Algebra Qualifying Exam
January 21, 1999

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be a finite abelian group.

   (a) State what it means for G to be an *elementary abelian* p-group, where p is a prime number.

   (b) If G is an elementary abelian p-group, explain fully in what sense can G be regarded as an $\mathbb{F}_p$-vector space, where $\mathbb{F}_p$ is the field of p elements.

2. Let $G = \langle x, y \rangle$ be a finite group, where x, y are *involutions*. Prove that G has a normal subgroup of index 2. (Look at $H = \langle xy \rangle$.)

3. Let G be a group acting on the set Ω. Assume that $\omega \in \Omega$, set $H = \mathrm{Stab}_G(\omega)$, and assume that K is a subgroup of G acting transitively on Ω. Prove that $G = KH$.

4. Let R be a commutative ring and assume that $M, M_1, M_2, \ldots, M_r$ are maximal ideals of R with $M_1 M_2 \cdots M_r \subseteq M$. Prove that for some i, $M_i = M$.

5. Let $R = \{\frac{a}{b} \in \mathbb{Q} | 2 \not| b\}$, a subring of the rational number field $\mathbb{Q}$. Show that R has a unique maximal ideal, and find it.

6. Let R be a ring and let M be an R-module. Assume that $M_1, M_2 \subseteq M$ with $M = M_1 \oplus M_2$. Prove or give a counterexample to the assertion: *If $N \subseteq M$ is a submodule, then*

$$N = N \cap M_1 \oplus N \cap M_2.$$

7. Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{C}$. Given that $m_{\alpha,\mathbb{Q}}(x) = x^4 - 4x^2 + 2$, and that the roots of $f(x) = m_{\alpha,\mathbb{Q}}(x)$ are $\alpha = \alpha_1 = \sqrt{2+\sqrt{2}}, \alpha_2 = -\sqrt{2+\sqrt{2}}, \alpha_3 = \sqrt{2-\sqrt{2}}, \alpha_4 = -\sqrt{2-\sqrt{2}}$, answer the following:
   (a) Compute the degree of the splitting field $\mathbb{K}$ over $\mathbb{Q}$ of $f(x)$.
   (b) Show that the Galois group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ is cyclic.

8. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q $(= p^r)$ elements, where p is prime, and let $\mathbb{K} = \mathbb{F}_{q^3} \supseteq \mathbb{F}$. Say that elements $\alpha, \beta \in \mathbb{K}$ are *equivalent* if they have the same minimimal polynomial over $\mathbb{F}$. Clearly this is an equivalence relation on $\mathbb{K}$. Compute the number of equivalence classes in $\mathbb{K}$ as a function of q. (Hint: this is *extremely* easy.)

9. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space over the field $\mathbb{F}$. Suppose that T has the following invariant factors:

   $$1 + x,\ x^2(1 + x),\ x^2(1 + x)(1 + x + x^2).$$

   Answer the following questions:
   (a) What is $\dim_{\mathbb{F}} V$?
   (b) Is T injective?
   (c) What is the minimal polynomial of T
   (d) Does T have a Jordan canonical form over $\mathbb{F}$ with respect to an appropriate basis of V? (If this depends on the field give an example of a field $\mathbb{F}$, for which the answer is "yes," and find the Jordan canonical form.)

10. Let $\mathbb{F}$ be a field. If V is a finite-dimensional $\mathbb{F}$-vector space and if $T : V \to V$ is a linear transformation, we have the notion of *minimal polynomial* $m_T(x) \in \mathbb{F}[x]$ of T. Likewise, if $\mathbb{K} \supseteq \mathbb{F}$ is a finite field extension, and if $\alpha \in \mathbb{K}$, then we also have the notion of *minimal polynomial* $m_\alpha(x) \in \mathbb{F}[x]$ of the field elements α. These notions of minimal polynomial share many similarities *except* that $m_\alpha(x)$ is *always* irreducible, whereas $m_T(x)$ *need not* be irreducible. Prove this.