Topology Qualifying Exam

January 27, 2014

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

1. Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map. Suppose that $f,\ g : Y \longrightarrow \widetilde{X}$ are continuous maps such that $\pi \circ f$ and $\pi \circ g$ are equal and assume that f and g agree at $y_0 \in Y$. Show that if Y is connected, then $f = g$.

2. Let $R = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ and consider the quotient space $X = R/\sim$ where $z \sim e^{2\pi i/3}z$ for $|z| = 1$ and $z \sim e^{2\pi i/5}z$ for $|z| = 2$. Thus X is obtained from the annulus by identifying certain points on its two boundary circles. Describe the fundamental group $\pi_1(X, *)$. (Hint: You may cut R along the circle of radius $3/2$ and apply the van Kampen theorem.)

3. Consider the two spaces $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $Y = \mathbb{C}P^2 \vee \mathbb{C}P^1$.

   (a) Compute the homology groups $H_*(X; \mathbb{Z})$ and $H_*(Y; \mathbb{Z})$.

   (b) Prove that X and Y are not homotopy equivalent.

4. (a) State the Poincaré duality theorem.

   (b) Let M be a compact, conneted n-dimensional oriented manifold without boundary and let $f : S^n \longrightarrow M$ be a continuous map of non-zero degree, i.e., the morphism
   $$H_n(f) : H_n(S^n; \mathbb{Z}) \longrightarrow H_n(M; \mathbb{Z})$$
   is non-trivial. Show that the rational homology groups $H_*(M; \mathbb{Q})$ of M and of the n-sphere S^n are the same.

5. In this problem all manifolds are assumed to be smooth.

   (a) Define what it means for two submanifolds Y and Z of a manifold X to be transversal.

   (b) Recall that an affine subspace of $\mathbb{R}^n$ is a translate of a linear subspace, and that affine subspaces are thus trivially seen to be submanifolds. Characterize, with proof, which affine subspaces of $\mathbb{R}^3$ are transversal to the unit sphere $\{(x, y, z) | x^2 + y^2 + z^2 = 1\}$.

   (c) Recall that if Y and Z are transveral submanifolds of X, then $Y \cap Z$ is a submanifold of Y with codimension in Y equal to the codimension of Z in X. Show using an example in which X is $\mathbb{R}^3$, Y is the unit sphere, and Z is an affine subspace of $\mathbb{R}^3$ that the conclusion need not hold in the absence of transversality.

6. (a) Give the interesection-theoretic definition of Euler characteristic applicable to compact smooth manifolds.

   (b) State and prove a general theorem about the Euler characteristic of manifolds which admit a fixed-point free map homotopic to the identity map.

   (c) For which of the following manifolds does the theorem of part (b) allow one to compute the Euler characteristic: S^2, $S^1 \times S^1 \times S^1$, S^3?

7. Prove that if Z_0 and Z_1 are compact, cobordant, p-dimensional submanifolds of X and ω is a closed p-form on X, then

$$\int_{Z_0} \omega = \int_{Z_1} \omega$$

Here all manifolds are assumed to be smooth and oriented, and cobordant refers to oriented cobordism.

8. (a) Define what it means for a space to be locally compact.

(b) Define compactification, and tell what special properties the one-point compactification and the Stone-Čech compactification have among all compactifications in terms of maps between compactifications of a given space.

(c) Give the construction of the one-point compactification and prove that it is, in fact, a compactification with the special property you noted in (b). Give any additional hypotheses on the space needed for the construction of the one-point compactification.