Algebra Qualifying Exam

April 17, 1989

Please work exactly two of the problems in each of the four sections below. Clearly indicate which problems you wish to have graded.

Groups

1. Let G be a group of order 28. Suppose that G has a normal Sylow 2-subgroup. Prove that G is abelian.

2. Let G be a group, F a free group. Let $\phi : G \to F$ be a surjective homomorphism. Show that G contains a subgroup isomorphic to F.

3. Let P be a finite p-group, for some prime p. Let A be a non-trivial normal subgroup of P. Show that $A \cap Z(P) \neq 1$; in particular, show that $Z(P) \neq 1$.

Rings and Modules

1. Let R be a commutative ring with identity. Show that

$$\{x \in R;\ x^n = 0 \text{ for some } 0 \leq n \in \mathbb{Z}\} = \bigcap\{P;\ P \text{ a prime ideal of } R\}.$$

2. Let R be a ring with identity, I a minimal left ideal of R. Show that either $I^2 = 0$ or that there exists $e \in R$ satisfying

$$I = Re, \quad e^2 = e.$$

3. Let R be a commutative ring with identity having a unique maximal ideal I. Let M be a finitely generated left R-module. Suppose that $IM = M$. Show that $M = 0$.

[Recall that $IM = \{\sum i_j m_j;\ i_j \in I,\ m_j \in M\}$.]

Field and Galois Theory

1. (a) What is the Galois group of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$? Explicitly describe the elements of this group. Explicitly determine the Galois correspondence.

(b) If a, b are non-zero rational numbers, show that $\mathbb{Q}(\sqrt{3}, \sqrt{5}) = \mathbb{Q}(a\sqrt{3} + b\sqrt{5})$.

2. Let $\mathbb{F}_p$ denote the field of p elements. Let $f \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree d.

(a) Show that f has a root in a field of order p^d.

(b) Show that f divides $x^{p^d} - x$.

3. Let $F \subseteq K$ be a Galois extension. Let $f \in F[x]$ be a polynomial irreducible over F. Suppose that $f = f_1 \cdots f_t$, with $f_i \in K[x]$ and the f_i irreducible over K for all i. Show that the f_i all have the same degree.

Linear Algebra

1. Describe completely, but without proof, how the structure theorem for modules over a PID can be used to provide canonical form theorems for a linear map on a finite dimensional vector space over a field.

2. Let V be a ~~finite~~ $2n$ dimensional vector space over the field of real numbers $\mathbb{R}$. Let $A : V \to V$ be a linear map. Suppose that $A^{1989} - I = 0$. Show that V is the direct sum of n A-invariant 2-dimensional subspaces.

3. Let A be an $n \times n$ matrix over an algebraically closed field. Show that there exists an invertible $n \times n$ matrix X with $A^T = X^{-1}AX$.