Ph.D. Qualifying Exam in Real Analysis
September 20, 1993

In this exam, $(X, \mathcal{A}, \mu)$ is an arbitrary measure space. In some problems we also suppose $\mu(X) < \infty$.

You have three hours to do as many of the following ten problems as you can. Seven correct solutions will suffice to pass.

1. (a) What does it mean to say that a function $g : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable?

   (b) Use your definition to prove that if $f_n : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable for $n \in \mathbb{N}$ and if $f(x) = \underline{\lim}_{n\to\infty} f_n(x)\ \forall x \in X$, then f too is $\mathcal{A}$-measurable.

2. Let $\mu(X) = 1$. Suppose $A, B \in \mathcal{A}$ satisfy $\mu(A \cap B) = \mu(A)\mu(B)$. Prove that $\mu(A' \cap B') = \mu(A')\mu(B')$. Here E' denotes the complement $X \setminus E$ if $E \subset X$.

3. Suppose $f : X \to \mathbb{C}$ is in $L_1(\mu)$. Prove that

$$|\int f d\mu| \leq \int |f| d\mu.$$

   [You may use the linearity of the integral on the complex space $L_1(\mu)$ and the fact that if $h : X \to [0, \infty]$ is $\mathcal{A}$-measurable, then $\int h d\mu \geq 0$.]

4. Let ν be another measure on $(X, \mathcal{A})$ with $\nu(X) < \infty$. Prove that the following two statements are equivalent.

   (a) $A \in \mathcal{A}$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$.

   (b) $\forall \epsilon > 0 \exists \delta > 0$ such that $\nu(B) < \epsilon$ whenever $B \in \mathcal{A}$ and $\mu(B) < \delta$.

5. Let $a < b$ in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ a monotone nondecreasing function.

(a) How do you know that f is differentiable a.e. on $[a, b]$? (No proof is required.)

(b) Use Fatou's Lemma to prove that

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

(c) Must equality actually hold in (b) when f is continuous? Explain.

6. Let λ^* be Lebesgue's outer measure on $\mathbb{R}$ and let $E \subset \mathbb{R}$ be λ^*-measurable. Must it be true that

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A') \quad \forall A \subset \mathbb{R}?$$

Explain. [Here A' is the complement $\mathbb{R} \setminus A$.]

7. Let f and $f_1, f_2, \ldots$ be $\mathcal{A}$-measurable functions from X to $\mathbb{C}$. For $\delta > 0$, define $S_n(\delta) = \{x \in X : |f(x) - f_n(x)| \geq \delta\}$ for $n = 1, 2, \ldots$

(a) Prove that if $\mu(X) < \infty$ and $f_n \to f$ μ - a.e. on X, then

$$\lim_{n\to\infty} \mu(S_n(\delta)) = 0 \quad \forall \delta > 0.$$

(b) Show by an example how (a) may fail if the hypothesis $\mu(X) < \infty$ is omitted.

8. Continue the notation in problem 7. Suppose

$$\lim_{n\to\infty} \mu(S_n(\delta)) = 0 \quad \forall \delta > 0.$$

Prove that there exist $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $f_{n_k} \to f$ μ - a.e. as $k \to \infty$. [Hint: Make $\mu(S_{n_k}(1/k)) < 2^{-k}$ and consider $A_j = \bigcup_{k=j}^{\infty} S_{n_k}(1/k)$.]

9. Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are finite measure spaces.

(a) Define the product measure space $(X \times Y, \mathcal{A} \times \mathcal{B}, \mu \times \nu)$.

(b) Prove that if $f : X \times Y \to [0, \infty]$ is $\mathcal{A} \times \mathcal{B}$-measurable, then

$$\int_Y \int_X f \, d\mu d\nu = \int_X \int_Y f \, d\nu d\mu.$$

10. Let $\mu(X) < \infty$. Suppose $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ satisfies

$$\lim_{m,n \to \infty} \mu(A_m \Delta A_n) = 0$$

where $C \Delta D = (C \cup D) \setminus (C \cap D)$ for any two sets C and D.

(a) Prove that $\exists A \in \mathcal{A}$ for which $\lim_{n \to \infty} \mu(A \Delta A_n) = 0$. [Hint: Use the fact that $L_1(\mu)$ is complete.]

(b) Must the set $A = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} A_n$ work in (a)? Explain.