Name ______________________

TOPOLOGY QUALIFYING EXAM
Spring 1999
(Maginnis and Strecker)

Choose and work any 6 of the following 15 problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a space always means a topological space.

1. Let $f : X \to Y$ be any function, let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$. Prove or disprove each of the following:

(a) $f[\bigcap \mathcal{A}] = \bigcap\{f[A] | A \in \mathcal{A}\}$

(b) $f^{-1}[\bigcap \mathcal{B}] = \bigcap\{f^{-1}[B] | B \in \mathcal{B}\}$

2. (a) State the Axiom of Choice.

(b) State the Well-Ordering Theorem.

(c) Either use the Axiom of Choice to prove the Well-Ordering Theorem or use the Well-Ordering Theorem to prove the Axiom of Choice.

3. Let X be a well-ordered set in the order topology, and assume X has a maximal element. Prove X is a compact space.

4. Prove that every closed subset of a metrizable space is a countable intersection of open sets.

5. Let X be a connected and locally path connected space. Prove X is path connected.

6. Let I denote the closed unit interval $[0, 1]$ in $\mathbb{R}$ (the real numbers with its usual topology). Prove or disprove that in $\mathbb{R}^{\mathbb{R}}$ with its usual product topology $(I^\circ)^{\mathbb{R}} = (I^{\mathbb{R}})^\circ$, (where A° denotes the interior of A).

7. Let $X \xrightarrow{f} Y$ be any continuous function.

(a) Prove that there is a factorization of f $X \xrightarrow{f} Y = X \xrightarrow{q} Z \xrightarrow{m} Y$ where q is a quotient map and m is a one-to-one continuous function.

(b) Prove that the factorization $f = m \circ q$ of part (a) is essentially unique in the sense that if $X \xrightarrow{f} Y = X \xrightarrow{\widehat{q}} \widehat{Z} \xrightarrow{\widehat{m}} Y$ is also a factorization of f with $\widehat{q}$ a quotient map and $\widehat{m}$ one-to-one and continuous, then there is a unique homeomorphism $Z \xrightarrow{h} \widehat{Z}$ such that $h \circ q = \widehat{q}$ and $\widehat{m} \circ h = m$.

8. Prove that a filter F on a space X converges to a point $x \in X$ if and only if the net f based on F converges to x.

9. Let $(X, \leq)$ be a linearly ordered set and let τ be the topology on X inherited from the order. If A is a subset of X, there are two natural ways to topologize A. The first, τ_1, is as a subspace of (X, τ). The second, τ_2, is as an ordered space with the order on A inherited from $(X, \leq)$. Prove or disprove that in all cases $(A, \tau_1) = (A, \tau_2)$.

10. Let A be a connected subset of a connected space X, and let C be a component of $X - A$. Prove $X - C$ is connected.

11. Prove that a Hausdorff space with a basis consisting of sets that are both open and closed is totally disconnected.

12. Prove or disprove.

(a) Every quotient map is an open map.

(b) Every open map is a quotient map.

13. Prove that a compact subset of a Hausdorff space is closed.

14. Find a flaw in the following purported proof of the statement:

Every discrete subspace of a topological space that has no isolated points is nowhere dense.

Purported proof:

Suppose X is a space with no isolated points and D is a discrete subspace of X. If D is not nowhere dense, then there is a nonempty open set U with $U \subseteq \overline{D}$. Thus there is some $d \in D \bigcap U$. Since D is discrete, there is some open set $W \subseteq U$ such that $W \bigcap D = \{d\}$. Since X has no isolated points $W \neq \{d\}$. Thus $W \setminus \{d\}$ is a nonempty open set contained in $\overline{D}$. Thus there is some $\widehat{d} \in (W \setminus \{d\}) \bigcap D \subseteq W \bigcap D = \{d\}$, which is a contradiction. Thus D must be nowhere dense. $\square$

15. Assume X is a normal space. Let $\beta(X)$ be the Čech-Stone compactification of X, and let $y \in \beta(X) - X$. Prove that y is not the limit of a sequence of points of X.