Topology Qualifying Exam
Spring 1990

Work 9 of the following problems. Do not turn in more than 9.

For a subset A contained in a topological space X, let A^- denote the closure of A in X, let A° denote the interior of A in X, and let $Fr(A)$ denote the frontier of A in X. All Euclidean spaces will have the usual topology, and all product spaces will have the product topology.

1. For a topological space X, prove that "o–" and "–o" are both idempotent operations on the subsets of X; that is, $A^{o-o-} = A^{o-}$ and $A^{-o-o} = A^{-o}$ for all $A \subseteq X$.

2. (a) Define "net".

(b) Give an example of a net that is not a sequence.

3. Prove that $[0, 1]$ (with its usual topology) is compact.

4. (a) Characterize the compact subsets of $\mathbb{R}$.

(b) State and prove a maximum value theorem from calculus.

(c) Using maximum, minimum, and intermediate value theorems from calculus, prove every continuous open function $f : [0, 1] \to [0, 1]$ is surjective.

5. Prove that there is no smallest ($\subseteq$ - minimal) base for the usual topology on $\mathbb{R}$.

6. Prove that if $e : X \to Y$ is an embedding and $h : Y \to Z$ is a homeomorphism, then $h \circ e$ is an embedding.

7. Give an example of two topologies σ and τ on the set of integers $\mathbb{Z}$ for which $\sigma \subsetneq \tau$ and $(\mathbb{Z}, \sigma)$ is homeomorphic to $(\mathbb{Z}, \tau)$.

8. (a) Prove that the product of two connected spaces is connected.

(b) Is the product of uncountably many connected spaces necessarily connected?

9. Prove that if $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are quotient maps, then $X \xrightarrow{g \cdot f} Z$ is a quotient map.

10. Show that $\mathbb{Q} \times \mathbb{Q} \cap S^1$ is dense in S^1.

11. Prove that if A is a compact subset of a regular (non Hausdorff) space X, then A^- is compact.

12. Prove that any uncountable subset of $\mathbb{R}$ has a limit point.

13. Give statements of the following theorems:

(a) Urysohn's Lemma.

(b) Urysohn's Metrization Theorem.

(c) Tychonoff's Theorem.

(d) Baire Category Theorem.

14. Prove that the circle (S^1) is not a retract of the plane ($\mathbb{R}^2$).

15. Prove or disprove that every separable space is second countable.

16. Prove that the following are equivalent for any topological space Y:

(a) Y is Hausdorff.

(b) The diagonal $\Delta_Y = \{(y, y)|y \in Y\}$ is closed subset of $Y \times Y$.

(c) For each space Z and each pair of continuous functions $f, g : Z \to Y$ that agree on a dense subset of Z, it follows that $f = g$.