Complex Variables Qualifying Exam
Spring 1990

In what follows $\mathbb{R}$ is the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{N}$ the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{D}$ the open disc cented at 0 in $\mathbb{C}$, $\mathbb{T}$ the boundary of $\mathbb{D}$.

1. What is the "reflection principle" for holomorphic functions? Prove the simplest version of this principle, i.e., that involving reflection in the real axis.

2. Show that if f is holomorphic and zero-free in the open set U, then $|f|^p$ is subharmonic in U for every real p. **Hint:** If $f = e^g$, the problem is easy. (For extra credit: Is the result true if f is permitted to have zeros?)

3. Write an essay on the role of simple-connectivity in complex analysis. Touch on the following points:

(a) A definition of simple-connectivity appropriate to regions in $\mathbb{C}$,

(b) several important equivalences of your definiton,

(c) connection with holomorphic logarithms and holomorphic roots of holomorphic functions,

(d) Cauchy's Integral Theorem and existence of primitives,

(e) special role of the regions $\mathbb{D}$ and $\mathbb{C}$,

(f) existence of harmonic conjugates and solvability of the Dirichlet problem.

4. ϕ is meromorphic in $\mathbb{C}$. Explain why (or why not) there must exist entire functions f and g such that $\phi = \frac{f}{g}$.

5. Describe pictorially the region $\Omega := \mathbb{C} \setminus \{z : \mathrm{Re} z = \mathrm{Im} z \geq 1\}$ and find explicitly a conformal map of it onto $\mathbb{D}$.

6. Suppose f has a pole at z_0, that $0 \leq \alpha < \beta \leq 2\pi$ and $T_r(\alpha, \beta) := \{re^{i\theta} + z_0 : \alpha \leq \theta \leq \beta\}$. Evaluate $\lim_{r\to 0} \int_{T_r(\alpha,\beta)} f(z)dz$ in terms of the Laurent coefficients of f at z_0.

7. Let $f(z)$ denote any holomorphic square-root of z in $D_1 := \{z \in \mathbb{C} : |z - 1| < 1\}$ and let F be an analytic continuation of f along a curve from 1 to -1. Show that $F(-1)$ is either i or $-i$.

8. Consider the (concentric) annulus $A := \{z \in \mathbb{C} : 1 < |z| < 2\}$. What linear fractional transformations map A to another annulus $A^* := \{z \in \mathbb{C} : r < |z - a| < s\}$? **Hint:** Build the map as a composite of simpler ones.

9. (a) Define: z_0 is an n-th order zero of the holomorphic function f.

(b) Define: z_0 is an essential singularity of the holomorphic function F.

(c) Can the poles of a holomorphic function have an accumulation point? Explain.

(d) Just what kind of set can be the set of poles of a meromorphic function?

(e) f is entire and $f(\mathbb{Z}) = 0$. Explain why $\frac{f(z)}{\sin(2\pi z)}$ is an entire function.

10. Show that $\mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) = \frac{1-|z|^2}{|e^{it}-z|^2}$ and that its integral over $t \in [0, 2\pi]$ is 2π if $z \in \mathbb{D}$. What is the value of this integral for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$?