Qualifying Exam: Geometry and Topology

August 21, 2012, 6:00 p.m. to 9:00 p.m.
Examiners: Prof. Ricardo Castaño-Bernard and Prof. David Yetter

Name: ______________________________

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number. Problems that are completely solved and thoroughly justified will be given more credit than scattered attempts leading to partial answers.

1. Let $z, z' \in \mathbb{C}^{n+1} - \{0\}$ and define the equivalence relation: $z \sim z'$ if and only if $z = \lambda z'$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The *complex projective space* is defined to be the quotient, $\mathbb{CP}^n := \mathbb{C}^{n+1} - \{0\}/\sim$.

   (a) Define local coordinates for $\mathbb{CP}^n$, and use them to prove it is a C^∞ manifold.

   (b) Show that on the unit sphere $S^3 \subset \mathbb{C}^2$ there is an S^1-action such that
   $$S^3/S^1 = \mathbb{CP}^1.$$

   (c) Use the map $H : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}$, $H(z_0, z_1) = (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$ to show that $\mathbb{CP}^1$ is diffeomorphic to S^2.

2. a) Write the statement of the Inverse Function Theorem for $f : M \to N$, where M and N are manifolds. Give the definition of the co-derivative $f^*(q)$ of at $q = f(p) \in N$ (also called pullback).

   b) Define the tangent space T_pM. Find $T_{(x,y,0)}S^2$, for all $(x, y, 0) \in S^2$ where S^2 is the standard 2-sphere in $\mathbb{R}^3$.

   c) Describe the surface C, $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Describe the intersection $C \cap S^2$ and show that it is not transversal. Define a small perturbation of C thats makes C transversal to S^2.

3. (a) Let M be a smooth manifold. Define the vector spaces $H^k_{dR}(M)$. Calculate the de Rham cohomology of the twice-punctured disc: $M = D - \{p, q\}$;

   (b) On $\mathbb{R}^4$ define $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and $H = x_1y_2 - x_2y_1$.

   i) Find a vector field X on $\mathbb{R}^4$ satisfying the equation:
   $$dH(\,\cdot\,) = \omega(X,\,\cdot\,).$$

   ii) Find the flow ϕ_t of X, compute $\phi_t^*\omega$

4. Let $\mathbb{C}[z]$ be the ring of polynomials in one variable with complex coefficients and let S be a subset of $\mathbb{C}[z]$. Define

$$V(S) = \{p \in \mathbb{C} \mid f(p) = 0, \text{ for all } f \in S\}.$$

The Zariski topology on $\mathbb{C}$ is the topology $\mathcal{Z}$, whose closed sets are $V(S)$ for all S.

(a) Show that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ is the ideal generated by S.

(b) Let X be an arbitrary subset of $\mathbb{C}$. Show that the closure of X is $V(S_X)$, where

$$S_X = \{f \in \mathbb{C}[z] \mid f(X) = 0\}.$$

Hint: Show that $V(S) \cap V(T) = V(S \cup T)$.