GEOMETRY OF MANIFOLDS QUALIFYING EXAM

Spring 2002

(Auckly & Miller)

1. Let $\alpha = z^3\, dx \wedge dy - y\, dx \wedge dz \in \Gamma(\wedge^1\mathbb{R}^3)$. Let $X = \partial_x + x\partial_z \in \Gamma(T\mathbb{R}^3)$ Compute:

   a) $d\alpha$

   b) $i_X\alpha$

   c) $L_X dx$

   d) $L_X dy$

   e) $L_X dz$

   f) $L_X\alpha$

   g) $\displaystyle\int_{S^2}\alpha$

   [Here S^2 is oriented with $i_{(x\partial_x+y\partial_y+z\partial_z)}(dx \wedge dy \wedge dz)$.]

2. Find $\int_{\sum} dy \wedge dx + dz \wedge dy + dx \wedge dz$ when

$$\sum = \{(x, y, z) \in \mathbb{R}^3 | z = 1 - (x^2 + y^2)^{2002},\ z \geq 0\}$$

   and $\Omega_{\sum}|_{(0,0,1)=dx\wedge dy}$.

3. Let $X = \mathbb{R}P^2 \vee S^1$ ($\vee$ is the 1 point union.)

   a) Compute $\pi_1(X)$.

   b) Construct a 2-fold cover of X, say $\widehat{X}$, with $H_2(\widehat{X};, \mathbb{Z}) \neq 0$.

   c) Compute $H_*(X; \mathbb{Z})$.

   d) Compute $H_*(\widehat{X}; \mathbb{Z})$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$; $f(x, y) = x^2 - y^2$. Let $g = dx^2 + dy^2$.

   a) Compute $\operatorname{grad} f$.

   b) Let $\alpha_n, \beta_n, \gamma_n : \mathbb{R} \to \mathbb{R}^2$ be integral curves of $\operatorname{grad} f$ with $\alpha_n(0) = (\frac{1}{n^2}, 1)$, $\beta_n(0) = (\frac{1}{n}, \frac{1}{n})$, $\gamma_n(0) = (1, \frac{1}{n^2})$. Find expressions for α_n, β_n and γ_n.

   c) Prove that $\alpha_n(\mathbb{R}) = \beta_n(\mathbb{R}) = \gamma_n(\mathbb{R})$.

   d) Compute $\lim_{n\to\infty} \alpha_n(t)$, $\lim_{n\to\infty} \beta_n(t)$ and $\lim_{n\to\infty} \gamma_n(t)$.

5. Let $0 < a < b$. The equations

$$\begin{aligned} x &= (b + a\cos\psi)\cos\theta \\ y &= (b + a\cos\psi)\sin\theta \\ z &= a\sin\psi, \quad \theta, \psi \in [0, 2\pi] \end{aligned}$$

describe a surface in $\mathbb{R}^3$.

a) What is this surface?

b) Calculate the Gaussian curvature.

c) Write the equations for geodesics on this surface.

6. Let $\varphi : M \to N$ be a smooth map between connected, oriented, closed n-dimensional manifolds. Prove that:

$$\left(\int_M \varphi^*\alpha\right)\left(\int_N \beta\right) = \left(\int_M \varphi^*\beta\right)\left(\int_N \alpha\right)$$

for all α, $\beta \in \Gamma(\wedge^n N)$.

Hint: Think about $H^n(M)$ and $H^n(N)$.

7. A vector bundle map, $J : TM \to TM$ is called an almost $\mathbb{C}$-structure if $J^2 = -\,\mathrm{id}$.

a) If a manifold, M, admits an almost complex structure, what can be said about $\dim M$? Why?

b) Prove that any manifold admitting an almost complex structure is orientable.

Hint: Let g be a Riemannian metric on M and define $w(X,Y) = g(X, J(Y)) - g(Y, J(X))$. Use w to construct an orientation.

8. Let $X = -y\partial_x + x\partial_y + \partial_z$, $Y = z\partial_x + \partial_y$. Let $B = \mathrm{span}\,\{X, Y\}$.

a) Is B integrable? If B is integrable, find the integral manifold through $(1, 0, 1)$.

b) Find the flow of X.

c) Find the flow of Y.