Numerical Analysis Qualifying Exam
Spring 1993

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1. (a) Assume that $a \neq 0$ and $b^2 - 4ac > 0$ and consider the equation $ax^2 + bx + c = 0$. The roots can be computed with the quadratic formulas

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

Improve these formulas so that it is good even in the case $|b| \approx \sqrt{b^2 - 4ac}$.

(b) Improve the following formula for numberical computation:

$$\ln(1 + x) - \ln x, \quad \text{where } x \text{ is large}$$

2. Suppose $f \in C^2(R)$, and $f(p) = 0$ implies $f'(p) \neq 0$.

(a) Show if $f(p) = 0$, then there is a δ such that if $|x_0 - p| < \delta$, then Newton's method starting at x_0 converges to p.

(b) Show that if p_1, p_2 are successive zeros of f (i.e. $f(x) \neq 0$ for $x \in (p_1, p_2)$) and p_3 is another zero of f, then there is an $x_0 \in (p_1, p_2)$ such that Newton's method starting from x_0 converges to p_3. (You may just use a geometrical way to show it)

3. Suppose that the Lagrange interpolation formula for the function f at the $n + 1$ distinct nodes $x_0, x_1, \ldots, x_n$ is given by

$$P_n(x) = \sum_{j=0}^{n} l_{j,n}(x) f(x_j),$$

where the Lagrange polynomial coefficients are given by

$$l_{j,n}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}.$$

Show that for any $n \geq 1$,

$$\sum_{j=0}^{n} x_j l_{j,n}(x) = x.$$

4. Suppose a numerical formula I_h (like a numerical integration formula) using step size h is used to approximate a mathematical expression I (like a definite integral). If the error of the formula is given by

$$I_h - I = kh^p + O(h^{p+2}), \quad \text{where } k, p \text{ are constants}$$

(a) describe Richarson extrapolation which uses $I_h, I_{h/2}$ to generate a more accurate numerical formula $\widetilde{I}_{h/2}$.

(b) Apply Richarson extrapolation to the trapezoidal rule

$$I(f) = \int_a^b f(x)dx \approx I_h(f) = \frac{h}{2}(f(a) + f(b)), \quad h = b - a$$

to derive a more accurate integration formula. Identify this more accurate integration formula by giving its familar name. (Hint: $I_{h/2}$ would use two subintervals)

5. Establish a finite difference formula to approximate $\frac{\partial f(x,y)}{\partial x}$ using $f(x,y), f(x-h,y), f(x-2h,y)$. Be as accurate as possible and derive an expression of the truncation error. Assume $f(x,y)$ is smooth enough.

6. Give an **upper bound** for the relative error in the solution of the system of linear equations

$$Ax = b$$

with symmetric matrix A given by

$$A = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 6 & 2 \\ 2 & 2 & 8 \end{bmatrix}$$

when the relative error in $\mathbf{b}$ is less that $4 \cdot 10^{-4}$, i.e.

$$\frac{\| \delta b \|}{\| b \|} < 4 \cdot 10^{-4}.$$

Use spectral norms, i.e., use

$$\| b \| = \left(\sum_j |b_j^2| \right)^{1/2}.$$

7. Let $A = (a_{ij})$ be an $n \times n$ matrix. An iterative scheme for the solution of the linear system $Ax = b$ is described by

$$\text{given } x_i^{(0)}, \quad i = 1, \ldots, n;$$

$$a_{ii} y_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} y_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}$$

$$x_i^{(k+1)} = \omega y_i^{(k+1)} + (1-\omega) x_i^{(k)}, \quad i = 1, \ldots, n; \quad k = 0, 1, \ldots$$

(a) Write the iterative scheme in the form

$$x^{(k+1)} = T x^{(k)} + c$$

(Hint: consider the splitting $A = D - L - U$).

(b) For the particular case

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

varify that the choice $\omega = 8/7$ gives the best rate of convergence.

8. Given $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ (T means transpose), define $v = x + \text{sign}(x_1) \| x \|_2 e_1$, where $e_1 = (1, 0, \ldots, 0)^T$. The Householder matrix (Householder transformation) with v (Householder vector) is given by

$$P = I - 2\frac{vv^T}{v^T v},$$

which is orthogonal and symmetric.

(a) Verify that $Px = -\text{sign}(x_1) \| x \|_2 e_1$.

(b) Describe how the Householder matrices can be used to construct an orthogonal matrix Q for a given matrix $A \in R^{n \times n}$ such that

$$A = QR,$$

where R is upper tirangular.