Complex Variables Qualifying Exam
Fall 1994

---

1. Let U be open $\subset \mathbb{C}$, $a \in U$, f holomorphic in $U\backslash\{a\}$.

(i) What does it mean to say f has a pole at a?

(ii) Show that if a is a pole, then $f(U\backslash\{a\})$ is a "neighborhood of ∞", that is, its complement in $\mathbb{C}$ is compact.

HINT: Consider $1/f(z)$ for z near a.

(iii) Suppose f has a simple pole at a. Prove that

$$\lim_{r\to 0}\int_{\gamma_r} f(z)dz = i\pi Res(f,a),$$

where $\gamma_r(t) := re^{it} + a$ for $r > 0$ and $0 \le t \le \pi$.

2. The holomorphic function f has an isolated singularity at z_0 and for some a, $M \in \mathbb{R}$ satisfies

$$|f(z)| \le M|z - z_0|^a$$

near z_0. Show that

(i) z_0 is a removable singularity if $a > -1$ and

(ii) if a satisfies $-n < a \le -1$ for some $n \in \mathbb{N}$, then z_0 is a pole of order at most $n - 1$.

HINT: Use the Cauchy estimates.

3. Suppose $\int_\gamma f = 0$ for all piecewise smooth loops γ in a region Ω and for all holomorphic functions f in Ω. Show how to **construct** a holomorphic logarithm for any given zero-free holomorphic function in Ω.

4. Explain why the identity function $f(z) := z$ has no holomorphic logarithm in $\Omega := \{z \in \mathbb{C} : 0 < |z| < 1\}$.

5. Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ satisfies $e^{f(z)} = z$ for all $z \in \Omega$ and is continuous on Ω. Show that f is in fact holomorphic in Ω.

6. f is holomorphic by not constant in a neighborhood of a and $f(a) = 0$. Show that a must be an isolated zero; that is, if $r > 0$ is small enough, f has no zero in $\{0 < |z - a| < r\}$.

HINT: Look at the Taylor series of f at a.

7. (i) The entire function f satisfies

$$z^{-n}f(z) \text{ is bounded for some } n \in \mathbb{N}.$$

Show that f is a polynomial of degree no greater than n.

HINT: Cauchy estimates.

(ii) The entire function F satisfies

$$|F(z)| \to \infty \text{ as } |z| \to \infty.$$

Show that F is a polynomial.

HINT: F has only finitely many zeros (proof?). So for an appropriate polynomial P, $f := P/F$ satisfies the hypothesis of (i) and it has no zeros.