TOPOLOGY
QUALIFYING EXAMINATION
SPRING 1983

(MUENZENBERGER - SUMMERHILL)

Do 9 of the following 16 problems.

1. Let A be a subspace of a topological space X. Prove the following.

   (i) $\mathrm{Fr}\, A = \bar{A} \setminus A^{\circ}$.

   (ii) $\overline{X \setminus A} = X \setminus A^{\circ}$.

2. (i) Prove that if X is a locally connected separable space, then any open set in X is the union of countably many disjoint open connected sets.

   (ii) Prove that every open set in the reals is the union of countably many disjoint open intervals.

   (iii) Find an open set in the plane which is not the union of countably many disjoint open discs.

3. Show that the following three spaces (with the Euclidean subspace topology) are homotopy equivalent.

   (i) The unit circle S^1.

   (ii) The punctured plane $\mathbb{R}^2 - \{(0,0)\}$.

   (iii) Three space with a line removed,

   $$\mathbb{R}^3 - \{(x,0,0) \mid x \in \mathbb{R}\}.$$

4. Prove or disprove.

   (i) The quotient space of a Hausdorff space is Hausdorff.

   (ii) The property of being a regular T_1 space is invariant under continuous surjections which are both open and closed.

5. Tell whether each of the following properties is hereditary. For each negative response, provide a counter example (but no proofs).

   (i) Second countability.

   (ii) Local connectivity.

   (iii) Normality.

   (iv) Regularity with second countability.

6. Tell whether each of the following properties is preserved by continuous surjections in the Hausdorff category. For each negative response, provide a counter example (but no proofs).

   (i) The property of being a continuum.

   (ii) Second countability.

   (iii) Local compactness.

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   (iv) The property of being a topologically complete metric space.

   $(\mathbb{Q}, \text{discrete}) \to (\mathbb{Q}, \text{usual})$

7. Prove that the comb

$$C = \{(\tfrac{1}{n}, y) \mid 0 \le y \le 1; n = 1, 2, \ldots\} \cup \{(x,0) \mid 0 \le x \le 1\} \cup \{(0,y) \mid 0 \le y \le$$

   is not a retract of the square

   $(X, \text{discrete}) \to (X, \text{non-discrete } T_2)$

$$S = \{(x,y) \mid 0 \le x, y \le 1\}$$

   where both C and S have the Euclidean subspace topology.

8. Let A be a connected subspace of a connected space X. Prove that if C is a component of $X - A$, then $X - C$ is connected.

9. Prove that the intersection of any countable family of open dense sets in a locally compact Hausdorff space X is dense.

10. Prove that if $A \times B$ is a compact subset of a product $X \times Y$ contained in an open set W in $X \times Y$, then there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that

$$A \times B \subseteq U \times V \subseteq W.$$

11. Let ω_1 denote the first uncountable ordinal number and let

$$\Omega = \{\gamma \mid \gamma \text{ is an ordinal number and } 0 \le \gamma \le \omega_1\}$$

   where $<$ is the usual ordering on the class of ordinal numbers. Assume that $(\Omega, <)$ is a well ordered set and that any countable subset of $\Omega - \{\omega_1\}$ has a least upper bound in $\Omega - \{\omega_1\}$. Prove that if Ω has the order topology and $f: \Omega \to \mathbb{R}$ is a continuous real valued function, then there is $\alpha \in \Omega - \{\omega_1\}$ such that $f(\gamma) = f(\omega_1)$ for all γ such that $\alpha < \gamma \le \omega_1$.

12. Show directly that any normal T_1 space X with a countable basis can be embedded as a subspace of the Hilbert cube. (You may assume Urysohn's Lemma.)

13. Prove that a paracompact Hausdorff space X is regular.

14. Using Zorn's Lemma show that every point finite open cover $\mathcal{U}$ of a space X has an irreducible subcover $\mathcal{W}$.

15. Prove that a product of connected spaces is connected.

16. Let

$$\mathbb{C} = \{\textstyle\sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i = 0 \text{ or } a_i = 2 \text{ for all } i = 1,2,\ldots\}$$

denote the Cantor set with the Euclidean subspace topology.

(i) Show that $\mathbb{C}$ is homeomorphic to the countable product $\Pi_{i=1}^{\infty} \{0,2\}$ of discrete two point spaces.

(ii) Construct a continuous open surjection $\Pi_{i=1}^{\infty} \{0,2\} \to [0,1]$.

(iii) Construct a continuous open surjection $\Pi_{i=1}^{\infty} \{0,2\} \to [0,1] \times [0,1]$.

(iv) Conclude that there is a continuous surjection $[0,1] \to [0,1] \times [0,1]$.