KSU Quals — Analysis

2015 June—2021 August

Contents

1 2021 August

1. Let $a < b$ be two points on the real line and let $f(x)$ be a function, thrice differentiable on the interval $[a, b]$. Prove that there is a point c between a and b, such that
$$\frac{f(b) - f(a)}{b - a} = \frac{f'(a) + f'(b)}{2} - f'''(c)\frac{(b-a)^2}{12}.$$
Hint: Consider the auxiliary function
$$g(x) = f(x) - f(a) - (x - a)\frac{f'(a) + f'(x)}{2} - k(x - a)^3,$$
where k is a suitable constant.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by
$$f(x, y) = x^4 + x^2y^2 + xy^3 + y^4.$$
Let $S \subset \mathbb{R}^2$ be the set of solutions of the equation $f(x, y) = 1$. Prove that every point in S has a neighborhood, where the equation can be solved for x in terms of y or vice versa.

3. Let (X, d) be a compact metric space, and let $\mathcal{F}$ be a family of real-valued functions on X. Assume that the family $\mathcal{F}$ is *pointwise* equicontinuous: for every $x \in X$ and for every $\epsilon > 0$ there is $\delta > 0$, such that for every function f from the family $\mathcal{F}$ it holds that
$$|f(x) - f(y)| < \epsilon,$$
whenever $y \in X$ is such that $d(x, y) < \delta$.
Prove that the family $\mathcal{F}$ is *uniformly* equicontinuous: for every $\epsilon > 0$ there is $\delta > 0$, such that for every function f from the family $\mathcal{F}$ it holds that
$$|f(x) - f(y)| < \epsilon,$$
whenever $x, y \in X$ are such that $d(x, y) < \delta$.

4. Consider the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function satisfying $f(0) = \frac{1}{2}$ and $f(\frac{1}{2}) = 0$. Prove that $f(z) = \dfrac{2z - 1}{z - 2}$, $\forall z \in \mathbb{D}$.

5. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, and for every positive integer n, there exists a positive constant C_n and a neighborhood V_n of 0, such that
$$|f(z)| \leq C_n|z|^n, \quad \forall z \in V_n.$$
Prove that f is the constant zero function.

6. Let Γ be the circle of radius 2 centered at i, parametrized counterclockwise. Compute the complex line integral
$$\oint_\Gamma \frac{\sin(\pi z)}{z^4 + 3z^3 + 2z^2}\, \mathrm{d}z.$$

2 2020 August

1. Show that both series

$$\sum_{n=1}^{\infty} x^n(1-x) \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^n x^n(1-x)$$

are convergent on $[0,1]$, but only one converges uniformly. Which one? Why?

2. Let $f : \mathbb{R} \to \mathbb{R}$. Show that each of the following conditions implies that f is Borel measurable:

   a) f is increasing

   b) f is lower semi-continuous, i.e., $f(x) \leq \liminf_{y\to x} f(y)$, for all $x \in \mathbb{R}$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Show that f is not injective.

4. Let $f(z)$ and $g(z)$ be two entire functions, such that

$$|f(z)| \leq |g(z)|, \quad z \in \mathbb{C}.$$

Prove that there exists a constant $c \in \mathbb{C}$, such that

$$f(z) = cg(z), \quad z \in \mathbb{C}.$$

5. How many distinct roots does the polynomial

$$p(z) = z^7 + 10z^4 + 7$$

have in the disk $|z| \leq 1$?

6. Use residues to compute

$$\int_0^{\infty} \frac{x\sin(2x)}{4+x^2}\, dx.$$

3 2020 June

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0) \end{cases}$$

   a) Is f differentiable at $(0,0)$?

   b) Are the partial derivatives $D_i f$, $i = 1,2$ continuous at $(0,0)$?

2. Let the sequence of functions be defined by $f_n(x) = nxe^{-nx}$ on $[0,+\infty)$. Determine the pointwise limit on the given interval (if it exists) and an interval on which the convergence is uniform (if any).
   Does the sequence of derivatives (f_n') converge uniformly on $[0,+\infty)$?

3. Let $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}$ is measurable. Show that f is measurable if and only if the function $g : \mathbb{R} \to \mathbb{R}$ is measurable, where $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ otherwise.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) = z|z|.$$

   Where is $f'(z)$ defined? Where is $f(z)$ analytic?

5. Construct a map that maps the half-strip

$$S = \{z : |\mathfrak{Re}(z)| < 1,\ \mathfrak{Im}(z) > 0\}$$

   conformally onto the open unit disk

$$\mathbb{D} = \{z : |z| < 1\}.$$

6. Let $f(z)$ be analytic in the punctured disk

$$D = \{z : |z| < 1,\ z \neq 1/2\}.$$

   Suppose that $f(z)$ has a simple pole at $z = 1/2$ and that

$$\operatorname*{Res}_{z=1/2} f(z) = 1.$$

   Determine the coefficient a_{-2} in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad 1/2 < |z| < 1.$$

4 2019 August

1. Prove the L^1 Chebyshev inequality: for any real $s > 0$,

$$|\{x : |f(x)| > s\}| \leq \frac{1}{s} \int |f|.$$

2. What is the Lebesgue measure of the set of rationals in the line? Give a proof of your assertion.

3. Suppose $\{f_n,\ n \geq 1\}$ is a family of real-valued functions on a compact interval I that are Hölder continuous with exponent α and constant M: i.e., for all $n \geq 1$ and all $x, y \in I$,

$$|f_n(x) - f_n(y)| \leq M|x - y|^\alpha.$$

Suppose also that the set $\{f_n(x_0) \mid n \geq 1\}$ is bounded for some fixed $x_0 \in I$. Prove that $(f_n)_{n=1}^\infty$ has a subsequence converging uniformly to a function f that is Hölder continuous with the same exponent α and constant M.

4. Let $U \in \mathbb{R}^n$ be an open set, $a \in U$, and $f : U \to \mathbb{R}^m$. Prove that the following statements are equivalent:
   a) The mapping f is differentiable at a.
   b) Every component function $f_i : U \to \mathbb{R}$ of f, $1 \leq i \leq m$, is differentiable at a.

5. Let f be an entire function such that $f(z) = f(z+2\pi)$ and $f(z) = f(z+2\pi i)$ for all $z \in \mathbb{C}$. Prove that f is constant.

6. Assume a and b are complex with $|a| \neq 1$. Evaluate, distinguishing cases

$$\int_\gamma \left(\frac{z-b}{z-a}\right)^2 \mathrm{d}z,$$

where $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$

5 2019 June

1. Assume a function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 1, in the sense that $f(tx) = tf(x)$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$.
   a) Show that f has directional derivatives at 0 in all directions.
   b) Prove that f is differentiable at 0 if and only if f is linear.
2. Find all functions f that are holomorphic in the disk $D(0; 1)$ and such that
$$f(1/n) = n^2 f(1/n))^3, \quad \text{for } n = 2, 3, 4, \ldots$$
3. Let f be an entire function. Prove that if $f(z)$ is real for all z with $|z| = 1$, then f is constant.
4. Prove that the family of all polynomials $P(x)$ of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.
5. What is the Lebesgue measure of the Cantor set?
6. Prove that a non-negative measurable function has integral equal to zero if and only if it is zero almost everywhere.

6 2018 August

1. Let (M, d) be a metric space. Show that $\rho(x, y) = \sqrt{d(x, y)}$ also defines a metric. Is the identity map $i : (M, d) \to (M, \rho)$, $i(x) = x$ continuous?

2. The function $f : M \to \mathbb{R}$ is called lower semicontinuous if for all $\alpha \in \mathbb{R}$ the set $\{x : f(x) > \alpha\}$ is open. Show that if f is lower semicontinuous and M is compact then
   a) f is bounded below, and
   b) f attains a minimum value.

3. Let $f_n(x) = \sum_{j=1}^{n} \frac{1}{n} f(x + \frac{j}{n})$, where f is a continuous function on $\mathbb{R}$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function.

4. Suppose $f, g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous functions.
   a) Show that the set $B = \{x \in \mathbb{R}^n : f(x) = g(x)\}$ is closed in $\mathbb{R}^n$
   b) Let $p = 1$. Prove that the set $C = \{x \in \mathbb{R}^n : f(x) > g(x)\}$ is open in $\mathbb{R}^n$.

5. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula
$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$
   a) Is f continuous at $(0, 0)$?
   b) Show that partial derivatives $D_1 f(0, 0)$ and $D_2 f(0, 0)$ exist and are equal to 0.
   c) Is f differentiable at $(0, 0)$?

6. Consider the following equation for $x \in \mathbb{R}$ with $y = (y_1, y_2) \in \mathbb{R}^2$ as a parameter:
$$x^3 y_1 + x^2 y_1 y_2 + x + y_1^2 y_2 = 0.$$
   a) Prove that there are neighborhoods V of $(-1, 1)$ and U of 1 such that for every $y \in V$, the above equation has a unique solution $x = \psi(y)$ in U.
   b) Find $D_1\psi(-1, 1)$ and $D_2\psi(-1, 1)$.
   c) Prove that there do *not* exist neighborhoods V of $(-1, 1)$ and U' of -1 such that for every $y \in V$ the equation has a unique solution $x = x(y) \in U'$. *Hint:* Explicitly determine the three solutions for x in the special case where $y_1 = -1$.

7. Find the number of zeroes of the function $f(z) = z^7 - 8z^2 + 2$ in the annulus $1 < |z| < 2$.

8. Does there exist an entire function f such that $f(\frac{1}{n}) = \frac{n}{n+1}$? *Hint:* Use the Identity Theorem.

9. Use the contour integral to compute $\displaystyle\int_0^\infty \frac{x^2}{x^4 + 5x^2 + 4}\,dx$.

7 2018 June

1. Let (X, d) be a metric space. Let $f : X \to X$ be a continuous map. Assume that for all $x, y \in X$,
$$d(f(x), f(y)) < d(x, y).$$
   a) Show that f has at most one fixed point.
   b) Show that if X is compact, f has exactly one fixed point.

2. Let $K > 0$. The function $f : [a, b] \to \mathbb{R}$ is K-Lipschitz if for all $x, y \in [a, b]$:
$$|f(x) - f(y)| \leq K|x - y|$$
   a) Assume that f has a bounded derivative on (a, b). Show that there exists K such that f is K-Lipschitz.
   b) For every K, give an example of the function that is K-Lipschitz, but not differentiable.

3. Prove or disprove:
   a) The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.
   b) The product of two uniformly continuous functions on $[0, 1]$ is also uniformly continuous.

4. Let I be a rectangle in $\mathbb{R}^2$ and suppose f is continuous on I. Prove that there exists a point $x_0 \in I$ such that
$$\int_I f(x)\,\mathrm{d}x = f(x_0)\,\mathrm{vol}(I),$$
where $\mathrm{vol}(I)$ is the n-dimensional volume of the rectangle.

5. Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula
$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$
   (a) Is f continuous at $(0, 0)$?
   (b) Show that both partial derivatives $D_1 f(0, 0)$ and $D_2 f(0, 0)$ exist and compute them.
   (c) Is f differentiable at $(0, 0)$?

6. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula
$$F(x_1, x_2) = e^{x_1}\left(\cos(x_2), \sin(x_2)\right).$$
   (a) Find the image of F.

(b) Prove that for every $x \in \mathbb{R}^2$ there exists a neighborhood U in $\mathbb{R}^2$ such that $F : U \to F(U)$ is a diffeomorphism, but that F is not injective on all of $\mathbb{R}^2$.

(c) Let $x = (0, \frac{\pi}{3})$, $y = F(x)$ and let H be the continuous inverse of F, defined in a neighborhood of y, such that $H(y) = x$. Give an explicit formula for H.

7. Calculate the integral $\int_C \frac{\cos z}{z^3+4z}\,\mathrm{d}z$, where C is counterclockwise oriented circle of radius 2 with center at the point $z = i$.

8. Let $\mathbb{D} = \{|z| < 1\}$. Consider the set of holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{3}{4}) = 0$. What are the possible values of $f'(\frac{3}{4})$?

9. Let $f(z)$ be an entire function that does not take negative real values. Show that f is constant. (Hint. consider $\sqrt{f}$).

8 2017 August

1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be Cauchy sequences in a metric space (X, d). Show that $(d(x_n, y_n))_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $f(x) < f'(x)$ for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$.

3. Let $f(x)$ be continuous real-valued function on $[a, b]$ such that $\int_a^b (f(x))^2 \, \mathrm{d}x = 0$. Show that $f \equiv 0$.

4. For $n \in \mathbb{N}$, define $f_n : [1, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{n+1}{n} e^{-nx}$. Show that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuous function.

5. Let
$$f(x, y) = \begin{cases} \frac{xy^3}{x^3+y^6} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Show that for every unit vector $\mathbf{u}$, the directional derivative of f in the direction $\mathbf{u}$ at the point $(0, 0)$ exists.
   (b) Is f continuous at $(0, 0)$?
   (c) Is f differentiable at $(0, 0)$?

6. Consider the system of equations
$$\begin{aligned} x^2 + y^2 + z^2 &= 2 \\ \sin(xyz) &= 0. \end{aligned}$$
   (a) Show that there is a neighborhood of $(1, 0, 1)$ on which the solution to the system of equations can be written as $(x, y) = f(z)$, where f is a vector-valued function.
   (b) Is there an $S \subset \mathbb{R}$ and a vector-valued function $f : S \to \mathbb{R}^2$ such that for all $x, y, z \in \mathbb{R}$, $(x, y) = f(z)$ iff x, y, z satisfy the system?
   (c) Does the system define x and z uniquely from y in some neighborhood of $(1, 0, 1)$?

7. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic ($\mathbb{D}$ is a unit disk). Is the function $\Re f(\bar{z})$ harmonic? Prove or give counterexample.

8. Let f be an entire function such that $\Re f > -1$. Show that f is constant. (Recall that function is entire if it is holomorphic in $\mathbb{C}$).

9. Use residues to calculate the integral
$$\int_0^{\infty} \frac{1}{1 + x^4} \, \mathrm{d}x.$$

9 2017 June

1. Let (a_n) be a Cauchy sequence in a metric space (M, d). Show that if (a_n) has a convergent subsequence, then it actually converges.

2. Let $a_n \geq 0$ and $\displaystyle\sum_{n=1}^{\infty} a_n < \infty$.
   (a) Show that $\liminf_{n\to\infty} na_n = 0$
   (b) Give an example showing that $\limsup_{n\to\infty} na_n > 0$ is possible.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that f takes on no value more than twice. Show that f takes on some value exactly once.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \begin{cases} \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2} & \text{if } y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Show that f is continuous at $(0, 0)$.
   (b) Calculate all the directional derivatives of f at $(0, 0)$.
   (c) State the definition of differentiability for a function $f : \mathbb{R}^2 \to \mathbb{R}$.
   (d) Show that f is not differentiable at $(0, 0)$. Hint: violate the definition.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.
   (a) Find all points where the map is locally one-to-one. Let S be the set of these points.
   (b) Is T one-to-one on S?
   (c) Determine the range of T.

6. Let $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. Suppose that $f : U \to \mathbb{R}$ is such that both partial derivatives of f are zero at every point in U. Must f be constant? Justify your answer.

7. Let a, b be given complex numbers, $|a| < |b|$. Let $|a| < r < |b|$. Calculate
$$\int_{C_r} \frac{1}{(z - a)(z - b)} \, \mathrm{d}z,$$
   where C_r is the circle of radius r with center 0.

8. Assume that a function f is holomorphic in an open subset $U \subset \mathbb{C}$. Is the function $g = (\Re f)(\Im f)$ always harmonic in U? Prove the statement or give a counterexample.

9. Let $\mathbb{D} = \{|z| < 1\}$. Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that $f(\frac{1}{2}) = \frac{3}{4}$, $f'(\frac{1}{2}) = \frac{2}{3}$? (Hint: use Schwarz's Lemma)

10 2016 August

1. Let $K \subset \mathbb{R}$ be a set with the following property: every continuous function $f : K \to \mathbb{R}$ is bounded. Prove that K is closed and bounded (hence compact).

2. Let a_n be a sequence of positive real numbers, such that
$$\sum_{n=1}^{\infty} a_n$$
diverges. Prove that
$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$$
also diverges.

3. Recall that the Dirichlet function $f : [0,1] \to \mathbb{R}$ is defined by
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$
Show that the Dirichlet function is not Riemann integrable.

4. Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by the formula
$$f(x,y) = \begin{cases} \frac{\sin(xy^2)}{x^2+y^6}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$
Prove that f is not continuous.

5. Let $E \subset \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}$ a function. Suppose that all partial derivatives $D_1 f, \ldots, D_n f$ are bounded in E. Prove that f is continuous in E.

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = (x_1^2 - x_2)(3x_1^2 - x_2)$. Prove that f has $(0,0)$ as a critical point but not as a local extremum.

   Hint: consider $f(0,t)$ and $f(t, 2t^2)$ for t near 0.

7. Let $\mu(z)$ denote the Möbius transformation which maps 1 to 0, i to 1, and -1 to ∞. What is the μ-image of the half-disk $\{z : |z| < 1,\ \Im(z) > 0\}$?

8. Let $f(z)$ be an entire function such that $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is of the form $f(z) = cz$, where c is a complex constant.

9. Find the Laurent series of the function
$$f(z) = \frac{z}{z^2 - 1}$$
in the annulus $\{z : 0 < |z-1| < 2\}$ and in the annulus $\{z : |z-1| > 2\}$.

11 2016 June

Note: The uploaded pdf only has 2 problems for this exam. This is probably a mistake in what file was uploaded.

1. Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence of real numbers. Let $b \in \mathbb{R}$ be such that

$$\forall n \geq 1,\ a_n \neq b \qquad \text{and} \qquad \lim_{n \to \infty} a_n \neq b.$$

   Show that there must be a $d > 0$ such that $\forall n \geq 1$, $|a_n - b| > d$.

2.
   - Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} x_n$ converges, but $\sum_{n=1}^{\infty} x_n^2$ diverges. Prove that $\sum_{n=1}^{\infty} x_n$ must converge conditionally.
   - Let (x_j) and (y_j) be sequences of real numbers such that $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ are both convergent. Prove that the series $\sum_{j=1}^{\infty} \sqrt{|x_j y_j|}$ is also absolutely convergent. Hint: a possible solution uses the Limit Comparison Test.

12 2016 January

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that
$$|f(x) - f(y)| \leq |x - y|^\alpha$$
for some $\alpha > 0$.
   (i) Show that f is uniformly continuous.
   (ii) Show that if $\alpha > 1$ then f must be constant.
   Hint: is f differentiable?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$. Prove that f is bounded.

3. Show that the characteristic function of the rationals
$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$
is not Riemann integrable over any interval $[a, b]$ in $\mathbb{R}$.

4. Let (f_n) be a sequence of functions $f_n : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$, and suppose that there exist constants $M_n \geq 0$ such that
$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \text{and} \quad \sum_{n=1}^{\infty} M_n < \infty.$$
Prove that the series $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

5. Describe the set of points at which the Implicit Function Theorem guarantees that the curve $x^4 + xy^6 - 3y^4 = c$ is locally the graph of a function.

6. Let $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   a) Do the first partial derivatives exist at the origin?
   b) Is the function differentiable at the origin?

7. Let
$$f(z) = \frac{z}{z^2 + 6z + 8}.$$
Write the power series expansion of $f(z)$ centered at $z_0 = 0$ in the annulus $2 < |z| < 4$.

8. Calculate using residues
$$\int_0^\infty \frac{x^3 \sin(x)}{(1 + x^2)^2}\, \mathrm{d}x.$$
Hint: Consider $f(z) = \frac{z^3 e^{iz}}{(1+z^2)^2}$.

9. Find the number of roots (counting multiplicities) of the polynomial $p(z) = 3z^4 - z^3 + 8z^2 - 2z + 1$ in the annulus $\{z : 1 < |z| < 2\}$.
Hint: Use Rouché's theorem twice.

13 2015 August

Instructions: Attempt at most six problems.

Section I

1. Assume that (a_n) is a convergent sequence in a metric space (X, d). Show that there is a subsequence (a_{n_k}) such that the series $\sum_{k=1}^{\infty} d(a_{n_k}, a_{n_{k+1}})$ converges.

2. Using the definition of uniform continuity, show that any uniformly continuous function $f : (0, 1) \to \mathbb{R}$ is bounded.

3. Let $r_1, r_2, \ldots, r_n$ be real numbers in $[0, 1]$ where $n \in \mathbb{N}^+$. Let $f : [0, 1] \to \mathbb{R}$ be the characteristic function of $\{r_1, r_2, \ldots, r_n\}$; i.e.,

$$f(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{otherwise.} \end{cases}$$

   Using the definition of the Riemann integral, prove that f is Riemann integrable.

Section II

4. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and

$$\int_9^1 f(x) x^n \, \mathrm{d}x = 0 \quad (n = 0, 1, 2, \ldots).$$

   Prove that f is identically zero on $[0, 1]$. Hint: prove that $\int_0^1 f^2(x) \, \mathrm{d}x = 0$.

5. Consider the set of points (x, y) in the real plane that satisfy $x + \sin(xy) = 0$.
   (a) Is there a neighborhood of the origin on which this set is the graph of a function $y = f(x)$?
   (b) Is there a neighborhood of the origin on which this set is the graph of a function $x = f(y)$?

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

   (a) Show that f is continuous at $(0, 0)$.
   (b) Show that the directional derivatives $D_u f$ exist at $(0, 0)$ (u a unit vector) and compute them.
   (c) Show that f is not differentiable at $(0, 0)$.

Section III

7. Given two Laurent Series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

8. Use residues to compute the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{2+\cos\theta}.$$

9. Find all points z where the function $f(z) = \Re(z) \cdot \Im(z)$ is complex differentiable.

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Instructions: Attempt at most 6 problems.

Section I

1. Let (X, d) be a metric space. Prove that if a Cauchy sequence in X has a convergent subsequence then the sequence converges.
2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and is periodic with period 1, i.e.,
$$f(x + 1) = f(x) \text{ for all } x \in \mathbb{R}.$$
Prove that f is uniformly continuous.
3. Use the definition of the Riemann integral to prove that if $a < b < c$ are real numbers and f is Riemann integrable on both $[a, b]$ and on $[b, c]$, then f is Riemann integrable on $[a, c]$.

Section II

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$
   a) Show that f is continuous at $(0, 0)$.
   b) Show that the directional derivatives $D_u f$ exist at $(0, 0)$, and compute them.
   c) Show that f is not differentiable at $(0, 0)$.
5. Can the equation $(x^2 + y^2 + 2z^2)^{1/2} = \cos z$ be solved uniquely for y from x and z in a neighborhood of $(0, 1, 0)$? For z in terms of x and y?
6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that there is a sequence p_n of polynomials such that for every $R > 0$, the sequence converges uniformly to f on the interval $[-R, R]$.

Section III

7. Find the Laurent series for $f(x) = \dfrac{z^2 + 1}{z(z - 3)}$ in the annulus $0 < |z| < 3$.
8. Let G be a connected open subset of $\mathbb{C}$ and f and g analytic functions in G such that $f(z)g(z) = 0$ for all $z \in G$. Prove that either $f \equiv 0$ or $g \equiv 0$.
9. Let u and v be real harmonic functions and suppose that v is the harmonic conjugate of u. Show that
$$\frac{u}{u^2 + v^2} \quad \text{and} \quad \frac{-v}{u^2 + v^2}$$
are both harmonic, assuming $u^2 + v^2 \neq 0$.

15 Sample

1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals that converges to 0. Prove that there is a subsequence $(a_{n_k})_{n=1}^{\infty}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely.

2. Show that if f is a nonnegative continuous function defined on $[0, 1]$ satisfying $\int_0^1 f(x)\,\mathrm{d}x = 0$, then $f \equiv 0$ on $[0, 1]$.

3. (a) Let K be a compact subset of $\mathbb{R}$ and let $f : K \to \mathbb{R}$ be continuous. Show that f attains its maximum value: i.e., there is a point $a \in K$ such that $\forall x \in K,\ f(x) \leq f(a)$.

   (b) Suppose $f : \mathbb{R} \to (0, \infty)$ is a continuous function with limit $\lim\limits_{x\to\pm\infty} f(x) = 0$. Show that f attains its maximum value.

4. Show that the system of equations (note that these are not linear)

$$\begin{aligned} 3x + 7 - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

   cannot be solved for x, y, z in terms of u but can be solved for each of the other sets of three variables in terms of the remaining one.

5. Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n}$.

   - Show that the series converges pointwise on $[0, \infty)$. To what function?
   - Does the series converge uniformly on $[0, 1]$? On $[1, \infty)$?

6. Does there exist a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$\int_{-\pi}^{\pi} x f(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{-\pi}^{\pi} x^n f(x)\,\mathrm{d}x = 0$$

   for $n = 0, 2, 3, 4, \ldots$? Give an example or prove that no such f exists. Hint: calculate the Fourier coefficients of f using the power series expansion for e^x.

7. Compute $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$.

8. Suppose that f is a complex-valued analytic function in the open unit disk $\mathbb{D}$ such that $|f|$ is constant. Prove that f is constant.

9. Find a conformal map from the strip $\{z \in \mathbb{C} : |\Re(z)| < 1\}$ onto the open disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.