Differential Equations Qualifying Exam
Fall 1991

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1. Solve the following initial-boundary value problem

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < 1, t > 0 \\ u(x,0) = x(1-x) & 0 \le x \le 1 \\ u_t(x,0) = 0 & 0 \le x \le 1 \\ u(0,t) = u(1,t) = 0 & t > 0 \end{cases}$$

2. The motion of an undamped, unforced spring-mass system is governed by the equation

$$m\frac{d^2x}{dt^2} + kx = 0.$$

The energy of the system is $E(x,v) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ where $v = \frac{dx}{dt}$.

(a) Show $\frac{dE}{dt} = 0$.

(b) Show that if $\Phi(x,v)$ is a C^1 function such that $\frac{d\Phi}{dt} = 0$ then $\Phi = f(E(x,v))$ for some function f.

3. (a) Formulate the maximum principle for

(1) Laplace's Equation $\Delta u = 0$.

(2) Cauchy problems for the heat equation

$$u_t - \Delta u = 0$$

$$u(x,0) = f(x) \text{ where } x \in \mathbb{R}^n.$$

(b) Assume that u is a harmonic function in $\mathbb{R}^n$ and that u attains its minimum at some point $x_0 \in \mathbb{R}^n$. What can you conclude about u? Give your argument.

4. (a) Show that if $u \in C^1(\mathbb{R}^n)$ and $\Delta u = 0$ then u has the mean value property.

(b) Show that if $u \in C^1(\mathbb{R}^n)$ and d $\Delta u \ge 0$ then $u(x) \le \frac{1}{|S_r|}\int_{S_r} u(x+y)d\sigma(y)$ where S_r is the "sphere" of radius r, $|x| = r$, and $|S_r|$ is the "area" of S_r.

(c) Formulate a maximum principle for C^1 functions that satisfy $\Delta u \ge 0$. Give your argument.

5. (a) Show that $\frac{1}{4\pi r}$ is a weak solution of $\Delta u = \delta$ on $\mathbb{R}^3$ where δ is the Dirac delta function.

(b) Give the definition of the Sobolev space $W_0^{m,p}(\Omega)$ for a bounded open set Ω in $\mathbb{R}^n$.

6. Let $f \in C^1(\mathbb{R}^1)$ be such that $f(P_0) = 0, f(P_1) = 0$ and $f(x) > 0$ for $P_0 < x < P_1$. Suppose

$$\frac{dP}{dt} = f(P)$$

$$P(0) = Q$$

with $P_0 < Q < P_1$. Show $\lim_{t\to\infty} P(t) = P_1$ and $\lim_{t\to-\infty} P(t) = P_0$.

7. Let $\phi_1, \dots, \phi_n, \dots$ be the complete sequence of eigenfunctions of $\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$ where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$. Show that

$$G(x, y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n}$$

is the Green's function of the Laplace equation with homogeneous Dirichlet boundary condition.

8. Let $u(x, t)$ be the solution to the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} = c^2 \Delta u & x \in \mathbb{R}^n, t > 0 \\ u(x, 0) = f(x) & x \in \mathbb{R}^n \\ u_t(x, 0) = g(x) & x \in \mathbb{R}^n \end{cases}$$

where f and g are continuous functions satisfying $\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|^{\alpha}} = A$, $\lim_{\|x\| \to \infty} \frac{g(x)}{\|x\|^{\alpha - 1}} = B$ for some constant $\alpha > 1$. Prove that

$$\lim_{t \to \infty} \frac{u(x, t)}{t^{\alpha}} = C_n$$

exists where C_n is a constant which depends on the dimension n and find C_3.

9. Let $\Delta u + u = 0$ in $\mathbb{R}^3$. Can $u(x) \geq C$ for $\| x \| \to \infty$ for some positive constant C? Give your argument.