Name: ______________________________

# Complex Analysis Qualifying Exam

## August, 2002

**Instructions:** Below you will find 8 problems. Each problem is worth 10 points. Only the best 6 scores will be added.
**Time:** 2 hours.

Notations: $\mathbb{R}$ = set of all real numbers; $\mathbb{C}$ = set of all complex numbers; $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (the unit disk); $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (the unit circle).

**1.** Let $f$ be a holomorphic function on the open unit disk $\mathbb{D}$ with $|f(z)| < 1$ for all $z \in \mathbb{D}$. Assume there exist $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, and such that $f(z_1) = z_1$ and $f(z_2) = z_2$. Show that $f(z) = z$ for all $z \in \mathbb{D}$.

**2.** Prove that if $F(z)$ and $G(z)$ are entire functions with the same zeros and with the same multiplicities at each zero, then there is an entire function $\varphi(z)$ such that

$$F(z) = e^{\varphi(z)} G(z), \quad \forall\, z \in \mathbb{C}.$$

**3.** (i) Suppose $h : \mathbb{D} \to \mathbb{R}$ is *harmonic*, that is,

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

Prove there exists an analytic function $f : \mathbb{D} \to \mathbb{C}$, such that $h = \operatorname{Re} f$.

(ii) Is this true if $h$ (and $f$) is only defined on $\mathbb{D} \setminus \{0\}$?

**4.** Let $\Omega \subset \mathbb{C}$ be an open set which contains the closed unit disk $\mathbb{D} \cup \mathbb{T}$, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $f(0) = 1$ and

$$|f(z)| > 1, \ \ \forall\, z \in \mathbb{T}.$$

Prove that $f$ has at least one zero in $\mathbb{D}$.

**5.** Compute

$$\int_0^\infty \frac{dx}{1+x^5}.$$

Hint: For every $R > 0$, use the boundary of the circular sector

$$C_R = \{re^{i\theta} \, : \, 0 < r < R, \ 0 < \theta < 2\pi/5\}$$

as contour.

**6.** Let $P$ and $Q$ be polynomials, with $\deg Q \geq 2 + \deg P$. Define the functions $f(z) = \dfrac{P(z)}{Q(z)}$ and $g(z) = \pi f(z) \cot(\pi z)$. Let $M = \{w_1, \ldots, w_p\}$ be the set of poles of $f$. Prove that, if $M \cap \mathbb{Z} = \varnothing$, then

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{p} \operatorname{Res}(g; w_j).$$

Hint: For every integer $m \geq 1$, integrate over the boundary of the square

$$S_m = \big\{x + yi \, : \, x, y \in \mathbb{R}, \ \max(|x|, |y|) \leq m + \frac{1}{2}\big\}.$$

**7.** Define the strip $\Sigma = \{z \in \mathbb{C} \, : \, -1 < \operatorname{Im} z < 1\}$, and let $f : \Sigma \to \mathbb{C}$ be a holomorphic function. Consider the set

$$S = \{z \in \mathbb{R} \, : \, f(z) \in \mathbb{R}\}.$$

Prove that, either $S$ is *discrete* (i.e. it has no accumulation points), or $S = \mathbb{R}$.

Hint: Consider the function $g(z) = \overline{f(\bar{z})}$.

**8.** Map the region

$$\Omega = \{z \in \mathbb{C} \, : \, |z| < 2, \ |z - 1| > 1\}$$

conformally onto the upper half-plane

$$U = \{z \in \mathbb{C} \, : \, \operatorname{Im} z > 0\}.$$