Qualifying Examination in Differential Equations, Fall '96

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(Try to get as many points as you can.)

 # 1. 1) Find all the solutions of the equations:

(a) (*1 point*) $x\,\frac{\partial u}{\partial y} + y\,\frac{\partial u}{\partial x} = 0\,;$

(b) (*2 points*) $x\,\frac{\partial u}{\partial y} + y\,\frac{\partial u}{\partial x} = 1\,;$

(c) (*2 points*) $x\,\frac{\partial u}{\partial y} + y\,\frac{\partial u}{\partial x} = u\,;$

2) (*2 points*) Solve the Cauchy problem:

$$x\,\frac{\partial u}{\partial y} + y\,\frac{\partial u}{\partial x} = u\,, \qquad u(x,y) = \sin(x-y) \qquad \text{on the line} \quad y = 1-x\,.$$

 # 2. (*4 points*) What is a *well-posed* problem? Give an example.

 # 3. (*4 points*) State a maximum principle for the solutions of the heat equation.

 # 4. (a) (*2 points*) Give an example of a function which is not differentialble in the classical sense but has a distributional (generalized) derivative in L^2_{loc}.

(b) (*3 points*) Give a definition of the Sobolev space $W^{m,p}(\Omega)$ (the same as $W^m_p(\Omega)$), where $1 \le p \le +\infty$, m is a positive integer, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary.

 # 5. (*5 points*) Find the solution of the initial boundary value problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial x} = 0\,, \qquad t > 0, \quad 0 < x < 1\,, \\ &u(t,0) = u(t,1) = 0, \\ &u(0,x) = e^{-x}\,\sin(11\pi x)\,. \end{aligned}$$

 # 6. (*5 points*) Solve the following boundary value problem in the square $\Omega = \{(x,y):\ 0 < x,\, y < 1\}$:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\,\pi^2\,u = \sin(\pi x)\,\sin(\pi y)\,,\ (x,y) \in \Omega, \qquad u\Big|_{(x,y)\in\partial\Omega} = 0\,.$$

 # 7. (*5 points*) Let $u(x)$ be the solution of the following boundary value problem in the disk $\Omega = \{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < 1\}$:

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = 0\,, \quad (x,y) \in \Omega; \qquad u(x,y)\Big|_{(x,y)\in\partial\Omega} = \frac{y}{\sqrt{|x|^2+|y|^2}}\,.$$

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Without solving the problem, show that $-1 \le u(x,y) \le 1$ for all $(x,y) \in \Omega$.

 # 8. (*5 points*) For a solution $\phi(t,x)$ of the Cauchy problem

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \phi(0,x) = f(x), \quad \frac{\partial \phi}{\partial t}(0,x) = 0,$$

show that if $f \in L^p(\mathbb{R}^1)$, for some p, $1 \le p \le +\infty$, then, $\phi(t,\cdot) \in L^p(\mathbb{R}^1)$, as well, and estimate $\sup_t \|\phi(t,\cdot)\|_{L^p}$ in terms of the L^p-norm of f.

 # 9. (*10 points*) Consider the following hyperbolic equation on $\mathbb{R}^1$:

$$\frac{\partial^2 \phi}{\partial t^2} - (1 - \frac{1}{2}\cos^2 x)\frac{\partial^2 \phi}{\partial x^2} = 0\,.$$

Let u and v be the solutions of this equation corresponding to the initial conditions

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = u_1(x)\,,$$

and

$$v(0,x) = v_0(x), \quad \frac{\partial v}{\partial t}(0,x) = v_1(x)\,.$$

Show that if

$$u_0(x) = v_0(x), \qquad u_1(x) = v_1(x)\,, \qquad \text{for all } x \text{ in the interval } [-1,1]\,,$$

then $u(t,x) = v(t,x)$ for all (t,x) in the triangle $\{0 \le t \le 1,\ |x| \le 1-t\}$. (You may assume that the solutions u and v, and all the initial data, are as smooth as you wish.)

 # 10. (*10 points*) Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Consider the following initial boundary value problem:

$$\begin{aligned} u_t &= a^2\,\Delta u \quad t>0,\ x\in\Omega \\ u(0,x) &= f(x), \quad x\in\Omega \\ \tau\,\frac{\partial u}{\partial \nu} + \sigma\,u &= 0, \quad (t,x)\in[0,T]\times\partial\Omega\,, \end{aligned}$$

where a, τ and σ are positive constants. Show that

$$\int_\Omega |u(t,x)|^2\,dx$$

does not increase with t. Use this fact to prove the uniqueness of the (appropriately smooth) solutions.

Show that the energy,

$$E(t) = \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} \frac{\sigma}{\tau}\,|u|^2\,ds\,,$$

does not increase with t.