TOPOLOGY QUALIFYING EXAM
Spring 2001
(Maginnis and Muenzenberger)

Choose and work any 6 of the following 10 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems.** A space always means a topological space below.

1. Assume that every open cover of the space X has a countable subcover. Let $A \subseteq X$ be an uncountable subset. Prove A has a limit point.

2. Let $f : X \to Y$ be an open, continuous, surjective map. Let $R \subseteq X \times X$ be the set $R = \{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\}$. Assume R is closed in the product topology on $X \times X$. Prove Y is a Hausdorff space.

3(a) Prove an open subspace of a separable space is separable.

(b) Give an example of a subspace of a separable space which is not separable.

4. Let X be a complete metric space, and let $\{C_n \subseteq X | n \in \mathbb{N}\}$ be a nested sequence of nonempty, closed, bounded subsets of X, with $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$. The diameter of a bounded subset of a metric space is defined as the least upper bound of the distances between points, $\operatorname{diam}(C) = \operatorname{lub}\{d(x,y) | x, y \in C\}$. Assume $\lim_{n\to\infty} \operatorname{diam}(C_n) = 0$. Prove the intersection $\bigcap_{n=1}^{\infty} C_n$ is <u>not</u> empty.

5. Let X be a locally compact Hausdorff space. Prove a subset $U \subseteq X$ is open if and only if for every compact subspace $C \subseteq X$, $U \cap C$ is an open subset of C.

6. Let X be a connected normal T_2 space having more than one point. Prove that X is uncountable.

7. Let X be a connected space, and assume $X - \{x_0\}$ is not connected for some point $x_0 \in X$. Let $X - \{x_0\} = U \cup V$ be a separation, so that U and V are disjoint, nonempty, open (and closed) subsets of $X - \{x_0\}$. Prove $A = U \cup \{x_0\}$ is a connected subspace of X.

8. Prove a compact metric space has a countable basis.

9. Prove a space X is compact if and only if every net in X has a cluster point.

10. Let X be a paracompact space, and let Y be a compact Hausdorff space. Use the Tube Lemma to prove that $X \times Y$, in the product topology, is paracompact.