Real and Complex Analysis Qualifying Exam – Spring 2015

Hakobyan & Poggi-Corradini

Instructions: Do 8 out of the 10 problems below. Pick 4 problems from # 1-5 and pick 4 problems from # 6-10.
Notations: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk). The Lebesgue measure on $\mathbb{R}$ is denoted by m.

1. Is there a Borel set $A \subset \mathbb{R}$ such that $0 < m(A \cap I) < m(I)$, for every interval $I \subset \mathbb{R}$? Justify your answer.

2. Show that if $f \in L^1(\mathbb{R})$ then the function $F(x) = \int_{-\infty}^{x} f(t)\,dt$ is continuous. (Note: Do not assume absolute continuity of the integral, unless you prove it.)

3. Compute the following limit and justify your calculation.

$$\lim_{n\to\infty} \int_0^1 \frac{1+\frac{n}{2}x}{(1+x)^{\frac{n}{2}}}\,dx.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function.

(a) Define $\|f\|_\infty$.
(b) Is there a function f s.t. $f \in L^p, \forall p > 1$, but $f \notin L^\infty$. Justify your answer.
(c) Is there a countable dense subset in $L^\infty(\mathbb{R})$? Justify your answer.

5. Let H be a Hilbert space. Show that if $\{e_i\}_{i=1}^\infty$ is an orthonormal sequence in H, then for every $x \in H$ the following (Bessel's) inequality holds

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \le \|x\|^2.$$

6. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Define

$$\ell = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in [0,\infty],$$

and put

$$R = \begin{cases} 1/\ell & \text{if } 0 < \ell < \infty \\ 0 & \text{if } \ell = \infty \\ \infty & \text{if } \ell = 0 \end{cases}$$

Prove that the power series $\sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$ is

(i) *convergent*, if $|z| < R$;
(ii) *divergent*, if $|z| > R$.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire non-constant function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

8. How many roots (counting multiplicities) does

$$z^4 + 3z - 1$$

have in the annulus $\Omega = \{z : 1 < |z| < 2\}$?

9. Let f be an analytic function defined on the strip $-1 < \operatorname{Im} z < 1$. Consider the set

$$S = \{x \in \mathbb{R} : f(x) \in \mathbb{R}\}.$$

Prove that, either S is *discrete* (i.e. it has no accumulation points), or $S = \mathbb{R}$.

HINT: Consider the function $g(z) = \overline{f(\bar{z})}$.

10. Is there a conformal map from the slit strip

$$\Omega := \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 1\} \setminus (-\infty, 0]$$

onto the unit disk? If yes, find one. If no, prove that there is none.