Algebra Qualifying Exam
August 23, 2001

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.
Note: All rings on this exam are associative and have multiplicative identity 1. All integral domains are assumed to be commutative.

1. Suppose that G is a group and that $N_1, N_2 \trianglelefteq G$ are two normal subgroups with $G/N_1 \cong G/N_2$. Must it follow that $N_1 \cong N_2$? Prove or supply a counterexample.
2. Let G be a group, and let H be a cyclic normal subgroup of G. Prove that the commutator subgroup G' is a subgroup of the centralizer $C_G(H) = \{g \in G \mid gh = hg, \text{ for all } h \in H\}$.
3. Let G be a finite group and let P be a p-Sylow subgroup of G. If for all $x \in G$ we have $P \cap P^x = 1$ or P, show that $|\mathrm{Syl}_p(G)| \equiv 1 \pmod{|P|}$.
4. $\mathbb{K}$ be a field and let $\mathcal{O} \subseteq \mathbb{K}$ be a subring. Assume that for all $0 \neq \alpha \in \mathbb{K}$, we have either $\alpha \in \mathcal{O}$ or $\alpha^{-1} \in \mathcal{O}$. Prove that the set of nonunits of $\mathcal{O}$ forms an ideal of $\mathcal{O}$.
5. Let $\mathbb{F}$ be a field and let R be the ring of all 2×2 matrices over $\mathbb{F}$. Prove that, up to isomorphism, R has only one irreducible left R-module M, viz.,
$$M = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{F} \right\},$$
acted on by ordinary matrix multiplication.
6. Let $\mathbb{F}$ be a field, let V be a finite-dimensional $\mathbb{F}$-vector space, and let $T : V \to V$ be a linear transformation. If the minimal polynomial $m_T(x)$ splits into *linear* factors, prove that there exists an ordered basis of V relative to which T is represented by an upper triangular matrix.
7. Let V be an n-dimensional vector space and let V^* be the dual space. Let $\{f_1, f_2, \ldots f_r\}$ be linearly independent functionals in V^*, and let $W := \cap_{i=1}^r \mathrm{ann}(f_i)$, where $\mathrm{ann}(f_i)$ is the subspace of V annihilated by f_i. Prove that $\dim W = n - r$.
8. Let $\mathbb{F} \subseteq \mathbb{E} \subseteq \mathbb{K}$ be fields, let $\alpha \in \mathbb{K}$, and let $f(x)$ be the minimal polynomial of α over $\mathbb{F}$. Assume that $[\mathbb{E} : \mathbb{F}]$ and $\deg f(x)$ are relatively prime. Prove that $f(x)$ is also the minimal polynomial of α over $\mathbb{E}$.
9. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree at least 2, and let $\alpha \in \mathbb{C}$ be a root of $f(x)$. Give an example to show that the Galois group of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ can be trivial.
10. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial and let $\mathbb{K} \subseteq \mathbb{C}$ be the splitting field of $f(x)$. Show that $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ must act transitively on the roots of $f(x)$.