NUMERICAL ANALYSIS QUALIFYING EXAM
APRIL 30, 1984

1. Let A be the matrix

$$A = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 2 \\ 2 & 2 & 0 & 2 \end{bmatrix}$$

Use Gauss Elimination with partial pivoting to factor A as

$$A = PLU,$$

where: P is a permutation matrix.
L is a unit lower triangular matrix.
U is an upper triangular matrix.

2. Suppose we want to compute some integrals and the only numerical integration program we have available is the composite Simpson's rule. For each of the following integrals, tell whether straightforward use of Simpson's rule would work well and, if it would not, explain how the problem could be transformed to remove the difficulty.

(a) $\int_{.001}^{2} \frac{e^x}{x}\,dx$

(b) $\int_{0}^{1} \frac{\cos x}{\sqrt{1 - x^2}}\,dx$

(c) $\int_{1}^{2} \frac{\sin x}{x - 3}$

3. Consider the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

Use a Householder transformation to obtain a tridiagonal matrix which is similar to the above matrix.

4. Derive the 2-point Gauss-Chebyshev quadrature formula and use it to approximate

$$\int_{-1}^{1} \frac{\cos x \, dx}{\sqrt{1 - x^2}} .$$

5. Find the polynomial $p(x)$ of degree ≤ 4 which best approximates x^5 on $[-1,1]$ with respect to the norm $\| \ \|_\infty$. Justify your answer by citing the appropriate theorem(s).

6. Fit a cubic spline $S(x)$ through the three points $(x_0,y_0) = (1,0)$, $(x_1,y_1) = (2,1)$, $(x_2,y_2) = (3,0)$.

Use the natural boundary conditions

$$S''(x_0) = S''(x_2) = 0.$$

7. Consider a real, symmetric, non-singular $n \times n$ matrix A. Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ and suppose u_i is an eigenvector for λ_i for each i such that $\{u_1, \ldots, u_n\}$ is an orthonormal basis for R^n. Let $\|v\| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$ be the usual 2-norm on R^n. Given $x^{(0)} \in R^n$, $x^{(0)} \neq 0$, we define a sequence $(x^{(k)})_k$ by:

$$y^{(k+1)} = A\, x^{(k)}$$

$$x^{(k+1)} = \frac{y^{(k+1)}}{\|y^{(k+1)}\|}$$

(a) Show that $y^{(k+1)}$ is never 0, so that $x^{(k+1)}$ is defined.

(b) Show that $x^{(k)} = \dfrac{A^k x^{(0)}}{\|A^k x^{(0)}\|}$.

(c) Suppose $|\lambda_1| > |\lambda_i|$ for $i = 2, \ldots, n$. Prove that, if $\beta_k = (y^{(k)})^T x^{(k-1)}$ then

$$\lim_{k \to \infty} \beta_k = \lambda_1$$

(d) Suppose $\lambda_1 > 0$ and $\lambda_1 > |\lambda_i|$ for $i = 2,\ldots,n$. Show that the sequence $(x^{(k)})_k$, generated above, converges to an eigenvector of A for the eigenvalue λ_1.

8. Construct the appropriate divided difference table and use it to construct the polynomial $p(x)$ of degree ≤ 4 which satisfies the following conditions.

$$p(1) = 1$$

$$p'(1) = 3$$

$$p''(1) = -1$$

$$p(2) = 2$$

$$p(4) = 0\,.$$

9. Let $\langle\ ,\ \rangle$ be an inner product on $C[a,b]$ and let $p_0(x), p_1(x), \ldots, p_n(x), \ldots$ be the sequence of orthonormal polynomials. Given $f \in C[a,b]$, show that (using $\|\ \|$ corresponding to $\langle\ ,\ \rangle$)

$$\left\|f - \sum_{j=0}^{N} a_j p_j\right\| \leq \|f - q\| \quad \text{for all} \quad q \in P_N$$

if and only if $a_j = \langle f,p_j\rangle$ for $0 \leq j \leq N$.

(Hint: Show that if $q = \sum_{j=0}^{N} \beta_j p_j$ is any polynomial in P_N, then

$$\|f - q\|^2 = \|f\|^2 + \sum_{j=0}^{N} (\beta_j - \langle f,p_j\rangle)^2 - \sum_{j=0}^{N} \langle f,p_j\rangle^2.)$$

10. Let $\|\ \|$ be a norm on $\mathbb{C}^n$ and also denote by $\|\ \|$ the corresponding operator norm on $n \times n$ complex matrices. If T is a linear operator on $\mathbb{C}^n$ the spectral radius formula says .

$$\lim_{n\to\infty} \|T^n\|^{1/n} = r_\sigma(T).$$

Using this, prove that $\lim\limits_{n\to\infty} T^n = 0$ if and only if $r_\sigma(T) < 1$

(Hint: write $\|T^n\| = (r_\sigma(A) + \epsilon_n)^n$ with $\epsilon_n \to 0$).

11. Let A be a non-singular $n \times n$ matrix and consider the problem of solving the equation

$$Ax = b$$

by iteration. Recall that we split A as $A = M - N$ and then, having chosen a starting value $x^{(0)}$, iterate according to the scheme

$$Mx^{(m+1)} = Nx^{(m)} + b$$

(a) What conditions on M and N will insure that the sequence $(x^{(m)})_m$ converges to the solution of $Ax = b$?

(b) Describe the matrices M and N for the Jacobi iteration.

(c) Describe the matrices M and N for the Gauss-Seidel iteration.

(d) State conditions on A which guarantee convergence of the Gauss-Seidel iteration.

(e) Use the conditions stated in (d) to conclude that the Gauss-Seidel method converges for the system

$$\begin{bmatrix} 4 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

12. Consider an interval $[0,b]$ and the initial value problem

$$y' = y$$
$$y(0) = 1$$

Given $h > 0$, let $N = \frac{b}{h}$ and let $\rho_0, \ldots, \rho_N$ be chosen. Generate u_j via the Euler scheme

$$u_{j+1} - u_j = hu_j + h\rho_{j+1}, \; j = 0, \ldots, N-1.$$
$$u_0 = 1 + \rho_0.$$

(a) Define <u>truncation error</u> for the scheme.

(b) Define <u>consistency</u> for this scheme and show the above scheme is consistent.

(c) If we take all $\rho_j = 0$ the scheme becomes

$$u_{j+1} - u_j = hu_j$$
$$u_0 = 1$$

and the solution is $u_j = (1 + h)^j$. Show that if $j = N$ (which depends on h), then, as $N \to \infty$, $u_N \to y(b)$.

13. We wish to compute

$$y = \frac{\sqrt{1 + 4x^2} - 1}{x^2}$$

by first computing $\sqrt{1 + 4x^2}$, subtracting 1 and dividing the result by x^2. We do this on the TI-59 calculator for several small values of x and the results are given in the table.

x	y
$(1.0)10^{-1}$	1.98039027
$(1.0)10^{-3}$	1.999998
$(1.0)10^{-5}$	1.99
$(1.0)10^{-6}$	1.0
$(9.0)10^{-7}$	1.2345679

(a) On theoretical grounds, what should y approach as $x \to 0$?

(b) Explain the inaccuracy which appears in the table.

(c) Explain how you would obtain accurate values of y for small x and use your calculator to obtain accurate values of y for $x = (1.0)10^{-6}$, $(1.0)10^{-7}$.

Some Possibly useful information:

Householder: $x \neq 0$, n dimensional vector

$$S = (x_1^2 + \dots + x_n^2)^{1/2}$$

$$k = \pm S$$

$$K^2 = \frac{1}{2}(k^2 - kx_1)$$

$$u = x - ke_1$$

$$P = I - \frac{uu^*}{2K^2}$$

Recursions for orthogonal polynomials

(Legendre) $P_{r+1}(x) = \frac{2r+1}{r+1} xP_r(x) - \frac{r}{r+1} P_{r-1}(x)$

$P_0(x) = 1,\ P_1(x) = x$

(Chebychev) $T_{r+1}(x) = 2x\, T_r(x) - T_{r-1}(x)$

$T_0(x) = 1,\ T_1(x) = x$