DE QUALIFYING EXAM

1. Find $u(\rho,\phi,z,t)$ from the conditions:

$$\begin{cases} u_{tt} = \Delta u, \ t > 0, \ 0 \le z \le \ell, \ 0 \le \rho \le a, \ 0 \le \phi \le 2\pi. \\ u\big|_{z=0} = 1, \ u\big|_{z=\ell} = J_0(x_{01} \frac{\rho}{a}), \text{ where } J_0(x_{0j}) = 0. \\ u\big|_{\rho=a} = 0 \\ u\big|_{t=0} = 0, \ u_t\big|_{t=0} = 0. \end{cases}$$

2. A dielectric ball of radius a is placed into an exterior homogeneous electrostatic field $E = e_3$, where e_3 is the unit vector along the x_3 axis. Find the field around the ball.

3. Let $\begin{cases} \frac{\partial S}{\partial x} - xy \frac{\partial S}{\partial y} = 0 \\ S\big|_{x=0} = \sin y, \ y \in R^1. \end{cases}$ Find $S(x,y)$.

4. Prove that eq. $Ax = y$ is solvable for any y provided that $A: H \to H$, $A = I + T$, T is compact linear operator, H is a Hilbert space, $(Tx,x) > -q(x,x)$, $q < 1$.

5. $\begin{cases} u_t = u_{xx}, \ t \ge 0, \ -\infty < x < \infty \\ u\big|_{t=0} = f(x) \end{cases}$

Assume $f(x) \in C_0^\infty(R^1)$, $\int_{-\infty}^{\infty} f dx = 1$, $f(0) = -1$. Is $u(0,t)$ positive or negative for $t \to +\infty$? Same question about $u(1,t)$.

6. Assume $\Delta u = 0$ in $\mathcal{D}$, $x \in \mathcal{D}$, $d = \operatorname{dist}(x, \partial\mathcal{D})$, $\mathcal{D} \subset \mathbb{R}^3$.

Prove or disprove: $\left|\frac{\partial u}{\partial x_j}\right| \leq \frac{3M}{d}$, $M = \max\limits_{y \in \mathcal{D}} |u(y)|$.

7. Can $y = x \sin x$ be a solution to the equation $y'' + a_1(x)y' + a_0(x)y = 0$, $-1 \leq x \leq 1$ with some continuous coefficients $a_1(x)$, $a_0(x)$?

8. Is the solution to $\begin{cases} \frac{dy}{dx} = x + y^2 \\ y(0) = 0 \end{cases}$ defined on the interval $[0,4]$?

9. Is the solution to the equation

$$\begin{cases} \dot{x}_1 = -x_1(t) + f(t), & x_1(0) = x_2(0) = 1 \\ \dot{x}_2 = x_1(t) - x_2(t) \end{cases}$$

$f(t) = \frac{1}{1 + t^2}$ bounded as $t \to +\infty$?

10. Compute $\frac{d^2}{dx^2} e^{-a|x|}$, $a = \text{const} > 0$ in the sense of distributions.

11. Assume: $\begin{cases} y' \leq e^{-t}y + e^{-2t}, \; t \geq 0. \\ y(0) = 0 \end{cases}$ Does it imply that $y(1000) \leq 4$?

12. Let $y'' + q(x)y = 0$, q is real-valued, continuous, $1 \leq q \leq 4$. Prove or disprove: if $y \not\equiv 0$, $y(t_1) = y(t_2) = 0$, $t_1 < t_2$, then $\frac{\pi}{2} \leq t_2 - t_1 \leq \pi$.