Complex Analysis Qualifying Exam
Spring 1988

Throughout $\mathbf{C}$ denotes the complex plane and $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ the open unit disc therein. $\mathbf{N}$ is the positive integers, $\mathbf{Z}$ all the integers and $\mathbf{R}$ the real numbers. I denotes the identity function: $I(z) = z$ for all z.

1. f is holomorphic and bounded in $\mathbf{D}$ and $f(z) \to 0$ as $z \to 1$ along the upper arc of the circle $|z - \frac{1}{2}| = \frac{1}{2}$. Show that $f(x) \to 0$ as $x \to 1$ along $[0, 1[$. **Hints:** For each $N \in \mathbf{N}$ let $f_N(z) := z^N f(z)$, $F_N(z) := f_N(z)\overline{f_N(\overline{z})}$. Show that F_N is holomorphic and for $\varepsilon > 0$ there is an N such that $|F_N| \leq \varepsilon$ on the whole circle $|z - \frac{1}{2}| = \frac{1}{2}$, and infer that $|f(x)|^2 \leq \varepsilon x^{-2N}$ for $x \in]0, 1[$ by the Maximum Modulus Principle.

2. f is holomorphic and bounded by M in $\mathbf{D}$ and has zeros at the distinct points $a_1, \ldots, a_N \in \mathbf{D}$. Prove that

$$|f(z)| \leq M \prod_{j=1}^{N} \left| \frac{z - a_j}{1 - \overline{a}_j z} \right| \quad \forall z \in \mathbf{D}.$$

Is this an improvement over the hypothesized inequality $|f(z)| \leq M$? **Hint:** $f(z) \prod_{j=1}^{N} \frac{1 - \overline{a}_j z}{z - a_j}$, appreance to the contrary, is holomorphic in $\mathbf{D}$ and bounded by M. Why?

3. If f is *one-to-one* and holomorphic in the open set U except for isolated singularities, then f has no essential singularities and at most one pole.

4. f is continuous in $\mathbf{D}$ and holomorphic in $\mathbf{D} \setminus [-1, 1]$. Show that f is actually holomorphic in $\mathbf{D}$.

5. Show that if c is a non-removable isolated singularity of the holomorphic function f, then c is an essential singularity of the function e^f.

6. Let $\mathbf{A}$ denote the *disc algebra*: $C(\overline{\mathbf{D}}) \cap H(\mathbf{D})$. Find all the homomorphisms ϕ of $\mathbf{A}$ into $\mathbf{C}$.

Hints: The number $c := \phi(I)$ plays a special role. Show that for each f the number $\phi(f)$ lies in $f(\overline{\mathbf{D}})$ and determines ϕ first on the polynomials.

7. State necessary and sufficient conditions on a sequence $(b_n)_{n \in N} \subset \mathbf{D}$ in order that

(i) there exists a non-zero holomorphic function on $\mathbf{D}$ with zeros at each b_n;

(ii) there exists a bounded, non-zero holomorphic function on $\mathbf{D}$ with zeros at each b_n.

In case (ii) how would you construct such a function (outline only, no proof).

8. Let Ω be an open connected subset of $\mathbf{C}$, f_n holomorphic in Ω, and suppose that $\{f_n\}$ converges to a non-constant function f uniformly on each compact subset of Ω. Show that

(i) $\overline{\lim}_{n \to \infty} f_n(K) \subset f(K)$ for every compact $K \subset \Omega$ and

(ii) $\underline{\lim}_{n \to \infty} f_n(G) \supset f(G)$ for every open $G \subset \Omega$.

Suppose in addition that there exists an $M < \infty$ such that every $w \in \mathbf{C}$ and $n \in \mathbf{N}$

$$\text{card } [f_n^{-1}(w)] \leq M.$$

Show that then

(iii) card $[f^{-1}(w)] \leq M$ for all $w \in \mathbf{C}$.

9. Let f be holomorphic in a neighborhood of $\overline{D}$. Suppose $z_0 \in \partial\mathbf{D}$ satisfies $|f(z_0)| = \max|f(\overline{D})|$. Show that $f'(z_0) \neq 0$, unless f is constant in $\mathbf{D}$. **Hints:** WLOG, $1 = z_0 = |f(z_0)|$. Then f non-constant plus the Maximum Modulus Principle gives $f(\mathbf{D}) \subset \mathbf{D}$. If $f(0) = 0$, then use Schwarz to argue $\left|\frac{f(1)-f(x)}{1-x}\right| \geq \frac{1-x}{1-x} = 1$ for $x \in [0, 1[$, so $|f'(1)| \geq 1$. In general, let $a = f(0) \in \mathbf{D}$, form $T(w) := \frac{w-a}{1-\overline{a}w}$, $F := T \circ f$ and apply the preceding. Compute $F'(1)$ by the chain rule and express $f'(1)$ in terms of it.

10. (i) State the Harnack inequalities for harmonic functions.

(ii) Let $h : \mathbf{C} \to \mathbf{R}$ be harmonic and non-constant. Show that h has at least one zero. **Hint:** One method is to use (i).

(iii) Let $h : \mathbf{C} \to \mathbf{R}$ be harmonic and non-constant. Show that $h(\mathbf{C}) = \mathbf{R}$. **Hint:** Use (ii). Alternatively, since $h(\mathbf{C})$ is necessarily an interval in $\mathbf{R}$, if not all of $\mathbf{R}$, it is bounded above or below and with harmonic conjugate of h we should be able to use Liouville.