**TOPOLOGY QUALIFYING EXAM    Fall 1994**
(Wu and Strecker)

Choose and work any 6 of the following 15 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems.** A space always mean a topological space below.

**1.** State the axiom of choice and give another statement which is equivalent to the axiom of choice. Then prove one of the implications for the equivalence.

**2.** Prove that the following statements are equivalent for any space $X$:

i) $X$ is Hausdorff;

ii) for any space $Y$ and any pair of continuous maps $f, g : Y \to X$ that agree on a dense subset of $Y$, then $f = g$.

iii) the diagonal $\Delta \subseteq X \times X$ defined by $\Delta := \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is a closed subset of $X \times X$;

**3.** Prove or give a counterexample for each of the following implications: separable $\overset{(a)}{\Rightarrow}$ second countable $\overset{(b)}{\Rightarrow}$ first countable $\overset{(c)}{\Rightarrow}$ separable $\overset{(d)}{\Rightarrow}$ first countable $\overset{(e)}{\Rightarrow}$ second countable $\overset{(f)}{\Rightarrow}$ separable

**4.** Prove that every quotient of a locally connected space is locally connected.

**5.** Give a proof or counterexample for each of the following implications:

i) $X$ is locally connected $\Rightarrow$ the connected components of $X$ are open;

ii) the connected components of $X$ are open $\Rightarrow$ $X$ is locally connected.

**6.** Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of spaces and let $x$ be a fixed point in $\prod_{\alpha \in A} X_\alpha$. Show that

$$D = \left\{ y \in \prod_{\alpha \in A} X_\alpha \mid y \text{ and } x \text{ differ in at most finitely many coordinates} \right\}$$

is dense in $\prod_{\alpha \in A} X_\alpha$.

**7.** Prove or disprove that any two spaces that are simultaneously discrete and indiscrete must be homeomorphic.

**8.** Let $X$ be a $T_1$ regular space with a countable basis and let $U$ be open in $X$. Show that

i) $U$ is a countable union of closed sets of $X$;

ii) There is a continuous function $f : X \to [0,1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.

**9.** Let $X$ be a $T_1$ completely regular space and $Y$ a compactification of $X$; let $\beta(X)$ be the Stone-Čech compactification of $X$. Show that there exists a continuous surjective closed map $g : \beta(X) \to Y$ that equals the identity on $X$.

**10.** Prove that the following statements are equivalent for any space $X$:

i) $X$ is Tychonoff (i.e., $T_1$ and completely regular);

ii) $X$ can be embedded in a compact Hausdorff space.

**11.** Find an error in the following purported "proof" of the jactitation that $2^{\mathbb{R}}$ is metrizable, where $2 = \{0,1\}$ is a two point discrete space and $\mathbb{R}$ is the set of real numbers.
"<u>PROOF</u>": Consider the inclusion $\mathbb{N} \subseteq \mathbb{R}$, where $\mathbb{N}$ is the set of natural numbers. This induces a "natural" embedding $2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{R}}$ by the map $f \mapsto \hat{f}$ where

$$\hat{f}(r) = \begin{cases} f(r), & \text{if } r \in \mathbb{N} \\ 0, & \text{if } r \in \mathbb{R} \setminus \mathbb{N}. \end{cases}$$

Let $U$ be an open subset of $2^{\mathbb{R}}$. By the definition of the product topology, the projection of $U$ is $\{0,1\} = 2$ in all but finitely many coordinates. Thus $U \cap 2^{\mathbb{N}} \neq \emptyset$. So $2^{\mathbb{N}}$ is dense in $2^{\mathbb{R}}$. But $2^{\mathbb{N}}$ is compact and $2^{\mathbb{R}}$ is Hausdorff, so $2^{\mathbb{N}}$ is closed in $2^{\mathbb{R}}$. Thus $2^{\mathbb{N}} = 2^{\mathbb{R}}$. But $2^{\mathbb{N}}$ is metrizable since it is a countable product of metrizable spaces. Hence $2^{\mathbb{R}}$ is metrizable.

**12.** Let $\mathfrak{A}$ be a locally finite collection of subsets in the space $X$. Show that

$$\overline{\bigcup_{A \in \mathfrak{A}} A} = \bigcup_{A \in \mathfrak{A}} \overline{A}.$$

**13.** Prove or disprove each of the following:

i) Every compact subset of a Hausdorff space is closed;

ii) every closed subset of a Hausdorff space is compact.

**14.** Let $E^n$ denote the Euclidean $n$-dimensional space. For what value of $n$ is it true that $E^n \setminus \{p\}$ is connected and simply connected? Prove your answer.

**15.** Show that in a locally compact Hausdorff space, countable intersections of dense open sets are dense.