Real Analysis Qualifying Exam
Fall 1994

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1. Define Lebesgue outer measure, λ^*, on [0,1].

2. Let (X, S, μ) be a finite measure space and suppose that $\{f_n\}$ is a sequence of measurable functions which converges to a finite function f a.e.. Let $\varepsilon > 0$ and set $A_N = \{x : \sup_{n \geq N} |f_n(x) - f(x)| > \varepsilon\}$. Prove that $\mu(A_N) \to 0$ as $N \to \infty$.

3. Suppose $f \in L^p([0,\infty))$ where $1 \leq p \leq 2$. For $x \geq 0$ set $g(x) = \int_x^{x^2} f(t)dt$. Show that $\lim_{x\to\infty} \frac{g(x)}{x} = 0$.

4. Let f be a C^∞ function (that is, f and all its derivatives exist and are continuous) from $\mathbb{R}$ to $\mathbb{R}$ with the property that for each $x \in \mathbb{R}$ there exists a $k \in \mathbb{N}$ (depending on x) such that $\frac{\partial^k f}{\partial x^k}(x) = 0$. Show that there exists an interval $I \subseteq \mathbb{R}$, $I \neq \varnothing$ such that $f|_I$ is a polynomial.

5. (a) Construct a bounded, Lebesgue integrable function $g(x)$ on [0,1] such that $\int_0^1 |f(x)-g(x)|dx > 0$ for every Riemann integrable function $f(x)$ on [0,1].

(b) Can you construct such a $g(x)$ so that $\int_0^1 |f(x)-g(x)|dx > 10^{-5}$ for every Riemann integrable function $f(x)$ on [0,1]?

6. Suppose f is a Lebesgue measurable function on [0,1]. Is it true that if $f' = 0$ a.e. then f cannot be strictly increasing?

7. Suppose $\Phi : [0,\infty) \to [0,\infty)$ is strictly increasing continuously differentiable function with $\Phi(0) = 0$. Suppose that $f \in L^1(X, \mathcal{M}, \mu)$ and for $\lambda > 0$ set $m(\lambda) = \mu(\{x \in X : |f(x)| > \lambda\})$. Prove that $\int_X \Phi(f(x))d\mu(x) = \int_0^\infty m(\lambda)d\Phi(\lambda)$.

8. Suppose μ is a complex measure on $\mathbb{R}^n$ with the property that $\int_{\mathbb{R}^n} f(x)d\mu \geq 0$ whenever $f \geq 0$ is a continuous function on $\mathbb{R}^n$ with compact support. Show that μ is a positive measure.

9. Let v, μ be complex measures on $(X, \mathcal{M})$. Suppose $v << \mu$. Show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|\mu|(E) < \delta$ then $|v(E)| < \varepsilon$.

10. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and μ is positive and σ-finite.

(a) Suppose $f \in L^1(X, \mathcal{M}, \mu)$ and suppose $\mathcal{A} \subseteq \mathcal{M}$ is also a σ-algebra. Prove that there exists a function $g \in L^1(X, \mathcal{A}, \mu)$ such that $\int_E g d\mu = \int_E f d\mu$ for all $E \in \mathcal{A}$.

(b) If g and f are related as in part (a) we write $g = E(f|\mathcal{A})$. Suppose that $\mathcal{B} \subset \mathcal{A}$ is also a σ-algebra on X. Show that $E(E(h|\mathcal{A})|\mathcal{B}) = E(h|\mathcal{B})$ whenever $h \in L^1(X, \mathcal{M}, \mu)$.