Algebra Qualifying Exam (Old and New)
August 28, 2004

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet. You should have at least one problem from each of the five sets $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$, $\{9,10\}$ among your choice. **Note:** All rings in this exam are associative and with 1 and all integral domains are commutative.

$\mathbb{Z}$ and $\mathbb{Q}$ are the sets of integers and rational numbers respectively.

1. Prove that a finite group of order 30 has a normal subgroup of order 15.

2. Let G be a group. Assume that the group, $\mathrm{Auto}(G)$, of all automorphisms of G is a cyclic group. Show that G is an abelian group.

3. Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients and $P \subseteq \mathbb{Z}[x]$ be a prime ideal with $P \cap \mathbb{Z} = \{0\}$. Prove that P is a principal ideal.

4. For the ring $\mathbb{Z}[i]$ of all Gaussian integers of the form $a + bi$ with $a, b \in \mathbb{Z}$, describe all irreducible elements of $\mathbb{Z}[i]$.

5. Let M be a Noetherian left module over a ring R, and let $\phi : M \to M$ be a surjective R-module homomorphism. Show that ϕ is an isomorphism of R-modules.

6. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space over a field F. Consider V as a module over the ring of polynomials $F[x]$ with the action $f(x) \cdot v = f(T)v$. Show that V is an irreducible $F[x]$-module if and only if the characteristic polynomial of T is an irreducible polynomial in $F[x]$.

7. Let F be a finite field and E be a finite algebraic extension of F. Show that E is Galois over F and that the Galois group $\mathrm{Gal}(E/F)$ is cyclic.

8. Let $F \subseteq K \subseteq L$ be field extensions. Let $\alpha \in L$ be algebraic over F and $f(x) \in K[x]$ be the (monic) minimal polynomial of α over K.
  (a) Prove that all roots of $f(x)$, in any extension field of K, are algebraic over F; and
  (b) Prove that each of the coefficients of the polynomial $f(x)$ is algebraic over F as well.

9. Let A be a complex square matrix and assume that $A^4 = A$. Prove that A is diagonalizable over the field of complex numbers.

10. Let A and B be two $n \times n$ matrices over an algebraically closed field F and $AB = BA$. Show that there exists an invertible $n \times n$ matrix P such that both PAP^{-1} and PBP^{-1} are upper triangular matrices.