Analysis Qualifying Exam - Fall 09

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Instructions: Do all ten problems. Start each problem on a separate page and clearly indicate the problem number.

Notation: $\mathbb{N}$ is the positive integers, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{U} := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, U is a non-empty open subset of $\mathbb{C}$, ∂ denotes topological boundary, $H(U)$ are the holomorphic functions in U, n and N are always an integer, a simplex is the convex hull of 3 points in $\mathbb{C}$.

1. The meromorphic function f has an inessential singularity at a. Show that $\displaystyle\lim_{z\to a} \frac{(z-a)f'(z)}{f(z)}$ exists and is an integer n. Show that $|n|$ is the multiplicity of a as a zero or pole of f according to whether a is a removable or a polar singularity.

   Hint: What does a being a zero or pole of multiplicity m mean in terms of $(z-a)^m$ being a factor of $f(z)$?

2. f is continuous in the open convex set $U \subset \mathbb{C}$ and $\int_{\partial\Delta} f = 0$ for every simplex $\Delta \subset U$. Construct a primitive F for f and directly verify that it satisfies $F' = f$.

3. $f_n \in H(U)$ and $\sum |f_n|$ converges locally uniformly in U. Show that $\sum |f_n'|$ is also locally uniformly convergent in U.

   Hint: For each closed disk $K \subset U$, $f \in H(U)$, majorize f' on K in terms of f (Cauchy formula).

4. (a) State two properties enjoyed by all holomorphic functions in regions in $\mathbb{C}$ but not by all differentiable functions on open intervals in $\mathbb{R}$. Give examples.

   (b) State two properties enjoyed by all differentiable functions on open intervals in $\mathbb{R}$ but not by all holomorphic functions in regions in $\mathbb{C}$. Give examples.

5. Compute $\displaystyle\int_0^\infty \frac{\sqrt{x}}{1+x^2}\,dx$.

   Hint: Use a holomorphic square-root function in, say, $\mathbb{C} \setminus i(-\infty, 0]$, $\mathbb{C}$ minus the non-positive y-axis. Be clear how you handle the origin.

6. Let E be a Lebesgue measurable set in $\mathbb{R}^n$. Prove that

   $$E = A_1 \cup N_1 = A_2 \setminus N_2$$

   where A_1 is an F_σ set, A_2 is a G_δ set, and $m(N_1) = m(N_2) = 0$ (m denotes Lebesgue measure in $\mathbb{R}^n$).

   Hint: Recall that m is a *regular* measure. What does that mean?

7. (a) State the Fubini-Tonelli Theorem.

   (b) Show by example that the equality of the iterated integrals can not be inferred without the hypothesis of σ-finiteness.

   (c) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces (not necessarily σ-finite). Prove that if $f \in L^1(X)$ and $g \in L^1(Y)$, then the function h defined by $h(x, y) = f(x)g(y)$ is $\mathcal{M} \bigotimes \mathcal{N}$-measurable, $h \in L^1(X \times Y)$ and $\int_{X \times Y} h \, d(\mu \times \nu) = (\int_X f \, d\mu)(\int_Y g \, d\nu)$.

8. State the Monotone Convergence Theorem, Fatou's lemma, the Dominated Convergence Theorem, and Egoroff's Theorem.

9. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Prove that if f_n, g_n, f, $g \in L^1(X)$, $n \in \mathbb{N}$, are such that $g_n \to g$ a.e., $f_n \to f$ a.e., $|f_n| \leq g_n$ a.e., and $\int_X g_n \, d\mu \to \int_X g \, d\mu$, then $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

   (b) Suppose $1 \leq p < \infty$. Prove that if f_n, $f \in L^p(X)$, $n \in \mathbb{N}$, are such $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in $L^p(X)$.
   Hint: Use part (9a).

10. (a) Prove that a Hilbert space $\mathcal{H}$ is separable if and only if every orthonormal basis of $\mathcal{H}$ is countable.

    (b) Prove the Riemann-Lebesgue lemma: Every $f \in L^1([0, 2\pi])$ satisfies

$$\lim_{n \to \infty} \int_0^{2\pi} f(x) \cos(nx) \, dx = \lim_{n \to \infty} \int_0^{2\pi} f(x) \sin(nx) \, dx = 0.$$