Real Analysis Qualifying Exam
Spring 1995

In problems 1 through 4, (X, M, μ) is a measure space. In problems 9 and 10, $|E|$ denotes the Lebesgue measure of the measurable set E.

1. Let $w : X \to [0, \infty)$ be measurable, and let $v(E) = \int_E w d\mu$ for $E \in M$.

Prove: (a) v is a measure on M and (b) $\int f dv = \int f w d\mu$ for each nonnegative measurable function f on X.

2. Suppose $0 < p < \infty$, (f_n) is a sequence of measurable functions on X, $f_n \to f$ a.e. and $\| f_n \|_p \to \| f \|_p < \infty$. Prove $\| f_n - f \|_p \to 0$.

Hint: Find $\alpha > 0$ s.t. $|f_n - f|^p \le \alpha(|f_n|^p + |f|^p)$ and apply Fatou's lemma.

3. Let $I \subseteq \mathbb{R}$ be an open interval, and let f be a function on $X \times I$ such that

(i) $\forall t \in I, f(\cdot, t) \in L_1(\mu)$, and

(ii) $\exists g \in L_1(\mu)$ such that $\left| \frac{\partial f}{\partial t}(x,t) \right| \le g(x) \quad \forall (x,t) \in X \times I$.

Prove:

$$\frac{d}{dt} \int f(x,t) d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) d\mu(x).$$

4. Suppose $\mu(X) < \infty$. A family $\mathcal{F}$ of measurable functions on X is said to be uniformly integrable if given $\varepsilon > 0$, $\exists \delta > 0$ such that

$$E \in M, \quad \mu(E) < \delta \text{ and } f \in \mathcal{F} \Rightarrow \int_E |f| d\mu < \varepsilon.$$

Prove Vitali's theorem: Suppose

(i) $\{f_n\} \subset L_1(\mu)$ is uniformly integrable,

(ii) $f_n \to f$ a.e. for some function f, and

(iii) $|f| < \infty$ a.e.

Thus $f \in L_1(\mu)$ and $\| f_n - f \|_1 \to 0$.

5. Show that for $1 \le p \le \infty$, the closed unit ball of $\ell_p(\mathbb{N})$ is not compact.

6. Let $1 \le p \le \infty$, $f \in (\mathbb{R})$ and $g \in L_p(\mathbb{R})$. Prove $\| f * g \|_p \le \| f \|_1 \cdot \| g \|_p$.

7. Suppose $f \in L_2(\mathbb{R}^+)$. Prove that

$$x^{-\frac{1}{2}} \int_0^x f dt \to 0 \text{ as } x \to 0^+.$$

8. Fix a Lebesgue measurable function f on $\mathbb{R}^+$ and define

$$\phi(p) = \left\{ \int_0^\infty |f(x)|^p e^{-x} dx \right\}^{\frac{1}{p}} \quad (0 < p < \infty).$$

Prove: (a) $p < q \Rightarrow \phi(p) \le (q)$.

(b) If $\phi(p) = \phi(q) < \infty$ for some $0 < p < q < \infty$, then $|f| = const$ a.e.

9. (a) Fix $0 < \varepsilon < 1$. Construct a closed set $K \subset [0,1]$ such that $|K| > 1 - \varepsilon$ and K contains no rationals.

(b) Does there exist a Borel set $E \subset [0,1]$ such that $0 < |E \cap I| < |I|$ for each nonempty open interval $I \subset [0,1]$?

10. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set such that

(i) $(E + k) \cap E = \emptyset \quad \forall k \in \mathbb{Z} \setminus \{0\}$ and (ii) $E + \mathbb{Z} = \mathbb{R}$.

Prove that $|E| = 1$. Hint: Let $I = (0,1]$. Then $I = \bigcup_k I \cup (E + k)$ and $E = \bigcup_k (I + k) \cup E$.