REAL ANALYSIS
Ph D QUALIFYING EXAM
APRIL 29, 1985

Do as much as you can but remember that five correct solutions will count more than ten half-solutions

Throughout this exam $(X, \mathcal{A}, \mu)$ denotes an arbitrary measure space and λ denotes Lebesgue outer measure on the real line $\mathbb{R}$

In 1 and 2 discuss the truth value of the lettered assertions. Quote theorems, give proofs or counter examples, etc.

1. Let E be a λ-measurable subset of $\mathbb{R}$.

   (a) If $\lambda(E) > 0$, then E contains a nonvoid open interval.

   (b) If E is closed but contains no rational number, then $\lambda(E) = 0$.

   (c) If $\lambda(E) = 0$, then E is of first Baire category in $\mathbb{R}$.

2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of μ-integrable functions on X. This means that each f_n is complex-valued, $\mathcal{A}$-measurable, and $\int |f_n| d\mu < \infty$.

   (a) If $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges μ-a.e.

   (b) If $\lim_{n\to\infty} f_n(x) = 0$ for __all__ $x \in X$, then $\lim_{n\to\infty} \int f_n d\mu = 0$.

   (c) If $\lim_{n\to\infty} \int |f_n| d\mu = 0$, then $\lim_{n\to\infty} f_n(x) = 0$ μ-a.e.

3. Prove that if $(f_n)_{n=1}^{\infty}$ is a sequence of μ-integrable functions on X such that

$$\lim_{m,n\to\infty} \int |f_m - f_n| d\mu = 0,$$

then there exist a subsequence $(f_{n_k})_{k=1}^{\infty}$ and a μ-integrable function f such that

   (a) $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ μ-a.e.

   and

   (b) $\lim_{n\to\infty} \int |f - f_n| d\mu = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Suppose that

$$\int_{-\infty}^{\infty} f(x)g(x)dx = 0$$

for each continuous $g : \mathbb{R} \to \mathbb{R}$ such that $\overline{\{x \in \mathbb{R} : g(x) \neq 0\}}$ is bounded. Give a detailed proof that $f = 0$ almost everywhere.

5. Prove that there exist Banach spaces that are not reflexive.

6. Prove that if $f \in L_1(\mathbb{R})$ and g is defined on $\mathbb{R}$ by

$$g(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}dt,$$

then

(a) g is continuous on $\mathbb{R}$ and

(b) $\lim_{|x| \to \infty} g(x) = 0$.

[Hint: Show that the set of f for which (a) and (b) hold is both dense and closed in L_1]

7. Let $a < b$ be real numbers and let $f \in L_1([a,b])$ Define

$$F(x) = \int_a^x f(t)dt \qquad (a \leq x \leq b)$$

and

$$V = \sup \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|$$

where this supremum is taken over all (finite) subdivisions $\{a = x_0 < x_1 < \dots < x_n = b\}$ of $[a,b]$. Prove that

$$V = \int_a^b |f(t)|dt.$$

8. For $f \in C([0,1])$ and $n \in \mathbb{N}$ define

$$J_n(f) = \int_0^1 f(t) \frac{\sin nt}{t} dt.$$

Prove that there exists such an f such that

$$\overline{\lim_{n\to\infty}} |J_n(f)| = \infty.$$

9. Let $a < b$ be real numbers and let $\phi : [a,b] \to \mathbb{R}$ be Borel measurable and satisfy

$$\lambda(E) = 0 \Longrightarrow \lambda(\phi^{-1}(E)) = 0.$$

(a) Prove that there is a Lebesgue integrable function $w : \mathbb{R} \to \mathbb{R}$ such that

$$\int_a^b f(\phi(t))\,dt = \int_{-\infty}^{\infty} f(x)w(x)\,dx$$

for all bounded Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$.

(b) If $[a,b] = [0,2\pi]$ and $\phi(t) = \cos t$, then what is w?

10. Let $f,g \in L_1([0,1])$. Prove that the formula

$$h(x) = \int_0^x f(x - t)g(t)\,dt$$

defines h almost everywhere on $[0,1]$, that $h \in L_1([0,1])$, and that

$$||h||_1 \leqq ||f||_1 \cdot ||g||_1$$

where $||\phi||_1 = \int_0^1 |\phi(t)|\,dt$ for $\phi \in L_1([0,1])$.