Topology Qualifying Exam
Spring 1991

Work 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than 6 problems.

1. Prove that a continuous map from a compact space to a Hausdorff space is closed.

2. (a) True - False

1. Every connected space is locally connected.
2. For $i \in \{0, 1, 2, 3, 4\}$ every product of T_i-spaces is T_i.
3. For $i \in \{0, 1, 2, 3, 4\}$ every closed subspace of a T_i-space is T_i.
4. If $(X_i \xrightarrow{f_i} Y_i)_{i \in I}$ is a set of topological embeddings, then $\Pi_I X_i \xrightarrow{\Pi f_i} \Pi_I Y_i$ is an embedding.
5. The statement "The product of any family of nonempty sets is nonempty" is equivalent to the Axiom of Choice.

(b) For each false entry give a counterexample (no proofs).

3. Show that no two of the intervals of $\mathbb{R}[0, 1], (0, 1)$, and $[0, 1)$ (with their usual subspace topologies) are homeomorphic.

4. Prove that if x is any point of a compact Hausdorff space, then x has a neighborhood base consisting of closed sets.

5. Let $f : X \to Y$ be a quotient map, and assume that X is locally pathwise connected (i.e., each point has a neighborhood base consisting of pathwise connected sets). Prove that Y is locally pathwise connected.

6. (a) Consider a function $f : X \to \Pi_{\alpha \in A} Y_\alpha$ and the family of associated coordinate functions $f_\alpha : X \to Y_\alpha$. Prove that f is continuous if and only if every f_α is continuous, assuming we give the product set the product topology.

(b) Give a counterexample to the above statement if we give the product set the "box" topology (where all products of open sets are open).

7. Prove or disprove: If $f : X \to Y$ is one-to-one and continuous and $A \subseteq X$, then $f[Fr(A)] \subseteq Fr(f[A])$, where $Fr(A) = \overline{A} \cap \overline{X - A}$.

8. Let X be the subspace of the plane $(\mathbb{R} \times \mathbb{R})$ that consists of all lines parallel to the x-axis that cross the y-axis at positive integral heights, i.e.,

$$X = \{a, b) | a \in \mathbb{R}, b \in \mathbb{Z}, \text{ and } b \geq 1\},$$

and let Y be the subspace of the plane that consists of all lines through the origin that have positive integral slopes, i.e.,

$$Y = \{(0, 0)\} \cup \{(a, b) \in \mathbb{R} \times \mathbb{R} | \frac{b}{a} \in \mathbb{Z} \text{ and } \frac{b}{a} \geq 1\}.$$

Find an error in the following "proof" that Y is a quotient space of X:

Define $f : X \to Y$ by $f(a, b) = (a, ab)$. [Note that the restriction of f to the horizontal line at height n maps this line homeomorphically onto the line contained in Y of slope n]. f is clearly continuous on each of the lines that make up X, so that since X is the disjoint union of these lines, f is continuous on X. Also, since the restriction of f to each of the lines that make up X is a homeomorphism (and so is open) f is open. Since every open continuous surjection is a quotient map, f must be a quotient map.

9. Prove that a paracompact Hausdorff space is normal.

10. Let $\mathbb{P}$ be the irrational numbers with the usual (subspace) topology. Show that the intersection of any countable family of dense open subsets of $\mathbb{P}$ is dense in $\mathbb{P}$.

11. Prove that the product of connected spaces is connected.

12. Let S be a well-ordered uncountable set that has the property that for every $x \in S$ the subset $S_x = \{y \in S | y < x\}$ is countable. Give S the order topology. Prove two of the following:

(a) S is first countable.

(b) S is **not** second countable.

(c) S has the property that each of its infinite subsets has a limit point.

(d) S is normal.