REAL ANALYSIS
QUALIFYING EXAM
OCTOBER 22, 1982

You have three hours to solve as many of the following ten problems as you can.

1. a) Define Lebesgue outer measure λ on $\mathbb{R}$.

   Without assuming any properties of λ other than your definition, prove that

   b) $\lambda([0,1]) = 1$ and

   c) there exists a closed set $F \subset [0,1]$ such that F contains <u>no</u> rational number and $\lambda(F) > 1/2$.

2. Let $(E_n)_{n=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ such that $\overline{\lim_{n\to\infty}} \lambda(E_n) = 1$ and let $0 < \varepsilon < 1$. Prove that there exists a subsequence $(E_{n_k})_{k=1}^{\infty}$ for which

   $$\lambda\left(\bigcap_{k=1}^{\infty} E_{n_k}\right) > 1 - \varepsilon.$$

3. Let (X, A, μ) be a measure space with $\mu(X) = 1$. Define $\log 0 = -\infty$ and $\log\infty = \infty$. Let $f: X \to [0,\infty]$ be A-measurable. Prove that if the integral on the left is defined, then

   $$\int (\log f(x))\, d\mu(x) \leqq \log\left(\int f(x)\, d\mu(x)\right)$$

   and that equality obtains if and only if f is μ - a.e. equal to a constant. [Hint: Check that $\log t \leqq t - 1$ and put $t = f(x)/\int f d\mu$.]

4. Let $f \in L_1(\mathbb{R})$. Define $\phi: \mathbb{R} \to \mathbb{C}$ by

   $$\phi(t) = \int_{-\infty}^{\infty} \frac{f(x)}{1 + x^2 t^2}\, dx.$$

   Prove that

   a) $\lim_{|t|\to\infty} |\phi(t)| = 0.$

   b) ϕ is differentiable at each $t \in \mathbb{R}$, and

   c) $\phi'(t) = -2t \int_{-\infty}^{\infty} \frac{x^2 f(x)}{(1 + x^2 t^2)^2}\, dx$

5. Let $(f_n)_1^\infty$ be a sequence in L_1 on a measure space $(X, \mathcal{A}, \mu)$. Suppose that $f = \lim_n f_n$ exists a.e. on X and that

$$||f_n||_1 \leq C \quad \text{and} \quad \int_X \log|f_n| d\mu \geq -C \quad (n = 1, 2, \ldots)$$

where C is a finite constant. Prove that both f and $\log|f|$ belong to L_1.

Suggestion: Apply Fatou's Lemma to $(|f_n|)_1^\infty$ and $(g_n)_1^\infty$, where $g_n = |f_n| - \log|f_n|$.

6. a) State Fubini's Theorem.

b) Let $f, g \in L_1(\mathbb{R})$. Prove that the formula

$$h(x) = \int_{-\infty}^{\infty} f(xy)\, g(y) \sin y \, dy$$

defines a function h at almost every $x \in \mathbb{R}$ and that $h \in L_1(\mathbb{R})$. [You may presume that the function $(x,y) \to f(xy)$ is measurable.]

7. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let ϕ be a real-valued measurable function on X. Define

$$\nu(B) = \mu(\phi^{-1}(B)) \qquad (B \in \mathcal{B}),$$

where $\mathcal{B}$ denotes the σ-algebra of all Borel sets in $\mathbb{R}$. Prove the following:

a) ν is a measure on $(\mathbb{R}, \mathcal{B})$.

b) If f is a non-negative, simple, Borel function on $\mathbb{R}$, then

$$\int_{\mathbb{R}} f \, d\nu = \int_X f \circ \phi \, d\mu.$$

c) The above formula holds for all non-negative Borel functions f on $\mathbb{R}$.

8. Let μ be a regular Borel measure on the plane $\mathbb{R}^2$. Define ν on the Borel sets $\mathcal{B}$ of $\mathbb{R}$ by

$$\nu(A) = \mu(A \times \mathbb{R}).$$

Prove that there exists a mapping $B \to f_B$ of $\mathcal{B}$ into $L_1^+(\nu)$ such that

a) $\mu(A \times B) = \int_A f_B \, d\nu$ and

b) $f_{B_1} + f_{B_2} = f_{B_1 \cup B_2} + f_{B_1 \cap B_2}$ whenever $A, B, B_1, B_2 \in \mathcal{B}$.

9. Prove that if $f\colon \mathbb{R} \to \mathbb{C}$ is continuous with $f(x+1) = f(x)$ for all $x \in \mathbb{R}$ and if $\xi \in \mathbb{R}$ is irrational, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(j\xi) = \int_0^1 f(x)\,dx.$$

[Hint: Show first that the set F of all f for which the conclusion obtains is a linear space that contains all f of the form

$$f(x) = e^{2\pi ikx} \qquad (k \in \mathbb{Z}).]$$

10. Prove that $f \in L_1(\mathbb{R})$ implies $\hat{f} \in C_0(\mathbb{R})$, where

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{itx}dx \qquad (t \in \mathbb{R}).$$

[<u>Suggestions</u>: (i) Consider the case where f is the characteristic function of a bounded interval. (ii) Use the fact that the step functions (in $L_1(\mathbb{R})$) are dense in $L_1(\mathbb{R})$.]