ALGEBRA
QUALIFYING EXAM
SEPTEMBER 1985

I. Groups

1. Let A be a finite abelian group of order n. Assume that for each divisor m of n the equation $x^m = 1$ has m solutions, where 1 is the identity in A. Prove that A is cyclic.

2. Let A be the abelian group with presentation

$$A = \langle a_1, a_2 \mid 2a_1 - a_2 = 0, -2a_1 + 2a_2 = 0 \rangle.$$

Find the structure of A.

3. Let G be a finite group and let P be a 2-Sylow subgroup. Let $M \leq P$ be a subgroup of index 2 in P. Assume that $x \in P - M$ is not conjugate in G to any element of M. Then show that $x \notin G'$, the commutator subgroup of G. (Hint: Look at the permutation representation of G induced on the cosets of M. What is the cycle structure of x ?)

4. (You may assume the conclusion of exercise (3).) Let G be a finite simple group with dihedral 2-Sylow subgroups. Prove that G has a single class of involutions.

5. Let G be a simple group of order 60. Prove that G has exactly 5 2-Sylow subgroups. (Thus $G \cong A_5$, the alternating group on 5 symbols.)

II. Rings and Modules

1. Let R be an integral domain such that every ideal is a free R-module. Prove that R is a principal ideal domain.

2. Let R be a principal ideal domain.

   (a) Prove that any non-zero prime ideal in R is maximal.

   (b) Using (a), prove that the polynomial ring $\mathbb{Z}[x]$ is not a principal ideal domain.

3. Let R be a ring and let M be a left R-module. Prove that the following are equivalent:

   (i) $M = \sum M_i$, where $\{M_i\}_{i \in I}$ is a collection of irreducible R-submodules of M;

   (ii) $M = \bigoplus_{\alpha \in J} M_\alpha$, where $\{M_\alpha\}_{\alpha \in J}$ is a collection of irreducible R-submodules of M.

4. Let R be a ring with identity. Define the <u>Jacobson</u> <u>radical</u> $J(R)$ by setting

   $$J(R) = \{r \in R \mid rM = 0 \text{ for every irreducible left } R\text{-module } M\}.$$

   (a) Prove that $J(R)$ is a 2-sided ideal of R.

   (b) Prove that $J(R/J(R)) = 0$.

III. Linear Algebra

1. Let T be a linear transformation on the finite dimensional Q-vector space V. If $T^2 + I = 0$ prove that $\dim_Q V$ is even.

2. Let F be any field over which the polynomial $x^2 + x + 1$ is irreducible. Prove that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ is not similar to any upper triangular matrix, with entries in the field F.

3. Let $T : V \to V$ be a linear transformation on the finite dimensional F-vector space V. Let $W \subseteq V$ be a T invariant subspace of V.

   (a) Prove that there exists a unique linear transformation $\bar{T} : V/W \to V/W$ such that $\bar{T}(v + W) = T(v) + W$.

   (b) Prove that $m_{\bar{T}}(x) \mid m_T(x)$.

4. Let $T : V \to V$ be a linear transformation on the finite dimensional F-vector space V. Let $m_T(x) = \sum_{i=0}^{n} \alpha_i x^i$. Prove that T is invertible if and only if $\alpha_0 \neq 0$.

IV. Fields

1. Let ζ be the complex number $\zeta = e^{2\pi i/n}$. Prove that $[Q[\zeta] : Q] = 2$ if and only if $n = 3,4$ or 6.

2. Let F be a finite field, and let $F^* = F - \{0\}$. Prove that with respect to multiplication, F^* is a cyclic group. (You may use the result of exercise (1) of I.)

3. Prove that the Galois group of $x^4 - 5$ cannot be abelian. (Bear in mind that every subgroup of an abelian group is normal.)

4. Let K be a splitting field over $\mathbb{C}[x]$ for the polynomial $Y^3 - (x - 1)(x - 2)(x - 3)$. Prove that the genus of K over $\mathbb{C}$ is 2. (Just kidding!)

5. Let p be a prime and let ζ be the complex number $\zeta = e^{2\pi i/p}$. Prove that $\mathrm{Gal}(Q[\zeta]/Q) \cong Z_{p-1}$, a cyclic group of order $p - 1$.