Algebra Qualifying Exam
Spring 1993

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a group of order $5 \cdot 7 \cdot 11$. Prove that $7 \parallel Z(G)|$, where $Z(G)$ denotes the *center* of G.

2. Let G be a finite group and let $\mathcal{C}$ be a conjugacy class of elements in G. If $|\mathcal{C}| = \frac{1}{2}|G|$, show that every element of $\mathcal{C}$ is an involution (i.e., an element of order 2).

3. Let F be a field. Prove that the groups $(F, +)$, $(F^\times, \times)$ cannot be isomorphic. (Here, $F^\times = F - \{0\}$).

4. Let R be a commutative ring, and let $x \in R$. Define what it means for x to be an *irreducible* element, and define what it means for x to be *prime*. If R is a *unique factorization domain*, show that x is irreducible if and only if x is prime.

5. Let F be a field and let R be the ring

$$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in F \right\}.$$

Define the R-modules

$$M_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in F \right\}, M_2 = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} | b \in F \right\},$$

where the module action is matrix multiplicaton. Prove that $M_1 \not\cong_R M_2$.

6. Let V a vector space and let $S \subseteq V$ be a finite subset which generates V. Prove that S contains a basis of V.

7. Let V be an n-dimensional vector space over a field F, and let $1_V \neq T : V \to V$ be a linear transformation satisfying $\ker(T - 1_V) \supseteq \operatorname{im}(T - 1_V)$. Compute the characteristic and minimal polynomials of T.

8. Let $f(x), g(x) \in F[x]$, and let $d(x)$ be the greatest common divisor of $f(x)$ and $g(x)$ in $F[x]$. If $F \subseteq K$ is any field extension, show that $d(x)$ is still the greatest common divisor of $f(x)$ and $g(x)$ in $K[x]$.

9. Let F_q be the finite field of q elements, and let K be an extension of f_q, of degree n.

(a) Prove that the map $\tau_q : K \to K, \tau_q(x) = x^q$ is an automorphism of K, and that F_q is precisely the subfield of fixed elements of τ_q.

(b) Show that $Gal(K/F_q)$ is cyclic, generated by the element τ_q.