QUALIFYING EXAMINATION IN ALGEBRA
SUMMER, 1998

Groups.

1. Classify all groups of order $7 * 11 * 13$.

2. Let G be a finite group and let S be a p-Sylow subgroup of G. Suppose S is abelian. Let x, y be elements of S and assume y is conjugate to x in G --i.e. $y = g^{-1}xg$ for some g in G. Show that y is conjugate to x in $N_G(S)$ --that is $y = h^{-1}xh$ for some h in $N_G(S)$.

3. Let $G = \langle x, y, z \mid x^2 = y^2 = z^2 = 1,\ (xy)^3 = (yz)^3 = 1,\ xz = zx \rangle$

Show that G is a finite group. (Extra credit: What is the order of G?)

4. (a) Define what it means for a group to act on a set.

   (b) Show that a group of order 120 cannot be simple.

5. Let G be a finite group and let f be an automorphism of G. Assume f is an involution (i.e. f^2 is the identity automorphism) and that $f(g) \neq g$ for any non-identity g in G. Show that G is abelian and $f(g) = g^{-1}$.

Rings and Modules.

1. Prove the "Chinese Remainder Theorem": If n is a positive integer, $n = ab$ with a and b relatively prime, then there exists a ring isomorphism:

2. Let R be the ring $\mathbb{Z}[\sqrt{3}]$. Show that R contains infinitely many units.

3. Prove or disprove:

(a) $\mathbb{R}[x]/(x^2+x+1)\mathbb{R}[x] \cong \mathbb{R}[x]/(x^2+1)\mathbb{R}[x]$

(b) $\mathbb{R}[x]/(x^2+x+1)\mathbb{R}[x] \cong \mathbb{R}[x]/(x^2+x)\mathbb{R}[x]$

4. Let F be a finite field. Show that $F[x]$ contains irreducible polynomials of arbitrarily high degree.

5. Let M be an R-module.

(a) Define what it means for M to be irreducible.

(b) Show that if M is an irreducible R module, then

$\mathrm{End}_R(M) = \{ R\text{-homomorphisms } f: M \longrightarrow M \}$ is a division ring.

Linear Algebra.

1. Show that there exists a 2-by-2 matrix A with real entries satisfying

$$A \neq I, \; A^{17} = I.$$

2. Determine the Jordan form of the real matrix

$$\begin{pmatrix} 1 & 0 & 1 & -2 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

3. Let V be a 2-dimensional vector over $\mathbb{R}$ and let $A: V \to V$ be a linear map. Suppose A has no real eigenvalues. Then there exists a basis for V with respect to which A has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

or some a, b in $\mathbb{R}$.

4. Suppose A and B have the same minimal polynomials and the same characteristic polynomials. Let

$$\chi(x) = \prod_{i=1}^{t} (x-a_i)^{m_i}$$

be the characteristic polynomial, where the a_i run through the distinct eigenvalues of A and B. Suppose each m_i lies between 1 and 3 inclusively. Prove that A and B are similar.

5. Let $SL_3(5)$ be the group of 3-by-3 matrices over a field of 5 elements having determinant 1. Determine the order of $SL_3(5)$.

Fields and Galois Theory

1. Let F be a finite field and K a finite extension of F with $[K:F] = n$. Show

(a) K is normal over F

(b) $Gal(K/F)$ is cyclic.

2. Prove the following result: Let N be a normal extension field of the field K. Let f be an irreducible polynomial of $K[x]$. Assume N contains a root of f. Then f factors completely into linear factors over N.

3. Let $F = GF(2)$, $f = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Let K be a splitting field for f over F. What is $[K:F]$?

4. (a) Define what it means for an extension field K to be separable over F.

(b) Show that if char $F = 0$, then any extension field K over F is separable over F.

5. State the Fundamental Theorem of Galois Theory and state without proof the main ideas involved in its proof.