KSU Quals — Geometry of Manifolds (Older system)

1994 Fall—2003 Fall

Contents

1 2003 Fall (Old System)

Examiners: Auckly & Vidussi

1. Let $\pi : S^2 \to \mathbb{R}P^2$ be the standard covering projection. Prove that there is no map $f : \mathbb{R}P^2 \to S^2$ so that $\pi \circ f = \mathrm{id}$.

2. Recall that

$$\begin{aligned} \mathrm{d}\alpha(X_0, \ldots, X_p) = & \sum_{k=0}^{p} (-1)^k X_k \alpha(X_0, \ldots, \widehat{X_k}, \ldots, X_p) \\ & + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p). \end{aligned}$$

   Prove that $\mathrm{d}\alpha(X_0, \ldots, X_p) = \sum_{k=0}^{p} (-1)^k (\nabla_{X_k}\alpha)(X_0, \ldots, \widehat{X_k}, \ldots, X_p)$.

3. a) Give the definition of a Lie group.
   b) Give the definition of a Lie algebra.
   c) Give the definition of a representation of a Lie group, $\mu : G \to \mathrm{Aut}(V)$.
   d) Give the definition of a representation of a Lie algebra, $\dot{\mu} : \mathfrak{g} \to \mathrm{End}(V)$.
   e) Define the Lie algebra of a Lie group.
   f) Describe how a representation of a Lie group induces a representation of the corresponding Lie algebra and prove that the induced representation is a Lie algebra representation.

4. Prove that the holonomy of a simply connected Riemannian manifold is connected.

5. Let $X = \dfrac{\partial}{\partial x}$ and $Y = \dfrac{\partial}{\partial x} + (x^2 + 1)\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$.
   a) Compute $[X, Y]$.
   b) Compute the flow of X.
   c) Compute the flow of Y.
   d) Let $F^Z : \mathbb{R}^M \to M$ be the flow of a vector field Z. If $F^Z_s \circ F^W_t = F^W_t \circ F^Z_s$ for all s and t, what can you say about $[Z, W]$? Why?
   e) Is there a function $f_Y : \mathbb{R}^2 \to \mathbb{R}$ so that $F^{fX}_t \circ F^Y_s = F^Y_s \circ F^{fX}_t$ for all s and t? Why?

6. Let $f : \mathbb{R}^3 \to \mathbb{R} : f(x, y, z) = xy - z$.
   $\Sigma = f^{-1}(0) \wedge \{(x, y, z) \mid x^2 + y^2 \leq 1\}$
   a) Verify that Σ is a manifold.

b) Compare the orientation induced on Σ using $\nabla f/|\nabla f|$ and $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ with the orientation $\mathrm{d}x \wedge \mathrm{d}y$.

c) Compute $\displaystyle\int_\Sigma \frac{|\nabla f \circ \kappa|}{|\nabla f|} \mathrm{d}x\wedge\mathrm{d}y$ when Σ is oriented by $\mathrm{d}x\wedge\mathrm{d}y$. What does this represent?

7. The connected sum $M_1 \# M_2$ of two oriented n-manifolds M_1, M_2 is defined as $(M_1 \setminus \operatorname{int} B^n) \bigcup_{S^{n-1}} (M_2 \setminus \operatorname{int} B^n)$, where B^n is a ball in $M_1(M_2)$ and S^{n-1} is its boundary.

a) Show that if $n \geq 3$, then $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$.

b) Compute the fundamental group of $T^2 \# T^2$ (where T^2 is the 2-dimensional torus). Hint: What is $\pi_1(T^2 \setminus \operatorname{int} D^2)$?

8. a) Show that there exists a natural map $S^1 \times S^3 \to U(2)$ with discrete fiber by using the Lie group structure of S^1 and S^3.

b) What is the fiber?

c) Using the result above, what is $\pi_1 U(2)$?

2 2003 Spring

Examiners: Auckly & Vidussi

1. a) State the definition of a Lie algebra.

   b) Let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$; $\mathrm{ad}(X)(Y) = [X, Y]$ be the adjoint representation. Prove that

   $$\mathrm{Tr}\big(\mathrm{ad}([X, Y])\,\mathrm{ad}(Z)\big) = \mathrm{Tr}\big(\mathrm{ad}([Z, X])\,\mathrm{ad}(Y)\big).$$

2. Let $X = S^1 \times I/\sim$ with $(z, 0) \sim (z^3, 1) \quad \forall x \in S^1$.

   a) Construct a CW decomposition of X.

   b) Compute $\pi_1(X)$.

   c) Compute $H_*(X)$.

3. Let ∇ be the Levi-Civita connection on a Riemannian manifold. Define $Hf(X, Y) = X(Y f) - (\nabla_X Y)f$.

   a) Prove that Hf is symmetric i.e., $Hf(X, Y) = Hf(Y, X)$.

   b) Prove that Hf is tensorial i.e., $Hf(\varphi X, \psi Y) = \varphi\psi Hf(X, Y)$ for $\varphi, \psi \in C^\infty(M)$.

4. Let: $X = x^2y\partial_x - \partial_z$, $Y = xy^2\partial_y - \partial_z$, $Z = (1 + x^2)\partial_x - y(1 + x^2)\partial_z$.

   a) Find the integral curves of X.

   b) Define what it means for a distribution to be integrable at a point.

   c) Let E be the distribution spanned by X and Y, and let F be the distribution spanned by Y and Z. Test both distributions for integrability near the point $(1, 2, 3)$.

5. Compute $\displaystyle\int_{S^2} x^2z\,\mathrm{d}x\,\mathrm{d}y$ where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with orientation $\mathrm{d}x \wedge \mathrm{d}y$ at $(0, 0, 1)$.

6. Prove that $T^*(S^2 \times S^2)$ and $\mathbb{R}^8$ are not homeomorphic.

7. Let $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^3 + z^4 + w^5 = 6\}$.

   a) Prove that W is a manifold.

   b) What is $\dim_{\mathbb{R}} W$?

8. a) Prove that a real line bundle is trivial if and only if it admits a global nonvanishing section.

   b) Prove that the vector bundle $E \otimes E^*$ (where E is any vector bundle over a manifold M) is orientable.

9. Compute the sectional curvature of the metric $g = \frac{1}{y^4}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$.

3 2002 Fall

Examiners: Auckly and Crane

1. (A) What is $H^2(\mathbb{R}^3)$?

   (B) Write Stokes' Formula.

   (C) What kind of differential form can one integrate on a 3-surface embedded in a 5-manifold?

   (D) Which of the following topological surfaces are simply connected?
   a) S^3 b) RP^3 c) $S^1 \times S^2$ d) $\mathbb{C} - \{0\}$

   (E) Which lie algebras are contractible as topological spaces?

2. Compute $H^*(RP^2 \times S^2, \mathbb{Z})$ and $\pi_1(RP^2 \times S^2)$.

3. Let $F = \mathbb{R} \times (0, 2\pi)$ and let the metric g be given by $\dfrac{2z^2+1}{z^2+1}\,\mathrm{d}z^2 + (z^2+1)\,\mathrm{d}\theta^2$.

   (A) Compute $\nabla_{\partial_\theta}\partial_\theta$ for the Levi-Civita connection corresponding to g.

   (B) Compute the sectional curvature of (F, g).

4. Give an example of a topological space, every point of which has a neighborhood homeomorphic to $(0, 1)$, which is not a manifold.

5. Let (X, X_0) be a pointed topological space, G a lie group and $C^0[(X, X^0), (G, 1)]$ the set of continuous pointed maps. This forms a group under pointwise multiplication. Since π_1 is a functor, we obtain a map $\widetilde{\pi} : C^0[(X, X^0), (G, 1)] \to \mathrm{Hom}(\pi_1(X, X_0), \pi_1(G, 1))$ given by $\widetilde{\pi}, (u)([\ell(t)]) \to [u(\ell(t))]$ where $\ell(t)$ is a loop in (X, X_0). Prove that $\widetilde{\pi}$ is a group homomorphism when $\mathrm{Hom}[\pi_1(X), \pi_1(G)]$ is viewed as a group under pointwise multiplication. (Recall that π_1 of a lie group is abelian so this is in fact a group.)

6. Prove that if $p : G_1 \to G_2$ is a smooth homomorphism of connected lie groups which induces an isomorphism of lie algebras, then p is a covering projection. [Recall that a covering projection is a surjective continuous map such that for any $x \in G_2$ there is a neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets V_α and $p|_{V_\alpha}$ is a homeomorphism for each α onto U.]

7. Let X be a complete vector field on a manifold M. This means that X has a global flow, $\varphi : \mathbb{R} \times M \to M$. Let $f : M \to \mathbb{R}$ be a smooth function with $X(f)(p) = 1$ for all $p \in M$. Prove that $\phi_t(f^{-1}(a)) = f^{-1}(a + t)$ for all $a, t \in \mathbb{R}$. Give an example to show that the vector field must be complete for this to hold.

4 2002 Spring

Examiners: Auckly & Miller

1. Let $\alpha = z^3\,\mathrm{d}x \wedge \mathrm{d}y - y\,\mathrm{d}x \wedge \mathrm{d}z \in \Gamma(\wedge^1\mathbb{R}^3)$. Let $X = \partial_x + x\partial_z \in \Gamma(T\mathbb{R}^3)$.
Compute:

   a) $\mathrm{d}\alpha$

   b) $i_X\alpha$

   c) $L_X\,\mathrm{d}x$

   d) $L_X\,\mathrm{d}y$

   e) $L_X\,\mathrm{d}z$

   f) $L_X\alpha$

   g) $\displaystyle\int_{S^2}\alpha$
   [Here S^2 is oriented with $i_{(x\partial_x+y\partial_y+z\partial_z)}(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$.]

2. Find $\int_\Sigma \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}z$ when

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid z = 1 - (x^2 + y^2)^{2002},\ z \geq 0\}$$

   and $\Omega_\Sigma|_{(0,0,1)=\mathrm{d}x\wedge\mathrm{d}y}$.

3. Let $X = \mathbb{R}P^2 \vee S^1$ ($\vee$ is the 1 point union.)

   a) Compute $\pi_1(X)$.

   b) Construct a 2-fold cover of X, say $\widehat{X}$, with $H_2(\widehat{X};\mathbb{Z}) \neq 0$.

   c) Compute $H_*(X;\mathbb{Z})$.

   d) Compute $H_*(\widehat{X};\mathbb{Z})$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$; $f(x, y) = x^2 - y^2$. Let $g = \mathrm{d}x^2 + \mathrm{d}y^2$.

   a) Compute $\operatorname{grad} f$.

   b) Let $\alpha_n, \beta_n, \gamma_n : \mathbb{R} \to \mathbb{R}^2$ be integral curves of $\operatorname{grad} f$ with $\alpha_n(0) = (\frac{1}{n^2}, 1)$, $\beta_n(0) = (\frac{1}{n}, \frac{1}{n})$, $\gamma_n(0) = (1, \frac{1}{n^2})$. Find expressions for α_n, β_n and γ_n.

   c) Prove that $\alpha_n(\mathbb{R}) = \beta_n(\mathbb{R}) = \gamma_n(\mathbb{R})$.

   d) Compute $\lim_{n\to\infty}\alpha_n(t)$, $\lim_{n\to\infty}\beta_n(t)$ and $\lim_{n\to\infty}\gamma_n(t)$.

5. Let $0 < a < b$. The equations

$$\begin{aligned} x &= (b + a\cos\psi)\cos\theta \\ y &= (b + a\cos\psi)\sin\theta \\ z &= a\sin\psi, \quad \theta, \psi \in [0, 2\pi] \end{aligned}$$

   describe a surface in $\mathbb{R}^3$.

   a) What is this surface?

   b) Calculate the Gaussian curvature.

c) Write the equations for geodesics on this surface.

6. Let $\varphi : M \to N$ be a smooth map between connected, oriented, closed n-dimensional manifolds. Prove that:

$$\left(\int_M \varphi^*\alpha\right)\left(\int_N \beta\right) = \left(\int_M \varphi^*\beta\right)\left(\int_N \alpha\right)$$

for all $\alpha, \beta \in \Gamma(\wedge^n N)$.
Hint: Think about $H^n(M)$ and $H^n(N)$.

7. A vector bundle map, $J : TM \to TM$ is called an almost $\mathbb{C}$-structure if $J^2 = -\operatorname{id}$.

   a) If a manifold, M, admits an almost complex structure, what can be said about $\dim M$? Why?

   b) Prove that any manifold admitting an almost complex structure is orientable.
   Hint: Let g be a Riemannian metric on M and define $w(X, Y) = g(X, J(Y)) - g(Y, J(X))$. Use w to construct an orientation.

8. Let $X = -y\partial_x + x\partial_y + \partial_z$, $Y = z\partial_x + \partial_y$. Let $B = \operatorname{span}\{X, Y\}$.

   a) Is B integrable? If B is integrable, find the integral manifold through $(1, 0, 1)$.

   b) Find the flow of X.

   c) Find the flow of Y.

5 2001 Fall

Examiners: Auckly & Miller

1. Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (x + z^2, y - z^2) = (u, v)$ where (x, y, z) are coordinates on $\mathbb{R}^3$ and (u, v) are coordinates on $\mathbb{R}^2$. Let $\alpha = \mathrm{d}u + u\,\mathrm{d}v$. Find
   a) $f^*(\alpha)$
   b) $f^*(\mathrm{d}\alpha)$
   c) $\int_S f^*(\mathrm{d}\alpha)$ where S is the surface $x^2 + y^2 + z^2 = 1$, $z \geq 0$ oriented with the upward pointing normal.

2. a) Construct an example of a covering projection, $P : E \to X$ with $\mathrm{Deck}(E) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.
   b) Compute $H^\infty(X : \mathbb{R})$ for your example.

3. On $\mathbb{R}^3$ let $X = xz^2 \dfrac{\partial}{\partial x} + y \dfrac{\partial}{\partial y} + z \dfrac{\partial}{\partial z}$.
   a) Calculate $\displaystyle\int_B L_X(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$ where $B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ with orientation given by $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
   b) Compute the flow of X, $F : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$.

4. Consider a 2-dimensional Riemannian manifold (M, g) where M is an open subset of $\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$. Suppose g is such that $e_1 = \dfrac{\partial}{\partial x}$ and $e_2 = f(x, y) \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$ is an orthonormal frame.
   a) Find the coframe (θ^1, θ^2) dual to (e_1, e_2).
   b) Find a 1-form w so that $\mathrm{d}\theta^1 = w \wedge \theta^2$ and $\mathrm{d}\theta^2 = -w \wedge \theta^1$.
   c) Find a function $K(x, y)$ so that $\mathrm{d}w = -K\theta^1 \wedge \theta^2$.
   d) Show how to choose $f(x, y)$ so that $K(x, y) = -1$ for all (x, y).

5. Let K be a simplicial complex.
   a) Define the Euler characteristic of K.
   b) Let $p : L \to K$ be an n-fold simplicial cover. Show that $\chi(L) = n\chi(K)$.
   c) Determine all **closed** 2-manifolds that cover T^2.
   d) Give an example of a non-trivial cover $q : T^2 \to T^2$.
   e) Compute $\chi(S^n)$ as a function of n.
   f) Let G be a finite group that acts freely on S^{2k}. Assume that the map $\pi : S^{2k} \to S^{2k}/G$ is a covering projection. Prove that $G \cong \mathbb{Z}_2$ or G is trivial.

6. a) Given that X is a non-vanishing vector field on M, prove that there is a diffeomorphism, $f : M \to M$ without fixed points (i.e., x such that $f(x) = x$) that is homotopic to $\mathrm{id} : M \to M$.

b) Given that $f : S^n \to S^n$ has no fixed points, show that f is homotopic to $p : S^n \to S^n$; $p(x) = -x$.

c) Compute $H^n(p) : H^n(S^n, \mathbb{R}) \to H^n(S^n, \mathbb{R})$.

d) What can be said about the dimension of a sphere that admits a non-vanishing vector field.

e) Let G be a Lie group that acts freely (i.e., $x \cdot g = x$ implies $g = 1$) on S^{2k}. Prove that $\dim G = 0$.

7. Let G be a Lie group.

a) Define the Le algebra of G, say $\mathfrak{g}$.

b) Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Explain how to construct a distribution on G using $\mathfrak{h}$ and prove that this distribution is integrable.

c) Let H be the maximal connected leaf of the distribution from part b) passing through 1. Construct an atlas on H.

d) Prove that H is closed under multiplication.
Hint: Consider $L_g H := \{gh \mid h \in H\}$.

6 2001 Spring

Examiners: Auckly & Miller

1. On $\mathbb{R}^2$ let

$$X = y\,\frac{\partial}{\partial x} + y^2 x\,\frac{\partial}{\partial y}\,,$$
$$Y = (x+y)\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\,,$$
$$\alpha = -\,\mathrm{d}x + x\,\mathrm{d}y.$$

   Calculate: a) $[X,Y]$, b) $L_Y\alpha$, c) $L_Y(\mathrm{d}\alpha)$.

2. a) Construct an atlas for $S^2 \times S^1$.

   b) Parametrize S^2 by (θ,φ) where $x = \cos\theta\cos\varphi$, $y = \sin\theta\cos\varphi$, $z = \sin\varphi$. Parametrize $S^2 \times S^1$ by $(\theta_1,\varphi_1,\alpha)$ in a similar way ($\alpha \in (\theta, 2\pi)$) and define $f : S^2 \times S^1 \to S^2$; $f(\theta_1,\varphi_1,\alpha) = (2\theta_1,\varphi_1)$. Notice that $\mu = \frac{1}{4\pi}\sin\varphi\,\mathrm{d}\theta\wedge\mathrm{d}\varphi$ is the normalized area form on S^2.
   Compute $\displaystyle\int_{S^2\times S^1} \frac{1}{2\pi}\,\mathrm{d}\alpha \wedge f^*\mu$. (Use the orientation $\mathrm{d}\alpha\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi$.)

   c) Show that the answer remains unchanged if μ is replaced by $\mu + \mathrm{d}\omega$.

   d) What does this integral represent geometrically?

3. Let $g = \mathrm{d}t^2 + \sin^2 t\,\mathrm{d}\alpha^2 + \cos^2 t\,\mathrm{d}\beta^2$.

   a) Pick an orthonormal coframe, θ^k, suitable for computation with the metric, g.

   b) Compute the connection form, ω, relative to the coframe that you chose in part a).

   c) Compute the curvature form, Ω, relative to the same coframe.

4. Let E and F be distributions on (M,g) so that $X \in E$ and $Y \in F$ implies $g(X,Y) = 0$ and so that $X, Y \in \Gamma(E)$ implies $\nabla_X Y \in \Gamma(E)$ and $X, Y \in \Gamma(F)$ implies that $\nabla_X Y \in \Gamma(F)$.

   a) State Frobenius' Theorem.

   b) Show that E and F are integrable.

   c) Show that any point of M has a product neighborhood $V \times U$ so that $g|_{V\times U} = g|_v \oplus g|_u$.

5. Let X and Y be the vector fields on $\mathbb{R}^4 - \{0\} \times \mathbb{R}^2$ given by

$$X = x^2\frac{\partial}{\partial x^2} + x^3\frac{\partial}{\partial x^3}$$
$$Y = x^3\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^4}.$$

   Let E be the subbundle of the tangent bundle generated by X and Y.

a) Use the Frobenius Theorem to show that E is integrable.

b) Find parametric equations for the integral manifold containing the point $(1, 2, 3, 4)$.

6. a) State Sard's Theorem. Let $f : S^2 \to \mathbb{R}$ be a smooth map.

   b) Show that for any $y_0 \in \mathbb{R}$ and any $\epsilon > 0$ there is a $y \in B_\epsilon(y_0)$ so that $f^{-1}(y)$ is a finite disjoint union of circles.

7. a) Give an example of a space, X, with $\pi_1(Y) = \langle a, b, c \mid a^3 = 1,\ b^4 = 1,\ c^5 = 1,\ (abc)^2 = 1\rangle$.

   b) Give an example of a space, Y, with $\pi_1(Y) = \langle a, b \mid aba^{-1}b = 1\rangle$.

   c) Closed, Hausdorff, separable 2-manifolds have been classified. State the classification theorem.

   d) There is a closed 2-manifold with the fundamental group from part b). Which 2-manifold has this fundamental group?

7 2000 Fall

Instructions: For Part A, Short answers. Do all of them.
For Part B, Choose 3 (and only 3).

Part A.

1. What is the fundamental group of
   a) $S^2 \times S^2$
   b) $T^*(S^3 \times S^1)$ the total space of the cotangent bundle of $S^3 \times S^1$
   c) R^3 with 2 parallel lines deleted
2. A Riemannian metric on a manifold is a cross section of what bundle?
3. In R^3 with the standard euclidean flat metric, describe the flow and integral curves of a covariant-constant vector field. (Covariant constant means all covariant derivatives vanish).
4. Let Γ be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ in R^3 calculate $\int_\Gamma z\,\mathrm{d}x \wedge \mathrm{d}y - y\,\mathrm{d}z \wedge \mathrm{d}x$.
5. Find the scalar curvature of the surface $z = x^2 + y^2$ at $(0,0,0)$.
6. Can we integrate a 3-form on a surface in a 4-manifold? Why or why not?
7. a) What is the fiber dimension of the bundle of 5-forms on S^7?
   b) What is the fiber dimension of the bundle of 7-forms on S^7?
8. Are all vector spaces
   a) parallelizable? Why or why not?
   b) Simply connected? Why or why not?
9. What is the scalar curvature of the euclidean plane in polar coordinates.

Part B.

1. Let G be a Lie group. Prove G is orientable.
2. Compute the Levi-Civita connection at a point on the standard unit 2-sphere in R^3 in latitude-longitude coordinates.
3. Compute the De-Rham cohomology of $S^1 \times S^2$.
4. Show an explicit isomorphism between Lie algebras $so(3)$ and $su(2)$.
5. Let C be the 2-dimensional subbundle of the tangent bundle to R^4 determined by $V_1 = x_2 \dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_3}$ and $V_2 = \dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_3}$. Use the Frobenius theorem to determine if C is integrable.

8 1999 Fall

Examiners: Auckly & Miller

1. On $\mathbb{R}^2$ let

$$\begin{aligned} X &= x^2 y \frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y} \\ Y &= y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\ \alpha &= -y\,\mathrm{d}x + x\,\mathrm{d}y. \end{aligned}$$

   Calculate: a) $[X, Y]$ b) $L_Y \alpha$ c) $L_Y\,\mathrm{d}\alpha$

2. On $\mathbb{R}^2$ consider the metric $g = (1+x^2)\,\mathrm{d}x^2 + \frac{1}{2}(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + \mathrm{d}y^2$.
   a) Compute $\nabla_{\frac{\partial}{\partial x}}(y\,\mathrm{d}x)$.
   b) Calculate the sectional curvature.

3. On $\{(x,y) \mid x^2+y^2<1\}$, let $g = \dfrac{\mathrm{d}x^2+\mathrm{d}y^2}{(1-x^2-y^2)^2}$ be a metric.
   a) Show that $\phi(z) = \dfrac{az+b}{\overline{b}z+\overline{a}}$ is an isometry if $|a|^2 - |b|^2 = 1$ and $z = x+iy$.
   Hint: Show that if $w = \phi(z)$ is analytic then $\mathrm{d}w \otimes \mathrm{d}\overline{w} = |\phi'(z)|^2\,\mathrm{d}z \otimes \mathrm{d}\overline{z}$.
   b) Using this metric, compute the radius and the area of a circle with Euclidean radius $R < 1$ centered at the origin.

4. Let α be a differential 2-form on the 2-sphere S^2 with $\int_{S^2}\alpha = 1$. Suppose $f : S^3 \to S^2$ is smooth, S^3 the 3-sphere.
   a) Show that there exists a 1-form θ on S^3 such that $f^*\alpha = \mathrm{d}\theta$.
   b) Define $Q(f) = \int_{S^3}\theta \wedge \mathrm{d}\theta$. Show that this is independent of the choice of α and θ.
   Hint: First show that for α fixed it is independent of choice of θ.

5. Suppose that $F : N \to M$ is a smooth covering mapping and that M is a Riemannian manifold with metric g.
   a) Show that there exists a unique metric on N so that F is a local isometry.
   b) Suppose that N is connected and compact. Determine the relation between $\mathrm{vol}(M)$ and $\mathrm{vol}(N)$ in terms of the fundamental groups of M and N.

6. Suppose M is a smooth oriented n-dimensional manifold and X is a complete vector field which generates a 1-parameter group of diffeomorphisms $(F_t)_t$. Suppose that μ is a differential n-form on M and that U is a relatively compact ($\overline{U}$ is compact) open subset of M.

a) Show that $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{F_t(U)} \mu = \int_U L_X \mu$.

b) For $M = R^3$ and $\mu = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, the usual volume element, calculate an expression for $L_X\mu$ for any vector field $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$. Thus obtain a formula for $\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}(F_t(U))$.

7. Prove that smooth connected manifolds are topologically homogeneous. That is, given $p, q \in M$ there is a diffeomorphism $f : M \to M$ so that $f(p) = q$.

8. On $\mathbb{R}^2$ with coordinates (x^1, x^2) let a connection (Γ^i_{jk}) be given by $\Gamma^1_{11} = \frac{\partial f}{\partial x^1}$ and $\Gamma^2_{22} = \frac{\partial f}{\partial x^2}$ and all other $\Gamma^i_{jk} = 0$. Here $f : \mathbb{R}^2 \to R$ is some given function.

   Let $P_0(x^1_0, x^1_0)$ be a given point. If $v : [a, b] \to \mathbb{R}^2$ is a smooth curve such that $v(a) = v(b) = P_0$, let T_v be the 2×2 matrix which represents (with respect to $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right)$) the holonomy transformation $T_{p_0}\mathbb{R}^2 \to T_{p_0}\mathbb{R}^2$ of parallel transport around v.

   a) Show that T_v is a diagonal matrix with determinant equal to 1.

   b) For $f(x^1, x^2) = x^1 x^2$ let $v_c : [0, 4] \to \mathbb{R}^4$ be given by

$$v_c(t) = \begin{cases} ((\ln c)t,\ 0) & 0 \le t < 1, \\ (\ln c,\ t-1) & 1 \le t < 2, \\ ((\ln c)(3-t),\ 1) & 2 < t < 3, \\ (0,\ 4-t) & 3 \le 4 \le 4 \end{cases}$$

   Find T_{v_c}.

9 1999 Spring

Examiners: Miller and Auckly

Instructions: Work the first four and as many others as you can.

1. On $\mathbb{R}^3$ with coordinates (x, y, z), let

$$\begin{aligned} X &= y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y} \\ Y &= x^2 y\,\frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + y\,\frac{\partial}{\partial z} \\ \alpha &= (x^3 + y^3 + z^3)(\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}y \wedge \mathrm{d}z) \end{aligned}$$

   a) calculate the Lie bracket $[X, Y]$.
   b) describe the flow of the vector field Y through the point (x_0, y_0, z_0).
   c) calculate $\mathrm{d}\alpha$ as a function times the usual volume element.
   d) if $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f(s,t) = (s, t, s^2 + t^2)$, compute $f^*\alpha$.

2. On $\{(x, y) \mid x, y \in \mathbb{R},\ y > 0\}$ let g be the metric $g = y\,\mathrm{d}x^2 + \mathrm{d}y^2$. Determine the differential equations for parallel transport of a vector field $\xi = \xi^1\,\frac{\partial}{\partial x} + \xi^2\,\frac{\partial}{\partial y}$ along a curve $x = x(t)$, $y = y(t)$.

3. Let $C = \{(x, y, z) \mid x^2 + y^2 = 1\}$ with the orientation $\Omega_C = x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z$. Compute

$$\int_C \frac{(x+1)}{(1+z^2)(x^2+y^2)}(x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z).$$

4. Construct a cell complex X, with

$$\begin{aligned} \pi_1(X) &= \langle a, b \mid a^2 = b^3 \rangle \\ H_0(X) &= Z \end{aligned}$$

   and

$$H_2(X) = Z \oplus Z_2.$$

5. On a Riemannian manifold define the scalar curvature to be $S = -\sum_{n,k} g(R(e_n, e_k)e_n, e_k)$ where (e_n) is an orthonormal basis.
   a) Prove that S is an independent of the choice of basis.
   b) Let $S^n_r = \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}$, with the induced metric. Compute the scalar curvature of $S^2_a \times S^3_b$.

6. Let M be a simply connected manifold, ω a closed differential 2-form, X a vector field and H a smooth real valued function on M. Suppose they are related by $dH = i_X\omega = \omega(X, -)$. Further suppose that Y is a second vector field such that $L_Y\omega = 0$ and $Y(H) = 0$.

   a) Show that there is a smooth function f on M such that $i_Y\omega = \mathrm{d}f$.

   b) Show that the function f of part a) is constant along the flows of X.
   Note: If you do not remember the formula giving the Lie derivative L_Y acting on differential forms in terms of i_Y and d, you may ask at the cost of a penalty.

7. Let G be a Lie group and $\pi_G : P \to M$ be a principal G-bundle, and H be a closed subgroup of G. We say that the structural group of P may be reduced to H if and only if there is a principal H-bundle $\pi_H : Q \to M$ and a bundle map $i : Q \to P$ so that $i(q \cdot h) = i(q) \cdot h$. Let $E = P \times_G (G/H) = P \times (G/H)/\sim$ where $(p, [g]) \sim (pk, [k^{-1}g])$, $p \in P$, $g, k \in G$. Prove that $E \to M$ admits a global section if and only if the structural group of P may be reduced to H.

8. On $\mathbb{R}^3$ with coordinates (x, y, z) let $\alpha = x\,\mathrm{d}y + \mathrm{d}z$ and $E = \{v \in T\mathbb{R}^3 \mid \alpha(v) = 0\}$. Prove or disprove:

   a) There is a codimension 2 foliation $\mathcal{F}$ of $\mathbb{R}^3$ so that any leaf N of $\mathcal{F}$ satisfies $TN \subseteq E \mid N$.

   b) There is a codimension 1 foliation $\mathcal{F}$ of $\mathbb{R}^3$ so that any leaf of $\mathcal{F}$ satisfies $TN \subseteq E \mid N$.

9. Let $SO_3 = \{A : \mathbb{R}^3 \to \mathbb{R}^3 \mid A \text{ is linear, } A^*A = I \text{ and } \det A = 1\}$. Prove that

$$\{A \in SO_3 \mid A^* = A,\ A \neq I\}$$

   is a compact manifold. What is its dimension?

10 1998 Fall

Examiners: Auckly & Miller

1. (A) Define the deRham cohomology groups of a differential manifold.

   (B) Determine all of the deRham cohomology groups of $S^2 \times S^2$. For those that are nonzero specify representatives for generators.

2. Describe in detail the flows of the vector field
$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} \text{ on } \mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$
   Describe the behavior of the orbits as $t \to +\infty$.

3. On $\{(x, y) \mid x, y \in \mathbb{R},\ 0 < y < \pi\} \subset \mathbb{R}^2$ let g be the metric
$$g = \mathrm{d}x^2 + \cos y(\mathrm{d}x \otimes \mathrm{d}y + \mathrm{d}y \otimes \mathrm{d}x) + \mathrm{d}y^2$$

   (A) Compute $\left[x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},\ xy\frac{\partial}{\partial x}\right]$

   (B) Compute $g\left(\nabla_{\sin y\frac{\partial}{\partial y}}\left(\cos y\frac{\partial}{\partial x}\right),\ \frac{\partial}{\partial y}\right)$

4. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}/\sim$ where $\sim$ is the equivalence relation generated by $(x, y) \sim (-x, -y)$ if $x^2 + y^2 = 1$ or $x^2 + y^2 = 4$. Determine the fundamental group of D.

5. Suppose that f and $g : M \to N$ are smooth mappings between two n-dimensional manifolds and that w is a closed n-form on N. If f and g are homotopic show that $\displaystyle\int_M f^*w = \int_M g^*w$.

6. Let B be a smooth vector subbundle of TM, the tangent bundle of the manifold M.

   (A) Define what we mean when we say that B is integrable.

   (B) State the Frobenius theorem which gives necessary and sufficient conditions for B to be integrable in terms of the bracket of vector fields.

   (C) Suppose that α is a smooth 1-form and $\alpha(m) \neq 0$ for all $m \in M$. If $B = \displaystyle\bigcup_{m\in M} \ker(\alpha(m))$ show that B is integrable if and only if $\mathrm{d}\alpha|B = 0$.

7. Prove that $O_n = \{A \mid A \text{ is an } n \times n \text{ real matrix and } A^TA = I\}$ is a manifold. What is the dimension of O_n?
   Hint: Consider the mapping $f : GL(n, \mathbb{R}) \to$ *symmetric matrices* by $f(A) = A^TA$.

8. Prove or disprove the statements

   (A) There is a Lie group G which is diffeomorphic to S^2, the 2-sphere.

(B) There is a Lie group G which is diffeomorphic to S^3, the 3-sphere.

9. Prove that a simply connected 2-manifold with nonpositive curvature can have at most one geodesic (parametrized by arc length) from a point A to a second point B.
   Hint: Consider the Gauss-Bonnet Theorem.

11 1998 Spring

Examiners: Yetter and Miller

Instructions: For part A, answers with a brief explanation will suffice; in part B, detailed calculations or proofs are expected unless a sketch of a proof is explicitly requested. Do all 6 questions in part A. For part B, choose 4 and only four of the problems.

Part A

1. What is the fundamental group of
   a) S^3 (the 3-sphere)
   b) $S^2 \times S^1$
   c) $T^*(\mathbb{RP}^2)$, the total space of the cotangent bundle to the real projective plane

2. Describe in detail the flows of the vector field on $\mathbb{R}^3$ given by

$$(x^2 + y^2 - 4)\frac{\partial}{\partial z}$$

3. a) Give an example of a simply connected compact manifold with non-trivial tangent bundle.
   b) Give an example of a simply connected compact manifold with trivial tangent bundle.
   c) Give an example of a non-simply connected compact manifold.

   Note: Manifold means manifold without boundary.

4. a) If $M = S^3$, what is the dimension of the total space of the second exterior bundle $\Lambda^2(M)$?
   b) What are sections of the second exterior bundle called?

5. Let Z be the vector space of closed 1-forms on $S^1 \times S^1$ and B be the vector space of exact 1-forms on $S^1 \times S^1$. What is the dimension of Z/B?

6. Let $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ with the induced orientation. Calculate

$$\int_{S^2} z\, \mathrm{d}x \wedge \mathrm{d}y - y\, \mathrm{d}x \wedge \mathrm{d}z + x\, \mathrm{d}y \wedge \mathrm{d}z.$$

Part B.

1. Let G be a Lie group
   a) Define its Lie algebra.

b) Describe the correspondence between the Lie algebra and the 1-parameter subgroups of G.

c) Define the exponential mapping for G.

d) Derive the form of the exponential mapping for the group of non-singular $n \times n$ real matrices.

2. Prove that S^n for $n > 1$ is an orientable manifold.

3. Let $p \in \mathbb{RP}^2$, and let $X = \mathbb{RP}^2 \times \{0, 1\}/ \equiv$ where $\equiv$ is the equivalence relation generated by $(p, 0) \equiv (p, 1)$. Compute in detail $\pi_1(X)$.

4. Let (M, g) be a Riemannian manifold, and let X be a smooth vector field on M. Define a function on M by $f(m) = \|X(m)\|^2$ and let Y be the gradient vector field of f. Show that $X(m) = 0$ implies $Y(m) = 0$.

5. Let B be the 2-dimensional subbundle of the tangent bundle to $\mathbb{R}^3$ whose fibre at each point is spanned by the vector fields $X_1 = z\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$ and $X_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Use the Frobenius Theorem to determine whether B is integrable. Here (x, y, z) is the standard Euclidean coordinate system.

6. Consider a 2-dimensional Riemannian manifold (M, g), where M is an open subset of $\mathbb{R}^2$ and $g = f^2(\mathrm{d}x^2 + \mathrm{d}y^2)$ for some positive function $f(y)$ depending only on y.

a) Write down the simplest moving orthonormal frame you can think of.

$$e_1 = \qquad\qquad e_2 =$$

b) The moving coframe dual to (e_1, e_2) is

$$\theta^1 = \qquad\qquad ; \theta^2 =$$

c) Now find a 1-form ω such that $\mathrm{d}\theta^1 = \omega \wedge \theta^2$ and $\mathrm{d}\theta^2 = -\omega \wedge \theta^1$.

d) Find K so that $\mathrm{d}\omega = -K\theta^1 \wedge \theta^2$.

e) What is the Gaussian curvature when $\mathrm{d}s^2 = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$?

12 1997 Fall

Instructions: For part A, do all nine questions. For part B, choose four and only four of the problems.

Part A.

1. What is the fundamental group of
   a) $\mathbb{RP}^2$ (the real projective plane)
   b) $S^1 \times S^1$
   c) $T(M)$, the total space of the tangent bundle to a simply connected smooth manifold, M.

2. Describe in detail the flows of the vector field on $\mathbb{R}^2$ given by

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

3. Let ω be the 1-form on $\mathbb{R}^2$ given by $x(x-1)(y-1)\,\mathrm{d}x$, and let R be the region

$$\{(x, y) \mid 0 \le x \le 1,\, 0 \le y \le 1\}.$$

   Find $\int_R \mathrm{d}\omega$.

4. a) Give an example of a compact orientable manifold with non-trivial tangent bundle.
   b) Give an example of a compact orientable manifold with trivial tangent bundle.
   c) Give an example of a compact non-orientable manifold.

5. If $M = S^1 \times S^4$, what is the dimension of the fibres of the third exterior bundle $\Lambda^3(M)$?

6. How many non-zero vector spaces of differential forms are there in the deRham complex of $S^2 \times S^2$?

7. What is the scalar curvature of the surface $3x + 2y - z = 0$ in $\mathbb{R}^3$ at the point $(0, 0, 0)$?

8. Give an example of a locally Euclidean topological space which is not a topological manifold.

9. State the deRham Theorem.

Part B.

1. On $\mathbb{R}^3$ with the standard Euclidean coordinates (x, y, z), consider the 2-form $\alpha = f(x, y, z)\,\mathrm{d}x\wedge\mathrm{d}y + yz\,\mathrm{d}x\wedge\mathrm{d}z + x^2\,\mathrm{d}y\wedge\mathrm{d}z$. Choose a function $f(x, y, z)$ so that $\mathrm{d}\alpha = 0$ and $\alpha\big|_{z=1} = \mathrm{d}x\wedge\mathrm{d}y$.

2. a) Define the deRham cohomology groups of a differentible manifold.

   b) Calculate the deRham cohomology groups of the circle S^1 directly from the definition in part (a).

3. Give a detailed computation of the fundamental group of the closed compact surface of genus 2 (a.k.a the "two-holed torus").

4. a) Write down the deRham cohomology groups for the 4-sphere S^4.

   b) Suppose that ω is a differential 2-form on S^4 and that $\mathrm{d}\omega = 0$. Show that

      i. $\omega \wedge \omega = \mathrm{d}\phi$ for some 3-form ϕ.

      ii. $\int_{S^4} \omega \wedge \omega = 0$.

      iii. There is at least one point $x \in S^4$ such that $\omega \wedge \omega(x) = 0$.

5. a) Define what we mean by a Lie group.

   b) If G is a Lie group, define its Lie algebra g.

   c) Apply the construction of b) to determine the Lie algebra of $SO(3)$, including a derivation of the bracket.

   d) Show that the tangent bundle to a Lie group is equivalent to a trivial (product) bundle.

6. Let (M, g) be a Riemannian manifold and $V(M)$ be the smooth vector fields over M.

   a) For $X, Y \in V(M)$ define the Riemannian curvature operator $R(X, Y) : V(M) \to V(M)$.

   b) Show that if $M = \mathbb{R}^n$ and g is the Euclidean metric, then $R(X, Y)Z = 0$ for all vector fields X, Y, Z.

   c) Suppose that $R(X, Y)Z = 0$ for all vector fields X, Y, Z on an arbitrary Riemannian manifold (M, g). *Sketch* a proof that shows that for $x \in M$ there is a coordinate system $(x_1, \ldots, x_n)$ around x such that

$$g = \sum_{i=1}^{n} \mathrm{d}x^i \otimes \mathrm{d}x^i.$$

13 1997 Spring

Examiners: Yetter and Wu

Instructions: Work out problem 1 and then choose 5 (and only 5) additional problems among the remaining ones.

1. Answer the following questions and give a brief explanation or counterexample:
   (i) (a) Give an examples of orientable, connected, simply-connected smooth 2-manifold with trivial tangent bundle.
   (b) Give an examples of orientable, connected, simply-connected smooth 2-manifold with non-trivial tangent bundle.
   (ii) What is the dimension of the total space of the exterior bundle $\Lambda^3(M)$ if M is a smooth 6-manifold?
   (iii) Give an example of a compact non-orientable manifold.
   (iv) Consider the vector field on $\mathbb{R}^2 \setminus \{(0,0)\}$ given in polar coordinates by $(r-1)\,\mathrm{d}\theta$ Describe in detail its flows. Is this vector field complete?
   (v) (a) What is the fundamental group of $S^1 \times S^1$?
   (b) What is the fundamental group of $S^1 \times S^2$?
   (c) What is the fundamental group of $\mathbb{R}^2 \setminus \{(0,1),(0,-1)\}$?
   (vi) Define a connection on the tangent bundle of a manifold. Define the Levi-Civita connection.
   (vii) Give an example of a space which is locally euclidean, but is not a manifold.
2. Let Γ be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ in $\mathbb{R}^3$. Calculate
$$\int_\Gamma z\,\mathrm{d}x \wedge \mathrm{d}y - y\,\mathrm{d}z \wedge \mathrm{d}x.$$
3. **(i)** Describe the natural Lie algebra structure on the set of vector fields on a smooth manifold M.
   (ii) In the case where M is a Lie group, use the group law and the Lie algebra structure in **i)** to construct a Lie algebra structure on the tangent fibre at the identity $T_e(M)$.
4. Suppose G is a connected compact Lie group. Show that the fundamental group of G, $\pi_1(G)$ is abelian.
5. Find the scalar curvature of the surface $z = x^2 + y^2$ at $(0,0,0)$.
6. Show that every 1-form on $\mathbb{R}^1$ is *exact.* Show that every *closed* 1-form on $\mathbb{R}^3$ is *exact.*

7. Let $\varphi : M \to N$ be a (smooth) map. Then the vector field X on M and Y on N are said to be φ-related if $\mathrm{d}\varphi_m(X_m) = Y_{\varphi(m)}$ for all $m \in M$.

   Let X_1, X_2 be vector fields on M and Y_1, Y_2 vector fields on N. Assume that X_1 is φ-related to Y_1 and X_2 is φ-related to Y_2. Show that $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.

8. Suppose $f : X^d \to \mathbb{R}^{d+1}$ is a (smooth) embedding of the d-dimensional manifold X into $\mathbb{R}^{d+1}$. A normal vector field along (X, f) is a smooth map $N : X \to T(\mathbb{R}^{d+1})$ such that for each $p \in X$, $N(p) \in T_{f(p)}\mathbb{R}^{d+1}$ and it $(N(p))$ is orthogonal to the subspace $\mathrm{d}f(T_pX) \subset T_{f(p)}\mathbb{R}^{d+1}$. Prove that the manifold X is orientable if and only if there is a smooth *nowhere-vanishing* normal vector field along (X, f).

9. Prove that S^n $(n \geq 1)$ is orientable.

10. Let $f : M \to N$ be a smooth map such that for all $m \in M$, $\mathrm{d}f_m : T_mM \to T_{f(m)}N$ is *surjective*. Show that for any $n \in N$, $f^{-1}(n) \subset M$ is a smooth submanifold of M. What is the dimension of $f^{-1}(n)$?

14 1996 Spring

Instructions: Work out problem 1 and then choose 4 (and only 4) additional problems among the remaining ones.

1. Answer the following questions and give a brief explanation or counterexample:
   (i) (a) Give 2 non-diffeomorphic examples of orientable connected 2-dimensional compact smooth manifolds.
   (b) Give an example of a non-orientable connected 2-dimensional compact smooth manifold.
   (ii) What is the dimension of the total space of the exterior bundle $\Lambda^2(M)$ if M is a smooth 5-manifold?
   (iii) Consider the 1-form $\omega = (x^2 + y^2 - 1)\,\mathrm{d}x$ on $\mathbb{R}^2$. Let D be the standard unit disk, $D = \{(x, y) \mid x^2 + y^2 \le 1\}$. Find $\int_D \mathrm{d}\omega$.
   (iv) Let $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$ be the unit sphere in $\mathbb{R}^4$.
   (a) What is the fundamental group of S^3?
   (b) What is the fundamental group of $S^3 \setminus \{(0, 0, 0, 1)\}$?
   (c) What is the fundamental group of $S^3 \setminus \{(0, 0, 0, 1), (0, 0, 0, -1)\}$?
   (v) Let M be a compact **connected** n-dimensional smooth manifold. Is there a nowhere vanishing n-form on M? What if M is also **simply connected**?
   (vi) Let θ be a closed 5-form on $\mathbb{R}^6$. Is θ exact?
   (vii) Describe in detail the flows of the vector field $X = (x^2 + y^2 - 1)\frac{\partial}{\partial z}$ on $\mathbb{R}^3$.
   (viii) Let $\gamma(t)$ be an integral curve of a complete vector field X on a smooth manifold M. Suppose that $\gamma(t_0) = 0$ for some t_0. What can you say about $\gamma(t)$?

2. Let ω be the 2-form on $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ given by the formula

$$\omega = \frac{x_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_2\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}.$$

   Show that ω is closed but not exact. (Hint: Consider the restriction to the unit sphere $S^2 \subset \mathbb{R}^3$.)

3. Compute the de Rham cohomology of S^2.

4. What is the fundamental group of
   i) $\mathbb{R}P^2$ (real projective space)?
   ii) the 2-sphere S^2 with 3 distinct points removed?
   Use generators and relations if you like.

5. Denote by $S^1 = \{e^{it} \in \mathbb{C} \mid t \in \mathbb{R}\} \subset \mathbb{C}$ the unit circle and $\mathbb{T} = S^1 \times S^1$ the 2-torus. Fix any real number $\alpha \in \mathbb{R}$ and let $\varphi_\alpha : \mathbb{R} \to \mathbb{T}$ be the map defined by

$$\varphi_\alpha(t) = (e^{2\pi i t}, e^{2\pi i \alpha t}).$$

   Show that the image $\varphi_\alpha(\mathbb{R}) \subset \mathbb{T}$ is either compact or dense in $\mathbb{T}$.

6. Let M be a compact smooth n-dimensional oriented manifold without boundary. Show that for any $(n-1)$-form ω on M, there is $p \in M$ such that $\mathrm{d}\omega(p) = 0$.

7. Let $f : S^n \to S^n$ be a smooth map such that $f(x) = f(-x)$. Show that the degree of f is even.

15 1995 Spring

Instructions: For part A, do all parts. For part B, calculate – do 3 of the 5.

Part A.

1. A Riemannian metric is a cross section of what bundle on a manifold?
2. Is the two handled torus (connected sum of two $S^1 \times S^1$'s) parallelizable?
3. Give an example of a compact manifold which is not orientable.
4. If we are given a 3-form on the unit ball in R^3, when will Stoke's theorem allow us to rewrite its integral as an integral on S^2?
5. Describe the universal covering space of (a) S^2, (b) $S^2 \times S^1$.
6. Consider the vector field on $R^3 (y^2 + z^2 + 1)\frac{\partial}{\partial x}$. Describe the family of its flows.
7. What is the dimension of the fiber of the bundle of 7-forms on a 9-manifold?
8. a) Give an example of a lie group which is contractible as topological space and has dimension seven.

   b) Give an example of a lie group which is not contractible as a topological space. Is its lie algebra contractible?

Part B.

1. Let an atlas for the 2-sphere be given by choosing stereographic projection from two antipodal points. Pick a geodesic joining the two as 0-ray and write polar coordinates on each patch.
   a) Find the transition function $(r, \theta) \to (r', \theta')$.
   b) Write the round metric of radius 1 in each patch.
   c) Find the $\{\theta^{\theta}_{r}\}$ component of the Levi-Civita connection in one patch.
2. Write generators and relations for π_1 of the once punctured torus.
3. Give a set of generators for the lie algebra $su(2)$, and compute the bracket of each pair.
4. Use Stokes theorem to compute $\int_{S^2} z\,\mathrm{d}x \wedge \mathrm{d}y - y\,\mathrm{d}x \wedge \mathrm{d}z$ on the unit sphere in R^3.
5. Find the scalar curvature of the surface $z = x^2 - y^2$ at the point $(0, 0, 0)$.

16 1994 Fall

Instructions: For part A, short answers. Answer all. For part B, do any two.

Part A

1. What is the fundamental group of
   a) S^3
   b) $S^1 \times S^1$
   c) The Euclidean plane with two pts. deleted.
2. A differential 1-form is a section of what bundle?
3. What is the topology of the underlying manifold of the lie group $SU(2)$? What is the topology of the lie algebra $\mathcal{SU}(2)$?
4. State Stokes' theorem.
5. a) What kind of differential form can we integrate on a 4-manifold?
   b) What kind of differential form can we integrate on a surface in a 4-manifold?
6. a) Give an example of a compact surface whose tangent bundle is trivial.
   b) Give an example of a compact surface whose tangent bundle is not trivial.
7. Suppose we have a non-zero vector field on R^n all of whose covariant derivatives with respect to the standard metric and Levi-Civita connection vanish. Describe the family of its integral curves.
8. If M is a 4-dimensional manifold, what is the dimension of the fibers of its bundle of differential 2-forms?
9. Consider the complex of differential forms used to define the de Rham cohomology of a 5-manifold. How many of these spaces are non-vanishing?

Part B

1. a) Write the metric tensor for the Euclidean plane in polar coordinates.
   b) Compute the $\theta^\theta r$ components of the Levi-Civita connection for the Euclidean plane in polar coordinates.
2. Give an example of a topological space every point of which has a neighborhood homeomorphic to R^2 which is not a manifold.
3. Prove the Jacobi identity holds for $\mathcal{SU}(3)$.
4. Compute the DeRham cohomology of the Torus T^2.
5. Use Stokes' theorem to compute the area of the unit ball in R^2.