Real Analysis Qualifying Exam
Fall 1987

Do as many of the following ten problems as time permits. If the problem is phrased as an assertion, then you are to prove that assertion.

1. If f and g are complex-valued Lebesgue measurable function on $\mathbb{R}$ with f integrable and g bounded, then the formula

$$f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy$$

defines a function $f*g$ that is uniformly continuous on $\mathbb{R}$. [HINT: Approximate f by a function h that is continuous with compact support.]

2. Let λ denote Lebesgue measure on $\mathbb{R}$. If A and B are λ-measurable subsets of $\mathbb{R}$ with $0 < \lambda(A) < \infty$ and $\lambda(B) > 0$, then the function $x \to \lambda(A \cup (B+x))$ from $\mathbb{R}$ into $\mathbb{R}$ in continuous and not identically 0, so the set $A - B := \{a - b : a \in A, b \in B\}$ has nonvoid interior. [HINT: Apply problem 1 to the characteristic functions of A and $-B$. Also, $\int h \neq 0$ implies $h \neq 0$.]

3. Let λ denote Lebesgue measure on $\mathbb{R}$.

(a) If $B \subset \mathbb{R}$ is λ-measurable with $\lambda(B) > 0$ and if D is a countable dense subset of $\mathbb{R}$, then the set $D + B := \{d + b : d \in D, b \in B\}$ is almost all of $\mathbb{R}$, that is, $\lambda(\mathbb{R} \backslash (D+B)) = 0$. [Hint: Use problem 2.]

(b) The set $D + B$ in (a) might not equal $\mathbb{R}$. Give an example.

4. Let I be a nonvoid open interval of $\mathbb{R}$ and let $\phi : I \to \mathbb{R}$ be convex, that is,

$$\phi((1-\alpha)s + \alpha t) \leqq (1-\alpha)\phi(s) + \alpha\phi(t)$$

whenever $s, t \in I$ and $0 \leq \alpha \leq 1$. Show that:

(a) $s < c < t$ in I implies

$$\frac{\phi(c) - \phi(s)}{c - s} \leqq \frac{\phi(t) - \phi(s)}{t - s} \leqq \frac{\phi(t) - \phi(c)}{t - c}.$$

[HINT: $c = (1-\alpha)s + \alpha t$ where $\alpha = \frac{c-s}{t-s}$.]

(b) If $c \in I$ and

$$m = \inf\left\{\frac{\phi(t) - \phi(c)}{t - c} : t \in I, c < t\right\},$$

then $m \in \mathbb{R}$ and $m(u - c) + \phi(c) \leqq \phi(u)$ for every $u \in I$.

(c) [Jensen's Inequality]. If $(x, \mathcal{M}, \mu)$ is a measure space with $\mu(X) = 1$ and if $f : X \to I$ is μ-integrable, then $\int \phi \circ f d\mu$ is meaningful and $\phi(\int f d\mu) \leq \int \phi \circ f d\mu$. You may use the fact that ϕ is continuous on I, but prove it if time permits.

[HINT: In (b), take $c = \int f d\mu$, $u = f(x)$, and integrate.]

5. Consider the function $f(x) := x$ on $[0, 2\pi[$. Write down its Fourier series with respect to the orthonormal basis $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n=-\infty}^{\infty}$ in $L^2([0, 2\pi[$, Legesgue measure). Use this to calculate the value of $\sum_{n=1}^{\infty} n^{-2}$.

6. Let $(X, \mathcal{M}, \mu)$ be a finite (positive) measure space, $M(X, \mathcal{M}, \mu)$ the C-valued, $\mathcal{M}$-measurable functions on X. Say that, for $f_n, f \in M, f_n$ converges to f in μ-measure if $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. For $f, g \in M$ define

$$d(f, g) := \inf\{\varepsilon > 0 : \mu(\{|f - g| \geq \varepsilon\}) \leq \varepsilon\}.$$

(a) Show that d is a pseudo-metric.

(b) Show that $f_n \to f$ in μ-measure iff $d(f_n, f) \to 0$.

(c) Show that d is complete.

(d) Show that $\rho(f, g) := \int_X \frac{|f-g|}{1+|f-g|} d\mu$ defines a pseudo-metric in M that is equivalent to d.

7. Let $(X, \mathcal{M}, \mu)$ be a (positive) measure space. Call $A \in \mathcal{M}$ an atom if there is no measurable $B \subset A$ with $0 < \mu(B) < \mu(A)$. Suppose that there are no atoms.

(a) Show that if $A \in \mathcal{M}$, $0 < \mu(A) < \infty$ and $\varepsilon > 0$, then $\exists$ measurable $B \subset A$ with $0 < \mu(B) < \varepsilon$.

(b) Show that if $A \in \mathcal{M}, 0 < \beta < \mu(A) < \infty$, then A contains a measurable subset of measure β.

HINTS: Inductively define classes $\mathcal{H}_n \subset \mathcal{M}$, sets $H_n \in \mathcal{H}_n$ and numbers h_n by

$$H_0 := \phi, \mathcal{H}_0 := \{H_0\}, h_0 := 0;$$

$$\mathcal{H}_n := \{H \in \mathcal{M} : H \subset A \backslash \bigcup_{k<n} H_k \ \& \quad \mu(H) + \mu\left(\bigcup_{k<n} H_k\right) \leq \beta\},$$

$$h_n := \sup\{\mu(H) : H \in \mathcal{H}_n\}, \quad \mu(H_n) > h_n - \frac{1}{n}.$$

Then consider $\cup_k H_k$.

(c) Show that if $\alpha_j \in \mathbb{R}^+$, $A \in \mathcal{M}$ and $\sum_{j=1}^{\infty} a_j = \mu(A) < \infty$, then A can be written as a disjoint union of $A_j \in \mathcal{M}$ with $\mu(A_j) = \alpha_j$ for each j.

8. Let X be a locally compact Hausdorff space, $\mathcal{M}$ the class of its Borel sets and μ a regular Borel measure on $\mathcal{M}$. Suppose that μ is continuous in the sense that $\mu(\{x\}) = 0$ for each $x \in X$. Show that then there are no atoms (See problem 7 for the definition of this term.)

HINT: The concept of the support of μ is useful.

9. Let X be a normed linear space, X^* its dual space, and M a closed linear subspace of X. Define $M^\perp := \{f \in X^* : f(x) = 0$ for all $x \in M\}$. Show that $M = \{x \in X : f(x) = 0$ for all $f \in M^\perp\}$.

10. Let $a < b$ in $\mathbb{R}$, let $\phi : [a, b] \to \mathbb{R}$ be continuous, let $[\alpha, \beta] = \phi([a, b])$ and let λ denote Lebesgue measure. Suppose that $\lambda(\phi(E)) = 0$ whenever $E \subset [a, b]$ and $\lambda(E) = 0$. Then there exists a Borel measurable $w : [a, b] \to [0, \infty[$ such that $f \in L_1([\alpha, \beta])$ implies $(f \circ \phi)w \in L_1([a, b])$ and

$$\int_\alpha^\beta f d\lambda = \int_a^b (f \circ \phi) w d\lambda.$$

[HINT: Let $\mu(B) := \lambda(\phi^{-1}(B))$ for Borel sets $B \subset [\alpha, \beta]$. Consider f equal to the characteristic function of B. Maybe $w = g \circ \phi$.]