1. For the following questions, suppose $\mathbf{u} = \langle 1, -1, 0 \rangle$ and $\mathbf{v} = \langle 0, -1, 1 \rangle$.

(a) (5 points) Evaluate $\mathbf{u} - 3\mathbf{v}$.

**Solution:** $= \langle 1, -1, 0 \rangle - 3\langle 0, -1, 1 \rangle = \boxed{\langle 1, 2, -3 \rangle}$

(b) (5 points) Evaluate $\mathbf{u} \cdot \mathbf{v}$.

**Solution:** $= 1(0) + (-1)(-1) + 0(1) = \boxed{1}$

(c) (5 points) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

**Solution:**

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$
$$\implies \theta = \cos^{-1}\frac{1}{2} = \boxed{\frac{\pi}{3}}$$

(d) (5 points) Evaluate $\mathbf{u} \times \mathbf{v}$.

**Solution:** $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = \boxed{\langle -1, -1, -1 \rangle}$

(e) (5 points) Find the volume of the parallelopiped spanned by $\mathbf{u}$, $\mathbf{v}$ and the vector $\langle 3, 1, -2 \rangle$.

**Solution:**

$$= \left|(\mathbf{u} \times \mathbf{v}) \cdot \langle 3, 1, -2 \rangle\right| = \left|\langle -1, -1, -1 \rangle \cdot \langle 3, 1, -2 \rangle\right| = |-3 - 1 + 2| = \boxed{2}$$

(f) (5 points) Find the distance between the point $Q = (1, 0, 0)$ and the line with direction vector $\mathbf{v}$ which passes through the point $P = (0, 1, 0)$.

**Solution:** The formula for distance can be derived by considering the area of the parallelogram spanned by $\mathbf{v}$ and $\overrightarrow{PQ}$:

$$A = \left\| \mathbf{v} \times \overrightarrow{PQ} \right\| = \|\mathbf{v}\|\, d \implies d = \frac{\left\| \mathbf{v} \times \overrightarrow{PQ} \right\|}{\|\mathbf{v}\|}$$

Computing,

$$\begin{aligned} \overrightarrow{PQ} &= \langle 1, -1, 0 \rangle = \mathbf{u} \\ \left\| \mathbf{v} \times \overrightarrow{PQ} \right\| &= \|\mathbf{v} \times \mathbf{u}\| = \left\| \langle 1, 1, 1 \rangle \right\| = \sqrt{3} \\ \|\mathbf{v}\| &= \sqrt{2} \end{aligned}$$

so $d = \boxed{\sqrt{\frac{3}{2}}}$

2. Solve the problems regarding the points $P = (-2, 0, 1)$, $Q = (0, 1, 1)$ and $R = (-1, 1, 0)$.

(a) (10 points) Find a normal vector to the plane containing $P$, $Q$, and $R$.

**Solution:**

$$\overrightarrow{PQ} = \langle 2, 1, 0 \rangle$$
$$\overrightarrow{PR} = \langle 1, 1, -1 \rangle$$
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \boxed{\langle -1, 2, 1 \rangle} =: \mathbf{n}$$

(b) (10 points) Write an equation for the plane containing $P$, $Q$ and $R$.

**Solution:** Using the point $P$,

$$\begin{aligned} \langle -1, 2, 1 \rangle \cdot \langle x+2, y, z-1 \rangle &= 0 \\ -(x+2) + 2y + z - 1 &= 0 \\ -x + 2y + z - 3 &= 0 \end{aligned}$$

Any of these works.

(c) (10 points) Write an equation for a line passing through $Q$ and perpendicular to the plane found in part (b).

**Solution:** Such a line's direction vector is a scalar of $\mathbf{n}$, so a possible equation is

$$\mathbf{r}(t) = \langle 0, 1, 1 \rangle + t \langle -1, 2, 1 \rangle$$

(d) (10 points) Suppose $S$ is any point on the line found in part (c). Find the vector projection of $\overrightarrow{PS}$ onto $\overrightarrow{PQ}$. Explain your response.

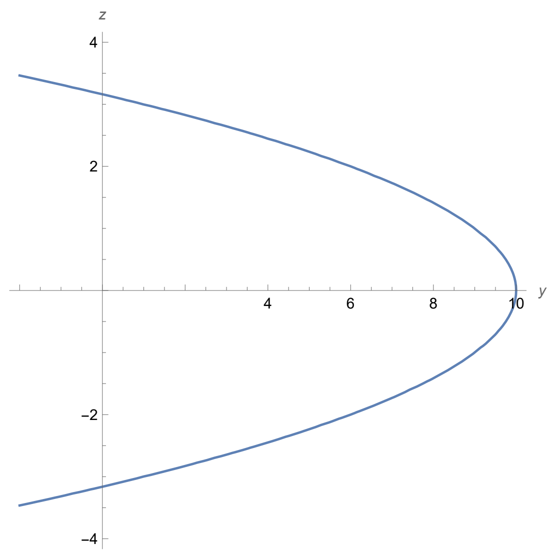
**Solution:** Since $S$ is on the line perpendicular to the plane and passing through $Q$, it is effectively "above" $Q$. Projecting the point $S$ onto the plane will send it to the point $Q$. Thus, the projection of $\overrightarrow{PS}$ onto $\overrightarrow{PQ}$ is $\boxed{\overrightarrow{PQ}}$ itself.

3. Sketch and describe the indicated traces of the quadric surface

$$x^2 + y + z^2 = 11$$

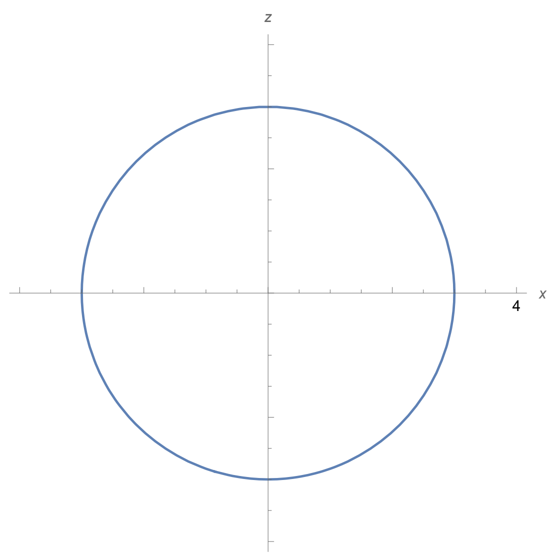
(a) (5 points) The $x = 1$ trace.

**Solution:** Setting $x = 1$ gives the trace: $y = -z^2 + 10$ which is a parabola:




(b) (5 points) The $y = 2$ trace.

**Solution:** Setting $y = 2$ gives the trace: $x^2 + z^2 = 9$, which is a circle of radius 3.



4. (10 points) Give the inequalities in Cartesian coordinates that describe the region below given in spherical coordinates

$$\rho \leq 3, \qquad \frac{\pi}{2} \leq \varphi \leq \pi, \qquad 0 \leq \theta \leq \pi$$

**Solution:**

$$x^2 + y^2 + z^2 \leq 9, \qquad -3 \leq z \leq 0, \qquad y \geq 0$$

5. (10 points) If it exists, find

$$\lim_{t \to 0} \mathbf{r}(t)$$

for the vector valued function

$$\mathbf{r}(t) = \left\langle \frac{e^t - 1}{t}, \frac{t+1}{t^2+1}, \ln\big(t^2+1\big) \right\rangle$$

**Solution:** Evaluating the limits for the second and third component functions is straightforward. Evaluating limit for the first component function requires L'Hôpital:

$$\lim_{t \to 0} \frac{e^t - 1}{t} \overset{L'H}{=} \lim_{t \to 0} \frac{e^t}{1} = 1$$

Thus the answer is

$$\boxed{\langle 1, 1, 0 \rangle}$$