Algebra Qualifying Exam
Spring 1992

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Prove that no group of order 120 can be a simple group.

2. Let G be a group, and let H, K be *solvable*- subgroups of G with K normal. Prove that HK is a solvable subgroup of G.

3. Let G be a finite group of order greater than 3, and let $\mathcal{C}$ be a conjugacy class of elements in G. If $|\mathcal{C}| = \frac{1}{3}|G|$, show that every element of $\mathcal{C}$ is an element of order 3.

4. Let R be a unique factorization domain in which every prime ideal is maximal. Prove that every prime ideal is principal. (In fact, it turns out that *every* ideal is principal, but you are not asked to prove this.)

5. Let R be a ring. Prove that the following three conditions are equivalent for the left R-module M.

(i) Any increasing chain of submodules

$$M_1 \subseteq M_2 \subseteq \ldots,$$

(ii) Any submodule of M is finitely generated.

(iii) Any family of submodules of M has a maximal member with respect to inclusion.

6. Let F be a field and let R be the ring

$$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in F \right\}.$$

Define the left R-modules

$$M_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in F \right\}, M_2 = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} | b \in F \right\}.$$

(The module action is matrix multiplication.) Prove that M_1 and M_2 are **not** isomorphic as left R-modules.

7. Let V be a vector space of dimension n, and let $1_V \neq T : V \to V$ be a linear transformation of V. If $\operatorname{im}(T - 1_V) \subseteq \ker(T - 1_V)$, compute the minimal and characteristic polynomials of T on V.

8. Let F_q be the finite field of q elements, and let K be an extension of F_q, of degree n.

(a) Prove that the map $\tau_q : K \to K$, $\tau_q(x) = x^q$ is an automorphism of K, and that F_q is precisely the subfield of fixed elements of τ_q.

(b) Compute $Gal(K/F_q)$.

9. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$.

(a) Construct a splitting field $K \supseteq \mathbb{Q}$ for $f(x)$ over $\mathbb{Q}$.

(b) If $G = Gal(K/\mathbb{Q})$, prove that no noidentity element of G can fix two roots of $f(x)$.