TOPOLOGY
QUALIFYING EXAMINATION
FALL - 1986
(Strecker - Summerhill)

Do 9 and only 9 of the following 15 problems.

1. Let $f : A \to B$, $C = (C_n)_{n \in \omega}$ be a family of subsets of A and $D = (D_n)_{n \in \omega}$ be a family of subsets of B. Check "true" or "false" for each of the following assertions. For each false one, indicate further hypotheses that will make it true. (NO PROOFS ARE NECESSARY.)

Assertion	True or False	Further Hypotheses (if needed)
$f[\cup C] = \cup_{n \in \omega}\{f[C_n]\}$		
$f[\cap C] = \cap_{n \in \omega}\{f[C_n]\}$		
$f[A - C_0] = B - f[C_0]$		
$f^{-1}[\cup D] = \cup_{n \in \omega}\{f^{-1}[D_n]\}$		
$f^{-1}[\cap D] = \cap_{n \in \omega}\{f^{-1}[D_n]\}$		
$f^{-1}[B - D_0] = A - f^{-1}[D_0]$		

2. Consider the first projection function $\pi_1 : X \times Y \to X$.

   a. Prove that π_1 is open and continuous.

   b. Prove that if Y is compact, then π_1 is closed.

3. Find a flaw in the "Proof" of the following:

"Theorem" All real-valued functions are continuous.

"Lemma" $f : X \to \mathbb{R}$ is continuous iff g_f defined by $x \mapsto (x, f(x))$ is continuous.

"Proof" If f is continuous, then g_f is determined by continuous functions, f and the identity, so it must be continuous. If g_f is continuous then since f is g_f followed by the second projection, it must be continuous.

"Proof of Theorem" For any $f : X \to \mathbb{R}$, g_f is one-to-one. Thus it is continuous iff its inverse is open. However the inverse of g_f is the first projection. But all projection functions are well-known to be open surjections. Hence g_f^{-1} is open, so g_f is continuous. Hence, by the lemma, f is continuous.

4. Prove that:

a. if A is a connected subset of a space and $A \subseteq B \subseteq \bar{A}$, then B is connected.

b. every component of a space is closed.

5. Prove that every closed continuous surjection is a quotient map.

6. Let Y be a locally compact Hausdorff space and let $\{U_i\}$ be a countable family of dense open sets. Show that $\cap U_i$ is dense in Y.

7. a. Compute the fundamental group of the figure 8.

b. Give an example of an open subset of E^3 (Euclidean 3-space) that is homotopically equivalent to S^1, and prove the equivalence.

8. Let Y be compact and let A be a closed subset of X. If U is open in $Y \times X$ containing $Y \times A$, show that there is an open set W in X containing A such that

$$U \supseteq Y \times W \supseteq Y \times A.$$

9. Let A be a closed subset of a normal space X and let U be an open subset of $X \times [0,1]$ containing $(X \times \{0\}) \cup (A \times I)$. Show that there is a map $f : X \times I \to U$ which is the identity on $(X \times \{0\}) \cup (A \times I)$.

10. Let X be locally connected and let $Y \subseteq X$. If C is a component of Y, show

a. $IntC = C \cap IntY$.

b. $FrC \subset FrY$.

c. If Y is closed, then $FrC = C \cap FrY$.

11. Show that a compact Hausdorff space is normal.

12. Explain the relationship between 2nd countable, Lindelof, and separable in two situations: (1) For the class of Hausdorff spaces and (2) for the class of metric spaces. You need not give proofs, but you must give statements of theorems or complete descriptions of counter examples.

13. State two of the following and prove that one of them implies the other.

a. Zorn's lemma.

b. Hausdorff Maximality Principle.

c. The Well-Ordering Principle.

d. The Axiom of Choice

14. A covering $\{U_\alpha\}_{\alpha\in A}$ of a topological space X is shrinkable if there is a covering $\{V_\alpha\}_{\alpha\in A}$ of X such that $\overline{V}_\alpha \subseteq U_\alpha$ for each $\alpha \in A$ and $V_\alpha \neq \emptyset$ whenever $U_\alpha \neq \emptyset$. Show that a Hausdorff space is normal iff each point-finite covering is shrinkable.

15. Let $\ell_2 = \{\{x_i\} | x_i$ is a real number and $\sum_{i=1}^{+\infty} x_i^2 < +\infty\}$ and let $d : \ell_2 \times \ell_2 \to \mathbb{R}$ be defined by

$$d(\{x_i\}, \{y_i\}) = \sqrt{\sum_{i=1}^{+\infty} (x_i - y_i)^2}.$$

a. Show that d is a metric.

b. Show that (ℓ_2, d) is complete.