Real and Complex Analysis Qualifying Exam.

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Instructions: The exam consists of 9 problems. Each problem is worth 10 points.
Time: 3 hours.
Notation: $\mathbf{R}$ = reals, $\mathbf{C}$ = complexes, $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, Ω is a region = open, connected subset of $\mathbf{R^2} = \mathbf{C}$, $\mathbf{H}(\Omega)$ the set of holomorphic functions in Ω, $|E|$ is the Lebesgue measure of $E \subset \mathbf{R}$, and (μ, X) is an abstract measure space.
Note: Any fact from the hints that you use you are expected to prove.

1. **Problem 1**.

   a) Compute
   $$\int_0^{2\pi} \frac{dt}{\cos t - 2}.$$

   b) Find the following residues (n is a positive integer):
   $$Res_{z=\infty}\; z^n e^{10/z}, \qquad\qquad Res_{z=0}\; \frac{e^{z^2}}{z^{2n+1}}.$$

2. **Problem 2**.

   Let E_j, $j = 1, ..., m$ be measurable subsets of $[0, 1]$. Assume also that $q \leq m$ and each $x \in [0, 1]$ belongs to at least q sets E_j. Prove that there exists j such that $|E_j| \geq q/m$.

3. **Problem 3**.

   Consider $1 \leq s < r < p < \infty$. Prove that $f \in L^s(\mu, X) \cap L^p(\mu, X)$ implies $f \in L^r(\mu, X)$.

4. **Problem 4**.

   Let $f \in L^1(\mu, X)$. Prove that $\forall \epsilon > 0\ \exists \delta = \delta(\epsilon) > 0$ such that
   $$\Big| \int_A f\, d\mu \Big| < \epsilon,$$
   provided $A \subset X$ is measurable and $\mu(A) < \delta$.

5. **Problem 5**.

   Is there an $f \in \mathbf{H}(\mathbf{D})$ such that $\lim\limits_{|z| \to 1} |f(z)| = \infty$, that is, such that
   $$\inf\{|f(z)| : 1 > |z| \geq 1 - 1/n\} \to \infty \qquad \text{as} \qquad n \to \infty?$$

   Hint: f can have only finitely many zeros. Consider $1/f$.

6. **Problem 6**.

   a) Let $f : \Omega \to \mathbf{C}$ be holomorphic. True or false: if e^f is constant, then f is constant. Give a proof or a counterexample. Answer this question if f is only continuous.

   b) According to Liouville a bounded function which is holomorphic in $\mathbf{C}$ is constant. Is this true if "holomorphic" is weakened to "harmonic"?

   Hint: Use a).

7. **Problem 7**.

   Let I be a non-empty compact interval in $\mathbf{R}$ and let $f : I \to \mathbf{C}$ be **real-analytic**, by which is meant that in a neighborhood of each point of I, f is represented by a convergent power series. Show that I lies in an open subset U of $\mathbf{C}$ into which f may be extended as a holomorphic function.

   Hint: Cover I with finitely many discs and use the uniqueness theorem for holomorphic functions to synthesize a single extension.

8. **Problem 8**.

   Let $(r_k)_{k=1}^{\infty}$ be a $1-1$ enumeration of the sequence of all rationals in $(0, 1]$, and let $r_k = p_k/q_k$ with relatively prime positive integers p_k, q_k. Define $f_k(x) := e^{-(p_k - xq_k)^2}$ for $x \in [0, 1]$. Prove that $f_k \to 0$ in measure as $k \to \infty$, but the pointwise limit of $(f_k(x))_{k=1}^{\infty}$ does not exist at any point $x \in [0, 1]$.

   Hint: Given x, prove that there exist two sub-sequences $(k_n)_{n=1}^{\infty}$, $(l_n)_{n=1}^{\infty}$ such that $|r_{k_n} - x|$ is bounded away from 0 (hence $f_{k_n}(x) \to 0$), and $|r_{l_n} - x| = |p_{l_n}/q_{l_n} - x| \leq 1/q_{l_n}$ (hence $f_{k_n}(x) \geq 1/e$).

9. **Problem 9**. Let $\mu(X) < \infty$, and let $f \geq 0$ on X be measurable. Prove that f is $\mu-$integrable on X if and only if

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X : f(x) \geq 2^n\}) < \infty.$$