Topology Qualifying Exam
Summer 1991

Work 9 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than 9 problems.

1. Prove that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x)|x \in X\}$ is closed in $X \times X$.

2. (a) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Construct a counterexample for an infinite union.

3. Prove that if A is a retract of a Hausdorff space X, then A is closed in X.

4. Give an example of a quotient map which is neither open nor closed. Prove all of your assertions.

5. (a) Prove that every compact metric space is closed and bounded.

(b) Give an example that shows that not every closed, bounded metric space is compact.

6. Prove that $I \times I$ in the dictionary order topology is locally connected but not locally path connected. Here I denotes the unit interval $[0,1]$.

7. Let X be a compact Hausdorff space. Prove that if every point of X is a limit point, then X is uncountable.

8. Let G be a topological group such that all points are closed. Prove that G is a regualr space.

9. Let M be a compact, connected, orientable 3-dimensional manifold, and assume that $H_1(M;\mathbb{Z})$ is finite. Show that $H_2(M;\mathbb{Z}) = 0$.

10. Compute the homology with $\mathbb{Z}$-coefficients of the space $\mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4$.

11. Prove that there does not exist a homeomorphism from $\mathbb{C}P^2$ to itself which reverses the orientation.

12. Let A be a 3×3 matrix of positive real numbers. Prove that A has a positive real eigenvalue.

13. Assume that A is a retract of X, and that $\pi_1(A)$ is a normal subgroup of $\pi_1(X)$. Prove that $\pi_1(X) \simeq \pi_1(A) \times \frac{\pi_1(X)}{\pi_1(A)}$.

14. Let $X = S^1 \vee S^2$ be the wedge of a circle and a sphere.

(a) Compute $\pi_1(X)$ and $\pi_2(X)$, giving explicit statements of any theorems you use.

(b) Describe the action of $\pi_1(X)$ on $\pi_2(X)$.

15. (a) State (do not prove) a theorem relating the coverings of a given space X and the subgroups of $\pi_1(X)$. Be explicit with the hypotheses and conclusions.

(b) Applying the theorem for $X = T^2$, describe the homeomorphism types of spaces which cover the torus.

16. Prove that if X is a suspension of some space Y, then all cup products in the cohomology ring of X of elements of positive degree are zero.