Topology Qualifying Exam
Fall 1988

Work 9 of the following problems. Do not turn in more than 9.

1. Let A be a subset of space X and let $\overline{A}$, $Fr(A)$, and A° denote the closure, frontier, and interior of A in X, respectively. The set A is called poor if and only if $Fr(A) = A$, and the set A is called thin if and only if $\overline{A}^\circ = \emptyset$. Prove the following

(i) If A is poor, then A is thin.

(ii) If A is closed, then A is poor if and only if A is thin.

Give an example of space X in which some subset A is thin but not poor.

2. (a) True-False

(i) The composition of quotient maps is a quotient map.

(ii) The product of metrizable spaces is metrizable.

(iii) $f : X \to Y$ is a topological embedding if and only if f is one-to-one and X has the coarset (=weakest) topology making f continuous.

(iv) A space is T_1 if and only if it is locally T_1; i.e., each pt has a base of T_1 neighborhoods.

(v) A space is T_2 if and only if it is locally T_2; i.e., each pt has a base of T_2 neighborhoods.

(vi) Every metrizable space is normal.

(vii) Every locally compact Hausdorff space is completely regular.

(viii) Every subspace of a separable Hausdorff space is separable and Hausdorff.

(b) For each false entry, give a counter example (no proofs).

3. A family of functions $(f_i : A \to B_i)_{i\in I}$ with a common domian is called a mono-source provided that for each set C and each pair of functions $h, k : C \to A$, $f_i \circ h = f_i \circ k$ for each $i \in I$ implies that $h = k$.

Prove that $(f_i : A \to B)_{i\in I}$ is a mono-source if and only if it separates points; i.e., for each $a, a' \in A$ with $a \neq a'$ there is some $j \in I$ such that $f_j(a) \neq f_j(a')$.

4. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be initial provided that for each topological space (Z, μ), each set-function $g : Z \to X$ is continuous whenever $f \circ g : (Z, \mu) \to (Y, \sigma)$ is continuous.

Prove that:

(a) The composition of initial maps is initial.

(b) the product of initial maps is initial; i.e., each of $(X_i, \tau_i) \xrightarrow{f_i} (Y_i, \sigma_i)$ is initial, then so is the unique induced map

$$(\Pi X_i, \tau) \xrightarrow{f=\langle f_i\rangle} (\Pi Y_i, \sigma)$$

where τ and σ are the product topologies.

5. Prove that each non-degenerate completely regular T_1 connected space is uncountable.

6. Prove that any open subset of a locally connected, second countable space X can be written as the union of a finite or infinite sequence of disjoint open, connected subsets of X.

7. Consider the ordinal spaces $[1, \omega]$ and $[1, \omega_1]$ where ω is the first inifinite ordinal and ω_1 is the first uncountable ordinal. Recall that every continuous real-valued function on $[1, \omega_1)$ (or on $[1, \omega_1]$) is constant on some tail. Prove that the Stone-Čech compactification of the deleted Tychonoff plank $[1, \omega_1] \times [1, \omega] - \{(\omega_1, \omega)\}$ is the Tychonoff plank $[1, \omega_1] \times [1, \omega]$.

8. Give 4 examples, 1 compact, 1 non-compact, 1 non-locally connected, 1 non-locally compact, of spaces homotopically equivalent to S^1, but not homeomorphic to S^1.

9. Characterize the sequences (x_n, y_n) in the Moore plane Γ (tangent disc space) which converge to $(0, 0)$.

10. Show that $\prod_{\alpha \in A} X_\alpha$ is connected if and only if each X_α is connected.

11. Let $f : X \to Y$ be perfect. Show that if Y is paracompact, so is X.

12. Find a flaw in the "Proof" of the following:

Theorem. All real-valued functions are continuous.

Lemma. $f : X \to R$ is continuous if and only if g_f defined by $x \mapsto (x, f(x))$ is continuous.

Proof. If f is continuous, then g_f is determined by continuous functions, f and identify, so it must be continuous. If g_f is continuous then f is g_f followed by the second projection, so it must be continuous.

Proof of Theorem For any $f : X \to R$, g_f is one-to-one. Thus it is continuous if and only if its inverse is open. However the inverse of g_f is the first projection. But all projection functions are well-known to be open surjections. Hence g_f^{-1} is open, so g_f is continuous. Hence, by the lemma, f is continuous.

13. Prove that there is no smallest base for the usual topology on R.

14. Prove that the following are equivalent for any topological space Y:

(a) Y is Hausdorff.

(b) the diagonal $\Delta_Y = \{(y, y) | y \in Y\}$ is a closed subset of $Y \times Y$.

(c) for each space Z and each pair of continuous functions $f, g : Z \to Y$ that agree on a dense subset of Z, it follows that $f = g$.

15. (a) Prove that if $A \times B$ is compact subset of $X \times Y$ contained in an open set W in $X \times Y$, then there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that

$$A \times B \subseteq U \times V \subseteq W.$$

(b) Prove that the product of two compact spaces is compact.