Algebra Qualifying Exam
January 19, 2006

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. Write solutions to each problem on separate pages and write your name on top.
Note: All rings are assumed to be associative and with multiplicative identity 1; all integral domains are assumed to be commutative. The integers and the rational numbers are denoted by $\mathbf{Z}$ and $\mathbf{Q}$, respectively.

1. Let G be a finite group of order $105 = 3 \times 5 \times 7$.

   (a) Prove that G is not simple.

   (b) Show that G contains a normal subgroup of order 5.

2. Let G be the abelian group $\mathbf{Z}/4\mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})^2 \times (\mathbf{Z}/9\mathbf{Z})^2$.

   (a) Determine the isomorphism classes of subgroups of order 36 inside G.

   (b) Determine the actual number of subgroups of order 36 inside G.

3. Let R be a commutative ring with 1, let $n \geq 2$ be a fixed integer, and suppose that $x^n = x$ for all $x \in R$.

   (a) Prove that R contains no nonzero nilpotent elements.

   (b) If P is a prime ideal of R, show that R/P is a finite field containing at most n elements.

4. Let R be the subring of the set of polynomials $f(x)$ with real coefficients for which $f(x)$ has no linear term, i.e., $R = \mathbf{R} \oplus x^2\mathbf{R}[x]$.

   (a) Prove that any nonzero nonunit $f(x)$ in R can be factored into a product of irreducible elements in R.

   (b) Prove that R is not a unique factorization domain. (Hint: Note that both, x^2 and x^3, are irreducible in R.)

5. Let R be a commutative ring with 1. An R-module M is called faithful if $rM = 0$ for $r \in R$ implies $r = 0$. Let M be a finitely generated, faithful R-module, and let I be an ideal of R. Prove that if $IM := \operatorname{span}(\{i\,m \mid i \in I \text{ and } m \in M\}) = M$ then $I = R$. (Hint: If $v_1, \ldots, v_n$ are generators of M, construct a matrix A with entries in I such that $(v_1, \ldots, v_n)(E_n - A) = 0$, where E_n is the identity matrix. Use that for every $n \times n$ matrix B with entries in a commutative ring one has $B\widetilde{B} = \det(B) \cdot E_n$, where $\widetilde{B}$ is the matrix containing the cofactors of B as entries.)

6. Let A be a square 2×2 matrix with entries in the complex numbers. Prove that there exists a complex matrix B with $A = B^2$ if and only if A is not similar to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

7. Let K be an algebraic closed field of prime characteristic p and let V be an K-vector space of dimension precisely p. Suppose A and B are endomorphisms of V such that $AB - BA = B$. If B is invertible, prove that V has a basis $\{v_1, v_2, \ldots, v_p\}$ of eigenvectors of A such $Bv_i = v_{i+1}$ for $1 \leq i \leq p - 1$ and $Bv_p = \lambda v_1$ for some nonzero $\lambda \in K$.

8. Let $T : V \longrightarrow V$ be a linear transformation of a finite dimensional vector space V over a field F. Assume that T is "indecomposable" in the sense that if $V = W_1 \oplus W_2$ is the direct sum of two subspaces, both invariant under T, then either $W_1 = 0$ or $W_2 = 0$. Prove that the minimal polynomial for T, $f(x) \in F[x]$, is a power $f(x) = (g(x))^n$ of some irreducible polynomial $g(x) \in F[x]$.

9. Prove that the field $\mathbf{Q}(\sqrt[3]{2})$ is not a subfield of any Galois extension of the rational numbers with abelian Galois group.

10. Find the Galois group over $\mathbf{Q}$ of $f(x) = x^3 + 10\,x + 5$. Explain.