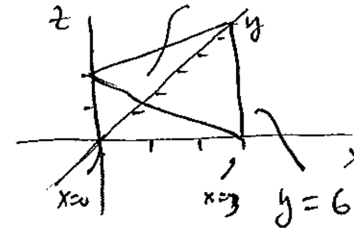
Short answer questions (8 points each):

1. Express the triple integral of the function $xy + yz$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + 3z = 6$ as an iterated integral. Do not evaluate the iterated integral – doing so will waste time needed to complete other questions.

$z = \frac{1}{3}(6-2x-y)$

$$\int_{6}^{3} \int_{0}^{6-2x} \int_{0}^{\frac{1}{3}(6-2x-y)} xy+yz \, dz \, dy \, dx$$



$y = 6-2x$ in xy plane

2. Find the Jacobian determinant of the transformation $(u, v) \to (u^2 - v^2, \frac{1}{2}uv)$

$$\begin{vmatrix} 2u & -2v \\ \frac{1}{2}v & \frac{1}{2}u \end{vmatrix} = 2u(\tfrac{1}{2}u) - (-2v)(\tfrac{1}{2}v)$$

$$= u^2 + v^2$$

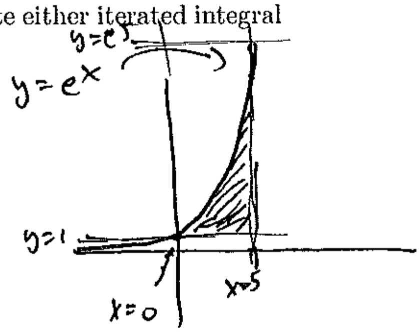
Short answer questions, continued.

3. Consider the iterated integral

$$\int_1^{e^5} \int_{\ln y}^{5} x \, dx dy$$

Draw the region over which this iterated integral represents the double integral of the function y and use your drawing to express the same double integral as an iterated integral with the order of integration reversed. You do not need to evaluate either iterated integral – doing so will waste time needed to complete other questions.

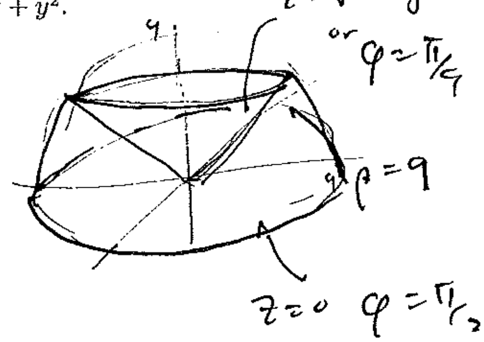
$x = \ln y \Leftrightarrow y = e^x$



$$\int_0^5 \int_1^{e^x} x \, dy \, dx$$

4. Express the volume of the region described in the next sentence as an iterated integral by using spherical coordinates. Do not evaluate the iterated integral – doing so will, you guessed it, waste time needed to complete other questions.

The region lies in the first octant (all rectangular coordinates non-negative) inside the sphere of radius 9 about the origin and below the cone $z = \sqrt{x^2 + y^2}$.



$0 \leq \rho \leq 9$

$\pi/4 \leq \varphi \leq \pi/2$

$0 \leq \theta \leq 2\pi$

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^9 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Long questions: point values in parentheses follow the question number

5. (18) Find and classify the critical points of

$$H(x,y) = y^3 + 4xy + 2x^2$$

at c.p.'s

$H_x(x,y) = 4y + 4x = 0$

$H_y(x,y) = 3y^2 + 4x = 0$

So at c.p.'s $x = -y$ $\qquad 3y^2 - 4y = 0$

$y(3y-4) = 0$

$y = 0$ or $y = 4/3$

So the c.p.'s are $(0,0)$ and $(-4/3, 4/3)$

$H_{xx}(x,y) = 4$

$H_{xy}(x,y) = H_{yx}(x,y) = 4$

$H_{yy}(x,y) = 6y.$

So the Hessian determinant is

$$D = \begin{vmatrix} 4 & 4 \\ 4 & 6y \end{vmatrix} = 24y - 16$$

$D(0,0) = 24 \cdot 0 - 16 < 0$ $\quad \therefore \quad (0,0)$ is a saddle point

$D(-4/3, 4/3) = 24 \cdot \frac{4}{3} - 16 = 16 > 0$

and $H_{xx}(-4/3, 4/3) = 4 > 0$ $\quad \therefore \quad (-4/3, 4/3)$ is a local minimum

6. (18) Use the method of Lagrange multipliers to find the maximum and minimum values of $P(x, y) = 2x^2 + y$ on $\{(x, y)|x^2 + \frac{y^2}{4} = 1\}$, and the points at which they occur.

Let $g(x,y) = x^2 + \frac{y^2}{4}$

then at extrema $x^2 + \frac{y^2}{4} = 1$ and

$\nabla P = \lambda \nabla g \qquad \langle 4x, 1 \rangle = \lambda \langle 2x, y/2 \rangle$

$4x = 2\lambda x \qquad x = 0$ or $\lambda = 2$

$1 = \frac{1}{2}\lambda y$

$x = 0 \Rightarrow \frac{y^2}{4} = 1 \qquad y = \pm 2 \qquad (0,2), (0,-2)$

$\lambda = 2 \Rightarrow 1 = y \qquad x^2 + 1/4 = 1$

$x^2 = 3/4 \qquad x = \pm \sqrt{3}/2 \qquad (\frac{\sqrt{3}}{2}, 1) \; (-\frac{\sqrt{3}}{2}, 1)$

So $P(0, 2) = 2 \cdot 0^2 + 2 = 2$

$P(0, -2) = 2 \cdot 0^2 + (-2) = -2$ — minimum

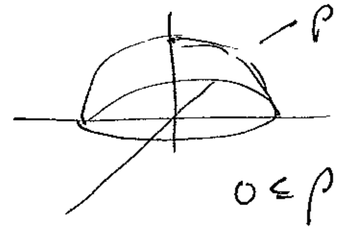
$\left.\begin{array}{l} P(\frac{\sqrt{3}}{2}, 1) = 2 \cdot \frac{3}{4} + 1 = 5/2 \\ P(-\frac{\sqrt{3}}{2}, 1) = 2 \cdot \frac{3}{4} + 1 = 5/2 \end{array}\right] \text{maximum}$

7. (18) Let R be the region bounded below by the xy-plane and above by the sphere of radius 3 about the origin. Use spherical coordinates to express the following triple integral as an iterated integral and evaluate the resulting iterated integral.

$$\iiint_R \frac{z}{x^2+y^2+z^2}\,dV$$

$$\frac{z}{x^2+y^2+z^2} = \frac{\rho\cos\varphi}{\rho^2}$$

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$



$0 \le \rho \le 3$

$0 \le \varphi \le \pi/2$

$0 \le \theta \le 2\pi$

So

$$\iiint_R \frac{z}{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} \rho\cos\varphi\sin\varphi \, d\rho\, d\varphi\, d\theta$$

since bounds are constant and the integrand factors as 1 var functions

$$= \int_0^3 \rho\,d\rho \int_0^{\pi/2} \cos\varphi\sin\varphi\,d\varphi \int_0^{2\pi} d\theta$$

In middle integral let

$u = \sin\varphi \qquad \varphi = 0 \Rightarrow u = 0$

$du = \cos\varphi\,d\varphi \qquad \varphi = \pi/2 \Rightarrow u = 1$

$$= \int_0^3 \rho\,d\rho \int_0^1 u\,du \int_0^{2\pi} d\theta$$

$$= \frac{\rho^2}{2}\Big|_0^3 \cdot \frac{u^2}{2}\Big|_0^1 \cdot 2\pi$$

$$= \frac{9}{2}\cdot\frac{1}{2}\cdot 2\pi = \frac{9\pi}{2}$$

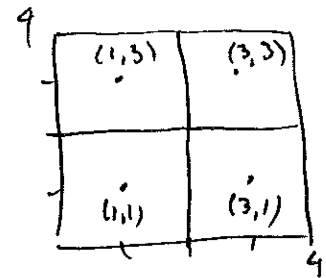
8. (16)

(a) Let R be the square $[0,4] \times [0,4]$ in the xy-plane. Give the midpoint Riemann sum approximation to the double integral

$$\iint_R x^2 + 4y \, dV$$

corresponding to the subdivision of R into four 2 by 2 squares.

$$R_p(f) = (1^2 + 4\cdot 1)\cdot 2\cdot 2 + (3^2 + 4\cdot 1)\cdot 2 \cdot 2$$
$$+ (1^2 + 4\cdot 3)\cdot 2\cdot 2 + (3^2 + 4\cdot 3)\cdot 2\cdot 2$$



$$= 4[1 + 4 + 9 + 4 + 1 + 12 + 9 + 12]$$

$$= 4 \cdot 52 = 208$$

(b) Find the double integral of part (a) by iterated integration.

$$\int_0^4 \int_0^4 x^2 + 4y \, dx \, dy = \int_0^4 \frac{x^3}{3} + 4yx \Big|_{x=0}^{x=4} dy$$

$$= \int_0^4 \frac{64}{3} + 16y \, dy = \frac{64}{3}\cdot y + 8y^2 \Big|_0^4$$

$$= \frac{256}{3} + 128 = \frac{2\cdot 128}{3} + \frac{3\cdot 128}{3}$$

$$= \frac{640}{3}$$

9. (18) Find the volume of the region lying inside both the cylinder about the z-axis of radius 2 and the ellipsoid $\frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{64} = 1$ by expressing the appropriate triple integral as a interated integral using cylindrical coordainates and evaluating the iterated integral.

cylinder : $r = 2$

ellipsoid $\frac{r^2}{16} + \frac{z^2}{64} = 1$

so top & bottom are

$$z^2 = 64\left(1 - \frac{r^2}{16}\right)$$

$$z^2 = 64 - 4r^2$$

$$z = \pm\sqrt{64 - 4r^2}$$

$$= \pm 2\sqrt{16 - r^2}$$

So the region is

$$-2\sqrt{16-r^2} \le z \le 2\sqrt{16-r^2}$$

$$0 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

So $$V = \int_0^{2\pi}\int_0^{2}\int_{-2\sqrt{16-r^2}}^{2\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} 4r\sqrt{16-r^2}\,dr\,d\theta$$

$$= 2\pi \int_0^2 4r\sqrt{16-r^2}\,dr$$

let $u = 16 - r^2$

$du = -2r\,dr$

$-2du = 4r\,dr$

$r = 0 \Rightarrow u = 16$

$r = 2 \Rightarrow u = 12$

$$= 2\pi \int_{12}^{16} 2u^{1/2}\,du$$

(neg sign used to reverse order of integration)

$$= 4\pi \cdot \frac{2}{3} u^{3/2} \Big|_{12}^{16}$$

$$= \frac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$