Algebra Qualifying Exam

September 26, 1996

Instructions: You are given 10 problems from which you are to do 8. Note: All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Suppose that $G/Z(G)$ is cyclic. Show that, in fact, G is abelian.

2. Let G be a group.

   (i) State what it means for G to be a *solvable group.*

   (ii) Let G be a group, $K \lhd G$ be a normal subgroup of G. Show that G is solvable if and only if both K and G/K are solvable groups.

3. Let R be a commutative ring. Recall that an ideal $I \subseteq R$ is *finitely generated* if there exist elements $x_1, x_2, \ldots, x_k \in I$ such that $I = (x_1, x_2, \ldots, x_k)$ $(= \{\sum r_i x_i | \; r_1, r_2, \ldots, r_k \in R\})$. Recall next that R satisfies the *ascending chain condition*, or is *Noetherian* if any sequence $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ of ideals eventually stabilizes, i.e., there exists an integer M such that $m \geq M$ implies that $I_m = I_M$. Now prove that the following are equivalent for the commutative ring R:

   (i) Every ideal $I \subseteq R$ is finitely generated.

   (ii) R satisfies the ascending chain condition.

4. Let $R = \{\frac{a}{b} \in \mathbb{Q} | \; 2 \not| \; b\}$, a subring of the rational number field $\mathbb{Q}$. Prove that R has a unique proper maximal ideal, viz., the one generated by the element $2 \in R$.

5. Let R be a ring and let M be an irreducible R-module. Prove that $M \cong R/\mathcal{M}$, where $\mathcal{M} \subseteq R$ is a maximal left ideal.

6. Let V be a finite dimensional vector space with dual space V^*. If $W \subseteq V$ is a subspace, set $\mathrm{Ann}(W) = \{f \in V^* | \; f(w) = 0$ for all $w \in$

$W\}$, the *annihilator* of W in V^*. If W_1, W_2 are subspaces of V, show that $\mathrm{Ann}(W_1 + W_2) = \mathrm{Ann}(W_1) \cap \mathrm{Ann}(W_2)$.

7. Let $T : V \to V$ be a linear transformation on the vector space V, over the field $\mathbb{F}$. Assume that T has the following property: whenever $W \subseteq V$ is a T-invariant subspace of V then there exists another T-invariant subspace $W' \subseteq V$ with the property that $V = W \oplus W'$. Must T be diagonalizable? Prove, or give a counterexample.

8. Let $\mathbb{F} \subseteq \mathbb{K}$ be an algebraic extension of fields. If $\alpha \in \mathbb{K}$, prove that the minimal polynomial $m_{\alpha,\mathbb{F}}(x)$ of α over $\mathbb{F}$ is an *irreducible* polynomial.

9. Let $\mathbb{K} = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. Prove that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$. (Hint: show that if $m(x)$ is the minimal polynomial of $\sqrt{2+\sqrt{2}}$ over $\mathbb{Q}$, then $m(x)$ splits completely in $\mathbb{K}[x]$.

10. Let p be a prime and let $\mathbb{F} = \mathbb{F}_p$ be the finite field of order p. Let $f(x) = x^2 + x + 1 \in \mathbb{F}[x]$ and let $\mathbb{K} \supseteq \mathbb{F}$ be the splitting field of $f(x)$ over $\mathbb{F}$. Compute $[\mathbb{K} : \mathbb{F}]$ in the cases:

    (i) $p = 2$,

    (ii) $p = 3$.