Complex Analysis Qualifying Exam
Fall 1983

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1. Prove that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic on $\mathbb{C} \backslash [-1, 1]$, then f is an entire function.

2. Let $f(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n} z^n, z \varepsilon \mathbb{C}$.

a) For which values of z does $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ converge?

b) Find the largest region on which f is analytic.

c) Can f be analytically continued to domain including $z = 1$? *You must prove all your answers.*

3. Let f be analytic on $D_2 = \{z \varepsilon \mathbb{C} : |z| < 2\}$. Suppose $|f(z)| < 1$ if $|z| = 1$. Show that f has exactly one fixed point inside the unit circle.

4. Use Cauchy's Residue Theorem to evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

5. (a) Prove that for $n = 1, 2, \ldots$

$$\frac{1}{\sqrt{n}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-nt^2} dt$$

by using the identity $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$, z not an integer.

(b) Find an analytic extension of $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n}}$, $|z| < 1$ to $\mathbb{C} \backslash [1, \infty)$. *Justify all steps.*

6. Let P and Q be polynomials with (degree Q) $\geq$ (degree P) $= 2$. Calculate the sum of the residues of P/Q.

7. Consider the class of all entire functions F such that if $f \varepsilon F$ then for all $z \varepsilon \mathbb{C}$.

$$f(z) = f(z + 1)$$

and

$$f(z) = f(z + i).$$

Characterize F.

8. Find a harmonic function $u(x, y)$ in $D = \{z \varepsilon \mathbb{C} : |z| < 1\}$ with boundary values $u(x, y) = x^2$.

9. (i) Give an example (as simple as possible) of an entire function with zeros precisely at:

(a) $n!, n = 1, 2, \ldots$

(b) $\sqrt{n} \log n, n = 2, 3, \ldots$

(c) $\log n, n = 2, 3, \ldots$

(ii) Give an example of an entire function of order one with zeros precisely at the positive even integers.

Justify all answers!

10. Prove that an entire function of order $3/2$ must have infinitely many zeros.