Real and Complex Analysis Qualifying Exam

January 2006

Throughout, $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{R} :=$ real numbers, $\mathbb{C} :=$ the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, Ω is an open connected subset of $\mathbb{C}$, $H(\Omega)$ is the set of holomorphic functions in Ω, and (X, μ) is a measure space.

Problem 1.

(a) Suppose $\mu(X) < \infty$ and $1 \leq r \leq s \leq \infty$. Prove that $L^s(X) \subseteq L^r(X)$.

(b) Give an example to show that if $\mu(X) = \infty$ the result in (a) may not hold.

Problem 2. Let $A, B \subseteq \mathbb{R}$ be Lebesgue measurable sets. Let $h(x) = |(A - x) \bigcap B|$. (Here $|\ |$ is Lebesgue measure on $\mathbb{R}$.) Prove that h is measurable and $\displaystyle\int_{\mathbb{R}} h(x)dx = |A||B|$.

Hint: Express $h(x)$ in terms of an integral.

Problem 3. Let $\mathcal{F}$ be a σ-algebra on a set X and $\eta : \mathcal{F} \to [0, \infty]$ a function defined on $\mathcal{F}$ with the properties:

(i) If $A, B \in \mathcal{F}$ and $A \bigcap B = \emptyset$, then $\eta(A \bigcup B) = \eta(A) + \eta(B)$.

(ii) If $A_n \in \mathcal{F}$, $n \in \mathbb{N}$ and $A_1 \supseteq A_2 \supseteq \ldots$ with $\displaystyle\bigcap_{n=1}^{\infty} A_n = \varnothing$, then $\displaystyle\lim_{n\to\infty} \eta(A_n) = 0$.

Prove that η is a measure.

Problem 4. Let h be a bounded measurable function on $\mathbb{R}$ which has the property that $\displaystyle\lim_{n\to\infty} \int_E h(nx)dx = 0$ for every measurable set E of finite measure. Prove that for every $f \in L^1(\mathbb{R})$, $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} f(x)h(nx)dx = 0$.

Problem 5. (i) Show that if $L : \Omega \to \mathbb{C}$ is continuous and $e^{L(z)} = z$ for all $z \in \Omega$, then L is holomorphic in Ω.

(ii) Prove that there is no continuous logarithm in $\mathbb{D} \setminus \{0\}$. That is, no continuous function $L : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ exists which satisfies $e^{L(z)} = z$ for all $z \in \mathbb{D} \setminus \{0\}$.

Problem 6. $f, g \in H(\mathbb{C})$ and $|f| \leq |g|$.

(i) Show that if z_0 is a zero of g having multiplicity m, then z_0 is a zero of f having multiplicity at least m.

(ii) Show that f is a constant multiple of g.

Problem 7. Compute

$$(i) \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^4} \qquad\qquad (ii) \quad \int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}}\, d\theta.$$

Problem 8. Let $S := \{z \in \mathbb{C} : \ 0 < \operatorname{Re} z < 2\}$ and for each $z \in S$ let $l_z(t) := tz + 1 - t$, $0 \leq t \leq 1$. Without assuming a primitive of f exists, show that for each $f \in H(S)$, $F(z) := \displaystyle\int_{l_z} f$ defines a holomorphic function in S such that $F' = f$. Explain why the conclusion fails whenever f is not holomorphic.