Geometry of Manifolds Fall 2001 Qualifying Exam Auckly & Miller

Notes: Work as many of the following problems as you can. Good luck.

- **1.** Define $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (x + z^2, y z^2) = (u, v)$ where (x, y, z) are coordinates on \mathbb{R}^3 and (u, v) are coordinates on \mathbb{R}^2 . Let $\alpha = du + u dv$. Find
 - a) $f^*(\alpha)$
 - b) $f^*(d\alpha)$
 - c) $\int_S f^*(d\alpha)$ where S is the surface $x^2 + y^2 + z^2 = 1$, $z \ge 0$ oriented with the upward pointing normal.
- **2.** a) Construct an example of a covering projection, $P: E \to X$ with $\operatorname{Deck}(E) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.
 - b) Compute $H^{\infty}(X : \mathbb{R})$ for your example.
- **3.** On \mathbb{R}^3 let $X = xz^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$.
 - a) Calculate $\int_B L_X(dx \wedge dy \wedge dz)$ where $B = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$ with orientation given by $dx \wedge dy \wedge dz$.
 - b) Compute the flow of $X, F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$.
- **4.** Consider a 2-dimensional Riemannian manifold (M,g) where M is an open subset of $\mathbb{R}^2 = \{(x,y)|x \in \mathbb{R}, y \in \mathbb{R}\}$. Suppose g is such that $e_1 = \frac{\partial}{\partial x}$ and $e_2 = f(x,y)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ is an orthonormal frame.
 - a) Find the coframe (θ^1, θ^2) dual to (e_1, e_2) .
 - b) Find a 1-form w so that $d\theta^1 = w \wedge \theta^2$ and $d\theta^2 = -w \wedge \theta^1$.
 - c) Find a function K(x,y) so that $dw = -K\theta^1 \wedge \theta^2$.
 - d) Show how to choose f(x,y) so that K(x,y) = -1 for all (x,y).

- **5.** Let K be a simplicial complex.
 - a) Define the Euler characteristic of K.
 - b) Let $p: L \to K$ be an *n*-fold simplicial cover. Show that $\chi(L) = n\chi(K)$.
 - c) Determine all **closed** 2-manifolds that cover T^2 .
 - d) Give an example of a non-trivial cover $q: T^2 \to T^2$.
 - e) Compute $\chi(S^n)$ as a function of n.
 - f) Let G be a finite group that acts freely on S^{2k} . Assume that the map $\pi: S^{2k} \to S^{2k}/G$ is a covering projection. Prove that $G \cong \mathbb{Z}_2$ or G is trivial.
- **6.** a) Given that X is a non-vanishing vector field on M, prove that there is a diffeomorphism, $f: M \to M$ without fixed points (i.e., x such that f(x) = x) that is homotopic to id: $M \to M$.
 - b) Given that $f: S^n \to S^n$ has no fixed points, show that f is homotopic to $p: S^n \to S^n$; p(x) = -x.
 - c) Compute $H^n(p): H^n(S^n, \mathbb{R}) \to H^n(S^n, \mathbb{R})$.
 - d) What can be said about the dimension of a sphere that admits a non-vanishing vector field.
 - e) Let G be a Lie group that acts freely (i.e., $x \cdot g = x$ implies g = 1) on S^{2k} . Prove that dim G = 0.
- **7** Let G be a Lie group.
 - a) Define the Lie algebra of G, say \mathfrak{g} .
 - b) Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . Explain how to construct a distribution on G using \mathfrak{h} and prove that this distribution is integrable.
 - c) Let H be the maximal connected leaf of the distribution from part b) passing through 1. Construct an atlas on H.
 - d) Prove that H is closed under multiplication. Hint: Consider $L_gH:=\{gh|h\in H\}$.