Name ______________________________

REAL ANALYSIS QUALIFYING EXAM
Spring 1999
(Saeki & Vaninsky)

Do all eight. Throughout, $(X, \mathcal{A}, \mu)$ is a measure space.

1. Let $1 \le p \le \infty$ and define $L^1 + L^P := \{g + h : g \in L^1, h \in L^P\}$. Prove $L^r \subset L^1 + L^P$ for $r \in [1, p]$.

2. Let $w : X \to [0, \infty]$ be measurable, and let $\nu(E) := \int_E w \, d\mu$ for $E \in \mathcal{A}$. Prove: (a) ν is a measure on $\mathcal{A}$, and (b) $\int f \, d\nu = \int f w \, d\mu$ for each nonnegative measurable function f on X.

3. Let I be an open interval in $\mathbb{R}$, and let f be a function on $X \times I$ such that

(i) $\forall t \in I, f(\cdot, t) \in L^1(\mu)$,

(ii) $\forall x \in X, f(x, \cdot)$ is differentiable,

(iii) $\exists g \in L^1(\mu)$ such that $\left| \frac{\partial t}{\partial t}(x,t) \right| \le g(x)$ everywhere.

Prove that

$$\frac{d}{dt} \int f(x,t) d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

4. (a) Fix $0 < \varepsilon < 1$. Construct a closed set $K \subset [0,1]$ such that K contains no rationals and $|K| > 1 - \varepsilon$.

(b) Construct a Borel set $E \subset [0,1]$ such that $0 < |E \cap I| < |I|$ for each nonempty open interval $I \subset [0,1]$.

5. Suppose $f \in L^1(\mathbb{R}), g \in L^P(\mathbb{R}), 1 \le p \le \infty$. Prove $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

6. Let $(a_n) \in \ell^2$. Prove that

$$\exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \left| \sum_{n=0}^{\infty} a_n e^{int} \right| dt \right) \le \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

7. Prove the completeness of $L^P(\mu)$ for $1 \le p \le \infty$.

8. Let $\varphi \in L^2(\mathbb{R})$ and let V be the closed linear span (in L^2) of the $\varphi(x - n)$ with $n \in \mathbb{Z}$. Also let w be a 2π-periodic bounded measurable function on $\mathbb{R}$. Prove that $\widehat{\varphi} w = \widehat{f}$ for some $f \in V$.