MATH 222 CALCULUS 3 SUMMER 2014: EXAM 3

Name: ______________________________________

Instructor: ____________________________________

To receive credit you must show your work.

Problem 1. (10 points)Evaluate by changing to polar coordinates:

$$\int_{\sqrt{3}}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{1}{\sqrt{x^2+y^2}} dydx.$$

Answer: Let $x = rcos(t)$, $y = rsin(t)$. Then

$$\begin{aligned}
\int_{\sqrt{3}}^{2}\int_{0}^{\sqrt{4-x^2}} \frac{1}{\sqrt{x^2+y^2}} dydx &= \int_{0}^{\pi/6}\int_{\sqrt{3}/cos(t)}^{2} \frac{1}{r} rdrdt \\
&= \int_{0}^{\pi/6} (2 - \frac{\sqrt{3}}{cos(t)})dt = \int_{0}^{\pi/6} 2dt - \int_{0}^{\pi/6} \frac{\sqrt{3}}{cos(t)}dt \\
&= \frac{\pi}{3} - \int_{0}^{\pi/6} \frac{\sqrt{3}}{cos(t)}dt.
\end{aligned}$$

Let $u = sin(t)$. Then

$$\begin{aligned}
\frac{\pi}{3} - \int_{0}^{\pi/6} \frac{\sqrt{3}}{cos(t)}dt &= \frac{\pi}{3} - \int_{0}^{1/2} \frac{\sqrt{3}du}{1-u^2} \\
&= \frac{\pi}{3} - \int_{0}^{1/2} \frac{\sqrt{3}du}{2(1-u)} - \int_{0}^{1/2} \frac{\sqrt{3}du}{2(1+u)} \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2} ln\frac{|u+1|}{|u-1|}\Big|_{0}^{1/2} \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2} ln3.
\end{aligned}$$

Problem 2. (15 points) Evaluate

$$\int\int_{\mathcal{D}} e^{x+y} dA,$$

where $\mathcal{D}$ is bounded by $y = x - 2$, $y = 8 - x$, $x = 0$, $y = 2$ and $y = 4$.

Answer:

$$\begin{aligned}\int\int_{\mathcal{D}} e^{x+y} dA &= \int_2^3 \int_0^{y+2} e^{x+y} dxdy + \int_3^4 \int_0^{8-y} e^{x+y} dxdy \\ &= \int_2^3 (e^{y+2+y} - e^y) dy + \int_3^4 (e^{8-y+y} - e^y) dy \\ &= (e^{2y+2}/2 - e^y)|_2^3 + (e^8 y - e^y)|_3^4 = \frac{3}{2}e^8 - \frac{1}{2}e^6 - e^4 + e^2.\end{aligned}$$

Problem 3. (15 points) Evaluate the average of the function $f(x, y, z) = x^2 + y^2$ in the bounded reign $x^2 + y^2 \leq z \leq 4$.

Answer:

$$\int\int\int_{\mathcal{W}} dV = \int\int_{(x,y)\in\mathcal{R}} \int_{x^2+y^2}^{4} dzdxdy = \int\int_{(x,y)\in\mathcal{R}} (4 - x^2 - y^2)dxdy.$$

where $\mathcal{R}$ indicates the region $x^2 + y^2 \leq 4$ on xy-plane. Then let $x = rcos(t)$, $y = rsin(t)$, we have

$$\int\int_{(x,y)\in\mathcal{R}} (4 - x^2 - y^2)dxdy = \int_0^{2\pi}\int_0^2 (4 - r^2)rdrdt = 8\pi.$$

$$\int\int\int_{\mathcal{W}} (x^2 + y^2)dV = \int\int_{(x,y)\in\mathcal{R}} \int_{x^2+y^2}^{4} (x^2 + y^2)dzdxdy = \int\int_{(x,y)\in\mathcal{R}} (x^2 + y^2)(4 - x^2 - y^2)dxdy.$$

where $\mathcal{R}$ indicates the region $x^2 + y^2 \leq 4$ on xy-plane. Then let $x = rcos(t)$, $y = rsin(t)$, we have

$$\int\int_{(x,y)\in\mathcal{R}} (x^2 + y^2)(4 - x^2 - y^2)dxdy = \int_0^{2\pi}\int_0^2 r^2(4 - r^2)rdrdt = \frac{32\pi}{3}.$$

Thus the average is $\frac{32\pi}{3}/8\pi = \frac{4}{3}$.

Problem 4. (20 points) Evaluate

$$\int\int_{\mathcal{R}} (x+y)^2 e^{x^2-y^2} dxdy$$

where $\mathcal{R}$ is the square bounded by $x - y = 1$, $x - y = -1$, $x + y = 1$ and $x + y = -1$.

Answer: Let $u = x + y$, $v = y - x$. Then the region can be expressed as $-1 \le u \le 1$ and $-1 \le v \le 1$. Since $x = (u - v)/2$, $y = (u + v)/2$, the jacobian is $1/2$. Therefore

$$\begin{aligned}
\int\int_{\mathcal{R}} (x+y)^2 e^{x^2-y^2} dxdy &= \int\int_{(x,y)\in\mathcal{R}} u^2 e^{-uv} \frac{1}{2} dudv \\
&= \int_{-1}^{1}\int_{-1}^{1} u^2 e^{-uv} \frac{1}{2} dudv \\
&= \frac{1}{2}\int_{-1}^{1} u(e^u - e^{-u}) du \\
&= \frac{1}{2}(ue^u - e^u + ue^{-u} + e^{-u})|_{-1}^{1} = 2e^{-1}.
\end{aligned}$$

Problem 5. (20 points) Evaluate

$$\int_0^1 \int_{y=x^{2/5}}^1 e^{y^6} x dy dx.$$

Answer:

$$\begin{aligned}\int_0^1 \int_{y=x^{2/5}}^1 x e^{y^6} dy dx &= \int_0^1 \int_0^{x=y^{5/2}} x e^{y^6} dx dy \\ &= \int_0^1 \frac{x^2}{2} e^{y^6} |_0^{x=y^{5/2}} dy \\ &= \int_0^1 \left(\frac{(y^{5/2})^2}{2} e^{y^6} - 0\right) dy = \int_0^1 \frac{y^5}{2} e^{y^6} dy.\end{aligned}$$

Let $u = y^6$, $du = 6y^5 dy$. Then

$$\begin{aligned}\int_0^1 \frac{y^5}{2} e^{y^6} dy &= \int_{y=0}^{y=1} \frac{1}{12} e^u du \\ &= \frac{1}{12} e^u |_{y=0}^{y=1} = \frac{1}{12} e^{y^6} |_0^1 = \frac{1}{12}(e-1).\end{aligned}$$

Problem 6. (20 points) Evaluate

$$\int\int_{\mathcal{D}} (x+y)dxdy$$

where $\mathcal{D}$ is the region bounded by $x+y=1$, $x+y=4$, $y=2x$ and $y=x$.

Answer: $x=\frac{u}{v+1}$, $y=\frac{uv}{v+1}$. Then the Jacobian is $\frac{u}{(v+1)^2}$. Since $x+y=u$, $y/x=v$, we have $1\le u\le 4$ and $1\le v\le 2$. Therefore

$$\begin{aligned}\int\int_{\mathcal{D}} (x+y)dxdy &= \int_{1\le u\le 4}\int_{1\le v\le 2} \frac{u}{(v+1)^2}\left(\frac{u}{v+1}+\frac{uv}{v+1}\right)dudv \\ &= \int_1^2\int_1^4 \frac{u^2}{(v+1)^2}dudv \\ &= \int_1^2 \frac{21}{(v+1)^2}dv = 7/2.\end{aligned}$$