NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 1998

(do at lest 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Explain the difficulty in numerical evaluation of the error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ for large positive x using the Maclaurin series of $\mathrm{erf}(x)$, and suggest an alternative way to overcome the difficulty.

2. State and prove a theorem about the order of convergence of Newton's method for finding roots of a function of one variable.

3. Show that there exists $-1 \leq x_1 \leq \cdots \leq x_n \leq 1$ and $w_1, \ldots, w_n$ so that

$$\sum_{i=1}^{n} w_i p(x_i) = \int_{-1}^{1} p(x) dx$$

for all polynomials $p(x)$ with $\deg(p) \leq 2n - 1$.

4. (a) Derive the numerical differentiation formula

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi).$$

(b) Consider the effect of rounding errors and the truncation error in the formula in item (a) for estimating f''. If $|f^{(4)}(x)| \leq M_4$ and each evaluation of f is in error by at most $\frac{1}{2} \times 10^{-k}$, show that a choice of h near to

$$h = \left(\frac{24 \times 10^{-k}}{M_4} \right)^{1/4}$$

may be expected to minimize the total error.

5. Let A be an m by m invertible matrix. Consider a factorization of the form $A = QHQ^*$ where Q is a unitary matrix and H is a Hessenberg matrix. Show that the columns of Q are uniquely determined up to complex unit factors if its first column is determined up to a complex unit factor.

6. Suppose an m by m Hessenberg matrix A is normal: $A^*A = AA^*$, and suppose $a_{j+1,j} \neq 0$ for $j = 1, \ldots, m-1$. Show that the eigenvalues of of A are all distinct.

7. Show that Gaussian elimination with partial pivoting applied to any invertible matrix A generates an LU decomposition of the matrix with permuted columns, i.e.,

$$PA = LU$$

where P is a permutation matrix, L is a lower-triangular matrix whose diagonal entries are all ones, and U is an upper-triangular matrix.

8. Prove the following theorem:

   Let $A \in \mathcal{C}^{m\times m}$ be a hermitian positive definite matrix. The solution of the systems $Ax = b$ via Cholesky factorization is backward stable, generating a computed solution $\tilde{x}$ that satisfies

$$(A + \Delta A)\tilde{x} = b$$

   for some m by m matrix $\Delta A \in \mathcal{C}^{m\times m}$ that satisfies

$$\frac{\|\Delta A\|}{\|A\|} = O(u)$$

   where u stands for the computer unit round. (**Hint:** You can use any theorems on the backward stability of Cholesky factorization and back substitutions.)