Geometry/Manifolds QUALIFYING EXAM Spring 1997

(Yetter and Wu)

Work out problem **1** and then choose **5** (and only 5) additional problems among the remaining ones. Start each problem on a new sheet of paper. **Do not turn in more than 6 problems including the required number 1.** A space always means a topological space below.

1. This one is required!

Answer the following questions and give a brief explanation or counterexample:

i) (a) Give an examples of orientable, connected, simply-connected smooth 2-manifold with trivial tangent bundle.

(b) Give an examples of orientable, connected, simply-connected smooth 2-manifold with non-trivial tangent bundle.

ii) What is the dimension of the total space of the exterior bundle $\Lambda^3(M)$ if M is a smooth 6-manifold?

iii) Give an example of a compact non-orientable manifold.

iv) Consider the vectorfield on $\mathbb{R}^2 \setminus \{(0,0)\}$ given in polar coordinates by $(r-1)d\theta$. Describe in detail its flows. Is this vectorfield complete?

v) (a) What is the fundamental group of $S^1 \times S^1$?

(b) What is the fundamental group of $S^1 \times S^2$?

(c) What is the fundamental group of $\mathbb{R}^2 \setminus \{(0,1),(0,-1)\}$?

vi) Define a connection on the tangent bundle of a manifold. Define the Levi-Civita connection.

vii) Give an example of a space which is locally euclidean, but is not a manifold.

(continued on next page)

Choose 5 problems from below:

2. Let Γ be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ in $\mathbb{R}^3$ calculate

$$\int_\Gamma z dx \wedge dy - y dz \wedge dx$$

3.

i) Describe the natural Lie algebra structure on the set of vectorfields on a smooth manifold M.

ii) In the case where M is a Lie group, use the group law and the Lie algebra structure in **i)** to construct a Lie algebra structure on the tangent fibre at the identity $T_e(M)$.

4. Suppose G is a connected compact Lie group. Show that the fundamental group of G, $\pi_1(G)$ is abelian.

5. Find the scalar curvature of the surface $z = x^2 + y^2$ at $(0, 0, 0)$.

6. Show that every 1-form on $\mathbb{R}^1$ is *exact*. Show that every *closed* 1-form on $\mathbb{R}^3$ is *exact*.

7. Let $\varphi : M \to N$ be a (smooth) map. Then the vector field X on M and Y on N are said to be φ-related if $d\varphi_m(X_m) = Y_{\varphi(m)}$ for all $m \in M$.

Let X_1, X_2 be vector fields on M and Y_1, Y_2 vector fields on N. Assume that X_1 is φ-related to Y_1 and X_2 is φ-related to Y_2. Show that $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.

8. Suppose $f : X^d \to \mathbb{R}^{d+1}$ is a (smooth) embedding of the d-dimensional manifold X into $\mathbb{R}^{d+1}$. A normal vector field along (X, f) is a smooth map $N : X \to T(\mathbb{R}^{d+1})$ such that for each $p \in X$, $N(p) \in T_{f(p)}\mathbb{R}^{d+1}$ and it $(N(p))$ is orthogonal to the subspace $df(T_pX) \subset T_{f(p)}\mathbb{R}^{d+1}$. Prove that the manifold X is orientable if and only if there is a smooth *nowhere-vanishing* normal vector field along (X, f).

9. Prove that S^n $(n \geq 1)$ is orientable.

10. Let $f : M \to N$ be a smooth map such that for all $m \in M$, $df_m : T_mM \to T_{f(m)}N$ is *surjective*. Show that for any $n \in N$, $f^{-1}(n) \subset M$ is a smooth submanifold of M. What is the dimension of $f^{-1}(n)$?