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Real and Complex Analysis Qualifying Exam

New System

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Instructions: Below you will find 8 problems. Each problem is worth 10 points. **Time:** 2 hours.

Notations: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle); λ = Lebesgue measure on $\mathbb{R}$.

1. Let $f : [0,1] \to \mathbb{C}$ be a continuous function. Define the function $F : \mathbb{C} \setminus [0,1] \to \mathbb{C}$ by

$$F(z) = \int_0^1 \frac{f(t)}{t-z}\, dt, \quad z \in \mathbb{C} \setminus [0,1].$$

Prove that F is holomorphic.

2. Consider the annulus $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$. Show there does not exist a sequence $(P_n)_{n=1}^{\infty}$ of polynomials in one variable, such that

$$P_n(z) \xrightarrow[n\to\infty]{\text{uniformly}} \frac{1}{z}, \quad \forall\, z \in \Omega.$$

3. Compute

$$\int_0^\infty \frac{dx}{1+x^7}.$$

Hint: For (large) $R > 1$, use the boundary of the circular sector

$$C_R = \{re^{i\theta} : 0 < r < R,\ 0 < \theta < 2\pi/7\}$$

as contour.

4. Let $f : [0,\infty) \to \mathbb{C}$ be a Lebesgue measurable function. Assume there exists real numbers $a, k > 0$, such that

$$|f(x)| \le a e^{-kx}, \quad \forall\, x \ge 0.$$

Consider the half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > k\}$.

(i) Prove that, for every $z \in H$, the function $[0, \infty) \ni t \longmapsto e^{itz} f(t)$ is Lebesgue integrable.

(ii) Prove that the function $F : H \ni z \longmapsto \displaystyle\int_0^\infty e^{itz} f(t)\, dt \in \mathbb{C}$ is holomorphic.

5. Call a subset of $\mathbb{R}$ *negligible*, if it is Lebesgue measurable, and has Lebesgue measure zero. Prove that, for $A \subset \mathbb{R}$, the following are equivalent:

(i) A is negligible;

(ii) there is a sequence $(D_n)_{n=1}^\infty$ of open sets in $\mathbb{R}$ with $\lim\limits_{n\to\infty} \lambda(D_n) = 0$, and $A \subset \bigcap\limits_{n=1}^\infty D_n$.

6. Let $p > q \geq 1$ be real numbers.

(i) Prove the inclusion $L^p\big([0,1], \lambda\big) \subset L^q\big([0,1], \lambda\big)$.

(ii) Show (by example) that the inclusion in (i) is strict.

(iii) Give an example of a measure space $(X, \mathcal{A}, \mu)$, for which one has the inclusion $L^p(X, \mathcal{A}, \mu) \supset L^q(X, \mathcal{A}, \mu)$.

7. Let $p \geq 1$ be a real number, and let $(f_n)_{n=1}^\infty \subset L^p(\mathbb{R}, \lambda)$ be a sequence with $\lim\limits_{n\to\infty} \|f_n\|_p = 0$. Prove there exist integers $1 \leq n_1 < n_2 < \dots$, such that $\lim\limits_{k\to\infty} f_{n_k} = 0$, λ-a.e.

8. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with the property:

$$f(z + m + ni) = f(z), \;\; \forall\, z \in \mathbb{C},\ m, n \in \mathbb{Z}.$$

Prove that f is constant.