Name ______________________

REAL ANALYSIS QUALIFYING EXAM
Fall 1999
(Saeki & Moore)

Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ **denotes a measure space,** μ **denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.**

1. Suppose $1 \leq p \leq \infty$. Show that the closed unit ball of $\ell_p(\mathbb{N})$ is not compact.

2. Let $1 \leq p \leq 2$ and $f \in L^p([0, \infty])$. For $x \geq 0$ set $g(x) = \int_x^{x^2} f(t)dt$. Show that $\lim_{x \to \infty} \frac{g(x)}{x} = 0$.

3. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions on X such that $\lim_{n \to \infty} f_n(x) = f(x)$ exists a.e. and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu < \infty.$$

Prove that $\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ for every measurable set $E \subseteq X$.

4. Prove that if A and B are Borel sets in topological spaces X and Y respectively, then $A \times B$ is a Borel set in $X \times Y$.

5. Let $f \in L^1(\mu)$. Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\left| \int_E f \, d\mu \right| < \varepsilon$ whenever $\mu(E) < \delta$.

6. Prove that $L^p(\mu)$ is complete for $1 \leq p \leq \infty$.

7. Assume now that $(X, \mathcal{M}, \mu)$ is a $\alpha -$ finite measure space. Let $f : x \to [0, \infty)$, $0 < p < \infty$. Show $\int_X f^p \, d\mu = \int^i nfty_0 pt^{p-1} \mu(\{x \in X | f(x) > t\}) dt$.

8. Suppose f is defined on $X \times (0, 1)$ and that for each fixed $t \in (0, 1)$, $f(0, t) \in L^1(\mu)$. Suppose also that $\frac{\partial f}{\partial t}(x, t)$ exists for every $(x, t) \in X \times (0, 1)$ and $\frac{\partial}{\partial t}$ is bounded on $X \times (0, 1)$. Show that

$$\frac{d}{dt} \int_X f(x, t) d\mu(x) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x).$$