Real Analysis Qualifying Exam
September 19, 1996

Let (X, M, μ) be a measure space.

1. Let $0 < a < 1$. Prove that there exists a compact set $K \subset [0,1] \backslash \mathbb{Q}$ such that $\lambda(K) = a$, where λ is Lebesgue measure on $\mathbb{R}$.

2. Let f be a real-valued function on X such that $\{x : f(x) > a\}$ is measurable for each $a \in \mathbb{R}$. Prove that $f^{-1}(E)$ is measurable for each Borel set $E \subset \mathbb{R}$.

3. Let Y be a topological space and let $\varphi : X \to Y$ be measurable. Define $\nu(E) := \mu(\varphi^{-1}(E))$ for each $E \in \mathcal{B}_Y$ (the Borel subsets of Y). Prove:

a) ν is a Borel measure on Y.

b) Let $f : Y \to \mathbb{C}$ be Borel measurable. Then $f \in L^1(\nu)$ if and only if $f \circ \varphi \in L^1(\mu)$, in which case

$$\int_Y f d\nu = \int_X f \circ \varphi d\mu.$$

4. Prove the completeness of $L^P(\mu)$ for $1 \leq p < \infty$.

5. Let $0 < p < \infty$ and $f_n, f \in L^P(\mu)$. Suppose $||f_n||_P \to ||f||_P$ and $f_n \to f a.e.$ Prove that $||f_n - f||_P \to 0$.

Hint: $|f_n - f|^P \leq 2^P(|f_n|^P + |f|^P)$.

6. Let $f : X \times [0,1] \to \mathbb{C}$. State (nontrivial) conditions on f that guarantee

$$(*) \qquad \frac{d}{dt}\int f(x,t)\,d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)\,d\mu(x) \qquad \forall t \in (0,1)$$

and then prove $(*)$.

7. Let f and g be Borel functions on $\mathbb{R}^+$ such that

$$\int_0^\infty (|f(x)| + |g(x)|)\frac{dx}{x} < \infty.$$

Define

$$(f * g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}$$

for $x > 0$ whenever the integral in the right-hand side exists.

Prove: (i) $|f * g| < \infty$ Lebesgue-almost everywhere, (ii) $f * g$ is Borel measurable, and

$$(iii) \qquad \int_0^\infty |(f * g)(x)|\frac{dx}{x} \le \int_0^\infty |f(x)|\frac{dx}{x} \cdot \int_0^\infty |g(y)|\frac{dy}{y}.$$

8. Let f be a 2π-periodic differentiable function on $\mathbb{R}$ with $\int_0^{2\pi} |f'(t)|^2 dt < \infty$. Prove that

$$\sum_{n=-\infty}^{\infty} n^2 |\hat{f}(n)|^2 < \infty.$$