Real and Complex Analysis Qualifying Exam.

New System-August 2004

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Instructions: The exam consists of 8 problems. Each problem is worth 10 points.
Time: 3 hours.
Notation: $\mathbf{R}$ = reals, $\mathbf{C}$ = complexes, $\mathbf{Q}$ = rationals, $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, Ω is a region = open, connected subset of $\mathbf{R^2} = \mathbf{C}$, $\mathbf{H}(\Omega)$ the set of holomorphic functions in Ω, $C(\bar{\Omega})$ the set of continuous $\mathbf{C}$-valued functions on the closure of Ω. $|E|$ is the Lebesgue measure of $E \subset \mathbf{R}$, 1_E is the indicator function of $E \subset \mathbf{R}$ and $L^1(\mathbf{R^2})$ refers to planar Lebesgue measure.

1. **Problem 1**. Compute the Lebesgue integrals.

   a)
   $$\int_0^\infty e^{-[x]}dx,$$
   where $[x]$ stands for the integer part of x,

   b)
   $$\int_0^{\pi/2} f(x)dx,$$
   where
   $$f(x) := \begin{cases} \cos x & \text{for } x \in \mathbf{R} \setminus \mathbf{Q} \\ \sin x & \text{for } x \in \mathbf{Q}. \end{cases}$$

2. **Problem 2**.

   a) Does
   $$f(x) := \begin{cases} x \sin \frac{1}{x} & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$$
   have bounded variation?

   Hint: Consider
   $$\sum_{l=1}^{N} |f(x_{2l+1}) - f(x_{2l})|$$
   for appropriate subdivisions.

   b) Compute $Var_0^{50}(e^x)$, that is, the total variation of the function e^x over $[0, 50]$.

3. **Problem 3**. $f \in \mathbf{H}(\mathbf{D} \setminus \{0\})$ and the point 0 is either a pole of order $k \in \mathbf{N}$ or a removable singularity whose removal results in a zero of order k. Show that 0 is a first-order pole of f'/f and that the residue is $-k$ or k, respectively.

   Hint: Factor f.

4. **Problem 4**. $f \in \mathbf{H}(\mathbf{C})$ is not constant. Prove that $f(\mathbf{C})$ must be dense in $\mathbf{C}$.

   Hint: Remember the canonical proof of the Casorati-Weierstrass Theorem.

5. **Problem 5**.

   a) Ω is a bounded region, $f \in C(\bar{\Omega}) \cap \mathbf{H}(\Omega)$, f is zero-free, $|f|$ constant on $\partial\Omega$. Prove that f is constant.

   Hint: $1/f$ satisfies the same hypothesis.

   b) Can we drop the assumption of boundedness of Ω?

6. **Problem 6**. Does there exist a set $E \subset \mathbf{R}$, such that for every interval $I \subset \mathbf{R}$, we have
$$\frac{|E \cap I|}{|I|} = \frac{1}{2} \ ?$$

   Hint: Use the Lebesgue differentiation theorem on function 1_E.

7. **Problem 7**.

   a) Prove that
$$\int_0^1 dx \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dy \neq \int_0^1 dy \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} dx.$$

   Hint: Use
$$\frac{x^2 - y^2}{(x^2+y^2)^2} = \frac{\partial}{\partial y}\Big(\frac{y}{x^2+y^2}\Big).$$

   b) To save Fubini's theorem, use polar coordinates to give a direct proof that the area integral
$$\int_0^1 \int_0^1 \frac{|x^2 - y^2|}{(x^2+y^2)^2} dx dy$$
   is not finite.

8. **Problem 8**. $f \in \mathbf{H}(\mathbf{C}) \cap L^1(\mathbf{R^2})$. Prove that $f \equiv 0$.

   Hint: Justify the identity
$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$
   and integrate it with respect to rdr to relate the size of f with its $L^1(\mathbf{R^2})$-norm.