Algebra Qualifying Exam
January 27, 1998

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be a finite group of even order. Prove G contains an odd number of involutions. (An *involution* is an element of order 2.)

2. Let G be a group and assume that for all $1 \neq x \in G$, $C_G(x)$ is abelian. Define a relation $\sim$ on G by $x \sim y$ if and only if $xy = yx$.
   (a) Prove that $\sim$ is an equivalence relation on the nonidentity elements $G^\sharp$ of G.
   (b) If $\mathcal{C}$ is an equivalence class in $G^\sharp$, prove that $\{1\} \cup \mathcal{C}$ is a subgroup of G.

3. Let G be a finite group acting transitively on the set X. Fix $x \in X$, set $G_x = \mathrm{Stab}_G(x)$ and let $P \in \mathrm{Syl}_p(G_x)$, where p is prime. Prove that $N_G(P)$ acts transitively on $\mathrm{Fix}(P)$.

4. Let $\mathbb{F}$ be a field and form the quotient ring
$$R \;=\; \mathbb{F}[x,y]/(x^2+y^2)\mathbb{F}[x,y]$$
where x and y are indeterminate over $\mathbb{F}$. Prove that R is an integral domain if and only if -1 is not a square in $\mathbb{F}$.

5. Let R be a ring and let M be an irreducible R-module. Prove that if $0 \neq m \in M$, then the set
$$\mathrm{Ann}_R(m) \;=\; \{r \in R \mid rm = 0\}$$
is a maximal left ideal in R.

6. Suppose that V is a vector space over the field $\mathbb{F}$ and that $T : V \to V$ is a linear transformation having invariant factors
$$x-2,\, x(x-2),\, x^2(x-2)^2,\, x^2(x-2)^2(x^3-1).$$

(a) What is the nullity of T if char $\mathbb{F} \neq 2$?

(b) What is the nullity of T if char $\mathbb{F} = 2$?

(c) What is the dimension of the set of vectors fixed by T if char $\mathbb{F} \neq 3$?

(d) What is the dimension of the set of vectors fixed by T if char $\mathbb{F} = 3$?

7. Let M be a Noetherian R-module and let $\phi : M \to M$ be a surjective R-module homomorphism. Prove that ϕ is injective. (Hint: consider the increasing sequence of submodules $0 \subseteq \ker \phi \subseteq \ker \phi^2 \subseteq \cdots$.)

8. Let R be a unique factorization domain in which every prime ideal is maximal. Prove that every prime ideal is a principal ideal. (In fact, it turns out that *every* ideal is principal, but you don't need to show this.)

9. Let $\mathbb{F}$ be a field and let $f(x), g(x) \in \mathbb{F}[x]$. Suppose that $f(x)$ and $g(x)$ share a common root in some extension field $\mathbb{K} \supseteq \mathbb{F}$. Show that the greatest common divisor of $f(x)$ and $g(x)$ in $\mathbb{F}[x]$ cannot be a unit.

10. Let p be prime and let $\zeta = e^{2\pi i/p}$. Compute the degree of $m_{\zeta+\zeta^{-1}}(x)$.