Numerical Analysis Qualifying Exam
Fall 1987

Hand in at most ten problems. You must work at least one from each of the five sections.

I. Differentiation, Integration, and General Topics.

1. Write the formula for approximating the derivative of a function f at x_0 using the points $x_0, x_0 + h$, and $x_0 + 2h$, $h > 0$, and determine a bound for the error of this approximation assuming that $f^{(4)}$ exists and is continuous on an interval $[a, b]$ containing the three points.

2. Show that the simple quadrature defined by interpolation using polynomials of degree $\leq n$ has degree of precision at least n. Give an example of a simple interpolating quadrature involving polynomials of degree $\leq n$ that has degree of precision $n + 1$.

II. Root Finding.

3. The equation $x + \ln x = 0$ has a root $\hat{x}$ near 0.5. Which of the following iteration schemes will produce iterates that converge to the root? Which scheme converges the fastest?

   (a) $x_n = -\ln(x_{n-1})$.

   (b) $x_n = e^{-x_{n-1}}$.

   (c) $x_n = \frac{x_{n-1}+e^{-x_{n-1}}}{2}$.

4. (a) Define Aitken's δ^2-method of accelerating the convergence of a sequence.

   (b) State sufficient conditions for a linearly convergent sequence $\{x_n\}$ to satisy

$$\lim_{n\to+\infty} \frac{x_n' - x}{x_n - x} = 0,$$

   where $\{x_n'\}$ is the associated accelerated sequence, and prove this result.

5. Outline the method of steepest descent for determining an approximation to the location of a minimum for a real valued function f of several variables. Describe conditions under which this method might converge very slowly.

III. Approximation Theory.

6. Find the interpolating polynomial determined by the five points $(0, 1), (1, 3), (2, 1), (3, 3)$, and $(4, 1)$.

7. Define the continuous least squares approximating polynomial of degree at most n for a function f and state and prove the uniqueness theorem for this approximation.

8. Let f be a function defined on $[-1, 1]$ and consider the following two sets of points:

$$\{-0.9510565, -0.5877853, 0.0000000, 0.5877853, 0.9510565\}$$

   and

$$\{-1.0000000, -0.5000000, 0.0000000, 0.5000000, 1.0000000\}$$

   Consider the interpolating polynomials defined by each of these sets with respect to f. Which would you expect to be more accurate on $[-1, 1]$. Explain your answer.

IV. Linear Algebra.

9. Let A be the n by n matrix

$$\begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & 1 & -2 & \vdots & \\ & & \vdots & \vdots & -1 \\ 0 & & & -1 & -2 \end{bmatrix}.$$

Show that the Jacobi iteration procedure converges to a solution of $Ax = b$ for any vector b and any starting vector x_0. (Hint: You may assume that the matrix

$$\begin{bmatrix} 0 & -1 & & & 0 \\ -1 & 0 & -1 & & \\ & -1 & 0 & \vdots & \\ & & \vdots & \vdots & -1 \\ 0 & & & -1 & 0 \end{bmatrix}.$$

has eigenvectors

$$v_k = \left(\sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, dots, \sin \frac{nk\pi}{n+1} \right)^T.$$

10. State as much information as you can about the location and type of the eigenvalues of the matrix

$$\begin{bmatrix} 1 & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & 2 & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & 3 & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & 4 & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & 5 \end{bmatrix}.$$

11. Perform a basic QR step (do not use any kind of shifting) on the upper Hessenberg matrix

$$\begin{bmatrix} 3 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

V. Differential Equations.

12. Define the concepts of stability and consistency for a finite difference scheme associated with a differential equation.

$$y' = f(x, y)$$

on an interval $[a, b]$ and give an example of an unstable and consistant scheme. Clearly state any relavant theorems you use for the example.

13. Consider the BVP

$$\begin{array}{rcl} y'' & = & xy + y' \\ y(0) & = & \alpha \\ y(1) & = & \beta \\ x & \varepsilon & [0,1] \end{array}$$

Show that the solution to the discrete equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = x_i y_i + \frac{y_{i+1} - y_{i-1}}{2h}$$

for $1 \le i \le N-1$, $h = \frac{\beta - \alpha}{N}$, $y_0 = \alpha$, and $y_N = \beta$ attains its maximum at y_0 or y_N. That is,

$$|y_k| \le \max\{|y_0|, |y_N|\}.$$

14. Show that any solution of the system $y' = Ay$ where

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is stable, but that no solution of the corresponding explicit difference equation

$$y_{k+1} = y_k + hAy_k$$

is stable for any h. Can the same be said of the solution of the corresponding implicity difference equation

$$y_{k+1} = y_k + hAy_{k+1}?$$

Explain your answers!