Analysis Qualifying Exam, Fall 2008

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Instruction: Pick five problems from #1 − 6, and pick five problems from #7 − 12. Start each problem with a separate page, and clearly label the problem number.

Notations: $\mathbb{R}$ denotes the reals, $\mathbb{C}$ the complexes, $\mathbb{D}$ the open disk $\{z \in \mathbb{C},\ |z| < 1\}$, Ω an open connected, non-void subset of $\mathbb{C}$, and $H(\Omega)$ the holomorphic functions on Ω, and $\mathbb{T}$ the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

1. Let $a \in \mathbb{C} \setminus \{0\}$. What is the radius of convergence of the power series $\sum_{n=0}^{\infty} a^n z^{n^2}$?

2. (a) Let M and R be positive numbers, and f be a holomorphic function in $R\mathbb{D}$ and bounded by M. Show that

$$|f(w) - f(0)| \leq 2MR^{-1}|w|, \quad w \in R\mathbb{D}.$$

   Hint: Apply Schwarz' Lemma to an appropriate function.

   (b) If F is holomorphic and bounded in $\mathbb{C}$, use (a) to infer (Liouville's Theorem) that F is constant.

3. Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

   (a) Show that this series converges absolutely in $\bar{\mathbb{D}}$.

   (b) Show that f has no holomorphic extension to any neighborhood of $z = 1$.

   Hint: Look at the power series for $f''(z)$ in $\mathbb{D}$ and consider *real* $z \nearrow 1$.

4. For positive real a, compute $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+a^2} dx$.

5. Let K be a compact subset of Ω, an open connected subset of $\mathbb{C}$. Let $u : \Omega \to \mathbb{R}$ be harmonic, $c \in \mathbb{R}$, and $u \leq c$ in $\Omega \setminus K$. Show that $u \leq c$ throughout Ω.

6. Assume that U, V are open subsets of $\mathbb{C}$, and $g_n, g : U \to V$ are holomorphic, $f_n, f : V \to \mathbb{C}$ are holomorphic, $f_n \to f$ and $g_n \to g$ locally uniformly. Prove that $f_n \circ g_n \to f \circ g$ locally uniformly.

   Can the holomorphy hypothesis be weakened to mere continuity?

7. (a) State Fubini's theorem.

   (b) State Lebesgue's dominated convergence theorem.

8. Decide which space is bigger, $L^1([0,1])$ or $L^2([0,1])$? Explain why.

9. Let $dm(z)$ be the normalized Lebesgue area measure on $\mathbb{D}$, that is, $dm(z) = \frac{dA(z)}{2\pi}$, and let $B = \{f \in H(\mathbb{D}), \int_{\mathbb{D}} |f(z)|^2 dm(z) < \infty\}$. Define a Hilbert inner product on B by

$$< f, g >= \int_{\mathbb{D}} f(z)\overline{g(z)} dm(z)$$

   Then prove the space B is complete with respect to the metric defined by the above inner product.

10. Let $f, g \in L^1(\mathbb{R})$. Define the convolution $f * g$ and show that $f * g \in L^1(\mathbb{R})$.

11. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function with $f(0) = 0$. Show that for any $t \in [0,1]$,

$$|f(t)| \leq [\int_0^1 f'(x)^2 dx]^{1/2}.$$

12. Let f be holomorphic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$. Show that

$$\lim_{r \to 1^-} ||f_r||_p = ||f||_p.$$

   Here $f_r(z) = f(rz)$ and for $1 \leq p < \infty$, the p-norm $||g||_p$ on the circle $\mathbb{T}$ is defined by

$$||g||_p = \{\int_{\mathbb{T}} |g|^p dm\}^{1/p},$$

   and dm is the normalized Lebesgue measure on $\mathbb{T}$, that is, $m(\mathbb{T}) = 1$.