Name ____________________

TOPOLOGY QUALIFYING EXAM
Spring 2000
(Lee and Strecker)

Choose and work any 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than six problems. In the problems below, a "space" always means a "topological space".

1. Prove or disprove each of the following:

a) If A is a connected subset of space X and $Q \subseteq X$ such that $A \subseteq Q \subseteq \overline{A}$. then Q must be connected.

b) Each component of a space must be closed.

c) Each component of a space must be open.

2. Prove or disprove each of the following:

a) Every subspace of a compact Hausdorff space is locally compact.

b) Every subspace of a connected Hausdorff space is locally connected.

3. Prove that each compact subset of a Hausdorff space must be closed but a compact subset of a T_1-space need not be closed.

4. a) Complete the following sentence so that it is a true statement:

Given a set X, a family $\mathcal{B}$ of subsets of X is a base for a topology on X if, and only if ____________________.

b) Construct another true sentence that describes a subbase for a topology on X.

5. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let τ be the product topology on $X_1 \times X_2$. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. There is a "natural" subspace topology σ on $Y_1 \times Y_2$ that is induced from τ and the fact that $Y_1 \times Y_2$ is a subset of $X_1 \times X_2$. There is also a "natural" product topology ρ on $Y_1 \times Y_2$ that comes from the subspace topologies (Y_1, ρ_1) and (Y_2, ρ_2) induced by τ_1 and τ_2 respectively. Prove or disprove that $\sigma = \rho$.

6. Let $A = \{x | x \text{ is an ordinal number that is less than or equal to } \omega_1\}$, where ω_1 is the first uncountable ordinal number and let τ be the order-topology on A induced by the usual order. Prove one of the following:

a) The space (A, τ) is a compact Hausdorff space.

b) If $f : (A, \tau) \to \mathbb{R}$ is a real-valued continuous function (where $\mathbb{R}$ is assumed to have its usual topology) then there exists some $x_0 < \omega_1$ such that $f(x) = f(x_0)$ for all $x \geq x_0$ in A.

c) The Stone-Čech compactification of $A \backslash \{\omega_1\}$ (with its usual subspace topology) is equivalent to (A, τ).

7. Prove that for any set X the cardinality of X is strictly smaller than the cardinality of $\mathcal{P}(X)$, the set of all subsets of X.

8. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be **final** provided that for each topological space (Z, μ) each set-function $g : Y \to Z$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \mu)$ is continuous. Prove any two of the following statements:

a) The composition of final (continuous) maps is final.

b) The "second factor" of a final map is final: i.e., if

$$(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{h} (W, s)$$

are continuous maps and $h \circ f$ is final, then h is final.

c) Each quotient map is final.

9. Work one of the following:

a) Give an example of two topologies σ and τ on the set of integers $\mathbb{Z}$ for which $\sigma \subsetneqq \tau$ and $(\mathbb{Z}, \sigma)$ is homeomorphic to $(\mathbb{Z}, \tau)$.

b) Give an example of a point-finite open covering of the real line with the usual topology that is not locally finite.

10. Prove one of the following statements:

a) Each compact Hausdorff space is a Baire space.

b) The set of rational numbers is not a G_δ-set in $\mathbb{R}$ (where $\mathbb{R}$ has its usual topology).

11. Prove or disprove one of the following:

a) For each open covering $\{U_\alpha | \alpha \in A\}$ of a paracompact space X, there is a partition of unity subordinated to $\{U_\alpha\}_{\alpha\in A}$.

b) Each locally compact Hausdorff space is paracompact.

12. Prove one of the following theorems:

a) Urysohn Lemma.

b) Tychonoff Theorem on Product of Compact Spaces.

c) Urysohn Metrization Theorem.

13. Let X, Y and Z be topological spaces and X be locally compact Hausdorff. Let $C(X,Y)$ be the space of continuous functions from X into Y with compact-open topology. For a function

$$F : X \times Z \to Y$$

let $\hat{F}$ be the function from Z into the set of all functions from X to Y defined by

$$\hat{F}(z)(x) = F(x,z).$$

Prove one of the following:

a) If F is continuous, then $\hat{F}$ is continuous from Z into $C(X,Y)$.

b) If $\hat{F}$ is continuous from Z into $C(X,Y)$ then F is continuous.

14. For any $\varepsilon > 0$ and any $(a,b) \in \mathbb{R}^2$ let an <u>ε-plus centered at (a,b)</u> be the set $+_\varepsilon(a,b) = \{(x,b) \in \mathbb{R}^2 \big| |x-a| < \varepsilon\} \cup \{(a,y) \in \mathbb{R}^2 \big| |y-b| < \varepsilon\}$. Then call a set $U \subseteq \mathbb{R}^2$ <u>plus-open</u> if for each $(a,b) \in U$ there is some $\varepsilon > 0$ such that $+_\varepsilon(a,b) \subseteq U$.

a) Prove that the plus-open subsets of $\mathbb{R}^2$ form a topology for $\mathbb{R}^2$, (called the plus-topology for $\mathbb{R}^2$).

b) Prove that the plus-topology for $\mathbb{R}^2$ is <u>strictly</u> finer than the usual topology for $\mathbb{R}^2$.