Complex Analysis Qualifying Exam
Fall 1987

NOTATION: For $a \in \mathbf{C}$, $0 < r < \infty$, $D(a,r) := \{z \in \mathbf{C} : |z-a| < r\}$, $C(a,r) := \partial D(a,r)$ and $\mathbf{D} := D(0,1)$. For $G \subset \mathbf{C}$, $C(G)$ denotes the set of continuous $\mathbf{C}$-valued functions on G and if G is open, $H(G)$ denotes the set of holomorphic functinos in G.

1. (a) Show by direct calculation that if G is an open set in $\mathbf{C}$ star-shaped with respect to a point z_1 and $f \in C(G)$ satisfies $\int_{[z_1,z_2,z_3,z_1]} f = 0$ for all points z_2, z_3 such that convex hull $\{z_1, z_2, z_3\} \subset G$, then f has a primitive in G; that is, there exists a complex-differentiable F in G with $F' = f$.

(b) Give an example, with proof, of a non-star shaped region G and an $f \in C(G)$ which satisfies the hypothesis in (a) and which is not F' for any complex-differentiable F.

2. Let f be meromoprhic in $\mathbf{C}$, $R_n \uparrow \infty$. Suppose that $\sup |f(C(0,R_n))| \to 0$ as $n \to \infty$. For each $\zeta \in \mathbf{C}$ denote by $S_n(\zeta)$ the sum of the residues of the function $z \mapsto \frac{f(z)}{z-\zeta}$ at the poles of f inside $D(0,R_n)$. Prove that $\lim_{n\to\infty} S_n(\zeta) = f(\zeta)$ uniformly for ζ in any compact set which is disjoint from the poles of f.

3. Let f be holomorphic on some neighborhood of $\mathbf{D}^-$. If $|f(z)| < 1$ for all $|z| = 1$, then there exists a unique $z \in \mathbf{D}$ such that $f(z) = z$.

4. Let $I = [0,1]$ and let $f \in C(\mathbf{D} \times I)$. Suppose that for each $t \in I, z \mapsto f(z,t)$ is holomorphic and has a unique zero $z(t)$ in $\mathbf{D}$. Prove that $t \mapsto z(t)$ is continuous on I.

5. Let Γ_n be the rectangular path

$$\left[n + \frac{1}{2} + ni, \quad -n - \frac{1}{2} + ni, \quad -n - \frac{1}{2} - ni, \quad n + \frac{1}{2} - ni, \quad n + \frac{1}{2} + ni\right].$$

(a) Evaluate the integral

$$\int_{\Gamma_n} \frac{\pi}{(z+a)^2} \cot(\pi z) dz$$

for $a \in \mathbf{C} \backslash \mathbf{Z}$.

(b) Show that for such a

$$\lim_{n\to\infty} \int_{\Gamma_n} \frac{\pi}{(z+a)^2} \cot(\pi z) dz = 0.$$

HINT: Use the fact that for $z = x+iy$, $|\cos z|^2 = \cos^2 x + \sin h^2 y$ and $|\sin z|^2 = \sin^2 x + \sin h^2 y$ to show that $|\cot(\pi z)| \le 2$ for z on Γ_n if n is sufficiently large.

(c) Use (a), (b) to deduce that

$$\frac{\pi^2}{\sin^2(\pi a)} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} \quad \forall a \in \mathbf{C} \backslash \mathbf{Z}.$$

(d) From (c) infer that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

6. State and prove the Liouville theorem.

7. Let $S := \mathbf{R}\times]a,b[$, with $a, b \in \mathbf{R}$ and $a < b$. Let $h \in C(\overline{S})$ be holomorphic in S. Suppose that $\sup_y \int_{-\infty}^{\infty} |h(x+iy)|dx < \infty$. Show that then $\int_{-\infty}^{\infty} h(x+iy)dx$ is independent of y.

Extra Credit: What is the situation if "h holomorphic" is weakened to "h harmonic".

HINTS: Adroit use of Cauchy's theorem. Alternatively, set $H_n(z) := \int_{-n}^{n} h(z+x)dx$ and show that the sequence of holomorphic functions $\{H_n\}_{n=1}^{\infty}$ converges locally uniformly in S. Examine the limit function on various horizontal lines in S.

8. Let $G \subset \mathbf{C}$ be a bounded region, $f \in H(G)$, and $z_0 \in \partial G$. Suppose that $\limsup_{z\to w} |f(z)| \le 1$ for all $w \in \partial G\backslash\{z_0\}$ and that $\lim_{z\to z_0} |f(z)| \cdot |z - z_0|^{\varepsilon} = 0$ for each $\varepsilon > 0$. Prove that $|f(z)| \le 1$ for all $z \in G$.

9. Show that if f is holomorphic in the open set U and one-to-one in $U\backslash A$, where the points of A are isolated in U, then f is one-to-one in U.

HINT: This is a result about continuous open maps.

10. Let $f, g \in H(\mathbf{D})$. Prove:

(a) $|f|$ is subharmonic on $\mathbf{D}$.

(b) If $|f| + |g|$ attains a maximum at some point of $\mathbf{D}$, then both f and g are constants.