Topology Qualifying Exam

January 30, 2015

Instructions: Do all eight problems. Start each problem on a separate page and clearly indicate the problem number.

1. (a) For abelian groups A and B and any pair of spaces describe explicitly the map
$$C^*(X,A)\otimes C^*(Y,B)\to C^*(X\times Y, A\otimes B),$$
where $C^*(-,A)$, $C^*(-,B)$ denote the singular cochains functors with coefficients in A and B respectively.

   (b) Explain how the above map induces a map in cohomology $H^k(X,A)\otimes H^\ell(Y,B)\to H^{k+\ell}(X\times Y, A\otimes B)$.

   (c) Is it true that for any pair of spaces and for any integer $k\geq 0$, the induced map
$$\bigoplus_{i=0}^{k} H^i(X,\mathbb{Q})\otimes H^{k-i}(Y,\mathbb{Q})\to H^k(X\times Y,\mathbb{Q})$$
is an isomorphism? Give a counterexample or state a general theorem that implies this statement. ($\mathbb{Q}$ is the field of rationals.)

2. Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Compute

   (a) $\pi_1(X)$;

   (b) Describe the universal cover of X.

3. Let $X=\mathbb{R}P^3$ and Y be a space obtained from a circle by attaching a 2-disc along a degree 6 map. Compute the groups $H_*(X)$, $H_*(Y)$, $H_*(X\times Y)$, $H^*(X)$, $H^*(Y)$, $H^*(X\times Y)$.

4. Let $X=\mathbb{R}^2\setminus\{0\}$. Define an action of $\mathbb{Z}$ on X by letting the generator of $\mathbb{Z}$ act as a hyperbolic rotation:
$$(x,y)\mapsto (2x,\frac{y}{2}).$$

   (a) Show that $X\to X/\mathbb{Z}$ is a covering map.

   (b) Prove that $X/\mathbb{Z}$ is not Hausdorff.

   (c) Which of the following properties does $X/\mathbb{Z}$ have: connectedness, path-connectedness, compactness, separability, T_0-ness, T_1-ness, metrizability?

5. Consider $\mathbb{R}^3$ with coordinates (x, y, z).

   (a) Write down explicit formulas for the vector fields X and Y which represent the infinitesimal generators of uniform rotations about the x- and y-axes respectively.

   (b) Compute the Lie bracket of X and Y.

   (c) Determine the flow for the vector field $[X, Y]$.

6. Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with projection map $\pi : \mathbb{R}^4 \longrightarrow T^4$.

   (a) Show that there exist 1-forms dy_i on T^4 such that $\pi^* dy_i = dx_i$ for $i = 1, \ldots, 4$, where the x_i are the standard coordinate functions on $\mathbb{R}^4$.

   (b) Does there exists a differentiable map $F : T^4 \longrightarrow T^4$ such that

$$[F^*(dy_1 \wedge dy_2)] = [dy_3 \wedge dy_4] \quad \text{and} \quad [F^*(dy_2 \wedge dy_3)] = [dy_1 \wedge dy_2]$$

   where $[\omega]$ denotes the de Rham cohomology class of a differental form ω? You have to justify your answer.

7. Let M be a smooth compact manifold. Show that there is no submersion $F : M \longrightarrow \mathbb{R}^k$, $k > 0$.

8. On $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) consider $\omega = \frac{1}{\pi}(x_1 dx_2 + x_3 dx_4)$. Let $i : S^3 \longrightarrow \mathbb{R}^4$ be the embedding of the unit 3-sphere. Compute

$$\int_{S^3} i^*(\omega \wedge d\omega).$$