Algebra Qualifying Exam
August 26, 2006

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. You should have at least one problem from each of the five sets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$, $\{9,10\}$ among your choice. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative. $\mathbb{Q}$ and $\mathbb{C}$ are the fields of rational and complex numbers, respectively.

---

1. Let G be a finite group and p be a fixed prime number that divides the order of G. Define $G(p) = \{g \in G \mid o(g) = p^r \text{ for some } r \geq 0\}$. Prove that the subset $G(p)$ of G is a subgroup of G if and only if G has a normal Sylow p-subgroup.

2. Let G be finite group and X be a finite set on which G acts. For each $x \in X$, let $G_x = \{g \in G \mid gx = x\}$. Similarly, for each $g \in G$ define $X^g = \{x \in X \mid gx = x\}$.

(a) Prove the identity $\sum_{x \in X} |G_x| = \sum_{g \in G} |X^g|$;

(b) If G acts on X transitively and $|X| > 1$, then G has an element which does not fix any element of X.

3. Let R be a ring (not necessarily commutative) with identity 1. Let $a, b \in R$ such that ab is nilpotent (i.e., $(ab)^m = 0$ for some $m > 0$).

(a) Show that $(1 - ab)$ has a multiplicative inverse and express the inverse in terms of a and b;

(b) Show that $(1 - ba)$ also has a multiplicative inverse and find a relation between the inverses of $(1 - ab)$ and $(1 - ba)$.

4. Let R be a unique factorization domain and $R[x]$ be the ring of polynomials with coefficients in R. Note that for any finite subset S of R, the greatest common divisor GCD(S) is defined up to a unit factor in R. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$ be primitive, i.e., GCD$(\{a_0, a_1, \ldots, a_n\}) = 1$. Let $g(x) \in R[x]$. If $a \in R$ is a nonzero element in R such that $f(x)g(x) = ah(x)$ for some $h(x) \in R[x]$, show that there is a polynomial $p(x) \in R[x]$ such that $g(x) = ap(x)$. (You **cannot** use the fact that $R[x]$ is a UFD.)

5. Let F be a field and $R = F[x, y]$ be the ring of polynomials with two variables x and y. Let R^2 be the free R-module of rank 2 written as column vectors with entries in R. Consider the following sequences of R-module maps

$$(*) \qquad 0 \longrightarrow R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \xrightarrow{d_0} F \longrightarrow 0$$

Here $d_2(r) = \binom{-yr}{xr}$, $d_1\binom{r_1}{r_2} = xr_1 + r_2 y$, and $d_0(f(x, y)) = f(0, 0)$ for $f(x, y) \in R = F[x, y]$. Recall that a sequence of maps $\cdots \xrightarrow{\phi} M \xrightarrow{\psi} \cdots$ is called exact at M if $\ker \psi = \text{image}(\phi)$. Show that above sequence (*) is exact at the two R's and at R^2.

6. Let R be a ring with 1. A left R-module P is called projective if for any surjective R-module homomorphism $\phi : E \to F$, the map $\psi : \hom_R(P, E) \to \hom_R(P, F)$ defined by $\psi(f) = \phi \circ f$ is surjective. Prove that any free R-module is projective.

7. Let p be a fixed prime and $\mathbb{F}_{p^r}$ the finite field with p^r elements. For two positive integers r_1 and r_2 such that $r_1 \mid r_2$, $\mathbb{F}_{p^{r_1}}$ is a subfield of $\mathbb{F}_{p^{r_2}}$.

(a) For $r_1 \mid r_2$, describe elements of the Galois group $\text{Gal}(\mathbb{F}_{p^{r_2}}/\mathbb{F}_{p^{r_1}})$ explicitly as maps;

(b) If $r_1 \mid r_2$ and $r_2 \mid r_3$, describe explicit relations and correspondences of elements between the two groups $\text{Gal}(\mathbb{F}_{p^{r_3}}/\mathbb{F}_{p^{r_1}})$ and $\text{Gal}(\mathbb{F}_{p^{r_3}}/\mathbb{F}_{p^{r_2}})$.

8. Let p be a prime number and $f(x) = x^p - 3$. Compute the degree of the splitting field of $f(x)$ over $\mathbb{Q}$ and show how you derived the answer.

9. Let F be a field (not necessarily algebraically closed) of any characteristic. An $n \times n$-matrix A is called nilpotent if $A^m = 0$ for some $m > 0$. Show that if A is nilpotent, then there exists an invertible matrix P with entries in F such that PAP^{-1} is in Jordan canonical form.

10. Let A be a 3×3 matrix with entries in $\mathbb{C}$. Show that A is nilpotent if and only if $\text{tr}(A) = \text{tr}(A^2) = \text{tr}(A^3) = 0$. (Hint: If all eigenvalues of A are zero then A is nilpotent.)