Algebra Qualifying Exam
Fall 1988

Instruction: You are to do two problems from each of the four major-areas (**Group Theory, Linear Algebra, Field and Galois Theory**, and **Rings and Modules**). Even if you attempt to do more, you will only be graded on two problems from each section. For the graders' convenience, please circle the problem numbers of those problems that you do from each of the four major areas:

I. Group Theory

1. Let G be a finite group of the order $2n$, where n is an odd integer. Prove that G has a normal subgroup of order n.

2. Let G be a nonabelian group of order p^2q, where p and q are primes. Prove that either $p|q-1$ or $q|p-1$.

3. Let $\mathbf{F}_q$ denote the finite field of q elements, and let G be the group of invertible *upper triangular* n by n matrices with elements in $\mathbf{F}_q$. Prove that G is solvable.

4. Let G be a *finite* subgroup of the multiplicative group $\mathbf{C}^\times$ of nonzero complex numbers. Prove that G is a cyclic group.

II. Linear Algebra

1. Let $\mathbf{C}$ be the complex field, and let A be an $n \times n$ matrix satisfying $A^k = I$ for some positive integer k. Prove that A is diagonalizable.

2. Let $\mathbf{F}$ be a field of characteristic p, and let V be a finite dimensional vector space over $\mathbf{F}$. Assume that T is a linear transformation on V such that $T^p = I$. Prove that there exists $0 \neq v \in V$ such that $T(v) = v$.

3. Let $A \in M_3(\mathbf{Q})$ be the matrix

$$A = \begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Compute the invariant factors of A and hence the *rational canonical form* of A.

4. Let $0 = V_0 \to V_1 \to V_2 \to \cdots \to V_{r-1} \to V_r = 0$ be an exact sequence of vector spaces. (Thus $\ker(V_i \to V_{i+1}) =$ image $(V_{i-1} \to V_i), 1 \leq i \leq r$.) Prove that $\sum_{i=0}^{r}(-1)^i dim(V_i) = 0$. (Use the *Rank-Nullity theorem, together with induction.*)

III. Field and Galois Theory

1. Let $\mathbf{F}$ be a subfield of the real field $\mathbf{R}$, and assume that $\sqrt{\alpha} \in \mathbf{F}$ for each $0 \leq \alpha \in \mathbf{F}$. Prove that $[\mathbf{F} : \mathbf{Q}] = \infty$, where $\mathbf{Q}$ is the rational field.

2. Let $n \geq 3$ be an integer, and let $\omega = \sqrt[n]{2}$, the real n-th root of 2. Let $\mathbf{E} = \mathbf{Q}(\omega)$, where $\mathbf{Q}$ is the rational number field. Compute the Galois group of $\mathbf{E}$ over $\mathbf{Q}$. (Note: Even through $\mathbf{E} \supseteq \mathbf{Q}$ might not be a Galois extension, the question still makes sense.)

3. Compute the extension degree $[\mathbf{Q}(\sqrt{2} + \sqrt{5}) : \mathbf{Q}]$.

4. Let $\mathbf{F}$ be a field, and let $\mathbf{E} = \mathbf{F}[x]$ where x is transcendental over $\mathbf{F}$. Assume that $\mathbf{F} \subseteq \mathbf{K} \subseteq \mathbf{E}$, where $\mathbf{K} \neq \mathbf{F}$ is an extension field of $\mathbf{F}$. Prove that $\mathbf{E}$ is algebraic over $\mathbf{K}$.

IV. Rings and Modules

1. Let R be a ring, and let M be an irreducible left R-module. If $0 \neq m \in M$, and if $Ann_R(m) = \{r \in R | rm = 0\}$, prove that $Ann_R(m)$ is a maximal left ideal of R.

2. Let R be a ring and define the *Jacobson radical* of R by setting $J(R) = \{r \in R | rM = 0$ for every irreducible left R-module $M\}$. Prove that $J(R) \subseteq \cap \mathcal{M}$, where the intersection is over the set of all maximal left ideals of R.

3. Let R be a unique factorization domain, and let $(0) \neq I \subseteq R$ be an ideal of R. Show that there are only finitely many principal ideals of R which contain I.

4. Let F be a field and let R be the ring

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in F \right\}.$$

Prove that R isn't semisimple.