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# Real Analysis Qualifying Exam

## January, 2002

**Instructions:** Below you will find 8 problems. Each problem is worth 10 points. Only the best 6 scores will be added.
**Time:** 2 hours.

NOTATIONS: $\mathbb{R}$ = set of all real numbers; $\lambda$ = Lebesgue measure on $\mathbb{R}$; $L^p(I, d\lambda)$ = the space of real-valued Lebesgue measurable functions on $I$ with $\int_I |f|^p \, d\lambda < \infty$ ($I$ = any interval, $p > 0$).

**1.** Give an example of a set $A \subset \mathbb{R}$, which is not Lebesgue measurable.

**2.** Let $K$ be a compact Hausdorff space, and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on $K$, such that for every $x \in K$ one has:

(i) $f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots$

(ii) $\lim_{n\to\infty} f_n(x) = 0$.

Prove that $\lim_{n\to\infty} f_n = 0$ uniformly.

**3.** Suppose $(f_n)_{n=1}^{\infty}$ is a sequence in $L^1(\mathbb{R}, d\lambda)$, such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n| \, d\lambda = 0. \qquad (*)$$

(a) Prove that there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$, with $\lim_{k\to\infty} f_{n_k} = 0$, a.e.

(b) Given an example of a sequence $(f_n)_{n=1}^{\infty}$, satisfying $(*)$, but which is not convergent a.e. to 0.

**4.** Let $C[0,1]$ denote the Banach space of all real-valued continuous functions on $[0,1]$. (The norm on $C[0,1]$ is defined by $\|f\| = \max_{t\in[0,1]} |f(t)|$.)

(a) Prove that any finite dimensional linear subspace $V$ of $C[0,1]$ is closed in the norm topology.

(b) Prove that if $V$ is a linear subspace of $C[0,1]$, which has a countable infinite linear basis, then $V$ is not closed in the norm topology. (Hint: Use Baire's Theorem.)

**5.** Let $f \in L^2([-\pi,\pi], d\lambda)$ be a function with the property:

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0, \quad \forall\, n \in \mathbb{N}.$$

Prove that $f(-x) = f(x)$, a.e.

**6.** Let $f \in L^1(\mathbb{R}, d\lambda)$, and let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^1(\mathbb{R}, d\lambda)$, with

(i) $\lim_{n\to\infty} f_n = f$, a.e.

(ii) $\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1$.

Prove that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$.

**7.** Let $f \in L^1(\mathbb{R}, d\lambda)$. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) \sin nx \, dx = 0.$$

**8.** Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space, and let $\{E_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{M}$ be a disjoint collection. Prove that the set

$$S = \{\lambda \in \Lambda \,:\, \mu(E_\lambda) > 0\}$$

is at most countable.