Real Analysis Qualifying Exam
Fall 1997

Unless otherwise stated, $(X, \mathcal{A}, \mu)$ is an arbitrary measure space.

1. Let $E \subset \mathbb{R}$ be a Borel set. Prove that

$$E' = \{(x, y) \in \mathbb{R}^2 : x + y \in E\}$$

is a Borel set.

2. Let f be a measurable function on X and $p > 0$. Prove

$$\int |f|^p d\mu = p \int_0^\infty t^{p-1} \mu(\{|f| > t\}) dt.$$

3. Suppose that f_n, $f \in L^1(\mu)$ and $\| f - f_n \|_1 \to 0$. Prove

$$\limsup \int \log |f_n| d\mu \leq \int \limsup \log |f_n| d\mu,$$

where $\log x = -\infty$ for $x = 0$.

4. Suppose that μ is a positive finite Borel measure on a Hausdorff space X, and for each open set $V \subset X$,

$$\mu(V) = \sup\{\mu(K) : K \text{ is compact }, K \subset V\}.$$

Prove: Given a Borel set $E \subset X$ and $\varepsilon > 0$, there exist a compact set K and an open set V such that

$$K \subset E \subset V \text{ and } \mu(V \backslash K) < \varepsilon.$$

5. If f is complex function on $[a, b]$, its total variation $F : [a, b] \to [0, \infty]$ is defined by

$$F(x) = \sup \left\{ \sum_{j=1}^N |f(t_j) - f(t_{j-1})| : a = t_0 < t_1 < \cdots < t_N = x \right\}.$$

Given an example of a continuous function f on $[a, b]$ for which $F(x) = \infty$ for any $x \in (a, b]$.

6. Let $f_n(t) = e^{i(n+\frac{1}{2})t}, t \in [0, 2\pi]$. Is it true that $\{f_n\}_{n \in \mathbb{Z}}$ is a complete orthonormal system in $L^2[0, 2\pi]$?

(You may use the fact that the functions e_n, $e_n(t) = e^{int}$, $n \in \mathbb{Z}$, form a complete orthonormal system in $L^2[0, 2\pi]$.)

7. Construct a nonzero positive Borel measure μ on $[0, 1]$ which is singular with respect to Lebesgue measure and such that $\mu(\{t\}) = 0$ for any $t \in [0, 1]$.

8. Let $\{f_n\}$ be a sequence of functions in $L^1[0, 1]$ such that $f_n(t) \to 0$ for any $t \in [0, 1]$. Is it true that $\int_0^1 f_n(t) dt \to 0$?