Numerical Analysis Qualifying Exam
Fall 1997

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1. Explain the difficulty in evaluating $f(x) = x^{-1}\,(1 - \cos x)$ when the absolute value of x is small and find a method without using the Taylor series to overcome the difficulty.

2. Prove the following theorem: If $f \in C^2(a,b)$, $f'(x)\ f''(x) \neq 0$, and $f(x)$ has a zero in (a,b), then the zero is unique in (a,b), and the Newton iteration will converge to it if the starting value x_0 and the first approximation x_1 are both in (a,b). (You may just do a special case where $f'(x) < 0$, $f''(x) < 0$ in (a,b))

3. Prove the following theorem for Gaussian quadrature:

Let $I(f) = \int_a^b f(x)w(x)dx$, where $w(x)$ is a positive weight function, be approximated by a quadrature formula $I_n(f) = \sum_{i=1}^n A_i f(x_i)$, where $x_i \in [a,b]$, $i = 1:n$ are distinct. Then the quadrature formula $I_n(f)$ has a maximum degree of precision of $2n-1$. This is attained if and only if $x_i, x_2, \ldots, x_n$ are the zeros of $p_n(x)$, the nth orthogonal polynomial, with the inner product

$$(f,g) = \int_a^b f(x)g(x)w(x)dx.$$

(Hint: if $f(x)$ is a polynomial of degree m with $m \geq n$, then $f[x_1, x_2, \ldots, x_n, x]$ is a polynomial of degree $m-n$)

4. Let $f(x)$ be a piecewise constant function on $[a,b]$:

$$f(x) = \begin{cases} \alpha & a \leq x \leq c \\ \beta & c < x \leq b \end{cases}$$

where $a < c < b$ and $\alpha \neq \beta$. Let $x_0 < x_1 < \cdots < x_n$ be $n+1$ points and j be an integer with $0 \leq j \leq n-1$ such that $a < x_0 < \cdots < x_j < c < x_{j+1} < \cdots < x_n < b$ and that $P_n(x)$ is the polynomial of degree less than or equal to n interpolating f at the $n+1$ points x_i, $i = 0, \ldots, n$. Show that $P_n(x)$ is monotone in the interval $[x_j, x_{j+1}]$. (Hint: Count the number of zeros of $\frac{dP_n(x)}{dx}$.)

5. Suppose $A \in R^{n\times n}, b \in R^n$, and $PA = LU$, where P is a permutation matrix, L is lower triangular, and U is upper triangular. A, b, P, L, U are known. Determine the purpose of the following algorithm:

For $j = 1:k$

  solve for y in $Ly = Pb$ overwrite b with y

  solve for x in $Ux = b$ overwrite b with x

end

(Hint: consider $k = 1$ first)

6. Let A be an invertible $n \times n$ matrix, and let u and v be two vectors in R^n. Find the necessary and sufficient conditions on u and v in order that the matrix

$$B = \begin{bmatrix} A & u \\ v^t & 0 \end{bmatrix}.$$

be invertible, and give the inverse in terms of A, u, v when it exists. (Hint: multiply B by a suitable matrix to have a simpler matrix to handle)

7. Suppose that an n by n nonsingular matrix A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding right eigenvectors $u_1, \ldots, u_n$ and left eigenvectors $v_1, \ldots, v_n$. Suppose

$$c_i = \| u_i \|_2 \| v_i \|_2 / |v_i^T u_i|, i = 1, \ldots, n.$$

Show that the solution of $Ax = b$ satisfies

$$\| x \|_2 \leq \| b \|_2 \sum_{i=1}^{n} \frac{c_i}{|\lambda_i|}.$$

(Hint: write x in terms of u_i)

8. State Schur's theorem and use it to show: Let A be an $n \times n$ complex matrix, it can be similar to a upper triangular matrix with |off-diag. entries| $\leq \varepsilon$, (Hint: on the result of Schur's, construct suitable diagonal matrices)