SPECIAL COMPLEX ANALYSIS
Qualifying Exam
for Khaled Al-Agha
July 1, 1992
(Burckel)

In the following $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{D}$ the set $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ the set of all holomorphic functions on $\mathbb{D}$.

1. Show that if $g : \Omega \to \mathbb{C}$ is continuous and e^g is holomorphic, then g is holomorphic. (That is, a **continuous** logarithm of a holomorphic function is necessarily holomorphic.)

2. Show directly (without reference to the concept of simple-connectivity) that every zero-free function $f \in H(\mathbb{D})$ has a holomorphic logarithm; that is, $\exists g \in H(\mathbb{D})$ such that $f = e^g$.

3. Show directly (without reference to the concept of simple-connectivity) that the identity function, $I(z) = z$, in $\mathbb{C} \setminus \{0\}$ has no continuous logarithm. **Hint:** Problem 1 may be useful.

4. (a) f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$. Show that f is uniformly approximable on $\overline{\mathbb{D}}$ by polynomials. **Hint:** First approximate f uniformly on $\overline{\mathbb{D}}$ by a function f_r which is holomorphic in $D(0, 1/r)$, $0 < r < 1$.

   (b) State and prove the converse of (a).

5. State

   (a) the Maximum Modulus Principle for holomorphic functions,

   (b) the Open Map Theorem for holomorphic functions.

   (c) Show that (a) can be deduced from (b).

6. Show that $\displaystyle\int_{\partial\mathbb{D}} \frac{e^{\pi z}}{4z^2+1}\, dz = \pi i.$

8. Let Ω be a bounded region in $\mathbb{C}$, $f : \overline{\Omega} \to \mathbb{C}$ a continuous non-constant function which is holomorphic in Ω and maps $\partial\Omega$ into $\mathbb{T}$.

(i) Show that $0 \in f(\Omega)$.

(ii) Show that $f(\Omega) = \mathbb{D}$.

Hint: To get "$\supset$", apply (i) to $\phi \circ f$ for certain holomorphic maps ϕ of $\mathbb{D}$ into $\mathbb{D}$.

9. f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$ and $\operatorname{diam} f(\mathbb{T}) \leq 1$. Show that $\operatorname{diam} f(r\mathbb{T}) \leq r$ for each $0 \leq r \leq 1$.

Hint: $\operatorname{diam} f(r\mathbb{T}) := \max\{|f(ru_1) - f(ru_2)| : u_1, u_2 \in \mathbb{T}\}$. If this is achieved at u_1, u_2, consider the holomorphic function $F(z) := f(zu_1) - f(zu_2)$.

10. $h : \mathbb{C} \to \mathbb{R}$ is harmonic and non-constant.

(i) Prove that h is not bounded above.

(ii) Prove that h is not bounded below.

(iii) Prove that $h(\mathbb{C}) = \mathbb{R}$.