COMPLEX ANALYSIS QUALIFYING EXAM

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(Burckel & Bennett)

Do any 7 of the 9 problems. The notation is: $\mathbb{N}$ for the natural numbers $(1, 2, 3, \ldots)$, $\mathbb{R}$ the reals, $\mathbb{C}$ the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. For open $U \subset \mathbb{C}$, $H(U)$ is the set of all holomorphic functions on U, $C(\overline{\mathbb{D}})$ the set of all continuous complex-valued functions on $\overline{\mathbb{D}}$.

1. Let S denote the open sector in the upper half-plane having bounding rays $\{t \in \mathbb{R} : t \geq 0\}$ and $\{(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)t : t \geq 0\}$ and let L denote the lens-shaped region $\{z \in \mathbb{D} : |z-1| < 2 \,\&\, |z+1| < 2\}$. Exhibit a Moebius (=fractional-linear) transformation of S onto L or prove that none exists.

2. The function $f \in H(\mathbb{D})$ is defined by $f(z) := \sum_{n=1}^{\infty} z^n$, $z \in \mathbb{D}$. Find the Laurent series for its analytic continuation into the annulus $\mathbb{C} \backslash \overline{\mathbb{D}}$.

3. The piecewise-smooth closed curve Γ is defined on $[0, 4]$ by
$$\Gamma(t) := \begin{cases} -ie^{2\pi i t} & 0 \leq t \leq 1 \\ -i + 4(t-1) & 1 \leq t \leq 2 \\ -i + 4e^{\pi i (t-2)} & 2 \leq t \leq 3 \\ -i + 4(t-4) & 3 \leq t \leq 4. \end{cases}$$

   Sketch this curve and compute $\int_\Gamma \csc(z) dz$.

4. (i) Formulate a Morera theorem in $\mathbb{D}$ that involves only rectangles parallel to the axes. (A proof is not required.)

   (ii) Using (i), outline a proof that "straight lines are removable for holomorphic functions", meaning that $C(\overline{\mathbb{D}}) \cap H(\mathbb{D} \backslash \mathbb{R}) \subset H(\mathbb{D})$.

5. U open $\subset \mathbb{C}$, $g : U \to \mathbb{C}$ is continuous and g^n is holomorphic (for some $n \in \mathbb{N}$). Show that g itself is holomorphic in U.
   Hint: First consider the case that g is zero-free; work locally.

6. f is holomorphic and non-constant in the bounded region Ω and $\lim_{z \to b} |f(z)| = 1$ for each b in the boundary of Ω.

   (i) Show that $f(\Omega) \subset \mathbb{D}$.

   (ii) Show that f has a zero in Ω.

   (iii) Use (ii) to show that $\mathbb{D} \subset f(\Omega)$. [In summary, $f(\Omega) = \mathbb{D}$.]
   Hint: Apply (ii) to $\varphi \circ f$ for various conformal automorphisms φ of $\mathbb{D}$.

7. Say that f_n converges continuously to f if $f_n(z_n) \to f(z)$ whenever $z_n \to z$.

   (i) Show that for f_n, $f \in H(\Omega)$ this concept is equivalent to local uniform convergence of f_n to f.

(ii) Use (i) to show that if f_n, g_n, f, g are all holomorphic self-maps of Ω and $f_n \to f$, $g_n \to g$ (locally uniformly), then $f_n \circ g_n \to f \circ g$.

8. $h : \mathbb{D} \to \mathbb{R}$ is harmonic and $f := \dfrac{\partial h}{\partial x} - i\dfrac{\partial h}{\partial y}$.

(i) Show that f is holomorphic in $\mathbb{D}$. (You may wish to use the definition "harmonic $\Leftrightarrow$ Laplacian identically 0".)

(ii) Show that if F is an antiderivative of f, then h and $\operatorname{Re} F$ differ only by a constant.

(iii) $h : \overline{\mathbb{D}} \to \mathbb{R}$ is continuous and h is harmonic in $\mathbb{D}$. How would you find a harmonic conjugate k for h in $\mathbb{D}$?

9. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $\operatorname{Im} f(z) > 0 \quad \forall z \in \mathbb{C}$. Prove that f is constant.