NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 2004

(do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

(1) Show that in the bisection method solving an equation $f(x) = 0$ where $f \in C([a, b])$ and $f(a)*f(b) < 0$, if one wants an accuracy of ϵ in the result, that is the iteration is stopped when $|a_n - b_n| < \epsilon$, then the number of steps necessary to achieve this is no more than

$$1 + \frac{\ln(\frac{b-a}{\epsilon})}{\ln 2}$$

where the interval $[a, b]$ is the one on which the bisection method applies with $a_0 = a$, and $b_0 = b$.

(2) Let α be a root of multiplicity m for the equation $f = 0$, where f is sufficiently smooth near α. Show that if the "multiply-relaxed" Newton method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

converges to α, it does so at least quadratically.

(3) Suppose that k and n are positive integers with $k < n$ and that f is a real valued function continuous on the interval $[-1, 2]$. For each integer $m \geq n$, S_m is a piecewise polynomial approximation to f on $[0, 1]$ defined as follows: First, set up a mesh $\{x_j\}_{j\in\mathbf{Z}} \bigcap [-1, 2]$ where $x_j = jh$ with $h = 1/m$. Then on each subinterval $[x_j, x_{j+1}] \bigcap [0, 1]$ define $S_m(x) = p_j(x)$ where $p_j(x)$ is the polynomial of degree at most n that interpolates f at the $n + 1$ consecutive points $x_{j-k}, \ldots, x_{j-k+n}$. Show that S_m converges to f uniformly on $[0, 1]$ as $m \to \infty$. (**Hint:** Use the Lagrange interpolation formula and change the variable x to s by $x = x_{j-k} + sh$.)

(4) Let q_k, $k = 0, 1, \ldots, n$ be a set of orthogonal polynomials on $(-1, 1)$ with weight function $w(x) = 1 - |x|$, where q_k has degree k and leading term x^k.

(a). Find q_0, q_1 and q_2.

(b). Find the Gaussian quadrature formula for

$$\int_{-1}^{1} (1 - |x|)f(x)dx$$

using the roots of q_2 and verify its degree of precision.

(c). Show that the Gaussian quadrature rule

$$\int_{-1}^{1} (1 - |x|)f(x)dx \approx G_n(f)k = \sum_{k=1}^{n} A_k f(x_k)$$

has all positive coefficients A_k.

(5) Two matrices $A, B \in \mathbb{C}^{n\times n}$ are unitary equivalent if $A = QBQ^*$ for some unitary matrix $Q \in \mathbb{C}^{n\times n}$. Is it true or false that A and B are unitary equivalent if and only if they have the same singular values? Prove or show a counterexample.

(6) Assume that the linear system

$$\begin{aligned} r_{11}x + r_{12}y &= b_1 \\ r_{22}y &= b_2 \end{aligned}$$

where r_{ij} and b_i are floating point numbers is solved by back substitution using floating point arithmetic with the machine accuracy ϵ. Show that the back substitution algorithm is backward stable in the sense that the computed solution $\tilde{x}$ and $\tilde{y}$ satisfy

$$\begin{aligned} \tilde{r}_{11}\tilde{x} + \tilde{r}_{12}\tilde{y} &= b_1 \\ \tilde{r}_{22}\tilde{y} &= b_2 \end{aligned}$$

for some $\tilde{r}_{11}$, $\tilde{r}_{12}$, and $\tilde{r}_{22}$ that satisfy

$$|\tilde{r}_{ij} - r_{ij}|/|r_{ij}| \le 2\epsilon + O(\epsilon^2).$$

(7) Assume that A is a symmetric $n \times n$ matrix. Let μ and $\mathbf{x}$ be an approximate eigenvalue and an approximate eigenvector respectively with $\|x\|_2 = 1$. Let $\mathbf{r}$ be the residual in the sense that $\mathbf{r} = A\mathbf{x} - \mu\mathbf{x}$. Show that there exists an eigenvalue λ of A such that $|\mu - \lambda| \le \|\mathbf{r}\|_2$.

(8) Show that the Jacobi iteration converges for 2 by 2 symmetric positive definite systems.