FALL 2008

Algebra Basic Exam

Instructions: You are given six problems from which you are to do four. Please, indicate those four problems which you would like to be graded by circling the problem numbers on the problem sheet. Please, use separate sheets of paper for each problem and give a clear exposition of your arguments.

1. Let $\mathbb{Z}_n$ denote the group of integers modulo the positive integer n. Prove that the product of groups $\mathbb{Z}_n \times \mathbb{Z}_m$ is isomorphic to the group $\mathbb{Z}_{nm}$ if and only if the greatest common divisor of n and m is equal to 1.

2. Let H be a normal subgroup of a group G. Show that

(a) For any subgroup K of G, the sets

$$KH \stackrel{\text{def}}{=} \{xy \mid (x,y) \in K \times H\} \quad \text{and} \quad HK \stackrel{\text{def}}{=} \{yx \mid (x,y) \in K \times H\}$$

are subgroups of the group G.

(b) If the group K is also a normal subgroup of G, then $H \cap K$ and HK are normal subgroups of the group G.

(c) If the group K is normal and the intersection $H \cap K$ is trivial (that is it consits of the unit element of the group G), then $xy = yx$ for any $(x, y) \in K \times H$.

3. Using induction argument, show that for any set $\{g_1, \ldots, g_n\}$ of elements of a group G,

(a) $(g_1 \cdot \ldots \cdot g_n)^{-1} = g_n^{-1} \cdot \ldots \cdot g_1^{-1}$ and

(b) $x^{-1}(g_1 \cdot \ldots \cdot g_n)x$ for all $x \in G$.

4. Let A be an $n \times n$ matrix with entrees from a field $\mathbb{F}$, and let I_n denote the $n \times n$ identity matrix. Show that if $A - \lambda I_n$ is a *nilpotent* matrix for some $\lambda \in \mathbb{F}$ (that is $(A - \lambda I_n)^m = 0$ for a positive integer m), then either $A = \lambda I_n$, or the matrix A is not *diagonalizable* (that is there is no invertible matrix B such that $B^{-1}AB$ is a diagonal matrix).

5. Let $\mathbb{R}[x]$ be the ring of polynomials over the field $\mathbb{R}$ of real numbers. Let $x^2 + ax + b$ be an irreducible polynomial with real coefficients.

(a) Show that the quotient ring $\mathbb{R}[x]/(x^2 + ax + b)$ is isomorphic to the field $\mathbb{C}$ of complex numbers.

(b) Describe all isomorphisms $\mathbb{R}[x]/(x^2 + ax + b) \xrightarrow{\varphi} \mathbb{C}$ of $\mathbb{R}$-*algebras*; that is isomorphisms φ such that $\varphi(\lambda \mathfrak{f}) = \lambda\varphi(\mathfrak{f})$ for every $\lambda \in \mathbb{R}$ and $\mathfrak{f} \in \mathbb{R}[x]/(x^2 + ax + b)$.

6. Let $\mathbb{F}$ be a field, $\mathbb{F}[x]$ the $\mathbb{F}$-vector space of polynomials with coefficients in $\mathbb{F}$ and $\frac{d}{dx}$ the derivation $\mathbb{F}[x] \longrightarrow \mathbb{F}[x]$ which is defined by the formula $\frac{d}{dx}(\sum_{n \geq 0} a_n x^n) = \sum_{n \geq 1} n a_n x^{n-1}$.

(a) Show that $\frac{d}{dx}$ is a linear transformation.

(b) Find eigenvalues and eigenvectors of $\frac{d}{dx}$

(b1) in the case $\mathbb{F} = \mathbb{R}$,

(b2) in the case $\mathbb{F} = \mathbb{Z}_7$.