Geometry/Topology Qualifying Exam
Fall 2007

Do as many problems as you can in the time permitted. Make your calculations and proofs as complete as possible.

1. (a) Define what it means for a topological space to be compact.
   (b) Prove that a closed subspace of a compact space is compact.
   (c) State Tychonoff's Theorem
   (d) Prove Tychonoff's Theorem in the special case of a finite family of spaces.
2. Prove that any closed subset of a metrizable space is a countable intersection of open sets.
3. Use the Meyer-Vietoris Theorem to prove that $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$.
4. Given a group G, describe with proof the construction of a CW complex X_G such that $\pi_1(X_G) \cong G$.
5. Prove that for all $n \in \mathbb{Z}$, there exists a 4-manifold with $\chi(M) = n$.
6. Consider the differential form

$$\eta = dx \wedge dy + 2x dx \wedge dz$$

   on $\mathbb{R}^3$.

   (a) Is η exact?
   (b) Use Stokes theorem to find $\int_X \eta$, where X is the surface

$$\{(x, y, z) | x^2/9 + y^2/4 + z^2 = 1, x \geq 0\}$$

7. Using the definition of the Euler characteristic of an compact orientable smooth manifold X as the self-intersection number of the diagonal $\Delta \subset X \times X$,

   (a) Prove that the Euler characteristic does not depend on the choice of orientation.
   (b) Prove that the Euler characteristic of an odd-dimensional compact orientable smooth manifold is zero.

8. Prove the General Position Lemma: If X and Y are submanifolds or $\mathbb{R}^n$, then for almost every $\alpha \in \mathbb{R}^n$, the translate $X + \alpha$ intersects Y transversely.