TOPOLOGY
QUALIFYING EXAMINATION
SPRING - 1985
(Muenzenberger - Strecker)

Do 8 of the following 16 problems.

1. Prove that if $f_1, f_2 : X \to Y$ are homotopic maps and $g_1, g_2 : W \to Z$ are homotopic maps, then $f_1 \times g_1$ and $f_2 \times g_2$ are homotopic.

2. Prove that if A is a compact subset of a regular (non-Hausdorff) space, then $\bar{A}$ is compact.

3. Let $f : A \to B$, $\mathcal{C} = (C_n)_{n \in \mathbb{N}}$ be a family of subsets of A and $\mathcal{D} = (D_n)_{n \in \mathbb{N}}$ be a family of subsets of B. Check "true" or "false" for each of the following assertions. For each false one, indicate further hypotheses that will make it true. (NO PROOFS ARE NECESSARY.)

ASSERTION	TRUE OR FALSE	FURTHER HYPOTHESES (if needed)
$f[\bigcup \mathcal{C}] = \bigcup_{n \in \mathbb{N}} \{f[C_n]\}$		
$f[\bigcap \mathcal{C}] = \bigcap_{n \in \mathbb{N}} \{f[C_n]\}$		
$f[A - C_0] = B - f[C_0]$		
$f^{-1}[\bigcup \mathcal{D}] = \bigcup_{n \in \mathbb{N}} \{f^{-1}[D_n]\}$		
$f^{-1}[\bigcap \mathcal{D}] = \bigcap_{n \in \mathbb{N}} \{f^{-1}[D_n]\}$		
$f^{-1}[B - D_0] = A - f^{-1}[D_0]$		

4. Prove that if (X,τ) is a metrizable space, then there exists a bounded metric ρ such that the topology determined by ρ is τ.

5. (a) Define "nowhere dense".

   (b) Prove that in a metrizable space X without isolated points, the closure of a discrete set in X must be nowhere dense in X.

6. Prove that for any topological space (X,τ) the family of all subsets A of X with the property that A is the interior of its closure, forms a base for some topology on X.

7. For each of the following, give a proof or a counterexample:

   (a) Every open subspace of a separable space is separable.

   (b) Every first countable separable space is second countable.

   (c) Let $(X,\leq)$ be a linearly ordered set and let τ be the topology induced by $\leq$ on X. If A is a subset of X, then the subspace topology on A is the same as the topology generated by the order on X restricted to A.

8. (a) State Urysohn's Lemma, Tychonoff's Product Theorem and Tietze's Extension Theorem.

   (b) Sketch a proof of one of the above.

9. Prove that the quotient of a locally connected space is locally connected.

10. Prove that the product of connected spaces is connected.

11. Prove that if a filter F is contained in a unique ultrafilter G, then $F = G$.

12. Given a net $\delta : \Lambda \to X \times Y$ prove or disprove each of the following:

    (a) If each of $\pi_1 \circ \delta$ and $\pi_2 \circ \delta$ has a cluster point, then so does δ.

    (b) If δ has a cluster point, then so do $\pi_1 \circ \delta$ and $\pi_2 \circ \delta$.

13. Let $\hat{T}$ denote the compact surface obtained by removing an open disc from a torus T. Compute the fundamental group of $\hat{T}$.

14. State and prove the Cantor-Berstein Theorem.

15. Prove that if $\{C_n \mid n \in \mathbb{Z}^+\}$ is a nest of continua $C_1 \supseteq C_2 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} C_n$ is a continuum.

16. Prove that the Axiom of Choice is equivalent to the assertion that the product of any set-indexed family of nonempty sets is nonempty.