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Topology Qualifying Exam
Old System – Fall 2003
Miller & Strecker

Choose and work any 7 of the following 11 problems. Start each problem on a new sheet of paper. **Do not turn in more than seven problems.** A "space" always means a topological space below.

1. Prove the equivalence of any two of the following three statements:

a) The Axiom of Choice.

b) If each linearly ordered subset of a nonempty partially ordered set $(X, \leq)$ has an upper bound, then $(X, \leq)$ has a maximal element.

c) The cartesian product of any set of nonempty sets is nonempty.

2. Suppose that every point of a topological space X has a path connected open neighborhood. Show that the path connected components of X coincide with the connected components of X.

3. Prove or disprove (with a counterexample) each of the following:

a) If A is a connected subset of a space X and $Q \subseteq X$ such that $A \subseteq Q \subseteq \overline{A}$, then Q must be connected.

b) Each component of a space must be closed.

c) Each component of a space must be open.

4. a) Suppose that X is a topological space and $\{O_j | \ 1 \leq j \leq n\}$ is a finite collection of open dense subsets of X. Show that $\displaystyle\bigcap_{j=1}^{n} O_j$ is dense in X.

b) Now suppose that X is locally compact and Hausdorff and that $(O_n)_{n \in Z_+}$ is a countable collection of open dense subsets of X. Show that $\displaystyle\bigcap_{n \in Z_+} O_n$ is dense in X.

5. Given an example of a first countable Hausdorff space which is not metrizable. Of course, you must prove that your space has the desired properties.

6. Let X be a topological space and R an equivalence relation on X. Suppose that $C \subseteq X$ closed implies that $R[C] = \{x \in X | \exists c \in C \text{ such that } cRx\}$ is closed. Suppose that X is normal. Show that X/R with the quotient topology is normal.

7. If Ω is the first uncountable ordinal, prove that the interval of ordinals $[0, \Omega]$ with the order topology is a compact Hausdorff space.

8. Suppose that $p : E \to X$ is a covering map and that E is simply connected. For $x_0 \in X$ construct a lifting correspondence $\Phi : \pi_1(X, x_0) \to p^{-1}[\{x_0\}]$ and prove that it is a bijection.

9. Let S^2 denote the 2-dimensional sphere with its usual topology and let $x_1, x_2, \ldots, x_n$ be n distinct points on S^2. Determine the fundamental group of $S^2 - \{x_1, x_2, \ldots, x_n\}$. You must justify your answer.

10. State the Urysohn Lemma and use it to give a complete proof that a space is regular and second countable iff it can be embedded as a supspace of the Hilbert cube $= I^\omega$ (where I is the unit interval).

11. Suppose X is a topological space and $X = \displaystyle\bigcup_{n=1}^{\infty} S_n$ where each S_n with the subspace topology is homeomorphic with the unit circle. Furthermore, suppose

a) $\exists x_0 \in X$ such that $S_m \cap S_n = \{x_0\}$ for $m \neq n$.

b) $C \subseteq X$ is closed in X if and only if $C \cap S_n$ is closed in S_n $\forall n$.

Show that X is not first countable.