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Geometry of Manifolds Qualifying Exam
Old System – Fall 2003
Auckly & Vidussi

1. Let $\pi : S^2 \to \mathbb{R}P^2$ be the standard covering projection. Prove that there is no map $f : \mathbb{R}P^2 \to S^2$ so that $\pi \circ f = id$.

2. Recall that

$$d\alpha(X_0, \ldots, X_p) = \sum_{k=0}^{P} (-1)^k X_k \alpha(X_0, \ldots, \widehat{X}_k, \ldots X_p) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], \ldots \widehat{X}_i, \ldots \widehat{X}_j \ldots X_p)$$

Prove that $d\alpha(X_0, \ldots, X_p) = \sum_{k=0}^{P} (-1)^k (\nabla_{X_k} \alpha)(X_0, \ldots \widehat{X}_k, \ldots X_p)$.

3. (a) Give the definition of a Lie group.

(b) Give the definition of a Lie algebra.

(c) Give the definition of a representation of a Lie group, $\mu : G \to \mathrm{Aut}\,(V)$.

(d) Give the definition of a representation of a Lie algebra, $\dot{\mu} : \mathbf{g} \to \mathrm{End}\,(V)$.

(e) Define the Lie algebra of a Lie group.

(f) Describe how a representation of a Lie group induces a representation of the corresponding Lie algebra and prove that the induced representation is a Lie algebra representation.

4. Prove that the holonomy of a simply connected Riemannian manifold is connected.

5. Let $X = \dfrac{\partial}{\partial x}$ and $Y = \dfrac{\partial}{\partial x} + (x^2+1)\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(a) Compute $[X, Y]$.

(b) Compute the flow of X.

(c) Compute the flow of Y.

(d) Let $F^Z : \mathbb{R} \times M \to M$ be the flow of a vector field Z. If $F^Z_s \circ F^W_t = F^W_t \circ F^Z_s$ for all s and t, what can you say about $[Z, W]$? Why?

(e) Is there a function $f_Y : \mathbb{R}^2 \to \mathbb{R}$ so that $F^{fX}_t \circ F^Y_s = F^Y_s \circ F^{fX}_t$ for all s and t? Why?

6. Let $f : \mathbb{R}^3 \to \mathbb{R} : f(x, y, z) = xy - z$.
$\Sigma = f^{-1}(0) \wedge \{(x, y, z) | x^2 + y^2 \leq 1\}$

(a) Verify that Σ is a manifold.

(b) Compare the orientation induced on Σ using $\nabla f / |\nabla f|$ and $dx \wedge dy \wedge dz$ with the orientation $dx \wedge dy$.

(c) Compute $\displaystyle\int_\Sigma \frac{|\nabla f \circ \kappa|}{|\nabla f|} dx \wedge dy$ when Σ is oriented by $dx \wedge dy$. What does this represent?

7. The connected sum $M_1 \# M_2$ of two oriented n-manifolds M_1, M_2 is defined as $(M_1 \setminus \operatorname{int} B^n) \bigcup_{S^{n-1}} (M_2 \setminus \operatorname{int} B^n)$, where B^n is a ball in $M_1 (M_2)$ and S^{n-1} is its boundary.

(a) Show that if $n \geq 3$, then $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$.

(b) Compute the fundamental group of $T^2 \# T^2$ (where T^2 is the 2-dimensional torus).

[Hint: What is $\pi_1(T^2 \setminus \operatorname{int} D^2)$?]

8. (a) Show that there exists a natural map $S^1 \times S^3 \to U(2)$ with discrete fiber by using the Lie group structure of S^1 and S^3.

(b) What is the fiber?

(c) Using the result above, what is $\pi_1 U(2)$?