Geometry/Topology Qualifying Exam
Fall 2006

Do as many problems as you can in the time permitted. Make your calculations and proofs as complete as possible.

1. Determine, with proof or counterexample, whether each of the following conditions is necessary, sufficient or necessary and sufficient for a topological space X to be connected:

   (a) X is path connected.

   (b) X contains a dense connected set Y.

2. (a) Prove that if X is Hausdorff, any two disjoint compact sets C_1 and C_2 can be separated by disjoint open sets (i.e. there exist disjoint opens U_1 and U_2 with $C_i \subset U_i$ for $i = 1, 2$).

   (b) Derive as a corollary the result that a compact Hausdorff space is normal.

3. (a) Give an explicit triangulation of the Mobius strip.

   (b) Use the result of part a to calculate the cohomology of the Mobius strip.

4. Let X be a CW-complex with cells given by

   0-cells p_0, p_1

   1-cells A_0, A_1 with $\partial A_0 = p_0 p_0^{-1}$ and $\partial A_1 = p_0 p_1^{-1}$

   2-cells C_0, C_1 with $\partial C_0 = A_0^4$ and $\partial C_1 = A_1^{-1} A_0^2 A_1$

   Find $\pi_1(X)$, $H_\bullet(X, \mathbb{Z})$, and $\chi(X)$.

5. Give an explicit description of the De Rham cohomology in dimension 2 of S^2.

6. Give an example of a smooth map of manifolds with the property that the inverse image of one point is a smooth manifold, and the inverse image of a different point is not.

7. Given a region in R^2 bounded by a smooth simple closed curve $(x(\theta), y(\theta))$ derive an expression for the area of the region as an integral over θ.

8. (a) Prove or give a counterexample: the wedge product of any differential form with itself is zero.

   (b) The same as the previous part, but for a form of odd degree.

9. Suppose X is a topological space, with open subsets A, B and $\overline{B} \subset A$.

   Suppose, moreover that both X and A have trivial homology in dimensions $n-1$ and $n+2$, and homology free of rank 2 in dimensions n and $n+1$, with the maps induced by the inclusion $\iota_A : A \to X$ given by

$$H_n(\iota_A) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

   and

$$H_{n+1}(\iota_A) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

   in terms of a suitable basis.

   (a) Assuming homology is given with integer coefficients, find $H_{n-1}(X-B, A-B)$, $H_n(X-B, A-B)$, and $H_{n+1}(X-B, A-B)$.

   (b) Assuming homlogy is given with rational coefficients, find $H_{n-1}(X-B, A-B)$, $H_n(X-B, A-B)$, and $H_{n+1}(X-B, A-B)$.

   (c) Still assuming rational coefficients, assume that $n > 2$, that X is connected, and thus $H_0(X) = \mathbb{Q}$, but that all homology in dimensions other than 0, n and $n+1$ vanishes. Find the rational homology of X^3 and $X \times [0, 1]$.