Algebra Qualifying Exam (New)
January 29, 2004

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. You should have at least one problem from each of the five sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, $\{7, 8\}$, $\{9, 10\}$ among your choice. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

$\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers respectively.

1. Let G be a finite group that has exactly 50 Sylow 7-subgroups. Let P be one of the Sylow 7-subgroups and $N = N_G(P)$ is the normalizer of P in G. Show that N is a maximal subgroup of G (i.e., $N \neq G$ and for subgroup H of G, if $N \subseteq H \subseteq G$, then either $H = N$ or $H = G$).

2. Let G be a finite nilpotent group and let $x, y \in G$ be fixed elements. Define $z = [x, y] = x^{-1}y^{-1}xy$. Assume that for any normal subgroup N of G, $x \in N$ whenever $z \in N$. Show that $x = e$ (the identity of G).

3. Let F be a field and $R = F[x, y]$ be the ring of polynomials with two indeterminates x and y. Let $I = xR$ be the principal ideal of R generated by x. Set

$$S = \{a + b \mid a \in F, \quad b \in I\}.$$

(a) Prove that S is a subring of R and I is an ideal of S.
(b) Prove that I, as an ideal of S, is not finitely generated.

4. Let R be a commutative ring with 1 and $m \subseteq R$ be a maximal ideal. Define $m^2 = \{\sum_i^n a_i b_i \mid n \in \mathbb{N}, a_i, b_i \in m, i = 1, \ldots, n\}$.
(a) Show that m^2 is an ideal of R and $m \supseteq m^2$.
(b) Prove that, in the quotient ring $R/(m^2)$, the only idempotents e ($e^2 = e$) are 0 and 1 only.

5. Let R be a ring with 1 and S be a subring of R. For any (unitary) S-module M, define $F(M) = \hom_S(R, M)$, the set of all S-module homomorphisms $f : R \to M$. Note that $F(M)$ is an abelian group with the point-wise addition.
(a) Prove that the map $R \times F(M) \to F(M)$ defined by $(rf)(x) = f(xr)$ for all $r, x \in R$ and $f \in F(M)$ defines a left R-module one $F(M)$.
(b) Prove that if N is a left R-module and $\phi : N \to M$ is an injective homomorphism of S-modules, then the map $\psi : N \to F(M)$ defined by $\psi(n)(x) = \phi(xn)$ ($x \in R$, $n \in N$), is an injective homomorphism of R-modules.

6. Let D be a principal ideal domain. It is known that any submodule of a free D-module is free. Without using the fundamental theorem of finitely generated modules over a PID, prove that any finitely generated torsion free

D-module is free. A D-module M is called torsion free if there is no element $0 \neq m \in M$ such that $rm = 0$ for some $0 \neq r \in D$.

7. Let V be an n-dimensional vector space over an algebraically closed field F. Let $S, T : V \to V$ be two linear transformations such that $S \circ T = T \circ S$. Assume that the characteristic polynomial $f_S(x)$ of S has n distinct roots in F. Prove that all eigenvectors of S are also eigenvectors of T and T is diagonalizable.

8. Let F be a field and A be an $n \times n$-matrix over F. The characteristic polynomial of A is defined to be the polynomial $f_A(x) = \det(xI_n - A)$. Write $f_A(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Show that $A^m = 0$ for some m if and only if $a_i = 0$ for all $i = 1, 2, \ldots, n$.

9. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ be the smallest subfield of real numbers containing $\sqrt{2}$ and $\sqrt[3]{2}$. Show that $F = \mathbb{Q}(\alpha)$ with $\alpha = \sqrt{2} + \sqrt[3]{2}$.

10. Let F be a field and $E = F(\alpha) \supseteq F$ be a finite separable extension contained in an algebraic closure $\bar{F}$ of F. Define $\mathrm{Tr}_{E/F}(\alpha) = \sum_{\sigma \in \mathrm{Gal}_F(\alpha)} \sigma(\alpha)$. Here $\mathrm{Gal}_F(\alpha)$ is the Galois group of the splitting field of α over F. Show that

(a). $\mathrm{Tr}_{E/F}(\alpha) \in F$, and

(b). $\mathrm{Tr}_{E/F}(\alpha) \in F$ is the trace of the F-linear transformation $m_\alpha : E \to E$ defined by $m_\alpha(x) = \alpha x$ for all $x \in E$.