Short answer questions (12 points each):

1. The vector field $\vec{H}(x, y) = \langle 2x + 2y, 2x + 2y \rangle$ is conservative. Find a potential function h.

h must satisfy

$$h(x,y) = \int 2x + 2y \, dx = x^2 + 2xy + A(y)$$

and $$h(x,y) = \int 2x + 2y \, dy = 2xy + y^2 + B(x)$$

so letting $A(y) = y^2$, $B(x) = x^2$ gives

$h(x,y) = x^2 + 2xy + y^2$ is a potential function for $\vec{H}$

2. Find the curl and divergence of the vector field $\vec{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle$.

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & yz^2 & zx^2 \end{vmatrix} = (0 - 2yz)\vec{i} - (2xz - 0)\vec{j} + (0 - 2xy)\vec{k}$$

$$= \langle -2yz, -2xz, -2xy \rangle$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} xy^2 + \frac{\partial}{\partial y} yz^2 + \frac{\partial}{\partial z} zx^2$$

$$= y^2 + z^2 + x^2$$

More short answer questions:

3. Find the function $\vec{T}(t)$ which expresses the unit tangent vector to the trajectory

$$\vec{r}(t) = \langle 13\cos(t), 5\sin(t), 12\sin(t)\rangle$$

as a function of time.

$\vec{r}\,'(t) = \langle -13\sin t,\ 5\cos t,\ 12\cos t\rangle$

$\|\vec{r}\,'(t)\| = \sqrt{169\sin^2 t + 25\cos^2 t + 144\cos^2 t}$

$= \sqrt{169\sin^2 t + 169\cos^2 t} = \sqrt{169} = 13$

So $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{1}{13}\langle -13\sin t, 5\cos t, 12\cos t\rangle$

$= \langle -\sin t, \frac{5}{13}\cos t, \frac{12}{13}\cos t\rangle$

4. Express the work done by a force described by the vector field $\vec{F}(x, y, z) = \langle -y, 2x + z, z\rangle$ moving a particle along the circular helix $\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$ from $(3, 0, 0)$ to $(-3, 0, 7\pi)$ as an ordinary integral. You do not need to evaluate the integral. (You may assume the force is given in newtons and distances in the coordiante system are given in meters, if you wish to have definite units).

$\vec{F}(\vec{r}(t)) = \langle -3\sin t, 6\cos t + t, t\rangle$

$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 1\rangle$

So

$$W = \int_C \vec{F}\cdot d\vec{s} = \int_0^{7\pi} \langle -3\sin t, 6\cos t + t, t\rangle \cdot \langle -3\sin t, 3\cos t, 1\rangle\, dt$$

$$= \int_0^{7\pi} 9\sin^2 t + 18\cos^2 t + 3t\cos t + t\, dt$$

either answer is fine, the second would be slightly easier to evaluate by hand

$$= \int_0^{7\pi} 9 + 9\cos^2 t + 3t\cos t + t\, dt$$

Yet more short answer questions:

5. Find the flux of the vector field $\vec{F}(x, y, z) = x\vec{\imath} - z\vec{k}$ through the sphere of radius 2 about the origin. Let's try using the divergence thm:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z}(-z)$$
$$= 1 + 0 - 1 = 0$$

So $\text{flux} = \iiint_R 0 \, dV = 0$

R the region inside the sphere

6. Find the Jacobian determinant of the transformation

$$(u, v) \mapsto (u^2 - v^2, u^2 + v^2)$$

$$\begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv - (-4uv) = 8uv$$

7. Give a vector equation for the plane passing through the point $(1, \frac{1}{2}, -3)$ and perpendicular to the line $\vec{r}(t) = \langle 1 + t, 2 - t, 3t \rangle$.

$= \langle 1, 2, 0 \rangle + t \langle 1, -1, 3 \rangle$

use the direction of the line as the normal to the plane:

$$\langle x, y, z \rangle \cdot \langle 1, -1, 3 \rangle = \langle 1, \tfrac{1}{2}, -3 \rangle \langle 1, -1, 3 \rangle$$
$$x - y + 3z = 1 - \tfrac{1}{2} - 9 = -\tfrac{17}{2}$$

$$x - y + 3z = -\tfrac{17}{2}$$

Matching (20 points)

8. Ten formulas are given in the left column. The ten lettered items in the right column are descriptions or other formulas, each of which applies to exactly one item in the left column in the sense of correctly describing it or being equal to it. Put the letter of the item in the right column which applies to each item in the left column in the blank provided next to the corresponding item in the left column. Please uses block capital letters.

F $\iint_{\partial R} \langle 0, y, 0\rangle \bullet d\vec{S}$

B $\iint_{\Sigma} \vec{F} \bullet d\vec{S}$

G $2\cos(t)\vec{\imath} + t^2\vec{\jmath} + 2\sin(t)\vec{k}$

H $2\cos(x)\vec{\imath} + z^2\vec{\jmath} + 2\sin(y)\vec{k}$

D $\int_C \vec{F} \bullet d\vec{s}$

E $\int_C f ds$

J $x\vec{\imath} + y\vec{\jmath} - \vec{k}$

I $4\vec{\imath} + \vec{\jmath} - 2\vec{k}$

A $\int_{\partial R} \langle x, 0\rangle \bullet d\vec{S}$

C $\iint_{\Sigma} f dS$

A. 0

B. useful in finding the flux of a fluid flow through a surface

C. useful in finding the average value of a function on a surface

D. useful in finding the work done by a force varying from place to place moving a particle along a curve

E. useful in finding the average value of a function on a curve

F. gives the volume of a region in 3-space.

G. a curve lying on the cylinder $x^2 + z^2 = 4$

H. a vector field which is neither irrotational nor represents the flow of an incompressible fluid

I. an irrotational vector field, which could also represent the flow of an incompressible fluid

J. an irrotational vector field which could not represent the flow of an incompressible fluid

Long questions (24 points each)

9. Find the length of the curve given parametrically by

$$\vec{r}(t) = \left\langle \cos t + t\sin t, \sin t - t\cos t, \frac{1}{3}t^3 \right\rangle \quad t \in [0, \pi]$$

(Hint: if you think you should have been given an integral table for this one, you've done something wrong. Find it and fix it.)

$$\vec{r}\,'(t) = \langle -\sin t + \sin t + t\cos t,\ \cos t - \cos t + t\sin t,\ t^2 \rangle$$

$$= \langle t\cos t,\ t\sin t,\ t^2 \rangle$$

So

$$\|\vec{r}\,'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t + t^4}$$

$$= \sqrt{t^2 + t^4} = t\sqrt{1+t^2}$$

So

$$\ell = \int_0^{\pi} t\sqrt{1+t^2}\,dt$$

let $u = 1 + t^2$

$du = 2t\,dt$

$t\,dt = \frac{1}{2}du$

$t = 0 \Rightarrow u = 1$

$t = \pi \Rightarrow u = 1 + \pi^2$

$$= \int_1^{1+\pi^2} \frac{1}{2} u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_{1}^{1+\pi^2}$$

$$= \frac{1}{3}(\pi^2+1)^{3/2} - \frac{1}{3}$$

10. Recall that Green's Theorem can be used to express the area of a region as a line integral over the boundary of the region (Rogwaski gave a formula in the text, and two more in an exercise, but any vector field for which the integrand in the double integral of Green's theorem equals the constant function 1 will work). Express the area inside the limaçon given in polar coordinates by $r = 3 + 2\cos\theta$ as a line integral and rewrite the line integral as an ordinary one-variable integral. Do not evaluate the integral (Hint: if done correctly, your answer should be an integral which would have been a feasible, but very long, boring and tedious Calc II exercise).

As a further hint, the boundary of the limaçon is given parametrically by

$$\vec{r}(t) = \langle (3+2\cos t)\cos t, (3+2\cos t)\sin t\rangle \quad t \in [0, 2\pi]$$

Let's use the one Rogawski gave $\vec{F} = \langle -\tfrac{1}{2}y, \tfrac{1}{2}x\rangle$

so $\vec{F}(\vec{r}(t)) = \langle -\tfrac{1}{2}(3+2\cos t)\sin t, \tfrac{1}{2}(3+2\cos t)\cos t\rangle$

$\vec{r}\,'(t) = \langle -2\sin t\cos t - 3\sin t - 2\cos t\sin t, -2\sin^2 t + 3\cos t + 2\cos^2 t\rangle$

$= \langle -3\sin t - 4\cos t\sin t, 3\cos t + 2\cos^2 t - 2\sin^2 t\rangle$

So

$$\text{Area} = \oint_{\partial R} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \langle -\tfrac{3}{2}\sin t - \cos t\sin t, \tfrac{3}{2}\cos t + \cos^2 t\rangle \cdot \langle -3\sin t - 4\cos t\sin t, 3\cos t + 2\cos^2 t - 2\sin^2 t\rangle \, dt$$

$$= \int_0^{2\pi} \tfrac{9}{2}\sin^2 t + 6\cos t\sin^2 t + 3\cos t\sin^2 t + 4\cos^2 t\sin^2 t + \tfrac{9}{2}\cos^2 t + 3\cos^3 t - 3\cos\sin^2 t + 3\cos^3 t + 2\cos^4 t - 2\cos^2 t\sin^2 t \, dt$$

$$= \int_0^{2\pi} \tfrac{9}{2} + 6\cos t\sin^2 t + 2\cos^2 t\sin^2 t + 6\cos^3 t + 2\cos^4 t \, dt$$

$$= \int_0^{2\pi} \tfrac{9}{2} + 6\cos t + 2\cos^2 t \, dt$$

11. Find the maximum and minimum values of the function

$$f(x, y) = x^3 - 6x^2 + 9x - 6y^2$$

on the closed ball $\{(x, y)|x^2 + y^2 \leq 4\}$.

(Hint: You will need to use different methods to locate the points inside the ball and the points on the boundary at which extrema might occur.)

Find critical points: $f_x = 3x^2 - 12x + 9 = 0$ at c.p. $3(x-3)(x-1) = 0$

$x = 3$ or $x = 1$

$f_y = -12y$ $\therefore y = 0$ so c.p.'s are

$(3, 0)$, $(1, 0)$ but $(3, 0)$ is outside the region.

On boundary use Lagrange multipliers

boundary is $g(x,y) = x^2 + y^2 = 4$

So at boundary extrema $\nabla f = \lambda \nabla g$

$\langle 3x^2 - 12x + 9, -12y \rangle = \lambda \langle 2x, 2y \rangle$

$\therefore -12y = 2\lambda y$

$y = 0$ or $\lambda = -6$

if $y = 0$

$x^2 + 0^2 = 4$

$x = \pm 2$

$(2, 0)$

$(-2, 0)$

if $\lambda = -6$

$3x^2 - 12x + 9 = -12x$

$3x^2 + 9 = 0$

no solutions

Test

$f(2, 0) = 8 - 24 + 18 = 2$

$f(1, 0) = 1 - 6 + 9 = 4$ max

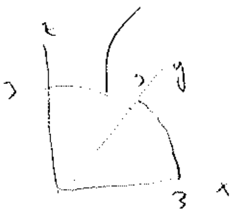
$f(-2, 0) = -8 - 24 - 18 = -50$ min.

12. Write iterated integrals equal to the triple integral

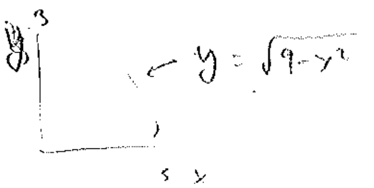
$x^2+y^2+z^2=9$

$z=\sqrt{9-x^2-y^2}=\sqrt{9-r^2}$

$$\iiint_R \frac{z}{x^2+y^2+z^2}\,dV$$



where R is region lying inside the sphere of radius 3 about the origin and in the first octant (all rectangular coordinates non-negative) using each of rectangular, cylindrical and spherical coordinates. Briefly explain which you would prefer to evaluate and why. You do not need to evaluate any of the interated integrals.



Rectangular:

$$\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2-y^2}} \frac{z}{x^2+y^2+z^2}\,dz\,dy\,dx$$

Cylindrical

$0\le z\le\sqrt{9-r^2}$

$0\le r\le 3$

$0\le\theta\le\pi/2$

$x^2+y^2+z^2=r^2+z^2$

$dV=r\,dz\,dr\,d\theta$

$$\int_0^{\pi/2}\int_0^3\int_0^{\sqrt{9-r^2}} \frac{zr}{r^2+z^2}\,dz\,dr\,d\theta$$

Spherical

$0\le\rho\le 3$

$0\le\varphi\le\pi/2$

$0\le\theta\le\pi/2$

$\dfrac{z}{x^2+y^2+z^2}=\dfrac{\rho\cos\varphi}{\rho^2}$

$dV=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \frac{\rho\cos\varphi}{\rho^2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\cos\varphi\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Spherical is clearly the easiest to evaluate, since it decomposes as integrating a constant, integrating a polynomial and an easy trig integral