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Complex Analysis Qualifying Exam

January, 2000

Instructions: Below you will find 8 problems. Each problem is worth 10 points. Do as many problems as you can.
Time: 2 hours.

NOTATIONS: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $T = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle).

1. Let f be a holomorphic function on the open unit disk $\mathbb{D}$ with $|f(z)| < 1$ for all $z \in \mathbb{D}$. If $f(z_1) = z_1$ and $f(z_2) = z_2$ with $z_1 \neq z_2$, show that $f(z) = z$ for all $z \in \mathbb{D}$.

2. Use the Residue Formula to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx.$$

3. Let C be the unit circle, oriented counter clockwise. For any z in the complex plane, $|z| \neq 1$, evaluate

$$\int_C \frac{\overline{\zeta}}{\zeta - z} d\zeta.$$

4. How many roots (counting multiplicities) does

$$z^4 + 3z - 1$$

have in the annulus $\Omega = \{z : 1 < |z| < 2\}$.

5. Let G be the circle in the complex plane which passes through 0 and 1, such that the angle from the real axis to G is $\pi/4$ (measured counterclockwise). Let Ω be the region inside G and above the real axis. Find a conformal map of Ω onto the upper half-plane. You may express the conformal map as a composition of simpler maps.

6. Given an entire function g, describe all entire functions f with the property:

$$|f(z)| \leq |g(z)|, \ \forall\, z \in \mathbb{C}.$$

7. Let $\Omega \subset \mathbb{C}$ be an open set which contains the open unit disk $\mathbb{D}$, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $f(0) = 1$ and

$$|f(z)| > 1, \ \forall\, z \in \mathbb{T}.$$

Prove that f has at least one zero in $\mathbb{D}$.

8. Suppose $f, g : \mathbb{D} \to \mathbb{C}$ are holomorphic, without zeros. Assume

$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}, \ \ n = 2, 3, 4, \ldots$$

Find another simple relation between f and g.