Geometry of Manifolds Qualifying Examination

Spring 1998

(Yetter and Miller)

This examination is in two parts: in part A, answers with a brief explanation will suffice; in part B, detailed calculations or proofs are expected unless a sketch of a proof is explicitly requested.

Part A Do all six (6) questions in part A.

1. What is the fundamental group of
   (a) $\mathbf{S^3}$ (the 3-sphere)
   (b) $\mathbf{S^2} \times \mathbf{S^1}$
   (c) $T^*(\mathbb{RP}^2)$, the total space of the cotangent bundle to the real projective plane
2. Describe in detail the flows of the vectorfield on $\mathbb{R}^\mathbf{3}$ given by
$$(x^2 + y^2 - 4)\frac{\partial}{\partial z}$$
3. (a) Give an example of a simply connected compact manifold with non-trivial tangent bundle.
   (b) Give an example of a simply connected compact manifold with trivial tangent bundle.
   (c) Give an example of a non-simply connected compact manifold.

   Note: Manifold means manifold without boundary.
4. (a) If $M = \mathbf{S^3}$, what is the dimension of the total space of the second exterior bundle $\Lambda^2(M)$?
   (b) What are sections of the second exterior bundle called?
5. Let Z be the vectorspace of closed 1-forms on $\mathbf{S^1} \times \mathbf{S^1}$ and B be the vectorspace of exact 1-forms on $\mathbf{S^1} \times \mathbf{S^1}$. What is the dimension of Z/B?
6. Let $\mathbf{S^2} = \{(x, y, z)|x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$. with the induced orientation. Calculate
$$\int_{\mathbf{S^2}} zdx \wedge dy - ydx \wedge dz + xdy \wedge dz$$

Part B. Choose four (4) and only four of the following problems.

1. Let G be a Lie group
   (a) Define its Lie algebra.
   (b) Describe the correspondence between the Lie algebra and the 1-parameter subgroups of G.
   (c) Define the exponential mapping for G.
   (d) Derive the from of the exponential mapping for the group of non-singular $n \times n$ real matrices.

2. Prove that $\mathbf{S^n}$ for $n > 1$ is an orientable manifold.

3. Let $p \in \mathbb{RP}^2$, and let $X = \mathbb{RP}^2 \times \{0, 1\}/\equiv$ where $\equiv$ is the equivalence relation generated by $(p, 0) \equiv (p, 1)$. Compute in detail $\pi_1(X)$.

4. Let (M, g) be a Riemannian manifold, and let X be a smooth vectorfield on M. Define a function on M by $f(m) = \|X(m)\|^2$ and let Y be the gradient vectorfield of f. Show that $X(m) = 0$ implies $Y(m) = 0$.

5. Let B be the 2-dimensional subbundle of the tangent bundle to $\mathbb{R}^3$ whose fibre at each point is spanned by the vectorfields $X_1 = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Use the Frobenius Theorem to determine whether B is integrable. Here (x, y, z) is the standard Euclidean coordinate system.

6. Consider a 2-dimensional Riemannian manifold (M, g), where M is an open subset of $\mathbb{R}^2$ and $g = f^2(dx^2 + dy^2)$ for some positive function $f(y)$ depending only on y.
   (a) Write down the simplest moving orthonormal frame you can think of.

   $$e_1 = \qquad\qquad e_2 =$$

   (b) The moving coframe dual to (e_1, e_2) is

   $$\theta^1 = \qquad ; \qquad \theta^2 =$$

   (c) Now find a 1-form ω such that $d\theta^1 = \omega \wedge \theta^2$ and $d\theta^2 = -\omega \wedge \theta^1$.
   (d) Find K so that $d\omega = -K\theta^1 \wedge \theta^2$.
   (e) What is the Gaussian curvature when $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$?