Real and Complex Analysis Qualifying Exam

Fall 2007

(Burckel and Moore)

Throughout, $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{R} :=$ real numbers and $\mathbb{C} :=$ the complex numbers.

Problem 1. R is an open rectangle with sides parallel to the coordinate axes, $f : \bar{R} \to \mathbb{C}$ is continuous and satisfies

$$\int_{x_0}^{x} f(s+iy_0)ds - \int_{x_0}^{x} f(s+iy)ds + i\int_{y_0}^{y} f(x+it)dt - i\int_{y_0}^{y} f(x_0+it)dt = 0$$

for all $x_0 + iy_0, x + iy \in R$. Find a continuous function $F : R \to \mathbb{C}$ such that $D_1F = f = -iD_2F$ in R.

Problem 2. (i) Prove there is no holomorphic logarithm in the region $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

(ii) Improve (i) by showing that there is no logarithm is A which is even continuous.

Problem 3. (i) $\emptyset \neq U$ is an open subset of $\mathbb{C}$, f_n are holomorphic functions in U and $f_n \to f$ uniformly on each compact subset of U. Show that f is differentiable in U.

(ii) Show that the conclusion of (i) fails if U is an open subset of $\mathbb{R}$ and each f_n is a differentiable function in U.

Problem 4. Show that $f(x) = \dfrac{\cos x}{1+x^4}$ is (absolutely) integrable over $\mathbb{R}$ and calculate $\displaystyle\int_{\mathbb{R}} f$.

Problem 5. Let $f_n : X \to \mathbb{R}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose E is a measurable subset of X such that for each $x \in E$,

$$\sup_{k\in\mathbb{N}} |f_k(x)| < \infty.$$

Suppose also that for each $\alpha > 0$ there exists a positive integer k_α such that for every $k \geq k_\alpha$, $\mu(\{x \in E : |f_k(x)| \leq \alpha\}) \leq \dfrac{\alpha}{k}$. Prove that $\mu(E) = 0$.

Problem 6. Suppose $(X, \mathcal{M}, \mu)$ is a measure space, μ is a positive measure, and $f_n \in L^p(X)$ for $n \in \mathbb{N}$, and $f \in L^p(X)$, where $1 \leq p < \infty$. Prove:

(i) If $\|f - f_n\|_p \to 0$ as $n \to \infty$, then $\|f_n\|_p \to \|f\|_p$.

(ii) If $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$ then $\|f - f_n\|_p \to 0$.

Problem 7. Let h be a bounded Lebesgue measurable function on $[0, 1]$ which has the property that $\displaystyle\lim_{n\to\infty} \int_I h(nx)dx = 0$ for every interval $I \subset [0, 1]$. Prove that for every $f \in L^1([0, 1])$,

$$\lim_{n\to\infty} \int_0^1 f(x)h(nx)dx = 0.$$

Problem 8. Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $f : X \to [0, \infty)$ an $\mathcal{M}$-measurable function. Suppose $G : [0, \infty) \to [0, \infty)$ is increasing. Prove that

$$\int_X G(f(x))d\mu(x) \geq \int_0^\infty G'(t)\mu(\{x \in X : f(x) > t\})dt.$$