Algebra Qualifying Exam
August 22, 2000

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings on this exam are associative and have multiplicative identity 1. All integral domains are assumed to be commutative.

1. Prove that a finite group of order $224 = 2^5 \cdot 7$ cannot be a simple group.
2. Let G be a finite group, and let $\phi : G \to G$ be a homomorphism satisfying $\phi^2 = 1_G$. Assume also that $\phi(g) = g$ if and only if $g = e$, the identity element of G.
   (a) Show that for all $g \in G$, there exists $h \in G$ with $g = h^{-1} \cdot f(h)$.
   (b) Prove that G must be abelian.
3. Let G be a group, and let H be a cyclic normal subgroup of G. Prove that the commutator subgroup G' is a subgroup of the centralizer $C_G(H) = \{g \in G \,|\, gh = hg, \text{ for all } h \in H\}$. (Hint: note that $\mathrm{Aut}(H)$ is an abelian group.)
4. Let R be a ring and assume that $R = \bigoplus_{i=1}^n I_i$, where $I_1, \ldots I_n$ are minimal left ideals of R. If M is an irreducible left R-module, prove that $M \cong I_j$ for some index j, $1 \leq j \leq n$.
5. Let R be a unique factorization domain and assume that for every pair of elements $a, b \in R$, their greatest common divisor d can be expressed in the form $d = ra + sb$, for suitable $r, s \in R$. Prove that R must be a principal ideal domain.
6. Let R be a simple ring and let $C = Z(R)$, the center of R. Prove that C is a field. (Hint: if $0 \neq a \in C$, consider the mapping $R \to R$, given by $r \mapsto ar$. Why must this mapping be bijective?)
7. Let A be a nilpotent $n \times n$ matrix, and assume that its Jordan canonical form consists of a single Jordan block. Find the Jordan canonical form of A^2. (Hint: there are two cases, depending on the parity of n.)
8. Let V be an $\mathbb{F}$-vector space, where $\mathbb{F}$ is an algebraically closed field. Assume that $T : V \to V$ is a linear transformation such that for any T-invariant subspace $W \subseteq V$, there is a T-invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$. Prove that T must be diagonalizable.
9. Let $\mathbb{F}$ be any field and let $\mathbb{F}[x]$ be the ring of polynomials in the indeterminate x, with coefficients in the field $\mathbb{F}$. Prove that there are infinitely many irreducible polynomials in $\mathbb{F}[x]$. (Hint: Euclid's proof of the corresponding result for the ring $\mathbb{Z}$ of integers works here, too.)
10. Let $\alpha = \sqrt{2 + \sqrt{2}}$, and prove that the Galois group $\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})$ is cyclic of order 4.