Real and Complex Analysis Qualifying Exam.

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Instructions: The exam consists of 8 problems. Each problem is worth 10 points.
Time: 3 hours.
Notation: $\mathbf{N} := \{1,2,3,...\}$, $\mathbf{R}$:= reals, $\mathbf{C}$:= complexes, $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, $\bar{\mathbf{D}} := \{z \in \mathbf{C} : |z| \leq 1\}$, and (μ, X) is an abstract measure space.
Note: Any fact from the hints that you use you are expected to prove.

1. **Problem 1**.

   a) State and prove Schwarz' Lemma for holomorphic self-maps of $\mathbf{D}$.

   Hint: Consider $f(z)/z$ and use (you may but need **not** prove) the Maximum Principle.

   b) From a) infer that any conformal automorphism of $\mathbf{D}$ that fixes 0 is a rotation.

2. **Problem 2**.

   a) Show that for any $a \in \mathbf{D}$ the function

   $$f_a(z) := \frac{a - z}{1 - \bar{a}z}$$

   maps $\bar{\mathbf{D}}$ bijectively onto $\bar{\mathbf{D}}$ and is its own inverse.

   b) Show that every conformal automorphism f of $\mathbf{D}$ extends to a homeomorphism of $\bar{\mathbf{D}}$ by showing even more, namely that $f = uf_a$ for some $a \in \mathbf{D}$, $u \in \bar{\mathbf{D}} \setminus \mathbf{D}$.

   Hint: a), together with problem 1 a).

   c) Show that a and u in b) are uniquely determined by f.

   Hint: Compute $f'(0)$.

3. **Problem 3**.

   The series $\sum_{n=0}^{\infty} c_n z^n$ converges in $\mathbf{D}$ and the function $f(z)$ that it defines vanishes at $1/k$ for each $k \in \mathbf{N}$. Show that $f = 0$.

   Hint: Show by induction that all coefficients are 0.

4. **Problem 4**.

   a) State the Open-Mapping Theorem for holomorphic functions.

   b) State the Maximum Modulus Principle for holomorphic functions.

   c) Give a short deduction of b) from a).

   d) Show that the image of any closed subset of $\mathbf{C}$ under any non-constant polynomial is a closed set.

   e) From a) and d) deduce the Fundamental Theorem of Algebra.

   Hint: Connectedness.

5. **Problem 5**.

   Let $\mu(X) = +\infty$, where μ is Lebesgue measure on $\mathbf{R}$. Construct a function $f \in L^p(\mu, X)\ \forall p \geq 1$ such that $f \notin L^\infty(\mu, X)$.

6. **Problem 6**.

   Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu, X)$, $\|f\|_\infty > 0$, and $\alpha_n := \int\limits_X |f(x)|^n d\mu$, $n \in \mathbf{N}$. Prove that $\lim\limits_{n\to\infty} \alpha_{n+1}/\alpha_n = \|f\|_\infty$.

   Hint. Prove first that $\|f\|_n \to \|f\|_\infty$ as $n \to \infty$. Then use (you may but need **not** prove) the fact that for **any** sequence of positive numbers $(\alpha_n)_{n=1}^\infty$, $\lim\limits_{n\to\infty} \alpha_{n+1}/\alpha_n = \lim\limits_{n\to\infty} \alpha_n^{1/n}$, provided the first limit exists.

7. **Problem 7**.

   Define $h_c := \sum\limits_{n=1}^{\infty} n^c 1_{(1/(n+1),1/n]}$, $c > 0$. Prove that

   a) h_c is Lebesgue integrable on $\mathbf{R}$ if $c \in (0, 1)$.

   b) h_1 is of weak-type L but not Lebesgue integrable on $\mathbf{R}$.

   c) h_c is not of weak-type L if $c > 1$.

8. **Problem 8**.

   Choose intervals $W_n \subset (0, 1)$ in such away that $\cup_n W_n$ is dense in $(0, 1)$, and the set $K := (0, 1) \setminus \cup_n W_n$ has a positive measure.