Numerical Analysis Qualifying Exam
Fall 1991

---

1. A rectagular parallelpiped has sides 31 mm (millimeter), 42 mm and 53 mm, measured only to the nearest millimeter. Give a practical upper and lower bounds for the total surface area of the parallelpiped.

2. The computation of the sequence $p_n = (1/3)^n$ is a stabel mathematical problem. The sequence can also be generated by the following recurrence relation:

$$p_0 = 1, p_1 = 1/3, p_n = \frac{5}{3}p_{n-1} - \frac{4}{9}p_{n-2}, n = 2, 3, \ldots$$

Indicate with proof if this recurrence relation is stable.

3. Suppose that the equation $f(x) = 0$ can be rearranged as $x = h(x)$ for some function $h \in C^2$.

(1) Show that if, for some non-zero value of a parameter ω, the sequence generated by

$$x_{n+1} = x_n + \omega[h(x_n) - x_n]$$

converges to a number α, then α must be a fixed point of h.

(2) Taking $g(x) = x + \omega[h(x) - x]$, deduce that the above iteration is second order if

$$\omega = \frac{1}{1 - h'(\alpha)}, \quad h'(\alpha) \neq 1$$

(3) Based on the discussion, can you suggest a practical way of choosing a suitable ω during the iteration? (you do not need to give a proof for it)

4. Suppose $f(x)$ have an $(n+1)$st derivative in $[a, b]$ and $P_n(x)$ is the interpolation polynomial with respect to $n+1$ distinct points $x_i, i = 0, 1, \ldots, n, x_i \in [a, b]$ (i.e. $P_n(x_i) = f(x_i)$). Show that for any $x \in [a, b]$, there exists a $\xi = \xi(x)$, with

$$\min(x_0, x_1, \ldots, x_n, x) < \xi < \max(x_0, x_1, \ldots, x_n, x),$$

such that

$$f(x) - P_n(x) \equiv R_n(x) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi).$$

where

$$\omega_n(x) \equiv (x - x_0)(x - x_1) \ldots (x - x_n).$$

5. Given the following theorem: a quadrature formula

$$I_n\{f\} = \sum_{j=1}^{n} \alpha_j f(x_j) \qquad (1)$$