Analysis Qualifying exam - Spring 09

Burckel & Naibo

Instructions: Do all ten problems. Start each problem on a separate page and clearly indicate the problem number.

Notation: $\mathbb{N}$ is the positive integers, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{u \in \mathbb{C} : |u| = 1\}$, Ω is an open connected subset of $\mathbb{C}$ (=: a region), $H(\Omega)$ is the set of all holomorphic (= analytic) functions in Ω.

1. P is a polynomial of degree at most $n \in \mathbb{N}$ and $\sup\{|P(u)| : u \in \mathbb{T}\} = 1$. Show that $|P(z)| \leq |z|^n$ for all $z \in \mathbb{C} \setminus \mathbb{D}$ and determine for what P equality holds at some $z \neq 0$.

   Hint: Consider the function (polynomial!) $f(w) := w^n P(1/w)$.

2. State five equivalent definitions of Ω being simply-connected.

3. $f_n \in H(\Omega)$ and $f_n \to f$ uniformly in each compact subset of Ω. Show that f is holomorphic in Ω.

   Hint: First show that f is continuous. Then use Morera's theorem.

4. Ω is a convex region, $g \in H(\Omega)$, $f \in H(\Omega)$ is zero-free.

   (a) *Construct* a primitive h for g (and prove that $h' = g$).
   Hint: What role does convexity play?

   (b) Use (4a) to construct a holomorphic logarithm for f, that is, $L \in H(\Omega)$ such that $e^L = f$.

5. Show that $f(x) := \dfrac{\cos x}{4 + x^2}$ is absolutely integrable over $\mathbb{R}$ and compute its integral.

   Hint: When integrating over a circle, e^{iz} is better than $\cos z$.

6. Let E be a Lebesgue measurable set in $\mathbb{R}^n$. Prove that

$$E = A_1 \cup N_1 = A_2 \setminus N_2$$

   where A_1 is an F_σ set, A_2 is a G_δ set, and $m(N_1) = m(N_2) = 0$ (m denotes Lebesgue measure in $\mathbb{R}^n$).

   Hint: Recall that m is a regular measure, what does that mean?

7. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space. For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{C}$ be a measurable function. Consider the following types of convergence for the sequence $\{f_n\}$: uniform convergence (uc), convergence in measure (μ-c), convergence in $L^p(X,\mu)$ (L^p-c), and almost everywhere convergence (a.e.-c). Complete the following diagrams by drawing an arrow ($\longrightarrow$) if one type of convergence implies another and by drawing a dashed arrow ($\dashrightarrow$) if one type of convergence implies another for some subsequence. You do not have to give proofs.

	General measure space			Measure space with $\mu(X) < \infty$	
	uc			uc	
L^p-c		a.e.-c	L^p-c		a.e.-c
	μ-c			μ-c	

General measure space **Measure space with $\mu(X) < \infty$**

(b) State the Monotone Convergence Theorem, Fatou's lemma, the Dominated Convergence Theorem, and Egoroff's Theorem.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space in which measurable functions f_n and f are given. Prove that if $f_n \to f$ in measure, $|f_n| \leq |g|$ a.e. for all n, and $g \in L^p(X,\mu)$, then $f_n, f \in L^p(X,\mu)$ and $f_n \to f$ in $L^p(X,\mu)$.

9. Let E be a Lebesgue measurable set in $\mathbb{R}$ such that $0 < m(E) < \infty$ (m denotes Lebesgue measure in $\mathbb{R}$). Prove that if f is a nonnegative Lebesgue measurable function then $g(x) = \int_E f(x-t)\,dt$ is Lebesgue measurable and that $g \in L^1(\mathbb{R}, m)$ if and only if $f \in L^1(\mathbb{R}, m)$.

10. (a) Give four equivalent definitions of an orthonormal basis in a Hilbert space.

(b) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces such that $L^2(X,\mu)$ and $L^2(Y,\nu)$ are separable. If $\{u_m\}_{m\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are orthonormal bases for $L^2(X,\mu)$ and $L^2(Y,\nu)$, respectively, prove that $\{u_m v_n\}$ is an orthonormal basis for $L^2(X \times Y, \mu \times \nu)$.