PDE Qualifying Exam - Spring 2005 (Old System)

1. Let $g_n : B_R(0) \to I\!R$, where $B_R(0) \subset I\!R^n$ is the ball of radius R centered at 0, be a sequence of harmonic functions satisfying that $g_n(x_0)$ converges for some $x_0 \in B_R(0)$. Show that g_n converges to a harmonic function in $B_R(0)$.

   (*Hint: Recall that any function that is harmonic in $\mathcal{D}'(\Omega)$ is harmonic in Ω, combine this with Harnack's inequality.*)

2. Show that under suitable assumptions on the function $f : I\!R \to I\!R$, the bounded solution to

$$(P1)\begin{cases} u_t = u_{xx}, & \text{for } x > 0,\ t > 0 \\ u_x(0,t) = 0 & t > 0 \\ u(x,0) = f(x), & 0 < x, \end{cases}$$

   Is given by

$$u(x,t) = \int_0^\infty G(x,\xi,t) f(\xi)\, d\xi,$$

   where $G(x,\xi,t) = K(x-\xi,t) - K(x+\xi,t)$ and K is the fundamental solution of the heat equation.

   (*Hint: extend f in a suitable way to $(-\infty, 0)$ and solve the initial value problem for the extended f.*)

   State a sufficient condition on the growth of f for existence and uniqueness of solution to (P1) to hold.

3. Show that the problem

$$(P2)\begin{cases} u'(t) = f(u(t)), \\ u(0) = 0 \end{cases}$$

   need not be uniquely solvable in any neighborhood of $t = 0$ if f is continuous but not Lipschitz.

4. Suppose u, v are two solutions of

$$\frac{d^2 f}{dt^2} + 3t\frac{df}{dt} - f = 0$$

   with $W(u,v) \neq 0$ for all t where W denotes the Wronskian. Show that the zeros of u and v interlace, i.e. if $v(p) = 0 = v(q)$, $p < q$, then there exists an r, $p \le r \le q$ with $u(r) = 0$ and similarly there is a zero of v between every pair of zeros of u.

5. Let $J_0(t) = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t\cos(\theta))\, d\theta$. Show that $J_0(t)$ solves the initial value problem

$$ty'' + y' + ty = 0, \qquad y(0) = 1.$$

(Hint: write everything in terms of power series.)

6. Suppose u solves

$$\begin{aligned} &u_{xx} + 3u_{xt} + 2u_{tt} = 0 \\ &u(x,0) = F(x) \\ &u_t(x,0) = G(x) \end{aligned}$$

where $F, G \in C^2(-\infty,\infty)$. Suppose also $F(x) = 0 = G(x)$ for $-1 \le x \le 1$. For what region of $\{(x,t) : t > 0\}$ can you conclude $u(x,t) = 0$?