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COMPLEX VARIABLES QUALIFYING EXAM
Fall 1998
(Burckel & Nagy)

Do any 6 of the 8 problems. Standard notation throughout is: $\mathbb{Z}$ is the set of integers; $\mathbb{C}$ is the set of complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; Ω is a non-void open, connected subset of $\mathbb{C}$. $H(\Omega)$ is the set of all holomorphic functions in Ω.

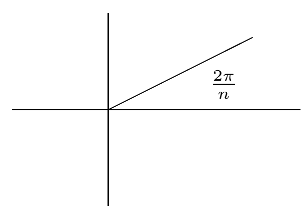
1. $f \in H(\Omega)$ is zero-free.

(a) Show that if Ω is convex, then $\exists g \in H(\Omega)$ such that $f = e^g$.

(b) Give an example of an Ω and an f for which no such g exists.

2. Compute $\displaystyle\int_0^\infty \frac{dx}{1+x^n}$ for each integer $n \geq 2$.

Hint: Use the contour



3. $f : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is bounded and has the property that for each fixed $w \in \mathbb{D}$, $f(w,z)$ is a holomorphic function of z, and for each $z \in \mathbb{D}$, $f(w,z)$ a holomorphic function of w. Show that f is (jointly) continuous on $\mathbb{D} \times \mathbb{D}$.

Hint: Use Cauchy's integral representation in both variables simultaneously.

4. $f \in H(\mathbb{C})$ and $f(\mathbb{C}) \subset \{z \in \mathbb{C} : -1 < Re z < 1\}$. Show that f is constant.

5. The usual topology on $H(\Omega)$ is the c.c. topology: the topology of uniform convergence on compact subsets of Ω.

(a) Show that $f \mapsto f'$ is a continuous mapping of $H(\Omega)$ into itself.

(b) Give an example of an Ω for which the mapping $f \mapsto f'$ is not surjective.

6. $f \in H(\Omega)$, A, B, k are positive real constants and $|f(z)| \leq A + B|z|^k$ for all z. Show that f is a polynomial.

7. (a) P, Q are polynomials, $\deg(Q \geq 2+\deg(P)$, $f = \frac{P}{Q}$, $S = \{p_1, \ldots, p_k\}$ are the poles of f and they satisfy $S \cap \mathbb{Z} = \omega$. Show that

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{j=1}^{k} Res(g; p_j)$$

where $g(z) = \pi f(z)\cot(\pi z)$ for all $z \in \mathbb{C}\backslash\mathbb{Z}$.

<u>Hint</u>: Consider integrating over the contour which is a square having vertices $\pm(n+\frac{1}{2}) \pm (n+\frac{1}{2})i$, $n \in \mathbb{Z}$.

(b) Use a variation on the idea in (a) to compute $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$.

8. For $f : \Omega \to \mathbb{C}$ three fundamental properties are equivalent:

(a) f is locally of power-series type.

(b) f satisfies Cauchy's integral formula for circles.

(c) f satisfies the Cauchy-Riemann equation(s).

State these properties precisely and prove one of the six implications.