Numerical Analysis Qualifying Exam
Spring 1992

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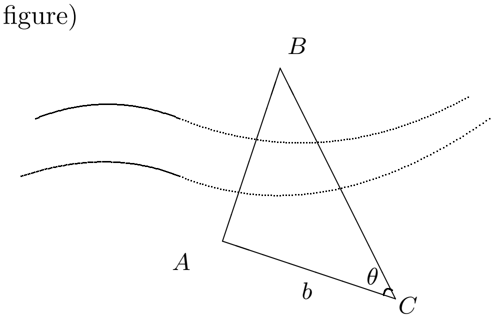
1. A cannon is located at A on the bank of a river. An enemy position is on the other bank of the river at B. To estimate the distance from A to B, a soldier chose a position C on the bank of his side such that AC is perpendicular to AB. He measured the distance from A to C, which is b m (meter), with relative error less than or equal to 0.001. He also measured the angle θ between AC and CB, with relative error ≤ 0.01. Give an approximation of the distance from A to B and give an estimate of the relative error of the approximation in terms of the known quantities. (see figure)



2. Assume $f \in C^3[a,b], |f(x)| \leq M, |f'''(x)| \leq N$ for $x \in [a,b]$. Given the numerical difference formula

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f'''(\xi)$$

where $x_0 \in (a,b), |\xi - x_0| \leq h$, h is sufficiently small, and the first term in the previuos formula is used to approximate $f'(x_0)$. Assume that the relative error due to round-off error in the evaluation of f is bounded by ε and no round-off error in the evaluation of h.

(a) Find a bound for the total absolute error of the computed approximation of $f'(x_0)$ in terms of M, N and h.

(b) Find the value of h which minimized the bound.

(c) Is the difference formular stable with respect to round-off error? (Hint: take $h \to 0$)

3. (a) Suppose we want the numerical integration formula

$$\int_a^b f(x)dx \approx A_0 f(x_0) + \cdots + A_n f(x_n)$$

to be exact for all polynomials of degree $\leq n$, where $x_0, \ldots, x_n$ are given distinct points on $[a,b]$. Give the expressions for $A_0, \ldots, A_n$ in terms of the Lagrange polynomial coefficients

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}.$$

(b) Suppose $x_j \in [a,b]$ for $j = 0, 1, \ldots, n$. Show that the error in the above formula is bounded by

$$\frac{1}{(n+1)!}(b-a)^{n+2} \sup_{x \in [a,b]} |f^{(n+1)}(x)|$$

for $f \in C^{n+1}[a,b]$.

4. Let $f(x)$ be given on the points $a = x_0 < x_1 < \cdots < x_n = b$. State the properties which define a cubic spline $S(x)$ for these data with $S''(a) = S''(b) = 0$. Show that

$$\int_a^b [g''(x)]^2 \, dx \geq \int_a^b [S''(x)]^2 \, dx$$

where $g(x)$ is any twicely-continuously-differentiable function that interpolates $f(x)$ at x_j, for $j = 0, 1, \ldots, n$. This indicates that the cubic spline is the least oscillatory one among twicely-continuously-differentiable interpolate functions. (**Hint:** Write $g(x) = S(x) + r(x)$)

5. Describe a modification of the Newton's method, which converges quadratically for a double root p of the equation $f(x) = 0$ if $f(x)$ is sufficiently smooth in a neighborhood of p and if the initial guess is sufficiently close to p. Justify your answer (you may quote a theorem on order of convergence)

6. (a) Show that the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

is invertible, but cannot be written as

$$A = LU$$

with L lower triangular and U upper triangular.

(b) How would you solve a linear system

$$Ax = b$$

for x, given any matrix A having this property.

7. Given an n by n linear system $Ax = b$, we do a splitting for A as $A = M - N$, where M is nonsingular, and construct an iterative method for solving the linear system:

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b$$

Suppose that the iterative matrix $T = M^{-1}N$ is symmetric, positive definite with eigenvalues having the property:

$$\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n.$$

Show that for a general starting value $x^{(0)}$, and for large k

$$\| x - x^{(k)} \|_2 \approx C\lambda_1^k,$$

or

$$\| x - x^{(k)} \|_2 \approx C\rho(T)^k,$$

where $\rho(T)$ is the spectral radius of T, C is a non-negative constant satisfying $C \leq \| x - x^{(0)} \|_2$, and x is the exact solution of the linear system. When does the iterative method converge for any starting value $x^{(0)}$?

8. Given a matrix

$$A = \begin{bmatrix} a_1 & \varepsilon & & & & & & \\ 1 & a_2 & 1 & & & & & \\ & \varepsilon & a_3 & \varepsilon & & & & \\ & & 1 & a_4 & 1 & & & \\ & & & \varepsilon & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & 1 \\ & & & & & & \varepsilon & a_{2n+1} \end{bmatrix}$$

with distinct $a_i, i = 1, 2, \ldots, 2n + 1$. Show that for sufficiently small ε the matrix A has $2n + 1$ distinct eigenvalues $\lambda_i, i = 1, 2 \ldots, 2n + 1$ such that $|\lambda_i - a_i| \leq K\varepsilon$ with a constant K independent of ε.

(Hint: Write $A = B + \varepsilon E$ with B, E independent of ε. You may quote a theorem on perturbation of eignvalues and a theorem that the roots of any polynomial are continuous functions of its coefficients)