Name ________________________

GEOMETRY OF MANIFOLDS QUALIFYING EXAM
Spring 1999
(Miller and Auckly)

WORK THE FIRST FOUR AND AS MANY OF THE OTHERS AS YOU CAN.

1. On $\mathbb{R}^3$ with coordinates (x, y, z), let

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$Y = x^2y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

$$\alpha = (x^3 + y^3 + z^3)(dx \wedge dy + dz \wedge dx + dy \wedge dz)$$

a) calculate the Lie bracket $[X, Y]$.

b) describe the flow of the vector field Y through the point (x_0, y_0, z_0).

c) calculate $d\alpha$ as a function times the usual volume element.

d) if $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(s, t) = (s, t, s^2 + t^2)$, compute $f^*\alpha$.

2. On $\{(x, y)|x, y \in \mathbb{R}, y > 0\}$ let g be the metric $g = y\,dx^2 + dy^2$. Determine the differential equations for parallel transport of a vector field $\xi = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y}$ along a curve $x = x(t)$, $y = y(t)$.

3. Let $C = \{(x, y, z)|x^2 + y^2 = 1\}$ with the orientation $\Omega_C = x\,dy \wedge dz - y\,dx \wedge dz$. Compute

$$\int_C \frac{(x+1)}{(1+z^2)(x^2+y^2)}(x\,dy \wedge dz - y\,dx \wedge dz)$$

4. Construct a cell complex X, with

$\pi_1(X) = \langle a, b | a^2 = b^3 \rangle$

$H_0(X) = Z$ and $H_2(X) = Z \oplus Z_2$.

5. On a Riemannian manifold define the scalar curvature to be $S = -\sum_{n,k} g(R(e_n, e_k)e_n, e_k)$ where (e_n) is an orthonormal basis.

a) Prove that S is independent of the choice of basis.

b) Let $S_r^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$, with the induced metric. Compute the scalar curvature of $S_a^2 \times S_b^3$.

6. Let M be a simply connected manifold, ω a closed differential 2-form, X a vector field and H a smooth real valued function on M. Suppose they are related by $dH = i_X\omega = \omega(X, \quad)$. Further suppose that Y is a second vector field such that $L_Y\omega = 0$ and $Y(H) = 0$.

a) Show that there is a smooth function f on M such that $i_Y\omega = df$.

b) Show that the function f of part a) is constant along the flows of X.

Note: If you do not remember the formula giving the Lie derivative L_Y acting on differential forms in terms of i_Y and d, you may ask at the cost of a penalty.

7. Let G be a Lie group and $\pi_G : P \to M$ be a principal G-bundle, and H be a closed subgroup of G. We say that the structural group of P may be reduced to H if and only if there is a principal H-bundle $\pi_H : Q \to M$ and a bundle map $i : Q \to P$ so that $i(q \cdot h) = i(q) \cdot h$. Let $E = P \times_G (G/H) = P \times (G/H)/\sim$ where $(p, [g]) \sim (pk, [k^{-1}g])$, $p \in P, g, k \in G$. Prove that $E \to M$ admits a global section if and only if the structural group of P may be reduced to H.

8. On $\mathbb{R}^3$ with coordinates (x, y, z) let $\alpha = x\,dy + dz$ and $E = \{v \in T\mathbb{R}^3 | \alpha(v) = 0\}$.

Prove or disprove:

a) There is a codimension 2 foiliation $\mathcal{F}$ of $\mathbb{R}^3$ so that any leaf N of $\mathcal{F}$ satisfies $TN \subseteq E \mid N$.

b) There is a codimension 1 foiliation $\mathcal{F}$ of $\mathbb{R}^3$ so that any leaf of $\mathcal{F}$ satisfies $TN \subseteq E \mid N$.

9. Let $S0_3 = \{A : \mathbb{R}^3 \to \mathbb{R}^3 | A \text{ is linear, } A^*A = I \text{ and } \det A = 1\}$. Prove that

$$\{A \in S0_3 \mid A^* = A, A \neq I\}$$

is a compact manifold. What is its dimension?