Groups. Spring 1983

1. Write down the isomorphism types of all abelian groups of order $3^2 \cdot 5^4$.

2. Sketch a proof that any finitely generated abelian torsion group is finite.

3. Let A and B be subgroups of a group. Prove that

$$|AB| = |A| \cdot |B| / |A \cap B|$$

4. Is $G = \langle a, b, c \mid ac = ca,\ (ab)^3 = (bc)^3 = 1,\ a^2 = b^2 = c^2 = 1 \rangle$ a finite group?

5. Let G be a subgroup of Sym(5) generated by two subgroups of order 5. Show $G \cong \mathrm{Alt}(5)$. (You are allowed to use the following hint: If H is a group of order $h \equiv 2 \pmod 4$, then H contains a normal subgroup of index 2.).

Fields and Galois Theory.

1. (a) Define what it means to say that a field F is algebraically closed.

(b) Prove that a finite field cannot be algebraically closed.

2. Let F be a field and let ξ be a primitive seventh root of unity — that is, $\xi \neq 1$, but $\xi^7 = 1$. Find $[F(\xi):F]$ if
(a) $F = Q$, the field of rational numbers.
(b) $F = GF(2)$, the field with two elements.

3. (a) Find the lattice of all subfields of $Q(\sqrt{3}, \sqrt{5})$. (b) Find the monic irreducible polynomial satisfied by $\sqrt{3} + \sqrt{5}$.

4. Sketch a proof that θ is algebraic over F if and only if $[F(\theta):F]$ is finite.

Rings and Modules.

1. Show that the ring $\mathbb{Z}[\sqrt{5}]$ contains infinitely many units.

2. Let F be an algebraic extension of the rational numbers $\mathbb{Q}$. Show that for $\alpha \in F$, the following two conditions are equivalent

   (a) α satisfies a monic polynomial in $\mathbb{Z}[x]$

   (b) The subring $\mathbb{Z}[\alpha]$ of F is a finitely generated $\mathbb{Z}$-module.

3. An R-module M is said to be completely reducible if and only if every submodule A is a direct summand, that is, there exists a submodule B such that $M = A \oplus B$.

   (1) Given an example of a ring R and a completely reducible R-module M and submodule A for which the complementing submodule B is not unique

   (2) Give an example of an R-module M which is not completely reducible.

Rings and Modules. (continued)

4. Let R be a finite ring with no nilpotent elements. Show that R is a direct sum of of fields.

Linear Algebra.

1. Let F be a field, and I the ideal generated by x^3 in $F[x]$ and let V be the vectorspace $F[x]/I$.

   (a) Show that $B = \{1+I, x+I, x^2+I\}$ is a basis for B.

   (b) If $T: V \to V$ is the linear operator on V induced by left multiplication by $1+X$, write out the matrix ${}_B[T]_B = A$

   (c) Is T invertible?

2. Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial but which are not similar.

3. Find the rational canonical form of

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

4. Construct the Jordan form of a linear operator T with minimal polynomial $x^2(x-1)^2$ and characteristic polynomial $x^3(x-1)^4$

5. (a) Define what is meant by a bilinear form B on a vector space V.

(b) Let V be the vector space of 2×2 matrices over a field F. For $A, B \in V$, define

$$B(A, B) = \text{trace}(AB).$$

Is $B: V \times V \longrightarrow F$, defined in this way, a bilinear form? (Indicate your reasoning!)