Topology Qualifying Exam
Spring 1988

Do 9 of the following 17 problems.

1. Prove that the following statements are equivalent for any topological space X:

a) X is Hausdorff

b) for any space Y and for each pair of continuous functions $f, g : Y \to X$ that agree on a dense subset of Y, $f = g$.

2. Prove that the product of normal T_1 spaces must be T_1 and completely regular, but need not be normal.

3. Prove or give a counterexample for each of the following implications:

separable $\overset{(a)}{\Longrightarrow}$ second countable $\overset{(b)}{\Longrightarrow}$ first countable $\overset{(c)}{\Longrightarrow}$ separable $\overset{(d)}{\Longrightarrow}$ first countable $\overset{(e)}{\Longrightarrow}$ second countable $\overset{(f)}{\Longrightarrow}$ separable.

4. Let A be a subset of a topological space X. Show that the following statements are equivalent:

a) $x \in \overline{A}$

b) there is a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to x.

c) There is a net $\alpha : \Lambda \to A$ that converges to x.

5. Let $(X_\alpha)_{\alpha \in A}$ be a collection of topological spaces and let x be fixed point in $\prod_{\alpha \in A} X_\alpha$. Show that

$$D = \left\{ y \in \prod_{\alpha \in A} X)\alpha |\ y \text{ and } x \text{ differ in at most finitely many coordinates} \right\}$$

is dense in $\prod_{\alpha \in A} X_\alpha$.

6. Let $\{A_\alpha\}$ be a point-finite covering of a Hausdorff space X. Show that $\{A_\alpha\}$ has a subcovering $\{B_\gamma\}$ so that no subset of $\{B_\gamma\}$ covers X.

7. Prove that the following statements are equivalent for any topological space X:

a) X is Tychonoff (i.e., T_1 and completely regular)

b) X can be embedded in a compact Hausdorff space.

8. Show that for $n \geq 1$ the n-sphere, S^n, is homeomorphic to the one-point compactification of Euclidean n-space, E^n.

9. Find an error in the following purported "proof" that 2^R is metrizable, where $2 = \{0, 1\}$ is a two point discrete space, and R is the set of real numbers.

Proof. Consider the inclusion $N \subseteq R$, where N is the natural numbers. This induces a "natural" embedding $2^N \hookrightarrow 2^R$ where $f \mapsto \hat{f}$ and

$$\hat{f}(r) = \begin{cases} f(r) & \text{if } r \in N \\ 0 & \text{if } r \in R - N. \end{cases}.$$

But 2^N is the Cantor space C. Thus C is embedded in 2^R. Let U be an open subset of 2^R. By the definition of product topology, the projection of U is $\{0, 1\} = 2$ in all but finitely many coordinates. Thus $U \cap 2^N \neq \emptyset$. So 2^N is dense in 2^R. But 2^N is compact and 2^R is Hausdorff, so 2^N is closed in 2^R. Thus $2^N = 2^R$. But C is metrizable, and so 2^R is metrizable.

10. Prove that every quotient of a locally connected space is locally connected.

11. Let E^n denote Euclidean n-dimensional space. For what values of n is it true that $E^n - \{p\}$ is simply connected for any point $p \in E^n$. Prove your answer.

12. Let E^n denote Euclidean n-dimensional space. For what values of n is it true that $E^n - C$ is connected for any countable subset C. Prove your answer.

13. Show that in a locally compact Hausdorff space, countable intersections of dense open sets are dense and open.

14. A covering $\{U_\alpha\}_{\alpha \in A}$ of a topological space X is called shrinkable if there is a covering $\{V_\alpha\}_{\alpha \in A}$ of X such that $\overline{V_\alpha} \subseteq U_\alpha$ for each $\alpha \in A$ and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$. Show that a Hausdorff space is normal if and only if each point-finite covering is shrinkable.

15. Prove that every separable metrizable space can be embedded in a countable product of closed unit intervals.

16.

a) Give an example of an ultrafilter on the set $\mathbb{R}$ of real numbers.

b) Give an example of a filter on $\mathbb{R}$ that is not an ultrafilter.

c) Prove that every filter on $\mathbb{R}$ is contained in an ultrafilter on $\mathbb{R}$.

17. Show that every compact connected Hausdorff space is the continuous image of $[0, 1]$.