Topology Qualifying Exam
Fall 1991

Work 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than 6 problems. Assume that all products have the product topology.

1. Prove that $[0, 1]$, with the usual topology, is connected.

2. Prove that if A is a retract of a Hausdorff space X, then A is closed in X.

3. Prove that a quotient of a locally connected space is locally connected.

4. Let $f : X \to Y$ be an open, continuous surjection. Prove that Y is Hausdorff if and only if the set

$$C = \{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\}$$

is closed subset of $X \times X$.

5. Let S^1 have the usual topology. Prove that $(\mathbb{Q} \times \mathbb{Q}) \cap S^1$ is dense in S^1.

6. Let $\mathbb{E}$ denote the set of real numbers with the Sorgenfrey topology, which has basis consisting of all half-open intervals of the form $[x, y)$. Prove that any compact subset of $\mathbb{E}$ is countable.

7. Prove that the first projection $\pi_1 : X \times Y \to X$ is closed, if Y is compact.

8. Let $\mathbb{R}^2$ have the usual topology. Prove that if U is a convex open subset of $\mathbb{R}^2$, then

$$\overline{U}^0 = U,$$

where "– " indicates closure and "^{0}" indicates interior.

9. Let X be a compact Hausdorff space. Prove that if every point of X is a limit point of X, then X is uncountable.

10. State the Axiom of Choice and Zorn's Lemma, and prove that Zorn's Lemma implies the Axiom of Choice.

11. Let X be a metric space. Show that if every family of pairwise disjoint non-empty open subsets of X is countable, then X is separable.

12. A space is called *functionally Hausdorff* if for every pair of distinct points x and y in X, there exists a continuous function $f : X \to [0, 1]$ with $f(x) = 0$ and $f(y) = 1$. Either prove or disprove that every product of functionally Hausdorff spaces is functionally Hausdorff.