Qualifying exam, geometry and topology

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Problem 1. Let
$$\{\{2k-1, 2k, 2k+1\} |\ k \in \mathbb{Z}\}$$
be a basis for a topology τ on $\mathbb{Z}$. Is $(\mathbb{Z}, \tau)$ path-connected?

Problem 2. Let T^2 be a 2-torus, let $p : T^2 \to F$ be a covering map and let $q : E \to F$ be a covering map (where E is assumed connected). Give, with a proof, a list of all possibilities for E up to homeomorphism.

Problem 3. Let D^n be the unit ball in $\mathbb{R}^n$. Let X be a CW-complex with $X^{(0)} = \{p\}$, $X^{(1)} = X^{(0)} \cup (\{a, b\} \times D^1)$, $X^{(2)} = X^{(1)} \cup (\{E, F\} \times D^2)$ with
$$(E, (\cos \pi\theta, \sin \pi\theta)) \sim \begin{cases} (a, 2\theta + 1) & \theta \in [-1, 0] \\ (a, 2\theta - 1) & \theta \in [0, 1]. \end{cases}$$
and
$$(F, (\cos \pi\theta, \sin \pi\theta)) \sim \begin{cases} (b, 3\theta + 1) & \theta \in [-1, -1/3] \\ (b, 3\theta) & \theta \in [-1/3, 1/3] \\ (b, 3\theta - 2) & \theta \in [1/3, 1]. \end{cases}$$
and $X = X^{(3)} = X^{(2)} \cup (\{L\} \times D^2)$ with
$$(L, (x, y, z)) \sim (E, (x, y))$$
for $x^2 + y^2 + z^2 = 1$.

(a) Compute $H_*(X)$.
(b) Compute $\pi_1(X)$.
(c*) Describe the universal cover of X.

Problem 4. For a non-empty topological space X define the *cone* CX as the quotient $(X \times [0, 1])/(X \times \{1\})$. Define the *suspension* ΣX as the quotient $(X \times [0, 1])/(X \times \{0, 1\})$.

(a) Show that CX is contractible.
(b) Use Mayer–Vietoris sequence to show that for all p there is a natural isomorphism $\tilde{H}_p(\Sigma X) \simeq \tilde{H}_{p-1}(X)$.

Problem 5. (a) Let X be a 3-dimensional CW-complex such that $H^0(X, \mathbb{R}) \simeq H^1(X, \mathbb{R}) \simeq \mathbb{R}$, $H^2(X, \mathbb{R}) \simeq \mathbb{R}^2$, $H^3(X, \mathbb{R}) = 0$. Calculate its cohomology ring $H^*(X, \mathbb{R})$.

(b) Give an example of such a CW-complex.

Problem 6. Let $X = x\partial_x - y\partial_y + \partial_z$ be a vector field on $\mathbb{R}^3$. Let

$$\alpha = f(x,y,z)\, dy \wedge dz + g(x,y,z)\, dz \wedge dx + h(x,y,z)\, dx \wedge dy$$

be a smooth 2-form on $\mathbb{R}^3$.

(a) Compute the Lie derivative $L_X\alpha$.

(b) Compute the flow of X and find a system of straightening coordinates for X.

(c) If

$$L_X\alpha = xy^2\, dy \wedge dz + x^3\, dz \wedge dx + z\, dx \wedge dy$$

and for all x and y we have

$$\alpha(x,y,0) = dy \wedge dz,$$

what is α?

Problem 7. Consider the surface

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid x^3 + y^3 + z^3 = 27\}$$

and orient it by a 2-form Ω such that $\Omega(0,0,3) = dx \wedge dy$. Let

$$\alpha = (1 + x(x^2+y^2+z^2)^{-3/2})\, dy \wedge dz$$

and let $f : \Sigma \to \Sigma$ be given by $f(x,y,z) = (y,z,x)$.

(a) Compare

$$I_1 = \int_\Sigma \alpha, \quad I_2 = \int_\Sigma f^*\alpha \text{ and } I_3 = \int_\Sigma f^*f^*\alpha.$$

(b) Compute I_1. (Hint: compute $I_1 + I_2 + I_3$ first.)

Problem 8. Let $f : T^3 \to S^2$ be a smooth map. (T^3 is a 3-torus.)

(a) Give a definition of the set of regular values of f in S^2 (denote it by R_f).

(b) What can you say about $f^{-1}(p)$ for $p \in R_f$?

(c*) Given $p, q \in R_f$ with $p \neq q$, show that there is a submanifold $W \subset T^3$ such that $\partial W = f^{-1}(p) \cup f^{-1}(q)$. (Hint: modify the typical argument to establish general position of a function. In other words given a function $F : M \times P \to N$ viewed as a family of functions $M \to N$ parameterized by P, consider a regular value of projection onto the second factor $F^{-1}(n_0) \to P$. By a suitable choice of coordinates one may assume that $p = (0,-1)$ and $q = (0,1)$ in $\mathbb{R}^2$, and then construct a suitable $F : M \times P \to N$ to solve this problem.)