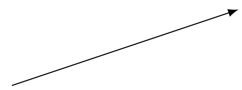
1. Here is a vector which you can assume has unit length:



Call this vector $\mathbf{u}$. Now using the same base point draw a vector $\mathbf{w}$ (and label it) so that the following are all satisfied:

(a) $|\mathbf{w}| = 1$.

(b) $\mathbf{u} \times \mathbf{w}$ points away from you

(c) $\mathbf{u} \times \mathbf{w} \approx \sqrt{2}/2$. (Try to make it as close as you can.)

Next using the same base point again draw a vector $\mathbf{v}$ (and label it) so that the following are all satisfied:

(a) $|\mathbf{v}| = 1$.

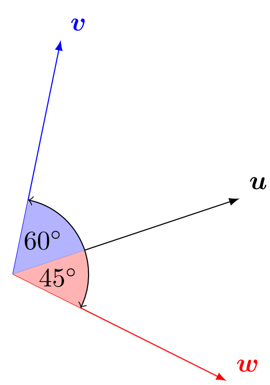
(b) $\mathbf{u} \cdot \mathbf{v} \approx 1/2$. (Again, do your best to get equality.)

(c) $\mathbf{u} \times \mathbf{v}$ points toward you.

Solution:

$$\mathbf{u} \times \mathbf{w} = \|\mathbf{u}\|\|\mathbf{w}\| \sin\theta = \sqrt{2}/2 \implies \theta = 45^\circ$$
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta = 1/2 \implies \theta = 60^\circ$$

so the final configuration looks like



2. Short answers... Intuition and Understanding

(a) If you are driving, then what device (or devices) in your car will be the best way to change your normal acceleration?

Solution: Steering wheel

(b) What is the curvature of a circle with radius 81?

Solution: $\kappa = \frac{1}{81}$

(c) $f(x,y)$ is defined to equal 2 for all points on the disk $(x-1)^2 + (y-2)^2 \le 9$, to equal -3 for all points on the disk $(x+4)^2+(y+7)^2 \le 4$, and to equal 0 everywhere else. Compute:

$$\int_{x=-90}^{100} \int_{y=-100}^{90} f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Solution: $= 2 \cdot \pi(3^2) + -3\pi(2^2) = \boxed{6\pi}$

(d) Will the surface integral

$$\iint_S f(x,y,z)\,\mathrm{d}S$$

typically give you the surface area of S? Explain your answer in one sentence or less.

Solution: No. Only if $f(x,y,z) = 1$ will the integral compute the surface area of S.

(e) What is the average value of the function $f(x,y) = 3+2y$ on the rectangle $1 \le x \le 5$, $2 \le y \le 4$?

Solution:

$$= \frac{\int_1^5 \int_2^4 3 + 2y\,\mathrm{d}y\,\mathrm{d}x}{(5-1)(4-2)} = \frac{4[3y+y^2]_2^4}{2 \cdot 4} = \boxed{9}$$

3. Short answers... Definitions and Theorems

(a) Suppose that $\nabla f(0,0) = \langle 0,0 \rangle$, and $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both negative. Do you need anything else to conclude that $(0,0)$ is a local maximum? (If yes, then what? If no, then why not?)

Solution: Yes, need to know that the discriminant > 0. In particular, that $f_{xx}(0,0)f_{yy}(0,0) > f_{x,y}(0,0)^2$.

(b) What does it mean (definition!) for a vector field $\mathbf{F}(x,y,z)$ to be incompressible?

Solution: $\operatorname{div} \mathbf{F} = 0$.

(c) According to the theorem that we learned, if f is a continuous function on a set Ω, then what condition or conditions on Ω will guarantee that f attains an absolute maximum and absolute minimum?

Solution: Ω must be closed and bounded

(d) Assume that you have been given a differentiable vector field defined on the first octant. How can you quickly tell if it is conservative?

Solution: If it is conservative, one should produce a potential function. If it is not conservative, one should show it violates the cross partials property.

4. A certain differentiable function satisfies:

   (a) $f(7,-9)=1$, and $f(-2,4)=6$

   (b) $\nabla f(7,-9)=(5,3)$, and $\nabla f=(-2,4)=(8,-\pi)$.

   At each of the two points in question (i.e. at $(7,-9)$ and at $(-2,4)$) answer the following questions:

   (a) In what direction is the function increasing the fastest?

   Solution: For $(7,-9)$: $\langle 5,3\rangle$; For $(-2,4)$: $\langle 8,-\pi\rangle$.

   (b) What is the rate of change in that direction?

   Solution: For $(7,-9)$: $\sqrt{25+9}=\sqrt{34}$; For $(-2,4)$: $\sqrt{64+\pi^2}$.

   (c) What is the directional derivative in the direction of $\langle 3,-4\rangle$? (Note: just to be completely clear about semantics here, you are supposed to give the same directional derivative at each point. I did not ask for the directional derivative in the direction of the point $(3,-4)$.)

   Solution: Let $\mathbf{u}=\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle$.
   For $(7,-9)$: $D_{\mathbf{u}}f(7,-9)=\nabla f(7,-9)\cdot\mathbf{u}=\langle 5,3\rangle\cdot\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle=\frac{3}{5}$.
   For $(-2,4)$: $D_{\mathbf{u}}f(-2,4)=\nabla f(-2,4)\cdot\mathbf{u}=\langle 8,-\pi\rangle\cdot\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle=\frac{24+4\pi}{5}$.

   (d) What is the tangent plane and/or the linear approximation at each of the two points?

   Solution: For $(7,-9)$: $L(x,y)=1+5(x-7)+3(y+9)$
   For $(-2,4)$: $L(x,y)=6+8(x+2)-\pi(y-4)$

5. Find the maximum and minimum of the function

$$f(x,y) = 4x^2 - x + 4y^2 - 2y$$

on the set

$$g(x,y) = x^2 + y^2 \leq 45.$$

Show your work carefully, and explain what you are doing. (No essays, please. Just a few short words in the right places will suffice.)

Solution: First, consider $g < 45$. Setting $\nabla f = 0$, and solving

$$\nabla f = \langle 8x - 1, 8y - 2 \rangle = \mathbf{0}$$

gives the critical point $(\frac{1}{8}, \frac{1}{4})$. Next, considering $g = 45$, we solve $\nabla f = \lambda \nabla g$:

$$\begin{cases} 8x - 1 = 2\lambda x \\ 8y - 2 = 2\lambda y \\ x^2 + y^2 = 45 \end{cases}$$

The first two equations give

$$x = \frac{1}{8 - 2\lambda} \qquad y = \frac{2}{8 - 2\lambda}$$

Substituting into the the third equation gives

$$\frac{5}{(8 - 2\lambda)^2} = 45 \implies \frac{1}{(8 - 2\lambda)^2} = 9 \implies \frac{1}{8 - 2\lambda} = \pm 3$$

which yields the two constrained critical points: $(3, 6)$, $(-3, -6)$. Evaluating f on the found critical points:

$$\begin{aligned} f(\frac{1}{8}, \frac{1}{4}) &= -\frac{5}{16} \\ f(3, 6) &= 165 \\ f(-3, -6) &= 195 \end{aligned}$$

Thus the maximum is 195, and the minimum is $-\frac{5}{16}$.

6. Let S be the part of the set
$$z = x^2 + y^2$$
which is between the planes $z = 4$ and $z = 9$.
Express the surface area for S as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ which has **constant** bounds of integration. (i.e. it should be over a rectangular solid or a rectangle in the domain in which you are finally integrating.) You do **NOT** need to find this integral.

Solution: The set S can be parametrized by
$$G(r, \theta) = (r\cos\theta, r\sin\theta, r^2) \qquad 2 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi$$
We can then compute
$$\begin{aligned} \mathbf{N} = G_r \times G_\theta &= \langle \cos\theta, \sin\theta, 2r\rangle \times \langle -r\sin\theta, r\cos\theta, 0\rangle \\ &= \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r\right\rangle \end{aligned}$$
and
$$\|\mathbf{N}\| = \sqrt{4r^4 + r^2}$$
The surface area of S can thus be computed as the surface integral
$$\iint_S 1\,\mathrm{d}S = \boxed{\int_0^{2\pi}\int_2^3 \sqrt{4r^4 + r^2}\,\mathrm{d}r\,\mathrm{d}\theta}$$

7. Let C be the curve given by

$$\mathbf{r}(t) = \left(t \cdot \cos(5\pi t), t + \sin(5\pi t), \frac{t^3}{4+t^2} \right),$$

with $0 \leq t \leq 2$. Compute the following integral:

$$\int_C \left\langle z, 3y^2, x \right\rangle \cdot \mathrm{d}\mathbf{r}.$$

Solution: Note that the vector field $\mathbf{F} = \left\langle z, 3y^2, x \right\rangle$ is conservative, with potential function $f = xz + y^3$. Hence this line integral can be computed as

$$\begin{aligned} \int_C \left\langle z, 3y^2, x \right\rangle \cdot \mathrm{d}\mathbf{r} &= f(\mathbf{r}(2)) - f(\mathbf{r}(0)) \\ &= f(2,2,1) - f(0,0,0) \\ &= 10 - 0 = \boxed{10} \end{aligned}$$

8. Let Q be the set of points within the set:

$$\{(x, y, z) : x^2 + y^2 + z^2 \le 4, \quad \text{and } 0 \le x\}$$

and let ∂Q be the boundary of this set. If $\mathbf{n}$ is the outward unit normal to this region, then compute:

$$\iint_{\partial Q} (x^2, \cos(z^4), \sin(y^4)) \cdot \mathbf{n}\, \mathrm{d}S.$$

Solution: By the divergence theorem,

$$\iint_{\partial Q} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_Q \operatorname{div} \mathbf{F}\, \mathrm{d}V$$

so

$$\iint_{\partial Q} (x^2, \cos(z^4), \sin(y^4)) \cdot \mathbf{n}\, \mathrm{d}S = \iiint_Q 2x\, \mathrm{d}V$$

Evaluating this integral in spherical coordinates,

$$\begin{aligned}
\iiint_Q 2x\, \mathrm{d}V &= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^2 2\rho \sin\varphi \cos\theta \cdot \rho^2 \sin\varphi\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\theta \\
&= \int_0^2 2\rho^3\, \mathrm{d}\rho \cdot \int_0^{\pi} \sin^2\varphi\, \mathrm{d}\varphi \cdot \int_{-\pi/2}^{\pi/2} \cos\theta\, \mathrm{d}\theta \\
&= \left[\frac{\rho^4}{2}\right]_0^2 \cdot \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2\varphi\, \mathrm{d}\varphi \cdot [\sin\theta]_{-\pi/2}^{\pi/2} \\
&= 8 \cdot \left[\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right]_0^{\pi} \cdot 2 \\
&= \boxed{8\pi}
\end{aligned}$$

9. Let E be the subset of

$$z = x^2 + y^2$$

which also satisfies

$$z \leq 25, \qquad x \leq 0, \quad \text{and} \quad y \geq 0.$$

Express

$$\iint_E x^2 \, \mathrm{d}S$$

as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. You do **NOT** need to find this integral.

Solution: The set E can be parametrized by

$$G(r, \theta) = (r\cos\theta, r\sin\theta, r^2) \qquad 0 \leq r \leq 5,\ \frac{\pi}{2} \leq \theta \leq \pi$$

Then we can compute

$$\begin{aligned} \mathbf{N} = G_r \times G_\theta &= \langle \cos\theta, \sin\theta, 2r \rangle \times \langle -r\sin\theta, r\cos\theta, 0 \rangle \\ &= \left\langle -2r^2\cos\theta, -2r^2\sin\theta, r \right\rangle \end{aligned}$$

and

$$\|\mathbf{N}\| = \sqrt{4r^4 + r^2}$$

Which allows to rewrite the integral as

$$\iint_E x^2 \, \mathrm{d}S = \boxed{\int_{\pi/2}^{\pi} \int_0^5 r^2 \cos^2\theta \cdot \sqrt{4r^4 + r^2} \, \mathrm{d}r \, \mathrm{d}\theta}$$

10. Let E be the part of the set

$$\sqrt{x^2+y^2} \leq z \leq 5$$

that also satisfies

$$y \leq 0.$$

Find

$$\iiint_E y \, \mathrm{d}V.$$

Solution: E is half of a solid cone. The integral can be written in cylindrical coordinates as

$$\begin{aligned}
\int_0^5 \int_r^5 \int_\pi^{2\pi} r\sin\theta \cdot r \, \mathrm{d}\theta \, \mathrm{d}z \, \mathrm{d}r &= \int_0^5 \int_r^5 r^2 \, \mathrm{d}z \, \mathrm{d}r \cdot \int_\pi^{2\pi} \sin\theta \, \mathrm{d}\theta \\
&= \int_0^5 r^2(5-r) \, \mathrm{d}r \cdot [-\cos\theta]_\pi^{2\pi} \\
&= \left[\frac{5}{3}r^3 - \frac{1}{4}r^4\right]_0^5 \cdot -2 \\
&= 5^4\left(\frac{1}{3} - \frac{1}{4}\right) \cdot -2 = \boxed{-\frac{5^4}{6}}
\end{aligned}$$