

1. (10 points) §2.1: Let  $P = (0, 2)$ ,  $Q = (1, 3)$ ,  $R = (-4, -6)$ .

(a) Find and express in both component form and by using standard unit vectors:

$$\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PQ} - \overrightarrow{PR}$$

**Solution:**

$$\begin{aligned}\overrightarrow{PQ} &= \langle 1, 1 \rangle = \mathbf{i} + \mathbf{j} \\ \overrightarrow{PR} &= \langle -4, -8 \rangle = -4\mathbf{i} - 8\mathbf{j} \\ \overrightarrow{PQ} - \overrightarrow{PR} &= \langle 1, 1 \rangle - \langle -4, -8 \rangle = \langle 5, 9 \rangle = 5\mathbf{i} + 9\mathbf{j}\end{aligned}$$

- (b) Find a unit vector in direction  $\overrightarrow{PR}$ .

**Solution:**

$$\begin{aligned}\frac{\overrightarrow{PR}}{\|\overrightarrow{PR}\|} &= \frac{\langle -4, -8 \rangle}{\sqrt{16 + 64}} \\ &= \frac{1}{\sqrt{80}} \langle -4, -8 \rangle \\ &= \frac{1}{4\sqrt{5}} \langle -4, -8 \rangle \\ &= \left\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle\end{aligned}$$

- (c) Find  $\mathbf{v}$  with the same direction as  $\overrightarrow{PQ} - \overrightarrow{PR}$  and length 10.

**Solution:**

$$\begin{aligned}10 \cdot \frac{\overrightarrow{PQ} - \overrightarrow{PR}}{\|\overrightarrow{PQ} - \overrightarrow{PR}\|} &= 10 \frac{\langle 5, 9 \rangle}{\sqrt{25 + 81}} \\ &= \frac{10}{\sqrt{106}} \langle 5, 9 \rangle \\ &= \left\langle \frac{50}{\sqrt{106}}, \frac{90}{\sqrt{106}} \right\rangle\end{aligned}$$

2. (10 points) §2.2:

- (a) Give an equation for  $S$ , the set of points of distance 5 from the point  $C = (2, -4, -6)$  in  $\mathbb{R}^3$ .

**Solution:**

$$S := \boxed{(x - 2)^2 + (y + 4)^2 + (z + 6)^2 = 25}$$

- (b) Give an equation to describe the intersection of  $S$  and the plane  $y = -3$ .

**Solution:** Setting  $y = -3$  to the description for  $S$  determined in the previous part,

$$(x - 2)^2 + (-3 + 4)^2 + (z + 6)^2 = 25$$

$$\boxed{(x - 2)^2 + (z + 6)^2 = 24}$$

- (c) Given  $P = (2, -9, -6)$  in  $S$ , find the unique point  $Q$  of distance 10 from  $P$  which is also in  $S$ .

**Solution:** The sphere is centered at  $C = (2, -4, -6)$  and of radius 5. The given point  $P = (2, -9, -6)$  is therefore the left-most point (most negative in the  $y$ -direction) on the sphere. The point  $Q$  we are looking for is thus the right-most point (most positive in the  $y$ -direction):  $\boxed{(2, 1, -6)}$

- (d) Find a unit-length vector in the direction of  $\overrightarrow{PQ}$ .

**Solution:**  $\overrightarrow{PQ} = \langle 0, 10, 0 \rangle$ , so  $\boxed{\langle 0, 1, 0 \rangle}$

3. (10 points) §2.3: Let  $\mathbf{u} = \langle 1, 4, -3 \rangle$  and  $\mathbf{v} = \langle 2, -5, 2 \rangle$ .

(a) Find  $\mathbf{u} \cdot \mathbf{v}$ . Find the angle  $\theta \in [0, \pi]$  between  $\mathbf{u}, \mathbf{v}$ .

**Solution:**  $\mathbf{u} \cdot \mathbf{v} = 2 - 20 - 6 = -24$ . Using the formula

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|},$$

and computing

$$\|\mathbf{v}\| = \sqrt{4 + 25 + 4} = \sqrt{33}$$

$$\|\mathbf{u}\| = \sqrt{1 + 16 + 9} = \sqrt{26},$$

we obtain

$$\cos \theta = \frac{-24}{\sqrt{26} \cdot \sqrt{33}} \implies \theta = \arccos\left(\frac{-24}{\sqrt{858}}\right)$$

(b) Find  $\|\text{proj}_{\mathbf{u}}(\mathbf{v})\|$ . Find  $\text{proj}_{\mathbf{u}}(\mathbf{v})$

**Solution:**

$$\begin{aligned} \text{proj}_{\mathbf{u}}(\mathbf{v}) &= \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \\ &= \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} \\ &= -\frac{24}{26} \langle 1, 4, -3 \rangle \\ &= -\frac{12}{13} \langle 1, 4, -3 \rangle = \left\langle -\frac{12}{13}, -\frac{48}{13}, \frac{36}{13} \right\rangle \end{aligned}$$

and

$$\|\text{proj}_{\mathbf{u}}(\mathbf{v})\| = \left| \frac{-12}{13} \right| \sqrt{1 + 16 + 9} = \frac{12}{13} \sqrt{26}$$

- (c) Are  $\mathbf{v}$  and the standard basis vector  $\mathbf{k}$  orthogonal? Why or why not?

**Solution:**

$$\mathbf{v} \cdot \mathbf{k} = \langle 2, -5, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2 \neq 0$$

so NO,  $\mathbf{v}$  and  $\mathbf{k}$  are not orthogonal.

- (d) Find the work done by a force  $\mathbf{F} = \langle 1, -4, -3 \rangle$  applied to move an object in a straight line from the terminal point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ . Let units be in feet and lbs.

**Solution:**

$$\begin{aligned} \mathbf{W} &= \mathbf{F} \cdot \overrightarrow{uv} \text{ where } \overrightarrow{uv} = \langle 2 - 1, -5 - 4, -2 + 3 \rangle = \langle 1, -9, 5 \rangle \\ &= \langle 1, -4, -3 \rangle \cdot \langle 1, -9, 5 \rangle \\ &= 1 + 36 - 15 = 22 \text{ ft} \cdot \text{lbs.} \end{aligned}$$

4. (10 points) §2.4: Let  $\mathbf{u} = \langle 4, 1, 3 \rangle$  and  $\mathbf{v} = \langle -5, 1, 6 \rangle$ .

(a) Find  $\mathbf{u} \times \mathbf{v}$  and  $\|\mathbf{u} \times \mathbf{v}\|$ .

**Solution:**

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 3 \\ -5 & 1 & 6 \end{vmatrix} \\ &= 3\mathbf{i} - 39\mathbf{j} + 9\mathbf{k} \\ &= \langle 3, -39, 9 \rangle\end{aligned}$$

with

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{9 + 39^2 + 81} = \sqrt{1611} \approx 40.13726.$$

- (b) Find a unit-length vector orthogonal to both  $\mathbf{u}$  and  $\mathbf{v}$ .

**Solution:**

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{\langle 3, -39, 9 \rangle}{\sqrt{1611}}.$$

- (c) Let  $P = \langle 2, -5, 7 \rangle$ . Find the area of the triangle spanned by  $P$  and the terminal points of  $\mathbf{u}$ ,  $\mathbf{v}$

**Solution:** Let  $P = \langle 2, -5, 7 \rangle$ . Then

$$\text{Area}(\triangle Pvu) = \frac{1}{2} \|\vec{Pv} \times \vec{Pu}\|$$

with  $u = \langle 4, 1, 3 \rangle$ ,  $v = \langle -5, 1, 6 \rangle$  gives  $\vec{Pv} = \langle -7, 6, -1 \rangle$ ,  $\vec{Pu} = \langle 2, 6, -4 \rangle$  and

$$\begin{aligned}\vec{Pv} \times \vec{Pu} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & 6 & -1 \\ 2 & 6 & -4 \end{vmatrix} \\ &= -18\mathbf{i} - 30\mathbf{j} - 54\mathbf{k}\end{aligned}$$

The area of the triangle is thus

$$\frac{1}{2} \|\vec{Pv} \times \vec{Pu}\| = \frac{1}{2} \sqrt{18^2 + 30^2 + 54^2} = \boxed{\frac{1}{2} \sqrt{4140} \approx 32.1714}$$

- (d) Find the volume of the parallelopiped spanned by  $\mathbf{u}$ ,  $\mathbf{v}$  and  $-\mathbf{j}$ , where  $\mathbf{j} = \langle 0, 1, 0 \rangle$ .

**Solution:**

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times -\mathbf{j}) &= \begin{vmatrix} 4 & 1 & 3 \\ -5 & 1 & 6 \\ 0 & -1 & 0 \end{vmatrix} \\ &= 4 \cdot 6 - 1 \cdot 0 + 3(5) \quad (\text{expanding along the top row}) \\ &= 39\end{aligned}$$

- (e) Are  $\mathbf{u}$  and  $\langle -3, 4, -1 \rangle$  orthogonal? Why or why not?

**Solution:**

$$\begin{aligned}\mathbf{u} \cdot \langle -3, 4, -1 \rangle &= \langle 4, 1, 3 \rangle \cdot \langle -3, 4, -1 \rangle \\ &= -12 + 4 - 3 = -11 \neq 0\end{aligned}$$

so NO, they are not orthogonal.

5. (10 points) §2.5: Let  $P = (1, 1, -1)$ ,  $Q = (-7, -6, 4)$ , and  $R = (-3, 2, 8)$ .

(a) Describe the line  $\mathcal{L}$  through  $P$  and  $Q$  in vector, parametric, and symmetric form.

**Solution:**

**Vector form**

$$\begin{aligned}\mathcal{L} &= \{P + t \overrightarrow{PQ} \mid t \in \mathbb{R}\} \\ &= \langle 1, 1, -1 \rangle + t \langle -8, -7, 5 \rangle\end{aligned}$$

**Parametric form**

$$\begin{aligned}x &= 1 - 8t, \\ y &= 1 - 7t, \\ z &= -1 + 5t\end{aligned}$$

**Symmetric form**

$$\frac{x - 1}{-8} = \frac{y - 1}{-7} = \frac{z + 1}{5}.$$

(b) Find the distance from  $R$  to the line  $\mathcal{L}$  through  $P$  and  $Q$ .

**Solution:** The distance between  $R$  and  $\mathcal{L}$  is given by the length of the vector  $\overrightarrow{RP} - \text{proj}_{\overrightarrow{PQ}} \overrightarrow{RP}$ , which is the portion of  $\overrightarrow{RP}$  orthogonal to  $\overrightarrow{PQ}$  (draw a picture to see this). We thus compute:

$$\begin{aligned}\text{proj}_{\overrightarrow{PQ}} \overrightarrow{RP} &= \frac{\langle -4, 1, 9 \rangle \cdot \langle -8, -7, 5 \rangle}{64 + 49 + 25} \cdot \langle -8, -7, 5 \rangle \\ &= \frac{70}{138} \langle -8, -7, 5 \rangle \\ &= \frac{35}{69} \langle -8, -7, 5 \rangle \\ \overrightarrow{RP} - \text{proj}_{\overrightarrow{PQ}} \overrightarrow{RP} &= \langle -4, 1, 9 \rangle - \frac{35}{69} \langle -8, -7, 5 \rangle = \left\langle \frac{4}{69}, \frac{314}{69}, \frac{446}{69} \right\rangle\end{aligned}$$

which has length  $\frac{1}{69} \sqrt{4^2 + 314^2 + 446^2} = \frac{1}{69} \sqrt{297528} = \boxed{\frac{14}{69} \sqrt{1518}}$

- (c) Describe the plane  $\Pi$  through  $P, Q$  and  $R$  in vector, scalar, or general form (any form is acceptable).

**Solution:**

$$\overrightarrow{PQ} = \langle -8, -7, 5 \rangle, \overrightarrow{PR} = \langle -4, 1, 9 \rangle \text{ gives}$$

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -7 & 5 \\ -4 & 1 & 9 \end{vmatrix}$$

$$= \langle -68, 52, -36 \rangle$$

gives

$$\langle -68, 52, -36 \rangle \cdot \langle x - 1, y - 1, z + 1 \rangle = 0$$

so  $\Pi$  is given by

$$-68x + 52y - 36z - 20 = 0.$$

- (d) Classify the relationship between the line  $\mathcal{L}$  through  $P, Q$  and the line  $\mathcal{M}$  through  $R$  and  $(-10, -5, 2)$ . Are  $\mathcal{L}$  and  $\mathcal{M}$ : equal, parallel but not equal, intersect but not equal, or skew?

**Solution:** Let

$$\mathcal{L} = \mathbf{P} + t\mathbf{PQ} = \langle 1, 1, -1 \rangle + t\langle -8, -7, 5 \rangle$$

$$\mathcal{M} = \mathbf{R} + t\mathbf{RA} = \langle -3, 2, 8 \rangle + u\langle -7, -7, -6 \rangle$$

where  $A := (-10, -5, 2)$ . Since  $\langle -8, -7, 5 \rangle \neq k\langle -7, -7, -6 \rangle$  for any  $k \in \mathbb{R}$ , the two lines are not parallel.

Next we check whether the lines have a point of intersection. Setting the component functions of the two lines equal to each other, we get the system of equations

$$\begin{cases} 1 - 8t = -3 - 7u \\ 1 - 7t = 2 - 7u \\ -1 + 5t = 8 - 6u \end{cases}$$

Solving the first two equations yields the solution  $t = 5, u = \frac{36}{7}$ . Plugging this solution into the third equation yields 24 for the LHS and  $-\frac{160}{7}$  for the RHS, so the two lines do not intersect.

We have found that  $\mathcal{L}$  and  $\mathcal{M}$  are not parallel, and do not intersect. Therefore,  $\mathcal{L}$  and  $\mathcal{M}$  must be skew.



- (e) Find the distance from  $(2, -2, 2)$  to the plane  $\Pi$ .

**Solution:** Let  $B = (2, -2, 2)$ . We can take the line  $L'$  parallel to  $\overrightarrow{PQ} \times \overrightarrow{PR}$  through  $B$  until it hits  $\Pi$ : That is, we look for a simultaneous solution to

$$x = 2 - 68t, \quad y = -2 + 52t, \quad z = 2 - 36t$$

and

$$-68x + 52y - 36z - 20 = 0$$

Plugging in gives

$$\begin{aligned} -68(2 - 68t) + 52(-2 + 52t) - 36(2 - 36t) - 20 &= 0 \\ (68)^2t + (52)^2t + (36)^2t &= 20 + 136 + 104 + 72 \\ \implies t &= \frac{332}{68^2 + 52^2 + 36^2} = \frac{83}{2156} \end{aligned}$$

Thus the point on  $\Pi$  closest to  $B$  is given by

$$Q' := (2, -2, 2) + \frac{332}{68^2 + 52^2 + 36^2} \cdot \langle -68, 52, -36 \rangle$$

and we can compute the distance between  $\Pi$  and  $B$  by computing  $\|\overrightarrow{BQ'}\|$ :

$$\begin{aligned} \|\overrightarrow{BQ'}\| &= \frac{332}{68^2 + 52^2 + 36^2} \cdot \sqrt{68^2 + 52^2 + 36^2} \\ &= \frac{332}{\sqrt{68^2 + 52^2 + 36^2}} = \boxed{\frac{83}{7\sqrt{11}}} \end{aligned}$$

6. (10 points) §2.6:

(a) Classify the following quadric surface and identify the axis of the surface:

$$49x^2 - 392x + 16y^2 - 32y - 784z + 2336 = 0$$

**Solution:** Completing the square:

$$0 = 49x^2 - 392x - 16y^2 + 32y - 784z + 2336$$

$$0 = 49x^2 - 392x + 784 - 16y^2 + 32y - 16 - 784z + 1568$$

$$0 = 49x^2 - 8 \cdot 49x + 16 \cdot 49 - 16y^2 + 32y - 16 - 16 \cdot 49z + 32 \cdot 49$$

$$16 \cdot 49z - 32 \cdot 49 = 49(x^2 - 8x + 16) - 16(y^2 - 2y + 1)$$

$$16 \cdot 49(z - 2) = 49(x - 4)^2 - 16(y - 1)^2$$

$$(z - 2) = \frac{(x - 4)^2}{16} - \frac{(y - 1)^2}{49}$$

gives a hyperbolic paraboloid.

(b) Give the trace in the  $z = 5$  plane.

**Solution:** Setting  $z = 5$ , we have

$$\begin{aligned} 3 &= \frac{(x - 4)^2}{16} - \frac{(y - 1)^2}{49} \\ \implies 1 &= \frac{(x - 4)^2}{48} - \frac{(y - 1)^2}{147} \end{aligned}$$

(which is a hyperbola)

7. (10 points) §2.7:

(a) Identify the following surfaces in spherical coordinates, and sketch a graph of the surface:

i.  $\phi = \frac{\pi}{2}$ .

**Solution:** This is the  $xy$ -plane  $\{z = 0\}$ .

ii.  $\rho = \cos \theta \sin \phi$ . Hint: Multiply both sides by  $\rho$ .

**Solution:** Multiplying both sides by  $\rho$  gives the equation  $\rho^2 = \rho \cos \theta \sin \phi$  which when converted to Cartesian coordinates becomes

$$\begin{aligned}x^2 + y^2 + z^2 &= x \\x^2 - x + y^2 + z^2 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + y^2 + z^2 &= 0 \\ \left(x - \frac{1}{2}\right)^2 + y^2 + z^2 &= \frac{1}{4}\end{aligned}$$

gives a sphere centered at  $(\frac{1}{2}, 0, 0)$  with radius  $\frac{1}{2}$ .

(b) Convert the following points from rectangular to both cylindrical and spherical coordinates, respectively:

i.  $(3, 1, 5)$ .

**Solution:**

**Rectangular**  $(3, 1, 5)$

**Cylindrical**  $(r, \theta, z) = (\sqrt{10}, \arctan(\frac{1}{3}), 5)$

**Spherical**  $(\rho, \theta, \phi) = \left(\sqrt{35}, \arctan(\frac{1}{3}), \arccos\left(\frac{5}{\sqrt{35}}\right)\right)$

ii.  $(-2, 1, 7)$ .

**Solution:**

**Rectangular**  $(-2, 1, 7)$

**Cylindrical**  $(r, \theta, z) = \left(\sqrt{5}, \arccos\left(-\frac{2}{\sqrt{5}}\right), 7\right)$

**Spherical**  $(\rho, \theta, \phi) = \left(\sqrt{54}, \arctan\left(-\frac{2}{\sqrt{5}}\right), \arccos\left(\frac{7}{\sqrt{54}}\right)\right)$

8. (10 points) §3.1: Let  $\mathbf{r}(t) = \langle e^{-4t}, e^{\frac{1}{2t}}, \ln(2t) - 7 \rangle$ .

(a) Find  $\lim_{t \rightarrow 10} \mathbf{r}(t)$ .

**Solution:** All component functions are continuous at  $t = 10$ . Thus we have

$$\lim_{t \rightarrow 10} \mathbf{r}(t) = \langle e^{-40}, e^{1/20}, \ln(20) - 7 \rangle.$$

(b) Is  $\mathbf{r}(t)$  continuous at  $t = 10$ ?

**Solution:** Yes,  $\mathbf{r}(t)$  is continuous in each component on its domain, which includes  $t = 10$ .

(c) Are there any domain restrictions on  $\mathbf{r}(t)$ ?

**Solution:** The second component function requires  $2t \neq 0$  and the third component function requires that  $2t > 0$ . Hence  $\text{Dom}(\mathbf{r}(t)) = \{t > 0\}$ .

(d) Are there any values for  $t$  at which  $\mathbf{r}(t)$  is not continuous?

**Solution:** No,  $\mathbf{r}(t)$  is continuous on  $\text{Dom}(\mathbf{r}(t))$ .

(e) Let  $C$  be the space curve given by  $\mathbf{r}(t) = \langle t, t^3, 20 \rangle$ . Describe the curve: What shape is it?

**Solution:** Cubic curve in the plane  $\{z = 20\}$ .

(f)  $C$  is contained in a unique plane in  $\mathbb{R}^3$ . Which plane is  $C$  contained in?

**Solution:** The plane  $\{z = 20\}$ . Note that the third coordinate is constant.

9. (10 points) §3.2:

(a) Using the limit definition of the derivative, find  $\mathbf{r}'(t)$  for

$$\mathbf{r}(t) = \langle t^2, -4t \rangle.$$

**Solution:**

$$\begin{aligned}\mathbf{r}'(t) &:= \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\langle (t+h)^2, -4(t+h) \rangle - \langle t^2, -4t \rangle}{h} \\ &= \lim_{h \rightarrow 0} \frac{\langle 2th + h^2, -4h \rangle}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \langle 2t + h, -4 \rangle = \boxed{\langle 2t, -4 \rangle}\end{aligned}$$

(b) Using your formula from part (a), find  $\mathbf{r}'(5)$ .

**Solution:**  $\mathbf{r}'(5) = \langle 10, -4 \rangle.$

(c) Let  $\mathbf{u}(t) = \langle t, t^2, t^3 \rangle$  and  $\mathbf{w}(t) = \langle 2t + 3, \ln(t), 12 \rangle$ . Find

$$\int \mathbf{u}(t) \times \mathbf{w}(t) \, dt$$

**Solution:**

$$\begin{aligned} \mathbf{u} \times \mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 2t + 3 & \ln(t) & 12 \end{vmatrix} \\ &= \langle 12t^2 - \ln(t) \cdot t^3, -[12t - 2t^4 - 3t^3], t \ln(t) - (2t^3 + 3t^2) \rangle \\ &= \langle 12t^2 - \ln(t) \cdot t^3, -12t + 2t^4 + 3t^3, t \ln(t) - 2t^3 - 3t^2 \rangle \end{aligned}$$

so

$$\begin{aligned} \int \mathbf{u} \times \mathbf{w} \, dt &= \int \langle 12t^2 - t^3 \ln(t), -12t + 2t^4 + 3t^3, t \ln(t) - 2t^3 - 3t^2 \rangle \, dt \\ &= \left\langle \frac{12t^3}{3} + C_1 - \int t^3 \ln(t) \, dt, -6t^2 + \frac{2}{5}t^5 + \frac{3}{4}t^4 + C_2, \right. \\ &\quad \left. \int t \ln(t) \, dt - \frac{1}{2}t^4 - t^3 + C_3 \right\rangle \end{aligned}$$

The remaining integrals are evaluated using integration-by-parts:

$$\begin{aligned} \int t^3 \ln(t) \, dt &= \ln(t) \cdot \frac{t^4}{4} - \frac{1}{4} \int t^3 \, dt \\ &= \frac{\ln(t)t^4}{4} - \frac{1}{16}t^4 + C \end{aligned}$$

using  $u = \ln(t)$ ,  $dv = t^3 \, dt$ , and

$$\begin{aligned} \int t \ln(t) \, dt &= \ln(t) \cdot \frac{t^2}{2} - \frac{1}{2} \int t \, dt \\ &= \frac{\ln(t)t^2}{2} - \frac{1}{4}t^2 + C' \end{aligned}$$

using  $u = \ln(t)$ ,  $dv = t \, dt$ . Thus we have

$$\left\langle 4t^3 - \frac{\ln(t)t^4}{4} + \frac{1}{16}t^4, -6t^2 + \frac{2}{5}t^5 + \frac{3}{4}t^4, \right. \\ \left. \frac{\ln(t)t^2}{2} - \frac{1}{4}t^2 - \frac{1}{2}t^4 - t^3 \right\rangle + \langle C_1, C_2, C_3 \rangle$$

(d) Find  $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{w}(t)]$

**Solution:** In the previous part, we computed

$$\mathbf{u} \times \mathbf{w} = \langle 12t^2 - t^3 \ln(t), -12t + 2t^4 + 3t^3, t \ln(t) - 2t^3 - 3t^2 \rangle$$

Its derivative is thus

$$= \langle 24t - t^2 - 3t^2 \ln(t), -12 + 8t^3 + 6t, 1 + \ln(t) - 6t^2 - 6t \rangle$$

10. (10 points) §3.3: Let  $\mathbf{r}(t) = \langle 3 \cos(2t), 3 \sin(2t), 12t \rangle$ .

(a) Find the arc length of  $\mathbf{r}(t)$  over  $t \in [0, 4\pi]$ .

**Solution:** We evaluate  $\int_0^{4\pi} \|\mathbf{r}'(t)\| dt$ . First we compute

$$\begin{aligned}\mathbf{r}'(t) &= \langle -6 \sin(2t), 6 \cos(2t), 12 \rangle, \\ \|\mathbf{r}'(t)\| &= \sqrt{36(\sin^2(2t) + \cos^2(2t)) + 144} \\ &= \sqrt{180} = 6\sqrt{5}\end{aligned}$$

Thus

$$\int_0^{4\pi} \|\mathbf{r}'(t)\| dt = \int_0^{4\pi} 6\sqrt{5} dt = \boxed{24\pi\sqrt{5}}$$

(b) Give an arc length parametrization of  $\mathbf{r}(t)$  for  $t > 0$ .

**Solution:** The arc length function is

$$s(t) = \int_0^t \|\mathbf{r}'(u)\| du = 6\sqrt{5}t$$

by the computation in the previous part. The inverse of the arc length function is thus

$$t(s) = \frac{s}{6\sqrt{5}}.$$

Hence

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(t(s)) \\ &= \left\langle 3 \cos\left(2 \frac{s}{6\sqrt{5}}\right), 3 \sin\left(2 \frac{s}{6\sqrt{5}}\right), 12 \frac{s}{6\sqrt{5}} \right\rangle \\ &= \boxed{\left\langle 3 \cos\left(\frac{1}{3\sqrt{5}}\right), 3 \sin\left(\frac{s}{3\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle}\end{aligned}$$



- (c) Find the principal unit normal vector  $\mathbf{N}(t)$  at  $t = 4\pi$ .

**Solution:** A series of calculations are required. We use the definitions that

$$\mathbf{T}(t) := \frac{\mathbf{r}'}{\|\mathbf{r}'\|} \quad \text{and} \quad \mathbf{N}(t) := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

Then

$$\begin{aligned} \mathbf{r}(t) &= \langle 3 \cos(2t), 3 \sin(2t), 12t \rangle \\ \mathbf{T}(t) &:= \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{6\sqrt{5}} \langle -6 \sin(2t), 6 \cos(2t), 12 \rangle \\ \mathbf{T}'(t) &= \frac{1}{6\sqrt{5}} \langle -12 \cos(2t), -12 \sin(2t), 0 \rangle \\ \|\mathbf{T}'\| &= \frac{1}{6\sqrt{5}} \cdot 12 = \frac{2}{\sqrt{5}} \\ \mathbf{N}(t) &:= \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{1}{12} \langle -12 \cos(2t), -12 \sin(2t), 0 \rangle \\ &= \langle -\cos(2t), -\sin(2t), 0 \rangle \end{aligned}$$

$$\text{Thus } \mathbf{N}(4\pi) = \boxed{\langle -1, 0, 0 \rangle}$$