Topology Qualifying Exam

January 23, 2010

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

1. A space X is called *locally path-connected* if for every point x in X and neighborhood U of x there exists an open neighborhood $V \subset U$ of x such that V is path-connected.

   (a) Prove that a connected and locally path-connected space is path-connected.

   (b) Give an example of a path-connected, but not locally path-connected space.

2. The group $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ in standard way. Describe the space of orbits and its induced quotient space topology.

3. Consider the map $f\colon \mathrm{Mat}(2\times 2, \mathbb{R}) \to \mathrm{Mat}(2\times 2, \mathbb{R})$ that sends $X \mapsto X^2 - 3X$. Describe the set of critical points of f.

4. Prove that any continuous map $\mathbb{R}P^2 \times \mathbb{R}P^2 \longrightarrow S^1$ is nullhomotpic.

5. (a) Compute the singular cohomology groups of $S^2 \times S^4$.

   (b) Prove that the spaces $S^2 \times S^4$ and $\mathbb{C}P^6$ are not homotopy equivalent by comparing their cohomology rings.

6. Let $\omega = dx_1 \wedge dx_2 \wedge dx_3$, $X = x_1\partial/\partial x_1 + x_2\partial/\partial x_2 + x_3\partial/\partial x_3$ be respectively a 3-form and a vector field on $\mathbb{R}^3$.

   (a) Compute $\eta = \iota_X\omega$. Is η closed or exact?

   (b) Consider the map $F\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3$ given by $F(x) = \dfrac{x}{|x|}$. Compute $F^*\eta$.

   (c) Prove without making any computation that $F^*\eta$ is closed but not exact. (Hint: consider F as a composition $\mathbb{R}^3 \setminus \{0\} \to S^2 \hookrightarrow \mathbb{R}^3$.)

7. Suppose M and N are smooth, oriented, connected closed n-manifolds, and $F\colon M \to N$ is a smooth map. If $\int_M F^*\omega \neq 0$ for some $\omega \in \Omega^n(N)$, show that F is surjective.

8. (a) Let X be an n-dimensional manifold and let X' be obtained by removing an open n-ball B from X such that the closure $\overline{B}$ of B is contained in a larger n-ball $C \subset X$. Assuming the Euler characteristic of X is defined, express the Euler characteristic $\chi(X')$ in terms of $\chi(X)$ and the dimension n.

   (b) Let X and Y be two n-dimensional manifolds such that their Euler characteristics are defined. A *connected sum* $X\#Y$ is obtained by removing open n-balls in X and Y as in (a) and then identifying the boundaries of the removed n-balls in the resulting spaces X' and Y' by a homeomorphism. Express the Euler characteristic $\chi(X\#Y)$ in terms of $\chi(X)$ and $\chi(Y)$ and the dimension n.