Numerical Analysis Qualifying Exam
Spring 1989

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1. Establish a finite difference formula to approximate $\frac{\partial f(x,y)}{\partial x}$ using $f(x,y), f(x-h,y), f(x-2h,y)$. Be as accurate as possible and derive an expression of the truncation error. Assume $f(x,y)$ is smooth enough. Then explain how one might improve the accuracy using Richardson's extrapolation.

2. A quadrature formula for $I(f) = \int_a^b f(x)dx$ is given by

$$I_n(f) = h\sum_{j=1}^{n} f(a+jh), \quad h = \frac{b-a}{n}.$$

(a) Derive an error estimate for the formula (stating the condition on $f(x)$).

(b) Apply the integral formula to $\int_0^1 \ln x dx$ and derive the error by direct calculation of $I(f)$ and $I_n(f)$ for large n. Compare the error in this case with the error in (a). (Hint: $n! \approx (n/e)^n\sqrt{2\pi n}$).

3. Consider the integral

$$E_n = \int_0^1 x^n e^{x-1} dx, \quad n = 1, 2, 3, \ldots$$

Show

(a) $E_n = 1 - nE_{n-1}$.

(b) $E_1 = 1/e$.

(c) $E_n > 0$.

(d) $E_n < E_{n-1}$.

(e) $E_n \to 0$ as $n \to \infty$.

Explain why the iterative scheme (a) is unstable to round-off error. Can you suggest an improvement of the numerical method to compute E_{10}?

4. (a) Write the Newton's divided difference interpolation formula for

$$f(x) = \frac{1}{x}, \quad \text{with } n = 2, x_0 = 2, x_1 = 3, x_2 = 4.$$

(b) Derive an error formula for the interpolation polynomial $P_n(x)$, which interpolates $f(x)$ at $n+1$ distinct points $x_0, x_1, \ldots, x_n$.

(c) For $f(x) = 1/x, x_0 = 2, x_j = x_0 + jh, j = 1, 2, \ldots, n, h = 2/n, x_n = 4$, show

$$\max_{a \le x \le 4} |f(x) - P_n(x)| \to 0, \text{ as } n \to \infty.$$

(d) Is it always true that

$$\max_{a \le x \le b} |f(x) - P_n(x)| \to 0, \text{ as } n \to \infty,$$

for any $f(x) \in C^\infty[a,b], x_0 = a, x_j = a + jh, j = 1, 2, \ldots, n, h = (b-a)/n, x_n = b$?

5. Prove that the Jacobi iterative method applied to $Ax = b$ converges for any starting vector $x^{(0)}$ if A is strictly column diagonally dominant. i.e.

$$|a_{jj}| > \sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{jj}|, \text{ for } j = 1, 2, \dots, n.$$

6. A complex matrix A is called normal if $AA^* = A^*A(A^*$ denotes the conjugate transpose of $A)$. Show

(a) $A_{n\times n}$ is normal if and only if there is a unitary matrix U, such that

$$U^*AU = \text{ diag } (\lambda_1, \lambda_2, \dots, \lambda_n).$$

(Hint: use an appropriate theorem and show a normal upper triangle matrix is diagonal).

(b) If A is normal, then $\| A \|_2 = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

7. Consider the Initial Value Problem

$$\dot{y} = f(y, t), \quad y(t_0) = g_0 \text{ given }, \quad t_0 \le t \le b, \quad b \text{ fixed.} \tag{1}$$

f is assumed to be Lipschitz in y and continuous in t. We seek to solve the IVP by means of the explicit one-step method

$$y_{n+1} = y_n + h\phi(y_n, t_n, h), \quad t_n = t_0 + n\Delta t \le b, \quad \Delta t = \frac{b - t_0}{N}, \tag{2}$$

with starting value y_0 not necessarily equal to g_0. Do the following:

(a) Define what is meant by **stability** of the method.

(b) State and justify conditions on $\phi(y, t, h)$ which make the scheme stable.

(c) Define **convergence** of the above one-step method.

(d) State conditions which quarantee convergence.

Local truncation error d_n and global truncation error e_n are defined by

$$d_n \equiv y(t_n) + h\phi(y(t_n), t_n, h) - y(t_{n+1}), \text{ and}$$

$$e_n \equiv y_n - y(t_n), \text{ respectively,}$$

where $y(t)$ is the exact solution of the IVP (1).

(e) For the explicit Euler scheme (i.e. $\phi(y, t, h) = f(y, t)$) show that

$$|d_n| \le Dh^2, \text{ where } D = \max |\ddot{y}(t)/2|,$$

and that

$$|e_n| \le (1 + hL)^n |e_0| + Dh^2 \frac{[(1 + hL)^n - 1]}{hL}.$$

Conclude that the explicity Euler scheme is convergent.

8. Consider the discrete analog of the eigenvalue problem

$$y'' + \lambda y = 0, 0 < x < \pi,$$

$$y(0) = y(\pi) = 0,$$

given by

$$\frac{y_{i+1} + y_{i-1} - 2y_i}{(\Delta x)^2} + \lambda y_i = 0,$$

$$y_0 = y_N = 0,$$

defined on the uniform mesh $0 = x_0 < x_1 < \cdots < x_N = \pi$. Compute the eigenvalues of the discrete problem by solving the finite difference equation. How do these eigenvalues compare with those of the continuous problems?