Algebra Qualifying Exam
Fall 1992

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a group of order $2n$, where n is odd. Prove that G has a normal subgroup of index 2.

2. Let G be a group of order $5 \cdot 7 \cdot 11$. Prove that $7 \parallel Z(G)|$, where $Z(G)$ denotes the *center* of G.

3. Let F be a field and let R be the ring

$$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a \in F \right\}.$$

Define the R-modules

$$M_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} | a \in F \right\}, M_2 = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} | b \in F \right\}.$$

Prove that $M_1 \ncong M_2$.

4. Let R be a ring and let M be an irreducible left R-module. Prove that there exists a maximal left ideal I such that $R/I \cong M$ as left R-modules.

5. Let R be a ring. An ideal $P \subseteq R$ is called *primary* if for any $a, b \in R$ with $ab \in P$, $a \notin P$, then $b^n \in P$ for some positive integer n. Show that P is primary if and only if every zero-divisor of R/P is nilpotent.

6. Let V be an n-dimensional vector space over a field F, and let

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$$

be a chain of subspaces of V, with $\dim(V_i/V_{i+1}) = 1$ for $i = 0, 1, \ldots, n-1$. Suppose that $T : V \to V$ is linear transformation satisfying $T(V_i) \subseteq V_{i+1}$ for all $i = 0, 1, \ldots, n-1$. Compute the characteristic polynomial of T.

7. Let V be a complex vector space and let $T \in End_C$. If $T^{1992} = 1_V$, prove that T is diagonalizable.

8. Let $f(x), g(x) \in F[x]$, and assume that $F \subseteq K$ is an extension of fields. If $f(x)$ divides $g(x)$ in $K[x]$, prove that $f(x)$ divides $g(x)$ in $F[x]$.

9. Let $K \subseteq C$ be the splitting field over Q for the polynomial $x^6 - 1 \in Q[x]$. Compute the Galois group of K over Q and show exactly how it operates on K.