QUALIFYING EXAM
APRIL 21, 1986

Do at most nine of the following 16 problems.

1. Let (X,d) be a complete metric space and let $T : X \to X$ be a contractive map (i.e., there exists a real number $\alpha < 1$ such that $d(T(x),T(y)) \leq \alpha d(x,y)$ for all x,y in X). Show that T has exactly one fixed point.

2. Prove that every locally compact Hausdorff space is completely regular.

3. Show that the one point compactification of the real line is the 1-dimensional unit sphere S^1.

4. Prove that if $A \times B$ is a compact subset of $X \times Y$ contained in an open set W in $X \times Y$, then there exists open sets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq W$.

5. Show that the countable product of metric spaces is a metric space. Also, prove that each second countable normal space can be embedded in the countable power of $[0,1]$.

6. Call a space "singularly locally compact" if each of its points has a compact neighborhood and "basically locally compact" if each of its points has a neighborhood base consisting of compact sets. Prove that for Hausdorff spaces these two concepts are equivalent.

7. Prove that (a) a connected metrizable space X with more than one point is uncountable, and (b) such a space that is also separable has exactly the cardinality of the real line.

8. Prove or disprove: (a) If X can be embedded in Y and Y can be embedded in X, then X and Y are homeomorphic. (b) Every T_0 space is homeomorphic to a subspace of some product of the Sierpinski space S (S has two points and exactly three open sets).

9. Let A be a closed subset of a normal space X and let $f', g' : A \to S^n$ (the n-dimensional sphere) be homotopic continuous maps. Show that if f' extends to a map $f : X \to S^n$, then g' extends to a map g which is homotopic to f.

10. Show that every map $f : X \to Y$ can be factored

$$X \xrightarrow{p_f} Z_f \xrightarrow{g_f} Y$$

where p_f is an identification map and g_f is injective. Moreover, show that the factorization can be chosen to be "functorial". That is, if

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

is a commutative diagram of continuous maps, then there exists $\pi : Z_f \to Z_{f'}$ such that the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "p_f"] \arrow[d, "\phi"'] & Z_f \arrow[r, "g_f"] \arrow[d, "\pi"] & Y \arrow[d, "\psi"] \\ X' \arrow[r, "p_{f'}"] & Z_{f'} \arrow[r, "g_{f'}"] & Y \end{tikzcd}$$

11. Prove or disprove: (a) A locally connected space is locally path connected. (b) A path connected space is locally path connected. (c) In a locally path connected, connected space, the path component of any point is a dense open set.

12. Prove or disprove: (a) S^1 is homotopically equivalent to the punctured plane. (b) S^1 is homotopically equivalent to S^0.

13. For a collection of subsets B in a space X, let $st(y,\mathcal{B}) = \cup\{B \in \mathcal{B} \mid y \in B\}$. Show that in a normal space X, each neighborhood-finite open covering $\mathcal{U}$ has a refinement B such that the covering $\{st(y,B)\}_{y\in Y}$ refines $\mathcal{U}$.

14. Show that the unit interval is the continuous image of the Cantor set.

15. Show that $\pi_1(X \times Y, (x_0,y_0))$ is isomorphic to $\pi_1(X,x_0) \oplus \pi_1(Y,y_0)$.

16. Prove that if R is any relation on the set X, then there is a maximal subset Z of X for which $Z \times Z \subseteq R$.