Analysis Qualifying Exam - August 2012
(Korten / Volok)

1. Let $\{f_k\}$ and f be non-negative measurable functions on E such that $f_k \to f$ μ–a.e. in E. Prove that if $\int_E f_k d\mu \leq M$ for all k, then $\int_E f d\mu \leq M$.

2. Use Fubini's theorem to prove that $\int_{\mathbb{R}^n} \exp(-|x|^2)dx = \pi^{n/2}$.

3. Let f, $\{f_k\} \in L^p$. Show that if $||f_k - f||_p \to 0$ then $||f_k||_p \to ||f||_p$.

4. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$, χ its indicator function, λ_n the Lebesgue measure in $\mathbb{R}^n$. Let $K(x) = \frac{\chi(x)}{\lambda_n(B(0,1))}$ for $x \in \mathbb{R}^n$, and $K_\epsilon(x) = \frac{1}{\epsilon^n} K(x/\epsilon)$. Let $L^1_{loc}(\mathbb{R}^n)$ stand for the set of Lebesgue measurable functions that are integrable over each compact subset of $\mathbb{R}^n$. Prove in detail that for every $f \in L^1_{loc}(\mathbb{R}^n)$ and every Lebesgue point x of f

$$\lim_{\epsilon \to 0} (f * K_\epsilon)(x) = f(x).$$

(Def.: A point x at which $\lim_{r \to 0} \frac{1}{\lambda_n(B(0,r))} \int_{B(0,r)} |f(y) - f(x)| dy = 0$ is called a *Lebesgue point* of f.).

5. Let $f(z)$ be a bounded entire function, and let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be its Taylor expansion at the origin. Use the Cauchy Integral Formula to show that

$$c_1 = c_2 = \cdots = 0,$$

thus deducing Liouville theorem.

6. Let $a \in \mathbb{D} = \{z : |z| < 1\}$ and let

$$f(z) = \frac{z - a}{1 - z\bar{a}}.$$

Use the Maximum Modulus Principle to prove that f maps $\mathbb{D}$ into itself.

7. How many roots does the polynomial $p(z) = z^3 + 3z + 1$ have in the annulus $\{z : 1 < |z| < 2\}$?

8. Calculate using residues

$$\int_0^\infty \frac{x^2}{1+x^4}\,dx.$$