

1. (16 points) Computation

- (a) Let  $f(x, y) = x^3 \exp(2x^2 + 3xy + y^2)$ . Find all of the first partial derivatives. (In case you haven't seen it before, “ $\exp(u)$ ” is the same thing as  $e^u$ .)

**Solution:**

$$\begin{aligned}f_x &= 3x^2 \exp(2x^2 + 3xy + y^2) + x^3 \exp(2x^2 + 3xy + y^2)(4x + 3y) \\f_y &= x^3 \exp(2x^2 + 3xy + y^2)(3x + 2y)\end{aligned}$$

- (b) Let  $g(x, y) = \frac{x}{\sqrt{4x^2 + y^2}}$ . Find the first partial derivative with respect to  $x$  and simplify it.

**Solution:**

$$\begin{aligned}g_x &= \frac{\sqrt{4x^2 + y^2} \cdot 1 - x \cdot \frac{1}{2\sqrt{4x^2 + y^2}} \cdot 8x}{4x^2 + y^2} \\&= \frac{4x^2 + y^2 - 4x^2}{(4x^2 + y^2)^{3/2}} \\&= \boxed{\frac{y^2}{(4x^2 + y^2)^{3/2}}}\end{aligned}$$

2. (12 points) A certain differentiable function satisfies:

(a)  $f(2, 5) = -3$ , and  $f(-4, 1) = \pi$ .

(b)  $\nabla f(2, 5) = \langle -4, 7 \rangle$ , and  $\nabla f(-4, 1) = \langle \sqrt{6}, e^{-2} \rangle$ .

At each of the two points in question (i.e. at  $(2, 5)$  and at  $(-4, 1)$ ) answer the following questions:

- (a) In what direction is the function increasing the fastest and what is the rate of change in that direction?

**Solution:**

At  $(2, 5)$ , the function is increasing the fastest in the direction  $\nabla f(2, 5) = \langle -4, 7 \rangle$ , with rate of change  $\|\langle -4, 7 \rangle\| = \sqrt{16 + 49} = \sqrt{65}$ .

At  $(-4, 1)$ , the function is increasing the fastest in the direction  $\nabla f(-4, 1) = \langle \sqrt{6}, e^{-2} \rangle$ , with rate of change  $\|\langle \sqrt{6}, e^{-2} \rangle\| = \sqrt{6 + e^{-4}}$ .

- (b) What is the directional derivative in the direction of the vector  $\langle 4, -3 \rangle$ ?

**Solution:** The unit vector in the direction  $\langle 4, -3 \rangle$  is  $\langle 4/5, -3/5 \rangle$ . So

$$\begin{aligned} D_{\langle 4/5, -3/5 \rangle} f(2, 5) &= \nabla f(2, 5) \cdot \langle 4, -3 \rangle \cdot \frac{1}{5} \\ &= \langle -4, 7 \rangle \cdot \langle 4, -3 \rangle \cdot \frac{1}{5} \\ &= (-16 - 21) \cdot \frac{1}{5} = \boxed{\frac{-37}{5}} \end{aligned}$$

and

$$\begin{aligned} D_{\langle 4/5, -3/5 \rangle} f(-4, 1) &= \nabla f(-4, 1) \cdot \langle 4, -3 \rangle \cdot \frac{1}{5} \\ &= \langle \sqrt{6}, e^{-2} \rangle \cdot \langle 4, -3 \rangle \cdot \frac{1}{5} \\ &= \boxed{(4\sqrt{6} - 3e^{-2}) \cdot \frac{1}{5}} \end{aligned}$$

- (c) What is the tangent plane and/or the linear approximation at each of the two points?

**Solution:**

$$\begin{aligned} (2, 5) : \quad z &= -3 + -4(x - 2) + 7(y - 5) \\ (-4, 1) : \quad z &= \pi + \sqrt{6}(x + 4) + e^{-2}(y - 1) \end{aligned}$$

3. (12 points) Set up **but do not solve** the following problems. As part of setting these problems up, you should list the unknowns and the equations that you would need to use to find them. You **should also do** all of the **derivative** calculations, but the **algebra** is totally unmanageable, so do **not** attempt it!

- (a) Maximize  $f(x, y) = x^2 \cos(y)$   
 Subject to  $g(x, y) = x^6 + y^6 = 64$ .

**Solution:**

$$\begin{aligned}\nabla f &= \lambda \nabla g \\ \langle 2x \cos y, -x^2 \sin y \rangle &= \lambda \langle 6x^5, 6y^5 \rangle\end{aligned}$$

The system to solve is:

$$\begin{cases} 2x \cos y = 6\lambda x^5 \\ -x^2 \sin y = 6\lambda y^5 \\ x^6 + y^6 = 64 \end{cases}$$

- (b) Maximize  $F(x, y, z) = \cos(xy^2z^3)$   
 Subject to  $G(x, y, z) = x + 2y + 3z = 0$   
 and  $H(x, y, z) = x^2 + z^2 = 25$ .

**Solution:**

$$\nabla F = \langle -y^2z^3 \sin(xy^2z^3), -2xyz^3 \sin(xy^2z^3), -3xy^2z^2 \sin(xy^2z^3) \rangle$$

$$\nabla G = \langle 1, 2, 3 \rangle$$

$$\nabla H = \langle 2x, 0, 2z \rangle$$

Setting  $\nabla F = \lambda \nabla G + \mu \nabla H$  gives the system to solve:

$$\begin{cases} -y^2z^3 \sin(xy^2z^3) = \lambda + 2\mu x \\ -2xyz^3 \sin(xy^2z^3) = 2\lambda \\ -3xy^2z^2 \sin(xy^2z^3) = 3\lambda + 2\mu z \\ x + 2y + 3z = 0 \\ x^2 + z^2 = 25 \end{cases}$$

4. (14 points) For the function  $f(x, y) = x^3 + xy - y^2$  find and classify all of the critical points.

**Solution:**

$$\nabla f = 0 \implies \begin{cases} 3x^2 + y = 0 \\ x - 2y = 0 \end{cases}$$

Solving this system gives two critical points:  $(0, 0)$  and  $(-\frac{1}{6}, -\frac{1}{12})$ . To classify them, the discriminant is

$$f_{xx}f_{yy} - (f_{xy})^2 = 6x(-2) - (1)^2 = -12x - 1$$

For  $(0, 0)$ , the discriminant gives  $-1$ , so  $(0, 0)$  is a saddle point.

For  $(-\frac{1}{6}, -\frac{1}{12})$ , the discriminant gives  $1$ , and  $f_{yy} = -2 < 0$ , so  $(-\frac{1}{6}, -\frac{1}{12})$  is a local maximum.

5. (20 points) Find the maximum and the minimum of the function

$$f(x, y) = x^2 + 2x + y^2 - 6y$$

in the region given by

$$g(x, y) = x^2 + y^2 \leq 40.$$

Show your work carefully in this problem, and let us know what you are doing.

**Solution:** First we deal with the case  $g(x, y) < 40$  by solving for  $\nabla f = 0$ :

$$\langle 2x + 2, 2y - 6 \rangle = 0 \implies x = -2, y = 3$$

We will save the evaluating for the end.

Next we deal with the case  $g(x, y) = 40$  by solving for  $\nabla f = \lambda \nabla g$ :

$$\begin{aligned} \langle 2x + 2, 2y - 6 \rangle &= \lambda \langle 2x, 2y \rangle \\ \implies \begin{cases} 2x + 2 = 2\lambda x \\ 2y - 6 = 2\lambda y \end{cases} \end{aligned}$$

Solving the system yields the values

$$x = \frac{-1}{1 - \lambda} \quad y = \frac{3}{1 - \lambda}$$

which substituting into the constraint gives

$$\frac{1}{(1 - \lambda)^2} + \frac{9}{(1 - \lambda)^2} = 40 \implies \frac{1}{1 - \lambda} = \pm 2$$

This gives the points  $(-2, 6)$  and  $(2, -6)$ . Evaluating at all found points:

$$f(-2, 3) = -9$$

$$f(-2, 6) = 0$$

$$f(2, -6) = 80$$

Hence the minimum and maximum in the given region are  $-9$  and  $80$ , respectively.

6. (8 points) Suppose that  $x = r \cos \theta$  and  $y = r \sin \theta$  (the usual polar coordinates) and  $f(x, y) = x^2 y$ . Express

$$\frac{\partial f}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta}$$

as functions of  $r$  and  $\theta$ . (Hint/Comment: Do this however you like.)

**Solution:**

$$\begin{aligned} f &= x^2 y = r^3 \cos^2 \theta \sin \theta \\ f_r &= 3r^2 \cos^2 \theta \sin \theta \\ f_\theta &= r^3 (2 \cos \theta (-\sin \theta) \sin \theta + \cos^2 \theta \cos \theta) \\ &= r^3 (-2 \cos \theta \sin^2 \theta + \cos^3 \theta) \end{aligned}$$

7. (18 points) Short answers ...

- (a) If  $f$  is a function of  $x$  and  $y$ , and  $x$  and  $y$  are each functions of  $r$ ,  $s$ , and  $t$ , then use the chain rule to express  $\frac{\partial f}{\partial s}$ .

**Solution:**

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$$

- (b) Find the average value of the function  $f(x, y) = x^2y$  on the rectangle  $0 \leq x \leq 3$ ,  $0 \leq y \leq 4$ .

**Solution:**

$$\frac{1}{12} \int_0^3 \int_0^4 x^2 y \, dy \, dx = \frac{1}{12} \left[ \frac{x^3}{3} \right]_0^3 \left[ \frac{y^2}{2} \right]_0^4 = \frac{1}{12} (9)(8) = 6$$

- (c) According to the theorem that we learned, what should you require of a set  $S$  to guarantee that any continuous function  $f$  will attain an absolute maximum and an absolute minimum on  $S$ ?

**Solution:** The set  $S$  must be closed and bounded.

- (d) For the set  $5x^2 + y^3 + 2z^6 - 3xy^2z^2 = 11$  write down the tangent plane at the point  $(-1, -2, 1)$ .

**Solution:** Letting  $f = 5x^2 + y^3 + 2z^6 - 3xy^2z^2 - 11$ ,

$$\begin{aligned} \nabla f &= \langle 10x - 3y^2z^2, 3y^2 - 6xyz^2, 12z^5 - 6xy^2z \rangle \\ \nabla f(-1, -2, 1) &= \langle -22, 0, 36 \rangle \end{aligned}$$

$$0 = -22(x + 1) + 0(y + 2) + 36(z - 1)$$