1. Answer the following questions concerning the vector valued function

$$\mathbf{r}(t) = \left\langle 4t^3, \pi\cos(\pi t), \frac{1}{t+1} \right\rangle$$

(a) (5 points) Evaluate $\mathbf{r}'(t)$.

Solution:

$$\mathbf{r} = \left\langle 12t^2, -\pi^2\sin(\pi t), -\frac{1}{(t+1)^2} \right\rangle$$

(b) (5 points) Evaluate

$$\int_0^2 \mathbf{r}(t)\,\mathrm{d}t.$$

Solution:

$$\begin{aligned}\int_0^2 \mathbf{r}(t)\,\mathrm{d}t &= \left\langle t^4\Big|_0^2,\ \sin(\pi t)\Big|_0^2,\ \ln|t+1|\Big|_0^2 \right\rangle \\ &= \langle 16, 0, \ln 3\rangle\end{aligned}$$

2. Consider the vector valued function

$$\mathbf{r}(t) = \left\langle e^t, 1 - 2e^t, 2e^t + 1 \right\rangle$$

for $0 \le t \le \ln(2)$.

(a) (10 points) Find the arc-length function $s(t)$ of $\mathbf{r}(t)$.

Solution: We compute $s(t) = \int_1^t \|\mathbf{r}'(u)\| \, \mathrm{d}u$.

$$\mathbf{r}'(t) = \left\langle e^t, -2e^t, 2e^t \right\rangle$$
$$\|\mathbf{r}'(t)\| = \sqrt{e^{2t} + 4e^{2t} + 4e^{2t}} = \sqrt{9e^{2t}} = 3e^t$$

Hence

$$s(t) = \int_1^t 3e^u \, \mathrm{d}u = 3e^u \Big|_1^t = \boxed{3e^t - 3e}$$

(b) (5 points) Find the arc-length parametrization $\mathbf{r}(s)$.

Solution: Since $s = 3e^t - 3e$ we have $t = \ln\left(\frac{s+3e}{3}\right)$. Substituting gives

$$\boxed{\mathbf{r}(s) = \left\langle \frac{1}{3}s + 3e, 1 - \frac{2}{3}(s + 3e), \frac{2}{3}(s + 3e) + 1 \right\rangle}$$

(c) (5 points) Recalling that curvature is $\kappa = \|\mathbf{r}''(s)\|$, find the curvature of the curve paramerized by $\mathbf{r}(t)$. Explain your answer by identifying the curve.

Solution: Taking derivatives with respect to s,

$$\mathbf{r}'(s) = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right\rangle$$
$$\mathbf{r}''(s) = \langle 0, 0, 0 \rangle$$

so

$$\kappa = \| \langle 0, 0, 0 \rangle \| = \boxed{0}$$

The curve is a line.

3. Find the limit, if it exists. If the limit does not exist, explain why.

(a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - xy}{x^4 + y^2x^2}$$

Solution: We simplify before converting to polar:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2y - xy}{x^4 + y^2x^2} &= \lim_{(x,y)\to(0,0)} \frac{xy(x-1)}{x^2(x^2+y^2)} \\
&= \lim_{r\to 0} \frac{r^2\cos\theta\sin\theta(r\cos\theta - 1)}{r^4\cos^2\theta} \\
&= \lim_{r\to 0} \frac{\sin\theta(r\cos\theta - 1)}{r^2\cos\theta}
\end{aligned}$$

This funciton still depends on θ, so the limit $\boxed{\text{does not exist}}$

(b) (5 points)

$$\lim_{(x,y)\to(\pi/4,\pi/4)} \frac{\cos(x) - \sin(y)}{\cos^2(x) - \sin^2(y)}$$

Solution:

$$\begin{aligned}
\lim_{(x,y)\to(\pi/4,\pi/4)} \frac{\cos(x) - \sin(y)}{\cos^2(x) - \sin^2(y)} &= \lim_{(x,y)\to(\pi/4,\pi/4)} \frac{1}{\cos x + \sin y} \\
&= \frac{1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = \boxed{\frac{1}{\sqrt{2}}}
\end{aligned}$$

(c) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4 + x^2y^2 + y^4}$$

Solution: Approaching along $y = 0$, the limit reduces to $\lim_{x\to 0} \frac{0}{x^4} = 0$.

Approaching along $x = y$, the limit reduces to $\lim_{x\to 0} \frac{x^4}{3x^4} = \frac{1}{3}$.

Thus the limit $\boxed{\text{does not exist}}$

4. Evaluate the partial derivatives, if they exist. If they do not exist, explain why

(a) (5 points) $f_y(3,6)$ for $f(x,y) = xe^{y-6} + \ln(xy)$.

Solution:

$$\begin{aligned} f_y &= xe^{y-6} + \frac{1}{xy}x \\ &= xe^{y-6} + \frac{1}{y} \\ f_y(3,6) &= \boxed{3e^3 + \frac{1}{6}} \end{aligned}$$

(b) (5 points) $\frac{\partial^3 f}{\partial x \partial y \partial z}(1,2,3)$ for $f(x,y,z) = xy + yz + xz + xyz$.

Solution: The only term with all three variables is the last. The other terms will become 0 when the appropriate partial derivative is taken. We only need to concern ourselves with

$$\frac{\partial^3}{\partial x \partial y \partial z}(xyz) = \boxed{1}$$

(c) (5 points) $f_z(0,3,0)$ for

$$f(x,y,z) = y + \sqrt{xyz + x^2 + z^2}$$

Solution:

$$f_z = \frac{xy + 2z}{2\sqrt{xyz + x^2 + z^2}}$$

Evaluating at $(0,3,0)$ gives $\frac{0}{0}$ hence $f_z(0,3,0)$ $\boxed{\text{does not exist}}$

5. Consider the function $f(x, y) = xy - 2x + y^2$.

   (a) (5 points) Give the linearization $L(x, y)$ of $f(x, y)$ at $(-1, 1)$.

Solution: The linearization centered at $(-1, 1)$ is

$$L(x, y) = f(-1, 1) + f_x(-1, 1)(x + 1) + f_y(-1, 1)(y - 1)$$

Some side calculation gives

$$\begin{aligned} f(-1, 1) &= 2 \\ f_x(x, y) &= y - 2 \\ f_x(-1, 1) &= -1 \\ f_y(x, y) &= x + 2y \\ f_y(-1, 1) &= 1 \end{aligned}$$

so the linearization is

$$\begin{aligned} L(x, y) &= f(-1, 1) + f_x(-1, 1)(x + 1) + f_y(-1, 1)(y - 1) \\ &= 2 - (x + 1) + (y - 1) \end{aligned}$$

$$\boxed{L(x, y) = -x + y}$$

   (b) (5 points) Use your result from part (a) to write the equation of the tangent plane to the graph of $f(x, y)$ at $(-1, 1)$.

Solution:

$$z = -x + y$$

6. Let

$$f(x, y, z) = xy + yz + xz + 1$$

(a) (5 points) Find the gradient of f.

Solution:

$$\nabla f = \langle y + z, x + z, y + x\rangle$$

(b) (5 points) Let $\mathbf{u} = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle$ and find the directional derivative $D_{\mathbf{u}}f(\sqrt{3}, \sqrt{3}, \sqrt{3})$.

Solution:

$$\begin{aligned} D_{\mathbf{u}}f(\sqrt{3}, \sqrt{3}, \sqrt{3}) &= \nabla f(\sqrt{3}, \sqrt{3}, \sqrt{3}) \cdot \mathbf{u} \\ &= \left\langle 2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\rangle \\ &= 2\sqrt{3}\frac{\sqrt{3}}{3} \cdot 3 = \boxed{6} \end{aligned}$$

(c) (5 points) Give an example of a vector $\mathbf{v}$ for which f is increasing in the direction of $\mathbf{v}$ starting at $(-1, 1, 1)$.

Solution:

$$\nabla f(-1, 1, 1) = \boxed{\langle 2, 0, 0\rangle =: \mathbf{v}}$$

7. Consider the function

$$f(x, y, z) = x^2 + y^2 + z$$

(a) (10 points) Use the chain rule to calculate $\frac{\partial f}{\partial \varphi}$ at the point $(\rho, \theta, \phi) = (1, 0, \pi/2)$ in spherical coordinates.

Solution:

$$\begin{aligned} \frac{\partial f}{\partial \varphi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \varphi} \\ &= 2x \cdot \rho \cos\theta \cos\varphi + 2y \cdot \rho \sin\theta \cos\varphi + 1 \cdot -\rho \sin\varphi \\ &= 0 + 0 + -1 = \boxed{-1} \end{aligned}$$

(b) (5 points) Confirm your result by expressing f in spherical coordinates and taking its partial derivative with respect to φ.

Solution:

$$\begin{aligned} f &= \rho^2 \cos^2\theta \sin^2\varphi + \rho^2 \sin^2\theta \sin^2\varphi + \rho\cos\varphi \\ &= \rho^2 \sin^2\varphi + \rho\cos\varphi \\ f_\varphi &= \rho^2 \cdot 2\sin\varphi\cos\varphi - \rho\sin\varphi \\ f_\varphi(1, 0, \pi/2) &= 1 \cdot 2\sin\pi/2\cos\pi/2 - 1 \cdot \sin\pi/2 \\ &= 0 - 1 \cdot 1 = \boxed{-1} \end{aligned}$$