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Complex Analysis Qualifying Exam

August, 1999

Instructions: Below you will find 8 problems. Each problem is worth 10 points. Only the best 6 scores will be added.
Time: 2 hours.

NOTATIONS: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle).

1. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and

$$|f(z)| < 1, \text{ for all } z \in \mathbb{D}$$

(i) Prove that

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(z)} f(w)} \right| \leq \left| \frac{z - w}{1 - \bar{z} w} \right|, \text{ for all } z, w \in \mathbb{D}.$$

(ii) Use (i) to prove that

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \text{ for all } z \in \mathbb{D}.$$

HINT: Schwarz Lemma.

2. Use the Residue Formula to prove that

$$\int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.$$

3. Let $\phi : \mathbb{T} \to \mathbb{C}$ be a continuous function. Define the function $f : \mathbb{D} \to \mathbb{C}$ by

$$f(z) = \int_0^{2\pi} \frac{\phi(e^{it})\, dt}{1 - z e^{-it}}, \text{ for all } z \in \mathbb{D}.$$

Prove that f is analytic.

4. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function with the property that $\left|f(\frac{1}{n})\right| \leq e^{-n}$, for any integer $n \geq 1$. Prove that f is identically zero.

HINT: Expand f in a power series.

5. Let $f : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ be a bounded analytic function. Prove that f extends analytically to the whole unit disk $\mathbb{D}$.

6. Let f be an entire function, which is bounded in the strip $0 \leq \operatorname{Re} z < 1$, and satisfies:

$$f(z+1) = 2f(z), \text{ for all } z \in \mathbb{C}.$$

Prove there exists a constant $a \in \mathbb{C}$ such that

$$f(z) = a2^z, \text{ for all } z \in \mathbb{C}.$$

7. Let f be an analytic function defined on the strip $-1 < \operatorname{Im} z < 1$. Consider the set

$$S = \{x \in \mathbb{R} \,:\, f(x) \in \mathbb{R}\}.$$

Prove that, either S is *discrete* (i.e. it has no accumulation points), or $S = \mathbb{R}$.

HINT: Consider the function $g(z) = \overline{f(\bar{z})}$.

8. (i) Suppose $u : \mathbb{D} \to \mathbb{R}$ is continuously twice differentiable and is *harmonic*, that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Prove there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$, such that $u = \operatorname{Re} f$.

(ii) Is this true if u (and f) is only defined on $\mathbb{D} \setminus \{0\}$?