Numerical Analysis Qualifying Exam
Spring 1987

Hand in at most ten problems. You must work at least one from each of the five sections.

I. Differentialtion, Integration, and General Topics.

1. A centered difference method for approximating a second derivative uses

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2},$$

$h>0$, for the approximation. If $f^{(4)}$ is defined and continuous on an interval $[a,b]$ containing $x-h$ and $x+h$, show that the error in the approximation is given by

$$-\frac{h^2}{12}f^{(4)}(c)$$

for some $c\varepsilon(x-h,x+h)$.

2. Let $E(f)=\sum_{k=0}^{n}\alpha_k f(x_k)$ be a simple quadrature formula which uses $n+1$ distinct nodes. If E has degree of procision at least n, show that E is an interpolatory quadrature.

3. Suppose we are given a program which approximates $\int_b^a f(x)dx$ using composite Simpson's rule. The user supplies the integrand $f(x)$, the limits a, b, and the number of panels N. The program runs on a computer which uses nine decimal digit floating point calculations. If we seek the value of $\int_0^1 \sqrt{x}\exp x dx$ we get the following results:

N (Panels)	Approximate Integral
5	1.253107
10	1.254730
20	1.255311
40	1.255517
100	1.255601
400	1.255627
800	1.255629

The actual value of the integral, correct to 7 figures is 1.255630.

(a) Why should we have expected possible trouble in directly using Simpson's rule on this integral?

(b) What should be done to use the program more efficiently to calculate the value of this integral?

4. Explain how to calculate accurately the value of

$$f(x)=\frac{6\sin x-6x+x^3}{x^5}$$

for very small positive x. Why is there any difficulty at all?

II. Root Finding.

5. (a) Describe Newton's method for functions of several variables. State without proof the quadratic convergence theorem for this method. (Use the asymptotic convergence rate difinition).

(b) Let

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{x} & \text{if } x < 0 \end{cases}$$

Apply Newton's method for any starting value and explain the results in light of the convergence theorem you stated (a).

6. (a) Describe Aitken's δ^2-method of accelerating the convergence of a sequence.

(b) Let $\{x_n\}$ converge to x with $x_n \neq x$ for all $n = 1, 2, \ldots$. Show that if the convergence is linear, then

$$\lim_{n \to +\infty} \frac{x'_n - x}{x_n - x} = 0$$

where $\{x'_n\}$ is the associated accelerated sequence.

7. Give an example to show that the process of determining a root of a polynomial is, in general, ill-conditioned.

III. Approximation Theory.

8. (a) Describe the interpolating polynomial P_n for a real valued function f and distinct points $\{x_0, \ldots, x_n\}$ in the domain of f.

(b) Let f be infinitely differentiable on $[a, b]$ and suppose there exists a real number M such that $|f^{(n)}(x) \leq M$ for all $x \varepsilon [a, b]$ and all $n \geq 0$. For $n \geq 0$, let P_n interpolate f at some set of $n+1$ distinct points in $[a, b]$. Show that $P_n \to f$ uniformly on $[a, b]$.

(c) For an arbitray function f defined on $[a, b]$ and for evenly spaced nodes $x_k = a + \frac{k}{n}(b-a)$, $0 \leq k \leq n$ in $[a, b]$, let P_n interpolate f. Does $p_n \to f$ uniformly on $[a, b]$? State the appropriate theorem or give a counterexample. You need not give a proof for this part.

9. Of all polynomials of degree ≤ 3, find the one which best approximates $p(x) = 2x^4 - 3x^2 + x + 1$ on the interval $[-1, 1]$ with respect to the uniform norm $\| \quad \|_\infty$. Use the fact that if $q_n(x)$ is the polynomial in P_n which best approximates x^{n+1} on $[-1, 1]$, then $x^{n+1} - q_n(x) = 2^n T_{n+1}(x)$ where $T_{n+1}(x)$ is the Chebyshev polynomial.

10. Let V be a real vector space, W a finite dimensional subspace of V having dimension n, and let $<,>$ be a symmetric, positive semidefinite, bilinear form on V such that $<,>$ is positive definite on W. Let $p_1, \ldots, p_n$ be an orthonormal basis for W and let $\| v \| = \sqrt{<v, v>}$ for $v \varepsilon V$.

(a) Let $v \varepsilon V$ and let $w = \sum_{i=1}^n \beta_i p_i \varepsilon W$. Show that

$$\| v - w^2 \|^2 = \| v \|^2 + \sum_{j=0}^n (\beta_j - <v, p_j>)^2 - \sum_{j=0}^n <v, p_j>^2 .$$

(b) Given $v \varepsilon V$, show that there is a unique $w \varepsilon W$ that minimizes $\| v - w \|$ and that it is given by

$$w = \sum_{j=1}^n <v, p_j> p_j .$$

IV. Linear Algebra.

11. Assume that the equation $x = Ax_b$ has a unique solution and consider the iterative scheme

$$x_{i+1} = Ax_i + b$$

where A is an n by n complex matrix and x and b are complex n-vectors.

(a) Prove that if the sequence $\{x_i\}$ converges for an arbitrary starting vector x_0, then $\tau_\sigma(A) < 1$, where $\tau_\sigma(A)$ is the spectral radius of A.

(b) Prove that if $\tau_\sigma(A) < 1$, then $\{x_i\}$ converges for all starting values x_0. (Hint: Use the spectral radius formula.)

12. Outline in detail the Q-R method with Francis Q-R step. State, without proof, any relevant theorems associated with this method.

13. Does the Gauss-Seidel method converget for the linear equation

$$\begin{bmatrix} 4 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \end{bmatrix} ?$$

V. Differential Equations.

14. Let $f : [a, b] \to R$ satisfy a Lipschitz condition,

$$|f(x, y) - f(x, y')| \leq L|y - y'|$$

for all $x\varepsilon[a, b]$ and $y\varepsilon R$. Show that Euler's method for the numerical solution of the initial value problem

$$y' = f(x, y)$$

$$y(0) = y_0$$

is stable. Begin by stating the meaning of "stable" in this context.

15. Explain the meaning of the term "parasitic solution" asd it relates to the numerical solution of ordinary differential equations. Relate it to the concept of stability and to the root condition.