# ANALYSIS QUALIFYING EXAM - SPRING 2014

BURCKEL & HAKOBYAN

Notation:
$\mathbb{N} := \{1, 2, \cdots\}$, $\mathbb{R} :=$ reals, $\mathbb{C} :=$ complex numbers.
$\Omega :=$ a non-empty, open, connected subset of $\mathbb{C}$.
$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $H(\Omega) :=$ all holomorphic (= analytic) functions on $\Omega$.
Everywhere below, $X$ will denote a measurable space and $\mu$ a positive measure on $X$.

Name: ______________________________________________

**Problem 1.** For $a > 1$ find

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}.$$

Hint: Express cosine in terms of the exponential function.

**Problem 2.** What is the largest region $\Omega$ in which the series

$$\sum_{n=1}^{\infty} \frac{z^2}{n^2 z^2 + 4}$$

converges locally uniformly.
Hint: First show that convergence occurs at each $z \notin \{\pm\frac{2i}{n} : n \in \mathbb{N}\}$.

**Problem 3.** $f_n \in H(D)$, $f_n(z) \to 0$ for all $z \in \mathbb{D}$ and for some $C \in \mathbb{N}$, $|f_n| \le C$ for all $n \in \mathbb{N}$.
*(i)* Show that $f_n \to 0$ uniformly in $r\mathbb{D}$ for each $r < 1$.
*(ii)* Show that $f_n \to 0$ *uniformly* in $\mathbb{D}$ need not hold.

**Problem 4.** Is there an $f \in H(\mathbb{C})$ satisfying $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3}$ for all $n \in \mathbb{N}$? Justify your answer.

**Problem 5.**
*(i)* Prove that there there is no continuous logarithm in $\mathbb{C} \setminus \{0\}$.
*(ii)* For $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ define

$$L(re^{i\theta}) := \log r + i\theta$$

and prove that $L$ is a continuous function of $(r, \theta)$ satisfying $e^{L(z)} = z$ for all $z \neq 0$.
*(iii)* Explain why *(i)* does not contradict *(ii)*.

**Problem 6.** Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable sets in $X$, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

Then $\mu$ - almost every $x \in X$ belongs to at most finitely many of the sets $E_k$.
Hint: Consider the collection of points $x$ which belong to $E_k$ for infinitely many $k$'s.

**Problem 7.**
*(i)* Define the Borel $\sigma$-algebra of $\mathbb{R}$.
*(ii)* Is there a Lebesgue measurable set in $\mathbb{R}$ which is not Borel measurable? Justify your answer.

**Problem 8.**
Suppose
- $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\mu)$, and
- $\lim_{n\to\infty} f_n(x) = f(x)$, for $\mu$ a.e. $x \in X$

Show that $f \in L^p(\mu)$ and $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^p(\mu)$.

**Problem 9.** Show that $L^p([0,1])$ is separable if $1 \leq p < \infty$, but $L^\infty([0,1])$ is not separable. The underlying measure here is the Lebesgue measure.

**Problem 10.** Is there a measurable set $E \subset \mathbb{R}^2$ such that for every $x \in \mathbb{R}^2$ and every $r > 0$

$$\frac{1}{3}\pi r^2 \leq m(E \cap B(x,r)) \leq \frac{3}{4}\pi r^2?$$

Justify your answer. Here $m$ is the Lebesgue measure in $\mathbb{R}^2$.