Final Exam Make Up
Math 222 Fall 2018
December 12, 2018
Name: Solutions

Recitation Time and Instructor's Initials: ______________

You may not use any type of calculator whatsoever. (Cell phones off and away!) You are not allowed to have any other notes, and the test is closed book. Use the backs of pages for scrapwork, and if you write anything on the back of a page which you want to be graded, then you should indicate that fact (on the front). Except for the last page which is the **cheat sheet** (and which you should not hand in) do not unstaple or remove pages from the exam.

By taking this exam you are agreeing to abide by KSU's Academic Integrity Policy.

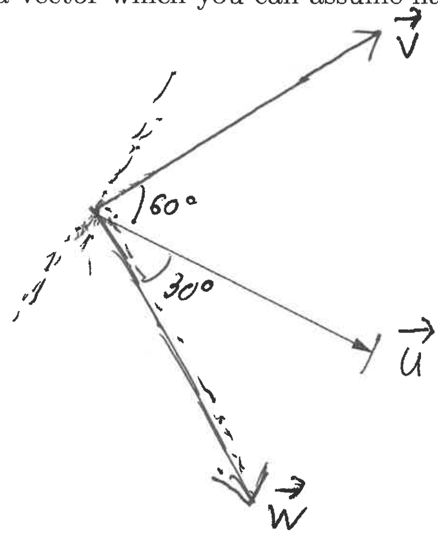
**Simple or standard simplifications should be made.** Box your final answers when it is reasonable. You must **show your work** for every problem, and in order to get credit or partial credit, your work must make sense!

**MAY THE FORCE BE WITH YOU, YOUNG JEDIS!!!**

| Problem | Possible | Score | Problem | Possible | Score |
|---|---|---|---|---|---|
| 1 | 14 | | 6 | 14 | |
| 2 | 18 | | 7 | 14 | |
| 3 | 16 | | 8 | 14 | |
| 4 | 16 | | 9 | 14 | |
| 5 | 16 | | 10 | 14 | |
| Total | 80 | | | 70 | |

**Rewrite and sign the following using a pen:** I give my word that I will do my best to prevent any of the information on this test leaking to anyone in this course who has yet to take it.

1. Here is a vector which you can assume has unit length:




Call this vector $\vec{u}$. Now using the same base point draw a vector $\vec{w}$ (and label it) so that the following are all satisfied:

(a) $|\vec{w}| = 1$.

(b) $\vec{u} \times \vec{w}$ points away from you.

(c) $|\vec{u} \times \vec{w}| \approx 1/2$. (Try to make it as close as you can.)

$\sin \theta = \frac{1}{2} \Rightarrow 30°$

Next using the same base point again draw a vector $\vec{v}$ (and label it) so that the following are all satisfied:

(a) $|\vec{v}| = 1$.

(b) $\vec{u} \bullet \vec{v} \approx 1/2$. (Again, do your best to get equality.) $\cos \theta = \frac{1}{2} \Rightarrow 60°$

(c) $\vec{u} \times \vec{v}$ points toward you.

2. Short answers ... Intuition and Understanding

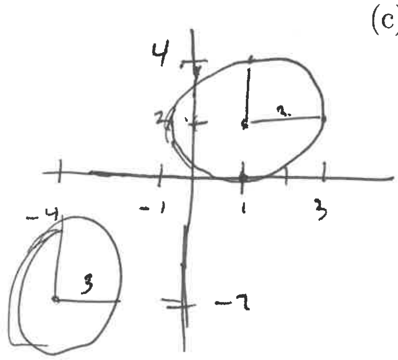
(a) If you are driving, then what device (or devices) in your car will be the best way to change your normal acceleration?

Steering wheel

(b) What is the curvature of a circle with radius 25?

$\kappa = \frac{1}{25}$

(c) $f(x, y)$ is defined to equal 3 for all points on the disk $(x-1)^2 + (y-2)^2 \le 4$, to equal $-2$ for all points on the disk $(x+4)^2 + (y+7)^2 \le 9$, and to equal 0 everywhere else. Compute:




$$\int_{x=-100}^{100} \int_{y=-90}^{90} f(x, y)\, dy\, dx\ .$$

$= 3 \cdot \pi\, 2^2 + (-2)\, \pi \cdot 3^2$

$= 12\pi - 18\pi = \boxed{-6\pi}$

(d) Will the surface integral

$$\iint_S f(x, y, z)\, dS$$

typically give you the surface area of $S$? Explain your answer in one sentence or less.

No. Only $f(x,y,z) = 1$ will compute surface area of $S$.

$\iint 1\, dA$

$=$ Area $= 4 \cdot 3$

(e) What is the average value of the function $f(x, y) = 1 + 2x$ on the rectangle $1 \le x \le 5,\ 3 \le y \le 6$?

$\int_3^6 \int_1^5 (1+2x)\, dx\, dy = \int_3^6 1\, dy \cdot \int_1^5 (1+2x)\, dx = 3 \cdot [x + x^2]_1^5$

$= 3(5 + 25 - (1+1))$

$= 3(30 - 2)$

$= 3(28)$

~~28~~

avg val $= \frac{28 \cdot \cancel{3}}{4 \cdot \cancel{3}} = \boxed{7}$

3. Short answers ... Definitions and Theorems

(a) Suppose that $\nabla f(0,0) = \langle 0,0 \rangle$, and $f_{xx}(0,0)$ and $f_{yy}(0,0)$ are both positive. Do you need anything else to conclude that $(0,0)$ is a local minimum? (If yes, then what? If no, then why not?)

Need $f_{xx} > 0$

(b) What does it mean (definition!) for a vector field $\vec{F}(x,y,z)$ to be incompressible?

$\text{div}\, \vec{F} = 0$

(c) According to the theorem that we learned, if $f$ is a continuous function on a set, $\Omega$, then what condition or conditions on $\Omega$ will guarantee that $f$ attains an absolute maximum and absolute minimum?

$\Omega$ closed and bounded

(d) Assume that you have been given a differentiable vector field defined on the first octant. How can you quickly tell if it is conservative?

$\text{curl}\, \vec{F} = 0$ on a simply connected domain

(Hence has a potential function)

4. A certain differentiable function satisfies:

(a) $f(9,7) = 1$, and $f(2,-4) = 6$.

(b) $\nabla f(9,7) = (5,3)$, and $\nabla f(2,-4) = (-\pi, 8)$.

At each of the two points in question (i.e. at $(9,7)$ and at $(2,-4)$) answer the following questions:

(a) In what direction is the function increasing the fastest?

At $(9,7)$: $\langle 5,3\rangle$

At $(2,-4)$ : $\langle -\pi, 8\rangle$

(b) What is the rate of change in that direction?

For $(9,7)$: $\sqrt{25+9} = \sqrt{34}$

For $(2,-4)$: $\sqrt{\pi^2+64}$

(c) What is the directional derivative in the direction of $\langle 3,-4\rangle$? $= \vec{v}$ (Note: just to be completely clear about semantics here, you are supposed to give the same directional derivative at each point. I did not ask for the directional derivative in the direction of the point $(3,-4)$.)

$\vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

$D_{\vec{u}} f(9,7) = \nabla f(9,7) \cdot \vec{u} = \langle 5,3\rangle \cdot \langle \frac{3}{5}, -\frac{4}{5}\rangle = 3 + \frac{-12}{5} = \boxed{\frac{3}{5}}$

$D_{\vec{u}} f(2,-4) = \nabla f(2,-4) \cdot \vec{u} = \langle -\pi, 8\rangle \cdot \langle \frac{3}{5}, \frac{-4}{5}\rangle = \boxed{-\frac{3\pi}{5} - \frac{32}{5}}$

(d) What is the tangent plane and/or the linear approximation at each of the two points?

$L_{(9,7)}(x,y) = f(9,7) + f_x(9,7)(x-9) + f_y(9,7)(y-7)$

$\boxed{L_{(9,7)}(x,y) = 1 + 5(x-9) + 3(y-7)}$

$\boxed{L_{(-2,4)}(x,y) = 6 + -\pi(x+2) + 8(y-4)}$

5. Find the maximum and minimum of the function

$$f(x, y) = 8x^2 - 4x + \frac{y^2}{3}$$

on the set

$$g(x, y) = 4x^2 + \frac{y^2}{9} \leq 4.$$

Show your work, and explain what you are doing. (No essays, please. Just a few short words in the right places will suffice.)

First $g < 4$: Solve $\nabla f = 0$.

$\nabla f = \langle 16x - 4, \frac{2}{3}y \rangle = 0$

$\Rightarrow \begin{cases} x = \frac{1}{4} \\ y = 0 \end{cases} \Rightarrow$ crit point $(\frac{1}{4}, 0)$

Now consider $g = 4$: Solve $\nabla f = \lambda \nabla g$:

$\begin{cases} 16x - 4 = \lambda\, 8x \\ \frac{2}{3}y = \lambda \frac{2}{9} y \\ 4x^2 + \frac{y^2}{9} = 4 \end{cases}$

$16x - 8x\lambda = 4$

$x(16 - 8\lambda) = 4 \quad \Rightarrow x = \frac{1}{4 - 2\lambda}$

$3y = \lambda y \Rightarrow \lambda = 3$ or $y = 0$

If $y = 0$, $\quad 4x^2 = 4 \Rightarrow x = \pm 1 \Rightarrow (1, 0), (-1, 0)$

If $\lambda = 3$, $\quad x = \frac{1}{4-6} = -\frac{1}{2}$

so $\quad 4 \cdot \frac{1}{4} + \frac{y^2}{9} = 4$

$\Rightarrow \frac{y^2}{9} = 3 \Rightarrow y^2 = 27 \Rightarrow y = \pm 3\sqrt{3}$

$(-\frac{1}{2}, 3\sqrt{3}), (-\frac{1}{2}, -3\sqrt{3})$

$f(\frac{1}{4}, 0) = \frac{8}{16} - 1 = -\frac{1}{2}$

$f(1, 0) = 8 - 4 = 4$

$f(-1, 0) = 8 + 4 = 12$

$f(-\frac{1}{2}, 3\sqrt{3}) = 2 + 2 + 9 = 13$

$f(-\frac{1}{2}, -3\sqrt{3}) = 2 + 2 + 9 = 13$

Thus $\boxed{\text{max} = 13, \quad \text{min} = -\frac{1}{2}}$

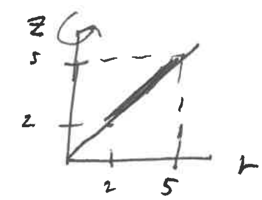
6. Let $S$ be the part of the set

$$z = \sqrt{x^2 + y^2}$$

$z = r$

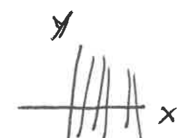
which is between the planes $z = 2$ and $z = 5$ and which has $x \geq 0$.

Express the surface area for $S$ as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ which has **constant** bounds of integration. (i.e. it should be over a rectangular solid or a rectangle in the domain in which you are finally integrating.) You do **NOT** need to find this integral.




Parametrization for S:

$g(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$ $\quad r \in [2, 5]$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$




$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$g_r = \langle \cos\theta, \sin\theta, 1 \rangle$

$g_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$g_r \times g_\theta = \langle -r\cos\theta, -r\sin\theta, r \rangle$

$\|\vec{N}\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = r\sqrt{2}$

$$\iint_S 1\, dS = \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_2^5 r\sqrt{2}\, dr\, d\theta}$$

7. Let $C$ be the curve given by

$$\vec{r}(t) = \left(1 + \sin^2(2t),\ \sin(2t)\right) ,$$

with $0 \leq t \leq \pi/4$. Compute the following integral:

$$\int_C \left(2x + \pi\cos(\pi x)e^{2y},\ 3y^2 + 2\sin(\pi x)e^{2y}\right) \cdot d\vec{r} .$$

$f = x^2 + \sin(\pi x)e^{2y} + y^3$ is the potential function.

So this is a line integral over a conservative v.f.

$\vec{r}(0) = (1, 0)$

$\vec{r}(\frac{\pi}{4}) = (1 + \sin^2(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$

$= (2, 1)$

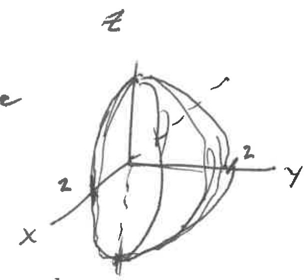
By FTC for conservative vector fields,

Hence $\int_C \vec{F} \cdot d\vec{r} = f(2,1) - f(1,0)$

$= 4 + 0 + 1 - (1 + 0 + 0)$

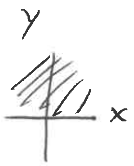
$= 5 - 1 = \boxed{4}$

ball
~~sphere~~ of radius 2,
right hemisphere




8. Let $Q$ be the set of points within the set:

$$Q = \{ (x, y, z) \ : \ x^2 + y^2 + z^2 \leq 4, \ \text{and } 0 \leq y \}$$




and let $\partial Q$ be the boundary of this set. If $\vec{n}$ is the outward unit normal to this region, then compute:

$$\iint_{\partial Q} (ze^{3y}, y^2, y\sin(2y)) \cdot \vec{n}\, dS \, . \quad = \iiint_Q \operatorname{div} F \, dV$$

by divergence thm.

$$\operatorname{div} F = 0 + 2y + 0 = 2y$$

spherical

$\rho \in [0,2]$

$\theta \in [0, \pi]$

$\varphi \in [0, \pi]$

$$\iiint_Q 2y \, dV = \int_0^{\pi}\int_0^{\pi}\int_0^{2} 2\rho \sin\varphi \sin\theta \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^2 2\rho^3 \, d\rho \cdot \int_0^{\pi} \sin\theta \, d\theta \cdot \int_0^{\pi} \sin^2\varphi \, d\varphi$$

$\sin^2\varphi = \frac{1}{2} - \frac{1}{2}\cos 2\varphi$

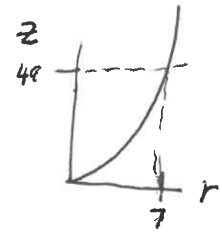
$$= \left[\frac{\rho^4}{2}\right]_0^2 \cdot \left[-\cos\theta\right]_0^{\pi} \cdot \left[\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right]_0^{\pi}$$

$$= 8 \cdot (+1+1) \cdot \left(\frac{\pi}{2} - 0 - 0\right)$$

$$= +16 \cdot \frac{\pi}{2} = \boxed{+8\pi}$$

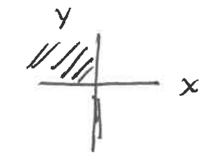
9. Let $E$ be the subset of

$$\overset{r^2}{z = x^2 + y^2}$$




which also satisfies

$$z \le 49, \qquad x \le 0, \qquad \text{and} \qquad y \ge 0.$$




$\theta \in [\frac{\pi}{2}, \pi]$

Express

$$\iint_E (x^2 + y^3)\, dS$$

as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. You do **NOT** need to find this integral.

Cylindrical, scalar surface integral

E parametrized by: $g(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\rangle$ $\qquad r \in [0,7]$, $\theta \in [\frac{\pi}{2}, \pi]$

$$g_r = \langle \cos\theta, \sin\theta, 2r\rangle$$

$$g_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$g_r \times g_\theta = \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle$$

$$\|\vec{n}\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2}$$

So $$\iint_E (x^2+y^3)\, dS = \boxed{\int_{\frac{\pi}{2}}^{\pi} \int_0^7 (r^2\cos^2\theta + r^3\sin^3\theta)\sqrt{4r^4+r^2}\; dr\, d\theta}$$

10. Let $E$ be the part of the set

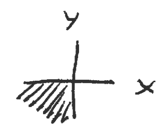
$0 \leq r^2 \leq z \leq 4$

$$0 \leq x^2 + y^2 \leq z \leq 4$$

that also satisfies

$$x \leq 0 \;\text{ and }\; y \leq 0.$$

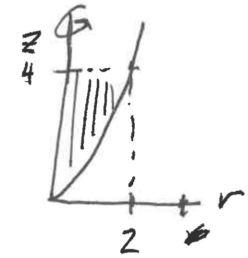
$\theta \in [\pi, \frac{3\pi}{2}]$




Express

$$\iiint_E (2x + 3y)\, dV$$

as an iterated integral (i.e. a double or triple integral) over a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. You do **NOT** need to find this integral.




E : volume of revolution. Solid paraboloid

Volume integral. Use cylindrical

$r \in [0, 2]$

$z \in [r^2, 4]$ ← z depends on r, z-integral inside r-integral

$\theta \in [\pi, \frac{3\pi}{2}]$

$$\iiint_E (2x+3y)\, dV = \boxed{\int_{\pi}^{\frac{3\pi}{2}} \int_0^2 \int_{r^2}^4 (2r\cos\theta + 3r\sin\theta)\, r \cdot dz\, dr\, d\theta}$$

↑ jacobian

**Integral Definitions and Basic Formulas:**

Line Integrals:

$$\int_C \vec{F}(\vec{r}) \bullet d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}'(t)\, dt\,, \qquad \text{Orientation Matters!}$$

$$\int_C f(\vec{r})\, ds = \int_a^b f(\vec{r}(t))\, ||\vec{r}'(t)||\, dt\,, \qquad \text{Orientation Doesn't Matter!}$$

$$\int_C f(\vec{r})\, dx = \int_a^b f(x(t), y(t), z(t))\, x'(t)\, dt\,, \qquad \text{Orientation Matters!}$$

Surface Integrals: With $\vec{N} := \vec{r}_u \times \vec{r}_v \neq 0$, and with $\vec{r}(R) = S$ we have

$$\int\int_S \vec{F} \bullet \vec{n}\, dS = \int\int_R \vec{F}(\vec{r}(u,v)) \bullet \vec{N}(u,v)\, du\, dv \quad \text{Orientation Matters!}$$

$$\int\int_S f(\vec{r})\, dS = \int\int_R f(\vec{r}(u,v))\, ||\vec{N}(u,v)||\, du\, dv \quad \text{Orientation Doesn't Matter!}$$

Green's Theorem: If $D$ is a region in the plane, and $\partial D$ has positive orientation (i.e. has counter-clockwise orientation), then

$$\int_{\partial D} P(x,y)\, dx + Q(x,y)\, dy = \int\int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,.$$

Stokes's Theorem:

$$\int\int_S (\nabla \times \vec{F}) \bullet \vec{n}\, dA = \int_{\partial S} \vec{F} \bullet d\vec{r}\,.$$

(If $\vec{n}$ is pointing right at you then orient $\partial S$ in a positive fashion (i.e. counter-clockwise fashion, typically) to make the identity hold.)

**Spherical Coordinates:**

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \,.$$

**Second Derivative Test:** Suppose the second partial derivatives of $f$ are continuous on a disk with center $(a, b)$ and suppose $\nabla f(a, b) = (0, 0)$. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2 \,.$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $(a, b)$ is a local maximum.

(c) If $D < 0$, then $(a, b)$ is a saddle point.

**Curves:**

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$\text{Unit Tangent Vector} \quad \vec{T}(t) := \frac{\vec{v}(t)}{||\vec{v}(t)||} = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$$

$$\text{Principle Unit Normal} \quad \vec{N}(t) := \frac{\vec{T}\,'(t)}{\left|\left|\vec{T}\,'(t)\right|\right|}$$

$$\text{Binormal} \quad \vec{B}(t) := \vec{T}(t) \times \vec{N}(t)$$

Cheat Sheet Bonus:

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}, \quad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}, \quad \sin(2\theta) = 2\sin\theta\cos\theta \,.$$