Topology Qualifying Exam
Fall 1989

Work 9 of the following problems. Do not turn in more than 9.

For a subset A contained in a topological space X, let A^- denote the closure of A in X, let A° denote the interior of A in X, and let $Fr(A)$ denote the frontier of A in X. All Euclidean spaces will have the usual topology, and all product spaces will have the product topology.

1. For a topological space X, prove that "o–" and "–o" are both idempotent operations on the subsets of X; that is, $A^{o-o-} = A^{o-}$ and $A^{-o-o} = A^{-o}$ for all $A \subseteq X$.

2. Let X and Y be topological spaces. Prove that a bijection $f : X \to Y$ is a homeomorphism if and only if $f[A^-] = f[A]^-$ for all $A \subseteq X$.

3. Prove that the line L defined by the equation $x = 0$ separates the plane $\mathbb{R}^2$ into exactly two components and is the frontier of each of them.

4. Prove that $[0, 1]$ (with its usual topology) is compact.

5. Let X be the unit square in the plane. Give precise descriptions of four quotient maps with domain X whose images are:

a) the sphere S^2;

b) the torus, $S^1 \times S^1$;

c) the cylinder, $S^1 \times I$;

d) a Möbius band M in $\mathbb{R}^3$.

6. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be initial provided that for each topological space (Z, μ), each set-function $g : Z \to X$ is continuous whenever $f \circ g : (Z, \mu) \to (Y, \sigma)$ is continuous. Prove that:

a) The composition of initial maps is initial.

b) The "first factor" of an initial map is initial; i.e., if $(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{h} (W, \nu)$ are continuous maps and $h \circ f$ is initial, then f is initial.

7. (a) True-False.

(i) Every compact Hausdorff space is metrizable.

(ii) If $f, g : X \to Y$ are homotopically equivalent and $h, k : Z \to W$ are homotopically equivalent, then $f \times h$ and $g \times k$ are homotopically equivalent.

(iii) The product of connected spaces is connected.

(iv) Every retract of a locally connected space is locally connected.

(v) A space is T_2 if and only if it is locally T_2; i.e., each pt has a base of T_2 neighborhoods.

(vi) Every metrizable space is normal.

(vii) Every locally compact Hausdorff space is completely reuglar.

(viii) Every subspace of a separable Hausdorff space is separable and Hausdorff.

(b) For each false entry, give a counter example (no proofs).

8. Does there exist a subset C of the unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ such that

a) C is closed in S^1

b) C contains exactly one point from each pair of diametrically opposite points on S^1.

9. Compute the fundamental group of the compact surface $\hat{T}$ obtained by removing an open disc from a torus T.

10. Prove that every contractible space is pathwise connected.

11. Prove that if X is Lindelöf and Y is compact, then $X \times Y$ is Lindelöf.

12. Suppose that X and Y are spaces for which the second projection $\pi_2 : X \times Y \to Y$ is closed. Suppose also that $q : Y \to Y'$. is a quotient map. Prove or disprove that the second projection $\pi_2' : X \times Y' \to Y'$ is closed.

13. a) Give an example of an ultrafilter on the set $\mathbb{R}$ of real numbers.

b) Give an example of a filter on $\mathbb{R}$ that is not an ultrafilter.

c) Prove that every filter on $\mathbb{R}$ is contained in an ultrafilter on $\mathbb{R}$.

14. Prove that if a normal space X contians a closed copy of $[0, \infty)$, then X does not have the fixed point properly.

15. Suppose that there is an continuous open surjection $f : [0, 1] \to X$ where X is a nondegenerate IIausdorff space.

a) Show that f need not be injective.

b) Show that X is homeomorphic to $[0, 1]$.

16. Let A be a nonempty proper subset of a continuum X. If C is component of A, show that $C \cap Fr(A) \neq 0$.