Complex Analysis Qualifying Exam
Fall 1984

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1. Let f be analytic on the open unit disc $D = D(0,1)$ in the sense that f' exists and is continuous on D. Give an *ab inito* proof that

$$\int_\gamma f(z)dz = 0$$

for every closed smooth curve γ in D.

HINT: Consider the derivative of

$$I(r) = \int_{r\gamma} f(z)dz,$$

where $(r\gamma)(t) = r \cdot \gamma(t)$.

2. Evaluate

$$\oint_{|z|=1} \frac{dz}{z^2 \sin z}$$

by means of the Cauchy Redidue Theorem.

3. Let f be analytic and bounded on the punctured disc

$$D_0 = \{z \varepsilon \mathbb{C} : 0 < |z - z_0| < R\}.$$

Prove that f has a removable singularity at z_0.

4. Prove that each polynomial P of the form

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

satisfies

$$\sup\{|P(z)| : |z| \leq 1\} \geq 1.$$

5. Suppose that f is an entire function such that

$$Ref(z) \leq |z|^n$$

for some natural number n and all z with sufficiently large $|z|$. Prove that f is a polynomial.

6. Let f be an entire function of finite order. Suppose f has only finitely many zeros. Prove that $f = Qe^P$ for some polynomials P and Q.

HINT: Use problem 5.

7. Let f be a continuous function on $(0,1] \times D$, where $D = D(0,1)$. Suppose:

(i) For each t in $[0,1]$, $f(t,z) = 0$ has a unique soultion z in D, and

(ii) For each *rational* number t in $[0,1]$, $f(t,\cdot)$ is analytic on D.

Prove that there exists a continuous function $t \to z(t) : [0,1] \to D$ such that $f(t, z(t)) = 0$ for all t in $[0,1]$.

HINT: First show that for *every* t in $[0,1]$, $f(t,\cdot)$ is analytic on D.

8. The Riemann zeta function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \quad (\mathrm{Re}\, z > 1),$$

where $n^{-z} = \exp(-z \log n)$ for all n. Prove that ζ is well-defined and analytic for $\mathrm{Re}\, z > 1$.

9. Prove or disprove the existence of a double sequence $(a_{k,n})$ of complex numbers such that

(a) $\sum_{k=1}^{\infty} |a_{k,n}| \leq C$ for all $n = 1, 2, \ldots$; and

(b) $\lim_n \sin\left(\sum_{k=1}^{\infty} a_{k,n} z^k\right) = 1$ for all z in K,

where C is a finite constant and $K = \{1/2 + l/m : m = 3, 4, \ldots\}$.

HINT: Montel's theorem.

10. Let G be a nonempty open connected subset of $\mathbb{C}$. State at least three conditions each of which is equivalent to the simple connectedness of G, and prove their equivalence.