Algebra Qualifying Exam
April 2, 1990

Instructions: You are to do two problems from each of the four major areas (**Group Theory, Linear Algebra, Field and Galois Theory**, and **Rings and Modules**). Even if you attempt to do more, you will only be graded on two problems from each section. For the graders' convenience, please circle the problem numbers of those problems that you do from each of the four major areas:

I. Group Theory 1. 2. 3. 4.
II. Linear Algebra 1. 2. 3. 4.
III. Field and Galois Theory 1. 2. 3. 4.
IV. Rings and Modules 1. 2. 3. 4.

I. Group Theory

1. Prove that any group of order 51 must be cyclic.
2. Let the finite group G act transitively on the set Ω. If $g \in G$, set $\mathrm{Fix}(g) = \{\omega \in \Omega |\ g(\omega) = \omega\}$. Prove that
$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| \ = \ 1.$$
(*Hint*: compute $|\{(\omega, g) |\ \omega \in \Omega, g \in G,\ g(\omega) = \omega\}|$.)
3. Let $|G| = pq^2$, where p, q are distinct primes. Prove that either a p-Sylow subgroup is normal in G or that a q-Sylow subgroup is normal in G.

4. Let $\sigma = (1\ 2\ \dots\ n) \in S_n$ and prove that $<\sigma> = C_{S_n}(\sigma)$.

II. Linear Algebra

1. Let V be a rational vector space and let $T \in \mathrm{End}_{\mathbf{Q}}(V)$. If $T^3 = 5 \cdot 1_V$, prove that dim V is a multiple of 3.

2. Let $\mathbf{C}$ be the field of complex numbers, and let $A \in \mathrm{M}_3(\mathbf{Q})$ be the matrix
$$A = \begin{bmatrix} 8 & 0 & -6 \\ 3 & 2 & -3 \\ 9 & 0 & -7 \end{bmatrix}.$$
   (a) Compute the invariant factors of A.
   (b) Compute the Jordan cononical form of A.

3. Let V be a finite dimensional vector space and let $T_1, T_2 \in \mathrm{End}(V)$ be commuting linear transformations. If T_1 and T_2 are *individually* diagonalizable, show that they are *simultaneously* diagonalizable.

4. State and prove the *Rank Nullity Theorem* from elementary linear algebra.

III. Field and Galois Theory

1. Prove or give a counterexample to the following. Let $f(x), g(x) \in \mathbf{F}[x]$ be relatively prime irreducible polynomials. If $\mathbf{K}$ is a splitting field over $\mathbf{F}$ for $f(x), g(x)$, and if $\alpha, \beta \in \mathbf{K}$ are roots of $f(x), g(x)$ respectively, then $[\mathbf{F}(\alpha, \beta) : \mathbf{F}] = \deg f(x) \deg g(x)$.

2. Give an example of an irreducible polynomial $f(x) \in \mathbf{Q}(x)$ such that $\mathrm{Gal}(f(x))$ is nonabelian. Prove that your example works.

3. Let $a_1, a_2, \ldots, a_n \in \mathbf{Q}$, and consider the field $\mathbf{K} = \mathbf{Q}(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n})$. Prove that $\mathcal{Gal}(\mathbf{K}/\mathbf{Q})$ is an elementary abelian 2-group. (Hint: show that every element of the Galois group is an involution.)

4. Let $\mathbf{F}$ be a subfield of the real field $\mathbf{R}$, and assume that $\sqrt{\alpha} \in \mathbf{F}$ for each $0 \leq \alpha \in \mathbf{F}$. Prove that $[\mathbf{F} : \mathbf{Q}] = \infty$, where $\mathbf{Q}$ is the rational field.

IV. Rings and Modules

1. Let R be a commutative ring in which every ideal is a free R-module. Prove that R is a *principal ideal domain.*

2. Prove that $\mathbf{Z}/(2)$ is *not* a free $\mathbf{Z}$-module, but is a free $\mathbf{Z}/(2)$-module.

3. Let $\epsilon : M \to M'$ be a module homomorphism, and assume that there exists a homomorphism $\sigma : M' \to M$ such that $\epsilon \circ \sigma = 1_{M'}$. Prove that $M = \mathrm{im}(\sigma) \oplus \ker(\epsilon)$.

4. Let R be a commutative ring, and let $x \in R$. Define what it means for x to be *irreducible* element, and define what it means for x to be *prime.* If R is a *unique factorization domain*, show that x is irreducible if and only if x is prime.