REAL ANALYSIS QUALIFYING EXAMINATION
Spring 1998 (Saeki and Peller)

Unless otherwise stated, let $(X, \mathcal{A}, \mu)$ be a measure space.

1. (a) What does it mean that $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable?

(b) Prove that if $f, g : X \to \mathbb{C}$ are both $\mathcal{A}$-measurable, then so is $f + g$.

2. Let S be the smallest σ-algebra of subsets of $\mathbb{R}^2$ that contains

$$\mathcal{F} := \{I \times J : I \text{ and } J \text{ are bounded open intervals of } \mathbb{R}\}.$$

Prove that $S = \mathcal{B}(\mathbb{R}^2)$, the Borel subsets of $\mathbb{R}^2$.

3. Let (f_n) be a sequence of real-valued measurable functions on X that converges to some function f at each point of X. Prove that f is measurable and that for each $\alpha \in \mathbb{R}$, we have

$$\mu(\{f > \alpha\}) \leq \liminf_{n\to\infty} \mu(\{f_n > \alpha\}).$$

4. Let $f \in L^1(\mu)$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$E \in \mathcal{A} \text{ and } \mu(E) < \delta \Rightarrow \int_E |f| d\mu < \varepsilon.$$

5. Let $E_n \in \mathcal{A}$ for each $n \in \mathbb{N}$. Prove that

$$\mu\left(\bigcup_1^\infty E_n\right) \leq \sum_1^\infty \mu(E_n).$$

6. Let $f : X \to [0, \infty]$ be measurable and $0 < p < \infty$. Prove

$$\int_X f^p d\mu = p \int_0^\infty t^{p-1} \mu(\{f > t\}) dt.$$

7. Prove the completeness of $L^p(\mu)$ for $1 \leq p < \infty$.

8. Suppose f_n, $f \in L^1(\mu)$ and $||f_n - f||_1 \to 0$. Prove that

$$\limsup_{n\to\infty} \int \log|f_n| d\mu \leq \int \log|f| d\mu.$$

[Consider an appropriate subsequence of $(|f_n| - \log|f_n|)_1^\infty$.]