Real Analysis Qualifying Exam
Fall 1992

In what follows $(X, \mathcal{A}, \mu)$ is an arbitrary measure space and λ is Lebesgue outer measure on $\mathbb{R}$.

1. (a) Find a necessary and sufficient condition that $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha + \beta| = |\alpha| + |\beta|$.

(b) Find a necessary and sufficient condition that $f, g \in L_1(\mu)$ satisfy:

$$\int |f+g| d\mu = \int |f| d\mu + \int |g| d\mu.$$

2. Suppose

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Show that

$$A = \{x : x \in A_n \text{ for infinitely many } n\}$$

has $\mu(A) = 0$.

3. Let F be a subset of $[0,1]$ that is not Lebesgue measureable.

(a) Is it possible that $\lambda(F) = 0$? Why?

(b) It is possible that $\lambda(F) = 1$? Why?

4. Suppose f is continuous on [0,1]. Show that

$$\text{Riemann} \int_0^1 f dx = \text{ Lebesgue } \int_0^1 f dx.$$

5. Let ν be a finite measure that is absolutely continuous with respect to a measure μ that is regular. Prove that ν is regular.

6. (a) Fix $\alpha > 0$. Show that it is impossible to construct a bounded, Lebesgue measureable function f such that

$$\int_0^1 |f - g| d\lambda > \alpha$$

for every Riemann integrable function g.

(b) Can such a f be found if we take $\alpha = 0$? Why?

7. Let X, Y be topological spaces, each having a countable base for its topology. Let

$$\mathcal{B}(X) \times \mathcal{B}(Y)$$

be the smallest σ-algebra of subsets of $X \times Y$ that contains every $R \times S$ where $R \in \mathcal{B}(X)$ and $S \in \mathcal{B}(Y)$. Prove that

$$\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y).$$

8. State which types of convergence imply which other types of convergence. Prove all of your assertions and prove counterexamples.

(A) Pointwise a.e. convergence

(B) Convergence in L^1

(C) Convergence in measure