Partial Differential Equations Qualifying Exam
Fall 1989

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1. Show that the Cauchy problem of Laplace equation

$$\begin{cases} u_{tt} + u_{xx} & = 0 \\ u(0,x) & = \phi(x) \quad t > 0, x \in R^1 \\ u_t(o,x) & = \psi(x) \end{cases}$$

is not well posed.

2. A semidisc of radius a is well isolated from the surrounding. The temperature at ADB and segment $\overline{AB}$ are kept at constant c_2 and c_1 respectively. Find the stationary temperature distribution u in this semidisc. (See Fig. 1.)

3. (i) Under the boundary condition $\left(au + b\frac{\partial u}{\partial b}\right)_{|\partial\Omega} = 0$ where $a \neq 0 \neq b$ are constants, show that the Laplacian Δ is selfadjoint operator with all eigenvalue being negative and that $-\Delta$ is positive definite operator.

(ii) Can you use the result from part (i) to show that the equation

$$\begin{cases} \Delta u = f \text{ in } \Omega \\ \left(au + b\frac{\partial u}{\partial n}\right)_{|\partial\Omega} = 0 \end{cases}$$

exists a unique solution u where $f \in L^2(\Omega)$ and $a \cdot b$ are nonzero constants.

4. Find the solution $u(x,y,z)$ for the equation $xu_x + yu_y + u_z = u$ with the Cauchy data $u(x,y,0) = \phi(x,y)$ where ϕ is given C^1-function.

5. Let $\phi(x)$ be a bounded function in $\mathbb{R}^n$, define its average over $\mathbb{R}^n$ as

$$\phi_{av} = \lim_{r\to\infty} \frac{1}{|B_r|} \int_{B_r} \phi(x)dx$$

where B_r is the ball $\| x \| \leq r$. For heat conduction in $\mathbb{R}^n$ without sources, show that the average of temperature remains constant in time.

6. (a) Define a weak solution of $\Delta u = 0$.

(b) Define a strong solution of $\Delta u = 0$.

(c) Show any weak solution is also a strong solution.

7. Let $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ and le t$B_t = \{x : |x| \leq 1 - t\}$ for $0 \leq t \leq 1$ let $E(t) = \frac{1}{2}\int_{B_t} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] dx$. Show $\frac{dE}{dt} \leq 0$.

8. Find the C^2 function u on $D = \{x^2 + y^2 \leq 1\}$ satisfying $u(e^{i\theta}) = \theta^2 - \pi^2 (-\pi < \theta \leq \pi)$ that minimizes

$$\int_D |\nabla u|^2 dxdy.$$