Instructions: Wait to open the exam until instructed to do so. Then answer the questions in the spaces provided on the question sheets. Please explain your responses in full detail. You will have 1 hour to complete this exam.

Question	Points	Score
1	20	
2	15	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Name: ______________________________

Recitation Instructor: ______________________________

Recitation Time: ______________________________

1. Consider the function

$$f(x, y) = x^2 - x + 2y + 4y^2$$

on the domain $\mathcal{D}$ defined by

$$\frac{x^2}{4} + y^2 \le 1.$$

(a) (5 points) Find the critical points of f in the interior of $\mathcal{D}$.

$f_x = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f_y = 2 + 8y = 0 \Rightarrow y = -\frac{1}{4}$

Crit point: $\boxed{\left(\frac{1}{2}, -\frac{1}{4}\right)}$

Means "What kind of critical point?"

(b) (5 points) Describe the local behavior of f near the critical points.

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= 2 \cdot 8 - 0$

$= 16 > 0$

$f_{xx} = 2 > 0$

$\Rightarrow$ the crit point $\left(\frac{1}{2}, -\frac{1}{4}\right)$ is a local min.

(c) (10 points) Find the global maximum value and the global minimum value for f on $\mathcal{D}$ if they exist. Explain your response.

The domain D is an ellipse. This might prevent us from using the method of parameterizing the boundary, unless we know the formula to do so (see next page for this method).

We can instead opt to use the Lagrange multiplier method.
Our constraint function g(x, y) is given by the inequality that defines D, except replacing "<=" with "=".
Strict inequality, "<", corresponds to the interior of the region D.
Equality, "=", corresponds to the boundary of D.

Set $g = \frac{x^2}{4} + y^2 - 1$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} \langle 2x - 1, 2 + 8y \rangle = \lambda \langle \frac{x}{2}, 2y \rangle \\ g = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x - 1 = \frac{1}{2}\lambda x & \Rightarrow (2 - \frac{1}{2}\lambda)x = 1 \Rightarrow x = \frac{2}{4-\lambda} = -2y \\ 8y + 2 = 2\lambda y & \Rightarrow (8 - 2\lambda)y = -2 \Rightarrow y = \frac{1}{\lambda - 4} \\ \frac{x^2}{4} + y^2 = 1 & \Rightarrow \frac{(-2y)^2}{4} + y^2 = 1 \end{cases}$$

$$\Rightarrow 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow \boxed{y = \pm\frac{1}{\sqrt{2}} \Rightarrow x = \mp\sqrt{2}}$$

Crit points on the boundary of D:

$f = x^2 - x + 2y + 4y^2$

$f(\sqrt{2}, -\frac{1}{\sqrt{2}}) = 2 - \sqrt{2} + (-\sqrt{2}) + 4(\frac{1}{2}) = 4 - 2\sqrt{2}$

$f(-\sqrt{2}, \frac{1}{\sqrt{2}}) = 2 + \sqrt{2} + \sqrt{2} + 4(\frac{1}{2}) = \boxed{4 + 2\sqrt{2} \leftarrow \text{global max value (on } \mathcal{D})}$

$f(\frac{1}{2}, -\frac{1}{4}) = \frac{1}{4} - \frac{1}{2} + (-\frac{1}{2}) + \frac{1}{4} = \frac{1}{2} - 1 = \boxed{-\frac{1}{2} \leftarrow \text{global min value}}$

We have found the min and max values that occur on the boundary of D.
Comparing with the local min in the interior of D, we see that -1/2 < 4 - 2sqrt(2), so the global min is -1/2.

Alternative approach: Parametrizing the boundary

$$\frac{x^2}{4}+y^2=1 \longrightarrow \left\{\left(2\cos\theta,\ \sin\theta\right) \,\middle|\, 0\le\theta\le 2\pi\right\}$$

$$f(x,y)=x^2-x+2y+4y^2$$

$$f(2\cos\theta,\sin\theta)=4\cos^2\theta-2\cos\theta+2\sin\theta+4\sin^2\theta$$

$$=4 \qquad +2\sin\theta-2\cos\theta$$

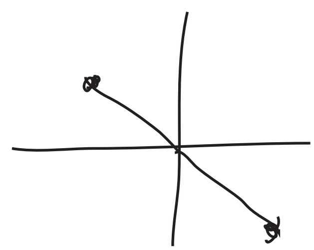
Rename $f(\theta)$

Objective: Find critical points on the boundary.

$$f'(\theta)=2\cos\theta+2\sin\theta=0$$

$$\Rightarrow \cos\theta=-\sin\theta$$

$$\Rightarrow \theta=\frac{3\pi}{4},\ \frac{7\pi}{4}$$



$$\left(2\cdot -\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2}\right) = \left(-\sqrt{2},\ \frac{1}{\sqrt{2}}\right)$$

$$\left(2\cdot \frac{\sqrt{2}}{2},\ -\frac{\sqrt{2}}{2}\right) = \left(\sqrt{2},\ -\frac{1}{\sqrt{2}}\right)$$

The rest (testing the boundary and interior points to determine global min/max) follows similarly to what is on the previous page.

2. (15 points) Use Lagrange multipliers to find the critical points of the function

$$f(x, y) = x - y$$

on the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

Identify the global maximum and minimum values of f on the ellipse.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} \langle 1, -1 \rangle = \lambda \langle \frac{1}{2}x, 2y \rangle \\ g = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 1 = \frac{1}{2}\lambda x & \Rightarrow \frac{x}{2} = \frac{1}{\lambda} \\ -1 = 2\lambda y & \Rightarrow y = -\frac{1}{2\lambda} \\ \frac{x^2}{4} + y^2 = 1 & \Rightarrow \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 1 \end{cases}$$

$$\frac{4}{\lambda^2} + \frac{1}{\lambda^2} = 4 \Rightarrow \frac{5}{\lambda^2} = 4 \Rightarrow \lambda^2 = \frac{5}{4}$$

$$\Rightarrow \lambda = \pm\frac{\sqrt{5}}{2}$$

$$x = \frac{2}{\lambda} = \pm\frac{4}{\sqrt{5}}$$

$$y = \mp\frac{1}{\sqrt{5}}$$

$$\boxed{\text{Crit points: } \left(\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right), \left(-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}$$

$$f\left(\frac{4}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \leftarrow \text{global max value}$$

$$f\left(-\frac{4}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = -\frac{4}{\sqrt{5}} - \frac{1}{\sqrt{5}} = \frac{-5}{\sqrt{5}} = -\sqrt{5} \leftarrow \text{global min value}$$

3. (15 points) Calculate the integral

$$\iiint_{\mathcal{B}} \pi^2 x^2 z \cos(\pi x y z)\ \mathrm{d}V$$

where $\mathcal{B} = [0,1] \times [0,2] \times [0,3]$.

$$\int_0^1 \int_0^2 \int_0^3 \pi^2 x^2 z \cos(\pi x y z)\, dz\, dy\, dx$$

Reorder the integral to simplify evaluation

$$= \pi^{\cancel{2}} \int_0^1 \int_0^3 \underbrace{\int_0^2 x^2 z \cos(\pi x y z)\, dy}_{=}\, dz\, dx$$

$$\frac{x^{\cancel{2}} \cancel{z}}{\cancel{\pi} \cancel{x} \cancel{z}} \sin(\pi x y z) \Big]_{y=0}^{2}$$

$$= x \sin(2\pi x z) - x \sin(0) \xrightarrow{} 0$$

Note, x and z are not necessarily integers, so cannot just say that sin(2 pi x z) = 0

$$= \cancel{\pi} \int_0^1 \underbrace{\int_0^3 x \sin(2\pi x z)\, dz}_{=}\, dx$$

$$\frac{-\cancel{x}}{2\cancel{\pi}\cancel{x}} \cos(2\pi x z) \Big]_{z=0}^{3}$$

$$= -\frac{1}{2} \cos(6\pi x) + \frac{1}{2} \cos(0) \xrightarrow{} 1$$

$$= \int_0^1 -\frac{1}{2} \cos(6\pi x) + \frac{1}{2}\, dx$$

$$= \frac{-1}{2 \cdot 6 \cdot \pi} \sin(6\pi x) + \frac{1}{2} x \Big]_0^1$$

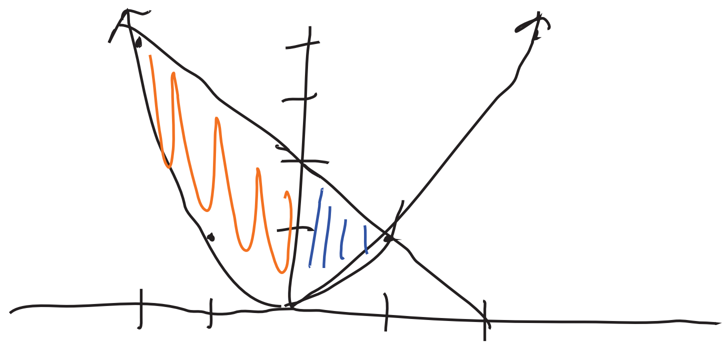
$$0 + \frac{1}{2} - (0 + 0)$$

$$= \boxed{\frac{1}{2}}$$

4. (15 points) Let $\mathcal{D}$ be the region bounded by $y = x^2$, the y-axis and $y = 2 - x$. Evaluate

$$\iint_{\mathcal{D}} 12x \; dA$$

The region D is ambiguous.



$$\int_0^1 \int_{x^2}^{2-x} 12x \, dy \, dx$$

$$= \int_0^1 12x(2 - x - x^2) \, dx$$

$$= 12 \int_0^1 2x - x^2 - x^3 \, dx$$

$$= 12 \cdot \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= 12 \cdot \left[1 - \frac{1}{3} - \frac{1}{4} - (0 - 0 - 0) \right]$$

$$= 12 \left(\frac{12}{12} - \frac{4}{12} - \frac{3}{12} \right)$$

$$= \boxed{5}$$

$$\int_{-2}^0 \int_{x^2}^{2-x} 12x \, dy \, dx$$

$$= \int_{-2}^0 12x(2 - x - x^2) \, dx$$

$$= 12 \int_{-2}^0 2x - x^2 - x^3 \, dx$$

$$= 12 \cdot \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_{-2}^0$$

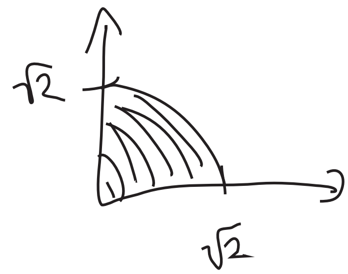
$$= 12 \left[0 - \left(4 + \frac{8}{3} - 4\right) \right]$$

$$= 12 \cdot -\frac{8}{3}$$

$$= \boxed{-32}$$

5. Consider the region $\mathcal{E}$ of points (x, y, z) satisfying

First quadrant $0 \le x,\ 0 \le y,\ x^2 + y^2 \le 2,\ 1 \le z \le 3$



(a) (10 points) Express the triple integral,

$$\iiint_{\mathcal{E}} \underbrace{3x}_{3r\cos\theta}\ \mathrm{d}V$$

as an iterated integral using cylindrical coordinates.

$$\int_1^3 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} 3r\cos\theta \cdot r\,dr\,d\theta\,dz$$

$0 \le \theta \le \frac{\pi}{2}$

$0 \le r \le \sqrt{2}$

(b) (5 points) Evaluate the integral.

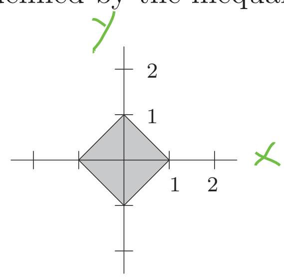
$$= 3\int_1^3 1\,dz \cdot \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta \cdot \int_0^{\sqrt{2}} r^2\,dr$$

$$= 3\cdot\left[x\right]_1^3 \cdot \sin\theta\Big]_0^{\frac{\pi}{2}} \cdot \frac{r^3}{3}\Big]_0^{\sqrt{2}}$$

$$= 3\cdot(3-1)\cdot(1-0)\cdot\left(\frac{\sqrt{8}}{3} - 0\right)$$

$$= 2\sqrt{8} = \boxed{4\sqrt{2}}$$

6. Let $\mathcal{R}$ be the square in the plane defined by the inequalities $|x+y| \leq 1$ and $|y-x| \leq 1$ which is illustrated below:

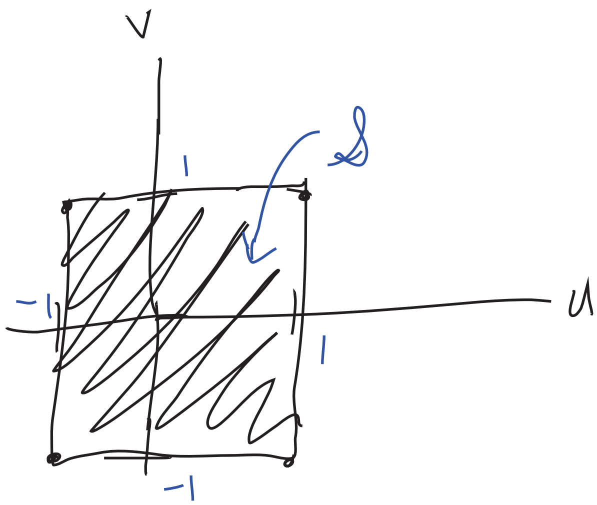


(a) (5 points) Describe the domain $\mathcal{S}$ which maps onto $\mathcal{R}$ by the transformation

$$T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right).$$

$$\begin{array}{cc} \shortparallel & \shortparallel \\ x & y \end{array}$$

$$\left.\begin{array}{l} x+y=u \\ x-y=v \end{array}\right\} T^{-1}(x,y)$$



$T^{-1}(1,0)=(1,1)$

$T^{-1}(0,1)=(1,-1)$

$T^{-1}(-1,0)=(-1,-1)$

$T^{-1}(0,-1)=(-1,1)$

A square of side length 2, centered at the origin.

(b) (5 points) Compute the Jacobian of T.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4}$$

$$= \boxed{-\frac{1}{2}}$$

(c) (10 points) Use the change of variables formula to compute the double integral

$$\iint_{\mathcal{R}} \cos\left(\frac{\pi}{2}(x+y)\right) \, \mathrm{d}A$$

$$= \iint_{\mathcal{S}} \cos\left(\frac{\pi}{2} u\right) \cdot |J| \, dA$$

$$= \int_{-1}^{1} \int_{-1}^{1} \frac{1}{2} \cos\left(\frac{\pi}{2} u\right) du \, dv$$

$$= \frac{1}{2} \int_{-1}^{1} 1 \, dv \cdot \int_{-1}^{1} \cos\left(\frac{\pi}{2} u\right) du$$

$$= \frac{1}{2} \cdot 2 \cdot \left[\frac{2}{\pi} \sin\left(\frac{\pi}{2} u\right)\right]_{-1}^{1}$$

$$= \frac{2}{\pi} \left[\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi} \left[1 - (-1)\right]$$

$$= \boxed{\frac{4}{\pi}}$$

Derivative formulas

$$\begin{aligned} \text{Directional derivative} &: D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}, \\ \text{Discriminant} &: D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2 \end{aligned}$$

Coordinate systems

Polar	Cylindrical	Spherical
$x = r\cos(\theta)$	$x = r\cos(\theta)$	$x = \rho\cos(\theta)\sin(\phi)$
$y = r\sin(\theta)$	$y = r\sin(\theta)$	$y = \rho\sin(\theta)\sin(\phi)$
	$z = z$	$z = \rho\cos(\phi)$
$r = \sqrt{x^2+y^2}$	$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{x^2+y^2+z^2}$
$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$	$\tan(\theta) = \dfrac{y}{x}$
	$z = z$	$\cot(\phi) = \dfrac{z}{\sqrt{x^2+y^2}}$
$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$

Change of variables

$$T : \mathcal{S} \to \mathcal{R}$$

$$G(u,v) = (x(u,v), y(u,v))$$

$$\mathrm{Jac}(T) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\int_{\mathcal{R}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathcal{S}} f(x(u,v), y(u,v))\;|\mathrm{Jac}(T)|\;\mathrm{d}u\,\mathrm{d}v$$