Numerical Analysis Qualifying Exam
Fall 1989

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1. Suppose we want to compute the value $J_{100}(1)$ by using the values $J_0(1)$ and $J_1(1)$ (which are assumed known) and the recursion.

$$J_{m+1}(1) = 2mJ_m(1) - J_{m-1}(1).$$

Is this calculation numerically stable? Explain in detail.

2. Let $g : [a, b] \to R$ be a $C^{(1)}$ function and suppose $p \in (a, b)$ satisfies the conditions:

$$p = g(p), \quad |g'(p)| < 1.$$

Prove that there exists a $\delta > 0$ such that if $|x_0 - p| < \delta$, then the sequence $\{x_n\}_n$ generated by the iteration

$$x_{n+1} = g(x_n)$$

converges to p.

3. Given a set of data points $\{(x_i, y_i), i = 1, \ldots, m\}$, it is desired to fit the data with a polynomial of degree $n < m - 1$ using the method of least squares. Thus, if we let

$$p(x) = \sum_{k=0}^{n} a_k x^k,$$

then we must choose the coefficients $a_0, a_1, \ldots, a_n$ so as to minimize the expression

$$E = \sum_{i=1}^{m} (y_i - p(x_i))^2.$$

(a) Show that the coefficients $a_0, a_1, \ldots, a_n$ must satisfy the matrix equation (called the normal equation):

$$\begin{bmatrix} \sum x_i^0 & \sum x_i^1 & \ldots & \sum x_i^n \\ \sum x_i^1 & \sum x_i^2 & \ldots & \sum x_i^{n+1} \\ \vdots & \vdots & & \vdots \\ \sum x_i^n & \sum x_i^{n+1} & \ldots & \sum x_i^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y_i x_i^0 \\ \sum y_i x_i^1 \\ \vdots \\ \sum y_i x_i^n \end{bmatrix},$$

where

$$\sum \equiv \sum_{i=1}^{m}.$$

(b) Show that the coefficient matrix of the above system is nonsingular and hence the system has a unique solution. (**Hint:** If we denote the coefficient matrix by $\mathbf{C}$, then if $\mathbf{C}$ were singular there would be a vector $\mathbf{b}$ with $\mathbf{Cb} = \mathbf{0}$. Then show that the polynomial

$$q_n(x) = b_0 + b_1 x + \cdots + b_n x^n$$

has more than n roots.)

4. An iteration method

$$x_{n+1} = g(x_n), \quad n \geq 0, \tag{1}$$

is to be used to find a fixed point of $g(x)$, such that $x = g(x)$.

(a) If α is a fixed point of $g(x), g(x), g'(x), \ldots, g^{(p)}(x)$ are continuous for all x near α for some $p \geq 2$. Furthermore, assume

$$g'(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0,$$

and the initial quess x_0 is sufficiently close to α. Show that the iteration (1) has order of convergence p, i.e.

$$|x_{n+1} - \alpha| \leq c|x_n - \alpha|^p, \quad c: \text{ constant}.$$

(b) A modification of Newton's method for solving the equation $f(x) = 0$ is Steffenson's method, defined as follows:

Choose a starting point x_0 and then iterate as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)},$$

where

$$D(x) = \frac{f(x + f(x)) - f(x)}{f(x)}.$$

Assuming the sequence $\{x_n\}_n$ converges to the root α and assuming $f'(\alpha) \neq 0$, show that the convergence is second order. (Hint: Write the iteration as $x = g(x)$. Use $f(x) = (x - \alpha)h(x)$ with $h(\alpha) \neq 0$, then compute the formula for $g(x)$ in terms of $h(x)$).

5. If A is a Hermitian matrix show

$$\| A \|_2 = \rho(A),$$

where $\| A \|_2$ is the Euclidean norm of A and $\rho(A)$ is the spectral radius of A. (Hint: use an appropriate theorem and then show a upper triangular Hermitian matrix is diagonal).

6. Consider the discrete analog of the eigenvalue problem

$$y'' + \lambda y = 0, \quad 0 < x < \pi,$$

$$y(0) = y(\pi) = 0,$$

given by

$$\frac{y_{i+1} + y_{i-1} 2y_i}{(\Delta x)^2} + \lambda y_i = 0,$$

$$y_0 = y_N = 0,$$

defined on the uniform mesh $0 = x_0 < x_1 < \cdots < x_N = \pi$. Compute the eigenvalues of the discrete problem by solving the finite difference equation. How do these eigenvalues compare with those of the continuous problems?

7. Suppose we want to solve the quadratic equation $x^2 + bx + c = 0$ for x, where b, c are from measurement and $b^2 - 4ac > 0$. Suppose b^*, c^* are given to us as approximate values of b, c. The absolute error of them is Err $(b^*) \leq \varepsilon_1$, Err $(c^*) \leq \varepsilon_2$ for some small $\varepsilon_1, \varepsilon_2$. Round-off error is negligible. Give an estimate of the absolute error and relative error of the solution x in terms of known quantities.

8. Let A and B have order n, with A nonsingular. Consider solving the linear system

$$Az_1 + Bz_2 = b_1, \quad Bz_1 + Az_2 = b_2$$

with $z_1, z_2, b_1, b_2 \in \mathbf{R}^n$.

(a) Find necessary and sufficient conditions for convergence of the iteration method

$$Az_1^{(m+1)} = b_1 - Bz_2^{(m)} \quad Az_2^{(m+1)} = b_2 - Bz_1^{(m)} \quad m \geq 0.$$

(b) Repeat part (a) for the iteration method

$$Az_1^{(m+1)} = b_1 - Bz_2^{(m)} \quad Az_2^{(m+1)} = b_2 - Bz_1^{(m+1)} \quad m \geq 0.$$

Compare the convergence rates of the two methods.

9. If 8-point Gauss-Legendre quadrature is used to compute

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}} dx.$$

The computed answer is 1.36468684, while the correct answer to nine figures 1.57079633.

(a) What is likely to be the cause of such a large error?

(b) What can be done to allow use of the 8-point Gaussian rule to compute the answer accurately?

10. Let A be a Hermitian matrix, x a non-zero vector. For a given $\lambda \in \mathbf{C}$ let $\eta(\lambda) = Ax - \lambda x$. Show that $\| \eta(\lambda) \|$ is minimized by taking

$$\lambda = \frac{x^* Ax}{x^* x}.$$