Differential Equations Qualifying Exam
Spring 1990

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1. (i) Formulate maximum principle for Cauchy's problem for parabolic equations.

(ii) Give an example of its applications without proof.

2. Solve the exterior Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{for } |x| > 1 \\ u(x) = c & \text{for } |x| = 1 \end{cases} \quad x \in \mathbb{R}^3.$$

3. (i) State the Huygens Principle. How does it depend on the space dimension?

(ii) Define the energy integral $E(t)$ for the wave equation

$$\begin{cases} u_{tt} & = c^2 \Delta u \\ u(0,x) & = \phi(x), \quad u_t(0,x) = \psi(x) \\ u(t,x)|_{x\in\partial\Omega} & = 0 \end{cases}$$

Show that the $E'(t) = 0$.

4. (i) For $u \in L^1(\Omega)$ define the weak derivative $D_i u$. $(1 \le i \le n)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

(ii) Define the Sobolev space $W^{2,p}(\Omega)$ where $p \ge 1$ and give the definition of the norm $\| u \|_{W^{2,p}}$.

5. A semidisc of radius a is well isolated from the surrounding medium. The temperature at ADB and segment $\overline{AB}$ are kept at $u = c_2$ and $u = c_1$ respectively. Find the stationary temperature distribution u in the semidisc. (See Fig. 1.)

6. Solve the following equation in the upper half-plane

$$\begin{cases} \Delta u(x,y) & = 0, \quad -\infty < x < \infty, \quad 0 < y < \infty \\ u(x,0) & = f(x) \end{cases}$$

7. (i) Give the definition of the well posed problem in the sense of Hadamard.

(ii) Prove that the solution of the Dirichlet problem

$$\begin{cases} \Delta u & = 0 \\ u|_{\partial\Omega} & = f(x) \end{cases}$$

depends on f continuously.

8. Solve the following equation

$$\begin{cases} xu_y - yu_x & = u \\ u(x,0) & = h(x) \end{cases} \quad (x,y) \in \mathbb{R}^2$$

where $u = u(x,y), h$ is a given function.