Real and Complex Analysis Qualifying Exam - Spring 2015

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Instructions: Do 8 out of the 10 problems below. Pick 4 problems from # 1-5 and pick 4 problems from # 6-10.

Notations: \mathbb{R} = set of all real numbers; \mathbb{C} = set of all complex numbers; \mathbb{D} = $\{z \in \mathbb{C} : |z| < 1\}$ (the unit disk). The Lebesgue measure on \mathbb{R} is denoted by m.

- 1. Is there a Borel set $A \subset \mathbb{R}$ such that $0 < m(A \cap I) < m(I)$, for every interval $I \subset \mathbb{R}$? Justify your answer.
- 2. Show that if $f \in L^1(\mathbb{R})$ then the function $F(x) = \int_{-\infty}^x f(t) dt$ is continuous. (Note: Do not assume absolute continuity of the integral, unless you prove it.)
- 3. Compute the following limit and justify your calculation.

$$\lim_{n \to \infty} \int_0^1 \frac{1 + \frac{n}{2} x}{(1 + x)^{\frac{n}{2}}} dx.$$

- **4.** Let $f: \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function.
 - (a) Define $||f||_{\infty}$.
 - (b) Is there a function f s.t. $f \in L^p, \forall p > 1$, but $f \notin L^{\infty}$. Justify your answer.
 - (c) Is there a countable dense subset in $L^{\infty}(\mathbb{R})$? Justify your answer.
- **5.** Let H be a Hilbert space. Show that if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in H, then for every $x \in H$ the following (Bessel's) inequality holds

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \le ||x||^2.$$

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6. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. Define

$$\ell = \limsup_{n \to \infty} \sqrt[n]{|a_n|} \in [0, \infty],$$

and put

$$R = \begin{cases} 1/\ell & \text{if } 0 < \ell < \infty \\ 0 & \text{if } \ell = \infty \\ \infty & \text{if } \ell = 0 \end{cases}$$

Prove that the power series $\sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{C}$ is

- (i) convergent, if |z| < R;
- (ii) divergent, if |z| > R.

7. Let $f:\mathbb{C}\to\mathbb{C}$ be an entire non-constant function. Show that $f(\mathbb{C})$ is dense in \mathbb{C} .

8. How many roots (counting multiplicities) does

$$z^4 + 3z - 1$$

have in the annulus $\Omega = \{z: 1 < |z| < 2\}$?

9. Let f be an analytic function defined on the strip $-1 < \operatorname{Im} z < 1$. Consider the set

$$S = \{x \in \mathbb{R} : f(x) \in \mathbb{R}\}.$$

Prove that, either S is discrete (i.e. it has no accumulation points), or $S = \mathbb{R}$. HINT: Consider the function $g(z) = \overline{f(\overline{z})}$.

10. Is there a conformal map from the slit strip

$$\Omega := \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 1\} \setminus (-\infty, 0]$$

onto the unit disk? If yes, find one. If no, prove that there is none.

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