QUALIFYING EXAM

Spring 1990

<u>Algebra</u>

1. a. Give an example of an abelian group

   b. Give an example of a non-abelian group.

2. Let G be a group and let $x \in G$. Define the set

$$C_G(x) = \{y \in G \mid xy = yx\}.$$

   Show that $C_G(x)$ is a subgroup of G.

3. Show that no group can be the union of two proper subgroups.

4. Let R be a ring which has a multiplicative identity 1. Show that if $1'$ is any other multiplicative identity then $1 = 1'$.

5. Let $\mathbb{Z}$ denote the ring of integers. Prove that

$$\frac{\mathbb{Z}}{(6)} \cong \frac{\mathbb{Z}}{(2)} \oplus \frac{\mathbb{Z}}{(3)} .$$

6. Let m be a positive integer and let $\mathbb{Z}$ be the ring of integers. Prove that $\frac{\mathbb{Z}}{(p)}$ is a field if and only if p is a prime number.

7. Let $\mathbb{Q}$ be the field of rational numbers, let p be a prime number, and let

$$\mathbb{Z}_{(p)} = \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \text{ are relatively prime and } p \text{ does not divide } b\}.$$

   Prove that $\mathbb{Z}_{(p)}$ is a subring of $\mathbb{Q}$

8. Let Q be the field of rational numbers and let $Q[x]$ denote the ring of polynomials in the indeterminate x with coefficients in Q. Let $(x^2 + 1)$ denote the ideal in Q generated by the polynomial $x^2 + 1$. Prove that $\dfrac{Q[x]}{(x^2 + 1)}$ is a field.

9. Let Q be the field of rational numbers and let $V = \{(a,b,c) \mid a, b, c \in Q\}$ be the vector space whose operations are given by

$$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + b_1, a_2 + b_2, c_1 + c_2)$$
$$r(a,b,c) = (ra, rb, rc)$$

for $r, a, b, c, a_1, b_1, c_1, a_2, b_2, c_2 \in Q$. Now let

$$W = \{(a, b, c) \in V \mid 2a - 3b + \tfrac{1}{3}c = 0\}.$$

Prove that W is a vector subspace of V.

10. Let V be a 2-dimensional vector space over a field F and let $T : V \to V$ be a linear transformation such that

(i) $T \neq 0_V$

(ii) $T^2 = 0_V$.

Prove that there exists a <u>unique</u> 1-dimensional subspace $W \subsetneq V$ such that $T(w) = 0$ for all $w \in W$.

[illegible]

Do any 4 of the following.

1. Use an ϵ-δ argument to show $f(x) = \frac{1}{x^2}$ is continuous on the open interval $(0,\infty)$.

2. [illegible] set of real numbers is uncountable.

3. Suppose f is a bounded real valued function on the closed interval $[0, 1]$ and that the set of discontinuities of f form a countable subset of $[0, 1]$. Prove f is Riemann integrable on $[0, 1]$.

4. Suppose f is a continuous real valued function on the closed interval $[0, 1]$ and $\int_0^1 f^2(x)\,dx = 0$. Prove $f(x) = 0$ for all x in $[0, 1]$.

5. Suppose f is a differentiable real valued function on $(0, \infty)$ and the limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f'(x)$ both exist and are both finite. Prove $\lim_{x\to+\infty} f'(x) = 0$.

6. Let f be a continuous function on $[0,\pi]$. Prove

$$\lim_{n\to\infty} \int_0^{\pi} f(x)\sin(nx)\,dx = 0.$$

7. [illegible] interval of convergence for the power series

$$\sum_{n=1}^{\infty} [illegible]\ .$$

8. [illegible] $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. ($\ln = \log_e$)