Short answer questions (8 points each):

1. Find both first partial derivatives of $f(x, y) = x\sin(x + y)$

$f_x(x,y) = \sin(x+y) + x\cos(x+y)$

$f_y(x,y) = x\cos(x+y)$

2. Find the gradient of $g(x, y, z) = xy - z^2$ at $(2, -1, 3)$

$\nabla g(x,y,z) = \langle y, x, -2z \rangle$

$\nabla g(2,-1,3) = \langle -1, 2, -6 \rangle$

3. Explain the role of the determinant of the Hessian (denoted D by Stewart) at a critical point in the two-variable second-derivative test.

The sign of D distinguishes cases in which all concavities at the point are the same $(D > 0)$, from cases where both positive and negative concavity occurs in different directions $(D < 0)$. If $D = 0$ no information is given

Short answer questions, continued.

4. Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$$

Does not exist

restricting to y-axis (x=0) gives $\lim_{y\to 0} \frac{2y^2}{y^2} = 2$

but restricting to the x-axis (y=0) gives

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1$$

5. Find the indicated limit, or explain why it does not exist:

$$\lim_{(x,y)\to(1,1)} \frac{x^2+2y^2}{x^2+y^2} = \frac{1^2+2\cdot 1^2}{1^2+1^2} = \frac{3}{2}$$

since all arithmetic operations are continuous on their domains

Yet more short answer questions.

6. Write two (different) iterated integrals that are equal to the double integral below if $R = [0, 1] \times [1, 3]$ and tell which one will be easier to evaluate, and why.

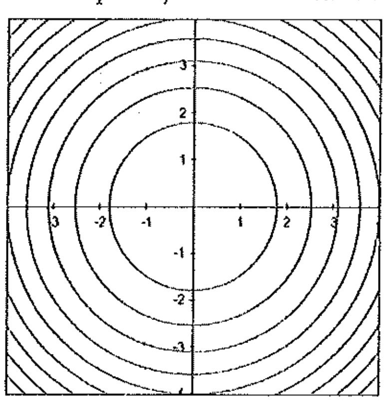
$$\iint_R \frac{x}{xy+4}\, dA$$

Your answer should be two iterated integrals and a brief explanation. Do not exaluate the integrals, though starting to do so may help you decide which is easier and tell why. Finishing the exaluation of either integral will be a waste of time and will not earn additional points.

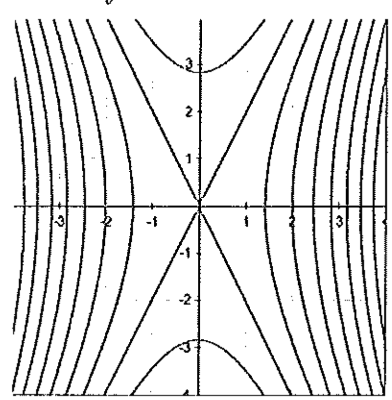
$$\int_1^3 \int_0^1 \frac{x}{xy+4}\, dx\, dy \qquad \int_0^1 \int_1^3 \frac{x}{xy+4}\, dy\, dx$$

The first requires dividing, the second is easier since the substitution $u = xy+4$ $du = x\,dy$ simplifies it.

7. In each of the contour plots below, the origin is a critical point of the function. One is a saddle point, one is a local minimum. Which is which, and how do you know?



local min



saddle point

Near a saddle point the level set of the saddle is always two curves intersecting at the saddle point.

Near a local min (at least of the sort the 2ND derivative test detects) level sets for higher levels surround the min.

Long questions (20 points each)

8. Find and classify the critical points of

$$h(x, y) = 2x^3 - 15x^2 + 36x + 2y^3 - 6y$$

$$h_x(x,y) = 6x^2 - 30x + 36 \overset{\text{at c.p.}}{=} 0$$

$$h_y(x,y) = 6y^2 - 6 = 0$$

$$6(x^2 - 5x + 6) = 0$$
$$6(x-2)(x-3) = 0$$
$$x = 2 \text{ or } x = 3$$

$$h_{xx}(x,y) = 12x - 30$$

$$h_{xy}(x,y) = h_{yx}(x,y) = 0$$

$$h_{yy}(x,y) = 12y$$

$$6(y-1)(y+1) = 0$$
$$y = 1 \text{ or } y = -1$$

c.p's:
(2,−1)
(2, 1)
(3,−1)
(3, 1)

$$D = \begin{vmatrix} 12x-30 & 0 \\ 0 & 12y \end{vmatrix} = 12y(12x-30)$$

$D(2,-1) = -12(24-30) = 72 > 0$ (all conc. same)

$h_{yy}(2,-1) = -12 < 0$ conc. down in y directi

(2,−1) is a local maximum

$D(2,1) = 12(24-30) = -72 < 0$

(2,1) is a saddle point

$D(3,-1) = -12(36-30) = -72 < 0$

(3,−1) is a saddle point

$D(3,1) = 12(36-30) = 72 > 0$ (all conc same)

$h_{yy}(3,1) = 12 > 0$ conc up in y direction.

(3,1) is a local minimum

9. Use the method of Lagrange multipliers to find the maximum and minimum values of $w(x, y, z) = 3x + 4y - 12z$ on $\{(x, y, z)|x^2 + y^2 + z^2 = 169\}$, and the points at which they occur.

constraint: $g(x,y,z) = x^2+y^2+z^2 = 169$

$\nabla w(x,y,z) = \langle 3, 4, -12 \rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$

So $\nabla w = \lambda \nabla g$ gives

$3 = 2\lambda x$ $\qquad \frac{1}{2\lambda} = \frac{x}{3} = \frac{y}{4} = \frac{z}{-12}$

$4 = 2\lambda y$ $\qquad y = \frac{4}{3}x \quad z = -4x$

$-12 = 2\lambda z$

$x^2 + y^2 + z^2 = 169$

$x^2 + \left(\frac{4}{3}x\right)^2 + (-4x)^2 = 169$

$x^2 + \frac{16}{9}x^2 + 16x^2 = 169$

$\frac{9}{9}x^2 + \frac{16}{9}x^2 + \frac{144}{9}x^2 = 169$

$\frac{169}{9}x^2 = 169$

$x^2 = 9$

$\therefore x = \pm 3, \quad y = \pm 4, \quad z = \mp 12$

$w(3, 4, -12) = 3 \cdot 3 + 4 \cdot 4 - 12(-12) = 9 + 16 + 144 = 169$ ← max

$w(-3, -4, 12) = 3(-3) + 4(-4) - 12(12) = -9 - 16 - 144 = -169$ ← min

10. Evaluate

$$\iint_{\{(x,y)\,|\,-1\leq x\leq 4,\,0\leq y\leq 1\}} x^2 - xy + y^2 \, dA$$

$$\int_0^1 \int_{-1}^{4} x^2 - xy + y^2 \, dx \, dy$$

$$= \int_0^1 \frac{x^3}{3} - \frac{x^2 y}{2} + xy^2 \Big|_{x=-1}^{x=4} dy$$

$$= \int_0^1 \frac{64}{3} - \frac{16y}{2} + 4y^2 - \left[\frac{-1}{3} - \frac{y}{2} - y^2\right] dy$$

$$= \int_0^1 \frac{65}{3} - \frac{15y}{2} + 5y^2 \, dy$$

$$= \frac{65}{3}y - \frac{15y^2}{4} + \frac{5y^3}{3} \Big|_0^1$$

$$= \frac{65}{3} - \frac{15}{4} + \frac{5}{3} - [0 - 0 + 0]$$

$$= \frac{260}{12} - \frac{45}{12} + \frac{20}{12} = \frac{235}{12}$$

$$\begin{array}{r} 280 \\ -45 \\ \hline 235 \end{array}$$