# QUALIFYING EXAM

August, 1979

ALGEBRA - Do 5 problems

1. a. Define what it means for a group to be abelian.

   b. Give an example of an abelian group and an example of a non-abelian group.

   c. Prove that any group with exactly four elements is abelian.

2. a. Define an equivalence relation on a set $S$.

   b. Let $G$ be a group and $H$ a subgroup of $G$. If $x, y \in G$, we say that $x$ and $y$ are "congruent mod H," (written $x \equiv y \pmod{H}$) if $xy^{-1} \in H$. Prove that "congruence mod H" is an equivalence relation on $G$.

3. Let $f: G_1 \to G_2$ be a surjective (i.e. onto) homomorphism from $G_1$ to $G_2$. Assume that $G_2 \neq \{1\}$. Prove, or give a counter-example to each assertion below.

   a. If $G_1$ is abelian, then so is $G_2$.

   b. If $G_1$ is nonabelian, then so is $G_2$.

   c. If $G_1$ is infinite, then so is $G_2$.

4. Solve completely the linear system of equations

$$x_1 + x_2 - x_3 = 5$$

$$x_1 + 2x_2 + x_3 = 4$$

5. Let $V_1$ and $V_2$ be vector spaces over a field $F$, and let $T$ be a mapping from $V_1$ to $V_2$.

   a. Define what it means for $T$ to be a linear transformation. Give an example of vector spaces $V_1, V_2$ over $F$ and a linear transformation $T: V_1 \to V_2$.

   b. Assume that $\dim V_1 = n < \infty$. State (don't prove) the "rank-nullity" theorem.

6. Let $R$ be a ring and let $I \subseteq R$. Define what it means for $I$ to be

   a. a (2-sided) ideal of $R$.

   b. a left ideal of $R$.

   c. Give an example of a ring $R$ and a left ideal $I$ that is not a 2-sided ideal of $R$.

   d. If $I \subseteq R$ is a 2-sided ideal, what are the ring operations in the quotient ring $R/I$? (Continued on page 2)

6. d. (Continued)

   That is, given $r + I$, $s + I \in R/I$ how are the "product" and "sum" of these elements defined in terms of the corresponding operations in $R$? Prove that these operations are well-defined.

7. State (without proof) the "Fundamental Theorem of Arithmetic" for the ring $\mathbb{Z}$ of integers.

   Give an example of a ring $R \neq \mathbb{Z}$, for which a suitable generalization of the Fundamental Theorem of Arithmetic is valid.

8. Let $R$ be a commutative-ring-with-a-one (i.e. $R$ has a multiplicative identity element).

   a. If $M \subseteq R$ is a maximal ideal in $R$, describe (don't prove anything) how to obtain a field from $R$ and $M$.

   b. Give an example of this construction.

9. Let $\mathbb{R}$, $\mathbb{C}$ be the fields of real and complex numbers, respectively, and let $x$ be an indeterminate over $\mathbb{R}$. Show how to define an isomorphism between the fields $\mathbb{R}[x]/(x^2 + 1)$ and $\mathbb{C}$. Simply construct the mapping; you don't have to show that it affords an isomorphism.

ANALYSIS - Do 5 problems

1. Let $X$ and $Y$ be metric spaces and $f:X \to Y$ a function. Define: $f$ is uniformly continuous. Give an example, with $X = Y = \mathbb{R}$, of a continuous function $f$ which is not uniformly continuous.

2. Suppose $f$ is a real valued function defined on $(0,1)$, and $f'$ existsand is bounded on $(0,1)$. Prove that

   a. $f$ is bounded on $(0,1)$.

   b. $\lim_{n\to+\infty} f(\frac{1}{n})$ exists.

3. Suppose $f$ is a continuous function on the closed interval $[a,b]$ and $\int_a^c f(x)dx = 0$ for every rational number $c$, $a < c < b$.

   Prove that $f(x) = 0$ for all $x$ in $[a,b]$.

4. Let $\{f_n\}$ be a sequence of real valued continuous functions defined on a set $S \subset \mathbb{R}$. Suppose $\{f_n\}$ converges uniformly to $f$ on $S$. Prove that $f$ is continuous on $S$.

5. Let $\{[a_i,b_i]\}_{i=1}^{+\infty}$ be a sequence of closed intervals such that $[a_i,b_i] \supset [a_{i+1},b_{i+1}]$ for each $i = 1, 2, \ldots$ Prove the $\bigcap [a_i,b_i]$ is nonempty. (Hint: Use the least upper bound property.

6. Determine the interval of convergence of the power series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{2^n n} .$$