# DIFFERENTIAL EQUATIONS QUALIFYING EXAM, FALL 1998

Department of Mathematics, Kansas State University

**1)** Solve the Cauchy problem

$$\cos(y)\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+2\tan(y)u-2\tan(y)=0, u(x,0)=h(x)$$

**2)** **a)** Find a $2\pi$-periodic in $x$ solution of the equation

$$u_t-u_{xx}=0$$

that satisfies the initial condition

$$u(0,x)=\phi_N(x),$$

where $N\geq 1$ is an integer and the function $\phi_N$ is defined on $[-\pi,\,\pi]$ as follows

$$\phi_N(x)=\begin{cases} N, \text{ if } -1/2N<x<1/2N \\ 0, \qquad \text{otherwise,}\end{cases}$$

and extended periodically to the whole line. How many solutions are there? In what sense?

**b)** Let $u_N(t,x)$ denote the solution of the above problem such that $u_N(t,\cdot)$ is a strongly continuous function of $t\in[0,T]$ with values in $L^2([-\pi,\,\pi])$. Prove that such solution does exist and is unique. Is $u_N(t,x)$ is a bounded function of $x$ for every $t\geq 0$?

**c)** In what sense does there exist the limit $u_\infty=\lim_{N\to\infty}u_N$ ? Find $u_\infty$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

**3)** State and prove the maximum principle for the equation

$$u_t - \Delta u + e^u = 0$$

in $\mathbb{R}^n$.

Use the maximum principle to state and prove a uniqueness theorem.

**4)** Determine the type of the equation

$$\sum_{i,j=1}^{4} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{4} \frac{\partial u}{\partial x_i} - u = 0,$$

where $A_{ij}$ is the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

State and prove the finite domain of dependence property for this equation.

**5)** For the equation $-\Delta u + u = f$ on the torus $\mathbb{T}^n$, prove the estimate

$$\sum_{j,k=1}^{n} \|u_{x_j x_k}\|_{L^2(\mathbb{T}^n)} \le C(\|f\|_{L^2(\mathbb{T}^n)} + \|u\|_{L^2(\mathbb{T}^n)})$$

with a constant $C$ independent of $f$.

**6)** Find all distributional solutions of the equation

$$xy'(x) = 1.$$

**7)** State and prove a finite dimensional version of Fredholm's Alternative.

**8)** Consider the Cauchy problem for the wave equation on $\mathbb{R}^n$:

$$u_{tt} - \Delta u = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0.$$

In the 1-dimensional case ($n = 1$) show that there exists a constant $C > 0$ such that

$$\sup_t \|u(t,\cdot)\|_{L^1(\mathbb{R}^1)} \le C\, \|f\|_{L^1(\mathbb{R}^1)},$$

for all $f \in L^1(\mathbb{R}^1)$.

In the 3-dimensional case ($n = 3$) prove that there is no constant such that the above inequality holds. (Hint: consider spherically-symmetric solutions, use the change of variables $u(t,r) = r^\alpha v(t,r)$ with an appropriate $\alpha$ to obtain the 1-D wave equation for $v$, and choose $v(0,r)$ to have a support in an annulus.)