Algebra Qualifying Exam
Fall 1994

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let P be a finite p-group for a given prime number p. For any $x \in P$ show that either $\langle x \rangle$ is a normal subgroup of P, or there exists $g \in P$ such that $[x, gxg^{-1}] = 1$ and $x \neq gxg^{-1}$.

2. Let G be a finite groups and p a prime number dividing the order of G. $Syl_p(G)$ is the set of all Sylow p-subgroups of G.

(a) Show that $N_P(Q) = P \cap Q$ for $P, Q \in Syl_p(G)$. Here $N_P(Q) = \{x \in P | xQx^{-1} = Q\}$ is the normalizer of Q in P.

(b) Show that is there is $P \in Syl_p(G)$ such that $P \cap xPx^{-1} = P$ or $\{1\}$ for any $x \in G$, then $|Syl_p(G)| \equiv 1 \pmod{|P|}$.

3. Prove that every prime ideal in a PID is maximal. Then give an example of an integral domain, in which there is a prime ideal which is not maximal.

4. Recall that a ring is called Noetherian if every ascending chain of ideals terminates. Show that any PID is Noetherian. Give an example of a Noetherian integral domain which is not a PID.

5. Let F by any field and $f(x) \in F[x]$. If E is the splitting field of $f(x)$ over F, show that $[E : F] \leq (\deg(f(x)))!$. Give an example for which the equality holds.

6. Let $\alpha = \sqrt{3} + \sqrt[3]{2}$. It is known that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 6$.

(a) Calculate the minimal polynomial of α over the field $\mathbb{Q}$ of rational numbers.

(b) Is the field $\mathbb{Q}(\alpha)$ Galois over $\mathbb{Q}$? If yes, determine the Galois group. If not, justify your answer.

7. Let R be a ring. If M is a left R-module, show that for any R-submodule N of M, the set $\{x \in R | xM \subseteq N\}$ is a two-sided ideal of R.

8. Let R be a ring. Prove Schur's Lemma: For any two simple left R-modules M and N, any R-module homomorphism $\phi : M \to N$ is either identically zero or an isomorphism.

9. Let F by any fixed field and V a vector space over F (of any dimension). Suppose that $A : V \to V$ is a nilpotent linear transformation (i.e., $A^n = 0$, for some n). Show that A has at least one eigenvector with eigenvalue in F.

10. Let $A = \begin{bmatrix} 5 & 1 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ be a 3×3 matrix. Show that there exist complex matrices D and N such that D is diagonalizable, N is nilpotenet, $DN = ND$, and $A = D + N$.