# NUMERICAL ANALYSIS QUALIFYING EXAM
## Spring, 1998
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(do at lest 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Show that

   (a) $1 + x \le e^x$, $\forall$ *real* $x$.

   (b) $e^x \le 1 + 1.01x$, $\forall\ 0 \le x \le 0.01$.

   And use the results to show

   $$(1+u)^n \le 1 + 1.01nu \quad \text{ if } \ 0 \le nu \le 0.01.$$

2. By constructing a fixed-point iteration, find the value of $x$ given by

   $$x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$$

   where $p$ is a positive number. Prove the convergence of the fixed-point iteration.

3. Construct a near-minimax polynomial of degree $\le 2$ for the function $g(t) = e^t$ on the interval $t \in [0, 1]$ and estimate its maximum error. You can express the result in terms of the expenential function. (hint: $\cos(n+1)\theta = \cos\theta\cos n\theta - \cos(n-1)\theta$)

4. Given the trapezoidal rule and Simpson's rule as:

   $$\int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}[f(x_0) + f(x_0+h)] - \frac{h^3}{12}f^{(2)}(\xi),$$

   $$\int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_0+h) + f(x_0+2h)] - \frac{h^5}{90}f^{(4)}(\xi),$$

   **(1)** Describe the composite trapezoidal rule and composite Simpson's rule for evaluating integrals $\int_a^b f(x)dx$ using $n$ subintervals.

   **(2)** Derive an estimate for the error in the composite trapezoidal rule in terms of the length of the subintervals into which $[a, b]$ is divided.

   **(3)** Derive the asymptotic error formula for the composite Simpson's rule

   $$E_n(f) \doteq -\frac{h^4}{180}[f^{(3)}(b) - f^{(3)}(a)],$$

   where $h = (b-a)/n$.

5. Let $U$ and $V$ be two $3 \times 3$ matrices such that

   $$UV = [w_{ij}] = \begin{bmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where } \ w_{11}w_{22} \ne w_{12}w_{21}.$$

Show that either the last row of $U$ is the zero row vector or the last column of $V$ is the zero column vector. (Hint: If $U$ is singular then there exists a nonzero 3-tuple vector $p$ such that $p^T U = 0$)

6. Let $A$ be an $n \times n$ symmetric real positive definite matrix. Show that there exist $2^n$ real lower-triangular matrices $L$ such that $A = LL^T$?

7. Assume that $\mathbf{w} \in \mathbf{R}^n$, and that $\|\mathbf{w}\|_2 = 1$. What are the eigenvalues, eigenvectors, and determinant of a Householder matrix
$I - 2\mathbf{w}\mathbf{w}^T$?

8. By using the singular value decomposition, show that any square real matrix $A$ can be written as $A = QS$ where $Q$ is an orthogonal matrix and $S$ is a semipositive definite matrix.