Complex Variables Qualifying Exam
Fall 1996
Bennett and Burckel

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and for open $U \subset \mathbb{C}$, let $H(U)$ denote the set of holomorphic functions on U.

Do all 8 questions.

1. Let $z_0 \in U$ open and f holomorphic on U. Let $r = \sup\{\rho : |z - z_0| < \rho \Rightarrow z \in U\}$. Show that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for some choice of coefficients a_n, where the series converges in $\{z : |z - z_0| < r\}$.

2. Let f be holomorphic in $\mathbb{D} \setminus \{0\}$ and $\int_{\mathbb{D}\setminus\{0\}} |f|^2 d\lambda_2 < \infty$ where λ_2 denotes Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$. Show that f extends to be holomorphic on $\mathbb{D}$.

3. Show that if U is an open subset of $\mathbb{C}$, $g : U \to \mathbb{C}$ is continuous and e^g is holomorphic, then g is holomorphic. (Hint: It suffices to treat the case U as a disk).

4. The function f is entire, f' has no zeros, and $\lim_{|z|\to\infty} |f(z)| = \infty$. Show that for some $a, b \in \mathbb{C}$, $f(z) = az + b$ for all z.

5. Evaluate $\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2 + 1}\, dx.$

6. Construct an entire function f whose only zeros are simple zeros at the Gaussian integers $\{a + bi : a, b \in \mathbb{Z}\}$.

7. The functions $B_n(z) = \prod_{k=1}^{n} \frac{z - \frac{1}{k}}{1 - \frac{z}{k}} = \prod_{k=1}^{n} \frac{1 - kz}{z - k}$ are called Blaschke products.

a) Show $|B_n(z)| \leq 1$ for $z \in \mathbb{D}$.

b) Suppose $1 \leq n_1 < n_2 < \ldots$. If $\lim_{j\to\infty} B_{n_j}(z) = f(z)$ exists for each $z \in \mathbb{D}$, what can you say about the limit function $f(z)$?

c) Show $\lim_{n\to\infty} B_n(z)$ exists for all $z \in \mathbb{D}$. (Hint: Use (a) and (b).)

8. Find a conformal mapping from $\mathbb{D}$ onto $\mathbb{D} \cap \{Im(z) > 0\}$.