Algebra Qualifying Exam
January 21, 2003

Instructions: You are given six problems to work in a two-hour period.

Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

(1) Let G be a finite group and H be a subgroup of G such that the index of H in G is the smallest prime factor of the order of G. Show that H has to be normal.

(2) Let D be an integral domain containing a field F as a subring. Suppose every element of D is algebraic over F. Prove that D is a field. Then show that any finite integral domain D contains a finite field F as a subring and therefore is a field.

(3) Let A be an integral domain and $A[x]$ the ring of polynomials of variable x with coefficients in A. Determine all ring automorphisms $\phi : A[x] \to A[x]$ that are A-linear.

(4) Let R be a ring with 1 and M be a left R-module. If N and P are two R-submodules of M, define the set
$$(N : P) = \{r \in R \mid rP \subseteq N\}.$$
Show that $(N : P)$ is a two-sided ideal of R.

(5) Let $\mathbb{F}_q$ be a finite field with q elements. Then
  (a) Compute that number of distinct invertible 2×2-matrices with entries in $\mathbb{F}_q$.
  (b) For any n-dimensional vector space V-over $\mathbb{F}_q$, compute the number of 2-dimensional subspaces of V.

(6) Let K be a field of characteristic $\neq 2$, let $f(x) \in K[x]$ be an irreducible separable polynomial, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the roots of $f(x)$ in a splitting field E over K. Set
$$\delta = \prod_{1 \leq j < i \leq n} (\alpha_i - \alpha_j) \in E,$$
If $G = \mathrm{Gal(E/K)}$ and if $H = \mathrm{Gal(E/F(\delta))}$, show that $[G : H] \leq 2$. (Hint: show that $\delta^2 \in K$.)

(7) Let V be an n-dimensional vector space over an algebraically closed field k and $T : V \to V$ be a linear transformation. Show that there are two linear transformations $S : V \to V$ and $N : V \to V$ such that (a) S is diagonalizable, (b) N is nilpotent, (c) $SN = NS$, and (d) $T = S + N$.

(8) Let T_1 and T_2 be two diagonalizable linear transformations of a finite dimensional vector space V over a field F. Show that if $T_1T_2 = T_2T_1$, then both T_1T_2 and $T_1 + T_2$ are diagonalizable.