Fall 2009

Algebra Qualifying Exam
August 25, 2009

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Write solutions to each problem on separate pages and write your name on top.

Note: All rings are assumed to be associative and with multiplicative identity 1; all integral domains are assumed to be commutative. The integers and the rational numbers are denoted by $\mathbb{Z}$ and $\mathbb{Q}$, respectively.

(1) Classify all groups of order 165 up to isomorphism.

(2) Let $G = F(x_1, \ldots, x_n)$ be the free group generated by the set $\{x_1, \ldots, x_n\}$. Assume that $k_1, \ldots, k_n$ are nonzero integers with at least one $k_j \neq \pm 1$. Prove that the subgroup

$$U = \langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle$$

is *strictly* contained in G and is *isomorphic* to G.

(3) Let $f(x) = x^4 + 2x^3 - 4x - 4$, $g(x) = x^4 + 4$. Let $K = \mathbb{Q}[x]/(f) \oplus \mathbb{Q}[x]/(g)$, let $L = \mathbb{Q}[x]/(fg)$. Are the rings K and L isomorphic?

(4) Let R be an uniform factorization domain that is not a field, and is such that $R^* \cup \{0\}$ is a subring of R. (R^* denotes the set of units of R.) Prove that R contains infinitely many prime elements which are not associated.

(5) Let A and D be square matrices. Prove the following formula for the expansion of the determinant of a block matrix:

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\ \det(D - CA^{-1}B),$$

where A is assumed to be invertible.

(6) Let p be a prime, $k \in \mathbb{Z}/p\mathbb{Z}$. Prove that the set

$$M_k = \left\{ \begin{pmatrix} a & kb \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{Z}/p\mathbb{Z} \right\}$$

together with the usual matrix addition and multiplication is a field if and only if $-k$ is not a square in $\mathbb{Z}/p\mathbb{Z}$.

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(7) (a) Prove that for finite-dimensional vector spaces V and W there is an *natural* isomorphism

$$\lambda : V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$$

of vector spaces.

(b) Show that this is not true in general for infinite-dimensional vector spaces.

(Natural means that λ is a natural transformation, i.e. that if $f : V_1 \longrightarrow V_2$ and $g : W_1 \longrightarrow W_2$ are maps between vector spaces then $\lambda \circ (\mathrm{id} \otimes g) = \bar{g} \circ \lambda$ and $\lambda \circ (f^* \otimes \mathrm{id}) = \bar{f} \circ \lambda$, where $\bar{f}$, $\bar{g}$ are the maps on the Hom-spaces induced by f and g.)

(8) (a) Prove that the additive group $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

(b) Prove that the additive group $\mathbb{R}$ is a free $\mathbb{Q}$-module.

(9) (a) Find the splitting field for $f(x) = x^4 - 10x^2 + 21$ and the basis of this splitting field over $\mathbb{Q}$.

(b) Find the Galois group of this polynomial and determine all intermediate fields between $\mathbb{Q}$ and its splitting field.

(10) Let L/K be a finite field extension such that $\mathrm{char}(K)$ does not divide $[L : K]$. Show that L/K is separable.