Algebra Qualifying Exam
January 29, 2002

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.

Note: All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be an arbitary group and $H_1, H_2, \ldots, H_n$ be subgroups of G of with finite index in G. Show that there is a normal subgroup K of finite index such that $K \subseteq H_i$ for all $i = 1, 2, \ldots, n$.

2. Let $\mathbb{F}$ be a field and G be the group

$$G = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{F} \right\}$$

with usual matrix multiplication. Prove that the group G is nilpotent.

3. Let R be a commutative ring. Show that if $I_1, \cdots, I_n$ are ideals with $\cap_{i=1}^n I_i \subseteq P$ for some prime ideal P, then there is an i such that $I_i \subseteq P$.

4. Let p be a fixed prime number. Define $\mathbb{Z}(p) = \{ap^{-n} \in \mathbb{Q} \mid a \in \mathbb{Z}, n \in \mathbb{Z}\}$.

   (a) Show that $\mathbb{Z}(p)$ is a subgring of $\mathbb{Q}$ the field of rational numbers $\mathbb{Q}$ containing $\mathbb{Z}$ as a subring.

   (b) Assume that A is a ring and that $\phi : \mathbb{Z} \to A$ is a ring homomorphism such that $\phi(p)$ has an muliplicative inverse in A. Show that there exists a unique ring homomorphism $\psi : \mathbb{Z}(p) \to A$ such that $\psi(x) = \phi(x)$ for all $x \in \mathbb{Z}$.

5. Let A be a commutative ring, and let $A[x]$ be the ring of all polynomials $f(x) = \sum_{i=0}^n a_i x^n$ in the indeterminate x with coefficients $a_i \in A$. Show that for any two polynomials $f(x)$ and $g(x)$ such that the leading coefficient of $g(x)$ is invertible in A, then there is a unique pair of polynomials in $q(x)$ and $r(x)$ such that

$$f(x) \; = \; g(x)q(x) \; + \; r(x),$$

where $\deg(r(x)) < \deg(g(x))$. Can you conclude that the ring $A[x]$ is an Euclidean domain? Justify your conclusion.

6. Let D be a unique factorization domain and $Q(D)$ be the field of factions (quotient field of D). Suppose a, b, f, g are in D with $b \neq 0$ and $g \neq 0$ such that $\frac{a}{b} = \frac{f}{g} \in Q(D)$. If a and b are relatively prime, show that $a|f$ and $b|g$.

7. Let $\mathbb{F}$ be field and $R = M_n(\mathbb{F})$ be the ring of all $n \times n$ matrices with entries in $\mathbb{F}$. Show that the vector space $\mathbb{F}^n$ of $n \times 1$ column vectors with coefficients in $\mathbb{F}$ is an irreducible module of R (relative to ordinary matrix multiplication). Then show that any finite-dimensional R-module is a finite-dimensional vector space of dimension divisible by n.

8. Let $\mathbb{F}_q$ be a finite field with q elements. Compute the order of the group $\mathrm{GL}_n(q)$ of all invertible linear transformations of the vector space $\mathbb{F}_q^n$.

9. Let $\mathbb{F}$ be a field, let x be an indeterminate over $\mathbb{F}$, and set $R = \mathbb{F}[x]/(x^5(x-2)^6)$. Describe all R-modules of $\mathbb{F}$-dimension 25 up to R-module isomorphism.

10. Let q be a prime power. Show that the finite field $\mathbb{F}_{q^r}$ is isomorphic to a subfield of $\mathbb{F}_{q^n}$ if and only if r divides n. If r divides n, then show that $\mathbb{F}_{q^n}$ is a Galois extension of $\mathbb{F}_{q^r}$ and compute the Galois group of $\mathbb{F}_{q^n}$ over $\mathbb{F}_{q^r}$.