Algebra Qualifying Exam
August 25, 1998

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

1. Let G be a group of order greater than 2. Show that G has a non-trivial automorphism.

2. Let G be a group acting transitively on a set Ω. Fix $\omega \in \Omega$ and let $H = \mathrm{Stab}_G(\omega)$. If we let G act on $\Omega \times \Omega$ by $g \cdot (\omega_1, \omega_2) = (g \cdot \omega_1, g \cdot \omega_2),\ g \in G,\ \omega_1, \omega_2 \in \Omega$, show that the G-orbits on $\Omega \times \Omega$ are in bijective correspondence with the H-orbits on Ω.

3. The group G is called a CA-group if for every $e \neq x \in G,\ \mathrm{C}_G(x)$ is abelian. Prove that if G is a CA-group, then
   (a) the relation $x \sim y$ if and only if $xy = yx$ is an equivalence relation on $G^{\#}$;
   (b) If $\mathcal{C}$ is an equivalence class in $G^{\#}$, then $H = \{e\} \cup \mathcal{C}$ is a subgroup of G.

4. Let R be a *u.f.d.* in which every prime ideal is maximal. Prove that every prime ideal is principal.

5. Let p be a prime and let $R = \{\frac{a}{b} \in \mathbb{Q} \mid p \not| \ b\}$. If M is the principal ideal in R generated by p, prove that M is the *unique* maximal ideal in R. (*Hint:* Show that any element not in M is a unit in R.)

6. Let V be a vector space over the field $\mathbb{F}$ and let $T : V \to V$ be an *idempotent* linear transformation: $T^2 = T$. Prove that if $W \subseteq V$ is a T-invariant subspace of V, then there exists a T-invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$.

7. Let $0 \to M' \xrightarrow{\mu} M \xrightarrow{\epsilon} M'' \to 0$ be a short exact sequences of R-modules, where R is a ring. If M'' is R-free, show that $M \cong \mu(M') \oplus M_0$, where $M_0 \cong_R M''$.

8. Compute the Galois group over the rational field $\mathbb{Q}$ of the field $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/12}$.

9. Compute the field extension degree $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}]$.

10. Let $\mathbb{F} = \mathbb{F}_q$ be the finite field of q elements, and let $\mathbb{K} = \mathbb{F}_{q^2} \supseteq \mathbb{F}$ be a quadratic extension. Define the *Frobenius automorphism* $F : \mathbb{K} \to \mathbb{K}$ by setting $F(\alpha) = \alpha^q, \ \alpha \in \mathbb{K}$. If we define $N : \mathbb{K}^\times \to \mathbb{F}^\times$ by setting $N(\alpha) = \alpha F(\alpha), \ \alpha \in \mathbb{K}^\times$, show that N is a surjective homomorphism of groups whose kernel has order $q+1$.