Real and Complex Analysis Qualifying Exam

January 2007

Instructions: Work as many problems as possible. Throughout, $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{R} :=$ real numbers, $\mathbb{C} :=$ the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and (X, μ) is a measure space.

Problem 1. Let $\mu(X) = +\infty$, and let $1 \leq s < p < \infty$. Prove that $f \in L^s(\mu, X)$, $f \in L^p(\mu, X)$ imply $f \in L^r(\mu, X)$ for $s < r < p$.

Problem 2. F is holomorphic in $\mathbb{D} \setminus \{0\}$ and $\lim\limits_{z \to 0} |F(x)| = \infty$. Show that $0 \in U$ open $\subset \mathbb{D} \Rightarrow F$ maps $U \setminus \{0\}$ onto the complement of a compact set.

Problem 3. Suppose A and B are measurable subsets of $\mathbb{R}$, having finite positive measure. Show that the convolution $1_A * 1_B$ is continuous and not identically zero. Use this to prove that $A + B$ contains a segment.

Problem 4. Suppose u is harmonic on a region Ω. Define $\nabla u = (u_x, u_y)$. Show that either ∇u has isolated zeros on Ω or that u is constant on Ω.

Hint: Consider the function $f = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$.

Problem 5. Explain why in the definition of a sign-changing measure μ, ($E = \cup_{k=1}^{\infty} E_k$, $E_k \subset X$ are pairwise disjoint, imply $\mu(E) = \sum\limits_{k=1}^{\infty} \mu(E_k)$), we can assume that the series is absolutely convergent.

Problem 6. Compute $\displaystyle\int_0^{\infty} \frac{\log x}{x^4 + 1} dx$.

Problem 7. Let f be an integrable function on X. Prove that for every $\epsilon > 0$, there exists $\delta > 0$, such that $\left| \int\limits_A f d\mu \right| < \epsilon$ for any measurable $A \subset X$ satisfying $\mu(A) < \delta$.

Problem 8. Find a conformal map from the strip $\{z : 0 < \mathrm{Re} z < 1\}$ onto the half disk $\{z : \mathrm{Im} z > 0,\ 0 < |z| < 1\}$.