Name ______________________________

REAL ANALYSIS QUALIFYING EXAM
Fall 1998
(Saeki & Vaninsky)

Do all of the problems below.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. Let $\varphi : X \to \Omega$, where Ω is a topological space.

(a) What does it meant that φ is measurable?

(b) Suppose φ is measurable and E is a Borel subset of Ω. Prove that $\varphi^{-1}(E)$ is measurable.

2. Let $f \in L^1(\mu)$. Prove that $\left| \int f \, d\mu \right| \leq \int |f| d\mu$.

3. Let $1 \leq p < \infty$. Prove the completeness of $L^P(\mu)$.

4. Prove the Dominated Convergence Theorem by applying Fatou's Lemma.

5. Let $\varphi : X \to \Omega$ be measurable, where Ω is a topological space. Define $\nu(E) := \mu(\varphi^{-1}(E))$ for each $E \in \mathcal{B}_\Omega$ (the Borel subsets of Ω). Prove:

(a) ν is a Borel measure.

(b) Let $f : \Omega \to [0, \infty]$ be Borel measurable. Then

$$\int_\Omega f \, d\nu = \int_X f \circ \varphi \, d\mu.$$

6. Let $f : X \times [0,1] \to \mathbb{C}$. State (nontrivial) conditions on f that guarantee

$$(*) \qquad \frac{d}{dt} \int f(x,t) \, d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) \, d\mu(x) \quad \forall \, t \in (0,1)$$

and then prove ($*$).

7. Let $L^\infty(\mu) + L^1(\mu) = \{g + h : g \in L^\infty(\mu), \; h \in L^1(\mu)\}$. Prove that

$$L^P(\mu) \subset L^\infty(\mu) + L^1(\mu) \quad \forall p \in [1, \infty].$$

8. Let $\varepsilon > 0$. Construct a compact set $K \subset [0,1]\mathbf{Q}$ such that $|K| > 1 - \varepsilon$, where $|K|$ is the Lebesgue measure of κ.