Algebra Qualifying Exam
September 14, 1995

Instructions: You are given 10 problems from which you are to do 8. Please indicate 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Your reasoning and proof should be literally clear. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative. $\mathbb{Q}$ and $\mathbb{C}$ are the sets of rational and complex numbers respectively.

1. Let G be a group and H a subgroup of G. Then G acts on the set, G/H, of all left cosets of H in G and this action defines a group homomorphism $\phi : G \to \mathrm{Sym}(G/H)$. Here $\mathrm{Sym}(G/H)$ is the group of all permutations on the set G/H. Show that $\ker(\phi)$ is the largest subgroup of H that is normal in G.

2. Let G be a group of order 385. Show that every element of order 7 in G is in the center of the group G.

3. Let R be a ring with identity $1 \neq 0$. Suppose that R has a central element $e \neq 0, 1$ such that $e^2 = e$. Show that there exist two rings R_1 and R_2 both with identity $1 \neq 0$ such that $R \cong R_1 \times R_2$.

4. Let $f : R \to S$ be a homomorphism of commutative rings R and S.

   (a). Show that $f^{-1}(P)$ is a prime ideal of R if P is a prime ideal of S.

   (b). Is it possible to change the word "prime" into "maximal" in the above statement? Prove or give a counter example.

5. Let R be a commutative ring and N be the set of all nilpotent elements in R. (An element x in a ring is called nilpotent if $x^n = 0$ for some nonnegative integer n.)

   (a). Show that N is an ideal of R.

   (b). Is the statement (a) still correct without the commutativity condition on R? Prove or give an example.

6. Let R be a ring and M, N be left R-modules. Show that if N is a free R-module, then for any onto R-module homomorphism $\phi : M \to N$ there exists an R-module homomorphism $\psi : N \to M$ such that $\phi \circ \psi = \mathrm{Id}_N$ and then prove that $M = \ker(\phi) \oplus \psi(N)$.

7. Let R be a commutative ring and M be a free R-module of rank n. Show that for any idea I of R and any R-module homomorphism $\phi : M \to M$ such that $\phi(M) \subseteq IM$, then there exists $a_0, a_1, \dots, a_{n-1} \in I$ such that $\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_1\phi + a_0\mathrm{Id} = 0$ as a homomorphism $M \to M$.

   Can you prove the above statement for M being just finitely generated? If yes, explain briefly how to prove.

8. Let $\mathrm{GL}_n(\mathbb{C})$ be the group of all $n \times n$ invertible matrices with entries in complex numbers $\mathbb{C}$. Show that every element of finite order in $\mathrm{GL}_n(\mathbb{C})$ is conjugate to a diagonal matrix in $\mathrm{GL}_n(\mathbb{C})$

9. Let $n > 1$ be a positive integer. Calculate the degree of the splitting field of $f(x) = x^n - 2$ over the field of rational numbers $\mathbb{Q}$.

10. Let $\mathbb{F}$ be a finite field of characteristic p. Show that the multiplicative group $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ is a cyclic group.