TOPOLOGY
QUALIFYING EXAMINATION
SPRING 1984
(Muenzenberger - Summerhill)

Do 8 of the following 15 problems

1. Prove that if $f: X \to Y$ is 1-1 and continuous, then $f(Fr(A)) \subseteq Fr(f(A))$ for all $A \subseteq X$.

2. Prove that if X is connected and locally path connected, then X is path connected.

3. Prove that a space X is countably compact if and only if every countable family of closed subsets of X which has the finite intersection property has nonempty intersection.

4. Prove that if A is a compact subset of a regular (non Hausdorff) space X, then $\bar{A}$ is compact.

5. Consider the equivalence relation $\sim$ on I defined by

$$x \sim y \Leftrightarrow x = y \text{ or } \{x,y\} = \{0,1\}$$

where $x, y \in I$. Prove that the quotient space $I/\sim$ is homeomorphic to S^1.

6. Prove that every completely regular T_1 space can be embedded in a Tychonoff cube.

7. Prove (without using the Axiom of Choice) that the product of two compact spaces is compact.

8. Suppose that (K,h) is a compactification of a space X and suppose further that $K - h(X)$ is a singleton. Prove that U is open in K if and only if either U is open in $h(X)$ or else $K - U$ is a closed, compact subset of $h(X)$.

9. Prove that $(\mathbb{Q}\times\mathbb{Q}) \cap S^1$ is dense in S^1.

10. Prove that if a normal space X contains a closed copy of $[0,\infty)$, then X does not have the fixed point property.

11. True - False. For each false statement, provide a counterexample or reason (no proofs required).

____ a. Metrizability is productive.

____ b. In any metric space, compactness is equivalent to being closed and bounded.

____ c. Normality is neither hereditary nor productive.

____ d. The Tychonoff corkscrew is a completely regular space that is not normal.

____ e. Paracompactness is closed hereditary.

____ f. Every regular Lindelöf space is normal.

____ g. Tychonoff's Theorem is equivalent to the Axiom of Choice (in Zermelo-Fraenkel less Choice).

____ h. Every Peano space is arcwise connected.

____ i. The image of a second countable space under a closed, continuous function is first countable.

____ j. The image of a locally compact space under an open, continuous function is locally compact.

12. Prove that X is compact if and only if every $\subsetneq$-chain of nonvoid closed sets has nonvoid intersection.

13. Prove one of the following two theorems.

(a) If X is normal, then every point finite open cover $\{U_\alpha | \alpha \in A\}$ is shrinkable; that is, there is an open cover $\{V_\alpha | \alpha \in A\}$ such that $\overline{V}_\alpha \subsetneq U_\alpha$ for each $\alpha \in A$.

(b) If X contains a dense set D and a closed, relatively discrete subspace S with $|S| \geq 2^{|D|}$, then X is not normal.

14. Prove that every connected, locally compact, paracompact, Hausdorff space is Lindelöf.

15. Let $\mathbb{R}^3$ denote Euclidean 3-dimensional space, and let

$$B^3 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 \leq 1\}$$

and $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$. Prove that $\mathbb{R}^3 - B^3$ has the homotopy type of S^2.