Algebra Qualifying Exam
Fall 1990

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a group and let N_1, N_2 be normal subgroups of G with $N_1 \cong N_2$. Prove, or give a counterexample to the assertion

$$G/N_1 \cong G/N_2.$$

2. Let $G = GL_2(p)$, p prime, be the group of invertible 2×2 matrices over the field F_p. Using the fact that $|G| = p(p-1)(p^2-1)$, compute the number of Sylow p-subgroups of G.

3. Let V a vector space and let $S \subseteq V$ be a finite subset which generates V. Prove that S contains a basis of V.

4. Let $\mathbb{C}$ be the field of complex numbers, and let $A \in M_3(\mathbb{C})$ be the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -1 \\ -3 & -3 & -2 \end{bmatrix}.$$

(a) Compute the invariant factors of A.

(b) Compute the Jordan canocical form of A.

5. Let F_q be the finite field of q elements, and let K be an extension of F_q, of degree n.

(a) Prove that the map $\tau_q : K \to K, \tau_q(x) = x^q$ is an automorphism of K, and that F_q is precisely the subfield of fixed elements of τ_q.

(b) Compute $Gal(K/F_q)$.

6. Consider the matrix of rational entries

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

If $R = \mathbb{Q}[A]$ is the ring of polynomials in A with rational coefficients, prove that R is a field. (Hint: Consider the homomorphism $\mathbb{Q}[x] \to \mathbb{Q}[A]$, where $x \mapsto A$.)

7. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

(a) Construct a splitting field $K \supseteq \mathbb{Q}$ for $f(x)$ over $\mathbb{Q}$.

(b) Find an element $\alpha \in K, \alpha \notin \mathbb{Q}$, such that $\mathbb{Q}[\alpha]$ is a normal extension of $\mathbb{Q}$.

8. Let R be a principal ideal domain, and let $0 \neq I \subset R$ be a prime ideal. Prove that I is a maximal ideal.

9. Let D be an integral domain.

(a) State what it means for D to be a Euclidean domain.

(b) Prove that a Euclidean domain is a principal ideal domain.