Topology Qualifying Exam

August ..., 2010

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

1. (a) Give definition of a locally finite family of subsets in a topological space.

   (b) Give an example of a family of subsets in $\mathbb{R}^1$ which is infinite and locally finite.

   (c) Prove or disprove: any locally finite family of subsets in a compact space is always finite.

2. Let $f : X \longrightarrow Y$ be a continuous, surjective map from a space X to a connected space Y. Assume that $f^{-1}(y)$ is connected for each $y \in Y$.

   (a) Show that if f is a quotient map (i.e., Y is homeomorph to the quotient space obtained from X by identifying points with the same image), then X is connected.

   (b) Give an example to show that if X is not a quotient map, then X need not be connected.

3. Using the Mayer-Vietoris theorem compute the de Rham cohomology of the space $\mathbb{R}^3 \setminus \mathbb{S}^1$, where

$$\mathbb{S}^1 = \{(x, y, 0) \mid x^2 + y^2 = 1\}.$$

4. (a) Show that there exists no covering map $\varphi : S^2 \longrightarrow T^2$ between the 2-sphere and the 2-torus.

   (b) Describe a two-fold covering map

$$\pi : T^2 \setminus \{p_1, p_2, p_3, p_4\} \longrightarrow S^2 \setminus \{p_1, p_2, p_3, p_4\},$$

   where the p_i and q_i are some pairwise distinct points in S^2 and T^2, respectively.

5. Compute the integral of the form $\omega = dx \wedge dy$ over the following surfaces (specify which orientation you consider):

   (a) The sphere $x^2 + y^2 + z^2 = 1$.

   (b) The semisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

6. Show that for any continuous map $f : S^4 \longrightarrow S^2 \times S^2$, the induced map in cohomology $f^* : H^4(S^2 \times S^2) \longrightarrow H^4(S^4)$ is trivial.

7. Define a map $F : \mathbb{R}^2 \to \mathbb{R}^1$ by $F(x, y) = x^2 + xy + y^2 + x + y$.

   (a) Find all the critical points and critical values of F.

   (b) Find all the critical points and values of $F|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{R}^1$.

8. Let X be a finite, connected CW-complex. Prove that $H_1(X; \mathbb{Z}/p\mathbb{Z}) = 0$ for all primes $p \geq 2$ if and only if $H_1(X; \mathbb{Z}) = 0$.