REU topic: Invariants of 3-manifolds

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Outline

Bottom Tangles

Definition

A **bottom tangle** is an embedding of leftward oriented arc components in $\mathbb{R}^2 \times [0,1)$. Each component has fixed points on that interval $[0,1)$, such that one component starts and ends before the next component. Considered up to continuous deformation.

Example

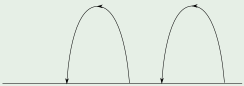


Figure: 2-Component untangle

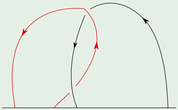


Figure: Clasp

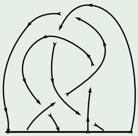
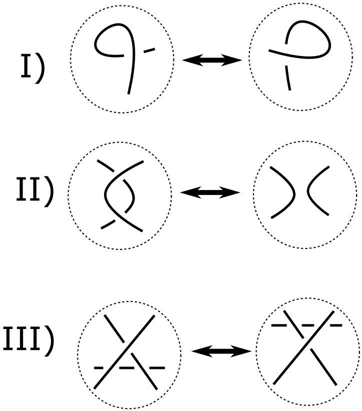


Figure: Borromean Tangle

Reidemeister moves



Outline

Notion of Positive, Negative Crossings

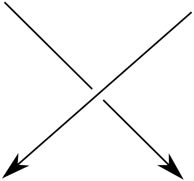


Figure: Positive crossing

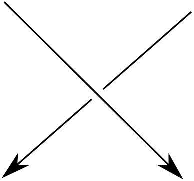


Figure: Negative crossing

Linking Matrix

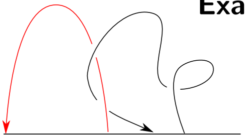
A matrix that summarizes data about the crossings in a bottom tangle.

Definition (Linking Matrix)

Let T be an m-component bottom tangle. The **linking matrix** is an $m \times m$ matrix, where

$$a_{ij} = \begin{cases} \text{\# of pos. crossings} - \text{\# of neg. crossings} & i = j \\ \frac{\text{\# of pos. crossings} - \text{\# of neg. crossings}}{2} & i \neq j. \end{cases}$$

Example:



$$\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$$

Zero Linking Matrix Tangles (ZLMTs)

- The class of bottom tangles that a linking matrix whose entries are all zero.
- Between any two components, there are an equal number of positive and negative crossings and similarly for self-crossings for each individual component.
- Ex: untangles, Borromean tangle

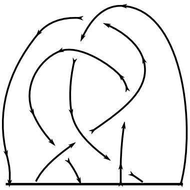


Figure: Borromean Tangle

Outline

Result 1

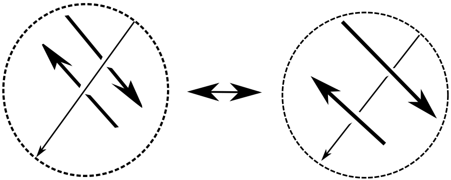


Figure: The BRO move

Statement

The application of the BRO move, with the Reidemeister moves untangles any ZLMT.

Outline

Geometric proof

Outline of proof:

1. In a ZLMT, if two components are not untangled, then there exists two oppositely signed undercrossings.
2. These two undercrossings can be made adjacent, so that the BRO move can be applied.
3. 1–2 untangles any two components from one another in a ZLMT.
4. BRO move untangles a single component from itself.

Showing Existence

Crossings can be categorized as:

1. Positive / Negative
2. strand going (Over / Under) a given strand
3. strand going (Left / Right) from a given strand's perspective

- For any strand in any tangle, there are an equal number of left and right crossings from that strand's perspective (because of inside/outside).
- Recall, for a ZLMT, there are an equal number of positive and negative crossings between any two components or in a single component.

We can conclude with these two facts that, in a ZLMT, if a given component is not already entirely over another component, there exist a pair of oppositely signed undercrossings.

Adjacency

- The BRO move requires these crossings to be consecutive.
- The Reidemeister-2,3 moves – slide crossings through each other.
- Additional crossings can be ignored.
- Slide the strand so that the pair is consecutive.
- BRO move is applied, strand is slid back.

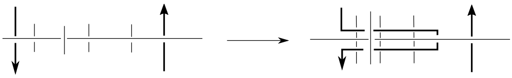
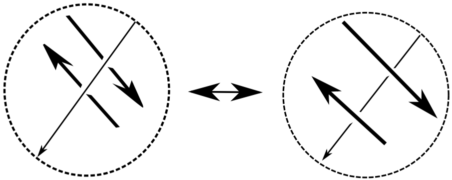


Figure: Sliding to Make Consecutive Crossings

Adjacency, Cont'd



Check we're always working with a ZLMT: Sliding produces an equal number of positive and negative crossings, keeping the linking matrix invariant. The BRO move switches a positive, negative pair.

In summary: We only need to find pairs of oppositely signed undercrossings. Can ignore whether they're consecutive.

Untying components from each other

The intuition: If you can make a component (hence its crossings) entirely over another, they are untangled.

- If two components are not already separated, by our existence argument, there will be pairs of positive and negative undercrossings.
- BRO move turns them to overcrossings.
- Inducting on pairs of undercrossings, we can untie any two components through the method we've described.
- We can therefore separate all components of a ZLMT.

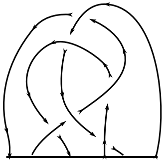


Figure: Borromean Tangle

Untying a Single Component

- Individual components may not be trivial.
- The intuition: make it so that you are going downhill the whole time, when traversing.
- The goal: Each crossing is going over the first time you approach it.
- This leaves twists, which cancel out.

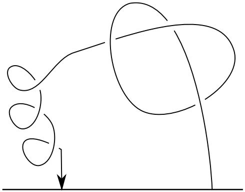


Figure: Component with zero linking number

Untying a Single Component Cont'd

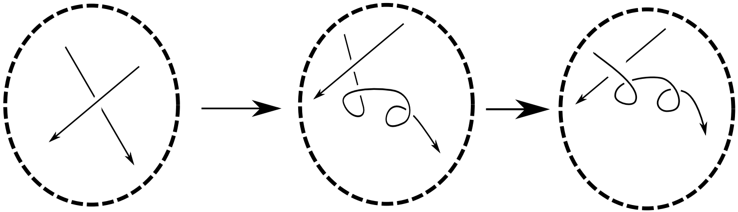


Figure: BRO move for self crossings

If there exists a crossing that goes under the first time you approach it, add in a positive and negative twist as shown above and apply the BRO move. This preserves the ZLMT while making every crossing go over the first time.

Untying a Single Component Cont'd

By induction, you are left with a component that has zero linking and goes downhill the whole way. Then Reidemeister moves would suffice to deform the tangle into the trivial untangle.

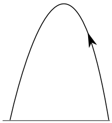


Figure: Untangled Component

Outline

Significance

Can show a property holds true for all ZLMTs by showing it holds for the untangle and that the property remains after the BRO move is applied to an arbitrary ZLMT.

Outline

1. Result 1: Untangling ZLMTs
   - Bottom Tangles
   - ZLMTs
   - Statement
   - Proof
   - Significance
2. Result 2: $\Gamma(\text{ZLMT}) \in \bar{U}_q^{\otimes m,\text{ev}}$
   - Tensor products
   - The Universal Invariant
   - Quantum groups
   - Statement
3. Result 3: Lemma 3.1
   - Quantum Functionals
   - 3-manifolds
4. Result 4: Hennings/Chern Simons Relation
   - Integral Homology 3-spheres
   - Hennings/Chern-Simons Definition
5. Conclusion

Tensor Products

On vector spaces: Combines vector spaces V, W as $V \otimes W$, with a basis of size $\dim V \cdot \dim W$ of the form $\{v_i \otimes w_j\}_{i=1\ldots n,\ j=1\ldots m}$
Compare: Direct product – basis of size $\dim V + \dim W$ of the form $\{v_1, \ldots, v_n, w_1, \ldots, w_m\}$.

$$(cv) \otimes w = v \otimes (cw) = cv \otimes w$$

$$v \otimes (w + z) = v \otimes w + v \otimes z$$

Notation:

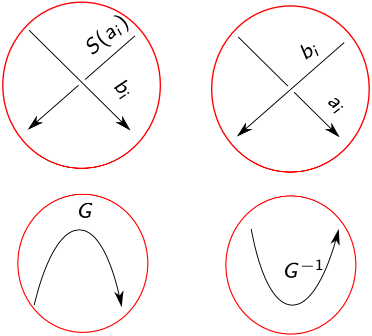
$$\omega^{\otimes m} = \underbrace{\omega \otimes \omega \otimes \cdots \otimes \omega}_{m \text{ tensors}}$$

Outline

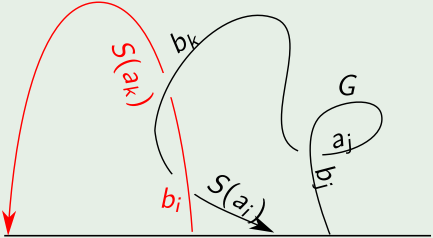
The Universal Invariant: $\Gamma(T)$

Description: $\Gamma(T)$

There exists a process of assigning algebraic elements to any bottom tangle in the form of $\Gamma(T)$, which is invariant under the Reidemeister moves.



Example



$$\Gamma(T) = \sum_{i,j,k} S(a_k)b_i \otimes S(a_i)b_k a_j G b_j$$

Hopf Algebras

A vector space with some extra maps:

$$m \colon A \otimes A \to A$$

$$i \colon \mathbb{C} \to A$$

$$\Delta \colon A \to A \otimes A$$

$$S \colon A \to A$$

$$\epsilon \colon A \to \mathbb{C}$$

Outline

Quantum groups

Description

Generated from a basis of elements: $\{E_i, K_i, K_i^{-1}, F_i\}$. Satisfying the following relations:

$$K_i K_j = K_j K_i$$

$$K_i E_j = q^{a_{ij}} E_j K_i$$

$$K_i F_j = q^{-a_{ij}} F_j K_i$$

$$E_i F_j = F_j E_i + \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

The R-matrix

$$\begin{aligned}
R &= D \sum_{t_1,\dots,t_N=0}^{\infty} \prod_{r=1}^{N} q^{t_r(t_r+1)/2}(1-q_{\beta_r}^2)^{t_r}[t_r]_{\beta_r}! E_{\beta_r}^{(t_r)} \otimes F_{\beta_r}^{(t_r)} \\
&:= \sum_{i,k} D_i' E_k \otimes D_i'' F_k \\
&= \sum_i a_i \otimes b_i \\
R^{-1} &= \sum_i S(a_i) \otimes b_i
\end{aligned}$$

Outline

$\Gamma(T) \in \overline{U_q}^{\otimes m,\text{ev}}$ for a ZLMT

Statement

Let T be a ZLMT. Then $\Gamma(T) \in \overline{U_q}^{\otimes m,\text{ev}}$.

We want to be able to narrow down where the universal invariant of a ZLMT lives. It turns out we can narrow it down to the subalgebra $\overline{U_q}^{\otimes m,\text{ev}}$, which is generated by $\{E, KF, K^2\}$.
The BRO move argument proves useful here, since we must only show that applying the BRO move keeps $\Gamma(T)$ in the subalgebra.

Let T be a ZLMT and T' denote the tangle generated by one bro move being applied to T. Notice the discrepancies between the two universal invariants:

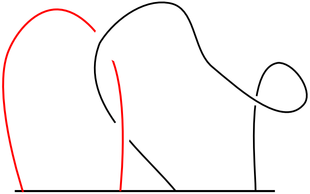
$$\Gamma(T) = \sum \cdots S(E_k)S(D_i')YD_j'E_l \cdots \otimes \cdots \otimes \cdots D_i''F_kD_j''F_l \cdots$$
$$\Gamma(T') = \sum \cdots D_i''F_kYD_j''F_l \cdots \otimes \cdots \otimes \cdots D_i'E_kS(E_l)S(D_j') \cdots ,$$

Where Y denotes an arbitrary product of elements of the algebra.

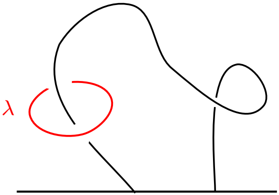
Outline

Quantum Functionals

- Linear functionals from the quantum group to $\mathbb{C}$
- We need a function to act in a specific way so that the same element is generated no matter where the link is cut, so that we get an invariant of links when closing off components by these functionals
- Used to label closed off components of a tangle as shown below



$\Gamma(T) = \sum_{i,j,k} S(a_k)b_i \otimes S(a_i)b_k a_j G b_j$



$\sum_{i,j,k} \lambda(S(a_k)b_i)S(a_i)b_k a_j G b_j$

Outline

3-manifolds

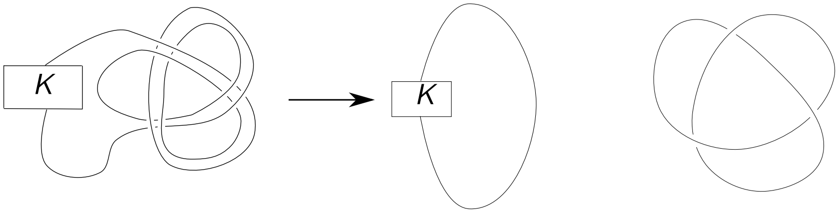
A 3-manifold manifold is a topological space which locally looks like Euclidean 3-space, so it can be thought of as a possible "shape" of the universe.

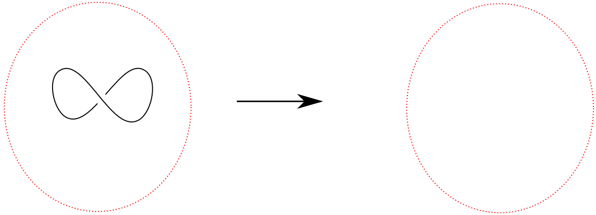
Surgery

A process of obtaining a 3-manifold by removing a tubular neighborhood surrounding a link and then gluing the meridian back in to the longitude and vice-versa

Kirby Moves

Moves on links which result the same 3-manifold when surgery is applied.





Important types of Quantum Functionals

We will use three important types of quantum functionals:

λ is a type of quantum trace
μ is the left and right cointegral
ω is a linear combination of quantum traces

We will see μ and ω as parts of the 3-manifold invariants which will be discussed (So they are invariant under the Kirby moves).

Correspondence between tangle and algebra

$\Delta : A \to A \otimes A$ is a function in our algebra that corresponds to doubling strands.

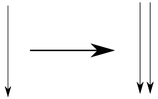


Figure: Doubling of strands

$$\Delta(a) = a' \otimes a''$$

Correspondence, cont'd.

Consider two elements from our algebra:

- r is the element corresponding to the universal invariant of a positive twist
- Let $u = u' \otimes u''$ be the element corresponding to the universal invariant of the clasp element.

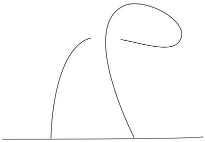


Figure: Positive Twist

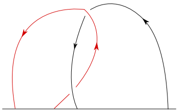


Figure: Clasp

Lemma 3.1

Lemma 3.1

Let $z_\lambda = (\lambda \otimes 1)(u)$ and let $|\epsilon| = 1$. Then

$$\frac{\omega(z_\lambda r^\epsilon)}{\omega(r^\epsilon)} = \frac{\mu(z_\lambda r^\epsilon)}{\mu(r^\epsilon)}.$$

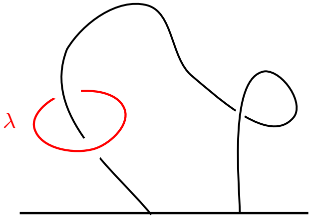


Figure: Pictoral depiction of $z_\lambda r^+$

Intuition of Lemma 3.1

Our idea is that handle slides might demonstrate the following equality.

$$\omega(z_\lambda r^\epsilon) \stackrel{?}{=} \lambda(r^{-\epsilon})\omega(r^\epsilon)$$
$$\mu(z_\lambda r^\epsilon) \stackrel{?}{=} \lambda(r^{-\epsilon})\mu(r^\epsilon)$$

Lemma 3.1 Restated

If $\frac{\omega(z_\lambda r^\epsilon)}{\omega(r^\epsilon)} = \frac{\mu(z_\lambda r^\epsilon)}{\mu(r^\epsilon)}$, then

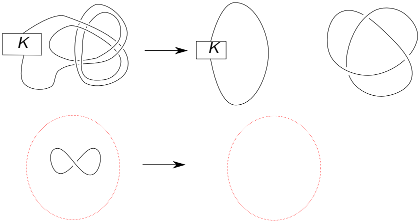
$$\omega(z_\lambda r^\epsilon)\mu(r^\epsilon) = \mu(z_\lambda r^\epsilon)\omega(r^\epsilon).$$

Proof of Lemma 3.1

First note the following: Let ϕ and ψ be quantum functionals, where ϕ is a linear combination of quantum traces and $\psi = \mu$ or ω. Then:

$$\phi(\xi)\psi(a) = \phi(\xi a')\psi(a'') \tag{1}$$

for ξ in the center and $\Delta(a) = a' \otimes a''$.



Proof, cont'd.

Then we have:

$$\phi(\xi)\psi(a) = \phi(\xi a')\psi(a'')$$
$$\Longrightarrow \lambda(r^{-\epsilon})\psi(r^{\epsilon}) = \lambda(r^{-\epsilon}(r'_{\epsilon}))\psi((r''_{\epsilon}))$$

Proof, cont'd.

So what is $\Delta(r^\epsilon) = r'_\epsilon \otimes r''_\epsilon$?
It turns out $\Delta(r^\epsilon) = (r^\epsilon \otimes r^\epsilon)u = (r^\epsilon u') \otimes (r^\epsilon u'')$, so:

$$\begin{aligned} \Longrightarrow \lambda(r^{-\epsilon}(r'_\epsilon))\psi((r''_\epsilon)) \\ = \lambda(r^{-\epsilon}(r^\epsilon u'))\psi(r^\epsilon u'') \\ = \lambda(u')\psi(r^\epsilon u'') \end{aligned}$$

Proof, cont'd

Now, it's just a matter of relabeling:

$$
\begin{aligned}
&\lambda(u')\psi(r^{\epsilon}u'') \\
&= \psi(r^{\epsilon}\lambda(u')u'') \\
&= \psi(r^{\epsilon}(\lambda \otimes 1)(u)) \\
&= \psi(z_{\lambda}r^{\epsilon})
\end{aligned}
$$

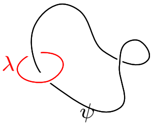


Figure: Pictoral depiction

Proof, cont'd

Since ψ represents either μ or ω, we have :

$$\omega(z_\lambda r^\epsilon) = \lambda(r^{-\epsilon})\omega(r^\epsilon)$$
$$\mu(z_\lambda r^\epsilon) = \lambda(r^{-\epsilon})\mu(r^\epsilon)$$
$$\implies \omega(z_\lambda r^\epsilon)\mu(r^\epsilon) = \lambda(r^{-\epsilon})\omega(r^\epsilon)\mu(r^\epsilon) = \mu(z_\lambda r^\epsilon)\omega(r^\epsilon)$$

What A Journey This Has Been

Lemma 3.1

$$\frac{\omega(z_\lambda r^\epsilon)}{\omega(r^\epsilon)} = \frac{\mu(z_\lambda r^\epsilon)}{\mu(r^\epsilon)}.$$

Remark:

Recall our result about ZLMTs living in $U_q^{\otimes m,\text{ev}}$. This result states that, if T is a single-component ZLMT, then $\Gamma(T)$ is a linear combination of z_λ's. This will useful later in our proof.

Outline

Integral Homology 3-spheres

We will be working on integral homology 3-spheres. There are really only two crucial facts we need:

1. Every integral homology 3-sphere is the result of surgery on a framed link with diagonal linking matrix.
2. This diagonal is made up of 1's and -1's.

Outline

Hennings/Chern-Simons Definition

$$\text{C.S. Invariant} = \frac{\omega^{\otimes m}(\Gamma(T))}{\omega(r)^{\sigma_-}\omega(r^{-1})^{\sigma_+}}$$

$$\text{Hennings Invariant} = \frac{\mu^{\otimes m}(\Gamma(T))}{\mu(r)^{\sigma_-}\mu(r^{-1})^{\sigma_+}}$$

where σ_+ and σ_- are the number of positive and negative twists in our link, respectively.

Proof, cont'd

Let T_0 be the tangle T after changing the self-linking of each component to zero by removing either a positive or negative twist:

$$\frac{\mu^{\otimes m}(\Gamma(T))}{\mu(r)^i \mu(r^{-1})^{m-i}} = \frac{\mu^{\otimes m}[(r^{\epsilon_1} \otimes r^{\epsilon_2} \otimes \cdots \otimes r^{\epsilon_m})\Gamma(T_0)]}{\mu(r^{\epsilon_1})\mu(r^{\epsilon_2})\cdots\mu(r^{\epsilon_m})}$$

This argument works switching in ω in place of μ, so what we really need to show is the following lemma:

Final Lemma

Lemma

Let T_0 be a ZLMT with m components and let $|\epsilon_i| = 1$ for all $1 \leq i \leq m$. Then

$$\frac{\mu^{\otimes m}[(r^{\epsilon_1} \otimes r^{\epsilon_2} \otimes \cdots \otimes r^{\epsilon_m})\Gamma(T_0)]}{\mu(r^{\epsilon_1})\mu(r^{\epsilon_2})\cdots\mu(r^{\epsilon_m})} = \frac{\omega^{\otimes m}[(r^{\epsilon_1} \otimes r^{\epsilon_2} \otimes \cdots \otimes r^{\epsilon_m})\Gamma(T_0)]}{\omega(r^{\epsilon_1})\omega(r^{\epsilon_2})\cdots\omega(r^{\epsilon_m})}$$

Lemma, cont'd.

We can ultimately break this down into a case of a one-tensor ZLMT, T_0:

$$\frac{\mu(r^\epsilon \Gamma(T_0))}{\mu(r^\epsilon)}$$

Now recall Lemma 3.1:

Lemma 3.1

$$\frac{\mu(z_\lambda r^\epsilon)}{\mu(r^\epsilon)} = \frac{\omega(z_\lambda r^\epsilon)}{\omega(r^\epsilon)}.$$

It was stated earlier that $\Gamma(T_0)$ is a linear combination of z_λ's, so we can apply this lemma.

I Swear, We're Almost There

Then we can turn this component into

$$\frac{\omega(r^{\epsilon_i}\Gamma(T_0))}{\omega(r^{\epsilon_i})}$$

and inductively we get

$$= \left[\frac{\omega(r^{\epsilon_1}T_1)}{\omega(r^{\epsilon_1})} \otimes \frac{\omega(r^{\epsilon_2}T_2)}{\omega(r^{\epsilon_2})} \otimes \cdots \otimes \frac{\omega(r^{\epsilon_m}T_m)}{\omega(r^{\epsilon_m})}\right]$$

which is what we needed.

Finally, Theorem Proved

And, finally, we can say

$$\left[\frac{\omega(r^{n_1}T_1)}{\omega(r^{\epsilon_1})} \otimes \frac{\omega(r^{n_2}T_2)}{\omega(r^{\epsilon_2})} \otimes \cdots \otimes \frac{\omega(r^{n_m}T_m)}{\omega(r^{\epsilon_m})}\right]$$

$$= \frac{\omega^{\otimes m}[(r^{n_1} \otimes r^{n_2} \otimes \cdots \otimes r^{n_m})\Gamma(T_0)]}{\omega(r^{\epsilon_1})\omega(r^{\epsilon_2})\cdots\omega(r^{\epsilon_m})}$$

$$= \frac{\omega^{\otimes m}(\Gamma(T))}{\omega(r)^{\sigma_-}\omega(r^{-1})^{\sigma_+}}$$

which is our C.S. invariant.

Conclusion

We have shown that the Hennings Invariant and Chern-Simons Invariant are equal on integral homology 3-spheres for the general quantum group. Possible next steps:

- The work we have done simplifies the argument of Chen, Yu, and Zhang. This work could also be used to simplify the argument of Habiro and Le.
- We would also like to see what kind of relationship could be built between the Hennings and Chern-Simons invariants on rational homology 3-spheres.

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