Qualifying Exam: Algebra

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Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Write solutions to each problem on separate pages and write your name on the top.

Note: All rings are assumed to be associative and with multiplicative identity 1. The integers, the rational numbers and the complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$, respectively.

1. Let $f_1 : G \to F$ and $f_2 : H \to F$ be two group homomorphisms. Prove that there is a group, $G \prod_F H$, together with group homomorphisms $p_1 : G \prod_F H \to G$ and $p_2 : G \prod_F H \to H$ satisfying the following two conditions:
   (i) $f_1 \circ p_1 = f_2 \circ p_2$ and,
   (ii) for any group E and any group homomorphisms $\phi_1 : E \to G$ and $\phi_2 : E \to H$ such that $f_1 \circ \phi_1 = f_2 \circ \phi_2$, there exists a unique group homomorphism $\psi : E \to G \prod_F H$ such that $\phi_1 = p_1 \circ \psi$ and $\phi_2 = p_2 \circ \psi$.

2. Let F be a (commutative) field and $B_n(F)$ be the set of all invertible upper triangular $n \times n$-matrices with entries in F. With respect to the matrix multiplication, $B_n(F)$ is a group.
   (a) Prove that $B_3(F)$ is a solvable group.
   (b) Prove that $B_3(F)$ is not a nilpotent group if F has more than two elements.

3. Let R be a ring with identity and consider R as a left R-module over itself under the left multiplication. The set $\mathrm{Hom}_R(R, R)$ of R-module homomorphisms is a ring with pointwise addition and composition of maps as the multiplication. Show that there is a ring isomorphism $\mathrm{Hom}_R(R, R) \cong R^{\mathrm{op}}$. Here, R^{op} is the ring with the same addition as that of R and a new multiplication defined by $a \circ b = ba$ for all $a, b \in R$.

4. Let R be a commutative ring with 1. An ideal J of R is said to be *primary* if, for all r, $s \in R$, $rs \in J$ implies that either $r \in J$ or $s^n \in J$ for some integer $n \geq 1$.
   (a) If P is a prime ideal of R and if I is an ideal of R such that the ideal IP is primary, prove that either $I \subseteq P$ or $P = IP \subseteq I$.
   (b) Assume now that *every* ideal of R is primary. Prove that R has a unique maximal ideal consisting precisely of non-invertible elements of R.

5. Let R be a ring with identity. An R-module P is called *projective* if for any surjective R-module homomorphism $\phi : M \to N$ and any R-module homomorphism $p : P \to N$, there is an R-module homomorphism $\psi : P \to M$ such that $p = \phi \circ \psi$. Prove that any free R-module is projective.

6. Let $\mathbb{Q}$ be the $\mathbb{Z}$-module with $\mathbb{Z}$ acting on $\mathbb{Q}$ by multiplication. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as abelian groups.

7. Let A be an $n \times n$ matrix over a field K and assume that the characteristic polynomial of A has distinct roots in the algebraic closure of K. Prove that any two $n \times n$ matrices over K which commute with A must commute with each other.

8. Let F be a field and $R = F[x]$ be the polynomial ring. For any $n \times n$-matrix A with entries in F, the vector space $V = F^n$ of column vectors becomes an R-module, denoted by V_A, such that x acts on V by multiplying the matrix A (from the left). Prove that V_A is an indecomposable R-module if and only if A has a single elementary divisor. If A has a single invariant factor, must V_A be indecomposable? Recall that a module is called decomposable if it is a direct sum of two proper submodules and called indecomposable if it is not decomposable.

9. Let $F \supseteq K$ be an algebraic field extension. If D is a subring of F containing K, show that D is a field.

10. (a) Determine the minimal polynomial of a primitive 6^{th} root of unity and compute its roots.
    (b) Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$ and α be a root of $f(x)$ in $\mathbb{C}$. Prove that $\mathbb{Q}(\alpha)$ is a Galois extension over $\mathbb{Q}$ by using (a).
    (c) Determine the Galois group of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Justify your answer.