Complex Variables Qualifying Exam
Fall 1995

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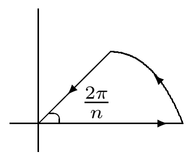
1. Find a function $f : \mathbb{C} \to \mathbb{C}$ such that f is complex differentiable exactly on the real axis.
Hint: Set $\frac{\partial f}{\partial \overline{z}} = z - \overline{z}$).

2. Compute $\int_0^\infty \frac{dx}{1+x^n}$ for all $n \geq 2$.
Hint: Use the contour below



3. Suppose f is entire and $|f(z)| \leq A + B|z|^k$ for some constants $A, B, k > 0$. Show that f is a polynomial.

4. Suppose f is an entire function and Range $(f) \subset \{-1 < \mathrm{Re}(z) < 1\}$. Show f is constant.

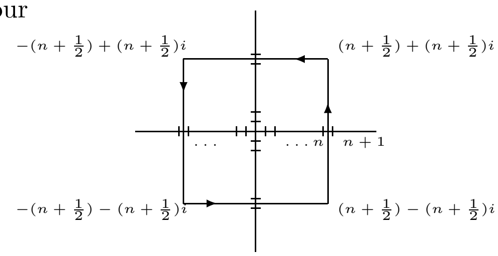
5. Construct an entire function having simple zeros at the points $\{a + bi : a, b \in \mathbb{Z}\}$.

6. Let $D = \{z : |z| < 1\}$ and let $f : D \to D$ be an analytic univalent (one-to-one) function. Let $\Omega = f(D)$. If $f(z) = \sum_{n=0}^\infty a_n z^n$ show that the area of Ω is equal to $\pi \sum_{n=1}^\infty n|a_n|^2$. (Hint: Consider the Jacobian of the mapping.)

7. (a) Let $f(z) = \frac{P(z)}{Q(z)}$ where P, Q are polynomials with $\deg(Q) \geq 2+\deg(P)$. Let $M = \{u_1, \ldots, u_k\}$ be the poles of f and assume $M \cap \mathbb{Z} = \oslash$. Show that

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{\ell=1}^{k} \mathrm{Res}(g; u_\ell) \quad \text{where } g(z) = \pi f(z)\cot(\pi z)$$

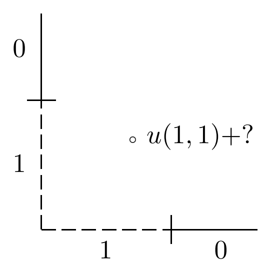
Hint: Consider the contour



(b) Use a variation on (a) to compute $\sum_{n=1}^\infty \frac{1}{n^2}$.

8. Let u be a harmonic function in the first quadrant satisfying the illustrated boundary conditions

$$u(x,0) = \begin{cases} 1 & x < 1 \\ 0 & x \geq 1 \end{cases} \qquad u(0,y) = \begin{cases} 1 & y < 1 \\ 0 & y \geq 1 \end{cases}$$



Compute $u(1,1)$.