Partial Differential Equations
Qualifying Examination
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Do any 7 problems.

1. (i) State carefully the maximum principle for the solution of Cauchy problem

$$u_t = \Delta u, \quad x \in \mathrm{R}^n, \quad t > 0$$
$$u(0,x) = f(x) \quad f \quad \text{is bounded, continuous in } \mathrm{R}^n$$

   (ii) Write out a counter example that the solution of Cauchy problem of heat equation is not unique.

2. Suppose $\Delta u = 0$ in R^3. $u \in C^2(\mathrm{R}^3)$, $u \geq 0$. What can you say about u? Can you find all such u?

3. Suppose $\Delta u = 0$ in $D \subset \mathrm{R}^3$, D is a bounded domain, $\Gamma = \partial D$ is smooth. Assume that $u = u_N = 0$ on Γ, u_N is the normal derivative. Then $u = 0$ in D, true or false?

4. In a wave propagation with external force under elastic constraint on boundary $\partial\Omega$ as

$$\begin{cases} u_{tt} = c^2 \Delta u + f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ \left(\dfrac{\partial u}{\partial n} + \sigma u\right)_{|\partial\Omega \times [0,T]} = p(x,t) \end{cases} \qquad \begin{array}{l} x \in \Omega \subset \mathrm{R}^n, \quad 0 < t < T \\ f, \phi, \psi, p \text{ are smooth.} \end{array}$$

Show that the solution $u(x,t)$ is stable with respect to the external force $f(x,t)$.

5. Consider the Conservation law

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

(i) What is the break time and locally classical solution?
(ii) Find the shock wave to this problem.

6. (i) Show that all eigenvalues of

$$\begin{aligned} \Delta u &= \lambda u \\ u_{|\partial\Omega} &= 0 \end{aligned} \qquad \Omega \subset \mathbf{R}^u, \quad \partial\Omega \quad \text{is smooth}$$

are negative and that $\phi_n \perp \phi_m$ where ϕ_n , ϕ_m are eigenfunctions corresponding to $\lambda_n \neq \lambda_m$.
(ii) State the Strum-Liouville theory for this problem.
(iii) Using the method of separation of variable to solve heat equation

$$\begin{cases} u_t - a^2 \Delta u = 0 & t > 0, x \in \Omega \subset \mathbf{R}^n \\ u(x,t)_{|x \in \partial\Omega} = 0 & t > 0 \\ u(x,0) = f(x) & x \in \Omega \end{cases}$$

in terms of eigenfunction ϕ_n s.

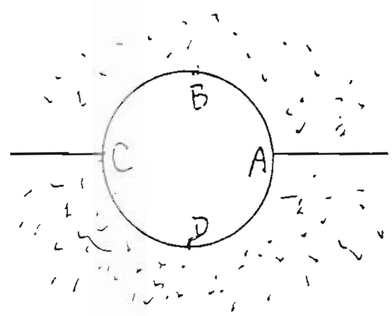
7. A continuous function $u(x)$ is said to satisfy the MVP (Mean Value Property) if

$$u(y) = \frac{1}{|\partial B_R|} \int_{\partial B_R} u(s) ds$$

holds for every small ball B centered at y with radius R. Show that a continuous function u satisfying MVP must be harmonic function.

8. Suppose $\Delta u + \Phi(u) = 0$ in R^3. $C_1(|u|+1)^p \leq \Phi(u) \leq C_2(|u|^p+1)$, $C_1 > 0$, C_i are constants, $i = 1,2$. Suppose $p > 2$. Can $|u| = 0\left(\frac{1}{|x|^\alpha}\right)$ $\alpha > 1$, $|x| \to \infty$ be true?

9. An infinite Conducting plate has in it a circular hole of unit radius. Temperatures of 20 ° C and 80 ° C are applied to arcs ABC, ADC and maintained indefinitely. Find the Steady State temperature distribution of the plate.



[HINT: Perform a suitable Möbius Transform first.]

10. Let $\Delta u = 0$ in R^3.
    (a) Suppose $|u(z, \theta_0)| \leq C\exp(-ar)$, $a = \text{Const} > 0$. $C = \text{Const} > 0$ $r = \| x \|$ $\theta = \frac{x}{r} \in S^2$. $\theta_0 \in S^2$ is fixed. Does this imply that $u \equiv 0$?
    (b) Suppose $\Delta u + u = 0$ in R^3. $|u| \leq C(1+r^2)^{-1}$. Does this imply $u \equiv 0$?