Numerical Analysis Qualifying Exam
Fall 1988

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1. State and prove a theorem giving necessary and sufficient conditions for the series

$$\sum_{k=0}^{\infty} A^k$$

to converge, where A is a real matrix.

2. Suppose we say x^* is a good approximation to x if the relative error $(x^* - x)/x$ is much less than 1.

(a) if x^* is a good approximation to x and y^* is a good approximation to y and all of x^*, x, y^*, y are nonzeros, show that x^*/y^* is a good approximation to x/y.

(b) if x^* is a good approximation to x and y^* is a good approximaiton to y, explain why $x^* - y^*$ may not be a good approximation to $x - y$. Hint: A good way of stating that x^* is a good approximation to x is to say $x^* = x(1 + \delta)$ where $|\delta| << 1$.

3. Of all polynomials of degree ≤ 1, find the one which best approximates $f(x) = x^2 + 1$ on the interval $[0, 1]$, in the ∞-norm. Be sure to justify your answer (by stating and using an appropriate theorem).

4. Tell whether the following matrix is positive definite.

$$\begin{bmatrix} 8 & -2 & 3 & 3 \\ -2 & 6 & 1 & 1 \\ 3 & 1 & 9 & 0 \\ 3 & 1 & 0 & 8 \end{bmatrix}$$

5. Suppose we want to use the Gauss-Seidel iteration to solve $Ax = b$ where

$$A = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 2 & 6 & 3 & 1 \\ 1 & 1 & 5 & 2 \\ 1 & 1 & 4 & 6 \end{bmatrix}$$

Will the Gauss-Seidel iteration converge? (justify your answer).

6. Suppose that $x = g(x)$ has a root at $x = \alpha$ and in the interval

$$|x - \alpha| < \rho \tag{1}$$

$g(x)$ satisfies

$$|g(x) - g(\alpha)| \leq \lambda |x - \alpha|,$$

with $\lambda < 1$.

(a) Prove that for the iteration scheme

$$x_{k+1} = g(x_k), k = 0, 1, 2, \ldots,$$

with any x_0 in (1),

(i) all the iterates x_k lie in the interval (1),

(ii) the iterates x_k converge to α.

(b) Consider the perturbed iteration scheme

$$X_{k+1} = g(X_k) + \delta_k, k = 0, 1, 2, \ldots,$$

where δ_k satifies

$$|\delta_k| \leq \delta, k = 0, 1, 2, \ldots$$

Prove that if X_0 is any point in the interval

$$|x - \alpha| \leq \rho_0,$$

where ρ_0 satisfies

$$0 < \rho_0 \leq \rho - \frac{\delta}{1-\lambda},$$

then the iterates X_k satisfy

$$|\alpha - X_k| \leq \rho,$$

and

$$|\alpha - X_k| \leq \frac{\delta}{1-\lambda} + \lambda^k \left(\rho_0 - \frac{\delta}{1-\lambda}\right),$$

and $\lambda^k \to 0$ and $k \to \infty$.

7. Let A be a symmetric, positive definite real matrix, $b \in R^n$, and define $f : R^n \to R$ by

$$f(x) = \frac{1}{2}x^T Ax - b^T x.$$

Show that f has a unique minimum and that it occurs where x satisfies

$$Ax = b.$$

8. Let V be a real vector space with inner product $<>$. Let $E \subset V$ be a finite dimensional linear subspace. Prove: if $v \in V$, there is a unique $e^* \in E$ such that for all $e \in E$ we have

$$\| v - e^* \| \leq \| v - e \|,$$

where $\| z \| = \sqrt{< z, z >}$.

9. Suppose $I(f) = Af(a) + Bf(b)$ is a 2-point quadrature rule on $C[-1, 1]$. Suppose further that I has degree of precision 3. Find A, a, B, and b.

10. Let $f \in C[a, b]$ and for $n \geq 0$ let $Q_n(x)$ be the least-squares polynomial for f of degree $\leq n$. Recall this means of all polynomials $P(x)$ of degree $\leq n$

$$\int_a^b |f(x) - Q_n(x)|^2 dx \leq \int_a^b |f(x) - P(x)|^2 dx.$$

Prove:$\lim_{n\to\infty} \int_a^b |f(x) - Q_n(x)|^2 dx = 0.$

Hint: if $a_n = \int_a^b |f(x) - Q_n(x)|^2 dx$, show the sequence $\{a_n\}_n$ is non-increasing. The Weierstrass Approximation Theorem may then be helpful.

11. Let $I(f) = \sum_{i=1}^{n} w_i f(x_i)$ be an n-point quadrature rule on $[a, b]$. Show that I cannot give the exact answer for every polynomial of degree $\leq 2n$; this says an n-point rule cannot have degree of precision $\geq 2n$.