

Math 221 Calculus 2  
Professor John Maginnis

Midterm Exam 2  
February 28, 2017

Your name: Solutions

Rec. Instr.: \_\_\_\_\_

Rec. Time: \_\_\_\_\_

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators. This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/6	/6	/13	/12
Problem	5	6		Total
Points	/10	/13		/60

1. Express the improper integral as a limit, and evaluate.

$$\int_0^4 \frac{dx}{\sqrt{x}} = \boxed{\lim_{R \rightarrow 0^+} \int_R^4 \frac{dx}{\sqrt{x}}} \quad ; \quad \frac{1}{\sqrt{x}} = x^{-1/2}$$

$$\lim_{R \rightarrow 0^+} \left[ \frac{x^{1/2}}{1/2} \right]_R^4 = \lim_{R \rightarrow 0^+} (4 - 2\sqrt{R})$$
$$= 4 - 0 = \boxed{4}$$

2. A force of 400 pounds will stretch a spring two feet from its equilibrium or natural length. Find the work (in foot-pounds) required to stretch the spring from two feet to three feet beyond its equilibrium length.

$$\boxed{F = Kx}$$

$$400 = K(2)$$

$$K = 200$$

(the spring constant  
in Hooke's Law)

$$F = 200x$$

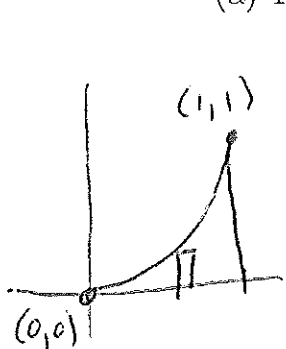
$$W = \int_2^3 200x \, dx$$

$$W = \left[ 100x^2 \right]_2^3$$

$$W = 900 - 400 = \boxed{500 \text{ ft-lbs}}$$

3. Find the centroid of the region under  $y = x^3$  for  $0 \leq x \leq 1$ .

(a) Find the area bounded by  $y = x^3$ ,  $x = 1$ , and  $y = 0$ .



$$A = \int_0^1 x^3 dx = \left[ \frac{1}{4} x^4 \right]_0^1 = \boxed{\frac{1}{4}}$$

Note  $A = \int_0^1 (1 - \sqrt[3]{y}) dy = \left[ y - \frac{3}{4} y^{4/3} \right]_0^1 = 1 - \frac{3}{4} = \boxed{\frac{1}{4}}$

(b) Find the moment  $M_x$  with respect to the  $x$ -axis.

$$M_x = \frac{1}{2} \int_0^1 (x^3)^2 dx = \frac{1}{2} \int_0^1 x^6 dx = \left[ \frac{1}{14} x^7 \right]_0^1 = \boxed{\frac{1}{14}}$$

Note  $M_x = \int_0^1 y (1 - \sqrt[3]{y}) dy = \int_0^1 y - y^{4/3} dy = \left[ \frac{1}{2} y^2 - \frac{3}{7} y^{7/3} \right]_0^1 = \frac{1}{2} - \frac{3}{7} = \boxed{\frac{1}{14}}$

(c) Find the moment  $M_y$  with respect to the  $y$ -axis.

$$M_y = \int_0^1 x(x^3) dx = \int_0^1 x^4 dx = \left[ \frac{1}{5} x^5 \right]_0^1 = \boxed{\frac{1}{5}}$$

Note  $M_y = \frac{1}{2} \int_0^1 (1^2 - (\sqrt[3]{y})^2) dy = \frac{1}{2} \int_0^1 1 - y^{2/3} dy = \frac{1}{2} \left[ y - \frac{3}{5} y^{5/3} \right]_0^1 = \frac{1 - 3/5}{2} = \boxed{\frac{1}{5}}$

(d) Compute the centroid  $(\bar{x}, \bar{y})$ .

$$\bar{x} = \frac{M_y}{A} = \frac{1/5}{1/4} = \boxed{\frac{4}{5}}, \quad \bar{y} = \frac{M_x}{A} = \frac{1/14}{1/4} = \boxed{\frac{2}{7}}$$

4. (a) Find the derivative  $\frac{dy}{dx}$  of the inverse hyperbolic function.

$$y = \cosh^{-1}(5x^2)$$

$$u = 5x^2$$

Chain Rule with  $\frac{du}{dx} = 10x$ .

Note if  $y = \cosh^{-1}(u)$ , then

$$\frac{dy}{du} = \frac{1}{\sqrt{u^2 - 1}}$$

$$\frac{dy}{dx} = \frac{10x}{\sqrt{25x^4 - 1}}$$

- (b) Evaluate the integral of the hyperbolic function.

$$\int \sinh^2(x) \cosh^3(x) dx$$

Substitution  $u = \sinh(x)$

$$du = \cosh(x) dx$$

$$\text{Note } \cosh^2(x) - \sinh^2(x) = 1,$$

$$\cosh^2(x) = 1 + \sinh^2(x) = 1 + u^2$$

$$\int u^2 \cdot (1 + u^2) du = \int (u^2 + u^4) du =$$

$$\frac{1}{3} u^3 + \frac{1}{5} u^5 + C = \frac{1}{3} \sinh^3(x) + \frac{1}{5} \sinh^5(x) + C$$

5. Find the arc length of the curve.

$$y = \ln(\cos(x)) \text{ for } 0 \leq x \leq \frac{\pi}{4}$$

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = \frac{-\sin x}{\cos x} = -\tan(x)$$

$$\left(\frac{dy}{dx}\right)^2 = \tan^2(x)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2(x) = \sec^2(x)$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\sec^2(x)} = \sec(x)$$

(using  $0 \leq x \leq \pi/4$ , so  $\sec(x) > 0$ )

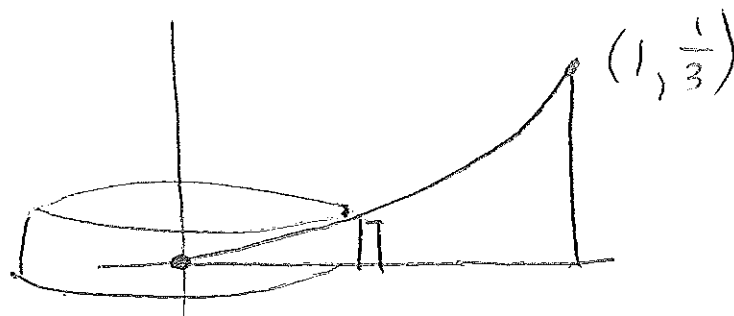
$$s = \int_0^{\pi/4} \sec(x) dx = \left[ \ln|\sec x + \tan x| \right]_0^{\pi/4}$$

$$s = \ln(\sqrt{2} + 1) - \ln(1 + 0)$$

$$\boxed{s = \ln(\sqrt{2} + 1)}$$

6. Use the method of cylindrical shells to find the volume of revolution formed by rotating around the  $y$ -axis the region under the curve

$$y = \frac{-x}{x^2 - 4} \quad \text{for } 0 \leq x \leq 1.$$



$$dV = 2\pi R H dx$$

$$R = x$$

$$H = y = \frac{-x}{x^2 - 4}$$

$$V = \int_0^1 2\pi (x) \left( \frac{-x}{x^2 - 4} \right) dx = 2\pi \int_0^1 \frac{-x^2}{x^2 - 4} dx$$

Long Division yields  $\frac{-x^2}{x^2 - 4} = -1 + \frac{-4}{x^2 - 4}$ .

Partial Fractions  $\frac{-4}{x^2 - 4} = \frac{A}{x-2} + \frac{B}{x+2}$ ,  $-4 = A(x+2) + B(x-2)$

If  $x=2$ ,  $-4=4A$ ,  $A=-1$ .

If  $x=-2$ ,  $-4=-4B$ ,  $B=1$ .

$$V = 2\pi \int_0^1 -1 + \frac{-1}{x-2} + \frac{1}{x+2} dx = 2\pi \left[ -x - \ln|x-2| + \ln|x+2| \right]_0^1$$

$$V = 2\pi \left( -1 - \ln(1) + \ln(3) - (0 - \ln(2) + \ln(2)) \right)$$

$$V = \boxed{2\pi (\ln(3) - 1)}$$