Numerical Analysis Qualifying Exam
Spring 1988

---

1. Let A be an $n \times n$ matrix, and let A° be the conjugate transpose of A, and $\rho(A)$ be the spectral radius of A. Recall that

$$\| A \|_n = \sup_{\|x\|_n = 1} \| Ax \|_n \text{ and } \| x \|_n = \left\{ \sum_{i=1}^{n} |x_i|^n \right\}^{\frac{1}{n}}$$

Prove:

(a)

$$\| A \|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|.$$

(b)

$$\| A \|_2^\circ = \sqrt{\rho(A^\circ A)}.$$

2. Let A be a strictly diagonally dominant matrix. Prove the Jacobi iteration method converges for A independently of the choice of initial vector.

3. Consider the application of the iteration

$$z_{k+1} = \frac{z_k^2 + 2}{3}$$

to the equation $x^2 - 3x + 2 = 0$.

(a) Does the method converge? Why?

(b) Show that $x_k \to 1$ as $k \to \infty$ if $-2 < z_0 < 2$. (First prove that z_{k+1} is between z_k and 1 when $k \geq 1$.)

(c) Show that $z_{k+1} = 2$ if $z_0 = \pm 2$, $k = 0, 1, \ldots$, but that the convergence to the root $\alpha = 2$ for any other value of z_0 is impossible.

4. (a) Define a Householder transformatin.

(b) Show that for any square matrix A, there is a unitary matrix U such that $U^0 AU$ is upper triangular. (U^0 is the conjugate transpose of U).

5. Suppose $A = (A_{ij})$ is an $n \times n$ matirx and that

$$|a_{ij}| \geq C \sum_{j \neq i} |a_{ij}|$$

for $i = 1, \ldots, n$ where $C > 1$. Let $K(A) = \rho(A)\rho(A^{-1})$ where $\rho(A)$ is the spectral radius of A. Find a bound for $K(A)$ in terms of C and the a_{ij}'s.

6. State the Weierstrase approximation theorem, define Bernstein polynomials, and explain how the latter are used in the proof of the former.

7. Assume

$$f(0) = 1, f(1) = 3, f(2), \text{ and } f(3) = 1.$$

(a) Write a formula for the polynomial, $P(x)$ of degree at most 3 that interpolates f at these points.

(b) Derive an expression for the error $f(x) - P(x)$ at an arbitrary point x. Be sure to indicate the conditions assumed on f.

8. Prove: Of all nth degree monic polynomials $P_n(x)$, The Chebyshev polynomial

$$T_n(x) = \frac{1}{2^n} \cos(n \cos^{-1} x)$$

has the smallest maximum norm on the interval $[-1, 1]$, that is,

$$\max_{-1 \le x \le 1} |T_n(x)| \le \max_{-1 \le x \le 1} |P_n(x)|.$$

9. Assume that $f(x)$ has a contiuous fourth derivative on an open interval containing the interval $[c - h, c + h]$. Derive an expression for the error in approximating $f''(c)$ by

$$\frac{f(c + h) + f(c - h) - 2f(c)}{h^2}.$$

10. Use undetermined coefficients to find coefficients H_1, H_2, and H_3 so that the approximation

$$\int_c^b f(x)dx \approx H_1 f\left(\frac{2a + b}{3}\right) + H_2 f\left(\frac{a + b}{2}\right) + H_3 f\left(\frac{a + 3b}{4}\right)$$

is exact for polynomials of degree less than or equal to 2.

11. Show that the numberical integration method

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right)$$

to approximating the solution to the ordinary differential equation $y' = f(x, y), y_o = y(x_0)$ is exact for polynomials of degree 0 or 1. Is the method stable? Explain.