Algebra Qualifying Exam (Old and New)
August 26, 2003

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. At least one problem from each of the five sets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$, $\{9,10\}$. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

$\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers respectively.

1. Let p be a prime integer. Set

$$\mathbb{Z}_p = \{\frac{a}{p^i} \in \mathbb{Q} \mid i, a \in \mathbb{Z}\}.$$

Then $\mathbb{Z}_p$ is an additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$ as a subgroup. Define $\mathbb{Z}(p^\infty) = \mathbb{Z}_p/\mathbb{Z}$ as the quotient group. For each $x \in \mathbb{Z}_p$, we use $\bar{x}$ to denote its image in $\mathbb{Z}(p^\infty)$. Prove the following:

(a) $\mathbb{Z}(p^\infty)$ is an infinite group in which every element has order p^n for some $n \geq 0$;

(b) Every finite subgroup of $\mathbb{Z}(p^\infty)$ is a cyclic group;

(c) For infinite subset X of the set $\{\overline{\frac{1}{p}}, \overline{\frac{1}{p^2}}, \dots,\}$, the subgroup $\langle X \rangle$ generated by X is $\mathbb{Z}(p^\infty)$ and then prove that any subgroup H of $\mathbb{Z}(p^\infty)$ is either finite cyclic or $\mathbb{Z}(p^\infty)$;

2. Let $f : G \to G$ be a group endomorphism satisfying the *normal condition*: $f(gag^{-1}) = gf(a)g^{-1}$ for all $a, g \in G$. Suppose that G satisfies the ascending and descending chain conditions on the normal subgroups, i.e, any ascending (and descending) chain of normal subgroups terminates. Show that there is a positive integer n such that $G = \ker(f^n) \times \operatorname{im}(f^n)$, where f^n is the composition of f with itself n times.

3. Let A be a commutative ring. For any ideal I of A, define

$$\sqrt{I} = \{x \in A \mid x^n \in I \text{ for some positive integer } n\}.$$

(a) Show that $\sqrt{I}$ is an ideal in A.

(b) An ideal P in A is called primary if, for any $a, b \in A$, $ab \in P$ with $a \notin P$ implies $b^n \in P$ for some positive integer n. Show that if an ideal P of A is primary then $\sqrt{P}$ is a prime ideal in A.

4. Let B be a commutative ring and $A \subseteq B$ be a subring such that B is finitely generated A-algebra, i.e., there is a surjective A-algebra homomorphism from a polynomial ring $A[x_1, \ldots, x_n]$ onto B for some n. Show that every element B is integral if and only if B is finitely generated as an A-module (via left multiplication). (Recall that an element b of B is called integral over A if there is a polynomial $f(x) = x^m + a_1 x^{m-1} + \cdots + a_m$ in $A[x]$ such that $f(b) = 0$.)

5. Let R be an associative ring (with 1) and M be a free left R-module. Prove that for any surjective homomorphism $f : N \to E$ of left R-modules, the map $\hom_R(M, N) \to \hom_R(M, E)$ defined by $\phi \to f \circ \phi$ is also surjective.

6. Let M be an associative ring with 1 and M be an irreducible left R-module. Show that there is a maximal left ideal I of R such that R/I is isomorphic to M as a left R-module.

7. Let k be a field (not necessarily algebraically closed) and V a finite dimensional k-vector space. Suppose $\phi : V \to V$ is a linear transformation such that its minimal polynomial can be factored over k into the form $(x - a_1)^{m_1} \cdots (x - a_r)^{m_r}$. One makes V into a $k[x]$-module with the action of x on V given by ϕ. Show that V is an indecomposable $k[x]$-module if and only if the minimal polynomial is of form $(x - a)^n$ with $n = \dim_k(V)$.

8. Let k a field (not necessarily algebraically closed) and V be a k-vector space (possibly infinite dimensional). Let $\phi : V \to V$ be a linear transformation such that $\psi = \phi - 1$ is a nilpotent linear transformation. Show that ϕ has an eigenvector in V.

9. Let K be a finite field and F a finite algebraic extension of K. Show that F is a Galois extension of K and the Galois group $Gal(F/K)$ is a cyclic group.

10. Prove that an algebraic extension $F \subseteq k$ is normal if and only if every irreducible polynomial in $F[X]$ factors in $k[X]$ as product of irreducible polynomials, all of which have the same degree.