Real Analysis Qualifying Exam
Spring 1989

In the following, $(X, \mathcal{A}, \mu)$ is a measure space and if X is a topological space, then $\mathcal{B}(X)$ is the σ-algebra of all Borel subsets of X.

1. (a) What does it mean to say that $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable?

(b) Prove that if $f : g : X \to \mathbb{C}$ are both $\mathcal{A}$-measurable, then so is $f + g$.

2. Suppose $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ and $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Define

$$B = \{x \in X : \{n \in N : x \in A_n\} \text{ is infinite}\}.$$

Prove that $B \in \mathcal{A}$ and $\mu(B) = 0$.

3. Suppose $\mu(X) < \infty$ and $\phi : X \to \mathbb{R}$ is $\mathcal{A}$-measurable. Define ν on $\mathcal{B}(\mathbb{R})$ by $\nu(B) = \mu(\phi^{-1}(B))$. Prove that ν is a finite measure and that if $f : \mathbb{R} \to \mathbb{C}$ is a bounded Borel measurable function, then $f \circ \phi$ is $\mathcal{A}$-measurable and

$$\int_X f \circ \phi d\mu = \int_{\mathbb{R}} f d\nu.$$

4. (a) Let $\mathcal{S}$ be the smallest σ-algebra of subsets of $\mathbb{R}^2$ that contains the family $\mathcal{F} = \{A \times B : A$ and B are bounded open intervals of $\mathbb{R}\}$. Prove that $\mathcal{S} = \mathcal{B}(\mathbb{R}^2)$.

(b) For $E \subset \mathbb{R}^2$ and $y \in \mathbb{R}$ put

$$E^y = \{x \in \mathbb{R} : (x, y) \in E\}.$$

Prove that if $E \in \mathcal{B}(\mathbb{R}^2)$, then $E^y \in \mathcal{B}(\mathbb{R})$ for all $y \in \mathbb{R}$.

5. Show that if B is a Borel subset of $\mathbb{R}^2$ and almost every vertical cross-section of B has (one-dimensional) Lebesgue measure 0, then almost every horizontal cross-section of B has Lebesgue measure 0.

6. Prove that if $p \in \mathbb{N}$ and $\mu : \mathcal{B}(\mathbb{R}^p) \to [0, \infty]$ is a measure such that $\mu(K) < \infty$ for every compact $K \subset \mathbb{R}^p$, then μ is regular. [HINT: Use a Riesz Representation Theorem to find a "regular relative" of μ.]

7. Let $p \in \mathbb{N}$ and let $(\mu_n)_{n=0}^{\infty}$ be a sequence Borel probability measures on $\mathbb{R}^p$. Suppose that

$$\lim_{n\to\infty} \int g d\mu_n = \int g d\mu_0$$

whenever $g : \mathbb{R}^p \to \mathbb{C}$ is continuous with compact support. Prove the following:

(a) If $\varepsilon > 0$, then there exists a compact $K \subset \mathbb{R}^p$ such that $\mu_n(K') < \varepsilon$ for all $n \geqq 0$. Here $K' = \mathbb{R}^p\ K$.

(b) If $f : \mathbb{R}^p \to \mathbb{C}$ is bounded and continuous, then

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu_0.$$

(c) If $B \subset \mathbb{R}^p$ is a Borel set whose closure B^- and interior B^0 satisfy $\mu_0(B^-) = \mu_0(B^0)$, then

$$\lim_{n\to\infty} \mu_n(B) = \mu_0(B).$$

8. For $\mu \in M(\mathbb{R})$, define $\widehat{\mu}$ on $\mathbb{R}$ by

$$\widehat{\mu}(t) = \int e^{-itx} d\mu(x).$$

Prove that if $\int |x| d|\mu|(x) < \infty$, then $\widehat{\mu}$ is differentiable on $\mathbb{R}$.

9. For $\mu \in M(\mathbb{T})$, define $\widehat{\mu}$ on $\mathbb{Z}$ by

$$\widehat{\mu}(n) = \int e^{-inx} d\mu(x).$$

Prove that if $\lim_{n\to+\infty} \widehat{\mu}(n) = 0$, then $\lim_{n\to-\infty} \widehat{\mu}(n) = 0$. [HINT: Don't forget that μ is a *complex* measure. Remember complex conjugation. Approximate a Radon-Nikodym derivative by trigonometric polynomials.]

10. Let $\mathcal{A}_0$ be an *algebra* of subsets of X (i.e., $\mathcal{A}_0$ is closed under finite unions, relative complements and contains $\emptyset$ and X), $\mathcal{A}$ the smallest σ-algebra which contains $\mathcal{A}_0$, and μ a finite positive measure on $\mathcal{A}$. Show that $\mathcal{A}_0$ is μ-dense in $\mathcal{A}$ in this sense: for $A \in \mathcal{A}$ and $\varepsilon > 0$ $\exists A_0 \in \mathcal{A}_0$ such that $\mu(A \Delta A_0) < \varepsilon$. Here $A \Delta A_0$ means $(A \backslash A_0) \cup (A_0 \backslash A)$. [HINTS: Look at the class of all A in $\mathcal{A}$ which are so approximable. Matters may be facilitated by translating everything into statements about characteristic functions and the L_1-metric.]