REAL ANALYSIS QUALIFYING EXAM
November 7, 1983

1. (a) Give a definition of Lebesgue outer measure λ on $\mathbb{R}$.

   (b) Use your definition in (a) to prove that $\lambda([0,1]) = 1$.

   (c) Prove that if $\varepsilon > 0$, then there is a closed subset A of $[0,1]$ containing no rational number such that $\lambda(A) > 1 - \varepsilon$.

2. Let $f:\mathbb{R} \to \mathbb{C}$ be continuous. Suppose that

$$\int_{\mathbb{R}} f(x)g(x)\,dx = 0$$

whenever $g:\mathbb{R} \to \mathbb{C}$ is continuous with compact support. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions on $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n(x)|\,dx < \infty.$$

Prove that there is a Lebesgue integrable function f on $\mathbb{R}$ such that

(a) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ a.e. on $\mathbb{R}$,

(b) $\lim_{N\to\infty} \int_{\mathbb{R}} \left|f(x) - \sum_{n=1}^{N} f_n(x)\right| dx = 0$,

and

(c) $\int_{\mathbb{R}} f(x)\,dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)\,dx$.

4. State the following:

   (a) The Riesz representation theorem for nonnegative linear functionals.

   (b) Fubini's theorem on multiple integrals.

   (c) The Radon-Nikodým Theorem.

   (d) The Hahn-Banach Theorem.

5. Let $(X, \mathcal{M})$ be a measurable space and let μ and ν be finite measures on it. Prove that the following two assertions are equivalent:

   (a) $\mu(E) = 0$ whenever $E \in \mathcal{M}$ and $\nu(E) = 0$.

   (b) To each $\varepsilon > 0$ corresponds some $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever $E \in \mathcal{M}$ and $\nu(E) < \delta$.

6. Let X be an infinite compact Hausdorff space. Prove that the Banach space $C(X)$ is not reflexive. [Hint: For an appropriate $a \in X$, define L on the space $M(X)$ of complex regular Borel measures on X by $L(\mu) = \mu(\{a\})$.]

7. Let $f, g \in L_1(\mathbb{R})$. Prove that the formula

$$h(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt$$

   defines h almost everywhere on $\mathbb{R}$, that $h \in L_1(\mathbb{R})$, and that $||h||_1 \leqq ||f||_1 \cdot ||g||_1$.

8. Let $(X, \mathcal{M})$ be a measurable space and let $f: X \to [0, \infty]$ be $\mathcal{M}$-measurable. Prove that there exist sequences $(a_n)_{n=1}^{\infty} \subset [0, \infty[$ and $(A_n)_{n=1}^{\infty} \subset \mathcal{M}$ such that

$$f(x) = \sum_{n=1}^{\infty} a_n \xi_{A_n}(x)$$

   for all $x \in X$. Here ξ_B is the characteristic function of B.

9. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\mathcal{F}$ be a nonvoid subfamily of $\mathcal{M}$. Prove that there is a set $B \in \mathcal{M}$ satisfying both

   (a) $\mu(F \setminus B) = 0$ for all $F \in \mathcal{F}$

and

   (b) if $A \in \mathcal{M}$ and $\mu(F \setminus A) = 0$ for all $F \in \mathcal{F}$, then $\mu(B \setminus A) = 0$.

[Hint: Consider the number $\beta = \sup\{\mu(U) : U$ is a countable union of members of $\mathcal{F}\}$.]

10. (a) Let $(n_j)_{j=1}^{\infty}$ be a sequence of integers such that $|n_1| < |n_2| < \ldots$. Prove that the set E of all $x \in \mathbb{R}$ such that

$$\lim_{j\to\infty} e^{in_j x}$$

exists must have Lebesgue measure 0. [Hint: Let $f(x)$ be the above limit if $x \in E$ and $f(x) = 0$ otherwise. Consider the Fourier coefficients of f.]

(b) Prove that if $n_j = j!$, then E is uncountable.