ALGEBRA QUALIFYING EXAM, JAN. 26, 2008

Instructions Please choose 8 from the following 10 problems, and solve them as best you can. Indicate the 8 problems that you would like to submit, by circling their numbers on this "problem sheet".

1. Let G be a group of order $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Show that G has a normal subgroup of order 5.

2. Let G be a non-abelian simple group. Show that the center of the group $Aut(G)$ of automorphisms of G is the identity group.

3. Let G be a finite group and let X be a faithful G-set (i.e. a set on which G acts, and such that the identity element of G is the only element of G which acts trivially on X). Let $A \leq G$ be a subgroup of G, and let B be a subgroup of G such that A and B commute element-wise. Suppose that A acts transitively on X. Show:

(a) $|B|$ is a divisor of $|X|$.
(b) If A is abelian then $|A| = |X|$.

4. Let A and B be two 3×3 complex matrices. Prove that A and B are similar if and only if they have the same characteristic polynomial and the same minimal polynomial.

5. Let R be a commutative ring with multiplicative identity, and assume that every ideal of R is a prime ideal. Show that R is a field.

6. Let $\overline{\mathbb{Q}}$ be the algebraic closure of the field $\mathbb{Q}$ of rational numbers, and let $\alpha \in \overline{\mathbb{Q}}$ with $\alpha \notin \mathbb{Q}$. It can be shown – using Zorn's Lemma – that there is a subfield M of $\overline{\mathbb{Q}}$ which is maximal with respect to the property that $\alpha \notin M$.

(a) Prove that any finite galois extension K/M has cyclic automorphism group. [Hint: a finite group with a unique maximal subgroup is cyclic].
(b) Prove that each finite field extension K/M is Galois.

7. Let A be a commutative ring with 1, let $a \in \mathcal{U}(A)$ be a unit, and let $b \in A$ be an arbitrary element of A. Regard A as a subring of the polynomial ring $A[x]$ in the usual way (as the set of constant polynomials). Prove that there is a unique automorphism σ of $A[x]$ having the two properties:

. $\sigma(x) = ax + b$, and
. the restriction $\sigma\,|_A$ of σ to A is the identity map on A.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

8. Let R be an associative (but not necessarily commutative) ring with 1, and let M be an irreducible left R-module. Show that there is a maximal left ideal I of R such that R/I is isomorphic to M as left R-modules.

9. Let F be a field and let V be a vector space over F. (We do *not* assume that V is finite-dimensional.) Let $\phi : V \to V$ be a nilpotent F-linear transformation. Show that ϕ has an eigenvector in V.

10. Let $\phi : V \to V$ be an F-linear endomorphism of a finite-dimensional vector space V over F. Consider V as a module for the polynomial ring $F[x]$, via the evaluation map $F[x] \to End_F(V)$ given by $x \mapsto \phi$. (That is; the polynomial $f(x)$ acts on V as $f(\phi)$.) Show that V is irreducible as a module for $F[x]$ if and only if the characteristic polynomial of ϕ is irreducible as a polynomial in $F[x]$.