Real and Complex Analysis Qualifying Exam – Fall 2014
Hakobyan & Poggi-Corradini

Instructions: Below you will find 10 problems. Do as many as you can.
Notations: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk).

1. Let f be analytic in $\mathbb{D}$ and assume that $f(0) = 0$.

(a) Show that the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \in \mathbb{D} \setminus \{0\} \\ f'(0) & \text{if } z = 0 \end{cases}$$

is analytic in all of $\mathbb{D}$.

(b) Assume further that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Use the function g in (a) to show that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

2. Suppose $g(t)$ is a continuous complex-valued function on $[0, 2\pi]$ with $g(0) = g(2\pi)$. Use the geometric series to prove that

$$\int_0^{2\pi} g(t) \frac{e^{it} + z}{e^{it} - z} dt$$

can be expanded in a power series about $z_0 = 1/2$, i.e., in powers of $(z - 1/2)$.

3. Let Ω be the intersection of the disks $\{z : |z - 3i| < 5\}$ and $\{z : |z + 3i| < 5\}$. Find a conformal map f of Ω onto the unit disk $\mathbb{D}$ so that $f(0) = 0$ and $f'(0) > 0$. You can leave f as a composition of maps.

4. Let f be entire and bounded on the strip $0 \leq \operatorname{Re} z \leq 1$, with the property that

$$f(z + 1) = f(z)/2 \qquad \forall z \in \mathbb{C}.$$

Prove that $f(z) = a2^{-z}$ for some $a \in \mathbb{C}$.
(Hint: You may want to use Liouville's theorem.)

5. Assume that the polynomial $P(z) = z + a_2 z^2 + \cdots + a_n z^n$ is one-to-one in $\mathbb{D}$. A point $c \in \mathbb{C}$ such that $P'(c) = 0$ is called a critical point.

(a) Explain why P does not have any critical points in $\mathbb{D}$.

(b) Deduce from (a) that $|a_n| \leq 1/n$ by factoring P' into its $n - 1$ roots.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are measurable functions. Prove that the sets

$$\{x : f(x) < g(x)\} \text{ and } \{x : f(x) = g(x)\}$$

are measurable.

7. Below m denotes the Lebesgue measure on $\mathbb{R}$.

(i). Show that for every $\varepsilon > 0$ there is an open dense set U such that $m(U) < \varepsilon$.

(ii). Show that for every $\varepsilon > 0$ there is a compact, nowhere dense set $E \subset \mathbb{R}$ such that $m(E) > 1 - \varepsilon$.

8. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, μ is a positive measure, $f_n \in L^p(X)$ for $n \in \mathbb{N}$ and $f \in L^p(X)$, where $1 \le p < \infty$. Prove the following implications:

(i). If $\|f - f_n\|_p \to 0$ as $n \to \infty$ then $\|f_n\|_p \xrightarrow[n\to\infty]{} \|f\|_p$.

(ii). If $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ then $\|f - f_n\|_p \to 0$.

9. Is it true that every closed and bounded set in $L^2([0,1])$ is compact? Justify your answer.

10. Suppose $0 < r < p < s < \infty$. Show that $L^r \cap L^s \subset L^p$. Here L^p, $p > 0$, denotes the space of p - integrable functions on the real line $\mathbb{R}$ equipped with the Lebesgue measure.
(Hint: Write $p = \lambda r + (1 - \lambda)s$, for some $\lambda > 0$.)