Differential Equations Qualifying Exam
Fall 1995

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1. (i) State a definition of fundamental solution of wave equations

$$u_{tt}(t,x) = a^2 \Delta u(t,x), \quad t > 0, x \in R^n.$$

(ii) Give the definitions of the Sobolev spaces $W^{m,p}(\Omega), W_0^{m,p}(\Omega)$ where $p \geq 1$, and Ω is a bounded domain with smooth boundary.

2. Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 1 & t > 0, x \in R^1 \\ u(0,x) = \begin{cases} 1, -1 \leq x \leq 1 \\ 0, |x| > 1 \end{cases} & \\ u_t(0,x) = 0. & \end{cases}$$

3. (i) Formulate the Dirichlet principle.

(ii) Give a proof of the Dirichlet principle.

4. A function is said to be a homogeneous function of degree $m \geq 0$, where m is an integer, if

$$f(\alpha x_1, \alpha x_2, \ldots, \alpha x_n) = \alpha^m f(x_1, x_2, \ldots, x_n)$$

holds true for any scalar α. Show that $f(x_1, \ldots, x_n)$ is such a function if and only if

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = m \cdot f.$$

5. Prove that the only bounded harmonic functions in R^n are constant functions.

6. Solve the exterior Dirichlet problem in R^3:

$$\Delta u(x) = 0, \text{for } |x| > 1; \quad u(x) = c \text{ for } |x| = 1.$$

7. Find a solution to the heat equation

$$\begin{cases} u_t(t,x) = \Delta u(t,x); & x = (x_1, x_2), 0 \leq x_1^2 + x_2^2 < 1 \\ u(0,x) = 1 - r^2; & r = \sqrt{x_1^2 + x_2^2} \\ u(t,x)|_{r=1} = 0. & \end{cases}$$