Geometry of Manifolds Qualifying Exam
Auckly and Crane, Fall 2002

1. (A) What is $H^2(\mathbb{R}^3)$?

(B) Write Stokes' Formula.

(C) What kind of differential form can one integrate on a 3-surface embedded in a 5-manifold?

(D) Which of the following topological spaces are simply connected?

a) S^3 b) RP^3 c)$S^1 \times S^2$ d) $\mathbb{C} - \{0\}$

(E) Which lie algebras are contractible as topological spaces?

2. Compute $H^*(RP^2 \times S^2, \mathbb{Z})$ and $\pi_1(RP^2 \times S^2)$.

3. Let $F = \mathbb{R} \times (0, 2\pi)$ and let the metric g be given by $\dfrac{2z^2+1}{z^2+1}dz^2 + (z^2+1)d\theta^2$.

(A) Compute $\nabla_{\partial_\theta}\partial_\theta$ for the Levi-Civita connection corresponding to g.

(B) Compute the sectional curvature of (F, g).

4. Give an example of a topological space, every point of which has a neighborhood homeomorphic to $(0, 1)$, which is not a manifold.

5. Let (X, X_0) be a pointed topological space, G a lie group and $C^0[(X, X^0), (G, 1)]$ the set of continuous pointed maps. This forms a group under pointwise multiplication. Since π_1 is a functor, we obtain a map $\widetilde{\pi} : C^0[(X, X^0), (G, 1)] \to \mathrm{Hom}(\pi_1(X, X_0), \pi_1(G, 1))$ given by $\widetilde{\pi}, (u)([\ell(t)]) \to [u(\ell(t))]$ where $\ell(t)$ is a loop in (X, X_0). Prove that $\widetilde{\pi}$ is a group homomorphism when $\mathrm{Hom}[\pi_1(X), \pi_1(G)]$ is viewed as a group under pointwise multiplication. (Recall that π_1 of a lie group is abelian so this is in fact a group.)

6. Prove that if $p : G_1 \to G_2$ is a smooth homomorphism of connected lie groups which induces an isomorphism of lie algebras, then p is a covering projection. [Recall that a covering projection is a surjective continuous map such that for any $x \in G_2$ there is a neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets V_α and $p|_{V_\alpha}$ is a homeomorphism for each α onto U.]

7. Let X be a complete vector field on a manifold M. This means that X has a global flow, $\varphi : \mathbb{R} \times M \to M$. Let $f : M \to \mathbb{R}$ be a smooth function with $X(f)(p) = 1$ for all $p \in M$. Prove that $\phi_t(f^{-1}(a)) = f^{-1}(a+t)$ for all a, $t \in \mathbb{R}$. Give an example to show that the vector field must be complete for this to hold.