Algebra Qualifying Exam
(Old and New System)
August 25, 2005

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Write solutions to each problem on separate pages and write your name on top.

Note: All rings are assumed to be associative and with multiplicative identity 1; all integral domains are assumed to be commutative. The integers and the rational numbers are denoted by $\mathbf{Z}$ and $\mathbf{Q}$, respectively.

1. Prove that a group G of order 315 cannot be simple. (Hint: Investigate the normalizer of a Sylow 3-subgroup.)

2. Let H be a normal subgroup of a finite group G. Let $S < G$ be a conjugacy class of elements in G, and assume that $S < H$. Prove that S is a union of n conjugacy classes in H, all having the same cardinality, where n equals the index $[G : H.C_G(x)]$ of the group generated by H and the centralizer in G of any element $x \in S$.

3. Let R be a unique factorization domain. Assume that for any pair of elements a, $b \in R$, the ideal $I = (a, b)$ they generate is a principal ideal. Prove that R is a principal ideal domain.

4. Let $f(x)$ and $g(x)$ be nonconstant polynomials with coefficients in a field F. Assume that $f(x)$ is irreducible in $F[x]$, and denote by n the degree of $f(x)$. Prove that the degree of every irreducible factor of the composition $f(g(x))$ is a multiple of n.

5. Let $L = \mathbf{F}_3(x, y)$ be the field of rational functions in the variables x and y over the finite field $\mathbf{F}_3$ with three elements. Let $K = \mathbf{F}_3(x^3, y^3) \subset L$. Prove that the field extension L/K is not simple. (Hint: Show that $[L : K] = 9$, but any simple extension of K by an element of L has degree at most 3.)

6. Construct a Galois extension F of the field $\mathbf{Q}$ of rational numbers with Galois group $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ and determine explicitly a primitive element for F. (Hint: Look for a subfield of a cyclotomic field, i.e., a field obtained by adjoining a root of unity.)

7. Let R be an integral domain and I be a principal ideal of R. Show that the R-module $I \otimes_R I$ is torsion free, i.e., there are no nonzero elements $r \in R$ and $m \in I \otimes_R I$ such that $rm = 0$.

8. Let $0 \longrightarrow N_1 \longrightarrow P_1 \xrightarrow{\varphi_1} M \longrightarrow 0$ and $0 \longrightarrow N_2 \longrightarrow P_2 \xrightarrow{\varphi_2} M \longrightarrow 0$ be two short exact sequences of left modules over a ring R. Assume that both P_1 and P_2 are projective R-modules. Prove that the direct sum $P_1 \oplus N_2$ is isomorphic to $P_2 \oplus N_1$. (Hint: Consider the subset of the direct sum $P_1 \oplus P_2$ consisting of those pairs (x, y) such that $\varphi_1(x) = \varphi_2(y)$.)

9. Let A be a square $n \times n$ matrix with entries in the rational numbers. Assume that $A^3 + A^2 - I = 0$. Prove that $\det(A)$ is nonzero and that n is a multiple of three.

10. Let $\mathrm{GL}_5(\mathbf{Q})$ be the group of invertible 5×5 matrices with rational entries and matrix multiplication as product. Find representatives for all conjugacy classes of elements of order 10 in $\mathrm{GL}_5(\mathbf{Q})$.