Topology Qualifying Exam
Fall 1997

Choose and work and 6 of the following problems. Start each new problem on a new sheet of paper. Do not turn in more than six problems. Below a "space" always means a "topological space".

1. Prove or disprove:

(a) Closed subspaces of path connected spaces are path connected.

(b) If $f : X \to Y$ is continuous and X is path connected, then $f[X]$ is path connected.

2. Let $\mathcal{A}$ be a collection of subsets of the topological space X such that $X = \cup\mathcal{A}$. Consider the function $f : X \to Y$; suppose that $f|A$ is continuous for each $A \in \mathcal{A}$.

(a) Show that if $\mathcal{A}$ is finite and each member of $\mathcal{A}$ is closed, then f is continuous.

(b) Give an example to show that the word "finite" in part (a) cannot be changed to "countable".

3. Let A and B be disjoint compact subsets in the Hausdorff space X. Show that there are disjoint open subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

4. Let Y be an ordered set with the order topology. Let $f, g : X \to Y$ be continuous.

(a) Let $h : X \to Y$ be the function given by

$$h(x) := \min\{f(x), g(x)\}.$$

Show that h is continuous.

(b) Show that the set $\{x \in X | f(x) \leq g(x)\}$ is closed in X.

5. Let X be a complete metric space and $f : X \to \mathbb{R}$ a continuous real-valued function on X. Show that every nonempty open subset of X contains a nonempty open subset on which f is bounded.

6. Let $f : X \to Y$ be a continuous surjective map, where X is compact and Y is Hausdorff. Show that f is a quotient map.

7. A space X is said to be *completely regular* if one-point sets are closed and if for each point x_0 and each closed subset A not containing x_0, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f[A] \subset \{0\}$.

Show that every locally compact Hausdorff space is completely regular.

8. If $f : X \to Y$ and $g : Y \to X$ are continuous functions such that gof is the identity function on X, prove that f is topological embedding and that g is a quotient map.

9. If f and g are real-valued continuous functions with the same domain, prove that $f + g$ is continuous, where $(f + g)(x) \equiv f(x) + g(x)$ for any x in the domain.

10. Prove that a filter $\mathcal{G}$ on a set X is an ultrafilter if and only if for each subset A of X, either $A \in \mathcal{G}$ or $X \backslash A \in \mathcal{G}$.

11. Prove or disprove:

(a) Every compact subset of a Hausdorff space is closed.

(b) Every closed subset of a Hausdorff space is compact.

12. Show that a metrizable space X has a countable dense subset if and only if it has a countable basis.

13. Prove or disprove that closed subspaces of normal spaces are normal.

14. Let Y be a metric space and let $f_n : X \to Y$ be a sequence of continuous functions and $f : X \to Y$ a (not necessarily continuous) function. Suppose that $\{f_n\}$ is equicontinuous and $f_n(x) \to f(x)$ for each $x \in X$ (point-wise convergence). Show that f is continuous.