Algebra Qualifying exam, January 2011

Instructions: You are given ten problems from which you are supposed to do eight. Please, indicate those eight problems which you would like to be graded by circling the problem number on the problem sheet. Please, write solutions to each problem on separate pages and write your name on top.

Note. All rings in this exam are associative, unital (that is with 1) and non-zero; and all ring homomorphisms map unit elements to unit elements.

Notations. We denote by $\mathbb{C}$ the field of complex numbers, by $\mathbb{Z}$ the ring of integers and by $\mathbb{Z}_+$ the monoid of nonnegative integers with respect to $+$.

1. Polynomials in many variables. Let R be a commutative ring with the unit element 1_R, and X a set. We consider a set $\mathbf{Maps}_0(X, \mathbb{Z}_+)$ of functions $X \xrightarrow{\mathfrak{n}} \mathbb{Z}_+$ with *finite support* (that is $\mathfrak{n}(x) \neq 0$ only for a finite number of $x \in X$). The set $\mathbf{Maps}_0(X, \mathbb{Z}_+)$ is a monoid with respect to addition, which we turn into multiplicative monoid $\mathbf{Mon}(X)$ of by formally exponentiating its elements: the elements of $\mathbf{Mon}(X)$ (which are called *monomials*) are symbols $x^{\mathfrak{n}}$, where $\mathfrak{n}$ runs through $\mathbf{Maps}_0(X, \mathbb{Z}_+)$ and the multiplication is given by $x^{\mathfrak{n}} \cdot x^{\mathfrak{m}} \stackrel{\text{def}}{=} x^{\mathfrak{n}+\mathfrak{m}}$.

The ring $R[X]$ of *polynomials in the set of variables* X with coefficients in R is defined as follows: its elements are expressions $\{\sum_{\mathfrak{n}} a_{\mathfrak{n}} x^{\mathfrak{n}}\}$, where $a_{\mathfrak{n}} \in R$, and $a_{\mathfrak{n}} \neq 0$ only for a finite number of $\mathfrak{n}$. We define addition and multiplication by

$$(\sum_{\mathfrak{n}} a_{\mathfrak{n}} x^{\mathfrak{n}}) + (\sum_{\mathfrak{n}} b_{\mathfrak{n}} x^{\mathfrak{n}}) \stackrel{\text{def}}{=} \sum_{\mathfrak{n}} (a_{\mathfrak{n}} + b_{\mathfrak{n}}) x^{\mathfrak{n}} \quad \text{and}$$
$$(\sum_{\mathfrak{n}} a_{\mathfrak{n}} x^{\mathfrak{n}}) \cdot (\sum_{\mathfrak{n}} b_{\mathfrak{n}} x^{\mathfrak{n}}) \stackrel{\text{def}}{=} \sum_{\mathfrak{n}} (\sum_{\mathfrak{m}+\mathfrak{l}=\mathfrak{n}} a_{\mathfrak{m}} b_{\mathfrak{l}}) x^{\mathfrak{n}}.$$

(a) Show that $R[X]$ is a commutative ring with unit and there is a natural unital ring homomorphism $R \longrightarrow R[X]$.

(b) Prove that, for any unital ring $\mathcal{A}$, there is a natural bijective correspondence between the set $\mathbf{Rings}(R[X], \mathcal{A})$ of unital ring homomorphisms $R[X] \longrightarrow \mathcal{A}$ and the product $\mathbf{Rings}(R, \mathcal{A}) \times \mathbf{Maps}(X, |\mathcal{A}|)$ of the set of ring homomorphisms from R to $\mathcal{A}$ by the set of maps from X to the set $|\mathcal{A}|$ of elements of $\mathcal{A}$.

(c) Show that

(c1) If X is a set with n elements, then $R[X]$ is naturally isomorphic to the usual polynomial ring in n variables with coefficients in R.

(c2) For an arbitrary set X, the ring $R[X]$ is the union of its subrings $R[T]$, where T runs through the family of all finite subsets of X.

2. Polynomials and localizations. Let R be a commutative ring with the unit element 1_R and S a subset of elements of R. Let $X_S = \{x_s \mid s \in S\}$ be set (of indeterminates) indexed by elements of S. We denote by $S^{-1}R$ the quotient of the polynomial ring

$R[X_S]$ by the ideal $\mathfrak{I}_S$ generated by the polynomials $\{sx_s - 1_R \mid s \in S\}$ and call it the *localization* of R at S. Let

$$R \xrightarrow{\varphi_S} S^{-1}R \tag{1}$$

be the composition of the embedding $R \longrightarrow R[X_S]$ and the epimorphism

$$R[X_S] \longrightarrow S^{-1}R \overset{\text{def}}{=} R[X_S]/\mathfrak{I}_S.$$

(a) Show that the ring homomorphism (1) has the following universal property:

$\varphi_S(t)$ is an invertible element for each $t \in S$, and if $R \xrightarrow{\psi} \mathcal{B}$ is a unital ring homomorphism such that $\psi(t)$ is an invertible element for all $t \in S$, then there exists a unique ring homomorphism $S^{-1}R \xrightarrow{\bar{\psi}} \mathcal{B}$ such that $\psi = \bar{\psi} \circ \varphi_S$.

(b) Show that the universal property in (a) determines the ring $S^{-1}R$ uniquely up to isomorphism.

3. Localizations at finitely generated sets. Let S be a *multiplicative* set of elements of a commutative ring R (that is $S \cdot S \subseteq S$) generated by a finite set of elements $\{s_1, \ldots, s_n\}$.

Show that $S^{-1}R$ is isomorphic to the quotient of the ring $R[t]$ of polynomials in one variable by the principal ideal $R[x](\mathfrak{s} \cdot x - 1_R)$, where $\mathfrak{s}$ is the product $s_1 \cdot \ldots \cdot s_n$.

4. Noetherian modules and rings. A left module $\mathcal{M}$ over an associative ring R is called *noetherian* if any set of its submodules has a maximal (with respect to the inclusion) element. Equivalently, any increasing chain of submodules of $\mathcal{M}$ stabilizes.

(a) Let $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ be an exact sequence of left R-modules. The module $\mathcal{M}$ is noetherian if and only if the modules $\mathcal{M}'$ and $\mathcal{M}''$ are noetherian.

(b) Let $\{\mathcal{M}_i \mid 1 \le i \le n\}$ be left R-modules. Prove that their direct sum, $\bigoplus_{1 \le i \le n} \mathcal{M}_i$, is a noetherian module if and only if each module $\mathcal{M}_i$, $1 \le i \le n$, is noetherian.

(**Hint:** use the problem (a).)

(c) **Modules over left noetherian rings.** A ring R is called *left noetherian* if it is noetherian as a left R-module (that is any increasing chain of left ideals of R stabilizes).
Prove that a left module over a left noetherian ring R is noetherian if and only if it is finitely generated.

(**Hint:** apply (b) to a direct sum of copies of R and then use (a).)

5. Endomorphisms of a noetherian module. Prove that an endomorphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{M}$ of a noetherian R-module is an isomorphism if and only if it is surjective.

(**Hint:** consider the increasing chain $\{Ker(\mathfrak{f}^n) \mid n \ge 1\}$ of kernels of powers of the endomorphism $\mathfrak{f}$ and, using the fact that it stabilizes, deduce that $Ker(\mathfrak{f}) = 0$.)

6. Trace and representations of the Weyl ring. Let R be a commutative ring with the unit 1_R and $M_n(R)$ the ring of $n \times n$ matrices with entries from the ring R.

The *trace* of a matrix $A = (a_{ij}) \in M_n(R)$ is, by definition, the sum of its diagonal entries:

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

(a) Show that $tr(A+B) = tr(A) + tr(B)$ and $tr(AB) = tr(BA)$ for all $A, B \in M_n(R)$.

(b) Deduce from (a) that, if A is an invertible element of the ring $M_n(R)$, then $tr(ABA^{-1}) = tr(B)$ for any $B \in M_n(R)$.

(c) **Representations of the Weyl ring.** The Weyl ring (or *the ring of quantum mechanics*) is the ring $A_{\mathbb{Z}}^1 \stackrel{\text{def}}{=} \mathbb{Z}\langle x, y\rangle/(xy - yx - 1)$. In other words, the ring $A_{\mathbb{Z}}^1$ is generated by elements x, y subject to the relation $xy - yx = 1$.

Prove the following fact:

Suppose that R is a ring of *zero characteristic*, which means that $m \cdot 1_R \neq 0$ for all non-zero integers m. Then there is no unital ring homomorphism from $A_{\mathbb{Z}}^1$ to $M_n(R)$.

(In particular, there are no finite dimensional representations of the Weyl ring over fields of zero characteristic.)

(**Hint:** Assume that there exists a homomorphism from $A_{\mathbb{Z}}^1$ to $M_n(R)$ and apply trace to the image of $xy - yx$.)

7. Playing with relations. Let G be the group generated by elements a and b subject to the relations $a^7 = b^8 = a^2b^3a^5b^5 = e$, where e is the unit element of G.

Show that the group G is cyclic and determine its order.

8. Irreducible and prime elements. Recall that a non-invertible element r of a commutative ring R is called *irreducible* if $r = ab$ implies that either a, or b is invertible. An element p of R is called *prime* if $p|ab$ implies that either $p|a$, or $p|b$.

(a) Show that

(a1) If R is a *domain* (that is it does not have zero divisors), then every prime element is irreducible.

(a2) If R is the ring $K[x]$ of polynomials with coefficients in a field K, then an element of R is irreducible if and only if it is prime.

(b) Let K be a field and R the subring of the polynomial ring $K[x]$ consisting of all polynomials with x-coefficient equal to 0; that is $R = K + x^2K[x]$.

(b1) Prove that the elements x^2 and x^3 are irreducible, but, not prime in the ring R.

(b2) Show that the intersection $xK[x] \bigcap R$ of the principal ideal $xK[x]$ of the ring $K[x]$ with the subring R is not a principal ideal of the ring R.

9. Minimal polynomial and invariant subspaces. Let $V \xrightarrow{T} V$ be an endomorphism of a finite dimensional vector space V over a field F and $\mathfrak{m}_T(x) \in F[x]$ its minimal polynomial.

(a) If $\mathfrak{m}_T(x)$ has a nonconstant polynomial factor of degree n, show that V has a nonzero subspace W of dimension $\leq n$ which is *invariant* under T, that is $T(W) \subset W$.

(b) Conversely, if V has a nonzero subspace W of dimension n with $T(W) \subset W$, show that $\mathfrak{m}_T(x)$ has a non-constant polynomial factor of degree $\leq n$.

10. Galois group. Let $f(x) \in \mathbf{Q}[x]$ be the minimal polynomial of a number $\alpha \in \mathbb{C}$. Assume that the Galois group $\mathrm{Gal}(F/\mathbf{Q})$ of the splitting field F of $f(x)$ is abelian. Prove that if $|\alpha| = 1$, then also $|\beta| = 1$ for every complex root β of $f(x)$.

(**Hint:** Consider the complex conjugation.)