GEOMETRY/TOPOLOGY QUALIFYING EXAM
FALL 2005 [YETTER & HANSEN]

Do as many of the following as time permits. For questions with multiple parts, you may attempt any or all parts, if necessary assuming the result of an earlier part to complete a later part. **Begin each question on a new sheet of paper, and order your papers by question number when you turn them in.**
In the following $\mathbb{R}^n$ is given the usual Euclidean topology, and smooth means C^∞.

Problem 1. Let f be a smooth function on an open set $U \subset \mathbb{R}^k$. For each $x \in U$, let $H(x)$ be the Hessian matrix of f, whether x is critical or not. Prove that f is a Morse function if and only if

$$det(H)^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i} \right)^2 > 0$$

Suppose that f_t is a (smoothly) homotopic family of functions on $\mathbb{R}^k$. Use the previous result to show that if f_0 is Morse on a neighborhood of a compact set K, then so is f_t for all t sufficiently small.

Use your previous result to show that Morse functions on compact manifolds are stable, that is given any (smoothly) homotopic family of functions $f_t : X \to \mathbb{R}$, with X a compact smooth manifold, f_0 Morse implies f_t Morse for all sufficiently small t.

Problem 2. Let X be any topological space. Prove or disprove (with a counter example) each of the following:

a): The connected components of X are closed.
b): The connected components of X are open.
c): The path-connected components of X are closed.
d): The path-connected components of X are open.
e): A path-connected space is connected.
f): An open subset in $\mathbb{R}^n$ is connected if and only if it is path-connected.
g): A closed subset in $\mathbb{R}^n$ is connected if and only if it is path-connected.

Problem 3. Let X be a locally compact space (so in particular X is a Hausdorff space). Let X^+ be the disjoint union of X and a single extra point $\{+\}$. Let τ^+ be the family of subsets of X^+ consisting of the open subsets of X and all sets of the form $\{+\} \cup (X \setminus K)$, where K is a compact subset of X.

Prove that τ^+ defines a topology on X^+.

Prove that X^+ with the topology τ^+ is compact, and that the inclusion $i : X \to X^+$ is an embedding (in the sense of point-set topology).

Prove that if Y is another compact (Hausdorff) space such that there exists an embedding $f : X \to Y$ with $Y \setminus f(X)$ being a single point, then X^+ and Y are homeomorphic.

Prove that $\mathbb{R}^+$ is homeomorphic to S^1.

Problem 4. A closed oriented 3–manifold X is called an integral homology sphere (respectively a rational homology sphere) if $H_*(X;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$ (respectively $H_*(X;\mathbb{Q}) = H_*(S^3;\mathbb{Q})$). (A closed manifold is a compact manifold without boundary.) Prove that a connected, closed, oriented 3–manifold is an integral (resp. rational) homology sphere if and only if $H_1(X;\mathbb{Z}) = 0$ (resp. $H_1(X;\mathbb{Q}) = 0$).

Let $g \in \{0, 1, 2, \ldots\}$, let F_g be the closed oriented surface of genus g and let $X(g) = F_g \times S^1$. Prove that the fundamental group $\pi_1(X)$ of $X = X(g)$ has a presentation with generators

$$h, a_1, b_1, \ldots, a_g, b_g$$

and relations

1) h belongs to the center of $\pi_1(X)$,
2) $[a_1, b_1] \cdots [a_g, b_g] = 1$.

Here $[x, y] = xyx^{-1}y^{-1}$ is the commutator of $x, y \in \pi_1(X)$.

Determine $H_1(X;\mathbb{Z})$ and $H_1(X;\mathbb{Q})$. Is X an integral homology sphere? Is X a rational homology sphere?

Problem 5. Let $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the closed unit disk of $\mathbb{R}^2$ with center $(0,0)$. Consider the following cell decompositions of D. In the first cell decomposition ϵ_1 the interior of D is a 2-cell and each points on the boundary ∂D of D is a 0-cell. In the second cell decomposition ϵ_2 the center $(0,0)$ of D and each point of ∂D are 0-cells, and each open radius of D, i.e. each subset $\{tx \mid t \in]0,1[\}$, $x \in \partial D$, is a 1-cell.

Is ϵ_1 a CW-complex? If not, give a precise explanation of why ϵ_1 fails to be a CW-decomposition of D.

Is ϵ_2 a CW-complex? If not, give a precise explanation of why ϵ_2 fails to be a CW-decomposition of D.

Determine a CW-decomposition of $\mathbb{R}P^n$, n any positive integer.

Problem 6. State the modern Stokes' Theorem.
State as many classical analogs of the modern Stokes' Theorem as you can, and for each give a derivation from the modern Stokes' Theorem.
Define a differential form on the plane by $\omega = xdy - ydx$, and let S^1 be the unit circle in $\mathbb{R}^2$.

- **a):** What order of form is ω?
- **b):** Compute

$$\int_{S^1} \omega$$

  directly.
- **c):** Use Stokes' Theorem to compute

$$\int_{S^1} \omega.$$

Problem 7. Let X, Y, and Z be compact smooth manifolds. Recall that two smooth maps $f : X \to Z$ and $g : Y \to Z$ are said to be transversal when

$$Im(df_x) + Im(dg_y) = T_z(Z)$$

whenever $f(x) = g(y) = z$.

Prove that f and g transversal implies that the pullback of f and g,

$$P_{f,g} = \{(x,y) | x \in X, y \in Y, f(x) = g(y)\}$$

is a smooth manifold.

Express the dimension of $P_{f,g}$ in terms of the dimensions of X, Y, and Z.

Problem 8. Compute π_1, H_* and H^* for each of the following spaces:

- **a):** $\mathbb{R}^3$
- **b):** $S^1 \times S^1$
- **c):** $\mathbb{RP}^2$