REAL ANALYSIS QUALIFYING EXAM
Old System - Fall 2003

Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. LEt $\{f_n\}$ be a sequence of functions in $L^1[0,1]$ such that $f_n(t) \to 0$ for each $t \in [0,1]$. Is it true that $\int_0^1 f_n(t)dt \to 0$?

2.(a) Suppose $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that $A = \{x : x \in A_n$ for infinitely many $n\}$ has $\mu(A) = 0$.

3. Suppose f is continuous on $[0,1]$. Show that

$$\text{Riemann} \int_0^1 f\,dx = \text{ Lebesgue } \int_0^1 f\,dx.$$

4. Suppose $\mu(x) < \infty$ and $\mathcal{B}$ is a σ-algebra contained in $\mathcal{A}$. Let $L^P(\mathcal{B}) := \{f \in L^P(\mu) : f$ is $\mathcal{B}$-measurable for $1 \le p \le \infty$. Prove:

(a) Given $f \in L^2(\mu)$, there is a unique (a.e. sense) $f^* \in L^2(\mathcal{B})$ such that

$$\int g f^* d\mu = \int g f d\mu \qquad \forall g \in L^2(\mathcal{B}).$$

(b) For $f \in L^1 \cap L^2(\mu)$, f^* in (a) satisfies $\|f^*\| - 1 \le \|f\|_4$.

(c) Given $f \in L^1(\mu)$, there is a unique $f^* \in L^1(\mathcal{B})$ such that

$$\int g f^* d\mu = \int g f d\mu \qquad \forall g \in L^\infty(\mathcal{B})$$

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5. If μ is an arbitrary positive measure and $f \in L^1(\mu)$ prove that $\{x : f(x) \neq 0\}$ has σ-finite measure.

6. Suppose μ is a positive measure on x, $\mu(x) < \infty$, $f \in L^\infty(\mu)$, $\|f\|_\infty > 0$. Set $\alpha_n = \int_x |f|^n d\mu$. Prove that

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

7. Let $f_n(x) = e^{inx}$. Prove that

(a) $\{f_n\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2([-\pi,\pi])$ with respect to the measure $\dfrac{dx}{2\pi}$.

(b) No subsequence of $\{f_n\}_{n=1}^\infty$ converges pointwise a.e. on $[-\pi,\pi]$.

8. Suppose $1 \leq p \leq \infty$, $f \in L^P(\mathbb{R})$. Prove that $\|f(x+h)-f(x)\|_{L^P} \to 0$ as $h \to 0$.