NUMERICAL ANALYSIS QUALIFYING EXAM
Fall, 1996

(do at lest 3 problems from problems 1-4, and do at least 3 problems from problems 5-8, you may do as many as you can)

1. Prove the following theorem: If $f \in C^2(a,b)$, $f^{'}(x)f^{''}(x) \neq 0$, and $f(x)$ has a zero in (a,b), then the zero is unique in (a,b), and the Newton iteration will converge to it if the starting value x_0 and the first approximation x_1 are both in (a,b). (You may just do a special case where $f^{'}(x) < 0, f^{''}(x) < 0$ in (a,b))

2. Suppose a numerical integration formula I_n using n subintervals to approximate the definite integral $I = \int_a^b f(x)dx$ has an error given by $I - I_n \doteq \frac{c}{n^p}$ where c, p are constants. Derive the computable estimate
$$\frac{I_{2n} - I_n}{I_{4n} - I_{2n}} \doteq 2^p$$
This gives a practical means of checking the value of p, using three successive values I_n, I_{2n}, and I_{4n}.

3. By **considering the proof** of
$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)}(x-x_0)(x-x_1)\ldots(x-x_n),$$
where $p_n(x)$ is the polynomial of degree less than or equal to n,which interpolates $f(x)$ at $n+1$ nodes $x_0, x_1, \ldots, x_n$. Find that the error formula for
$$f(x) - p_m(x),$$
where $p_m(x)$ is a polynomial of degree greater than n, which interpolates $f(x)$ at $n+1$ nodes $x_0, x_1, \ldots, x_n$.

4. Prove the following theorem: Define a set of functions
$$P_M^n \equiv \{p \in P^n | \max_{x\in[a,b]} |p(x)| \leq M\}$$
where P^n is the linear space of the polynomials of degree less than or equal to n. Then there is a constant $C > 0$ such that for every $p \in P_M^n$ and $x \in [a,b]$ and any positive integer k,
$$|\frac{d^k p(x)}{dx^k}| \leq C.$$
(Hint: Chebyshev polynomials of degree 0, 1, $\cdots, n$ form a basis for P^n)

5. A matrix norm is defined as
$$\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$$
Prove or disprove: $\|AB\|_\infty = \|A\|_\infty \|B\|_\infty$. What about the special case: $\|A^2\|_\infty = \|A\|_\infty \|A\|_\infty$?

6. Find the explicit form for the iterative matrix in the Gauss-Seidel iterative method for solving a linear system $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

7. Suppose A is an invertible matrix and that B is a matrix with $\|B - A^{-1}\| \leq \delta\|A^{-1}\|$. Let $\{\mathbf{x}_n\}_{n=0}^{\infty}$ be the sequence of vectors generated by the algorithm

   (i) $\mathbf{r}_n = \mathbf{b} - A\mathbf{x}_n$

   (ii) $\mathbf{x}_{n+1} = \mathbf{x}_n + B\mathbf{r}_n$

   with a given starting value $\mathbf{x}_0$. Give a sufficient condition on the size of δ for the sequence to converge to the solution of the linear system $A\mathbf{x} = \mathbf{b}$ for arbitrary starting value $\mathbf{x}_0$. Prove that your condition is correct.

8. Describe an algorithm that reduces a square real matrix to a lower Hessenberg matrix without changing its eigenvalues. (A matrix A is lower Hessenberg if $a_{ij} = 0$ provided $j - i > 1$.)