Algebra Qualifying exam
January 2010

February 7, 2011

Instructions: You are given ten problems from which you are supposed to do eight. Please, indicate those eight problems which you whould like to be graded by circling the problem number on the problem sheet.

Note. All rings in this exam are associative, unital and non-zero; and all ring homomorphisms map unit elements to unit elements.

1. Recall that a left module M over an associative ring R is called *simple* if it is nonzero and and has no proper non-zero submodules.

(a) Show that a left R-module M is simple if and only if it is isomorphic to the quotient module R/I for some maximal left ideal I.

(b) Let I_1 and I_2 be maximal left ideals of R. Show that the simple R-modules R/I_1 and R/I_2 are isomorphic to one another if and only if there exists an element $r \in R - I_1$ such that $I_2 = (I_1 : r)$, where

$$(I_1 : r) \stackrel{\text{def}}{=} \{a \in R \mid ar \in I_1\}.$$

2. Let R be an associative ring. Prove that its proper two-sided ideals (that is, those not equal to R itself) are precisely the kernels of ring homomorphisms from R to other associative rings.

3. Is the polynomial $p(x) = 5x^3 + 18x + 12$ irreducible in $\mathbb{Q}[x]$? Explain.

4. Let V and W be vector spaces over a field F. Let A and B be invertible elements of $End_F(V)$ and $End_F(W)$, respectively. Show that $A \otimes_F B$ is an invertible element of $End_F(V \otimes_F V)$.

5. (a) Let S and T be subgroups of the group $\mathbb{Z}$ of integers and φ_+ the group homomorphism

$$S \oplus T \longrightarrow \mathbb{Z}, \quad (x, y) \mapsto x + y.$$

Show that φ_+ is a monomorphism if and only if either S or T is the zero subgroup. In particular, $\mathbb{Z}$ cannot be represented as a direct sum of non-trivial subgroups.

(b) Let R be a *domain* – an associative ring without zero divisors (that is if $a \neq 0 \neq b$, then $ab \neq 0$). Let S and T be two-sided ideals of R and $S \oplus T \xrightarrow{\varphi_+} R$ is defined as in 5(a) above: $(x, y) \mapsto x + y$.

Show that the map φ_+ is a monomorphism if and only if either S or T is a zero ideal.

(c) Is the assertion (a) a particular case of the assertion (b)? Explain.

6. Recall that a left module P over an associative ring R is called *projective* if, for every epimorphism $M \xrightarrow{f} N$ of R-modules, the map

$$Hom_R(P, M) \longrightarrow Hom_R(N, P), \quad \xi \mapsto f \circ \xi,$$

is surjective. Prove the following assertions:

(a) Every free left R-module is projective.

(b) A module P is projective if and only if every epimorphism $M \xrightarrow{f} P$ *splits*; that is there exists a morphism $P \xrightarrow{g} M$ such that the composition $f \circ g$ is the identical map.

(c) Deduce from (a) and (b) that an R-module P is projective if and only if it is a direct summand of a free R-module. That is there exists an R-module L such that $P \oplus L$ is isomorphic to a free R-module.

7. a) Prove that there are no simple groups of order 200.

b) How many elements of order 7 are there in a simple group of order 168?

8. Find the Galois group of $p(x) = x^3 - 2$ over the field $F = \mathbb{Z}_5$.

9. Let A be a linear operator on a finite-dimensional vector space V over a field F. Let W be an invariant subspace of A (by definition, that means that

for any $w \in W$, one has $Aw \in W$). Prove that the characteristic polynomial of the restriction of A to W divides the characteristic polynomial of A.

10. Let F be a field extension of K, such that K is a fixed field of a finite subgroup G of the group of automorphisms of F. Prove that there are only finitely many subfields E of F such that $K \subset E \subset F$.