Numerical Analysis Qualifying Exam
Spring 1991

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1. Consider evaluating $\cos x$ for large x by using the Taylor approximation,

$$\cos x \approx 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

If one uses it to evuluate $\cos 2\pi = 1$, determine n so that the Taylor approximation error is less than .0005. Suppose one does the computation using 4-digit rounding, what trouble will one encounter? How should $\cos x$ be evaluated for large values of x?

2. Suppose $f \in C^2(R)$, and $f(p) = 0$ implies $f'(p) \neq 0$.

(1) Show if $f(p) = 0$, then there is a δ such that if $|x_0 - p| < \delta$, then Newton's method starting at x_0 converges to p.

(2) Show that if p_1, p_2 are succesive zeros of f (i.e. $f(x) \neq 0$ for $x \in (p_1, p_2)$) and p_3 is another zero of f, then there is an $x_0 \in (p_1, p_2)$ such that Newton's method starting from x_0 converges to p_3.

3. Suppose $A \in R^{n\times n}$, A^T (the transpose of A) is diagonally dominant, i.e.,

$$|a_{ii}| \geq \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}|,$$

and A is nonsingular, show that $A = LU$ with L being a unit lower triangular matrix, i.e., Gauss elimination can be performed without pivoting, and $|l_{ij}| \leq 1$, where l_{ij} are entries in L.

4. Suppose $B \in R^{n\times n}$ is symmetric, positive definite.

(1) Define $\| x \| \equiv \sqrt{x^t B x}, \forall x \in R^n$ (where x^t is the transpose of x). Show that this defines a norm in R^n (it is called an elliptical norm).

(2) A norm in R^n is monotonic if

$$|x_i| \leq |y_i|, i = 1, 2, \ldots, n, \text{ implies } \| x \| \leq \| y \| .$$

Construct an example to show that elliptical norms are not monotonic in general.

5. Suppose $A \in R^{m\times n}$, with $m < n$, and $w \in R^n$. Define

$$B = \begin{bmatrix} A \\ w^T \end{bmatrix},$$

Show $\sigma_1(B) \geq \sigma_1(A)$ and $\sigma_{m+1}(B) \leq \sigma_m(A)$. Thus, the condition grows if a row is added to A. (Recall that the 2-norm condition number of A is defined as $\sigma_1(A)/\sigma_m(A)$, where $\sigma_1(A)$ and $\sigma_m(A)$ are the largest and smallest singular values of A respectively).

6. Suppose $A \in R^{n\times n}$, and all its off-diagonal entries are small compared to some diagonal entries. (For example, A may be a matrix obtained during the procedure of the Jacobi method) Gerschgorin theorem can be used to give a good approximate location of some eigenvalues. The Wilkinson Correction Procedure sharpens the approximation with a little more work by multiplying the ith row of A by a small number α and multiplying the ith column of A by α^{-1}. Suppose

$$R_i = \left\{ z \in C : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha |a_{ij}| \right\}$$

is disjoint from all the disks

$$\left\{ z \in C : |z - a_{kk}| \leq \alpha^{-1} |a_{ki}| + \sum_{\substack{j=1 \\ j \neq k,i}}^{n} |a_{kj}| \right\}, \quad \forall k \neq i.$$

Show that R_i contains precisely one eigenvalue of A (notice the approximate location of this eigenvalue has been sharpened by the procedure).

7. Find, with proof, the monic polynomial of degree of 4, $P(x)$, such that

$$\max_{-1 \leq x \leq 1} |P(x)|$$

is minimized.

8.

(1) Find the first three monic orthogonal polynomials on the interval $[0, 1]$ with respect to weight function $\ln(1/x)$.

(2) Suppose the answer to (1) are given by

$$\psi_0(x) = 1, \psi_1(x) = x - \frac{1}{4}, \psi_2(x) = x^2 - \frac{5}{7}x + \frac{17}{252}.$$

Derive the two-point Gaussian quadrature formula for

$$I(f) = \int_0^1 f(x) \ln\left(\frac{1}{x}\right) dx$$

in which the weight function is $w(x) = \ln(1/x)$. What is the error of the quadrature formula (assuming that f is smooth enough)?