Topology Qualifying Exam
Fall 1990

Work 6 of the following problems. Start each problem on a new sheet of paper. Do not turn in more than 6 problems.

1. If (X, τ) and (Y, σ) are topological spaces and $f : X \to Y$ is a function, give a detailed set-theoretic proof that the following are equivalent.

(i) For each $U \in \sigma$, $f^{-1}[U] \in \tau$.

(ii) For each $A \subseteq X$, $f[\overline{A}^{\tau}] \subseteq \overline{f[A]}^{\sigma}$.

2. Prove that $[0, 1]$, with its usual topology, is connected.

3. Show that a space X that has the fixed point properly is connected.

4. (a) True-False.

(i) An open and closed one-to-one function between topological spaces must be an embedding.

(ii) Each space that is locally-Hausdorff (in the sense that each point has neighborhood base of Hausdorff subspaces) must be Hausdorff.

(iii) Each quotient of a locally connected space must be locally connected.

(iv) Each locally compact Hausdorff space is completely regular.

(v) The product of metrizable spaces is metrizable.

(b) For each false entry, give a counterexample of other explanation (no proofs).

5. (a) State the Axiom of Choice.

(b) Give another statement that is equivalent to the Axiom of Choice.

6. Prove that if A is a compact subset of a regular (not necessarily Hausdorff) space X, then $\overline{A}$ is compact.

7. Give an example of two topologies σ and τ on the set of integers $\mathbb{Z}$ for which $\sigma \subsetneq \tau$ and $(\mathbb{Z}, \sigma)$ is homeomorphic to $(\mathbb{Z}, \tau)$.

8. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is said to be final provided that for each topological space (Z, μ) each set-function $g : Y \to Z$ is continuous whenever $g \circ f : (X, \tau) \to (Z, \mu)$ is continuous. Prove that:

(a) The composition of final (continuous) maps is final.

(b) The "second factor" of a final map is final, i.e., if $(X, \tau) \xrightarrow{f} (Y, \sigma) \xrightarrow{h} (w, s)$ are continuous maps and $h \circ f$ is final, then h is final.

9. (a) Give an example of topological space X that has both a Stone Čech compactification, βX, and an Alexandorff compactification, αX, but for which αX and βX are not homoemorphic.

(b) Give a reason why αX and βX, in part (a), are not homeomorphic.

10. (a) If sequential limits in a space X are unique, must X be Hausdorff?

(b) Prove that your answer to (a) is correct.

11. Show that the Moore Plane (tangent disc space) is not normal.

12. Prove that if A is a connected subset of connected space X and if C is a component of $X - A$, then $X - C$ is connected.