COMPLEX VARIABLES

Qualifying Exam

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In the following $\mathbb{C}$ is the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{N}$ the natural numbers, $\mathbb{R}$ the real numbers.

1. Give 2 different proofs (from complex analysis) of the Fundamental Theorem of Algebra.

2. Integrate $\dfrac{1}{z^3 - z}$ around the curve drawn in Figure 1.

3. Suppose $u(x, y)$ is harmonic in $\mathbb{D}$ and has the boundary value indicated in Figure 2. Find a numerical value for $u(1/2, 0)$. (Hint: Use a conformal map and the circumferential mean value theorem.)

4. (a) Define what is meant by the analytic continuation of an analytic function along a curve γ.

   (b) Show that the derivative of the analytic continuation of f along γ in part (a) is the analytic continuation of the derivative f' along γ.

5. (a) f is continuous in $\bar{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. Show that f is uniformly approximable on $\bar{\mathbb{D}}$ by polynomials.

   (b) State and prove the converse of (a).

6. Show that if f has an inessential singularity at a, then $\lim_{z \to a}(z - a)\dfrac{f'(z)}{f(z)}$ exists, is an integer n, and that $|n|$ is the order of a as a zero of f if $n \geq 0$, the order of a as a pole of f if $n < 0$. (Hint: Factor f at a.)

7. Name five important 19th century analysts and give a complete statement of one theorem due to each.

8. f is holomorphic in $\mathbb{D}_+ := \{z \in \mathbb{D} : \operatorname{Im} z > 0\}$, is continuous in $\bar{\mathbb{D}}_+$ and is real-valued on $[-1, 1]$. Show that f extends to a holomorphic function in $\mathbb{D}$.

9. f is holomorphic and zero-free in the simply connected region Ω. Show how to construct a holomorphic logarithm function for f, that is, a holomorphic function g in Ω such that $f = e^g$.

Figure 1

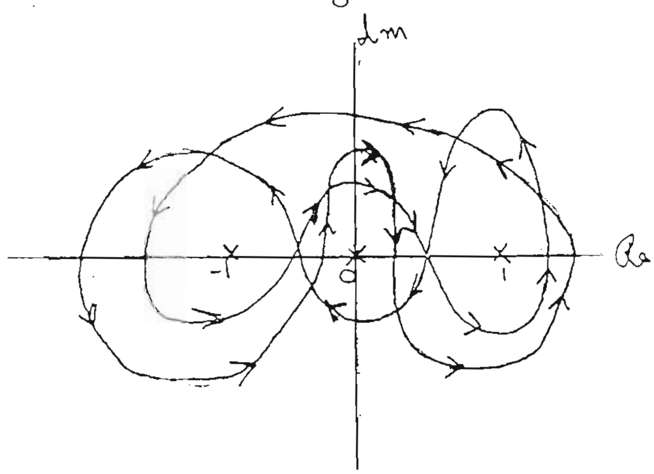

Figure 2