REAL ANALYSIS QUALIFYING EXAM
SPRING 2001
(Bennett & Moore)

Answer all eight questions. Throughout, $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Let ν, μ be positive measures on $(X, \mathcal{M})$. Show that the following are equivalent:

(a) $\nu << \mu$

(b) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(B) < \varepsilon$ whenever $B \in \mathcal{M}$ and $\mu(B) < \delta$.

2. Suppose $f \in L^p(\mathbb{R}), 1 \leq p < \infty$. Prove that $\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|^p \, dx = 0$.

3. A set $\mathcal{A} \subseteq L^1(\mu)$ is called uniformly integrable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

Prove Vitali's theorem:

If (i) $\mu(X) < \infty$, (ii) $\{f_n\}$ is uniformly integrable, and (iii) $f_n \to f$ a.e., where $|f| < \infty$ a.e., then $\|f_n - f\|_1 \to 0$.

4. Prove or disprove: If $U \subseteq \mathbb{R}$ is open, then $|\overline{U} \backslash U| = 0$.

5. Suppose $\{a_n\}$ is a decreasing sequence of positive numbers and $\sum_{n=1}^{\infty} a_n < \infty$.

Show that $\lim_{n \to \infty} n a_n = 0$.

6. Evaluate $\lim_{A \to \infty} \int_0^A \frac{\sin(x)}{x} \, dx$. (Hint: $\int_0^\infty e^{-xt} dt = \frac{1}{x}$).

7. Suppose T is a linear operator on $L^2(X, \mathcal{M}, \mu)$ with $\|Tf\|_2 = \|f\|_2$. Show $\langle Tf, Tg \rangle = \langle f, g \rangle$ for all $f, g \in L^2$.

8. Suppose $g_\alpha \in L^2(X, \mathcal{M}, \mu)$ are such that $\left| \int_X f(x) g_\alpha(x) \, d\mu(x) \right| \leq \|f\|_2^3$ for all α and $f \in L^2(X, \mathcal{M}, \mu)$. Show that there exists an $M > 0$ such that $\|g_\alpha\|_2 \leq M < \infty$ for every α.