Complex Variables Qualifying Exam
Spring 1996

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1. Let $G = \{a + bi : a, b \in \mathbb{Z}\}$. Does $\sum_{\substack{z \in G \\ z \neq 0}} \frac{1}{|z|^2}$ converge or diverge?

2. Suppose f is even and entire. Show $g(z) = f(\sqrt{z})$ is entire.

3. Suppose f is entire and there exist $\eta_1, \eta_2 > 0$ such that $|z| > \eta_1$ implies $|f(z)| > \eta_2$. Show f is a polynomial.

4. Evaluate $\int_0^\infty x(x^2+1)^{-2} \sin(x) dx$.

5. Find a conformal mapping from $\{x + iy : |y| < \frac{\pi}{2}\}$ onto $\{u + iv : v < u^2\}$. (Hint: What is the action of $w = z^2$ on horizontal lines?)

6. Write the function $\frac{1}{1-z} + \frac{3}{3-z}$ as a Laurent series centered at $z_0 = 0$ valid in some region which contains the point $z = 2$. What is the domain of convergence for this series?

7. Prove that the equation $e^z = 2z + 1$ has exactly one root in $|z| < 1$.

8. It is desired to approximate $\frac{1}{z}$ on $|z| = 1$ by a function $f(z)$ which is analytic on $|z| \leq 1$. Show that the maximum error is at least one, that is, show $\max_{|z|=1} |\frac{1}{z} - f(z)| \geq 1$.

9. Suppose $f : U \to S$ is conformal, onto, and one-to-one, where U is the unit disk and S is a square centered at 0. Suppose also $f(0) = 0$. Prove that $f(iz) = if(z)$ and that if $f(z) = \sum_{n=0}^\infty c_n z^n$, then $c_n = 0$ unless $n - 1$ is a multiple of 4. (Hint: Consider $f^{-1}(if(z))$.)

10. Show that $f(z) = \sum_{n=1}^\infty \frac{1}{n^z}$ defines an analytic function in the domian Rez > 1.