TOPOLOGY QUALIFYING EXAM

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Do nine but no more than nine of the following 15 problems:

1. Prove or give a counter example:

   The product of locally connected spaces is locally connected.

2. a) Let $f:X \to Y$ and $g:X \to Y$ be continuous, with Y Hausdorff. If D is dense in X and $f|D = g|D$, show that $f = g$.

   b) Prove or disprove this assertion if Y is not assumed to be Hausdorff.

3. Let C be the Cantor set in $[0,1]$; i.e.,

   $$C = \{ \sum_{i=1}^{\infty} \frac{n_i}{3^i} \mid n_i = 0,2\}.$$ Prove that C is homeomorphic

   to the countable power of a two-point discrete space $\{0,2\}$.

4. Prove or disprove:

   A completely regular T_1 space is connected if and only if its Stone-Čech compactification is connected.

5. Prove that any retract of a product of real lines must be locally connected.

6. Prove or disprove:

   If X is compact and metrizable, then any distance preserving function $f:X \to X$ must be surjective.

7. Prove that the following are equivalent:

   a) X is countably compact.

   b) Every decreasing sequence of closed nonempty subsets of X has a nonempty intersection.

8. Prove that the three-point space (X,τ) with $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}\}$ is "universal" in the sense that every topological space is a subspace of some power of it.

9. Prove or give a counter example for each of the following:

   a) Every quotient of a locally compact space is locally compact.

   b) Every separable space is second countable.

   c) Every subspace of a second countable space is separable.

   d) Every compact Hausdorff space is metrizable.

10. Give four examples, one compact, one locally compact but non-compact, one non-locally compact, and one non-locally connected, of spaces homotopically equivalent to the circle S^1 but not homeomorphic to S^1.

11. Let X be a compact Hausdorff space. Prove that X is metrizable if and only if the diagonal $\Delta \subseteq X \times X$ is a G_δ.

12. Prove:

    a) The inverse limit of any inverse spectrum of compact Hausdorff spaces is compact Hausdorff.

    b) The direct limit of any direct spectrum of locally connected spaces is locally connected.

13. Prove that if $A \times B$ is a compact subset of $X \times Y$ contained in an open set W of $X \times Y$, then there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that

$$A \times B \subseteq U \times V \subseteq W.$$

14. Call a metric space X an <u>absolute neighborhood extensor</u> (ANE) if for each metric space X, each closed subset A of X, and each continuous map $f:A \to Y$, there is a neighborhood N of A in X and a continuous map $\hat{f}:N \to X$ such that $\hat{f}|A = f$. If Y is an ANE, A is a compact subset of a metric space X, and $f,g:A \to Y$ are homotopic, show that f and g extend to maps (on a neighborhood of A) $\hat{f},\hat{g}:N \to Y$ which are homotopic.

15. Let A be a closed subset of the space X, and let $f:A \to Y$ be a closed continuous function. Prove that if X and Y are paracompact then X attached to Y by f $(X \cup_f Y)$ must be paracompact.