Final Exam
Math 222
July 31
Summer 2015

Name:

Instructor's Name:

Problem(1) [12 points]: Express the region R given by the following inequalities in terms of **Spherical** Coordinates :
$x \geq 0, x^2 + y^2 + z^2 \leq 16$ and $z \geq \sqrt{x^2 + y^2}$

$x^2+y^2+z^2 \leq 16$ gives, $0 \leq \rho^2 \leq 16 \Rightarrow 0 \leq \rho \leq 4$

$x \geq 0$ gives $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$z \geq \sqrt{x^2+y^2}$ gives $0 \leq \phi \leq \frac{\pi}{4}$

$\therefore$ The region in spherical coordinates is

$$\boxed{\begin{gathered} 0 \leq \rho \leq 4 \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \phi \leq \frac{\pi}{4} \end{gathered}}$$

Problem(2) [12 points]: Find the curvature of the curve given by $r(\vec{t}) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$.

We have, $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k} = \langle t, t^2, t^3 \rangle$

$$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$\vec{r}\,''(t) = \langle 0, 2, 6t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= \vec{i}\,(12t^2 - 6t^2) - \vec{j}\,(6t - 0) + \vec{k}\,(2 - 0)$$

$$= \langle 6t^2, -6t, 2 \rangle$$

$$\|\vec{r}\,'(t) \times \vec{r}\,''(t)\| = \sqrt{36t^4 + 36t^2 + 4}$$

$$\therefore \text{Curvature } (\kappa(t)) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$$

$$= \boxed{\frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}}$$

Problem(3) [18 points]: Determine and classify all the critical points of the function $f(x,y) = x^3 + y^2 - 3xy + 15$.

$f_x = 3x^2 - 3y$

$f_y = 2y - 3x$

Setting $f_x = 0$ and $f_y = 0$ we get

$3x^2 - 3y = 0$ — ① and $2y - 3x = 0$ — ②

Solving eqⁿ ① and ② we get

$f_{xx} = 6x$

$3x^2 - 3\left(\frac{3x}{2}\right) = 0$

$f_{yy} = 2$

$3x\left(x - \frac{3}{2}\right) = 0$

$f_{xy} = -3$

$\Rightarrow$ Either $x = 0$ or $x = 3/2$

$x = 0$ gives $y = 0$ and $x = 3/2$ gives $y = \frac{9}{4}$

$\therefore$ $(0,0)$ and $\left(\frac{3}{2}, \frac{9}{4}\right)$ are critical points.

$D = 6x \cdot 2 - (-3)^2 = 12x - 9$

At $(0,0)$, $D = 0 - 9 = -9$

$\therefore$ $\boxed{(0,0) \text{ is saddle point}}$

At $\left(\frac{3}{2}, \frac{9}{4}\right)$, $D = 12 \cdot \frac{3}{2} - 9 = 9 > 0$

and $f_{xx}\left(\frac{3}{2}, \frac{9}{4}\right) = 6 \cdot \frac{3}{2} = 9 > 0$

Hence $\boxed{\left(\frac{3}{2}, \frac{9}{4}\right) \text{ is local minimum}}$

Problem(4) [18 points]: Let $f(x, y, z) = xyz$. Find the directional derivative of $f(x, y, z)$ at the point $P(1, 2, 1)$ in the direction of the vector $\vec{v} = 3\vec{i} + 0\vec{j} + 4\vec{k}$.

$$f(x,y,z) = xyz$$

$$\nabla f(x,y,z) = \langle yz,\ zx,\ xy \rangle$$

$$\nabla f(1,2,1) = \langle 2,\ 1,\ 2 \rangle$$

$$\vec{V} = \langle 3, 0, 4 \rangle$$

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|} = \frac{\langle 3, 0, 4 \rangle}{\sqrt{9+0+16}} = \frac{1}{5}\langle 3, 0, 4 \rangle$$

$\therefore$ The directional derivative is

$$D_{\vec{V}} f(P) = \nabla f(P) \cdot \vec{u}$$

$$= \langle 2, 1, 2 \rangle \cdot \frac{1}{5}\langle 3, 0, 4 \rangle$$

$$= \frac{1}{5}(6+0+8)$$

$$= \boxed{\frac{14}{5}}$$

Problem(5) [12 points]: Let $f(x,y) = x + 2y$. Evaluate the integral:

$$\iint_D f(x,y)\, dA$$

Where D is the region bounded by parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solving $y = 2x^2$ and $y = 1 + x^2$ we get,

$$2x^2 = 1 + x^2$$
$$x^2 = 1$$
$$\Rightarrow x = \pm 1$$

$$\therefore \iint_D f(x,y)\, dA$$

$$= \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{x=-1}^{1} \left(xy + y^2\right)\Big|_{y=2x^2}^{1+x^2} dx = \int_{-1}^{1} \left[x\left(1+x^2-2x^2\right) + \left(1+x^2\right)^2 - (2x^2)^2\right] dx$$

$$= \int_{-1}^{1} (x - x^3 + 1 + 2x^2 + x^4 - 4x^4)\, dx$$

$$= \int_{-1}^{1} \left(1 + x + 2x^2 - x^3 - 3x^4\right) dx$$

$$= x + \frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \frac{3x^5}{5}\Big|_{-1}^{1} = 1 + \frac{1}{2} + \frac{2}{3} - \frac{1}{4} - \frac{3}{5} - \left(-1 + \frac{1}{2} - \frac{2}{3} - \frac{1}{4} + \frac{3}{5}\right)$$

$$= 1 + \cancel{\frac{1}{2}} + \frac{2}{3} - \cancel{\frac{1}{4}} - \frac{3}{5} + 1 - \cancel{\frac{1}{2}} + \frac{2}{3} + \cancel{\frac{1}{4}} - \frac{3}{5}$$

$$= 2 + \frac{4}{3} - \frac{6}{5} = \frac{30 + 20 - 18}{15} = \boxed{\frac{32}{15}}$$

Problem(6) [18 points] Set up the following integral (you do **NOT** need to evaluate the integral):

$$\iiint_W z\, dV.$$

Where W is the region bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z^2 = x^2 + y^2, z \geq 0$ (**Hint**: The region is shaped like an ice-cream cone.)

We write W in spherical coordinates,

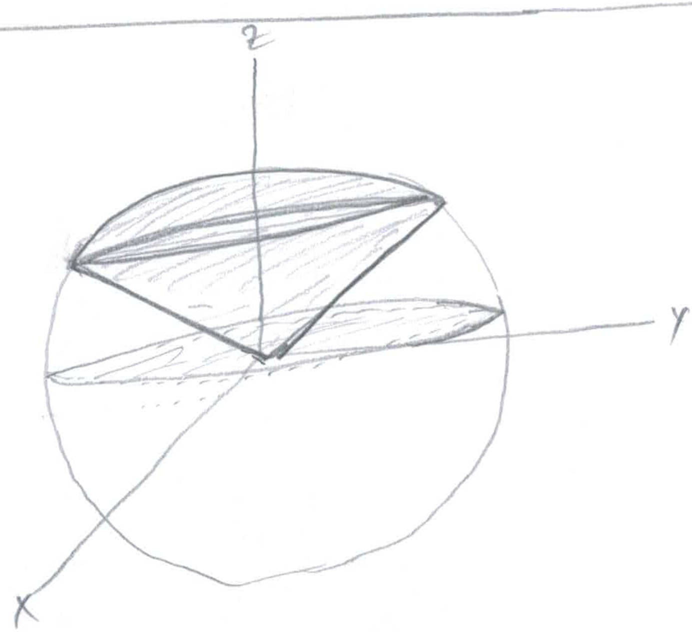
W: $0 \leq \rho \leq 4$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \frac{\pi}{4}$

$$\therefore \iiint_W z\, dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{4} \rho\cos\phi\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \boxed{\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=0}^{4} \rho^3 \sin\phi \cos\phi\, d\rho\, d\phi\, d\theta}$$



Problem(7) **[12 points]** Find a potential function for $\vec{F} = (2xyz^{-1}, z + x^2z^{-1}, y - x^2yz^{-2})$.

Let V be a potential function of $\vec{F}$

$\therefore \quad \vec{F} = \nabla V = \langle 2xyz^{-1}, z + x^2z^{-1}, y - x^2yz^{-2} \rangle$

$$\frac{\partial V}{\partial x} = 2xyz^{-1} \Rightarrow V = xyz^{-1} + g(y,z)$$

$$\frac{\partial V}{\partial y} = z + x^2z^{-1} \Rightarrow V = yz + x^2yz^{-1} + h(x,z)$$

$$\frac{\partial V}{\partial z} = y - x^2yz^{-2} \Rightarrow V = yz + x^2yz^{-1} + f(x,y)$$

So, we must have,

$$xyz^{-1} + g(y,z) = yz + x^2yz^{-1} + h(x,z)$$
$$= yz + x^2yz^{-1} + f(x,y)$$

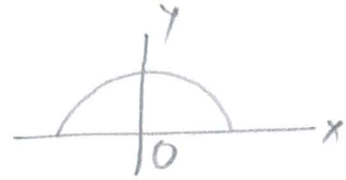
$$\Rightarrow g(y,z) = yz, \quad f(x,y) = 0, \quad h(x,z) = 0$$

$$\therefore \boxed{V = yz + x^2yz^{-1} + c}$$

Problem(8) [18 points] Set up the line integral (you do **NOT** need to evaluate the integral) :

$$\int_C f ds.$$

Where $f(x, y) = 2 + 2y^2$ and path C : $x^2 + y^2 = 1, y \geq 0$.



The parametric eqⁿ of C:

$\vec{r}(t) = \langle \cos t, \sin t \rangle, \; 0 \leq t \leq \pi$

$f(\vec{r}(t)) = 2 + 2\sin^2 t$

$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$\|\vec{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$$\therefore \int_C f ds = \int_{t=0}^{\pi} f(\vec{r}(t)) \|\vec{r}'(t)\| dt$$

$$= \boxed{\int_{t=0}^{\pi} (2 + 2\sin^2 t) dt}$$

Problem(9) [18 points] Let S be a surface parametrized by $G(u,v) = (ucosv, usinv, v), 0 \leq u \leq 1, 0 \leq v \leq 2\pi$ and $\vec{F}(x,y,z) = (0,0,z^2)$. Set up the surface integral (you do **NOT** need to evaluate the integral) :

$$\iint_S \vec{F} \cdot \vec{ds}$$

Orientation: upward-pointing normal.(**Hint** the z-component of the normal vector should be positive).

We have, $G(u,v) = (u\cos v, u\sin v, v)$

$\vec{T}_u = \frac{\partial G}{\partial u} = \langle \cos v, \sin v, 0 \rangle$

$\vec{T}_v = \frac{\partial G}{\partial v} = \langle -u\sin v, u\cos v, 1 \rangle$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}$$

$= \vec{i}(\sin v - 0) - \vec{j}(\cos v - 0) + \vec{k}(u\cos^2 v + u\sin^2 v)$

$= \langle \sin v, -\cos v, u \rangle$

Thus, $\vec{n}(u,v) = \langle \sin v, -\cos v, u \rangle$ ($\because$ upward-pointing normal)

$F(G(u,v)) = \langle 0, 0, v^2 \rangle$

$\therefore \iint_S \vec{F} \cdot d\vec{s} = \iint_D F(G(u,v)) \cdot n(u,v)\, du\, dv$

$= \int_{v=0}^{2\pi} \int_{u=0}^{1} \langle 0, 0, v^2 \rangle \cdot \langle \sin v, -\cos v, u \rangle \, du\, dv$

$= \boxed{\int_{v=0}^{2\pi} \int_{u=0}^{1} v^2 u \, du\, dv}$

Problem(10) **[12 points]**(Short answer porblems) Compute the following:
a) Find the curl of the gradient of fuction $f(x, y, z) = xy^2e^z$.

$\nabla \times \nabla f = \boxed{\vec{0}}$

b) Find the curvature of the circle $x^2 + y^2 = 64$

$K = \frac{1}{R} = \boxed{\frac{1}{8}}$

c) Find the vector line integral

$$\int_C \vec{F} \cdot d\vec{s}$$

Where $\vec{F} = \nabla V = (yz, xz, xy)$ and Path $C : x^2 + y^2 = 144$, clockwise.

By, Fundamental Theorem of Conservative vector field,

$\int_C \vec{F} \cdot d\vec{s} = \boxed{0}$