Algebra Qualifying Exam
Fall 1993

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let p be an odd prime. If the congruence $x^2 \equiv -1(\mod p)$ has a solution, show that $p \equiv 1(\mod 4)$.

2. Prove, or give a counterexample to the assertion that any torsion-free abelian group is free.

3. Let G be a group of order $2p$, where p is an odd prime. Suppose that G has a normal Sylow 2-subgroup. Show that G is cyclic.

4. Prove, or give a counterexample.

(a) Each ideal of $Z[x]$ is principal.

(b) If I is a maximal ideal of Z, then $I[x]$ is maximal ideal of $Z[x]$. Here, $I[x]$ is the ideal of $Z[x]$ consisting of polynomials with coefficients in I.

5. Consider the ring $R = Z[\sqrt{5}] = \{a + b\sqrt{5} | a, b \in Z\}$. Show that the element $3 \in R$ is irreducible but not prime. (Hint: Note that $3|(4+\sqrt{5})(4-\sqrt{5})$.)

6. Let $f(x) = x^4 + 1$. Is $f(x)$ irreducible over

(a) $\mathbb{R}$?

(b) $\mathbb{Q}$?

(c) $\mathbb{C}$?

(d) F_{16}? (Finite field of 16 elements.)

(e) F_7? (Finite field of 7 elements.)

7. Let $f(x)$ be an irreducible polynomial of degree 3 in $\mathbb{Q}[x]$, and assume that $f(x)$ has a non-real root. Prove that if K is a splitting field over $\mathbb{Q}$ for $f(x)$, then $Gal(K/\mathbb{Q}) \cong S_3$.

8. Give as long a list as possible of square matrices

(a) Each matrix has characteristic polynomial $(x-2)^4(x-3)$.

(b) Each matrix has minimal polynomial $(x-2)^2(x-3)$.

(c) No two matrices on the list are similar.

9. Let V be a n-dimensional vector space over the complex field C. Assume that $S, T : V \to V$ are linear transformations such that $ST = TS$. Show that T and S have a common eigenvector in V. Must they also have a common eigenvalue?