Name ______________________________

TOPOLOGY QUALIFYING EXAM
Fall 1999
(Maginnis and Strecker)

Choose and work any 6 of the following 14 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Prove or give a counterexample: If $f : X \to Y$ is an injective continuous function and Y is a Hausdorff space, then X is a Hausdorff space.

2. Let $f : X \to Y$ be a quotient map such that for all $y \in Y$, $f^{-1}[\{y\}]$ is connected. Let $W \subseteq Y$ be an open connected subset. Prove $f^{-1}[W]$ is connected.

3. A space X is called an R_0-space if for every pair of points $\{x, y\} \subseteq X$, x is in the closure $\overline{\{y\}}$ if and only if y is in the closure $\overline{\{x\}}$. Prove or give a counterexample for each of the following statements.

a) Every T_0-space is R_0.

b) Every R_0-space is T_0.

c) A space is T_1 iff it is both T_0 and R_0.

4. Let $\mathbb{R}$ be the real numbers with the usual topology. Prove that any open set in $\mathbb{R}$ is a countable union of pairwise disjoint intervals.

5. Let ω_1 be the first uncountable ordinal number, let X be the set of all ordinals less than or equal to ω_1, and let $Y = X - \{\omega_1\}$. (Y is often called the minimal uncountable well-ordered set).

a) Prove that no sequence in Y converges to ω_1.

b) Give either a net in Y or a filter in Y that converges to ω_1.

6. Let X be a compact Hausdorff space, and let $f : X \to X$ be a continuous function. Prove that there exists a nonempty subset $A \subseteq X$ such that $f[A] = A$.

7. Let X be a normal (and Hausdorff) space with a countable basis. Prove that every subspace of X is normal.

8. Prove or give a counterexample for each of the following statements.

a) A compact subset of a T_1-space is closed.

b) A closed subset of a paracompact space is paracompact.

9. Let $\mathbb{R}$ be the real numbers with the usual topology. Prove there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous precisely at the rational numbers. (i.e., continuous at the rationals and discontinuous at the irrationals.)

10. Let $f : X \to Y$ be a continuous function.

a) Prove that there exists a space Z, a continuous surjection $s : X \to Z$, and an embedding $e : Z \to Y$ such that f is the composite $F = e \circ s$.

b) Prove that if $f = e \circ s$ and $f = \hat{e} \circ \hat{s}$, where $X \xrightarrow{s} Z \xrightarrow{e} Y$ and $X \xrightarrow{\hat{s}} \hat{Z} \xrightarrow{\hat{e}} Y$, are two factorizations of f as a composite of a continuous surjection and an embedding, then there exists a unique homeomorphism $h : Z \to \hat{Z}$ with $h \circ s = \hat{s}$ and $\hat{e} \circ h = e$.

11. Let X be a Hausdorff space such that every infinite subset has a limit point. Prove that every countable open cover of X has a finite subcover.

12 Let X be the countable product $X = \mathbb{R}^{\mathbb{N}} = \prod_{n=1}^{\infty} \mathbb{R}$, where $\mathbb{R}$ is the real numbers with the usual topology. Prove that X, with the product topology, is locally connected but is not locally compact.

13. Let X and Y be spaces which are homotopy equivalent, and assume X is connected. Prove Y is connected. (Recall that X and Y are homotopy equivalent if there exist four continuous functions $f : X \to Y$, $g : Y \to X$, $H_1 : X \times I \to X$, and $H_2 : Y \times I \to Y$ where I is the interval $[0,1]$, satisfying $H_1(x,0) = g(f(x))$, $H_1(x,1) = x$, $H_2(y,0) = f(g(y))$, and $H_2(y,1) = y$, for all $x \in X$ and all $y \in Y$.)

14. Prove or give a counterexample for each of the following:

a) Every bounded subset of a well ordered set has both a greatest lower bound and a least upper bound.

b) Every linearly ordered set (also called totally ordered) in which every bounded subset has both a greatest lower bound and a least upper bounded is well ordered.

c) Every linearly ordered subset of a partially ordered set has an upper bound.

d) In any well ordered set, the minimal element is the only element not having an immediate predecessor.