REAL VARIABLES COMPONENT OF QUALIFYING EXAM

Fall 1990

1. Let $(X,\mathcal{B})$ be a measurable space.

   (a) State the definition of a $\mathcal{B}$-**measurable function** on X.

   (b) Show that the sum of two real-valued measurable functions is measurable.

2. (a) What is meant by $(X,\mathcal{B},\mu)$ being a **measure space**?

   Let $(X,\mathcal{B},\mu)$ be a measure space.

   (b) Show that if $B_n \in \mathcal{B}$ for each $n \in \mathbb{N}$, then $\mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n\in\mathbb{N}} \mu(B_n)$.

3. f is Borel-measurable on $\mathbb{R}$ and integrable with respect to Lebesgue measure λ.

   (a) Prove that $F(x) := \int_0^x f d\lambda$ defines a bounded and continuous function on $\mathbb{R}$.

   (b) Prove that if F is 0 λ-almost everywhere, then so is f.

   HINT: Show that the set-function $\mu(B) := \int_B f d\lambda$ (B Borel) is identically 0, then consider a B appropriately related to f.

   (c) Show that a Borel-measurable homomorphism ϕ of $\mathbb{R}$ into $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ – i.e., a Borel-measurable **character** of the additive group $\mathbb{R}$ – is necessarily continuous.

   HINT: $f(x) \cdot \int_0^a f = \int_x^{a+x} f$ for all $a, x \in \mathbb{R}$ (why?). Choose a appropriately.

4. Construct Borel-measurable $B \subset \mathbb{R}$ such that $0 < \lambda(B \cap I) < \lambda(I)$ for every non-void open interval I, λ being Lebesgue measure.

5. Let $(X,\mathcal{M})$ be a measurable space. Let $\mathcal{O}$, $\mathcal{B}$, $\mathcal{L}$ be the classes of open, Borel-measurable, and Lebesgue-measurable subsets of $\mathbb{R}$, respectively. What implications among the following statements are valid for $f : X \to \mathbb{R}$? For those that fail offer a counterexample.

   (i) f is $\mathcal{M}$-measurable.

   (ii) $f^{-1}(\mathcal{O}) \subset \mathcal{M}$.

   (iii) $f^{-1}(\mathcal{B}) \subset \mathcal{M}$.

   (iv) $f^{-1}(\mathcal{L}) \subset \mathcal{M}$.

6. Define $f_n(t) := \sin(nt)$, $n \in \mathbb{N}$, $t \in [-\pi,\pi]$. Prove that $\{f_n : n \in \mathbb{N}\}$ is closed and bounded in the (complete) metric space $L_2[-\pi,\pi]$, but is not compact.

7. Let μ be a non-negative measure on the σ-algebra $\mathcal{M}$. Say that μ is **semi-finite** if $E \in \mathcal{M}$ & $\mu(E) = \infty \Rightarrow \exists F \in \mathcal{M} \cap E$ with $0 < \mu(F) < \infty$. Show that when semi-finiteness prevails, for all $E \in \mathcal{M}$, $\mu(E) = \sup\{\mu(F) : F \in \mathcal{M} \cap E \text{ \& } \mu(F) < \infty\}$.

   HINT: Use exhaustion: a maximal family $\mathcal{F}$ of disjoint $\mathcal{M} \cap E$ sets of positive, finite measure cannot have $\sum_{\mathcal{F}} \mu(F) < \mu(E)$.

8. Let K be a non-void compact subset of $\mathbb{R}$. Use a countable dense subset $\{k_n : n \in \mathbb{N}\}$ of K and the linear functional $f \mapsto \sum_{n\in\mathbb{N}} 2^{-n} f(k_n)$ to construct a Borel measure on $\mathbb{R}$ whose support is K.