Geometry of Manifolds
Fall 2001 Qualifying Exam
Auckly & Miller

Notes: Work as many of the following problems as you can. Good luck.

1. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (x + z^2, y - z^2) = (u, v)$ where (x, y, z) are coordinates on $\mathbb{R}^3$ and (u, v) are coordinates on $\mathbb{R}^2$. Let $\alpha = du + u\,dv$. Find

a) $f^*(\alpha)$

b) $f^*(d\alpha)$

c) $\int_S f^*(d\alpha)$ where S is the surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$ oriented with the upward pointing normal.

2. a) Construct an example of a covering projection, $P : E \to X$ with $\mathrm{Deck}(E) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

b) Compute $H^\infty(X : \mathbb{R})$ for your example.

3. On $\mathbb{R}^3$ let $X = xz^2\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$.

a) Calculate $\displaystyle\int_B L_X(dx \wedge dy \wedge dz)$ where $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$ with orientation given by $dx \wedge dy \wedge dz$.

b) Compute the flow of X, $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$.

4. Consider a 2-dimensional Riemannian manifold (M, g) where M is an open subset of $\mathbb{R}^2 = \{(x, y) | x \in \mathbb{R}, y \in \mathbb{R}\}$. Suppose g is such that $e_1 = \dfrac{\partial}{\partial x}$ and $e_2 = f(x, y)\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$ is an orthonormal frame.

a) Find the coframe (θ^1, θ^2) dual to (e_1, e_2).

b) Find a 1-form w so that $d\theta^1 = w \wedge \theta^2$ and $d\theta^2 = -w \wedge \theta^1$.

c) Find a function $K(x, y)$ so that $dw = -K\theta^1 \wedge \theta^2$.

d) Show how to choose $f(x, y)$ so that $K(x, y) = -1$ for all (x, y).

5. Let K be a simplicial complex.

a) Define the Euler characteristic of K.

b) Let $p : L \to K$ be an n-fold simplicial cover. Show that $\chi(L) = n\chi(K)$.

c) Determine all **closed** 2-manifolds that cover T^2.

d) Give an example of a non-trivial cover $q : T^2 \to T^2$.

e) Compute $\chi(S^n)$ as a function of n.

f) Let G be a finite group that acts freely on S^{2k}. Assume that the map $\pi : S^{2k} \to S^{2k}/G$ is a covering projection. Prove that $G \cong \mathbb{Z}_2$ or G is trivial.

6. a) Given that X is a non-vanishing vector field on M, prove that there is a diffeomorphism, $f : M \to M$ without fixed points (i.e., x such that $f(x) = x$) that is homotopic to $\mathrm{id} : M \to M$.

b) Given that $f : S^n \to S^n$ has no fixed points, show that f is homotopic to $p : S^n \to S^n$; $p(x) = -x$.

c) Compute $H^n(p) : H^n(S^n, \mathbb{R}) \to H^n(S^n, \mathbb{R})$.

d) What can be said about the dimension of a sphere that admits a non-vanishing vector field.

e) Let G be a Lie group that acts freely (i.e., $x \cdot g = x$ implies $g = 1$) on S^{2k}. Prove that $\dim G = 0$.

7 Let G be a Lie group.

a) Define the Lie algebra of G, say $\mathfrak{g}$.

b) Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Explain how to construct a distribution on G using $\mathfrak{h}$ and prove that this distribution is integrable.

c) Let H be the maximal connected leaf of the distribution from part b) passing through 1. Construct an atlas on H.

d) Prove that H is closed under multiplication.
Hint: Consider $L_g H := \{gh | h \in H\}$.