COMPLEX VARIABLES QUALIFYING EXAMINATION - Spring 1998
(Bennett and Burckel)

Let $\mathbb{R}$ denote the real line, $\mathbb{C}$ the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, Ω a non-void, open, connected subset of $\mathbb{C}$, $C(\Omega)$ the continuous $\mathbb{C}$-valued functions on Ω, $H(\Omega)$ the complex-differentiable function on Ω.

1. Let S be the open square $]0,1[\times]0,1[$ and identify $(x,y) \in \mathbb{R}^2$ with $x+iy \in \mathbb{C}$.

(i) What does it mean for a function $f : S \to \mathbb{R}^2$ to be $\mathbb{R}$-differentiable at $(x_0, y_0) \in S$?

(ii) If f is $\mathbb{R}$-differentiable at (x_0, y_0), what property of its $\mathbb{R}$-derivative will make f also $\mathbb{C}$-differentiable at $x_0 + iy_0$?

2. Suppose Ω is starlike with respect to its point a. Show that for every $f \in H(\Omega)$ the companion function F defined by

$$F(z) := \int_{[a,z]} f \qquad \forall z \in \Omega$$

is also holomorphic in Ω and satisfies $F' = f$.

3. What is the *topology of local uniform convergence* in $C(\Omega)$? Is this a metric topology? Show that:

(i) $C(\Omega)$ is complete in this topology.

(ii) $H(\Omega)$ is a closed subset of $C(\Omega)$.

Hint: For (ii) Morera's theorem is useful.

4. Prove that $f \mapsto f'$ is a continuous mapping of $H(\Omega)$ into itself (in the topology of Problem 3). Give an example of an Ω for which this map is not surjective.

Hint: For the continuity, exploit Cauchy's integral formula.

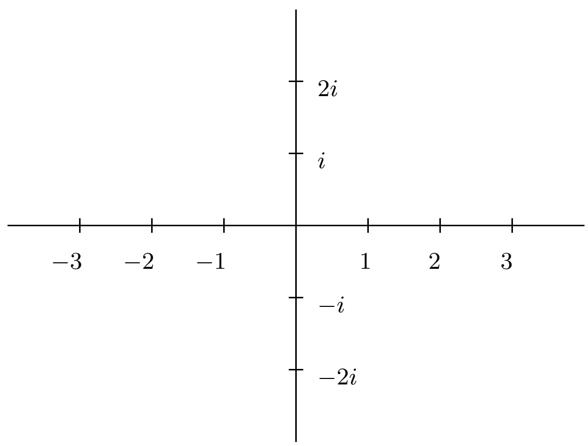
5. Show that if $f \in H(\Omega)$ is one-to-one, then f' is zero-free in Ω. Is the converse true?

6. $h : \mathbb{C} \to \mathbb{R}$ is harmonic and not constant. Prove that h has a zero.

Hint: If $h > 0$ throughout $\mathbb{C}$, employ Harnack's inequalities.

7. Compute $\displaystyle\int_\Gamma \frac{z^2+1}{z^2-1}dz$,

where Γ is the indicated path.



8. The *cross-ratio* of an ordered quadruple of distinct complex numbers is $[z_1, z_2, z_3, z_4] := \dfrac{(z_1-z_2)(z_3-z_4)}{z_1-z_4)(z_3-z_2)}$. Show that $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$ if and only if there is a Möbius transformation (i.e., a linear fractional transformation) that maps each z_j to w_j.

9. Suppose $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence 1. Show that the function $f(z) := \displaystyle\sum_{n=0}^{\infty} c_n z^n$ which it defines in $\mathbb{D}$ is holomorphic. Can you find such an f which can be continuously extended to $\overline{\mathbb{D}}$? Disprove or give an example.

10. Prove that the zeros of a non-constant polynomial depend continuously on its coefficients in the following sense: Given $P(z) = c_0 + c_1 z + \ldots + c_n z^n$ $(n > 0, c_n \neq 0)$ whose (distinct) zeros are $z_1, \ldots, z_r$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever complex numbers satisfy $|b_j - c_j| < \delta$ for all j, the polynomial $Q(z) := b_0 + b_1 z + \ldots + b_n z^n$ will have at least one zero in each of the disks $D(z_j, \varepsilon) := \{z \in \mathbb{C} : |z - z_j| < \varepsilon\}$ and all its zeros in the union $\displaystyle\bigcup_{j=1}^{r} D(z_j, \varepsilon)$ of these disks.