ALGEBRA QUALIFYING EXAM, JANUARY 2009

Instructions Please choose 8 from the following 10 problems, and solve them as best you can. Indicate the 8 problems that you would like to submit, by circling their numbers on this "problem sheet".

1. Let G be a finite p-group, and let H be a proper subgroup of G. Show that H is contained in a normal subgroup K of G such that the index $|G:K|$ of K in G is p.

2. Let P and Q be p-Sylow subgroups of a finite group G. Prove:
(a) If $Q \subseteq N_G(P)$ (the normalizer of P in G), then $Q = P$.
(b) One has $N_G(N_G(P)) = N_G(P)$.

3. Let G be a group (not necessarily finite), F a field, V a vector space over F, and $\mathbf{B}$ a basis for V over F. Suppose that there is a bijection $\beta : G \to \mathbf{B}$ (to be denoted $g \mapsto v_g$). Show that G is isomorphic to a subgroup of the group $GL(V)$ of invertible F-linear automorphisms of V.

4. Let R be a ring (not necessarily having a unit element). Then R is said to be *Boolean* if $x^2 = x$ for all $x \in R$. Show that a prime ideal $\wp$ in a Boolean ring R is necessarily maximal.

5. A module M is called *irreducible* if the only submodules of M are 0 and M. Prove that a module over a principal ideal domain R is irreducible if and only if it is generated by an element x with $Ann(x) = \langle p \rangle$, where p is a prime element of R. (The annihilator $Ann(x)$ of an element x of a module over a ring R is the set $Ann(x) = \{r \in R \mid rx = 0\} \subset R$.)

6. Show that an $n \times n$-matrix A with entries in an algebraic closed field is nilpotent (i.e., there exists a positive integer n with $A^n = 0$) if and only if

$$tr(A) = tr(A^2) = \cdots = tr(A^n) = 0.$$

($tr(X)$ denotes the *trace* of a matrix X, the sum of its diagonal elements.)

7. Construct a splitting field K for the polynomial $(x^2-2)(x^2+x+1)$ over $\mathbb{Q}$, and list its subfields.

8. Let M and N be modules over a commutative ring R. Show that $M \otimes N$ is isomorphic to $N \otimes M$ as R-modules.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

9. Let F be a field and let A and B be $n \times n$ matrices over F. Let E be an extension field of F, and suppose that there is an $n \times n$ invertible matrix C over E such that $B = C^{-1}AC$. Then in fact, such a matrix C may be chosen so that its entries are in F. Why is that ?

(10). Let V be a vector space over a field F (and where it is not assumed that the dimension of V is finite). Let $T \in End_F(V)$ such that T has a minimal polynomial of the form $(x - \lambda_1) \cdots (x - \lambda_n)$, where the elements λ_i of F are pairwise distinct. Show that V is the direct sum of the "eigenspaces" $V_i = \{v \in V \mid T(v) = \lambda_i v\}$.