1. (a) (6 points) Write an integral that calculates the length of the curve $y = x + \sin x$, $0 \le x \le \pi$. **Do not evaluate the integral.**

Solution:

$$\int_0^\pi \sqrt{1 + (1 + \cos x)^2}\, \mathrm{d}x$$

(b) (8 points) Find the surface area of the surface obtained by rotating the curve $y = \sqrt{x}$, $1 \le x \le 4$ around the x-axis. **Evaluate the integral.**

Solution:

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x}} \\ (y')^2 &= \frac{1}{4x} \end{aligned}$$

Then

$$\begin{aligned} SA &= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, \mathrm{d}x \\ &= \int_1^4 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, \mathrm{d}x \\ &= 2\pi \int_1^4 \sqrt{x + \frac{1}{4}}\, \mathrm{d}x \\ &= 2\pi\frac{2}{3} \cdot \left(x + \frac{1}{4}\right)^{3/2}\Bigg|_1^4 \\ &= \boxed{\frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right]} \end{aligned}$$

2. (14 points) Find the center of mass (centroid) $(\bar{x}, \bar{y})$ of the region bounded by $y = e^{-x}$, $x = 0$, $x = 1$ and the x-axis.

Solution:

$$M = \int_0^1 e^{-x}\,\mathrm{d}x = -e^{-x}\Big|_0^1 = \boxed{-e^{-1}+1}$$

$$M_x = \frac{1}{2}\int_0^1 (e^{-x})^2\,\mathrm{d}x = \frac{1}{2}\int_0^1 e^{-2x}\,\mathrm{d}x = -\frac{1}{4}\left[e^{-2x}\right]_0^1 = \boxed{-\tfrac{1}{4}(e^{-2}-1)}$$

$$M_y = \int_0^1 xe^{-x}\,\mathrm{d}x \qquad \begin{bmatrix} & D & I \\ + & x & e^{-x} \\ - & 1 & -e^{-x} \\ + & 0 & e^{-x} \end{bmatrix}$$

$$= -xe^{-x} - e^{-x}$$

$$= (-x-1)^{e^{-x}}\Big|_0^1 = \boxed{-2e^{-1}+1}$$

Thus

$$\bar{x} = \frac{M_y}{M} = \frac{-2e^{-1}+1}{-e^{-1}+1} = \boxed{\frac{-2+e}{-1+e}}$$

$$\bar{y} = \frac{M_x}{M} = \frac{-\frac{1}{4}(e^{-2}-1)}{-e^{-1}+1}$$

$$= -\frac{1}{4}\frac{1-e^2}{-e+e^2} = \frac{1}{4e}\frac{1-e^2}{1-e} = \boxed{\frac{1+e}{4e}}$$

3. (a) (7 points) A spring requires 10J to stretch it 2m from its rest length. How much work is required to stretch the spring from 2m to 4m from its rest length.

Solution:

$$10 = \int_0^2 kx \, \mathrm{d}x = k\frac{x^2}{2}\bigg|_0^2 = 2k \implies k = 5$$

$$W = \int_2^4 5x \, \mathrm{d}x = 5\left[\frac{x^2}{2}\right]_2^4 = 5(8-2) = \boxed{30\text{J}}$$

(b) (8 points) Find the work required to pump all the liquid out of a cylindrical tank that has a base of radius 10ft and height 50ft. Use the fact that the density of the liquid is ρ lb/ft^3.

Solution:

$$\begin{aligned}
W &= Fd \\
&= V\rho d \\
&= \rho\pi r^2 \, \mathrm{d}y(50-y) \\
&= \int_0^{50} \rho\pi 100(50-y) \, \mathrm{d}y \\
&= 100\pi\rho \left[50y - \frac{y^2}{2}\right]_0^{50} \\
&= 100\pi\rho\frac{50^2}{2} = 50^3\pi\rho = \boxed{125000\pi\rho \text{ lb ft}}
\end{aligned}$$

4. Evaluate the following

(a) (7 points) $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(\sin(x^2))$, where $\tanh^{-1}$ is the inverse function of tanh.

Solution:

$$\frac{1}{1-(\sin x^2)^2}\cdot\cos(x^2)\cdot 2x$$

(b) (7 points) $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,\mathrm{d}x$, using the substitution $x = 3\sinh\theta$.

Solution:

$$=\int \frac{1}{\sqrt{9(1+\sinh^2\theta)}}\cdot 3\cosh\theta\,\mathrm{d}\theta$$

$$=\int \frac{\cosh\theta\,\mathrm{d}\theta}{\sqrt{\cosh^2\theta}}=\int 1\,\mathrm{d}\theta=\theta=\boxed{\sinh^{-1}\left(\frac{x}{3}\right)+C}$$

5. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3y^2.$$

(a) (3 points) Find the constant solutions.

Solution:

$$0 = x^3y^2 \implies \boxed{y = 0}$$

(b) (8 points) Find the general solution to the differential equation

Solution:

$$\int y^{-2}\,\mathrm{d}y = \int x^3\,\mathrm{d}x$$

$$-y^{-1} = \frac{x^4}{4} + C_1 = \frac{x^4 + 4C_1}{4} = \frac{x^4 + C_2}{4}$$

$$\boxed{y(x) = -\frac{4}{x^4 + C}}$$

(c) (3 points) Find the particular solution satisfying $y(0) = 2$.

Solution:

$$y(0) = -\frac{4}{0 + C} = 2 \implies -4 = 2C \implies C = -2$$

$$\boxed{y(x) = -\frac{4}{x^4 - 2}}$$

6. (a) (7 points) Evaluate the limit of the sequence $\lim_n \frac{\ln(n)}{n^2}$.

Solution:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^2} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = \boxed{0}$$

(b) (8 points) Use the squeeze theorem to calculate $\lim_n \frac{2n - \cos(n)}{n}$.

Solution:

$$\begin{array}{ccccc} -1 & \leq & \cos(n) & \leq & 1 \\ -\frac{1}{n} & \leq & -\frac{\cos(n)}{n} & \leq & \frac{1}{n} \\ 2 - \frac{1}{n} & \leq & 2 - \frac{\cos(n)}{n} & \leq & 2 + \frac{1}{n} \end{array}$$

and

$$\lim_{n\to\infty} \left(2 - \frac{1}{n}\right) = 2$$
$$\lim_{n\to\infty} \left(2 + \frac{1}{n}\right) = 2$$

Thus by squeeze theorem,

$$\lim_{n\to\infty} \frac{2n - \cos(n)}{n} = \lim_{n\to\infty} \left(2 - \frac{\cos(n)}{n}\right) = \boxed{2}$$

7. Evaluate the series:

(a) (7 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 5 + 2^n}{3^n}$

Solution:

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n 5 + 2^n}{3^n} &= 5\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\
&= 5 \cdot \frac{1}{1-(-\frac{1}{3})} + \frac{1}{1-\frac{2}{3}} \\
&= 5 \cdot \frac{1}{\frac{4}{3}} + \frac{1}{\frac{1}{3}} \\
&= \frac{15}{4} + 3 = \boxed{\frac{27}{4}}
\end{aligned}$$

(b) (7 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(n-1)}$

Solution:

$$\frac{1}{n(n-1)} = \frac{-1}{n} + \frac{1}{n-1}$$

so we consider

$$\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

The k-th partial sum is

$$\begin{aligned}
S_k &= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{k-2}} - \cancel{\frac{1}{k-1}}\right) + \left(\cancel{\frac{1}{k-1}} - \frac{1}{k}\right) \\
&= \frac{1}{2} - \frac{1}{k}
\end{aligned}$$

and the series sums to

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(\frac{1}{2} - \frac{1}{k}\right) = \boxed{\frac{1}{2}}$$