Kansas State University | Department of Mathematics

Qualifying Exam: Geometry and topology

Spring 2012. January 17, 6:00 p.m. to 9:00 p.m.
Examiners: Prof. Ricardo Castaño-Bernard and Prof. David Yetter

Name: ______________________________

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number. Problems that are completely solved and thoroughly justified will be given more credit than scattered attempts leading to partial answers.

1. (a) State Tychonoff's theorem.
   (b) Prove the finite case of Tychonoff's theorem completely.
   (c) Name one of the concepts that can be used to prove the full version of Tychonoff's theorem that is not needed in the finite case.

2. (a) Give the definition of the derivative $f_*(p)$ of a smooth map between manifolds $f : M \to N$ at $p \in M$. Show that if $f_*(p)$ is an isomorphism, there is a neighborhood U of p such that for every $p' \in U$, $f_*(p')$ is an isomorphism.
   (b) Let $M(k, \mathbb{R})$ be the set of $k \times k$ matrices. Show that the set of 2×2 matrices of rank 1 is a 3-dimensional submanifold of $M(2, \mathbb{R})$.
   (c) Let $C = \{y^3 - x^2 = 0\} \subset \mathbb{R}^2$ and $f : \mathbb{R} \to \mathbb{R}^2$ be the map $f(t) = (cos(t), sin(t))$. Show that f is transversal to C.

3. (a) Let M be a smooth manifold. Define $\bigwedge^k T^*M$ and show it is a smooth manifold.
   (b) Let $M = \mathbb{R}^3$. Show that
   $$\theta = (x_1 + x_3^3)dx_1 \wedge dx_2 + x_1x_2dx_1 \wedge dx_3 + x_1dx_2 \wedge dx_3$$
   is a section of the bundle $\bigwedge^2 T^*M$. Compute $d\theta$ and give an explicit description of the locus $Z \subseteq \mathbb{R}^3$ where $d\theta$ intersects the zero section of $\bigwedge^3 T^*M$.
   (c) Let $X = (x_1 - x_2)\partial_{x_1} + (x_1 + x_2)\partial_{x_2}$ and θ as above. Compute the Lie derivative $L_X\theta$. Hint: you can use Cartan's identity, $L_Y = d \circ i_Y + i_Y \circ d$, where $i_Y\alpha = \alpha(Y, \cdot)$, for any vector field Y and any p-form α.

4. (a) Give the definition of (C^*, ∂), the singular chain complex with real coefficients, and (Ω^*, d) the de Rham complex.
   (b) Use Stokes' theorem to define a homomorphism between these two complexes in part (a).
   (c) Give a sketch of how to prove that the homomorphism of part (b) gives rise to an isomorphism $H^k_{dR}(M) \cong H^k(M, \mathbb{R})$.
   (d) Let $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. Compute the de Rham cohomology of $\mathbb{R}^3 - S^1$. Hint: use Mayer-Vietoris sequence. You could also use part (c).

5. (a) State the Seifert-Van Kampen Theorem for fundamental groupoids.
   (b) What additional hypothesis is needed in the Seifert-Van Kampen theorem for fundamental groups?
   (c) Use the Seifert-Van Kampen Theorem for fundamental groupoids to prove that if X, Y, Z and W are contractible spaces, with X and Y open subspaces of some larger ambient space (possibly simply their union), and in $X \cup Y$

$$X \cap Y \cong Z \coprod W$$

   then $\pi_1(X \cup Y) \cong \mathbf{Z}$.
   (d) Explain why the Seifert-Van Kampen Theorem for fundamental groups is inadequate to prove the statement in part (c)

6. Consider a CW complex X with one 0-cell, two 1-cells a and b, and three 2-cells A, B, and C. The 1-cells are attached to the 0-cell so that the 1-skeleton is $S^1 \vee S^1 \cong S^1/\{1, -1\}$. Considering S^1 to be the unit circle in the complex plane and the 2-cells to the copies of the unit disk in the complex plane, the 2-cells are attached by mapping the boundary of A (resp. B) to the first (resp. second) circle in the wedge (which we take to be a (resp. b) and the 0-cell) by $e^{i\theta} \mapsto e^{2i\theta}$, and mapping the boundary of C to the 1-skeleton considered as the quotient of S^1 by identifying 1 and -1 by the map $e^{i\theta} \mapsto [e^{3i\theta}]$.

   (a) Find the homology of X.
   (b) Justify the assertion that $\pi_1(X) \cong \mathfrak{S}_3$ (the symmetric group on three elements)
   (c) Describe a different CW complex in which only the attaching map of C has been changed, which has a different fundamental group, but the same homology.
   (d) State the Hurewicz theorem relating π_1 and H_1 and explain why your answer to part (a) and what you proved in (b) illustrate the theorem.

7. (a) State the Künneth theorem (formula) for cohomology.

   (b) Under what hypotheses does an extension of the Künneth theorem determine the multiplication on the cohomology ring of a product of two spaces from that on the two spaces? State the theorem.

   (c) Use the well-known cohomology groups of spheres to determine

$$H^\bullet(S^2 \times S^2 \times S^4),$$

   as a graded ring. That is, in addition to computing the cohomology groups, describe completely the multiplication on the cohomology ring.

8. Consider the finite topological space $X = \{0, x, 1\}$ with topology

$$\{\emptyset, \{x\}, \{0, x\}, \{x, 1\}, X\}.$$

   Which of the following topological properties does this space have: compactness, connectedness, path connectedness, separability, contractibility, T_0-ness, T_1-ness, Hausdorffness, metrizability?