

Complex Analysis Qualifying Exam

Fall 1985

1. Let U be open $\subset \mathbb{C}$, $f : U \times [0, 1] \rightarrow \mathbb{C}$ a continuous function such that $z \mapsto f(z, t)$ is holomorphic in U for each $t \in [0, 1]$.

(a) Show that then $D_1 f(z, t)$ is a continuous function on $U \times [0, 1]$.

(b) Show that the function

$$F(z) = \int_0^1 f(z, t) dt, \quad z \in U$$

is holomorphic in U and that

$$F'(z) = \int_0^1 D_1 f(z, t) dt, \quad z \in U.$$

2. Suppose that f is holomorphic in $V \setminus \{a\}$ for some neighborhood V of $a \in \mathbb{C}$.

(a) Define: f has a pole of order $k > 0$ at a .

(b) Show that if f has a pole of order k at a , then Residue of f at $a = \frac{1}{(k-1)!} \lim_{z \rightarrow a} \frac{d^k}{dz^k} [(z-a)^k f(z)]$.

3. If f is entire and periodic, then f has a fixed point.

HINT: Consider the range of the entire function $f(z) - z$.

4. Say that the continuous function $f : U \rightarrow \mathbb{C}$ has the *restricted (circumferential) mean value* property if for each $z \in U$ there is at least one $r = r(z) > 0$ such that $\{w \in \mathbb{C} : |w - z| \leq r\} \subset U$ and

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta.$$

(i) Show that if U is bounded and connected and f has this property, then f has no maximum in U unless it is constant.

HINT: Suppose that the subset M of U where the maximum is realized is not empty. It is relatively closed in U and if $M \neq U$, there will be a point $z \in M$ closest to $\mathbb{C} \setminus U$.

(ii) Suppose that U is bounded region for which the Dirichlet problem is solvable and that $g : \bar{U} \rightarrow \mathbb{C}$ is a continuous function which enjoys the restricted mean value property in U . Show that g is harmonic in U .

HINT: If h is the solution of the Dirichlet problem with boundary data $g|_{\partial U}$, consider $f = h - g$ and $f = g - h$.

5. Write an essay on "simple connectivity for planar regions". Cover the following aspects: topological characterizations involving both internal and external properties; homotopy; analytic characterizations. Define the index of a closed curve and explain its relation to simple connectivity. What special role do Jordan curves play? Elucidate the relation to Cauchy's integral theorem and the contributions of Bernhard Riemann and Carl Runge. Sketch the proofs of 4 of the equivalent definitions of simple connectivity.

6. Let $D_0 = \{0 < |z| < 1\}$ and $f, g \in H(D_0)$. Suppose that both fg and $f + g$ have poles at $z = 0$. Prove that at least one of f and g has a pole at $z = 0$.

7. Let f be holomorphic on $\{0 < |z| < 1\}$ and have an essential singularity at $z = 0$. Prove that if g is an entire function such that $g \circ f$ has a removable singularity at $z = 0$, then g is a constant.
8. Prove the following form of the maximum modulus principle:
- (a) If Ω is a bounded region in \mathbb{C} , $M \in \mathbb{R}$, f is holomorphic in Ω and $\overline{\lim}_{z \rightarrow w} |f(z)| \leq M$ for every $w \in \partial\Omega$, then $|f(z)| \leq M$ for every $z \in \Omega$.
 - (b) Give an example to show that the conclusion fails if the boundedness of Ω is not hypothesized.
 - (c) Suppose the hypotheses in (a) hold for all but one point $w_0 \in \partial\Omega$ but there we at least have $\lim_{z \rightarrow w_0} |f(z)| < \infty$. Show that the conclusion still follows.
- HINT:** For $d := \text{diameter of } \Omega$ and for positive integers n , apply (a), with appropriate new constants M , to the functions $F_n(z) := [f(z)]^n \left(\frac{z-w_0}{d}\right)$.
9. Calculate $\int_0^\infty \frac{\cos bx}{(x^2+a^2)^2} dx$, where $a \neq 0$ and b are real numbers. (Hint: WLOG: $a > 0, b \geq 0$.)
10. Consider a finite (Blaschke) product $B(z) = \prod_{k=1}^n \frac{z-a_k}{1-\overline{a_k}z}$, $a_k, z \in ID := \{z \in \mathbb{C} : |z| < 1\}$. Show that every point of ID is taken as a value n times (counted according to multiplicity) by the function B .