TOPOLOGY QUALIFYING EXAM Spring 1998

(Strecker and Wu)

Choose and work any **6** of the following problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems.** Below a "space" always means a "topological space".

1. Let $(X, \leq)$ be a linearly ordered set and let τ be the order topology on X determined by $\leq$. Let Y be a subset of X. Then there is a "natural" topology σ on Y that is induced by the inherited linear ordering on Y and another "natural" topology ρ on Y that is the subspace topology induced by τ. Then prove or disprove that $\sigma = \rho$.

2. Prove or disprove each of the following:

i) The intersection of any family of topologies on a set X is a topology on X.

ii) The union of two topologies on a set X is a topology on X.

3. Prove that every second countable space is Lindelöf. (If you used the Axiom of Choice, show precisely where it is used. If you do not use the Axiom of Choice, state so.)

4. Prove or disprove each of the following:

i) $\mathbb{R}$ in the standard topology cannot be written as a countable union of open sets;

ii) $\mathbb{R}$ in the standard topology cannot be written as a countable union of closed sets;

iii) $\mathbb{R}$ in the standard topology cannot be written as a countable union of sets with empty interiors;

iv) $\mathbb{R}$ in the standard topology cannot be written as a countable union of closed sets with empty interiors.

5. Prove or disprove each of the following:

i) If A is a connected subset of a space X and $Q \subseteq X$ such that $A \subseteq Q \subseteq \overline{A}$, then Q must be connected.

ii) Each component of a space must be closed.

iii) Each component of a space must be open.

6. Let ρ_1, ρ_2 be two metrics on the set X such that both metric spaces (X, ρ_1) and (X, ρ_2) are complete. If X is totally bounded with respect to the metric ρ_1, is it also totally bounded with respect to the metric ρ_2? Prove you assertion.

7.

i) State the Tychonoff Product Theorem (TPT)

ii) If AXC is the Axiom of Choice, show either $AXC \Rightarrow TPT$ or $TPT \Rightarrow AXC$.

8. Prove or disprove that the product of two normal spaces is normal.

9. Let X be a normal T_1 space and $W = \cup_{n \in \mathbb{Z}} C_n \subseteq X$, where each C_n is closed and W is open in X. Show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for all $x \in W$ and $f(x) = 0$ for all $x \notin W$.

10. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let τ be the product topology on $X_1 \times X_2$. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. There is a "natural" subspace topology σ on $Y_1 \times Y_2$ that is induced from τ and the fact that $Y_1 \times Y_2$ is a subset of $X_1 \times X_2$. There is also a "natural" product topology ρ on $Y_1 \times Y_2$ that comes from the subspace topologies (Y_1, ρ_1) and (Y_2, ρ_2) induced by τ_1 and τ_2 respectively. Then prove or disprove that $\sigma = \rho$.

11. Let X be a Hausdorff space. Suppose there exists a countable open covering $\{U_n\}_{n=1}^{\infty}$ of X such that for each n, $\overline{U_n}$ is compact and $\overline{U_n} \subseteq U_{n+1}$. Show that X is paracompact.

12. Prove or disprove that the contimuous image of a compact space is compact.