NUMERICAL ANALYSIS QUALIFYING EXAM
Spring, 2003

(do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Find analytically the solution of this difference equation with the given initial values:

$$\begin{cases} x_{n+1} = -0.2x_n + 0.99x_{n-1} \\ x_0 = 1, \qquad x_1 = 0.9 \end{cases}$$

   Determine whether a computation using the difference equation is stable. Justify your answer.

2. Let α be a double root of the equation $f = 0$, where f is sufficiently smooth near α. Show that if the "doubly-relaxed" Newton method

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

   converges to α, it does so at least quadratically.

3. Suppose that k and n are positive integers with $k < n$ and that f is a real valued function continuous on the interval $[-1, 2]$. For each integer $m \geq n$, S_m is a piecewise polynomial approximation to f on $[0, 1]$ defined as follows: First, set up a mesh $\{x_j\}_{j\in\mathbf{Z}} \bigcap [-1, 2]$ where $x_j = jh$ with $h = 1/m$. Then on each subinterval $[x_j, x_{j+1}] \bigcap [0, 1]$ define $S_m(x) = p_j(x)$ where $p_j(x)$ is the polynomial of degree at most n that interpolate f at the $n + 1$ consecutive points $x_{j-k}, \ldots, x_{j-k+n}$. Show that S_m converges to f uniformly on $[0, 1]$ as $m \to \infty$. (**Hint:** Use the Lagrange interpolation formula and change the variable x to s by $x = x_{j-k} + sh$.)

4. Let q_k, $k = 0, 1, \ldots, n$ be a set of orthogonal polynomials on $(-1, 1)$ with weight function $w(x) = |x|$, where q_k has degree k and leading term x^k.
   (a). Find q_0, q_1 and q_2.
   (b). Find the Gaussian quadrature formula for

$$\int_{-1}^{1} |x|f(x)dx$$

   using the roots of q_2 and verify its degree of precision.
   (c). Show that the Gaussian quadrature rule

$$\int_{-1}^{1} |x|f(x)dx \approx G_n(f)k = \sum_{k=1}^{n} A_k f(x_k)$$

   has all positive coefficients A_k.

5. Let A and B be two $n \times n$ real matrices. Show that $H = A + iB$ is Hermite positive definite if and only if $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ is symmetric positive definite.

6. Let $\mathbf{q}_j$, $j = 1, \ldots, n$ be the columns of an $n \times n$ orthogonal matrix Q, and K be the $n \times n$ Krylov matrix whose first column is $\mathbf{q}_1$ and the jth column is $A^{j-1}\mathbf{q}_1$ for $1 < j \leq n$. Show that $H = Q^T A Q$ is an unreduced Hessenberg matrix if and only if $Q^T K = R$ is nonsingular and upper triangular. (**Hint:** A Hessenberg matrix H is unreduced if $r_{j+1,j} \neq 0$ for $1 \leq j \leq n-1$.)

7. Suppose that a method solving linear system $A\mathbf{x} = \mathbf{b}$ yields a numerical solution $\hat{\mathbf{x}}$ that is the exact solution of the linear system $(A + \delta A)\hat{\mathbf{x}} = \mathbf{b}$ where δA is some $n \times n$ matrix.

   (a) Give s suitable description for the method to be backwardly stable.

   (b) Assume μ is a positive number with $\|\delta A\| \leq \mu \|A\|$. Show that the relative error of $\hat{\mathbf{x}}$ is bounded by
   $$\frac{\mu\kappa(A)}{1 - \mu\kappa(A)}, \quad \text{if} \quad \mu\kappa(A) < 1$$
   where $\kappa(A)$ is the condition number of A under the norm $\|\cdot\|$.

8. Show that the Jacobi iteration converges for 2 by 2 symmetric positive definite systems.