Algebra Qualifying Exam: August 29, 2014

Instructions: This exam consists of ten problems; answer as many as you can. Show all work for each problem. Here $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ is the rational numbers, $\mathbb{R}$ is the real numbers, and $\mathbb{C}$ is the complex numbers.

1. Let p be a prime integer, and let G be the multiplicative subgroup of the complex numbers $G = \{ z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some positive integer n} \}$.
   (a) Show that every proper subgroup of G is finite.
   (b) Show that G has no maximal subgroups. (A proper subgroup is maximal if it is not properly contained in any proper subgroup.)

2. Use Zorn's Lemma (or any other version of the Axiom of Choice) to prove that any ring R with a multiplicative identity $1 \in R$ has at least one maximal ideal. Explain why a similar argument does not show that every group has a maximal subgroup.

3. Note that $f(x) = x^7 - 1$ factors modulo two (over the field $\mathrm{F} = \mathbb{Z}/2\mathbb{Z}$ of two elements) as $x^7 - 1 = (x+1)(x^3+x+1)(x^3+x^2+1)$. Use this to prove that the multiplicative group $GL_3(F)$ of invertible 3×3 matrices with entries in F has exactly two conjugacy classes of elements of order 7. Write down a representative of each conjugacy class.

4. Let $K \subseteq F$ be two fields such that F is a finite dimensional vector space over the subfield K. Prove that for any element $\alpha \in F$, there exists a polynomial $f(x) \in K[x]$ with coefficients in K such that α is a root, $f(\alpha) = 0$.

5. Let R be a ring which is a PID (principal ideal domain) and let M be an R-module. For each element $x \in M$, the annihilator is $A(x) = \{ a \in R \mid ax = 0 \}$. The torsion submodule of M is $N = Tor(M) = \{ x \in M \mid A(x) \neq (0) \}$.
   (a) Show that N is a submodule of M.
   (b) Show that the torsion submodule of the quotient module M/N is the zero submodule.

6. Let G be a simple group of order $168 = 2^3 \times 3 \times 7$. Prove that G contains a nonabelian group of order 21.

7. Let A be a square $n \times n$ matrix with entries in the rational numbers. Assume that $A^3 = 3A^2 - 2A + I$, where I is the $n \times n$ identity matrix.

   (a) Prove that the determinant $\det(A) \neq 0$.

   (b) Prove that n is a multiple of 3.

8. Let $\omega = e^{\frac{2\pi i}{10}}$ be the primitive complex tenth root of unity. Determine all subfields of the extension field over the rational numbers $F = \mathbb{Q}(\omega)$.

9. Let $P \subseteq \mathbb{Z}[x]$ be a prime ideal in the ring of polynomials with integer coefficients. Assume that P contains no nonzero constants (so that $P \cap \mathbb{Z} = (0)$). Prove P is a principal ideal. Hint: the content of an integer polynomial is the GCD of its coefficients.

10. Let G be a finite group acting transitively on a finite set X. Let H be a normal subgroup of G. Assume that the index $[G : H]$ of the subgroup is relatively prime to the cardinality $|X|$. Prove that H acts transitively on the set X.