TOPOLOGY
QUALIFYING EXAMINATION
FALL - 1984
(Muenzenberger - Strecker)

Do 8 of the following 15 problems.

1. Let A be a subset of a space X and let $\bar{A}$, $Fr(A)$, and A^0 denote the closure, frontier, and interior of A in X, respectively. The set A is <u>poor</u> iff $Fr(A) = A$, and the set A is <u>thin</u> iff $\bar{A}^0 = \emptyset$. Prove the following.

   (a) If A is poor, then A is thin.

   (b) If A is closed, then A is poor iff A is thin.

   (c) Give an example of a space X in which some subset is thin but not poor.

2. Given a set X let $r : P(X) \to P(X)$ (where $P(X) = \{Y \mid Y \subseteq X\}$) such that for $A, B \in P(X)$

   (a) $r(A) \supseteq A$,

   (b) $r \circ r(A) = r(A)$,

   (c) $r(A \cup B) = r(A) \cup r(B)$, and

   (d) $r(\emptyset) = \emptyset$.

   Prove that $\{C \subseteq X \mid r(X - C) = X - C\}$ is a topology on X.

3. Let $(X, \mathcal{T})$ be a compact Hausdorff space. Prove that there is no finer compact topology on X as well as no coarser Hausdorff topology on X.

4. Give examples (but no proofs) of:

   (a) a non-normal completely regular Hausdorff space.

   (b) A regular Hausdorff space that is not completely regular.

   (c) A connected space that is not locally connected.

   (d) A compact Hausdorff space that is not first countable.

   (e) A locally Hausdorff space that is not Hausdorff.

5. Prove that $[0,1]$ is connected.

6. Prove that every infinite Hausdorff space has an infinite discrete subspace.

7. (a) Prove that if $f : X \to Y$ and $g : X \to Y$ are continuous functions and Y is Hausdorff, then the subset of X on which f and g agree is closed.

   (b) Show that the statement in (a) above is false if the condition that Y is Hausdorff is deleted.

8. If Ω is the first uncountable ordinal, prove that the ordinal space $[0,\Omega]$ (with the order topology) is compact.

9. State two of the following and prove that one of them implies the other.

   (a) Zorn's Lemma.

   (b) The Hausdorff Maximality Principle.

   (c) The Well Ordering Principle.

   (d) The Axiom of Choice.

10. Show that if X is a Tychonoff space, then any continuous function $f : X \to [0,1]$ extends to a continuous function

$$F : \beta X \to [0,1].$$

11. Prove or disprove.

    (a) Every quotient of a locally connected space is locally connected.

    (b) Every quotient of a locally compact space is locally compact.

12. A subset U of $\mathbb{R} \times \mathbb{R}$ is called radially open if and only if U contains an open line segment in each direction about each of its points.

    (a) Show that the radially open sets form a topology on the plane.

    (b) Prove or disprove that the topology of part (a) is the usual topology of the plane.

13. State and prove one of the following.

    (a) Urysohn's Lemma.

    (b) Urysohn's Metrization Theorem.

    (c) Tietze's Extension Theorem.

14. Prove that every metrizable space is paracompact.

15. Let $\hat{T}$ denote the compact surface obtained by removing an open disc from a torus T. Compute the fundamental group of $\hat{T}$.