Real Analysis Qualifying Exam
Spring 1994

Throughout $(X, \mathcal{M}, \mu)$ denotes a measure space, μ denotes a positive measure unless otherwise specified, and all functions are assumed to be measurable.

1. Suppose $\mu(X) < \infty$ and that $f_n \to f$ in measure. Prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f$ a.e.

2. Let g be a bounded function which has the property that for every measurable set E, $\lim_{n\to\infty} \int_E g(nx)dx = 0$. Show that for every $f \in L^1(X, \mathcal{M}, \mu), \lim_{n\to\infty} \int_X f(x)g(nx)dx = 0$.

3. Suppose $f \in L^1(\mu)$, $g \in L^p(\mu)$, $1 \leq p \leq \infty$. Prove that $\| f * g \|_p \leq \| f \|_1 \| g \|_p$.

4. Show that any orthonormal set in a separable Hilbert space is at most countable.

5. (a) Construct a closed set $K \subseteq [0,1]$ such that $|K| > \frac{1}{2}$ and K contains no rational.

(b) Can you construct such a K so that $|K \cap I| \leq \frac{9}{10}|I|$ for every interval $I \subseteq [0,1]$?

6. Suppose $T : B \to C$ is a bounded linear tranformation between the Banach spaces B and C. Note that T induces a map $T^* : C^* \to B^*$ given by $T^*(f) = f \circ T$. Hence, this induces a map $T^{**} : B^{**} \to C^{**}$. Consider also the natural embedding $i_B : B \hookrightarrow B^{**}$ given by $b \to \widehat{b}$ where $\widehat{b}(S) = S(b)$ for $S \in B^*$. Prove that "$T^{**}|_B = T$", that is, show that $T^{**} \circ i_B = i_c \circ T$ where i_c is the natural embedding $i_c : C \hookrightarrow C^{**}$.

7. Suppose $\{f_n\}$ is a sequence of functions which has $f_n \to f$ a.e. and $\| f_n \|_1 \to \| f \|_1 < \infty$. Prove that $f_n \to f$ in L^1.

8. Let D be the unit disk in the complex plane. Assume the following facts from complex analysis:

1. Given any continuous function f on ∂D, there exists a unique harmonic function u on D, continuous on $\overline{D}$, such that $u|_{\partial D} = f$.

2. The sum of any two harmonic functions is harmonic, as is the multiplication of harmonic function by a constant.

3. If u is assoiciated to f as in 1, then

$$\sup_{z\in D} |u(z)| \leq \sup_{z\in\partial D} |f(z)|.$$

4. If f is real valued, so is u. If $f \equiv 1$ then $u \equiv 1$.

Fix $z_0 \in D$. Prove that there exists a positive measure w_{z_0} on ∂D such that

$$u(z_0) = \int_{\partial D} f(z) dw_{z_0}(z)$$

if f and u are related as in 1.

Remark: w_{z_0} is called harmonic measure. You don't even have to know the defintion of harmonic function to solve this problem.

9. Let $\{a_n\} \in \ell^2$. Prove that

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\sum_{n=1}^{\infty} a_n e^{in\theta}\right| d\theta\right) \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$$

10. Suppose $f_n(x)$ is a sequence of functions on the interval [0,1] which satisfy the distrubution estimate:

$$|\{x \in [0,1] \,\|\, f_n(x)| > \lambda\}| \leq e^{-\lambda/n}.$$

Prove that $\lim_{n\to\infty} \sup \frac{|f_n(x)|}{n \log n} \leq 1$ a.e.