Partial Differential Equations Qualifying Exam
Spring 1996

The bundle #1: The Laplace Equation

1.1 Consider the following problem:

$$-\Delta u(x) = f(x), \quad x \in B_R(0) = \{|x| < R\}, \tag{1a}$$

$$u(x) = 0, \quad |x| = R, \tag{1b}$$

where $x = (x_1, \ldots, x_n), n \geq 1, |x| = \sqrt{x_1^2 + \cdots + x_n^2}, R > 0$.

How would you define the solution of problem (1) for $f \in L^2(B_R(0))$? State a theorem on the existence and uniqueness of solution to (1) with $f \in L^2(B_R(0))$.

Choose the appropriate function space for the solutions of (1) with f's in $L^2(B_R(0))$, and show that the solution depends continuously on f.

1.2 The Poisson formula,

$$u(x) = \frac{1}{|\mathbb{S}^{n-1}|} \cdot \frac{R^2 - |x|^2}{R} \cdot \int_{|y|=R} \frac{\varphi(y)}{|x-y|^n} dS_y, \tag{2}$$

where $|\mathbb{S}^{n-1}|$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$, gives the solution of the following Dirichlet problem in the ball $B_R(0)$,

$$-\Delta u(x) = 0, \quad x \in B_R(0) = \{|x| < R\}, \tag{3a}$$

$$u(y) = \varphi(y), \quad |y| = R. \tag{3b}$$

Use (2) to prove the following result (*Harnack's inequality*):

If u is a nonnegative harmonic function in the ball $B_R(0)$, and $u \in C(\overline{B_R(0)}) \cap C^2(B_R(0))$, then

$$R^{n-2} \cdot \frac{R - |x|}{(R+|x|)^{n-1}} \cdot u(0) \leq u(x) \leq R^{n-2} \cdot \frac{R + |x|}{(R-|x|)^{n-1}} \cdot u(0), \tag{4}$$

for any $x \in B_R(0)$.

Use (4) to prove the Liouville theorem:

If the function u is harmonic in $\mathbb{R}^n$ and bounded from below, then $u \equiv const$.

1.3 Let Γ_α, $0 < \alpha < 2\pi$, be the sector on the plane defined in the polar coordinates (r, θ) by the equation

$$\Gamma_\alpha = \{(r, \theta) : r > 0, 0 < \theta < \alpha\}.$$

Denote by $\partial\Gamma_\alpha$ the boundary of Γ_α (formed by the two rays, $\{r \geq 0, \theta = 0\}$ and $\{r \geq 0, \theta = \alpha\}$).

Use the separation of variables method to find nontrivial solutions u of the Dirichlet problem

$$-\Delta u(x) = 0, \quad x \in \Gamma_\alpha, \tag{5a}$$

$$u(x) = 0, \quad x \in \partial\Gamma_\alpha. \tag{5b}$$

In the case $\pi < \alpha < 2\pi$, choose the solution u with the properties:

$$u \in W_2^1(\Gamma_{\alpha,R}), \text{ but } u \notin W_2^2(\Gamma_{\alpha,R}). \tag{6}$$

for any $R > 0$, where $\Gamma_{\alpha,R}$ is the intersection of Γ_α with the disc $\{0 \leq r < R\}$, and W_2^1, W_2^2 are the standard Sobolev spaces (sometimes the notation $W^{1,2}$, $W^{2,2}$ is used for these spaces).

The bundle #2: The Heat Equation

2.1 Consider the Cauchy problem:

$$\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = 0, \quad t > 0, x \in \mathbb{R}^n, \tag{7a}$$

$$u(0,x) = f(x). \tag{7b}$$

Assuming that $f(\cdot)$ is smooth, rapidly decreasing function, use the Fourier transform in x to find the solution u of (7).

(You may need the equality $\int_{-\infty}^{+\infty} e^{-s^2} ds = \sqrt{\pi}$.)

Show that if $f \in L^2(\mathbb{R}^n)$, then the formula you obtained for $u(t,x)$ defines a function with the following properties:

a) $u(t,\cdot) \in L^2(\mathbb{R}^n)$ for all $t > 0$;

b) the mapping $t \mapsto u(t,\cdot) \in L^2(\mathbb{R}^n)$ from $[0,+\infty)$ into $L^2(\mathbb{R}^n)$ is continuous, and $u(t,\cdot) \to f(\cdot)$ in $L^2(\mathbb{R}^n)$ as $t \searrow 0$;

c) $u(t,x)$ satifies (7a) in the distributional sense.

2.2 In the case $n = 1$, find the solution of (7) corresponding to the initial condition

$$f(x) = e^{ax^2},$$

where $a = const > 0$. Find the life span of this solution (the life span is the largest time interval where the solution is finite).

2.3 Show that if $u(0,x) = f(x)$ is an even (odd) function of x, then, for every $t > 0$, the solution $u(t,x)$ of (7) is an even(odd) function of x as well.

Use this to solve the following initial boundary-value problem on the half- line:

$$\frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = 0, \quad t > 0, 0 < x < +\infty, \tag{8a}$$

$$u(0,x) = f(x) \equiv 1, 0 < x < +\infty, \tag{8b}$$

$$u(t,0) = 0, t > 0. \tag{8c}$$

The bundle #3: The Wave Equation

3.1 Consider the Cauchy problem

$$\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) = 0, \quad -\infty < t < +\infty, \quad x \in \mathbb{R}^n, \tag{9a}$$

$$u(0,x) = 0, \quad x \in \mathbb{R}^n, \tag{9b}$$

$$\frac{\partial}{\partial t}u(t,x)|_{t=0} = f(x), \quad x \in \mathbb{R}^n. \tag{9c}$$

Assume that you know the operator $U(t)$ that solves this problem, i.e.,

$$U(t) : f(\cdot) \mapsto (U(t)f)(\cdot) = u(t,\cdot), \tag{10}$$

where $u(t,)$ is the solution of (9).

Use this operator to express the solution of the more general problem,

$$\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) = h(t,x), \quad -\infty < t < +\infty, \quad x \in \mathbb{R}^n, \tag{11a}$$

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$$u(0, x) = g(x), \quad x \in \mathbb{R}^n, \tag{11b}$$

$$\frac{\partial}{\partial t} u(t, x)|_{t=0} = f(x), \quad x \in \mathbb{R}^n, \tag{11c}$$

in terms of the data: f, g, and h.

Hint: Looking at the Fourier transformation of the solution of (11) may help you.

3.2 Consider the problem (11). If the initial data f and g are chosen from the function spaces $H^1(\mathbb{R}^n) = W_2^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively, then what is the "right" function space for h in order that problem (11) be well-posed?

Write down the appropriate energy estimate that allows to prove the uniqueness of the solution for (11). Derive this estimate for the solution of (11) under the assumptions that the initial data f and g, the external force h, and the solution u are all sufficiently smooth and decrease fast at the spatial infinity.

3.3 What is the Huygens principle for the wave equation?

Solve problem (9) explicitly in the case $n = 1$ and

$$f(x) = \begin{cases} 1, \text{if } |x| \leq 1, \\ 0, \text{if } |x| > 1. \end{cases}$$

Explain how your solution illustrates the Huygens principle.

3.4 Use the method of separation of variables to solve the following initial boundary-value problem:

$$\frac{\partial^2}{\partial t^2} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = 0, \quad t > 0, \quad 0 < x < 1, \tag{12a}$$

$$u(0, x) = \sin(\pi x), \quad 0 \leq x \leq 1, \tag{12b}$$

$$\frac{\partial}{\partial t} u(t, x)|_{t=0} \equiv 1, \quad 0 \leq x \leq 1, \tag{12c}$$

$$u(t, 0) = u(t, 1) = 0, \quad t \geq 0. \tag{12d}$$