PDE Qualifying Exam - Fall 2003 (Old System)

1. Let $f \in \mathcal{H}^1(\mathbb{R}^n)$, and write $\tau_h f(x) = f(x-h)$. Show that

$$\frac{f - \tau_h f}{h} \to f'$$

in $L^2(\mathbb{R}^n)$, as $h \to 0$.

2. For $x \in I\!R$, let

$$t_a(x) = \begin{cases} \frac{1}{2a} & \text{for } |x| < a, \\ 0 & \text{for } |x| > a. \end{cases}$$

Show that $\lim_{a \to 0} t_a(x) = \delta_0$ in the sense of distributions. Is the function $t_a(x)$ weakly differentiable, i.e. is it true that $t_a \in W^{1,1}_{\text{loc}}(I\!R^n)$? Find the distributional derivative of the distribution associated to $t_a(x)$.

3. Let u_n be a sequence of harmonic functions in a domain $D \subset I\!R^n$ such that $\lim_{n\to\infty} u_n = u$ uniformly in D. Show that u is harmonic in D.

4. Let $U \subset I\!R^n$ be bounded domain with a smooth (say, C^2) boundary, and let $0 < T \in I\!R$. Benote by U_T the cylinder $U \times (0,T)$, and by Γ_T the parabolic boundary of U_T, $(\overline{U} \times \{0\}) \cup (\partial U \times (0,T))$. We say that $v \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ is a subsolution to the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

a) Show that $\max_{U_T} v = \max_{\Gamma_T} v$.

b) Let $\phi : I\!R \to I\!R)$ be a smooth and convex function such that $\phi'(0) \geq 0$, and and let u be a solution to the heat equation in D such that $0 \leq u$ on Γ_T. Show that $\phi(u)$ is a subsolution to the heat equation.

5. Show that there are no solutions to the equation

$$u_x + u_y = u$$

that satisfy the condition $u = 1$ on the line $x = y$.

6. Consider the problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < l,\ t > 0 \qquad (E)$$

$$u(0,t) = u(1,t) = 0 \quad \text{for } t > 0$$

$$u(x,0) = f(x) \quad \text{for } 0 < x < 1,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for } 0 < x < l.$$

Assume $f \in C^2(0,l),\ g \in C^1(0,l)$.

a) Show that by means of a linear change of variables the equation (E) can be transformed into

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \qquad (E').$$

b) Use (E') to deduce an expression of $u(x,t)$ valid in the region R given by $0 < x + ct < l,\ 0 < x - ct < l$.

c) Is the solution u unique in R? How do changes in the boundary data $u(0,t),\ u(1,t)$ $t > 0$ affect the solution in the region R?