Numerical Analusis Qualifying Exam
Fall 1995

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1. The flow velocity in a cross section of a square duct is given by:

$$u(y,z) = \sum_{k=1,3,\ldots}^{\infty} (-1)^{(k-1)/2} \left[1 - \frac{\cosh(k\pi z/2)}{\cosh(k\pi/2)}\right] \frac{\cos(k\pi y/2)}{k^3}, \cosh(x) = \frac{e^x + e^{-1}}{2}. \qquad (1)$$

The flow is in x-direction and u is a function of y, z in a cross section, where y, z are between -1 and 1.

(a) Assume that the series is an alternating series for large k's. Estimate the number of terms n used to approximate the series such that the error is less than or equal to 10^{-10} (a reasonable estimate for the worst case is enough).

(b) A program will cause overflows if magnitude of an intermediate result in the computation is greater than a certian number. Overflow does not give a numerical result. On the other hand, if the magnitude of an intermediate result is less than a certain number, underflow occurs and the computer will assign zero to the result. Reformulate the general term in the series to avoid overflow (underflow is allowed).

2. Let $f(x) = (x-a)^2 h(x)$, where a is real and $h(x)$ is a smooth function such that $h(a) \neq 0$. Show that the rate of convergence of Newton's method to the root a is only linear but the rate of convergence of the method

$$x_{k+1} = x_k - 2\frac{f(x_k)}{f'(x_k)}$$

is quadratic. Extend this to the case that $f(x) = (x-a)^m h(x)$ (m is a positive integer) without proof.

3. Use the Chebysev Equioscillation Theorem to prove the following result.

For a fixed integer $n > 0$, consider the minimization problems for $-1 \leq x \leq 1$:

$$\tau_n = \inf_{\deg(Q) \leq n-1} \| x^n + Q \|_\infty$$

with $Q(x)$ a polynomial. The minimum τ_n is attained uniquely by letting

$$Q(x) = \frac{1}{2^{n-1}} T_n(x) - x^n$$

where $T_n(x)$ is the Chebysev polynomial

$$T_n(x) = \cos(n \cos^{-1} x).$$

The minimum is $\tau_n = \frac{1}{2^{n-1}}$.

4. Describe the composite trapezoidal rule and composite Simpson's rule for evaluating integrals $\int_a^b f(x)dx$ using the trapezoidal rule and Simpson's rule:

$$\int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}\left[f(x_0) + f(x_0+h)\right] - \frac{h^3}{12} f^{(2)}(\xi),$$

$$\int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_0+h) + f(x_0+2h)\right] - \frac{h^5}{90} f^{(4)}(\xi),$$

(1) Derive an estimate for the error in the composite trapezoidal rule in terms of the length of the subintervals into which $[a, b]$ is divided.

(2) Derive the asymptotic error formula for the composite Simpson's rule

$$E_n(f) \doteq -\frac{h^4}{180}\left[f^{(3)}(b) - f^{(3)}(a)\right]$$

5. Consider the numerical differentiation formula

$$f'(x) \doteq \frac{f(z+h) - f(x-h)}{2h}$$

(1) Derive the formula with a remainder:

$$f'(x) - \frac{f(x+h) - f(x-h)}{2h} = -\frac{h^2}{6} f^{(3)}(\xi)$$

(2) Assume M is an upper bound of $f^{(3)}(x)$, ε is an upper bound of the absolute round off errors that occur when f is evaluated. Derive an upper bound for the total error. Discuss the effects of the truncation error and the round off error.

6. Prove that

$$\sigma_{\max} = \max_{x \in R^n, y \in R^m, x, y \neq 0} \frac{y^T A x}{\| x \|_2 \| y \|_2}, \quad A \in R^{m \times n}$$

where $\sigma_{\max}$ is the largest singular value of A. (Hint: consider the singular value decomposition of A)

7. Let A be a non-singular $n \times n$ matrix, consider the following iterative scheme for solving the linear system $Ax = b$: Choose a non-singular matrix B, let $x_0 = Bb$ and set

$$\Gamma_{n-1} = b - Ax_{n-1},$$

$$x_n = x_{n-1} + B\Gamma_{n-1}, n = 1, 2, \ldots$$

(1) Show that

$$\Gamma_n = (I - AB)^{n+1} b, n = 0, 1, \ldots$$

(2) State and prove a criterion (in terms of A, B) for convergence of the sequence $\{x_n\}$ to the solution of $Ax = b$.

8. Assume that $w, u, v \in R^n$, and that $\| w \|_2 = 1$. What are the eigenvalues, eigenvectors, and determinant of a Householder matrix $I - 2ww^T$? More generally, what can you say about the eigenvalues and eigenvectors of $I + uv^T$ for given vectors u and v? (The case $v^T u = 0$ is special.)