# KSU Quals — Algebra

2015 June—2021 August

# Contents

# 1 2021 August

1. Prove that $S_4$ is generated by (1234) and (1243).

2. Let $G$ be a group of order 30, and let $P$ and $Q$ be its Sylow subgroups of orders 5 and 3 respectively. Show that either $P$ or $Q$ is normal. Deduce that $G$ contains a cyclic subgroup of order 15.

3. Let $R$ be a commutative ring with identity $1 \neq 0$. Show that an ideal $M$ in $R$ is maximal if and only if the quotient ring $R/M$ is a field.

4. Let $M \subset \mathbb{C}[x, y]$ be the module over the ring $\mathbb{C}[x, y]$ consisting of all polynomials that are sums of monomials of degree least 3. For example, the polynomial $xy^2 + 2x^2y^3 - y^{10}$ is in $M$, but the polynomial $x + 2x^2y^3 - y^{10}$ is not in $M$ since the degree of the monomial $x$ is 1. Find a finite set of generators for $M$.

5. Let $F$ be the splitting field of $x^5 + 1$ over $\mathbb{Q}$. Show that $F$ is a Galois extension of $\mathbb{Q}$ and find $\mathrm{Gal}(F/\mathbb{Q})$.

6. Let $U \subset V \subset W$ be vector spaces over a field $F$ with $\dim U = 1$, $\dim V = 2$, $\dim W = 3$. Let $T : W \to W$ be a linear transformation such that $T(W) \subset U$ and $T(V) = 0$. Find the Jordan Normal Form of $T$ assuming $T \neq 0$.

# 2 2021 June

1. Let $G$ be any group. Prove that the map from $G$ to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if $G$ is abelian.

2. Show that if $H$ and $K$ are subgroups of a group, then $HK$ is a subgroup if and only if $HK = KH$.

3. Let $R$ be the quadratic integer ring $\mathbb{Z}[\sqrt{-5}]$. Recall that two elements $a$ and $b$ of a commutative ring are associate if $a = bc$ where $c$ is an invertible element. Prove that $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are irreducibles in $R$, no two of which are associate in $R$, and that $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ are two distinct factorizations of 6 into irreducibles in $R$.

4. Let $R$ be the subset of the ring of $2 \times 2$ matrices with integer coefficients that consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with even $b$, $c$. Let $M = \mathbb{Z} \times \mathbb{Z}$ viewed as a module over $R$ with the standard linear action:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$
Is $M$ a cyclic module? If yes, provide a generator, if no, prove it is not.

5. Let $E \subset F$ be a Galois extension with Galois group $S_3 \times \mathbb{Z}_3$.
   a) Find the number of distinct fields $K$ with $E \subset K \subset F$ such that $E \subset K$ is an extension of degree 3.
   b) Find the number of distinct fields $K$ with $E \subset K \subset F$ such that $E \subset K$ is a Galois extension of degree 3.

6. Let $A$ be a complex $n \times n$ matrix such that $A^2 = \mathrm{Id}$.
   a) Prove that $A$ is diagonalizable.
   b) Describe all possibilities for the characteristic polynomial of $A$.

# 3 2020 August

1. Let $J = J_n(0)$ be the Jordan block matrix $n \times n$ with eigenvalue 0. Find the Jordan canonical form of $J^2$. (**Hint:** consider odd and even cases separately.)

2. Describe all groups of order 9 up to isomorphism. For one of these groups exhibit two non-commuting automorphisms.

3. Let $G$ be a group. By a maximal proper subgroup of $G$ we mean a subgroup $M \subset G$ such that $M \neq G$ and the only subgroups of $G$ containing $M$ are $M$ and $G$.

   a) Describe all the maximal subgroups of the dihedral group of order 10.

   b) Show that if a maximal subgroup $M \subset G$ is normal, then the index of $M$ in $G$ is finite and prime.

4. Let $K$ be the splitting field of the polynomial $x^4 - x^2 - 1$ over $\mathbb{Q}$. Compute the Galois group of the extension $K/\mathbb{Q}$.

5. Recall that an $R$-module $N$ is called *Noetherian* if for any growing tower of submodules

   $$L_1 \subset L_2 \subset \cdots \subset L_k \subset \cdots \subset M$$

   there exists a positive integer $n$ such that $L_n = L_{n+1} = \ldots$.

   Let $R$ be a commutative ring with identity, $M$ be an $R$-module, and $f : M \to M$ be a surjective $R$-module homomorphism.

   a) Prove that if in addition $M$ is Noetherian then $f$ is an isomorphism.

   b) Provide an example of $R$, $M$ and $f$, where $M$ is not Noetherian over $R$ and $f$ is not an isomorphism.

6. Describe all zero-divisors and all units in the quotient rings

   a) $R := \mathbb{Q}[x]/(x^2 - 1)$,

   b) $S := \mathbb{Q}[x]/(x^2 + 1)$.

# 4 2020 June

1. Prove or disprove the following: a group is abelian if and only if every one of its subgroups is normal.

2. Describe all groups of order 55 up to isomorphism.

3. Let $R = \mathbb{Z}[\sqrt{-6}]$.

   a) Show that 2 and $\sqrt{-6}$ are irreducibles in $R$. Hint: use the norm.

   b) Show that $R$ is not a Unique Factorization Domain (hence not a P.I.D.)

   c) Give an explicit ideal in $R$ which is not principal.

4. Let $R$ be a Principal Ideal Domain, $p \in R$ be a prime, and $a \in R \setminus \{0\}$ be a non-zero element. Let $n$ be the maximal integer such that $a$ is divisible by $p^n$. Consider the $R$-modules $M := R/(a)$ and

$$p^k M := \{c \in M \mid \exists b \in M,\ c = p^k b\}$$

   for non-negative integers $k$. Prove that

$$p^k M = \begin{cases} (p^k)/(a) & k < n, \\ (p^n)/(a) & k \geq n \end{cases}$$

5. Find the Jordan Canonical Form of the matrix

$$M = \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Does there exist a matrix $L$ with the same characteristic and minimal polynomials as $M$, but not similar to $M$?

6. Let $n$ be a positive integer, $p_n := x^4 + n$, and $E_n$ be the splitting field of $p_n$ over $\mathbb{Q}$.

   a) Find all positive integers $n$ such that $p_n$ is irreducible over $\mathbb{Q}$.

   b) Show that $E_n$ coincides with the splitting field of $x^n - 4n$ for all positive integers $n$.

   c) Find all positive integers $n$ such that $[E_n : \mathbb{Q}] = 4$.

# 5 2019 August

1. a) Let $p$ be a prime number. Show that the dihedral group $D_{2p}$ with $2p$ elements has exactly one non-trivial normal subgroup.

   b) Show that if $n$ is not prime, then $D_{2n}$ has more than one non-trivial normal subgroup.

2. Suppose $G$ is a group of order 45 containing an element of order 9. Prove that $G$ is a cyclic group.

3. Let $R$ be a commutative ring with identity. Suppose $P, I_1, I_2, \ldots, I_n$ are ideals in $R$ with $P$ a prime ideal. Prove that if $I_1 I_2 \cdots I_n \subseteq P$ then $I_j \subseteq P$ for some $1 \leq j \leq n$.

4. Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ with basis $\{e_1, \ldots, e_n\}$ and $S : V \to V$ a linear operator defined by the property $S(e_i) = e_{i+1}$ for $1 \leq i \leq n-1$ and $S(e_n) = e_1$.

   a) What are the characteristic and minimal polynomials of $S$?

   b) What is the Jordan canonical form of $S$?

5. Let $R$ be a PID, and $M$ a finitely generated $R$-module. Recall that the torsion submodule of $M$ is
$$\operatorname{Tor}(M) = \{m \in M : \text{there is a nonzero } r \in R \text{ such that } rm = 0\}$$

   a) Show that if $\operatorname{Hom}(M, R) = 0$, then $M$ is a torsion module (i.e., $M = \operatorname{Tor}(M)$).

   b) Prove that there is an $R$-module isomorphism
$$\operatorname{Hom}(M, R) \cong \frac{M}{\operatorname{Tor}(M)}$$

6. Let $F$ be a field of characteristic zero, and $f(x) \in F[x]$ an irreducible polynomial of degree 4.

   a) What are the possible degrees of the splitting field of $f(x)$? Explain your response.

   b) Let $K$ be a splitting field of $f(x)$ such that $K$ is a quadratic extension of the intermediate field $E$ which is not Galois over $F$. Suppose also that every non-trivial intermediate field has even degree over $F$. What is the Galois group of $K$ over $F$?

# 6 2019 June

1. Suppose $G$ is a finite abelian group and there is no non-trivial isomorphism $\psi : G \to G$ which has order 2. Prove that $G$ is either the trivial group $\{1\}$ or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

2. Suppose $V$ is an $n$-dimensional vector space over $F$ with $n \geq 1$ and $T : V \to V$ is a linear operator with $\ker(T) = \operatorname{im}(T)$.

   a) Show $\dim V$ is even.

   b) Find the characteristic polynomial of $T$.

   c) Find the minimal polynomial of $T$.

   d) what is the Jordan normal form of $T$? Explain your answer.

3. Let $R$ be a PID with $I$ and $J$ ideals in $R$. Prove that $IJ = I \cap J$ if and only if $I + J = R$.

4. Let $R$ be a commutative ring with unit and $M$ an $R$-module.

   a) For $m \in M$, define the annihilator $\operatorname{Ann}_R(m)$ of $m$ as the subset

   $$\{r \in R : rm = 0\}$$

   Prove that $\operatorname{Ann}_R(m)$ is an ideal in $R$.

   b) Suppose $M$ is generated by $m$ as an $R$-module. If $\operatorname{Ann}_R(m)$ is a maximal ideal show that $M$ has no non-trivial submodules.

5. Let $F$ be a field, $f(x)$ the polynomial $x^4 + 1 \in F[x]$ and $K$ its splitting field.

   a) Prove that $f(x)$ is separable if and only if the characteristic of $F$ does not equal 2.

   b) Prove that $[K : F] \leq 4$.

6. Prove that $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}$ is a Galois extension and find its Galois group.

# 7 2018 August

1. Consider two permutations in $S_{100}$: let $\sigma = (1\ 2)$ be the transposition exchanging 1 and 2 and let $\tau = (2\ 3\ \dots\ 100)$ be the cycle that sends 2 to 3, 3 to 4, etc. and 100 to 2.
   a) Where does $\tau^{20}$ send 18?
   b) Find all permutations that you get by conjugating $\sigma$ by powers of $\tau$.
2. a) Let $I$ be the ideal in $\mathbb{R}[x]$ generated by $x^3 + 2x^2 + x$ and $x^3 + 3x^2 + 3x + 1$. Can this ideal be generated by one element?
   b) Is $I$ a prime ideal?
3. Let $I$ and $J$ be ideals in a PID $R$. Suppose $M = R/I$ and $N = R/J$. Show that if $\mathrm{Hom}(M, N) = 0$ then the ideal
$$I + J = \{a + b : a \in I,\, b \in J\}$$
   coincides with the ring $R$.
4. Let $U$, $V$, and $W$ be complex vector spaces. Let $A : U \to V$ and $B : V \to W$ be linear transformations. Given that the dimension of the kernel of $A$ is 2, the dimension of the kernel of $BA$ is 3, and the dimenison of $V$ is 4, list all the possibilities for the rank of $BA$.
5. Show that if two complex matrices are similar, then they have the same characteristic polynomial.
6. Let $F \subset E$ be a Galois extension of fields with the Galois group $D_8$ (the dihedral group with 8 elements). Find the number of distinct fields $K$ such that $F \subset K \subset E$ and $K$ is an Galois extension of $F$ of index 4. Justify your answer.

# 8 2018 June

1. Let a group $G$ with 35 elements act on a set $X$ with 18 elements. Prove that there is an element $x \in X$ whose stabilizer is $G$.

2. a) Let $G$ be a finite[1] abelian group. Show that the following are equivalent:

      i. $G$ is not cyclic.

      ii. There is a prime number $q$ such that there are more than $q-1$ elements of order $q$ in $G$.

   b) Let $p$ be prime. Show that the multiplicative group of units in $\mathbb{Z}_p$ is cyclic.

   c) Is it true that the multiplicative group of units in $\mathbb{Z}_n$ is cyclic for any $n$? Prove it or provide a counterexample.

3. a) Is $\mathbb{Z}[x, y]$ a Euclidean domain? Prove your answer.

   b) Is $\mathbb{Z}[x, y]$ a principal ideal domain? Prove your answer.

   c) Is $\mathbb{Z}[x, y]$ a unique factorization domain? You don't have to prove your answer.

4. Consider the ring $R = \mathbb{C}[x]$. Denote by $M = \mathbb{C}[x, y]/(x^2 + y^2)$. Then $M$ is a module over $R$ with the module structure given by multiplication modulo $x^2 + y^2$. Show that $M$ is finitely generated over $R$ and find a finite set of generators.

5. Let $A$ be an $n \times n$ matrix of a complex linear transformation such that $A^2 = A$. Prove that $A$ is diagonalizable.

6. Let $p, q$ be prime numbers. Let $E$ be the field with $p^q$ elements and let $F \subset E$ be the field with $p$ elements.

   a) Find the index $[E : F]$.

   b) Let $\alpha$ be any element of $E$ that does not lie in $F$. Prove that the degree of the minimal polynomial of $\alpha$ is $q$.

   c) Prove that $E/F$ is a Galois extension and find its Galois group.

[1] The exam forgot to include this condition.

# 9 2017 August

1. For $G$ a group, we say that $g \in G$ is an involution if $g^2 = e$ where $e$ is the identity element. Suppose $G$ is a finite group such that all elements of $G$ are involutions.

   a) Prove that $|G| = 2^k$ for some integer $k \geq 0$.

   b) Prove that $G$ is commutative.

2. Let $I$ and $J$ be ideals of $R$. We write

$$I + J = \{i + j : i \in I,\ j \in J\},$$

   and

$$I * J = \{ij : i \in I,\ j \in J\}.$$

   a) Is $I + J$ necessarily an ideal of $R$? Prove or provide a counterexample.

   b) Is $I * J$ necessarily an ideal of $R$? Prove or provide a counterexmple.

3. Let $A : V \to V$ be a linear transformation of a finite-dimensional vector space $V$ satisfying the property $A^2 = A$.

   a) Prove that $\operatorname{Im} A \cap \ker A = \{0\}$.

   b) Prove that $V = \operatorname{Im} A \oplus \ker A$.

   c) Suppose $\dim V = n$ and $\operatorname{rank} A = k$. What is the Jordan form of $A$?

4. Provide an example of three fields $F \subset E \subset L$ such that $E/F$ and $L/E$ are Galois, but $L/F$ is not.

   **Hint:** One can use subfields of the splitting field of $x^4 + 2$ for this problem.

5. Let $R$ be a ring with identity and let $M$ be a left $R$-module. Recall that the *annihilator* of $M$ in $R$ is

$$\operatorname{ann}_R(M) := \{r \in R : \forall m \in M,\ rm = 0\}.$$

   a) Prove that $\operatorname{ann}_R(M)$ is a two-sided ideal of $R$.

   b) Note that an abelian group is a $\mathbb{Z}$-module. How many possibilties, up to isomorphism, are there for an abelian group $M$ of order 400 with $\operatorname{ann}_{\mathbb{Z}}(M)$ the ideal generated by $20 \in \mathbb{Z}$?

6. a) How many distinct actions of the group $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$?

   b) How many distinct **transitive** group actions of $\mathbb{Z}$ are there on the set $\{1, 2, 3, 4\}$? (Recall that an action of a group $G$ on a set $X$ is *transitive* if for every $x \in X$ we have $\{gx : g \in G\} = X$.)

# 10 2017 June

1. Let $F$ be a field, and consider the ring $F[x]$ of polynomials in one variable with coefficients in $F$.
   a) Show that $F[x]$ is a vector space of infinite dimension over $F$.
   b) Construct a linear transformation $\phi : F[x] \to F[x]$ which is injective, but not surjective.
   c) Construct a linear transformation $\psi : F[x] \to F[x]$ which is surjective, but not injective.

2. a) Let $K/F$ be a field extension, and $M$ and $N$ be square matrices over $F$. Show that $M$ and $N$ are similar over $F$ if and only if they are similar over $K$.

   **Hint:** What do you know about the Rational Canonical Form?

   b) Let $M$ be a square matrix over $\mathbb{R}$. Show that $M$ is similar to its transpose.

   **Hint:** Use Part (a), with $K = \mathbb{C}$ and the Jordan Canonical Form.

3. Let $p$ be a positive prime integer. Consider the set

$$G := \{\theta \in \mathbb{C} : \exists n \in \mathbb{Z}_{\geq 0}, \theta^{p^n} = 1\}$$

   a) Show that $G$ is an infinite group under multiplication.
   b) Show that every proper subgroup of $G$ is finite.

4. Let $N \lhd G$ be a normal subgroup of a group $G$, and let $P < N$ be a Sylow subgroup of $N$ for some prime number $p$. Show that $G = N_G(P)N$, i.e. that any element $g \in G$ can be written as a product $g = hn$, where $n \in N$ and $h$ is such that $h^{-1}Ph = P$.

   **Hint:** Given a $g \in G$, what can one say about the conjugate subgroup $g^{-1}Pg$? Where does it lie?

5. a) Show that the polynomials $x^4 + 2$ and $x^4 - 8$ have the same splitting field $K \subset \mathbb{C}$ over $\mathbb{Q}$.
   b) Find $[K : \mathbb{Q}]$ and compute the Galois group $G$ of the extension.

   **Hint:** Use that $G$ is a subgroup of $S_4$.

   c) How many subfields $E \subset K$ such that $[K : E] = 2$ exist? Identify all of them.

6. Let $R$ be a commutative ring with 1. Denote

$$I := \{r \in R : \exists n \in \mathbb{Z}_{>0},\ r^n = 0\}.$$

   a) Show that $I$ is an ideal.
   b) Show that if $I$ is maximal, then for every $x \in R$ either $x \in I$ or $x$ is a unit.

# 11 2016 August

1. Let $R = k[x, y]$ be the ring of polynomials in two variables over a field $k$. Consider the ideal $I = (x, y)$. View it as a module over $R$.
   a) Check that the map $F : R^2 \to I$ given by $F : (f, g) \mapsto xf + yg$ is a homomorphism of modules.
   b) Check that $F$ is surjective and that $\ker F$ is isomorphic to $R$ as an $R$-module.
2. Let $I \subset \mathbb{C}[x]$ be the ideal generated by $x^3 + x^2 - 2x$. Consider the factor space $V = \mathbb{C}[x]/I$.
   a) Find the dimension and a basis of $V$ as a vector space over $\mathbb{C}$.
   b) Consider the operation $\varphi : V \to V$ given by multiplication by $x$. Compute the matrix of $\varphi$ in the basis constructed in the previous part. What are the rank and nullity of $\varphi$?
   c) Determine the eigenvalues of the operator $\varphi$.
3. Recall, that the center $Z$ of a group $G$ is defined by

   $$Z := \{c \in G : \forall g \in G, cg = gc\}$$

   a) Prove that $Z$ is a subgroup of $G$.
   b) Consider the action of $G$ on itself by conjugation, i.e. an element $g \in G$ acts on an element $h \in G$ by $h \mapsto g^{-1}hg$.

      Show that an element of $h \in G$ belongs to $Z$ if and only if the $G$-orbit of $h$ under the conjugation action consists of one element.
   c) Suppose that $G$ is of order $p^k$, where $p$ is a prime number. Show that the center $Z$ contains more than one element.

   Hint: use the Class Equation or the orbits of the conjugation action, and divisibility by $p$.
4. Let $H$ be a normal subgroup of a group $G$ of index 4. Show that there are either exactly 3 or exactly 5 subgroups of $G$ containing $H$ (including $G$ and $H$ themselves).
5. Let $E$ be the splitting field of $f(x) := x^4 + 7$ over $\mathbb{Q}$.
   a) Find all zeros of $f(x)$ in $\mathbb{C}$. (Hint: Use de Moivre's formula.)
   b) Prove that $\sqrt[4]{28} \in E$, $\sqrt[4]{28}i \in E$, and then that $i \in E$.
   c) Show that $\mathbb{Q}(\sqrt[4]{28})$ is a subfield of $E$ of degree 4 over $\mathbb{Q}$.
   d) Show that $E = \mathbb{Q}(\sqrt[4]{28}, i)$, $[E : \mathbb{Q}] = 8$, and find a basis for $E$ over $\mathbb{Q}$.
6. Let $R$ be a commutative ring with unity and $I$, $J$ be ideals in $R$ such that $I + J = R$. (Recall $I + J = \{a + b : a \in I, b \in J\}$.) Prove that

   $$R/(I \cap J) \simeq R/I \times R/J.$$

# 12 2016 June

1. a) Let $H$ and $K$ be normal subgroups of a finite group $G$. Suppose that $H \cap K = \{1\}$ and the order of $G$ equals the product of orders of $H$ and $K$. Show that $G \simeq H \times K$.

   b) Let $p$ and $q$ be positive prime integers, such that $p < q$ and $p \nmid q - 1$. Show that all groups of order $pq$ are isomorphic to each other.
   Hint: use Sylow's theorems and part a).

2. a) Consider the field $\mathbb{Q}(\sqrt[3]{-3})$. Show that this field is NOT a normal extension of $\mathbb{Q}$. (Here $\sqrt[3]{-3}$ is the real cubic root of $-3$.)

   b) Let $F$ be the Galois closure of the field $\mathbb{Q}(\sqrt[3]{-3})$. Show that $F$ is isomorphic to $\mathbb{Q}[x]/(x^6 + 3)$.
   Hint: we know that $\sqrt[3]{-3} \in F$. Use the geometry of complex numbers to show that also $\sqrt{-3} \in F$, and use it to deduce that $\sqrt[6]{-3} \in F$.

3. Let $k[x, y]$ be the ring of polynomials in two variables, where $k$ is a field of characteristic not equal to 2. Consider the ideal $I = (x^2 - y, x^2 + y + 2)$.

   a) Assume that $-1$ is a square in $k$. Show that $I = I_+ \cap I_- = I_+ I_-$, where $I_+ = (x + \sqrt{-1}, y + 1)$, $I_- = (x - \sqrt{-1}, y + 1)$. Is $I$ a prime ideal? Provide a proof.

   b) Assume that $-1$ is NOT a square in $k$. Show that $I$ is a maximal ideal in $k[x, y]$. Show that $K := k[x, y]/I$ is a field and is isomorphic to $k[x]/(x^2 + 1)$.

4. Let $M \in M_{n \times m}(k)$ be an $n \times m$ matrix with entries from a field $k$. Define the row rank of $M$ as the dimension of the subspace in $k^m$ spanned by the rows of $M$, and the column rank of $M$ as the dimension of the subspace in $k^n$ spanned by the columns of $M$. Show that these ranks are equal.

5. Let $A$ be a linear transformation of a complex 4-dimensional vector space, such that $A$ is NOT diagonalizable, and it satisfies the property $A^3 = A^2$.

   a) What eigenvalues might $A$ have?

   b) What Jordan blocks can the Jordan canonical form of $A$ have?

   c) What Jordan canonical form might $A$ have?

   Provide proofs.

6. Let $R$ be a principal ideal domain and let $I \subset R$ be a non-zero ideal in $R$. Prove that $I$ is isomorphic to $R$ as an $R$-module.

# 13 2016 January

1. $G$ is a group of order $351 = 3^3 \cdot 13$.
   (i) What are the orders of the Sylow subgroups of $G$? Must $G$ have a subgroup of order 9?
   (ii) Show that $G$ is not simple.
   (iii) Suppose that the order of $G$ is 351 and $G$ is cyclic. How many subgroups does $G$ have?

2. Suppose that $G$ and $H$ are groups and $\phi : G \to H$ is a *surjective* group homomorphism.
   (i) Prove that $K$, the kernel of $\phi$, is a normal subgroup of $G$.
   (ii) Prove the first isomorphism theorem, that is $G/K \simeq H$.
   (iii) Suppose that $B_i$ is a normal subgroup of $A_i$ for $i = 1, \ldots, n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \simeq A_1/B_1 \times \cdots \times A_n/B_n.$$

3. Let $K$ be the splitting field of $f(x) = x^3 - 5$ over $\mathbb{Q}$ and $G$ its Galois group.
   (i) Find $[K : \mathbb{Q}]$.
   (ii) Describe the elements of $G$.
   (iii) Find the proper subgroups of $G$ and the corresponding subfields of $K$ under the Galois correspondence.

4. Suppose that $V$ is an $n$-dimensional vector space $V$ over a field $F$, and $W$ is an $m$-dimensional subspace of $V$.
   (i) Prove that a basis for $W$ can be extended to a base for $V$.
   (ii) Prove that $V/W$ is an $F$-vector space of dimension $n - m$.

5. Suppose that $R$ is a commutative ring with unity.
   (i) Prove that the ideal in $R[x]$ generated by $x$ is a prime ideal iff $R$ is an integral domain.
   (ii) Prove that $\mathbb{Q}[x]$ is a principal ideal domain. Give a generator for the ideal generated by two polynomials $x^2 - 4x + 3$ and $x^2 - 9$.
   (iii) Show that $\mathbb{Z}[x]$ is not a principal ideal domain.

6. Let $A$ be a linear transformation of a complex finite-dimensional vector space satisfying the property $A^2 = A$.
   (i) Prove that $\operatorname{Im} A \cap \operatorname{Ker} A = \{0\}$.
   (ii) Prove that $V = \operatorname{Im} A \oplus \operatorname{Ker} A$.
   (iii) Describe how the Jordan canonical form for $A$ looks like.

# 14 2015 August

1. Let $H$ and $K$ be subgroups of a group $G$ and let $HK = \{hk : h \in H,\ k \in K\}$.
   (i) Prove that if $HK = KH$ then $HK$ is a subgroup of $G$.
   (ii) Prove that if $H, K \lhd G$ and $H \cap K = \{1\}$ then $hk = kh \quad \forall h \in H,\ k \in K$ and $HK \simeq H \times K$.

2. Prove that a group of order $132 = 2^2 \cdot 3 \cdot 11$ is not simple.

3. Let $R$ be a commutative ring with unity.
   (i) Prove that $R$ is a field iff the only ideals of $R$ are $\{0\}$ and $R$.
   (ii) Prove that $(x)$ is a maximal ideal in the polynomial ring $R[x]$ iff $R$ is a field.

4. Let $K$ be the splitting field of $f(x) = x^4 - 3$ over $\mathbb{Q}$.
   (i) Find $[K : \mathbb{Q}]$ and $[K : \mathbb{Q}(\sqrt{3})]$.
   (ii) Find the group of automorphisms of $K$ that fix $\mathbb{Q}(\sqrt{3})$. Find its proper subgroups and the corresponding fields between $\mathbb{Q}(\sqrt{3})$ and $K$ under the Galois correspondence.

5. Suppose that $V$ and $W$ are finite dimensional $F$-vector spaces and $T : V \to W$ a linear transformation.
   (i) Prove that the kernel $\mathrm{Ker}(T)$ and image $T(V)$ are subspaces of $V$ and $W$ respectively.
   (ii) State and prove the relationship (rank-nullity theorem) between the dimensions.

6. Let $R$ be an integral domain and $M$ a unital (unitary) left $R$-module. For a submodule $N$ of $M$ or ideal $I$ of $R$ define the annihilator

$$\mathrm{Ann}_R(N) = \{a \in R : an = 0 \quad \forall n \in N\}, \quad \mathscr{A}nn_M(I) = \{m \in M : cm = 0 \quad \forall c \in I\}.$$

   i) Prove that $\mathrm{Ann}_R(N)$ is an ideal of $R$ and $\mathscr{A}nn_M(I)$ is a submodule of $M$.
   ii) If $M$ is a free $R$-module what is $\mathrm{Ann}_R(N)$ and $\mathscr{A}nn_M(I)$?
   iii) For the $\mathbb{Z}$-module $M = \mathbb{Z}_{12} \times \mathbb{Z}_{15} \times \mathbb{Z}_{50}$ what is $\mathrm{Ann}_{\mathbb{Z}}(M)$? What is $\mathscr{A}nn_M(3\mathbb{Z})$? (Here $\mathbb{Z}_n$ denotes the integers mod $n$ and the module action on $M$ is just $r(a, b, c) = (ra, rb, rc)$.)

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1. Let $G$ be a group of order $245 = 5 \cdot 7^2$.
   (i) How many Sylow subgroups does $G$ have?
   (ii) How many different abelian $G$ are there up to isomorphism?
   (iii) Can $G$ be non-abelian? Explain.

2. Prove Cayley's Theorem; namely that any group of order $n$ is isomorphic to a subgroup of the permutation group $S_n$.

3. Let $K$ be the splitting field of $f(x) = (x^2 - 2)(x^2 + 3)$ over $\mathbb{Q}$ and $G$ its Galois group.
   (i) Find $[K : \mathbb{Q}]$ and identify $G$.
   (ii) Find the proper subgroups of $G$ and the corresponding subfields of $K$ under the Galois correspondence.

4. Suppose that $U$ and $W$ are finite dimensional subspaces of an $F$-vector space $V$. Prove that
$$\dim_F(U + W) = \dim_F U + \dim_F W - \dim_F(U \cap W).$$

5. Suppose that $R$ is a commutative ring with unity and $I$ is an ideal of $R$.
   (i) Prove that $R/I$ is a field iff $I$ is a maximal ideal.
   (ii) Is $\mathbb{Z}_5[x]/(x^2 + 1)$ a field?

6. Find the Jordan canonical form for $A = \begin{bmatrix} -1 & 0 & 9 \\ 7 & 6 & -25 \\ 1 & 1 & -2 \end{bmatrix}$ over $\mathbb{C}$. Are there $3 \times 3$ matrices with the same eigenvalues as $A$ which are not similar to $A$ over $\mathbb{C}$? Give the Jordan canonical form for each similarity class.

# 16 Sample

1. Recall that the *centralizer* of a subgroup $H$ of a group $G$ is the set

$$C_c(H) = \{g \in G \mid \forall h \in H \ ghg^{-1} = h\}.$$

   a) Prove that if $H$ is normal in $G$ then $C_c(H)$ is normal in $G$.

   b) Assume $H$ is normal. Prove that $G/C_c(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, the group of automorphisms of $H$.

2. Let $T$ be an invertible linear operator on a finite-dimensional vector space. Prove that $T^{-1}$ is diagonalizable if and only if $T$ is diagonalizable.

3. Let $M_{n\times n}(\mathbb{F})$ denote the ring of $n \times n$ matrices with entries in a field $\mathbb{F}$. Prove that this ring has no two-sided ideals except $M_{n\times n}(\mathbb{F})$ and $\{0\}$.

4. Find the splitting field and Galois group of $f(x) = (x^4 - 4)$ and identify the intermediate fields corresponding to its subgroups.

5. Prove that if $G$ is a group of order $p^n$, where $p$ is a prime number, then the center of $G$, $Z(G)$, cannot equal $\{e\}$. Hint: use the class equation.

6. Let $I$ be a non-zero ideal of a commutative ring $R$ with identity. Prove that $I$ is a free $R$-module if and only if $I = Ra$ for some $a \in R$ that is not a zero divisor.