Algebra Qualifying Exam (Old and New)
January 22, 2005

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. You should have at least one problem from each of the five sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, $\{9, 10\}$ among your choice. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

$\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers respectively.

1. Let G be a finite group of order $5075 = 5^2 \cdot 7 \cdot 29$. Prove that G has a unique Sylow 29-subgroup. (Note that $5^2 \cdot 7 = 1 + 6 \cdot 29$).

2. Let p be a prime integer and G be a subgroup of the symmetric group S_p. Assume that G acts transitively on the set $\{1, 2, \cdots, p\}$. Let H be a non-trivial normal subgroup of G. Prove that H also acts transitively on the set $\{1, 2, \cdots, p\}$.

3. Let R be a ring with identity and $M_n(R)$ be the ring of all $n \times n$ matrixes. Then $M_n(R)$ has a Jordan product $A \circ B = AB + BA$, where AB is the usual matrix multiplication. A Jordan homomorphism $\phi : M_n(R) \to M_n(R)$ is an abelian group homomorphism such that $\phi(A \circ B) = \phi(A) \circ \phi(B)$. Define $\phi(A) = A^t$ for all $A \in M_n(R)$, where A^t is the transpose of the matrix A. Show that ϕ is a Jordan homomorphism if R is a commutative ring. Is ϕ still a Jordan homomorphism when R is not commutative? If not, give an example and then modify the definition of ϕ to get a Jordan homomorphism involving the transpose operation and a ring anti-automorphism of R.

4. Let R be an integral domain and $I \subseteq R$ be a proper ideal. Note that the subset $S = \{1 + a \mid a \in I\} \subseteq R$ is multiplicatively closed. It is known that the subset $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$ is a sub-ring of the field of fractions of R. Show that the subset $S^{-1}I = \{\frac{a}{s} \mid a \in I, s \in S\}$ is an ideal of $S^{-1}R$ and is contained in every maximal ideal of $S^{-1}R$.

5. Let R be a PID and F be its field of fractions. Then F is an R-module with multiplication action. Show that if $R \subseteq M \subseteq F$ is an finitely generated R-submodule of F, then there is a $b \in R$ such that $M = R\frac{1}{b}$.

6. Let R be a ring (with identity) and M be a left R-module. Given a filtration $0 = M^0 \subset M^1 \subset \cdots \subset M^n = M$ of finite length of R-submodules, define $G = \{\phi \in \hom_R(M, M) \mid (\phi - 1)M^i \subseteq M^{i-1}, i = 1, \ldots, n\}$. Prove the following:
(a) G is a group under the composition of maps;
(b) G is a nilpotent group.

(*Hint:* Using $\phi\psi - \psi\phi = (\phi - 1)(\psi - 1) - (\psi - 1)(\phi - 1)$ for any two R-module homomorphisms $\phi, \psi : M \to M$ to show that if $(\phi - 1)M^i \subseteq M^{i-k}$ and $(\psi - 1)M^i \subseteq M^{i-l}$, then $([\phi, \psi] - 1)M^i \subseteq M^{i-k-l}$.)

7. Let F be a finite field of 2^n elements. Assume that F contains an element x satisfying $x^9 = 1$ and $x^3 \neq 1$. Prove that n is a multiple of 6.

8. Let $f(x) \in F[x]$ be a non-zero monic polynomial over a field F and E be a splitting field of $f(x)$ over F. Assume that all zeros of $f(x)$ in E are distinct and closed under multiplication. Prove that either $f(x) = x$, $f(x) = x^n - 1$, or $f(x) = x^n - x$ for some integer $n \geq 1$.

9. Let V be a finite dimensional vector space over a field F and $f, g : V \to F$ be two linear functionals with $\ker(f) \subseteq \ker(g)$. Prove that there exists a scalar $c \in F$ such that $g(v) = cf(v)$ for all $v \in V$.

10. Let A and B be two $n \times n$ matrix over a field F. Prove that A and B are similar if and only if the two $2n \times 2n$ matrices $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$ are similar.