Complex Analysis Qualifying Exam, January, 2001

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Instructions: Below you will find 8 problems. Do as many as you can.

Notations: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle); for U open in $\mathbb{C}$, $H(U)$ is the set of holomorphic functions in U.

1. Let f be a holomorphic function on the open unit disk $\mathbb{D}$, and suppose that $|f(z)| \leq M$ for all $z \in \mathbb{D}$. Prove that the number of zeros of f in $\{|z| \leq 1/4\}$ does not exceed

$$\frac{\log \left| \frac{M}{f(0)} \right|}{\log 4}.$$

Hint: Consider the Blaschke factors $\frac{z - z_j}{1 - \overline{z_j} z}$, where z_j are the zeros of f.

2. Let f be entire, $r > 0$, and $f(0) = 1$. Show that there is z with $|z| = r$ such that

$$e^{-z} f(z) \in [1, \infty).$$

Hint: Use the Argument Principle.

3. Recall that a normal family on $\mathbb{D}$ is a set $\mathcal{F}$ of functions $f : \mathbb{D} \to \mathbb{C}$ such that any sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets (here we take the usual metric on $\mathbb{C}$). Let $\mathcal{F}$ be the family of all the analytic functions on $\mathbb{D}$ with the property that $f(0) = 1$ and $f(z) \notin (-\infty, 0]$ for all $z \in \mathbb{D}$. Show that $\mathcal{F}$ is a normal family. Give a counter-example if the condition $f(0) = 1$ is omitted.

4. Find an explicit formula for a function u harmonic in the half-disk $D = \mathbb{D} \cap \{\operatorname{Re} z > 0\}$ such that $\lim_{z \to \zeta} u(z) = 0$ when $\zeta = it, -1 < t < 1$ and $\lim_{z \to \zeta} u(z) = 1$ when $\zeta \in \mathbb{T} \cap \{\operatorname{Re} z > 0\}$. Hint: Use the fact that the angle subtended by the diameter from a point on the half-circle is always equal to $\pi/2$ and the fact that $\operatorname{Arg} = \operatorname{Im} \log$ is harmonic whenever it is well-defined.

5. Let $f \in H(U)$ for some open set U. Suppose $f = e^g$ for some continuous function g. Prove that g is holomorphic in U.

6. Let f be holomorphic in $\{z \in \mathbb{C} : 0 < r < |z| < R\}$, and suppose $\operatorname{Re} f$ is constant on each circle $\{z \in \mathbb{C} : |z| = \rho\}$ with $r < \rho < R$. Show that f is constant. Hint: Consider $F(z) = f(e^z)$.

7. Suppose f is meromorphic in $\mathbb{C}$ and vanishes at infinity. Show that if $r > 0$ is large enough

$$\int_0^{2\pi} \frac{f(re^{it})}{1 - (z/r)e^{-it}} \frac{dt}{2\pi}$$

equals 0 for $|z| < r$ and equals $-f(z)$ for $|z| > r$. Hint: Consider $f(1/z)$.

8. Recall that $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$ and compute

$$\int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}.$$