S06 Geometry qual

Name ____________________

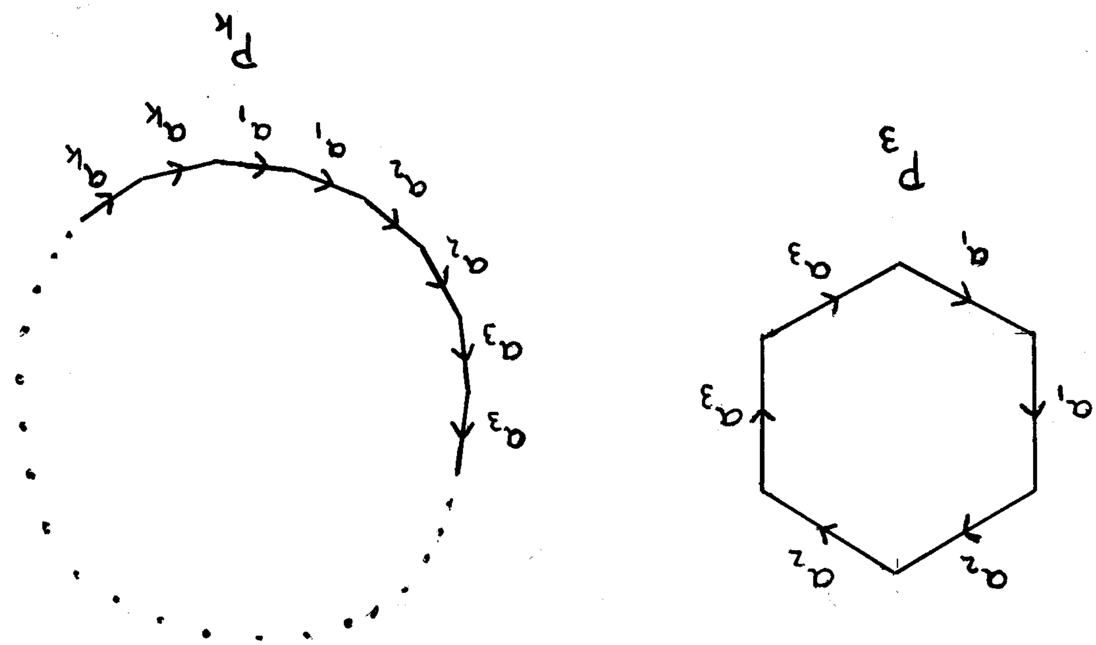
1. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linear independent over $\mathbb{R}$ and let
$$\Gamma = \mathrm{Span}_{\mathbb{Z}}\{\omega_1, \omega_2\}.$$
Let $\sim$ be the equivalence relation on $\mathbb{C}$ defined by $z \sim w$ if and only if $w - z \in \Gamma$. Let $X = \mathbb{C}/_{\sim}$ be the set of equivalence classes with the quotient topology.
   (a) Prove that X is a torus, i.e. prove that X is homeomorphic to $S^1 \times S^1$.
   (b) Prove that the canonical projection $\pi : \mathbb{C} \to X$ is an open mapping.
   (c) Prove that $\pi : \mathbb{C} \to X$ is a covering projection.

2. Verify Stokes' theorem for $\alpha = zx\, dy$ on
$$\Sigma = \{(x, y, z) | z = x^2 + y^2,\ z \le 1\},$$
with orientation $\Omega = dx \wedge dy$.

3. Let α be a nowhere zero 1-form on $\mathbb{R}^3$.
   (a) Given that there are smooth functions, $\lambda : \mathbb{R}^3 \to (0, \infty)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\alpha = \lambda df$, prove that $\alpha \wedge d\alpha = 0$.
   (b) Given that $\alpha \wedge d\alpha = 0$, prove that $\ker(\alpha) := \{X \in T\mathbb{R}^3 | \alpha(X) = 0\}$ is involutive. Conclude that there are smooth functions, $\lambda : \mathbb{R}^3 \to (0, \infty)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\alpha = \lambda df$.

4. Let $\Psi_t : \mathbb{R}^3 \to \mathbb{R}^3$ be right-handed rotation about the z-axis through t radians. Let $Y = y\partial_z - z\partial_y$.
   (a) Compute the infinitesimal generator of Ψ_t. (Call it X.)
   (b) Compute the flow of Y. (Call it Φ_t.)
   (c) Compute $\Psi_{-t} \circ \Phi_{-t} \circ \Psi_t \circ \Phi_t$ and the corresponding infinitesimal generator.
   (d) Compute $[X, Y]$.

5. In this problem all homology groups are with coefficients in $\mathbb{Z}$. Let k be a strictly positive integer and let N_k be the space constructed from a polygon P_k with 2k sides (a $2k$-gon) by identifying pairs of edges of P_k as shown in the figure below, where P_3 is shown together with a part of P_k for a general k.



(a) Prove that N_k has a cell decomposition with one 0-cell, k 1-cells and one 2-cell, and prove that N_k together with this cell decomposition is a CW-complex.

(b) What is the fundamental group of N_k? Why?

(c) Prove that N_1 is homeomorphic to the quotient space obtained from the 2-sphere S^2 by identifying antipodal points, i.e. by identifying x and $-x$ for all $x \in S^2$.

(d) What is the relation between $\pi_1(N_k)$ and the first homology group $H_1(N_k)$?

(e) Calculate the homology groups $H_*(N_k)$ of N_k. Compute the Euler characteristic of N_k.

(f) Prove that N_k is a topological 2-manifold.

(g) Is N_k orientable or nonorientable? Explain your answer.