1. (16 points) Computation

(a) Let $f(x,y) = (2x^2 + 3xy + y^2)\exp\bigl(x^3\bigr)$. Find all of the first partial derivatives. (In case you haven't seen it before, "$\exp(u)$" is the same thing as e^u.)

Solution:

$$\begin{aligned} f_x &= (4x + 3y)\exp\bigl(x^3\bigr) + (2x^2 + 3xy + y^2)\exp\bigl(x^3\bigr)(3x^2) \\ f_y &= \exp\bigl(x^3\bigr)(3x + 2y) \end{aligned}$$

(b) Let $g(x,y) = \dfrac{x^2}{\sqrt{2x^2 + y^2}}$. Find the first partial derivative with respect to x and simplify it.

Solution:

$$\begin{aligned} g_x &= \frac{\sqrt{2x^2 + y^2} \cdot 2x - x^2 \cdot \frac{1}{2\sqrt{2x^2+y^2}} \cdot 4x}{2x^2 + y^2} \\ &= \frac{(2x^2 + y^2)(2x) - 2x^3}{(2x^2 + y^2)^{3/2}} \\ &= \boxed{\frac{2x^3 + 2xy^2}{(2x^2 + y^2)^{3/2}}} \end{aligned}$$

2. (12 points) A certain differentiable function satisfies:

   (a) $f(2,5) = -7$, and $f(-1,4) = \pi$.

   (b) $\nabla f(2,5) = (-8,9)$, and $\nabla f(-1,4) = (\sqrt{6}, e^{-2})$.

   At each of the two points in question (i.e. at $(2,5)$ and at $(-1,4)$) answer the following questions:

   (a) In what direction is the function increasing the fastest and what is the rate of change in that direction?

   Solution:
   At $(2,5)$, the function is increasing the fastest in the direction $\nabla f(2,5) = \langle -8, 9\rangle$, with rate of change $\| \langle -8,9\rangle \| = \sqrt{64+81} = \sqrt{145}$.
   At $(-1,4)$, the function is increasing the fastest in the direction $\nabla f(-1,4) = \left\langle \sqrt{6}, e^{-2} \right\rangle$, with rate of change $\| \left\langle \sqrt{6}, e^{-2} \right\rangle \| = \sqrt{6+e^{-4}}$.

   (b) What is the directional derivative in the direction of the vector $\langle 6, -8\rangle$?

   Solution: The unit vector in the direction $\langle 6,-8\rangle$ is $\left\langle 3/5, -4/5 \right\rangle$. So

$$\begin{aligned} D_{\langle 3/5,-4/5\rangle} f(2,5) &= \nabla f(2,5) \cdot \langle 3, -4\rangle \cdot \frac{1}{5} \\ &= \langle -8, 9\rangle \cdot \langle 3,-4\rangle \cdot \frac{1}{5} \\ &= (-24-36)\cdot\frac{1}{5} = \boxed{-12} \end{aligned}$$

   and

$$\begin{aligned} D_{\langle 3/5,-4/5\rangle} f(-1,4) &= \nabla f(-1,4) \cdot \langle 3, -4\rangle \cdot \frac{1}{5} \\ &= \left\langle \sqrt{6}, e^{-2}\right\rangle \cdot \langle 3,-4\rangle \cdot \frac{1}{5} \\ &= \boxed{(3\sqrt{6} - 4e^{-2})\cdot\frac{1}{5}} \end{aligned}$$

   (c) What is the tangent plane and/or the linear approximation at each of the two points?

   Solution:

$$\begin{aligned} (2,5): &\quad z = -7 + -8(x-2) + 9(y-5) \\ (-1,4): &\quad z = \pi + \sqrt{6}(x+1) + e^{-2}(y-4) \end{aligned}$$

3. (12 points) Set up **but do not solve** the following problems. As part of setting these problems up, you should list the unknowns and the equations that you would need to use to find them. You **should also do** all of the **derivative** calculations, but the **algebra** is totally unmanageable, so do **not** attempt it!

(a) Maximize $f(x, y) = x^2 \cos(2y)$
Subject to $g(x, y) = x^4 + y^6 = 2$.

Solution:

$$\nabla f = \lambda \nabla g$$
$$\big\langle 2x\cos(2y), -2x^2\sin(2y)\big\rangle = \lambda\left\langle 4x^3, 6y^5\right\rangle$$

The system to solve is:

$$\begin{cases} 2x\cos(2y) = 4\lambda x^3 \\ -2x^2\sin(2y) = 6\lambda y^5 \\ x^4 + y^6 = 2 \end{cases}$$

(b) Maximize $F(x, y, z) = \cos\big(xy^2 + yz^2 + zx^2\big)$
Subject to $G(x, y, z) = 2x + 3y + 4z = 0$
and $H(x, y, z) = x^4 + z^4 = 625$.

Solution:

$$\begin{aligned} \nabla F &= -\sin\big(xy^2 + yz^2 + zx^2\big)\left\langle (y^2 + 2xz), (2xy + z^2), (2yz + x^2)\right\rangle \\ \nabla G &= \langle 2, 3, 4\rangle \\ \nabla H &= \left\langle 4x^3, 0, 4z^3\right\rangle \end{aligned}$$

Setting $\nabla F = \lambda \nabla G + \mu \nabla H$ gives the system to solve:

$$\begin{cases} -\sin\big(xy^2 + yz^2 + zx^2\big)(y^2 + 2xz) = 2\lambda + 4\mu x^3 \\ -\sin\big(xy^2 + yz^2 + zx^2\big)(2xy + z^2) = 3\lambda \\ -\sin\big(xy^2 + yz^2 + zx^2\big)(2yz + x^2) = 4\lambda + 4\mu z^3 \\ 2x + 3y + 4z = 0 \\ x^4 + z^4 = 625 \end{cases}$$

4. (14 points) For the function $f(x, y) = 4x^2 - 2xy - y^3$ find and classify all of the critical points.

Solution:

$$\nabla f = 0 \implies \begin{cases} 8x - 2y = 0 \\ -2x - 3y^2 = 0 \end{cases}$$

Solving this system gives two critical points: $(0, 0)$ and $(-\frac{1}{24}, -\frac{1}{6})$. To classify them, the discriminant is

$$f_{xx}f_{yy} - (f_{xy})^2 = 8(-6y) - (-2)^2 = -48y - 4$$

For $(0, 0)$, the discriminant gives -4, so $(0, 0)$ is a saddle point.

For $(-\frac{1}{24}, -\frac{1}{6})$, the discriminant gives 4, and $f_{xx} = 8 > 0$, so $(-\frac{1}{24}, -\frac{1}{6})$ is a local minimum.

5. (20 points) Find the maximum and the minimum of the function

$$f(x, y) = x^2 + 2x + y^2 + 6y$$

in the region given by

$$g(x, y) = x^2 + y^2 \leq 40.$$

Show your work carefully in this problem, and let us know what you are doing.

Solution: Use Lagrange multipliers with $g(x, y) = x^2 + y^2 = k$, for $k \in [0, 40]$.

$$\begin{aligned} \nabla f &= \langle 2x + 2, 2y + 6 \rangle \\ \nabla g &= \langle 2x, 2y \rangle \end{aligned}$$

so we solve the system of equations

$$\begin{cases} 2x + 2 = 2\lambda x \\ 2y + 6 = 2\lambda y \\ x^2 + y^2 = k \end{cases}$$

Solving the first two equations for λ yields

$$x = \frac{1}{\lambda - 1} \qquad y = \frac{3}{\lambda - 1}$$

Substituting into the third gives

$$\frac{10}{(\lambda - 1)^2} = k \implies \frac{1}{\lambda - 1} = \pm\sqrt{\frac{k}{10}}$$

so for a fixed k, the optimization problem yields two solutions:

$$(x, y) = \left(\pm\sqrt{\frac{k}{10}}, \pm 3\sqrt{\frac{k}{10}}\right)$$

To determine max/min values, we plug these points into f:

$$\begin{aligned} f &= x^2 + y^2 + 2x + 6y \\ &= k + 2\left(\pm\sqrt{\frac{k}{10}}\right) + 6\left(\pm 3\sqrt{\frac{k}{10}}\right) \\ &= k \pm 20\sqrt{\frac{k}{10}} \\ &= k \pm 2\sqrt{10}\sqrt{k} \end{aligned}$$

We want to optimize f with respect to $k \in [0, 40]$, which requires us to evaluate k at critical points and at end points. Note there are two cases for f.

- If $f = k + 2\sqrt{10}\sqrt{k}$, then $f' = 1 + \frac{\sqrt{10}}{\sqrt{k}} = 0$ which yields no critical points.
  - Evaluating at $k = 0$, $f = 0$
  - Evaluating at $k = 40$, $f = 40 + 2\sqrt{400} = 80$
- If $f = k - 2\sqrt{10}\sqrt{k}$, then $f' = 1 - \frac{\sqrt{10}}{\sqrt{k}} = 0 \implies k = 10$ is a critical point.
  - Evaluating at $k = 0$, $f = 0$
  - Evaluating at $k = 10$, $f = 10 - 2\sqrt{100} = -10$
  - Evaluating at $k = 40$, $f = 40 - 2\sqrt{400} = 0$

Thus the maximum for this function in the specified region is 80 (which occurs at $(2, 6)$) and the minimum is -10 (which occurs at $(-1, -3)$).

6. (8 points) Suppose that $x = r\cos\theta$ and $y = r\sin\theta$ (the usual polar coordinates) and $f(x, y) = x^2y^2$. Express

$$\frac{\partial f}{\partial r} \quad \text{and} \quad \frac{\partial f}{\partial \theta}$$

as functions of r and θ. (Hint/Comment: Do this however you like.)

Solution:

$$\begin{aligned}
f &= r^4\cos^2\theta\sin^2\theta \\
f_r &= 4r^3\cos^2\theta\sin^2\theta \\
f_\theta &= r^4\left(2\cos\theta(-\sin\theta)\sin^2\theta + \cos^2\theta\cdot 2\sin\theta\cos\theta\right) \\
&= r^4(-2\cos\theta\sin^3\theta + 2\sin\theta\cos^3\theta)
\end{aligned}$$

7. (18 points) Short answers ...

(a) If f is a function of x and y, and x and y are each functions of r, s, and t, then use the chain rule to express $\dfrac{\partial f}{\partial t}$.

Solution:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

(b) Find the average value of the function $f(x, y) = xy^2$ on the rectangle $0 \le x \le 4$, $0 \le y \le 3$.

Solution:

$$\frac{1}{12}\int_0^3 \int_0^4 xy^2 \,\mathrm{d}x\,\mathrm{d}y = \frac{1}{12}\left[\frac{y^3}{3}\right]_0^3 \left[\frac{x^2}{2}\right]_0^4 = \frac{1}{12}(9)(8) = 6$$

(c) According to the theorem that we learned, what should you require of a set S to guarantee that any continuous function f will attain an absolute maximum and an absolute minimum on S?

Solution: The set S must be closed and bounded.

(d) For the set $5x^2 + 2y^3 + 2z^6 - 3xy^2z^2 = 3$ write down the tangent plane at the point $(-1, -2, 1)$.

Solution: Letting $f = 5x^2 + 2y^3 + 2z^6 - 3xy^2z^2 - 3$,

$$\nabla f = \left\langle 10x - 3y^2z^2, 6y^2 - 6xyz^2, 12z^5 - 6xy^2z \right\rangle$$

$$\nabla f(-1, -2, 1) = \langle -22, 12, 36 \rangle$$

$$\boxed{0 = -22(x + 1) + 12(y + 2) + 36(z - 1)}$$