REAL ANALYSIS QUALIFYING EXAM
Fall 2001

Answer as many as possible. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. (a) Does $\int_0^\infty \frac{\sin x}{x} dx$ exist as an (improper) Riemann integral? Prove your answer.

(b) Does $\int_0^\infty \frac{\sin x}{x} dx$ exist as a Lebesgue integral? Prove your answer.

2. Suppose λ denotes Lebesgue measure on $\mathbb{R}$ and A is a Borel set with $\lambda(A) > 0$. Show that for every $0 < r < 1$, there is a bounded open interval I with $\lambda(A \cap I) > r\lambda(I)$.

3. Let μ be a complex measure on $\mathcal{M}$. Show that there exists a set $A \subset \mathcal{M}$ such that

(i) $B \subset A$ implies $\mathrm{Re}(\mu(B)) \geq \mathrm{Im}(\mu(B))$

(ii) $B \subset A^c$ implies $\mathrm{Im}(\mu(B)) \geq \mathrm{Re}(\mu(B))$.

4. Prove that $(L^\infty([0,1]))^* \neq L^1([0,1])$. Hint: One way to prove this is to show there is a bounded linear functional $\Lambda \neq 0$ such that $\Lambda|_{C([0,1])} \equiv 0$. (Here, $C([0,1])$ denotes continuous functions on the interval $[0,1]$.) Of course, you should explain why the existence of this functional proves that $(L^\infty([0,1]))^* \neq L^1([0,1])$.

5. Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space, μ positive, and $f : X \to [0, \infty)$ an $\mathcal{M}$ measurable function. Suppose $G : [0,\infty) \to [0,\infty)$ is increasing and absolutely continuous, $G(0) = 0$.

Prove that
$$\int_X G(f(x)) d\mu = \int_0^\infty G'(t)\, \mu(\{x : f(x) > t\}) dt.$$

6. Suppose ν, μ are positive measures with ν finite. Show that the following two statements are equivalent.

(a) $\nu << \mu$

(b) For every $\varepsilon > 0$, there exists a $\delta > 0$ that $\nu(B) < \varepsilon$ whenever $B \in \mathcal{M}$ and $\mu(B) < \delta$.

7. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions on X such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu < \infty$.

Prove that $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$ for every measurable set $E \subseteq X$.

8. Suppose f is a complex measurable function on X, μ is a positive measure on X and $\varphi(p) = \int_X |f|^p d\mu$, $0 < p < \infty$. Let $E = \{p : \varphi(p) < \infty\}$.

Show that if $r < p < s$, $r \in E$, $s \in E$ then $p \in E$. Show that the function $\log \varphi$ is convex on the interior of E.