun (located and fixed in the origin). This shows that the underlying force attractive is. The new part, coming from angular momentum, on the other hand is a decreasing function of distance. Thus, the related force is repelling.

We can make a drawing of the energy. See Figure 43.

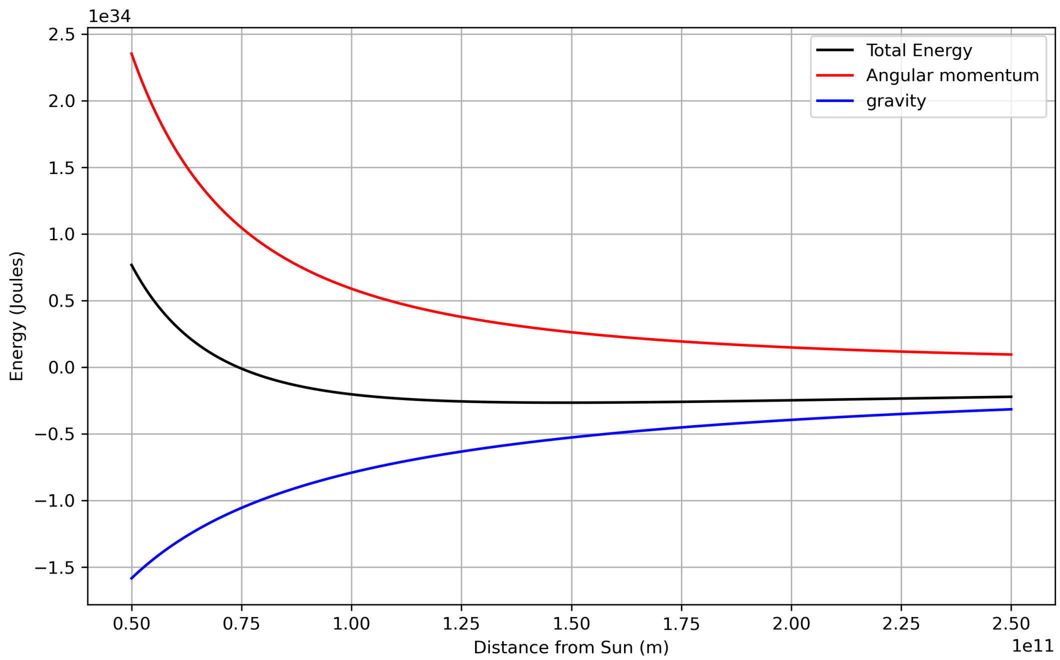


Figure 43: Energies related to our planet, with a minimum around $1.5e11m$.

The blue line is the potential energy of gravity. The red one stems from the kinetic energy associated with the angular velocity. The black line is the sum of the two, a kind of effective potential:

$$U_{eff} = \frac{l^2}{2mr^2} - -G\frac{mM}{r} \tag{164}$$

We see, that the energy can not be just any value: the kinetic energy of our quasi-one-dimensional particle ($\frac{1}{2}m\dot{r}^2$) can not be negative and the total potential energy has, according to Figure 43 a clear minimum. The total energy can not be below this minimum. On the other hand: there is no maximum.

Ellipsoidal orbits We are left with the task of showing that planets 'circle' the sun in an ellipse. From the above, we now know that this must mean that the total energy is smaller than zero: $E < 0$. We will not go over the details of the derivation, but leave that for another course.

The outcome of the analysis would be the following expression for the orbit in case of an ellipse:

$$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{165}$$

This is an ellipse with semi major and minor-axis a and b, respectively. The center of the ellipse is located at $(-ea, 0)$. Note that the sun is in the origin and that seen from the center of the ellipse, the origin is at one of the

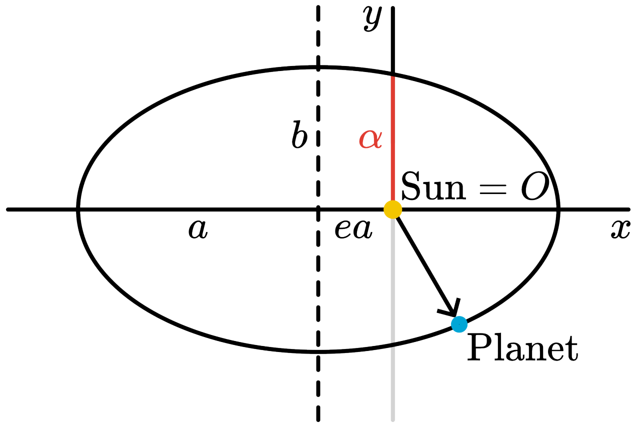


Figure 44: Ellips in Cartesian coordinates.

focal points of the ellipse. Consequently, the orbit is not symmetric as viewed from the sun. We notice this on earth: the summer and winter (when the sun is closest respectively furthest from the sun) are not symmetric, even if we take the tilted axis of the earth into account.

The half and short long axis are given by:

$$a = \frac{\alpha}{1-e^2} = \frac{GMm}{2|E|} \tag{166}$$

$$b = a\alpha = \frac{l^2}{2m|E|} \tag{167}$$

with

$$e = \sqrt{1 + \frac{2El^2}{(GMm)^2 m}} \tag{168}$$

and

$$\alpha \equiv \frac{l^2}{2GMm^2} \tag{169}$$

This type of curve is know as the conic sections. That is, they can be found by intersecting a cone with a plane. See the animation below, where a plane is at various positions and at various angles intesecting a cone.

Figure 45: Conic sections animation created by Sara van der Werf

Note that in the definition of e, the total energy of the system plays a role. This energy can be negative (see Figure 43). The minimum value of the effective potential energy is easily computed. It is $U_{eff,min} = -\frac{1}{2}\frac{(GmM)^2 m}{l^2}$ and is realized when the planet is at a distance $r = \frac{l^2}{GMm^2}$. For this case we have $e = 0$ and the planet is moving in a circle around the sun, as we already argued above.

For $0 \leq e < 1$ the orbit is an ellipse as Kepler already had postulated (for these values of e the orbit is a closed one).

For $e = 1$, the orbit is a parabola: the object will eventually move to infinity where it has exactly zero radial velocity.

Finally, for $e > 1$ the trajectory is a hyperbola with the planet again moving to infinity.

Conclusion: according to Newton's laws of mechanics, combined with the Gravitation force proposed by Newton, planets must move in ellipses around their star.

This holds for our solar system, but for any other star with planets as well. Research has shown that there are hundreds of solar systems out in the universe with thousands of planets moving around their star. See e.g. https://exoplanets.nasa.gov/

Kepler 3 We are left with proving Kepler's third law:

$$\frac{T_A^2}{R_A^3} = \frac{T_B^2}{R_B^3} = const \tag{170}$$

Now that we know the orbit, this is not difficult. We concentrate on the motion during one lapse (one 'year'). From Keppler's 1st law we know that the area a planet sweeps out of its ellipse is given by

$$A(t) = \frac{l}{2m}t + C \tag{171}$$

where C is an integration constant. Furthermore, this way of writing makes that the area swept keeps increasing: after one round along the ellipse, we simply keep counting.

However, we can easily back out what happens after exactly one round, or one 'year'. The total area swept is then, of course, the area of the ellipse itself, that is: in one year (time T) the area swept is πab. Hence we conclude:

$$A(T) = \pi ab \Rightarrow \pi ab = \frac{l}{2m}T \tag{172}$$

If we put back what we found for a and b, we get

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \tag{173}$$

Thus, indeed Kepler was right. Moreover, we note that the constant is only depending on the mass of the sun. The same law will hold for other solar systems, but with a different constant.

In Figure 46 Kepler's third law is shown for our solar system. The red data points are based on the measured 'year' of each planet and the distance to the sun. The blue line is the prediction from Newton's theory.

Haley's comet
The planets aren't the only objects that move around the sun. Several icy, rocky smaller objects are trapped in a closed orbit around the sun. These objects, comets from the Greek word for 'long-haired star', are left-overs from when our solar system was formed, some 4.6 billion years ago. There are many comets in our solar system. More than 4500 have been identified, but there are probably much more. Usually the orbit of a comet, if its is a closed one, has a high eccentricity (i.e. close to 1). Moreover, their orbital period may be very long.

One of the best visible comets is Haley's comet. However, its orbital period is about 75 years. It last appeared in the inner parts of the Solar System in 1986. So, you will have to wait until mid-2061 to see it again.

2.4.8 Speed of the planets & dark matter

Starting from Kepler 3, we can compute the orbital speed of a planet around the sun

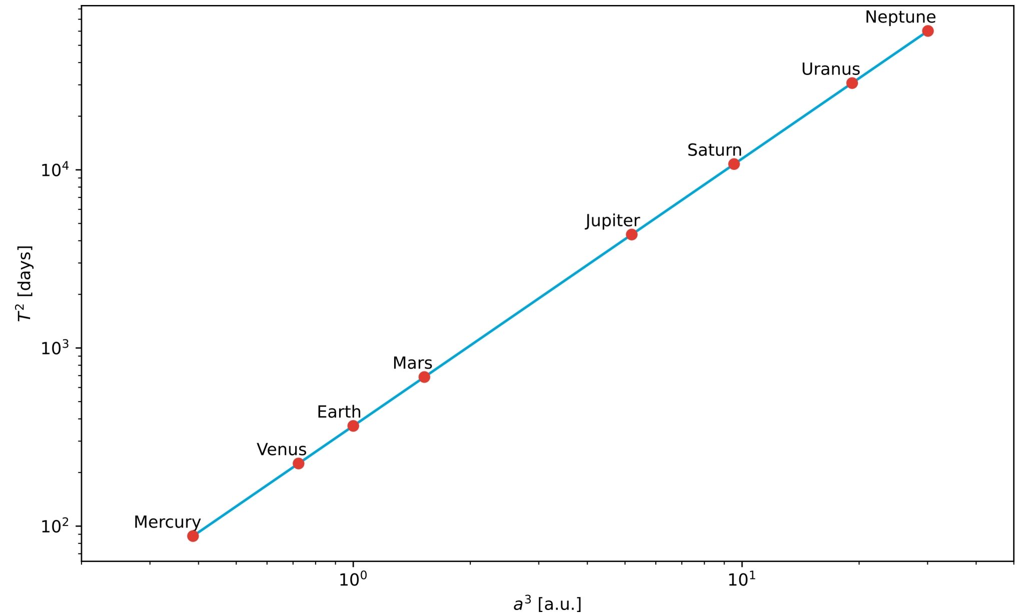


Figure 46: Kepler 3 for our solar system.

$$
\begin{aligned}
T^2 &= \frac{4\pi^2}{GM}a^3 \\
\omega^2 &= \frac{GM}{a^3}, \quad T = \frac{2\pi}{\omega}, \omega = \frac{v}{r}, a \approx r \\
\Rightarrow v &= \sqrt{\frac{GM}{r}}
\end{aligned}
\tag{174}
$$

Indeed if we measure the speed of the planets in the solar system this prediction holds, the velocity drops with the distance from the sun as $\propto r^{-1/2}$ (see figure). As M we use the mass of the sun here.

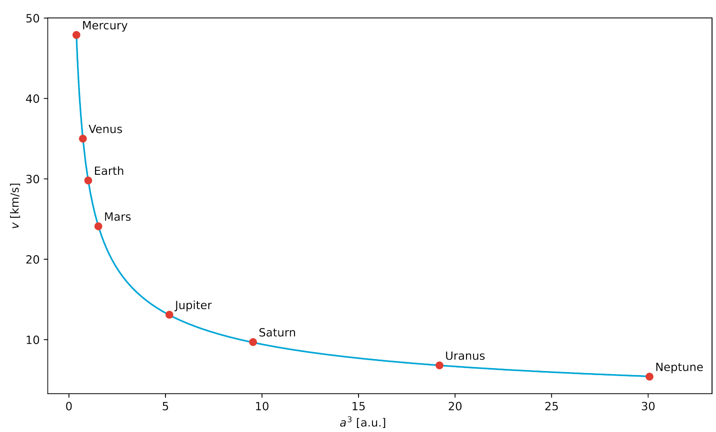


Figure 48: From LibreTexts Physics, licensed under CC BY-NC-SA 4.0.

The distance is measured in Astronomical Units [AU], the distance from the earth to the sun (about 8.3 light minutes). Note that the earth is moving with an unbelievable 30 km/s, that is 10^5 km/h! Do you notice any of that? We will use this motion later with the Michelson-Morley experiment.

If we plot the same speed versus distance curve not for the planets in our solar system, but for stars orbiting the center of our galaxy, the milky way, then the picture looks very different. The far away stars orbit at a much higher speed than expected and the form of the found curve does not match $\propto r^{-1/2}$.

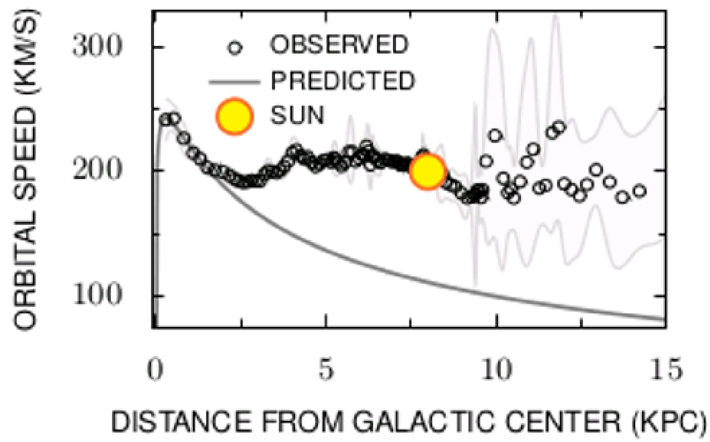


Figure 49: From Wikimedia Commons, licensed under CC-SA 3.0.

This mismatch is not understood to this day! The mass M here is calculated from the visible stars and the supermassive black holes at the center of the galaxy. But even if the mass is calculated wrongly, the shape of the dependency does not match. It turns out, this mismatch is observed in all galaxies! Apparently the law of gravity does not hold for large distances *or* there must be extra mass that increases the speed that we do not see. This mismatch has lead to the postulation of *dark matter* and an *alternative formulation* for the laws of gravity. This is the most disturbing problem in physics today; second is probably the interpretation of measurement in quantum mechanics (collapse of the wave function/Kopenhagen interpretation of Quantum Mechanics; multiverse theories).

The majority of all matter in the universe is believed to be *dark*. And we have no clue what it could be! Most scientist even think it must be non-baryonic, that is, other stuff than our well-known protons or neutrons. It remains most confusing.

The usual distance unit for distances in astronomy outside the solar system is not light years (ly), but parsec [pc], or kpc, or Mpc. One parsec is about 3.3 ly (or 10^{13} km). Note: stars visible to the eye are typically not more than a few hundred parsec away. The Milky Way is perfectly visible to the naked eye as a band/stripe of "milk" sprayed over the night sky. But you cannot see it anywhere close to Delft, there is much too much light from cities and greenhouses. Go to Scandinavia in the winter ("wintergatan") or any place remote where there are few people. The reason you see a "band" in the night sky, is that the Milky Way is a spiral galaxy, sort of pancake shaped, and you see the band in the direction of the pancake.

2.4.9 Examples, exercises and solutions

2.5 Conservation Laws / Galilean Transformation

In the previous chapters, we have seen that from Newton's three laws, we can obtain conservation laws. That means, under certain conditions (depending on the law), a specific quantity can not change.

These conservation equations are very important in physics. They tell us that no matter what happens, certain quantities will be present in the same amount: they are *conserved*.

Conservation of energy follows the concept of work and potential energy. Conservation of momentum is a direct consequence of N2 and N3, as we will see below. And finally, under certain conditions, angular momentum is also conserved. In this chapter we will summarize them. The reason is: their importance in physics. These laws are very general and in dealing with physics questions they give guidance and very strict rules that have to be obeyed. They form the foundation of physics that can not be violated. They provide strong guidance and point at possible directions to look for when analyzing problems in physics.

2.5.1 Conservation of Momentum

Consider two particles that mutually interact, that is they exert a force on each other. For each particle we can write down N2:

$$\left.\begin{array}{l} \frac{d\vec{p}_1}{dt} = \vec{F}_{21} \\ \frac{d\vec{p}_2}{dt} = \vec{F}_{12} = -\vec{F}_{21} \end{array}\right\} \rightarrow \frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0 \Rightarrow \vec{p}_1 + \vec{p}_2 = const \tag{175}$$

The total (linear) momentum is conserved if only internal forces are present; "action-reaction pairs" always cancel out.
This law has no exception: it must be obeyed at all times. The total momentum is constant, momentum lost by one must be gained by others.

2.5.2 Conservation of Energy

As we have seen when deriving the concept of potential energy, for a system with conservative forces the total amount of kinetic and potential energy of the system is constant. We can formulate that in a short way as:

$$\sum E_{kin} + \sum V = const \tag{176}$$

Again: energy can be redistributed but it can not disappear nor be formed out of nothing.

If non-conservative forces are present, the right hand side of the equation should be replaced by the work done by these forces.

$$\sum E_{kin} + \sum V = \sum W \tag{177}$$

In many cases this will lead to heat, a central quantity in thermodynamics and another form of energy. The "loss" of energy goes always to heat. With this 'generalization' we have a second law that must always hold. Energy can not be created nor destroyed. All it can do is change its appearance or move from one object to another.

2.5.3 Conservation of Angular Momentum

Also angular momentum can be conserved. According to its governing law $\frac{d\vec{l}}{dt} = \vec{r} \times \vec{F}$ it might seem that we can for two interacting particles again invoke N3 "action = -reaction" and the terms with the forces will cancel out. But we need to be a bit more careful, as outer products are involved which are bilinear. So, let's look at the derivation of "conservation of angular momentum" for two interacting particles:

$$\left.\begin{array}{l} \frac{d\vec{l}_1}{dt} = \vec{r}_1 \times \vec{F}_{21} \\ \frac{d\vec{l}_2}{dt} = \vec{r}_2 \times \vec{F}_{12} = -\vec{r}_2 \times \vec{F}_{21} \end{array}\right\} \rightarrow \frac{d}{dt}\left(\vec{l}_1 + \vec{l}_2\right) = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{21} \tag{178}$$

As we see, this is only zero if the vector $\vec{r}_1 - \vec{r}_2$ is parallel to the interaction forces (or zero). We called this a *central force*. Luckily, in many cases the interaction force works over the line connecting the two particles (e.g. gravity). In those cases, the angular momentum is conserved. Mathematically we can write this as:

$$\text{if } \vec{F}_{21} \,||\, (\vec{r}_1 - \vec{r}_2) \;\Rightarrow\; \vec{l}_1 + \vec{l}_2 = const \tag{179}$$

2.5.4 Conservation of Mass

Within the world of Classical Mechanics, mass is also a conserved quantity. Whatever you do, what ever the process the total mass in the system stays the same. We can not create nor destroy mass. From more modern physics we know that this is not true. On the one hand we can destroy mass. For instance, when an electron and a positron collide, they can annihilate each other resulting in two photons, i.e. 'light particles' that do not have mass. Similarly, we can create mass out of light. This is the inverse of the annihilation: pair production. If we have a photon of at least 1.022 MeV ($= 1.66\ 10^{-13}$J), then -under the right conditions- an electron-positron pair can be created.

Moreover, Albert Einstein showed that mass and energy are equivalent - expressed via his famous equation $E = mc^2$. His theory of Relativity showed us that in collisions at extreme velocities mass is not conserved: it can both be created or disappear. Rephrased: it is actually part of the energy conservation, mass is in that context just a form of energy.

Emmy Noether, symmetries and conservation laws

We discussed the conservation laws as consequences of Newton's Laws. That in itself is ok. However, there is a deeper understanding of nature that leads to these conservation laws. And from the conservation laws we can go to Newton's Laws, thus 'reversing the derivations' and starting from this new, different way of looking at nature.

What is it and how do we know? To answer this question we have to resort to Emmy Noether, a German mathematician. Noether made top contributions to abstract algebra. She proved, what we now call, Noether's first and second theorems, which are fundamental in mathematical physics. Noether is often named as one of the best if not the best female mathematicians ever lived. Her work on differential invariants in the calculus of variations has been called "one of the most important mathematical theorems ever proved in guiding the development of modern physics".

Noether shows, that if a dynamic system is invariant under a certain transformation, that is it has a symmetry, then there is a corresponding quantity that is conserved. Ok, pretty abstract. What does it mean, any examples? Yes! Here is one.

In physics we believe that it does not matter if we do an experiment now and repeated it exactly under the same conditions an hour later, the outcome will be the same. Or rephrased: if we translate it in time, the outcome is the same; the laws of physics are invariant. This is in mathematical terms a symmetry, a symmetry with respect to time. Noether's theorem then shows, that there is a conserved quantity and this quantity is energy. Hence, based on the idea that time itself has no effect on physical laws, we immediately arrive at conservation of energy.

Second example: we also believe that place or position in the universe doesn't matter. The physicals laws are not only always the same (time invariance), they are also the same everywhere (space invariance). From this symmetry, via Noether's work, we immediately get that momentum is a conserved quantity. Thus, these two invariances or symmetries -time and space - provide us directly with conservation of energy and momentum and from that we could easily derive Newton's second and third law. Much of modern physics is now build on the ideas put forward by Emmy Noether. That goes from quantum mechanics to quarks to string theory.

2.5.5 Galilean Transformation

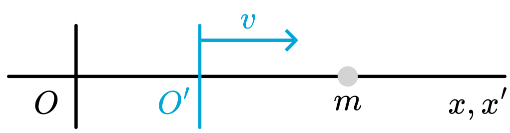
There is one important element of Classical Mechanics that we have to add: for which type of observer do Newton's Laws hold? The original thought was: for inertial observers. These are observers that are at rest with respect to an inertial frame of reference.

But this merely shifts the question to: what is an inertial frame of reference. One possible answer is: an inertial frame of reference is a frame in which Newton's Laws hold. That is: a particle on which, according to an observer in such a frame, no net force is acting will keep moving at a constant velocity.

All inertial frames of reference move at a constant velocity with respect to each other. They can not accelerated. To picture what it means, an inertial frame of reference or an inertial observer, we sometimes use the idea that such a frame or observer moves at a constant velocity with respect to the 'fixed' stars. And indeed, for a long time people believed that the stars were fixed in space. But from more modern times we do know, that this is not the case: stars are not fixed in space nor do they move at a constant velocity.

Later in the study of Classical Mechanics, we will see, that it is possible to do without the restriction that Newton's Law strictly speaking only hold in inertial frames. But for now, we will stick to inertial frames and look at the 'communication' between two observers in two different inertial frames.

An important requirement of any physical law is that it looks the same for all inertial observers. That doesn't mean that the outcome of using such a law is the same. As a trivial example, take two inertial observers S and S'. According to S, S' moves at a constant velocity, V, in the x-direction. S' observes a mass m that is not moving in the frame of reference of S'. For simplicity, we will assume that each observer is in its own origin.



S' rightfully concludes, based on Newton's 1st law that no force is acting on m. S agrees, but doesn't conclude that m is at rest. This is trival: both observers can use Netwon's second law which for this case states that $\frac{d\vec{p}}{dt} = 0 \rightarrow \vec{p} = const \rightarrow \vec{v} = const$. But the constant is not the same in both frames.

To make the above loose statements more precise. We have two coordinates systems CS and CS'. The transformation between both is given by a translation of the origin of S' with respect to that of S.

Communication Protocol We need to have a recipe, a protocol that translates information from S' to S and vice versa.

This protocol is called the *Galilean Transformation* between two inertial frames, S and S'.

According to observer S, S' is moving at a constant velocity V. Both observers have chosen their coordinate system such that x and x' are parallel. Moreover, at $t = t' = 0$, the origins O and O' coincide. The picture below illustrates this.

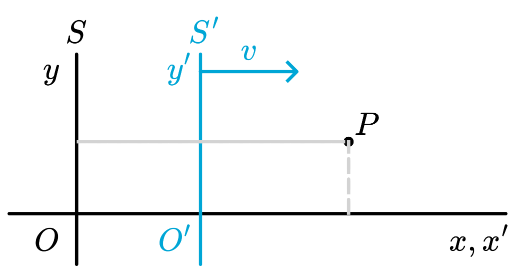


Figure 51: Two inertial observers S and S' and their coordinate systems.

Consider for simplicity a 2D point P with coordinates (x', y') and time t' for S'. What are the coordinates according to S? First of all: in classical mechanics, there is only one time, that is: $t = t'$. Until the days of Einstein this seemed self evident; we now know that nature is more complex.

For the spatial coordinates, we see immediately: $y = y'$. And for the x-coordinate S can do the following. To go to the x-coordinate of P, first S goes to the origin O' of S'. O' is a distance Vt from O. Thus, the distance to P along the x-axis is $Vt + x'$. If we sum the above up, we can formulate the relation between the coordinate system of the two observers. This transformation is the **Galilean Transformation**, or **GT** for short.

Galilean Transformation

$$\begin{aligned} x' &= x - Vt \\ y' &= y \\ t' &= t \end{aligned} \tag{180}$$

Velocity is relative; acceleration is absolute A direct consequence of the Galilean Transformation is that velocity is observer-dependent (not surprising), but for observers in inertial frames, observed velocities differ by a constant velocity vector.

In what follows we will derive the relations between velocity and acceleration as observed by S and S'. Note that we need to be precise in our notation: S' denotes quantities with a prime ('), but S does not. This is obvious for the coordinates as S uses x whereas S' will write x'. It is, however, also wise to use primes on the velocity: S will denote the x-component as: $v_x = \frac{dx}{dt}$. So, S' will note denote velocity by v, but by v'. Hence S' will write $v'_{x'} = \frac{dx'}{dt'}$. Now, obviously, $t' = t$ so we could drop the prime on time, but it is handy to do that in the second step.

First we look at velocity.

$$\begin{aligned} v'_{x'} &\equiv \frac{dx'}{dt'} \Rightarrow v'_{x'} = \frac{d(x - Vt)}{dt} = v_x - V \\ v'_{y'} &\equiv \frac{dy'}{dt'} \Rightarrow v'_{y'} = \frac{dy}{dt} = v_y \end{aligned} \tag{181}$$

Thus indeed velocity is 'relative': different observers find different values, but they do have a simple protocol to convert information from the other colleague to their own frame of reference.

Secondly, we look at acceleration.

$$\begin{aligned} a'_{x'} &\equiv \frac{dv'_{x'}}{dt'} \Rightarrow a'_{x'} = \frac{d(v_x - V)}{dt} = a_x \\ a'_{y'} &\equiv \frac{dv'_{y'}}{dt'} \Rightarrow a'_{y'} = \frac{dv_y}{dt} = a_y \end{aligned} \tag{182}$$

So, we conclude: acceleration is the same for both observers.

Consequently, N2 holds in both inertial systems if we postulate that $m' = m$. In other words: mass is an object property that does not depend on the observer.

Thus, two observers, each with its own inertial frame of reference, will both *see the same forces*: $F = ma = m'a' = F'$.

This finding is stated as: Newton's second law is *invariant* under Galilean Transformation. Invariant means that the form of the equation does not change if you apply the Galilean coordinate transformation. Later we will expand this to Lorentz invariant transformation in the context of special relativity. The concepts of invariance is very important in physics as hereby we can formulate laws that are the same for everybody (loosely speaking).

2.5.6 Exercises, examples & solutions

Worked Example In class you have seen the *Superballs* example. You can watch it here again. The explanation is not really correct there, we will reason the observations here.

If you let a smaller and a larger ball drop together, stacked on top of each other, the smaller ball will bounce back much stronger (higher) than if you let the small ball fall without stacking it on the lager ball. How can that happen?

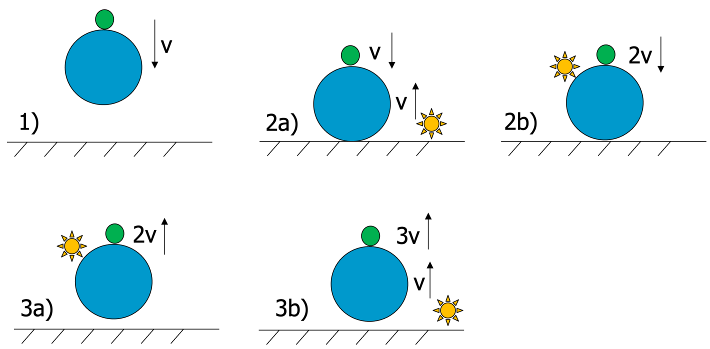


Figure 52: Bouncing balls.

To explain this we use the Galilean Transformation (GT).

- Situation 1). Both balls are falling with velocity $\vec{v}$ towards the ground.
- Situation 2a). The larger ball just hit the ground. As the mass of the ground is much larger than that of the large ball, it is (elastically) reflected, i.e. the direction of the velocity is reversed but the magnitude stays the same. The small ball is still moving downwards with $\vec{v}$.
- Situation 2b). We apply a GT of the observer (yellow star) from the ground to an observer moving with the larger ball. The observer moving with the larger ball sees the smaller ball moving with $2\vec{v}$ towards it.
- Situation 3a). The smaller ball hits the larger ball and is reflected due to its smaller mass. In the frame of the observer on the larger ball, the smaller ball now moves with $2\vec{v}$ away from it.
- Situation 3b). We apply a GT of the observer (yellow star) from the larger ball back to an observer on the ground. For the observer on the ground the larger ball has velocity $\vec{v}$ upwards from 2a), therefore the smaller ball has velocity $3\vec{v}$ upwards.

The smaller ball has now velocity $3\vec{v}$ instead of $\vec{v}$ if you drop it without the larger ball. NB: If you would use three balls instead of two, the third ball would have a velocity of $7\vec{v}$ using the same reasoning as above.

Figure 53: Bouncing of three (super)balls.

How much higher does the smaller ball fly with velocity $3\vec{v}$ compared to $\vec{v}$?

Answer We equate the kinetic energy when the ball is just reflected with the potential energy when the ball reached it maximal height before falling back.

$$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} \tag{183}$$

Therefore the ball with $3v$ flies 9 times higher.

We equate the kinetic energy when the ball is just reflected with the potential energy when the ball reached it maximal height before falling back.

$$\frac{1}{2}mv^2 = mgh \Rightarrow h = \frac{v^2}{2g} \tag{184}$$

Therefore the ball with $3v$ flies 9 times higher.

What is very fishy about this whole outcome?
In situation 1) the kinetic energy is $\frac{1}{2}m_s v^2 + \frac{1}{2}m_\ell v^2$, but in situation 3b) it is $\frac{1}{2}m_s(3v)^2 + \frac{1}{2}m_\ell v^2$ while the potential energy is zero in both cases. This clearly does not add up! But energy must be conserved under all circumstances!

The conclusion is, that we did make an approximation and did not solve the energy and momentum conservation equations for elastic collisions. Even for the case $M \gg m$ there is some momentum transfer. If you solve for the velocity of m after the collision with M, you obtain

$$v' = \frac{\frac{m}{M} - 1}{\frac{m}{M} + 1} v \tag{185}$$

For $M \gg m$ you indeed see $v' = -v$. Thus the smaller ball will have a smaller velocity than reasoned above *and* the larger ball with also have a smaller velocity (in the experiment you can clearly notice that it does not fly as high as when it drops without the small ball on top). In real life, the balls also deform which makes the collision inelastic.

8.1
Consider yourself biking at a constant velocity on an unlikely day with zero wind. Still, you experience a frictional force from the air, with the following observation: the faster you bike, the larger this force. An experimentalist is trying to measure the friction force of the air and relate it to your velocity. She finds that, by and large, these forces turn out to scale with the square of your velocity v_b

$$F_f \propto v_b^2 \tag{186}$$

Understanding the Galilean transformation, you immediately see that this can't be correct. In your frame of reference, your velocity is zero. And thus, the friction force would be zero. But that cannot be true: both observers should see the same forces. What you see is that the air is blowing at a speed $v_{air} - v_b$ past you. And indeed, the faster you bike, according to the experimentalist, the faster you see the air moving past you: velocity is relative.

You quickly realize that a proper description of the air friction must depend on the relative velocity between you and the air. *Relative* velocities are invariant under Galilean transformation:

$$F_f \propto (v_b - v_{air})^2 \tag{187}$$

8.2
Riding a bike while it rains. You have done this 100s of times. Your front gets soaked, while the backside of your coat stays dry. Or if you have a passenger on your carrier he/she will not get wet, while you take all the water. From a GT to the reference frame of the biker it is obvious why this is the case. The rain is not falling straight from the sky, but at an angle towards him.

NB: For Dutch bikers you have had this experiences with head wind and rain all your life.

Examples

Demo A ball is bouncing at a wall. The mass of the wall is much greater than that of the ball. So, acceleration of the wall or changes in momentum of the wall can be ignored.

On the left side, we see this from the perspective of an observer, S, standing next to the wall. The right side shows what observer S', who is traveling with the ball as it moves towards the wall, sees. Notice, that both S and S' are inertial observers. That is, they keep their velocity and are no part of the collision. What would Galilei say?

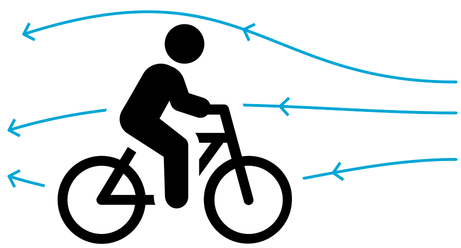


Figure 54: Air resistance on cyclist.

Exercise 1: A train is passing a station at a constant velocity V. At the platform, an observer S sees that in the middle of the train (train length $2L$), at $t = 0$ an object is released with a constant velocity u. The object moves towards the back of the train and, at some point in time, will hit the back.

Inside the train, observer S' sees the same phenomenon. Show that both find the same time for the object hitting the back of the train.

Exercise 2: A point particle of mass m is sitting on a horizontal frictionless table. Gravity is acting in the vertical downward direction.
According to your observation, m has zero velocity. But you see the table moving at a velocity $-v$ in the negative x-direction. The table doesn't stay flat, but has a bump of height H. What will happen to m?

Exercise 3: Finally, it is winter. And this time, there is lots of fresh snow! You get engaged in a great snowball fight. Your opponent has run out of 'ammunition' and runs away. She is at a distance $L = 2m$ when she starts running at a speed of 5m/s. You throw your last snowball at her at a speed of 10m/s.
Determine when and where the snowball hits her. Do that three times:

- Your perspective;
- Your opponent's perspective;
- The snowballs perspective.

Next, use the Galilei transformation and show that you could have used your perspective and GT to find the data for the other two perspectives.

Exercise

Solution to Exercise 1: A train is passing a station at a constant velocity
First we make a new sketch, now showing the two observers S and S' and their axis. We have made the velocity of the object red, the color of S. And we have given the coordinates of the front and back of the train in green as these are specified according to S'. We do this, as it is is crucial to realize that we have 'mixed' information.

The velocity of the object is u according to S. The observer in the train, S', sees a different velocity.
The observer in the train will denote the position of the front of the train by $x'_f = L$ and of the back $x'_b = -L$. Both are, according to S', fixed values. But S will se that differently.

According to S', the object moves with velocity $u' = u - V$. Note that this is a negative value, otherwise the object will not hit the back of the train.

S' will describe the trajectory of the object by: $x'(t) = x'_0 + u't$ with $x'_0 = 0$. Thus, the object will hit the back of the train at:

$$x'(T') = -L \rightarrow u'T' = -L \rightarrow T' = \frac{L}{-u'} \tag{188}$$

What does S observe? It will write for the trajectory of the object $x_o(t) = ut$ (where we used that the object was released in the middle of the train at $t = 0$ and both observers chose that as their origin).
According to S also the back of the train is moving. It follows a trajectory $x_b = -L + Vt$, since at $t = 0$ the back of the train was at position $x = -L$ according to S. The two will collide when

$$x_o(T) = x_b(T) \rightarrow uT = -L + VT \rightarrow T = \frac{L}{V - u} \tag{189}$$

Hence we have T and T' as times of collision. But we already found $u' = u - V$. If we substitute this in T' we get

$$T' = \frac{L}{-u'} = \frac{L}{V - u} = T \tag{190}$$

Thus, indeed both observers see the collision at the same moment.

Sneak Preview: much to our surprise, when we enter the world of Special Relativity, this will no longer be the case!

Solution to Exercise 2: A point particle of mass
The particle will 'collide' with the bump. This might cause the particle to start moving to the left. How to analyse this situation?

Perhaps it is easier when we view this from the point of view of an observer moving with the table.

Now we have a situation of a particle moving over a friction less table with velocity v. If we use conservation of energy, we can write down:

$$\frac{1}{2}mu^2 + mgh = E_0 = \frac{1}{2}mv^2 \tag{191}$$

where we have taken h as the height above the table and denote the velocity of m at some point by u. The initial height is zero and the initial velocity is v.

So, if the initial velocity is such that $\frac{1}{2}mv^2 > mgH$, the particle will go over the bump and come back to height $h = 0$. It will thus pass the bump and then continue moving with velocity v. For the original observer this means: the bump will pass the particle and after passing the particle is again laying still (but not at the same position!).

If, on the other hand v is such that $\frac{1}{2}mv^2 < mgH$, the particle will not reach the top of the bump: it has insufficient kinetic energy. In stead it will stop at some height $h^* = \frac{v^2}{2g}$ and then fall of the bump again. It will continue with velocity $-v$ at the flat part of the table. To the original observer this means that m first climbs the bump and returns to get a velocity $-2v$ on the flat part of the table.

The final possibility is $\frac{1}{2}mv^2 = mgH$. In that case the particle will exactly reach the top of the bump and stop there.

N.B. We have assumed that the bump is not too steep, because in such a case the particle will have a real collision with the bump. Think, for instance, of the bump as a sudden step. Then no matter how fast the particle is moving, it will not end up on the step, but bounce back.

Solution to Exercise 3: Finally, it is winter. And this time, there is lots of fresh snow! You get engaged in a great snowball fight. Your opponent has run out of 'ammunition' and runs away. She is at a distance
First, a sketch:

It is a 1-dimensional problem, so an x-axis will do. We denote the velocity of your opponent (as seen by you) by v_o and of the snowball v_s. The inertial system of you is S and you are sitting in the origin $\mathcal{O}$. Similarly, you opponents inertial system is S' with origin $\mathcal{O}'$ and finally the snowball has inertial system S" and the snowball sits in the origin $\mathcal{O}$".

1. Your perspective

$$x_s(t) = v_s t \tag{192}$$

$$x_o(t) = L + v_o t \tag{193}$$

require: $x_s(t^*) = x_o(t^*)$

$$\rightarrow t^* = \frac{L}{v_s - v_o} = 0.4s \rightarrow x^* = v_s t^* = 4m \tag{194}$$

2. Your opponent's perspective

$$v_s' = v_s - v_0 = 5m/s \tag{195}$$

require: $x_s'(t'^*) = 0$ since S' is in $x' = 0$. Thus

$$x_s'(t'^*) = -L + v_s' t'^* = 0 \rightarrow t'^* = \frac{L}{v_s'} = 0.4 \tag{196}$$

Same time of course. Position: your opponent concludes she is not moving and this she is hit at $x' = 0$.

3. The snowballs perspective.
According to the snowball $v"_o = v_o - v_s = -5m/s$. Thus,

$$x"_o = L + v"_o t \tag{197}$$

require: $x"_o(t"^*) = 0$

$$x"_o(t"^*) = L + v"_s t"^* \rightarrow t"^* = -\frac{L}{v"_o} = 0.4s \tag{198}$$

And, again the snowball will conclude that it all happened in its origin.

Galilei Transformation

We now have three different time/place coordinates for the event 'snowball hits opponent'.

$$\begin{aligned} S &: (x_h, t_h) = (4m, 0.4s) \\ S' &: (x'_h, t'_h) = (0m, 0.4s) \\ S" &: (x"_h, t"_h) = (0m, 0.4s) \end{aligned} \tag{199}$$

We could have found this directly from a GT.

a) from S to S': we need to take into account that at $t = 0$ the origins do not coincide. Instead $\mathcal{O}'$ is shifted over a distance L w.r.t. $\mathcal{O}$

$$\begin{aligned} x' &= x - L - v_o t \\ t' &= t \end{aligned} \tag{200}$$

Thus: $x'h = x_h - L - v_o t_h = 0$ and we get indeed $(x'_h, t'_h) = (0m, 0.4s)$

b) We do a similar exercise for S to $S"$:

$$\begin{aligned} x" &= x - v_s t \\ t" &= t \end{aligned} \tag{201}$$

Thus: $x"_h = x_h - v_s t_h = 0$ and we get $(x"_h, t"_h) = (0m, 0.4s)$

Answers

2.6 Oscillations

2.6.1 Periodic Motion

There are many, many examples of periodic systems. We see them in physics, like the orbit of planets around their star. We find them in biology (like the predator-prey systems), in chemistry (oscillating reactions like the Belousov-Zhabotinsky reaction), and in economics (like demand-supply fluctuations). They show up in daily life: the day-night rhythm, the tides, children on a swing, your heart-beat. Periodic motions are by definition motions that repeat themselves after a fixed period of time, usually called 'the period'.

A specific class of periodic motion is known as oscillatory motion, or simply oscillations. All oscillations are periodic, but not all periodic motions are oscillations. An oscillation involves movement back and forth around an equilibrium position. It is typically caused by a restoring force: a force that acts to return the system to equilibrium (in case of the mass spring system: $\vec{F} = -k\vec{u}$). However, due to inertia, the system overshoots this position. The restoring force then reverses direction, pushing the system back again, leading to continued oscillation.

A few simple examples will illustrate the above.

The merry-go-round The merry-go-round (Figure 57) is a periodic motion, but not an oscillation. The seats go round in a circular, periodic motion but there is no back & forth. This is in contrast to a swing. That is also a periodic motion, but it has the back and forth as well as a restoring force, which in this case is gravity.

Figure 57: Spinning carousel. By Oxana Mayer, from Wikimedia Commons, licensed under CC BY-SA 2.0.

Rabbits and Foxes As an example of a dynamic system that is periodic, we will take a look at the so-called predator-prey systems. These are well-known in biology and provide an interesting case. The idea is simple: the populations of rabbits growth as they multiply quickly. The idea in the prey-predator model is that growth rate is proportional to the population itself. For the rabbits that means that the derivative of the population of rabbits (with respect to time) is positive. If there are no foxes, the rabbit population will grow exponentially. Of course, in the real world that doesn't happen as sooner or later, the rabbits will ran out of food, resulting in starvation. However, we will assume here, that food is not limiting: but the number of foxes is. They stop the rabbit population from unbounded increasing. The more rabbits there are, the easier the foxes find food and the more foxes will survive childhood. A simple model reads as follows:

$$\begin{aligned} \frac{dr}{dt} &= \lambda_r r - \mu_r r \cdot f \\ \frac{df}{dt} &= -\lambda_f f + \mu_f r \cdot f \end{aligned} \tag{202}$$

here r and f represent the rabbit and fox population, resp. λ_r is the growth rate of the rabbits: the more rabbits, the larger the offspring. The higher λ_r the more babies per rabbit. μ_r, on the other hand, represents the effectiveness of the hunting foxes: the larger this value the more rabbits they kill. Of course: more rabbits, but also more foxes also means more kills. Similar arguments apply to λ_f and μ_f. Note that the term with λ_f carries a negative sign: the net increase of the fox population is negative if there is insufficient food, that is, by itself more foxes die then that are born if there is no food.

This is clearly a coupled and dynamic system. It is non-linear due to the product $r \cdot f$, making it much more difficult to solve analytically then linear versions. In literature, this kind of system is known as Lotka-Volterra or prey-predator models. Below is a plot of the numerical solution of the rabbit and fox population (for $(\lambda_r, \mu_r, \lambda_f, mu_f) = (0.2, 0.03, 0.1, 0.01)$ and initial conditions $(r_0,\ f_0)$ = (80, 2)).

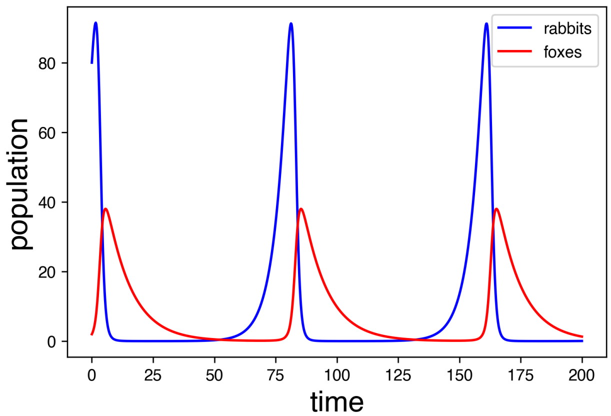


Figure 58: Periodic time evolution of the population of rabbits and foxes.

The solution is periodic. This can be illustrated better by plotting f against r. The animation below shows this (this kind of plot is called a phase plot).

Figure 59: Phase plot of the rabbit-fox prey-predator model. The red dot shows the population at different times. Note that the number of rabbits quickly increases when there are very few foxes. However, at some point the number of foxes also goes up and soon the start reducing the rabbits, while increasing in numbers themselves. That is not sustainable and when the number of rabbits is brought down substantially, als the number of foxes decreases, until both are almost extinct and the cycle repeats.

Wilberforce Oscillator As a second example we look at the Wilberforce pendulum. This is a spring, suspended vertically, to which a weight is fixed at the free end. The weight can go up and down but also rotate in a horizontal plane. A sketch is given below.

Image that we pull m a little down and let go. The spring will try to restore the position of the mass to the equilibrium position it was in prior to us pulling m down. Consequently, m will start oscillating in the vertical direction. However, something peculiar happens: the mass m also starts to rotate (around the vertical axis). And also this rotation turns out to be a back and forth oscillation. But that is not all: the two oscillations are coupled: they feed each other. If the vertical oscillation is at a maximum amplitude, the rotational motion is almost zero and vice-versa.

The system can be modeled with simple means. We will just postulate them. Later on, we will see where the terms come from.
First, we note that the mass has kinetic energy, in two forms: due to the vertical motion ($\frac{1}{2}m\dot{z}^2$) and due to the rotational motion ($\frac{1}{2}I\dot{\theta}^2$). Don't worry about the exact meaning for now.
Second, the mass has potential energy. We will ignore gravity (and pretend we do this experiment ins SpaceLab). A potential energy is associated with the vertical motion and is the spring energy: $V_z = \frac{1}{2}kz^2$, with z the vertical position of the mass with respect to the equilibrium position, which we took as $z = 0$. k is the spring constant and represents the strength of the spring. We will come back to this later.

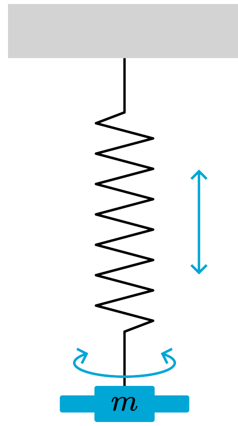


Figure 60: Wilberforce pendulum.

Figure 61: *
A Wilbertforce pendulum made by first year physics students

Then, we have potential energy associated with the rotation: $V_\theta = \frac{1}{2}\delta\theta^2$. θ represent the rotation angle, where we have taken $\theta = 0$ in the equilibrium position. δ is the torsional spring constant: it represents how strongly the spring tries to push back against rotation.
Finally, the vertical position and the rotation influence each other. That can be understood by realizing that if you shorten the spring, the spring material has to go somewhere. It can not only change its vertical length as that would mean that the total length of the spring would reduce. But that would compress the spring material and that is not possible for solid material (unless you apply incredibly large forces). The spring just increases its number of windings a bit. But that implies rotation. Similarly, if we only rotate the spring, it will try to adjust its length. As a consequence, there is also a potential energy involved in the influencing of z and θ of each other. It can be modeled as $V_{z\theta} = \epsilon z\theta$.

If we ignore friction, then we have a system that can be described in terms of energy:

$$\frac{1}{2}m\dot{z}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}kz^2 + \frac{1}{2}\delta\theta^2 + \epsilon z\theta = E_0 \tag{203}$$

From this, we can find 'N2', the equation of motion:

$$\begin{aligned} m\ddot{z} &= -kz - \epsilon\theta \\ I\ddot{\theta} &= -\delta\theta - \epsilon z \end{aligned} \tag{204}$$

Don't worry, if you don't follow this. The point here is, that we have a coupled system of two oscillators. This can be solved numerically.

We could use a simple numerical scheme like we have employed in Chapter 3. In the figure below $z(t)$ and $\theta(t)$ are shown using such a simple numerical scheme.

We indeed see the oscillating motion and that the vertical oscillation changes over to rotation and back again.

But there is something really disturbing: the amplitude of our oscillation is increasing and it seems to do so for every cycle. That cannot be true: It violates energy conservation. What did we do wrong? Well, our numerical method is just not good enough. If we use again a higher order method, we obtain the results in the figure below.

Now the amplitude of the oscillations stays nicely constant, obeying conservation of energy.

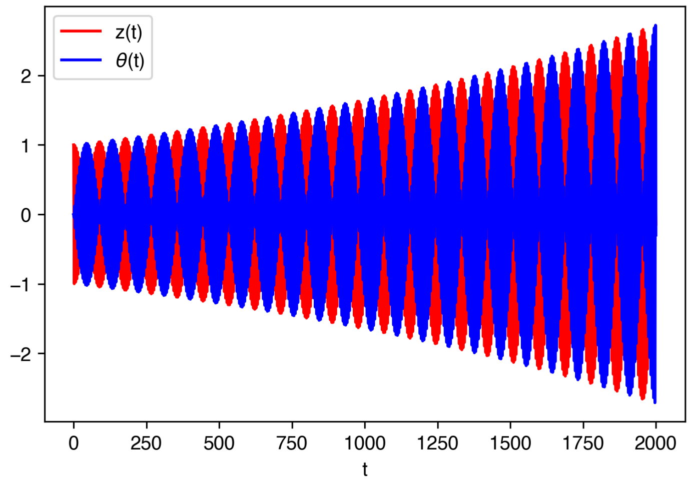


Figure 62: Numerical solution of the Wilberforce pendulum using a (too) simple numerical method.

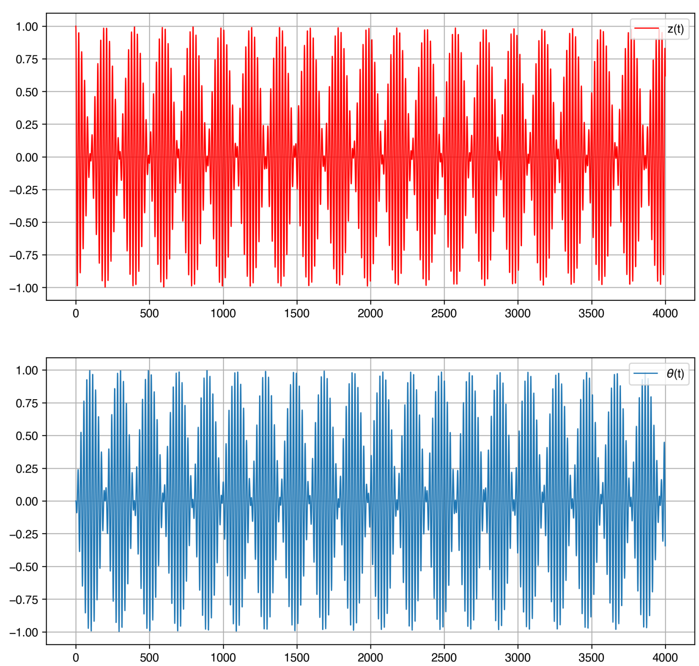


Figure 63: Numerical solution of the Wilberforce pendulum using a higher-order numerical method.

In the figure below a small animation can be seen: the marker in both graphs shows z and θ at the same time instant.

Figure 64: Animation of the Wilberforce pendulum using a higher-order numerical method.

The Wilberforce pendulum is clearly periodic. Moreover, it is an oscillation as there is back and forth motion around an equilibrium.

But, it does give us a **big warning**: (numerical) solutions always have to be **assessed** against the features and principles of the problem at hand. In this case, our first numerical solution could not be right: **it violated energy conservation**. We were able, right from the start, to formulate the problem in terms of energy. SInce we only had kinetic energy and potential energy we **knew up front** that the motion must be bounded!
That is why, we need a thorough understanding of physics. It is not sufficient to have the equations and put them in a 'solver'. It is the job of a physicist to understand and assess models, outcomes, etc against the laws of physics. Hence, we will dive into oscillations, starting from the beginning.

2.6.2 Harmonic Oscillation - archetype: Mass-Spring

The archetype of an oscillation is the mass-spring system. It is the simplest version (simpler than the pendulum as we will see). And it can be recognized in many systems. We consider the following: a mass is attached to a spring. The other end of the spring is fixed. The mass can only move in one direction: the x-direction. The spring has a natural or rest length l_0. That is the length of the sphere if no force is acting on it. It we pull the spring, it will exert a force that is proportional to the increase in length. Moreover, it is pointing in the direction opposite to the lengthening. In formula:

$$F_v = -k(l - l_0) = -k\Delta l \tag{205}$$

This is shown in the figure below.

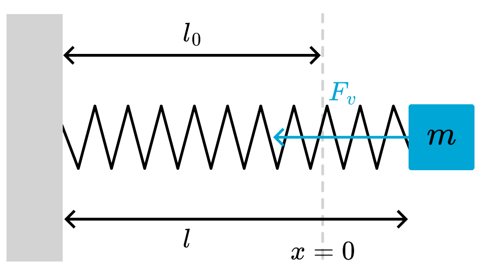


Figure 65: Mass-spring system: archetype of a (harmonic) oscillation.

The response of the spring is to exert a force on m proportional to its elongation (which may be negative). It is clearly a restoring force: no matter what we do pulling or pushing, the spring will always counteract.

It is not difficult to set up N2 for the mass-spring. There is only one force and the system is 1-dimensional. If we define the origin at the position of the mass when the spring is at its rest length, then Δl - the elongation of the spring - becomes x, the coordinate of the mass m. Thus N2 reads as:

$$m\ddot{x} = -kx \tag{206}$$

Or

$$m\ddot{x} + kx = 0 \tag{207}$$

To solve this, we need two initial condition. Let's take $t = 0 : x(0) = x_0, v(0) = 0$. We need to find a function $x(t)$ that upon differentiating twice it spits itself back but with an opposite sign. We do know two functions that do so: $x(t) = \sin(\omega_0 t)$ and $x(t) = \cos(\omega_0 t)$. Thus, the general solution of the above equation is known.

Harmonic Oscillator:

$$m\ddot{x} + kx = 0 \Leftrightarrow x(t) = A\sin\omega_0 t + B\cos\omega_0 t \tag{208}$$

If we insert the solution, we find

$$\omega_0^2 = \frac{k}{m} \tag{209}$$

This is called the natural frequency of the oscillator. Note, that it does not depend on the initial conditions. No matter what, the mass will always oscillate with this frequency.

It does make sense that the frequency is inversely proportional to m: we expect a heavy object will response slow to a force. Similarly, if the spring is strong, that is has a high spring constant k, it will move the mass around quickly.

If we substitute the initial condition, we can completely solve the motion of the mass:

$$\begin{aligned} &m\ddot{x} + kx = 0 \Rightarrow x(t) = A\sin\omega_0 t + B\cos\omega_0 t \Rightarrow \\ &\begin{cases} v(0 = \dot{x}(0) = 0 \rightarrow A\omega_0 \underbrace{\cos 0}_{=1} - B\omega_0 \underbrace{\sin 0}_{=0} = 0 \rightarrow A = 0 \\ x(0) = \Delta x \rightarrow B\cos 0 = \Delta x \end{cases} \\ &\Rightarrow x(t) = \Delta x \cos\sqrt{\frac{k}{m}}t \end{aligned} \tag{210}$$

A system is called a harmonic oscillator if and only if it obeys $m\ddot{x} + kx = 0$. You will find them in almost every branch of science and engineering. The reason why will become apparent in a moment.

Potential energy of a spring In the above, we have formulated the mass-spring system in Newton's second law. We can, however, also cast it in the form of energy. The force of the spring is conservative. We can easily prove this by finding the associated potential energy: $F_v = -\frac{dV}{dx}$.
Since $F_v = -kx$ we need to find a function $V(x)$ that satisfies $\frac{dV}{dx} = kx$. Let's do it:

$$\frac{dV}{dx} = kx \Rightarrow V(x) = \frac{1}{2}x^2 + C \tag{211}$$

We have the freedom to decide ourselves where we want the potential energy to be zero. Note: V is quadratic. It does make sense, to set the minimum of the potential energy such that if the mass is at the equilibrium position, the potential energy is zero, that is - take $C = 0$:

$$V(x) = \frac{1}{2}kx^2 \tag{212}$$

Thus the mass-spring system can also be described by

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E_0 \tag{213}$$

So, an other way of stating what a harmonic oscillator is: it is a system that obeys the above energy equation.

2.6.3 Behavior around an equilibrium point and harmonic oscillators

Now we will go back to paragraph 5.5.1, where we discussed the Taylor series expansion of the function $f(x)$:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \mathcal{O}(x^3) \tag{214}$$

We will apply it to a potential energy $V(x)$ of some system. We assume that the system has a stable equilibrium point at $x = x_0$, that is $\left[\frac{dV}{dx}\right]_{x=x_0} = 0$ and $\left[\frac{d^2V}{dx^2}\right]_{x=x_0} > 0$.
Thus, we can expand the potential as follows:

$$V(x) \approx V(x_0) + \frac{1}{2} \underbrace{\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}_{=k} (x - x_0)^2 + \mathcal{O}\left[(x - x_0)^3\right] \tag{215}$$

If we plug this in, in the energy equation and cut off after the quadratic term, we find

$$\frac{1}{2} m v^2 + V(x_0) + \frac{1}{2} \underbrace{\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}_{=k} (x - x_0)^2 = E_0 \tag{216}$$

or shortened by the abbreviation $\left[\frac{d^2V}{dx^2}\right]_{x=x_0} = k$

$$\frac{1}{2} m v^2 + V(x_0) + \frac{1}{2} k((x - x_0)^2 = E_0 \tag{217}$$

Move the constant $V(x_0)$ to the right hand side and change coordinate $s \equiv x - x_0 \to \dot{s} = \dot{x} = v$. This gives us:

$$\frac{1}{2} m \dot{s}^2 + \frac{1}{2} k s^2 = C \tag{218}$$

The harmonic oscillator!!! No wonder we find harmonic oscillators 'everywhere'. Any system that has a stable equilibrium point with a positive second derivative of its potential will start to oscillated as a harmonic one if we push it a little bit out of its equilibrium position. Doesn't matter how $V(x)$ exactly is. It doesn't have to be quadratic in x. But it will be pretty close to that, if we stay close enough to the equilibrium point. Hence, any small natural kick, any small amount noise will push a system out of its stable equilibrium point into an harmonic oscillating motion with a given, natural frequency given by $\omega_0^2 = \frac{\left[\frac{d^2V}{dx^2}\right]_{x=x_0}}{m}$.

2.6.4 Examples of Harmonic Oscillators

Torsion Pendulum We take a straight metal wire. Suspend one end at the ceiling and attach a disc of radius R and mass m at the other end.

The disk can rotate about a vertical axis. We call the rotation angle θ. The equilibrium position is $\theta = 0$. If we rotate the disc over a small angle, the wire will resist and apply a torque Γ on the disc trying to rotate the disc back to its equilibrium position, for which the torque, obviously is zero.

For small angles, the torque is proportional to the rotation angle and -of course -working in the direction opposite of the rotated angle. We can set up an angular momentum equation and find that it reads as:

$$I \frac{d^2\theta}{dt^2} = -k_t \theta \tag{219}$$

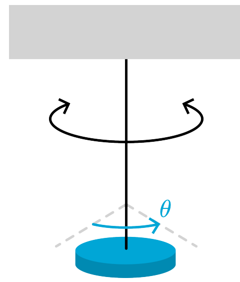


Figure 66: Torsion Pendulum.

In this equation, $I = \frac{1}{2}mR^2$ is the moment of inertia of the disc and k_t is the torsion constant of the wire. Don't worry about the exact meaning of the terms in the equation. For now, we focus on the equation itself:

$$I\frac{d^2\theta}{dt^2} + k_t\theta = 0 \Rightarrow \theta(t) = A\sin\omega_0 t + B\cos\omega_0 t \tag{220}$$

The torsion pendulum is a harmonic oscillator, $\omega_0^2 = \frac{k_t}{I}$, completely analogous to the archetype, mass-spring. Obviously, we thus can also write this in terms of energy:

$$\frac{1}{2}I\omega^2 + \frac{1}{2}k_t\theta^2 = E_0 \tag{221}$$

with $\omega \equiv \frac{d\theta}{dt}$, the angular velocity.

L-C circuit In Electronics alternating current (AC) circuits are building blocks of many complex systems. One of these is the L-C circuit, in which an inductor, L, and a capacitor, C, are in series coupled. See Figure 67.

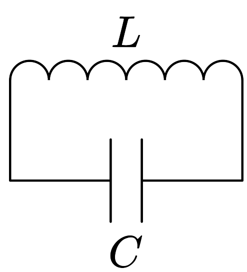


Figure 67: L-C circuit.

We could charge the capacitor and then close the circuit. What would happen? The capacitor will try to discharge via the inductor. Hence a current, I, starts flowing. In response, the inductor builds up a potential difference that is directly proportional to the rate of change of the current through the inductor.

Basic electronics shows that the voltage over the capacitor is coupled to the charge, Q_C, of the capacitor according to: $V_C = \frac{Q_C}{C}$. For the inductor we have: $V_L = L\frac{dI_L}{dt}$.

According to Kirchhoff's laws the current through both elements must be the same: $I_C = I_L$ and the sum of the voltages across them must be equal to zero: $V_c + V_L = 0$. If we put everything together, we get - using $I_C = \frac{dQ_c}{dt}$:

$$\begin{aligned}
V_L + V_C &= 0 \Rightarrow \\
\frac{dV_L}{dt} + \frac{dV_C}{dt} &= 0 \Rightarrow \\
L\frac{d^2I}{dt^2} + \frac{1}{C}I &= 0 \Rightarrow \\
\frac{d^2I}{dt^2} + \frac{1}{LC}I &= 0 \text{ Harmonic Oscillator!!!}
\end{aligned} \tag{222}$$

As we see, this LC-circuit will start to oscillate. In the animation below the current through the circuit and the voltage across the inductor are shown for $C = 1\mu F$ and $L = 1\mu H$.

Figure 68: Harmonic oscillation of an LC-circuit.

Musical Instruments Musical instruments produce sound waves. In many cases they do that via vibrations of strings, like the guitar, the violin, harp or piano. The strings of these instruments are displaced out of their equilibrium position. Due to the tension in these strings, there is a restoring force that is proportional to the displacement. Consequently, the string will start to oscillate in an harmonic way.

Not only strings, but also beams will exhibit this behavior, well-known example: a tuning fork. We will come back to waves at the end of this chapter.

2.6.5 The pendulum

Another example of oscillatory motion is the pendulum. In its most simple for it is a point-mass m, attached to a massless rod of length L. The rod is fixed to a pivotal point that allows it to swing freely.

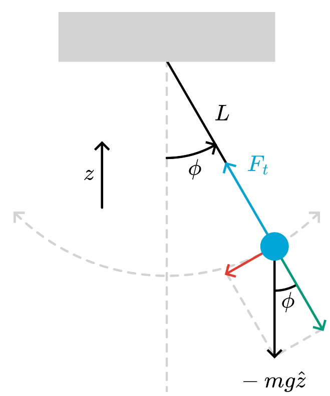


Figure 69: Sketch of a pendulum by Huygens (public domain).

On the mass, gravity is acting vertically downwards. Also the rod exerts a force on the mass. This force is always parallel to the rod and points to the pivotal point. It is the response of the rod to the component of gravity parallel to the rod (the dark blue arrow in Figure 69). It is good to realize, that this force makes sure that the distance from m to the pivotal point is always L. In other words, this force is a consequence of the fixed length L of the rod. It is the physics translation of the constraint: L is constant.

N2 for the pendulum: Equation of motion via N2 We will set up Newton's second Law for m.

$$m\frac{d\vec{v}}{dt} = -mg\hat{z} + \vec{F}_t \tag{223}$$

As stated above, the blue, parallel part of gravity is balanced by a tensional force in the rod. So, we don't need to worry about motion of m parallel to the rod. That leaves us with the direction perpendicular of the rod. In that direction only the red arrow works on m.

In the other direction only the red, perpendicular component of gravity acts on m. This component is equal to $-mg\sin\phi$. The velocity component in this direction is $v = r\frac{d\phi}{dt}$. Thus we get:

$$mL\frac{d^2\phi}{dt^2} = -mg\sin\phi \tag{224}$$

Or rewritten

$$mL\frac{d^2\phi}{dt^2} + mg\sin\phi = 0 \tag{225}$$

We do know from experience that the pendulum will swing back and forth in a periodic way. However, as we see from the above equation of motion: it is not a harmonic oscillator. The term with the sin prevents that.

But for small values of the angle ϕ, that is for small oscillations around the stable equilibrium $\phi_{eq} = 0$, we can approximate the sinus via a Taylor series and write:

$$\begin{aligned} \phi \ll 1 \Rightarrow \sin\phi &\approx \sin 0 + \frac{1}{1!}\cos 0\,\phi - \frac{1}{2!}\sin 0\,\phi^2 + ... \\ &\approx \phi \end{aligned} \tag{226}$$

Thus within this approximation we can write for the equation of motion of the pendulum:

$$mL\frac{d^2\phi}{dt^2} + mg\phi = 0 \Rightarrow \frac{d^2\phi}{dt^2} + \frac{g}{L}\phi = 0 \tag{227}$$

and that describes a harmonic oscillator.

We conclude that for small amplitudes of the oscillation, the pendulum is an harmonic oscillator and swings in a sin or cos way back and forth. Moreover, the oscillation has a frequency

$$\omega_{pendulum} = \sqrt{\frac{g}{L}} \tag{228}$$

Further, note that under this assumption, the period of the pendulum does not depend on the amplitude of the oscillation. That was already noted by Galileo Galileï.

N2 for the pendulum: Equation of motion via Angular Momentum Before we continue with the analysis of the pendulum, we will derive the equation of motion also via angular momentum considerations. On m gravity exerts a torque: $\Gamma = \vec{r} \times \vec{F}_g$ It has a magnitude $-Lmg\sin\phi$ and points into the screen. The angular momentum of m is given by $\vec{L} = \vec{r} \times \vec{p}$. This has magnitude $mL^2\frac{d\phi}{dt}$ and also points into the screen.

Thus N2 for angular momentum gives us:

$$\frac{d\vec{L}}{dt} = \vec{\Gamma} \Rightarrow mL^2\frac{d^2\phi}{dt^2} = -Lmg\sin\phi \tag{229}$$

Thus, angular momentum leads to the same equation of motion.

The Pendulum via energy conservation Alternatively, we can also use energy conservation to derive the equation governing the motion of the pendulum. There are, as discussed above, two forces acting on m. The first one is gravity, which is a conservative force with associated potential energy. We can write for this case $V_g = mgz$, taking $V_g(z=0) = 0$.

The second one is the force from the rod. But this one always acts perpendicular to the motion of m. Hence, it does not do any work and, thus, we don't need to worry about an associated potential.

We conclude that for the pendulum it holds that:

$$\frac{1}{2}mv^2 + mgz = E_0 \tag{230}$$

To solve this, we change from z to ϕ. z is, in terms of ϕ: $L - L\cos\phi$, see Figure 70.

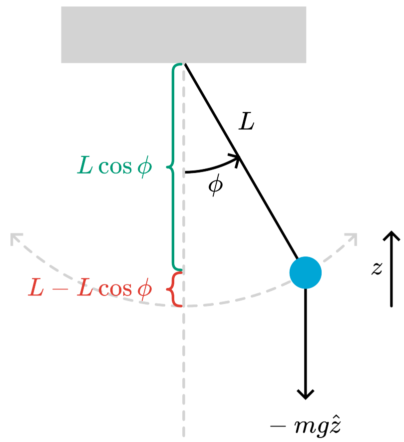


Figure 70: Potential energy of a pendulum.

Thus, our energy equation reads as:

$$\frac{1}{2}mv_\phi^2 + mgL\left(1-\cos\phi\right) = E_0 \tag{231}$$

or

$$\frac{1}{2}mv_\phi^2 - mgL\cos\phi = E_0 - mgL \tag{232}$$

Take the time-derivative and use $v_\phi = L\frac{d\phi}{dt}$ and we get

$$\begin{aligned} mv_\phi\frac{dv_\phi}{dt} + mgL\sin\phi\frac{d\phi}{dt} &= 0 \Rightarrow \\ mL\frac{d\phi}{dt}\frac{d}{dt}\left(L\frac{d\phi}{dt}\right) + mgL\sin\phi\frac{d\phi}{dt} &= 0 \Rightarrow \\ \frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi &= 0 \end{aligned} \tag{233}$$

And we have recovered the same equation of motion.

Pendulum for not so small angles In the above we have frequently used the approximation $\sin\phi \approx \phi$ for $\phi \ll 1$. What about the general case? Then we need to solve

$$\frac{d^2\phi}{dt^2} + \frac{g}{L}\sin\phi = 0$$
$$\text{with i.c. } \phi(0) = \phi_0 \text{ and } \frac{d\phi}{dt} = \dot{\phi}_0 \tag{234}$$

This equation is much more difficult to solve analytically and we will, therefore, use a numerical approach here. The animation below compares the motion of the pendulum numerically simulated to that of the pendulum when using the small amplitude approximation.

The animation shows: a green mass, that is the pendulum with a (fixed) small amplitude in the approximation $\sin\phi = \phi$. The blue one uses the same approximation even though ϕ is not small. Notice, that blue and green oscillate with exactly the same frequency. This is, of course, trivial as they obey the same harmonic oscillation equation and thus have the same frequency.

The red mass, on the other hand obeys the equation of motion of the pendulum leaving the term with $\sin\phi$. It is clear that the real pendulum (i.e. the red one) does not have the same frequency as the others. Moreover, its time trace (left part of the figure) is clearly not a true sinus.

Figure 71: Animation of the pendulum: red is the true pendulum, blue the small angle approximation applied to a large angle case and green the small angle approximation for a small angle.

In the widget below, you can vary the initial angle and observe that indeed for a small angle the red mass and the other two follow the same trajectory. But if you increase the initial angle, the red mass behaves differently: it oscillates slower and the time trace of angle as a function of time is no longer sinusoidal.

Warning

2.6.6 The damped harmonic oscillator

In the above, no friction of any form has been considered. However, in many practical cases friction will be present. For moving objects friction frequently depends on the velocity: the higher the velocity, the higher the frictional force. We will here consider the simplest version: a friction force that is directly proportional to the velocity: $F_f = -bv$ with b a positive constant. Thus, we need to add an additional force to our harmonic oscillator:

$$m\ddot{x} = -kx - b\dot{x} \tag{235}$$

or bringing all terms to the left hand side:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{236}$$

To solve this equation, it is easier not to try to look directly for sinus and cosines, but use the complex notation.

Intermezzo: complex exponential and sin, cos
In the 18$^{\text{th}}$ century, the study of complex numbers, i.e. $i = \sqrt{-1}$, revealed a surprising connection between the exponential function and trigonometry. It was Leonhard Euler (1707-1783) who derived:

$$e^{ix} = \cos x + i\sin x \tag{237}$$

If you want to understand this a bit further, write down the Taylor expansion of all three functions. By realizing that $i^2 = -1, i^3 = i^2 i = -i$, etc. You will see that Euler was right.

We can use the above equation to write $\sin x$ and $\cos x$ as exponential functions:

$$\begin{aligned}\sin x &= \frac{e^{ix} - e^{-ix}}{2} \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2}\end{aligned} \tag{238}$$

And we can also state that the real part of e^{ix} is equal to $\cos x$ and the imaginary part to $\sin x$.

The above turns out to be extremely useful in solving differential equations. For instance, rather than trying $\sin$ and $\cos$ as solutions for the harmonic equation $m\ddot{x} + kx = 0$, we could try $e^{i\omega t}$ (please note: x in the first part of this intermezzo is just a real number, whereas now x is the amplitude of the oscillator and is a function of t):

$$m\ddot{x} + kx = 0 \Rightarrow m\frac{d^2 Ae^{i\omega t}}{dt^2} + kAe^{i\omega t} = 0 \Rightarrow Am(i\omega)^2 e^{i\omega t} + Ake^{i\omega t} = 0 \Rightarrow Ae^{i\omega t}\left(-m\omega^2 + k\right) = 0 \quad \forall t \tag{239}$$

Thus we conclude that if $Ae^{i\omega t}$ is a solution of the harmonic equation, then $\omega^2 = \frac{k}{m}$. So, with that condition, $Ae^{i\omega t}$ is a solution with A an integration constant that will be fixed by proper initial conditions.

But let's be careful: the harmonic oscillator is governed by a second order differential equation, that is we have differentiated twice with respect to time. Thus we need to integrate also twice, leaving us with 2 rather than 1 integration constant. Hence, we have not found the general solution, yet.

That can be easily cured: if $Ae^{i\omega t}$ is a solution, the $Be^{-i\omega t}$ is one as well. And here is our second solution, with the same condition for ω. And our general solution is

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t} \tag{240}$$

Note that A and B are complex numbers and thus, we can always rewrite this equation back into $\sin(\omega t)$ and $\cos(\omega t)$. But in most cases we don't worry about that. We ar interested in the 'rule' for ω and once we have that, we can write our solution straightaway as $x(t) = C\cos(\omega t) + D\sin(\omega t)$.

Let's use this for the damped harmonic oscillator:

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{241}$$

Try $x(t) = Ae^{i\omega t}$.

$$\Rightarrow Ae^{i\omega t}\left(-\omega^2 m + i\omega b + k\right) = 0 \quad \forall t \Rightarrow \omega^2 m - i\omega b - k = 0 \Rightarrow \omega_{+,-} = \frac{+ib \pm \sqrt{-b^2 + 4mk}}{2m} \tag{242}$$

We immediately have our two solutions: one with the $+$sign, the other with $-$sign:

$$x(t) = Ae^{i\omega_+ t} + Be^{i\omega_- t} \tag{243}$$

with both ω's complex numbers.

Alternatively, we could have started with $x(t) = e^{\lambda t}$, but allowing λ to be a complex number. This somewhat easier and as we have seen above, ω is a complex number ($D = -b^2 + 4mk$ can be negative). This will gives us:

$$m\lambda^2 + b\lambda + k = 0 \Rightarrow \lambda_{+,-} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{244}$$

We see, that λ always has a negative real part. That makes sense: the negative real part shows that the solution is damped.

Our solution to the damped harmonic oscillator is thus:

$$x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} \tag{245}$$

The general solution of the (linearly) damped harmonic oscillator is:

$$m\ddot{x} + b\dot{x} + kx = 0 \Rightarrow x(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}\text{with}\lambda_{+,-} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{246}$$

We will investigate various cases.

Evolution of the damping Here we will have a quick look how the damping is evolving, that is we look at the root of the characteristic equation

$$\lambda_{1/2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \tag{247}$$

and see how it evolves as a function of the damping b in the complex plane.

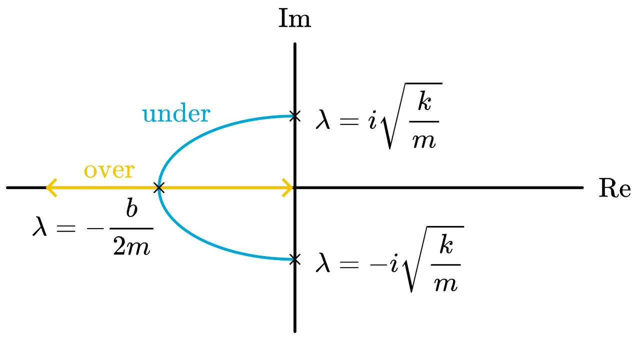


Figure 72: Evolution of λ as a function of b in the complex plane.

This gives quickly a qualitative view on the different regimes of the damping. The root $\lambda_{1/2}$ is in general complex. We start by looking at the value for roots $\lambda_{1/2}$ as a function of the damping b

- No damping: $b = 0$. The root is pure imaginary $\lambda_{1/2} = \pm i\sqrt{k/m}$ with two conjugate solutions on the imaginary axis. This gives pure oscillations.
- Some damping $0 < b < \sqrt{4mk}$. The root is complex, with real and imaginary part, the oscillation will damp out over time (shown in blue, underdamped regime).
- $b^2 = 4mk$. The roots collapse into one pure real root $\lambda = -b/2m$ (critically damped), no oscillation.
- Lots of damping $b > \sqrt{4mk}$. The root splits into two real roots, no oscillations (shown in yellow, overdamped regime).

The root walks over the shown graph from $b = 0$ on the imaginary axis to $b \to \infty$ over the blue and then yellow part of the graph. The yellow graph does not cross the imaginary axis.

From this plot you can directly see that the system is stable for $b > 0$, but unstable for $b = 0$ without the need to check the frequency that the system is driven with (for $b = 0$ driven with the resonance frequency results in an infinite amplitude - an instable system). How you can see that so quickly you will learn in the second year class *Systems and Signals*.

2.6.7 Driven Damped Harmonic Oscillator

Oscillators sometimes experience a driving force that can be periodic in itself. We will take here the case of a sinusoidal force with frequency ν. Once we understand this, forces consisting of more than one frequency (broader spectrum) can be understood using Fourier analysis (which you will learn about classes like *Systems and Signals* or

Fourier Analysis in math). There you will also learn to treat this system in more detail analytically. Here we will stick to a simple driving force of the form $F_{ext} = F_0 \sin(\nu t)$.

This gives for the equation of motion:

$$m\ddot{x} + b\dot{x} + kx = F_0 \sin(\nu t) \tag{248}$$

with initial conditions: at $t = 0$ the particle will have some position x_0 and some velocity v_0.

The solution of the driven damped harmonic oscillator equation of motion for the case $D = b^2 - 4mk < 0$ is:

$$x(t) = Ae^{-\frac{b}{2m}t} \sin\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t + \epsilon\right) + x_{max} \sin(\nu t + \alpha) \tag{249}$$

With A and ϵ determined by the initial conditions.

The two other parameters x_{max} and α are fixed. We will give only the expression for x_{max}:

$$x_{max} = \frac{F_0}{\sqrt{\left(\omega_0^2 - \nu^2\right)^2 + \frac{b^2}{m^2}\nu^2}} \tag{250}$$

For $t \to \infty$, the second part, i.e., the term from the driving force $x_{max} \sin(\nu t + \alpha)$, survives as the exponential decay will have damped the first term. The oscillation will have frequency ν of the driving force. As can be seen, the amplitude of the motion is for longer times x_{max}.

If the driving frequency $\nu \sim \omega_0$, the amplitude increases strongly. Especially for small damping, i.e., small b, the amplitude will increase to high values. This phenomenon is called *resonance*:

$$\text{if } b \to 0 \text{ and } \nu \to \omega_0 \text{ then } x_{max} \to \infty \text{ resonance} \tag{251}$$

2.6.8 Coupled Oscillators

In this course we mostly only consider one oscillator, but of course there could be many that are coupled in one way or another. Already Christiaan Huygens considered them.

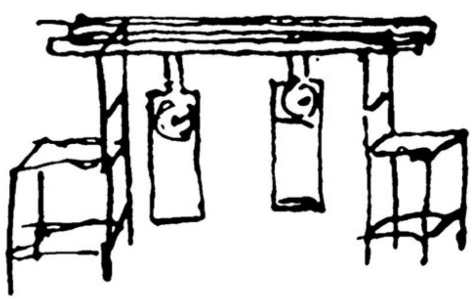


Figure 73: Huygens experiment of weakly coupled pendula.

There are 2 pendula suspended from a common connection, which rests on two chairs. If you set the pendula in motion, they will be initially *out of phase*, i.e. the relative position of the pendula is different. But over time their motion synchronises! What has happend? Apparently the two pendula are connected, *coupled*, via the suspension and act on each other, they are not independent, but influence the motion of the other pendulum.

The video below shows a modern day version of this phenomena.

Here the pendula are coupled via the ground. This influence is called *weak coupling*. In this course we cannot treat this coupling mathematically, but in the second year course on *Classical Mechanics* you will learn to study systems like these.

Figure 74: *
Weakly coupled metronomes.

2.6.9 Examples

1. Example of resonance: sound waves are exciting a glass. By changing the frequency of the sound waves to the resonance frequency, the glass starts oscillating with increasing amplitude until it finally breaks.

Warning

2. Driven harmonic oscillator with damping.

Warning

3. 1940: the Tacoma Narrows Bridge in the state Washington on the West coast of the USA is brought into resonance by the wind. The end result: click the movie to see it yourself.

Warning

4. Breaking a HDD hard disk with a song of Janet Jackson

Read here about this truly amazing piece of applied physics on a blog of Microsoft developer Raimond Chen.

5. The blue sky: Rayleigh scattering

Light from the sun (and stars) will have to travel through the atmosphere before reaching the ground level. On its way it will be subject to absorption and scattering.

When you look on a clear day into the sky its color is blue, everybody knows that. But few people know why. The reason is found in the scattering properties of the molecules: the probability of light being scattered by an air molecule is proportional to the wave length of the light to the power -4, or rephrased: proportional to f^4 (f the frequency of the light, the theory of molecular scattering was given first given by Lord Rayleigh). Thus, blue light of a wavelength of 450nm is compared to red light ($\lambda = 650$nm) $(650/450)^4 = 4.4$ times more likely to be scattered. Consequently, the blue end from the (white) sun light has a reduced probability to reach our eye directly in comparison with the red end. And thus most of the scattered light that reaches us is blue: the sky is blue.

We will look at scattering of light by considering a simple molecule made of a fixed nucleus with one electron orbiting it. The equation of motion of the electron can be written as that of a harmonic oscillator, with eigen frequency ω_0:

$$m\ddot{x} + kx = 0 \rightarrow \ddot{x} + \omega_0^2 x = 0 \tag{252}$$

When light passes the electron, the electron feels a force since light is an electro-magnetic wave. The electric field is the dominating force. For light of wave length λ, i.e. angular frequency $\omega = 2\pi f = 2\pi\frac{c}{\lambda}$, the electric field can be written as $E_0 \sin\omega t$. Such a field will produce a force $F_e = eE_0 \sin\omega t$ on the electron, modifying its equation of motion to:

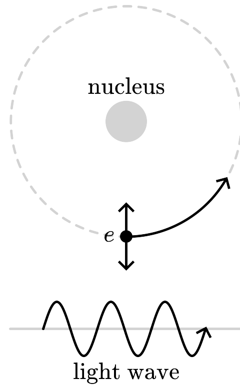


Figure 75: Simple model of electron-light scattering.

$$\ddot{x} + \omega_0^2 x = \frac{e}{m} E_0 \sin \omega t \tag{253}$$

We recognize this as the forced harmonic oscillator with solution

$$x(t) = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{eE_0}{m} \frac{\sin \omega t}{\omega_0^2 - \omega^2} \tag{254}$$

The important part is the last one: the extra motion cause by the passing electric field. This causes an additional acceleration of the electron: $a(t) = -\frac{eE_0}{m} \frac{\omega^2}{\omega_0^2 - \omega^2} \sin \omega t$.

The electron in its original orbit does not radiate. However, due to the extra acceleration the electron starts radiating. It sends out an electromagnetic field with the wave length of the incoming light and an intensity proportional to the square of the acceleration, $\langle a(t)^2 \rangle$, i.e.

$$I \propto \left[\frac{\omega^2}{\omega_0^2 - \omega^2} \right]^2 \tag{255}$$

As the eigen frequency ω_0 of the electrons in oxygen and nitrogen is much higher than the frequency ω of the incoming light we have that this is basically proportional to $\left(\frac{\omega}{\omega_0}\right)^4$. As this radiation by the electron obviously feeds on the incoming light, we find that the scattering of the light is proportional to the frequency of the incoming light to the power 4.

6. Second-harmonic generation

Of course the harmonic potential is only a first order approximation around an equilibrium. An example, for a non-linear force or anharmonic potential effect, is the generation of second-harmonic generation. If you shine high intensity light onto the electrons of a molecule, they are pushed out of equilibrium further and if the governing potential is anharmonic, the electric field response will not only include the incoming frequency ω but also *higher harmonics* $2\omega, 3\omega, \ldots$, but with much lower intensity. That the emitted frequencies are occurring in integer multiple of the incident frequency can be understood either from quantization of light into photons (and the conservation of energy) or from Fourier analysis of the periodic motion of the electron.

7. Erasmus Bridge & singing cables.

The bridge in Rotterdam, but also others, suffer from long cables that the wind can put into resonance. Their motion then generates acoustic waves in the audible spectrum. Listen here to the sound of the cables starting from 1:00 on the website for singing bridges!

2.6.10 Waves and oscillations

In the previous sections, we talked about oscillations of individual particles. Oscillations can also occur in a more collective mode. And there are plenty of examples: take for instance a violin or piano string. It is in essence an elastic string suspended between two fixed points. The string is under tension, that is: its natural length is (slightly) less than the distance between the two end points. As a consequence, equilibrium position of the string is a straight line and when brought out of equilibrium there is a net restoring force much like for the mass-spring system.

However, there are at least two important differences: (1) the restoring force is the net result from pulling on a small part of the string by its neighbor parts; (2) the entire string can oscillate in a direction perpendicular to the equilibrium position of the string, making the problem multi-dimensional.

We will give here an intuitive derivation of the equation of motion. Don't worry if you don't grasp it fully. This will come back in your studies further down the line.

In the figure below, a part of the string is drawn with special attention to a small part (the red line). On this small part the tension from the left and right side is pulling on the red part. This is visualized by the two blue arrows. In the inset, this is drawn at a larger scale. The two blue arrows are equal in magnitude (T) as the tension in the string is the same everywhere. But the direction in which the two blue forces are pulling is slightly different as the string is curved.

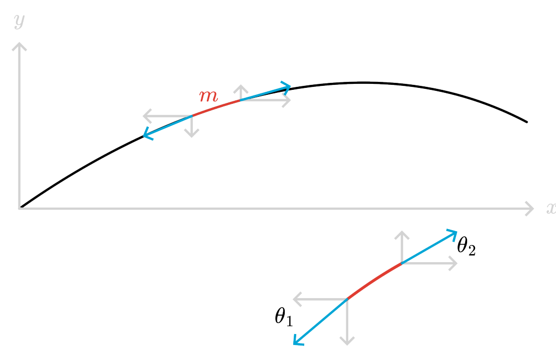


Figure 76: Forces on a small part of a string; inset shows an exaggeration of the vertical components of the forces.

If we call the angle of the blue forces with the x-axis θ, then $\theta_1 \neq \theta_2$. This makes that a net force is action on the small red piece. And according to Newton's second Law, the small red mass must accelerate.

Let's set up N2 for the red piece. The problem is 2-dimensional, so we set up N2 for the x and y-direction:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -T\cos\theta_1 + T\cos\theta_2 \\ m\frac{d^2y}{dt^2} &= -T\sin\theta_1 + T\sin\theta_2 \end{aligned} \tag{256}$$

Next, we simplify by only looking at situations where the angle θ_1 and θ_2 are small. The we can approximate the $\sin$ and $\cos$ terms: if $\theta \ll 1$ then $\sin\ theta \approx \theta$ and $\cos\theta \approx 1$ and we can write

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -T + T = 0 \\ m\frac{d^2y}{dt^2} &= -T\theta_1 + T\theta_2 \end{aligned} \tag{257}$$

Thus: for the x direction we don't need to worry, nothing interesting happening there.

For the y-direction we face that we have too many unknowns. We need relations between θ_1, θ_2, y and x. We are going to use again that $\theta \ll 1$ but know to make it seemingly more complex.

If $\theta \ll 1$ then $\tan\ theta \approx \theta$. And we are going to replace θ by $\tan\theta$. Is that smart??? Now we get trigonometry back in the equation!! Don't worry. We use the $\tan\theta$ in another way. It is also the direction of the tangent to the curve the spring is making at the point where we are looking. In formula:

$$\tan\theta = \frac{dy}{dx} \tag{258}$$

And this is the coupling between angles and coordinates that we have been looking for.

We are going to plug this in in N2 for the y-direction. But before doing so: the left position of the red piece is at position x. So instead of label '1' we will use subscript x. Similarly, the right end of the red piece is at $x + dx$. Thus we can write

$$m\frac{d^2y}{dt^2} = -T\left[\frac{dy}{dx}\right]_x + T\left[\frac{dy}{dx}\right]_{x+dx} \tag{259}$$

It looks still pretty messy but we are almost there. The mass of the red piece obviously scales with its length. So if we introduce μ as the mass of the string per unit length, we can write for the mass of the red piece: $m = \mu dx$. Our equation can now be written as

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{\left[\frac{dy}{dx}\right]_{x+dx} - \left[\frac{dy}{dx}\right]_x}{dx} \tag{260}$$

We recognize on the right hand side the second derivative of y with respect to x. Whereas on the left hand we see differentiating with respect to t.

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} \tag{261}$$

To make clear that we mean on the left hand side we mean: take the derivative only with respect to time we use ∂t instead of dt. Similarly on the right hand ∂x instead of dx. And we get our final result replacing $\frac{T}{\mu}$ by v^2

$$\frac{\partial^2 y}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2} \tag{262}$$

This equation is called the **wave equation** and you will find it back in many branches of science and engineering. To solve it, you need advance calculus and that will certainly come in future courses. Here we will look at some global aspects of the equation.

- units of v^2: \text{m}$\hat{2}$ / \text{s}$\hat{2}.Thus$v$ is a kind of velocity, at least based on its dimension.
- if $y(x,t)$ is such that it only depends on $x \pm vt$, that is $y(x,t) = y(x - vt)$ then no matter what y as function is, it is always a solution to the wave equation.

This is straightforward to prove: given $y(x,t) = y(x - vt)$ then call $s \equiv$

$$\frac{\partial y}{\partial t} = \frac{dy}{ds}\underbrace{\frac{\partial s}{\partial t}}_{=-\mathrm{v}} \tag{263}$$

Note the meaning of ∂t: differentiate $s = x - \mathrm{v}t$ as if x is a constant, not depending on t.

We can differentiate this once more:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(-\mathsf{v}\frac{dy}{ds}\right) = -\mathsf{v}\frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{\partial s}{\partial t} = \mathsf{v}^2\frac{d^2y}{ds^2} \tag{264}$$

Subsequently we look at $\frac{\partial^2 y}{\partial x^2}$:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{dy}{ds}\underbrace{\frac{\partial s}{\partial x}}_{=1}\right) = \frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{\partial s}{\partial x} = \frac{d^2y}{ds^2} \tag{265}$$

If we now substitute these two results in the wave equation we see:

$$\begin{split} \frac{\partial2̂ y}{\partial t2̂} - \mathsf{v}2̂ \frac{\partial2̂ y}{\partial x2̂} &= \mathsf{v}2̂ \frac{d2̂y}{ds2̂} - \mathsf{v}2̂ \frac{d2̂y}{ds2̂} = 0 \Rightarrow \

\frac{\partial2̂ y}{\partial t2̂} &= \mathsf{v}2̂ \frac{\partial2̂ y}{\partial x2̂}

\end{split}$ $Andweseethatourchoicefor$y(x,t) = y(x-\mathsf{v}t)$ automatically obeys the wave equation.

From the above we also learn that if the string has a certain 'amplitude' y at position x on time t a little later this same amplitude will show up at a position a bit further along the string. Argument: given x and t then at (x,t) the amplitude of the string is $y(x-\mathsf{v}t)$ and a little later, at $t+\Delta t$ we can look at position $x+\mathsf{v}\Delta t$: there $y(x+\Delta x, t+\Delta t)$ is

$$y(x+\Delta x, t+\Delta t) = y(x+\mathsf{v}\Delta t - \mathsf{v}(t+\Delta t)) = y(x-\mathsf{v}t) = y(x,t) \tag{266}$$

This actually means, that a traveling wave can be present in the string. We now this from our childhood when we probably all have been playing with a long rope making waves in it by quickly moving one end up and down.

The wave equation has as constant v^2. We have identified this as a velocity and we now understand that it is the velocity with which a wave travels. But since the equation contains the square of the velocity, we conclude that if we have a solution with $+\mathsf{v}$, then also a solution with $-\mathsf{v}$ holds. In other words: waves can travel in 2 directions and they do so with the same speed (in magnitude).

In the figure below, a wave is shown that starts as seemingly one hump. But it actually is two traveling waves on a rope.

Moreover, the rope has a fixed end at the left and a free one at the right. Notice the difference in reflection of the waves at both ends.

Figure 77: Forces on a small part of a string; inset shows an exaggeration of the vertical components of the forces.

Wave characteristics Waves are omnipresent. We find them in musical instruments e.g. the violin but also in flutes where the wave is directly in the air in the instrument. We have them in water and air: waves on the oceans, waves when we speak. There are waves in solid materials for instance after an earthquake. We use waves in telecommunication.

Why are waves so generally found? They are the analogue of the harmonic oscillator. And thus, many systems in that are brought a bit out of equilibrium will try to go back to equilibrium, over shoot it and end up in a wavy motion.

Wave Length Waves are often sinusoidal and if not, via Fourier Analysis they can be decomposed of a set of sinusoidal waves that built together the pattern we observe.

A sinusoidal wave is of the form

$$y(t) = A\sin(2\pi f t) \tag{267}$$

with f its frequency (and thus $\omega = 2\pi f$ its angular frequency).

As we have seen above, in general the wave is also a function of position:

$$y(x,t) \sim A\sin(x - vt) \tag{268}$$

How can we connect these two forms? First, we need to realize that the last equation has a dimensional issue: what is the sinus of say 7 meter? In other words, the argument of the sin-function should be dimensionless. So we write is in a different form, introducing the frequency in it:

$$y(x,t) = A\sin\left(\frac{2\pi f}{\mathrm{v}}x - 2\pi f t\right) \tag{269}$$

This seems unnecessary complicated. But it is not! The factor $\frac{f}{\mathrm{v}}$ has dimension 1 over length. If we call it $\frac{f}{\mathrm{v}} \equiv \frac{1}{\lambda}$ we can write

$$y(x,t) = A\sin\left[2\pi\left(\frac{x}{\lambda} - ft\right)\right] \tag{270}$$

Interpretation: for a fixed value of t the wave is periodic in space with period λ. This is what we already know: the wave has a wave length λ.

On the other hand: for a fixed position x the point at x oscillates with a frequency f and thus has a period $T = \frac{1}{f}$. Note that λ and f are coupled to each other:

$$\lambda \cdot f = \mathrm{v} \tag{271}$$

Standing waves versus traveling waves If we look at the motion of the string on a violin closely, we will not see traveling waves running from one side of the string to the other. Instead, we see all parts of the string moving up and down collectively: they have formed a standing wave. that is a wave that does not travel, but has a fixed, stationary shape whose amplitude various with time.

For a string with two ends fixed like on a piano or violin, the string can only show standing waves that 'fit'. These standing waves are sinusoidal and their wave length should be such that the beginning and end of the string don't oscillate. In the figure below four possibilities are shown.

Figure 78: *
Standing waves in a string.

We see that there is a simple relation between the length of the string, L and the possible wave length, λ of the standing waves:

$$\frac{n}{2}\lambda = L \Leftrightarrow \lambda = \frac{2L}{n} \text{ with } n \in N \tag{272}$$

Further we see that the smaller the wavelength, the faster the oscillation. This is due to the relation $\lambda \cdot f = \mathrm{v}$ that still holds: $f = \frac{\mathrm{v}}{\lambda} = \frac{n\mathrm{v}}{2L}$.

The traveling waves had as mathematical form $\sin(x - \mathrm{v}t)$. The standing waves take forms like $\sin\frac{x}{\lambda} \cdot \sin(2\pi f t)$. You will learn much more about this in e.g. Fourier Analysis classes.

Water waves and Sound waves It is not necessary that a wave is caused by a tension in the material that tries to restore the equilibrium position. The restoring force can be of a different nature. A well know example is the water waves that we see on lakes and seas. Here gravity is the restoring force: it tries to pull a crest down and push a through up. The water inertia causes overshoot resulting in oscillations, that we call waves. In dealing with waves, we usually don't use the frequency f, but instead the angular velocity $\omega = 2\pi f$. Similarly, frequently the wave length λ is replaced by the wavenumber $k \equiv \frac{2\pi}{\lambda}$. Note that these two quantities are also related to each other by the speed of the waves: $\lambda \cdot f = \frac{2\pi}{\lambda}\frac{\omega}{2\pi} = \frac{\omega}{k} =$ v.

For water waves (with large wave length) the angular momentum and the wave number are coupled to the depth, h, of the water:

$$\omega^2 = gk\tanh(kh) \tag{273}$$

From this we learn that waves on deep water travel much faster than on shallow water. This can be seen on our shores: the waves coming from the open sea are slowed down when the approach our beaches. But behind them the fast ones still come in. As a consequence, the wave gets squeezed in length and thus must get higher. This can be extreme with dramatic consequences: the Tsunami. The wave of the Tsunami is formed out in the open, where the sea is very deep. Here it travels at a very high speed which also means that it is a long wave. The Tsunami waves can travel ate velocities of 200m/s and have wave length of hundreds of kilometers. However at full see their amplitude is in the centimeter, decimeter range. A ship at full sea will hardly notice the passing Tsunami wave. But when the approach the shore, the front of the wave is slowed down to tens of m/s. As the back is still coming in at full speed the wave amplitude has to increase. And thus a huge wave in terms of amplitude storms towards the shore. A wall of water is seen coming, crushing everything in its way.

Sound waves are another type of waves that occur frequently. The can exist in solids, liquids and gasses. In contrast to the waves we have discussed so far, the amplitude is not perpendicular to the direction of traveling. It is what we call a longitudinal wave that oscillates in the same direction as it moves. The other waves are called transversal waves.

For sound waves it is the pressure that is the restoring force. The 'crest' is compressed material, the 'through' is an expansion part. newton was intrigued by sound waves and provided a theory for them. He found that the speed of sound in air, according to his theory, was about 290 m/s. In reality it is some 340 m/s. Newton was well aware of the mismatch. But he couldn't find a good explanation. It took another 100 years for Pierre Laplace corrected Newton's work and arrived at the correct answer. Newton did not know that sound is connected to adiabatic compression. He couldn't as the entire concept was not know. Laplace realized that Newton basically had made an isothermal solution and corrected this.

2.6.11 Exercises, examples & solutions

Exercises Here are some exercises that deals with oscillations. Make sure you practice IDEA.

Exercise 1: A massless spring (spring constant k) is suspended from the ceiling. The spring has an unstretched length l_0. At the other end is a point particle (mass m).

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.
- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

Exercise 2: Same question, but now two springs are used. Spring 1 has spring constant k; spring 2 has $2k$. Both have the same unstretched length l_0.

- The two springs are used in parallel, i.e., both are connected to the ceiling, and m is at the joint other end of the springs.
- Both springs are in series, i.e., spring 1 is suspended from the ceiling, and the other one is attached to the free. The particle is fixed to the free end of the second spring.

Exercise 3: A mass m is attached to two springs. The other ends of the springs are fixed and can not move. The distance between these point is $2l_0$. The mass can move only in the horizontal direction and there is no gravity. See the figure below for a sketch.

The springs are identical: both have rest length l_0 and spring constant k. Based on symmetry, we take the origin in the center of the figure.

We are going to repeat the same analysis as in the previous exercises.

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.
- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

Exercise 4: The same as above, but now the length between the two point where the spring are attached to is l_0 instead of $2l_0$.

Note: in the figure k, l_0 denotes the characteristics of the springs.

- Make a sketch of the situation and define your coordinate system.
- Find the equilibrium position of the mass m.

- Set up the equation of motion for m.
- Solve it for the initial condition that at $t = 0$ the mass m is at the equilibrium position and has a velocity v_0.

Solution to Exercise 1: A massless spring (spring constant
Sketch; $z = 0$ is when the mass is l_0 below the ceiling.

Equilibrium position of the mass m:

$$\sum F = 0 \rightarrow F_v - mg = 0 \tag{274}$$

Force of the spring: $F_v = -k(l - l_0) = -kz$. Thus

$$-kz_{eq} - mg = 0 \rightarrow z_{eq} = -\frac{mg}{k} \tag{275}$$

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kz - mg \tag{276}$$

Solution with $z(0) = z_{eq}$ and $v(0) = v_0$:

homogeneous part of the equation: $m\frac{dv}{dt} + kz = 0$

$$z_{hom}(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{277}$$

with $\omega_0^2 = \frac{k}{m}$

special solution: $z_s = -\frac{mg}{k}$

general solution:

$$z(t) = z_{hom}(t) + z_s(t) = z_{hom}(t) = A\cos\omega_0 t + B\sin\omega_0 t - \frac{mg}{k} \tag{278}$$

initial conditions:

$$z(0) = z_{eq} = -\frac{mg}{k} \rightarrow A = 0 \tag{279}$$

and

$$v(0) = v_0 \rightarrow v_0 = \omega_0 B \rightarrow B = \frac{v_0}{\omega_0} \tag{280}$$

Thus, the solution is

$$z(t) = -\frac{mg}{k} + \frac{v_0}{\omega_0}\sin\omega_o t \tag{281}$$

Solution to Exercise 2: Same question, but now two springs are used. Spring 1 has spring constant
Sketch; $z = 0$ is when the mass is at l_0 below the ceiling. Now we have 2 springs, one with spring constant k_1, the other with k_2. Both have the same rest length l_0

Equilibrium position of the mass m:

$$\sum F = 0 \rightarrow F_{v1} + F_{v2} - mg = 0 \tag{282}$$

Forces of the springs: $F_{v1} = -k_1(l - l_0) = -k_1 z$ and $F_{v2} = -k_2(l - l_0) = -k_2 z$. Thus

$$-k_1 z_{eq} - k_2 z_{eq} - mg = 0 \rightarrow z_{eq} = -\frac{mg}{k_1 + k_2} \tag{283}$$

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -(k_1 + k_2)z - mg \tag{284}$$

Thus we conclude, that the exercise is basically the same: all we have to do is replace k by $K_{tot} = k_1 + k_2$

$$m\frac{dv}{dt} = -k_{tot}z - mg \tag{285}$$

The solution with $z(0) = z_{eq}$ and $v(0) = v_0$ is thus

$$z(t) = -\frac{mg}{k_{tot}} + \frac{v_0}{\omega_0}\sin\omega_o t \tag{286}$$

with $\omega_0^2 = \frac{k_{tot}}{m}$

Solution to Exercise 3: A mass

Again, we have two springs acting on the mass. However, they are no on opposite sides. We expect on symmetry arguments that the equilibrium will be in the middle, i.e at $x = 0$.

If the mass is positioned to the right of $x = 0$, spring 1 is extended beyond its rest length and will pull in the negative x-direction:

$$F_{v1} = -k(l - l_0) = -kx \tag{287}$$

Spring 2 will than be shorter than its rest length and will push to the negative x-direction:

$$F_{v2} = k(l - L_0) = -kx \tag{288}$$

Thus, equilibrium is reached when

$$\sum F = F_{v1} + F_{v2} = 0 \to -2kx = 0 \to x_{eq} = 0 \tag{289}$$

as we anticipated.

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kx - kx = -2kx \tag{290}$$

Thus we conclude, that the exercise is basically the same: all we have to do is replace k by $k_{tot} = 2k$

$$m\frac{dv}{dt} = -2kx \tag{291}$$

General solution $x(t) = A\sin\omega_0 t + B\cos\omega_0 t$ with $\omega_0^2 = \frac{2k}{m}$.

Like in the previous exercises, it is now a matter of specifying the initial conditions and finding A and B.

Solution to Exercise 4: The same as above, but now the length between the two point where the spring are attached to is

Again, we have two springs acting on the mass. Now they don't fit both with their rest length. The will be compressed and try to lengthen. However, based on symmetry we still do expect that $x = 0$ is the equilibrium position.

If the mass is positioned to the right of $x = 0$, spring 1 stille too short and will push to the right:

$$F_{v1} = -k(l - l_0) = -k\left(\frac{l_0}{2} + x - l_0\right) = k\left(\frac{l_0}{2} - x\right) \tag{292}$$

Spring 2 will than be even shorter and will push to the negative x-direction:

$$F_{v2} = k\left(\frac{l_0}{2} - x - l_0\right) = -k\left(\frac{l_0}{2} + x\right) \tag{293}$$

Thus, equilibrium is reached when

$$\sum F = F_{v1} + F_{v2} = 0 \rightarrow k\left(\frac{l_0}{2} - x\right) - k\left(\frac{l_0}{2} + x\right) = -2kx = 0 \rightarrow x_{eq} = 0 \tag{294}$$

as we anticipated.

Equation of motion for m: set up N2

$$m\frac{dv}{dt} = -kx - kx = -2kx \tag{295}$$

Thus we conclude, $k_{tot} = 2k$, which is identical to the previous exercise!

Mass spring
Find a rubber band and use nothing but a mass (that you are not allowed to weigh) that you can tie one way or the other to the spring, a ruler, and the stopwatch/clock on your mobile.

Set up an experiment to find the mass m, the spring constant k, and the damping coefficient b.

Don't forget to make a physics analysis first, a plan of how to find both m and k.

Answers

Jupyter labs

1. Mass-spring system Exercise4.ipynb

2.7 Collisions

2.7.1 What are collisions?

In daily life we do understand what a collision is: the bumping of two objects into each other. From a physics point of view, we see it slightly different. The objects don't have to touch. It is sufficient if they undergo a mutual interaction *'with a beginning and an end'*. What do we mean by this?

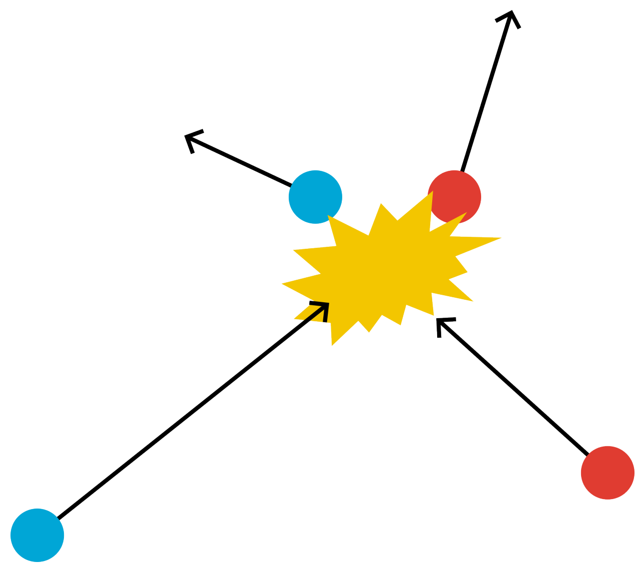


Figure 80: Collision of two particles.

Firstly, the mutual interaction means that the objects interact with each other through a mutual force, i.e. a force (pair) that obeys Newton's third law.
Secondly, we assume that this force works over a small distance only. Or re-phrased we will only consider the situation before the objects feel the force and compare that to after they have felt it. We don't bother about the details of the motion of the objects *during* their interaction. Hence, when we depict a collision as in Figure 80, we usually draw the situation before the collision, then some kind of 'comic way' of showing the collision and finally we draw the outcome of the collision, so after the interaction. In many cases, people leave the middle part out of their drawing.

There are two principle types of collisions to distinguish: *elastic* and *inelastic* collisions. For an elastic collision the kinetic energy is conserved, whereas for an inelastic that is not the case. In the latter case, energy can be converted into deformation or heat.

Since the objects interact under the influence of their mutual interaction, we have conservation of momentum:

$$\sum_i \vec{p}_i^{\,before} = \sum_i \vec{p}_i^{\,after} \tag{296}$$

Why? No external forces implies constant total momentum.

2.7.2 Elastic Collisions

For an elastic collision the kinetic energy is conserved by definition (next to the conservation of momentum). That is the sum of the kinetic energy before the collision is the same as the sum after the collision. This type of collision is also called *hard-ball collision*: as with colliding billiard balls no energy is dissipated into heat or deformation.

For elastic collisions the interaction force needs to be conservative. Then, a potential energy exists. And this energy is such that the objects have the same potential energy before as after the collision. Consequently energy conservation leads to:

$$E_{kin,before} + V_{before} = E_{kin,after} + \underbrace{V_{after}}_{=V_{before}} \Rightarrow E_{kin,before} = E_{kin,after} \tag{297}$$

Solving collision problems Given a collision experiment where the initial situation before the collision is known, how do we compute the situation after the collision? What will the velocities of the object be?

Consider a head-on collision of two point particles on a line as shown in Figure 81. One particle with mass $3m$ is initially at rest $(u = 0)$, the other with mass $2m$ has velocity $2v$. What are the velocities v', u' of the particles after the collision?

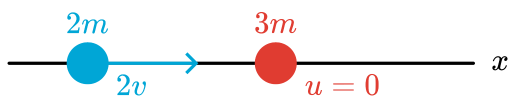


Figure 81: Example of a 1D elastic collision.

We can write down two equations using conservation of momentum and kinetic energy before and after the collision

$$\begin{array}{rcl} 2m(2v) + 0 & = & 2mv' + 3mu' \quad (*) \\ \frac{1}{2}2m(2v)^2 + 0 & = & \frac{1}{2}2mv'^2 + \frac{1}{2}3mu'^2 \end{array} \tag{298}$$

We have two equations and two unknowns (v', u'), therefore we can in principle solve this problem. The question is, what is the best strategy to do so? A strategy is needed especially since one equation involves the square of the velocity.

We first bring the velocities v, v' and u, u' to the same side.

$$\begin{array}{rcl} 2m(2v - v') & = & 3mu' \\ \frac{1}{2}2m(4v^2 - v'^2) & = & \frac{1}{2}3mu'^2 \end{array} \tag{299}$$

Now we divide the two equations (verify yourselves!), this makes the solution very easy as the squares of the velocities always divide out.

$$\Rightarrow 2v + v' = u' \quad (**) \tag{300}$$

We can use this to find both unknowns by smartly adding equations $(*)$ and $(**)$. Smartly in the sense that we can multiply either of the equations with a scalar in such way that one quantity can be discarded.

$$\begin{array}{lcl} 4v = 2v' + 3u' & \qquad & 4v = 2v' + 3u' \\ \underline{2v = -v' + u' | * 2} & & \underline{2v = -v' + u' | * -3} \\ 8v = 5u' & & -2v = 5v' \\ \Rightarrow u' = \frac{8}{5}v & & \Rightarrow v' = -\frac{2}{5}v \end{array} \tag{301}$$

This solution strategy always works. NB: you need to practice this. Although it is conceptually easy, we often see that students have problems when actually solving for the 2 unknowns.

Vpython simulation
Above we provided a Vpython simulation. Change the code in order to verify the above solution.

Actually, now we think about this strategy: isn't it strange, the kinetic energy equation is squared in our unknowns. Shouldn't we expect 2 solutions?

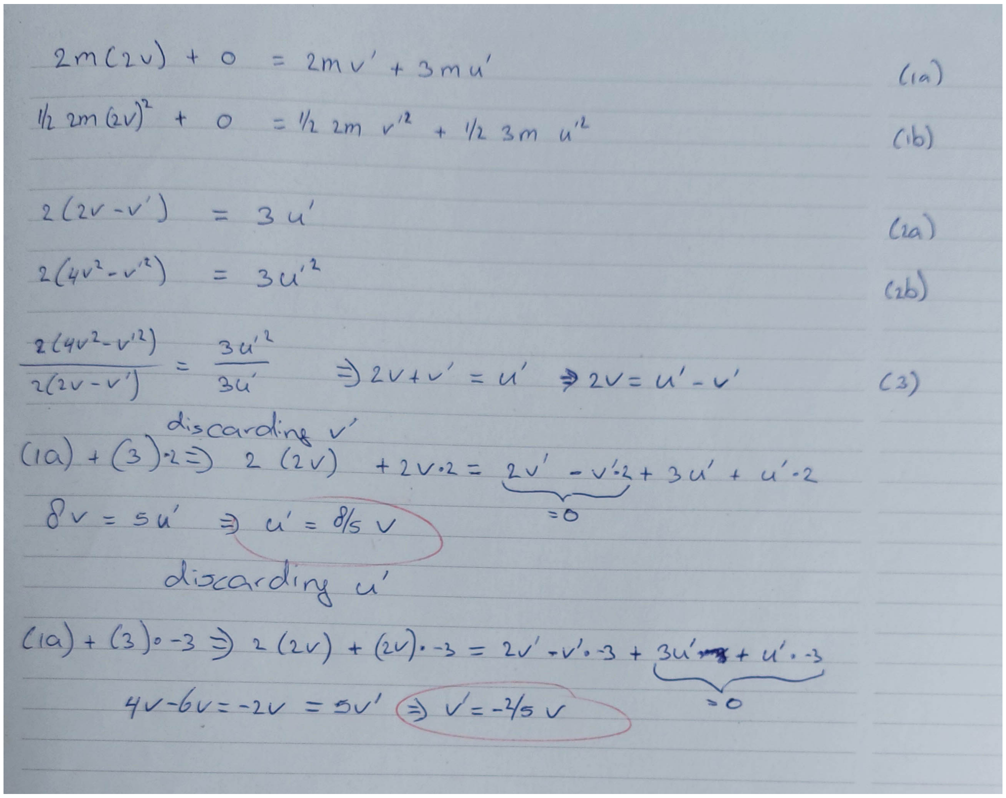


Figure 82: *
Solving

The answer is yes: there ought to be 2 solutions. Where is the second one? Note that when dividing the two equations, we have to make sure that we do not divide by 0. It is very well possible that we do so: suppose $u' = 0$, then also $2v - v' = 0$ and we can not do the division. But what does that mean: $u' = 0$ and $2v - v' = 0$? Well, of course, then we have after the collision that the incoming particle $2m$ still has velocity $2v$ and the other particle $3m$ is still at rest.

In retrospect: of course this must be one of the solutions to the problem. We haven't specified anything about the interaction force. But suppose it is absent, that is, the particles don't interact at all. Then of course the situation before the collision (a bit strange calling this a collision, but anyway), will still be present after the 'collision'. If nothing happens to the particles, then obviously the sum of the momentum as well as of the kinetic energy stays the same. This actually provides a great check of your work: do you recover the initial conditions?

Collisions in a plane Consider now a 2D elastic collision such that the two particles collide in the origin, Figure 83.

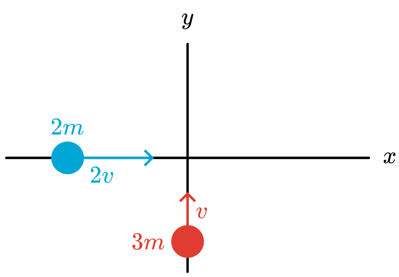


Figure 83: Example of a 2D elastic collision.

If we write down the equation of conservation of momentum (in x, y components) and of kinetic energy, we get

$$\begin{array}{lcr} 2m(2v)+0 & = & 2mv'_x + 3mu'_x \\ 0+3mv & = & 2mv'_y + 3mu'_y \\ \frac{1}{2}2m(2v)^2 + \frac{1}{2}3mv^2 & = & \frac{1}{2}2mv'^2 + \frac{1}{2}3mu'^2 \end{array} \tag{302}$$

Now we have **4** unknowns (v'_x, v'_y, u'_x, u'_y) but only **3** equations. The outcome seems not to be determined by the initial condition... Of course, that cannot be the case (Think shortly why?). The outcome is fully determined by the initial conditions, but we did not set up the equations in the best way because we did not first transform the problem into a 1D problem such that the collision is head-on.

We can use a Galilean Transformation to put one particle at rest. Here we set the blue particle to rest by subtracting $-2v$ from its velocity, that is we move with the blue particle (prior to the collision). The corresponding Galilean Transformation is

$$\begin{array}{lcl} x' & = & x - 2vt \\ y' & = & y \end{array} \tag{303}$$

The red particle now has velocity $(-2v, v)$. The problem is still 2D.

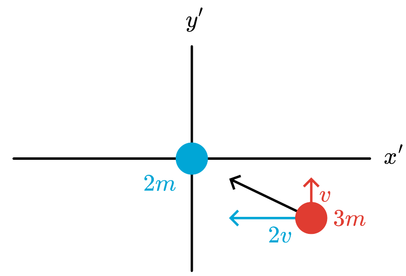


Figure 84: Applying the Galilean Transformation.

Next, we can rotate the coordinate system, to obtain a 1D head-on collision that we can solve as above.

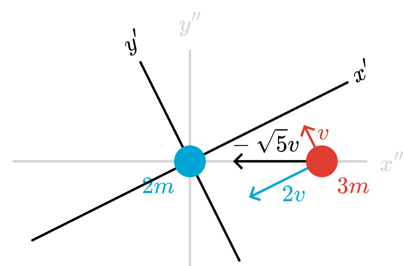


Figure 85: Rotating the coordinate system.

We see that we now have a 1-dimensional elastically collision with a particle of mass $3m$ coming in over the x"-axis with velocity $-\sqrt{5}v$. It will collide with a particle of mass $2m$ which is at rest. We know how to solve this problem and find the velocities of both particles after the collision. If we do this, we find that after the collision the velocity of the blue particle is $U''_{x''} = -\frac{6}{5}\sqrt{5}v$ and of the red particle $V''_{x''} = -\frac{1}{5}\sqrt{5}v$. Note that we have specified these velocity in the $(x", y")$ coordinate system.

Next steps would be to convert the velocities back to the initial coordinate frame. That is a bit cumbersome, but again conceptually easy. The final answer in the original frame of reference is:

$$\begin{array}{lll} 2m: & v'_x = -\frac{2}{5}v, & v'_y = \frac{6}{5}v \\ 3m: & u'_x = \frac{8}{5}v, & u'_y = \frac{1}{5}v \end{array} \tag{304}$$

Figure 86: *
The 3Blue1Brown series on linear algebra describes the linear transformations above in a mathematical way. Using linear algebra, above computations will become easier.

```
%pip install ipywidgets
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from IPython.display import display, HTML, Math
import ipywidgets as widgets

# Constants
m1_init = 1
m2_init = 1
v1x_init = 1
v1y_init = 0
v2x_init = 0
v2y_init = 1

x1_init = -1
y1_init = 0
x2_init = 0
y2_init = -1


# Time setup
dt = 0.05
time_max = 1
times_neg = np.arange( -time_max, dt, dt)
times_pos = np.arange(dt, time_max, dt)
times = np.arange( -time_max, time_max, dt)
Ntimes = len(times_neg) + len(times_pos)

# Widget slider mass 1
mass1_slider = widgets.FloatSlider(
    value=1,
    min=1,
    max=5,
    step=1,
    description='m1:',
    continuous_update=False
)

# Widget slider mass 2
mass2_slider = widgets.FloatSlider(
    value=1,
    min=1,
    max=5,
    step=1,
    description='m2:',
    continuous_update=False
)
```

```
# Widget slider velo v1x
velocity1x_slider = widgets.FloatSlider(
    value=1,
    min=1,
    max=5,
    step=1,
    description='v1x:',
    continuous_update=False
)

# Widget slider velo
velocity1y_slider = widgets.FloatSlider(
    value=0,
    min= -3,
    max=3,
    step=1,
    description='v1y:',
    continuous_update=False
)

# Widget slider velo v2x
velocity2x_slider = widgets.FloatSlider(
    value=0,
    min= -3,
    max=1,
    step=1,
    description='v2x:',
    continuous_update=False
)

# Widget slider velo
velocity2y_slider = widgets.FloatSlider(
    value=1,
    min=1,
    max=3,
    step=1,
    description='v2y:',
    continuous_update=False
)

def CalcCol(m1_init,m2_init,v1x_init,v1y_init,v2x_init,v2y_init):
    #initiallise m's and v's
    v1x, v1y = v1x_init, v1y_init
    v2x, v2y = v2x_init, v2y_init
    m1, m2 = m1_init, m2_init
    cos_2 = v2x/np.sqrt(v2x*v2x+v2y*v2y)
    sin_2 = v2y/np.sqrt(v2x*v2x+v2y*v2y)

    # new velocities after collision
    # velo center of gravity
    Vcg_x=(m1*v1x+m2*v2x)/(m1+m2);
    Vcg_y=(m1*v1y+m2*v2y)/(m1+m2);
    #relative velos before coll in COG
    u1x=v1x -Vcg_x;
```

```
    u1y=v1y -Vcg_y;
    u2x=v2x -Vcg_x;
    u2y=v2y -Vcg_y;
    u1=np.sqrt(u1x*u1x+u1y*u1y);
    u2=np.sqrt(u2x*u2x+u2y*u2y);
    cos_1=u1x/u1;
    sin_1=u1y/u1;
    cos_2=u2x/u2;
    sin_2=u2y/u2;

 #rotation matrix to rotatate to 1D picture -> particles moving over x -axis
    A11=cos_1;
    A12=sin_1;
    A21= -sin_1;
    A22=cos_1;
    uac1x=A11*u1x+A12*u1y;
    uac1y=A21*u1x+A22*u1y;
    uac2x=A11*u2x+A12*u2y;
    uac2y=A21*u2x+A22*u2y;

    #new velos: do a 1D collision
    wac2x=((1 -m1/m2)*uac2x+2*m1/m2*uac1x)/(1+m1/m2);
    wac1x=uac2x -uac1x+wac2x;
    wac1y=0;
    wac2y=0;
    #rotate back
    w1x=A11*wac1x -A12*wac1y;
    w1y= -A21*wac1x+A22*wac1y;
    w2x=A11*wac2x -A12*wac2y;
    w2y= -A21*wac2x+A22*wac2y;
    #transform back to lab frame
    vnew1_x=w1x+Vcg_x;
    vnew1_y=w1y+Vcg_y;
    vnew2_x=w2x+Vcg_x;
    vnew2_y=w2y+Vcg_y;
    if vnew1_x <0.0001:
        alpha_1 = 90
    else:
        alpha_1 = round(np.arctan(vnew1_y / vnew1_x)/np.pi*180/10)*10;
    if vnew2_x <0.0001:
        alpha_2 = 90
    else:
        alpha_2 = round(np.arctan(vnew2_y / vnew2_x)/np.pi*180/10)*10;

    return vnew1_x, vnew1_y, vnew2_x, vnew2_y, alpha_1, alpha_2

def generate_animation(m1_init,m2_init,v1x_init,v1y_init,v2x_init,v2y_init):
    v1x, v1y = v1x_init, v1y_init
    v2x, v2y = v2x_init, v2y_init
    x1 = v1x*( -time_max)
    x2 = v2x*( -time_max)
    y1 = v1y*( -time_max)
    y2 = v2y*( -time_max)
```

```
    m1, m2 = m1_init, m2_init

    u1_x, u1_y, u2_x, u2_y, a1, a2 = CalcCol(m1_init,m2_init,v1x_init,v1y_init,v2x_init,v2y_init)

    # Position history
    x1_list, y1_list = [], []
    x2_list, y2_list = [], []

    for t in times_neg:
        x1_t = v1x * t
        y1_t = v1y * t
        x1_list.append(x1_t)
        y1_list.append(y1_t)
        x2_t = v2x * t
        y2_t = v2y * t
        x2_list.append(x2_t)
        y2_list.append(y2_t)

    for t in times_pos:
        x1_t = u1_x * t
        y1_t = u1_y * t
        x1_list.append(x1_t)
        y1_list.append(y1_t)
        x2_t = u2_x * t
        y2_t = u2_y * t
        x2_list.append(x2_t)
        y2_list.append(y2_t)

    # Create figure and axes
    fig, ax = plt.subplots(figsize=(6, 6))
    ax.set_xlim( -6, 6)
    ax.set_ylim( -6, 6)
#    ax.set_yticks([])
    ax.set_title("2D Elastic Collision")
    ax.plot([ -6,6],[0,0], color='grey')
    ax.plot([0,0],[ -6,6], color='grey')

    p1, = ax.plot([], [], 'ro', markersize=6, label='Particle 1')
    p2, = ax.plot([], [], 'bo', markersize=6, label='Particle 2')
    p1_line_f, = ax.plot((x1_list[0],x1_list[0]),(y1_list[0],y1_list[0]),'r -')
    p1_line_a, = ax.plot((0,0),(0,0),'r -')
    p2_line_f, = ax.plot((x2_list[0],x2_list[0]),(y2_list[0],y2_list[0]),'b -')
    p2_line_a, = ax.plot((0,0),(0,0),'b -')
    ax.grid()
    ax.legend(loc='upper right')

    def init():
        p1.set_data([], [])
        p2.set_data([], [])
        return p1, p2

    def update(i):
        p1.set_data([x1_list[i]], [y1_list[i]])
        p2.set_data([x2_list[i]], [y2_list[i]])
```

```
        if i < len(times_neg):
            p1_line_f.set_data((x1_list[0],x1_list[i]),(y1_list[0],y1_list[i]))
            p2_line_f.set_data((x2_list[0],x2_list[i]),(y2_list[0],y2_list[i]))
        else:
            p1_line_a.set_data((0,x1_list[i]),(0,y1_list[i]))
            p2_line_a.set_data((0,x2_list[i]),(0,y2_list[i]))
        return p1, p2


    ani = animation.FuncAnimation(fig, update, frames=Ntimes, init_func=init,
                                  interval=50, blit=True)

    plt.close(fig)
    return HTML(ani.to_jshtml())

# Show slider and link it to animation
widgets.interact(generate_animation,
    m1_init = mass1_slider, m2_init = mass2_slider,
    v1x_init = velocity1x_slider, v1y_init = velocity1y_slider,
    v2x_init = velocity2x_slider, v2y_init = velocity2y_slider
    );
```

2.7.3 Collisions in the Center of Mass frame

There is a special frame of reference: the Center of Mass (CM) frame. Its origin is defined with respect to the *lab frame* as

$$\vec{R} = \frac{\sum m_i \vec{x}_i}{\sum m_i} \tag{305}$$

We will introduce this formally in the next chapter.

As the mass is conserved during a collision we have

1. $\sum m_i = const$ and
2. as the momentum is conserved $\sum m_i \dot{\vec{x}}_i = const$,

we see that the velocity of the CM does not change before and after collision

$$\dot{\vec{R}}_{before} = \dot{\vec{R}}_{after} \tag{306}$$

Instead of working in the lab frame we can use the CM frame. A sketch of the coordinates and vectors is given in the figure below.

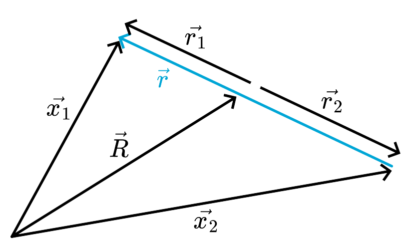


Figure 87: Center of mass.

For the relative coordinates $\vec{r}_i$ it holds that $\sum m_i\vec{r}_i = 0$. Considering two particles: The relative distance $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{x}_1 - \vec{x}_2$ is Galilei invariant.

Using this property we express the vectors $\vec{r}_1$ and $\vec{r}_2$ in terms of the relative distance vector $\vec{r}$ for $\vec{r}_1 = \frac{\mu}{m_1}\vec{r}$ and $\vec{r}_2 = -\frac{\mu}{m_2}\vec{r}$ with μ the so-called reduced mass (see next chapter). Therefore

$$\begin{aligned} m_1\vec{r}_1 &= \mu\dot{\vec{r}}_1 \\ m_2\vec{r}_2 &= -\mu\dot{\vec{r}}_2 \end{aligned} \tag{307}$$

This means the momenta of both particles are *always* equal in magnitude and opposed in direction in the CM frame. Only the orientation of the pair $\dot{\vec{r}}_{1,2}$ can change from before to after the collision.

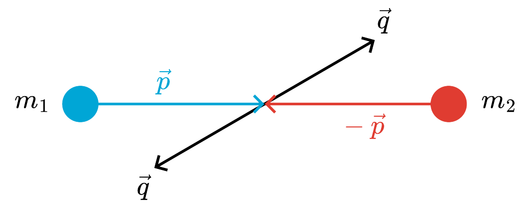


Figure 88: Collision in center of mass frame.

2.7.4 Computational

In an angle-free representation, the changed velocities are computed using the centers $\mathbf{x}_1$ and $\mathbf{x}_2$ at the time of contact as

$$\begin{aligned} \mathbf{v}'_1 &= \mathbf{v}_1 - \frac{2m_2}{m_1+m_2}\,\frac{\langle \mathbf{v}_1-\mathbf{v}_2,\,\mathbf{x}_1-\mathbf{x}_2\rangle}{\|\mathbf{x}_1-\mathbf{x}_2\|^2}\,(\mathbf{x}_1-\mathbf{x}_2), \\ \mathbf{v}'_2 &= \mathbf{v}_2 - \frac{2m_1}{m_1+m_2}\,\frac{\langle \mathbf{v}_2-\mathbf{v}_1,\,\mathbf{x}_2-\mathbf{x}_1\rangle}{\|\mathbf{x}_2-\mathbf{x}_1\|^2}\,(\mathbf{x}_2-\mathbf{x}_1) \end{aligned}$$

where the angle brackets indicate the inner product (or dot product) of two vectors.

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2.7.5 Inelastic Collisions

For inelastic collisions only the *momentum is conserved*, but *not the kinetic energy*. The kinetic energy after the collision is always less than before the collision. As the total energy is conserved, some kinetic energy is converted to deformation or heat.

The amount of "inelasticity" or loss of energy can be quantified by the *coefficient of restitution* e

$$e \equiv \frac{v_{rel,after}}{v_{rel,before}} \tag{308}$$

$$e^2 \equiv \frac{E_{kin,after}}{E_{kin,before}} \text{ in CM frame} \tag{309}$$

For $e = 0$ the collision is fully inelastic, for $e = 1$ it is fully elastic.

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2.7.6 Exercises, examples & solutions

Examples

Newton's Cradle Click on the image below for a video on Newton's cradle (gives you also the possibility to 'play' with different numerical solvers, from (too) simple to advanced).

Exercise 1: Colliding Superballs
Watch this video on bouncing superballs. We discussed this problem in this chapter.

Do you agree with the explanation in the movie?

We seem to violate the conservation of kinetic energy: there is much more kinetic energy after the collision than before! Can you figure out what is happening?

Tip
Look carefully at the bouncing of the blue ball with the earth. Is it really true that the velocity after bouncing is v and that the earth does not move? Probably not, as this violates conservation of momentum!

Elastic Collision

1D elastic collision Consider two particles, m_1 and m_2, moving along the x-axis. The two particles will elastically collide. We set mass 2 at a value of 1 (kg) and vary m_1. In the widget below, you can change the value of m_1 and of the velocities of both particles before the collision.

Solve the collision by using conservation of momentum and kinetic energy and compare your results with those of the widget.

Warning
here python code for jpnb: 1DelasticCollision.py

test below

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from math import atan2, degrees
from IPython.display import HTML

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Adjustable Parameters
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
m1 = 1   # mass of particle 1
m2 = 6   # mass of particle 2
v1x = 1  # x velocity of particle 1
v1y = 0  # y velocity of particle 1
v2y = 1  # y velocity of particle 2

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Constants and Initial Velocities
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
```

```
dt = 0.05
t_stop = 10
tcoll = 5
scale = 40

v1 = np.array([v1x, v1y], dtype=float)
v2 = np.array([0, v2y], dtype=float)

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Compute Collision Result
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
def compute_collision(m1, m2, v1, v2):
    Vcg = (m1 * v1 + m2 * v2) / (m1 + m2)
    u1 = v1 - Vcg
    u2 = v2 - Vcg

    angle = atan2(u1[1], u1[0])
    R = np.array([[np.cos(angle), np.sin(angle)],
                  [ -np.sin(angle), np.cos(angle)]])

    uac1 = R @ u1
    uac2 = R @ u2

    wac2x = ((1 - m1 / m2) * uac2[0] + 2 * m1 / m2 * uac1[0]) / (1 + m1 / m2)
    wac1x = uac2[0] - uac1[0] + wac2x

    wac1 = np.array([wac1x, 0])
    wac2 = np.array([wac2x, 0])

    R_inv = np.linalg.inv(R)
    w1 = R_inv @ wac1 + Vcg
    w2 = R_inv @ wac2 + Vcg

    return w1, w2

w1, w2 = compute_collision(m1, m2, v1, v2)
alpha_1 = round(degrees(atan2(w1[1], w1[0])) / 10) * 10
alpha_2 = round(degrees(atan2(w2[1], w2[0])) / 10) * 10

x1_init = -v1 * (t_stop - tcoll)
x2_init = -v2 * (t_stop - tcoll)

x1_coll = x1_init + v1 * tcoll
x2_coll = x2_init + v2 * tcoll

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Set Up Figure
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
fig, ax = plt.subplots(figsize=(6, 6))
ax.set_xlim( -300, 300)
ax.set_ylim( -300, 300)
ax.set_xticklabels([])
ax.set_yticklabels([])
ax.set_aspect('equal')
```

```
ax.grid()

p1, = ax.plot([], [], 'ro')
p2, = ax.plot([], [], 'bo')
path1, = ax.plot([], [], 'r - -', lw=1)
path2, = ax.plot([], [], 'b - -', lw=1)
angle_text = ax.text(0.02, 0.02, '', transform=ax.transAxes)

traj1, traj2 = [], []

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Animation Functions
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
def init():
    p1.set_data([], [])
    p2.set_data([], [])
    path1.set_data([], [])
    path2.set_data([], [])
    angle_text.set_text('')
    return p1, p2, path1, path2, angle_text

def update(frame):
    t = frame * dt
    if t < tcoll:
        pos1 = x1_init + v1 * t
        pos2 = x2_init + v2 * t
    else:
        pos1 = x1_coll + w1 * (t - tcoll)
        pos2 = x2_coll + w2 * (t - tcoll)

    traj1.append(pos1.copy())
    traj2.append(pos2.copy())

    p1.set_data([scale * pos1[0]], [scale * pos1[1]])
    p2.set_data([scale * pos2[0]], [scale * pos2[1]])

    traj1_np = np.array(traj1)
    traj2_np = np.array(traj2)

    path1.set_data(scale * traj1_np[:, 0], scale * traj1_np[:, 1])
    path2.set_data(scale * traj2_np[:, 0], scale * traj2_np[:, 1])

    if abs(t - t_stop) < dt:
        angle_text.set_text(f" \alpha = {alpha_1}°, \alpha = {alpha_2}°")

    return p1, p2, path1, path2, angle_text

# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
# Create and Display Animation
# - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
ani = FuncAnimation(fig, update, frames=int(t_stop / dt), init_func=init, blit=True, interval=50)
HTML(ani.to_jshtml())
```

2D elastic collision Next, we consider an elastic collision between m_1 and m_2, but now in a 2-dimensional setting.

In the widget below, the situation is animated. You can change the values of the initial velocity and masses. Can you (qualitatively) predict the outcome of the collision for a given set os parameters?

> **Warning**
> here python code for jpnb: 2DElasticCollision.py

Inelastic Collision PArticle m_1 is moving over the x-axis with unit velocity. Simultaneously, particle m_2 is moving over the y-axis also with unit velocity. Both particles will collide in the origin. The collision is inelastic with restitution coefficient e.

The widget below shows the trajectories of the particles and gives the velocities after the collision. Moreover, als the angle of the trajectories after the collision with the x-axis is given.

> **Warning**
> here python code for jpnb: 2DCollision.py

Can you solve this problem for a few values of the restitution coefficient? The 'easy ones' are for $e = 0$.

> **Exercise 2: Completely inelastic collision**
> Consider a particle with mass M being at rest in your frame of reference. A second particle of mass m comes in over the negative x-direction with velocity v. The collision is completely inelastic.
>
> Find the velocities after the collision.

> **Solution to Exercise 2: Completely inelastic collision**
> Given: the collision is completely inelastic. That means $e = 0$ or in words: after the collision the two particles stick together and move as one particle. Thus, we have only one unknown velocity after the collision.
>
> The problem is 1-dimensional and we can solve it by requiring conservation of momentum:
>
> $$\begin{split}
>
> \text{before ;;;;;;} mv &+ M \cdot 0 = (m+M)U \text{ ;;;;;; after} \ \Rightarrow U &= \frac{m}{m+M} v
>
> \end{split}$$

> **Exercise 3: Intuitive collisions**
> Consider two particles (m_1,m_2) with velocities (v_1,v_2) before head-to-head collision. What will the situation be after collision, tell so without calculations, if:
>
> 1. $m_1 = m_2$ and $v_1 = v; v_2 = 0$
> 2. $m_1 = m_2$ and $v_1 = v; v_2 = -v$
> 3. $m_1 = 2m_2$ and $v_1 = v; v_2 = 0$
> 4. $2m_1 = m_2$ and $v_1 = v; v_2 = 0$
> 5. $m_1 = 2m_2$ and $v_1 = v; v_2 = -v$

Exercise 4: A particle of mass $3m$ and velocity $2v$ will collide with a particle of mass $2m$ and velocity $-3v$. The problem is 1-dimensional.

- The collision is elastic. Find the velocities of the masses after the collision.
- The collision is completely inelastic. Find the velocities of the masses after the collision.
- The restitution coefficient is: e=1/5. Find the velocities of the masses after the collision.

Exercise 5: A particle of mass $2m$ moves over the x-axis with velocity v. It will collide with a particle of mass m that moves over the y-axis also with velocity v. The collision is completely inelastic.

Find the velocity of the particles after the collision and calculate the loss of kinetic energy.

Exercise 6: A tennis ball is dropped from a height of 1m (with zero initial velocity) on the tennis court. The restitution coefficient is $\frac{1}{2}\sqrt{2}$. After how many bounces does the tennis ball no longer reach a height of 0.25m. Friction with the air can be ignored.

Exercise 7: In Hollywood films often one of the persons is shot. That person (whether dead, wounded or 'just fine' for the hero) is blown of its feet and may fly a meter or more backwards.

The shooter, however, does not fly or fall backwards.

1. Show that if the victim moves backwards significantly, then the shooter shoot do at least the same.
2. A bullet weighs several grams and may have a velocity of several hundred m/s. Estimate what the backward velocity of a victim is. For comparison: when we walk, our velocity is 1 to 2 m/s. Conclusion?

Exercises

Solution to Exercise 4: A particle of mass

- $3m$ has velocity $-2v$ and $2m$ has velocity $3v$
- Both particles have zero velocity.
- $3m$ has velocity $-2/5v$ and $2m$ has velocity $3/5v$.

Solution to Exercise 5: A particle of mass

$$\vec{v}_{after} = \frac{2}{3}v\hat{x} + \frac{1}{3}v\hat{y} \tag{310}$$

$$\Delta E_{kin} = -\frac{2}{3}mv^2 \tag{311}$$

Solution to Exercise 6: A tennis ball is dropped from a height of 1m (with zero initial velocity) on the tennis court. The restitution coefficient is
After each bounce, the tennis ball reaches half of the height it had before the bounce. Thus after two bounces, the ball reaches 25cm and with the third bounce only 12.5cm.

Solution to Exercise 7: In Hollywood films often one of the persons is shot. That person (whether dead, wounded or 'just fine' for the hero) is blown of its feet and may fly a meter or more backwards.

1. We can consider the shooting as a collision. Bullets don't bounce back, they penetrate a body. So the victim 'gains' maximum momentum if the bullet stays in the body. Then according to conservation of momentum, we have for this inelastic collision:

$$m_b v_b + M_v \cdot 0 = (m_b + M_v)\,U \tag{312}$$

Thus the velocity of the victim after being shot is:

$$U = \frac{m_b}{m_b + M_v} v_b \tag{313}$$

For the shooter a similar argument holds: before the shot, bullet & shooter have zero momentum. After firing, the bullet has velocity v_b. Thus conservation of momentum requires:

$$0 = m_b v_b + M_s U_s \tag{314}$$

and we find for the velocity of the shooter:

$$U_s = -\frac{m_b}{M_s} v_b \tag{315}$$

Conclusion: as the mass of the bullet is negligible compared to that of a person both shooter and victim have similar velocities. As their mass is comparable, it is clear: from a physics point of view, if the victim is blown backward, than also the shooter is.

1. From the above we get, using $m_b \approx 10 \cdot 10^{-3}$kg, $v_b \approx 500$m/s and $M_v \approx 70$kg:

$$U_v = \frac{m_b}{m_b + M_v} v_b \approx 7cm/s \tag{316}$$

That is much too little to 'knock' someone over. Hollywood is good at 'dramatic effects', not so good at physics.

restitution coefficient
Is the restitution coefficient of a bouncing tennis ball a constant or does it depend on the velocity at bouncing? You can 'easily' find out yourself. What you need is a tennis ball, and your mobile with the phyphox app.

Experiment: drop a tennis ball with zero initial velocity from various height, H. Use the acoustic chronometer to measure the time between multiple bounces.

1. Show that the relation between height and time between two bounces is equal to $s = \frac{1}{8}gt^2$
2. Use your recordings to compute the height as function of number of bounces and compute the restitution coefficient e.
3. Plot e as a function H and you will have answered the above question.

Answers

2.8 Two Body Problem: Kepler revisited

Newton must have realized that his analysis of the Kepler laws was not 100% correct. After all, the sun is not fixed in space and even though its mass is much larger than any of the planets revolving it, it will have to move under the influence of the gravitational force by the planets. Take for example, the sun and earth as our system. By the account of Newton's third law, the Earth exerts also a force on the Sun. Therefore, the Sun has to move as well; thus, we must revisit the Earth-Sun analysis and incorporate that the Sun isn't fixed in space.

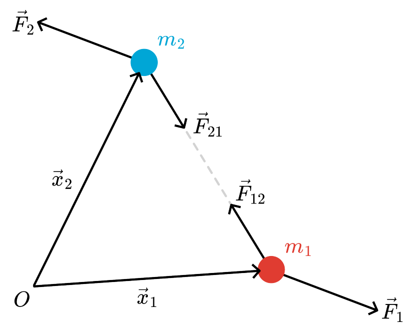


Figure 90: Two-particle system, with an action/reaction pair of forces.

The *two-body problem* is stated hereby as:

Particle m_1 feels an external force $\vec{F}_1$ and an interaction force from particle two, $\vec{F}_{21}$. Similarly for particle m_2: it feels an external force $\vec{F}_2$ and an interaction force from particle one, $\vec{F}_{12}$.

Consider the situation in the figure:

$$m_1\ddot{\vec{x}}_1 = \vec{F}_1 + \vec{F}_{21} \tag{317}$$

$$m_2\ddot{\vec{x}}_2 = \vec{F}_2 + \vec{F}_{12} \tag{318}$$

Add the two equations and use N3: $\vec{F}_{12} = -\vec{F}_{21}$:

$$m_1\ddot{\vec{x}}_1 + m_2\ddot{\vec{x}}_2 = \vec{F}_1 + \vec{F}_2 \Leftrightarrow \tag{319}$$

$$\dot{\vec{P}} = \vec{F}_1 + \vec{F}_2 \tag{320}$$

with $\vec{P} \equiv \vec{p}_1 + \vec{p}_2$. In words, it is as if a particle with (total) momentum $\vec{P}$ responds to the external forces but does not react to internal forces (the mutual interaction).

2.8.1 Center of Mass

It is now logical to assign the total mass $M = m_1 + m_2$ to this fictitious particle. It has momentum $\vec{p}_1 + \vec{p}_2$ which we can also couple to its mass M and assign a velocity $\vec{V}$ to it such that $\vec{P} = M\vec{V}$. Furthermore, if this fictitious mass has velocity $\vec{V}$, we can also assign a position to it. After all, $\vec{V} = \frac{d\vec{R}}{dt}$, which gives us the recipe for the position $\vec{R}$.

Its velocity $\vec{V}$ and position $\vec{R}$ then follow as:

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_1\frac{d\vec{x}_1}{dt} + m_2\frac{d\vec{x}_2}{dt}}{m_1 + m_2} \tag{321}$$

$$\Rightarrow \vec{R} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} + \vec{C}$$

In the last equation, we have an integration constant in the form of a vector, $\vec{C}$. We are free to choose it as we want: its precise value does not affect the velocity $\vec{V}$ nor the momentum $\vec{P}$ of our fictitious particle.

It makes sense, to choose: $\vec{C} = 0$ and thus define as position of the particle:

$$\vec{R} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} \tag{322}$$

Why?

We have a few arguments:

1. if the particles are actually each half of a real particle, we obviously require that $\vec{R}$ is the position of the real particle.
2. If the particles are separate by a small distance, we would like to have the fictitious particle somewhere in between the two. Moreover, if the two particles are identical, it makes sense to have the fictitious particle right in between them: the system is symmetric.

Where, in general is the position $\vec{R}$? That can be easily seen from the figure below.

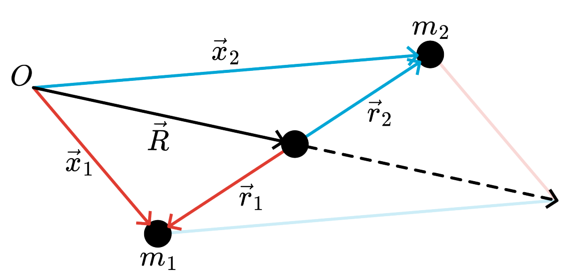


Figure 91: Center of Mass and relative coordinates.

We rewrite the definition of $\vec{R}$:

$$\vec{R} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2} = \vec{x}_1 + \frac{m_2}{m_1 + m_2}\left(\vec{x}_2 - \vec{x}_1\right) \tag{323}$$

Thus, the last part of the above equation tells us: first go to m_1 and then, 'walk' a fraction $\frac{m_2}{m_1+m_2}$ of the line connecting m_1 and m_2. If you have done that, you are at position $\vec{R}$.
Note: if $m_1 = m_2$ this recipe indeed brings us right between the two particles.
Further note: the position of M is always on the line from m_1 to m_2. If m_1 is much larger than m_2, it will be located close to m_1 and vice versa.

We call this position the **center of mass**, or CM for short. Reason: if we look at the response of our two particle system to the forces, it is as if there is a particle M at position $\vec{R}$ that has all the momentum of the system.

It turns out to be convenient to define relative coordinates with respect to the center of mass position (see also the figure above):

$$\vec{r}_1 \equiv \vec{x}_1 - \vec{R} \text{ and } \vec{r}_2 \equiv \vec{x}_2 - \vec{R} \tag{324}$$

Via the external forces, we can 'follow' the motion of the center of mass position, i.e. $\vec{R}$. From the CM as new origin, we can find the position of the two particles.

A helpful rule is found from:

$$\begin{aligned} &m_1\vec{r}_1 + m_2\vec{r}_2 = \\ &= m_1\left(\vec{x}_1 - \vec{R}\right) + m_2\left(\vec{x}_2 - \vec{R}\right) \\ &= m_1\vec{x}_1 + m_2\vec{x}_2 - (m_1+m_2)\vec{R} = 0 \end{aligned} \tag{325}$$

$$\Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \tag{326}$$

This has an important consequence: if we know $\vec{r}_1$, we know $\vec{r}_2$, and vice versa. Note: the directions of $\vec{r}_1$ and $\vec{r}_2$ are always opposed and the center of mass $\vec{R}$ is located somewhere on the connecting line between m_1 and m_2.

Note 2: in the case of no external forces $\vec{F}_1 = \vec{F}_2 = 0$ and only internal forces $\vec{F}_{12} \neq 0$ the CM moves according to N1 with constant velocity ($\dot{\vec{P}} = 0$).

2.8.2 Energy

In terms of relative coordinates, we can write the kinetic energy as a part associated with the CM and a part that describes the kinetic energy with respect to the CM, i.e., 'an internal kinetic energy.'

$$\begin{aligned} E_{kin} &\equiv \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 \\ &= \tfrac{1}{2}m_1\left(\dot{\vec{r}}_1 + \dot{\vec{R}}\right)^2 + \tfrac{1}{2}m_2\left(\dot{\vec{r}}_2 + \dot{\vec{R}}\right)^2 \\ &= \tfrac{1}{2}M\dot{\vec{R}}^2 + \tfrac{1}{2}m_1\dot{\vec{r}}_1^2 + \tfrac{1}{2}m_2\dot{\vec{r}}_2^2 \end{aligned} \tag{327}$$

For the potential energy, we may write:

$$V = \sum V_i + \frac{1}{2}\sum_{i\neq j}(V_{ij} + V_{ji}) \tag{328}$$

With V_i the potential related to the external force on particle i and V_{ij} the potential related to the mutual interaction force from particle i exerted on particle j (assuming that all forces are conservative).

2.8.3 Angular Momentum

The total angular momentum is, like the total momentum, defined as the sum of the angular momentum of the two particles:

$$\vec{L} = \vec{l}_1 + \vec{l}_2 = \vec{x}_1 \times \vec{p}_1 + \vec{x}_2 \times \vec{p}_2 \tag{329}$$

We can write this in the new coordinates:

$$\vec{L} = \vec{R} \times \vec{P} + \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = \vec{L}_{cm} + \vec{L}' \tag{330}$$

We find: that the total angular momentum can be seen as the contribution of the CM and the sum of the angular momentum of the individual particles as seen from the CM.

2.8.4 Reduced Mass

Suppose that there are no external forces. Then the equation of motion for both particles reads as:

$$\begin{aligned} m_1 \ddot{\vec{x}}_1 &= \vec{F}_{12} \\ m_2 \ddot{\vec{x}}_2 &= \vec{F}_{21} = -\vec{F}_{12} \end{aligned} \tag{331}$$

If we divide each equation by the corresponding mass and subtract one from the other we get:

$$\frac{d^2}{dt^2}(\vec{x}_1 - \vec{x}_2) = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12} \tag{332}$$

Note that the interaction force $\vec{F}_{12}$ is a function of the relative position of the particles, i.e., $\vec{x}_1 - \vec{x}_2 = \vec{r}_1 - \vec{r}_2$. Introduce $\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2 = \vec{x}_1 - \vec{x}_2$, then we obtain:

$$\frac{d^2}{dt^2}\vec{r}_{12} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}_{12}(\vec{r}_{12}) \tag{333}$$

As a final step, we introduce the *reduced mass* μ:

$$\frac{1}{\mu} \equiv \frac{1}{m_1} + \frac{1}{m_2} \Leftrightarrow \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{334}$$

And we can reduced the two-body problem to a single-body problem, by writing down the equation of motion for an imaginary particle with reduced mass.

$$\mu \frac{d^2 \vec{r}_{12}}{dt^2} = \vec{F}_{12} \tag{335}$$

If $m_1 \gg m_2$ we have $\mu \to m_2$. This is not surprising: when m_1 is much larger than m_2, it will look like m_1 is not changing its velocity at all. Or seen from the CM: is appears to be not moving. In this case, we can ignore particle 1 and replace it by a force coming out of a fixed position.

Back to the Two-Body Problem Once we solved the problem for the reduced mass, it is straightforward to go back to the two particles. It holds that:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \tag{336}$$

$$\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1 \quad \& \quad \vec{r}_2 = \vec{r}_1 - \vec{r}_{12} \tag{337}$$

$$\begin{aligned} \vec{r}_1 &= \tfrac{m_1}{m_1+m_2}\vec{r}_{12} \\ \vec{r}_2 &= -\tfrac{m_1}{m_1+m_2}\vec{r}_{12} \end{aligned} \tag{338}$$

Thus, if we have solved the motion of the reduced particle, then we can quickly find the motion of the two individual particles (seen from the CM frame).

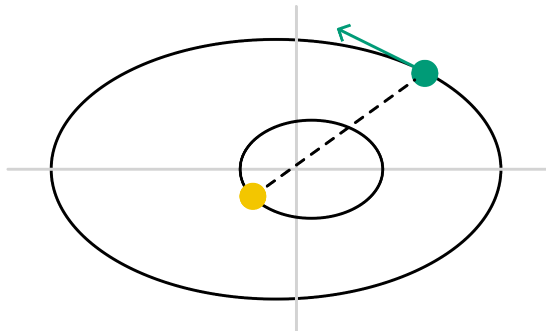


Figure 92: Kepler revisited.

2.8.5 Kepler Revisited

Now that we have seen how to deal with the two-body problem, we can return to the motion of the Earth around the Sun. This is obviously not a two-body problem, but a many-body problem with many planets.

However, we can approximate it to a two-body problem: we ignore all other planets and leave only the Sun and Earth. Hence, there are no external forces. Consequently, the CM of the Earth-Sun system moves at a constant velocity. And we can take the CM as our origin.

We have to solve the reduced mass problem to find the motion of both the Earth and the Sun:

$$\mu \frac{d^2\vec{r}_{12}}{dt^2} = -\frac{Gm_e m_s}{r_{12}^2}\hat{r}_{12} \tag{339}$$

Note: this equation is almost identical to the original Kepler problem. All that happened is that m_e on the left hand side got replaced by μ.

Everything else remains the same: the force is still central and conservative, etc.

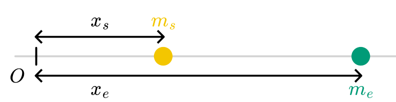


Figure 93: Position of CM in the sun-earth system.

Where is the CM located? We can easily find the center of mass of the Earth-Sun system. Chose the origin on the line through the Sun and the Earth (see fig.)

$$R = \frac{m_s x_s + m_e x_e}{m_s + m_e} = x_s + \frac{m_e}{m_s + m_e}(x_e - x_s) \approx x_s + 450km \tag{340}$$

In other words: the Sun and Earth rotate in an ellipsoidal trajectory around the center of mass that is 450 km out of the center of the Sun. Compare that to the radius of the Sun itself: $R_s = 7 \cdot 10^5$ km. No wonder Kepler didn't notice. The common CM and rotation point is called Barycenter in astronomy.

Exoplanets However, in modern times, this slight motion of stars is a way of trying to find orbiting planets around distant stars. Due to this small ellipsoidal trajectory, sometimes a star moves away from us, and sometimes it comes towards us. This moving away and towards us changes the apparent color of the light emission of molecules or atoms by the Doppler effect. This is a periodic motion, which lasts a 'year' of that solar system. Astronomers started looking out for periodic changes in the apparent color of the light of stars. One can also look for periodic changes in the brightness of a star (which is much, much harder than looking at spectral shifts of the light). If a planet is directly between the star and us, the intensity of the starlight decreases a bit. And they found one, and another one, and more and hundreds... Currently, more than 5,000 exoplanets have been found.

- Changing color of star light due to a period motion induced by a planet orbiting the star (movie from NASA).

Figure 94: *
with figure from nasa

Figure 95: *
from nasa

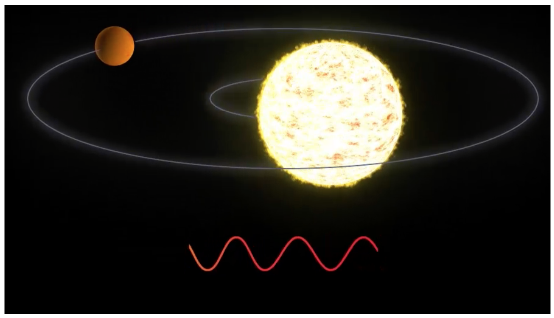


Figure 96: Finding planets via periodic changes in the velocity of a star (from NASA).

- Changing intensity of star light due to a period passage of a planet orbiting the star ((movie from NASA).

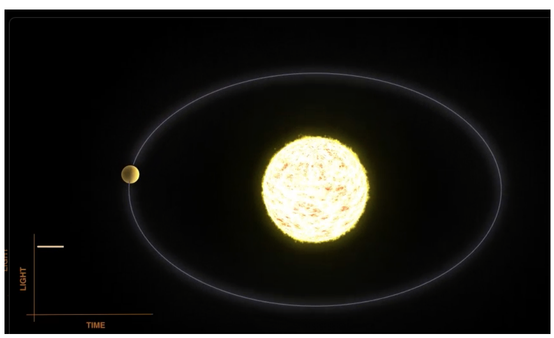


Figure 97: Finding planets via a periodic change in intensity of a star (from NASA).

- Changing intensity of star light due to a period passage of more than one planet orbiting the star (movie from NASA).

Exercise 1: In the table below, the mass and distance from the sun of the planets in our solar system are given (in terms of the earth mass and distance from the earth to the sun). Compute for each planet-sun pair the distance from the center of mass to the center of the sun. Given: distance CM to center of sun for the earth-sun system is 450km.

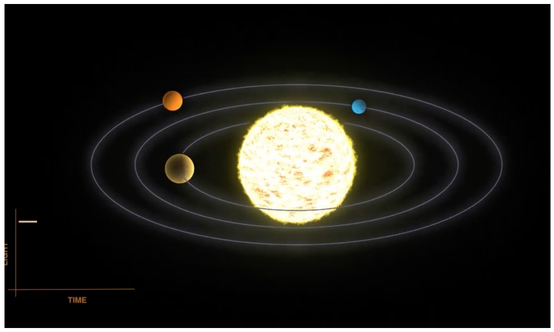


Figure 98: *
Finding multiple planets via a change in intensity of a star (from NASA).

planet	relative mass	relative distance to the sun
Mercurius	0.06	0.39
Venus	0.82	0.72
Earth	1.00	1.00
Mars	0.11	1.52
Jupiter	317.8	5.20
Saturnus	095.2	9.54
Uranus	14.6	19.22
Neptunus	17.2	30.06

Exercise 2: Two particles $m_1 = m$ and $m_2 = 2m$ are traveling both along the x-axis. At $t = 0$ the particles have both velocity $v_0 > 0$. Their positions at $t = 0$ are $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ with $x_{10} < x_{20}$. They repel each other with a force $F_r = \frac{k}{(x_2 - x_1)^2}$. Moreover, a constant external force F_e is acting on them. The problem is 1-dimensional.

- Find the velocity of the center of mass for $t > 0$
- Find the position of the center of mass for $t > 0$.

Exercise 3: Two particles m_1 = 3kg and m_2 = 2kg are connected via a massless rod of length L=50cm.

- Find the position of the center of mass of the system, measured from m_1
- Calculate the reduced mass of the two-particle system.

Exercise 4: Two bumper cars are approaching each other in a straight line. The two cars will collide head-on. The mass of car 1 (including the driver) is 200 kg, that of car 2 300kg. Car 1 has a velocity of 8m/s; car 2 of -4m/s.

- What is the velocity of the center of mass of the system?
- What is the reduced mass of the system?

- Transform the velocities of both carts to the center-of-mass frame.

Exercise 5: Two carts on a frictionless track move toward each other:

Cart 1: mass m_1 = 2kg, velocity v_1 = 4m/s
Cart 2: mass m_2 = 3kg, velocity v_2 = -2m/s

- What is the total kinetic energy in the lab frame?
- What is the velocity of the center of mass?
- What is the total kinetic energy in the center-of-mass frame?
- Verify that the CM frame kinetic energy equals the kinetic energy due to relative motion using the reduced mass.

Exercises

Solution to Exercise 1: In the table below, the mass and distance from the sun of the planets in our solar system are given (in terms of the earth mass and distance from the earth to the sun). Compute for each planet-sun pair the distance from the center of mass to the center of the sun. Given: distance CM to center of sun for the earth-sun system is 450km.

planet	relative mass	relative distance to the sun	distance CM to center of sun (km)
Mercurius	0.06	0.39	10
Venus	0.82	0.72	265
Earth	1.00	1.00	450
Mars	0.11	1.52	75
Jupiter	317.8	5.20	$743 \cdot 10$\hat{3}$
Saturnus	095.2	9.54	$409 \cdot 10$\hat{3}$
Uranus	14.6	19.22	$126 \cdot 10$\hat{3}$
Neptunus	17.2	30.06	$234 \cdot 10$\hat{3}$

Solution to Exercise 2: Two particles
We set up the equation of motion for the particles:

$$\begin{aligned} m_1 : m_1\dot{v}_1 &= F_e - F_r \\ m_1 : m_2\dot{v}_2 &= F_e + F_r \end{aligned} \tag{341}$$

Add these two equations:

$$M\dot{V} = m_1\dot{v}_1 + m_2\dot{v}_2 = 2F_e \rightarrow \dot{V} = \frac{2F_e}{m_1 + m_2} = \frac{2F_e}{3m} \tag{342}$$

As expected, we see that the repelling mutual force has no effect on the center of mass. We can solve this equation, using the initial condition the $MV(0) = m_1v_1(0) + m_2v_2(0) \rightarrow V(0) = \frac{mv_0+2mv_0}{m+2m} = v_0$

$$V(t) = \frac{2F_e}{3m}t + C_1 = \frac{2F_e}{3m}t + v_0 \tag{343}$$

As the next step we calculate $R(t)$:

$$\dot{R} \equiv V = v_0 + \frac{2F_e}{3m}t \rightarrow R(t) = v_0 t + \frac{F_e}{3m}t^2 + C_2 \tag{344}$$

The initial condition is: $R(0) = \frac{m_1 x_1(0) + m_2 x_2(0)}{m_1 + m_2} = \frac{1}{3}x_{10} + \frac{2}{3}x_{20}$.

This gives

$$R(t) = \frac{1}{3}x_{10} + \frac{2}{3}x_{20} + v_0 t + \frac{F_e}{3m}t^2 \tag{345}$$

Solution to Exercise 3: Two particles
The center of mass of two point masses is on the line connecting m_1 and m_2. We denote this line as the x-axis, with m_1 as the origin.

- The center of mass is than given by (with $m_1 = 3$kg, $m_2 = 2$kg, x_1=0 and $x_2 = x_1 + L = 0.5$m):

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0.2m \tag{346}$$

- The reduced mass is given by:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = \frac{6}{5}kg \tag{347}$$

Solution to Exercise 4: Two bumper cars are approaching each other in a straight line. The two cars will collide head-on. The mass of car 1 (including the driver) is 200 kg, that of car 2 300kg. Car 1 has a velocity of 8m/s; car 2 of -4m/s.
This is a 1-dimensional problem.

- The velocity of the center of mass is:

$$V_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{4}{5}m/s \tag{348}$$

- The reduced mass is given by:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} = 120kg \tag{349}$$

- In the CM frame the velocities of the cars are:

$$v_1' = v_1 - V_{cm} = 7.2m/s$$
$$v_2' = v_2 - V_{cm} = -4.8m/s \tag{350}$$

Solution to Exercise 5: Two carts on a frictionless track move toward each other:
Cart 1: mass m_1 = 2kg, velocity v_1 = 4m/s
Cart 2: mass m_2 = 3kg, velocity v_2 = -2m/s

- The total kinetic energy in the lab frame is

$$E_{kin} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = 22J \tag{351}$$

- The velocity of the center of mass is

$$V_{cm} \equiv \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = 0.4m/s \tag{352}$$

- The total kinetic energy in the center-of-mass frame is

$$E_{kin,CM} = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \tag{353}$$

with

$$v_1' = v_1 - V_{cm} = 3.6m/s$$
$$v_2' = v_2 - V_{cm} = -2.4m/s \tag{354}$$

Thus

$$E_{kin,CM} = 21.6J \tag{355}$$

- The reduced mass is

$$\mu \equiv \frac{m_1m_2}{m_1 + m_2} = 1.2kg \tag{356}$$

The relative velocity is

$$v_{rel} \equiv v_1 - v_2 = 6m/s \tag{357}$$

The kinetic energy associated with the motion of the reduced mass (i.e. the kinetic energy in the CM frame) is:

$$E_{kin,rel} \equiv \frac{1}{2}\mu v_{rel}^2 = 21.6J \tag{358}$$

as we expected.

Answers

2.8.6 Many-Body System

We have seen that we could reduce the two-body problem of sun-earth to a single body question via the concept of reduced mass. But that this strategy does not work for 3, 4, 5, ... bodies.

Linear Momentum We can, however, find some basic features of N-body problems. In the figure, a collection of N interacting particles is drawn.

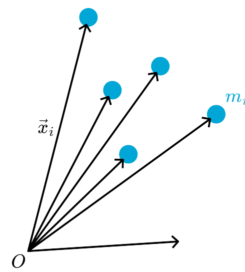


Figure 99: Many particle system.

Each particle has mass m_i and is at position $x_i(t)$. For each particle, we can set up N2:

$$m_i \ddot{\vec{x}}_i = \vec{F}_{i,ext} + \sum_{i \neq j} \vec{F}_{ji}. \tag{359}$$

Summing over all particles and using that all mutual interaction forces form "action = -reaction pairs", we get:

$$\sum_i m_i \ddot{\vec{x}}_i = \sum_i \vec{F}_{i,ext} \Leftrightarrow \sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_{i,ext} \tag{360}$$

The second part can be writen as:

$$\frac{d\vec{P}}{dt} = \sum_i \vec{F}_{i,ext} \text{ with } \vec{P} \equiv \sum_i \vec{p}_i \tag{361}$$

In other words: the total momentum changes due to external forces. If there are no external forces, then the total momentum is conserved. This happens quite a lot actually, if you consider e.g. collisions or scattering.

Center of Mass Analogous to the two-particle case, we see from the total momentum that we can pretend that there is a particle of total mass $M = \sum_i m_i$ that has momentum $\vec{P}$, i.e., it moves at velocity $\vec{V} \equiv \frac{\vec{P}}{M}$ and is located at position:

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{\sum m_i \frac{d\vec{x}_i}{dt}}{\sum m_i} \Rightarrow \vec{R} = \frac{\sum m_i \vec{x}_i}{\sum m_i} \tag{362}$$

Continuing with the analogy, we define relative coordinates:

$$\vec{r}_i \equiv \vec{x}_i - \vec{R} \tag{363}$$

and have a similar rule constraining the relative positions:

$$\sum m_i \vec{r}_i = 0 \tag{364}$$

Energy In terms of relative coordinates, we can write the kinetic energy as a part associated with the center of mass and a part that describes the kinetic energy with respect to the center of mass, i.e., 'an internal kinetic energy'.

$$
\begin{aligned}
E_{kin} &\equiv \sum \frac{1}{2} m_i v_i^2 \\
&= \frac{1}{2} M \dot{\vec{R}}^2 + \sum \frac{1}{2} m_i \dot{\vec{r}}_i^2 \\
&= E_{kin,cm} + E'_{kin}
\end{aligned}
\tag{365}
$$

For the potential energy, we may write:

$$
V = \sum V_i + \frac{1}{2} \sum_{i \neq j} (V_{ij} + V_{ji}) \tag{366}
$$

with V_i the potential related to the external force on particle i and V_{ij} the potential related to the mutual interaction force from particle i exerted on particle j (assuming that all forces are conservative).

Angular Momentum The total angular momentum is, like the total momentum, defined as the sum of the angular momentum of all particles:

$$
\vec{L} = \sum \vec{l}_i = \sum \vec{x}_i \times \vec{p}_i \tag{367}
$$

We can write this in the new coordinates:

$$
\vec{L} = \vec{R} \times \vec{P} + \sum \vec{r}_i \times \vec{p}_i = \vec{L}_{cm} + \vec{L}' \tag{368}
$$

Again, we find that the total angular momentum can be seen as the contribution of the center of mass and the sum of the angular momentum of all individual particles as seen from the center of mass.

The N-body problem is, of course, even more complex than the three-body problem. If we can solve it, it will be under very specific conditions only. However, a numerical approach can be done with great success. Moreover, current computers are so powerful that the system can contain hundred, thousands of particles up to billions depending on the type or particle-particle interaction.

All kind of computational techniques have been developed and various averaging techniques are employed in statistical techniques are introduced from the start. the reason is often, that a particular 'realisation' of all positions and velocities of all particles is needed nor sought for. A system is at its macro level described by averaged properties, the exact location of the individual atoms is not needed. You will find applications in cosmology all the way to molecular dynamics, trying to simulate the behavior of proteins or pharmaceuticals.

Exercise 6: Thee masses are forming an equilateral triangle with sides of 2m. Mass 1 (10kg) is positioned at $(x_1, y_1) = (-1m, 0))$. Mass 2 (6kg) is at $(x_2, y_2) = (1m, 0))$, while mass 3 (2kg) is at $(x_3, y_3) = (0, \sqrt{3})$.

- Calculate the position of the center of mass.

Exercises

- Calculate the velocity of the center of mass.
- Calculate the position of the center of mass as a function of time.

- Calculate the total angular momentum.
- Calculate the angular momentum associated with the center of mass and show that in this case this is equal to the total angular momentum.

The wheel is moving with a velocity V while it is rotating at the same time with angular velocity ω.

Calculate the total kinetic energy of this system. Hint: use the CM frame and connect that to the lab frame.

Exercise 9: A container of volume V_c and mass M_c contains Nitrogen gas. The number of molecules, N, is on the order of 10^{23}. The container is dropped from a height H. Gravity is acting on the molecules. Friction on the container is ignored.

Show that the container falls with the acceleration of gravity g.

Exercise 10: We consider a 2-dimensional problem: $N = 30$ particles move in the xy-plane. Each particle has a fixed velocity (v_x^i, v_y^i) with $i = 1..N$. The particle velocities have a magnitude ranging from 1 to 5 (m/s) randomly chosen for each particle. The direction of each velocity vector is also randomly chosen from 0 to 2π. The particles move inside a box with sides L=50m. Particles do not collide with each other, but they do collide with the walls of the container. The result of a collision is that the particle motion gets reflected.

- Write a python program that generates N particles starting all at $(x, y) = (0, 0)$.
- Compute the position of all particles after 1 second and compute the velocity and position of the center of mass.
- Write a loop that updates the particle velocities after a time step dt and recompute the velocity and position of the center of mass.
- Run the loop M times and plot the position of the center of mass in the xy-plane as a function of time.
- What happens if you change the number of particles from 30 to 3 or to 300?

Solution to Exercise 6: Thee masses are forming an equilateral triangle with sides of 2m. Mass 1 (10kg) is positioned at
The position of the center of mass is

$$\vec{R} \equiv \frac{\sum_i m_i \vec{x}_i}{\sum_i m_i} = \frac{(m_1 x_1)\hat{x} + (m_2 x_2)\hat{x} + (m_3 y_3)\hat{y}}{m_1 + m_2 + m_3} = -\frac{2}{9}[m]\hat{x} + \frac{1}{9}[m]\hat{y} \tag{369}$$

where $[m]$ indicates that the unit is meters.

Note: $\hat{x}$ and $\hat{y}$ do not carry units!

Answers

- Velocity of the center of mass:
$$\vec{V} = \frac{\sum_i m_i \vec{v}_i}{\sum_i m_i} \tag{370}$$
Since the velocities are all parallel to the x-axis, we can drop the vector notation. Substituting the data for mass and velocity, gives:
$$V_x = \frac{4mv + 6mv + 6mv + 4mv}{4m + 3m + 2m + m} = 2v \tag{371}$$
- Position of the center of mass:
$$\vec{V} = \frac{d\vec{R}}{dt} \rightarrow \vec{R}(t) = 2vt\hat{x} + \vec{c} \tag{372}$$

At $t = 0$ all particles at location $(0, y_0)$. Thus, we find

$$\vec{R}(t) = 2vt\hat{x} + y_0\hat{y} \tag{373}$$

- Total angular momentum:

$$\begin{aligned} \vec{L_{tot}} &= \sum_i \vec{l_i} \\ &= y_0 \cdot 4mv\hat{z} + y_0 \cdot 3m \cdot 2v\hat{z} + y_0 \cdot 2m \cdot 3v\hat{z} + y_0 \cdot m \cdot 4v\hat{z} \\ &= 20mvy_0\hat{z} \end{aligned} \tag{374}$$

- Angular momentum associated with the center of mass:

$$\vec{L} = \vec{R} \times M\vec{V} = y_0 10m \cdot 2v\hat{z} = 20mvy_0\hat{z} \tag{375}$$

which is indeed the same as the total angular momentum. This is in this case to be expected as the angular momentum seen from the CM frame is $\vec{L}' = 0$ as in the CM frame the position vector and momentum vector are parallel for all four particles.

Solution to Exercise 8: Eight point particles (each mass

We split the kinetic energy in the kinetic energy associated with the center of mass and the kinetic energy as seen from the CM frame:

$$E_{kin} = \frac{1}{2}MV^2 + E'_{kin} \tag{376}$$

Due to symmetry, the center of mass velocity is V.

In the CM frame, all particles rotate with ω and thus have a velocity of magnitude $v' = \omega R$. As all particles have the same mass, we have $M = 8m$. The kinetic energy is:

$$E_{kin} = \frac{1}{2}8V^2 + 8 \cdot \frac{1}{2}m\omega^2 R^2 = 4mV^2 + 4mR^2\omega^2 \tag{377}$$

Solution to Exercise 9: A container of volume

All nitrogen molecules feel gravity and have interaction with each other and with the wall of the container. If we write down the equation of motion for all molecules (labelled i) and the container we get:

$$\begin{aligned} M_c \ddot{\vec{x}}_c &= M_c\vec{g} + \sum_i \vec{F}_{molecule\ i\ on\ vessel\ wall} \\ m_i \ddot{\vec{x}}_i &= -m_i\vec{g} + \vec{F}_{vessel\ wall\ on\ molecule\ i} + \sum_{j\neq i} \vec{F}_{ji} \end{aligned} \tag{378}$$

with $\vec{F}_{molecule\ i\ on\ vessel\ wall}$ the force of molecule i on the vessel wall and $\vec{F}_{ji}$ the force from molecule j on molecule i. All these forces are internal forces and when summing over all particles (including the vessel) will cancel each other as they all obey N3.

Thus is we add the equations, we find:

$$\frac{d}{dt}\left(M_c \dot{\vec{x}}_c + \sum_i m_i \dot{\vec{x}}_i\right) = \left(M_c + \sum m_i\right)\vec{g} \tag{379}$$

On the left side, we recognize the total momentum which we can write in terms of the center of mass: $M_c \dot{\vec{x}}_c + \sum_i m_i \dot{\vec{x}}_i = M\vec{V}$.

And on the right hand side we see the total mass $M = M_c + \sum m_i$.

Thus, we conclude:

$$M\dot{\vec{V}} = M\vec{g} \rightarrow \dot{V} = -g \tag{380}$$

The entire container drops with acceleration $-g$.

Solution to Exercise 10: We consider a 2-dimensional problem:

2.8.11 Examples, exercises & solutions

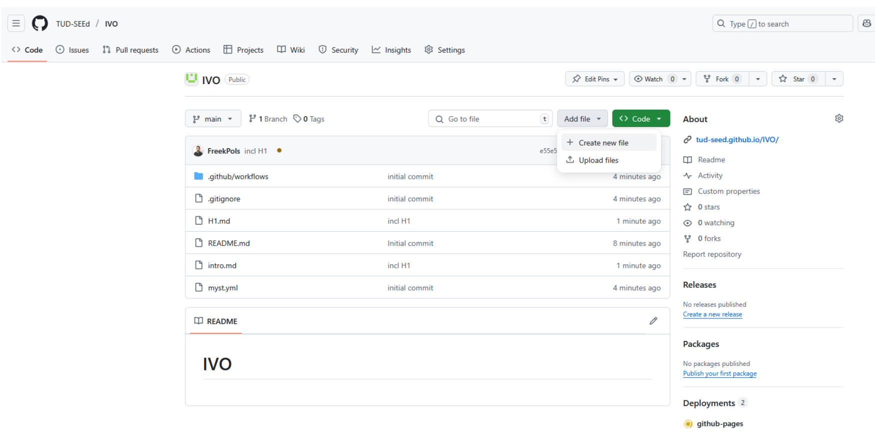
3 For developers

3.1 How to

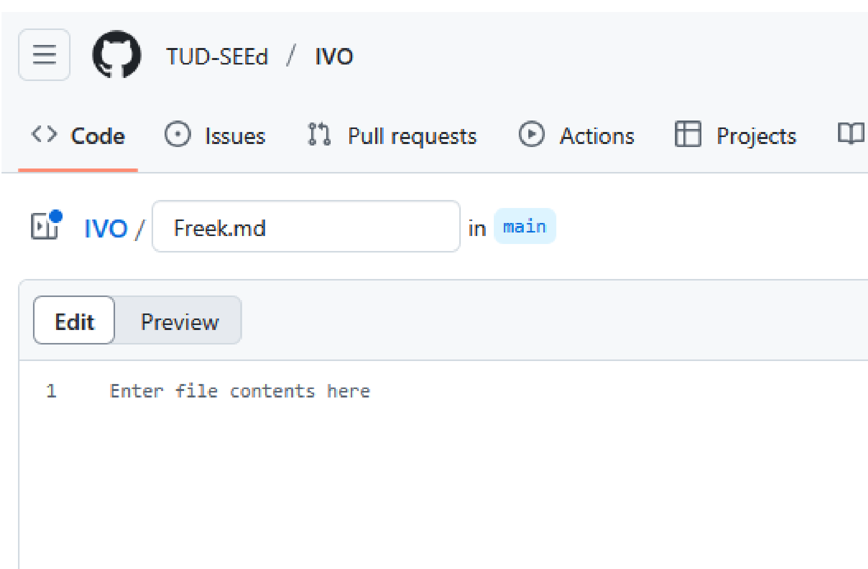
3.1.1 Introductie

Welkom bij Jupyter Book!

- Ga naar de website van Github en maak een account aan als je dat nog niet hebt.
- Geef je accountnaam door aan Freek, hij voegt jou toe aan het boek.
- Als je toegang hebt, kun je aan de slag met een eigen hoofdstuk maken of een bestaand hoofdstuk editen. De repo waar je toegang toe krijgt (voor dit specifieke boek) vind je hier.
- Ga naar de folder *content* en klik op *Add file* en *Create a new file*, zie hieronder.



- Geef je file een naam met als extensie *.md* bijv. *Freek.md*



- In die file kun je jouw inhoud stoppen / ontwikkelen.
- Maak een hoofdstuk titel (# Mijn eerste titel) en een section titel (## Mijn eerste sectie).

- Druk op de groene *Commit changes* knop om je aanpassingen door te zetten naar de repo. Je kunt de commit een passende titel geven (of niet).
- Let op! Het mechanica boek is gewijzigd tov de template, dusdanig dat je het hoofdstuk ook in de **Table of Contenst** moet zetten. Deze staat in de hoofdmap, in het bestand myst.yml

Wat er nu gebeurt is dat het boek opnieuw gemaakt wordt en via GitHub pages gepubliceerd. Na ongeveer 2 minuten kun je dus het resultaat op de website zien!

- Bekijk eens op de site van Jupyter Book naar wat je allemaal kunt toevoegen en pas dat aan in je eigen gemaakte hoofdstuk: klik daartoe op je gemaakte hoofdstuk en dan op het pennetje aan de rechterkant (*edit this file*)
- Je kunt natuurlijk ook de features bekijken in het volgende hoofdstuk.
- Succes!

Note
Goed om te weten... dit boek is gemaakt in MyST de meest recente versie van Jupyter Books.

3.1.2 Feedback / issue report / vragen

Rechtsboven op de page staat een knop met FEEDBACK. Wanneer je daar op klikt kom je op de issues pagina van de github van dit boek. Je kunt een nieuwe issue aanmaken (groene knop, *New issue*). Daarmee kom je bij een formulier die vraagt om een titel, en een beschrijving van het probleem. Je kunt verder iemand aanwijzen (*assignees*) om het probleem te koppelen aan iemand die het waarschijnlijk kan oplossen. Daarnaast is er de mogelijkheid om een label er aan te hangen (bijv. bug / invalid / help wanted).

Wanneer je de issue hebt gerapporteerd (Create) belandt deze in de to-do list en wordt het issue opgepakt wanneer daar tijd voor is.

Wil je tekeningen bij een specifiek onderwerp, tag dan *Hanna*. Beschrijf wat je voor tekening wilt, als dat onvoldoende helder is vanuit de vraagstelling zelf.

3.1.3 Opzetten van een lokale server

Wanneer je lokaal werkt en een push doet naar github, zal het boek opnieuw gebouwd worden en online te zien zijn. Een andere mogelijkheid is lokaal werken en je output (bijna) live te volgen. Wanneer je een document opslaat, wordt dit gedetecteerd en wordt alleen de pagina die je hebt gewijzigd opnieuw gemaakt.

Om direct te output van de wijzigingen te zien (lokaal), ga je via de terminal (anaconda prompt of de mac terminal) naar de folder waar het myst.yml bestand van dit boek staat. Typ in de terminal `myst start` (de eerste keer dat je het boek bouwt moet dit `myst init` zijn). Op dat moment worden de boeken geconverteerd naar een website, welke lokaal te zien is. Het adres wordt gegeven in de terminal, veelal is dat: `http://localhost:3000`. Via een webbrowser kan dit adres gekopieerd worden. Wanneer je een bestand opslaat, wordt deze binnen ~5 s zichtbaar.

3.1.4 Werken met GIT

Werken met Git heeft het voordeel dat je goed kunt samenwerken. Via de repository worden de bestanden gesynchroniseerd. Om hier goed gebruik van te maken is de volgende workflow handig:

Bij starten van nieuwe edits doe je een git pull, zie Figure 104.

Ben je klaar, dan commit & push je de wijzigingen naar de repository. Vergeet niet een samenvatting van de wijzigingen toe te voegen! Tussendoor kun je ook een push doen, om bijv. het resultaat online te bekijken.

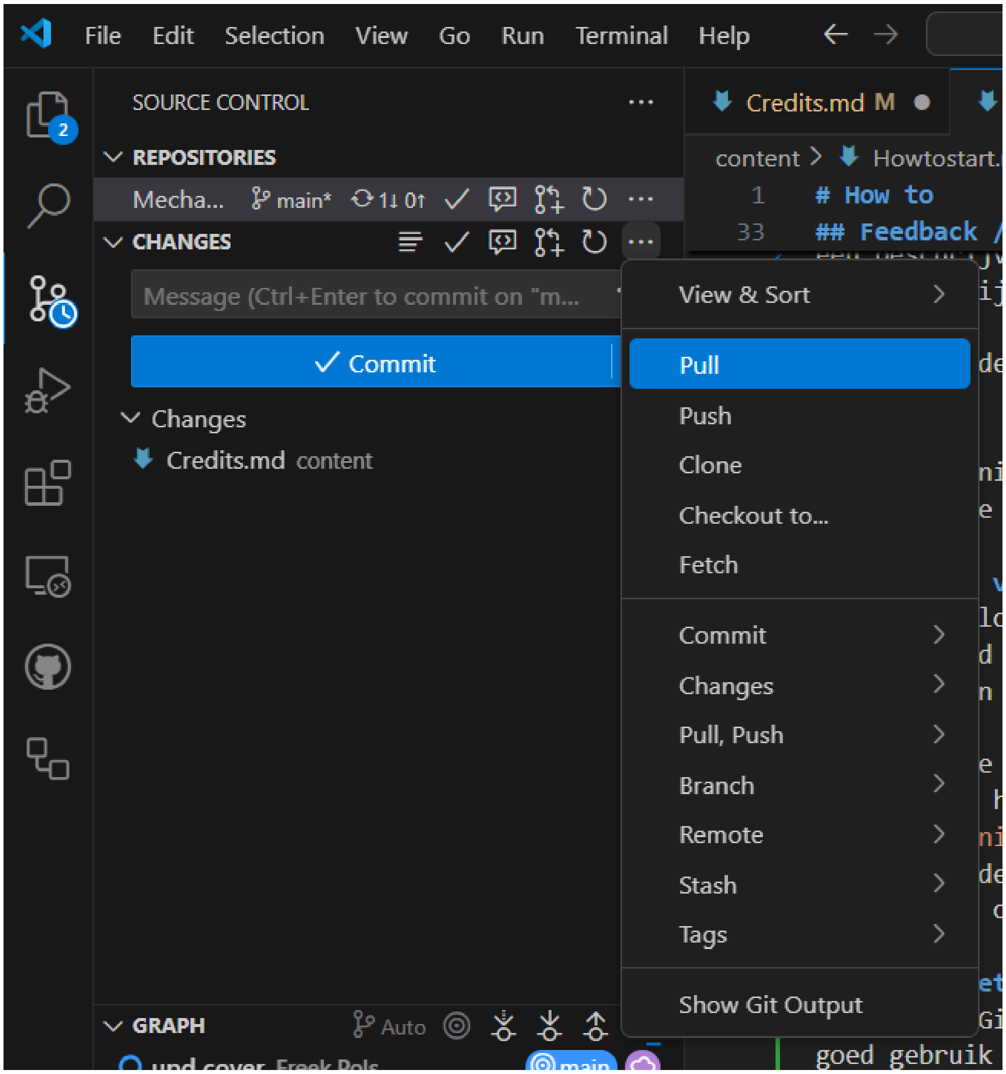


Figure 104: Bij de start doe je een pull.

3.1.5 Admonitions

Er zijn diverse admonitions mogelijk: danger / tips / exercises.

Het is ook mogelijk om eigen admonitions te maken. Voor nu zijn er: intermezzo en experiment.

Ik ben een intermezzo
Hier dan tekst.

Wil je een experiment doen?
Altijd.

Ik ben een example
Genoeg voorbeelden

Is er behoefte aan meer admonition types, laat het weten via een issue!

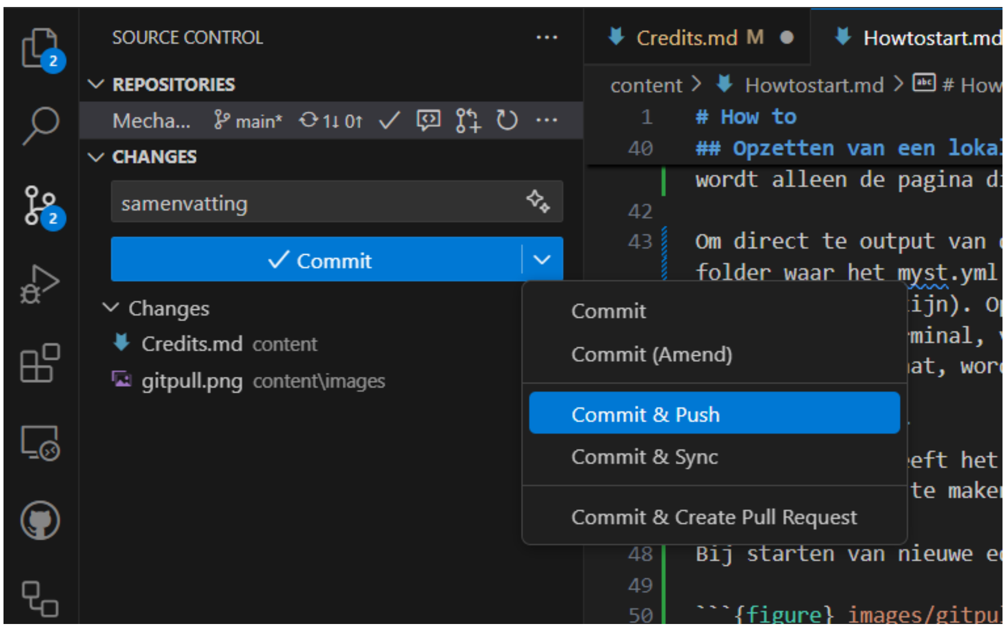


Figure 105: Aan het eind doe je een gitpush, de wijzigingen worden doorgestuurd naar de repository.

3.2 Markdown (Cheatsheet)

Markdown is een eenvoudige opmaaktaal: platte tekst die *opgemaakt* wordt met kleine stukjes 'code'. Die tekst is vervolgens snel te exporteren naar allerlei andere formats zoals pdf, word, html etc.

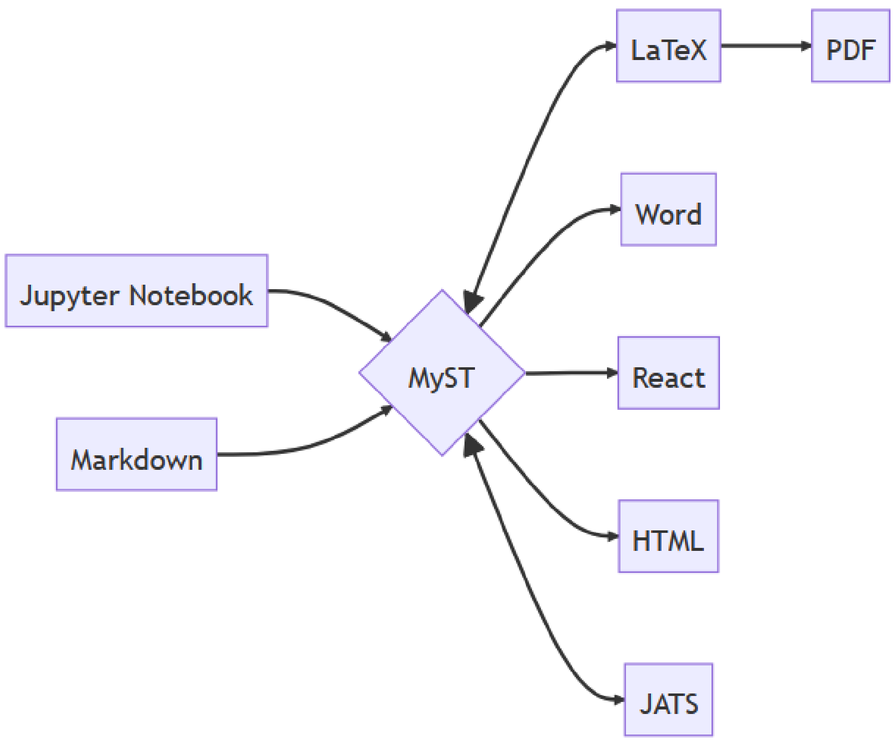


Figure 106: *
Een Jupyter Book gemaakt met MyST vraagt een collectie van markdown en jupyter notebooks die vervolgens geëxporteerd kunnen worden naar pdf, html maar ook word.

3.2.1 Structuur

We kunnen hier onderscheid maken in twee structuren: die van de inhoud van de boek (een collectie van verschillende documenten), en de (interne)structuur van de hoofdstukken.

Table of Contents De software waar we gebruik van maken bouwt zelf een inhoudsopgave (Table of contents, ookwel ToC). Dat gaat op alfabetische volgorde. Maar je kunt ook zelf de ToC specificeren. Dit kan wel het beste door offline te werken (myst init –toc), zie de documentatie van MyST.

Hoofdstukken Om onderscheid te maken tussen hoofdstuk, sectie en subsectie (en verder) wordt er gewerkt met aantal #, zie hieronder.

```
# H1 hoofdstuk
## H1.1 sectie
### H1.1.3 subsectie
```

Tip
Nummer je hoofdstukken en sectie niet! Dit gebeurt automatisch.

Een nieuwe regel krijg je door of een harde enter en een witregel, of door een \ achter de zin en een enter of door twee spaties achter de zin.

Nieuwe regel

3.2.2 Basic opmaak

Markdown is een opmaaktaal waarbij de formatting van de tekst gedaan wordt met kleine stukjes code (net als bij HTML).

Element	Syntax	Voorbeeld
Bold	`**dik gedrukte tekst**`	**Bold**
Italic	`*italics*`	*Italics*
Emphasis	`***emphasis***`	***emphasis***
in line Formule	`$F = m \cdot a$`	$F = m \cdot a$
Super en subscript	`H{sub}‘2‘O, and 4{sup}‘th‘ of July`	H_2O, and 4^{th} of July
Footnote	`- A footnote reference[m̂yref] \ [m̂yref]: This is an auto-numbered footnote definition.`	- A footnote reference[m̂yref] \ [m̂yref]: This is an auto-numbered footnote definition.

Lijsten optie 1

Lijsten optie 2

Afvinklijsten

3.2.3 Formules

Voor de betavakken zijn wiskundige vergelijkingen essentieel. Ook in JB's kun je vergelijkingen opnemen. Wat in LaTeX kan, kan in JB ook, bijv:

$$F_{res} = m \cdot a \tag{381}$$

Waarbij gelabelde vergelijkingen, zoals (381) naar verwezen kan worden.

`$$ Vergelijking $$`

Maar je kunt ook inline vergelijkingen opnemen zoals deze: $s = v_{gem}t$. Daarbij gebruik je een enkele dollar teken voor en na je `$ Vergelijking $`

Naam	Script	Symbolen
wortel	`\sqrt{4}`	$\sqrt{4}$
macht	`^{2x}`	^{2x}
breuk	`\frac{2}{3}`	$\frac{2}{3}$
subscript	`_{gem}`	$_{gem}$
superscript	`^{N}`	^{N}
vermenigvuldig	`\cdot`	$\cdot$

wat voorbeelden:

Naam	Script	Output
Afgeleide	`\frac{\Delta f}{\Delta t}`	$\frac{\Delta f}{\Delta t}$
Integraal	`\int_aˆb dx`	$\int_a^b dx$
sinus	`sin(x)`	$sin(x)$

: https://en.wikibooks.org/wiki/LaTeX/Mathematics

3.2.4 Admonitions

Je kunt speciale blokken toevoegen die gehighlight worden in de tekst. Zie bijvoorbeeld onderstaande waarschuwing.

> **Warning**
> Hier een waarschuwing

Daar zijn verschillende varianten van zoals:

- tip
- admonition
- warning
- note
- objective
- see also ...

> **De gouden...**
> Exercises zijn een speciaal soort admonition.

> **Exercise 1: Opdracht 1**
> Maak de som $4 + 2$

> **Solution to Exercise 1: Opdracht 1**
> 6

Opdrachten

3.2.5 Figuren

Een site / boek kan natuurlijk niet zonder figuren. Er zijn grofweg twee manieren om een figuur te maken

Snelle figuur, zonder opmaak mogelijkheden

— Snelle figuur — `![](link naar figuur)` —

Betere manier met meer controle:

Hier hebben we gebruik gemaakt van figuren die op het internet staan, maar je kunt ook figuren zelf toevoegen aan een folder (bijv. genaamd *Figuren*), waarbij je dan een relatief pad op geeft.

3.2.6 Tabellen

Tabellen worden gemaakt met scheidingsteken |

Of via ...

Methode 2 heeft als voordeel de mogelijkheid tot refereren.

3.2.7 Tabbladen

```
::::{tab -set}
:::{tab -item} Tab 1
Hier tekst in tab 1
:::

:::{tab -item} Tab 2
Hier tekst in tab 2
:::
::::
```

3.2.8 YouTube

Voor het embedden van YouTube filmpjes op de site heb je de embed YT link nodig. De code wordt dan:

> **YT in pdf**
> De embedded YT filmpjes worden niet opgenomen in de pdf. Een oplossing zou bijv. een qr code opnemen kunnen zijn.

3.2.9 Referenties

3.2.10 Replacing

To find and replace all HTML anchor tags like:
parsec
with Markdown-style links like:
parsec
You can use regular expressions in Visual Studio Code's Find and Replace:
FIND
`<a\s+href="([^"]+)">([^<]+)<\/a>`
REPLACE
`[$2]($1)`

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
from IPython.display import HTML

# Simulatieparameters
dt = 0.05
t_max = 10
t_values = np.arange(0, t_max, dt)

# Fysische parameters
vx = 1.0
Fy = 1.0
m = 1.0
ay = Fy / m

# Posities berekenen
x = vx * t_values
y = np.zeros_like(t_values)

x_burn_start = 2.0
x_burn_end = 4.0
i_start = np.argmax(x >= x_burn_start)
i_end = np.argmax(x >= x_burn_end)

for i in range(i_start, i_end+1):
    t_burn = t_values[i] - t_values[i_start]
    y[i] = 0.5 * ay * t_burn**2

vy_final = ay * (t_values[i_end] - t_values[i_start])
y0 = y[i_end]
t0 = t_values[i_end]
for i in range(i_end, len(t_values)):
    y[i] = y0 + vy_final * (t_values[i] - t0)

# Plot
fig, ax = plt.subplots(figsize=(8, 4))
ax.set_xlim(0, np.max(x)+1)
ax.set_ylim(0, np.max(y)+1)
ax.set_xlabel("x")
ax.set_ylabel("y")
ax.set_title(" Raket met stuwfase tussen x=2 en x=4")

# Raket (emoji als tekst)
rocket = ax.text(0, 0, '', fontsize=14)

# Trail
trail, = ax.plot([], [], 'r -', lw=1)

# Tijd
time_text = ax.text(0.98, 0.95, '', transform=ax.transAxes,
                    ha='right', va='top', fontsize=12)

# Init
```

```
def init():
    rocket.set_position((0, 0))
    trail.set_data([], [])
    time_text.set_text('')
    return rocket, trail, time_text

# Update
def update(frame):
    rocket.set_position((x[frame], y[frame]))
    trail.set_data(x[:frame+1], y[:frame+1])
    time_text.set_text(f"t = {t_values[frame]:.2f} s")
    return rocket, trail, time_text

# Animatie
ani = FuncAnimation(fig, update, frames=len(t_values),
                    init_func=init, interval=dt*1000, blit=True)

plt.close()
HTML(ani.to_jshtml())

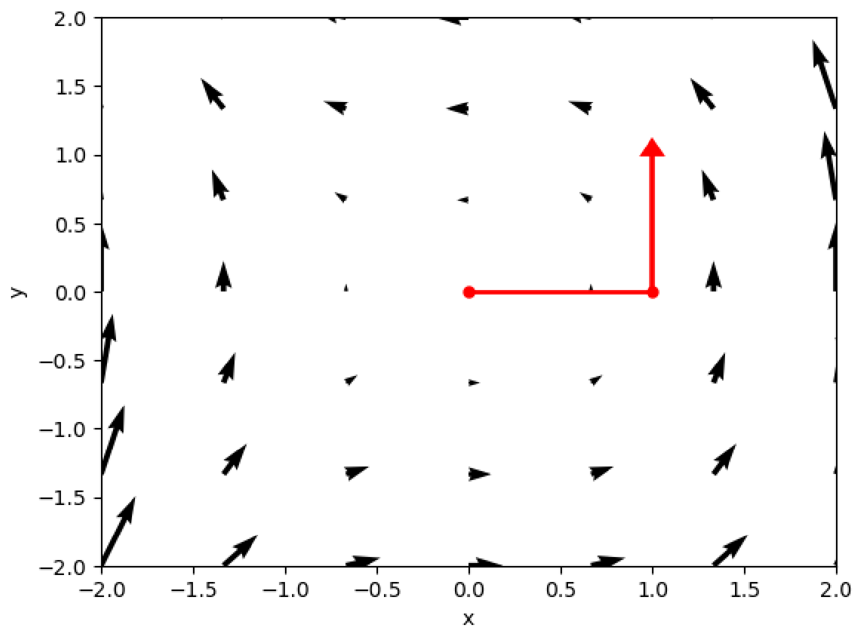
import numpy as np
import matplotlib.pyplot as plt

x = np.linspace( -2, 2, 7)
y = np.linspace( -2, 2, 7)
X, Y = np.meshgrid(x, y)
U = - Y
V = X**2

path_x = [0, 1]
path_y = [0, 0]


plt.figure()
plt.arrow(1, 0, 0, 1, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.plot(path_x, path_y, color='red', linewidth=2, marker='o', markersize=5)

plt.quiver(X, Y, U, V, color='k')
plt.xlim( -2, 2)
plt.ylim( -2, 2)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('images/force_field.png', dpi=300)
plt.show()
```



1. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} * 10 * 2^2 = 20\text{J}$
2. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 10 \cdot 2^2 = 20\text{J}$
3. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}(10)(2)^2 = 20\text{J}$
4. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}.(10).(2)^2 = 20\text{J}$
5. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}(10\text{kg})(2\text{m/s})^2 = 20\text{J}$
6. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2}.(10\text{kg}).(2\text{m/s})^2 = 20\text{J}$
7. $E_{kin} = \frac{1}{2}mv^2 = \frac{1}{2} \cdot 10 \bullet 2^2 = 20\text{J}$

```
import numpy as np
import matplotlib.pyplot as plt

t = np.linspace(0, 5, 100)
v_x = 30
a = 9.81

s_x = v_x * t
s_y = -0.5 * a * t**2

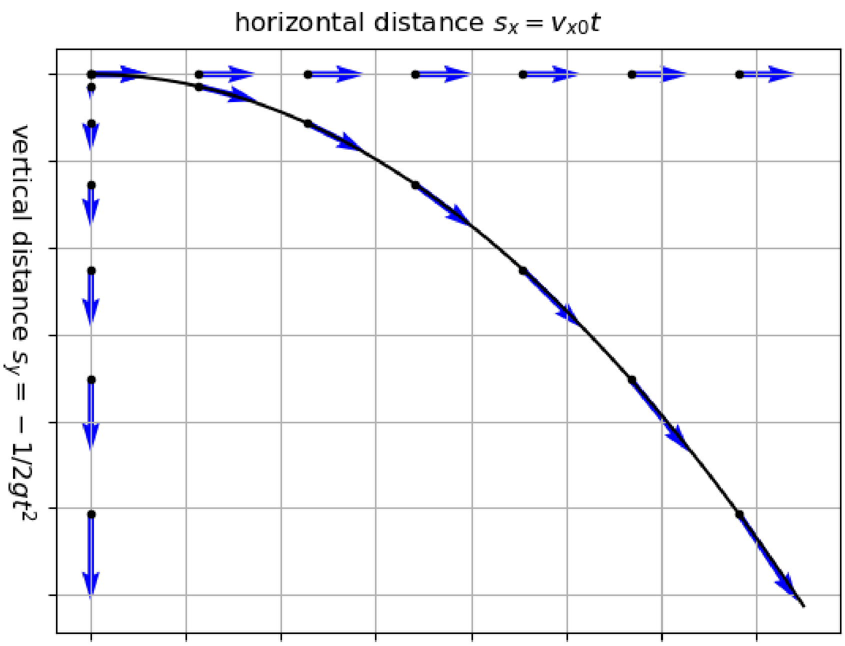

N = 15

plt.figure()
plt.plot(s_x, s_y, 'k -' )
plt.plot(s_x[::N], s_y[::N], 'k.' )
plt.plot(s_x[::N], s_y[::N]*0, 'k.' )
plt.plot(s_x[::N]*0, s_y[::N], 'k.' )

plt.quiver(s_x[::N], s_y[::N]*0, v_x, 0, color='blue', scale=400)
plt.quiver(s_x[::N]*0, s_y[::N], 0, -a*t[::N], color='blue', scale=400)
```

```
plt.quiver(s_x[::N], s_y[::N], v_x, -a*t[::N], color='blue', scale=400)


plt.gca().set_xticklabels([])
plt.gca().set_yticklabels([])
plt.grid(visible=True)
plt.text(30, 10, 'horizontal distance $s_x=v_{x0}t$', fontsize=12, color='black')
plt.text( -20, -100, 'vertical distance $s_y= -1/2gt^2$', fontsize=12, color='black',rotation= -90)
plt.savefig('../images/parmotionv.png', dpi=300)
plt.show()
```



Measure restitution coefficient
Use a pingpong ball and the app phyphox using the acoustic chronometer to determine the coefficient of restitution of the pingpongball. How does this coefficient changes with different surfaces?

```
import numpy as np
import matplotlib.pyplot as plt

N = 1.5
x = np.linspace( -N, N, 15)
y = np.linspace( -N, N, 15)
X, Y = np.meshgrid(x, y)
U = Y
V = -X

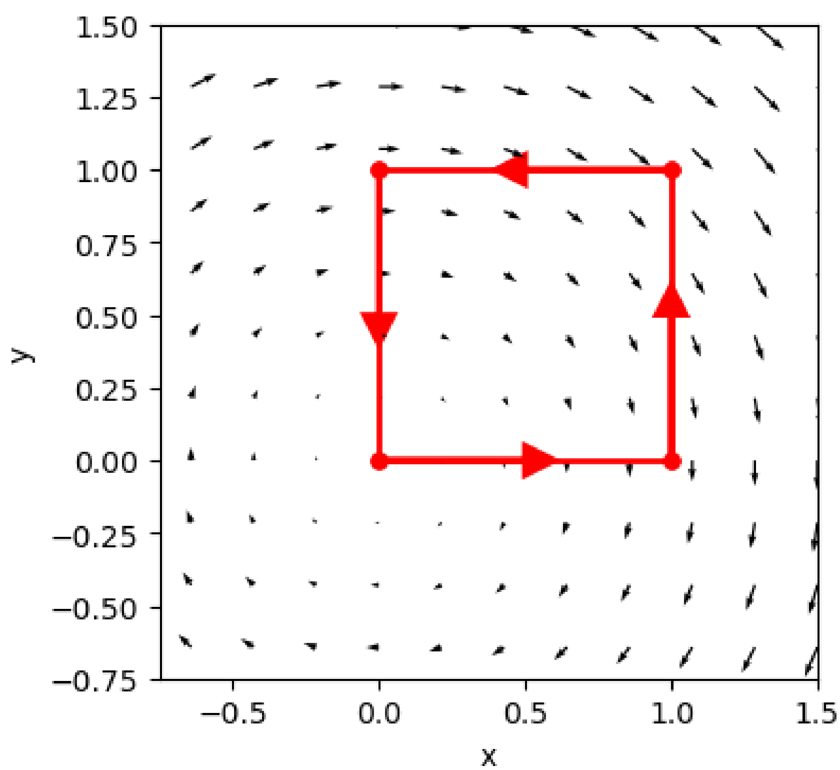

plt.figure(figsize=(4, 4))
```

```
plt.plot([0,1], [0,0], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([1,1], [0,1], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([1,0], [1,1], color='red', linewidth=2, marker='o', markersize=5)
plt.plot([0,0], [1,0], color='red', linewidth=2, marker='o', markersize=5)

plt.arrow(1, 0, 0, .5, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(0, 1, 0, -.5, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(0, 0, 0.5, 0, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)
plt.arrow(1, 1, -0.5, 0, head_width=0.1, head_length=0.1, fc='red', ec='red', linewidth=2)

#plt.plot(path_y, path_z, color='red', linewidth=2, marker='o', markersize=5)


plt.quiver(X, Y, U, V, color='k')
plt.xlim( -.5*N, N)
plt.ylim( -.5*N, N)
plt.xlabel('x')
plt.ylabel('y')
plt.savefig('../images/StokesTheoremExample.png', dpi=300)
plt.show()
```



References

T. Idema. *Introduction to particle and continuum mechanics*. TU Delft OPEN Publishing, 11 2023. doi:10.59490/tb.81. URL http://dx.doi.org/10.59490/tb.81.