Complex Analysis Qualifying Exam
Fall 1992

In the following $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{D}$ the set $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ the set of all holomorphic functions on $\mathbb{D}$.

1. Show that if $g : \Omega$ open $\subset \mathbb{C} \to \mathbb{C}$ is continuous and e^g is holomorphic, then g is holomorphic. (That is, a **continuous** logarithm of a holomorphic function is necessarliy holomorphic.)

2. Show directly (without reference to the concept of simple-connectivity) that every zero-free function $f \in H(\mathbb{D})$ has a holomorphic logarithm; that is, $\exists g \in H(\mathbb{D})$ such that $f = e^g$.

3. Show directly (without reference to the concept of simple-connectivity) that the identity function, $I(z) = z$, in $\mathbb{C}\backslash\{0\}$ has no continuous logarithm. **HINT:** Problem 1 may be useful.

4. (a) f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$. Show that f is uniformly approximable on $\overline{\mathbb{D}}$ by polynomials. **HINT:** First approximate f uniformly on $\overline{\mathbb{D}}$ by a function f_r which is holomorphic in $D(0, 1/r), 0 < r < 1$.

(b) State and prove the converse of (a).

5. State

(a) the Maximum Modulus Principle for holomorphic functions,

(b) the Open Map Theorem for holomorphic functions.

(c) Show that (a) can be deduced from (b).

6. Show that $\int_{\partial\mathbb{D}} \frac{e^{\pi z}}{4z^2+1} dz = \pi i$.

7. f is holomorphic in $A := \mathbb{D}\backslash\{0\}$ and satisfies $|f(z)| < |z|^{3/2}$ for all $z \in A$. Show that $|f(1/2)| \leq 1/4$. **HINT:** First see if the function $g(z) := f(z)/z$ can be holomorphically extended into $\mathbb{D}$. What will its value at 0 have to be?

8. f is holomorphic and one-to-one in the region Ω. Let $G = f(\Omega)$ and $g : G \to \Omega$ be the inverse of f. Prove that g is holomorphic. **HINT:** You will need to prove **en route** that f' is zero-free in Ω.

9. (a) Define (don't just name) the three kinds of isolated singularity which holomorphic functions can have and give an example of each.

(b) What does the Casorati-Weierstrass Theorem say about one of these kinds of singularities?

(c) What does the Great Picard Theorem say about one of these kinds of singularities?

(d) State Mittag-Leffler's Theorem regarding the principal parts of a meromorphic function.