ALGEBRA [illegible] S95

Do exactly 2 problems from each of the four sections: Groups, Rings and Modules, Linear Algebra, Fields and Galois Theory.

I. GROUPS

1. Let G be a group and let Z be a cyclic group of order 10. Show that if there exists a surjective homomorphism $G \to Z$, then G has a normal subgroup of index 5.

2. Let p and q be distinct primes and let G be a group of order p^2q. Show that G has a normal Sylow p-subgroup or a normal Sylow q-subgroup.

3. Let A be an abelian group of order 300. List the possible isomorphism types which A may have.

4. Let A be the abelian group with presentation

$$A = \langle a_1, a_2, a_3 \mid 2a_1 - a_2 = 0,\; -a_1 + 2a_2 - a_3 = 0,\; -a_2 + 2a_3 = 0 \rangle.$$

Calculate the structure of A.

II. RINGS AND MODULES

1. Let $\mathbb{Z}$ be the ring of integers, and let $f : \mathbb{Z} \to D$ be a surjective homomorphism of rings. If D is an integral domain but is not a field, prove that f is an isomorphism.

2. Let $\mathbb{Z}$ be the ring of integers and let x be an indeterminate over $\mathbb{Z}$. Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

3. Let R be an integral domain.

   (a) State what it means for R to be Noetherian.

   (b) Prove that every principal ideal domain is Noetherian.

4. Let R be a ring and set

$$J(R) = \{r \in R \mid rM = 0 \text{ for every simple left } R\text{-module } M\}$$

(the Jacobson radical of R). Prove that $J(R)$ is a 2-sided ideal of R and that $J(R/J(R)) = 0$.

III. LINEAR ALGEBRA

1. Let A be an $n \times m$ matrix and let B be an $m \times n$ matrix, with entries in a field F. If $n > m$ show that $\det(AB) = 0$.

2. Suppose that V is a $\mathbb{Q}$-vector space and that $T : V \to V$ is a linear transformation with $c_T(x) = (x^2 + x + 1)^2 (x - 2)^2$ (characteristic polynomial). Write down all possible rational canonical forms which T might have.

3. Let V be a finite dimensional real vector space. Assume that there exists a linear transformation $T : V \to V$ such that $T^2 + I = 0$. Prove that the dimension of V is even.

4. Let S be the real vector space of $n \times n$ real symmetric matrices.

   (a) Show that S has dimension $\frac{1}{2} n(n + 1)$.

   (b) Define an inner product on S by setting

   $$(A, B) = \operatorname{trace}(AB), \quad A, B \in S.$$

   Show that $(\ , \ )$ is a positive definite symmetric inner product on S.

IV. FIELD THEORY AND GALOIS THEORY

1. Let F be a field of characteristic 3 and let x be an indeterminate over F. Set $K = F(x)$ and let L be a splitting field over K of the polynomial $Y^3 - x \in K[Y]$. Show that L is not a separable extension of K.

2. Let $F \subseteq K$ be a field extension of finite degree and let $\alpha \in K$. Define the F-linear transformation

   $$\hat{\alpha} : K \to K$$

   by setting $\hat{\alpha}(\beta) = \alpha\beta$. If $m(x)$ is the minimal polynomial of $\hat{\alpha}$, show that $m(x)$ is an irreducible polynomial in $F[x]$.

3. Let K be the splitting field over the rational field $\mathbb{Q}$ of the polynomial $x^n - 2$, where $n > 2$. Prove that the Galois group of K over $\mathbb{Q}$ cannot be abelian.

4. Let p be a prime number and let n be a positive integer. Prove that there exists a finite field of order p^n. (Hint: let $q = p^n$ and consider the splitting field over $\mathbb{Z}/(p)$ of the polynomial $x^q - x$.)