COMPLEX VARIABLES
QUALIFYING EXAMINATION
Spring 1989

Do as many problems as you can but please label any partial solution as being just partial and indicate what else would need to be done to finish it.

1. Let f be a meromorphic function on the Riemann sphere. Show that the difference of the zeros and the poles of f, counted with multiplicity, is zero.
   (**Hint:** The difference of the zeros and the poles of f is the difference of the zeros and the poles of f in the unit disk plus the difference of the zeros and the poles of $f\left(\frac{1}{z}\right)$ in the unit disk. You may need to consider a disk other than the unit disk for certain functions.)

2. Let $\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$. Show $\zeta(z)$ is analytic in $\mathrm{Re}(z) > 1$.

3. Let Γ denote the unit circle $|z| = 1$ and suppose $r(z)$ is a rational function which has no poles on Γ. Define
$$G(z) := \frac{1}{2\pi i}\int_\Gamma \frac{r(\xi)}{\xi - z}d\xi, \quad |z| < 1$$
$$H(z) := \frac{1}{2\pi i}\int_\Gamma \frac{r(\xi)}{\xi - z}d\xi, \quad |z| > 1.$$
   Show that G and H each extend by continuity to $|z| = 1$ and that then $r = G - H$ on Γ.
   (**Hint:** Use the partial fraction decomposition for r. Taking limits under the integral is not a good idea.)

4. A **Möbius** (or **linear fractional**) transformation is a mapping of the form
$$w = \frac{az + b}{cz + d}$$
   for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Show that Möbius transformations map the family of straight lines and circles into the family of straight lines and circles.
   (**Hint:** Build the general Möbius transformation from appropriate elementary ones.)

5. Suppose $f = u + iv$ is analytic on $|z| < R$. Consider the circle $|z| = c < R$. Relate the normal and tangential derivatives of u and v along this circle. Deduce from the Cauchy Integral Theorem that
$$\int_{|z|=c} \frac{\partial u}{\partial \mathrm{n}}|dz| = 0.$$

6. (i) Suppose $0 < r < R$, F is holomorphic in $D(0,R)$, G is holomorphic in $D(0,1/r)$ with $G(0) = 0$ and that for some $\rho \in]r,R[$ the identity

$$F(z) + G(1/z) = 0$$

holds for all z satisfying $|z| = \rho$. Show that $F = G = 0$.

(ii) Use the result (i) [even if you didn't prove it] to prove the Fundamental Theorem of Algebra. **Hint:** If $P(z) = \sum_{j=0}^{d} a_j z^j$ is a zero-free polynomial over $\mathbb{C}$ of degree $d \geq 1$, then so is $Q(z) := \sum_{j=0}^{d} a_j z^{d-j}$. Look at $F := 1/P$ and $G(z) := -z^d/Q(z)$.

7. (i) Prove the following "factorization theorem":
G, H are open, connected subsets of $\mathbb{C}$, ϕ, g are holomorphic functions in G, $\phi(G) = H$ and g is constant on each fiber $\phi^{-1}(h)$ $(h \in H)$. Then $\exists$ a unique holomorphic function h on H such that $g = h \circ \phi$.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "g"'] & H \arrow[dl, dashed, "h"] \\ \mathbb{C} & \end{tikzcd}$$

Hint: Holomorphy off the set $\{z \in H : \phi^{-1}(z) \subset (\phi')^{-1}(0)\}$ is not too hard to prove, and this is a discrete subset of H.

(ii) Use the result (i) [even if you did not prove it] to show that if g is holomorphic and periodic with period 1 in the strip $G := \mathbb{R} \times]a,b[$ $(a, b \in \mathbb{R}, a < b)$, then $g(z) = h(e^{2\pi i z})$ for a function h which is holomorphic in an appropriate annulus.

(iii) Use the result (ii) [even if you did not prove it] to infer that $\exists$ unique $c_n \in \mathbb{C}$ such that

$$g(z) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n z} \qquad \forall z \in G,$$

with convergence uniform in $\mathbb{R} \times [a', b']$ whenever $a < a' < b' < b$. Show in fact that

$$c_n = \int_{[z_0, z_0+1]} g(\zeta) e^{-2\pi i n \zeta} d\zeta \qquad \forall n \in \mathbb{Z}$$

and any $z_0 \in G$.

8. Show that

$$|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} \left(\int_{-\pi}^{\pi} \left| f\left(e^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} \right)^{1/2} \qquad \forall z \in D(0,1)$$

whenever f is holomorphic in a neighborhood of $\overline{D}(0,1)$. Can the hypothesis be weakened to f continuous on $\overline{D}(0,1)$ and holomorphic in $D(0,1)$?