Complex Analysis Qualifying Exam
Spring 1992

$\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ the set of all complex numbers, $D(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$ for any $a \in \mathbb{C}$, $0 < r \in \mathbb{R}$, $\mathbb{D} := D(0,1)$.

1. $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Show that

$$\left|\frac{f(x) - f(w)}{1 - \overline{f(z)}f(w)}\right| \leq \left|\frac{z-w}{1-\overline{z}w}\right| \quad \forall z, w \in \mathbb{D}.$$

What can you say about f if there exists a pair (z,w) for which $z \neq w$ and equality holds above?

2. Let f be a one-to-one holomorphic function on a region Ω. Show that f' is zero-free.

3. (a) f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\overline{\overline{\mathbb{D}}}$. Show that f is uniformly approximable on $\overline{\mathbb{D}}$ by polynomials.

(b) State and prove the converse of (a).

4. Let A denote the algebra of continuous functions on $\overline{\mathbb{D}}$ which are holomorphic in $\mathbb{D}$. Find all $\mathbb{C}$-algebra homomorphisms $\Phi : A \to \mathbb{C}$ which are not identically 0. **HINT:** Show first that $\Phi(f) \in f(\overline{\mathbb{D}})$ for every f, then see what Φ does to polynomials. Is Φ continuous?

5. (a) State the Maximum Modulus Principle for holomorphic functions, and

(b) Give two different proofs of this principle.

6. Suppose that f is holomorphic on $\overline{\mathbb{D}}$, $|f(z)| < 1$ whenever $|z| = 1$, and that $\alpha \in \mathbb{D}$. Find the number of solutions in $\overline{\mathbb{D}}$ of the equation

$$f(z) = \left(\frac{z - \overline{\alpha}}{\alpha z - 1}\right)^2.$$

7. Suppose f is holomorphic in the disc $D(a,R)$. Prove that

$$|f(a)| \leq \frac{1}{\pi R^2} \int\int_{D(a,R)} |f(x+iy)| dx dy.$$

Is this inequality valid for harmonic functions? **HINT:** Polar coordinates.

8. (a) State and prove Harnack's Inequalities.

(b) Using (a) [whether or not you proved it] demonstrate the following: If $h : \mathbb{C} \to \mathbb{R}$ is harmonic but **not** constant, and

$$m(r) := \min_{\theta} h(re^{i\theta}), \quad M(r) := \max_{\theta} h(re^{i\theta}),$$

then

$$\overline{\lim_{r\to\infty}} \frac{m(r)}{r} < 0 < \lim_{r\to\infty} \frac{M(r)}{r}.$$

(c) For the h in (b), infer from (b) [whether you proved (b) or not] that h must have at least one zero, then go on to prove that $h(\mathbb{C}) = \mathbb{R}$.