ALGEBRA QUALIFYING EXAMINATION
FALL 1983

You are to do at least two problems from each of sections I-IV. Additional credit will be awarded for additional problems correctly solved.

I. Groups

1. Let $f: G_1 \to G_2$ be a homomorphism of groups and let H_2 be a subgroup of G_2. Prove that $f^{-1}(H_2)$ is a subgroup of G_1.

2. Let Z be a cyclic group of order $n > 0$, and let $H = \mathrm{Aut}(Z)$ be its group of automorphisms.

   (a) Show that $H \cong U(\mathbb{Z}/(n))$ (the group of units of the ring $\mathbb{Z}/(n)$); thus H has order $\phi(n)$.

   (b) Using (a) prove the number-theoretic congruence:

   $$(a,n) = 1 \Rightarrow a^{\phi(n)} \equiv 1 \pmod n$$

3. Prove that $(\mathbb{Q},+)$ is not a free abelian group

4. Prove that no group of order 120 can be a simple group

5. Let G be a group and let Ω be a set.

   (a) Define what it means for G to act on Ω

   (b) If G is a finite p-group (p = prime) and if Ω is finite with $p \nmid |\Omega|$, show that $\exists\, \omega \in \Omega$ such that $g \cdot \omega = \omega$ for all $g \in G$.

   (c) Conclude from (b) that any finite p-group has a nontrivial center

6. Prove that any group admits a nontrivial automorphism.

II. Rings and Modules

1. (a) Define a <u>Euclidean domain</u>.

   (b) Define a <u>principal ideal domain</u> (p.i.d.).

   (c) Prove that a Euclidean domain is a p.i.d.

   (d) Define a <u>unique factorization domain</u> (u.f.d.)

   (e) It is known that the polynomial ring $\mathbb{Z}[x]$ is a u.f.d. Prove that it is <u>not</u> a p.i.d

2. Let p be a prime number, and let $\mathbb{Z}_{(p)}$ be the subring of the rational numbers, given by

$$\mathbb{Z}_{(p)} = \{\tfrac{a}{b} \in \mathbb{Q} \mid \text{if } (a,b) = 1, \text{ then } p \nmid b\}.$$

Prove that $\mathbb{Z}_{(p)}$ contains a <u>unique</u> maximal ideal M, viz.,

$$M = \{\tfrac{a}{b} \in \mathbb{Z}_{(p)} \mid p \mid a\}.$$

3. Let R be a ring with identity and let M be a unital left R-module.

(a) Define what it means for M to be <u>irreducible</u>

(b) Define what it means for M to be <u>indecomposable</u>.

(c) Give an example of a ring R and a unital left R-module M which is indecomposable but not irreducible.

4. Let R be a ring with identity and let M be a unital left R-module. Assume that there exist a family $\{M_\alpha \mid \alpha \in A\}$ of irreducible submodules of M with $M = \sum_\alpha M_\alpha$. Prove that $\exists$ subset $A_0 \subseteq A$ such that $M = \bigoplus_{\alpha \in A_0} M_\alpha$. (This involves a Zorn's Lemma argument.)

5. Let R be a commutative ring with identity and let I be an ideal of R

(a) Define what it means for I to be a <u>prime ideal</u>.

(b) Prove that I is a prime ideal if and only if the quotient ring R/I is an integral domain.

(c) Let $R = \mathbb{Z}[x]$ and let I be the ideal

$$I = \{2f(x) + 3xg(x) \mid f(x), g(x) \in \mathbb{Z}[x]\}.$$

Prove that $R/I \cong \mathbb{Z}/(2)$, and therefore (why?) I is a prime ideal of R.

III. Linear Algebra.

1. Let V be a vector space over the field F, and let $\{v_1, \ldots, v_n\} \subseteq V$. If $w \in \langle v_1, \ldots, v_n\rangle$ but $w \notin \langle v_2, \ldots, v_n\rangle$, prove that $v_1 \in \langle w, v_2, \ldots, v_n\rangle$.

2. Let V be a finite dimensional vector space over F, and let T be a linear transformation on V.

   (a) Define the <u>minimal polynomial</u> $m_T(x) \in F[x]$ of T

   (b) Prove that $m_T(x)$ exists

3. Let V be an n-dimensional vector space over F and let T be a linear transformation on V. Prove that V admits a basis consisting of eigenvectors of T if and only if $m_T(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$ where $k \le n$ and where the α_i's are in F, and are distinct.

4. Give an example of a matrix A, with rational entries such that

   (a) $c_A(x) = (x - \frac{1}{2})^4 (x - 2)^3 (x + 6)^3$ (characteristic polynomial)

   (b) $m_A(x) = (x - \frac{1}{2})^2 (x - 2)^2 (x + 6)$

5. Let F be a field and let $a_0, a_1, \ldots, a_{n-1} \in F$. For the n-by-n matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & -a_0 \\ 1 & 0 & & -a_1 \\ 0 & 1 & & \cdot \\ \cdot & 0 & & \cdot \\ & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & -a_{n-1} \end{bmatrix},$$

show that $m_A(x) = \sum_{k=0}^{n} a_k x^k$, where $a_n = 1$.

IV. Fields and Galois Theory.

1. Let $F \subseteq K$ be fields.

   (a) Define $[K:F]$, the <u>degree</u> of K over F

   (b) Let $[K:F] = n$ and let $\alpha \in K$. Then α induces an F-linear transformation $T_\alpha : K \to K$, given by $T_\alpha(\beta) = \alpha\beta$, $\beta \in K$. Show that the minimal polynomial of T_α is an <u>irreducible</u> polynomial in $F[x]$.

2. Let $F \subseteq K$ be fields.

   (a) State what it means for K to be normal over F.

   (b) Is $\mathbb{Q}(\sqrt[3]{2})$ normal over $\mathbb{Q}$, where $\sqrt[3]{2}$ is the real cube root of 2 ? Why or why not ?

   (c) State what it means for K to be separable over F

   (d) Let x be an indeterminate over $\mathbb{Z}/(2)$; show that $\mathbb{Z}/(2)(x) \supseteq \mathbb{Z}/(2)(x^2)$ is not a separable field extension.

3. Let K be a splitting field over $\mathbb{Q}$ for the polynomial $f(x) = x^4 - 2$. Show that $Gal(K/\mathbb{Q}) \cong D_8$, the dihedral group of order 8

4. Let p be a prime and let $q = p^a$, a = positive integer. Let K be a splitting field over $F = \mathbb{Z}/(p)$ for the polynomial $f(x) = x^q - x$. Show that $Gal(K/F)$ is cyclic of order a and is generated by the F-automorphism $(\alpha \mapsto \alpha^p)$, $\alpha \in K$.

5. Let $F = \mathbb{C}(x)$, the field of rational functions where x is an indeterminate over $\mathbb{C}$ and where $\mathbb{C}$ is the complex field Let $f(T) = T^3 - x \in F[T]$ and let K be a splitting field over F for $f(T)$. Show that $Gal(K/F)$ is cyclic of order 3, and has generator determined by $(y \mapsto \zeta y)$ where y is a root of $f(T)$ and where $\zeta = e^{2\pi i/3}$.

6. Let $K \supseteq F$ be a (possibly infinite) normal field extension and let $\sigma: K \to K$ be a homomorphism which acts as the identity on F. Prove that σ is an isomorphism.