Name ______________________

REAL ANALYSIS QUALIFYING EXAM
Fall 2000
(Saeki & Moore)

Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Let $\mathcal{F}$ be a collection of subsets of a set Ω with the following properties:

(i) $\Omega \in \mathcal{F}$

(ii) $A, B \in \mathcal{F}$ implies $A - B \in \mathcal{F}$

(iii) $A_1, A_2, \ldots \in \mathcal{F}$ and $A_1 \subseteq A_2 \subseteq A_3 \ldots$ implies $\bigcup_{i=1}^{\infty} Ai \in \mathcal{F}$.

Prove that $\mathcal{F}$ is a σ-algebra.

2. If $f \in L^1(\mu)$, prove that $\{x : f(x) \neq 0\}$ has σ-finite measure.

3. Suppose f is a complex measurable function on x,
and set $\varphi(p) = \|f\|_p^P$ for $p > 0$.
Set $E = \{p > 0 : \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, $s + E$, prove that $p \in E$.

(b) By (a) e is connected, hence an interval. Is E open?

4. Prove that if $f \in L^1([-\pi, \pi])$ then $\lim_{n\to\infty} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = 0$.

5. Let $f_n(x) = e^{inx}$. Prove

(a) $\{f_n\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2[-\pi, \pi]$ with respect to the measure $\dfrac{dx}{2\pi}$.

(b) no subsequence of $\{f_n\}_{n=1}^{\infty}$ converges pointwise a.e. on $[-\pi, \pi]$.

6. Suppose $f \in L^1(\mu)$. Prove $|\int f \, d\mu| \leq \int |f| \, d\mu$. [You may use the linearity of integration on the complex space $L^1(\mu)$.]

7. Suppose $\mu(x) = 1$ and $f : x \to \mathbb{C}$ is measurable. Prove that $\|f\|_p \leq \|g\|_q$ whenever $0 < p < q \leq \infty$

8. Let $f : x \to [0, \infty]$ be a measurable function such that $\int_E f \, d\mu < \infty$ for each measurable set E with $\mu(E) < \infty$. Prove that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\int_E f \, d\mu < \varepsilon$ for every measurable set E with $\mu(E) < \delta$.