GEOMETRY QUALIFYING EXAM
SPRING 2001 (Auckly & Miller)

Work as many of the following problems as you can. You do not have to solve every problem to pass.

1. On $\mathbb{R}^2$ let

$$\begin{aligned} X &= y\frac{\partial}{\partial x} + y^2 x\frac{\partial}{\partial y}, \\ Y &= (x+y)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \\ \alpha &= -dx + x\,dy \end{aligned}$$

Calculate: a) $[X, Y]$, b) $L_Y \alpha$, c) $L_Y(d\alpha)$.

2. a) Construct an atlas for $S^2 \times S^1$.

b) Parametrize S^2 by (θ, φ) where $x = \cos\theta\cos\varphi$, $y = \sin\theta\cos\varphi$, $z = \sin\varphi$. Parametrize $S^2 \times S^1$ by $(\theta_1, \varphi_1, \alpha)$ in a similar way $(\alpha \in (\theta, 2\pi))$ and define $f : S^2 \times S^1 \to S^2$; $f(\theta_1, \varphi_1, \alpha) = (2\theta_1, \varphi_1)$. Notice that $\mu = \frac{1}{4\pi}\sin\varphi\, d\theta \wedge d\varphi$ is the normalized area form on S^2.

Compute $\displaystyle\int_{S^2 \times S^1} \frac{1}{2\pi} d\alpha \wedge f^*\mu$. (Use the orientation $d\alpha \wedge d\theta \wedge d\varphi$.)

c) Show that the answer remains unchanged if μ is replaced by $\mu + d\omega$.

d) What does this integral represent geometrically?

3. Let $g = dt^2 + \sin^2 t\, d\alpha^2 + \cos^2 t\, d\beta^2$.

a) Pick an orthonormal coframe, θ^k, suitable for computation with the metric, g.

b) Compute the connection form, ω, relative to the coframe that you chose in part a).

c) Compute the curvature form, Ω, relative to the same coframe.

4. Let E and F be distributions on (M, g) so that $X \in E$ and $Y \in F$ implies $g(X, Y) = 0$ and so that $X, Y \in \Gamma(E)$ implies $\nabla_X Y \in \Gamma(E)$ and $X, Y \in \Gamma(F)$ implies that $\nabla_X Y \in \Gamma(F)$.

a) State Frobenius' Theorem.

b) Show that E and F are integrable.

c) Show that any point of M has a product neighborhood $V \times U$ so that $g|_{V\times U} = g|_v \oplus g|_u$.

5. Let X and Y be the vector fields on $\mathbb{R}^4 - \{0\} \times \mathbb{R}^2$ given by

$$\begin{aligned} X &= x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3} \\ Y &= x^3\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^4} \end{aligned}$$

Let E be the subbundle of the tangent bundle generated by X and Y.

a) Use the Frobenius Theorem to show that E is integrable.

b) Find parametric equations for the integral manifold containing the point $(1, 2, 3, 4)$.

6. a) State Sard's Theorem. Let $f : S^2 \to \mathbb{R}$ be a smooth map.

b) Show that for any $y_0 \in \mathbb{R}$ and any $\varepsilon > 0$ there is a $y \in B_\varepsilon(y_0)$ so that $f^{-1}(y)$ is a finite disjoint union of circles.

7. a) Give an example of a space, X, with $\pi_1(Y) = \langle a, b, c | a^3 = 1,\ b^4 = 1,\ c^5 = 1,\ (abc)^2 = 1\rangle$.

b) Give an example of a space, Y, with $\pi_1(Y) = \langle a, b | aba^{-1}b = 1\rangle$.

c) Closed, Hausdorff, separable 2-manifolds have been classified. State the classification theorem.

d) There is a closed 2-manifold with the fundamental group from part b). Which 2-manifold has this fundamental group?