Name ____________________

GEOMETRY OF MANIFOLDS QUALIFYING EXAM

Fall 1998

(Auckly & Miller)

Work as many as you can in the 2 hours.
Best of luck.

1. (A) Define the deRham cohomology groups of a differential manifold.

(B) Determine all of the deRham cohomology groups of $S^2 \times S^2$. For those that are nonzero specify representatives for generators.

2. Describe in detail the flows of the vector field

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} \text{ on } \mathbb{R}^3 = \{(x,y,z)|x,y,z \in \mathbb{R}\}.$$

Describe the behavior of the orbits as $t \to +\infty$.

3. On $\{(x,y)|x,y \in \mathbb{R}, 0 < y < \pi\} \subset \mathbb{R}^2$ let g be the metric

$$g = dx^2 + \cos y(dx \otimes dy + dy \otimes dx) + dy^2$$

(A) Compute $\left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \quad xy\frac{\partial}{\partial x}\right]$

(B) Compute $g\left(\nabla_{\sin y\frac{\partial}{\partial y}}\left(\cos y\frac{\partial}{\partial x}\right), \frac{\partial}{\partial y}\right)$

4. Let $D = \{(x,y) \in \mathbb{R}^2|x^2+y^2 \le 4\}/\sim$ where $\sim$ is the equivalence relation generated by $(x,y) \sim (-x,-y)$ if $x^2+y^2=1$ or $x^2+y^2=4$. Determine the fundamental group of D.

5. Suppose that f and $g : M \to N$ are smooth mappings between two $n-$dimensional manifolds and that w is a closed $n-$form on N. If f and g are homotopic show that $\int_M f^* w = \int_M g^* w$.

6. Let B be a smooth vector subbundle of TM, the tangent bundle of the manifold M.

(A) Define what we mean when we say that B is integrable.

(B) State the Frobenius theorem which gives necessary and sufficient conditions for B to be integrable in terms of the bracket of vector fields.

(C) Suppose that α is a smooth 1-form and $\alpha(m) \neq 0$ for all $m \in M$. If $B = \bigcup_{m \in M} \text{kernel}(\alpha(m))$ show that B is integrable if and only if $d\alpha | B = 0$.

7. Prove that $O_n = \{A | A \text{ is an } n \times n \text{ real matrix and} A^T A = I\}$ is a manifold. What is the dimension of O_n?

Hint: Consider the mapping $f : GL(n, \mathbb{R}) \to$ *symmetric matrices* by $f(A) = A^T A$.

8. Prove or disprove the statements

(A) There is a Lie group G which is diffeomorphic to S^2, the 2-sphere.

(B) There is a Lie group G which is diffeomorphic to S^3, the 3-sphere.

9. Prove that a simply connected 2-manifold with nonpositive curvature can have at most one geodesic (parametrized by arc length) from a point A to a second point B.

Hint: Consider the Gauss-Bonnet Theorem.