ANALYSIS QUALIFYING EXAM - SPRING 2010

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Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

Notations:
:= means a defining equation.
$\mathbb{N}$ is the positive integers, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

(1) State precisely
  (a) The Riemann Mapping Theorem
  (b) Runge's Approximation Theorem
  (c) Picard's Great Theorem
  (d) Cauchy's Integral Theorem and Morera's converse of it.

(2) The continuous function $f : \mathbb{D} \to \mathbb{C}$ satisfies $\int_\Delta f = 0$ for every triangle $\Delta \subset \mathbb{D}$. Construct a holomorphic function F such that $F' = f$ in $\mathbb{D}$.

(3) True or false (Prove or give a counter-example): Every complex number is $\sin(z)$ for some $z \in \mathbb{C}$.
Hint: Since $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$, Picard's Great Theorem plays a role but doesn't by itself do the job.

(4) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that $|f'(z)| \le 4$ whenever $|z| \le 1/2$.

(5) If $E \subset \mathbb{R}$ is an open and dense subset of the real line, must its Lebesgue measure be infinite? (Prove, or give a counter-example.)

(6) Let $f, g \in L^1(\mathbb{R})$, with Lebesgue measure on $\mathbb{R}$. Define the convolution $f * g$ and show that $f * g \in L^1$.

(7) Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. A sequence $\{f_n\}$ of complex measurable functions on $\cancel{X}$ Ω is said to *converge in measure* to a complex measurable function f if for every $\epsilon > 0$,

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0 \text{ as } n \to \infty.$$

Assume that $\mu(\Omega) < \infty$. Prove or disprove (with a counter-example) the following statements
  (a) If $f_n \to f$ a.e. then $f_n \to f$ in measure.
  (b) If $f_n \to f$ in L^p, with $1 \le p \le \infty$, then $f_n \to f$ in measure.

(8) Compute the following limit and justify the calculations:

$$\lim_{n \to \infty} \int_0^\infty n \sin(2x/n)\,[x(1 + x^2)]^{-1} dx$$