Algebra Qualifying Exam
Sping 1987

Do two problems from each section.

Group Theory

1. Let G be a non-abelian group of order pq, where p and q are primes and $p < q$. Prove that $p|q-1$.

2. Prove that there does not exist a simple group of order 112.

3. Prove the Frattini lemma: if G is a finite group, K is a normal subgroup of G, and P is a Sylow p-subgroup of K, then $G = N_G(P)K$.

4. Prove that any finite nilpotent group G is the (internal) direct product of its Sylow subgroups. (One possible approach: use the fact that $N_G(H) \neq H$ for any proper subgroup H of G.)

Linear Algegra

1. Let V be a finite-dimensional $\mathbf{F}$-vector space and let $T \in \mathrm{End}_F(V)$. Assume that the minimal polynomial of T is of the form $f(x)g(x)$ where $f(x)$ and $g(x)$ are relatively prime polynomials in $\mathbf{F}[x]$. Prove that $V = \ker f(T) \oplus \ker g(T)$ (internal direct sum).

2. Let V be an $\mathbf{F}$-vector space and let W be a subspace of V. Let $\widehat{V}$ and $\widehat{W}$ denote the dual spaces of V and W, respectively. Prove that

$$\widehat{W} \cong \widehat{V}/Ann(W)$$

where $Ann(W)$ is $\{f \in \widehat{V} | W \subseteq \ker f\}$.

3. Let V be a vector space of dimension 7 over the rationals, and let $T \in \mathrm{End}_Q(V)$. Suppose that the characteristic polynomial of T is $(x-1)^5(x-2)^2$ and that the minimal polynomial of T is $(x-1)^4(x-2)$. List the possibilities for the Jordan canonical form of T, up to re-ordering of Jordan blocks.

4. Let V be a finite-dimensional vector space over an algebraically closed field $\mathbf{F}$, and let S and T be two *commuting* members of $\mathrm{End}_{\mathbf{F}}(V)$. Show that S and T have a common eigenvector (not necessarily for the same eigenvalue).

Rings and Modules

(In these problems, rings are assumed to have a multiplicative identity element "1", and modules are assumed to be unital. That is, if R is a ring and M is a (left) R-module, then $1 \cdot x = x$ for all $x \in M$).

1. By definition, and R-module is irreducible if it is non-zero and has no proper non-zero submodules. Let M and N be two irreducible R-modules. Prove that either $Hom_R(M, N) = 0$ or $M \cong N$. Show also that $Hom_R(M, M)$ is a division ring.

2. Show that if $\mathbf{F}$ is an infinite field and $f \in \mathbf{F}[x_1, \ldots, x_n] \neq 0$.

3. Let R be an integral domain and let ρ be a non-zero prime ideal of R. Show that R_ρ has a unique maximal ideal (where R_ρ denotes the "localization" of R at ρ).

4. Show that a module over a ring R is always a homomorphic image of a free R-module.

Fields and Galois Theory

1. Let E be a splitting field for the polynomial $x^3 - 5$ over $\mathbf{Q}$. Find all of the subfields of E.

2. Let F_0 be a field of order 4 (i.e., having precisely four elements). Let t be transcendenatal over F_0, and put $F = F_0(t)$, the function field in one variable over F_0. Finally, put $E = F(u)$ where $u^3 = t$.

(a) Show that E/F is normal and separable.

(b) Determine the galois group of the extension E/F.

3. Let K be an extension field of the rationals, of finite degree. Prove that K contains only a finite number of roots of unity.

4. Let E/F be an extension field and let $\alpha \in E$. Show that α is algebraic over F if and only if $[F(\alpha) : F]$ is finite.