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Geometry/Topology Qualifying Exam
Spring 2005 [Auckly & Yetter]

Solve as many of the following problems as you can. Start each problem on a new sheet of paper. Good luck!

1. A set $A \subseteq X$ is called a retract if there is a continuous map $r : X \to A$ so that $r\,|_A = id_A$.

a) Prove that $[-1, 1]$ is a retract of $\mathbb{R}$.

b) Prove that no proper subset of S^2 is a retract of S^2.
Hint: If $A \subset S^2 \backslash \{p\}$ is a retract of S^2, it is a retract of $S^2 \backslash \{p\}$.

2. Let X be the CW complex described by:

0-cells: p_1, p_2

1-cells: $A_1, A_2, A_3 :\ \partial A_1 = p_1 p_1^{-1},\ \partial A_2 = p_2 p_2^{-1},\ \partial A_3 = p_2 p_1^{-1}$

2-cells: F_1, F_2, F_3; $\partial F_1 = A_1^3,\ \ \partial F_2 = A_2, \partial F_3 = A_2$.

Compute $\pi_1(X)$, $H_*(X)$, and $H_*(X \times T^2)$.

3. Let $\psi : (0, \infty) \times (0, 2\pi) \times (0, \pi) \to \mathbb{R}^3$;
$\psi(r, \theta, \varphi) = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi)$
$\omega = z dx \wedge dy - \frac{1}{2} x dy \wedge dz$
$X = r \cos\theta \partial_\varphi, \quad Y = r^2 \partial_\theta$

Compute:

a) $d\omega$

b) $\psi_* X|_{(r_0, \theta_0, \varphi_0)}$

c) $\psi^* \omega$

d) $[X, Y]$

e) $\displaystyle\int_{f^{-1}(1)} \omega$

where $f : \mathbb{R}^3 \to \mathbb{R}$; $f(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$ and $f^{-1}(1)$ is oriented by $i_{grad(f)} dx \wedge dy \wedge dz$. (Do this in two ways, directly and by Stokes' theorem.)

4. Prove that a closed subspace of a normal space is normal.

5. Let $p : \widetilde{X} \to X$ be the universal covering. Prove that $\pi_2(p) : \pi_2(\widetilde{X}) \to \pi_2(X)$ is an isomorphism.

6. Let M be a compact smooth orientable n-manifold without boundary. Show that for any $(n-1)$-form ω on M there exists a point $p \in M$ such that $d\omega(p) = 0$.

7. Prove "The Stack of Records Theorem": If y is a regular value of a smooth map $f : X \to Y$ where X is compact and X and Y are smooth manifolds of the same dimension, then $f^{-1}(y)$ is a finite subset $\{x_1, \ldots, x_n\}$ of X, and there exists an open neighborhood U of y such that $f^{-1}(U)$ is a disjoint union of open sets V_i for $i = 1, \ldots, n$ in X and, for each i, $x_i \in V_i$ and $f\,|_{V_i}$ is a diffeomorphism with U.

8. Show that if $\mathbb{R}^n$ with its usual topology is also regarded as a group under vector addition, then the quotient group $\mathbb{R}^n/\mathbb{Z}^n$ is an n-manifold when given the quotient topology.