Real Analysis Qualifying Exam
Spring 1993

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1. State and prove the monotone convergence theorem for non-negative functions. Is it true if monotone increasing is replaced by monotone decreasing? Prove or give a counter-example.

2. Discuss the relationship between convergence in measure and convergence almost everywhere. Prove or give counter-examples for all assertions.

3. Suppose T is a linear functional on $\mathcal{C}(X)$ where X is a compact Hausdorf space. Show that $T1 = \parallel T \parallel_{op}$ if and only if T is a positive operator.

4. Prove that $| \int_\Omega fgh d\mu| \leq \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q d\mu\right)^{\frac{1}{q}} \left(\int_\Omega |h|^r d\mu\right)^{\frac{1}{r}}$ if $1/p + 1/r + 1/q = 1$.

5. Suppose f is a real valued nondecreasing function on $[a, b]$ which is differentiable almost everywhere. Show that f' is Legesgue measurable. Is it true that $\int_a^b f'(x)dx = f(b) - f(a)$? Prove or give a counter-example.

6. Show that a Banach space where the norm satisfies the parallelogram law, $\parallel x+y \parallel^2 + \parallel x-y \parallel^2 = 2 \parallel x \parallel^2 + 2 \parallel y \parallel^2$, is a Hilbert space.

7. Define $f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$. Show that $\parallel f * g \parallel_1 \leq \parallel f \parallel_1 \parallel g \parallel_1$.

8. Suppose $(\Omega, \mathcal{M}, \mu)$ is a measure space and that Ω is countable and $\mathcal{M}$ is the power set of Ω, (i.e. $\mathcal{M}$ is the set of all subsets of Ω). Must $L^p \subset L^q$ if $p \leq q$? Prove or give a counter-example.

9. Let $\mathcal{M}$ be a collection of subsets of a set Ω with the following properties:

i. $\Omega \in \mathcal{M}$

ii. $A, B \in \mathcal{M}$ implies that $A - B \in \mathcal{M}$

iii. $A_1, A_2, \ldots, A_n, \cdots \in \mathcal{M}$ and $A_1 \subset A_2 \subset \cdots \subset A_n \subset \ldots$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Show that $\mathcal{M}$ is a σ-field (also called a σ-algebra).