**Algebra Qualifying Exam, June 2016**

June 3rd, 2016

Last name ...............................

First name ..............................

KSU Email ..............................

**1. (10 pts)**

a) Let $H$ and $K$ be normal subgroups of a finite group $G$. Suppose that $H \cap K = \{1\}$ and the order of $G$ equals the product of orders of $H$ and $K$. Show that $G \simeq H \times K$.

b) Let $p$ and $q$ be positive prime integers, such that $p < q$ and $p \nmid q - 1$. Show that all groups of order $pq$ are isomorphic to each other.

Hint: use Sylow's theorems and part a).

**2. (10 pts)**

a) Consider the field $\mathbb{Q}(\sqrt[3]{-3})$. Show that this field is NOT a normal extension of $\mathbb{Q}$. (Here $\sqrt[3]{-3}$ is the real cubic root of $-3$.)

b) Let $F$ be the Galois closure of the field $\mathbb{Q}(\sqrt[3]{-3})$. Show that $F$ is isomorphic to $\mathbb{Q}[x]/(x^6+3)$.

Hint: we know that $\sqrt[3]{-3} \in F$. Use the geometry of complex numbers to show that also $\sqrt{-3} \in F$, and use it to deduce that $\sqrt[6]{-3} \in F$.

**3. (10 pts)**

Let $k[x, y]$ be the ring of polynomials in two variables, where $k$ is a field of characteristic not equal to 2. Consider the ideal $I = (x^2 - y, x^2 + y + 2)$.

a) Assume that $-1$ is a square in $k$. Show that $I = I_+ \cap I_- = I_+ I_-$, where $I_+ = (x + \sqrt{-1}, y + 1), I_- = (x - \sqrt{-1}, y + 1)$. Is $I$ a prime ideal? Provide a proof.

b) Assume that $-1$ is NOT a square in $k$. Show that $I$ is a maximal ideal in $k[x, y]$. Show that $K := k[x, y]/I$ is a field and is isomorphic to $k[x]/(x^2 + 1)$.

**4. (10 pts)**

Let $M \in M_{n\times m}(k)$ be an $n\times m$ matrix with entries from a field $k$. Define the row rank of $M$ as the dimension of the subspace in $k^m$ spanned by the rows of $M$, and the column rank of $M$ as the dimension of the subspace in $k^n$ spanned by the columns of $M$. Show that these ranks are equal.

**5. (10 pts)** Let $A$ be a linear transformation of a complex 4-dimensional vector space, such that $A$ in NOT diagonalizable, and it satisfies the property $A^3 = A^2$.

a) What eigenvalues might $A$ have?

b) What Jordan blocks can the Jordan canonical form of $A$ have?

c) What Jordan canonical form might $A$ have?

Provide proofs.

**6. (10 pts)**

Let $R$ be a principal ideal domain and let $I \subset R$ be a non-zero ideal in $R$. Prove that $I$ is isomorphic to $R$ as an $R$-module.