Spring 1984

ALGEBRA QUALIFYING EXAM

Do at least eight problems with at least two from each of the four sections.

Group Theory

1. Let $\phi: G \to H$ be a surjective homomorphism of groups, and let $K = \ker \phi$. If H_1 is a subgroup of H show that there is a unique subgroup G_1 of G such that

   (i) $K \leq G_1$,

   (ii) $\phi(G_1) = H_1$.

2. Let G be a group of order 56. Show that either

   (i) a 2-Sylow subgroup is normal, or

   (ii) a 7-Sylow subgroup is normal.

   (Extra credit: Give examples of groups G_1, G_2 of order 56 such that a 7-Sylow subgroup of G_1 is not normal and a 2-Sylow of G_2 is not normal.)

3. Let P be a finite p-group (p is prime), and let H be a proper subgroup of P. Show that $N_P(H) \supsetneqq H$.

4. Prove that no group can be written as the union of two proper subgroups. Give an example of a group which is a union of three proper subgroups.

5. Let A be an abelian group with generators a, b and relations $2a - b = 0$, $-a + 2b = 0$. Compute the structure of A.

6. Let G be the group with presentation $\langle a, b \mid a^2 = b^3 \rangle$. Show that G is infinite. (Hint: This is not hard at all! Let G_0 be the subgroup of $GL(2, \mathbb{Z})$ = 2×2 nonsingular matrices with integer entries, generated by $a_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $b_0 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

   Show that a_0, b_0 satisfy the given relation, and that G_0 is infinite.)

Rings and Modules

1. Let $\phi : R_1 \to R_2$ be a homomorphism of rings.

   a) If I_2 is an ideal of R_2, show that $\phi^{-1}(I_2)$ is an ideal of R_1.

   b) If I_1 is an ideal of R_1 show by example that $\phi(I_1)$ need not be an ideal of R_2.

2. Prove that "Chinese Remainder Theorem": If n is a positive integer with $n = ab$, a and b relatively prime, then there is an isomorphism of rings

$$\frac{\mathbb{Z}}{(n)} \cong \frac{\mathbb{Z}}{(a)} \times \frac{\mathbb{Z}}{(b)} .$$

3. Let R be a ring and let M be a left R-module. Let $\mathrm{Ann}(M) = \{r \in R \mid rM = 0\}$ be the annihilator of M.

   a) Show that $\mathrm{Ann}(M)$ is a 2-sided ideal of R

   b) If M is irreducible, and if R commutative, show that there is an isomorphism of R-modules

$$\frac{R}{\mathrm{Ann}(M)} \cong M$$

4. Let R be an integral domain such that every ideal of R is free. Prove that R is a principal ideal domain.

5. Let R be a ring and let M be a left R-module. Prove the so-called Noether isomorphism theorem: if M_1, M_2 are R-submodules of M then

$$\frac{M_1 + M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2} .$$

(Hint: Map $M_1 \to \dfrac{M_1 + M_2}{M_2}$ in the more or less obvious way.

Is the map surjective? What is the kernel?)

Linear Algebra

1. Let F be a field, and let V be a vector space over F.

   a) Define what it means for a subset $S \subseteq V$ to be a basis.

   b) Using Zorn's lemma, show that any vector space has a basis.

2. Let $\{v_1,\ldots,v_n\}$ be a basis for the vector space V over F. If $w \in V$ satisfies $w \notin \langle v_2,\ldots,v_n\rangle$ (where $\langle\ \rangle$ means F-span), show that $\{w,v_2,\ldots,v_n\}$ is a basis.

3. Let $T:V \to V$ be a linear transformation such that $T^2 = T$. Prove that the subspaces TV and $(I - T)V$ are T-invariant and that $V = TV \oplus (I - T)V$.

4. Give an example of a matrix A with rational entries such that

   minimal polynomial $= (x + 1)^2(x^2 + 1)^2(x^4 + x^3 + x^2 + x + 1)$,

   characteristic polynomail $= (x + 1)^3(x^2 + 1)^3(x^4 + x^3 + x^2 + x + 1)$

5. Let $T_1,T_2:V \to V$ be linear transformations, where V is a finite dimensional vector space over an algebraically closed field. If $T_1T_2 = T_2T_1$, prove that there exists a vector $v \in V$ which is an eigenvector for both T_1 and T_2.

Fields and Galois Theory

1. Let $F \subseteq K$ be fields and let $\alpha \in K$.

   a) State what it means for α to be algebraic over F.

   b) Prove that α is algebraic over F if $F[\alpha]$ is a finite dimension F-vector space.

2. Let F be a finite field, and let F^* be the non-zero elements of F, regarded as a multiplicative group. Show that F^* is a cyclic group. (Hint: If e = exponent of F^*, how many roots in F are there to the polynomial $x^e - 1$?)

3. Let $\sqrt[3]{2}$ be a real cube root of 2, and let ζ be the complex number $\zeta = \exp(\frac{2\pi i}{3})$. Let $K_1 = \mathbb{Q}[\sqrt[3]{2}]$, $K_2 = \mathbb{Q}[\zeta]$, $K_3 = \mathbb{Q}[\sqrt[3]{2},\zeta]$. Prove that K_1 is not normal over $\mathbb{Q}$ but that K_2, K_3 are normal over $\mathbb{Q}$.

4. Let $F \subseteq K$ be a separable normal extension of F, and let G be the Galois group of the extension. Let H be a subgroup of G and let L = field of invariants of H, i.e. $L = \{\alpha \in K | h\alpha = \alpha \text{ for all } h \in H\}$. Without using the fundamental theorem of Galois theory, prove that L is normal over F if and only if H is a normal subgroup of G.