Kansas State University | Department of Mathematics

Qualifying Exam: Algebra

Spring 2012. January 19, 6:00 p.m. to 9:00 p.m.
Examiners: Prof. Gerald Hoehn and Prof. Zongzhu Lin

Name: ____

Instructions: You should do all problems and provide as complete arguments as you can. Write solutions to different problems on separate pages with the problem number and your name on the top of each page.

Note: All rings are assumed to be associative and with multiplicative identity 1. Ring homomorphisms are assumed to map the multiplicative identities to each other. All modules over a ring are assumed to be unitary, i.e., 1 acts as identity. The integers, the rational numbers and the complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$, respectively.

(1) Let X be a set and $F\langle X\rangle$ be a free group over X. If $Y \subseteq X$ is a subset and H is the normal subgroup of $F\langle X\rangle$ generated by Y, show that $F\langle X\rangle/H \cong F\langle Y'\rangle$ as groups. Here $F\langle Y'\rangle$ is a free group over the set $Y' = X \setminus Y$. Recall that a free group $F\langle X\rangle$ over a set X is a group $F\langle X\rangle$ together with a map $\rho : X \to F\langle X\rangle$ of sets such that for any group G and any map $\varphi : X \to G$ of sets, there is a unique group homomorphism $\psi : F\langle X\rangle \to G$ such that $\varphi = \psi \circ \rho$.

(2) Let R be an integral domain and $S \subseteq R$ be a subset closed under multiplication and $0 \notin S$. If $Q(R)$ denotes the field of fractions of the integral domain R, define $S^{-1}R = \{\frac{r}{s} \mid r \in R,\ s \in S\}$ which is a subring of $Q(R)$ with the natural ring homomorphism $\rho : R \to S^{-1}R$ defined by $\rho(r) = \frac{rs}{s}$ for some $s \in S$. Show the following:

(a) If $p \in R$ is a prime element, then $\rho(p)$ is either invertible or prime in $S^{-1}R$ depending on whether p is a factor of some $s \in S$.

(b) If R is a unique factorization domain then $S^{-1}R$ is also a unique factorization domain.

(3) Let R be a ring and M be a left R-module. For any two submodules E, $N \subseteq M$, show that the subset

$$(N : E) = \{r \in R \mid rx \in N \text{ for all } x \in E\}$$

of R is a two-sided ideal of R.

(4) Let $f(x) = x^3 + 3x^2 - 5x - 15$ and $R = \mathbb{Q}[x]/\langle f(x)\rangle$ be the quotient ring of the polynomial ring $\mathbb{Q}[x]$ modulo the ideal generated by $f(x)$. Compute the automorphism group $\mathrm{Aut}(R)$ of the ring R.

(5) Let k be a field (not necessarily algebraically closed) and V be a vector space over the field k. Let S, $T : V \to V$ be two k-linear transformations.

(a) Show that the set of all eigenvalues of ST in the field k and the set of all eigenvalues of TS in the field k are equal if V is finite dimensional.

(b) Show, by example, that the set of all eigenvalues of ST in k and the set of all eigenvalues of TS in k can be different if V is infinite dimensional.