Numerical Analysis

Qualifying Exam, January 20, 2005

1. Suppose that the polynomial $P(x) = x^3 + ax^2 + bx + c$ has roots $\alpha_1, \alpha_2, \alpha_3$. Apply the iteration scheme using Newton's method with initial guess $x_0 = \dfrac{\alpha_1 + \alpha_2}{2}$, show that the next iteration point $x_1 = \alpha_3$.

2. The following is a quadrature formula for integration:

$$\int_a^b f(x)dx \approx \frac{4h}{3}[2f(x_1) - f(x_2) + 2f(x_3)], h = \frac{b-a}{4},$$

where $x_k = a + \dfrac{k}{4}(b-a), k = 1, 2, 3,$ evenly divide $[a, b]$ in four subintervals.

- **a.** What is the degree of precision of this quadrature formula? Give your arguments in details.
- **b.** Find the error estimate term $E(f)$ for above quadrature formula.

3. Denote $E_n = \displaystyle\int_0^1 x^n e^{x-1} dx, \quad , n = 1, 2, 3, \cdots$. Thus, applying integral by parts, $E_1 = 1/e$, and we have a recursion $E_n = 1 - nE_{n-1}$. This gives an algorithm to compute, say, E_{20}, from E_1.

- **a.** Is this recursive algorithm stable? Give your arguments.
- **b.** If you decide it is not stable, give an alternative stable algorithm to compute E_{20} using above recursion. And explain the stability of your algorithm.

4. Suppose that a function $f \in C^4[a, b]$, and $M = \max_{a \le x \le b} |f^{(4)}(x)|$.

- **a.** Describe the near minimax approximation p_3 to f of order 3 on the interval $[a, b]$.
- **b.** Find and prove the sharpest upper bound for the error $\max_{a \le x \le b} |f(x) - p_3(x)|$.

5. Find a range for the value k in the matrix A below, such that when solving the system $A\mathbf{x} = \mathbf{b}$ using the method of successive over relaxation (SOR) it will converge for arbitrary initial guess $\mathbf{x}^{(0)} \in \mathbf{R}^3$, and for arbitrary vector $\mathbf{b}$, with arbitrary choice of parameter $\omega \in (0, 2)$ for SOR. Here

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & k & 2 \\ 0 & 2 & 2 \end{pmatrix}.$$

6. Suppose that A is an $n \times n$ real matrix with n distinct real eigenvalues λ_j, $j = 1, 2, \ldots, n$. It is known that for a positive integer $k \le n$, μ is a number such that $|\mu - \lambda_k| < |\mu - \lambda_j|$ for $j \ne k$.

- **a.** Describe an iterative algorithm that computes the eigenvalue λ_k.
- **b.** Prove the convergence of the algorithm.

7. Let the factorization $PA = LU$ of a $n \times n$ nonsingular matrix A be computed by Gaussian elimination with partial pivoting, where P is a permutation matrix, L is lower triangular and R is upper triangular.

Let the growth factor for A be defined as the ratio

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|}.$$

Show that $\rho \le 2^{n-1}$.

8. A Householder reflector is a matrix of the form $F = I - 2\mathbf{v}\mathbf{v}^*$ where $\mathbf{v}$ is a (complex) unit vector of dimension n. Let $\mathbf{x}$ be a nonzero vector of dimension n.

Determine a Householder reflector F (i.e., determine a unit vector $\mathbf{v}$) so that

$$F\mathbf{x} = [\|\mathbf{x}\|, 0, \ldots, 0]^* .$$

In this question, the norm is the 2-norm, and the star represents the adjoint.