Kansas State University
Department of Mathematics

Real and Complex Analysis Qualifying Exam — Spring 2013

Notation: $\mathbb{N} := \{1, 2, 3, ...\}$, $\mathbb{R} :=$ the real numbers and $\mathbb{C} :=$ the complex numbers, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, Ω is a non-empty open connected subset of $\mathbb{C}$, and $H(\Omega)$ is the set of all holomorphic functions in Ω.

1. (i) Show that each $f \in H(\mathbb{D})$ satisfies

$$f(z) - f(0) = \int_0^1 z f'(tz) dt$$

for every $z \in \mathbb{D}$.

Hint: Is there a derivative with respect to t present?

(ii) If $F \in H(\Omega)$ and $F' = 0$, show that F is constant.

Hint: Fixing $z_0 \in \Omega$, use (i) to show that $\{z \in \Omega : F(z) = F(z_0)\}$ is open.

2. Suppose f is continuous in $\Omega \setminus \{0\}$ and $e^{f(z)} = z$ for every $z \in \Omega \setminus \{0\}$. Show that f is holomorphic in $\Omega \setminus \{0\}$ and compute f'.

Hint: f is necessarily one-to-one (why?) so given $z_0, z \in \Omega \setminus \{0\}$ with $z \neq z_0$, we may write

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{\frac{e^{f(z)} - e^{f(z_0)}}{f(z) - f(z_0)}}.$$

3. For some $\alpha > 0$, $S =: \{re^{i\theta} : r > 0, 0 < \theta < \alpha\}$, $f \in H(S)$, is bounded. Show that $\lim_{r \to \infty} f'(re^{i\theta}) = 0$ for each $0 < \theta < \alpha$.

Hint: Represent f' by Cauchy's integral formula in large disks with far-away centers.

4. Compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$$

where a, b are positive real numbers.

Hint: Integrate over large semicircles and use the Residue Theorem.

5. Let g be a bounded Lebesgue measurable function on $\mathbb{R}$ which has the property that $\lim_{n \to \infty} \int_I g(nx) dx = 0$ for every interval $I \subset [0, 1]$. Prove that for every $f \in L^1([0, 1])$,

$$\lim_{n \to \infty} \int_0^1 f(x) g(nx) dx = 0.$$

6. Let $f_n(x) = e^{inx}$. Prove

(a) $\{f_n\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2([-\pi,\pi])$ with respect to the measure $\dfrac{dx}{2\pi}$.
(b) No subsequence of $\{f_n\}_{n=1}^{\infty}$ converges pointwise a.e. on $[-\pi,\pi]$.

7. Let $\mathcal{F}$ be a collection of subsets of a set Ω with the following properties:
(i) $\Omega \in \mathcal{F}$
(ii) A,$B \in \mathcal{F}$ implies $A \setminus B \in \mathcal{F}$
(iii) $A_1, A_2, \dots \in \mathcal{F}$ and $A_1 \subset A_2 \subset A_3 \dots$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Prove that $\mathcal{F}$ is a σ-algebra.

8. For $\alpha > 0$ define the function f_α on $[0,1]$ by $f_\alpha(x) = x^\alpha \sin\frac{1}{x}$ if $x \neq 0$, $f_\alpha(x) = 0$ if $x = 0$. For what values of α is the function f_α of bounded variation?