Topology Qualifying Exam
Spring 1996

Choose and work any 6 of the following 14 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems**. A space always means a topological space below.

1. (a) Prove or disprove that if m and n are any cardinal numbers then $m + n = n + m$.

(b) Prove or disprove that if m and n are any ordinal numbers then $m + n = n + m$.

2. Prove or disprove that every filter on a nonempty set is contained in an ultra-filter on that set.

3. Prove that $\mathbb{R}$ and $\mathbb{R}^2$ with their usual topologies are not homeomorphic.

Use the following definition for problems 4 and 5.

For any topological property P a space is X called **locally** P if and only if every point x of X has a base of neighborhoods each of which has property P (as a subspace).

4. (a) Prove or disprove that locally compact spaces are hereditary [i.e., that every subspace of a locally compact space is locally compact].

(b) Prove or disprove that compact spaces are hereditary.

5. Prove that exactly one of the properties locally T_1 and locally T_2 is hereditary; i.e, that one is hereditary and the other is not.

6. Let X be a T_1 completely regular space. Prove that X is connected if and only if its Čech-Stone compactification $\beta(X)$ is connected.

7. List three different bases for the usual topology of the plane (i.e., $\mathbb{R}^2$).

8. A topological property P is called *divisible* iff every topological quotient of a space with property P has property P.

(a) Which of the following properties are divisible and which are not? Hausdorff, connected, locally connected, compact, metrizable, discrete.

(b) Prove one of your assertions from part (a).

9. A topological property P is called productive iff every topological product of a family of spaces with property P has propery P.

(a) Which of the following properties are productive and which are not? Hausdorff, connected, locally connected, compact, metrizable, discrete.

(b) Prove one of your assertions from part (a).

10. Find an incorrect statement in the proof of the following theorem and prove that it is an incorrect statement.

Theorem. If $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$ has the usual topology, then each continuous function $f : B^2 \to B^2$ has a fixed point.

Proof: Suppose that $f : B^2 \to B^2$ is a continuous function with no fixed points. Let $\pi_1 : B^2 \times B^2 \to B^2$ be first projection, let

$$\Delta = \{(z, z) | z \in B^2\}$$

be the diagonal in $B^2 \times B^2$, and let

$$F = \{(z, f(z)|z \in B^2\}$$

be the graph of f. Since f is continuous, $\pi_1|F : F \to B^2$ is a homeomorphism. Since B^2 is connected, F is therefore connected. However,

$$F \subseteq B^2 \times B^2 - \Delta$$

which is a contradiction since $B^2 \times B^2 - \Delta$ is connected.

11. (a) Prove or disprove that the subspace topology on the set of integers, $\mathbb{Z}$, in $\mathbb{R}$ (with its usual topology) is the sames as the discrete topology on $\mathbb{Z}$.

(b) Prove or disprove that the subspace topology on the set of rationals, $\mathbb{Q}$, in $\mathbb{R}$ (with its usual topology) is the same as the discrete topology on $\mathbb{Q}$.

12. Let (X, d) be a metric space. Show that the following are equivalent:

(i) X has a countable basis;

(ii) X is Lindelof;

(iii) X has a countable dense subset.

13. Let X be a regular space and assume that U is a countable open covering of X with the property that the closure of each of its members is paracompact. Prove that X is paracompact.

14. Prove that the topologist's comb

$$C = \left(\bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times [0,1] \right) \cup ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])$$

is not a retract of the square $S = [0,1] \times [0,1]$ (with both C and S having the usual Euclidean subspace topologies).