Algebra Qualifying Exam

January 16, 2007

Instructions. You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.

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1. (a) Show that a group of order $91 = 7 \times 13$ must be cyclic.
(b) Prove that a group of order $728 = 2^3 \times 7 \times 13$ cannot be simple.

2. Let p be a prime integer and let G be a nonabelian group of order p^3. Prove that the commutator subgroup G' has order p.

3. List all 5-Sylow subgroups of S_5, the group of permutations of 5 elements.

4. Let A be a square $n \times n$ matrix with entries in the rational numbers. Assume that $A^5 - 25A - 5I = 0$, where I is the $n \times n$ identity matrix. Prove that n is a multiple of 5.

5. Let L be an operator on an n-dimensional vector space over a field of characteristic 0. Assume that $\operatorname{tr}(L^k) = 0$ for all $1 \leqslant k \leqslant n$. Show that L is nilpotent.

6. Let $f(x) = x^3 - 3x + 1$. Prove that for any field F, either $f(x)$ is irreducible or splits (factors into linears) over F.

7. Let E be an algebraic extension of a field F. Show that every subring of E that contains F is a field.

8. Let $f(x) \in F[x]$ be an irreducible polynomial over a finite field F with q elements. Prove that a positive integer n is a multiple of $\deg(f)$ if and only if $f(x)$ divides $x^{q^n} - x$.

9. Let R be a commutative ring with a multiplicative identity. Assume that $x \in R$ is not a nilpotent element. Prove that there exists a prime ideal $\mathfrak{p} < R$ which does not contain the element x.

10. Let R be a ring (possibly without a multiplicative identity). Assume that R^2 is not equal to 0 (so there exist $a, b \in R$ with $ab \neq 0$). Also assume that R has no proper nonzero left ideals.

(a) Show that the set of elements $b \in R$ such that $ab = 0$ for all $a \in R$ is a left ideal.
(b) Prove that R has no zero divisors.
(c) Prove that R contains a (2-sided) multiplicative identity.