1. (15 points) Calculate the integral

$$\iiint_{\mathcal{B}} x^2 y \cos(xyz)\,\mathrm{d}V$$

where $\mathcal{B} = [0, \pi] \times [0, 1] \times [-1, 0]$.

Solution:

$$\begin{aligned}
&\int_0^{\pi} \int_0^1 \int_{-1}^0 x^2 y \cos(xyz)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_0^{\pi} \int_0^1 x\big[\sin(xyz)\big]_{z=-1}^{0}\,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_0^{\pi} \int_0^1 x \sin(xy)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad = \int_0^{\pi} -\big[\cos(xy)\big]_{y=0}^{1}\,\mathrm{d}x \\
&\quad = -\int_0^{\pi} \cos(x) - 1\,\mathrm{d}x \\
&\quad = -\big[\sin(x) - x\big]_0^{\pi} = -(-\pi - 0) = \boxed{\pi}
\end{aligned}$$

2. (15 points) Calculate the integral of

$$f(x, y) = (1 - x)^2$$

over the region

$$\mathcal{D} : 0 \leq x \leq 1 - y^2, \quad 0 \leq y$$

Solution: The region is the upper half of a sideways parabola.

$$\begin{aligned}
\int_0^1 \int_0^{1-y^2} (1-x)^2 \,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \int_0^{1-y^2} 1 - 2x + x^2 \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \left[x - x^2 + \frac{1}{3}x^3\right]_0^{1-y^2} \mathrm{d}y \\
&= \int_0^1 1 - y^2 - (1-y^2)^2 + \frac{1}{3}(1-y^2)^3 \,\mathrm{d}y \\
&= \int_0^1 1 - y^2 - (1 - 2y^2 + y^4) + \frac{1}{3}(1 - 3y^2 + 3y^4 - y^6) \,\mathrm{d}y \\
&= \int_0^1 \frac{1}{3} - \frac{1}{3}y^6 \,\mathrm{d}y \\
&= \frac{1}{3}\left[y - \frac{1}{7}y^7\right]_0^1 = \frac{1}{3}\left(1 - \frac{1}{7}\right) = \boxed{\frac{2}{7}}
\end{aligned}$$

3. Consider the region

$$\mathcal{W} : x^2 + y^2 + z^2 \leq 25, \quad x^2 + y^2 \geq 16$$

(a) (10 points) Express the volume $\mathcal{W}$ as an iterated integral using cylindrical coordinates.

Solution: The bounds can be expressed as

$$16 \leq r^2 \leq 25 - z^2 \implies 4 \leq r \leq \sqrt{25 - z^2}$$

This implies that z ranges from -3 to 3. Hence the volume is calculated by the integral

$$\boxed{\int_{-3}^{3} \int_{4}^{\sqrt{25-z^2}} \int_{0}^{2\pi} 1 \cdot r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z}$$

(b) (5 points) Evaluate the integral to obtain $\mathrm{Vol}(\mathcal{W})$.

Solution:

$$\begin{aligned}
\int_{-3}^{3} \int_{4}^{\sqrt{25-z^2}} \int_{0}^{2\pi} 1 \cdot r \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}z \\
&= 2\pi \int_{-3}^{3} \int_{4}^{\sqrt{25-z^2}} r \, \mathrm{d}r \, \mathrm{d}z \\
&= \pi \int_{-3}^{3} \left[r^2\right]_4^{\sqrt{25-z^2}} \mathrm{d}z \\
&= \pi \int_{-3}^{3} (9 - z^2) \, \mathrm{d}r \, \mathrm{d}z \\
&= \pi \left[9z - \frac{1}{3}z^3\right]_{-3}^{3} \\
&= \pi(27 - 9 - (-27 + 9)) = \boxed{36\pi}
\end{aligned}$$

4. Let $\mathcal{D}$ be the parallelogram in the plane with vertices $(0,0), (1,0), (2,3), (3,3)$.

   (a) (10 points) Find a linear map $G(u,v)$ which sends the unit square $[0,1] \times [0,1]$ to $\mathcal{D}$.

Solution: The desired mapping is

$$\begin{aligned}(0,0) &\mapsto (0,0)\\ (1,0) &\mapsto (1,0)\\ (0,1) &\mapsto (2,3)\\ (1,1) &\mapsto (3,3)\end{aligned}$$

hence is

$$G(u,v) = (u + 2v, 3v)$$

   (b) (5 points) Compute the Jacobian of G.

Solution:

$$\mathrm{Jac}(G) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1 \cdot 3 - 2 \cdot 0 = \boxed{3}$$

   (c) (5 points) Use the change of variables formula to compute the integral

$$\iint_{\mathcal{D}} e^{3x-2y}\,\mathrm{d}A$$

Solution:

$$\begin{aligned}\iint_{\mathcal{D}} e^{3x-2y}\,\mathrm{d}A &= \int_0^1 \int_0^1 e^{3(u+2v)-2(3v)} |3| \,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^1 \int_0^1 3e^{3u}\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_0^1 \mathrm{d}v \cdot \int_0^1 3e^{3u}\,\mathrm{d}u\\ &= \left[e^{3u}\right]_0^1 = \boxed{e^3 - 1}\end{aligned}$$

5. (15 points) Calculate

$$\int_{\mathcal{C}} e^{x^2+y^2+z^2}\,\mathrm{d}s$$

where $\mathcal{C}$ is the equator of a sphere, centered at the origin, of radius 3.

Solution: The path has the parameterization

$$\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 0\rangle \qquad t \in [0, 2\pi]$$

We compute

$$\begin{aligned}\mathbf{r}'(t) &= \langle -3\sin t, 3\cos t, 0\rangle \\ \|\mathbf{r}'\| &= \sqrt{9\sin^2 t + 9\cos^2 t} = 3\end{aligned}$$

so

$$\begin{aligned}\int_{\mathcal{C}} e^{x^2+y^2+z^2}\,\mathrm{d}s &= \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\,\mathrm{d}t \\ &= \int_0^{2\pi} e^9 \cdot 3\,\mathrm{d}t = \boxed{6\pi e^9}\end{aligned}$$

6. (10 points) Evaluate
$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$
where $\mathbf{F}(x, y, z) = \langle z, xyz, x \rangle$ and $\mathcal{C}$ is the curve parameterized by
$$\mathbf{r}(t) = (e^t, t, e^{-t})$$
for $0 \le t \le 2$.

Solution:

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\
&= \int_0^2 \left\langle e^{-t}, t, e^t \right\rangle \cdot \left\langle e^t, 1, -e^{-t} \right\rangle \mathrm{d}t \\
&= \int_0^2 1 + t - 1\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_0^2 = \boxed{2}
\end{aligned}$$

7. Consider the vector field

$$\mathbf{F}(x, y, z) = \langle y + z, x + z, x + y \rangle$$

(a) (5 points) Does $\mathbf{F}$ satisfy the cross-partials condition? Verify your response.

Solution: Yes. Denoting the component functions $\mathbf{F} = \langle P, Q, R \rangle$, we check that

- $R_y = Q_z$: $1 = 1$ ✓
- $P_z = R_x$: $1 = 1$ ✓
- $Q_x = P_y$: $1 = 1$ ✓

(b) (5 points) Find a potential for $\mathbf{F}$ if one exists. If not, explain why.

Solution:

$$xy + xz + yz$$