Algebra Qualifying Exam
August 25, 2007

Instructions: You are given 8 problems and do as many as you can. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative. $\mathbb{Q}$ and $\mathbb{C}$ are the fields of rational and complex numbers, respectively.

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1. Prove that the alternating group A_7 has no element of order 12.
2. Let G be a simple group of order $504 = 2^3 \times 3^2 \times 7$. Prove that G does not contain an element of order 21.
3. Let R be an integral domain with field of fractions K. Assume that there exists a nonzero element $t \in R$ such that $K = R[1/t]$. Prove that t is an element of every nonzero prime ideal of R.
4. Let $a \in \mathbb{C}$ be a complex root of the polynomial $f(x) = x^6 + 3$.
   (a) Prove that the roots of the polynomial $x^2 - x + 1$ are primitive 6th root of 1 and are in the field $\mathbb{Q}[a]$.
   (b) Prove that the field extension $\mathbb{Q} \subseteq \mathbb{Q}[a]$ is Galois, with Galois group isomorphic to the symmetric group S_3.
5. Let V and W be finite dimensional vector spaces over an algebraically closed field F. Let $A : V \to V$, $B : W \to W$, and $T : V \to W$ be linear transformations satisfying $T \cdot A = B \cdot T$. Assume $T \neq 0$, and denote $N = \ker(T)$.
   (a) Prove that $A(N) \subseteq N$.
   (b) Prove that there exists a scalar $\lambda \in F$ and a vector $v \in V$ with $v \notin N$ such that $Av - \lambda v \in N$. (Hint: Consider the quotient space V/N.)
   (c) Show that this scalar λ is an eigenvalue both for A and for B.
6. Let $\mathbb{Q} \subseteq E$ be a finite dimensional field extension of the rational numbers, and let $f(x) \in E[x]$ be a monic irreducible polynomial.
   (a) Prove that there exists a unique monic irreducible polynomial $g(x) \in \mathbb{Q}[x]$ such that $f(x)$ divides $g(x)$ in $E[x]$.
   (b) If we also assume that the extension $\mathbb{Q} \subseteq E$ is Galois, prove that the degree of $f(x)$ divides the degree of $g(x)$.
7. Let $G = GL_n(K)$ be the multiplicative group of $n \times n$ invertible matrices with entries in a field K. If the characteristic of the field K is not 2, prove that G has precisely n conjugacy classes of elements of order two.
8. For a ring R, an R-module M is called decomposable if M is isomorphic to $M_1 \oplus M_2$ for two non-zero R-modules; otherwise, M is called indecomposable. Let $R = F[t]/ < t^n >$ for a fixed positive integer n and a field F. Any R-module is an F-vector space. The dimension of an R-module M is defined to be the dimension of M as an F-vector space.
   (a) Prove that a finite dimensional R-module is indecomposable if and only if t acts on it as a linear transformation having a single Jordan block. Determine the possible size of the Jordan blocks.
   (b) Suppose that $n = 5$; find a way to describe all non-isomorphic 10-dimensional R-modules.