Partial Differential Equations Qualifying Exam
Fall 1997

Part 1: The Laplace Equation

1.1) Consider the following problem

$$-\Delta u(x) - 3u(x) = f(x), \quad x \in Q = \{x \in \mathbb{R}^3 |\ 0 < x_j < 2\pi, j = 1, 2, 3\}, \tag{1a}$$

$$\frac{\partial u(x)}{\partial \nu} = 0, \quad x \in \partial Q, \tag{1b}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative on the boundary ∂Q of the cube Q.

a) How would you define the solution of problem (1) for an $f \in L^2(Q)$? State a theorem on the existence and uniqueness of a solution to (1) with $f \in L^2(Q)$.

b) Choose the function space for the solutions of (1) with f's in an appropriate subspace of $L^2(Q)$, and show that the solution depends on continuously on f.

c) Solve problem (1) with $f(x) = \cos(x_1) \cdot \cos(x_2) \cdot \cos(x_3)$.

d) Explain the main difference between problem (1) and the following problem

$$-\Delta u(x) = f(x), \quad x \in Q = \{x \in \mathbb{R}^3 |\ 0 < x_j < 2\pi, j = 1, 2, 3\}, \tag{1'a}$$

$$\frac{\partial u(x)}{\partial \nu} = 0, \quad x \in \partial Q. \tag{1'b}$$

1.2) Let u be a C^2 function in a domain $\Omega \subset \mathbb{R}^n$. Assume that u is subharmonic in the sense that $\Delta u(x) \geq 0$ for $x \in \Omega$. Show that, for every $x \in \Omega$, and for every $r > 0$ such that the closed ball of radius r centered at x lies in Ω, the following inequality holds true

$$u(x) \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{|\xi|=1} u(x + r\xi) d_\xi \sum, \tag{2}$$

where $|\mathbb{S}^{n-1}|$ is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ and $d_\xi \sum$ is the natural measure on $|\mathbb{S}^{n-1}|$.

1.3) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Show that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and u is a solution of the problem

$$\Delta u(x) = u(x)^3, \quad x \in \Omega, \tag{3a}$$

$$u|_{x \in \partial\Omega} = 0 \tag{3b}$$

then $u(x) \equiv 0$ in Ω.

1.4) Give an example of a function f in the ball $B = \{x \in \mathbb{R}^n |\ |x| < 1\}$ such that $f \in W_2^1(B)$, but $f \notin W_2^2(B)$.

Part 2: The Heat Equation

2.1) Consider the following problem (the Cauchy problem for the heat equation on the torus):

$$\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) = 0, \quad t > 0, x \in \mathbb{R}^n, \tag{4a}$$

$$u(0, x) = f(x), x \in \mathbb{R}^n, \tag{4b}$$

$$u(t, x_1, \ldots, x_j + 2\pi, \ldots, x_n) = u(t, x_1, \ldots, x_j, \ldots, x_n), \quad j = 1, \ldots, n, t > 0, \tag{4c}$$

where f is assumed to be 2π-periodic in each of the variables.

a) Use Fourier series to find the solution u of (4) in terms of the Fourier coefficients of f.

b) Assume that f belongs to the 2π-periodic Sobolev space $\widetilde{W}_2^1$. The space $\widetilde{W}_2^1$ is defined as the closure of the set of C^∞-smooth functions 2π-periodic in each of the variables, in the norm

$$\| f \| = \left(\int_Q |f(x)|^2 + |\nabla f(x)|^2 dx \right)^{\frac{1}{2}},$$

where $Q = \{y \in \mathbb{R}^n |\ 0 < y_j < 2\pi\}$ (an elementary cell). Express the fact $f \in \widetilde{W}_2^1$ in terms of the Fourier coefficients of f.

c) Show that if $f \in \widetilde{W}_2^1$, then the formula you obtained for $u(t,x)$ defines a function with the following properties:

1) $u(t,\cdot) \in \widetilde{W}_2^1$ for all $t > 0$;

2) the mapping $t \mapsto u(t,\cdot) \in \widetilde{W}_2^1$ from $[0,+\infty)$ into $\widetilde{W}_2^1$ is continuous, and $u(t,\cdot) \to f(\cdot)$ in $\widetilde{W}_2^1$ as $t \searrow 0$;

3) $u(t,x)$ satisfies (7a) in the distributional sense.

2.2 a) In the case $n = 1$, find the solution $u(t,x)$ of (4) corresponding to the initial condition

$$f(x) = x - \pi, \text{for } 0 \le x < 2\pi, \text{and extended periodically to } \mathbb{R}^1.$$

You may leave the solution in the form of a Fourier series.

b) Show that $u(t,x)$ has the following properties:

1) $u(t,\cdot) \in L^2(0,2\pi)$ for all $t > 0$;

2) the mapping $t \mapsto u(t,\cdot) \in L^2(0,2\pi)$ from $[0,+\infty)$ into $L^2(0,2\pi)$ is continuous, and $u(t,\cdot) \to f(\cdot)$ in $L^2(0,2\pi)$ as $t \searrow 0$;

2.3 a) Find a formula for the solution of the initial value problem:

$$\frac{\partial}{\partial t} u(t,x) = \Delta u(t,x) - u(t,x), \quad t > 0, x \in \mathbb{R}^n, \tag{5a}$$

$$u(0,x) = g(x), \tag{5b}$$

where g is continuous and bounded.

b) Is the solution bounded?

c) Is it the only bounded solution?

Part 3: The Wave Equation

3.1 Consider the Cauchy problem

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = 0, \quad -\infty < t < +\infty, \quad x \in \mathbb{R}^n, \tag{6a}$$

$$u(0,x) = 0, \quad x \in \mathbb{R}^n, \tag{6b}$$

$$\frac{\partial}{\partial t} u(t,x)|_{t=0} = f(x), \quad x \in \mathbb{R}^n. \tag{6c}$$

Let f be a function with the following properties:

- $f \in C^\infty(\mathbb{R}^n \backslash \{0\})$;
- $|f(x)| \le C_0 |x|^\alpha$, for some $\alpha > -\frac{n}{2}$, when x is small: $0 < |x| \le \varepsilon$, some $\varepsilon > 0$;
- $|f(x)| \le C_\infty |x|^{-\beta}$, for some $\beta > \frac{n}{2}$, when x is large: $|x| > R$, some $R > 0$.

What can you say about the regularity properties of the solution $u(t,x)$ of (6)?

3.2 Consider the problem

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = h(t,x), \quad -\infty < t + \infty, \quad -\infty < x < +\infty, \tag{7a}$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}^1, \tag{7b}$$

$$\frac{\partial}{\partial t}u(t,x)|_{t=0} = f(x), \quad x \in \mathbb{R}^1. \tag{7c}$$

Assuming that f, g, h, and U are sufficiently smooth, integrate the equation (7a) over the isoceles triangle ("backward light cone") $C_{t_0,x_0} = \{(t,x)|0 < t < t_0, |x-y| < t_0 - t\}$ with the vertices $P = (t = t_0, x = x_0), Q = (t = 0, x + x_0 + t_0)$, and $R = (t = 0, x = x_0 - t_0)$. In order to simplify the left side, use integration by parts to get integrals over the sides of C_{t_0,x_0}, and then evaluate the integrals over the sides RP and QP. After this you should be able to obtain an expression for $u(t_0, x_0)$ in terms of f, g, and h.

Use the expression you obtained to explain the Huygens principle.

3.3

$$\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) + m^2 u(t,x) = h(t,x), \quad -\infty < t < +\infty, \quad x \in \mathbb{R}^n, \tag{8a}$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}^n, \tag{8b}$$

$$\frac{\partial}{\partial t}u(t,x)|_{t=0} = f(x), \quad x \in \mathbb{R}^n, \tag{8c}$$

where m is a positive constant.

a) If the initial data f and g are chosen from the function spaces $H^1(\mathbb{R}^n) = W_2^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively, then what is the "right" function space for h in order that problem (8) be well-posed?

b) Write down the appropriate energy estimate and show how this estimate allows to prove the uniqueness of solution of (8). Derive this estimate for the solution of (8) under the assumptions that the initial data f and g, the external force h, and the solution u are all sufficiently smooth and decrease fast at the spatial infinity.