F08 Geometry Qualifying Exam

Name ____________________

1. Let $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\psi_t(x,y) = (e^t x, e^{-t} y)$ and let $\omega = x^2\, dx \wedge dy$. Compute
   (a) the vector field $\dot{\psi}_0$.
   (b) $\psi_t^* \omega$.
   (c) $\frac{d}{dt}(\psi_t^*\omega)|_{t=0}$.
   (d) $L_{\dot{\psi}_0}\omega$.

2. Let $\omega \in \Omega^r(M)$ satisfy $\int_\Sigma \omega = 0$ for every smooth submanifold $\Sigma \subseteq M$ that is diffeomorphic to S^r. Prove that $d\omega = 0$.

3. (a) State the covering homotopy theorem for covering spaces.
   (b) Let $p : \tilde{X} \to X$ be a universal cover and prove that the induced map $p_* : \pi_2(\tilde{X}) \to \pi_2(X)$ is an isomorphism.
   (c) Compute $\pi_2(S^2 \vee \mathbb{R}P^2)$.

4. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$ be $f(w,x,y,z) = (w, x^3 - 3xw + y^2 - z^2)$. Compute the critical values of f.

5. (a) State the Eilenberg-Steenrod axioms for a homology theory.
   (b) Let $A_* := A_{-2} \to A_{-1} \to A_0 \to A_1 \to A_2$ and B_* be five-term exact sequences indexed by the same set and let $f_* : A_* \to B_*$ be a morphism of sequences. Assuming that f_k is an isomorphism for $k = -2, -1, 1, 2$, prove that it is injective for $k = 0$.
   (c) Using the Eilenberg-Steenrod axioms and the homology groups of spheres prove that $\mathbb{R}^3$ and $\mathbb{R}^4$ are not homeomorphic.

6. (a) A presentation of the symmetric group $\mathfrak{S}_3$ is given by $\langle \sigma, \tau | \sigma^3, \tau^2, (\sigma\tau)^2 \rangle$. Construct a space X having $\pi_1(X) = \mathfrak{S}_3$.
   (b) Compute the homology of the space X from the first part of this problem.
   (c) Classify up to homeomorphism every connected topological space that covers the space X from the first part of this problem.

7. Using the Mayer Vietoris sequence, homotopy axiom and cohomology of S^1 compute the cohomology groups of $\mathbb{R}^2 - \{p, q\}$ for $p \neq q$ and for T^2.

8. (a) Define what a smooth map between manifolds is.

   (b) Define the differential of a smooth map.

   (c) Let M denote the $n \times n$ real-valued matrices and let S denote the subspace of symmetric matrices. Using the map $f : M \to S$ given by $f(A) = A^t A$ prove that $O_n := \{A \in M | A^t A = I\}$ admits the structure of a smooth manifold in a natural way.