PDE Qualifying Exam - Fall 2004 (Old System)

1. A function $v \in C^2(\Omega)$ is said to be *subharmonic* in Ω if $\Delta v \geq 0$ in Ω.

   Show that if $x_0 \in \Omega$ and $r < d(x_0, \partial\Omega)$, then

$$v(x_0) \leq |B(x_0,r)|^{-1} \int_{B(x_0,r)} v(\xi)\, d\xi.$$

2. Solve by separation of variables

$$\begin{cases} u_t = u_{xx}, & \text{in } (0,1) \times (0,\infty) \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = u_0(x), & 0 < x < 1, \end{cases}$$

   where $u_0 \in L^2((0,1))$. Verify that u is a classical solution, and that $u(x,t) \to u_0$ in $L^2((0,1))$ when $t \to 0$. In addition, $u(x,t) \to \int_0^1 u_0(x)\, dx$ uniformly when $t \to \infty$. Give a physical interpretation of this fact.

3. Use the Fourier transform to solve the problem

$$\begin{cases} u_{tt} - u_{xx} = h(x,t), & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_t(x,0) = 0 & \text{in } \mathbb{R}. \end{cases}$$

4. Suppose that $y, z \in C^\infty[-1,1]$ and

$$\begin{gathered} (x^2+1)y'' + 2xy' + 3y = 0 \\ y(-1) = 0 = y(1) \end{gathered} \qquad\qquad \begin{gathered} (x^2+1)z'' + 2xz' + 4z = 0 \\ z(-1) = 0 = z(1) \end{gathered}$$

   Show that

$$\int_{-1}^{1} y(x)z(x)\, dx = 0.$$

5. Solve the initial value problem

$$\begin{gathered} yu_x + u_y = x \\ u(x,0) = x^2 \end{gathered}$$

6. Given $\dfrac{dx}{dt} = \cos(x - t)$, $x(0) = 1$, show $x(t) > t$ for all $t > 0$.