Algebra Qualifying Exam
Spring 1996

Note: All rings in this exam are associative and with 1 and all integral domains are commutative. $\mathbb{Q}$ and $\mathbb{C}$ are the fields of rational and complex numbers respectively.

1. Show that for any group G, the quotient group $G/Z(G)$ is never a nontrivial cyclic group. Here $Z(G)$ is the center of the group G.

2. Let G be a finite group and p a prime number dividing $|G|$. If P is a p-Sylow subgroup of G, show that $N_G(N_G(P)) = N_G(P)$. Here $N_G(H)$ is the normalizer of a subgroup H in the group G.

3. Let R be a commutative ring with 1 and $x \in R$. Suppose that x lies in every maximal ideal of R. Show that $1 - x$ is a unit of R.

4. This problem conisits of two parts:

(i) Give a definition to what a unique factorization domain (UFD) means.

(ii) Give an example of an integral domain that is not a UFD.

5. Let R be a ring with 1. Suppose that an R-module $M = M_1 \oplus M_2$ is a direct sum of two non-isomorphic irreducible submodules M_1 and M_2. Show that M_1 and M_2 are the only two proper submodules of M.

6. This problem consists of two parts:

(i) State the definition of what an indecomposable module over a ring means.

(ii) Give an example of a ring R and an indecomposable module M over R.

7. Let $T : V \to V$ be a linear transformation on a finite dimensional vector spaces over a field F. Suppose T has the following invariant factors:

$$1 + x, \quad x(1 + x), \quad x(1 + x)^2.$$

Answer the following questions:

(i) What is $\dim_F V$?

(ii) Is T onto?

(iii) Does T have a Jordan form over the field F with respect to an appropriate basis of V? If yes, write down such a matrix.

(iv) Is V indecomposable as an $F[T]$- module?

(v) What is the minimal polynomial of T?

8. Let V be the 4-dimensional vector space of all 2×2- matrices over a field F. We define the function $f(\cdot, \cdot) : V \times V \to F$ by $f(A, B) = \text{trace}\,(AB)$ for all $A, B \in V$.

(i) Show that F is a symmetric bilinear form;

(ii) Let $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the standard basis of V. Compute the matrix of the bilinear form f with respect to the standard basis.

(iii) Describe the radical of the bilinear form f. Here the radical of a bilinear form f is defined to be the set $\{v \in V | f(v, V) = 0\}$.

9. This problem consists of two parts:

(i) Define what it means for a field extension E over F to be separable.

(ii) Is the splitting field of the polynomial $p(x) = x^9 - x^3 + 1$ over $\mathbb{F}_3$ separable?

10. Show that, if $F \lneqq K \leq F(x)$ is tower of fields, where x is an indeterminate and $F(x)$ is the field of rational functions (i.e., x is transcendental over F), then K cannot be an algebraic extension over F.