Numerical Analysis Qualifying Exam
Fall 1992

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1. Consider evaluating $\exp(-x)$ for large x by using the Taylor approximation,

$$\exp(-x) \approx 1 - x + \frac{x^2}{2!} - \cdots + (-1)^n \frac{x^n}{n!}.$$

If one uses it to evaluate $\exp(-4)$, determine n so that the Taylor approximation error is less than .0005. Suppose one does the computation using 4-digit rounding, what trouble will one encounter? How should $\exp(-x)$ be evaluated for large positive values of x?

2. Let p be a root of the equation $f(x) = 0$, and assume that the Newton's method for this equation converges to p. Assume also that $f(x)$ has sufficiently high derivatives.

(a) If p is a single root, show that the Newton's method converges quadratically.

(b) If p is a multiple root, show that it converges only lineraly.

3. Suppose that the Lagrange interpolation formula for the function f at the $n+1$ distinct nodes $x_0, x_1, \ldots x_n$ is given by

$$P_n(x) = \sum_{j=0}^{n} l_{j,n}(x) f(x_j),$$

where the Lagrange polynomial coefficients are given by

$$l_{j,n}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}.$$

Show that for any $n \geq 1$,

$$\sum_{j=0}^{n} l_{j,n}(x) = 1.$$

4. The Trapezoidal Rule for an intergral $\int_a^b f(x)dx$ is based on linear interpolation and hence has degree of precision (DOP) at least one. The Simpson's Rule is based on quadratic interpolation with nodes $a, (a+b)/2, b$ and hence has DOP at least two. Based on the defination of DOP, explain, without quoting the theorems on the accuracy of the quadrature rules, why the Simpson's Rule has DOP three while the Trapezoidal Rule has DOP only one. (Hint: first consider special a, b)

5. Consider the numerical differentiation formula

$$f'(a) = \frac{1}{2h}\left[-3f(a) + 4f(a+h) - f(a+2h)\right] + \frac{h^2}{3} f'''(\xi),$$

where $a < \xi < a + 2h$. If f''' is bounded by M, and the absolute error in evaluation of f due to rounding is bounded by ε, discuss the effect of the rounding error and find the best choice for the value of h (ignoring the rounding error in h and all the other rounding errors except the ones appear in the evaluations of f).

6. Give an **upper bound** for the relative error in the solution of the system of linear equations

$$Ax = b$$

with symmetric matrix A given by

$$A = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 6 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

when the relative error in **b** is less that $4 \cdot 10^{-4}$, i.e.

$$\frac{\| \delta b \|}{\| b \|} < 4 \cdot 10^{-4}.$$

Use spectral norms, i.e., use

$$\| b \| = \left(\sum_j |b_j^2| \right)^{1/2}.$$

7. Let $A = (a_{ij})$ be an $n \times n$ matrix. An iterative scheme for the solution of the linear system $Ax = b$ is described by

$$\text{given } x_i^{(0)}, \quad i = 1, \ldots, n;$$

$$a_{ii} y_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} y_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

$$x_i^{(k+1)} = \omega y_i^{(k+1)} + (1 - \omega) x_i^{(k)}, \quad i = 1, \ldots, n; \quad k = 0, 1, \ldots$$

(a) Write the iterative scheme in the form

$$x^{(k+1)} = T x^{(k)} + c$$

(Hint: consider the splitting $A = D - L - U$).

(b) For the particular case

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

varify that the choice $\omega = 8/7$ gives the best rate of convergence.

8. Given $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ (T means transpose), define $v = x + \text{sign}(x_1) \| x \|_2 \, e_1$, where $e_1 = (1, 0, \ldots, 0)^T$. The Householder matrix (Householder transformation) with v (Householder vector) is given by

$$P = I - 2\frac{vv^T}{v^T v},$$

which is orthogonal and symmetric.

(a) Verify that $Px = -\text{sign}(x_1) \| x \|_2 \, e_1$.

(b) Describe how the Householder matrices can be used to construct an orthogonal matrix Q for a given matrix $A \in R^{n \times n}$ such that

$$A = QR,$$

where R is upper tirangular.