REAL ANALYSIS
Qualifying Exam
Saturday, April 2, 1994
Bennett & Moore

Do all ten.

1. Let X be a normed linear space and let X^* be its dual. Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

2. A nonnegative measurable function $w(x)$ is said to be an A_2 weight if it satisfies:

$$\sup_{\substack{I : I \subseteq \mathbb{R} \\ I \text{ an interval}}} \left(\frac{1}{|I|}\int_I w \, dx\right)\left(\frac{1}{|I|}\int_I \frac{1}{w} dx\right) < C$$

where $C < \infty$ is a constant.

If $w(x)$ is an A_2 weight, show that for every $a \in \mathbb{R}$

$$\liminf_{\epsilon \to 0} \left(\frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} w \, dx\right)\left(\frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} \frac{1}{w} dx\right) > 0.$$

3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$. Show that there exists an $f(x) \in L^1([0,1])$ such that $\int_0^1 x^n f(x) dx \geq a_n$ for every n.

4. Prove that if $f \in L^1(\mathbb{R})$ then $Mf \notin L^1(\mathbb{R})$ unless f is zero a.e. Here $Mf(x)$ is the Hardy-Littlewood maximal function

$$Mf(x) = \sup \frac{1}{|I|}\int_I |f(y)| dy$$

where the sup is taken over all intervals centered at x.

5. Suppose a sequence of real valued polynomials $\{p_n(x)\}_{n=1}^{\infty}$ on the interval $[-1,1]$ satisfies:

$$\int_{-1}^{1} p_n(x)p_m(x)dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n\,. \end{cases}$$

Let N be fixed and let

$$M = \text{span}\ \{p_1(x), p_2(x), \ldots, p_N(x)\}.$$

Show that for every $f \in L^2[-1,1]$, $\min\{\|f-g\|_2 : g \in M\}$ is attained for a unique $g = \sum_{i=1}^{N} c_i p_i(x)$. Find a formula for the coefficients c_i for the g which attains this minimum.

6. Suppose T is a bounded linear operator, $T : L^p \to L^q$ with $1 < p,\ q < \infty$. Define T^* by $\int (Tf)\bar{g}d\mu = \int f(\overline{T^*g})d\mu$ for all $f \in L^p$, $g \in L^{q'}$. Show T^* is a well-defined bounded linear operator, $T^* : L^{q'} \to L^{p'}$ where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

7. State and prove
   (a) Fatou's lemma.
   (b) Monotone convergence theorem.
   (c) Dominated convergence theorem.

8. Suppose $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which commutes with translation. Show

$$(Tf)^\wedge(\xi) = m(\xi)\hat{f}(\xi) \quad \text{for some } m \in L^\infty.$$

(Note $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx$ is the Fourier transform.)

9. Suppose $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} (1 + 4\pi^2|\xi|^2)|\hat{f}(\xi)|^2 d\xi < \infty$. Show f is continuous. (Note $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx$ is the Fourier transform.)

10. Show that any nonempty closed convex subset of a Hilbert space has an element of minimal norm. Is this statement true for every Banach space?