Complex Variables Qualifying Exam
Spring 1991

In what follows $\mathbb{R}$ is the real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{D}$ the open unit disc centered at 0 and $\mathbb{T}$ the boundary of $\mathbb{D}$. The set of functions holomorphic in Ω is denoted $H(\Omega)$.

1. Let f be a continuous function on $\mathbb{D}$ which satisfies $\int_\Delta f = 0$ for each triangle $\Delta \subset \mathbb{D}$ such that one side on Δ lies on $\mathbb{R}$ and another side of Δ is parallel to $i\mathbb{R}$. Show that f is holomorphic.

2. Prove that $1/z$ is not uniformly approximable on $\mathbb{T}$ by polynomials in z.

3. Let Ω be a region in $\mathbb{C}, f \in H(\Omega)\backslash\{0\}$, n a positive integer. Suppose that $|z|^n f(z)|$ attains a maximum over Ω at some point of Ω. Show that $0 \notin \Omega$.

4. Prove that for all z in the open right half-plane $\mathbb{H}$ the integral $\int_1^\infty e^{-t}t^{z-1}dt$ exists and defines a holomorphic function of $z \in \mathbb{H}$.

5. (i) If f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, then

$$|f(z)| \leq \frac{1}{\sqrt{1-|z|^2}} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{i\theta})\right|^2 d\theta\right]^{1/2} \qquad \forall z \in \mathbb{D}.$$

(ii) Use (i) to draw the same conclusion in case f is only continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$.

6. f and g are each holomorphic in a neighborhood of 0, $f(0) = 0$ with multiplicity $m, g(0) = 0$ with multiplicity n. What is the multiplicity of 0 as a zero of $f \circ g$?

7. (i) Use the function $f(t) := e^{it}, t \in [0, 2\pi]$, to show that the Mean-Value Theorem of differential calculus fails (generally) for complex-valued functions.

(ii) Prove that, in spite of (i), if F is holomorphic in a convex region Ω and $|F'| \leq M$, then

$$|F(z_2) - F(z_1)| \leq M|z_2 - z_1| \quad \forall z_1, z_2 \in \Omega.$$

8. Let Ω be a bounded region in $\mathbb{C}$, $f : \overline{\Omega} \to \mathbb{C}$ a continuous non-constant function which is holomorphic in Ω and maps $\partial\Omega$ into $\mathbb{T}$.

(i) Show that $0 \in f(\Omega)$.

(ii) Show that $f(\Omega) = \mathbb{D}$.

Hint: To get "$\supset$", apply (i) to $\phi \circ f$ for certian holomorphic maps ϕ of $\mathbb{D}$ into $\mathbb{D}$.

9. f is continuous on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$ and $\mathrm{diam} f(\mathbb{T}) \leq 1$. Show that $\mathrm{diam} f(r\mathbb{T}) \leq r$ for each $0 \leq r \leq 1$.

Hint: $\mathrm{diam} f(r\mathbb{T}) := \max\{|f(ru_1) - f(ru_2)| : u_1, u_2 \in \mathbb{T}\}$. If this is achieved at u_1, u_2 consider the holomorphic function $F(z) := f(zu_1) - f(zu_2)$.

10. $h : \mathbb{C} \to \mathbb{R}$ is harmonic and non-constant.

(i) Prove that h is not bounded above.

(ii) Prove that h is not bounded below.

(iii) Prove that $h(\mathbb{C}) = \mathbb{R}$.