Algebra Qualifying Exam
August 31, 1999

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.
Note: All rings on this exam are associative and have multiplicative identity 1. All all integral domains are assumed to be commutative.

1. Let P be a p-Sylow subgroup of the finite group G. Prove that $N_G(N_G(P)) = N_G(P)$.

2. Let G be a finite group and let $\mathcal{C}$ be a conjugacy class of elements in G. If $|\mathcal{C}| = \frac{1}{2}|G|$, show that every element of $\mathcal{C}$ is an involution (i.e., an element of order 2).

3. Let x be an element of p-power order in the finite group G, where p is prime. Assume that $|\{g^{-1}xg|\ g \in G\}| = p$. Show that x lies in a normal p-subgroup of G.

4. Prove, or give a counterexample to the assertions below:
   (a) $\mathbb{Z}[x]$ is a principal ideal domain.
   (b) If I is a maximal ideal of $\mathbb{Z}$, then $I[x]$ is a maximal ideal of $\mathbb{Z}[x]$.

5. Consider the commutative ring $R = \{a + b\sqrt{-5}|\ a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$. Show that the element $1 + 2\sqrt{-5}$ is irreducible but not prime in R.

6. Let R be a ring and let M be an irreducible left R-module. If K is the kernel of the action of R on M (*i.e.*, $K = \ker(R \to \mathrm{End}_{\mathbb{Z}}(M))$), prove that R/K is semisimple, *i.e.*, the Jacobson radical is trivial. (Hint: the problem itself is trivial.)

7. Let $\mathbb{F}$ be a field and let

$$0 \to V_1 \to V_2 \to V_3 \to V_4 \to 0$$

be an exact sequence of finite dimensional vector spaces over $\mathbb{F}$. Prove that

$$\dim V_1 - \dim V_2 + \dim V_3 - \dim V_4 = 0.$$

8. Let $\mathbb{F}$ be a field and let $T : V \to V$ be a linear transformation on V. Assume that T has elementary divisors $x - a, (x - a)^2, (x - a)^2, (x - b)^2, x - c, x - c$, where $a, b, c \in \mathbb{F}$ are distinct elements of $\mathbb{F}$.

   (i) What is the dimension of V?

   (ii) What is the minimal polynomial of T?

   (iii) What are the invariant factors of T?

   (iv) Compute the Jordan canonical form of T.

9. Let $\mathbb{F} \subseteq \mathbb{K}$ be fields such that the extension degree $[\mathbb{K} : \mathbb{F}] < \infty$. Prove that every element of $\mathbb{K}$ is algebraic over $\mathbb{F}$.

10. Let G be a finite *Hamiltonian* group, *i.e.*, one such that every subgroup of G is normal. Now assume that $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial whose Galois group is isomorphic to G. Prove that $\deg f(x) = |G|$.