Real Analysis Qualifying Exam
Spring 1987

The following numbered items form a list of logically related theorems, most depending on earlier ones. You are asked to *prove* as many of them as you can. In each case, you may use earlier ones in your proof, even if you did not suceed in proving the ones you cite.

For $f \in L_1(\mathbb{R})$ and $\mu \in M(\mathbb{R})$, define $\widehat{f}$ and $\widehat{\mu}$ on $\mathbb{R}$ by the rules

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-itx}dx$$

and

$$\widehat{\mu}(x) = \int_{-\infty}^{\infty} e^{-itx}d\mu(t).$$

1. For such f and μ, we have $\widehat{f} \in C_0(\mathbb{R})$ and $\widehat{\mu} \in C(\mathbb{R})$ with $\| \widehat{f} \|_u \leqq \| f \|_1$ and $\| \widehat{\mu} \|_u \leqq \| \mu \|$.

[HINT: First consider the case that f is the characteristic function of a bounded interval.]

2. If $a \in \mathbb{R}$ and $f(x) = e^{-|x|iax}$ for all $x \in \mathbb{R}$, then $f \in L_1(\mathbb{R})$ and

$$\widehat{f}(t) = \frac{2}{1+(t-a)^2} \text{ for all } t \in \mathbb{R}.$$

3. For $f, g \in L_2(\mathbb{R})$, the formula

$$f * g(x) = \int_{-\infty}^{\infty} f(x-u)g(u)du$$

defines a function $f * g$ at almost every $x \in \mathbb{R}$, $f * g \in L_1(\mathbb{R})$, and $\| f * g \|_1 \leq \| f \|_1 \| g \|_1$. [You may use without proof the measurability of the map $(x, u) \to f(x-u)g(u)$.]

4. If $f, g \in L_1(\mathbb{R})$, then $\widehat{f * g}(t) = \widehat{f}(t)\widehat{g}(t)$ for all $t \in \mathbb{R}$.

5. With pointwise operations, the set $\mathbb{A}(\mathbb{R}) = \{\widehat{f} : f \in L_1(\mathbb{R})\}$ is a dense subalgebra of $C_0(\mathbb{R})$ where $C_0(\mathbb{R})$ has the uniform norm.

6. If $f \in L_1(\mathbb{R})$, $\mu \in M(\mathbb{R})$, and $u \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f(x)\widehat{\mu}(x-u)dx = \int_{-\infty}^{\infty} \widehat{f}(t)e^{iut}d\mu(t).$$

7. If $\mu, \nu \in M(\mathbb{R})$ and $\widehat{\mu}(x) = \widehat{\nu}(x)$ for all $x \in \mathbb{R}$, then $\mu = \nu$.

For use in the next two problems, define $M_0(\mathbb{R}) = \{\mu \in M(\mathbb{R}) : \widehat{\mu} \in C_0(\mathbb{R})\}$.

8. If $\nu \in M(\mathbb{R})$, $\mu \in M_0(\mathbb{R})$, and $\nu << |\mu|$ (ν is absolutely continuous with respect to μ), then $\nu \in M_0(\mathbb{R})$.

[HINT: Approximate a Radon-Nikodyn derivative in $L_1(|\mu|)$ by an $\widehat{f} \in \mathbb{A}(\mathbb{R})$ and use 6.]

9. If $\mu \in M_0(\mathbb{R})$, then $\mu(\{a\}) = 0$ for all $a \in \mathbb{R}$. [HINT: Otherwise $\delta_a << \mu$.]

10. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is bounded and differentiable on $\mathbb{R}$ with f' also bounded on $\mathbb{R}$. Then for all $g \in L_1(\mathbb{R})$ the formula in 3 defines a function $f * g$ at each $x \in \mathbb{R}$, $f * g$ is differentiable at every point of $\mathbb{R}$, and $(f * g)' = f' * g$.