Real Analysis Qualifying Exam
September 3, 2002

Instructions: Do all 8 problems.

1. Suppose $(X, \mathcal{M}, \mu)$ is a measure space.

(a) State the definition of "$f : X \to \mathbb{C}$ is measurable".

(b) Prove that if $f, g : X \to \mathbb{C}$ are both measurable, then so is fg.

2. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, μ a σ-finite measure, and $f : X \to [0, \infty]$ is measurable. Suppose that $\int_A f\, d\mu = \mu(A)$ for each measurable set A with $\mu(A) < \infty$. Prove that $f = 1$ a.e.

3. Let $f, g \in L^1(\mathbb{R})$. Prove that $f * g$ is defined a.e. and $f * g \in L^1(\mathbb{R})$.

4. Use the Monotone Convergence Theorem to prove Fatou's Lemma.

5. Suppose $f \in L^p(\mathbb{R})$, $1 \le p < \infty$ and for $h \in \mathbb{R}$ define a function f_h by $f_h(x) = f(x + h)$.

Prove that $\lim_{h \to 0} \|f_h - f\|_p = 0$.

6. Let λ be Lebesgue measure on $\mathbb{R}$. Is it true that $\lambda(F \setminus \mathrm{int} F) = 0$ for every closed set $F \subseteq \mathbb{R}$?

7. Let $f \in L^1([0, 1])$. Consider the following two statements:

(a) For a.e. $x \in [0, 1]$, f is continuous at x.

(b) There exists a continuous function g such that $f = g$ a.e.

For each of the possible implications $(a) \Rightarrow (b)$ or $(b) \Rightarrow (a)$ either prove that the implication is true or give a counterexample.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{M}_0 \subseteq \mathcal{M}$ a sub σ- algebra of $\mathcal{M}$. Prove that given $f \in L^1(X, \mathcal{M}, \mu)$, there exists an $f_0 \in L^1(X, \mathcal{M}_0, \mu)$ such that $\int_X fg\, d\mu = \int_X f_0 g\, d\mu$ for all simple $\mathcal{M}_0$ measurable functions g.