Fall 2005 Numerical Analysis Qualifying Exam

Do at least 6 problems. You may do as many as possible.

1. Let

$$E_n = \int_0^1 \frac{x^n}{x+2} dx, \ n = 0, 1, 2, 3, \cdots.$$

It is routine to check that $E_0 = \ln(3/2)$ and that $E_n = \frac{1}{n} - 2E_{n-1}$ for $n \geq 1$.

- **1.** We apply the iteration scheme $E_n = \frac{1}{n} - 2E_{n-1}$ to calculate $E_1, E_2, E_3, \cdots, E_n, \cdots$. Show that this scheme is unstable.
- **2.** Suggest an alternative stable scheme to compute E_{20}.

2. The iteration sequence is defined as $x_{n+1} = \dfrac{x_n(x_n^2 + 3a)}{3x_n^2 + a}$, where a is a given positive number. Find the order of convergence of the iteration.

3. Apply the quadrature formula $K(f) = \dfrac{3h}{4}[f(0) + 3f(2h)]$ to compute the integral $I(f) = \displaystyle\int_0^a f(x)dx$, where $a = 3h$.

- **1.** Find the degree of precision of this quadrature formula.
- **2.** Determine the error term $E(f)$.

4. Consider the effect of rounding errors and the truncation error of using the rule

$$f''(x_1) \approx D^2(x_1) := \frac{f_2 - 2f_1 + f_0}{h^2},$$

where $f_k = f(x_k)$ and $x_k = x_0 + kh$. It is known that

$$f''(x_1) - D^2(x_1) = -\frac{h^2}{12} f^{(4)}(\xi),$$

where $\xi \in (x_0, x_2)$. If $|f^{(4)}(x)| \leq M_4$, and the f_k are in (absolute) error by at most ϵ, find an error bound for using the rule. For what value of h, the error bound is minimized?

5. Let A be an arbitrary nonsingular matrix. Show that if $Ax = b$ and $A\tilde{x} = \tilde{b}$, then

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|b - \tilde{b}\|}{\|b\|},$$

where b is a nonzero vector, $\| \cdot \|$ stands for a vector norm and its induced matrix norm, and where

$$\kappa(A) = \|A^{-1}\| \|A\|$$

is the condition number of A associated with the norm.

6. In stability analysis of linear algebra problems, one often deals with the following product:

$$\gamma = \prod_{j=1}^{n} (1 + \varepsilon_j)$$

where $|\varepsilon_j| \leq \varepsilon$. Show that if $n\varepsilon \leq 0.01$, then $|\gamma - 1| \leq 1.01 n\varepsilon$.

7. Let $A \in \mathbf{R}^{n \times n}$ be an upper Hessenberg matrix (i.e., $a_{ij} = 0$ if $i \geq j + 2$).

- (a) Show that the matrix remains being upper Hessenberg during the process of Gaussian elimination with partial pivoting.
- (b) Show that the factorization $PA = LU$ where P is a permutation matrix obtained, if possible, through Gaussian elimination with partial pivoting satisfies

  $$\max_{i,j} |u_{ij}| \leq n \max_{i,j} |a_{ij}|,$$

  where u_{ij} and a_{ij} are the i, j-entry of U and A respectively.

8. Let A be an $m \times m$ matrix and $\mathbf{b}$ be a vector of dimension m. The $m \times k$ matrix K_k is called the $m \times k$ Krylov matrix with respect to the pair $(A, \mathbf{b})$ if for $j = 1 : k$, the jth column of K_k is $A^{j-1}\mathbf{b}$. Show that if A is nonsingular and $R(K_l) = R(K_{l-1})$, then the solution of $A\mathbf{x} = \mathbf{b}$ satisfies $\mathbf{x} \in R(K_{l-1})$. Here, for any matrix A, $R(A)$ stands for the range of A (I.e., the linear space spanned by the columns of A.)