Algebra Qualifying Exam
February 23, 1995

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative. $\mathbb{Z}$ and $\mathbb{Q}$ are the sets of the integers and rational numbers respectively.

1. Let G be a finite group and p be the smallest prime divisor of $|G|$. If H is a subgroup of G of index p in G, show that H is a normal subgroup.

2. Let p and q be prime numbers. Show that any group of order p^2q is solvable.

3. Let $D = \mathbb{Z}[i]$, the ring of Gaussian integers. Compute the order of the quotient ring $D/(1+2i)D$.

4. Let $f : R \to S$ be a homomorphism of rings, and let $I \subseteq R$ be an ideal. Is it true that $f(I)$ is an ideal of S? Prove, or give a counterexample. What if f is assumed to be surjective?

5. Let B be a ring. An ideal I of R is called *nilpotent* if there exists a positive integer n such that $I^n = 0$ ($I^n = II \cdots I$). Show that $IM = \{0\}$ for any simple R-module M.

6. Let R be a ring and M an (left) R-module. An element m in M is called a *torsion element* if $rm = 0$ for some $0 \neq r \in R$. Let M_t be the set of all torsion elements in M. Show that, if R is an integral domain, then M_t is an R-submodule and the quotient module M/M_t has no torsion elements other than 0.

7. Let V be a finite dimensional vector space over an algebraically closed field F and $T : V \to V$ be a linear transformation. For each $a \in F$, we define $V_a = \{v \in \ \mid (T - aI)^n v = 0$ for some positive integer $n\}$, which is a T-invariant subspace of V. Here I is the identity linear transformation. Show the following:

   (a). $V_a \neq \{0\}$ if and only if a is an eigenvalue of T.

   (b). Let Π be the set of all eigenvalues of T. Then $V = \oplus_{a \in \Pi} V_a$.

8. Let V be a finite dimensional vector space over a field F and $A, B : V \to V$ be two commuting linear transformations. If both A and B are diagonalizable, then there exists a basis of V such that both A and B have diagonal matrices with respect to this basis.

9. Let E be the splitting field $f(x) = (x^3 - 2)(x^2 + x + 1)$ over $\mathbb{Q}$. Compute the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.

10. Let E be the splitting field of $f(x) = x^5 - 2$ over the field $\mathbb{F}_5$, the field of 5 elements. Is E a Galois extension over $\mathbb{F}_5$? Justify your answer. If your answer is "yes", compute the Galois group.