Real Analysis Qualifying Exam
Spring 1990

In what follows, $(X, \mathcal{A}, \mu)$ is a measure space and λ is Lebesgue measure on $\mathbb{R}$.

1. Suppose $f, g : X \to \mathbb{R}$ are $\mathcal{A}$-measurable.

(a) What does it mean to say that f is $\mathcal{A}$-measurable?

(b) Use your definition to prove that $f + g$ is also $\mathcal{A}$-measurable.

2. Suppose that $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable, $\mu(X) < \infty, |f(x)| \leq 1 \forall x \in X$, and $\int f d\mu = \mu(X)$. Prove that $f(x) = 1 \mu$-a.e.

3. Let $f \in L_1(\mu)$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$A \in \mathcal{A} \text{ and } \mu(A) < \delta \Rightarrow \int_A |f| d\mu < \varepsilon.$$

4. Apply the Monotone Convergence Theorem to prove Fatou's Lemma.

5. Suppose $(X, \mathcal{A}, \mu)$ is σ-finite and $f : X \to [0, \infty]$ is $\mathcal{A}$-measurable. Let

$$E = \{(x, t) \in X \times \mathbb{R} : 0 < t < f(x)\}.$$

Prove that $E \in \mathcal{A} \times \mathcal{B}(\mathbb{R})$ and $\mu \times \lambda(E) = \int f d\mu$.

6. Give examples of each of the following:

(a) A closed subset F of $[0, 1]$ that contains no rational number and had $\lambda(F) > 0$.

(b) A Borel measurable $f : [0, 1] \to [0, 1]$ such that if $g : [0, 1] \to \mathbb{R}$ is Riemann integrable, then

$$\int_0^{-1} |f(x) - g(x)| dx > 0.$$

7. Suppose $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable and $\int_a^x f(t) dt = 0$ for all $x \in [a, b]$. Prove that $f = 0$ a.e. on $[a, b]$.

8. Let $a < b$ in $\mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be increasing on $[a, b]$. Presuming it is known that f is differentiable a.e., prove that f' is Borel measurable and

$$\int_a^b f'(x) dx \leq f(b) - f(a).$$

[Hint: Extend f and use Fatou's Lemma.]