TOPOLOGY QUALIFYING EXAM

Fall 2001

Muenzenberger and Strecker

Choose and work any 6 of the following 12 problems. Start each problem on a new sheet of paper. **Do not turn in more than six problems.** A "space" always means a topological space below.

1. Assume that every open cover of the space X has a countable subcover. Let $A \subseteq X$ be an uncountable subset. Prove A has a limit point.

2. Prove a space X is compact if and only if every net in X has a cluster point.

3. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be **final** provided that for each topological space (Z, μ) each set-function $g : Y \to Z$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \mu)$ is continuous. Prove any two of the following statements:

   a) The composition of final (continuous) maps is final.

   b) The "second factor" of a final map is final: i.e., if $(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{h} (W, s)$ are continuous maps and $h \circ f$ is final, then h is final.

   c) Each quotient map is final.

4. Use Zorn's Lemma to prove that for each set X and relation R on X, there is a maximal $A \subseteq X$ such that $A \times A \subseteq R$.

5. Prove that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is closed; i.e., for each closed subset A of $X \times Y$, $\pi_1[A]$ is closed in X.

6. Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let τ be the product topology on $X_1 \times X_2$. Let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. There is a "natural" subspace topology σ on $Y_1 \times Y_2$ that is induced from τ and the fact that $Y_1 \times Y_2$ is a subset of $X_1 \times X_2$. There is also a "natural" product topology ρ on $Y_1 \times Y_2$ that comes from the subspace topologies (Y_1, ρ_1) and (Y_2, ρ_2) induced by τ_1 and τ_2 respectively. Prove or disprove that $\sigma = \rho$.

7. Prove that no continuous function $f : S^1 \to \mathbb{R}$ is $1-1$.

   $$[S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}]$$

   Note: S^1 and $\mathbb{R}$ have the usual topology.

8. Prove that a quotient of a locally connected space is locally connected.

9. Show that if X is a compact metrizable space, then every metric which generates the topology on X is complete.

10. Prove that every paracompact Hausdorff space is regular.

11. Show that the Moore Plane (tangent disc space) is not normal.

12. (a) What is the fundamental group of $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$?

    (b) What is the fundamental group of $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$?

    (c) Prove that there is no continuous function $r : B^2 \to S^1$ such that $r(x, y) = (x, y)$ for all $(x, y) \in S^1$.

    Note: B^2 and S^1 have the usual topology.