1. (16 points) Computation

(a) Let $f(x,y) = x^3 \exp\bigl(2x^2 + 3xy + y^2\bigr)$. Find all of the first partial derivatives. (In case you haven't seen it before, "$\exp(u)$" is the same thing as e^u.)

Solution:

$$\begin{aligned} f_x &= 3x^2 \exp\bigl(2x^2 + 3xy + y^2\bigr) + x^3 \exp\bigl(2x^2 + 3xy + y^2\bigr)(4x + 3y) \\ f_y &= x^3 \exp\bigl(2x^2 + 3xy + y^2\bigr)(3x + 2y) \end{aligned}$$

(b) Let $g(x,y) = \dfrac{x}{\sqrt{4x^2 + y^2}}$. Find the first partial derivative with respect to x and simplify it.

Solution:

$$\begin{aligned} g_x &= \frac{\sqrt{4x^2 + y^2} \cdot 1 - x \cdot \frac{1}{2\sqrt{4x^2+y^2}} \cdot 8x}{4x^2 + y^2} \\ &= \frac{4x^2 + y^2 - 4x^2}{(4x^2 + y^2)^{3/2}} \\ &= \boxed{\frac{y^2}{(4x^2 + y^2)^{3/2}}} \end{aligned}$$

2. (12 points) A certain differentiable function satisfies:

   (a) $f(2,5) = -3$, and $f(-4,1) = \pi$.
   (b) $\nabla f(2,5) = (-4,7)$, and $\nabla f(-4,1) = (\sqrt{6}, e^{-2})$.

   At each of the two points in question (i.e. at $(2,5)$ and at $(-4,1)$) answer the following questions:

   (a) In what direction is the function increasing the fastest and what is the rate of change in that direction?

   Solution:
   At $(2,5)$, the function is increasing the fastest in the direction $\nabla f(2,5) = \langle -4, 7\rangle$, with rate of change $\| \langle -4, 7\rangle \| = \sqrt{16+49} = \sqrt{65}$.
   At $(-4,1)$, the function is increasing the fastest in the direction $\nabla f(-4,1) = \left\langle \sqrt{6}, e^{-2} \right\rangle$, with rate of change $\| \left\langle \sqrt{6}, e^{-2} \right\rangle \| = \sqrt{6+e^{-4}}$.

   (b) What is the directional derivative in the direction of the vector $\langle 4, -3\rangle$?

   Solution: The unit vector in the direction $\langle 4, -3\rangle$ is $\left\langle 4/5, -3/5 \right\rangle$. So

$$\begin{aligned} D_{\left\langle 4/5,-3/5\right\rangle} f(2,5) &= \nabla f(2,5) \cdot \langle 4, -3\rangle \cdot \frac{1}{5} \\ &= \langle -4, 7\rangle \cdot \langle 4, -3\rangle \cdot \frac{1}{5} \\ &= (-16-21) \cdot \frac{1}{5} = \boxed{\frac{-37}{5}} \end{aligned}$$

   and

$$\begin{aligned} D_{\left\langle 4/5,-3/5\right\rangle} f(-4,1) &= \nabla f(-4,1) \cdot \langle 4, -3\rangle \cdot \frac{1}{5} \\ &= \left\langle \sqrt{6}, e^{-2} \right\rangle \cdot \langle 4, -3\rangle \cdot \frac{1}{5} \\ &= \boxed{(4\sqrt{6} - 3e^{-2}) \cdot \frac{1}{5}} \end{aligned}$$

   (c) What is the tangent plane and/or the linear approximation at each of the two points?

   Solution:

$$\begin{aligned} (2,5): &\quad z = -3 + -4(x-2) + 7(y-5) \\ (-4,1): &\quad z = \pi + \sqrt{6}(x+4) + e^{-2}(y-1) \end{aligned}$$

3. (12 points) Set up **but do not solve** the following problems. As part of setting these problems up, you should list the unknowns and the equations that you would need to use to find them. You **should also do** all of the **derivative** calculations, but the **algebra** is totally unmanageable, so do **not** attempt it!

   (a) Maximize $f(x, y) = x^2 \cos(y)$
   Subject to $g(x, y) = x^6 + y^6 = 64$.

   Solution:

$$\nabla f = \lambda \nabla g$$
$$\left\langle 2x \cos y, -x^2 \sin y \right\rangle = \lambda \left\langle 6x^5, 6y^5 \right\rangle$$

The system to solve is:

$$\begin{cases} 2x \cos y = 6\lambda x^5 \\ -x^2 \sin y = 6\lambda y^5 \\ x^6 + y^6 = 64 \end{cases}$$

   (b) Maximize $F(x, y, z) = \cos\left(xy^2z^3\right)$
   Subject to $G(x, y, z) = x + 2y + 3z = 0$
   and $H(x, y, z) = x^2 + z^2 = 25$.

   Solution:

$$\begin{aligned} \nabla F &= \left\langle -y^2z^3 \sin\left(xy^2z^3\right), -2xyz^3 \sin\left(xy^2z^3\right), -3xy^2z^2 \sin\left(xy^2z^3\right) \right\rangle \\ \nabla G &= \langle 1, 2, 3 \rangle \\ \nabla H &= \langle 2x, 0, 2z \rangle \end{aligned}$$

Setting $\nabla F = \lambda \nabla G + \mu \nabla H$ gives the system to solve:

$$\begin{cases} -y^2z^3 \sin\left(xy^2z^3\right) = \lambda + 2\mu x \\ -2xyz^3 \sin\left(xy^2z^3\right) = 2\lambda \\ -3xy^2z^2 \sin\left(xy^2z^3\right) = 3\lambda + 2\mu z \\ x + 2y + 3z = 0 \\ x^2 + z^2 = 25 \end{cases}$$

4. (14 points) For the function $f(x, y) = x^3 + xy - y^2$ find and classify all of the critical points.

Solution:

$$\nabla f = 0 \implies \begin{cases} 3x^2 + y = 0 \\ x - 2y = 0 \end{cases}$$

Solving this system gives two critical points: $(0, 0)$ and $(-\frac{1}{6}, -\frac{1}{12})$. To classify them, the discriminant is

$$f_{xx} f_{yy} - (f_{xy})^2 = 6x(-2) - (1)^2 = -12x - 1$$

For $(0, 0)$, the discriminant gives -1, so $(0, 0)$ is a saddle point.

For $(-\frac{1}{6}, -\frac{1}{12})$, the discriminant gives 1, and $f_{yy} = -2 < 0$, so $(-\frac{1}{6}, -\frac{1}{12})$ is a local maximum.

5. (20 points) Find the maximum and the minimum of the function

$$f(x, y) = x^2 + 2x + y^2 - 6y$$

in the region given by

$$g(x, y) = x^2 + y^2 \leq 40.$$

Show your work carefully in this problem, and let us know what you are doing.

Solution: Use Lagrange multipliers with $g(x, y) = x^2 + y^2 = k$, for $k \in [0, 40]$.

$$\begin{aligned} \nabla f &= \langle 2x + 2, 2y - 6 \rangle \\ \nabla g &= \langle 2x, 2y \rangle \end{aligned}$$

so we solve the system of equations

$$\begin{cases} 2x + 2 = 2\lambda x \\ 2y - 6 = 2\lambda y \\ x^2 + y^2 = k \end{cases}$$

Solving the first two equations for λ yields

$$x = \frac{1}{\lambda - 1} \qquad y = \frac{-3}{\lambda - 1}$$

Substituting into the third gives

$$\frac{10}{(\lambda - 1)^2} = k \implies \frac{1}{\lambda - 1} = \pm\sqrt{\frac{k}{10}}$$

so for a fixed k, the optimization problem yields two solutions:

$$(x, y) = \left(\pm\sqrt{\frac{k}{10}}, \mp 3\sqrt{\frac{k}{10}} \right)$$

To determine max/min values, we plug these points into f:

$$\begin{aligned} f &= x^2 + y^2 + 2x - 6y \\ &= k + 2\left(\pm\sqrt{\frac{k}{10}} \right) - 6\left(\mp 3\sqrt{\frac{k}{10}} \right) \\ &= k \pm 20\sqrt{\frac{k}{10}} \\ &= k \pm 2\sqrt{10}\sqrt{k} \end{aligned}$$

We want to optimize f with respect to $k \in [0, 40]$, which requires us to evaluate k at critical points and at end points. Note there are two cases for f.

- If $f = k + 2\sqrt{10}\sqrt{k}$, then $f' = 1 + \frac{\sqrt{10}}{\sqrt{k}} = 0$ which yields no critical points.
  - Evaluating at $k = 0$, $f = 0$
  - Evaluating at $k = 40$, $f = 40 + 2\sqrt{400} = 80$
- If $f = k - 2\sqrt{10}\sqrt{k}$, then $f' = 1 - \frac{\sqrt{10}}{\sqrt{k}} = 0 \implies k = 10$ is a critical point.
  - Evaluating at $k = 0$, $f = 0$
  - Evaluating at $k = 10$, $f = 10 - 2\sqrt{100} = -10$
  - Evaluating at $k = 40$, $f = 40 - 2\sqrt{400} = 0$

Thus the maximum for this function in the specified region is 80 (which occurs at $(2, -6)$) and the minimum is -10 (which occurs at $(-1, 3)$).

6. (8 points) Suppose that $x = r\cos\theta$ and $y = r\sin\theta$ (the usual polar coordinates) and $f(x,y) = x^2y$. Express

$$\frac{\partial f}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta}$$

as functions of r and θ. (Hint/Comment: Do this however you like.)

Solution:

$$\begin{aligned} f &= x^2y = r^3\cos^2\theta\sin\theta \\ f_r &= 3r^2\cos^2\theta\sin\theta \\ f_\theta &= r^3\left(2\cos\theta(-\sin\theta)\sin\theta + \cos^2\theta\cos\theta\right) \\ &= r^3(-2\cos\theta\sin^2\theta + \cos^3\theta) \end{aligned}$$

7. (18 points) Short answers . . .

(a) If f is a function of x and y, and x and y are each functions of r, s, and t, then use the chain rule to express $\dfrac{\partial f}{\partial s}$.

Solution:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

(b) Find the average value of the function $f(x,y) = x^2 y$ on the rectangle $0 \le x \le 3$, $0 \le y \le 4$.

Solution:

$$\frac{1}{12}\int_0^3 \int_0^4 x^2 y \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^3 \left[\frac{y^2}{2}\right]_0^4 = \frac{1}{12}(9)(8) = 6$$

(c) According to the theorem that we learned, what should you require of a set S to guarantee that any continuous function f will attain an absolute maximum and an absolute minimum on S?

Solution: The set S must be closed and bounded.

(d) For the set $5x^2 + y^3 + 2z^6 - 3xy^2z^2 = 11$ write down the tangent plane at the point $(-1,-2,1)$.

Solution: Letting $f = 5x^2 + y^3 + 2z^6 - 3xy^2z^2 - 11$,

$$\nabla f = \left\langle 10x - 3y^2z^2, 3y^2 - 6xyz^2, 12z^5 - 6xy^2z \right\rangle$$

$$\nabla f(-1,-2,1) = \langle -22, 0, 36 \rangle$$

$$\boxed{0 = -22(x+1) + 0(y+2) + 36(z-1)}$$