Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. Please explain your responses in full detail. You will have 1 hour to complete this exam.

Question	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Consider the function

$$f(x, y) = xy$$

on the domain $\mathcal{D}$ which is a triangle with vertices $(-1, -1), (2, -1)$ and $(-1, 2)$.

(a) (5 points) Find the critical points of f in the interior of $\mathcal{D}$.

Solution: The gradient of f is $\nabla f = \langle y, x \rangle$ which is zero if and only if $x = 0$ and $y = 0$. Thus $(0, 0)$ is the only critical point of f. It also lies in the interior of $\mathcal{D}$.

(b) (5 points) Describe the local behavior of f near the critical points.

Solution: The second partials of f are given by

$$\begin{aligned} f_{xx} &= 0 \\ f_{xy} &= 1 \\ f_{yy} &= 0 \end{aligned}$$

So $D < 0$ implying f has a saddle at $(0, 0)$.

(c) (10 points) Find the global maximum value and the global minimum value for f on $\mathcal{D}$ if they exist. Explain your response.

Solution: Since $(0,0)$ is a saddle, it is neither a global min or max. The boundary of $\mathcal{D}$ consists of lines $x = -1$ with $-1 \le y \le 2$, $y = -1$ with $-1 \le x \le 2$ and $y = 1 - x$ with $-1 \le x \le 2$. For the first line, we have

$$f(-1, y) = -y.$$

which has minimum -2 at $y = 2$ and max of 1 at $y = -1$. Using symmetry, there is a minimum of -2 on the second line segment at $x = 2$ and a max of 1 at $x = -1$. For the last line segment we have

$$f(x, 1 - x) = x - x^2$$

and taking derivatives gives $f'(x, 1 - x) = 1 - 2x$ which has a zero at $x = 1/2$. At this value, f is $1/4$ which is between the two extrema already found (note that we have already tested the endpoints of this last line segment). Thus the global min is -2 and the global max is 1.

2. (15 points) Use Lagrange multipliers to find the critical points of the function

$$f(x, y, z) = y^2 - z + x$$

on the unit sphere

$$x^2 + y^2 + z^2 = 1.$$

Identify the global maximum and minimum values of f on the sphere.

Solution: The gradient of f is $\nabla f = \langle 1, 2y, -1\rangle$ and, letting $g(x, y, z) = 1/2(x^2 + y^2 + z^2 - 1)$, the gradient of the constraint is $\nabla g = \langle x, y, z\rangle$. Thus $\nabla f = \lambda \nabla g$ if and only if $1 = \lambda x$, $2y = \lambda y$ and $1 = -\lambda z$. If $y \neq 0$ then $\lambda = 2$, $x = 1/2$ and $z = -1/2$. Putting this into the constraint implies that $y = \pm\sqrt{3}/2$ giving the two points

$$(x, y, z) = \left(1/2, \pm\sqrt{3}/2, -1/2\right)$$

On the other hand, if $y = 0$, then (since $\lambda \neq 0$), $x = \lambda^{-1}$ and $z = -\lambda^{-1}$. Putting this into the constraint gives $\lambda^2 = 2$ or $\lambda = \pm\sqrt{2}$ and we obtain the two points

$$(x, y, z) = \left(\pm\sqrt{2}/2, 0, \mp\sqrt{2}/2\right)$$

The value obtained by f on the first pair of points is $7/4$ while the values obtained by the second pair of points is $\pm\sqrt{2}$. Thus the global min is clearly $-\sqrt{2}$ while the max can be seen to be $7/4$.

3. (15 points) Calculate the integral

$$\iiint_{\mathcal{B}} y^2 z e^{xyz} + x\cos(z)\ \mathrm{d}V$$

where $\mathcal{B} = [-1,1] \times [0,1] \times [-1,0]$.

Solution: Note that the second summand is odd with respect to x and thus evaluates to zero. Thus

$$\begin{aligned}
\iiint_{\mathcal{B}} y^2 z e^{xyz} + x\cos(z)\ \mathrm{d}V &= \iiint_{\mathcal{B}} y^2 z e^{xyz}\ \mathrm{d}V \\
, &= \int_{-1}^{1}\int_{-1}^{0}\int_{0}^{1} y^2 z e^{xyz}\ \mathrm{d}x\ \mathrm{d}z\ \mathrm{d}y \\
&= \int_{-1}^{0}\int_{0}^{1} y e^{xyz}|_{-1}^{1}\ \mathrm{d}z\ \mathrm{d}y \\
&= \int_{-1}^{0}\int_{0}^{1} y e^{yz} - y e^{-yz}\ \mathrm{d}z\ \mathrm{d}y \\
&= \int_{0}^{1} e^{yz} + e^{-yz}|_{-1}^{0}\ \mathrm{d}y \\
&= \int_{0}^{1} 2 - e^{-y} - e^{y}\ \mathrm{d}y \\
&= 2y + e^{-y} - e^{y}|_{0}^{1} \\
&= 2 + 1/e - e.
\end{aligned}$$

4. (15 points) Let $\mathcal{D}$ be the region in the first quadrant bounded by the coordinate axes and $y = 1 - x^2$. Evaluate

$$\iint_{\mathcal{D}} 2\pi^2 x \cos(\pi y)\ \mathrm{d}A$$

Solution: We have that $\mathcal{D}$ is both a Type I and a Type II domain with bounds (for the Type I case) of $0 \le x \le 1$ and $0 \le y \le 1 - x^2$. Using Fubini, we have

$$\begin{aligned}
\iint_{\mathcal{D}} 2\pi^2 x \cos(\pi y)\ \mathrm{d}A &= \int_0^1 \int_0^{1-x^2} 2\pi^2 x \cos(\pi y)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 2\pi x \sin(\pi y)\big|_{y=0}^{1-x^2}\ \mathrm{d}x \\
&= \int_0^1 2\pi x \sin(\pi(1-x^2))\ \mathrm{d}x \\
&= \cos(\pi(1-x^2))\big|_0^1 \\
&= \cos(0) - \cos(\pi) = 2.
\end{aligned}$$

5. Consider the region $\mathcal{E}$ of points (x, y, z) satisfying

$$x^2 + y^2 \leq z^2, \ x^2 + y^2 + z^2 \leq 4, \ z \geq 0$$

(a) (10 points) Express the triple integral,

$$\iiint_{\mathcal{E}} z \ \mathrm{d}V$$

as an iterated integral using spherical coordinates.

Solution: The second inequality gives $0 \leq \rho \leq 2$ while the first gives $\rho^2 \sin^2 \phi \leq \rho^2 \cos^2 \phi$ implying $\tan^2 \phi \leq 1$. Now since ϕ is between 0 and $\pi/2$ (as $z \geq 0$), $\tan \phi$ is positive so this means $\tan \phi \leq 1$ which occurs when $0 \leq \phi \leq \pi/4$. Finally, there are no constraints on θ so that $0 \leq \theta \leq 2\pi$. Together, this gives

$$\iiint_{\mathcal{E}} z \ \mathrm{d}V = \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho \cos(\phi) \rho^2 \sin(\phi) \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi.$$

(b) (5 points) Evaluate the integral.

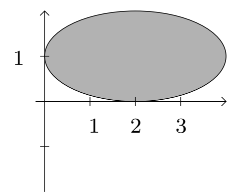
Solution: Evaluating gives

$$\begin{aligned}
\int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho \cos(\phi) \rho^2 \sin(\phi) \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi &= \int_0^{\pi/4} \int_0^{2\pi} \int_0^2 \rho^3 \cos(\phi) \sin(\phi) \ \mathrm{d}\rho \ \mathrm{d}\theta \ \mathrm{d}\phi, \\
&= 2\pi \int_0^{\pi/4} \rho^4/4|_0^2 \cos(\phi) \sin(\phi) \ \mathrm{d}z, \\
&= 4\pi \int_0^{\pi/4} \sin(2\phi) \ \mathrm{d}z, \\
&= -2\pi \cos(2\phi)|_0^{\pi/4} = 2\pi
\end{aligned}$$

6. Let $\mathcal{R}$ be the region inside the ellipse

$$\frac{(x-2)^2}{4} + (y-1)^2 \leq 1$$

which is illustrated below:



(a) (5 points) Assuming $r \geq 0$ and $0 \leq \theta \leq 2\pi$, describe the domain $\mathcal{S}$ in (r, θ)-coordinates which maps onto $\mathcal{R}$ by the transformation

$$T(r, \theta) = (2r\cos(\theta) + 2, r\sin(\theta) + 1) .$$

Solution: Putting this into the inequality we obtain $r^2 \cos^2\theta + r^2 \sin^2\theta = r^2 \leq 1$. So there is no constraint on θ and $0 \leq r \leq 1$. This is a box

$$\mathcal{S} = [0, 1] \times [0, 2\pi]$$

(b) (5 points) Compute the Jacobian of T.

Solution: We have

$$\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = 2r\cos^2\theta - (-2r\sin\theta)\sin\theta = 2r$$

(c) (10 points) Use the change of variables formula to compute the double integral

$$\iint_{\mathcal{R}} x^2 + 4y^2 \, \mathrm{d}A$$

Solution: Taking

$$I = \iint_{\mathcal{R}} x^2 + 4y^2 \, \mathrm{d}A$$

we have

$$\begin{aligned} I &= \iint_{\mathcal{S}} (4r^2 \cos(\theta) + 8r\cos(\theta) + 4 + 4r^2 \sin^2(\theta) + 8r\sin(\theta) + 4)|Jac(T)| \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^1 (4r^2 + 8r(\cos(\theta) + \sin(\theta)) + 8)(2r) \, \mathrm{d}r \, \mathrm{d}\theta \end{aligned}$$

Integrating the two middle terms with respect to θ first clearly gives zero for them so that

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 (8r^3 + 16r) \, \mathrm{d}r \, \mathrm{d}\theta \\ &= 2\pi(2r^4 + 8r^2)|_0^1 \\ &= 20\pi. \end{aligned}$$

Derivative formulas

$$\begin{aligned}\text{Directional derivative}: &\ D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u},\\ \text{Discriminant}: &\ D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2\end{aligned}$$

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\phi)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\phi)$
	$z = z$	$z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$	$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{x^2+y^2+z^2}$
$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$
	$z = z$	$\cot(\phi) = \dfrac{z}{\sqrt{x^2+y^2}}$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Change of variables

$$T : \mathcal{S} \to \mathcal{R}$$

$$T(u,v) = (x(u,v), y(u,v))$$

$$\mathrm{Jac}(T) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathcal{S}} f(x(u,v),y(u,v))\ |\mathrm{Jac}(T)|\ \mathrm{d}u\,\mathrm{d}v$$