TOPOLOGY QUALIFYING EXAM Spring 1995

(Strecker and Wu)

Choose and work any **6** of the following 15 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems.** A space always means a topological space below.

1. Let $f, g : X \to Y$ be continuous. Assume that Y is Hausdorff. Show that the set$\{x \in X \mid f(x) = g(x)\}$ is closed in X.

2. Consider the function $f : X \to Y$ between topological spaces and the following statements:

Statement 1. f is continuous iff for every subset A of X $f[cl_X[A]] \subseteq cl_Y f[A]$.

Statement 2. f is continuous iff for each open set U in Y the set $f^{-1}[U]$ is open in X.

Statement 3. f is continuous iff for each convergent net $\alpha : \Lambda \to X$ the net $f \circ \alpha : \Lambda \to Y$ is convergent.

Statement 4. f is continuous iff for each convergent filter $\mathcal{F}$ on X the filter on Y generated by $\{f[F] | F \in \mathcal{F}\}$ is convergent.

Choose one of the above statements as the definition of continuity and use it to show that one of the other statements is true.

3. Prove that every retract of the real line with the usual topology is a closed subset of the line.

4. Show that the following two statements about a topological space X are equivalent:

(a) X is connected.

(b) No two-point discrete space is a quotient space of X.

5. A space X is called *locally metrizable* if each point x of X has a neighborhood base consisting of metrizable subspaces. Show that a compact Hausdorff space X is metrizable if and only if it is locally metrizable.

6. Prove that the sequence $a : \mathbb{N} \to \mathbb{R}$ given by $a_n = \frac{1}{n}$ is **not** an ultranet in $\mathbb{R}$.

7. Prove that for any function f between spaces, the following are equivalent:

(a) f is a homeomorphism.

(b) f is a one-to-one quotient map.

(c) f is a surjective embedding.

8. Prove or disprove the following statements:

(a) A connected open set in a locally path connected space is path connected.

(b) A path connected open set in a locally connected space is connected.

9. Let C_n be the subspace of $\mathbb{R}^2$ defined by

$$C_n := \left\{(x,y) \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2}\right\}$$

and let $Y = \cup_{n=1}^{\infty} C_n \subseteq \mathbb{R}^2$ with the subspace topology. Let $X = C_1 \times \{1, 2, \ldots, \} \subseteq \mathbb{R}^2 \times \mathbb{R}$ with the subspace topology. Define $g : X \to Y$ by

$$g\left((x,y), n\right) := \left(\frac{x}{n}, \frac{y}{n}\right).$$

Show that g is continuous and surjective. Is g a quotient map?

10. Consider $X \times Y$ where Y is a compact space. Let $\{x_0\} \times Y \subseteq N$ where N is open in $X \times Y$. Show that there is a neighborhood W of x_0, W open in X such that

$$\{x_0\} \times Y \subseteq W \times Y \subseteq N.$$

11. (a) Give an example of an ultrafilter on the set $\mathbb{N}$ of natural numbers.
(b) Give an example of a filter on $\mathbb{N}$ that is not an ultrafilter.
(c) Prove that every filter on $\mathbb{N}$ in contained in an ultrafilter on $\mathbb{N}$

12. Prove or disprove the following statements:
(a) $\mathbb{R}$ (with the standard topology) is not a countable union of closed subsets each having empty interior.
(b) $\mathbb{R}$ (with the standard topology) is not a countable union of subsets each having empty interior.

13. Let X be a topological space and (Y, d) a metric space. Let $C(X, Y)$ be the set of continuous maps from X to Y. Show that the compact-open topology and the topology of uniform convergence on compact subsets on $C(X, Y)$ coincide.

14. Find an error in the following purported "proof" of the jactitation that if K is compact, $\mathbb{R}$ has the usual topology, and $f : K \to \mathbb{R}$ is continuous, then f is an open mapping.
"PROOF": Let U be open in K. Then $E = K \setminus U$ is closed. So E is compact. Hence $f[E]$ is a compact subset of $\mathbb{R}$. Thus $f[E]$ must be closed, so that $\mathbb{R} \setminus f[E] = W$ is open. But $W = f[U]$. Hence f preserves open sets and is thus an open mapping.

15. Let Ω_0 be the set of all ordinals less than the first uncountable ordinal, with the order topology. We know that Ω_0 is T_4 and that every real-valued function on Ω_0 is constant on some tail. Find the Stone-Čech compactification of Ω_0. [Give your reasoning].