Analysis Qualifying exam - Fall 10

Burckel & Naibo

Instructions: Do all ten problems. Start each problem on a separate page and clearly indicate the problem number.

Notation: $(X, \mathcal{M}, \mu)$ is a measure space, $\mathbb{N}$ is the positive integers, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, U an open non-empty subset of $\mathbb{C}$, $H(U)$ is the set of holomorphic functions in U, $:=$ means a definition equation, $\mathrm{Re}(z)$ means the real part of z.

1. (a) State Rouché's theorem.
   (b) State the Boundary Maximum Modulus Principle.
   (c) Derive (1b) from (1a). **Hint:** U is a bounded open set, f is continuous on $\overline{U}$, holomorphic in U and $M := \max|f(\partial U)|$. If $w \in \mathbb{C}$ and $|w| > M$, use Rouché's theorem to show that $f - w$ cannot have a zero.

2. Prove that if U is connected, then $H(U)$ is an integral domain; that is, $fg = 0$ only if $f = 0$ or $g = 0$. **Hint:** If $fg = 0$, then in any disk in U one of f or g has infinitely many zeros.

3. U is called *simply connected* if $\mathbb{C} \setminus U$ has no compact component. State two properties of $H(U)$ which are each equivalent to U being simply-connected but which make no reference to $\mathbb{C} \setminus U$.

4. (a) Show that $f(z) := \frac{e^{iz}-1}{z}$ defines a function in $H(\mathbb{C})$. What is $f(0)$?
   (b) Using Cauchy's theorem, show that for every $r > 0$

$$\int_{-r}^{r} f(x)\,dx - \pi i = -i\int_{0}^{\pi} \exp(ire^{ix})\,dx.$$

   (c) Show that

$$\int_0^{\pi} |\exp(ire^{ix})|\,dx = \int_0^{\pi} \exp(\mathrm{Re}(ire^{ix}))\,dx = 2\int_0^{\pi/2} \exp(-r\sin x)\,dx < 2\int_0^{\pi/2} e^{-2rx/\pi}\,dx.$$

   Hint: From the graph see that $\sin x \geq \frac{2}{\pi}x$ for $x \in [0, \pi/2]$.

   (d) Take imaginary parts in (4b) and infer that

$$\left|\int_{-r}^{r} \frac{\sin x}{x}\,dx - \pi\right| < \frac{\pi}{r} \qquad \forall r > 0$$

   and deduce the existence and the numerical value of $\lim_{r\to\infty} \int_{-r}^{r} \frac{\sin x}{x}\,dx$.

5. $f_n \in H(U)$, $n \in \mathbb{N}$, $\lim_{n\to\infty} f_n = f_0$ locally uniformly in U.

   (a) Show that f_0 is continuous.

   (b) Use Morera's and Cauchy's theorems for triangles to infer that f_0 is in fact holomorphic on U.

   (c) Use Cauchy's integral formula for circles to show anew that f_0 is holomorphic and moreover, $f_n' \to f_0'$ locally uniformly in U.

   (d) Does the analog of conclusion (5b) hold for differentiable functions on an open interval of $\mathbb{R}$?

6. (a) State Fatou's lemma, the monotone convergence theorem, and the dominated convergence theorem.

   (b) Let $f_n : X \to \mathbb{C}$, $n \in \mathbb{N}$, and $f : X \to \mathbb{C}$ be measurable functions. Prove that if $f_n \to f$ almost uniformly, then $f_n \to f$ a.e.$[\mu]$ and $f_n \to f$ in measure.

7. Suppose $(X, \mathcal{M}, \mu)$ is σ-finite and let $f \in L^1(X)$, $f \geq 0$. Prove that the set

$$G_f = \{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

   is $\mathcal{M} \times \mathcal{B}_{\mathbb{R}}$-measurable and that $(\mu \times m)(G_f) = \int_X f \, d\mu$. **Hint:** To show that G_f is measurable note that the function $F : X \times \mathbb{R} \to \mathbb{R}$ given by $F(x, y) = f(x) - y$ is the composition of $F_1 : X \times \mathbb{R} \to \mathbb{R}^2$ defined by $F_1(x, y) = (f(x), y)$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $F_2(z, y) = z - y$.

8. (a) If $\mu(X) < \infty$ and $0 < p < q \leq \infty$ prove that $L^q(X) \subset L^p(X)$ and $\|f\|_p \leq \|f\|_q \, \mu(X)^{\frac{1}{p} - \frac{1}{q}}$. Give an example of a measure space of infinite measure and indexes $0 < p < q \leq \infty$ for which $L^q(X)$ is not a subset of $L^p(X)$.

   (b) *Chebyshev's inequality.* Prove that if $0 < p < \infty$ and $f \in L^p(X)$ then for all $\lambda > 0$,

$$\mu\left(\{x : |f(x)| > \lambda\}\right) \leq \left(\frac{\|f\|_p}{\lambda}\right)^p.$$

9. Prove that if $f \in L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, then

$$\lim_{|h| \to 0} \|f(\cdot + h) - f(\cdot)\|_p = 0.$$

   Hint: First settle the case when f is a continuous function with compact support.

10. (a) Give four equivalent definitions of an orthonormal basis in a Hilbert space H.

    (b) Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Prove that if $\{e_n\}_{n \in \mathbb{N}}$ is an orthonormal sequence in H, then $\lim_{n\to\infty} \langle v, e_n \rangle = 0$ for every $v \in H$.

(c) Prove the Riemann-Lebesgue lemma: Every $f \in L^1([0, 2\pi])$ satisfies

$$\lim_{n\to\infty} \int_0^{2\pi} f(x)\cos(nx)\,dx = \lim_{n\to\infty} \int_0^{2\pi} f(x)\sin(nx)\,dx = 0.$$

Remark: Note that $L^1([0, 2\pi])$ is not a Hilbert space. Can nevertheless part (10b) be used?