Complex Variables Qualifying Exam
Fall 1982

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1. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+z^2}$ converges uniformly on compact subsets of $1 < |z| < 2$.

2. (Strong Form of Morera) f is continuous in $D = D(0,1)$ and $\int_{[0,x,x+iy,iy,0]} f = 0$ whenever $x + iy \varepsilon D$.

(i) Show that the function F defined in D by

$$F(z) = \int_{[0,Rez,z]} f$$

satisfies the Cauchy-Riemann Equation.

(ii) Conclude that f is holomorphic in D.

3. Calculate $\int_0^{\infty} \frac{\cos(mx)}{x^2+1} dx$ by the method of residues.

4. (Abstract Derivation) Let U be an open subset of $\mathbb{C}$, $H(U)$ the set of all holomorphic function on U. Suppose that $\Delta : H(U) \to H(U)$ is a *linear* map which satisfies

$$\Delta(fg) = \Delta(f) \cdot g + f \cdot \Delta(g) \quad \forall f, g \varepsilon H(u).$$

(i) Show that $\Delta(c) = 0$ for all constant functions c.

(ii) Let I denote the identity function on $U : I(z) = z$ for all $z \varepsilon U$. Show that

$$\Delta(f) = \Delta(I) \cdot f' \quad \forall f \varepsilon H(U).$$

HINT: Given $\lambda \varepsilon U$, $f \varepsilon H(U)$ write $f(z) = f(\lambda) + (z-\lambda)f'(\lambda) + (z-\lambda)^2 g(z)$ for some $g \varepsilon H(U)$, apply Δ and use (i).

5. Show that all the roots of $z^7 - 5z^3 + 12 = 0$, lie between the circles $|z| = 1$ and $|z| = 2$ by using Rouche's Theorem.

6. (Removable Sets) Let f be continuous in $D = D(0,1)$ and holomorphic in $D \backslash \mathbb{R}$. Show that actually f is holomorphic in D.

HINT: The strong form of Morera's Theorem (earlier problem) is applicable.

7. (i) State Schwarz' Lemma.

(ii) Use (i) to find all the conformal maps of $D(0,1)$ onto $D(0,1)$.

8. (Holomorphic Logarithms) f is a zero-free entire function. Show how to find an entire function g such that $f = e^g$.

9. f and g are entire funcitons which satisfy

$$|f(z)| \leq |g(z)| \forall z \varepsilon \mathbb{C}.$$

Show that f is a constant multiple of g. (Exercise appropriate caution at the zeros of g.)

10. Evaluate

$$\phi_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$$

where C is $|z| = 3$.