Qualifying Exam: Algebra

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Instructions: You should do all problems and provide as complete arguments as you can. Write solutions of different problems on separate pages with the problem number and your name on the top of each page.

Note: All rings are assumed to be associative with multiplicative identity 1. Ring homomorphisms are assumed to map the multiplicative identities to each other. All modules over any ring are assumed to be unitary, i.e., 1 acts as identity. The rings of integers, rational numbers and complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ respectively.

(1) A group G is called nilpotent if the lower descending central series

$$G^{(0)} \supset G^{(1)} \supset \cdots \supset G^{(n)} \supset \ldots$$

defined by

$$G^{(0)} = G, \quad G^{(n)} = [G^{(n-1)}, G],$$

terminates at identity, i.e., there is a positive integer N such that $G^{(N)} = \{e\}$. Here, for any two subgroups H, K of G, $[H, K]$ denotes the subgroup of G generated by the set of all commutators $\{[h, k] := hkh^{-}k^{-1} \mid h \in H, k \in K\}$.

Prove that a group G is nilpotent if and only if there is a series of subgroups of G

$$G_0 = G \supset G_1 \supset \cdots \supset G_N = \{e\}$$

such that the following two conditions are satisfied for any $i = 1, \ldots, N$: the subgroup G_i is normal in G and the quotient G_{i-1}/G_i is contained in the center of G/G_i.

(2) Let G be an arbitrary group. If G has a subgroup H of finite index, prove that H has a subgroup N, such that N is normal in G and the quotient group G/N is a finite group.

(3) Let D be a unique factorization domain. Prove that for any $d \in D$, $d \neq 0$, there are only finitely many principal ideals of D containing d.

(4) Let R be a ring with 1 and let $S = M_2(R)$ be the ring of all two 2×2-matrices with entries in R. Then S is a ring under the standard matrix addition and multiplication operations (paying attention to the order of multiplication of elements in R).

(a) Show that any 2-sided ideal J of S is of the form $M_2(I)$ for some 2-sided ideal I of R. Here $M_2(I)$ is the set of all 2×2 matrices with entries in I.

(b) Does your proof generalize to $n \times n$-matrices for general n? Why?

(5) Let A be an $n \times n$-matrix over the field $\mathbb{Q}$ and let $R = \mathbb{Q}[x]$ be the polynomial ring with indeterminate x. Let $V = \mathbb{Q}^n$ be the space of $n \times 1$ column matrices. Consider the action of R on V defined by the action of generator x as

$$x \cdot v = Av \quad \text{for any } v \in V.$$

Here Av is the standard matrix multiplication. Thus V becomes an R-module. Assume that there is an R-module isomorphism

$$V \cong R/\langle h_1 \rangle \oplus R/\langle h_2 \rangle \oplus R/\langle h_3 \rangle \oplus R/\langle h_4 \rangle,$$

such that

$$h_1 = x^2 + 2x + 1, \quad h_2 = (x+1)(x^2 - x + 1),$$
$$h_3 = (x+1)(x^2 + x + 1), \quad h_4 = (x+1)^5.$$

Determine the following:

(a) The size n of the matrix A;
(b) The set of elementary divisors (including repetitions) of the matrix A over $\mathbb{Q}$;
(c) The set of invariant factors (including repetitions) of A over $\mathbb{Q}$;
(d) The minimal polynomial of A over $\mathbb{Q}$;
(e) The characteristic polynomial of A, which is defined as $\det(xI - A)$;
(f) The set of n eigenvalues in the field of complex numbers indicating the multiplicities;
(g) The set of elementary divisors and invariant factors (including repetitions) over the field of complex numbers $\mathbb{C}$ (A is also a matrix with entries in $\mathbb{C}$);
(h) The Jordan canonical form of the matrix A over $\mathbb{C}$.

(6) For each field extension $E \supseteq F$ in the list below, answer questions (a)-(c). Give reasons to support your answers.

Questions:

(a) Determine whether the field extension is an algebraic extension, an algebraic Galois extension, a normal extension, a separable extension (There can be more than one "yes" answer!);
(b) Compute the degree of the extension;
(c) Compute the Galois group $\mathrm{Aut}_F(E)$ if it is a Galois extension.

List of field extensions:

(i) $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) \supseteq \mathbb{Q} = F$;
(ii) $E = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3}, \sqrt[5]{5}) \supseteq \mathbb{Q} = F$;
(iii) $E = F[t]/\langle t^p - x \rangle \supseteq F$, where K is a field of characteristic p and $F = K(x)$ is the field of rational functions with coefficients in K.