Partial Differential Equations
Qualifying Examination
April 10, 1989

Do any 7 problems.

1. Do any three of the following:
   (i) State the definition of well-posed problem.
   (ii) Write out an example to show the Cauchy problem for heat equation

$$u_t = u_{xx} \qquad x \in R^1, \quad 0 < t < \infty$$
$$u(0, x) = \phi(x) \in C^1(R^1)$$

   is not well-posed.
   (iii) Under what additional conditions is the PDE

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < t < \infty \qquad -\pi < x < \pi$$

   well-posed. Explain.
   (iv) Is the Neumann problem for Laplace's equation

$$\Delta u = 0 \quad \text{in} \quad \Omega$$
$$\frac{\partial u}{\partial n}_{|\partial\Omega} = f \quad \text{on} \quad \partial\Omega$$

   well-posed where $f \in C^1(\partial\Omega)$ and $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain? Explain.

2. (i) Show $\frac{1}{4\pi r}$ is a weak solution to equation $\Delta u = \delta$, the Dirac delta function, on $\mathbf{R}^3$.
   (ii) Give two equivalent definitions of Sobolev space $W^{m,p}(\Omega)$.

3. Let $X = \{u \in C^2(\Omega) \cap L^2(\Omega) \mid \frac{\partial u}{\partial u}_{|\partial\Omega} = 0, \int_\Omega u = 0\}$. Show that $-\Delta$ is a positive definite operator on X; $(-\Delta u, u) \geq a \| u \|^2_{L^2(\Omega)}$ for some positive constant a.

4. Solve $\Delta u(x,y) = 0$ on the strip $0 < y < \pi$ with the boundary value

$$u(x,\pi) = u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x < \ln 2 \\ 0 & \ln 2 < x \end{cases}$$

(See Fig. 1)

5. A traffic flow can be modeled by equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + u_m \left(1 - \dfrac{2\rho}{\rho_m}\right) \dfrac{\partial \rho}{\partial x} = 0 \\ \rho(0,x) = h(x) \end{cases}$$

where $\rho = \rho(t,x)$ is the traffic density,
$u_m =$ maximum permitted speed,
$\rho_m =$ maximum possible density of cars,
$h(x) =$ initial density.

(i) Find the density $\rho(t,x)$
(ii) Under what initial density of $h(x)$ may a catastrophe of traffic flow occur at some later time $t > 0$. Explain.

6. Solve the following equation by using the method of separation of variable

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u & 0 < x < \pi, \quad 0 < t \\ u(t,0) = u(t,\pi) = 0 \\ u(0,x) = x(1-x) \end{cases}$$

7. State and prove Dirichlet's principle for harmonic functions.

8. Prove the uniqueness of the solution to wave equation of mixed value problem

$$\begin{cases} u_{tt} = C^2 \Delta u & 0 < t < T, \quad x \in \Omega \\ \Omega \text{ is bounded domain in } \mathbb{R}^n \\ u(0,x) = f(x) \in C^1(\overline{\Omega}) \\ u_t(0,x) = g(x) \in C^1(\overline{\Omega}) \\ u(x,t)|_{[0,T]\times\partial\Omega} = 0. \end{cases}$$

9. Suppose $P_t(x,y)$ is the heat kernel on a bounded domain Ω in $\mathbf{R}^n$, i.e.

$$u(t,x) = \int_\Omega P_t(x,y) f(y)\,dy$$

solves equation

$$\begin{cases} u_t = \Delta u & x \in \Omega, \quad 0 < t < T \\ u(0,x) = f(x) \in C^1(\Omega) & \\ u(t,x) = 0 & x \in \partial\Omega, \quad 0 \le t \le T. \end{cases}$$

Write out the Green's function for Ω in terms of $P_t(x,y)$.

10. The vibration of a beam with infinite length can be described by the following equation

$$\begin{cases} u_{tt} + a^2 \dfrac{\partial^4 u}{\partial x^4} = 0 \\ u(0,x) = f(x), \quad u_t(0,x) = a g''(x), \quad f, g'' \in L^1(R^1) \end{cases}$$

where f, g'' are the initial vibration data. Use Fourier transform methods to solve this equation. HINT: Use the Fourier Cosine Transform:

$$\sin b\xi^2 \mapsto \sqrt{\frac{\pi}{8b}}\left(\cos\frac{x^2}{4b} - \sin\frac{x^2}{4b}\right)$$
$$\cos b\xi^2 \mapsto \sqrt{\frac{\pi}{8b}}\left(\cos\frac{x^2}{4b} + \sin\frac{x^2}{4b}\right)$$

11. (i) Let Ω be a bounded domain in $\mathbf{R}^n$. State the definition that a function $u \in L^1(\Omega)$ is weakly differentiable. (First order.)

(ii) We say a function $u \in L^1(\Omega)$ is strongly differentiable if there is a sequence $\{u_m\} \in C^\infty(\Omega)$ and a function $v \in L^1(\Omega)$ such that

$$u_m \to u, \quad \frac{\partial u_m}{\partial x_i} \to v \quad \text{in} \quad L^1(\Omega) \quad i = 1, 2, \ldots, n.$$

Show that a function u is weakly differentiable if and only if it is strongly differentiable.

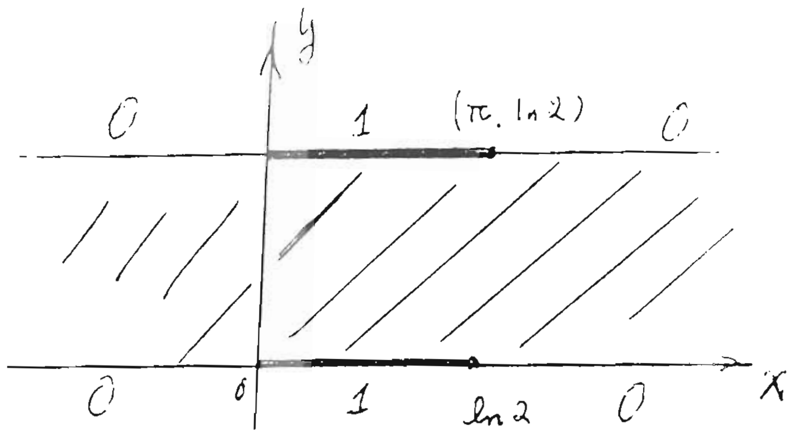

Fig. 1