Complex Analysis Qualifying Exam
Spring 1985

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1. Let $f\varepsilon C(G)$, where G is the open square defined by $-1<$ Rez <1 and $-1<Imz<1$. Suppose that

$$\int_\gamma f dz = 0$$

for all rectangular loops γ in G with one side contained in the real axis. Prove that $f\varepsilon H(G)$.

2. Evaluate

$$I_n = \int_0^\infty \frac{dx}{1=x^n} \text{ for } n=2,3,\ldots$$

3. Let G be a region in $\mathbb{C}$, and let (f_n) be a uniformly bounded sequence in $H(G)$. Suppose that the set

$$A=\{z\varepsilon G : \lim_n f_n(z)=0\}$$

has a limit point in G. Prove that the f_n converge to 0 uniformly on compact subsets of G.

4. Let G be a region in $\mathbb{C}$, f_n a sequence in $H(G)$, and $f\varepsilon H(G)$. Suppose that the f_n converge to f uniformly on compact subsets of G, that $\{z\varepsilon\mathbb{C} : |z-a|\leq r\}\subset G$, and that f has no zero on the circle $|z-a|=r$. Prove that there exists an integer N such that for each $n>N$, f_n and f have the same number of zeros in $D(a,r)$.

5. Let G be a region in $\mathbb{C}, f, g\varepsilon H(G)$, and $\varepsilon>0$. Suppose that $|f|\cdot|g|^\varepsilon$ attains a maximum on G. Prove that $|f|\cdot|g|^\varepsilon$ is a constant on G.

6. Let $G=\{z\varepsilon\mathbb{C} :$ Rez $>0\}$, $f\varepsilon C(\overline{G})$, and $f\varepsilon H(G)$. Suppose that f is bounded on the imaginary axis, and that $|f(z)|/(1+|z|^t)$ is bounded on G for each $t>0$. Prove that f is bounded on G.

7. Let $G\subset\mathbb{C}$ be open, and $\alpha\varepsilon\mathbb{C}$. Suppose that there exists $g\varepsilon H(G)$ such that $g(z)^2=z-\alpha$ for all z in G. Prove that α belongs to an unbounded component of $\mathbb{C}\backslash G$.

HINT: You may use the fact (without proof) that if B is a bounded component of $\mathbb{C}\backslash G$, then there exists a bounded open subset V of G such that $B\subset V$ and $\partial V\subset G$.

8. State and prove the Harnack inequality for harmonic functions.

9. Let G be a region in $\mathbb{C}$, and let u be a real-valued continuous function on G. Suppose that for each $a\varepsilon G$, there exists $R>0$ such that $D(a,R)\subset G$ and

$$2\pi u(a)\leq\int_{-\pi}^{\pi} u(a+re^{it})dt \quad \forall r\varepsilon[0,R).$$

Prove:

(a) If u attains a maximum on G, then u is a constant on G.

(b) u is subharmonic on G.

10. Let G be a region in $\mathbb{C}$. State four conditions each of which is equivalent to the simple connectivity of G. Also prove their equivalence.