# Topology Qualifying Exam
# Fall 1987

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Do 9 of the following 15 problems.

In the following problems, let $\mathbb{R}$ denote the real line with the usual topology, and let $\mathbb{N}$ denote the natural numbers.

**1.** Prove that any continuous bijection $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism.

**2.** Prove that in a metrizable space $X$ without isolated points, the closure of a discrete set in $X$ must be nowhere dense in $X$.

**3.** (a) Prove that every closed subset of a metrizable space $X$ is a $G_\delta$ in $X$.

(b) Give an example to show that a closed subset of a Hausdorff space $X$ is not necessarily a $G_\delta$ in $X$.

**4.** (a) Characterize the compact subsets of $\mathbb{R}$ and prove that your characterization is correct.

(b) State and prove a maximum value theorem from calculus.

(c) Using maximum, minimum, and intermediate value theorems from calculus, prove that every continuous open function $f : [0, 1] \to [0, 1]$ is surjective.

**5.** Let $X$ be the topological space whose underlying set is the set of real numbers and whose topology has as a basis the set of half open intervals of the form $[a, b)$ where $a < b$. Show that $X$ is

(a) first countable.

(b) separable.

(c) not second countable.

(d) not metrizable.

**6.** (a) True-False

(i) The composition of quotient maps is a quotient map.

(ii) The product of metrizable spaces is metrizable.

(iii) $f : X \to Y$ is a topological embedding iff $f$ is one-to-one and $X$ has the coarsest (=weakest) topology making $f$ continuous.

(iv) A space is $T_1$ iff it is locally $T_1$; i.e., each point has a base of $T_1$ neighborhoods.

(v) A space is $T_2$ iff it is locally $T_2$; i.e., each point has a base of $T_2$ neighborhoods.

(vi) Every metrizable space is normal.

(vii) Every locally compact Hausdorff space is completely regular.

(viii) Every subspace of a separable Hausdorff space is separable and Hausdorff.

(b) For each false entry, give a counter example (no proofs).

**7.** Let $S^1$ denote the unit circle in the plane (with the usual topology). Give 4 examples, 1 compact, 1 non-compact, 1 non-locally connected, 1 non-locally compact, of spaces homotopically equivalent to $S^1$, but not homeomorphic to $S^1$.

**8.** Let $f : A \to B$ where $A, B \subseteq \mathbb{R}$. For $x \in \mathbb{R}$, define

$$\operatorname{osc}(f, x) = \inf\{\operatorname{diam}\ f(A \cap U) |\ x \in U \quad \text{open in } \mathbb{R}\}$$

and

$$A^* = \{x \in \overline{A} | \operatorname{osc}(f, x) = 0\}.$$

Prove that if $f : A \to B$ is an order preserving homeomorphism and $A, B$ are dense in $\mathbb{R}$, then $A^* = \mathbb{R}$.

**9.** Prove that if $f : [0, 1] \to X$ is a continuous open surjection onto a nondegenerate Hausdorff space $X$, then $X$ is homeomorphic to $[0, 1]$.

**10.** Prove that if a filter $\mathcal{F}$ is contained in a unique ultrafilter $\mathcal{G}$, then $\mathcal{F} = \mathcal{G}$.

**11.** Prove that the following two statements about a $T_1$-space $X$ are equivalent.

(a) Every infinite subset of $X$ has an accumulation point in $X$.

(b) At least one member of every infinite open cover of $X$ can be discarded with the remaining sets still covering $X$.

**12.** Find the specific error in the following: "Proof" that the uncountably infinite power of a two point discrete space is metrizable. Let $D = \{0, 1\}$ have the discrete topology. For each $r \in \mathbb{R}$ define

$$f_r : D^N \to D \text{ by } f_r(g) = \begin{cases} g(r) & \text{if } r \in \mathbb{N} \\ 0 & \text{if } r \in \mathbb{R} - \mathbb{N}. \end{cases}$$

Then these functions are continuous and thus induce a continuous embedding $F : D^N \to D^R$. Let $U$ be open in $D^R$. Then $U$ restricts only finitely may coordinates. Thus $U \cap F[D^N] \neq \emptyset$, so $F[D^N]$ is dense in $D^R$. But $D^N$ is homeomorphic with the Cantor space, so $F[D^N]$ is compact. Since $D^R$ is Hausdorff, $F[D^N]$ must be closed in $D^R$. Hence $F[D^N] = D^R$, and since $D^N$ is metrizalbe, $D^R$ must be metrizable.

**13.** Prove that if $X$ is compact Hausdorff, then each quasicomponent of $X$ is connected.

**14.** Prove that if $\mathcal{C} = \{C_\alpha |\ \alpha \in \Lambda\}$ is a family of compact subsets of a Hausdorff space such that the finite intersections of members of $\mathcal{C}$ are connected, then $\bigcap \mathcal{C}$ is connected.

**15.** Show that every connected, locally compact, paracompact Hausdorff space is Lindelöf.