Algebra qualifying exam
Fall 2013

Guidelines: Complete as many problems as time allows, being certain to attempt at least one problem from each of the four sections. Begin each problem on a new page. Put your name on each page you submit and order them by question number when turning in the examination.

I. Finite groups.

1. Let $\mathbb{F}_3 \cong \mathbb{Z}/3\mathbb{Z}$ be the field of three elements. Consider the set S of 4 elements consisting of non-zero pairs (a, b) with $a, b \in \mathbb{F}_3$ defined up to a common multiple. The group $G = \mathrm{GL}_2(\mathbb{F}_3)$ of invertible 2×2 matrices with elements in the field $\mathbb{F}_3$ acts on S by left multiplication. Thus we have a group homomorphism

$$\phi : G \to S_4.$$

(a) What is the order of G?
(b) Identify the image and the kernel of the map ϕ.
(c) List all normal subgroups of G.

2. Let $p < q$ be two prime numbers. Let $N(p, q)$ be the number of non-isomorphic groups of order pq.
(a) What can you say about the number $N(p, q)$?
(b) List all groups up to isomorphism of order 15.

II. Rings and modules.

3. Let R be a commutative ring with 1. Let M, N, L be R-modules. Show that there is a natural isomorphism

$$\mathrm{Hom}(M \otimes N, L) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, L))$$

of $\mathbb{R}$-modules.

4. Recall that an element in a number field is called an *integer* if it is a root of a *monic* polynomial with integer coefficients.

(a) Let R be the ring of integers in the field $\mathbb{Q}(\sqrt{-19})$. Show that R is a principal ideal domain.

(b) Let S be the ring of integers in the field $\mathbb{Q}(\sqrt{-5})$. Show that S is *not* a principal ideal domain.

III. Field extensions.

5. Find the Galois group of the polynomial $x^5 - 2$ over $\mathbb{Q}$ and draw the lattice of subfields.

6. Let K be a finite extension of $\mathbb{Q}$. Show that there is an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. Find such an element for the field $K = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.

IV. Linear algebra.

7. Suppose ϕ is an endomorphism of a 10-dimensional vector space over $\mathbb{C}$ with the following properties.

(a) The characteristic polynomial of ϕ is $(x-2)^4(x^2-3)^3$.

(b) The minimal polynomial of ϕ is $(x-2)^2(x^2-3)^2$.

(c) The endomorphism $\phi - 2I$, where I is the identity map, is of rank 8.

Find the Jordan canonical form for ϕ.

8. Suppose ϕ is an endomorphism of a finite dimensional vector space V over $\mathbb{Q}$ such that $\phi^2 = \phi$. Prove that

(a) $\operatorname{Im} \phi \cap \ker \phi = 0$.

(b) $V = \operatorname{Im} \phi \oplus \ker \phi$.

(c) There is a basis of V in which the matrix of ϕ is diagonal with entries 0 or 1.

Which of the above remain true if we don't assume that V is finite dimensional?