Algebra Qualifying Exam
Fall 1991

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a group and let N_1, N_2 be normal subgroups of G with $N_1 \cong N_2$. Prove, or give a counterexample to the assertion
$$G/N_1 \cong G/N_2.$$

2. Let G be a finite group and let $\mathcal{C}$ be a conjugacy class of elements in G. If $|\mathcal{C}| = \frac{1}{2}|G|$, show that every element of $\mathcal{C}$ is an involution (i.e., an element of order 2).

3. Let R be a commutative ring, and let $x \in R$. Define what it means for x to be an *irreducible* element, and define what it means for x to be *prime*. If R is a *unique factorization domain*, show that x is irreducible if and only if x is prime.

4. Let R be a principal ideal domain, and let $0 \neq I \subset R$ be a prime ideal. Prove that I is a maximal ideal.

5. Let R be a ring and let M be an irreducible left R-module. Prove that there exists a maximal left ideal I such that $R/I \cong M$ as left R-modules.

6. Consider the map $T : \mathbb{C} \to \mathbb{C}$ defined by setting $T(\alpha) = (2+i)\alpha$. If we regard T as an $\mathbb{R}$-linear transformation of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{C}$, compute $\det(T)$.

7. Let V be an n-dimensional vector space over a field F, and let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. Let V^* denote the dual space of V, that is, V^* is the vector space $Hom_F(V, F)$ of all linear transformations $\lambda : V \to F$. Define elements $\lambda_1, \ldots, \lambda_n$ of V^* by setting
$$\lambda_i \left(\sum_{j=1}^{n} a_j x_j \right) = a_i,$$
$1 \leq i \leq n, a_j \in F$, and put $\mathcal{B}^* = \{\lambda_1, \ldots, \lambda_n\}$. Show that $\mathcal{B}^*$ is a basis of V^*.

8. Consider the matrix of rational entries
$$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$
If $R = \mathbb{Q}[A]$ is the ring of polynomials in A with rational coefficients, prove that R is a field. (Hint: Consider the homomorphism $\mathbb{Q}[x] \to \mathbb{Q}[A]$, where $x \mapsto A$.)

9. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

(a) Construct a splitting field $K \supseteq \mathbb{Q}$ for $f(x)$ over $\mathbb{Q}$.

(b) Find an element $\alpha \in K, \alpha \notin \mathbb{Q}$, such that $\mathbb{Q}[\alpha]$ is a normal extension of $\mathbb{Q}$.