Algebra Qualifying Exam
Fall 1997

All rings in this exam are associative with 1 and all integral domains are commutative.

1. Let $\mathbb{Q}$ be the field of rational numbers. Show that the additive group $A = (\mathbb{Q}, +)$ does not have any proper characteristic subgroups. Can you generalize the above result when $\mathbb{Q}$ is any division ring? If yes, please give a proof. If not, please give a counter example.

2. Let G a finite group. If, for any prime p and any p-subgroup H of G, $N_G(H)$ has index at most 2 in G, show that G has to be a nilpotent group.

3. Let S be an integral domain. We say that an integral domain R is an *integral extension* of S if $R \supseteq S$ and for each $r \in R$ there exists a monic polynomial $f(x) = x^n + s_1 x^{n-1} + \cdots + s_n \in S[x]$ with coefficients $s_i \in S$ and $n \geq 1$ such that $f(r) = 0$.

(a) Show that if R is an integral extension of a field S, then R is also a field.

(b) If an integral extension R of the integral domain S is a field, is S necessarily a field? Justify your answer.

4. Let S be the set of all nonzero integers which are sums of two squares of integers.

(a) Show that S is closed under multiplication.

(b) Show that the subring $\mathbb{Z}_S = \left\{ \frac{n}{s} | n \in \mathbb{Z}, s \in S \right\}$ of the set of rational numbers is the entire field $\mathbb{Q}$ of rational numbers.

5. Let R be a ring and M be a left R-module.

(a) Define what it means for M to be *irreducible*.

(b) Define what it means for M to be *indecomposable*.

(c) Describe, as complete as you can, the relations between the irreducibility and indecomposability of an R-module M by proving or giving examples to your conclusions.

6. Let F be a field and $R = F[t]/ < T^5 >$. Describe all finitely generated indecomposable R-modules up to R-module isomorphisms. (Hint: R-modules are $F[T]$-modules. But what kind of $F[T]$-modules are R-modules?)

7. Suppose that K is a splitting field of the polynomial $x^4 - x^5$ over the field $\mathbb{Q}$ of rational numbers. Compute $[K : \mathbb{Q}]$ and correctly justify your answer.

8. Let $E \supseteq F$ be a finite Galois extension of fields. Suppose that the Galois group $\mathrm{Gal}(E/F)$ is Abelian; show that for any $\alpha \in E$, $F(\alpha)$ is splitting field of a polynomial in $F[x]$.

9. Let V be a vector space over the field $\mathbb{Q}$ of rational numbers and $T : V \to V$ a linear transformation with the following invariant factors:

$$x + 1, \quad x^2 - 1, \quad x^4 - 1.$$

(a) Find $\dim V$.

(b) Find the nullity of T.

(c) Find the dimension of the subspace of all vectors fixed by T.

10. Let G be a solvable group and $K \neq \{1\}$ be a minimal finite normal subgroup of G. Show that there exists a prime number p and a positive integer r such that K is isomorphic to the additive group of the vector space $\mathbb{F}_p^r$ over the finite field $\mathbb{F}_p$ of p elements.