Spring 2011 Topology Qualifying Exam

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

(1) (a) Define what it means for a topological space to be connected.
(b) Define what it means for a topological space to be path connected.
(c) Prove that if a topological space X is path connected, then X is connected.
(d) Give an example of a topological space which is connected, but not path connected.

(2) (a) Show that the space $Mat_{\geq k}(m \times n)$ of $m \times n$ matrices of rank $\geq k$ is open in $Mat(m \times n) = \mathbb{R}^{mn}$.
(b) When is the set $Mat_k(m \times n)$ of matrices of rank k open in $Mat(m \times n)$?
(c) When is $Mat_k(m \times n)$ closed?

(3) Give an example of a smooth map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ that has a dense set of critical values [hint: use partition of unity].

(4) A *weak equivalence* between two vector bundles $\pi_1 \colon E_1 \to B$, $\pi_2 \colon E_2 \to B$ over the same base space B is a bundle map (F, f), where $F \colon E_1 \to E_2$ is an isomorphism on each fiber and $f \colon B \to B$ is a homeomorphism. Find two inequivalent, but weakly equivalent, vector bundles over the following spaces:
(a) the disjoint union of two circles;
(b) the wedge of two circles;
(c) the torus.

(5) (a) Compute $H^*_{dR}(\mathbb{R}^2 \setminus \{0\})$.
(b) Prove that $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$ is closed but not exact on $\mathbb{R}^2 \setminus \{0\}$.
(c) Let M be compact, connected, orientable smooth manifold of dimension 2, and let p be any point of M. Let U be a neighborhood of p diffeomorphic to $\mathbb{R}^2$ and let $V = M \setminus \{p\}$. Show that the connecting homomorphism $\delta \colon H^1_{dR}(U \cap V) \to H^2_{dR}(M)$ is an isomorphism [hint: show that $\delta[\omega] \neq 0$, where ω is the 1-form on $U \cap V \simeq \mathbb{R}^2 \setminus \{0\}$ as in 5b].
(d) For M and p as in 5c, show that the inclusion $M \setminus \{p\} \hookrightarrow M$ induces isomorphism in de Rham cohomology in degrees 0 and 1.

(6) Let X_n be the quotient space of $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by the equivalence relation $\equiv$ generated by

$$e^{i\theta} \equiv e^{i\omega} \text{ iff } e^{in\theta} = e^{in\omega}$$

(Note: the only equivalence classes with more than one element are the equivalence classes of points on the bounding copy of S^1.)

Prove that

$$\pi_1(X_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

and

$$H_k(X_n) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/n & k = 1 \\ 0 & k > 1 \end{cases}$$

(7) Let X_n be as in the previous problem. For this problem, you may assume the results of the previous problem even if you have not proved them.

(a) Find the homology of $X_n \times X_m$.

(b) Find the homology of $\Sigma X_n \times X_m$.

(c) Find the homology of $\Sigma(X_n \times X_m)$.

(8) (a) Find $H^*(\mathbb{RP}^2)$.

(b) Find $H^*(\mathbb{CP}^3)$.

In each case justify your work either by giving a CW complex decomposition of the space and computing the cellular cohomology, *or* by a calculation in which each step is justified by one of the Eilenberg-MacLane Axioms or the Meyer-Vietoris Theorem.