NAME Solutions   Rec. Instructor: ____________

Signature ____________________   Rec. Time ____________

CALCULUS II - EXAM 2
October 20, 2015

Show all work for full credit. No books, notes or calculators are permitted. The point value of each problem is given in the left-hand margin. You have 65 minutes.

Problem	Points	Points Possible	Problem	Points	Points Possible
1		10	7a		6
2		10	7b		6
3		12	8		4
4		12	9a		5
5		12	9b		5
6		12	9c		6
			Total Score		100

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right)+C$$

$$\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2}+a^2\sin^{-1}\frac{u}{a}\right)+C,$$

$$\int \sqrt{u^2\pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2\pm a^2}\pm a^2\ln|u+\sqrt{u^2\pm a^2}|\right)+C$$

Units of force: pounds, newtons; Gravitational acceleration: $g = 9.8m/sec^2$
Work = Force $\times$ Distance; Units of work: ft-lbs, newton-meters = joules;
Hooke's Law for springs: $F = kx$, where x is the distance stretched from rest position.

Moments: For the region between $y = f(x)$ and $y = g(x)$, with $a \le x \le b$,
$M_x = \frac{1}{2}\int_a^b f(x)^2 - g(x)^2\,dx$, $M_y = \int_a^b x(f(x)-g(x))\,dx$.

Taylor Remainder: $|R_n(x)| \le \frac{K}{(n+1)!}|x-a|^{n+1}$, with $K = \max_{a\le c\le x}|f^{(n+1)}(c)|$.

(10) 1. Calculate the length of the curve $y = x^2$, $0 \le x \le 1$. (Make use of an appropriate integral on the cover page if necessary.)

$$\frac{dy}{dx} = 2x, \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+(2x)^2}\, dx.$$

$$S = \int_0^1 \sqrt{1+(2x)^2}\, dx \qquad \text{Let } u = 2x, \quad du = 2\,dx$$

$$= \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du = \frac{1}{2}\cdot\frac{1}{2}\left[u\sqrt{1+u^2} + \ln\left|u+\sqrt{1+u^2}\right|\right]_0^2$$

$$= \frac{1}{4}\left[2\sqrt{5} + \ln|2+\sqrt{5}| - (0 + \ln(1))\right]$$

$$= \frac{1}{4}\left(2\sqrt{5} + \ln(2+\sqrt{5})\right)$$

(10) 2. Calculate the surface area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 1$, about the x-axis.

$$\frac{dy}{dx} = 3x^2, \quad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+9x^4}\, dx, \quad r = x^3$$

$$A = \int_0^1 2\pi r\, ds = \int_0^1 2\pi x^3\sqrt{1+9x^4}\, dx, \qquad \text{Let } u = 1+9x^4, \quad du = 36x^3\, dx$$

$$= 2\pi\,\frac{1}{36}\int_1^{10} u^{1/2}\, du = \frac{\pi}{\cancel{18}\,9}\,\frac{\cancel{2}}{3}\, u^{3/2}\Big|_1^{10}$$

$$= \frac{\pi}{27}\left(10^{3/2} - 1\right)$$

(12) 5. Find the second degree Taylor polynomial for the function $f(x) = xe^x$ about $x = 1$.

$f(x) = xe^x$ $\qquad$ $f(1) = e$

$f'(x) = xe^x + e^x$ $\qquad$ $f'(1) = e + e = 2e$

$f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$ $\qquad$ $f''(1) = e + 2e = 3e$

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= e + 2e(x-1) + \frac{3e}{2}(x-1)^2$$

(12) 6. Solve the differential equation with initial condition,

$$\frac{dy}{dx} = \frac{\ln x}{xy}, \qquad y(1) = 2, \quad (x > 0).$$

Your final answer should be in the form $y = f(x)$ for some function $f(x)$.

$$\int y\,dy = \int \frac{\ln x}{x}\,dx \qquad u = \ln x, \; du = \frac{1}{x}\,dx$$

$$= \int u\,du$$

$$\frac{y^2}{2} = \frac{u^2}{2} + C$$

$$y^2 = u^2 + C \quad (\text{new } C)$$

$$y^2 = (\ln x)^2 + C$$

$x=1, y=2$, $\quad 4 = (\ln 1)^2 + C = C$

$$y^2 = (\ln x)^2 + 4$$

$$y = \sqrt{(\ln x)^2 + 4}$$

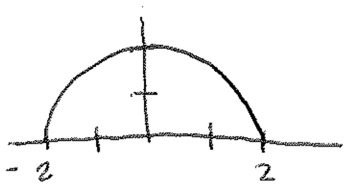
(Need positive square root so that $y(1) = 2$)

(12) 3. Calculate the amount of work required to pull a spring 1 foot beyond its rest length, if the force required to do so is 6 pounds.

$F = kx \Rightarrow 6 = k \cdot 1 \Rightarrow k = 6$ lb/ft.

$$W = \int_0^1 kx\,dx = k\frac{x^2}{2}\Big|_0^1 = \frac{k}{2} = \frac{6}{2} = 3 \text{ ft-lb}$$

(12) 4. Find the centroid of the region bounded by the semicircle $y = \sqrt{4 - x^2}$, $-2 \le x \le 2$ and the x-axis. (You may use symmetry and the area formula for a semicircle.)



By symmetry $\bar{x} = 0$

A = area below curve $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot 2^2 = 2\pi$

$$M_x = \frac{1}{2}\int_{-2}^{2} \left(\sqrt{4-x^2}\right)^2 - 0^2\,dx = \frac{1}{2}\int_{-2}^{2}(4 - x^2)\,dx$$

$$= \frac{1}{2}\left(4x - \frac{x^3}{3}\Big|_{-2}^{2}\right) = \frac{1}{2}\left(\left(8 - \frac{8}{3}\right) + \left(8 - \frac{8}{3}\right)\right) = 8 - \frac{8}{3} = 16/3$$

$$\bar{y} = \frac{M_x}{A} = \frac{16/3}{2\pi} = \frac{\cancel{16}\,8}{3}\cdot\frac{1}{\cancel{2}\pi} = \frac{8}{3\pi}$$

$$CM = \left(0, \frac{8}{3\pi}\right) = \text{Centroid}$$

7. Find the limit of the following sequences, or explain why they diverge.

(6) a) $\lim_{n\to\infty} \frac{n^2}{e^n} =$ $\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$ L'Hopital

$= \lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$ L'Hopital

$= \lim_{x\to\infty} \frac{2}{e^x}$

$= 0$

$\infty \cdot 0$ type

(6) b) $\lim_{n\to\infty} n\tan(\pi/n)$ $= \lim_{x\to\infty} \frac{\tan(\pi/x)}{1/x} = \frac{0}{0}$ type

$= \lim_{x\to\infty} \frac{\sec^2(\pi/x)\cdot \pi \cancel{(-1)x^{-2}}}{\cancel{(-1)x^{-2}}}$ L'Hopital

$= \lim_{x\to\infty} \pi \sec^2(\pi/x) = \pi \sec^2(0) = \pi$

(4) 8. Find a formula for the general term a_n of the series $\underset{a_1}{2} - \underset{a_2}{\frac{2^2}{2!}} + \underset{a_3}{\frac{2^3}{3!}} - \frac{2^4}{4!} + \cdots$
(You do not need to calculate any sum.)

$$a_n = \frac{(-1)^{n+1}\, 2^n}{n!}$$

9. Evaluate the following series, or state that it diverges and explain why it diverges.

(5) a) $\frac{3}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{18} + \frac{1}{54} + \cdots$ geometric with $r = \frac{1}{3} < 1$, so converges

$$= \frac{3}{2} \cdot \frac{1}{1 - 1/3}$$

$$= \frac{3}{2} \cdot \frac{1}{2/3} = \frac{9}{4}$$

(5) b) $\sum_{k=1}^{\infty} \frac{2k-1}{7k+5}$

$$\lim_{k \to \infty} \frac{2k-1}{7k+5} = \lim_{k \to \infty} \frac{(2k-1)\frac{1}{k}}{(7k+5)\frac{1}{k}}$$

$$= \lim_{k \to \infty} \frac{2 - 1/k}{7 + 5/k} = \frac{2}{7} \neq 0$$

By divergence test, the series diverges

(6) c) $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

$$\frac{2}{n^2-1} = \frac{A}{n-1} + \frac{B}{n+1}$$

$$2 = A(n+1) + B(n-1)$$

$n=1$ $\quad 2 = 2A \Rightarrow A = 1$

$n=-1$ $\quad 2 = -2B \Rightarrow B = -1$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} + \frac{-1}{n+1}\right) = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \cdots$$

$$= \frac{3}{2}$$