Relationship of the Hennings and Chern-Simons Invariant For Higher Rank Quantum Groups

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Abstract

The Hennings Invariant for the small quantum group associated to an arbitrary simple Lie algebra at a root of unity is shown to agree with the Chern-Simons (aka Jones-Witten or Reshetikhin-Turaev) invariant for the same Lie algebra and the same root of unity for all integer homology three-spheres, at roots of unity where both are defined. This partially generalizes work of Chen, Yu, Zhang and al.[1] which relates the Hennings and Chern Simons invariants for $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ for arbitrary rational homology three-spheres. It also complements recent work of Habiro and Le [2] equating the Chern-Simons invariant at roots of unity on integral homology three-spheres to a Hennings like invariant defined as a power series.

1. Introduction

Definition 1 *A **bottom tangle** is an embedding of leftward oriented arcs into $\mathbb{R}\times[0,1)$ such that each component begins and ends before the component to its left begins.*

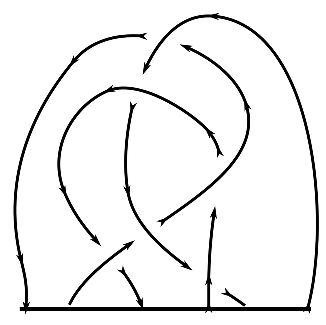


Figure 1: *A bottom tangle with 3 components.*

These bottom tangles are considered up to continuous deformation under the Reidemeister moves:

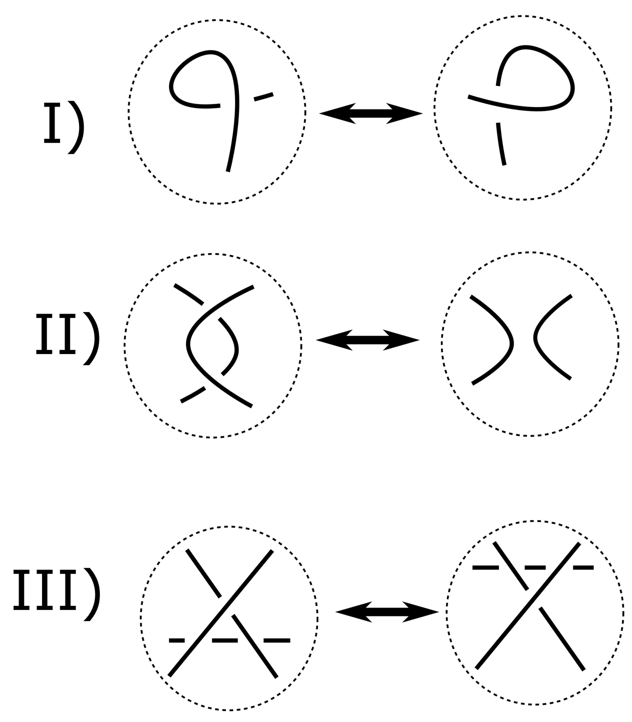


Figure 2: *Notice in Reidemeister I, a single twist cannot be undone. Consecutive oppositely oriented twists can be undone, however.*

Crossings carry a natural definition of **positive** and **negative**, defined below:

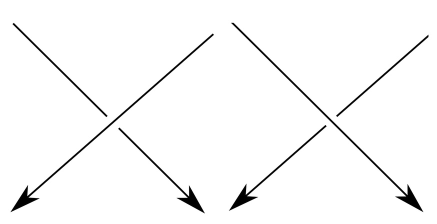
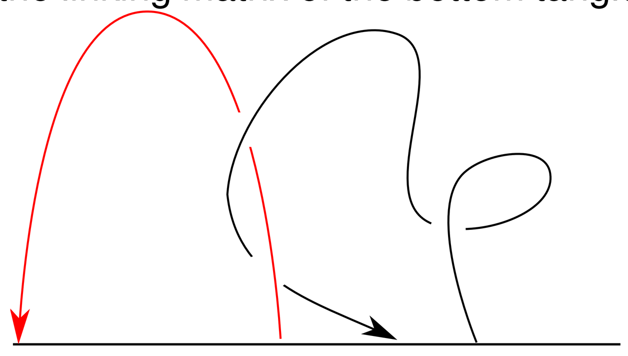


Figure 3: *Left image is positive, while right image is negative.*

Definition 2 *The **linking matrix** of an n-component tangle is an $n\times n$ matrix where the entry a_{ij} is the number of positive crossings minus the number of negative crossings between the i^{th} and j^{th} components which is then divided by 2 if $i\neq j$, but left alone if $i=j$.*

For example the linking matrix of the bottom tangle below is



$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

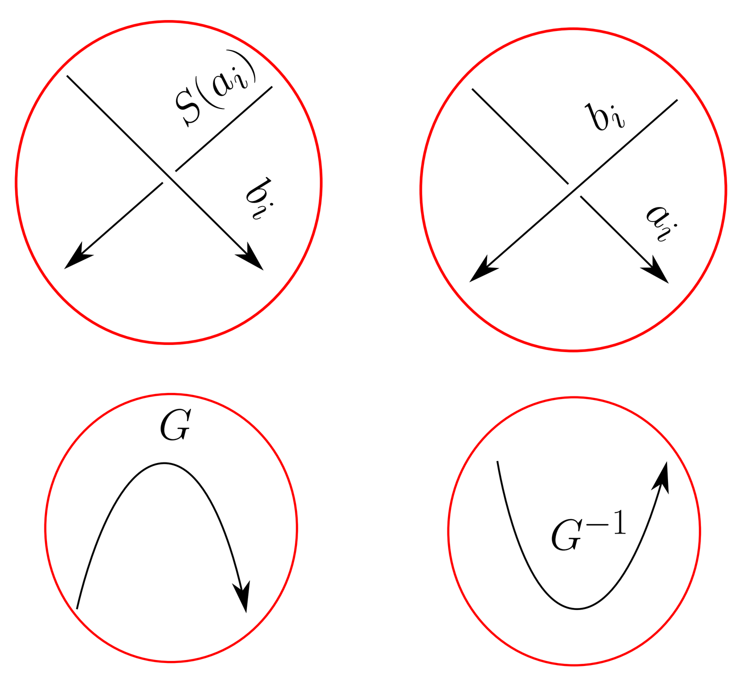
because one component has a positive twist in it, thus contributing the 1 to the matrix, the two -1's are due to the fact that the two components cross twice with negative crossings.

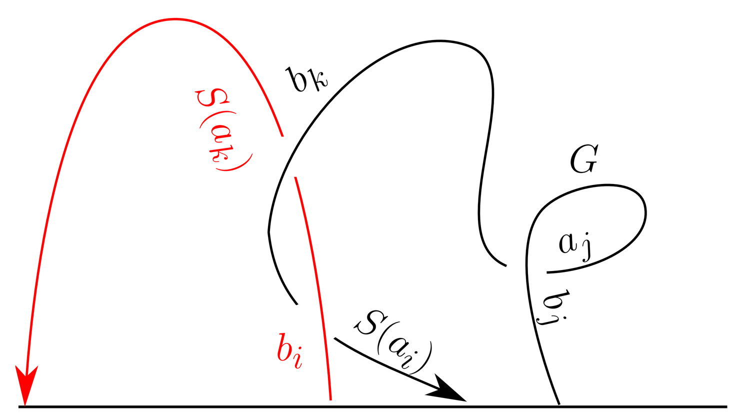
For our work we also need the definition of a zero linking matrix tangle or ZLMT for short.

Definition 3 *A **ZLMT** is a bottom tangle with all zero entries in its linking matrix (the above borromean tangle is an example). Note this does not imply the components can be unlinked nor does it imply the individual components are unknotted.*

2. The Hennings and Chern-Simons Invariants

To begin to discuss the invariants we are concerned with we must first introduce the universal invariant upon both of which are based. The universal invariant is an invariant of bottom tangles that is generated by assigning elements from $U_q(\mathfrak{g})$ to strands in positive and negative crossings and to cups and caps, as is shown below.





To get an invariant of 3-manifolds we need to close off by particular functionals to ensure that different tangles that produce the same 3 manifold through surgery yield the same output, the Hennings and Chern-Simons invariants differ in what functional is applied.

3. The BRO Move

Definition 4 *If a diagram for T contains a disk as in one side of Figure 5, where the two thick strands come from one component, the BRO move replaces T with T', which looks the same except with the disk replaced as indicated. Note that this move preserves the linking matrix.*

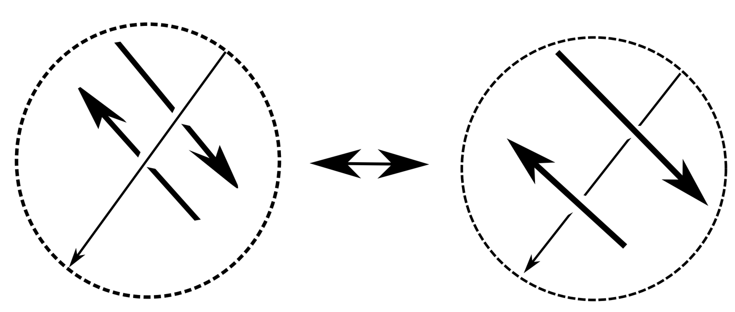


Figure 7: *The move is bidirectional and the two crossings must be consecutive from the perspective of the thin component*

Proposition 1 *Any Zero Linking Matrix Tangle (ZLMT) can be connected to the trivial tangle with the same number of components through isotopy and a sequence of moves as in Figure 2, where the two parallel strands are understood to be from the same component (the third strand may be from the same or a different component).*

Since T is a ZLMT, there are the same number of crossings where A is over B and positive as there are crossings where A is over B and negative. A similar statement can be made for under-crossings. Then we can find a pair of crossings as depicted in Figure 2, though these are not necessarily consecutive.

If they are consecutive, apply the BRO move as in Figure 2. If not, define a method of sliding in order to make them consecutive, as shown below:

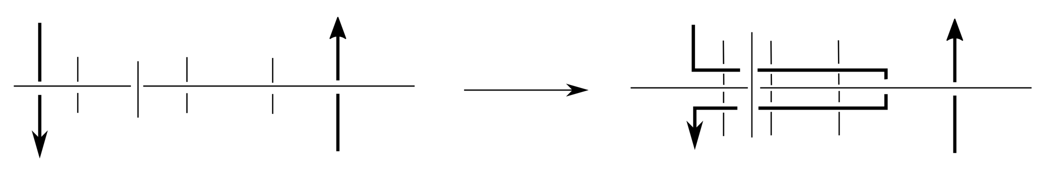


Figure 8: *Slide the crossings so they are consecutive, when introducing new crossings follow the behavior of the strand we are sliding along as pictured above, then after the bro move is applied simply slide back and follow the same behavior again.*

The BRO move can be applied and the crossings can be slid back to their original positions. This process can be iterated until the two components are separated, and therefore can separate all components.

Untying A Single Component:

We want the component to cross *over* the first time, so it is always travelling downhill. At any crossing where this does not occur, follow the procedure defined below:

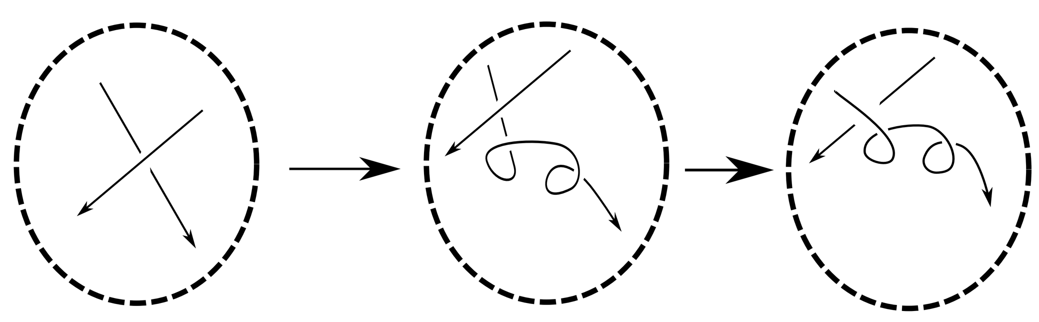


Figure 9: *Add a positive and negative twist to the under strand, first with a sign opposite the crossing we are trying to change and then a twist of the same sign, then apply the BRO move.*

<u>Note</u>: All twists will cancel out since linking matrix is preserved.

4. A Path to Verifying Equality

This gives us a natural inductive proof for demonstrating properties of ZLMTs. We use this in the following way:

Theorem 1 *If T is a zero linking matrix tangle with m components, then $\Gamma_{U_q}(T)\in \overline{U}_q^{\,\mathbf{ev},\otimes m}$ (in preparation).*

5. Relationship of Invariants

Proposition 2 *Let z_λ be a semisimple element. Then*

$$\omega(z_\lambda r)\mu(r)=\omega(r)\mu(z_\lambda r)$$

Using the above fact, combined with our previous work about the properties of the universal invariant of ZLMTs, we could therefore establish:

Proposition 3 *Let T be a ZLMT with m components and let $\epsilon_1,\epsilon_2,\ldots,\epsilon_m$ be nonzero integers with $|\epsilon_i| = 1$ for all $1\le i\le m$ and $|n_i|=|\epsilon_i|$. Then*

$$\frac{\mu^{\otimes m}[(r^{n_1}\otimes r^{n_2}\otimes\cdots\otimes r^{n_m})\Gamma_{u_q}(T)]}{\mu(r^{\epsilon_1})\mu(r^{\epsilon_2})\cdots\mu(r^{\epsilon_m})} = |n_1||n_2|\cdots|n_m|\frac{\omega^{\otimes m}[(r^{n_1}\otimes r^{n_2}\otimes\cdots\otimes r^{n_m})\Gamma_{u_q}(T)]}{\omega(r^{\epsilon_1})\omega(r^{\epsilon_2})\cdots\omega(r^{\epsilon_m})}$$

(in preparation).

This leads to our main theorem:

Theorem 2 *The Hennings invariant of a three-manifold is the product of the rank of the first homology (or zero if it is infinite) and the Chern-Simons invariant of that three-manifold (in preparation).*

References

[1] Qi Chen, Chih-Chien Yu, and Yu Zhang. Three-manifold invariantsassociated with restricted quantum groups. *Math. Z.*, 272(3-4): 987–999, 2012

[2] Kazuo Habiro and Thang T. Q. Le. Unified quantum invariantsfor integral homology spheres associated with simple Lie algebras. ArXiv:1503:03549v1, March 2015.