Complex Analysis Qualifying Exam
Fall 1991

$\mathbb{R}$ is the set of real numbers, $\mathbb{C}$, the set of complex numbers $D(a,\rho) := \{z \in \mathbb{C} : |z-a| < \rho\}$ for any $a \in \mathbb{C}, 0 < \rho \in \mathbb{R}, \mathbb{D} := D(0,1)$.

1. f is non-constant and holomorphic in the region Ω and $|f|$ assumes a minimum over Ω at z_0. Show that $f(z_0) = 0$.

2. f is holomorphic in $D(0,R)$ and bounded by $M < \infty$. Let $\sum_{n=0}^{\infty} c_n z^n$ be its Taylor series at 0. In terms of N, M and r estimate

$$\left| f(z) - \sum_{n=0}^{N} c_n z^n \right|$$

in $D(0,r)$, where $r < R$.

HINT: "Cauchy estimates" on the coefficients.

3. Let C be the unit circle $\partial\mathbb{D}$ parameterized in the customary way. Show that

$$\int_C \frac{e^{\pi z}}{4z^2+1} dz = \pi i.$$

4. Name four important 19th-century analysts and give a complete statement of one theorem due to each.

5. $S := \mathbb{R} \times]-1,1[$ and f is holomorphic and bounded in S. Suppose $\lim_{x\to+\infty} f(x) = 0$. Show that then $\lim_{x\ \to+\infty} f(x+iy) = 0$ for every $y \in\ ]-1,1[$.

HINT: For any sequence of real numbers $x_n \to +\infty$, the functions $f_n(z) := f(x_n + z)$ constitute a bounded sequence on S which converges to 0 pointwise on $\mathbb{R}$.

6. f is holomorphic and zero-free in the region Ω. Show how to construct a holomorphic logarithm for f, that is, a holomorphic function g in Ω such that $f = e^g$. Are any further hypotheses needed on Ω to accomplish this?

7. f is holomorphic in the annulus $A := \mathbb{D}\backslash\{0\}$ and satisfies $|f(z)| < |z|^{\pi/2}$ for all $z \in A$. Show that $|f(1/2)| \le 1/4$.

HINT: First see if the function $g(z) := f(z)/z$ can be holomorphically extended into $\mathbb{D}$. What will its value at 0 have to be?

8. f is continuous on $\overline{\mathbb{D}}$ holomorphic in $\mathbb{D}$ and vanishes on some arc of positive length on $\partial\mathbb{D}$. Show that $f = 0$ throughout $\mathbb{D}$.

9. f_n are holomorphic in the open subset U of $\mathbb{C}$ and $f_n \to f$ uniformly on each compact subset of U. Must f be holomorphic in U? If U is an open subset of $\mathbb{R}$ and the f_n are differentiable in U, must f be differentiable in U?