Ph.D. QUALIFYING EXAMINATION
REAL ANALYSIS (April 30, 1984)

1. Let $(X,\mathcal{A},\mu)$ be a measure space, $(Y,\mathcal{B})$ a measurable space, and $\phi: X \to Y$ a measurable mapping. Define

$$\nu(E) = \mu(\phi^{-1}(E)) \quad \text{for } E \in \mathcal{B}.$$

Prove:

(a) ν is a measure on $\mathcal{B}$.

(b) If f is a nonnegative measurable function on Y, then

$$\int f d\nu = \int f\circ\phi d\mu.$$

2. Let $f, f_1, f_2, \ldots$ be a sequence in $L^1(X,\mathcal{A},\mu)$ such that $f_n \to f$ as $n \to \infty$ in L^1-norm. Prove:

(a) There exists a subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for μ- almost all $x \in X$.

(b) Suppose that there exists $K \in \mathbb{R}$ such that

$$\int \log|f_n| d\mu \geq K \quad \text{for } n = 1,2,\ldots .$$

Prove that

$$\int \log|f| d\mu \geq K.$$

<u>Remark and Hint:</u> Notice $\log 0 = -\infty$ (by definition) and $t - \log t \geq 0$.

3. Let $f,g \in L^1(\mathbb{R})$ be Borel measurable and write

$$(f*g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Prove that $f*g$ is defined a.e. on $\mathbb{R}$, Borel measurable, and satisfies $||f*g||_1 \leq ||f||_1 \cdot ||g||_1$.

4. Let $f \in L^1(\mathbb{R})$ and

$$G(x) = \exp(-x^2) \quad \text{for } x \in \mathbb{R}.$$

Prove that $G*f$ is infinitely differentiable.

5. Use the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \quad (x > 0)$$

to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Time saver: You may use:

$$\frac{d}{dx}\,[-e^{-xt}(t \sin x + \cos x)] = (1 + t^2)e^{-xt}\sin x.$$

6. Let f be a complex function on $\mathbb{R}$ such that $fg \in L^1(\mathbb{R})$ for every $g \in L^2(\mathbb{R})$. Prove that $f \in L^2(\mathbb{R})$.

Hint: Closed Graph Theorem or Uniform Boundedness Principle.

7. Let μ be a nonnegative finite (regular) Borel measure on $\mathbb{R}$. Prove that if E is a Borel subset of $\mathbb{R}$, then the function f defined by

$$f(x) = \mu(E - x)$$

is Borel measurable.

Hint: If E is compact, then $\{x \in \mathbb{R} : f(x) < a\}$ is open for each $a \in \mathbb{R}$.

8. Let f be a complex function on $[0,1]$ which is continuous at 0 and in $L^1([0,1])$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)\,dx = f(0).$$

9. Suppose that $(c_n)_{n=0}^\infty \subset \mathbb{C}$ and $A \in \mathbb{R}$ satisfy

$$\left|\sum_{j=0}^n a_j c_j\right| \leq A \cdot \sup_{0\leq x\leq 1} \left|\sum_{j=0}^n a_j x^j\right|$$

whenever $n \in \mathbb{N}$ and $(a_j)_{j=0}^n \subset \mathbb{C}$. Prove that there exists a unique $\mu \in M([0,1])$ such that

$$c_n = \int_{[0,1]} x^n\,d\mu(x)$$

for all $n \geq 0$.

10. Let $\mu, \tau \in M(\mathbb{T})$ satisfy $\tau \ll \mu$. Prove that if $\lim_{n\to\infty} \hat{\mu}(n) = 0$, then $\lim_{n\to\infty} \hat{\tau}(n) = 0$. For μ and $n \in \mathbb{Z}$, $\hat{\mu}(n)$ is defined by

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-int}\,d\mu(t).$$