Algebra Qualifying Exam
January 20, 2000

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems that you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be a finite group, let p be a prime, and let S be a Sylow p-subgroup of G. Prove that if $N \triangleleft G$ is a normal subgroup of G, then $S \cap N$ is a Sylow p-subgroup of N.

2. Let G be a finite group of odd order and let $x \in G$ be a nonidentity element. Prove that x and x^{-1} are *not* conjugate in G.

3. Let G be a group acting transitively on the finite set X. Let $x \in X$ and denote by $G_x = \{g \in G|\ gx = x\}$ the *isotropy subgroup* of x. Let $H \triangleleft G$ be a normal subgroup of G; note that HG_x is a subgroup of G and that H acts on X. Prove that the number of distinct orbits of the action of H on X equals the index $[G : HG_x]$.

4. Let $I = (2, x) = \{2f(x) + xg(x)|\ f(x), g(x) \in \mathbb{Z}[x]\}$ be the ideal in the ring $R = \mathbb{Z}[x]$ of polynomials in the indeterminate x with integer coefficients. Prove that I is *not* a free R-module.

5. Let R be a Euclidean domain with respect to the function $d : R - \{0\} \to \mathbb{Z}^+$ $(= \{1, 2, \ldots\})$. Assume that d satisfies

   (a) $d(ab) = d(a)d(b)$, for all $a, b \in R - \{0\}$,

   (b) $d(a + b) \le \max\{d(a), d(b)\}$, for all $a, b, a + b \in R - \{0\}$.

   Prove that either R is a field, or that there exists a field $\mathbb{F} \subseteq R$ such that $R \cong \mathbb{F}[x]$, the ring of polynomials in the indeterminate x with coefficients in $\mathbb{F}$. [Hint: Let $\mathbb{F} = \{a \in R|\ d(a) = 1\}$.]

6. Let $A, B : V \to V$ be linear transformations on the finite dimensional vector space V over the complex numbers $\mathbb{C}$. Prove that if

$AB = BA$ then there exists a nonzero vector $0 \neq v \in V$ that is simultaneously an eigenvector for both A and B.

7. Let $T : V \to V$ be a linear transformation of the finite dimensional vector space V over the field $\mathbb{F}$. Define the usual $\mathbb{F}[x]$-module structure on V by setting $f(x) \cdot v = f(T)(v)$, $f(x) \in \mathbb{F}[x]$, $v \in V$. Prove that V is a cyclic $\mathbb{F}[x]$-module if and only if the characteristic polynomial of T equals the minimal polynomial of T.

8. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q $(= p^r)$ elements, where p is prime, and let $\mathbb{K} = \mathbb{F}_{q^4} \supseteq \mathbb{F}$. Say that elements $\alpha, \beta \in \mathbb{K}$ are *equivalent* if they have the same minimimal polynomial over $\mathbb{F}$. Clearly this is an equivalence relation on $\mathbb{K}$. Compute the number of equivalence classes in $\mathbb{K}$ as a function of q. (Hint: consider $\mathbb{F} \subseteq \mathbb{F}_{q^2} \subseteq \mathbb{F}_{q^4} = \mathbb{K}$.)

9. Let $\mathbb{F} \subseteq \mathbb{E} \subseteq \mathbb{K}$ be fields, where the extension degrees are finite, let $\alpha \in \mathbb{K}$, and let $f(x)$ be the minimal polynomial of α over $\mathbb{F}$. Assume that $[\mathbb{E} : \mathbb{F}]$ and $\deg f(x)$ are relatively prime. Prove that $f(x)$ is also the minimal polynomial of α over $\mathbb{E}$.

10. Let $n \geq 3$ be an integer and let $f(x) = x^n - 2 \in \mathbb{Q}[x]$. Prove that the Galois group of $f(x)$ is nonabelian but solvable.