REAL ANALYSIS QUALIFYING EXAM
SEPTEMBER 10, 1984

Complete solutions to half of the problems will be regarded more highly than half-solutions to all of the problems. Five complete solutions will be sufficient to pass this exam.

1. Prove that if $f \in L_1(\mathbb{R})$ and $\int_a^b f(x)dx = 0$ whenever $a < b$ in $\mathbb{R}$, then $f = 0$ a.e.

2. Let $(\Omega, \mathcal{A}, \mu)$ be any measure space. Suppose that $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$ satisfies $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Define $A = \{\omega \in \Omega : \{n \in \mathbb{N} : \omega \in A_n\}$ is infinite$\}$. Give a detailed proof that $A \in \mathcal{A}$ and $\mu(A) = 0$. [Hint: Consider $B_k = \bigcup_{n=k}^{\infty} A_n$.]

3. State (a) Lebesgue's Dominated Convergence Theorem and (b) Fatou's Lemma. Use one of (a) and (b) to prove the other one.

4. Suppose that $\phi : \mathbb{Z} \to \mathbb{C}$ satisfies $\sum_{n \in \mathbb{Z}} |\phi(n)|^2 < \infty$. Prove that there exists $f \in L_2([0,2\pi])$ such that $\frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx = \phi(n)$ for all $n \in \mathbb{Z}$ and $\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\phi(n)|^2$. [Hint: Consider $f_N(x) = \sum_{n=-N}^{N} \phi(n)e^{inx}$.]

5. Let $g : \mathbb{R} \to \mathbb{R}$ be monotone nondecreasing. Define an outer measure μ on $\mathbb{R}$ by

$$\mu(E) = \inf\{ \sum_{j=1}^{\infty} (g(b_j) - g(a_j)) :$$

$$a_j \leq b_j \text{ in } \mathbb{R},\ E \subset \bigcup_{j=1}^{\infty}]a_j, b_j[\}.$$

Prove that every Borel set $A \subset \mathbb{R}$ is μ-measurable. [Hint: Begin with $A =]c,\infty[$ where c is a point of continuity of g.]

6. Let $f \in L_1(\mathbb{R})$. Define g on $\mathbb{R}$ by

$$g(x) = \int_{\mathbb{R}} f(y)\mathrm{Tan}^{-1}(xy)dy$$

where Tan is the restriction of the tangent function tan to $]-\pi/2, \pi/2[$.

(a) Prove that g is odd and continuous on $\mathbb{R}$.

(b) Prove that g is differentiable at every $x \neq 0$.

(c) Prove that if $f(y) = y^{-2}$ for $y \geq 1$ and $f(y) = 0$ for $y < 1$, then $g'(0) = \infty$.

7. Let X be a (nonvoid) compact Hausdorff space and let $\emptyset \neq F \subseteq C(X)$. Suppose that

$$\sup\{|\int f d\mu| : f \in F\} < \infty$$

for every $\mu \in M(X)$. Prove that

$$\sup\{\|f\|_u : f \in F\} < \infty$$

where

$$\|f\|_u = \sup\{|f(x)| : x \in X\}.$$

8. Let $P =]0,\infty[$ denote the positive half-line. Define a Borel measure μ on P by

$$\mu(B) = \int_B \frac{dx}{x}.$$

Let $f, g \in L_1(\mu)$.

(a) Prove that the set A of all $x \in P$ for which the function $t \to f(x/t)g(t)$ is in $L_1(\mu)$ satisfies $\mu(P \setminus A) = 0$.

(b) Prove that the function h defined on P by

$$h(x) = \int_0^\infty f(x/t)g(t)d\mu(t)$$

for $x \in A$ and $h(x) = 0$ otherwise is in $L_1(\mu)$.

9. Suppose that (X, A, μ) is σ-finite and that $\{B_n\}_{n=1}^\infty \subseteq A$ satisfies $\mu(B_n) > 0$ and $B_{n+1} \subseteq B_n$ for all n, but $\bigcap_{n=1}^\infty B_n = \emptyset$.

(a) Prove that there exists a bounded linear functional ϕ on $L_\infty(X, A, \mu)$ such that $\phi(\xi_{B_n}) = 1$ for all n.

[Hint: Consider the linear subspace S of L_∞ consisting of all f for which there exists a constant $c(f) \in \mathbb{C}$ such that $f|B_n = c(f)$ μ-a.e. for some n, (depending on f).]

(b) Use (a) to prove that the Banach space $L_1(X, A, \mu)$ is <u>not</u> reflexive.

16. Let $\mu \in M(T)$. For $n \in \mathbb{Z}$, define

$$\hat{\mu}(n) = \int e^{-int}\, d\mu(t).$$

Suppose that $\lim_{n\to\infty} \hat{\mu}(n) = 0$. Prove that $\lim_{n\to\infty} \hat{\mu}(-n) = 0$.

[Hint: Approximate $d\bar{\mu}/d\mu$ by a trigonometric polynomial where $\bar{\mu}$ is the complex conjugate of μ.]