Real and Complex Analysis Qualifying Exam.

New System-August 2006

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Instructions: The exam consists of 8 problems. Each problem is worth 10 points.
Time: 3 hours.
Notation: $\mathbf{N} := \{1, 2, 3, ...\}$, $\mathbf{R}$:= reals, $\mathbf{C}$:= complexes, $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$.

1. Compute
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6}$$

2. Let E_j, $j = 1, ..., m$ be measurable subsets of $[0, 1]$. Assume also that $q \leq m$ and each $x \in [0, 1]$ belongs to at least q sets E_j. Prove that there exists j such that $|E_j| \geq q/m$.

3. F is holomorphic in $\mathbf{D} \setminus \{0\}$ and $\lim\limits_{z \to 0} |F(z)| = \infty$. Show that $0 \in U$ open $\subset \mathbf{D} \Rightarrow$ F maps $U \setminus \{0\}$ onto the complement of a compact set.

4. How many zeros does the polynomial $z^5 + 3z^2 - 1$ have in the annulus $1 < |z| < 2$?

5. Suppose $f : X \to \mathbf{R}$ is measurable and set $E_n = \{x \in X : |f(x)| > n\}$, $n \in \mathbf{N}$.

   (a) Suppose $f \in L^1(X)$. Prove that $\sum\limits_{n=1}^{\infty} \mu(E_n) < \infty$.

   (b) If $\mu(X) < \infty$ and $\sum\limits_{n=1}^{\infty} \mu(E_n) < \infty$, then $f \in L^1(X)$.

6. It is easy to guess the limits of
$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} dx, \qquad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx$$
   as $n \to \infty$. Prove that your guesses are correct.

7. Let $f(x) = 1$ if $x \in [0, 1]$, and zero otherwise. Define $h_c(x) := \sup_{n \in \mathbf{N}} n^c f(nx)$. Prove that h_c is Lebesgue integrable on $\mathbf{R}$ if $c \in (0, 1)$.

8. Explicitly find a conformal map from the unit disk to the strip $\{z \in \mathbb{C} | \, 0 < \operatorname{Re} z < 1\}$.