KSU Quals — Topology

2015 June—2021 August

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1. Let X be a totally ordered set. Give X the topology generated by the subbasis consisting of all sets of the form:

$$(a, \infty) = \{x \in X : x > a\}$$
$$(-\infty, a) = \{x \in X : x < a\}$$

   a) Show that each set of the form (a, b) is open in X and each set of the form $[a, b]$ is closed. (Here (a, b) and $[a, b]$ are defined as in $\mathbb{R}$)
   b) Show that X is Hausdorff.
   c) For any two points $a, b \in X$ with $a < b$, prove that $\overline{(a, b)} \subseteq [a, b]$. Give an example that proves the equality doesn't always hold.

2. a) Prove that if X is compact and non-empty, Y is connected and Hausdorff, and $f : X \to Y$ is a continuous open map, then f is surjective.
   b) Give an example to show the previous statement does not necessarily hold when X is not compact.

3. Let $X = S^1 \vee S^2$.
   a) Compute the fundamental group of X.
   b) Describe the universal covering of X.

4. a) Give the intersection theoretic definition of Euler characteristic applicable to smooth manifolds.
   b) Prove, using this definition and the homotopy invariance of intersection numbers, that if M is a smooth manifold, then $\chi(M \times S^1) = 0$.

5. a) State the theroem called Stokes Theorem applicable to differential forms and generalizing the classical theorem of the same name.
   b) Let $\Sigma = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4\}$ and $H = \Sigma \cap \{(x, y, z) \mid x \geq 0\}$. Use Stokes Theorem to evaluate the following integrtals of differential 2-forms:
      i.
$$\int_\Sigma x dy \wedge dz$$
      ii.
$$\int_\Sigma 2xy dx \wedge dz + x^2 dy \wedge dz$$
      iii.
$$\int_H 2xy dx \wedge dz + x^2 dy \wedge dz$$

6. a) Use a Meyer-Vietoris sequence and homotopy invariance of deRham cohomology to prove
Theorem For any manifold X, $H^n_{dR}(X \times S^1) \cong H^n_{dR}(X) \oplus H^{n-1}_{dR}(X)$, where $H^{-1}_{dR}(X)$ is understood to be 0.

b) Use the preceeding theorem and induction to compute the deRham cohomology of the k-dimensioanl torus (that is the product of k copioes of S^1).

2 2021 June

1. Let X be a set and p an arbitrary point in X.
   a) Show that
   $$\mathcal{T} = \{U \subseteq X \mid U = \emptyset \text{ or } p \in U\}$$
   is a topology on X. This is called the particular point topology.
   b) Show that $(X, \mathcal{T})$ is a connected topological space.
   c) When is $(X, \mathcal{T})$ compact? Justify your answer.

2. Let $\mathbb{T}^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus.
   a) What is the universal covering space of $\mathbb{T}^n$.
   b) Assume $n \geq 2$ and let $f : S^n \to \mathbb{T}^n$ be a continuous map. Show that f is null-homotopic.

3. a) State the Seifert-Van Kampen theorem.
   b) Let K be the space obtained from a square $[0,1] \times [0,1]$ by identifying opposite sides as follows: $(x, 0) \sim (x, 1)$ for all x, and $(0, y) \sim (1, 1-y)$ for all y. Compute $\pi_1(K)$ using the Seifert-Van Kampen theorem.

4. Recall that a property P of smooth maps is said to be stable if whenever X is a compact manifold, Y a manifold, and $h : X \times [0,1] \to Y$ a smooth homotopy, if $h(-,0) : X \to Y$ satisfies P, then there is an $\epsilon > 0$ such that $t < \epsilon$ implies $h(-,t) : X \to Y$ satisfies P.
   a) Without proof, which of the following properties are stable: local diffeomorphiosm, surjectivity, injectivity, transversality to a fixed submanifold Z of the target Y?
   b) If any of the listed propeties are not stable, give an example of a compact manifold and homotopy of maps from it which begins with a map exhibiting the property, but for which no $\epsilon > 0$ as required in the definition exists.
   c) If any of the listed properties are stable, give an example of a non-compact manifold and a homotopy of maps from it which begins with a map exhibiting the property, but for which no $\epsilon > 0$ as required in the definition exists.

5. Let $j : S^1 \times S^1 \to \mathbb{R}^4$ be the inclusion give in terms of angular coordinates θ and ϕ on the first and second factors, respectively, by
   $$(\theta, \phi) \mapsto (\cos(\theta), \sin(\theta), \cos(\phi), \sin(\phi)),$$
   and consider the differential 2-form
   $$\omega := x dy \wedge dz + y dx \wedge dz + y dz \wedge dt + z dy \wedge dt$$
   on $\mathbb{R}^4$.
   Find
   $$\int_{S^1 \times S^1} j^* \omega.$$

6. Without invoking the Künneth formula (though if you know it, it will tell you what the answer will be), but using only the well-known deRham cohomology groups of the point and of spheres, and the homotopy invariance of deRham cohomology and Mayer-Vietoris Theorem, compute the deRham cohomology of $S^k \times S^n$ for $k, n \geq 1$. (Hint: the case $k = n$ will be a bit different.)

3 2020 August

1. a) Define what it means for a topological space to be each of the following: discrete, Hausdorff, T_1.

   b) A space is *totally disconnected* if its only connected subsets are singletons. Prove that for finite topological spaces each property of part (a) is equivalent to total disconnecctedness.

   c) Give an example of a (necessarily infinite) Hausdorff space which is totally disconnected, but not discrete.

2. a) Define what it means for a topological space to be compact.

   b) Prove that any subspace of a Hausdorff space which is compact in the subspace topology is closed as a subset of the ambient space.

3. Prove that if $A \subset X$ is a deformation retract of X, then for any space Y there is a bijection between the set of homotopy classes of continuous maps from Y to X and the set of homotopy classes of continuous maps from Y to A.

4. a) Determine (with a proof) the fundamental group $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$ of the wedge sum of two projective planes.

   b) Prove that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ does not admit a 5-sheeted connected normal covering. (Hint: use the fact that the only group of order five is the cyclic one $\mathbb{Z}_5$.)

5. Let $p \subset \mathbb{R}^3$ be the plane $z = 1$, and let $\pi : \mathbb{R}^3 \setminus p \to \mathbb{R}2$ be the stereographic projection on the (x, y)-coordinate plane from the point $(0, 0, 1)$.

   a) Let $\omega = dx \wedge dy \in \Omega^2(\mathbb{R}^2)$. Compute $\pi^*\omega \in \Omega^2(\mathbb{R}^3 \setminus p)$ in the natural (x, y, z) coordaintes on $\mathbb{R}^3 \setminus p$.

   b) Is $\pi^*\omega$ closed? Is it exact?

6. Let $\ell \subset \mathbb{R}^3$ be a straight line and $A \in \mathbb{R}^3 \setminus \ell$ be a point in its complement. Compute the de Rham cohomology $H^*(\mathbb{R}^3 \setminus (\ell \cup \{A\}))$.

4 2020 June

1. a) Some authors call the subset of a topological space X a “clopen” if it is both open and closed. Prove that there is abijection between the set of clopens of a space X and the set of continuous functions from X to the two-point discrete space.

   b) Define what it means for a space to be connected.

   c) Use your result in part (a) to prove that a space X is connected if and only if there are exactly two continuous functions from X to the two-point discrete space.

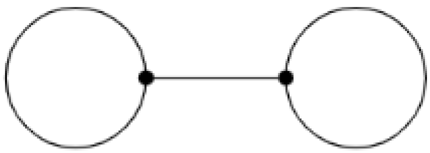
2. a) Define what it means for a topological space to be T_1 and what it means to be normal.

   b) Prove that a compact subspace of a normal T_1 space is normal and T_1 (in the subspace topology).

3. a) Using the fact that $\pi_1(S^1, *) \cong \mathbb{Z}$, the Seifert-VanKampen Theorem, and the fact that the fundamental groups are invariant under deformation retracts as starting points, give a fully justified calculation showing that the fundamental group of a bouquet of n circles, $\vee_{i=1}^{n}(S^1, *)$ is a free group on n-generators.

   b) Describe the universal covering space of a bouquet of two circles, $(S^1, *) \vee (S^1, *)$. (Hint: it is the geometric realization of an infinite graph.)

4. Let X be the following one-dimensional CW complex:



   a) Prove that X is homotopy equivalent to $S^1 \vee S^1$.

   b) Give an example of a connected 3-sheeted non-normal covering of $S^1 \vee S^1$.

   c) Give an example of a connected 3-sheeted non-normal covering of X.

   (Make sure that your pictures indicate clearly both the covering space and the covering map.)

5. Prove that a product $M \times N$ of two non-empty smooth manifolds M and N is orientable if and only if each factor is orientable.

6. Let ℓ be a straight line in $\mathbb{R}^3$. Compute the de Rham cohomology of $\mathbb{R}^3 \setminus \ell$.

5 2019 August

1. Consider the set $\mathbb{Z}$ of integer numbers endowed with the cofinite topology (only finite sets and the entire set are closed).
   a) Is this space compact? Prove your statement.
   b) Is it connected? Prove.
   c) Is it T_0? T_1? Hausdorff? regular? normal? metrizable? Explain.
2. Let $X = S^1 \wedge S^2 \wedge S^3$.
   a) Compute $\pi_1(X)$.
   b) Describe the universal covering of X.
3. Let $\Sigma = \mathbb{R}P^2 \# \mathbb{R}P^2$.
   a) Compute the Euler characteristic Σ.
   b) Does Σ admit a 2-sheeted covering by itsef? Describe such covering if your answer is yes, otherwise profe that it is not possible.
4. Let $g : N \to \mathbb{R}$ be a C^∞ function on the manifold N. Prove that a nonempty regular level set $S = g^{-1}(c)$ is a smooth proper submanifold of N of codimension 1.
5. Recall that the exterior derivative d on a manifold M is defined as an $\mathbb{R}$-linear map $\Omega^* M \to \Omega^* M$ such that $d \circ d = 0$, the form $d(f)$ is the differntial df of any smooth function f on M, and for any pair of differential forms $a, b \in \Omega^* M$ one has
$$d(a \wedge b) = d(a) \wedge b + (-1)^k a \cdot d(b),$$
   where k is the degree of the form a. Using these properties of d prove that if two differntial forms w and w' agree on a neighborhood of a point $p \in M$, then $d(w)|_p = d(w')|_p$.
6. The antipodal map $a : S^n \to S^n$ is the map $x \to -x$. Show that the antipodal map is orientation preserving if and only if n is odd. Deduce that $\mathbb{R}P^{2n+1}$ is orientable for each $n \geq 0$.

6 2019 June

1. Consider the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers endowed with the cofinite topology (only finite sets and the entire set are closed).
   a) Prove that it is compact.
   b) Prove that it is path-connected.
   c) Is it T_0? T_1? Hausdorff? regular? normal? connected? metrizable?

2. a) Give an exmaple of a connected 2-sheeted covering of the wedge of two circles and a 2-sphere $S_a^1 \vee S_b^1 \vee S^2$.
   b) Is your example a regular covering?

3. a) Compute $\pi_1(S^1 \times \mathbb{R}P^2)$.
   b) Let X be the space obtained from two copies of $S^1 \times \mathbb{R}P^2$ by identifying them along their circles $S^1 \times \{p\}$, where p is the base point of $\mathbb{R}P^2$. Compute $\pi_1(X)$.

4. Let X be the real number line $\mathbb{R}$ with the smooth structure. Let Y denote the same real line with the smooth structures given by this maximal atlas of the coordinate chart $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x^{\frac{1}{5}}$. Show that the two smooth structures on $\mathbb{R}$ are distinct, but the X is diffeomorphic to Y.

5. Let $f_1, \ldots, f_n$ be smooth functions on a neighborhood U of a point p. Show that there is a neighborhood $W \subset U$ of p on which $f_1 \ldots, f_n$ are coordinate functions if and only if $df_1 \wedge \cdots \wedge df_n|_p \neq 0$

6. Calculate the de Rham cohomology groups of an orientable surface of genus 2. (You may use the fact that the cohomology groups of a punctured torus are $\mathbb{R}$ in degree 0, $\mathbb{R}^2$ in degree 1, and 0 otherwise.)

7 2018 August

1. Recall that the long line L is the set $(\Omega+1)\times[0,1)\setminus\{(0,0)\}$ where Ω is the first uncountable ordinal (so $\Omega+1$ is the set of all ordinals from $0=\emptyset$ up to and including Ω, ordered by $\in$), with the order topology induced by the lexicographic order $(\alpha,x)\leq(\beta,y)$ whenever $\alpha<\beta$ or both $\alpha=\beta$ and $x\leq y$.

   a) Prove in detail that L is connected.

   b) Which of the following topological properties does L exhibit – Hausdorffness, separability, second countability, compactness, path connectedness? For each explain *briefly* why it does or does not exhibit the property.

   For part (b) your explanations need not be complete proofs, just an indication that you understand the key idea that could be exploited to give a proof, or citing a theorem that implies your claim.

2. a) State the Meyer-Vietoris theorem for deRham cohomology.

   b) Use the Meyer-Vietoris theorem, homotopy invariance, and the well-known deRham cohomology groups of spheres and contractible spaces and a suitable open cover to compute the deRham cohomology of the space obtained by removing a closed disk from the torus $S^1\times S^1$.

3. Let ω denote the form $xdz\wedge dy+y^3dx\wedge dy+z^2dy\wedge dz$ on $\mathbb{R}^3$. Calculate the integral of $\omega|_{S^2}$ over the standard unit sphere $S^2\subset\mathbb{R}^3$.

4. Find the fundamental group of the space obtained by attaching two disks (thought of as the unit disk in the complex plane) to the torus $S^1\times S^1$ (thinking of S^1 as the unit circle in the complex plane) by their boundaries, the first by the map $e^{i\theta}\mapsto(e^{im\theta},1)$, the second by the map $e^{i\theta}\mapsto(1,e^{in\theta})$ for m,n non-zero integers. A presentation by generators and relations will be acceptable, but it should be possible to identify the group as a reasonably familiar finite group for every pair m,n.

5. a) Define what it means for a space to be compact.

   b) Define what it means for a space to be sequentially compact.

   c) Prove that if X is compact then X is sequentially compact.

6. a) Define what it means for a smooth map between map between manifolds $f:X\to Y$ to be an immersion.

   b) What additional conditions must an immersion satisfy to be an embedding?

   c) Using the inverse function theorem as a starting point, prove that an immersion is locally injective (that is, for any $x\in X$ ther is a neighborhood of x on which the immersion f is one-to-one).

8 2018 June

1. a) Consider the space $X = \{0\} \cup \{\frac{1}{2^n} \mid n = 1, 2, 3, \ldots\}$ in the subspace topology induced by the usual metric (or equivalently order) topology on $\mathbb{R}$.

   b) Prove in detail that X is compact.

   c) Which of the following topological properties does X exhibit — Hausdorffness, separability, second countability, connectedness, contractibility? For each explain *briefly* why it does or does not exhibit the property.

2. Let $H^k(X)$ denote the k-th deRham cohomology group of a manifold (with or without boundary) X.
   Prove in detail that $H^k(X \coprod Y) \cong H^k(X) \times H^k(Y)$, where $\coprod$ denotes the disjoint union of manifolds.

3. Give a presentation for the fundamental group of the space obtained by removing a disk from the torus $S^1 \times S^1$ and a disk from the real projective plane $\mathbb{R}P^2$, and gluing the bounding circles by a map of degree 1.

4. a) Define what it means for a space to be connected.

   b) Define what it means for a space to be path connected.

   c) Prove that if a space X is path connected then X is connected.

   d) Give a counter-example to the converse implication.

5. a) Prove the "Stack of Records Theorem":
   If $y \in Y$ is a regular value of a smooth map $f : X \to Y$ with X compact and $\dim(X) = \dim(Y)$, then $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_n\}$ and moreover there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union of opens $V_1, \ldots, V_n$ in X with $x_i \in V_i$ and $f|_{V_i} : V_i \to U$ a diffeomorphism, for $i = 1, \ldots, n$.

   b) Give a counterexample to the same assertion with the compactness of X omitted, in which, not only is the preimage of y infinite, but there is no neighborhood U of y whose preimage is a disjoint union of open sets each mapped diffeomorphically to U by f.

6. Suppose that $X = \partial W$ where W is a compact manifold of dimension $k+1$. Let $f : X \to Y$ be a smooth map. Let ω be a closed k-form on Y. Prove that if f extends to all of W, then $\int_X f^*\omega = 0$.

9 2017 August

1. Consider the finite topological space $X = \{0, a, b, c\}$ with topology induced by the subbasis $\{\{a, 0, b\}, \{a, 0, c\}, \{b, 0, c\}\}$.
   a) List all of the open sets in the topology on X.
   b) Prove in detail that X is path connected.
   c) Which of the following topological properties does X exhibit – compactness, T_0-ness, T_1-ness, Hausdorffness, separability, contractibility?

2. The word "embedding" is used differently in point-set topology and in differential topology. Give both definitions, and prove that a smooth function between smooth manifolds which is an embedding in the sense of differential topology, when regarded as a continuous function between the underlying topological manifolds is an embedding in the sense of point-set topology.

3. The usual definition of the real projective plane, $\mathbb{RP}^2$ is the quotient space of $\mathbb{R}^3 \setminus \{\vec{0}\}$ by the equivalence relation $\vec{x} \equiv \vec{y}$ if and only if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\vec{x} = \lambda\vec{y}$.

   It can also be described as a quotient of the disjoint union of an open disk and an open Möbius strip by an equivalence relation that identifies an open annulus along the boundary of the disk with the annulus obtained from the Möbius strip by removing a circle which generates the fundamental group of the Möbius strip. Use the second description – you do **not** need to prove it is equivalent to the first – to find
   a) The fundamental group $\pi_1(\mathbb{RP}^2)$ using the Seifert-van Kampen Theorem.
   b) The deRham cohomology of $\mathbb{RP}^2$ by using an appropriate Meyer-Vietoris sequence.

4. Consider the differential 3-form

$$\omega = w^2 \; dx \wedge dy \wedge dz + xw \; dw \wedge dy \wedge dz + yw \; dw \wedge dx \wedge dz$$

   defined on $\mathbb{R}^4$.
   a) Find $\int_{S^3} \omega$, where S^3 is the standard unit sphere $\{(w, x, y, z) \mid x^2 + y^2 + z^2 + w^2 = 1\}$.
   b) Is there a differential 2-form η such that $\omega = d\eta$? Justify your answer.

5. a) Define what it means to be a covering space and what it means for a map to be nullhomotopic.
   b) Prove that if $|\pi_1(X)| < \infty$, then every map $f : X \to T^n$ is nullhomotopic (where $T^n = S^1 \times \cdots \times S^1$ is the n-torus).

6. a) Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is f carries closed sets to closed sets).
   b) Give an example of a continuous map $f : X \to Y$, where either X is noncompact, Y is not Hausdorff, or both, such that f fails to be closed.

10 2017 June

1. (a) Define what it means to be a covering space and what a lifting of a map is with respect to a covering map.
   (b) Show that given $p : \widetilde{X} \to X$ a covering map, every map $f : S^n \to X$ lifts to $\widetilde{X}$ provided $n > 1$.
   (c) Give an example where the above statement fails when $n = 1$.

2. A $2n$-dimensional smooth manifold is *symplectic* if it admits a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(X)$ is a volume form; in particular you may assume that
$$\int_X \omega^n > 0.$$
   (a) Show that ω^k is closed for each $k \in \{1, \dots, n\}$.
   (b) Show that ω^k cannot be exact for each $k \in \{1, \dots, n\}$.
   (c) What does this say about $H^{2k}_{\mathrm{DR}}(X)$ for any symplectic manifold X?

3. (a) Define what it means for a map between smooth manifolds to be a submersion.
   (b) Use the inverse function theorem to prove:
   If $f : X \to Y$ is a submersion, then for every $x \in X$, there exists charts $(U, h_U) \ni x$ and $(V, h_V) \ni f(x)$ on which $h_V(f(h_U^{-1}))$ is given by
$$(x_1, \dots, x_{\dim(Y)}, \dots, x_{\dim(X)}) \mapsto (x_1, \dots, x_{\dim(Y)}).$$

4. (a) Define what it means to be a metric space.
   (b) Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points $x_1, x_2 \in X$,
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$
   Show that f is an embedding (i.e. that f is a homeomorphism onto its image $f(X)$).

5. Recall that a map $f : X \to Y$ is *open* if $f(U)$ is open whenever $U \subset X$ is.
   (a) Define what it means to be a quotient map.
   (b) Suppose that $p : X \to X^*$ and $q : Y \to Y^*$ are each *open* quotient maps. Show that $p \times q : X \times Y \to X^* \times Y^*$ is a quotient map.

6. Consider the space $X_{m,n,k}$ obtained by attaching two 2-cells to
$$T^2 = \{(z, w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$$
via maps $\varphi_{m,N} : S^1 \to T^2$ and $\psi_k : S^1 \to T^2$ defined by $\varphi_{m,n}(z) = (z^m, \overline{z}^n)$ and $\psi_k(z) = (1, z^k)$ respectively. That is,
$$X_{m,n,k} = T^2 \bigsqcup \{D_1^2, D_2^2\} / \{z \sim \varphi_{m,n}(z), y \sim \psi_k(y) : z \in \partial D_1^2, y \in \partial D_2^2\}.$$
Calculate the fundamental group of $X_{m,n,k}$.

11 2016 August

1. a) Define the subspace topology.

   b) Let $f : X \to Y$ be a continuous map, let $X \times Y$ be the product space (with the product topology) and let

   $$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y,$$

   equipped with the subspace topology. Show that $\Gamma(f)$ is homeomorphic to X.

2. Let X be a topological space, and let $\sim$ be a relation on X.

   a) State what it means for $\sim$ to be an equivalence relation.

   b) Define $X/\sim$ and the quotient topology on it.

   c) True or false (with proof or counter-example): If X is compact, $X/\sim$ is compact.

   d) True or false (with proof or counter-example): If X is normal, $X/\sim$ is normal.

3. Recall that a map $f : X \to Y$ is *open* if $f(U)$ is open whenever $U \subseteq X$ is open. Recall that a *closed map* is one for which $f(A)$ is closed whenever $A \subseteq X$ is closed. Let

   $$S^2 := \{v \in \mathbb{R}^3 \mid |v| = 1\}.$$

   Give S^2 the subspace topology of the metric topology on $\mathbb{R}^3$. Let $g : S^2 \to S^2$ be continuous and open.

   a) Prove that g is a closed map.

   b) Prove that g is surjective.

4. a) Define what it means for a map to be nullhomotopic.

   b) Prove that if $|\pi_1(X)| < \infty$, then every map $f : X \to T^2$ is nullhomotopic (where $T^2 = S^1 \times S^1$ is the 2-torus).

5. Let

   $$SK := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ |z| \leq 1\}/\sim$$

   and

   $$K := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ |z| \leq 1\}/\sim$$

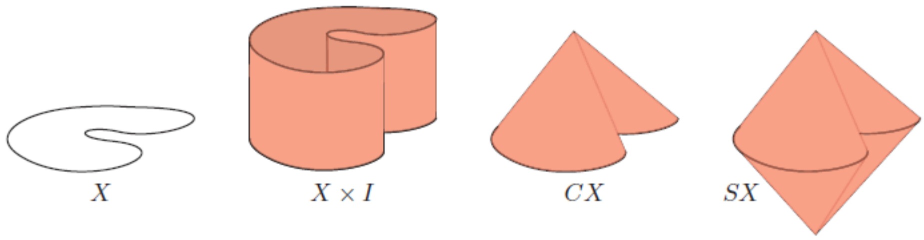
   where $(x, y, -1) \sim (x, -y, 1)$. Let $[(1, 0, 1)]$ be the base point for each space. Let $\iota : K \to SK$ be the inclusion.

   a) Compute $\pi_1(K)$, $\pi_1(SK)$, and the induced map $\iota_* : \pi_1(K) \to \pi_1(SK)$.

   b) Construct a non-cyclic abelian group H and a surjective homomorphism $f : \pi_1(K) \to H$.

   c) Prove that there is no continuous map $r : SK \to K$ so that $\iota \circ r = \mathrm{id}_K$.

6. **Definition:** Given a space X, the *cone* CX is the quotient space obtained by collapsing one end of the cylinder $X \times I$. That is $X \times I / \sim$ where $(x, 1) \sim (y, 1)$ for all $x, y \in X$. Furthermore, the *suspension* SX is the quotient space obtained by collapsing both ends of the cylinder $X \times I$ separately - or equivalently by identifying two copies of CX along $X \times \{0\}$.



Problem: Calculate the cohomology of SX in terms of the cohomology of X (Hint: It is helpful to note that CX is contractible).

12 2016 June

1. a) Define what it means for a topological space to have each of the following properties: compact, connected, path-connected, contractible.

   b) Let X be one of the following topological spaces: the Euclidean space $\mathbb{R}^n$ (for some $n \geq 1$), the Cantor subset of $[0,1]$, the subset $\bigsqcup_{n\geq 1}\left[\frac{1}{2n^2+1}, \frac{1}{2n^2-1}\right] \subset \mathbb{R}$, or the torus $S^1 \times S^1$. In each case, briefly justify whether or not X is compact, connected, path-connected, and/or contractible.

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be nonempty topological spaces. Let $A \subset X$ and $B \subset Y$ be proper subsets. If X and Y are path-connected, show that the complement of $A \times B$ in $X \times Y$ is path-connected.

3. Let $(X, \mathcal{T})$ be a compact space, and let $\{C_j\}_{j\in\mathcal{J}}$ be a family of closed subsets of X. Set $C = \bigcap_{j\in\mathcal{J}} C_j$, and let $U \subset X$ be an open subset containing C. Show that there exists $\{j_1, \ldots, j_n\} \subset \mathcal{J}$ such that
$$C_{j_1} \cap \cdots \cap C_{j_n} \subset U.$$

4. a) Calculate the cohomology of S^n for each $n \geq 0$.

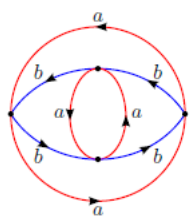
   b) Define what it means for a map to be a retraction.

   c) Show that there is no retraction of D^{n+1} onto its boundary S^n for each $n \geq 0$.

5. Consider the space $X_{m,n}$ obtained by attaching two 2-cells to
$$T^2 = \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$$
via maps $\varphi_m : S^1 \to T^2$ and $\psi_n : S^1 \to T^2$ defined by $\varphi_m(z) = (z^m, 1)$ and $\psi_n(z) = (1, z^n)$ respectively. That is,
$$X_{m,n} = T^2 \bigsqcup \{D_1^2, D_2^2\}/\{z \sim \varphi_m(z), y \sim \psi_n(y) : z \in \partial D_1^2, y \in \partial D_2^2\}.$$
Calculate the fundamental group of $X_{m,n}$.

6. a) Define what a deck transforation of a covering space is.

   b) For the following covering space of $S^1 \vee S^1$, describe the corresponding group of deck transformations.



13 2016 January

1. Consider the finite topological space $X = \{a, b, c, d\}$ with topology

$$\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}.$$

Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, seperability, contractibility, Hausdorffness, metrizability?

2. a) Let (X, d) be a metric space. Show that if $A \subseteq X$ is a compact subset, then it is a closed subset.

   b) Give two definitions of connectivity for topological spaces: one using subsets, and one using maps to the discrete space $\{0, 1\}$. Show that both definitions are equivalent.

   c) Give an example of a compact and connected space that is not path-connected.

3. For each $n \geq 1$, let $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$ be the n-th dimensional sphere, and let $\mathbf{b}_{n,0} = (1, 0, \ldots, 0) \in S^n$. Let

$$Y := \left(\coprod_{i=1}^{n} S^i \right) / (\mathbf{b}_{j,0} \sim \mathbf{b}_{k,0}) = S^1 \vee S^2 \vee \cdots \vee S^n.$$

   a) Find the fundamental group $\pi_1(Y, *)$ where $*$ is an arbitrary base point.

   b) Find the homology groups of Y.

4. Let $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. Given $n, m \in \mathbb{Z}$, let $\varphi_n : S^1 \to S^1$ be the map defined by $z \mapsto z^n$, and let $\varphi_m : S^1 \to S^1$ be the map defined by $z \mapsto z^m$. Finally, let

$$X = S^1 \coprod \{D_1^2, D_2^2\} / \{z \sim \varphi_n(z), y \sim \varphi_m(y) \mid z \in S^1 = \partial D_1^2, y \in S^1 = \partial D_2^2\}.$$

Compute the homology groups of X.

5. Let $\mathbb{T} = S^1 \times S^1$, the 2-dimensional torus, seen as

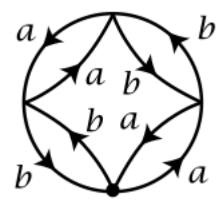
$$\mathbb{T} = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid \|z_i\| = 1, i = 1, 2\}.$$

Let $X = \mathbb{T} \times \mathbb{T} / ((z, 0) \sim (0, z))$, with the obvious topology. Briefly justify that X is path-connected, and compute its fundamental group.

6. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $(\widetilde{X}, \mathcal{T}_{\widetilde{X}})$ and $(\widetilde{Y}, \mathcal{T}_{\widetilde{Y}})$ be covering spaces of X and Y respectively. Show that $\widetilde{X} \times \widetilde{Y}$ is a covering space of $X \times Y$ (with the corresponding product topologies, of course).

14 2015 August

1. Consider the finite topological space $X = \{0, x, 1\}$ with topology $\{\emptyset, \{0\}, \{1\}, X\}$.
   (a) Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, contractibility, T_0-ness, Hausdorffness, metrizability?
   (b) Prove that if Y is a topological space, there is a bijection between the set of continuous functions from Y to X and the set
   $$\{(U, V) \mid U \text{ and } V \text{ are disjoint open subsets of } X\}.$$
2. A topological space X has the *fixed point property* if every continuous map $f : X \to X$ has at least one fixed point (i.e., there exists $x_f \in X$ such that $f(x_f) = x_f$).
   (a) Show that the sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ does not have the fixed point property.
   (b) Suppose that X has the fixed point property. Show that X is T_0, that is, for each pair of distinct points in X, at least one of the points has an open neighborhood which does not contain the other point (Hint: suppose otherwise and define a map $f : X \to X$ without fixed points).
3. (a) Define what it means for a topological space to be connected and to be path-connected.
   (b) Prove that if a topological space is path-connected then it is also connected.
4. A set of sets $\mathcal{C}$ is said to have *the finite intersection property* if whenever $\{C_1, \ldots, C_n\} \subset \mathcal{C}$ is a finite subset of $\mathcal{C}$, $\bigcap_{j=1}^n C_j$ is non-empty. For a given topological space X, prove that the following statements are equivalent.
   - X is compact.
   - For each set $\mathcal{C}$ of *closed* subsets of X, $\mathcal{C}$ has the finite intersection property if and only if $\bigcap_{C \in \mathcal{C}} C$ is non-empty.
5. (a) Define what a deck transformation (of a covering space) is, and define what a normal covering space is.
   (b) Consider the topological space $X = S^1 \vee S^1$. Describe the group of deck transformations of the following covering space of X.



6. Find the homology groups of the 3-torus, $S^1 \times S^1 \times S^1$.

15 2015 June

1. Consider the finite topological space $X = \{0, x, 1\}$ with topology $\{\emptyset, \{1\}, \{x, 1\}, X\}$.
   (a) Which of the following topological properties does this space exhibit: compactness, connectedness, path connectedness, separability, contractibility, Hausdorffness, metrizability?
   (b) Prove that if Y is a topological space, there is a bijection between the set of continuous functions from Y to X and the set of nested pairs of open sets $U \subset V$ in Y.
2. (a) Define what it means for a topological space to be compact.
   (b) State Tychonoff's Theorem
   (c) Prove the special case of Tychonoff's Theorem for two topological spaces.
3. Let Y be the space obtained from the torus $S^1 \times S^1$ by removing an open disk. Find the homology groups of Y.
4. Let $S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$. Let $\iota_i : S^1 \to D_i^2$ be the inclusions of S^1 onto the boundaries of two disjoint copies of the unit disk $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$. Finally, let $X_{n,m} = S^1 \coprod D_1^2 \coprod D_2^2 / \sim$, where $\sim$ is the equivalence relation generated by $z \sim \iota_1(z^n)$ and $z \sim \iota_2(z^m)$ for n and m integers, for all $z \in S^1$.
   (a) Prove that $X_{n,m}$ is path-connected.
   (b) Compute the fundamental group $\pi_1(X_{n,m}, *)$ in terms of n and M, where $*$ is an arbitrary base point.
5. Let $X = S^1 \times S^1 \times S^1$, and let $Y = \mathbb{R}P^2 \times \mathbb{R}P^2$.
   (a) Define what it means for a map to be nulhomotopic.
   (b) Show that any map $f : Y \to X$ is nulhomotopic.
6. Prove that if X and Y are path connected spaces, and $A \subset X$ and $B \subset Y$ are proper subsets, then the space $X \times Y \setminus A \times B$ is path connected in the subspace topology induced by the product topology on $X \times Y$.

16 Sample

1. Let $\pi : \widetilde{X} \to X$ be a covering map. Suppose that $f, g : Y \to \widetilde{X}$ are continuous maps such that $\pi \circ f$ and $\pi \circ g$ are equal and assume that f and g agree at $y_0 \in Y$. Show that if Y is connected, then $f = g$.

2. Let $R = \{z \in \mathbb{C} \mid 1 \le |z| \le 2\}$ and consider the quotient space $X = R/\sim$ where $z \sim e^{2\pi i/3} z$ for $|z| = 1$ and $z \sim e^{2\pi i/5} z$ for $|z| = 2$. Thus X is obtained from the annulus by identifying certain points on its two boundary circles. Describe the fundamental group $\pi_1(X, *)$. (Hint: You may cut R along the circle of radius $3/2$ and apply the van Kampen theorem.)

3. (a) Give the intersection-theoretic definition of Euler characteristic applicable to compact smooth manifolds.
   (b) State and prove a general theorem about the Euler characteristic of manifolds for which there exists a fixed-point-free self-map homotopic to the identity map.
   (c) For which of the following manifolds does the theorem of part (b) allow one to compute the Euler characteristic: S^2, $S^1 \times S^1 \times S^1$, S^3?

4. Prove that if Z_0 and Z_1 are compact oriented p-dimensional submanifolds, such that they are cobordant in X, that is, there is a compact oriented $p+1$-dimensional submanifold with boundary of X, whose boundary is $Z_0 \coprod -Z_1$ and ω is a closed p-form on X, then

$$\int_{Z_0} \omega = \int_{Z_1} \omega.$$

5. Consider the finite topological space with underlying set $\{a, b, c, d\}$ and topology [1]

$$\{\emptyset, \{a\}, \{c\}, \{a, b, c\}, \{c, d, a\}, \{a, b, c, d\}\}$$

   Which of the following topological properties does it satisfy: connected, path connected, compact, Hausdorff, T_0, T_1, simply connected? For each property briefly prove your assertion that the space does or does not satisfy it.

6. Let X be the subspace of $\mathbb{R}^2$ consisting of the union of the two sets $A = \{(0, y) \mid y \in [-1, 1]\}$ and $B = \{(x, \sin(1/x)) \mid x \in (0, \pi/2]\}$.
   (a) Show that X is a compact space.
   (b) Prove that X is connected but not path-connected.

[1] This doesn't satisfy the definiton of being a topology...