Topology Qualifying Exam
Spring 2009

You have been provided with blank paper. Attempt any six of the following eight problems. Your answers, especially proofs, should be as complete as possible. Please begin each numbered problem on a new sheet. Put your name on all pages turned in, and order them by problem number.

1. Prove that $X = \{0\} \cup \{\frac{1}{n} | n \in \mathbb{Z}^+\} \subset \mathbb{R}$ and a countable discrete space Y are not of the same homotopy type. (Hint: except for knowing what a homotopy and a homotopy type are, this is actually a point-set topology question.)

2. (a) State the simpler, special case of the Künneth Theorem that applies when $H_n(X)$ and $H_n(Y)$ are free abelian groups for all n.
   (b) Use the Künneth Theorem to find $H_\bullet(S^1 \times S^3)$.
   (c) Identifying $S^1 \vee S^3$ with the subspace $S^1 \times \{(1,0,0,0)\} \cup \{(1,0)\} \times S^3$ (we here assume both spheres are the 'round' sphere of radius 1 in the appropriate euclidean space), find $H_\bullet(S^1 \times S^3, S^1 \vee S^3)$.

3. (a) Without proof, what are the homology groups of the n-sphere, S^n?
   (b) Find the homology groups of the CW complex with one 0-cell a, one 1-cell b, and one 2-cell c, with each cell containing all lower dimensional cells in its closure, and $\partial b = 0$ and $\partial c = n \cdot b$ (for $n \in \mathbb{Z}^+$).
   (c) Use a Meyer-Vietoris argument to prove that $\tilde{H}_{\bullet+1}(\Sigma X) = \tilde{H}_\bullet(X)$.
   (d) Use your answers to the first three parts to provide a construction for path connected spaces $Y\{G_1, \ldots, G_N\}$ with $\tilde{H}_n(Y\{G_1, \ldots, G_N\}) \cong G_n$, for $n = 1 \ldots N$ and $G_1, \ldots, G_N$ an arbitrary finite sequence of finitely generated abelian groups.

4. Give an explicit decomposition of $\mathbb{CP}^3$ as a CW complex and use this decomposition to compute $H_\bullet(\mathbb{CP}^3)$.

5. (a) Define what it means for a topological space to be Hausdorff.
   (b) Define what it means for a topological space to be compact.
   (c) Prove that if A is a subspace of a Hausdorff space X, which is compact in the subspace topology, then A is a closed subset of X.

6. (a) Let $\phi : M \to N$ be a smooth map between manifolds. Define the tangent space T_pM. If $v \in T_pM$, define $\phi_*(v)$.
   (b) Let T^2 be a 2-torus. Give an explicit example of a smooth map $\phi : \mathbb{R} \to T^2$ which is an immersion but not an embedding.
   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x,y,z) = (xy - z^2, x^3 + y^2)$ find the values $p \in \mathbb{R}^2$ for which the subset $M_p = f^{-1}(p)$ of $\mathbb{R}^3$ is a smooth submanifold.

7. Let $\omega \in \Omega^r(M)$ satisfy $\int_\Sigma \omega = 0$ for every smooth submanifold $\Sigma \subseteq M$ that is diffeomorphic to S^r. Prove that $d\omega = 0$.

8. Let $D \subset \mathbb{R}^2$ be an open disc and let p, q points in D with $p \neq q$. Using the de Rham cohomology of D and S^1, the homotopy axiom and the Mayer Vietoris sequence, compute the cohomology of:

   (a) the twice-punctured disc: $D - \{p, q\}$;

   (b) a once-punctured 2-torus: $T^2 - \{x\}$, $x \in T^2$.

   (c) the genus 2 surface Σ_2.