Partial Differential Equations
Qualifying Examination
September 14, 1987

Do any 7 problems.

1. (a) What does it mean to say that a problem in partial differential equations is well-posed?

   (b) Given a region Ω in R^n and a second order linear PDE defined in Ω, what does it mean to say the PDE is elliptic in Ω?

   (c) Consider the Cauchy problem

$$\begin{aligned} u_t &= \Delta u, && x \in R^n, \quad t > 0 \\ u(0,x) &= f(x), && x \in R^n. \end{aligned}$$

   State the Maximum Principle for the solution of this problem (be sure to state clearly all relevant hypotheses).

2. Let $\mathcal{D} = \{u | u \in C^2(\overline{\Omega}), u(x) = f(x) \text{ for } x \in \partial\Omega\}$ where Ω is a bounded domain in R^n with smooth boundary. For $u \in \mathcal{D}$ let

$$J(u) = \int_\Omega |\nabla u|^2 dx.$$

   Suppose that $u \in \mathcal{D}$ satisfies the Dirichlet problem

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \\ u(x) &= f(x) && \text{on } \partial\Omega. \end{aligned}$$

   Prove that u minimizes J.

3. (a) Consider the initial value problem

$$\begin{aligned} u_t + uu_x &= 0 && -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), && -\infty < x < \infty. \end{aligned}$$

   Assume f is C^1. Show that unless f is nondecreasing on $(-\infty, \infty)$ there cannot be a C^1 function $u(x,t)$, $-\infty < x < \infty$, $t \geq 0$ which is a solution of the IVP everywhere in $-\infty < x < \infty$, $t \geq 0$.

(b) Show that the IVP

$$\begin{array}{ll} u_t + uu_x = 0 & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = 2x+1 & -\infty < x < \infty \end{array}$$

has a smooth solution by finding an explicit formula for the solution.

4. Consider the quasilinear system

$$u_t + uu_x + \frac{c^2}{\rho} = 0$$
$$\rho_t + u\rho_x + \rho u_x = 0$$

where u and ρ are unknown functions of x and t and c is a known function of ρ.

(a) Show that this is a hyperbolic system provided we assume $c(\rho) > 0$.

(b) Find the differential equations of the characteristic curves for this system.

5. Let Ω be the first quadrant in R^2. Define $f(x,y)$ for $(x,y) \in \partial\Omega$ by:

$$f(0,y) = \begin{cases} 1 & 0 \le y < 1 \\ 0 & 1 \le y \end{cases}$$

$$f(x,0) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & 1 \le x \end{cases}$$

Use complex variable methods to solve the Dirichlet problem on Ω with boundary data f.

6. Explain what is meant by "Huygens' Principle". For which dimensions does this principle hold?

7. Suppose the initial temperature in a spherical body of radius $a > 0$ is constant at U_0. For $t > 0$ the boundary is kept at temperature 0. Assume heat conduction is governed by

$$u_t = \Delta u$$

where u is the temperature function. Derive a series representation for the solution of this problem.

8. (a) State the mean value property of harmonic functions in R^n.

(b) Using the result in (a), state and prove the Maximum Principle for harmonic functions on a bounded domain in R^n.

(c) Use the Maximum Principle to show that the solution of the Dirichlet problem on a bounded domain Ω in R^n is unique (if it exists).

(d) Prove that the solution of the Dirichlet problem on a bounded domain Ω in R^n depends continuously on the boundary condition. Include a careful statement of what this means.

9. (a) Derive the Green function for Dirichlet's problem for the Laplace equation on the upper half-plane in R^2.

(b) Use (a) to derive the formula

$$u(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2+y^2}\,d\xi$$

for the solution of the Dirichlet problem

$$\begin{aligned} &\nabla^2 u = 0 \quad -\infty < x < \infty, \quad 0 < y < \infty \\ &u(x,0) = f(x), \end{aligned}$$

where f is continuous on $-\infty < x < \infty$.

10. Prove uniqueness of solutions for the problem

$$\begin{aligned} &u_{tt} = a^2 \Delta u + f(t,x) \quad \text{for} \quad x \in \Omega, \quad t > 0 \\ &u(0,x) = \phi(x), \quad u_t(0,x) = \psi(x) \quad \text{for} \quad x \in \Omega \\ &\tau \frac{\partial u}{\partial n} + \sigma u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is a region with smooth boundary in R^n, ϕ and ψ are C^1 on Ω, and σ and τ are positive constants. (Hint: Use the energy integral

$$E(t) = \int_\Omega (\tau u_t^2 + \tau a^2 |\nabla u|^2)\,dx + \int_{\partial\Omega} a^2\sigma^2 u^2\,ds.)$$

11. Use Fourier transforms to solve the Cauchy problem for the 1-dimensional heat equation with source term $f(x,t)$,

$$\begin{aligned} &u_t = u_{xx} + f(x,t) \\ &u(x,0) = \phi(x). \end{aligned}$$