REAL ANALYSIS
Qualifying Exam
February 21, 1996

Do eight problems. In 1-5, $(X, \mathcal{M}, \mu)$ is a measure space.

1. Let μ be σ-finite, and let $\{E_\alpha : \alpha \in A\}$ be a disjoint collection of measurable sets in X. Prove that $\mu(E_\alpha) > 0$ for at most countably many $\alpha \in A$.

2. Let $1 \leq p < \infty$. Prove the completeness of $L^p(\mu)$.

3. Let $\{w_n\}_1^\infty$ be an orthonormal set in $L^2(\mu)$ such that

$$C = \sup\{\|w_n\|_\infty : n \geq 1\} < \infty.$$

Prove that for each $f \in L^1(\mu)$, $\int f w_n d\mu \to 0$ as $n \to \infty$.

Hint: Approximate f by a simple measurable function s, then apply Bessel's inequality to s.

4. Let $f : X \to [0, \infty]$ be a measurable function such that $\int_E f d\mu < \infty$ for each measurable set E with $\mu(E) < \infty$. Prove that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\int_E f d\mu < \epsilon$ for every measurable set E with $\mu(E) < \delta$.

5. Let $f : X \to [0, \infty]$ be measurable and $0 < p < \infty$. Prove

$$\int f^p d\mu = p \int_0^\infty t^{p-1} \mu(\{f > t\}) dt.$$

6. Let μ be a finite (positive) Borel measure on a Hausdorff space X such that

$$\mu(V) = \sup\{\mu(K) : K \quad \text{is compact and} \quad K \subset V\}$$

for each open set $V \subset X$. Prove:

(a) Given a Borel set $E \subset X$ and $\epsilon > 0$, there exists a closed set F and an open set V such that

$$F \subset E \subset V \quad \text{and} \quad \mu(V \setminus F) < \epsilon.$$

(b) The closed set F in (a) can be chosen to be compact.

6. Let $f : X \times [0,1] \to \mathbb{C}$. State (nontrivial) conditions on f that guarantee

$$(*) \qquad \frac{d}{dt}\int f(x,t)d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)d\mu(x) \qquad \forall t \in (0,1)$$

and then prove $(*)$.

7. Let f and g be Borel functions on $\mathbb{R}^+$ such that

$$\int_0^\infty (|f(x)| + |g(x)|)\frac{dx}{x} < \infty.$$

Define

$$(f * g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}$$

for $x > 0$ whenever the integral in the right-hand side exists.

Prove: (i) $|f * g| < \infty$ Lebesgue-almost everywhere, (ii) $f * g$ is Borel measurable, and

$$(iii) \qquad \int_0^\infty |(f * g)(x)|\frac{dx}{x} \leq \int_0^\infty |f(x)|\frac{dx}{x} \cdot \int_0^\infty |g(y)|\frac{dy}{y}.$$

8. Let f be a 2π-periodic differentiable function on $\mathbb{R}$ with $\int_0^{2\pi} |f'(t)|^2 dt < \infty$. Prove that

$$\sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2 < \infty.$$

7. Suppose that f is a continuous function on $D := \{z \in \mathbb{C} : |z| < 1\}$, $g \in L^1([0, 2\pi])$, and $\|f_r - g\|_1 \to 0$ as $r \to 1$, where $f_r(t) = f(re^{it})$ for $0 \le r < 1$ and $0 \le t \le 2\pi$. Prove

$$\limsup_{r \to 1} \int_0^{2\pi} \log|f_r| dt \le \int_0^{2\pi} \log|g| dt,$$

where $\log x = -\infty$ for $x = 0$.

Hint: $x - \log x > 0$ for $0 \le x < \infty$.

8. Consider $F(x) := \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1} dt$, $x \in \mathbb{R}$.

(a) Justify differentiating F "behind the integral".

(b) Use (a) to show that the function

$$G(x) := \left(\int_0^x e^{-t^2} dt\right)^2 + F(x), \quad x \in \mathbb{R},$$

is constant. What is that constant?

(c) By letting $x \to +\infty$ in (b) deduce the value of $\int_0^\infty e^{-t^2} dt$.

9. True or false: Lebesgue measure λ in $\mathbb{R}^n$ has the property that $\lambda(\overline{U} \setminus U) = 0$ for every open set U.

10. X is a compact metric space, μ a positive finite Borel measure on X which annihilates every countable set, S_n are Borel and diameter $(S_n) \to 0$ as $n \to \infty$. Prove that $\mu(S_n) \to 0$ as $n \to \infty$.