# Algebra Qualifying Exam
# Fall 1989

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**Group Theory**

**1.** Let $\phi : G \to H$ be a surjective homorphism of groups, and let $K = \ker \phi$. If $H_1$ is a subgroup of $H$ show that there is a *unique* subgroup $G_1$ of $G$ such that

(i) $K \leq G_1$,

(ii) $\phi(G_1) = H_1$.

**2.** Let $G$ be a group of order 56. Show that either

(i) a 2-Sylow subgroup is normal, or

(ii) a 7-Sylow subgroup is normal.

(Extra credit: Give examples of groups $G_1, G_2$ of order 56 such that a 7-Sylow subgroup of $G_1$ is not normal and a 2-Sylow of $G_2$ is not normal.)

**3.** Let $P$ be a finite $p$-group ($p$ is prime), and let $H$ be a proper subgroup of $P$. Show that $N_p(H) \supsetneqq H$.

**4.** Prove that no group can be written as the union of two proper subgroups. Give an example of a group which is a union of three proper subgroups.

**5.** Let $A$ be an abelian group with generators $a, b$ and relations $2a - b = 0$, $-a + 2b = 0$. Compute the structure of $A$.

**6.** Let $G$ be the group with presentation $< a, b | a^2 = b^3 >$. Show that $G$ is infinite. (Hint: This is not hard at all! Let $G_0$ be the subgroup of $GL(2, \pi) = 2 \times 2$ nonsingular matrices with integer entries, generated by $a_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $b_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Show that $a_0, b_0$ satisfy the given relation, and that $G_0$ is infinite.)

**Rings and Modules**

**1.** Let $\phi : R_1 \to R_2$ be a homomorphism of rings.

(a) If $I_2$ is an ideal of $R_2$, show that $\phi^{-1}(I_2)$ is an ideal of $R_1$.

(b) If $I_1$ is an ideal of $R_1$, show by example that $\phi(I_1)$ need not be an ideal of $R_2$.

**2.** Prove that "Chinese Remainder Theorem": If $n$ is a positive integer with $n = ab$, $a$ and $b$ relatively prime, then there is an isomorphism of rings

$$\frac{\mathbb{Z}}{(n)} \cong \frac{\mathbb{Z}}{(a)} \times \frac{\mathbb{Z}}{(b)}.$$

**3.** Let $R$ be a ring and let $M$ be a left $R$-module. Let $\text{Ann}(M) = \{r \varepsilon R | rM = 0\}$ be the *annihilator* of $M$.

(a) Show that $\text{Ann}(M)$ is a 2-sided ideal of $R$

(b) If $M$ is irreducible, and if $R$ commutative, show that there is an isomorphism of $R$-modules

$$\frac{R}{\text{Ann}(M)} \cong M$$

**4.** Let $R$ be an integral domain such that every ideal of $R$ is free. Prove that $R$ is a principal ideal domain.

**5.** Let $R$ be a ring and let $M$ be a left $R$-module. Prove the so-called *Noether isomorphism theorem*: if $M_1, M_2$ are $R$-submodules of $M$ then

$$\frac{M_1 + M_2}{M_2} \cong \frac{M_1}{M_1 \cap M_2}.$$

(Hint: Map $M_1 \to \frac{M_1+M_2}{M_2}$ in the more or less obvious way. Is the map surjective? What is the kernel?)

**Linear Algebra**

**1.** Let $F$ be a field, and let $V$ be a vector space over $F$.

(a) Define what it means for a subset $S \subseteq V$ to be a *basis*.

(b) Using Zorn's lemma, show that any vector space has a basis.

**2.** Let $\{v_1, \ldots, v_n\}$ be a basis for the vector space $V$ over $F$. If $w \varepsilon V$ satisfies $w \notin < v_2, \ldots, v_n >$ (where $<>$ means $F$-span), show that $\{w, v_2, \ldots, v_n\}$ is a basis.

**3.** Let $T : V \to V$ be a linear transformation such that $T^2 = T$. Prove that the subspaces $TV$ and $(I - T)V$ are $T$-invariant and that $V = TV \oplus (I - T)V$.

**4.** Give an example of a matrix $A$ with rational entries such that minimal polymonial $= (x+1)^2(x^2+1)^2(x^4 + x^3 + x^2 + x + 1)$, characteristic polynomial $= (x + 1)^3(x^2 + 1)^3(x^4 + x^3 + x^2 + x + 1)$.

**5.** Let $T_1, T_2 : V \to V$ be linear transformations, where $V$ is a finite dimensional vector space over an algebraically closed field. If $T_1T_2 = T_2T_1$, prove that there exists a vector $v \varepsilon V$ which is an eigenvector for both $T_1$ and $T_2$.

**Fields and Galois Theory**

**1.** Let $F \subseteq K$ be fields and let $\alpha \varepsilon K$.

(a) State what it means for $\alpha$ to be *algebraic* over $F$.

(b) Prove that $\alpha$ is algebraic over $F$ if $F[\alpha]$ is a finite dimension $F$-vector space.

**2.** Let $F$ be a finite field, and let $F^*$ be the non-zero elements of $F$, regarded as a multiplicative group. Show that $F^*$ is a cyclic group. (Hint: If $e =$ exponent of $F^*$, how many roots in $F$ are ther to the polynomial $x^e - 1$?)

**3.** Let $\sqrt[3]{2}$ be a real cube root of 2, and let $\zeta$ be the complex number $\zeta = \exp\left(\frac{2\pi i}{3}\right)$. Let $K_1 = \Phi\left[\sqrt[3]{2}\right]$, $K_2 = \Phi[\zeta]$, $K_3 = \Phi\left[\sqrt[3]{2}, \zeta\right]$. Prove that $K_1$ is not normal over $\Phi$ but that $K_2, K_3$ are normal over $\Phi$.

**4.** Let $F \subseteq K$ be a seperable normal extension of $F_1$ and let $G$ be the Galois group of the extension. Let $H$ be a subgroup of $G$ and let $L=$ field of invariants of $H$, i.e. $L = \{\alpha \varepsilon K | h\alpha = \alpha \text{ for all } h \varepsilon H\}$. Without using the fundamental theorem of Galois theory, prove that $L$ is normal over $F$ if and only if $H$ is a normal subgroup of $G$.