Algebra Qualifying Exam
Spring 1994

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a finite group and p is a prime number. Define $G(p) = \{g \in G | o(g) = p^n \text{ for some } n\}$.

(a) Show that $G(p)$ is the union of all Sylow p-subgroups of G.

(b) Show that $G(p)$ is a subgroup if and only if G has a normal Sylow p-subgroup.

2. Show that if G is a finite p-group, then for any diviser d of $|G|$, G has a normal subgroup of order d.

3. Prove or disprove the following statements:

(a) An ideal I of a commutative ring R with 1 is maximal if and only if R/I is a field.

(b) An ideal I of a ring R with 1 is maximal if and only if R/I is a division ring.

4. A commutative ring R with 1 is called *local* if R has only one maximal ideal m. Show that in this case, the maximal ideal m is precisely the set of all non-units in R. Is it true in general that for any commutative ring the set of all non-units is an ideal?

5. Let R be a ring with 1. An element $e \in R$ is called a central idempotent if $e^2 = e$ and e is in the center of the ring R.

(a) Give an example of a ring R having a central idempotent different from 0 and 1.

(b) Let $e \in R$ be a central idempotent show that for any unitary R-module M, both eM and $(1-e)M$ are R-submodules of M and that $M = eM \oplus (1-e)M$.

6. Let V be an n-dimensional vector space over a field F and T: $V \to V$ be a linear transformation. Set $P = \{x \in V | Tx = x\}$ to be the subspace of T-fixed points and assume that $T(V) \subseteq P$. Calculate the characteristic polynomial and minimal polynomial of T in terms of n and $k = \dim \ker(T)$. Can T be diagonalized?

7. For V a vector space over the field F, let V^* denote the dual space of V, that is, V^* is the vector space $Hom_F(V, F)$ of all linear transformations $\lambda : V \to F$. If V is n-dimensional with a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$, define elements $\lambda_1, \ldots, \lambda_n$ of V^* by setting

$$\lambda_i \left(\sum_{j=1}^{n} a_j x_j \right) = a_i,$$

$1 \leq i \leq n, a_j \in F$, and put $\mathcal{B}^* = \{\lambda_1, \ldots, \lambda_n\}$.

(a) Show that $\mathcal{B}^*$ is a basis of V^*.

(b) If V is infinite dimensional with a basis $\{e_1, e_2, \ldots, e_n, \ldots\}$ and if the λ_i's are defined similarly as above for $i = 1, 2, \ldots$, prove or disprove the statement that $\{\lambda_1, \lambda_2, \ldots\}$ is a basis for V^*.

8. Give an example of a normal field extension which is not Galois.

9. Prove that any finite extension of degree n over a finite field is Galois. What is the Galois group?