Analysis Qualifying Exam - Fall 2011

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Instructions: Do all ten problems. Start each problem on a separate page and clearly indicate the problem number. In a multiple-part problem, a solution to a later part of the problem that uses a result from an earlier part may still earn full credit even if the earlier part was not done successfully.

Notation: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R}$ = the set of reals, $\mathbb{C}$ = the set of complexes, $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ for $a \in \mathbb{C}$, $r > 0$, $\mathbb{D} = D(0, 1)$, U = a non-empty open subset of $\mathbb{C}$, $H(U)$ = the set of all holomorphic (i.e., analytic) functions on U. By λ_1 we denote Lebesgue measure on $\mathbb{R}$ and by λ_2 Lebesgue measure on $\mathbb{R}^2$.

1. The function L is defined in $\mathbb{C}_- := \mathbb{C} \setminus (-\infty, 0]$ by
$$L(z) := \int_{[1,z]} \frac{1}{\xi} \, \mathrm{d}\xi, \quad \text{that is,} \quad \int_0^1 \frac{z-1}{1+t(z-1)} \, \mathrm{d}t.$$
Prove that L is holomorphic and $L'(z_0) = 1/z_0$ for all $z_0 \in \mathbb{C}_-$.

2. (a) Show that if f is holomorphic and zero-free in Ω and f'/f has a primitive (i.e., is the derivative of some $F \in H(\Omega)$), then f has a holomorphic logarithm. In fact, there is a $c \in \mathbb{C}$ such that $f = e^{c+F}$.

3. Prove that if $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converges in $\mathbb{D}$ and the zeros of f accumulate at 0, then $f = 0$ (i.e., all $c_n = 0$).

4. Use the Residue Theorem to integrate the rational function
$$R(z) := \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$$
over $\mathbb{R}$.

5. Prove that a Hilbert space is separable if and only if every orthonormal basis is countable.

6. Suppose $f_n \in H(U)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n = f_0$ locally uniformly in U.

(a) Show that f_0 is continuous in U.

(b) Use Cauchy's integral formula for a circle to show that f_0 is holomorphic in each $D(a,r) \subseteq U$ (i.e., holomorphic in U) and that $f_n' \to f_0'$ locally uniformly in U.

7. (a) State the Dominated Convergence Theorem.

   (b) State Fatou's Lemma.

   (c) Let $\{f_k\}$ be a sequence of nonnegative measurable functions on the measure space $(X, \mathcal{M}, \mu)$. Suppose that $f_k \to f$ on X and $f_k \leq f$ μ-a.e. on X. Prove that $\displaystyle\int_X f_k \, d\mu \to \int_X f \, d\mu$.

8. Suppose $1 \leq q < p < \infty$.

   (a) Prove the inclusion $L^p([0,1], \lambda_1) \subset L^q([0,1], \lambda_1)$.

   (b) Give an example to show that the inclusion in part (8a) is strict.

   (c) Give an example of a measure space $(X, \mathcal{M}, \mu)$ for which one has the inclusion $L^q(X, \mathcal{M}, \mu) \subseteq L^p(X, \mathcal{M}, \mu)$. Is the inclusion strict in this case?

9. (a) Show that if $\int_X |f| d\mu = 0$, then $f = 0$ μ-a.e. on X.

   (b) Let E be a λ_2-measurable subset of $\mathbb{R}^2$ such that for λ_1-a.e. $x \in \mathbb{R}$, $E_x := \{y \,|\, (x,y) \in E\}$ has λ_1 measure zero. Show that E itself has λ_2 measure zero and that for λ_1-almost every $y \in \mathbb{R}$, the set $E^y := \{x \in \mathbb{R} \,|\, (x,y) \in E\}$ has λ_1 measure zero.

10. Let E_j, $j = 1, \ldots, n$ be λ_1-measurable subsets of $[0,1]$. Assume that for some positive integer $q \leq n$ each $x \in [0,1]$ belongs to at least q of the sets E_j. Prove that there exists j such that $\lambda_1(E_j) \geq q/n$.

11. Let E be a λ_1 set of positive measure. Show that for every $\alpha < 1$ there is an open interval $I = I(\alpha)$ such that $\lambda_1(E \cap I) > \alpha \lambda_1(I)$.