Numerical Analysis Qualifying Exam
Spring 1997

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1. In some computers, division $1/a$ is done by using Newton's method. The operation becomes finding the root of the function $f(x) = a - 1/x$. Consider Newton's method for this function, write down the Newton's iteration for it (the operation only involves addition, subtraction and multiplication).

(a) Introducing the scaled residual $r_n = 1 - ax_n$, find the relation of r_{n+1} and r_n.

(b) Find the condition that guarantees the convergence.

(c) Using the results, find

$$\prod_{n=0}^{\infty}(1 + r^{2^n}), \text{ where } |r| < 1$$

(Hint: Let $r = r_0$, write x_n in terms of x_0 and r_0)

2. Suppose $S_{na}(x), S_{cl}(x)$ are the natural cubic spline interpolant and clamped cubic spline interpolant respectively for a function $f(x)$ with knots $t_0, t_1, \ldots, t_n$. That is, they both are cubic spline interpolating $f(x)$ at $a = t_0 < t_1 < \cdots < t_n = b$, besides $S''_{na}(t_0) = 0$, $S''_{na}(t_n) = 0$; $S'_{cl}(t_0) = f'(t_0)$, $S'_{cl}(t_n) = f'(t_n)$.Which of the two splines has a smaller $\int_a^b [S''(t)]^2 dt$? Justify your answer.

3. Construct a polynomial $f(x)$ with suitable degree such that $f[x_0, x_1, \ldots, x_n, x] = x^r$, where r is a natural number, $x_0 < x_1 < \cdots < x_n$ are real numbers, and $f[x_0, x_1, \ldots, x_n, x]$ is the Newton divided difference. (Hint: consider the error formula of an interpolation polynomial in terms of the divided difference)

4. Prove the following theorem for Gaussian quadrature:

Let $I(f) = \int_a^b f(x)w(x)dx$, where $w(x)$ is a positive weight function, be approximated by a quadrature formula $I_n(f) = \sum_{i=1}^n A_i f(x_i)$. Then the quadrature formula $I_n(f)$ has a maximum degree of precision of $2n - 1$. This is attained if and only if $x_1, x_2, \ldots, x_n$ are the zeros of $p_n(x)$, the nth orthogonal polynomial, with the inner product

$$(f, g) = \int_a^b f(x)g(x)w(x)dx.$$

(Hint: consider the form of $f[x_1, x_2, \ldots, x_n, x]$)

5. Can a matrix of $m \times n$ has a right inverse and a left inverse that are not equal? Justify your answer.

6. Given a linear system $Ax = b$ where $A \in C^{m \times n}, x \in C^{n \times 1}$, and $b \in C^{m \times 1}$. The system may have no solution (inconsistent) or have a unique solution, or have non-unique solutions. The **minimal solution** of the system is defined as follows:

let

$$\rho = inf\{\| Ax - b \|_2 : x \in C^n\}$$

Then the *minimal solution* is the element of least norm in the set $K = \{x : \| Ax - b \|_2 = \rho\}$.

Prove the theorem: The minimal solution of the equation $Ax = b$ is given by

$$x = A^+ b$$

where A^+ is the pseudoinverse of A, defined as $A^+ = Q^* D^+ P^*$, if the singular-value decomposition of A is $A = PDQ$, where P is an $m \times m$ unitary matrix, D is an $m \times n$ diagonal matrix, Q is an $n \times n$ unitary matrix, D^+ is an $n \times m$ diagonal matrix, P^* is the Hermitian transpose of P and

$$D = \begin{bmatrix} \sigma_1 & & & & & & \\ & \sigma_2 & & & & & \\ & & \cdots & & & & \\ & & & \sigma_r & & & \\ & & & & 0 & & \\ & & & & & \cdots & \\ & & & & & & 0 \end{bmatrix}, \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, r \leq \min(m, n),$$

$$D^+ = \begin{bmatrix} \sigma_1^{-1} & & & & & & \\ & \sigma_2^{-1} & & & & & \\ & & \cdots & & & & \\ & & & \sigma_r^{-1} & & & \\ & & & & 0 & & \\ & & & & & \cdots & \\ & & & & & & 0 \end{bmatrix}.$$

(Hint: start from the expression of ρ)

7. Find explicitly (i.e., find the numerical value of every entry of) the iterative matrix in the Gauss-Seidel iterative method for solving a linear system $Ax = b$ when

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & -1 \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}.$$

8. Consider a matrix A that does not have any zero off-diagonal entry. Prove that if an eigenvalue λ of A lies on the boundary of the union of the Gershgorin Circles of the matrix A, then the circumference of every Gershgorin Circle passes through λ. (Hint: consider the proof of the Gershgorin theorem: consider the eigenvector corresponding to λ, and the component x_k of the eigenvector which has the maximum magnitude, then consider other components)