Real Analysis Qualifying Exam
Spring 1989

In this exam λ denotes Lebesgue outer measure on $\mathbb{R}$ and $(X, \mathcal{A}, \mu)$ denotes a measure space.

1. (a) Define Lebesgue outer measure λ on $\mathbb{R}$.

(b) What does it mean to say that a subset E of $\mathbb{R}$ is λ-measurable?

(c) Use the fact that the family $\mathcal{M}_\lambda$ of all λ-measurable subsets of $\mathbb{R}$ is a σ-algebra to prove that every Borel subset of $\mathbb{R}$ is λ-measureable.

2. Prove the following assertions. You may use the fact that the restriction of λ to $\mathcal{M}_\lambda$ is countably additive if you wish.

(a) If $a < b$ in $\mathbb{R}$, then $\lambda([a, b]) = b - a$.

(b) There exists a compact set $K \subset [0, 1]$ such that $\lambda(K) > \frac{3}{4}$ and K contains no rational number.

4. (a) What does it mean to say that a function $f : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable?

(b) Suppose that $f_n : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable for $n = 1, 2, \ldots$ and define f on X by

$$f(x) = \varliminf_{n\to\infty} f_n(x)$$

for $x \in X$. Use your definiton in (a) to prove that f is $\mathcal{A}$-measurable.

5. Let $f : X \to [0, \infty[$ be $\mathcal{A}$-measurable. Define

$$A = \{(x, y) \in X \times \mathbb{R} : 0 \le y < f(x)\}$$

Prove that $A \in \mathcal{A} \times \mathcal{M}_\lambda$ and that if μ is σ-finite, then

$$\mu \times \lambda(A) = \int_X f d\mu.$$

6. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued $\mathcal{A}$-measurable functions on X that converges to some function f at each point of X. Prove that for each $a \in \mathbb{R}$ we have

$$\mu(\{f > a\}) \le \varliminf_{n\to\infty} \mu(\{f_n > a\}).$$

7. Let $f : X \to [0, \infty[$ be $\mathcal{A}$-measurable and suppose that μ is σ-finite. Define $m(t) = \mu(\{f > t\})$ for each $t \ge 0$. Prove that if $0 < p < \infty$, then

$$\int_X f^p d\mu = p \int_0^\infty t^{p-1} m(t) dt.$$

[Hint: $f^p(x) = \int_0^{f(x)} pt^{p-1} dt$.]

8. Prove the completeness of $L^p(\mu)$ for $1 \le p < \infty$.

9. Let $2 \le p < \infty$ and let $f, g \in L^p(\mu)$. Prove that

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \le \frac{1}{2}(\| f \|_p^p + \| g \|_p^p).$$

[Hints: First show that if $a, b \ge 0$, then $a^p + b^p \le (a^2 + b^2)^{p/2}$ and $\left(\frac{a^2+b^2}{2}\right)^{1/2} \le \left(\frac{a^p+b^p}{2}\right)^{1/p}$.]

10. Suppose $\mu(X) < \infty$ and let ν be another (positive) measure on $(X, \mathcal{A})$ with $\nu(X) < \infty$. Let $\nu = \nu_a + \nu_s$ be the Lebesgue decomposition of ν with respect to μ and let $w : X \to [0, \infty[$ be a Radon-Nikodym derivative of ν_a with respect to μ. Define w_0 on X by $w_0(x) = \min\{w(x), 1\}$. Prove the following:

(a) For each $\mathcal{A}$-measurable $f : X \to [0, \infty]$ we have

$$\int f w_0 d\mu \leq \min \left\{ \int f d\nu, \int f d\mu \right\}.$$

(b) Suppose that to each $\varepsilon > 0$ corresponds some $\mathcal{A}$-measurable $f : X \to]0, \infty[$ such that

$$\left(\int f d\nu \right) \left(\int f^{-1} d\mu \right) < \varepsilon$$

where $f^{-1} = 1/f$. Then ν and μ are mutually singular. [Hint for (b): Apply the Schwarz Inequality to $f^{1/2} \cdot f^{-1/2} = 1$ for the measure $w_0 d\mu$.]