GEOMETRY OF MANIFOLDS QUALIFYING EXAM

Spring 2003

(Auckly & Vidussi)

1.a) State the definition of a Lie algebra.

b) Let $ad : \mathbf{g} \to \mathrm{End}(\mathbf{g})$; $ad(X)(Y) = [X,Y]$ be the adjoint representation. Prove that

$$Tr(ad([X,Y])ad(Z)) = Tr(ad([Z,X])ad(Y)).$$

2. Let $X = S^1 \times I/\sim$ with $(z,0) \sim (z^3,1) \quad \forall z \in S^1$.

a) Construct a CW decomposition of X.

b) Compute $\pi_1(X)$.

c) Compute $H_*(X)$.

3. Let ∇ be the Levi-Civita connection on a Riemannian manifold. Define $Hf(X,Y) = X(Yf) - (\nabla_X Y)f$.

a) Prove that Hf is symmetric i.e., $Hf(X,Y) = Hf(Y,X)$.

b) Prove that Hf is tensorial i.e., $Hf(\varphi X, \psi Y) = \varphi\psi Hf(X,Y)$ for $\varphi, \psi \in C^\infty(M)$.

4. Let: $X = x^2 y \partial_x - \partial_z$, $Y = xy^2 \partial_y - \partial_z$, $Z = (1+x^2)\partial_x - y(1+x^2)\partial_z$.

a) Find the integral curves of X.

b) Define what it means for a distribution to be integrable at a point.

c) Let E be the distribution spanned by X and Y, and let F be the distribution spanned by Y and Z. Test both distributions for integrability near the point $(1,2,3)$.

5. Compute $\displaystyle\int_{S^2} x^2 z\,dxdy$ where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with orientation $dx \wedge dy$ at $(0, 0, 1)$.

6. Prove that $T^*(S^2 \times S^2)$ and $\mathbb{R}^8$ are not homeomorphic.

7. Let $W = \{x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^3 + z^4 + w^5 = 6\}$.

a) Prove that W is a manifold.

b) What is $\dim_{\mathbb{R}} W$?

8.a) Prove that a real line bundle is trivial if and only if it admits a global nonvanishing section.

b) Prove that the vector bundle $E \otimes E^*$ (where E is any vector bundle over a manifold M) is orientable.

9. Compute the sectional curvature of the metric $g = \frac{1}{y^4}(dx \otimes dx + dy \otimes dy)$.