Topology Qualifying Exam
Spring 1992

Work 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than six problems.

1. For a topological space (X, τ) define $I : \mathcal{P}(X) \to \mathcal{P}(X)$ by $I(A) = A^\circ$ and $C : \mathcal{P}(X) \to \mathcal{P}(X)$ by $C(A) = \overline{A}$. For a set A of X, consider the sequence A, $I(A), CI(A), ICI(A), CICI(A), \ldots$

a) For any space what is the largest number of distinct sets that this sequence can contain?

b) Find $A \subseteq \mathbb{R}$ for which this largest number is obtained (where $\mathbb{R}$ has the usual topology).

2. Prove that the projection of $[0,1] \times \mathbb{R}$ onto $\mathbb{R}$ is a closed map, where both have their usual topologies.

3. Prove that if $A \subseteq X \times X$, then there is a maximal subset Y of X with $Y \times Y \subseteq A$.

4. For any topological space X, define an equivalence relation $\sim$ on X by $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$. Prove that the resulting quotient space $X/\sim$ is T_0.

5. Let $\mathbb{E}$ denote the set of real numbers with the Sorgenfrey topology, which has basis consisting of all half-open intervals of the form $[x, y)$. Prove that $\mathbb{E} \times \mathbb{E}$ is not normal.

6. Let $f_i : [-1,1] \to \mathbb{R}$ be the function defined by $f_i(x) = (i + x)^2, i = -1, 0, 1$. Prove or disprove that the (unique) evaluation map $f : [-1,1] \to R^3$ with the property that $\pi_i \circ f = f_i$ is an embedding.

7. Prove that if ρ is a metric on M, then $\rho_1 : M \times M \to R$ defined by $\rho_1(x,y) = \frac{\rho(x,y)}{1+\rho(x,y)}$ is a metric on M which is equivalent to ρ.

8. Prove that if $\{A_\lambda | \lambda \in \Lambda\}$ is locally finite system of sets, then $\overline{\cup_{\lambda\in\Lambda} A_\lambda} = \cup_{\lambda\in\Lambda} \overline{A_\lambda}$.

9. Let f be a one-to-one onto function from the real line with the usual topology itself. If f and f^{-1} map connected subsets onto connected subsets, prove that f is a homeomorphism.

10. Let Ω_0 be the set of all ordinals less than the first uncountable ordinal with the order topology. We know that Ω_0 is T_4 and every continuous real-valued function on Ω_0 is constant on some tail. Find the Stone- Čech compactification of Ω_0. (Justify your answer!)

11. Let $\mathbb{P}$ be the irrational numbers with the usual (subspace) topology. Show that the intersection of any countable family of dense open subsets of $\mathbb{P}$ is dense in $\mathbb{P}$.

12. Let X be a locally compact Hausdorff space and let $C(X, Y)$ be the space of all continuous functions from X into Y with the compact-open topology. Prove that the map $P : C(X, Y) \times X \to Y$ defined by $P(f, x) = f(x)$ is continuous.