Real Analysis Qualifying Exam
Fall 1995

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1. Suppose $f \in L^2(\mu)$. Prove that

$$\sum_{n=1}^{\infty} n\mu(\{x \in X : |f(x)| > n\}) < \infty$$

2. Suppose f is a C^∞ function on $\mathbb{R}$ with the property that for every $x \in \mathbb{R}$ there exists an n (which many depend on x) so that $\frac{\partial^n f}{\partial x^n}(x) = 0$. Prove that there exists a nonempty open interval I such that $f|_I$ is a polynomial.

3. Suppose $\psi \in L^1([0, \infty))$ is decreasing function on $[0, \infty)$. For each $n = 1, 2, 3, \ldots$ set $f_n(x) = n\psi(nx)$. Show that $f_n(x) \to 0$ uniformly on the set $[1, \infty)$.

4. Suppose $\{f_n\}$ is a sequence of measurable functions on X and f is measurable on X. Discuss the relationship between the conditions $f_n \to f$ a.e. and $f_n \to f$ in measure. You may want to consider the cases $\mu(X) = \infty$ and $\mu(X) < \infty$ separately.

5. Suppose $\mu(X) < \infty$, $f_n \in L^1$, $f_n \to f$ in measure and there exists $g \in L^1(\mu)$ such that $|f_n| \le |g|$ a.e. Show that $f_n \to f$ in $L^1(\mu)$.

6. Let $T(x) = \sum_{k=-N}^{N} a_k e^{ikx}, x \in [-\pi, \pi]$. Prove that $|a_0| \le \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp |T(x)| dx$.

7. On (X, M, μ) define $L^\infty(\mu) + L^1(\mu) = \{h + g | h \in L^\infty(\mu), g \in L^1(\mu)\}$. Prove that $L^p(\mu) \subseteq L^\infty(\mu) + L^1(\mu), 1 \le p \le \infty$.

8. Let S be a linear subspace of $C[0, 1]$ which is closed as a subspace of $L^2[0, 1]$. Show that S is a closed subspace of $L^2([0, 1])$.