Name ______________________________

# REAL ANALYSIS QUALIFYING EXAM
Spring 2000
(Saeki & Moore)

**Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, $\mu$ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.**

**1.** Suppose $f \in L^1(\mu)$. Prove that given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\int_E f\,d\mu| < \varepsilon$ whenever $\mu(E) < \delta$.

**2.** Let $f$ be a measurable function on $X$ and $p > 0$. Prove

$$\int_x |f|^P d\mu = p\int_0^\infty t^{p-1}\mu(\{|f| > t\})dt.$$

**3.** Let $g$ be a bounded measurable functon on $\mathbb{R}$ which has the property that for every measurable set $E$, $\lim_{n\to\infty}\int_E g(nx)dx = 0$. Show that for every $f \in L^1(\mathbb{R})$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)g(nx)d\mu = 0.$$

**4.** Suppose $\{f_n\}$ is a sequence of measurable functions which has $f_n \to f$ a.e. and $\|f_n\|_1 \to \|f\|_1 < \infty$. Prove that $f_n \to f$ in $L^1$.

**5.** Let $v$ be another measure on $(x, \mathcal{M})$ with $v(x) < \infty$. Prove that the following two statements are equivalent.

(a) $A \in$, $\mu(A) = 0 \Rightarrow v(A) = 0$.

(b) $\forall\varepsilon > 0$ there exists $\delta > 0$ such that $v(b) < \varepsilon$ whenever $b \in \mathcal{M}$ and $\mu(b) < \delta$.

**6.** Supose $a \subseteq \mathcal{M}$ is also a $\sigma$-algebra on $x$. Suppose $f \in L^1(x, \mathcal{M}, \mu)$.

(a) Prove that there exists a function $g \in L^1(x, a, \mu)$ such that $\int_E g\,d\mu = \int_E f\,du$ for all $E \in a$.

(b) Give an example to show that in (a) we may not necessarily have $g = f$.

**7.** Prove that an orthnormal set in a separable Hilbert space is at most countable.

**8.** Let $T(x)$ be a trigonometric polynomial on $[-\pi, \pi]$. (Recall that this means $T(x)$ is a finite linear combination of elements of the set $\{e^{inx}\}_{n\in\mu}$). Prove that

$$|T(0)| \le \log\frac{1}{2\pi}\int_{-\pi}^{\pi}\exp|T(x)|dx.$$