Name ______________________________

TOPOLOGY QUALIFYING EXAM
Fall 1998
(Maginnis and Strecker)

Choose and work any 6 of the following 15 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Prove or disprove: For any space X with topology $\mathcal{J}$, the family $\mathcal{B} = \{A \subseteq X \mid A$ equals the interior of its closure$\}$ forms a base for some topology $\mathcal{J}'$ on X.

2. Prove that a quotient space of a locally connected space is locally connected.

3. Let $f : X \to Y$ be an open map between the spaces X and Y. Let $B \subseteq Y$ and $A = f^{-1}[B]$. Prove that the restriction $\overline{f} : A \to B$ (i.e., $\overline{f}(a) = f(a)$) is an open map from A to B.

4. (a) State the Axiom of Choice.

(b) State the Well-Ordering Theorem.

(c) Either use the Axiom of Choice to prove the Well-Ordering Theorem or use the Well-Ordering Theorem to prove the Axiom of Choice.

5. Prove that the plane $\mathbb{R}^2$ with its usual topology is not equal to a countable union of straight lines.

6. Prove that the Sorgenfrey line $X = \mathbb{R}$ with basis $\{[a,b) \mid a, b \in \mathbb{R}\}$ is a paracompact space.

7. Let $f : [a,b] \to \mathbb{R}$ be a real-valued function on a closed interval and let $G = \{(x, f(x)) \in \mathbb{R} \times \mathbb{R} \,\big|\, a \leq x \leq b\}$ be its graph. Prove or give a counterexample for the following.

(a) If G is connected, then f is continuous.

(b) If f is continuous, then G is connected.

8. Prove that the net based on an ultrafilter is an ultranet.

9. Let Y be a compact space. Prove that the projection map $\pi_1 : X \times Y \to X$ is a closed map.

10. Let A be a subset of a complete metric space X. Prove that A is totally bounded if and only if the closure $\overline{A}$ is compact.

11. Let X be a completely regular T_1 space (i.e., one point sets are closed, and for each closed set C and point $x \notin C$, there exists a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f[C] = \{0\}$). Prove that the Stone-Čech compactification $\beta(X)$ is connected if and only if X is connected.

12 Let $f : X \to Y$ be a continuous surjective map from a compact space X to a Hausdorff space Y. Prove that f is a quotient map.

13. Let D be a dense subset of a metric space X, and let Y be a complete metric space. Prove that any uniformly continuous function $f : D \to Y$ can be extended to a uniformly continuous function $F : X \to Y$ (i.e., $F|_D = f$).

14. Prove that a metric space is compact if and only if every sequence has a convergent subsequence.

15. Prove that each metric space is a normal space.