REAL ANALYSIS QUALIFYING EXAM
Spring 2003

Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Prove that $L^p(\mu)$ is complete for $1 \leq p \leq \infty$.

2.(a) Let $(X, \mathcal{M}, \mu)$ be a measure space. Define what it means for μ to be σ-finite.

(b) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be measure spaces. Show by example that we may have

$$\int_X \int_Y f(x,y) d\nu(y) d\mu(x) \neq \int_Y \int_X f(x,y) d\mu(x) d\nu(y)$$

for some $f \geq 0$ which is $\mathcal{M} \times \mathcal{A}$ measurable if we do not assume both μ and ν are σ-finite.

3. Let λ denote Lebesgue measure on $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be open. Is it true that $\lambda(\overline{U} \setminus U) = 0$?

4. Prove that the closed unit ball of $\ell^p(\mathbb{N})$, $1 \leq p \leq \infty$, is not compact.

5. Suppose H is a Hilbert space, $\{e_1, \ldots, e_n\}$ an orthonormal set in H and $f \in H$. Prove that the quantity $\|f - \sum_{j=1}^{n} a_j e_j\|$ is minimized by taking $a_j = \langle f, e_j \rangle$ for every j.

6. Let $w : X \to [0, \infty)$ be measurable, and let $v(E) = \int_E w d\mu$ for $E \in \mathcal{M}$.

Prove: (a) v is a measure on $\mathcal{M}$ and (b) $\int f dv = \int f w d\mu$ for each nonnegative measurable function f on X.

7. Suppose $1 \leq p \leq \infty$, $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Prove $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

8. Let $E \subset \mathbb{R}$ be a Borel set. Prove that $E' = \{(x,y) \in \mathbb{R}^2 : x + y \in E\}$ is a Borel set.