NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 2003

(Do at least 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Let $\mathbf{A}_n$ be a symmetric positive-definite matrix of the form

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{A}_{n-1} & \mathbf{b} \\ \mathbf{b}^T & a_{nn} \end{bmatrix}.$$

(a). Show that if $\mathbf{A}_{n-1}$ has a Cholesky factorization

$$\mathbf{A}_{n-1} = \mathbf{L}_{n-1}\mathbf{L}_{n-1}^T,$$

with $\mathbf{L}_{n-1}$ nonsingular, then there exist a vector $\mathbf{c}$ and a real number d such that

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{L}_{n-1} & 0 \\ \mathbf{c}^T & d \end{bmatrix} \begin{bmatrix} \mathbf{L}_{n-1}^T & \mathbf{c} \\ 0 & d \end{bmatrix}.$$

(b). Use the observation in part (a) in an inductive argument to prove that every symmetric positive-definite matrix has a Cholesky factorization.

2. Let A be nonsingular and suppose that $A = LU = \hat{L}\hat{U}$, where L and $\hat{L}$ are unit lower-triangular (i.e., their diagonal entries are all 1s) and U and $\hat{U}$ are upper-triangular. Show that $L = \hat{L}$ and $U = \hat{U}$.

3. For the linear system

$$\begin{aligned} x + \alpha y &= a \\ -\alpha x + y &= b \end{aligned}$$

(a). Write out the Jacobi method, the Gauss-Seidel method, and the SOR method for the system.

(b). Under what conditions on α will Jacobi and Gauss-Seidel converge?

(c). Under what conditions on α and ω will SOR converge?

4. Let A be a real symmetric matrix whose eigenvalues λ_j, $j = 1, \ldots, m$, satisfy

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m|.$$

Denote by $\mathbf{q}_j$ a unit eigenvector corresponding to λ_j. The power iteration is the following algorithm:

$\mathbf{v}^{(0)}$ is some vector with Euclidean norm $\|\mathbf{v}^{(0)}\| = 1$.
for $k = 1, 2, \ldots$

$\mathbf{w} = A\mathbf{v}^{(k-1)}$	apply A
$\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|$	normalize
$\lambda^{(k)} = (\mathbf{v}^{(k)})^T A\mathbf{v}^{(k)}$	Rayleigh quotient

Show that if $\mathbf{q}_1^T \mathbf{v}^{(0)} \neq 0$, then

$$\|\mathbf{v}^{(k)} - (\pm \mathbf{q}_1)\| = O(|\lambda_2/\lambda_1|^k),$$

and

$$|\lambda^{(k)} - \lambda_1| = O(|\lambda_2/\lambda_1|^{2k}).$$

The $\pm$ sign means that at each step k, one or the other choice of sign is to be taken, and then the indicated bound holds.

5. Suppose

$$\begin{aligned} \frac{dy}{dx} &= f(x,y), \\ y(x_0) &= y_0, \end{aligned}$$

and let

$$\begin{aligned} y_1(h) &= y_0 + hf(x_0, y_0) \\ y_2(h) &= y_0 + h\tfrac{f(x_0,y_0)+f(x_0+h,y_1(h))}{2} \end{aligned}$$

Assuming $y \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\mathbb{R} \times \mathbb{R})$, show that

$$|y_2(h) - y(x_0 + h)| = O(|h|^3). \tag{1}$$

6. The Chebyshev polynomials are defined by $T_n(x) = \cos(n \cos^{-1}(x))$.

   (a) Show the Chebyshev polynomials are actually polynomials.

   (b) Find a weight function $w(x)$ so the Chebyshev polynomials are orthogonal on $[-1, 1]$. Justify your work.

7. Approximate $\int_0^4 \frac{e^{-x}}{\sqrt{x}}\, dx$. Show your work and give a bound for the error in your approximation.

8. Given the data points $(-2, -4)$, $(-1, -1)$, $(0, 1)$, $(1, 4)$, approximate $y(0.5)$ using a natrual cubic spline interpolation.