Midterm Exam III
Math 222
July 16
Summer 2015

Name: Solution

Instructor's Name:

Problem(1) [12 points]: Let $f(x, y) = ysin(xy)$ and $R = [1, 2] \times [0, \pi]$. Evaluate the integral:

$$\iint_R f(x, y)\, dA$$

$$= \int_{y=0}^{\pi}\int_{x=1}^{2} y\sin(xy)\,dx\,dy$$

$$= \int_{y=0}^{\pi} -\cancel{y}\frac{\cos(xy)}{\cancel{y}}\Big|_1^2\,dy$$

$$= \int_{y=0}^{\pi} (\cos(y) - \cos(2y))\,dy$$

$$= \sin(y)\Big|_0^{\pi} - \frac{\sin(2y)}{2}\Big|_0^{\pi}$$

$$= \sin(\pi) - \sin(0) - \frac{1}{2}[\sin(2\pi) - \sin(0)]$$

$$= 0 - 0 - \frac{1}{2}(0 - 0)$$

$$= \boxed{0}$$

Problem(2) [12 points]: Evaulate the triple integral.

$$\iiint_B xyz^2 \, dx \, dy \, dz.$$

Where B is the rectalgular box given by B $= \{(x, y, z) : 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$.

$$= \int_0^1 \int_{-1}^2 \int_0^3 xyz^2 \, dz \, dy \, dx$$

$$= \left(\int_0^1 x \, dx\right)\left(\int_{-1}^2 y \, dy\right)\left(\int_0^3 z^2 \, dz\right)$$

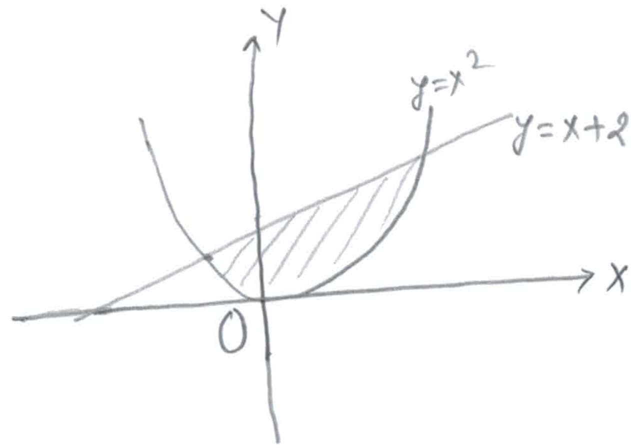
$$= \left(\frac{x^2}{2}\Big|_0^1\right)\left(\frac{y^2}{2}\Big|_{-1}^2\right)\left(\frac{z^3}{3}\Big|_0^3\right)$$

$$= \left(\frac{1}{2} - 0\right)\left(\frac{4}{2} - \frac{1}{2}\right)\left(\frac{27}{3} - 0\right)$$

$$= \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{27}{3}$$

$$= \boxed{\frac{27}{4}}$$

Problem(3) [18 points]: Integrate $f(x, y) = x$ over the region bounded by $y = x^2$ and $y = x + 2$



Solving: $y = x^2$

$y = x+2$

We get $x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

The region R : $-1 \le x \le 2$

$x^2 \le y \le x+2$

$$\therefore \iint_R f(x,y)\, dA$$

$$= \int_{x=-1}^{2} \int_{y=x^2}^{x+2} x \, dy \, dx$$

$$= \int_{x=-1}^{2} \left(xy \Big|_{x^2}^{x+2} \right) dx$$

$$= \int_{-1}^{2} x(x+2-x^2)\, dx = \int_{-1}^{2} (x^2+2x-x^3)\, dx$$

$$= \frac{x^3}{3} + x^2 - \frac{x^4}{4} \Big|_{-1}^{2} = \left(\frac{8}{3} + 4 - \frac{16}{4}\right) - \left(\frac{-1}{3} + 1 - \frac{1}{4}\right)$$

$$= \frac{8}{3} + \frac{1}{3} - 1 + \frac{1}{4}$$

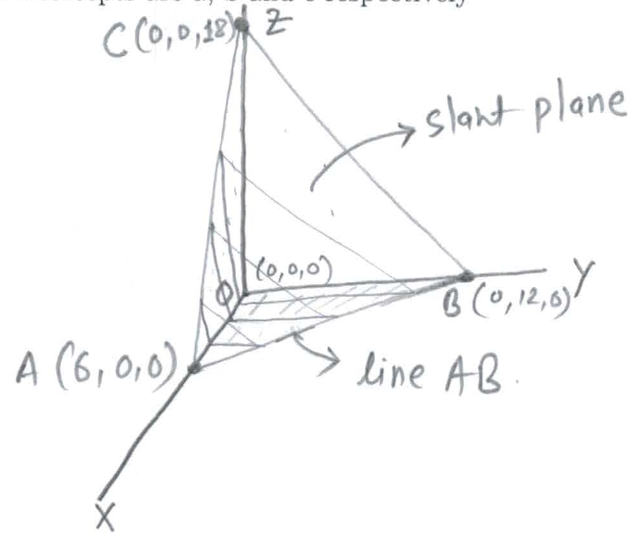
$$= 3 - 1 + \frac{1}{4}$$

$$= 2 + \frac{1}{4} = \boxed{\frac{9}{4}}$$

Problem(4) [18 points]: Let T be the tetrahedron with vertices (0, 0, 0), (6, 0, 0), (0, 12 ,0), and (0, 0, 18). Set up the following integral as an iterated integrals(But **do not** slove it).

(**Hint:** the equation of the plane whose x, y and z intercepts are a, b and c respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$)

$$\iiint_T x\,dV.$$



Eqⁿ of st. line AB

$$\frac{x}{6}+\frac{y}{12}=1$$

$$\Rightarrow y = 12-2x$$

Eqⁿ of slant plane ABC

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{x}{6}+\frac{y}{12}+\frac{z}{18}=1$$

$$\frac{z}{18}=1-\frac{x}{6}-\frac{y}{12}$$

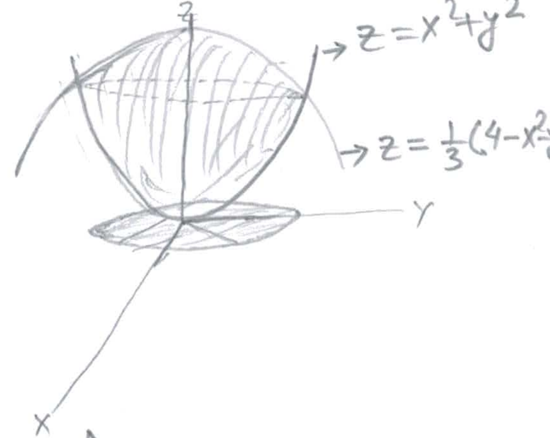
$$\Rightarrow z = 18-3x-\frac{3y}{2}$$

$\therefore$ T: $0 \le x \le 6$
$0 \le y \le 12-2x$
$0 \le z \le 18-3x-\frac{3y}{2}$

Thus, $\iiint_T x\,dV =$

$$\boxed{\int_{x=0}^{6}\int_{y=0}^{12-2x}\int_{z=0}^{18-3x-\frac{3y}{2}} x\,dz\,dy\,dx}\,.$$

Problem(5) [20 points]: Calculate the volume between paraboloids $z = x^2 + y^2$ and $z = \frac{1}{3}(4 - (x^2 + y^2))$



Solving $z = x^2+y^2$

$z = \frac{1}{3}(4 + (x^2+y^2))$

$$x^2+y^2 = \frac{1}{3}(4-(x^2+y^2))$$

$$3(x^2+y^2) = 4-(x^2+y^2)$$

$$x^2+y^2 = 1 \text{ (projection on XY plane)}$$

Using cylindrical Co-ordinates

$$W: \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad r^2 \le z \le \frac{1}{3}(4-r^2)$$

$$\therefore \text{Volume } (V) = \iiint_W 1\, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{\frac{1}{3}(4-r^2)} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_{r=0}^{1} (rz)\Big|_{z=r^2}^{\frac{1}{3}(4-r^2)} dr$$

$$= 2\pi \int_0^1 \left(\frac{4}{3}r - \frac{r^3}{3} - r^3\right) dr = 2\pi\left[\frac{4}{3}\frac{r^2}{2} - \frac{r^4}{12} - \frac{r^4}{4}\Big|_0^1\right]$$

$$= 2\pi\left[\frac{2}{3} - \frac{1}{12} - \frac{1}{4}\right]$$

$$= 2\pi\left[\frac{8-1-3}{12}\right]$$

$$= 2\pi \cdot \frac{4}{12} = \boxed{\frac{2\pi}{3}}$$

Problem(6) **[20 points]** For given function $f(x,y) = e^{\frac{(x+y)}{(x-y)}}$ and trapezoidal region R with vertices (1, 0), (2, 0), (0, -2), and (0, -1). Use an appropriate change of variables to evaluate the integral.

$$\iint_R f(x,y)\, dA$$

Define inverse map:

put $u = x+y$

$V = x-y$

Map from (u,v) plane to (x,y) plane:

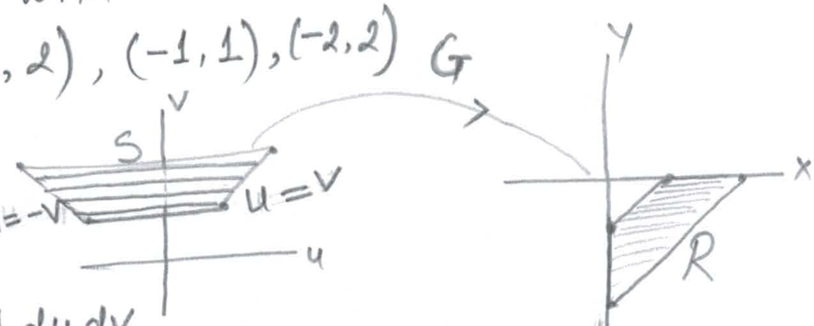
$$G(u,v) = (x(u,v), y(u,v)) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

Jacobian of $G(u,v)$

$$\mathrm{Jac}(G) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Domain of G: S: trapezoidal with vertices

(1, 1), (2, 2), (-1, 1), (-2, 2)



$$\therefore \iint_R f(x,y)\, dA = \int_{v=1}^{2}\int_{u=-v}^{v} e^{\frac{u}{v}} \left|-\frac{1}{2}\right| du\, dv$$

$$= \frac{1}{2}\int_{v=1}^{2} \frac{e^{\frac{u}{v}}}{\frac{1}{v}}\Big|_{-v}^{v} dv = \frac{1}{2}\int_{v=1}^{2} v\left(e - e^{-1}\right) dv$$

$$= \frac{1}{2}\left(e - e^{-1}\right)\left(\frac{v^2}{2}\right)\Big|_1^2$$

$$= \frac{1}{2}\left(e - e^{-1}\right)\left(2 - \frac{1}{2}\right)$$

$$= \boxed{\frac{3}{4}\left(e - e^{-1}\right)}$$