# NUMERICAL ANALYSIS QUALIFYING EXAM
# Fall, 2002

(do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Given $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$ with $x_1, \ldots, x_n$ distinct, prove there exists a unique polynomial $p(x)$ of degree $\leq 2n-1$ with

$$p(x_i) = y_i,\ p'(x_i) = z_i.$$

2. (a). Define $\theta_i = \arctan(2^{-i})$. Show that any angle $0 < \theta < \pi/2$ can be represented as $\theta = \widehat{\theta}_m + \epsilon$ where $\widehat{\theta}_m = \sum_{i=0}^{m} \alpha_i \theta_i$, and $\alpha_i = \pm 1$, and $|\epsilon| < \arctan(2^{-m})$.

   (b). Show that the recursion

$$\begin{aligned} X_0 &= 1 \\ Y_0 &= 0 \\ X_i &= X_{i-1} - \alpha_{i-1} Y_{i-1} 2^{-i+1} \\ Y_i &= Y_{i-1} + \alpha_{i-1} X_{i-1} 2^{-i+1} \end{aligned}$$

   will produce

$$\begin{aligned} X_m &= G_m \cos(\widehat{\theta}_{m-1}) \\ Y_m &= G_m \sin(\widehat{\theta}_{m-1}) \end{aligned}$$

   where $G_m$ is real, positive and does not depend on $\theta$.

3. Prove that Simpson's rule is exact when applied to the integration of cubic polynomials. Justify any error estimates you use,

4. IEEE-754 double precision artismetic represents numbers with a 53 bit mantissa and 11 bit exponent. Arithmetic operations return the exact answer rounded to the nearest representable number. If exact answer is precisely halfway between two representable numbers, round toward 0. Arithmetic operations never fail, but can return $\pm$`Inf` or `NaN`.

   Let `eps` be defined by the condition that $1 +$ `eps` is the smallest representable number $> 1$. What value will each of the following variable hold.

```
forward = (2 + eps - 2)/eps
backward = (2 - eps - 2)/(-eps)
diff = forward - backward
finv = 1/forward
binv = 1/backward
invdiff = finv - binv
small = 2 ∧ 512        (or 2**512 in FORTRAN notation)
medium = 2 ∧ 1024
large = 2 ∧ 2048
giant = 2 ∧ 4096
bigdiff = giant - large
```

5. (a). Define the condition number $\kappa(A)$ of a matrix with respect to a matrix norm $\|\cdot\|$ which is induced by a vector norm $\|\cdot\|$. Explain with detailed argument the role the condition number plays in estimating the error in the solution of $A\mathbf{x} = \mathbf{b}$ caused by an error in $\mathbf{b}$.
   (b). Let $A \in \mathbf{R}^{n\times n}$ be a nonsingular matrix. Prove that for each singular matrix $B$,

$$\kappa(A) \geq \frac{\|A\|}{\|A - B\|}.$$

6. For an arbitrary $m \times n$ real matrix $A$, show that

$$\lim_{\alpha\to 0+} (\alpha I + A^T A)^{-1} A^T = A^+$$

   where $A^+$ is the psuedoinverse of $A$. (**Hint: Apply singular value decomposition**)

7. Let $A$ be an $n \times n$ symmetric real positive definite matrix, and $\mathbf{b} \in \mathbf{R}^n$. Consider the following iteration

$$\mathbf{x}_{k+1} = \mathbf{x}_k + M^{-1}(\mathbf{b} - A\mathbf{x}_k)$$

   where $M$ is a real nonsingular matrix and $\mathbf{x}_0 \in \mathbf{R}^n$ is a given vector. Let $\mathbf{x} = A^{-1}\mathbf{b}$. Show that the error $\mathbf{e}_k = \mathbf{x} - \mathbf{x}_k$ satisfies

$$\langle A\mathbf{e}_{k+1}, \mathbf{e}_{k+1}\rangle = \langle A\mathbf{e}_k, \mathbf{e}_k\rangle - \langle FM^{-1}A\mathbf{e}_k, M^{-1}A\mathbf{e}_k\rangle$$

   where $F = M + M^T - A$, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbf{R}^n$. From this show that if $F$ is symmetric positive definite, then the sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$.

8. Let $U$ be a unitary matrix, $R$ an upper triangular matrix and $UR$ an upper Hessenberg matrix. State and prove a condition on $R$ under which $U$ is an upper Hessenberg matrix.