Geometry Manifolds Qualifying Exam
Spring 1996

Work out problem 1 and then choose 4 (and only 4) additional problems among the remaining ones. State each problem on a new sheet of paper. **Do not turn in more than 4 additional problems**. A space always means a topological space below.

This one is required!

1. Answer the following questions and give a brief explanation or counterexample:

i) (a) Give 2 non-diffeomorphic examples of orientable connected 2-dimensional compact smooth manifolds.

(b) Give an example of a non-orientable connected 2-dimensional compact smooth manifold.

ii) What is the dimension of the total space of the exterior bundle $\Lambda^2(M)$ if M is a smooth 5-manifold?

iii) Consider the 1-form $\omega = (x^2 + y^2 - 1)dx$ on $\mathbb{R}^2$. Let D be the standard unit disk, $D = \{(x, y)|x^2 + y^2 \leq 1\}$. Find $\int_D d\omega$.

iv) Let $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ be the unit sphere in $\mathbb{R}^4$.

(a) What is the fundamental group of S^3?

(b) What is the fundamental group of $S^3 \backslash \{(0, 0, 0, 1)\}$?

(c) What is the fundamental group of $S^3 \backslash \{(0, 0, 0, 1), (0, 0, 0, -1)\}$?

v) Let M be a compact **connected** n-dimensional smooth manifold. Is there a nowhere vanishing n-form on M? What if M is also **simply connected**?

vi) Let θ be a closed 5-form on $\mathbb{R}^6$. Is θ exact?

vii) Describe in detail the flows of the vector field $X = (x^2 + y^2 - 1)\frac{\partial}{\partial z}$ on $\mathbb{R}^3$.

viii) Let $\gamma(t)$ be an integral curve of a complete vector field X on a smooth manifold M. Suppose that $\gamma(t_0) = 0$ for some t_0. What can you say about $\gamma(t)$?

Choose 4 problems from below:

2. Let ω be the 2-form on $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$ given by the formula

$$\omega = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}.$$

Show that ω is closed but not exact. (Hint: Consider the restriction to the unit sphere $S^2 \subset \mathbb{R}^3$.)

3. Compute the de Rham cohomology of S^2.

4. What is the fundamental group of

i) $\mathbb{R}P^2$ (real projective space)?

ii) the 2-sphere S^2 with 3 distinct points removed?

Use generators and relations if you like.

5. Denote by $S^1 = \{e^{it} \in \mathbb{C} | t \in \mathbb{R}\} \subset \mathbb{C}$ the unit circle and $\mathbb{T} = S^1 \times S^1$ the 2-torus. Fix any real number $\alpha \in \mathbb{R}$ and let $\varphi_\alpha : \mathbb{R} \to \mathbb{T}$ be the map defined by

$$\varphi_\alpha(t) = (e^{2\pi i t}, e^{2\pi i \alpha t}).$$

Show that the image $\varphi_\alpha(\mathbb{R}) \subset \mathbb{T}$ is either compact or dense in $\mathbb{T}$.

6. Let M be a compact smooth n-dimensional oriented manifold without boundary. Show that for any $(n-1)$- form ω on M, there is $p \in M$ such that $d\omega(p) = 0$.

7. Let $f : S^n \to S^n$ be a smooth map such that $f(x) = f(-x)$. Show that the degree of f is even.