Geometry of Manifolds Qualifying Exam
Fall 1997

Part A. Do all nine (9) questions in part A.

1. What is the fundamental group of

(a) $\mathbb{RP}^2$ (the real projective plane)

(b) $S^1 \times S^1$

(c) $T(M)$, the total space of the tangent bundle to a simply connected smooth manifold, M.

2. Describe in detain the flows of the vectorfield on $\mathbb{R}^2$ given by

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

3. Let ω be the 1-form on $\mathbb{R}^2$ given by $x(x-1)(y-1)dx$, and let R be the region

$$\{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Find $\int_R d\omega$.

4. (a) Give an example of a compact orientable manifold with non-trivial tangent bundle.

(b) Give an example of a compact orientable manifold with trivial tangent bundle.

(c) Give an example of a compact non-orientable manifold.

5. If $M = S^1 \times S^4$, what is the dimension of the fibres of the third exterior bundle $\Lambda^3(M)$?

6. How many non-zero vectorspaces of differential forms are there in the deRham complex of $S^2 \times S^2$?

7. What is the scalar curvature of the surface $3x + 2y - z = 0$ in $\mathbb{R}^3$ at the point $(0, 0, 0)$?

8. Give an example of a locally Euclidean topological space which is not a topological manifold.

9. State the deRham Theorem.

Part B. Choose four (4) and only four of the following problems.

1. On $\mathbb{R}^3$ with standard Euclidean coordinates (x, y, z), consider the 2-form $\alpha = f(x, y, z)dx \wedge dy + yzdx \wedge dz + x^2 dy \wedge dz$. Choose a function $f(x, y, z)$ so that $d\alpha = 0$ and $\alpha|_{z=1} = dx \wedge dy$.

2. (a) Define the deRham cohomology groups of a differentiable manifold.

(b) Calculate the deRham cohomology groups of the circle S^1 directly from the definition in part (a).

3. Give a detailed computation of the fundamental group of the closed compact surface of genus 2 (a.k.a the "two-holed torus").

4. (a) Write down the deRham cohomology groups for the 4-sphere S^4.

(b) Suppose that ω is a differential 2-form on S^4 and that $d\omega = 0$. Show that

i. $\omega \wedge \omega = d\phi$ for some 3-form ϕ.

ii. $\int_{S^4} \omega \wedge \omega = 0$.

iii. There is at least one point $x \in S^4$ such that $\omega \wedge \omega(x) = 0$.

5. (a) Define what we mean by a Lie group.

(b) If G is a Lie group, define its Lie algebra g.

(c) Apply the construction of b) to determine the Lie algebra of $SO(3)$, including a derivation of the bracket.

(d) Show that the tangent bundle to a Lie group is equivalent to a trivial (product) bundle.

6. Let (M, g) be a Riemannian manifold and $V(M)$ be the smooth vectorfields over M.

(a) For $X, Y \in V(M)$ define the Riemannian curvature operator $R(X, Y) : V(M) \to V(M)$.

(b) Show that if $M = \mathbb{R}^n$ and g is the Euclidean metric, then $R(X, Y)Z = 0$ for all vectorfields X, Y, Z.

(c) Suppose that $R(X, Y)Z = 0$ for all vectorfields X, Y, Z on an arbitrary Reimannian manifold (M, g). *Sketch* a proof that shows that for $x \in M$ there is a coordinate system $(x_1, \ldots, x_n)$ around x such that

$$g = \sum_{i=1}^{n} dx^i \otimes dx^i$$