Qualifying Exam: Geometry and Topology

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Examiners: Prof. Ricardo Castaño-Bernard and Prof. David Yetter

Name: ______________________________

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number. Problems that are completely solved and thoroughly justified will be given more credit than scattered attempts leading to partial answers.

1. (a) Define what it means for a topological space to be compact.

   (b) A compactification of a non-compact topological space X is a compact topological space Y equipped with a continuous inclusion $\iota : X \to Y$ such that $\iota(X)$ is dense in Y. Show that if X is any non-compact topological space, there is a topology on $X \coprod \{*\}$ such that the obvious inclusion $i_X : X \to X \coprod \{*\}$ is a compactification of X.

   (c) Show that if $\iota : X \to Y$ is any compactification of X, there is a unique continuous map $\phi : Y \to X \coprod \{*\}$ such that $\phi(\iota) = i_X$, when $X \coprod \{*\}$ has the topology from part (b).

   (Note: Some authors include Hausdorffness in the definition of compactness. This question assumes you will follow the convention which does not.)

2. Let $z, z' \in \mathbb{C}^{n+1} - \{0\}$ and define the equivalence relation: $z \sim z'$ if and only if $z = \lambda z'$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The *complex projective space* is defined to be the quotient, $\mathbb{CP}^n := \mathbb{C}^{n+1} - \{0\}/\sim$.

   (a) Define local coordinates for $\mathbb{CP}^n$, and use them to prove it is a C^∞ manifold.

   (b) Show that on the unit sphere $S^3 \subset \mathbb{C}^2$ there is an S^1-action such that
   $$S^3/S^1 = \mathbb{CP}^1.$$

   (c) Use the map $H : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}$, $H(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$ to show that $\mathbb{CP}^1$ is diffeomorphic to S^2.

3. (a) State the Universal Coefficient Theorem for Homology

   (b) Suppose X is a space such that
   $$H_0(X) \cong \mathbb{Z},\ H_1(X) \cong \mathbb{Z}/4\mathbb{Z},\ H_2(X) \cong \mathbb{Z}/2\mathbb{Z},\ and H_3(X) \cong \mathbb{Z}$$
   with all other integral homology groups being zero.
   Use the Universal Coefficient Theorem to find $H_*(X, \mathbb{Z}/4\mathbb{Z})$.

4. (a) Give the definition of the derivative $f_*(p)$ of a smooth map between manifolds $f : M \to N$ at $p \in M$. Show that if p is a regular point there is a neighborhood U of p such that every $p' \in U$ is regular.

   (b) Compute the derivative of the map $f : A \mapsto AA^t$ where A is a square matrix. Use this to show that the orthogonal group, $O(n)$, is a smooth submanifold of $GL(n, \mathbb{R})$.

   (c) Consider the map $\mu : \mathbb{C}^2 \to \mathbb{R}^2$, $\mu(z_1, z_2) = (|z_1|^2, |z_2|^2)$. For each $z \in \mathbb{C}^2$, compute the derivative $\mu_*(z)$ and indicate its rank. Describe the set of critical points, $Crit(\mu) \subset \mathbb{C}$. Find the values $b \in \mu(\mathbb{C})$, for which the fibre $\mu^{-1}(b)$ is a smooth submanifold of $\mathbb{C}^2$ and give an explicit description of each fibre.

5. (a) Give an explicit construction of a 2-dimensional CW complex X whose fundamental group is the dihedral group $D_5 = \langle a, b | a^2,\ b^5\ abab \rangle$.

   (b) Compute the homology (with integer coefficients) of your space X from part (a).

   (c) State the Hurewicz Theorem relating π_1 and H_1 and illustrate explicitly that it holds for your space X.

6. (a) Let α be a k-form on M. Show that if $\int_C \alpha = 0$ for every C diffeomorphic to S^k then $d\alpha = 0$.

   (b) Show that $\omega = \frac{xdy - ydx}{x^2+y^2}$ represents a non-trivial class $[\omega] \in H^1_{dR}(\mathbb{R}^2 - (0,0))$. Use this to compute $H^*_{dR}(S^1)$.

7. (a) State the Künneth Theorem (Formula) for Cohomology.

   (b) Use the Künneth Theorem and the well-known cohomology groups of spheres to compute $H^*(S^3 \times S^2)$.

   (c) Explicitly describe the multiplication on the cohomology ring of $S^3 \times S^2$.

8. Let $\mathbb{C}[z]$ be the ring of polynomials in one variable with complex coefficients and let S be a subset of $\mathbb{C}[z]$. Define

$$V(S) = \{p \in \mathbb{C} \mid f(p) = 0,\ \text{for all } f \in S\}.$$

   The Zariski topology on $\mathbb{C}$ is the topology $\mathcal{Z}$, whose closed sets are $V(S)$ for all S.

   (a) Show that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by S.

   (b) Let X be an arbitrary subset of $\mathbb{C}$. Show that the closure of X is $V(S_X)$, where

$$S_X = \{f \in \mathbb{C}[z] \mid f(X) = 0\}.$$

   Hint: Show that $V(S) \cap V(T) = V(S \cup T)$.

   (c) Is $S^1 \subset \mathbb{C}$ closed in the Zariski topology? If not, what is its closure?

   (d) Prove that the topological space $(\mathbb{C}, \mathcal{Z})$ is not Hausdorff.