

Your name: Solutions

Rec. Instr.: _____

Rec. Time: _____

Show all your work in the space provided under each question. Please write neatly and present your answers in an organized way. You may use your one sheet of notes, but no books or calculators.

For each test of convergence that you use, either give the name of the test, or briefly describe what the test says.

This exam is worth 60 points. The chart below indicates how many points each problem is worth.

Problem	1	2	3	4
Points	/8	/10	/12	/10
Problem	5	6		Total
Points	/8	/12		/60

1. Determine whether the series converges or diverges. Explain.

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \quad \text{Use the Limit Comparison Test to}$$

compare to $\frac{n^2}{n^3} = \frac{1}{n}$. Note that the harmonic series

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-series test with $p=1$.

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{n^2}{n^3+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n^3}} = \frac{1}{1+0} = \boxed{1}.$$

Thus $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ also diverges.

2. Determine whether the series converges or diverges. Explain.

$$\sum_{n=0}^{\infty} \frac{\cos(n)}{3^n} \quad \text{Use the Absolute Convergence Test}$$

and the Comparison Test where $\left| \frac{\cos(n)}{3^n} \right| \leq \frac{1}{3^n}$.

Note that $\sum_{n=0}^{\infty} \frac{1}{3^n}$ is a convergent geometric series with ratio $r = \frac{1}{3} < 1$. Thus $\sum_{n=0}^{\infty} \left| \frac{\cos(n)}{3^n} \right|$ also

converges by the comparison test, and then

$\sum_{n=0}^{\infty} \frac{\cos(n)}{3^n}$ converges by the absolute convergence test.

3. Find the interval of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 2^n} \quad \text{Use the Ratio Test :}$$

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{n+1}}{(n+1) 2^{n+1}}}{\frac{(x-3)^n}{n \cdot 2^n}} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \left(\frac{2^n}{2^{n+1}} \right) \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right|$$

$$\rho = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \left(\frac{1}{2} \right) \cdot |x-3| = (1) \cdot \left(\frac{1}{2} \right) \cdot |x-3| = \boxed{\frac{|x-3|}{2}}$$

Thus $\rho < 1$ if $\frac{|x-3|}{2} < 1$, $|x-3| < 2$, $-2 < x-3 < 2$.

So the power series converges when $1 < x < 5$.

Check endpoints: If $x=5$, we have $\sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$

which diverges by the p-series test (the harmonic series diverges).

If $x=1$, we have $\sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

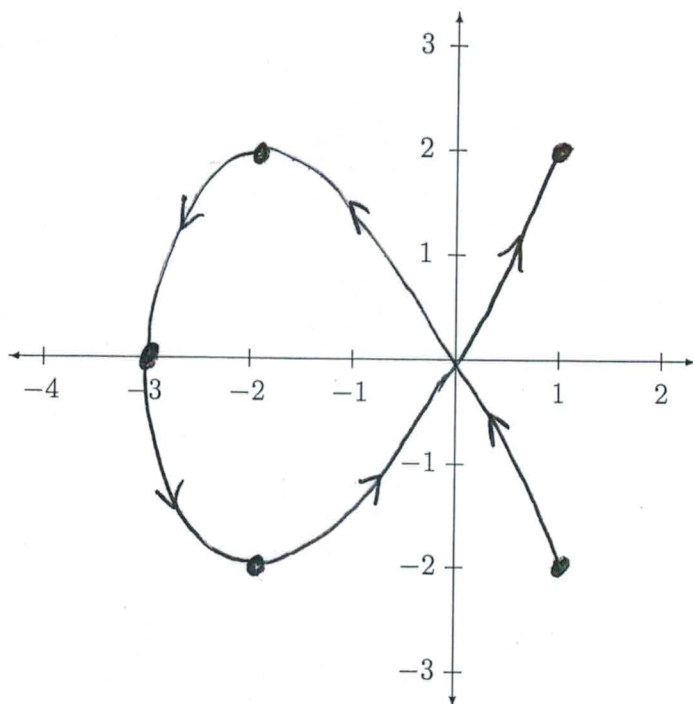
The alternating harmonic series converges by the Leibniz test (note $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$).

The interval of convergence is $\boxed{1 \leq x < 5}$.

4. A curve is given by the parametrization.

$$x = t^2 - 3 \quad \text{and} \quad y = t^3 - 3t$$

- (a) Graph the curve for $-2 \leq t \leq 2$ by plotting points when $t = -2, -1, 0, 1, 2$. Draw arrows indicating the direction of motion.



t	x	y
-2	1	-2
-1	-2	2
0	-3	0
1	-2	-2
2	1	2

- (b) Find the slope $\frac{dy}{dx}$ when $t = 2$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

When $t = 2$, the slope of the tangent line is $\frac{dy}{dx} = \frac{12 - 3}{4} = \boxed{\frac{9}{4}}$

5. Find the Maclaurin series (the Taylor series at $c = 0$) for the function. You need to find a formula for the general term. Hint: Do not take derivatives.

$$f(x) = \frac{x}{4+x^2} = \frac{x}{4} \left(\frac{1}{1 + \frac{x^2}{4}} \right) = \frac{x}{4} \left(\frac{1}{1 - \left(-\frac{x^2}{4} \right)} \right) =$$

$$\frac{x}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{4} \right)^n = \frac{x}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^{n+1}}}$$

Note we are using the formula for a geometric series $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

6. Use the integral test to determine whether the series converges or diverges. Note you must show that the function $f(x) = \frac{1}{x(\ln(x))^2}$ is decreasing.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \quad \int \frac{dx}{x(\ln(x))^2} = \int \frac{du}{u^2} = \frac{-1}{u} + C$$

Substitution $u = \ln(x)$
 $du = \frac{1}{x} dx$

$$= \boxed{\frac{-1}{\ln(x)} + C}$$

$$\int_2^{\infty} \frac{dx}{x(\ln(x))^2} = \lim_{R \rightarrow \infty} \left[\frac{-1}{\ln(x)} \right]_2^R = \lim_{R \rightarrow \infty} \left[\frac{-1}{\ln(R)} + \frac{1}{\ln(2)} \right]$$

$$= \frac{-1}{\infty} + \frac{1}{\ln(2)} = 0 + \frac{1}{\ln(2)} = \boxed{\frac{1}{\ln(2)}}$$

The improper integral converges,
 so the series converges.

If $f(x) = \frac{1}{x(\ln(x))^2}$ then $f'(x) = \frac{-((\ln(x))^2 + x(2)(\ln(x))(\frac{1}{x}))}{x^2(\ln(x))^4} < 0$

thus $f(x)$ is decreasing. for $x \geq 2$