Algebra Qualifying Exam
Spring 1997

All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be a group and let H be a subgroup of finite index in G. Show that the subgroup $N = \cap_{g \in G} gHg^{-1}$ has finite index in G.

2. Let G be a finite group and H a subgroup of G. Show that if $H \neq G$, then $G \neq \cup_{g \in G} gHg^{-1}$.

Find a counter-example to this statement of infinite groups by considering a matrix group over the field of complex numbers.

3. Let R be a commutative ring and let $I_1, I_2, \ldots, I_n$ be ideals of R. If P is a prime ideal of R and $\cap_{i=1}^{n} I_i \subseteq P$, then there is an i such that $I_i \subseteq P$.

4. Let R be a commutative ring. An ideal $Q \subseteq R$ is said to be a *primary* ideal if $ab \in Q$ and $a \notin Q$ implies that $b^n \in Q$ for some positive integer n. Prove that if $Q \subseteq R$ is a primary ideal, then the set $P = \{r \in R | r^m \in Q$ for some positive integer $m\}$, is the smallest prime ideal of R that also contains Q.

5. Let R be a ring and let M be a left R-module. Then $S = Hom_R(M, M)$ is also an associative ring with 1, relative to pointwise addition and composition of homomorphisms. Show that M is indecomposable if and only if S has no idempotents except 0 and 1. (An element e in a ring is called an idempotent if $e^2 = e$.)

6. Let R be a commutative ring with 1 and $S = M_n(R)$ be the ring of all $n \times n$-matrices with entries in R with matrix addition and multiplicaton. For any left R-module M, then $M^{\oplus n} = M \oplus M \oplus \cdots \oplus M(n$ terms) is a left S-module via $A \cdot \sum_i^n m_i = \sum_i^n \sum_j^n a_{ij} m_j \in M^{\oplus n}$, where $A = (a_{ij})$. For each pair of indices i, j we let $e_{ij} \in S$ be the matrix with a 1 in the (i, j)-position, and zero elsewhere.

(a) Show that for any left S-module N, then, $M = e_{11} N$ is a left R-module.

(b) Show that as S-modules, $N \cong M^{\oplus n}$.

7. Let V and W be two vector spaces over a field k. A bilinear form $f : V \times W \to k$ is called non-degenerate if for any $v \in V$ and $w \in W$, $f(v, W) = 0$ implies that $v = 0$ and $f(V, w) = 0$ implies that $w = 0$. Show that if V and W are finite dimensional, then a bilinear form f is non-degenerate if and only if $\dim_k V = \dim_k W = n$ and there exist bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ of V and W respectively, such that $f(v_i, w_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

8. Let V be a vector space over a field k and $T : V \to V$ be a linear transformation. Show that $f(AT)A = Af(TA)$ for any polynomial $f(x) \in k[x]$ and any linear transformation $A : V \to V$.

9. Let K be a Galois extension of a field k and let F be a subfield of K containing k. Show that the subgroup $H = \{g \in \text{Gal } (K/k) | g(F) = F\}$ is the normalizer of $\text{Gal}(K/F)$ in $\text{Gal}(K/k)$.

10. Let K be the splitting field of the polynomial $x^{p^2} - t \in F[x]$ over $F = \mathbb{F}_p(t)$ for a prime p and an indeterminate t. Prove that $[K : F] = p^2$.