Real Analysis Qualifying Exam
Spring 1992

In what follows $(X, \mathcal{A}, \mu)$ is an arbitrary measure space and λ is Lebesgue outer measure on $\mathbb{R}$. Prove all of your assertions.

1. (a) Find a necessary and sufficient condition that $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha+\beta| = |\alpha| + |\beta|$.

(b) Find a necessary and sufficient condition that $f, g \in L_1(\mu)$ satisfy.

$$\int |f+g| d\mu = \int |f| d\mu + \int |g| d\mu.$$

2. (a) Show why it is impossible to construct a bounded Lebesgue measurable $f : [0,1] \to \mathbb{R}$ for which there is some $\alpha > 0$ such that

$$\int_0^1 |f-g| d\lambda > \alpha$$

for every Riemann integrable $g : [0,1] \to \mathbb{R}$.

(b) Can such an f be found if we take $\alpha = 0$? Why?

3. (a) Construt Lebesgue measurable functions $f_n, f : [0,1] \to \mathbb{C} (n = 1, 2, \dots)$ such that $\lim_{n\to\infty} \int_0^1 |f - f_n| d\lambda = 0$ but we don't have $f_n \to f \lambda$ - a.e.

(b) What positive result about pointwise convergence follows from convergence in L_1-norm?

4. Let ν be a countably additive measure on a (not necessarily σ) algebra $\mathcal{E}$ of subsets of X. For $T \subset X$, define

$$\nu^*(T) = \inf \left\{ \sum_{n=1}^{\infty} \nu(E_n) : (E_n)_{n=1}^{\infty} \subset \mathcal{E} \text{ and } T \subset \bigcup_{n=1}^{\infty} E_n \right\}.$$

Prove that

(a) ν^* is an outer measure on X,

(b) $E \in \mathcal{E} \Rightarrow E$ is ν^*-measurable,

(c) $E \in \mathcal{E} \Rightarrow \nu^*(E) = \nu(E)$.

5. Let F be a subset of $[0,1]$ that is not Lebesgue measurable.

(a) Is it possible that $\lambda([0,1] \backslash F) = 0$? Why?

(b) Is it possible that $\lambda(F) = 1$? Why?

6. Let F be any subset of $[0,1]$ (possible nonmeasurable) and let $\{I_1, I_2, \dots, I_n\}$ be a partition of $[0,1]$ into subintervals. Must it be true that

$$\lambda(F) = \sum_{k=1}^{n} \lambda(F \cap I_k)?$$

Why?

7. Let $f_n, f : X \to \mathbb{C}$ be $\mathcal{A}$-measurable with $f_n \to f$ μ-a.e. as $n \to \infty$. Suppose there exists $g \in L_1^+(\mu)$ with $|f_n| \leq g$ μ-a.e. for every $n \geq 1$. Fix $\varepsilon > 0$ and define

$$A_N = \bigcup_{n=N}^{\infty} \{x \in X : |f(x) - f_n(x)| \geq \varepsilon\}.$$

Prove that $\mu(A_N) \to 0$ as $N \to \infty$.

8. Let C be Cantor's ternary set and let ψ be Lebesgue's singular function. Suppose that $x \in C$ but x is not the left-hand endpoint of an open interval disjoint from C. Prove that

$$D^+\psi(x) = +\infty.$$

Recall the definiton:

$$D^+\psi(x) = \overline{\lim_{t \to x^+}} \frac{\psi(t) - \psi(x)}{t - x}.$$