Topology Qualifying Exam
August 2000
(Muenzenberger and Strecker)

Instructions: Choose and work any 6 of the following 14 problems. Start each problem on a new sheet of paper. <u>Do not turn in more that six problems.</u> In the problems below, a space means a topological space.

1. Prove or disprove that the continuous image of a compact space is compact.

2.(a) True - False.

(i) An open and closed one-to-one function between topological spaces must be an embedding.

(ii) Each space that is locally-Hausdorff (in the sense that each point has a neighborhood base of Hausdorff subspaces) must be Hausdorff.

(iii) Each quotient of a locally connected space must be locally connected.

(iv) Each locally compact Hausdorff space is completely regular.

(v) The product of metrizable spaces is metrizable.

(vi) The product of continua is a continuum. (A continuum is a compact, connected, Hausdorff space.)

(vii) Every metrizable space is normal.

(viii) Every subspace of a separable space is separable.

(b) For each false entry, give a counterexample or other explanation (no proofs).

3. Let $(X_i, \tau_i)_{i\in I}$ be a family of topological spaces and let τ be the product topology on ΠX_i. Let $Y_i \subseteq X_i$ for each $i \in I$. There is a "natural" subspace topology σ on ΠY_i that is induced from τ and the fact that ΠY_i is a subset of ΠX_i. There is also a "natural" product topology ρ on ΠY_i that comes from the subspace topologies $(Y_i, \rho_i), i \in I$ induced by the topologies $(\tau_i, i \in I)$. Prove or disprove that $\sigma = \rho$.

4. Let $f : X \to Y$ be a continuous surjective map from a compact space X to a Hausdorff space Y. Prove that f is a quotient map.

5. Use Zorn's lemma to prove that for every set X and relation R on X there is a maximal $A \subseteq X$ such that $A \times A \subseteq R$.

6. Prove that the interval $[0, 1]$ with the usual topology is connected.

7. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be final provided that for each topological space (Z, μ) each set-function $g : Y \to Z$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \mu)$ is continuous. Prove any two of the following:

(a) The composition of final (continuous) maps is final.

(b) The "second factor" of a final map is final; i.e., if $(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{h} (W, \rho)$ are continuous maps and $h \circ f$ is final, then h is final.

(c) Every quotient map is final.

8. Prove or disprove each of the following:

a) Every subspace of a compact Hausdorff space is locally compact.

b) Every subspace of a connected Hausdorff space is locally connected.

9. Describe the fundamental groups of the following spaces, where each has the usual topology.

(a) The circle, S^1;

(b) The Mobius Strip, M;

(c) The figure eight, ∞;

(d) The torus, $S^1 \times S^1$;

(e) The projective plane, $\mathbb{P}$.

10. Prove that if a filter F is contained in a unique ultrafilter G, then $F = G$.

11. Prove that every continuous bijection $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, where $\mathbb{R}$ has the usual topology.

12. Prove that if A and B are compact subsets of a Hausdorff space X, then there are open disjoint sets U and V such that $A \subseteq U$ and $B \subseteq V$.

13. Prove that if X and Y are spaces where Y is compact and $\{x\} \times Y \subseteq W$, an open set in $X \times Y$, then there is U open in X and V open in Y such that $\{x\} \times Y \subseteq U \times V \subseteq W$.

14. Prove that if a normal space X contains a closed homemorphic image of $\mathbb{R}$ (with the usual topology), then there is a continuous function $f : X \to X$ such that $f(x) \neq x$ for all $x \in X$.