Qualifying Exam **Algebra** **January 28, 2015**

Instructions: Wait to open the exam until instructed to do so. Choose 8 out of the 10 questions to answer in the spaces provided on the question sheets. Please indicate these 8 problems by circling the problem numbers. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
	10	
	10	
	10	
	10	
	10	
	10	
	10	
	10	
Total:	80	

Name: ______________________________

1. (a) (3 points) Let H, K be normal subgroups of G and $H \cap K = \{e\}$. Show that

$$[H, K] = \{hkh^{-1}k^{-1} : h \in H, k \in K\}$$

equals $\{e\}$.

(b) (3 points) Show that if H, K are normal subgroups of G, $H \cap K = \{e\}$ and $HK = G$ then $G \cong H \times K$.

(c) (4 points) Suppose p and q are twin primes (i.e. p and q are prime and $|p - q| = 2$) and G is a group of order pq. Prove that G is abelian.

2. (10 points) Show that there does not exist a simple group of order 910.

3. Let R be a commutative ring with identity and $Max(R)$ the set of maximal ideals in R.
   (a) (5 points) Prove that $I = \{a \in R : a \text{ is nilpotent}\}$ and $J = \cap_{\mathfrak{m} \in Max(R)} \mathfrak{m}$ are ideals in R.
   (b) (5 points) Prove that $I \subseteq J$.

4. (10 points) A ring (or algebra) is Artinian if its ideals satisfy the descending chain condition. In other words, any descending chain of two sided ideals

$$I_1 \supseteq I_2 \supseteq \cdots$$

eventually stabilizes so that $I_k = I_{k+1}$ when $k > N$. Suppose k is a field and A is a finite dimensional k-algebra. Prove that A is Noetherian and Artinian.

5. (10 points) Suppose R is a ring with identity and M is an indecomposable left R-module. Prove that if N is a non-zero left R-module, there does not exist a surjective homomorphism $\phi : M \to R \oplus N$.

6. (10 points) Suppose M and N are finitely generated torsion modules over a principal ideal domain R and $\mathrm{Hom}_R(M, N) = \{0\}$. Prove that $M \otimes_R N = \{0\}$.

7. (10 points) Prove that if 1 is not an eigenvalue of $A \in GL_n(\mathbb{Q})$ and A has prime order p then $(p-1) \mid n$.

8. (10 points) Suppose K is a Galois extension of F and $Gal(K/F)$ is a finite p-group. Prove that there exists an irreducible degree p polynomial $f(x) \in F[x]$ with a root in K for which $E \cong F[x]/f(x)$ is a splitting field.

9. (10 points) Suppose F is field and $F(\alpha_1), \ldots, F(\alpha_n)$ are non-isomorphic, simple, algebraic field extensions. Suppose $\beta_i \in F(\alpha_i)$ for $1 \leq i \leq n$ and prove that there exists $f(x) \in F[x]$ such that $f(\alpha_i) = \beta_i$ for $1 \leq i \leq n$.

10. (10 points) Construct a splitting field K for $x^4 - 4$ over $\mathbb{Q}$ and describe its Galois group.