Topology Qualifying Exam
Fall 1992

---

1. Prove that the interval [0,1] with the usual topology is connected.

2. Prove or disprove: Every compact Hausdorff space is separable.

3. Let $\pi_1 : X \times Y \to X$ be the projection map, and let Y be compact. Prove π_1 is a closed map (where $X \times Y$ has the product topology).

4. Prove that a paracompact Hausdorff space is regular.

5. Use Zorn's lemma to prove that for every set X and relation R there is maximal $A \subseteq X$ such that $A \times A \subseteq R$.

6. Prove that the plane $\mathbb{R}^2$ is not a countable union of straight lines.

7. If Ω is the first uncountable ordinal, prove that $[0, \Omega]$ with the order topology is compact.

8. Prove that a connected normal Hausdorff space containing more than one point is uncountable.

9. Let A be a connected subset of a connected space X, and let C be a component of $X - A$. Prove that $X - C$ is connected.

10. Let $C(X, Y)$ be the set of continuous functions from X to Y, given the compact-open topology. Let $e : C(X, Y) \times X \to Y$ be the evaluation map $e(f, x) = f(x)$. Prove that if X is locally compact Hausdorff, then e is continuous.

11. Let $f : X \to Y$ be a continuous surjective map from a compact space X to a Hausdorff space Y. Prove that f is a quotient map.

12. Find an incorrect statement in the proof of the following theorem and prove that it is an incorrect statement.

Theorem. If $B^2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 1\}$ has the usual topology, then each continuous function $f : B^2 \to B^2$ has a fixed point.

Proof: Suppose that $f : B^2 \to B^2$ is a continuous function with no fixed points. Let $\pi_1 : B^2 \times B^2 \to B^2$ be first projection, let

$$\Delta = \{(z, z) | z \in B^2\}$$

be the diagonal in $B^2 \times B^2$, and let

$$F = \{(z, f(z) | z \in B^2\}$$

be the graph of f. Since f is continuous, $\pi_1|_F : F \to B^2$ is a homeomorphism. Since B^2 is connected, F is therefore connected. However,

$$F \subseteq B^2 \times B^2 - \Delta$$

which is a contradiction since $B^2 \times B^2 - \Delta$ is disconnected.

13. Prove that the topologist's comb

$$C = \left(\bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\} \times [0, 1] \right) \cup ([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1])$$

is **not** a retract of the square $S = [0, 1] \times [0, 1]$ (with both C and S having the usual Euclidean subspace topologies).