Algebra Qualifying Exam
January 23, 2001

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems that you would like to be graded by circling the problem numbers on the problem sheet. Please understand that one problem done completely correctly is worth quite a bit more than two problems each half completed.
Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

1. The group G is called a CA-group if for every $e \neq x \in G$, $\mathrm{C}_G(x)$ is abelian. Prove that if G is a CA-group, then
   (a) the relation $x \sim y$ if and only if $xy = yx$ is an equivalence relation on $G^{\#}$;
   (b) If $\mathcal{C}$ is an equivalence class in $G^{\#}$, then $H = \{e\} \cup \mathcal{C}$ is a subgroup of G.
2. Let G be a group and let $M, N \triangleleft G$. If $G = MN$, prove that $G/(M \cap N) \cong G/M \times G/N$.
3. Let $G = GL_2(p)$, p prime, be the group of invertible 2×2 matrices over the field $\mathbb{F}_p$. Using the fact that $|G| = p(p-1)(p^2-1)$, compute the number of Sylow p-subgroups of G.
4. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers and let $I \subseteq \mathbb{Z}[i]$ be an ideal. Assume that I is invariant under complex conjugation, i.e., $x \in I$ implies that $\overline{x} \in I$. Prove that I must be one of the types:
   (i) $I = a\mathbb{Z}[i]$,
   (ii) $I = ai\mathbb{Z}[i]$, or
   (iii) $I = a(1+i)\mathbb{Z}[i]$,

   where $a \in \mathbb{Z}$
5. Prove that no principal ideal in the polynomial ring $\mathbb{Z}[x]$ can be maximal.
6. Let $f : R \to R$ be a surjective homomorphism of the noetherian domain R. Prove that f is injective.
7. Let V be an n-dimensional vector space over a field $\mathbb{F}$, and let
   $$V \;=\; V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n \;=\; 0$$
   be a chain of subspaces of V, with $\dim(V_i/V_{i+1}) = 1$ for $i = 0, 1, \ldots, n-1$. Suppose that $T : V \to V$ is a linear transformation satisfying $T(V_i) \subseteq V_{i+1}$ for all $i = 0, 1, \ldots, n-1$. Compute the characteristic polynomial of T.
8. Let $\mathbb{C}$ be the field of complex numbers, and let $A \in \mathrm{M}_3(\mathbb{C})$ be the matrix
   $$A \;=\; \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & -1 \\ -3 & -3 & -2 \end{bmatrix}.$$
   (a) Compute the invariant factors of A.
   (b) Compute the Jordan cononical form of A.
9. Let V be an $\mathbb{F}$-vector space and let $T : V \to V$ be a linear transformation. Assume that T is *irreducible* in that V has no T-invariant subspaces. If we set
   $$C_V(T) = \{\text{linear transformations } S : V \to V \mid ST = TS\},$$
   prove that $C_V(T) = \mathbb{F}[T]$, *i.e.*, any linear transformation on V that commutes with T is a polynomial in T. [Hint: Show that V is a 1-dimensional vector space over the *field* $\mathbb{F}[T]$. How does this help?]
10. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with Galois group G over the rational field Q. Assume that every subgroup of G is normal. Prove that $\deg f(x) = |G|$.