Topology Qualifying Examination

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There are eight questions. Attempt to solve as many as time allows. You have been supplied with an adequate supply of paper. Please begin each question on a new page. Put your name on all pages you submit and order them by question number when turning in the examination.

Problem 1. Let X be a topological space. Let $\Delta \subset X \times X$ be the diagonal (that is, $\Delta := \{(x, x) \mid x \in X\}$). Prove that X is Hausdorff if and only if Δ is closed in $X \times X$.

Problem 2. Let X be the one-point topological space.

(a) Let A be an abelian group. Calculate the cohomology groups $H^n(X, A)$ for $n \geq 0$ *using only the definition of singular cohomology* .

(b) Now, let A be a commutative ring. Calculate the multiplication in the cohomology ring.

Problem 3. Let $T := \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Let a, b, c, and d be integers. Define the map $\varphi : T \to T$ by $(x, y) \mapsto (ax + by, cx + dy)$. For all $n \geq 0$ calculate the induced map $\varphi_* : H_n(T, \mathbb{Z}) \to H_n(T, \mathbb{Z})$.

Problem 4. (a) Define the degree of a smooth map between smooth n-manifolds.

(b) Let $T := \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Let a, b, c, and d be integers. Define the map $\varphi : T \to T$ by $(x, y) \mapsto (ax + by, cx + dy)$. Find the degree of φ .

Problem 5. Let $B \subset GL(n; \mathbb{C})$ be the group of invertible upper triangular $n \times n$ complex matrices (that is, matrices such that all entries strictly below the main diagonal are zeroes). Denote by e the identity matrix. Calculate $\pi_1(B, e)$.

Problem 6. Consider the differential form

$$\eta = dx \wedge dy - 3x dx \wedge dz$$

(a) Is η exact?

(b) Use Stokes theorem to find $\int_X \eta$, where X is the unit sphere about the origin.

Problem 7. (a) Define what it means for a smooth map between manifolds to be a submersion.

(b) State the local submersion theorem, that is the theorem which describes a local model for submersions.

(c) Using the local submersion theorem as a starting point, prove that if $f : M \to N$ is a submersion, and $y \in N$, then $f^{-1}(n)$ is a manifold.

Problem 8. (a) Prove that the unit interval $[0, 1]$ in the usual metric (or equivalently order) topology is connected.

(b) Prove that if a topological space X is path connected then X is connected.

(c) Give an example of a topological space which is connected, but not path connected.