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Complex Analysis Qualifying Exam

August, 2001

Instructions: Below you will find 8 problems. Each problem is worth 10 points. Only the best 6 scores will be added.
Time: 2 hours.

NOTATIONS: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle).

1. Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function, which is holomorphic on the open unit disk $\mathbb{D}$. Prove that there exists a sequence $(P_n)_{n\geq 1}$ of polynomials, such that

$$P_n \xrightarrow[n\to\infty]{} f, \text{ uniformly on the } \textit{whole} \text{ closed unit disk } \overline{\mathbb{D}}.$$

HINT: For every $r < 1$ define the function $f_r(z) = f(rz)$. Prove first that

$$f_r \xrightarrow[r\to 1]{} f, \text{ uniformly on } \overline{\mathbb{D}}.$$

Then use the fact that each f_r is holomorphic on an open disk D_r that contains $\overline{\mathbb{D}}$.

2. Let $\Omega \subset \mathbb{C}$ be an open set, which contains the closed unit disk $\overline{\mathbb{D}}$, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function which satisfies

$$|f(z)| > |f(0)|, \text{ for all } z \in \mathbb{T}.$$

Prove that f has at least one zero in $\mathbb{D}$.

3. Suppose $f, g : \mathbb{D} \to \mathbb{C}$ are holomorphic and

$$|f(z)| = |g(z)|, \text{ for all } z \in \mathbb{D}$$

Prove that every zero of g is also a zero of f, having the *same* multiplicity. Infer that there exists some $\lambda \in \mathbb{T}$, such that

$$f(z) = \lambda g(z), \text{ for all } z \in \mathbb{D}.$$

4. Compute

$$\int_0^\infty \frac{dx}{1+x^7}.$$

HINT: For (large) $R > 1$, use the boundary of the circular sector

$$C_R = \{re^{i\theta} \,:\, 0 < r < R,\ 0 < \theta < 2\pi/7\}$$

as contour.

5. Let $\Omega \subset \mathbb{C}$ be an open set, which contains the closed unit disk $\overline{\mathbb{D}}$, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function which satisfies

$$f(z) \neq 0, \text{ for all } z \in \mathbb{T}.$$

Prove that the number

$$Z_f := \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{f'(z)}{f(z)}\, dz$$

is equal to the number of zeros (counted with multiplicities) of f in $\mathbb{D}$. (In the above line integral, the unit circle $\mathbb{T}$ is oriented counterclockwise and traversed.)

6. Give an example of a harmonic function $h : \mathbb{C} \setminus \{0\} \to \mathbb{R}$, which is not the real part of any holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

7. How many zeros (counted with multiplicities) does

$$f(z) := z^4 + 3z - 1$$

have in the annulus $\Omega = \{z \in \mathbb{C} \,:\, 1 < |z| < 2\}$?

8. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let $f : \Omega \setminus \{a\} \to \mathbb{C}$ be a holomorphic function. Prove that the following are equivalent:

(i) a is an essential singularity of f.

(ii) There is no integer $n \geq 1$ such that

$$\lim_{z \to a} (z-a)^n f(z)$$

exists in $\mathbb{C}$.