Kansas State University | Department of Mathematics

Qualifying Exam: Analysis

Spring 2012. January 18, 6:00 p.m. to 9:00 p.m.
Examiners: Prof. Bob Burckel and Prof. Marianne Korten

Name: ____________________

1. Let E be a Lebesgue measurable set in $I\!R^n$, λ_n be the Lebesgue measure in $I\!R^n$, and $\{f_k\}$ be a sequence of measurable real-valued functions on E. Show that if there exists $\phi \in L^1(E)$ such that $|f_k| \leq \phi$ a.e. for all k, then
$$\int_E \limsup_{k\to\infty} f_k \geq \limsup_{k\to\infty} \int_E f_k.$$

2. Let $E \subset I\!R^2$ be Lebesgue measurable such that for λ_1 a.e. $x \in I\!R$, $E_x = \{y \in I\!R : (x, y) \in E\}$ is a λ_1 null set. Show that E is a λ_2 null set and for λ_1 a.e. $y \in I\!R$, $E^y = \{x \in I\!R : (x, y) \in E\}$ is a λ_1 null set too.

3. Prove that if $f_k \to f$ in L^p , $1 \leq p < \infty$, $g_k \to g$ pointwise, and $|g_k| \leq M$ for all k, then $f_k g_k \to fg$ in L^p.

4. Let $B(0, 1)$ be the unit ball in $I\!R^n$, χ its indicator function, λ_n the Lebesgue measure in $I\!R^n$. Let $K(x) = \frac{\chi(x)}{\lambda_n(B(0,1))}$ for $x \in I\!R^n$, and $K_\epsilon(x) = \frac{1}{\epsilon^n}K(x/\epsilon)$. Let $L^1_{loc}(I\!R^n)$ stand for the set of Lebesgue measurable functions that are integrable over each compact subset of $I\!R^n$. Prove in detail that for every $f \in L^1_{loc}(I\!R^n)$ and every Lebesgue point x of f
$$\lim_{\epsilon\to 0}(f * K_\epsilon)(x) = f(x).$$

5. **(i)** Using the identity $\cos t = (\exp it + \exp(-it))/2$, show that for $a > 1$
$$\int_0^{2\pi} \frac{1}{a + \cos t} dt = -2\pi i \int_{|z|=1} \frac{1}{z^2 + 2az + 1} dz,$$
where the unit circle is parameterized in the counterclockwise direction.

   (ii) Using **(i)** and the residue theorem, find the numerical value of this integral.

6. Let Ω be an open connected subset of the complex plane $\mathbb{C}$, and f be holomorphic in Ω. Suppose that f has a continuous logarithm F, that is, $F : \Omega \to \mathbb{C}$ is continuous and $f = \exp F$. Prove that necessarily F is *holomorphic* and find F'.

   Hints: You may assume f is not constant. Then if $z, z_0 \in \Omega$ and $0 < |z - z_0|$ is sufficiently small, $f(z) \neq f(z_0)$ (why?), so $F(z) \neq F(z_0)$ and we have (why?)

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} \left(\frac{\exp F(z) - \exp F(z_0)}{F(z) - F(z_0)} \right)^{-1}. \qquad (*)$$

   As $z \to z_0$ in $(*)$, $F(z) \to F(z_0)$ (why?) and the parenthetic quotient does what?

7. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Prove that there is no continuous logarithm in $\mathbb{D} \setminus \{0\}$, that is, no continuous function $L : \mathbb{D} \setminus \{0\} \to \mathbb{C}$ satisfies $\exp L(z) = z$ for all $z \in \mathbb{D} \setminus \{0\}$.

   Hint: According to the preceding problem, any such L would be holomorphic. What does this say about $\int_{|z|=1/2} \frac{1}{z} dz$?

8. The function f is holomorphic in $\mathbb{D} \setminus \{0\}$.

   (i) Define the residue of f at 0 and describe how to compute it.

   (ii) Show that for this number, call it c, the function $f(z) - c/z$ has a primitive, that is, $f(z) - c/z = F'(z)$ for some holomorphic function $F(z)$.

   Hint: $f(z)$ has a series representation in (positive and negative) integer powers z^n (why?). Which z^n have primitives in $\mathbb{D} \setminus \{0\}$? (Recall the preceding problem.)