Real Analysis Qualifying Exam
Spring 1991

In what follows, $(X, \mathcal{A}, \mu)$ is an arbitrary measure space and λ is Lebesgue measure on $\mathbb{R}$.

1. (a) What does it mean to say that a function $f : X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable?

(b) Prove that if $\mathcal{F}$ is a countable, nonvoid set of such functions f and if

$$g(x) = \sup\{f(x) : f \in \mathcal{F}\}$$

for each $x \in X$, then g is $\mathcal{A}$-measurable.

(c) Give an example of $X, \mathcal{A}$, and $\mathcal{F}$ to show that the assertion in (b) can fail if "countable" is omitted.

2. Let $(E_n)_{n=1}^{\infty} \subset \mathcal{A}$ and define

$$E = \{x \in X : \{n \in \mathbb{N} : x \in E_n\} \text{ is infinite }.$$

Prove that $E \in \mathcal{A}$ and if $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, then $\mu(E) = 0$. [Hint: Consider the sets $A_j = \cup_{n=j}^{\infty} E_n$.]

3. Let $f \in L_1(\mu)$ and $\varepsilon > 0$. Prove that there is some $\delta > 0$ such that

$$A \in \mathcal{A}, \mu(A) < \delta \Rightarrow |\int_A f d\mu| < \varepsilon.$$

4. Suppose $f \in L_p(\mathbb{R})$ and $p > 0$. Prove that

$$\lim_{t \to 0} \int_{\mathbb{R}} |f(x+t) - f(x)|^p dx = 0.$$

[Hint: Fist approximate f with a continuous function having compact support.]

5. Suppose $\mu(X) < \infty, f : X \to [0, \infty]$ is $\mathcal{A}$-measurable, $\int_X f d\mu < \infty$, and $\mathcal{B}$ is a sub-σ-algebra of $\mathcal{A}$. Prove that there exists a $\mathcal{B}$-measurable $g : X \to [0, \infty]$ such that

$$\int_B g d\mu = \int_B f d\mu \quad \forall B \in \mathcal{B}.$$

Prove also that

$$\int_X hg d\mu = \int_X hf d\mu$$

whenever $h : X \to [0, \infty]$ is $\mathcal{B}$-measurable.

6. Suppose $(g_n)_{n=1}^{\infty} \subset L_1([0,1])$, $g_n \geq 0$ a.e. $\forall n$, and the sequence $(\int_0^1 f g_n d\lambda)_{n=1}^{\infty}$ converges $\forall f \in C([0,1])$. Prove that there is a Borel measure ν on [0,1] such that

$$\lim_{n \to \infty} \int_0^1 f g_n d\lambda = \int_{[0,1]} f d\nu \quad \forall f \in C([0,1]).$$

7. For $f \in L_1(\mathbb{R})$, define its Fourier transform $\widehat{f}$ on $\mathbb{R}$ by $\widehat{f}(t) = \int_{\mathbb{R}} f(x) e^{-itx} dx$. Prove that if $f, g \in L_1(\mathbb{R})$, then

(a) $\widehat{f}$ is continuous on $\mathbb{R}$,

(b) $\widehat{f}$ is bounded,

(c) $\lim_{|t|\to\infty} \widehat{f}(t) = 0$ [Hint: First suppose f is a step function.],

(d) $\int_{\mathbb{R}} f(x)\widehat{g}(x)dx = \int_{\mathbb{R}} \widehat{f}(t)g(t)dt$.

8. Suppose that $g : \mathbb{R} \to \mathbb{C}$ is measurable and

$$\int_{\mathbb{R}} (1 + |y|)|g(y)|dy < \infty.$$

Define f on $\mathbb{R}$ by

$$f(x) = \int_{\mathbb{R}} g(y)\cos(xy)dy.$$

Prove that f is differentiable on $\mathbb{R}$ and

$$f'(x) = -\int_{\mathbb{R}} yg(y)\sin(xy)dy$$

for every $x \in \mathbb{R}$.

9. Give an explicit example of a Borel measure σ on $\mathbb{R}$ for which

(a) $\sigma(\mathbb{R}) = 1$,

(b) $\sigma(\{x\}) = 0 \forall x \in \mathbb{R}$, and

(c) for some compact set $P \subset \mathbb{R}$ we have $\lambda(P) = 0$ and $\sigma(P) = 1$ where λ is Lebesgue measure on $\mathbb{R}$.

10. Suppose that $f : [a, b] \to \mathbb{C}$ is absolutely continuous. Prove that the total variation $V_a^b f$ over $[a, b]$ of f is given by

$$V_a^b f = \int_a^b |f'(x)|dx.$$

[Hint: Approximate with appropriate step functions.]