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GEOMETRY OF MANIFOLDS QUALIFYING EXAM

Fall 1999

(Auckly & Miller)

Work as many as you can in the 2 hours.
Best of luck.

1. (A) On $\mathbb{R}^2$ let

$$\begin{aligned} X &= x^2y\tfrac{\partial}{\partial x} + (x+y)\tfrac{\partial}{\partial y} \\ Y &= y\tfrac{\partial}{\partial x} + x\tfrac{\partial}{\partial y} \\ \alpha &= -y\,dx + x\,dy \end{aligned}$$

Calculate:

a) $[X,Y]$ b) $L_Y\alpha$ c) $L_Y d\alpha$

2. On $\mathbb{R}^2$ consider the metric $g = (1+x^2)dx^2 + \frac{1}{2}(dx\otimes dy + dy\otimes dx) + dy^2$.

a) Compute $\nabla_{\frac{\partial}{\partial x}}(y\,dx)$.

b) Calculate the sectional curvature.

3. On $\{(x,y)\}\Big| x^2+y^2<1\}$, let $g = \dfrac{dx^2+dy^2}{(1-x^2-y^2)^2}$ be a metric.

a) Show that $\phi(z) = \dfrac{az+b}{\bar{b}z+\bar{a}}$ is an isometry if $|a|^2-|b|^2=1$ and $z=x+iy$.

Hint: Show that if $w=\phi(z)$ is analytic then $dw\otimes d\overline{w} = |\phi'(z)|^2 dz\otimes d\overline{z}$.

b) Using this metric, compute the radius and area of a circle with Euclidean radius $R<1$ centered at the origin.

4. Let α be a differential 2-form on the 2-sphere S^2 with $\int_{S^2}\alpha = 1$. Suppose $f: S^3\to S^2$ is smooth, S^3 the 3-sphere.

a) Show that there exists a 1-form θ on S^3 such that $f^*\alpha = d\theta$.

b) Define $Q(f) = \int_{S^3}\theta\wedge d\theta$. Show that this is independent of the choice of α and θ .

Hint: First show that for α fixed it is independent of choice of θ .

5. Suppose that $F : N \to M$ is a smooth covering mapping and that M is a Riemannian manifold with metric g.

a) Show that there exists a unique metric on N so that F is a local isometry.

b) Suppose that N is connected and compact. Determine the relation between volume (M) and volume (N) in terms of the fundamental groups of M and N.

6. Suppose M is a smooth oriented n-dimensional manifold and X is a complete vector field which generates the 1-parameter group of diffeomorphisms $(F_t)_t$. Suppose that μ is a differential n-form on M and that U is a relatively compact ($\overline{U}$ is compact) open subset of M.

a) Show that $\left.\dfrac{d}{dt}\right|_{t=0} \displaystyle\int_{F_t(U)} \mu = \int_U L_X \mu$.

b) For $M = R^3$ and $\mu = dx \wedge dy \wedge dz$, the usual volume element, calculate an expression for $L_X\mu$ for any vector field $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$. Thus obtain a formula for $\left.\frac{d}{dt}\right|_{t=0}$ volume $(F_t(U))$.

7. Prove that smooth connected manifolds are topologically homogeneous. That is, given $p, q \in M$ there is a diffeomorphism $f : M \to M$ so that $f(p) = q$.

8. On $\mathbb{R}^2$ with coordinates (x^1, x^2) let a connection (Γ^i_{jk}) be given by $\Gamma^1_{11} = \frac{\partial f}{\partial x^1}$ and $\Gamma^2_{22} = \frac{\partial f}{\partial x^2}$ and all other $\Gamma^i_{jk} = 0$ Here $f : \mathbb{R}^2 \to R$ is some given function.

Let $P_0(x^1_0, x^2_0)$ be a given point. If $v : [a, b] \to \mathbb{R}^2$ is a smooth curve such that $v(a) = v(b) = P_0$, let T_v be the 2×2 matrix which represents (with respect to $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$) the holonomy transformation $T_{p_0}\mathbb{R}^2 \to T_{p_0}\mathbb{R}^2$ of parallel transport around v.

a) Show that T_v is a diagonal matrix with determinant equal to 1.

b) For $f(x^1, x^2) = x^1x^2$ let
$v_c : [0, 4] \to \mathbb{R}^2$ by
$v_c(t) = ((\ln c)t, 0)$ if $0 \le t < 1$,
$v_c(t) = (\ln c, t - 1)$ if $1 \le t < 2$,
$v_c(t) = ((\ln c)(3 - t), 1)$ if $2 < t < 3$.
$v_c(t) = (0, 4 - t)$ if $3 \le t \le 4$.
Find T_{v_c}.