TOPOLOGY QUALIFYING EXAM
Spring 1997
(Maginnis and Strecker)

Choose and work any 6 of the following 14 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Prove that a space Y is Hausdorff if and only if for every space X and every pair of continuous functions $f : X \to Y$ and $g : X \to Y$, the set $\{x \in X | f(x) = g(x)\}$ is closed in X.

2. Prove that every continuous function $f : X \to Y$ between topological spaces has a factorization $f = g \circ h \circ k$, where k is a quotient map, h is a bijective continuous function, and g is a topological imbedding.

3. Let $x \in X$ be a cluster point of a net $\alpha : \Lambda \to X$. Prove that x is a cluster point of the filter generated by the net α.

4. Let I be the interval $[0, 1]$ in the usual topology, and give $I \times I$ the product topology. Let $f : I \to I \times I$ be a bijective function. Prove f is not continuous.

5. For a space X and a subset $A \subseteq X$, denote by $C(A)$ the closure $C(A) = \overline{A}$, and denote by $I(A)$ the interior $I(A) = A^0$. Consider the sequence A, $C(A)$, $IC(A)$, $CIC(A)$, $ICIC(A)$, $\ldots$

   (a) For any space, what is the largest number of distinct sets that this sequence can contain?

   (b) Find $A \subseteq \mathbb{R}$ for which this largest number is obtained (where the real numbers $\mathbb{R}$ have the usual topology).

6. State the Axiom of Choice, and prove that it is equivalent to the statement "The product of any set of nonempty sets is nonempty."

7. Provide an example of a function between spaces that preserves convergence of sequences, but fails to be continuous.

8. Prove that a metrizable space is compact if and only if it is sequentially compact (i.e., each of its sequences has a convergent subsequence).

9. Prove or disprove: The reals $\mathbb{R}$ (in the usually topology) can be expressed as a countable union of subsets having empty interior.

10. Let X be a countably compact space (i.e., every countable open covering has a finite subcovering), and let Y be a compact space. Prove that the product $X \times Y$ is countably compact.

11. Prove or disprove: Any subspace of a locally connected space is locally connected. What if the subspace is assumed to be open?

12. Prove that a connected space is path connected if and only if every path component is open.

13. Let $I = [0, 1]$, and give $I \times I$ the dictionary (i.e., lexicographic) order topology. Prove that $I \times I$ is not metrizable.

14. Let $X \subseteq Y$, and assume Y is a compact Hausdorff space and that X is dense in Y. Let $\beta(X)$ be the Stone-Čech compactification of X. Show that there exists a closed continuous surjection $g : \beta(X) \to Y$ such that $g(x) = x$ for all $x \in X$.