Fall 1971

Topology Examination October 1971

1) Suppose $\mathcal{U} = \{V_j \mid 1 \le j \le n\}$ is a finite open cover of a normal topological space. Give a direct construction of an open cover $\{W_j \mid 1 \le j \le n\}$ so that $\overline{W_j} \subseteq V_j$ for $1 \le j \le n$. Construct a partition of unity subordinate to $\mathcal{U}$.

2) Prove that a metric space is compact if and only if it is sequentially compact.

3) Prove that the one point compactification of a 2nd countable, [illegible] locally compact space is metrizable.

4) Suppose that $f: [0,1] \longrightarrow S^2 = \{(x_1,x_2,x_3) \mid x_1^2+x_2^2+x_3^2 = 1\} \subseteq \mathbb{R}^3$ is continuous and satisfies

a) $f(0) = f(1)$

b) $f \mid (0,1]$ is one-to-one

Show that

a) image $f \neq S^2$

b) There is a continuous homotopy $F: [0,1] \times [0,1] \longrightarrow S^2$ so that $F(x,1) = f(x)$, $F(x,0) = f(0)$, $F(0,t) = f(0) = F(1,t)$ for all t and x.

5) A topological space is said to be extremely disconnected if the closure of every open set is open.

a) Show that a space is extremely disconnected if and only if every two disjoint open sets have disjoint closures.

b) Let N be the natural numbers with the discrete topology and X be the Stone-Cech compactification of N. Show that X is extremely disconnected.

6) Let X be a set and $\leq$ a linear order on X. Assume there is a least element 0 and a greatest element 1 in X. Let $\mathcal{T}$ be the topology generated by sets of the form $\{y \mid y > x\}$, $\{y \mid y < x\}$ for $x \in X$. Show that $(X, \mathcal{T})$ is compact if and only if every subset of X has a least upper bound.

7) Let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$ for $n \geq 1$. Define an equivalence relation ρ_n on S^n by $x \rho_n y$ iff $x = \pm y$ and let P^n be the topological space S^n/ρ_n. Let $D^n = \{(x_1, x_2, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq 1\}$. Then S^{n-1} is a closed subset of D^n. If $\pi_n : S^n \to P^n$ is the natural projection, show that P^n is homeomorphic to $P^{n-1} \cup_{\pi_{n-1}} D^n$ for $n \geq 2$.

8) Show that $\{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\} \subseteq \mathbb{R}^2$ is connected but not path connected.

9) Define $f : [0,1] \times [0,1] \to \mathbb{R}$ by $f(x,y) = \sqrt{x^2 + \sqrt{xy}}$. Show that f is continuous.

10) Let X be a compact T_2 space and let $\mathcal{F} = \{f : X \to X \mid f$ is continuous and $\exists x \in X$ so that $f(x) = x\}$. Show that $\mathcal{F}$ is a closed subset of $C(X, X)$ with respect to the compact-open topology.