Name ______________________

COMPLEX VARIABLES QUALIFYING EXAM

Spring 1999

(Burckel and Nagy)

$\mathbb{N} :=$ **natural numbers (positive integers),** $\mathbb{R} :=$ **reals,** $\mathbb{C} :=$ **complexes,** $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$**,** $\mathbb{U} := \{z \in \mathbb{C} : Im\, z > 0\}$**.**

Do any 8 of the 10 problems.

1. Compute $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^4}$.

Hint: Integrate $f(z) := \dfrac{\pi \cot \pi z}{z^4}$ over bigger and bigger circles.

2. Suppose $f : \mathbb{D} \to \mathbb{C}$ is analytic, $f(0) = 1$, and $|f(z)| < \dfrac{1}{|z|}$ for all $z \in \mathbb{D}$, $z \neq 0$. Prove that f is identically equal to 1.

Hint: Schwarz.

3. Show that the function $u : \mathbb{R}^2 \setminus \{(0,0\} \to \mathbb{R}$, defined as $u(x,y) := \ln\left(\sqrt{x^2+y^2}\right)$ is harmonic, but has no harmonic conjugate.

4. Let f be an entire function. Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x,y) = Re\, f(x+iy)$. Prove that if u is a polynomial function (in two variables), then f is a polynomial.

5. Compute: $\displaystyle\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}}\, d\theta$.

6. Prove the following strong form of Morera's theorem: If f is continuous in $\mathbb{D}$ and $\int_{\partial R} f = 0$ for every rectangle R lying in $\mathbb{D}$ and having sides parallel to the coordinate axes, then f is holomorphic.

Hint: Via an integral, construct an F satisfying $D_1 F = -i D_2 F = f$.

7. f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D} \setminus \mathbb{R}$. Show that f is holomorphic in $\mathbb{D}$.

Hint: Use preceding exercise (whether or not you did it).

8. f is holomorphic in $\mathbb{D} \setminus \{0\}$ with a pole at 0. Show that the range of f contains the complement of a compact disk.

Hint: Show that $1/f$ is holomorphic near 0 and apply the Open Mapping Theorem.

9. Show that if f is holomorphic in a deleted neighborhood of a, then f' cannot have a *simple* pole at a.

10. Use Liouville's Theorem to show that the range of every non-constant entire function is dense in $\mathbb{C}$. What is the strongest conclusion about the size of the image set that is valid?