TOPOLOGY
QUALIFYING EXAMINATION
FALL 198[illegible]
(Strecker - [illegible])

Do [illegible] of the following 13 problems.

1. Prove that if $f: X \to Y$ is 1-1 and continuous, then for all $A \subseteq X$, $f(Fr(A)) \subseteq Fr(f(A))$ where $Fr(A) = \overline{A} \cap \overline{X - A}$.

2. (a) True - False.

____(i) Every compact Hausdorff space is metrizable.

____(ii) If $f, g: X \to Y$ are homotopically equivalent and $h, k: Y \to Z$ are homotopically equivalent, then $f \circ h$ and $g \circ k$ are homotopically equivalent.

____(iii) The product of connected spaces is connected.

____(iv) Every retract of a locally connected space is locally connected.

____(v) Every separable space is second countable.

____(vi) The Hausdorff Maximality Principle implies Zorn's Lemma.

____(vii) Every regular space is completely regular.

(b) For each false entry in (a), state a counterexample (no proofs required).

3. Prove that any continuous bijection $f: \mathbb{R} \to \mathbb{R}$ is a homeomorphism.

4. Describe the fundamental groups of the following spaces.

(a) The circle, S^1;

(b) The Möbius strip, M;

(c) The figure eight, ∞;

(d) The torus, $S^1 \times S^1$;

(e) The projective plane, $\mathbb{P}$.

5. A space X is called a <u>highly normal</u> space if and only if for every two disjoint closed sets A and B there are open sets U and V such that

$A \subseteq U$, $B \subseteq V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Prove that every normal space is highly normal.

6. Let X be a compact Hausdorff space and let $f: X \to X$ be continuous. A subset A of X is <u>invariant</u> if and only if $f(A) \subseteq A$. Prove the following.

   (a) Every nonempty, closed, invariant set $A \subseteq X$ contains a nonempty, closed, invariant set M that is minimal with respect to set-inclusion.

   (b) If M has the properties listed in (a), then $f(M) = M$.

7. Prove that the comb

$$C = \{(\tfrac{1}{n}, y) \mid 0 \le y \le 1; n = 1,2,\ldots\} \cup \{(x,0) \mid 0 \le x \le 1\} \cup \{(0,y) \mid 0 \le y \le 1\}$$

is not a retract of the square

$$S = \{(x,y) \mid 0 \le x,y \le 1\}$$

where both C and S have the Euclidean subspace topology.

8. Prove that every infinite Hausdorff space contains a copy of $\mathbb{N}$ (i.e., a countably infinite, discrete subspace).

9. Find an error in the following purported "proof" that $2^{\mathbb{R}}$ is metrizable, where $2 = \{0,1\}$ is a two point discrete space, and $\mathbb{R}$ is the set of real numbers.

   "Proof". Consider the inclusion $\mathbb{N} \subseteq \mathbb{R}$, where $\mathbb{N}$ is the natural numbers. This induces a "natural" embedding $2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{R}}$ where $f \mapsto \hat{f}$ and $\hat{f}(r) = \begin{cases} f(r) & \text{if } r \in \mathbb{N} \\ 0 & \text{if } r \in \mathbb{R} - \mathbb{N}. \end{cases}$

   But $2^{\mathbb{N}}$ is the Cantor space C. Thus C is embedded in $2^{\mathbb{R}}$. Let U be an open subset of $2^{\mathbb{R}}$. By the definition of product topology, the projection of U is $\{0,1\} = 2$ in all but finitely many coordinates. Thus $U \cap 2^{\mathbb{N}} \neq \emptyset$. So $2^{\mathbb{N}}$ is dense in $2^{\mathbb{R}}$. But $2^{\mathbb{N}}$ is compact and $2^{\mathbb{R}}$ is Hausdorff, so $2^{\mathbb{N}}$ is closed in $2^{\mathbb{R}}$. Thus $2^{\mathbb{N}} = 2^{\mathbb{R}}$. But C is metrizable, and so $2^{\mathbb{R}}$ is metrizable.

10. Use the following two facts to prove that a product of compact Hausdorff spaces is compact.

    (a) A compact Hausdorff space is completely regular.

    (b) A product of completely regular spaces is completely regular.

11. A topological space X is called extremally disconnected if and only if the closure of every open set is open. Prove that a space X is extremally disconnected if and only if every two disjoint open sets in X have disjoint closures.

12. Let ω_1 denote the first uncountable ordinal number and let

$$\Omega = \{\gamma \mid \gamma \text{ is an ordinal number and } 0 \leq \gamma \leq \omega_1\}$$

where $\leq$ is the usual ordering on the class of ordinal numbers. Assume that $(\Omega, \leq)$ is a well ordered set. Prove the following.

(a) If A is a countable subset of $\Omega - \{\omega_1\}$, then lub $A < \omega_1$.

(b) If Ω has the order topology and $f: \Omega \to \mathbb{R}$ is a continuous real valued function, then there is $\alpha \in \Omega - \{\omega_1\}$ such that $f(\gamma) = f(\omega_1)$ for all γ satisfying $\alpha \leq \gamma \leq \omega_1$.

13. Let $S = \{(x, \sin(\frac{\pi}{x})) \mid 0 < x \leq 1\}$, $L = \{(0,y) \mid -1 \leq y \leq 1\}$, and let $X = S \cup L$ have the Euclidean subspace topology. Let $f:[0,1] \to X$ be continuous. Prove the following.

(a) If $f[0,1] \cap S \neq \emptyset$, then $f[0,1] \subseteq S$.

(b) If $f[0,1] \cap L \neq \emptyset$, then $f[0,1] \subseteq L$.

14. Prove that the following two statements about a T_1-space X are equivalent.

(a) Every infinite subset of X has a cluster point in X.

(b) At least one member of every infinite open cover of X can be discarded with the remaining sets still covering X.

15. State and prove one of the following theorems.

(a) Urysohn's Lemma.

(b) Urysohn's Metrization Theorem.

(c) Tietze's Extension Theorem.

(d) Tychonoff's Theorem.