Partial Differential Equations Qualifying Exam
Spring 1992

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1. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $f \in C^1(\partial\Omega)$.

(i) Is the following Neumann problem well posed? Why?

$$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial n}|\partial\Omega = f \end{cases}$$

(ii) Find a necessary condition for this problem to have a solution and prove your result.

2. Suppose $u(x,t), x \in \mathbb{R}^1, t > 0$ is a solution of $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$. Suppose that on $I = [a,b] \subseteq \mathbb{R}, u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$. Show that $u \equiv 0$ in $\Omega = \left\{(x,t) = 0 \le t \le \frac{a+b}{2}, \left|x - \frac{a+b}{2}\right| \le \frac{a+b}{2} - t\right\}$. HINT: Consider the energy integral $E(t) = \frac{1}{2}\int_{B_t}(u_t)^2 + |\nabla u|^2 dx$, where

$$B_t = \left\{x : \left|x - \frac{a+b}{2}\right| \le \frac{a+b}{2} - t\right\}.$$

3. Consider the half disc Ω in $\mathbb{R}^2$ as in Fig. 1 below. Find a solution $u(x,y)$ of the Dirichlet problem

$$\begin{cases} \Delta u & = 0 \text{ in } \Omega \\ u(x,0) & = 0, -1 \le x \le 1\ . \\ u|_{\partial\Omega - [-1,1]} & = 1. \end{cases}$$

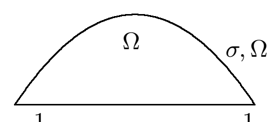

Fig.1

4. Suppose u is a continuous function on a domain $\Omega \subset \mathbb{R}^n$ which satisfies the mean value property, that is, whenever $B \subset \Omega$ is a ball centered at x then $u(x) = \frac{1}{|\partial B|}\int_{\partial B} u(y)dy$. Show that $u(x)$ is a harmonic function and is also C^∞ on Ω.

5. Let f be a C^2 function and ϕ be a C^1 function on R^1. Consider the conservation law of nonlinear flow

$$\begin{cases} u_t + (f(u))_x & = 0 \\ u(0,x) & = \phi(x), \end{cases}$$

where $u = u(t,x)$ is the density of this flow. Show that if f is convex and ϕ is decreasing then this flow must undergo a blow-up at some time $t_0 > 0$.

6. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Let $f \in C(\partial\Omega)$ be fixed. Consider the functional $Du = \int_\Omega |\nabla u|^2 dx$ where $u|_{\partial\Omega} = f$. Show that Du achieves a minimum when u is harmonic.

7. Suppose $\Delta u + u = 0$ in $\mathbb{R}^3$ and $|u(x)_{|\partial B_R}| \le \frac{C}{1+R^2}$ for some constant $c > 0$, where $B_R = \{x \in \mathbb{R}^3 : \| x \| = R\}$. Does $u \equiv 0$? Give your argument.

8. (i) Give an example that a function is weakly differentiable but not differentiable Give also the weak derivatives for your example.

(ii) Let Ω be an open bounded set with smooth boundary in $\mathbb{R}^3$. If a function $u \in W_0^{2,2}(\Omega)$, is $u(x)$ classically differentiable? Why?

9. Let H be a real Hilbert space and A a densely defined linear selfadjoint operator on H. Assume $\langle Au, u\rangle \ge c \| u \|^2, \forall u \in H$, where $c > 0$ is a constant. Let $f \in H$ be given.

(i) Show that if $u_0 \in H$ minimizes the functional $F(u) = \langle Au, u \rangle - 2\langle f, u \rangle$, then $Au_0 = f$.

(ii) In the case that $H = L^2(\Omega)$, $A = -\Delta$ under the homogeneous Dirichlet boundary condition. Specify the constant c above.