Real and Complex Analysis Qualifying Exam — Fall 2013

Notation: $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{R} :=$ the real numbers and $\mathbb{C} :=$ the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, Ω is a non-empty open connected subset of $\mathbb{C}$, and $H(\Omega)$ is the set of all holomorphic functions in Ω. $(X, \mathcal{M}, \mu)$ is a measure space. $|E|$ denotes the Lebesgue measure of the set E.

1. Let $\{f_n\}$ be a sequence of integrable functions on $(X, \mathcal{M}, \mu)$ such that $f_n \to f$ a.e., where $f \in L^1(\mu)$. Prove that

$$\int_X |f - f_n| d\mu \to 0$$

if and only if

$$\int_X |f_n| d\mu \to \int_X |f| d\mu.$$

2. Prove that the measure μ on X is σ-finite if and only if there exists an $f \in L^1(\mu)$ such that $f(x) > 0$, for almost every $x \in X$.

3. Suppose that $f \in L^1([0, 1])$. Consider the two statements:

(a) For almost every x, f is continuous at x.

(b) There exists a continuous function g such that $f = g$ almost every

For each of the possible implications $(a) \implies$ (b) and $(b) \implies (a)$, either prove the implication is true or provide a counterexample.

4. Given $\varepsilon > 0$, find a compact set $K \subset [0, 1]$ that contains no rationals and satisfies $|K| > 1 - \varepsilon$.

5. Suppose that $f \in L^1(\mu)$. Prove that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

6. Find a closed form in which "i" does not explicitly appear for the function $f(z)$ represented by the series

$$\sum_{n=0}^{\infty} \sin(n) z^n$$

and deduce the radius of convergence of this series.

7. The functions $f_n \in H(\mathbb{D})$ are all bounded by $C < \infty$ and $f_n(z) \to 0$ for each $z \in \mathbb{D}$.

(i) Show that $f_n \to 0$ uniformly in $\mathbb{D}$ may fail.

(ii) Show that $f_n \to 0$ uniformly in $r\mathbb{D}$ for each $0 \le r < 1$.

8. Suppose f is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ and $|f(0)| < \min\{|f(u)| : u \in \partial\mathbb{D}\}$. Prove that f has a zero in $\mathbb{D}$.

9. Suppose $f \in H(\mathbb{C})$ maps $\mathbb{R}$ into $\mathbb{R}$ and the open upper half plane into itself. Show that $f'(x) \ge 0$ for all $x \in \mathbb{R}$.

Hint: Cauchy - Riemann equations.

10. Evaluate

$$\int_0^\infty \frac{\cos x}{1+x^4}\,dx.$$

Hint: Consider the meromorphic function $\dfrac{e^{iz}}{1+z^4}$ and the semicircular contour with diameter $[-R, R]$.