Differential Equations Qualifying Exam
Fall, 2002 - Kapitanski & Korten
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1. Find the solution of the equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

such that $u(x,y,z) = xy$ on the plane $\{z = 0\}$.

2. Give an example of a function in some plane domain $\Omega \subset \mathbb{R}^2$ which belongs to $W^{1,2}(\Omega)$, but does not belong to $W^{2,2}(\Omega)$.

3. Find all distributional solutions of the equation $x^2\dfrac{d^2y(x)}{dx^2} = 0$.

4. Let $u(x,t)$ be a sufficiently smooth solution of the problem

$$\frac{1}{1+t^2}u_t - \Delta u = e^u$$

in some region of space-time $\mathbb{R}^n \times \mathbb{R}$ containing the cylinder

$$Q = \{(x,t) \,|\, |x| \le 1,\ 0 \le t \le 2\}.$$

Show that the minimum of $u(x,t)$ in Q can be attained only on the set

$$\Sigma = \{(x,t) \,|\, |x| \le 1 \text{ and } t = 0,\ \text{or } |x| = 1 \text{ and } 0 < t \le 2\}.$$

5. Let $u(x,t)$ be a 1-periodic in x finite energy solution of the equation

$$\frac{\partial^2 u}{\partial t^2} + \mu \cdot (1+t^2)\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,$$

where μ is a positive constant. [Recall, that "finite energy" implies that $u(x,t)$ is a continuous function of t with values in $W^{1,2}([0,1])$ and $u_t(x,t)$ is a continuous function of t with values in $L^2([0,1])$.]

Derive an energy estimate and use it to prove that

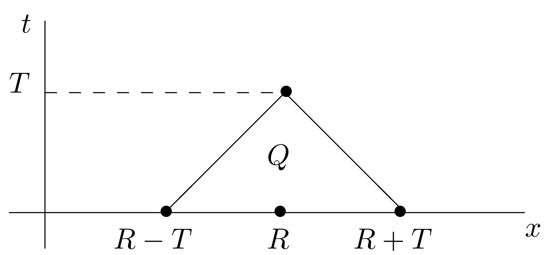
$$\int_0^1 |u_t(x,t)|^2\,dx \;\to\; 0$$

as $t \to +\infty$.

6. Consider the problem

$$\begin{cases} u_{tt} - u_{xx} + (1+t^2)u = h(x,t), & -\infty < x < +\infty \\ u(0,x) = 0,\;\; u_t(o,x) = 0. \end{cases}$$

Show that if $h(x,t) = 0$ inside the right triangle Q,



then $u(x,t) = 0$ in Q.