Algebra Qualifying exam, August 2010

Instructions: You are given ten problems from which you are supposed to do eight. Please, indicate those eight problems which you would like to be graded by circling the problem number on the problem sheet.

Note. All rings in this exam are associative, unital (that is with 1) and non-zero; and all ring homomorphisms map unit elements to unit elements.

1. Let R be an associative ring and $I,\ J$ its left ideals. Let IJ denote the set of all finite sums $\sum_i a_i b_i$, where $a_i \in I$ and $b_i \in J$.

Show that the map $(I, J) \longmapsto IJ$ turns the set $I_\ell(R)$ of left ideals of the ring R into a *monoid*; i.e. the operation is associative and there is the identity element.

2. A two-sided proper ideal $\mathfrak{P}$ of an associative ring R is called *prime*, if, for two left ideals I and J, the inclusion $IJ \subseteq \mathfrak{P}$ implies that either $I \subseteq \mathfrak{P}$, or $J \subseteq \mathfrak{P}$.

A two-sided ideal $\mathfrak{p}$ is called *completely prime*, if the set $R - \mathfrak{p}$ is closed under multiplication.

(a) Show that a two-sided ideal $\mathfrak{P}$ is prime iff for any two elements x and y of the ring R, the inclusion $xRy \subseteq \mathfrak{P}$ implies that either $x \in \mathfrak{P}$, or $y \in \mathfrak{P}$.

(b) Deduce from (a) that

(b1) Every completely prime two-sided ideal is prime.

(b2) If the ring R is commutative, then the notions of prime ideals and completely prime ideals coincide.

(c) An element x of the ring R is called *idempotent*, if $x^2 = x$.
Let $\mathfrak{p}$ be a completely prime ideal of the ring R and n a positive integer. Show that if x is an idempotent of the quotient ring $R/\mathfrak{p}^n$, then x is either 1, or 0.

3. Let $R \xrightarrow{\varphi} S$ be a ring homomorphism. For every R-module $\mathcal{M}$, consider the set $Hom_R(S, \mathcal{M})$ of R-module homomorphisms from S to $\mathcal{M}$.

(a) Prove that the map

$$S \times Hom_R(S, \mathcal{M}) \xrightarrow{\zeta^{\varphi}_{\mathcal{M}}} Hom_R(S, \mathcal{M})$$

defined by $a\xi(x) = \xi(x\varphi(a))$ for all $a, x \in S$ and $\xi \in Hom_R(S, \mathcal{M})$ is a left S-module structure on $Hom_R(S, \mathcal{M})$. We denote this module by $\varphi^!(\mathcal{M})$.

(b) Show that, for any R-module homomorphism $\mathcal{L} \xrightarrow{f} \mathcal{M}$, the map

$$Hom_R(S, \mathcal{L}) \xrightarrow{\varphi^!(f)} Hom_R(S, \mathcal{M})$$

defined by $\varphi^!(f)(\xi) = f \circ \xi$ for any $\xi \in Hom_R(S, \mathcal{L})$ is an S-module homomorphism

$$\varphi^!(\mathcal{L}) \xrightarrow{\varphi^!(f)} \varphi^!(\mathcal{M})$$

and the correspondence $f \longmapsto \varphi^!(f)$ is functorial; that is it maps identical morphisms to identical morphisms and $\varphi^!(g \circ f) = \varphi^!(g) \circ \varphi^!(f)$ for any composable pair (f, g) of R-module morphisms.

(c) For every S-module $\mathcal{L}$ and an R-module $\mathcal{M}$, construct explicitly a canonical isomorphism

$$Hom_S(\mathcal{L}, \varphi^!(\mathcal{M})) \xrightarrow{\sim} Hom_R(\varphi_*(\mathcal{L}), \mathcal{M}),$$

where $\varphi_*(\mathcal{L})$ is an R-module obtained from the S-module $\mathcal{L}$ via restriction of scalars.

(d^*) (for extra credit) Use (c) (and the fact that restriction of scalars preserves monomorphisms) to show that $\varphi^!$ maps injective modules to injective modules. Recall that a left R-module E is called *injective* if, for every monomorphism $M \xrightarrow{\mathfrak{j}} N$ of R-modules, the map

$$Hom_R(N, E) \longrightarrow Hom_R(M, E), \quad \xi \mapsto \xi \circ \mathfrak{j},$$

is surjective.

4. Recall that a left module P over an associative ring R is called *projective* if, for every epimorphism $M \xrightarrow{\mathfrak{e}} N$ of R-modules, the map

$$Hom_R(P, M) \longrightarrow Hom_R(P, N), \quad \xi \mapsto \mathfrak{e} \circ \xi,$$

is surjective. Prove the following assertions:

(a) Every free left R-module is projective.

(b) A module P is projective if and only if every epimorphism $M \xrightarrow{f} P$ *splits*; that is there exists a morphism $P \xrightarrow{g} M$ such that the composition $f \circ g$ is the identical map.

(c) Deduce from (a) and (b) that an R-module P is projective if and only if it is a direct summand of a free R-module. That is there exists an R-module L such that $P \oplus L$ is isomorphic to a free R-module.

(d) Let $R \xrightarrow{\varphi} S$ be a ring homomorphism. Prove that, for any projective R-module P, the S-module $\varphi^*(P) \overset{\text{def}}{=} S \otimes_R P$ is projective.

5. Let V be an n-dimensional vector space over a field k of zero characteristic. Let $V \xrightarrow{\mathfrak{f}} V$ a k-linear transformation. Let $A_{\mathfrak{f},\mathfrak{b}}$ be the matrix of $\mathfrak{f}$ for some basis $\mathfrak{b}$ of the vector space V.

(a) Show that the trace $tr(A_{\mathfrak{f},\mathfrak{b}})$ of the matrix $A_{\mathfrak{f},\mathfrak{b}}$ (which is, by definition, the sum of its diagonal entrees) is an invariant of the transformation $V \xrightarrow{\mathfrak{f}} V$ (that is it does not depend on the choice of a basis). So that we write $tr(\mathfrak{f})$ instead of $tr(A_{\mathfrak{f},\mathfrak{b}})$.

(b) Show that the following conditions are equivalent:

(i) The transformation $V \xrightarrow{\mathfrak{f}} V$ is nilpotent (that is some power of $\mathfrak{f}$ is the zero transformation).

(ii) $\mathfrak{f}^n = 0$ (where $n = dim_k V$.)

(iii) $tr(\mathfrak{f}^m) = 0$ for $1 \leq m \leq n$.

(**Hint** for (iii): one can replace V by $\bar{k} \otimes_k V$, where $\bar{k}$ is the algebraic closure of the field k, and $\mathfrak{f}$ by $\bar{k} \otimes_k V \xrightarrow{\bar{k}\otimes_k \mathfrak{f}} \bar{k} \otimes_k V$, reducing this way the problem to the case when the base field k is algebraically closed. Use the fact that, if the field is algebraically closed, a linear transformation is nilpotent iff its only eigenvalue is zero.)

6. Let R be an associative ring, $\mathcal{L}$ a right R-module and $\mathcal{M}$ a left R-module. The tensor product $\mathcal{L} \otimes_R \mathcal{M}$ is the quotient of the tensor product $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{M}$ over $\mathbb{Z}$ by the $\mathbb{Z}$-module generated by all elements of the form $xr \otimes y - x \otimes ry$, where $r \in R$, $x \in \mathcal{L}$, and $y \in \mathcal{M}$.

(a) Show that, for every $\mathbb{Z}$-module $\mathcal{V}$, the map

$$R \times Hom_{\mathbb{Z}}(\mathcal{L}, \mathcal{V}) \longrightarrow Hom_{\mathbb{Z}}(\mathcal{L}, \mathcal{V})$$

defined by $r\xi(z) = \xi(zr)$ for all $r \in R$, $z \in \mathcal{L}$, and $\xi \in Hom_{\mathbb{Z}}(\mathcal{L}, \mathcal{V})$, is an R-module structure.

(b) Prove that the map

$$Hom(\mathcal{L} \otimes_R \mathcal{M}, \mathcal{V}) \longrightarrow Hom_R(\mathcal{M}, Hom_R(\mathcal{L}, \mathcal{V}))$$

which assigns to every $\mathcal{L} \otimes_R \mathcal{M} \xrightarrow{\mathfrak{u}} \mathcal{V}$ the map $\mathcal{M} \xrightarrow{\mathfrak{u}'} Hom_R(\mathcal{L}, \mathcal{V})$ sending $x \in \mathcal{M}$ to the map $\mathcal{L} \xrightarrow{\mathfrak{u}'_x} \mathcal{V}$, $z \mapsto \mathfrak{u}(z \otimes x)$ is a $\mathbb{Z}$-module isomorphism.

7. Prove that a group of order pq, where p and q are primes, has a proper normal subgroup.

8. Let P be a Sylow p-subgroup of a finite group G. Let N be a normal subgroup of G. Prove that $P \cap N$ is a Sylow subgroup of N and PN/N a Sylow p-subgroup of G/N.

9. Determine the group of automorphisms of the field $\mathbb{Q}(2^{\frac{1}{3}})$.

10. Let $F = \mathbb{Z}_2(u)$ be the field of rational functions over the field $\mathbb{Z}_2$ of two elements. Show that the polynomial $x^2 - u$ is irreducible in $F[x]$ and it has two equal roots in its splitting field.