Algebra Qualifying Exam: January 24, 2014

Instructions: This exam consists of nine problems; answer as many as you can. Show all work for each problem. Here $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ is the rational numbers, and $\mathbb{R}$ is the real numbers.

1. Let G be a group of order $600 = 2^3 \times 3 \times 5^2$. Prove G is not a simple group.

2. Let G be the multiplicative group $GL_4(\mathbb{R})$ of invertible 4×4 matrices with entries in the real numbers. Find a representative from each conjugacy class of elements of order 4 in G, and find the characteristic polynomial for each of these representatives.

3. Prove that the polynomial $f(x) = x^7 - 10x^2 + 5$ is irreducible when the ring of coefficients is the Gaussian integers $\mathbb{Z}[i] = \{\ a + bi \mid a, b \in \mathbb{Z}\ \}$.

4. Let $I \subseteq R$ be an ideal of a commutative ring R with a multiplicative identity $1 \in R$. Assume that I is a free R-module. Prove that I is a principal ideal which is generated by an element which is not a zero divisor.

5. Let $R = (\mathbb{Z}/3\mathbb{Z})[x]$ be the ring of polynomials with coefficients in the integers modulo 3, and let $I = (x^5 + x^3)$ be the ideal of R generated by the polynomial $f(x) = x^5 + x^3$.

   (a) Prove that the quotient ring R/I is finite, and determine its cardinality.

   (b) Determine all of the ideals of the quotient ring R/I, and also determine which of these ideals are prime ideals.

6. Let F be the splitting field of the polynomial $f(x) = x^4 - 2x^2 - 1$ over the field of rational numbers $\mathbb{Q}$.

   (a) Find the dimension of F as a vector space over $\mathbb{Q}$.

   (b) Determine the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$.

   (c) Show that f(x) is irreducible over $\mathbb{Q}$. (One method uses part (b).)

7. Let $U \subseteq V$ be a vector subspace of a vector space V over a field F. Use Zorn's Lemma (or another equivalent version of the Axiom of Choice such as the Well Ordering Principle) to prove that there exists a subspace $W \subseteq V$ such that V is the internal direct sum $V = U \oplus W$. (Note there are other methods to prove this, but you will only receive full credit for an explicit use of Zorn's Lemma, or some version of the Axiom of Choice or Well Ordering Principle.)

8. Let $T : V \to W$ be a linear transformation of the two vector spaces over a field $\mathbb{F}$. Recall that the dual map $T^* : W^* \to V^*$ between the dual spaces is defined by $T^*(f) = f \circ T$, for a linear functional $f : W \to \mathbb{F}$.

   (a) If T is surjective, prove the dual map T^* is injective.

   (b) If T is injective, prove the dual map T^* is surjective. (One method uses problem 7.)

9. Let N be a normal subgroup of finite index in a group G. Let H be a finite subgroup of G. Assume that the index $[G : N]$ is relatively prime to the order of H. Prove that $H \subseteq N$.