Qualifying Exam - Spring 1976

Algebra - Do any five problems

1. Show that the field with 27 elements has a cyclic multiplicative group.

2. Let T be a linear transformation of an n-dimensional vector space into itself. If $T^k = 0$ for some k, show that a basis may be selected for V such that if $A = (a_{ij})$ is the matrix of T with respect to this basis (i.e., $T(v_i) = \sum a_{ij} v_j$), then $a_{ij} = 0$ for $j \geq i$. As a corollary, can you show that $T^n = 0$?

3. Let D be a commutative ring with 1. For $a \varepsilon D$, show that D/aD is an integral domain iff a is prime (that is, if $ar = bc$, then $b = aq$ or $c = aq$ for ring elements r,b,c). Give one or two nontrivial conditions on a,D (or both) which insure that D/aD is a field.

4. Let G be a subgroup of the symmetric group S_n. If G contains an odd permutation, show that G has even order and that exactly half of the elements of G are odd permutations.

5. If G is a cyclic group of order n, show that every subgroup of G is cyclic. Also show that if $m|n$, then G has a unique subgroup of order m.

6. Let R be a ring such that for all $r \varepsilon R$, $mr = 0$ for some fixed, square-free integer m. Show that R is the direct sum of ideals R_i and for each i there is a prime factor p_i of m such that for all $r \varepsilon R_i$, $p_i r = 0$.

Analysis - Do four of the following six exercises

1. Suppose $\{K_n\}_{n=1}^{\infty}$ is a collection of closed sets contained in the compact set K and $K \supset K_1 \supset K_2 \supset \ldots$. Prove $\bigcap_{n=1}^{\infty}$ is not empty.

2. Prove that if f is a continuous real-valued function defined on a compact set of real numbers, then f is uniformly continuous.

3. State and prove the chain rule for functions of one real variable.

5. Let K be algebraic over F where F has characteristic 0. Prove that K is separable over F.

6. Let $f(x)$ be a polynomial in $F[x]$ which has no multiple roots in any extension field of F. If K is the splitting field of $f(x)$ over F, and $G = \mathrm{Gal}(K/F)$ show that $f(x)$ is irreducible in $F[x]$ if and only if G transitively permutes the roots of $f(x)$ in K.

7. Let $F = \mathbb{Z}/(2)$. Let $f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$. Let K be a splitting field of $f(x)$ over F. Compute $[K:F]$

8 If $F \subseteq K$ with K a finite field, prove $\mathrm{Gal}(K/F)$ is abelian.

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