Topology Qualifying Exam
Fall 1996

Choose and work any 6 of the following 14 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems**. A space always means a topological space below.

1. Prove that the unit interval $[0,1]$ is connected.

2. Prove that if X is a space with the property that each filter on X has a cluster point, then X is compact.

3. Prove or disprove: There exists an infinite strictly decreasing sequence of ordinal numbers.

4. Prove or disprove:

(a) Every path connected space is connected.

(b) Every connected space is path connected.

5. Let $f : X \to Y$ be a surjective continuous function with compact Hausdorff domain. Prove that Y is Hausdorff if and only if f is closed.

6. For a topological space (X, τ) define $I : P(X) \to P(X)$ by $I(A) = A^\circ$ and $C : P(X) \to P(X)$ by $C(A) = \overline{A}$. For a set A of X, consider the sequence: $A, I(A), CI(A), ICI(A), CICI(A), \ldots$

(a) For any space, what is the largest number of distinct sets that this sequence can contain?

(b) Find $A \subseteq \mathbb{R}$ for which this largest number is obtained (where $\mathbb{R}$ has the usual topology).

7. Prove or disprove: The product of two normal spaces is normal.

8. Prove that a product of spaces $\Pi_{\alpha \in I} X_\alpha$ is locally connected if and only if each space X_α is locally connected and all but finitely many of the X_α are connected.

9. Prove that a metric space is compact if and only if it is complete and totally bounded.

10. Prove that if $f : X \to Y$ is a continuous function between Hausdorff spaces with $f[X]$ dense in Y, then $|Y| \leq 2^{2^{|x|}}$.

11. Let $f : X \to Y$ be a continuous open surjection. Assume that X is a Baire space. Prove that Y is a Baire space.

12. Prove or disprove: There is a completely regular T_1 space X such that the Čech-Stone compactification of X is homeomorphic to the Alexandroff compactification of X.

13. Find an error in the following purported proof that "Every compact T_2 space is metrizable."

Proof If X is compact T_2 it is locally compact T_2 and so it is completely regular. Since X is completely regular, it can be embedded in a Hilbert cube, $I^{\aleph_0}$. But $I^{\aleph_0}$ is metrizable, and metrizability is hereditary. Thus, since X is homeomorphic to a metrizable space, it must be metrizable.

14. Prove that every space that is paracompact and countably compact must be compact.