Math 221 Calculus 2   Midterm Exam 3
Professor John Maginnis   November 19, 2013

Your name: *Solutions*

Rec. Instr.: _____   Rec. Time: _____

Show all your work in the space provided under each question. Please write neatly
and present your answers in an organized way. You may use your one sheet of
notes, but no books or calculators. This exam is worth 60 points. The chart below
indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/10	/10	/10	/10
Problem	5	6		Total
Points	/10	/10		/60

1. Determine whether the series converges or diverges. Explain.

$$\sum_{n=0}^{\infty} \frac{n^2+4}{n^5+1}$$ $\underline{converges}$ by the

$\underline{\underline{Limit\ Comparison\ Test}}$, by comparing to

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$, which $\underline{converges}$ by the $p\underline{-series\ test}$

with $p = 3 > 1$ (or by the integral test),

$$L = \lim_{n \to \infty} \left(\frac{\frac{n^2+4}{n^5+1}}{\frac{1}{n^3}} \right) = \lim_{n \to \infty} \left(\frac{n^5 + 4n^3}{n^5 + 1} \right) =$$

$$\lim_{n \to \infty} \left(\frac{1 + \frac{4}{n^2}}{1 + \frac{1}{n^5}} \right) = \frac{1+0}{1+0} = \boxed{1}$$

$\underline{\underline{or}}$ use L'Hôpital's Rule,

$$\lim_{x \to \infty} \left(\frac{x^5 + 4x^3}{x^5 + 1} \right) = \frac{\infty}{\infty} = \lim_{x \to \infty} \left(\frac{5x^4 + 12x^2}{5x^4} \right) =$$

$$\lim_{x \to \infty} \left(1 + \frac{12}{5x^2} \right) = 1 + 0 = \boxed{1}$$

2

2. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{5^n}$$ converges since it absolutely converges:

The associated positive series

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{5^n} \right|$$ converges by the Comparison Test;

$$\left| \sin(n) \right| \leq 1, \text{ so } \left| \frac{\sin(n)}{5^n} \right| \leq \frac{1}{5^n}.$$

Note that $$\sum_{n=1}^{\infty} \frac{1}{5^n}$$ converges since it is a

geometric series with ratio $r = \frac{1}{5} < 1$.

3

3. Find the interval of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{2^n}{n+1}(x-1)^n \quad \text{converges for} \quad \boxed{\frac{1}{2} < x < \frac{3}{2}} :$$

$$\underline{\text{Ratio Test}} \quad \rho = \lim_{n \to \infty} \left| \frac{\frac{2^{n+1}}{n+2}(x-1)^{n+1}}{\frac{2^n}{n+1}(x-1)^n} \right| =$$

$$\lim_{n \to \infty} \left| \frac{2(n+1)(x-1)}{n+2} \right| = \lim_{n \to \infty} \left| \frac{2 + \frac{2}{n}}{1 + \frac{2}{n}}(x-1) \right| = \left| 2(x-1) \right| .$$

Then $\left| 2(x-1) \right| < 1$ for $-1 < 2x-2 < 1$,

$$1 < 2x < 3 , \quad \frac{1}{2} < x < \frac{3}{2} .$$

$\underline{\text{Check endpoints}}$: If $x = \frac{3}{2}$, $x-1 = \frac{1}{2}$, series is

$$\sum_{n=0}^{\infty} \frac{2^n}{n+1} \left(\frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots ; \text{ is the harmonic}$$

series which $\underline{\text{diverges}}$ by the p-series test $(p=1)$.

If $x = \frac{1}{2}$, $x-1 = \frac{-1}{2}$, $\sum_{n=0}^{\infty} \frac{2^n}{n+1} \left(\frac{-1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} =$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, the $\underline{\text{alternating}}$ harmonic series which

$\underline{\text{converges}}$ by the $\underline{\text{Leibniz Test}}$ (note $\frac{1}{n+1} > \frac{1}{n+2}$ for all n).

4

4. Find the Maclaurin series (the Taylor series at $c = 0$) for the function.

$$f(x) = \frac{x^2}{2 + x^3}$$

You need to find a formula for the general term. Hint: Do not take derivatives.

Use $\dfrac{1}{1-X} = 1 + X + X^2 + X^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} X^n$

(geometric series)

Then $\dfrac{x^2}{2+X^3} = \left(\dfrac{X^2}{2}\right) \cdot \dfrac{1}{1 + \dfrac{X^3}{2}} = \dfrac{X^2}{2}\left(1 - \dfrac{X^3}{2} + \dfrac{X^6}{4} - \dfrac{X^9}{8} \cdots\right)$
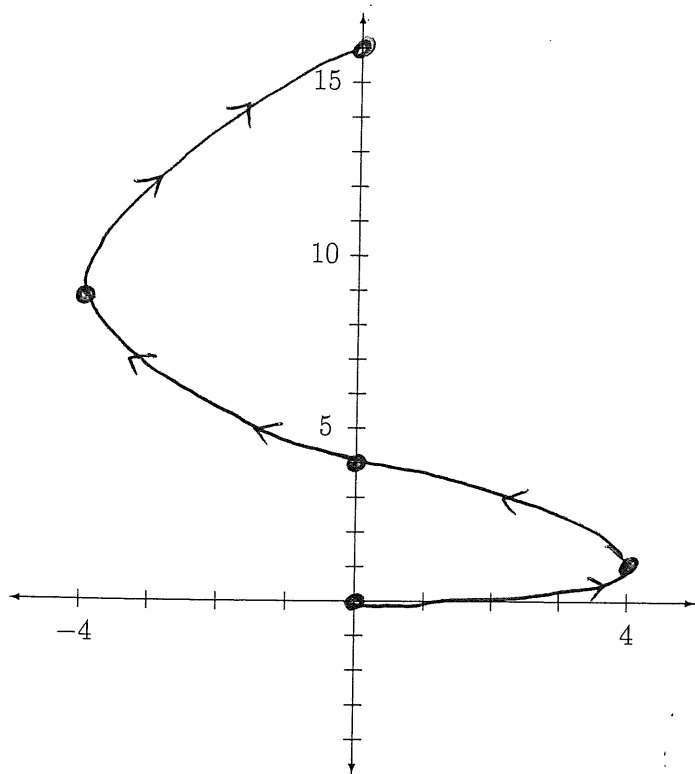
$= \dfrac{X^2}{2} \displaystyle\sum_{n=0}^{\infty} \left(\dfrac{-X^3}{2}\right)^n = \dfrac{X^2}{2} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^n} X^{3n}$

$$\boxed{= \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^{n+1}} X^{3n+2}}$$

5

5. A curve is given by the parametrization.

$$x = 4\sin\left(\frac{\pi}{2}\, t\right) \quad \text{and} \quad y = t^2$$

(a) Graph the curve for $0 \le t \le 4$ by plotting points when $t = 0, 1, 2, 3, 4$.

t	x	y
0	0	0
1	4	1
2	0	4
3	-4	9
4	0	16

(b) Find the slope $\frac{dy}{dx}$ when $t = 2$.

$$\begin{cases} \dfrac{dx}{dt} = 4\cos\left(\frac{\pi}{2}t\right)\cdot\frac{\pi}{2} \\ \qquad = 2\pi\cos\left(\frac{\pi}{2}t\right) \\ \dfrac{dy}{dt} = 2t \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2t}{2\pi\cos\left(\frac{\pi}{2}t\right)} = \frac{t}{\pi\cos\left(\frac{\pi}{2}t\right)}, \quad \text{If } t=2 \text{ then}$$

$$y'(2) = \frac{2}{\pi\cos(\pi)} = \boxed{\dfrac{-2}{\pi}}$$

6

6. Use the integral test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$

$$f(x) = \frac{1}{x \ln(x)} > 0 \quad \text{if } x > 1.$$

$$f'(x) = \frac{-(1 + \ln(x))}{(x \ln(x))^2} < 0 \quad \text{if } x > \frac{1}{e}.$$

$$\int_{2}^{\infty} \frac{dx}{x \ln(x)} = \lim_{R \to \infty} \int_{2}^{R} \frac{dx}{x \ln(x)}$$

$$u = \ln(x) \qquad \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C$$

$$du = \frac{1}{x} dx \qquad\qquad\qquad\qquad = \ln|\ln(x)| + C$$

$$\lim_{R \to \infty} \int_{2}^{R} \frac{dx}{x \ln x} = \lim_{R \to \infty} \left[\ln|\ln(x)| \right]_{2}^{R}$$

$$= \lim_{R \to \infty} \left[\ln|\ln R| - \ln(\ln 2) \right]$$

$$= \boxed{\infty}$$

Thus $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ <u>diverges</u>

7