Algebra Qualifying Exam
Spring 1991

All rings are assumed to have a multiplicative identity, denoted 1. The fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of *rational, real* and *complex* numbers, respectively.

1. Let G be a finite group, N a normal subgroup of G, and let $g \in G - N$. If p is a prime with $g^p \in N$, prove that the cyclic group $< g >$ has a subgroup of order p.

2. If p is prime, prove that the center of any non-identity finite p-group is non-trivial.

3. Prove that any simple group of order 60 is isomorphic to the alternating group A_5.

4. Let R be the polynomial ring $\mathbb{Z}[x]$, and let M be the ideal in R generated by the elements $2, x \in R$. Prove that M is a maximal ideal in R.

5. Let p be a prime and let $R = \left\{\frac{a}{b} \in \mathbb{Q} | p \nmid b\right\}$. If M is the principal ideal in R generated by p, prove that M is the *unique* maximal ideal in R. (*Hint:* Show that any element not in M is a unit in R.)

6. Let $f(x) = x^5 - 2 \in \mathbb{Q}[x]$, and let ω be a complex primitive fifth root of unity. Show that $\mathbb{Q}(\omega, \sqrt[5]{2})$ is a splitting field for $f(x)$.

7. Let $f(x) = x^5 - 1 \in \mathbb{Q}[x]$. Prove that the Galois group of $F(x)$ over $\mathbb{Q}$ is nonabelian.

8. Let V be an n-dimensional vector space over a field F, and let $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. Let V^* denote the dual space of V, that is, V^* is the vector space $Hom_F(V, F)$ of all linear transformations $\lambda : V \to F$. Define elements $\lambda_1, \ldots, \lambda_n$ of V^* by setting

$$\lambda_i \left(\sum_{j=1}^{n} a_j x_j \right) = a_i,$$

$1 \leq i \leq n, a_j \in F$, and put $\mathcal{B}^* = \{\lambda_1, \ldots, \lambda_n\}$. Show that $\mathcal{B}^*$ is a basis of V^*.

9. Let V be an n-dimensional vector space over a field F, and let

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$$

be a chain of subspaces of V, with $\dim(V_i/V_{i+1}) = 1$ for $i = 0, 1, \ldots, n-1$. Suppose that $T : V \to V$ is linear transformation satisfying $T(V_i) \subseteq V_{i+1}$ for all $i = 0, 1, \ldots, n-1$. Compute the characteristic polynomial of T.