1. Let $\mathbf{u} = \langle 1, 1, 0\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$ and $\mathbf{w} = \langle 0, 1, 1\rangle$.

   (a) (5 points) Compute the volume of the parallelopiped spanned by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

   $$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1 - 1 + 0 = -2$$

   So the volume is $\boxed{2}$

   (b) (5 points) Compute the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

   Solution:

   $$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} = \frac{1}{\sqrt{2} \cdot \sqrt{2}}$$

   Thus $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}}$

   (c) (5 points) Give the equation for the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the origin.

   Solution: The normal vector for the plane is

   $$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle 1, -1, -1\rangle$$

   Thus the equation for the plane is

   $$\boxed{\langle 1, -1, -1\rangle \cdot \langle x, y, z\rangle = 0} \quad \text{or} \quad \boxed{x - y - z = 0}$$

2. Consider the curve $\mathcal{C}$ given by the parametrization

$$\mathbf{r}(t) = \left\langle \sin(t), \cos(t), e^t \right\rangle \quad \text{for } 0 \le t \le \pi$$

(a) (5 points) Find the speed of $\mathbf{r}(t)$ as a function of t.

Solution:

$$s(t) = \|\mathbf{r}'(t)\| = \| \left\langle \cos t, -\sin t, e^t \right\rangle \| = \sqrt{\cos^2 t + \sin^2 t + e^{2t}} = \boxed{\sqrt{1+e^{2t}}}$$

(b) (5 points) Compute the scalar line integral

$$\int_{\mathcal{C}} 3x^2z^2 + 3y^2z^2 \, \mathrm{d}s.$$

Solution: Formula sheet:

$$\int_{\mathcal{C}} f(x,y,z)\, \mathrm{d}s = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, \mathrm{d}t$$

So

$$\begin{aligned}
\int_{\mathcal{C}} 3x^2z^2 + 3y^2z^2 \, \mathrm{d}s &= \int_0^{\pi} (3\sin^2 te^{2t} + 3\cos^2 te^{2t})\sqrt{1+e^{2t}}\, \mathrm{d}t \\
&= \int_0^{\pi} 3e^{2t}\sqrt{1+e^{2t}}\, \mathrm{d}t \qquad \text{u sub: } \left(\begin{array}{c} u = 1+e^{2t} \\ \mathrm{d}u = 2e^{2t}\, \mathrm{d}t \implies \mathrm{d}t = \dfrac{\mathrm{d}u}{2e^{2t}} \end{array} \right) \\
&= \int 3e^{2t} \cdot u^{1/2} \frac{\mathrm{d}u}{2e^{2t}} \\
&= \frac{3}{2} \int u^{1/2}\, \mathrm{d}u \\
&= u^{3/2} = \left[(1+e^{2t})^{3/2} \right]_0^{\pi} = \boxed{(1+e^{2\pi})^{3/2} - 2^{3/2}}
\end{aligned}$$

3. Calculate the following quantities if they exist. Otherwise, explain why they do not exist. Justify either response.

   (a) (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$$

Solution: Converting to polar gives

$$\lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to 0} r^2\cos^2\theta\sin^2\theta = \boxed{0}$$

   (b) (5 points) For

$$f(x,y,z) = \cos(z^2-y^2) + y\sin(x)$$

compute

$$f_{xy}(x,y,z)$$

Solution:

$$\frac{\partial^2}{\partial x\partial y}\cos(z^2-y^2) = 0$$

$$\frac{\partial^2}{\partial x\partial y} y\sin x = 0$$

thus

$$f_{xy} = 0 + \cos x = \boxed{\cos x}$$

(c) (5 points) For $f(x, y, z) = x + y^2 + z^3$ find the change in $f(x, y, z)$ as one moves in the direction of the unit vector $\mathbf{u} = \frac{\sqrt{3}}{3} \langle 1, 1, 1 \rangle$ starting at $\langle 2, 0, -1 \rangle$.

Solution: We know $D_{\mathbf{u}} f(p) = \nabla f_p \cdot \mathbf{u}$, so we compute

$$\begin{aligned} \nabla f &= \left\langle 1, 2y, 3z^2 \right\rangle \\ \nabla f|_p &= \langle 1, 0, 3 \rangle \\ \nabla f|_p \cdot \mathbf{u} &= \langle 1, 0, 3 \rangle \cdot \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle = \frac{1}{\sqrt{3}}(1 + 0 + 3) = \boxed{\frac{4}{\sqrt{3}}} \end{aligned}$$

(d) (5 points) Find the equation for the tangent plane to the surface

$$z = xy$$

at the point $(1, 2, 2)$.

Solution: The tangent plane at a point $p = (a, b)$ is given by

$$z = f(p) + f_x(p)(x - a) + f_y(p)(y - b)$$

so we compute

$$\begin{aligned} f_x &= y & f_y &= x \\ f_x(p) &= 2 & f_y(p) &= 1 \end{aligned}$$

giving

$$\boxed{z = 2 + 2(x - 1) + 1(y - 2)} \text{ or } \boxed{2x + y - z = 2}$$

4. Let

$$f(x, y) = x^3 - 12x + y^2$$

and $\mathcal{D}$ be the square $[-3, 3] \times [-3, 3]$.

(a) (5 points) Find the critical points of $f(x, y)$ in the interior of $\mathcal{D}$.

Solution: Solving $\nabla f = \left\langle 3x^2 - 12, 2y \right\rangle = \langle 0, 0 \rangle$ gives $x = \pm 2$ and $y = 0$. So the critical points for f are $\boxed{(2, 0), (-2, 0)}$

(b) (5 points) Describe the local behavior of $f(x, y)$ at the critical points found in part (a).

Solution: The discriminant is

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 12x$$

Evaluating the discriminant at the previously found critical points:

$$D_{(2,0)} = 24 > 0 \quad \text{and} \quad f_{xx}(2, 0) = 12 > 0 \quad \text{so } \boxed{(2, 0) \text{ is a local min}}$$

$$D_{(-2,0)} = -24 < 0 \quad \text{so } \boxed{(-2, 0) \text{ is a saddle point}}$$

(c) (5 points) Find the maximum value of f on $\mathcal{D}$.

Solution: Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ denote paths along the right, left, top, and bottom sides of the square $\mathcal{D}$, respectively.

Along γ_1, $f(3, y) = -9 + y^2$, with $y \in [-3, 3]$. Looking for critical points along γ_1, we compute $f' = 2y = 0 \implies y = 0$, so $(3, 0)$ is a critical point along γ_1. We test on the critical points and the end points of this path and obtain

$$f(3,0) = -9 \qquad f(3,-3) = 0 \qquad f(3,3) = 0$$

Along γ_2, $f(-3, y) = 9 + y^2$, with $y \in [-3, 3]$. Looking for critical points along γ_2, we compute $f' = 2y = 0 \implies y = 0$, so $(-3, 0)$ is a critical point along γ_2. We test on the critical points and the end points of this path and obtain

$$f(-3,0) = 9 \qquad f(-3,-3) = 18 \qquad f(-3,3) = 18$$

Along γ_3, $f(x, 3) = x^3 - 12x + 9$, with $x \in [-3, 3]$. Looking for critical points along γ_3, we compute $f' = 3x^2 - 12 = 0 \implies x^2 = 4 \implies = \pm 2$. So $(\pm 2, 3)$ are critical points along this path. We test on the critical points and the end points of this path and obtain

$$f(2,3) = -7 \qquad f(-2,3) = 25 \qquad f(-3,3) = 18 \qquad f(3,3) = 0$$

Finally, along γ_4, $f(x, -3) = x^3 - 12x + 9$, with $x \in [-3, 3]$. This yields critical points $(\pm 2, -3)$. As $f(x, y)$ will not distinguish the difference between $y = 3$ and $y = -3$, the values for γ_4 are the same as for γ_3:

$$f(2,-3) = -7 \qquad f(-2,-3) = 25 \qquad f(-3,-3) = 18 \qquad f(3,-3) = 0$$

Thus the maxiumum value for f on $\mathcal{D}$ is $\boxed{25}$ (which occurs at $(-2, \pm 3)$).

5. (10 points) Let $\mathcal{W} = [0,1] \times [-1,0] \times [0,2]$. Evaluate the triple integral

$$\iiint_{\mathcal{W}} (2x+z)e^y \,\mathrm{d}V$$

Solution:

$$\begin{aligned}
\int_0^2 \int_{-1}^0 \int_0^1 (2x+z)e^y \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{-1}^0 e^y \,\mathrm{d}y \cdot \int_0^2 \int_0^1 (2x+z)\,\mathrm{d}x\,\mathrm{d}z \\
&= e^y\Big|_{-1}^0 \cdot \int_0^2 \left[x^2 + xz\right]_{x=0}^1 \mathrm{d}z \\
&= \left(1 - \frac{1}{e}\right) \cdot \int_0^2 (1+z)\,\mathrm{d}z \\
&= \left(1 - \frac{1}{e}\right) \cdot \left[z + \frac{1}{2}z^2\right]_0^2 \\
&= \boxed{4\left(1 - \frac{1}{e}\right)}
\end{aligned}$$

6. Evaluate the following integrals.

   (a) (10 points) Let $\mathcal{D}$ be the region $x^2 + y^2 \leq 4$, $0 \leq y$, $x \leq 0$. Evaluate

$$\iint_{\mathcal{D}} 3x \,\mathrm{d}A.$$

Solution: The region is a quarter circle of radius 2 in the second quadrant. We convert to polar and integrate:

$$\begin{aligned}
\iint_{\mathcal{D}} 3x \,\mathrm{d}A &= \int_{\pi/2}^{\pi} \int_0^2 3r\cos\theta \; r \,\mathrm{d}r \,\mathrm{d}\theta \\
&= \int_0^2 3r^2 \,\mathrm{d}r \cdot \int_{\pi/2}^{\pi} \cos\theta \,\mathrm{d}\theta \\
&= r^3 \Big|_0^2 \cdot \sin\theta \Big|_{\pi/2}^{\pi} \\
&= (8-0)\cdot(0-1) \\
&= \boxed{-8}
\end{aligned}$$

   (b) (10 points) Let $\mathcal{D}$ be the region between the lines $y = -x$, $y = -1$ and $x = -1$. Compute the integral

$$\iint_{\mathcal{D}} 2y \,\mathrm{d}A.$$

Solution: The integral to set up is clear from a sketch of the region $\mathcal{D}$.

$$\begin{aligned}
\iint_{\mathcal{D}} 2y \,\mathrm{d}A &= \int_{-1}^{1} \int_{-1}^{-x} 2y \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{-1}^{1} y^2 \Big]_{-1}^{-x} \,\mathrm{d}x \\
&= \int_{-1}^{1} x^2 - 1 \,\mathrm{d}x \\
&= \frac{x^3}{3} - x \Big|_{-1}^{1} = \boxed{-\frac{4}{3}}
\end{aligned}$$

7. Let

$$\mathbf{F} = \langle 2x + yz, xz, xy \rangle .$$

(a) (5 points) If $\mathbf{F}$ is a conservative vector field, find a potential. Otherwise, explain why it is not conservative.

Solution: $\mathbf{F}$ is a conservative vector field, with potential function $f = x^2 + xyz$. One can check that $\nabla f = \mathbf{F}$.

(b) (5 points) Let $\mathcal{C}$ be the oriented curve with parametrization

$$\mathbf{r}(t) = \left\langle \sin^6(\pi t) + t + 1, e^t + e^{-t}, e^{t^2-1} - 1 \right\rangle$$

for $-1 \leq t \leq 1$. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Solution: Since $\mathbf{F}$ is conservative,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(-1)) \\ &= f(2, e + e^{-1}, 0) - f(0, e + e^{-1}, 0) \\ &= \boxed{4} \end{aligned}$$

(c) (5 points) Is there a vector potential for $\mathbf{F}$ (a vector field $\mathbf{A}$ that satisfies $\mathbf{F} = \operatorname{curl}(\mathbf{A})$)? Explain your response.

Solution: No. If $\mathbf{F}$ had a vector potential, then it would necessarily follow that $\operatorname{div}\mathbf{F} = 0$. However,

$$\operatorname{div}\mathbf{F} = 2 + 0 + 0 = 2 \neq 0$$

(d) (5 points) Let $\mathcal{S}$ be the sphere $x^2 + y^2 + z^2 = 9$ oriented outwardly. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

State any theorems used in the computation.

Solution: By the divergence theorem, we know

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_{\mathcal{E}} \operatorname{div}\mathbf{F}\,\mathrm{d}V$$

where $\mathcal{E}$ is the region for which $\partial\mathcal{E} = \mathcal{S}$. For this problem, $\mathcal{E}$ is the ball of radius 3 centered at the origin. Since $\operatorname{div} F = P_x + Q_y + R_z = 2 + 0 + 0 = 2$, we have

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_{\mathcal{E}} \operatorname{div}\mathbf{F}\,\mathrm{d}V \\ &= \iiint_{\mathcal{E}} 2\,\mathrm{d}V \\ &= 2 \cdot \frac{4}{3}\pi \cdot 3^2 = \boxed{24\pi}\end{aligned}$$

8. Let $\mathcal{D}$ be the lower half disc

$$\mathcal{D} = \{(x, y) : x^2 + y^2 \leq 1,\ y \leq 0\}.$$

The boundary of $\mathcal{D}$ consists of the line segment $\mathcal{C}_1$ along the x-axis oriented from $(1, 0)$ to $(-1, 0)$ and the semi-circle

$$\mathcal{C}_2 = \{(x, y) : y = -\sqrt{1 - x^2},\ -1 \leq x \leq 1\}$$

oriented counter-clockwise. Let $\mathbf{F}$ be the vector field

$$\mathbf{F} = \left\langle -yx^2, xy^2 \right\rangle.$$

(a) (5 points) Using polar coordinates, calculate the double integral

$$\iint_{\mathcal{D}} x^2 + y^2 \, \mathrm{d}A$$

Solution:

$$\begin{aligned} &= \int_{\pi}^{2\pi} \int_0^1 r^2 \cdot r \, \mathrm{d}r \, \mathrm{d}\theta = \int_{\pi}^{2\pi} 1 \, \mathrm{d}\theta \cdot \int_0^1 r^3 \, \mathrm{d}r \\ &= \pi \cdot \left[\frac{r^4}{4} \right]_0^1 = \boxed{\frac{\pi}{4}} \end{aligned}$$

(b) (5 points) Compute the line integral

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Solution: We konw that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t$$

The curve $\mathcal{C}_1$ can be parametrized by $\mathbf{r}(t) = \langle 1-2t, 0\rangle$ where $t \in [0,1]$. Computing $\mathbf{F}(\mathbf{r}(t)) = \langle 0, 0\rangle$, we get

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 0\,\mathrm{d}t = \boxed{0}$$

(c) (5 points) Using only Green's Theorem and the computations in parts (a) and (b), compute the vector line integral

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Solution: Let $\mathbf{F} = \langle P, Q\rangle$. By Green's theorem,

$$\begin{aligned}\iint_{\mathcal{D}} (Q_x - P_y)\,\mathrm{d}A &= \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= \int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.\end{aligned}$$

For this problem, we have $Q_x - P_y = y^2 + x^2$, which gives

$$\iint_{\mathcal{D}} (y^2 + x^2)\,\mathrm{d}A = \int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Substituting in the results from parts (a) and (b),

$$\frac{\pi}{4} = 0 + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Hence

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \boxed{\frac{\pi}{4}}$$

9. Let $\mathcal{S}$ be the cylinder

$$\{(x, y, z) : x^2 + y^2 = 1,\ 0 \leq z \leq 3\}$$

oriented outward and $\mathbf{F} = \langle zy, -zx, 0\rangle$.

(a) (5 points) Compute curl($\mathbf{F}$).

Solution:

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle$$

$$= \langle 0 + x, y - 0, -z - z\rangle = \boxed{\langle x, y, -2z\rangle}$$

(b) (5 points) Calculate

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

Solution: To make this match part (c), $\mathcal{S}$ consists only of the lateral surface of the cylinder. This surface can be parametrized by

$$G(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 3$$

with outward pointing normal vector

$$\mathbf{N} = G_\theta \times G_z = \langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0, 0, 1\rangle = \langle \cos\theta, \sin\theta, 0\rangle$$

Hence the vector surface integral can be calculated directly as

$$\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_0^3 \int_0^{2\pi} \langle \cos\theta, \sin\theta, -2z\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle \, \mathrm{d}\theta \, \mathrm{d}z$$

$$= \int_0^3 \int_0^{2\pi} 1 \, \mathrm{d}\theta \, \mathrm{d}z = \boxed{6\pi}$$

(c) (5 points) The boundary of $\mathcal{S}$ consists of a unit circle $\mathcal{C}_1$ on the xy-plane oriented counterclockwise and a unit circle $\mathcal{C}_2$ on the $z = 3$ plane oriented clockwise. Noting that the vector field is zero on the xy-plane, one easily sees that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

Using only this fact, Stokes' Theorem and your result from part (b), compute the vector line integral

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Solution: By Stokes' Theorem,

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

Using the result from part (b) and the fact that the contour integral over $\mathcal{C}_1$ is zero, we have

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \boxed{6\pi}$$