Algebra Qualifying Exam
Spring 1990

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a group, and let H, K be subgroups of G. If $K \triangleleft G$, prove that $HK/K \cong H/(H \cap K)$.

2. Let G be a finite group and let $\mathcal{C}$ be a conjugacy class of elements in G. If $|\mathcal{C}| = \frac{1}{2}|G|$, show that every element of $\mathcal{C}$ is an involution (i.e., an element of order 2).

3. Let R be a commutative ring, and let $x \in R$. Define what it means for x to be an *irreducible* element, and define what it means for x to be *prime*. If R is a *unique factorization domain*, show that x is irreducible if and only if x is prime.

4. State and prove the *Eisenstein criterion* for irreducibility of polynomials.

5. Let $K \subseteq \mathbb{C}$ be the splitting field over $\mathbb{Q}$ for the polynomial $x^6 - 1 \in \mathbb{Q}[x]$. Compute the Galois group of K over $\mathbb{Q}$ and show exactly how it operates on K.

6. Let $f(x), g(x) \in F[x]$, and assume that $F \subseteq K$ is an extension of fields. If $f(x)$ divided $g(x)$ in $K[x]$, prove that $f(x)$ divides $g(x)$ in $F[x]$.

7. Consider the map $T : \mathbb{C} \to \mathbb{C}$ defined by setting $T(\alpha) = (2+i)\alpha$. If we regard T as an $\mathbb{R}$-linear transformation of the 2-dimensional $\mathbb{R}$-vector space $\mathbb{C}$, compute $\det(T)$.

8. Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

be a real matrix. Find 3×3 real matrices N, D with $A = N + D, DN = ND$, where N is nilpotent and D is diagonalizable.

9. A band of 17 pirates decided to divide their gold coins into equal portions. When they found that they had 3 coins remaining, they agreed to give them to their Chinese cook Wun Tu. But 6 of the pirates were killed in a fight. Now when the treasure was divided equally among them, there were 4 coins left that they considered giving to Wun Tu. Before they could divide the coins, there was a shipwreck and only 6 pirates, the coins, and the cook were saved. This time equal divison left a remainder of 5 coins for the cook. Now Wun Tu took advantage of his culinary position to concoct a poison mushroom stew so that the entire fortune in gold coins became his own .What is the smallest number of coins that the cook would have finally received?