COMPLEX VARIABLES QUALIFYING EXAM
Fall 1997
(Burckel and Bennett)

1. f **is holomorphic in** $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$**.**

(i) Show that

$$F(z) := \overline{f(\overline{z})}, \quad z \in \mathbb{D}$$

is also holomorphic in $\mathbb{D}$**.**

(ii) Show that if f **is real-valued on** $\mathbb{D} \cap \mathbb{R}$**, then**

$$\overline{f(z)} = f(\overline{z}) \qquad \forall z \in \mathbb{D}.$$

(iii) If g **is holomorphic in all of** $\mathbb{C}$ **and real-valued on** $[-1, 1]$**, does it follow as in (ii) that**

$$\overline{g(z)} = g(\overline{z}) \qquad \forall z \in \mathbb{C} \quad ?$$

2. f **is holomorphic and zero-free in the region** Ω**,** $z_0 \in \Omega$**, and for each** $z \in \Omega$**,** γ_z **is a piecewise-smooth curve in** Ω **joining** z_0 **to** z**. A function** L_f **is then well defined by**

$$L_f(z) := \int_{\gamma_z} f'/f, \qquad z \in \Omega.$$

(i) Show that in case $\Omega = \mathbb{D}$ **this function is holomorphic and satisfies**

$$L_f' = f'/f.$$

(ii) Infer that $f = f(z_0)e^{L_f}$**.**

(iii) For what other Ω **besides** $\mathbb{D}$ **does every zero-free holomorphic function** f **on** Ω **satisfy (i) and (ii)?**

(iv) Show that in $\Omega := \{z \in \mathbb{D} : 1/2 < |z| < 2\}$ **with** $z_0 := 1$**, say, conclusion (i) may fail. Show in fact that for an appropriate zero-free holomorphic** f**, the function** f'/f **has no primitive.**

[It is an interesting fact, which you need not prove, that (ii) holds for every Ω**.]**

3. Show that $\zeta(z) := \sum_{n=1}^{\infty} n^{-z}$ **converges and defines a holomorphic function in Re**$(z) > 1$**.**

4. Let U be a domain in $\mathbb{C}$, $\mathcal{F} \subset C(U)$ an equicontinuous family, $(f_n) \subset \mathcal{F}$ a pointwise convergent sequence. Show that this sequence converges uniformly on each compact subset of U.

5. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, r, R positive real numbers.

(i) State Schwarz' Lemma for holomorphic maps $f : \mathbb{D} \to \mathbb{D}$, and formulate a version for maps from $r \cdot \mathbb{D}$ into $R \cdot \mathbb{D}$.

(ii) Use (i) to prove Liouville's theorem about entire functions.

6. Show that $\displaystyle\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{2\pi}}{4}$.

Hint: Use the contour

7. A *Möbius transformation* is a mapping from the extended complex plane into itself of the form $f(z) := \frac{az+b}{cz+d} \quad (a, b, c, d \in \mathbb{C},\ ad - bc \neq 0)$.

(i) Show that the set of all Möbius transformations forms a group under composition.

A Möbius transformation is called *hyperbolic* if it is conjugate to a *dilation* (i.e., to a g of form $g(z) = rz$, $r > 0$).

(ii) Show that every hyperbolic transformation f has two distinct fixed-points and that $f(C) = C$ for every circle C that passes through both these fixed-points.