Real Analysis Qualifying Exam
Fall 1991

In what follows $(X, \mathcal{A}, \mu)$ is an arbitrary measure space and λ is Lebesgue's outer measure on $\mathbb{R}$.

1. (a) What does it mean to say that a function $f : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable?

(b) Use your definition given in (a) to prove that if f and g are two such functions, then

$$\{x \in X : f(x) < g(x)\} \in \mathcal{A}$$

and

$$\{x \in X : f(x) = g(x)\} \in \mathcal{A}.$$

2. Suppose $f : X \to [0, \infty]$ is $\mathcal{A}$-measurable. Discuss how $\int_X f d\mu$ is defined. You should suppose your reader knows about measure spaces and measurable functions, but has never heard of simple functions or integrals.

3. What is meant by (a) σ-algebra, (b) Borel set, (c) Lebesgue measurable set? How are these three concepts related?

4. If $\mu(X) = 1$ and $0 < p < q < \infty$, prove that $\| f \|_p \leqq \| f \|_q$ for every $\mathcal{A}$-measurable $f : X \to \mathbb{C}$. [Hint: Apply Hölder's Inequality with $r = q/p > 1$.]

5. Prove (a) Fatou's Lemma by applying the Monotone Convergence Theorem, and also (b) the Monotone Convergence Theorem by applying Fatou's Lemma. Begin by clearly stating both results.

6. Prove the completeness of $Lp(\mu)$ for $1 \leq p < \infty$.

7. Suppose $(f_n)_1^\infty$ is a sequence in $L_1(\mu)$, $f \in L_1(\mu)$, and $\| f_n - f \|_1 \to 0$. Prove that

$$\lim_{n\to\infty} \sup \int \log |f_n| d\mu \leq \int \log |f| d\mu,$$

where $\log 0 = -\infty$. [Hints: First show that $(f_n)_1^\infty$ has a subsequence with converges pointwise a.e. and then apply Fatou's Lemma to an appropriate subsequence of $(|f_n| - \log |f_n|)_1^\infty$.]

8. Suppose μ is a σ-finite measure, and $\mathcal{C}$ is a subcollection of $\mathcal{A}$. Prove that there exists $G_0 \in \mathcal{A}$ such that

(i) If $B \in \mathcal{C}$, then $B \subset G_0$ a.e., that is, $\mu(B \backslash G_0) = 0$;

(ii) If $G \in \mathcal{A}$ is another set satisfying (i), then $G_0 \subset G$ a.e.

[Hints: Let $\mathcal{C}^\sim$ be the collection of all countable unions of sets chosen from $\mathcal{C}$. For each $E \in \mathcal{A}$ with $\mu(E) < \infty$, show that $\sup\{\mu(B \cap E) : B \in \mathcal{C}^\sim\}$ is attained at some $G = G_E \in \mathcal{C}^\sim$. Finally consider an appropriate countable union of such G's.]

9. Let $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\}$ and define f on S by

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Use Fubini's Theorem to prove that f is not Lebesgue integrable over S.

[Hint: $\frac{\partial}{\partial y} \frac{y}{x^2+y^2} = f(x, y)$.]