Algebra Qualifying Exam
August 23, 2001

Instructions: You are given 10 problems from which you are to do 8. Please indicate those 8 problems which you would like to be graded by circling the problem numbers on the problem sheet.
Note: All rings on this exam are associative and have multiplicative identity 1. All integral domains are assumed to be commutative.

1. Show that for any group G, the quotient group $G/Z(G)$ is never a nontrivial cyclic group. Here, $Z(G)$ is the center of the group G.

2. Let F be a field, and show that the matrix group

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in F,\ ac \neq 0 \right\}$$

   is a solvable group.

3. Let F be any field and G be a finite multiplicative subgroup of $F^{\times}$. Prove that if $|G| > 1$, then $\sum_{g \in G} g = 0$ in F.

4. Let A be a commutative ring. Assume that every element a of A is either invertible or nilpotent (i.e., $a^n = 0$ for some n depending on a). Show that A has a unique maximal ideal.

5. Let R be a ring with 1 and M an R-module. An element $x \in M$ is called *torsion* if there exists $r \in R$ and $r \neq 0$ such that $rx = 0$. Let M_t be the set of all torsion elements in M. Show that if R is an integral domain then M_t is a submodule of M and M/M_t is a torsion-free R-module for any R-module M. Give an example of a commutative ring R and an R-module M such that M_t is not a submodule.

6. Let A be a commutative ring and M be a finitely generated A-module. One form of Nakayama Lemma says that

   if $M = N + IM$, where $N \subseteq M$ is an A-submodule of M, and where I is an ideal of A contained in every maximal ideal of A, then $M = N$.

   Now assume that A is a commutative local ring (i.e., A has a unique maximal ideal m), and assume that $f : E \to F$ is a homomorphism of A-modules. Therefore, $f(mE) \subseteq mF$ and so f induces a homomorphism $\bar{f} : E/mE \to F/mF$. Use Nakayama's Lemma to show that if F is finitely generated as an A-module, then f is surjective if and only if $\bar{f}$ is surjective.

7. Let k be a field and let A be an k-algebra. A k-linear transformation $D : A \to A$ is a called a k-*derivation* if

   $$D(xy) = D(x)y + xD(y), \quad \text{for all } x, y \in A.$$

   Show that if D_1 and D_2 are k-derivations on A, then the composition $D_1 \circ D_2$ need not be a k-derivation, but that $D_1 \circ D_2 - D_2 \circ D_1$ is always a k-derivation on A.

8. Let $F \supseteq k$ be a finite extension of degree n and $f(x) \in k[x]$ be an irreducible polynomial of degree m. If m and n are relatively prime, then $f(x)$, as a polynomial over F, is still irreducible.

9. Let k be a finite field of p^r elements. If $f(x)$ is an irreducible polynomial in $k[x]$, show that the field $F = k[x]/k[x]f(x)$ contains all roots of $f(x)$ and that the Galois group $\mathrm{Gal}(F/k)$ permutes the set of roots of $f(x)$ transitively.

10. Let $T : V \to V$ be a linear transformation on the n-dimensional complex vector space V. Give V the usual $\mathbb{C}[x]$-module structure. Suppose that V is isomorphic as a $\mathbb{C}[x]$-module to

$$\mathbb{C}[x]/\mathbb{C}[x]f_1(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_2(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_3(x) \oplus \mathbb{C}[x]/\mathbb{C}[x]f_4(x),$$

where

$$\begin{aligned} f_1(x) &= (x-2)^6(x-3)^7(x-4)^3, \\ f_2(x) &= (x-2)^7(x-3)^9(x-4)^3, \\ f_3(x) &= (x-2)^6(x-3)^7(x-4)^3, \\ f_4(x) &= (x-2)^5(x-3)^5(x-4)^2. \end{aligned}$$

Now do the following:

(a) Compute n.

(b) List the characteristic polynomial and the minimal polynomial of T.

(c) List the invariant factors of T.

(d) List the elementary divisors of T.

(e) Write done the Jordan canonical matrix of T.