TOPOLOGY QUALIFYING EXAM   Fall 1995
(Crane and Wu)

Work out problem **1** and then choose **4** (and only 4) additional problems among the remaining ones. Start each problem on a new sheet of paper. **Do not turn in more than 4 additional problems.** A space always means a topological space below.

This one is required!

1. Answer the following questions and give a brief explanation or counterexample:

i) What is the smallest possible cardinality of a noncompact Hausdorff space?

ii) (a) Is the disjoint union of finitely many compact spaces compact?
(b) Is the Cartesian product of finitely many compact spaces compact?
(c) Is the Cartesian product of any family of compact spaces compact?
(d) Is the intersection of any family of compact subsets of a Hausdorff space compact?

iii) Is the intesection of two connected spaces connected?

iv) Is the Cartesian product of finitely many connected spaces connected?

v) Is the Cartesian product of uncountably many connected spaces connected?

vi) Is every space metrizable?

vii) Is the inverse of any continuous bijective function continuous?

viii) Is the composite of two continuous functions continuous?

ix) What is a quotient map?

x) Is a quotient map always an open map?

Choose 4 problems from below:

2. Let $S := \left\{(x,y) \in \mathbb{R}^2 \mid y = \sin\frac{1}{x}, x > 0\right\} \subset \mathbb{R}^2$. The closure $\overline{S}$ of S (in $\mathbb{R}^2$) is called the **topologist's sine curve**. It is endowed with the subspace topology. Show that the topologist's sine curve is connected, but not path connected. Is $\overline{S}$ locally connected?

3. Let $f, g : X \to Y$ be continuous. Assume that Y is Hausdorff. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.

4. Let (X, d) be a metric space. Show that the following are equivalent:

i) X has a countable basis;

ii) X is Lindelöf;

iii) X has a countable dense subset.

5. Prove or disprove the following statements:

i) The continuous image of a locally connected space is locally connected;

ii) The quotient of a locally connected space is locally connected.

6. Let $f : X \to Y$ be a map, where Y is a compact Hausdorff space and X any topological space. Show that f is continuous iff the graph of f,

$$G_f := \{(x, f(x)) \mid x \in X\} \subset X \times Y$$

is closed in $X \times Y$.

7. Show that if Y is compact,then the projection map

$$\pi_1 : X \times Y \longrightarrow X$$

is a closed map. Is the compactness of Y necessary?

8. Let X be a regular space with a countable basis and U a non-empty open subset. Show that

i) U is a countable union of closed subsets of X;

ii) There is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.

9. A space X is called *locally metrizable* if each point x of X has a neighborhood that is metrizable in the subspace topology. Show that a regular Lindelöf space is metrizable if it is locally metrizable.

10. Let $f : X \to \mathbb{R}$ be a continuous real-valued function on the compact Hausdorff space X. Show that every nonempty open subset of X contains a nonempty open set on which f is bounded.

11. Prove or disprove the following statements:
(a) $\mathbb{R}^2$ (with the standard topology) is a countable union of closed subsets each having empty interior.
(b) $\mathbb{R}^2$ (with the standard topology) is a countable union of subsets each having empty interior.

12. Let Y be a metric space and $f_n : X \to Y$ a sequence of continuous functions and $f : X \to Y$ a function (not necessariy continuous). Suppose $\{f_n\}$ is equicontinuous and $f_n(x) \to f(x)$ for each $x \in X$. Show that f is continuous.