Numerical Analysis Qualifying Exam
Spring 1996

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1. The Bessel function has the following recursion formula:

$$J_{m+1}(x) = 2mJ_m(x) - J_{m-1}(x).$$

Suppose we want to calculate the value $J_n(x)$ for a large value of n by using the values $J_0(x)$ and $J_1(x)$ (which are assumed known) and the previous recursion formula, is the calculation numerically stable? Explain in detail. (Hint: Assume that there is an error in either $J_1(x)$ or $J_0(x)$. Give an estimate of the resulted error in $J_m(x)$ that is enought to determine the stability or instability of the formula.)

2. The Bessel function of order 0 can be defined by the power series

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

It is known that it has the following asymptotic property that

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{x}{4}\right) + O(x^{-3\backslash 2}).$$

as $x \to \infty$. Explain why the power series is not a good tool to evaulate $J_0(x)$ for large values of x. Can you suggest a method to evaulate $J_0(x)$ for very large values of x?

3. (a) Write down the formula of Newton's method for solving $f(x) = 0$ (write x_{n+1} in terms of x_n)

(b) In Newton's method, an approximation of the error in x_n is give as:

$$\alpha - x_n \approx x_{n+1} - x_n, \quad \text{for large } n,$$

where α is a root, and $x_n \to \alpha$, $n \to \infty$. Give a derivation to justify it (hint: state with $f(x_n) = f(x_n) - f(\alpha) = f'(\xi_n)(x_n - \alpha)$)

4. (a) Write down the error formula for the interpolation polynomial $p_n(x)$ of a known smooth function $f(x)$ with nodes $x_0, x_1, \ldots, x_n$.

(b) Suppose we study the approximation of $f(x)$ in interval $[a, b]$. Starting from the error formula in (a), if one tries to reduce the maximum error by choosing suitable $x_i, i = 0, 1, \ldots, n$, one is getting a near-minimax polynomial. From this direction, derive the error bound for the (true) minimax polynomial $M_n(x)$:

$$\max_{a \le x \le b} |f(x) - M_n(x)| \le \frac{[(b-a)/2]^{n+1}}{(n+1)!2^n} \max_{a \le x \le b} |f^{(n+1)}(x)|.$$

(Hint: first consider $[a, b] = [-1, 1]$, The Chebyshev polynomials defined as

$$T_n(x) = \cos(n \cos^{-1} x), \quad -1 \le x \le 1$$

have the form of $T_n(x) = 2^{n-1}x^n+$ lower degree terms)

5. Approximate $I(f) = \int_0^{2h} f(x)dx$ by approximating $f(x)$ by $P_1(x)$, the linear interpolant to $f(x)$ at $x = 0$ and $x = 4h/3$. Give the resulting numerical integration formula. What is its degree of precision?

6. (a) Given the following n by n lower triangular matrix:

$$L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \dots & & & \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix}.$$

Write its inverse L^{-1} as the product of $n-1$ matrices. What special features does L^{-1} have?

(b) Using the result in the previous item, show that the inverse of a non-singular upper triangular matrix U with diagonal elements u_{ii}, $i = 1, \dots, n$, is also upper triangular.

(c) Suppose an invertible matrix A is factored as $A = L_1 U_1$ and as $A = L_2 U_2$ where L_1, L_2 are lower triangular with 1's on the diagonal and U_1, U_2 are upper triangular. Prove that $L_1 = L_2$ and $U_1 = U_2$ by using the results in the previous items.

7. Use the Gerschgorin Circle Theorem to show that for a linear system $Ax = b$ with A being diagonally dominant, i.e.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{ij}|,$$

the Jacobi Iterative Method converges for any initial choice. (Warning: You must use the Gerschgorin Circle Theorem in an essential way to get credit for the problem)

8. Let A be an n by n **symmetric** real matrix whose eigenvalues are λ_i with corresponding eigenvectors v_i, $i = 1, \dots, n$. Assume that the inequality $|\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_{n-1}| < |\lambda_n|$ holds. A power method that computes the largest eigenvalue λ_n can be described as follows,

choose an initial vector $x^{(0)}$

For $k = 1, 2, \dots$

$$\begin{aligned} \tilde{x}^{(k)} &= A x^{(k-1)} \\ \lambda^{(k)} &= (\tilde{x}^{(k)})^t x^{(k-1)} (x^t \text{ is the transpose}). \\ x^{(k)} &= \tilde{x}^{(k)} / \parallel \tilde{x}^{(k)} \parallel_2 \end{aligned}$$

Show that if the initial vector $x^{(0)}$ satisfies $(x^{(0)})^t v_n \neq 0$, then

$$\lambda^{(k)} = \lambda_n + O\left(\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^{2k}\right)$$

as $k \to \infty$.