TOPOLOGY

(Old) Qualifying Examination

Fall 1990

(Muenzenberger and Strecker)

Work 9 of the following problems. Start each problem on a new sheet of paper. Do **not** turn in more than 9 problems.

1. If (X,τ) and (Y,σ) are topological spaces and $f : X \to Y$ is a function, give a detailed set-theoretic proof that the following are equivalent.

   (i) For each $U \in \sigma$, $f^{-1}[U] \in \tau$.

   (ii) For each $A \subseteq X$, $f[\overline{A}^{\tau}] \subseteq \overline{f[A]}^{\sigma}$.

2. Prove that $[0,1]$, with its usual topology, is connected.

3. (a) True – False.

   (i) An open and closed one-to-one function between topological spaces must be an embedding.

   (ii) Each space that is locally-Hausdorff (in the sense that each point has neighborhood base of Hausdorff subspaces) must be Hausdorff.

   (iii) Each quotient of a locally connected space must be locally connected.

   (iv) Each locally compact Hausdorff space is completely regular.

   (v) The product of metrizable spaces is metrizable.

   (vi) The product of continua is a continuum. (A **continuum** is a compact, connected, Hausdorff space.)

   (vii) Every metrizable space is normal.

   (viii) Every subspace of a separable space is separable.

   (b) For each false entry, give a counterexample or other explanation (no proofs).

4. Prove that if A is a compact subset of a regular (not necessarily Hausdorff) space X, then $\overline{A}$ is compact.

5. Give an example of two topologies σ and τ on the set of integers $\mathbb{Z}$ for which $\sigma \subsetneq \tau$ and $(\mathbb{Z},\sigma)$ is homeomorphic to $(\mathbb{Z},\tau)$.

6. (a) State the Axiom of Choice.
   (b) Using the Axiom of Choice, give a detailed proof that if $S = \{A_n \mid n \in \mathbb{N}\}$ and $|A_n| = 2$ for each n, then $\bigcup_{n=0}^{\infty} A_n$ is countable.

7. For w and z on the unit circle S_1, define $w \sim z \Leftrightarrow w = z$ or $w = -z$. Prove that if S^1 has the usual topology, then the quotient space $S^1/\sim$ is homeomorphic to S^1.

8. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be **final** provided that for each topological space (Z, μ) each set-function $g : Y \to Z$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \mu)$ is continuous. Prove that:

   (a) The composition of final (continuous) maps is final.
   (b) The "second factor" of a final map is final; i.e., if $(X, \tau) \xrightarrow{\text{f}} (Y, \sigma) \xrightarrow{\text{h}} (w, s)$ are continuous maps and $h \circ f$ is final, then h is final.

9. (a) Give an example of a topological space X that has both a Stone Čech compactification, βX, and an Alexandroff compactification, αX, but for which αX and βX are not homeomorphic.
   (b) Give a reason why αX and βX, in part (a), are not homeomorphic.

10. Show that if X is a second countable space and if $\mathcal{B}$ is any base for X, then there is a countable subcollection of $\mathcal{B}$ that is also a base for X.

11. (a) If sequential limits in a space X are unique, must X be Hausdorff?
    (b) Prove that your answer to (a) is correct.

12. Show that if X is a compact metrizable space, then every metric which generates the topology on X is complete.

13. Prove that every paracompact Hausdorff space is regular.

14. Show that the Moore Plane (tangent disc space) is not normal.

15. Prove that if A is a connected subset of a connected space X and if C is a component of $X - A$, then $X - C$ is connected.

16. Describe the fundamental groups of the following spaces.

(a) The circle, S^1 ;

(b) The Mobius Strip, M ;

(c) The figure eight, ∞ ;

(d) The torus, $S^1 \times S^1$;

(e) The projective plane, $\mathbb{P}$.