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A. I. Kostrikin · I. R. Shafarevich (Eds.)

Algebra V

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Homological Algebra

S. I. Gelfand, Yu. I. Manin

Contents

Introduction

1. Homological algebra is rather young. Its subject descends from two areas of mathematics studied at the end of the previous century; these areas later became combinatorial topology and "modern algebra" (in the sense of van der Waerden) respectively. As the examples of main notions inherited from this early period, we can mention Betti numbers of a topological space and D. Hilbert's "syzygy theorem" (1890).

At present we easily recognize a general construction which underlies these notions. A topological space X is glued from cells (or simplices) of various dimensions i; the boundary of a cell is a linear combination of other cells. The i-th Betti number is the number of linearly independent chains with zero boundary modulo chains that are boundaries themselves; in other words, the i-th Betti number is the rank of the group $\operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i-1}$, where $\partial_i : C_i \to C_{i-1}$ is the boundary operator and C_i is the group of i-dimensional chains. "Syzygies" occur in a different problem. Let M be a graded module with a finite number of generators over the ring $A = k[x_1, \dots, x_n]$ of polynomials with coefficients in a fixed field k. Hilbert considered the case when M is an ideal in A generated by several forms (homogeneous polynomials). In general, generators of M can not be chosen to be independent. Fixing a set of r_0 generators we obtain a submodule in A^{r_0} consisting of coefficients of all relations among these generators. This submodule has a natural grading and is called "the first syzygy module" $Z_0(M)$ of the module M. For $i > 1$ let $Z_i(M) = Z_0(Z_{i-1}(M))$ (on each step we have a freedom in choosing the generators of $Z_{i-1}(M)$). The Hilbert theorem asserts that $Z_{n-1}(M)$ is a free module so that we can always assume $Z_n(M) = 0$.

The algebraic framework of both constructions is the notion of a complex; a complex is a sequence of modules and homomorphisms $\dots \to K_i \xrightarrow{\partial_i} K_{i-1} \to \dots$ with the condition $\partial_{i-1}\partial_i = 0$. The complex of chains of a topological space determines its homology $H_i(X) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i-1}$. The Hilbert complex consists of free modules. It is acyclic everywhere but at the end: $Z_i(M)$ is both the group of cycles and the group of boundaries in a free resolution of the module M:

$$0 \to A^{r_n} \to A^{r_{n-1}} \to \dots \to A^{r_1} \xrightarrow{\partial_1} A^{r_0} \xrightarrow{\partial} 0$$
$$M \simeq \operatorname{Ker} \partial_0 = A^{r_0} / \operatorname{Im} \partial_1.$$

Both the complex of chains of a space X and the resolution of a module M, are defined non-uniquely: they depend on the decomposition of X into cells or on the choice of generators of subsequent syzygy modules. The essence of the first theorems in homological algebra is that there is something that does not depend on this ambiguity in the choice of a complex, namely the Betti numbers (or the homology groups themselves) in the first case, and the maximal length of a complex (the last non-zero place) in the second case.

The first stage of homological algebra was marked by the acquisition of data. Combinatorial and, later, homotopic topology supplied plentiful examples of

– types of complexes;

– operations over complexes that reflect some geometrical constructions: the product of spaces led to the tensor product of complexes, the multiplication in cohomology led to the notion of a differential graded algebra, homotopy resulted in the algebraic notion of a homotopy between morphisms of complexes, the algebraic framework of the geometrical study of fiber spaces is the notion of a spectral sequence associated to a filtered complex, and so on and so forth;

– algebraic constructions imitating topological ones; examples are cohomology of groups, of Lie algebras, of associative algebras, etc.

2. The famous "Homological algebra" by H. Cartan and S. Eilenberg, published in 1956 (and written some time between 1950 and 1953) summarized the achievements of this first period, and introduced some very important new ideas which determined the development of this branch of algebra for many years ahead. It seems that the very name "homological algebra" became generally accepted only after the publication of this book.

First of all, this book contains a detailed study of the main algebraic formalism of (co)homology groups and of working instructions that do not depend on the origin of the complex. Second, this book gave a conceptially important answer to the question about the nature of homological invariants (as opposed to complexes themselves, which cannot be considered as invariants). This answer can be formulated as follows. The application of some basic operations over modules, such as tensor products, the formation of the module of homomorphisms, etc., to short exact sequences violates the exactness; for example, if the sequence $0 \to M' \to M \to M'' \to 0$ is exact, the sequence $0 \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0$ can have non-trivial cohomology at the left term. One can define the "torsion product" $\mathrm{Tor}_1(N, M'')$ in such a way that the complex

$$\begin{aligned}\mathrm{Tor}_1(N, M') \to \mathrm{Tor}_1(N, M'') \to \mathrm{Tor}_1(N, M'') \to \\ \to N \otimes M' \to N \otimes M \to N \otimes M'' \to 0\end{aligned}$$

is acyclic. However, to extend this complex further to the left one must introduce $\mathrm{Tor}_2(N, M'')$, etc.

These modules $\mathrm{Tor}_i(N, M)$ are the derived functors (in one of the arguments) of the functor $\otimes$. They are uniquely determined by the requirement that the exact triples are mapped to acyclic complexes. To compute these functors one can use, say, free resolutions of the module M and define $\mathrm{Tor}_i(M, N)$ as homology groups of the tensor product of such a resolution with the module N.

Hence, a homological invariant of the module N is the value on N of some higher derived functor which can be uniquely characterized by a list of properties and can be computed using resolutions.

This idea, which first originated in the algebraic context, immediately returned to topology in the extremely important paper by A. Grothendieck "Sur quelques questions d'algèbre homologique", published in 1957. In order to pursue the point of view of Cartan and Eilenberg, Grothendieck had to revise completely the system of basic notions of combinatorial topology. Before his paper it was clear that the (co)homology depends, first of all, on the space X, and the axioms of homology described the behavior of $H(X)$ in passing to an open subspace (the excision axiom), under homotopy, etc. However, spaces X look quite unlike modules over a ring, and in this context the groups $H(X)$ do not behave like the derived functors. Grothendieck stressed the role of a second "hidden" parameter of the cohomology theory, the group of coefficients. It occurs that if we consider the cohomology $H^i(X, \mathcal{F})$ of X with coefficients in an arbitrary sheaf of abelian groups $\mathcal{F}$ on X (at the beginning of the fifties this notion was introduced and studied in detail due to the needs of the theory of functions in several complex variables), we can almost completely "ignore" the space X! Namely, $H^i(X, \mathcal{F})$ becomes in this context the i-th derived functor of the functor $\mathcal{F} \to \Gamma(X, \mathcal{F})$ (the global sections functor) in the spirit of Cartan–Eilenberg.

This idea turned out to be extremely fruitful for topology (understood in a wide sense). Being widely developed and generalized by Grothendieck himself and by his students and collabarators, it led to algebraic topology of algebraic varieties over an arbitrary field (the "Weil program"). The jewel of this theory is P.Deligne's proof of Riemann–Weil conjectures. We must mention also the cohomological version of class field theory (Chevalley and Tate among others), the modern version of Hodge theory (Griffiths, Deligne,...), theory of perverse sheaves, and the general penetration of the homological language into various areas of mathematics.

3. In the sixties homological algebra was enriched by yet another important construction. We mean here the notions of derived and triangulated categories.

While earlier the main concern of a mathematician working with homology were homological invariants, in the last twenty years the role of complexes themselves was emphasized; the complexes are viewed as objects of a rather complicated and not very explicit category. The idea is that, say, a resolution of a module is not only a tool to compute various Ext's and Tor's, but, in a sense, a rightful representative of this module. What we only need is a method that enables us to identify all resolutions of a given module. In the same way the chain complex of a space together with a sufficient set of auxiliary structures, is an adequate substitute of this space.

Although the axioms and the initial constructions of the theory of derived and triangulated categories are rather cumbersome, the approach itself is rather flexible and in the last few years this approach turned out to be indispensable in topology, representation theory, theory of analytical spaces, not to mention, of course, algebraic geometry which initiated all this (the Grothendieck seminars, the Verdier thesis, the Hartshorne notes).

One of the paradoxes of homological algebra, which now slowly becomes to be understood, is that in some cases an appropriately chosen triangulated category is simpler than the abelian category studied before. For example, the derived category of coherent sheaves on a projective space is understood better than the category of sheaves themselves. Next, one triangulated category can have several abelian "cores". Such a phenomenon leads to various meaningful versions of classical duality theories.

4. This volume of the Encyclopaedia is not intended to be a complete survey of all known results in homological algebra. This task could not presumably be solved both because of authors' limitations and the huge amount of data involved.

The volume can be roughly divided into three parts. The introductory Chap. 1–3 contains the most classical aspects of the theory; even now the main technical methods of homological algebra are based on these ideas (complemented from time to time by new constructions). For example, a comparatively new subject is cyclic (co)homology.

Chapters 4 and 5 describe derived and triangulated categories.

Finally, Chap. 6–8 contain geometrical applications of the modern homological algebra to mixed Hodge structures, perverse sheaves and $\mathcal{D}$-modules. In other topological and algebraic geometry volumes of the Encyclopaedia the reader can find several parallel expositions and of additional material; in this volume we mostly emphasize the categorical and homological aspects of the theory.

The bibliography, inevitable quite incomplete, can help the interested reader to learn more about topics involved.

Let us remark also that the references in the text give section and subsection numbers, e.g. Chap. 2, 2.1, or Chap. 1, 1.5.1. In references inside the current chapter the chapter number is omitted.

Chapter 1
Complexes and Cohomology

§ 1. Complexes and the Exact Sequence

1.1. Complexes. A *chain complex* is a sequence of abelian groups and homomorphisms

$$C_.: \ \dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

with the property $d_n \circ d_{n+1} = 0$ for all n. Homomorphisms d_n are called *boundary operators* or *differentials.* A *cochain complex* is a sequence of abelian groups and homomorphisms

$$C^{\cdot}: \ \dots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C_{n+1} \xrightarrow{d_{n+1}} \dots$$

with the property $d^n \circ d_{n-1} = 0$. A chain complex can be considered as a cochain complex by reversing the enumeration: $C^n = C_{-n}$, $d^n = d_{-n}$. This is why we will usually consider only cochain complexes. A complex of A-modules is a complex for which C_n (respectively C^n) are modules over a ring A and d_n (resp. d^n) are homomorphisms of modules.

1.2. Homology and Cohomology. Since $d_n \circ d_{n+1} = 0$, we have $\operatorname{im} d_{n+1} \subset \ker d_n$. A *homology* of a chain complex is the group $H_n(C_.) = \ker d_n / \operatorname{im} d_{n+1}$. A *cohomology* of a cochain complex is the group $H^n(C^{\cdot}) = \ker d^n / \operatorname{im} d^{n-1}$. The standard terminology is as follows: elements of C_n are called *n-dimensional chains,* elements of C^n are *n-dimensional cochains,* elements of $\ker d_n = Z_n$ are *n-dimensional cycles,* elements of $\ker d^n = Z^n$ are *n-dimensional cocycles,* those of $\operatorname{im} d_{n+1} = B_n$ are *boundaries,* those of $\operatorname{im} d^{n-1} = B^n$ are *coboundaries.* If $C^{\cdot}$ is a complex of A-modules, its cohomology is an A-module. A complex is said to be *acyclic* (or an *exact sequence*) if $H^n(C^{\cdot}) = 0$ for all n.

1.3. Morphisms of Complexes. A morphism $f\colon C^{\cdot} \to D^{\cdot}$ is a family of group (module) homomorphisms $f^n\colon C^n \to D^n$ commuting with differentials: $f^{n+1} \circ d_C^n = d_D^n \circ f^n$. A morphism f induces a morphism of cohomology $H^{\cdot}(f) = \{H^n(f)\colon H^n(C^{\cdot}) \to H^n(D^{\cdot})\}$ by the formula $\{$*the class of a cocycle* $c\} \mapsto \{$*the class of a cocycle* $f(c)\}$.

A *homotopy* between morphisms of complexes $f, g\colon C^{\cdot} \to D^{\cdot}$ is a family of group homomorphisms $h^n\colon C^n \to D^{n+1}$ such that $f^n - g^n = h^{n+1} \circ d^n + d^{n-1} \circ h^n$. The class of morphisms homotopic to zero form "an ideal," i.e. it is stable under addition and the composition with an arbitrary morphism.

1.3.1. Lemma. *If f and g are homotopic then $H^n(f) = H^n(g)$ for each n.*

Indeed, if c is a cocycle then $f(c) = g(c) + d(h(c))$, so that the classes of $f(c)$ and $g(c)$ coincide.

1.4. Exact Triple of Complexes. A sequence of complexes and morphisms $O \to K^{\cdot} \to L^{\cdot} \to M^{\cdot} \to 0$ is said to be *exact* (or an *exact triple*) if for each n the sequence of groups (modules) $O \to K^n \to L^n \to M^n \to 0$ is exact.

1.5. Connecting Homomorphism. Let $O \to K^{\cdot} \to L^{\cdot} \to M^{\cdot} \to 0$ be an exact triple of complexes. For any n define a homomorphism $\delta^n = \delta^n(f,g) : H^n(M^{\cdot}) \to H^{n+1}(K^{\cdot})$ as follows. Let $m \in M^n$ be a cycle. Choose $l \in L^n$ such that $g^n(l) = m$. Then $g^{n+1}(d^n(l)) = 0$, so that $d^n(l)) = f^{n+1}(k)$ for some $k \in K^{n+1}$. It is clear that $d^{n+1}k = 0$. Set δ(the class of m) = (the class of k). Direct computations show that δ^n does not depend on the choices.

1.5.1. Theorem. *The cohomology sequence*

$$\begin{aligned} \dots \longrightarrow H^n(K^{\cdot}) \xrightarrow{H^n(f)} H^n(L^{\cdot}) \xrightarrow{H^n(g)} \\ \xrightarrow{H^n(g)} H^n(M^{\cdot}) \xrightarrow{\delta^n(f,g)} H^{n+1}(K^{\cdot}) \longrightarrow \dots \end{aligned}$$

is exact.

§ 2. Standard Complexes in Algebra and in Geometry

2.1. Simplicial Sets. Complexes in homological algebra are mostly either of topological nature or somehow appeal to topological intuition. A classical method to study a topological space is to consider its triangulation, i.e. to decompose it into simplexes: points, segments, triangles, tertrahedra, etc. The corresponding algebraic technique is the technique of simplicial sets.

2.1.1. Definition. A *geometrical n-dimensional simplex* is the topological space

$$\Delta_n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, \ x_i \geq 0\right\}.$$

The point e_i such that $x_i = 1$ is called its *i-th vertex*. To any nondecreasing mapping $f : [m] \to [n]$, where $[m] = 0, 1, \dots, m$, we associate the mapping Δ_f, called "the f-th face," as follows: Δ_f is a unique linear mapping that sends the vertex $e_i \in \Delta_m$ to the vertex $e_{f(i)} \in \Delta_n$ for $i = 0, 1, \dots, m$.

2.1.2. Definition. A *simplicial set* is a family of sets $X = (X_n)$, $n = 0, 1, \dots,$ and mappings $X(f) : X_n \to X_m$, one for each nondecreasing map $f : [m] \to [n]$, such that

$$X(\mathrm{id}) = \mathrm{id}, \quad X(g \circ f) = X(f) \circ X(g).$$

One can consider X_n as the set of indices enumerating a family of n-dimensional geometrical simplexes. Mappings $X(f)$ describe how to glue all these simplexes together in order to obtain one topological space.

A *simplicial mapping* $\varphi : X \to Y$ is a family $\varphi_n : X_n \to Y_n$ such that $Y(f)\varphi_n = \varphi_m X(f)$ for each nondecreasing $f : [m] \to [n]$.

2.1.3. Definition. *Geometric realization* of a simplicial set X is the topological space

$$|X| = \coprod_{n=0}^{\infty} (\Delta_n \times X_n)/R,$$

where the equivalence relation R is defined as follows: $(s, x) \in \Delta_n \times X_n$ is identified with $(t, y) \in \Delta_m \times X_m$ if there exists a nondecreasing mapping $f : [m] \to [n]$ with $Y = X(f)x$, $s = \Delta_f t$. The topology on $|X|$ is the weakest one for which the factorization by R is continuous.

2.2. Homology and Cohomology of Simplicial Sets. Let X be a simplicial set. Denote by $C_n(X, \mathbb{Z})$, $n > 0$, the free abelian group generated by the set X_n, and set $C_n = 0$ for $n < 0$. For any abelian group F set $C_n(X, F) = C_n(X, \mathbb{Z}) \otimes_{\mathbb{Z}} F$. Hence, elements of $C_n(X, F)$, called chains of X with coefficients in F, are formal linear combinations of the form $\sum_{x \in X_n} a(x)x$, $a(x) \in F$. The boundary operator is defined as follows. Let $\partial_n^i : [n-1] \to [n]$ be a unique decreasing mapping whose image does not contain i. We set $d_0 = 0$, and then

$$d_n\Big(\sum_{x \in X_n} a(x)x\Big) = \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x), \quad n \geq 1,$$

Cochains $C^n(X, F)$ are defined dually: $C^n(X, F)$ consists of functions on X_n with values in F, and

$$(d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i f(X(\partial_{n+1}^i)(x).$$

Set

$$H_n(X, F) = H_n(C_.(X, F)), \quad H^n(X, F) = H^n(C^{\cdot}(X, F)).$$

2.3. The Singular Complex. Let Y be a topological space. By a *singular n-simplex* of Y we mean a continuous mapping $\varphi : \Delta_n \to Y$. Define

$$X_n \text{ is the set of singular } n\text{-simplexes of } Y;$$
$$X(f)\varphi = \varphi \circ \Delta_f, \text{ where } f : [m] \to [n] \text{ does not decrease, } \Delta_f : \Delta_n \to \Delta_m.$$

(Co)homology Y with coefficients in an abelian group F is defined as $H_n(X, F)$ and $H^n(X, F)$ and denoted $H_n^{\mathrm{sing}}(X, F)$ and $H^n_{\mathrm{sing}}(X, F)$.

2.4. Coefficient Systems. In the definition of an n-chain of a simplicial set coefficients we can assume that coefficients at different simplexes are taken from different group. However, to define the boundary operator in this case one has to impose to these coefficient groups the following compatibility conditions.

2.4.1. Definition. a. A *homological coefficient system* $\mathcal{A}$ on a simplicial set X is a family of abelian group $\mathcal{A}_x$, one for each simplex $x \in X$, and a family of group homomorphisms $\mathcal{A}(f,x) : \mathcal{A}_x \to \mathcal{A}_{X(f)x}$, one for each pair $(x \in X_n,\ f : [m] \to [n])$, such that the following conditions are satisfied:

$$\mathcal{A}(\mathrm{id}, x) = \mathrm{id}; \quad \mathcal{A}(fg, x) = \mathcal{A}(g, X(f)x)\mathcal{A}(f, x).$$

b. A *cohomological coefficient system* $\mathcal{B}$ on a X is a similar family of abelian group $\{\mathcal{B}_x\}$, and a similar family of group homomorphisms $\mathcal{B}(f,x) : \mathcal{B}_{X(f)x} \to \mathcal{B}_x$ such that

$$\mathcal{B}(\mathrm{id}, x) = \mathrm{id}; \quad \mathcal{B}(fg, x) = \mathcal{B}(f, x)\mathcal{B}(g, X(f)x).$$

2.5. Homology and Cohomology with Coefficients. In the notation of 2.4, set

$$C_n(X, \mathcal{A}) = \Big\{ \sum_{x \in X_n} a(x)x \mid a(x) \in \mathcal{A}_x \Big\},$$

$$d_n\Big(\sum_{x \in X_n} a(x)x \Big) = \sum_{x \in X_n} \sum_{i=0}^{n} \mathcal{A}(\partial_n^i, x)(a(x))(-1)^i X(\partial_n^i x), \quad n \geq 1,$$

and similarly

$$C^n(X, \mathcal{B}) = \Big\{ \text{functions } f : X_n \to \coprod_{x \in X} \mathcal{B}_x, \quad f(x) \in \mathcal{B}_x \Big\},$$

$$(d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i \mathcal{B}(\partial_n^i, x)(f(X(\partial_{n+1}^i))), \quad x \in X_{n+1}.$$

If the groups $\mathcal{A}_x$ (resp. $\mathcal{B}_x$) do not depend on x, and all mappings $\mathcal{A}(f,x)$ (resp. $\mathcal{B}(f,x)$) are the identity homomorphisms, we recover the definition from 2.2. (Co)homology of $C_.(X, \mathcal{A})$ and $C^{\cdot}(X, \mathcal{B})$ are called the (co)homology of the simplicial set with the coefficient system.

2.6. Čech Cohomology with Coefficients in a Sheaf. Let Y be a topological space, $U = (U_\alpha)$, $\alpha \in A$, be its open or closed covering. The *nerve* of the covering U is the following simplicial set X:

$$X_n = \{(\alpha_0, \ldots, \alpha_n) \mid U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq \emptyset\} \subset A^{n+1};$$
$$X(f)(\alpha_0, \ldots, \alpha_n) = (\alpha_{f(0)}, \ldots, \alpha_{f(n)}) = \quad \text{for } f : [m] \to [n].$$

Let $\mathcal{F}$ be a sheaf of abelian groups on Y. It determines a cohomological coefficient system on the nerve of Y as follows:

$$\mathcal{F}_{\alpha_0, \ldots, \alpha_n} = \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_n}, \mathcal{F}),$$

$$\mathcal{F}(f,(\alpha_0,\dots,\alpha_n)) \quad \text{maps the section } \varphi \in \Gamma(U_{\alpha_{f(0)}} \cap \dots \cap U_{\alpha_{f(n)}}, \mathcal{F})$$
$$\text{to its restriction to } U_{\alpha_0} \cap \dots \cap U_{\alpha_n}.$$

Cohomology of X with this coefficient system is called the Čech cohomology of the covering U with coefficients in the sheaf $\mathcal{F}$.

2.7. Group Cohomology. Let G be a group. Define a simplicial set BG as follows:

$$(BG)_n = G^n;$$
$$\text{for } f:[m]\to[n], \quad BG(f)(g_1,\dots,g_n) = (h_1,\dots,h_m),$$
$$\text{where } h_i = \prod_{j=f(i-1)+1}^{f(i)} g_j \quad (= e \text{ if } f(i-1) = f(i)).$$

The geometric realization $|BG|$ is called the *classifying space* of the group G.

Let A be a left G-module, i.e. an additive group with the action of G by automorphisms. Such a module yields the following cohomological coefficient system $\mathcal{B}$ on BG:

$$\mathcal{B}_x = A \quad \text{for all } x;$$
$$\mathcal{B}(f,x)(a) = ha, \quad \text{where } h = \prod_{j=1}^{f(0)} g_j \quad (= e \text{ if } f(0) = 0)$$
$$\text{for } f:[m]\to[n], \quad x = (g_1,\dots,g_n) \in (BG)_n, \quad a \in A.$$

Using the above definitions we can describe the complex $C^\cdot(BG,\mathcal{B})$ (denoted also by $C^\cdot(G,A)$) explicitly:

$$C^0(G,A) = A;$$
$$C^n(G,A) = \text{function on } G^n \text{ with values in } A.$$

Next, for an n-cochain f,

$$df(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1} g_{n+1} f(g_1,\dots,g_n).$$

Cohomology of this complex are denoted $H^n(C,A)$. Similarly, using A one can construct the following homological coefficient system $\mathcal{A}$ on BG:

$$\mathcal{A}_x = A \quad \text{for all } x;$$
$$\mathcal{A}(f,x)a = h^{-1}a, \quad \text{where } h = \prod_{j=1}^{f(0)} g_j.$$

It gives the homology $H_n(C,A)$.

2.8. The de Rham Complex. In the above examples the transition from geometry to algebra was performed using combinatorics and simplicial decomposition. In the case when the topological space X has the structure of a smooth manifold, the ring of smooth differential forms is a complex. More precisely, let $\Omega^i(X)$ be the space of i-forms. The exterior derivative is given in local coordinates $(x^1, \ldots, x^n)$ by the formula

$$d\Big(\sum_{|I|=k} f_I dx^I\Big) = \sum_{|I|=k} \sum_i \frac{\partial f_I}{\partial x^i} dx^i \wedge dx^I,$$

where

$$I = (i_1, \ldots, i_k), \quad |I| = i_1 + \cdots + i_k, \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The cohomology of the complex $(\Omega^{\cdot}(X), d)$, denoted $H^{\cdot}_{\mathrm{DR}}(X)$, is called the *de Rham cohomology* of the manifold X.

The de Rham theorem established a canonical isomorphism

$$H^{\cdot}_{\mathrm{DR}}(X) = H^{\cdot}_{\mathrm{sing}}(X, \mathbb{R}).$$

On the level of chains this isomorphism associates to a differential i-form its integrals over smooth i-dimensional singular chains.

2.9. Lie Algebra Cohomology. Let us consider the de Rham complex of a connected Lie group G. The group G acts on this complex by the right shifts. Denote by $\Omega_{\mathrm{inv}}(G)$ the subcomplex consisting of G-invariant chains. It admits a purely algebraic description. Let $\mathfrak{g}$ be the Lie algebra of G considered as the Lie algebra of right-invariant vector fields on G. Then $\Omega^n_{\mathrm{inv}}(G) = L(\wedge^n \mathfrak{g}, \mathbb{R})$ is the space skew-symmetric n-linear real forms on $\mathfrak{g}$. The exterior derivative of an n-form considered as a polylinear function on vector fields (on an arbitrary smooth manifold) is given by the following Cartan formula:

$$\begin{aligned}(d\omega^n)&(\xi_1, \ldots, \xi_{n+1}) \\ &= \sum_{1<j<l<n} (-1)^{j+l-1} \omega^n([\xi_j, \xi_l], \xi_1, \ldots, \widehat{\xi_j}, \ldots, \widehat{\xi_l}, \ldots, \xi_{n+1}) \\ &\quad + \sum_{j=1}^{n+l} (-1)^j \xi_j [\omega^n(\xi_1, \ldots, \widehat{\xi_j}, \ldots, \xi_{n+1})]\end{aligned}$$

(here $\widehat{\ }$ means that the corresponding term is omitted). Applying this formula to $\Omega^n_{\mathrm{inv}}(G)$ we obtain the following formula for d on $C^{\cdot}(\mathfrak{g}) = L(\wedge^{\cdot} \mathfrak{g}, \mathbb{R})$:

$$(dc)(g_1, \ldots, g_{n+1}) = \sum_{1<j<l<n} (-1)^{j+l-1} c([g_j, g_l], g_1, \ldots, \widehat{g_j}, \ldots, \widehat{g_l}, \ldots, g_{n+1}).$$

Denote the cohomology of this complex by $H^{\cdot}(\mathfrak{g}, \mathbb{R})$. Merging the de Rham theorem with the averaging over a compact subgroup, we obtain the E. Cartan theorem: for a compact connected group G

$$H^{\cdot}_{\mathrm{sing}}(G, \mathbb{R}) = H^{\cdot}(\mathfrak{g}, \mathbb{R}).$$

The construction of $G^{\cdot}(\mathfrak{g})$ does not require the existence of the Lie group G associated to the Lie algebra $\mathfrak{g}$ and can be applied to an arbitrary Lie algebra over a field k.

More generally, let M be a $\mathfrak{g}$-module. Set $C^n(\mathfrak{g}, M) = L(\wedge^n \mathfrak{g}, M)$ and define the differential by the Cartan formula

$$\begin{aligned}(dc)(g_1, \dots, g_{n+1}) = & \sum_{1<j<l<n} (-1)^{j+l-1} c([g_j, g_l], g_1, \dots, \widehat{g_j}, \dots, \widehat{g_l}, \dots, g_{n+1}) \\ & + \sum_{j=1}^{n+1} (-1)^j g_j c(g_1, \dots, \widehat{g_j}, \dots, g_{n+1}).\end{aligned}$$

Denote the cohomology of this complex by $H^{\cdot}(\mathfrak{g}, M)$.

To define the homology $H_{\cdot}(\mathfrak{g}, M)$ we must use the complex $H_{\cdot}(\mathfrak{g}, M) = M \otimes \wedge^{\cdot} \mathfrak{g}$ with the differential

$$\begin{aligned}& d(m \otimes (g_l \wedge \dots \wedge g_n)) \\ & \quad = \sum_{1<j<l<n} (-1)^{j+l-1} m \otimes ([g_j, g_l], g_1 \wedge \dots \wedge \widehat{g_j} \wedge \dots \wedge \widehat{g_l} \wedge \dots \wedge g_n) \\ & \quad + \sum_{j=1}^{n+l} (-1)^{j+1} g_j m \otimes (g_j \wedge \dots \wedge \widehat{g_j} \wedge \dots \wedge g_{n+1}).\end{aligned}$$

2.10. The Hochschild Complex. Let k be a commutative ring with unity, A be an associative k-algebra with unity. Consider the following chain complex $T_{\cdot}(A)$:

$$T_n(A) = \overbrace{A \otimes_k \cdots \otimes_k A}^{n+2 \text{ times}}, \quad n \geq -1,$$

$$d(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^j (a_0 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}).$$

It can be considered as a complex of k-modules, as a complex of A-bimodules, and, finally, as a complex of A^e -modules where $A^e = A \otimes_k A^t$, A^t is the opposite ring of A (i.e. the ring A with the oppposte multiplication). This complex is acyclic since its identity mapping is homotopic to the zero mapping: the homotopy is given by the formula

$$a_0 \otimes \cdots \otimes a_{n+1} \to 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}.$$

Let M be an A-bimodule. Then we can consider the complexes of k-modules $M \otimes_{A^e} T_{\cdot}(A)$ and $\mathrm{Hom}_{A^e}(T_{\cdot}(A), M)$. The homology of these complexes are denoted by $H_n(A, M)$ and $H^n(A, M)$ respectively and called the Hochschild (co)homology of the algebra A with coefficient in the module M. We can get rid of the tensor product over A^e using the isomorphism

$$M \otimes_{A^e} T_n(A) = M \otimes_{A^e} A^e \otimes_k T_{n-2}(A) = M \otimes_{A^e} T_{n-2}(A).$$

In this setting $H_n(A, M)$ becomes the homology of the following complex $C_{\cdot}(A, M)$:

$$\text{the } n\text{-th term is } M \otimes A^{\otimes n};$$

$$\begin{aligned} d(m \otimes a_1 \otimes \cdots \otimes a_n) &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_n \\ &+ (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Similarly, is the cohomology of the complex $C^{\cdot}(A, M)$:

$$\text{the } n\text{-th term is } \operatorname{Hom}_k(A^{\otimes n}, M);$$

$$\begin{aligned} df(a_1 \otimes \cdots \otimes a_{n+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{i=0}^{n} (-1)^i f(a_1 \otimes \cdots \otimes (a_i a_{i+1}) \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1}. \end{aligned}$$

2.11. Cyclic Homology of an Algebra. Let us keep the setup of the previous subsection and take in this setup $k \supset \mathbb{Q}$, $M = A$. The cyclic shift acts on the terms of the complex $C_{\cdot}(A, A)$, whose homology is $H_n(A, A)$. Define

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

The operator t does not commute with the differential. However, if we set

$$d'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n,$$

then

$$d(1 - t) = (1 - t)d'.$$

Hence the image of $1 - t$ is a subcomplex of $C_n(A, A)$ so that we can define the quotient complex

$$\begin{aligned} C_n^{\lambda}(A) &= C_n(A, A) / \operatorname{im}(1 - t), \\ d^{\lambda} &= d \mod \operatorname{im}(1 - t). \end{aligned}$$

Its homology is called the cyclic homology of the algebra A and are denoted $H_n^{\lambda}(A)$ or $HC_n(A)$.

2.12. Cyclic Cohomology of an Algebra. To define it we have to consider the subcomplex $C_{\lambda}^{\cdot}(A) \subset C^{\cdot}(A, A^*)$ consisting of t-invariant cochains, i.e. of k-linear functionals $f : A^{\otimes n} \to A^*$ with the property

$$f(a_0 \otimes \cdots \otimes a_n) = (-1)^n f(a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}).$$

The coboundary operator is given by the last formula in 2.9.

Cohomology is denoted by $H^n_\lambda(A)$ or $HC^n(A)$.

2.13. (Co)Chain Complex of a Cell Decomposition. To use the singular (co)chain complex in the computation of (co)homology of a topological space X is non-economic because this complex is infinite-dimensional. Topologists often use (co)chains associated to a realization of X as a cell decomposition. Let us give the basic definitions.

A *cell decomposition* (or CW-complex) is a topological space X represented as a union $X = \bigcup\limits_{n=0}^{\infty} \bigcup\limits_{i\in I_n} e^n_i$ of disjoint sets e^n_i (cells) with mappings $f^n_i : B^n \to X$ of the closed unit ball into X such that the restriction of f^n_i to the interior $\operatorname{Int} B^n$ of B^n is a homeomorphism $f^n_i : \operatorname{Int} B^n \xrightarrow{\sim} e^n_i$, and the following conditions are satisfied:

a) The boundary $\dot{e}^n_i = \bar{e}^n_i \setminus e^n_i$ of any cell is contained in the union of a finite number of cell of smaller dimensions.

b) The set $Y \subset X$ is closed if and only if the preimage $(f^n_i)^{-1}(Y) \cap \bar{e}^n_i$ is closed in $\bar{e}^n_i$ for all n and all $i \in I_n$.

For a pair of cells e^n_i, e^{n-1}_j define the *incidence coefficient* $c(e^n_i, e^{n-1}_j)$ as follows. Let X^r be the union of all cells of dimension $\leq r$. Then X^{n-1}/X^{n-2} (X^{n-2} is contracted to a point in X^{n-1}) is the wedge of $(n-1)$-dimensional spheres S^{n-1} in the number equal to the cardinality of I_{n-1}, and the cell e^{n-1}_j distinguishes one sphere in this wedge (denote it by S). Consider the composite mapping

$$S^{n-1} = \dot{B}^n \xrightarrow{f^n_i|S^{n-1}} X^{n-1} \longrightarrow X^{n-1}/X^{n-2} \xrightarrow{\pi} S,$$

where π is the projection of the wedge onto one of its components. The resulting mapping $S^{n-1} \to \dot{B}^n = S^{n-1}$ determines an element of the group $\pi_{n-1}(S^{n-1})$, i.e. an integer (the degree of the mapping), and we define the incidence coefficient $c(e^n_i, e^{n-1}_j)$ to be equal to this integer.

Define now the group of integral n-dimensional chains as the free abelian group generated by e^n_i, $i \in I_n$, and define the differential by the formula

$$de^n_i = \sum_{j\in I_{n-1}} c(e^n_i, e^{n-1}_j) e^{n-1}_j.$$

By the condition a) above, this sum is finite.

Cochains, as well as chains and cochains with coefficients, can be defined similarly.

2.13.1. Theorem. *(Co)homology of a cell decomposition computed from cell (co)chains is canonically isomorphic to singular (co)homology.*

§ 3. Spectral Sequence

3.1. Definition of a Spectral Sequence. Together with the cohomology exact sequence (Theorem 1.5.1), the spectral sequence is one of the most powerful computational tools in homological algebra.

A spectral sequence of abelian groups is a family of abelian groups $E = (E_r^{p,q}, E^n)$, $p, q, r \in \mathbb{Z}$, $r \geq 1$, and a family of homomorphisms with some properties that we will describe shortly.

But first we say a few words about a convenient way to represent all these data.

The reader can imagine a stack of square-lined paper sheets, each square being numbered by a pair of integers $(p,q) \in \mathbb{Z}^2$. An object $E_r^{p,q}$ is assumed to live in the (p,q)-th square at the r-th sheet. Objects E^n live in the last, "transfinite" sheet, and occupy the entire diagonal $p+q=n$.

Now we describe homomorphisms and the conditions they satisfy.

a. On the r-th sheet we have homomorphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. For $r=1$ they act from a square to its right neighbor, for $r=2$ they act by a chess springer move (one square down and two squares to the right). For $r \geq 3$ we get a generalized springer move.

Condition: $d_r^2 = 0$; more explicitly, $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ for all p, q, r.

Using $(E_r^{p,q}, d_r^{p,q})$ we can construct cohomology of the r-th sheet:

$$H^{p,q}(E_r) = \ker d_r^{p,q} / \operatorname{im} d_r^{p+r,q-r+1}.$$

The following data are included into the definition of E:

b. Isomorphisms $\alpha_r^{p,q} : H^{p,q}(E_r) \to E_{r+1}^{p,q}$.

Usually we will assume that on the $(r+1)$-th sheet we have just cohomology of the r-th sheet and $\alpha_r^{p,q}$ are identities.

The main condition to isomorphisms $\alpha_r^{p,q}$ is the existence of limit objects $E_\infty^{p,q}$. The simplest way to guarantee this, which usually suffices in applications, is the following:

c. For any pair (p,q) there exists r_0 such that $d_r^{p,q} = 0$, $d_r^{p+q,q-r+1} = 0$ for $r \geq r_0$. In this case isomorphisms $\alpha_r^{p,q}$ identify all $E_r^{p,q}$ for $r \geq r_0$ and we will denote this object by $E_\infty^{p,q}$.

At this moment on the transfinite sheet $(r = \infty)$ we have objects $E_\infty^{p,q}$ and objects E^n along the diagonal $p+q=n$. The last collection of data relates these two classes of objects.

d. A decreasing *regular filtration* $\ldots \supset F^pE^n \supset F^{p+1}E^n \supset \ldots$ (i.e. $\cap F^pE^n = \{0\}$, $\cup F^pE^n = E^n$) on each E^n and isomorphisms $\beta^{p,q} : E_\infty^{p,q} \to F^pE^{p+q}/F^{p+1}E^{p+q}$ are given.

If these conditions are satisfied we say that the spectral sequence $(E_r^{p,q})$ *converges to* (E^n) or that (E^n) *is the limit of* $(E_r^{p,q})$.

Let us emphasize once more that the components of one spectral sequence E are all objects $(E_r^{p,q}, E^n)$, all homomorphisms $(d_r^{p,q}, \alpha_r^{p,q}, E^{p,q})$, and all filtrations on E^n.

A morphism $f : E \to E'$ of spectral sequences is a family of homomorphisms $f_r^{p,q} : E_r^{p,q} \to E'^{p,q}_r$, $f^n : E^n \to E'^n$ commuting with structural morphisms and are compatible with filtrations.

3.2. Remarks. a. Working with complexes that are bounded from one side we usually get spectral sequences with non-zero objects placed in only one quadrant (the region of the form, say $p > p_0$, $q > q_0$).

Let the only nonzero $E_r^{p,q}$ be those in the quadrant I. Then the condition **c** in 3.1 is automatically satisfied, since for $r > r_0(p,q)$ either the beginning or the end of arrows $d_r^{p,q}$, $d_r^{p+r,q-p+1}$ lies outside the quadrant. Moreover, filtrations on E^n are automatically finite: $F^pE^n = 0$ for $p < p_-(n)$, $F^pE^n = E^n$ for $p > p_+(n)$. The same holds if the only nonzero $E_r^{p,q}$ are those in the qudrant III.

b. The larger the number of zero objects $E_r^{p,q}$ and of zero morphisms $d_r^{p,q}$, the better a spectral sequence may serve as a computational tool. One special case has its own name: E is said to be *degenerate* at E_r if $d_{r'}^{p,q} = 0$ for $r \geq r'$ and for all p, q. In this case we have obviously $E_\infty^{p,q} = E_r^{p,q}$.

c. Normally spectral sequences are used in problems when one wants to learn something about E^n from $E_1^{p,q}$ or $E_2^{p,q}$.

Let us show, for example, that some invariants similar to the Euler characteristic can be computed explicitly even when we know nothing about differentials $d_r^{p,q}$. Let C be an abelian group and χ is a mapping that associates to each abelian group A an element $\chi(A) \in C$. Assume that $\chi(A) = \chi(A')$ for isomorphic groups A and A', and χ is additive, i.e. satisfies the conditions $\chi(A) = \chi(B) + \chi(A/B)$ for any group A and any subgroup $B \subset A$. For a finite complex $K^\cdot$ let $\chi(K^\cdot) = \sum(-1)^i\chi(K^i)$. Then it is easy to show (and also well known) that $\chi(K^\cdot) = \sum(-1)^i\chi(H^i(K^\cdot))$.

Let now E be a spectral sequence. Consider the complexes $(E_r^\cdot, d_r^\cdot)$, where $E_r^n = \oplus E_r^{p,q}, d_r^n = \oplus d_r^{p,q}$. Let us assume that for some $r = r_0$ all direct sums $\oplus E_r^{p,q}$ are finite and complexes $E_{r_0}^\cdot$ are bounded. Then the same is true for all $r > r_0$ and one can easily verify that for $r \geq r_0$ we have

$$\sum_n (-1)^n \chi(E^n) = \chi(E_r^\cdot).$$

3.3. Spectral Sequence of a Filtered Complex. Let $K^\cdot$ be a complex and let we are given a decreasing filtration of $K^\cdot$ by its subcomplexes $F^pK^\cdot$. This means that in K^n we have subobjects $\ldots \supset F^pK^n \supset F^{p+1}k^n \supset \ldots$ and $d^n(F^pK^n) \subset F^pK^{n+1}$.

Let us present two useful examples.

Denote

$$(F^pK^\cdot)^n = \tau_{\leq -p}(K^\cdot)^n = \begin{cases} K^n & \text{for } n < -p, \\ \ker d^{-p} & \text{for } n = -p, \\ 0 & \text{for } n > -p. \end{cases}$$

This is obviously a filtration which is called the *canonical* filtration. It kills cohomology of $K^{\cdot}$ one by one:

$$H^n(F^pK^{\cdot}) = \begin{cases} H^n(K^{\cdot}) & \text{for } n \leq -p, \\ 0 & \text{for } n > -p. \end{cases}$$

Next, let

$$(\widetilde{F^p}K^{\cdot}) = \sigma_{\geq p}(K^{\cdot})^n = \begin{cases} 0 & \text{for } n < p, \\ K^n & \text{for } n \geq p. \end{cases}$$

This filtration is called the *stupid* filtration: it also kills cohomology of $K^{\cdot}$ one by one, but while doing so it spoils them before killing:

$$H^n(\widetilde{F^p}K^{\cdot})^n = \begin{cases} 0 & \text{for } n < p, \\ \ker d^p & \text{for } n = p, \\ H^n(K^{\cdot}) & \text{for } n > p. \end{cases}$$

Now, given a filtered complex, we construct a spectral sequence.

a) *Construction of $E_r^{p,q}$ and of $d_r^{p,q}$.* Let

$$Z_r^{p,q} = d^{-1}(F^{p+r}K^{p+q+1}) \cap (F^pK^{p+q}).$$

This group "bounds from above" the cycles in K^{p+q} that belong to the p-th filtration subgroup: the differential d does not necessarily maps them to 0, but increases the filtration index by at least r.

The group $Z_r^{p,q}$ contains a trivial part, which is the sum of two following subgroups:

$$Z_{r-1}^{p+1,q-1} = d^{-1}(F^{p+r}K^{p+q+1}) \cap (F^{p+1}K^{p+q})$$
$$dZ_{r-1}^{p-r+1,q+r-2} = d(F^{p-r+1}K^{p+q-1}) \cap (F^pK^{p+q}).$$

Let

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}).$$

One can easily verify that d induces differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

b) *Construction of $\alpha_r^{p,q}$.* To construct $\alpha_r^{p,q}$ one must define homomorphisms

$$(Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1})/(Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}) \to Z(E_r^{p,q}),$$
$$(dZ_r^{p+r,q-r+1} + Z_{r-1}^{p+1,q-1})/(Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}) \to B(E_r^{p,q})$$

(with cycles and boundaries of d_r in the right-hand sides) and show that they are isomorphisms. This is a rather straightforward, although cumbersome, computations.

c) *Construction of E^n.* Let $E^n = H^n(K^{\cdot})$ and

$$F^pE^n = \text{the image of } H^n(F^pK^{\cdot})$$

under the natural morphism $F^pK^{\cdot} \to K^{\cdot}$.

3.3.1. Theorem. *Let us assume that for each n the filtration on K^n is finite and regular. Then the above spectral sequence $E_n^{p,q}$ converges to (E^n).*

3.4. Examples of Spectral Sequences. Here we compute spectral sequences related to the stupid and to the canonical filtration of a complex $K^\cdot$ (see the beginning of 3.3).

a) *Stupid filtration* $\widetilde{F^p}K^\cdot$. We have

$$Z_r^{p,q} = \begin{cases} 0 & \text{for } q < 0, \\ \ker d^{p+q} & \text{for } q \geq 0,\ r < q+1, \\ K^{p+q} & \text{for } q \geq 0,\ r \geq q+1. \end{cases}$$

$$E_r^{p,q} = \begin{cases} 0 & \text{for } q \neq 0, \\ K^p & \text{for } q = 0,\ r = 1, \\ H^p(K^\cdot) & \text{for } q = 0,\ r \geq 2, \text{ and } q = 0,\ r = \infty. \end{cases}$$

The differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+q,q-r+1}$ coincides with $d^p : K^p \to K^{p+1}$ for $q = 0$, $r = 1$ and is trivial in all other cases.

$E^n = H^n(K^\cdot)$; the filtration on E^n is trivial:

$$F^pE^n = \begin{cases} E^n & \text{for } p \leq n, \\ 0 & \text{for } p > n. \end{cases}$$

b) *Canonical filtration.* We have

$$E_1^{p,q} = \begin{cases} H^p(K^\cdot) & \text{for } q = -2p, \\ 0 & \text{for } q \neq -2p. \end{cases}$$

$$d_1^{p,q} = 0 \quad \text{for all } p, q.$$

Hence $E_r^{p,q} = E_1^{p,q}$ and $d_r = 0$ for all $r \geq 1$. Next, $E^n = H^n(K^\cdot)$ and the filtration on E^n is trivial:

$$F^pE^n = \begin{cases} E^n & \text{for } n \leq -p, \\ 0 & \text{for } n > -p. \end{cases}$$

3.5. Spectral Sequence of a Double Complex. *A double complex* $L = (L^{ij}, d_{\mathrm{I}}^{ij}, d_{\mathrm{II}}^{ij})$ is a family of objects L^{ij} and homomorphisms $d_{\mathrm{I}}^{ij} : L^{ij} \to L^{i+1,j}$, $d_{\mathrm{II}}^{ij} : L^{ij} \to L^{i,j+1}$ satisfying the conditions

$$d_{\mathrm{I}}^2 = 0, \quad d_{\mathrm{II}}^2 = 0, \quad d_{\mathrm{I}}d_{\mathrm{II}} + d_{\mathrm{II}}d_{\mathrm{I}} = 0 \tag{1}$$

Denote $(SL)^n = \oplus_{i+j=n} L^{ij}$ (in all situations we shall meet with later, these direct sums will be finite). The conditions (1) means that the operator

$$d = d_{\mathrm{I}} + d_{\mathrm{II}} : (SL)^n \to (SL)^{n+1}$$

satisfies the condition $d^2 = 0$ so that $((SL)^\cdot, d)$ is a complex called the *diagonal complex* of L.

Morphisms of double complexes are defined in an obvious way. Denote by $H_{\mathrm{I}}^{ij}(L^{\cdot,j})$ of the ordinary complex $L^{\cdot,j}$ with the differential $d_{\mathrm{I}}^{\cdot,j}$. The differential d_{II}^{ij} induces, clearly, morphisms $H_{\mathrm{I}}^{ij}(L) \to H_{\mathrm{I}}^{i,j+1}(L)$ which determine complexes $H_{\mathrm{I}}^{i,\cdot}(L)$. Denote the cohomology of this by $H_{\mathrm{II}}^{j}(H_{\mathrm{I}}^{i,\cdot}(L^{\cdot,\cdot}))$. Similarly one can define cohomology $H_{\mathrm{I}}^{i}(H_{\mathrm{II}}^{\cdot,j}(L^{\cdot,\cdot}))$.

3.5.1. Proposition. *Let L lie in the first quadrant (i.e. $L^{ij} = 0$ if either $i < 0$ or $j < 0$). Then there exist two spectral sequences ${}^{\mathrm{I}}E$ and ${}^{\mathrm{II}}E$ with the common limit $\{H^n(SL)\}$ whose E_2-terms are of the form*

$$ {}^{\mathrm{I}}E_2^{pq} = H_{\mathrm{I}}^{q}(H_{\mathrm{II}}^{\cdot,q}(L^{\cdot,\cdot})), \quad {}^{\mathrm{II}}E_2^{pq} = H_{\mathrm{II}}^{p}(H_{\mathrm{I}}^{q,\cdot}(L^{\cdot,\cdot})). $$

3.6. Serre-Hochschild Spectral Sequence. Let G be a group, H be a normal subgroup of G, A be a G-module. Then the group G/H acts on the cohomology $H^q(H,A)$. The *Serre-Hochschild spectral sequence* has $E_2^{pq} = H^p(G/H, H^q(H,A))$ and converges to $E^n = H^n(G,A)$.

3.7. Spectral Sequence of a Fibration. Let $p : E \to B$ be a morphism of topological spaces; assume that p is a fibration in the sense of Serre. Assume further that the base B is connected and simply connected. Let F be the fiber of p. Then there exists a spectral sequence (called the *Serre spectral sequence*) with

$$ E_2^{pq} = H^p(B, H^q(F)) $$

(cohomology of B with coefficients in the local system $H^q(F)$), which converges to $H^n(E)$.

Later (see Chap. 2; 4.5.2) we will describe a general method to construct spectral sequences. Both examples above are special cases of this method.

Bibliographic Hints

The results of this chapter constitute a part of the classical homological algebra as it was thought of in early sixties; further development of these ideas can be found in classical textbooks such that (Cartan, Eilenberg 1956; Hilton, Stammbach 1971; MacLane 1963). The exact cohomology sequence (Theorems 1.5.1) and the spectral sequence of a filtered complex (Theorem 3.3.1) are the main computational tools in homological algebra. The exact sequence (in the topological situation) was known already to Poincaré; the spectral sequence appeared first in (Leray 1946). The proof of Theorem 1.5.1 see in (Cartan, Eilenberg 1956) or in (Gelfand, Manin 1988). The literature about spectral sequences is plentiful; one of the most general schemes and numerous examples (mostly topological) can be found in (McCleary 1985). The proof of Theorem 3.3.1 see, e.g., in (Cartan, Eilenberg 1956). About the Lyndon-Serre-Hochschild spectral sequence from 3.6 see (Brown 1982). The original paper (Serre 1952) still is one of the best expositions of the Serre spectral sequence; see also (Fuchs, Fomenko, Gutenmacher 1969).

Each of the topic from Sect. 2 is a starting point of one of the applications of homological algebra to the corresponding branches of mathematics; more about these applications see the corresponding sections of Chap. 3 and in Sect. 5 of Chap. 4. About simplicial sets and their applications to topology see (Gabriel, Zisman 1967; May 1967). The proof of Theorem 2.12.1 see, e.g., in (Dold 1972).

Chapter 2
The Language of Categories

§ 1. Categories and Functors

1.1. Definition. A *category* $\mathcal{C}$ is the following collection of data:

a. A class $\mathrm{Ob}\,\mathcal{C}$ whose elements are called objects of the category $\mathcal{C}$.

b. A family of sets $\mathrm{Hom}(X,Y)$ (sometimes denoted also $\mathrm{Hom}_{\mathcal{C}}(X,Y)$), one for each ordered pair $X,Y \in \mathrm{Ob}\,\mathcal{C}$; elements of $\mathrm{Hom}(X,Y)$ are called *morphisms* (from X to Y) and denoted $\varphi : X \to Y$.

c. A family of mappings

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z),$$

one for each ordered triple of objects X, Y, Z. To a pair $\varphi : X \to Y$, $\psi : Y \to Z$ such a mapping associates a morphism denoted $\psi \circ \varphi$ or $\psi\varphi : X \to Z$, called the *composition* of φ and ψ.

These data must satisfy the following axioms.

A. Any morphism φ uniquely determines $X,Y \in \mathrm{Ob}\,\mathcal{C}$ such that $\varphi \in \mathrm{Hom}(X,Y)$. In other words, sets $\mathrm{Hom}(X,Y)$ are disjoint.

B. For any $X \in \mathrm{Ob}\,\mathcal{C}$ there exists the identity morphism $\mathrm{id}_X : X \to X$, which can be uniquely determined by the conditions $\mathrm{id}_X \circ \varphi = \varphi$, $\psi \circ \mathrm{id}_X = \psi$ each time these compositions are defined.

C. The composition is associative:

$$(\chi\psi)\varphi = \chi(\psi\varphi)$$

for any $\varphi : X \to Y$, $\psi : Y \to Z$, $\chi : Z \to U$.

Instead of $X \in \mathrm{Ob}\,\mathcal{C}$ we will write also $X \in \mathcal{C}$, and $\bigcup\limits_{X,Y \in \mathrm{Ob}\,\mathcal{C}} \mathrm{Hom}(X,Y)$ sometimes will be denoted $\mathrm{Mor}\,\mathcal{C}$.

1.2. Examples of Categories. a. An important class of categories are the categories whose objects are sets with some additional structure, and morphisms are mapping compatible with this additional structure. Examples are

Set: the category of sets and mappings of sets;

Top: the category of topological spaces and continuous mappings;

Diff: the category of infinitely differentiable (smooth) manifolds and infinitely differentiable mappings;

Ab: the category of abelian groups and their homomorphisms;

A-mod: the category of left modules over a fixed ring A and their homomorphisms;

Gr: the category of groups and their homomorphisms.

b. The *category* Δ:

$$\operatorname{Ob} \Delta = \{[n], \quad n = 0, 1, 2, \dots\},$$
$$\operatorname{Hom}_\Delta([m],[n]) \text{ is the set of nonincreasing mappings from } \{0,\dots,m\} \text{ to } \{0,\dots,n\}.$$

The *category of simplicial sets* $\Delta^\circ\mathbf{Set}$:

$\operatorname{Ob} \Delta^\circ\mathbf{Set}$ is the class of simplicial sets;

$\operatorname{Hom}(X, Y)$ is the set of simplicial mappings from X to Y.

The *category of simplicial objects* $\Delta^\circ\mathcal{C}$, where $\mathcal{C}$ is an abstract category. The reader is advised to try to define this category.

c. The *category of complexes of abelian groups* $\operatorname{Kom}(\mathbf{Ab})$:

$$\operatorname{Ob} \operatorname{Kom}(\mathbf{Ab}) = \{\text{cochain complexes } K^\cdot \text{ of abelian groups}\};$$
$$\operatorname{Hom}(K^\cdot, L^\cdot) = \{\text{morphisms of complexes } K^\cdot \to L^\cdot\}.$$

d. Sometimes it is convenient to interpret some classical structures as categories.

The *category of a patrtially ordered set.* Let I be a partially ordered set. Define the category $\mathcal{C}(I)$ as follows:

$$\operatorname{Ob} \mathcal{C}(I) = I;$$
$$\operatorname{Hom}(i, j) \begin{cases} \text{consists of one element} & \text{if } i \le j, \\ \text{empty} & \text{otherwise.} \end{cases}$$

A special case of the category $\mathcal{C}(I)$ is the

Category $\mathbf{Top}\, X$. Let X be a topological space. Denote

$$\operatorname{Ob} \mathbf{Top}\, X = \{\text{open subset } U \subset X\};$$
$$\operatorname{Hom}(U, V) \text{ is } \begin{cases} \text{the natural embedding } U \to V & \text{if } U \subset V, \\ \text{empty} & \text{if } U \not\subset V. \end{cases}$$

1.3. Definition. A *functor* F from the category $\mathcal{C}$ to the category $\mathcal{D}$ (denoted $F : \mathcal{C} \to \mathcal{D}$) consists of the following data:

a. A mapping $\operatorname{Ob} \mathcal{C} \to \operatorname{Ob} \mathcal{D} : X \mapsto F(X)$.

b. A mapping $\operatorname{Mor} \mathcal{C} \to \operatorname{Mor} \mathcal{D} : \varphi \mapsto F(\varphi)$ such that $F(\varphi) : F(X) \to F(Y)$ for $\varphi : X \to Y$.

These data must satisfy the following condition:

$$F(\psi\varphi) = F(\psi)F(\varphi)$$

for all $\varphi, \psi \in \operatorname{Mor} \mathcal{C}$ such that the composition $\psi\varphi$ is defined; in particular, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

1.4. Examples of Functors. a. *Geometric realization*:

$$|\cdot| : \Delta^\circ\mathbf{Set} \to \mathbf{Top}.$$

The values of this functor on objects of $\Delta^\circ\mathbf{Set}$ is defined in Chap. 1, 2.1.3, and the values on morphisms are defined in the obvious manner.

b. *Singular simplicial set:*

$$\mathrm{Sing}: \mathbf{Top} \to \Delta^\circ\mathbf{Set}$$
$$(\mathrm{Sing}\, Y_n) \quad \text{is the set of singular } n\text{-simplices of } Y$$

(see Chap. 1, 2.7),

$$(\mathrm{Sing}\, Y)(f)(\varphi) = \varphi \circ \Delta_f \quad \text{for } f : [m] \to [n], \quad \Delta_f : \Delta_m \to \Delta_n.$$

The value of Sing on a morphism (a continuous mapping) $a : Y \to Y'$ is defined using the composition: $\mathrm{Sing}(a)$ maps a singular simplex $\varphi : \Delta_n \to Y$ to the singular simplex $a \circ \varphi : \Delta_n \to Y'$.

c. *The n-th cohomology group:*

$$H^n : \mathrm{Kom}\, \mathbf{Ab} \to \mathbf{Ab}.$$

d. *The classifying space:*

$$B : \mathbf{Gr} \to \Delta^\circ\mathbf{Set}.$$

The definition of B is given in Chap. 1, n.2.7.

1.5. Remarks. a. The notion of a functor had appeared as a formalization of what was known as a "natural construction". In 1.3 and 1.4 you could see examples of such constructions.

b. The functors $F : \mathcal{C} \to \mathcal{D}$ we have defined in 1.3 are sometimes called *covariant functors*, and a *contravariant functor* G is defined as a pair of mappings $G : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$, $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{D}$ with the condition that G maps $\varphi : X \to Y$ to $G(\varphi) : G(Y) \to G(X)$ and $G(\varphi\psi) = G(\psi)G(\varphi)$. The modern way to perform this "inversion of arrows" is usually relegated to the initial category $\mathcal{C}$.

Formally, let us define the *dual category* $\mathcal{C}^\circ$ as follows: $\mathrm{Ob}\,\mathcal{C}^\circ = \mathrm{Ob}\,\mathcal{C}$, (but $X \in \mathrm{Ob}\,\mathcal{C}$ as an object of $\mathcal{C}^\circ$ will be denoted by X°), $\mathrm{Hom}_{\mathcal{C}^\circ}(X^\circ, Y^\circ) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ (to a morphism $\varphi : X \to Y$ we associate $\varphi^\circ : Y^\circ \to X^\circ$), and finally $(\psi\varphi)^\circ = \varphi^\circ\psi^\circ$, $\mathrm{id}_{X^\circ} = (\mathrm{id}_X)^\circ$.

Now we can define a "contravariant" functor $\mathcal{C} \to \mathcal{D}$ as a (covariant) functor $G : \mathcal{D}^\circ \to \mathcal{C}^\circ$.

The notation $\Delta^\circ\mathbf{Set}$ from 1.2 reminds us that each simplicial set X can be viewed as a functor $\widetilde{X} : \Delta^\circ \to \mathbf{Set}$:

$$\widetilde{X}([n]) = X_n, \quad \widetilde{X}(f) = X(f).$$

c. The functor $\mathrm{Hom}_{\mathcal{C}}$ can be naturally considered as depending on two arguments. Instead of introducing formally the corresponding notion of functors depending on several variables, the standard approach is to use the *direct product of categories.*

Let, say, $\mathcal{C}$ and $\mathcal{C}'$ be two categories. Denote

$$\mathrm{Ob}(\mathcal{C} \times \mathcal{C}') = \mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C}'.$$
$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X,X'),(Y,Y')) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}'}(X',Y'),$$
$$(\varphi\varphi') \circ (\psi\psi') = (\varphi \circ \psi, \varphi' \circ \psi'),$$
$$\mathrm{id}_{(X,X')} = (\mathrm{id}_X, \mathrm{id}_{X'}).$$

One can easily verify that $\mathcal{C} \times \mathcal{C}'$ is a category. Similarly one can define the product $\prod_{i\in I} \mathcal{C}_i$ of any family of categories. A functor in several variables is defined as a functor on the corresponding product of categories. Example:

$$\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^\circ \times \mathcal{C} \to \mathbf{Set}.$$

d. The *set-theoretic composition of functors* $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ is the functor $\mathcal{C} \xrightarrow{GF} \mathcal{D}$. The identity mappings $\mathrm{Id}_{\mathcal{C}} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Mor}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ is the functor. Hence we can consider any set of categories as a category with functors as morphisms.

1.6. More Examples of Functors. a. A *presheaf of sets* (abelian groups, etc.) on a topological space X is a functor

$$\mathcal{F} : (\mathbf{Top}\,X)^\circ \to \mathbf{Set} \quad (\mathbf{Ab}, \dots)$$

(the category $\mathbf{Top}\,X$ was defined in 1.2.d).

b. Let I be a partially ordered set, $\mathcal{C}(I)$ be the corresponding category (see 1.2.d). A functor $G : \mathcal{C}(I) \to \mathbf{Ab}$ is a family of abelian groups $G(i)$, $i \in I$, and homomorphisms $\varphi_{i,j} : G(i) \to G(j)$, one for each pair $i \le j$, such that $\varphi_{jk}\varphi_{ik} = \varphi_{ik}$ for $i \le j \le k$, $\varphi_{ii} = \mathrm{id}_{G(i)}$. Such families usually occur as raw material to construct projective and inductive limits.

c. *Forgetful functors.* A large class of standard functors can be obtained as follows: we must forget one or several structures on an object of the initial category. This procedure yields the functors "the set of element":

$$\mathbf{Top}, \mathbf{Diff}, \mathbf{Ab}, \mathbf{Gr} \to \mathbf{Set},$$

and the functors

$$\mathbf{Diff} \to \mathbf{Top}, \quad A\text{-mod} \to \mathbf{Ab}.$$

Next definition of this section is the definition of a morphism of functors (also called "natural transformations of natural constructions").

1.7. Definition. Let F, G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A *morphism of functors* from F to G (notation: $f : F \to G$) is a family of morphisms

$$f(X) : F(X) \to G(X),$$

one for each object $X \in \mathrm{Ob}\,\mathcal{C}$, such that the following condition is satisfied:

for any morphism $\varphi : X \to Y$ in the category $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "f(X)"] \arrow[d, "F(\varphi)"'] & G(X) \arrow[d, "G(\varphi)"] \\ F(Y) \arrow[r, "f(Y)"] & G(Y) \end{tikzcd}$$

is commutative.

The composition of morphisms of functors, as well as the identity morphism of functors, is defined in the obvious way. Hence, the functors $\mathcal{C} \to \mathcal{D}$ form the category which is denoted by $\mathbf{Funct}(\mathcal{C}, \mathcal{D})$.

1.8. Examples. a. Viewing simplicial sets $X, Y \in \Delta^\circ\mathbf{Set}$ as functors $\widetilde{X}, \widetilde{Y} : \Delta^\circ \to \mathbf{Set}$, we can identify simplicial mappings $f : X \to Y$ with morphisms of these functors.

b. Consider the category **Esc** (exact sequences of complexes) whose objects are exact triples S of complexes of abelian groups

$$S : 0 \to K^\cdot \xrightarrow{f} L^\cdot \xrightarrow{g} M^\cdot \to 0,$$

and morphisms are commutative diagrams of the form

$$\begin{tikzcd} 0 \arrow[r] & K^\cdot \arrow[r, "f"] \arrow[d, "p"] & L^\cdot \arrow[r, "g"] \arrow[d, "q"] & M^\cdot \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & \widetilde{K^\cdot} \arrow[r, "\widetilde{f}"] & \widetilde{L^\cdot} \arrow[r, "\widetilde{g}"] & \widetilde{M^\cdot} \arrow[r] & 0 \end{tikzcd}$$

where p, q, r are morphisms of complexes. Fix an integer n and consider two functors

$$F(S) = H^n(M^\cdot), \quad G(S) = H^{n+1}(K^\cdot).$$

The connecting homomorphisms $\delta^n(S)$ from 1.5 in Chap. 1 (denoted there $\delta^n(f, g)$) form a morphism of functors $\delta^n : F \to G$.

1.9. Several Definitions. The following notions from the category theory are quite useful.

A category $\mathcal{C}$ is said to be a *subcategory* of a category $\mathcal{D}$ if

a) $\mathrm{Ob}\, \mathcal{C} \subset \mathrm{Ob}\, \mathcal{D}$;
b) For any $X, Y \in \mathrm{Ob}\, \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \subset \mathrm{Hom}_{\mathcal{D}}(X, Y);$$

c) The composition of morphisms in $\mathcal{C}$ coincides with its composition in $\mathcal{D}$; for $X \in \mathrm{Ob}\, \mathcal{C}$ the identity morphism id_X in $\mathcal{C}$ coincides with id_X in $\mathcal{D}$.

The subcategory $\mathcal{C} \subset \mathcal{D}$ is said to be *full* if

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{D}}(X,Y)$$

for $X, Y \in \mathrm{Ob}\,\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* if for any $X, Y \in \mathrm{Ob}\,\mathcal{C}$ the mapping

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective, and *full* if this mapping is surjective.

In particular, the embedding of a full subcategory $\mathcal{C} \to \mathcal{D}$ is a fully faithful functor. Conversely, one can prove that any fully faithful functor can be obtained in such a way.

An object α of an arbitrary category $\mathcal{C}$ is said to be an *initial object* if for any $X \in \mathrm{Ob}\,\mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(\alpha, X)$ consists of one element. Similarly, an object ω of $\mathcal{C}$ is said to be a *final object* if for any $X \in \mathrm{Ob}\,\mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(X, \omega)$ consists of one element. It is clear that both the initial and the final objects of the category $\mathcal{C}$ (if they exist) are determined uniquely up to an isomorphism.

Example. In the category **Set** the initial element is the empty set, and the final element is (any) one-element set.

1.10. Isomorphism. Many mathematical problems can be formulated as classification problems (classification of simple groups, of singularities, etc.). Below we describe the categorical approach to these problems. The classification is usually the classification up to an isomorphism.

1.10.1. Definition. a. A morphism $\varphi : X \to Y$ in the category $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $\psi : Y \to X$ such that $\psi\varphi = \mathrm{id}_X$, $\varphi\psi = \mathrm{id}_Y$.

b. Objects X, Y of a category $\mathcal{C}$ are said to be *isomorphic* if there exists at least one isomorphism $\varphi : X \to Y$.

The reader can easily verify that the relation "to be isomorphic" is an equivalence relation in $\mathrm{Ob}\,\mathcal{C}$. Morphisms φ, ψ with the properties from the above definition are said to be mutually inverse. Each isomorphism uniquely determines the inverse isomorphism.

Applying this definition to the category of functors (see 1.5.d) we obtain an important notion of an *isomorphism of functors*. Namely, an isomorphism of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ is a morphism of functors $f : F \to G$ such that there exists the inverse morphism $g : G \to F$, i.e. $gf = \mathrm{id}_F$, $gf = \mathrm{id}_G$.

One can easily verify that instead of the existence of the inverse morphism of functors $g : G \to F$ we can require a more natural condition: for any $X \in \mathrm{Ob}\,\mathcal{C}$ the morphism $f(X) : F(X) \to G(X)$ is an isomorphism.

Applying Definition 1.10.1 to the "category of categories" we get a rather useless notion of an isomorphism of categories. A much more useful notion is the following one.

1.11. Definition. a. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that the functor GF is isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and GF is isomorphic to $\mathrm{Id}_{\mathcal{D}}$.

b. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there exists a functor that establishes the equivalence between these categories.

The functor G from **a** is sometimes called a *quasi-inverse* to F.

1.12. Example. Let $\mathbf{Vect}_k^n$ is the category of all n-dimensional vector spaces over a field k, and V_k^n is the category with one object k^n and linear mappings of k^n into itself as morphisms. The natural inclusion functor $V_k^n \to \mathbf{Vect}_k^n$ is an equivalence of categories.

This example is rather typical: a) equivalent categories have "the same" isomorphism classes of objects and "the same" morphisms between these classes; b) the functors quasi-inverse to an equivalence are usually non-unique and their construction requires the axiom of choice; in the example above we must choose a basis in each n-dimensional space.

In proving formally that a functor establishes an equivalence of categories the following result is often quite useful (Freyd's theorem).

1.13. Theorem. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if*

a) *F is a fully faithful functor,*

b) *any object $Y \in \mathrm{Ob}\,\mathcal{D}$ is isomorphic to an object of the form $F(X)$ for some $X \in \mathrm{Ob}\,\mathcal{C}$.*

1.14. Examples. A meaningful theorem about an equivalence of categories can often be thought of as the disclosure of two complementary description of a mathematical object. The following small classical theories, each formulated as an equivalence theorem, illustrate this statement.

a. *Galois theory.* Let k be a field, assumed for simplicity to be of characteristic zero. Denote by G the Galois group of the algebraic closure $\overline{k}/k$ with the Krull topology. A major part of the classical Galois theory can be formulated as follows: the category $(k\text{-Alg})^\circ$ dual to the category of commutative finite-dimensional semisimple k-algebras is equivalent to the category G-**Set** of finite topological G-sets.

b. *Poincaré theory of the fundamental group.* Let X be a pathwise connected Hausdorff topological space with the base point $x_0 \in X$. Denote by $\mathbf{Cov}_X$ the category whose objects are coverings $p : Y \to X$ of X, and morphisms $p_1 \to p_2$ are commutative diagrams

$$\begin{tikzcd} Y_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & Y_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

On the other hand, let $\pi_1 = \pi_1(X, x_0)$ be the fundamental group of X and π_1-**Set** be the category of left π_1-sets.

The theory of the fundamental group can be summarized by the following statement: *The category* $\mathbf{Cov}_X$ *is equivalent to the category* π_1-**Set**.

c. *Commutative Banach algebras.* Let **Ban** be the category whose objects are commutative Banach algebras with involution, and morphisms are homomorphisms of algebras preserving the norm and the involution.

On the other hand, denote by **Haus** the full subcategory of the category **Top** formed by compact Hausdorff topological spaces. One of the main results in the theory of commutative Banach algebras can be formulated as follows: *The categories* $(\mathbf{Ban})^\circ$ *and* **Haus** *are equivalent.*

d. *The Pontryagin duality.* Let $\mathcal{C}$ be the category of commutative locally compact topological groups (and continuous homomorphisms). The Pontryagin duality can be formulated as follows: *The category* $\mathcal{C}$ *is equivalent to the dual category* $\mathcal{C}^\circ$.

More precisely, let $F : \mathcal{C} \to \mathcal{C}^\circ$, $F^\circ : \mathcal{C}^\circ \to \mathcal{C}$ be the functors which associate to each group G the group G^* of its unitary characters. A more precise formulation of the duality theory is that *the functors* F *and* F° *are quasi-inverse to each other.*

1.15. Representable Functors. Let $\widehat{\mathcal{C}}$ be the category of functors (see 1.7)

$$\widehat{\mathcal{C}} = \mathbf{Funct}(\mathcal{C}^\circ, \mathbf{Set}).$$

For any $X \in \mathrm{Ob}\,\mathcal{C}$ consider the functor $h_X : \mathcal{C}^\circ \to \mathbf{Set}$ defined by the formula $h_X(Y^\circ) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ as an object of $\widehat{\mathcal{C}}$.

1.15.1. Definition. A functor $F \in \mathrm{Ob}\,\widehat{\mathcal{C}}$ is said to be *representable* if it is isomorphic to a functor of the form h_X for some $X \in \mathrm{Ob}\,\mathcal{C}$. In this case the object X is said to be *representing* the functor F.

Let $\varphi : X_1 \to X_2$ be a morphism in the category $\mathcal{C}$. To this morphism corresponds a morphism of functors $h_\varphi : h_{X_1} \to h_{X_2}$ which associates to an object $Y \in \mathrm{Ob}\,\mathcal{C}$ the mapping

$$h_\varphi(Y) : h_{X_1}(Y) \to h_{X_2}(Y)$$

transforming a morphism $\theta \in \mathrm{Hom}_{\mathcal{C}}(Y, X_1) = h_{X_1}(Y)$ to the composition $\varphi \circ \theta \in \mathrm{Hom}_{\mathcal{C}}(Y, X_2) = h_{X_2}(Y)$. We have clearly $h_{\psi\varphi} = h_\psi h_\varphi$.

1.16. Theorem. *In the above notations the mapping* $\varphi \to h_\varphi$ *determines an isomorphism of sets*

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, h_Y).$$

Moreover, this mapping determines an isomorphism of two functors $\mathcal{C}^\circ \times \mathcal{C} \to$ **Set** *(as functors in arguments* X *and* Y*). Therefore the functor* $h : \mathcal{C} \to \widehat{\mathcal{C}}$ *defined by the formulas* $h(X) = h_{X'}$, $h(\varphi) = h_{\varphi'}$ *determines an equivalence* *of the category $\mathcal{C}$ with the full subcategory of $\widehat{\mathcal{C}}$ consisitng of representable functors.*

1.16.1. Corollary. *If a functor F is representable, the representing object X is determined uniquely up to the canonical isomorphism.*

1.17. Examples: Direct and Fiber Product. Let us recall that the product $X \times Y$ of two sets is the set of ordered pairs $\{(x,y) \mid x \in X, y \in Y\}$. We give two definitions of the direct product of two objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ of an arbitrary category $\mathcal{C}$.

a. The direct product $X \times Y$ "is" the object Z representing the functor

$$U \mapsto (\text{the direct product } h_X(U) \times h_Y(U))$$

(if this functor is representable).

b. The direct product $X \times Y$ "is" an object Z together with projection morphisms $X \overset{p_X}{\leftarrow} Z \overset{p_Y}{\rightarrow} Y$ such that for any pair of morphisms $X \overset{p'_X}{\leftarrow} Z' \overset{p'_Y}{\rightarrow} Y'$ there exists a unique morphism $q : Z' \to Z$ such that $p'_X = p_X q$, $p'_Y = p_Y q$ (again if a triple (Z, p_X, p_Y) with this property exists). This second definition is the result of the preliminary description of the product in the category **Set**.

An easy generalization of this construction enables us to define the fiber product in the category theory language. Let us recall that if $\varphi : X \to S$, $\psi : Y \to S$ are two mappings of sets, the fiber product of X and Y over S is the following set of pairs:

$$X \times_S Y = \{(x,y) \in X \times Y \text{ such that } \varphi(x) = \psi(y)\} \subset X \times Y.$$

The object $X \times_S Y$ in the category can be represented in two ways.

a'. $X \times_S Y$ represents the functor $U \to X(U) \times_{S(U)} Y(U)$.

b'. $X \times_S Y$ "is" the ordinary product in the new category $\mathcal{C}_S$ whose objects are morphisms $\varphi : X \to S$, and morphisms from $\varphi : X \to S$ to $\psi : Y \to S$ are commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "\chi"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & S & \end{tikzcd}$$

where $\chi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

The diagram **b** in the category $\mathcal{C}_S$ is represented by the following diagram in the category $\mathcal{C}$:

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\varphi"] & S \end{tikzcd}$$

This category has an obvious universality property; it is called the *cartesian square.*

1.18. Inversion of Arrows. Any categorical construction defines the dual construction: we must apply the original construction in the dual category $\mathcal{C}^\circ$ and interpret the result in the category $\mathcal{C}$. In the other words, we must invert the directions of all arrows in the original definition. This procedure yields the construction of the amalgamated sum, or *coproduct*, or cocartesian square, corresponding to the diagram $X \overset{\varphi}{\leftarrow} S \overset{\psi}{\rightarrow} Y$:

$$\begin{tikzcd}
X \coprod_S Y & Y \arrow[l, "i_Y"'] \\
X \arrow[u, "i_X"] & S \arrow[l, "\varphi"] \arrow[u, "\psi"']
\end{tikzcd}$$

The universality property of this square is that any diagram of the form $X \overset{j_X}{\rightarrow} Z \overset{j_Y}{\leftarrow} Y$ with the property $j_X\varphi = j_Y\psi$ determines a unique morphism $q\colon X\coprod_S Y \to Z$ such that $j_X = qi_X$, $j_Y = qi_Y$.

In the remaining part of this section we introduce two important notions of the category theory that can be conveniently formulated in terms of the representable functors: the notion of the limit and the notion of the adjoint functor.

1.19. Limits. Fix a category J (often called the category of indices); in many cases the category J is finite (has a finite number of objects and a finite number of morphisms). Let us recall that by $\mathbf{Funct}(J, \mathcal{C})$ we denote the category of functors $F : J \to \mathcal{C}$ (see 1.7).

The *diagonal functor* $\Delta : \mathcal{C} \to \mathbf{Funct}(J, \mathcal{C})$ is defined as follows:
On objects:

$$\begin{aligned}\Delta X &= \{\text{the constant functor } J \to \mathcal{C} \text{ taking the value } X\},\\ &\text{i.e. } \Delta X(j) = X \text{ for } j \in \operatorname{Ob} J,\\ &\Delta X(\varphi) = \mathrm{id}_X \text{ for all morphisms } \varphi \text{ in } J.\end{aligned}$$

On morphisms:

for $\psi : X \to X'$ in $\mathcal{C}$ the morphism $\Delta\psi : \Delta X \to \Delta X'$ is defined as follows:
$$\Delta\psi(j) = \psi : X = \Delta X(j) \to X' = \Delta X'(j).$$

The mapping $\Delta\psi$ is clearly a morphism of functors and $\Delta(\psi\psi') = \Delta\psi\Delta\psi'$, so that Δ is in fact a functor from $\mathcal{C}$ to $\mathbf{Funct}(J, \mathcal{C})$.

1.19.1 Definition. Let $F : J \to \mathcal{C}$ be a functor. The *projective limit* of the functor F in the category $\mathcal{C}$ is the object $X \in \operatorname{Ob}\mathcal{C}$ representing the functor

$$Y \to \operatorname{Hom}_{\mathbf{Funct}(J,\mathcal{C})}(\Delta Y, F) : \mathcal{C}^\circ \to \mathbf{Set}.$$

The projective limit of F is denoted by $X = \lim\operatorname{proj} F$; sometimes it is called the direct limit, or simply the limit.

According to this definition, $X = \lim\operatorname{proj} F$ is characterized by the equality

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) = \mathrm{Hom}_{\mathbf{Funct}(J,\mathcal{C})}(\Delta Y, F). \qquad (1)$$

Theorem 1.16 shows that if $\lim\operatorname{proj} F$ exists, it is determined uniquely up to a unique isomorphism.

1.20. The Universal Property of the Limit. Any functor $F : J \to \mathcal{C}$ is determined by a family of objects $F(j) = X_j \in \mathrm{Ob}\,\mathcal{C}$, one for each $j \in \mathrm{Ob}\, J$, and a family of morphisms $F(\varphi) : X_j \to X_j$ in $\mathcal{C}$, one for each morphism $\varphi : j \to j'$ in J.

Let the limit $X = \lim\operatorname{proj} F$ exists. Set in (1) $Y = X$. To the identity morphism $\mathrm{id}_X : X \to X$ in $\mathcal{C}$ corresponds a morphism of functors $f : \Delta X \to F$; this morphism f is a family of morphisms $f(j) : X \to X_j$ in $\mathcal{C}$, one for each $j \in \mathrm{Ob}\, J$, such that

$$F(\varphi) f(j) = f(j') \quad \text{for each morphism } \varphi : j \to j' \quad \text{in } J. \qquad (2)$$

Next, an arbitrary morphism of functors $f : \Delta Y \to F$ it is a family of morphisms $g(j) : Y \to X_j$ in $\mathcal{C}$, one for each $j \in \mathrm{Ob}\, J$, such that

$$F(\varphi) g(j) = g(j') \quad \text{for each morphism } \varphi : j \to j' \quad \text{in } J. \qquad (3)$$

Formula (1) shows that the definition of $\lim\operatorname{proj} F$ can be formulated as the following universality property.

An object $X \in \mathrm{Ob}\,\mathcal{C}$ is the projective limit of the functor $F : J \to \mathcal{C}$ in the category $\mathcal{C}$ if a family of morphisms $f(j) : X \to X_j = F(j) \in \mathrm{Ob}\,\mathcal{C}$ is given, one for each $j \in \mathrm{Ob}\, J$, satisfying the condition (2), and such that for any family of morphisms $g(j) : Y \to X_j$ in $\mathcal{C}$, one for each $j \in \mathrm{Ob}\, J$, satisfying the condition (3), there exists a unique morphism $\psi : Y \to X$ in $\mathcal{C}$ satisfying the condition $g(j) = f(j) \circ \psi$.

1.21. The Dual Theory: Colimits. Let again J be a category of indices, $F : J \to \mathcal{C}$ be a functor. The *inductive limit* (direct limit, colimit) of the functor F in the category $\mathcal{C}$ is an object $X = \lim\operatorname{ind} F$ in $\mathcal{C}$ representing the functor

$$Y \to \mathrm{Hom}_{\mathbf{Funct}(J,\mathcal{C})}(F, \Delta Y) : \mathcal{C} \to \mathbf{Set}.$$

In the other words, $X = \lim\operatorname{ind} F$ if

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathbf{Funct}(J,\mathcal{C})}(F, \Delta Y)$$

functorially in Y.

The inductive limit can be defined by the universal property dual to the property in 1.20.

In classical categories $\mathbf{Set}, \mathbf{Gr}, \mathbf{Top}, \mathbf{Ab}$ finite limits and colimits always exist.

1.22. Limits over Partially Ordered Sets. An important (and historically the first one) special case of limits is the case when the index category J is the category $\mathcal{C}(I)$ for some partially ordered set I (see 1.5.d).

Let $\mathcal{C} = \mathbf{Set}$ be the category of sets. A functor $F : J \to \mathbf{Set}$ is a family of sets X_α, $\alpha \in I$ and of mapping $f_{\alpha\beta} : X_\alpha \to X_\beta$, such that $f_{\alpha\alpha} = \mathrm{id}$, $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$.

One can easily verify that $\lim\operatorname{ind} F$ in $\mathbf{Set}$ can be constructed as follows. A subset $L \subset I$ is said to be complete if $\alpha \in L$, $\beta \geq \alpha$ implies $\beta \in L$, i.e. together with any element L contains all larger elements. By a string we mean a family $\{x_\alpha \in X_\alpha,\ A \in L\}$ for some complete L such that $f_{\alpha\beta} x_\alpha = x_\beta$ for $\alpha \leq \beta$, $\alpha, \beta \in L$. Then $\lim\operatorname{ind} F$ is the set of equivalence classes with respect to the relation $\{x_\alpha,\ \alpha \in L\} \approx \{x'_\beta,\ \beta \in L'\}$ if and only if for any $\alpha \in L$, $\beta \in L'$ there exists $\gamma \in L$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ and $f_{\alpha\gamma} x_\alpha = f_{\beta\gamma} x'_\beta$.

Next, if $I' \subset I$ is a filtered subset (i.e. for any $\alpha \in I$ there exists $\beta \in I'$ with $\beta \geq \alpha$) then $\lim\operatorname{ind} F$ for $F : \mathcal{C}(I) \to \mathbf{Set}$ equals $\lim\operatorname{ind} F'$, where F' is the restriction of F to $\mathcal{C}(I') \subset \mathcal{C}(I)$.

Similar results hold for the categories $\mathbf{Gr}, \mathbf{Ab}, \mathbf{Top}$.

In the case when I is a directed partially ordered set (i.e. for any $\alpha, \beta \in I$ there exists $\gamma \in I$ which is larger than both α and β; a standard example is the set $\mathbb{Z}_+$ of positive integers), the limit $\lim\operatorname{proj} F$ for $F : \mathcal{C}(I) \to \mathcal{C}$ was originally called the *direct spectrum limit*, and the limit $\lim\operatorname{proj} F$ for $F : \mathcal{C}(I) \to \mathcal{C}^\circ$ was called the *inverse spectrum limit.*

1.23. Adjoint Functors. The last important notion of the category theory that we will introduce in this section is the notion of the adjoint functor.

Let $\mathcal{C}$, $\mathcal{D}$ be two categories, $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1.23.1. Lemma-Definition. *Let us assume that for any $Y \in \operatorname{Ob}\mathcal{D}$ the functor $T \to \operatorname{Hom}_\mathcal{D}(F(T), Y)$ from $\mathcal{C}^\circ$ to* $\mathbf{Set}$ *is representable, and let $X \in \operatorname{Ob}\mathcal{C}$ be the corresponding representing object. Then the mapping $Y \to X$ can be uniquely extended to a functor $G : \mathcal{D} \to \mathcal{C}$ which defined an isomorphism of bifunctors* $\operatorname{Hom}_\mathcal{C}(T, G(Y))$ *and* $\operatorname{Hom}_\mathcal{D}(F(T), Y))$ *from $\mathcal{C}^\circ \times \mathcal{D}$ to* $\mathbf{Set}$. *The functor G is called the* right adjoint *to F, and F is called the* left adjoint *to G.*

1.24. Adjunction Morphisms. Let F and G is a pair of adjoint functors, so that we have morphisms of sets

$$\alpha : \operatorname{Hom}_\mathcal{C}(T, G(Y)) \xrightarrow{\sim} \operatorname{Hom}_\mathcal{D}(F(T), Y)$$

that are functorial in X and Y. Substituting $X = G(Y)$ we obtain a morphism

$$\sigma_Y = \alpha(\mathrm{id}_{G(Y)}) : FG(Y) \to Y.$$

On the other hand, substituting $Y = F(X)$ we obtain a morphism

$$\tau_X = \alpha^{-1}(\mathrm{id}_{F(X)}) : X \to GF(X).$$

The reader can easily verify that the families $\{\sigma_Y\}$ and $\{\tau_X\}$ define morphisms of functors

$$\sigma : FG \to \mathrm{Id}_{\mathcal{D}}, \quad \tau : \mathrm{Id}_{\mathcal{C}} \to GF. \tag{4}$$

These morphisms of functors are called the *adjunction morphisms* corresponding to the given pair of adjoint functors F and G. They satisfy the following condition:

Compositions

$$F \xrightarrow{F\circ\tau} FGF \xrightarrow{\sigma\circ F} F, \quad G \xrightarrow{\tau\circ G} GFG \xrightarrow{G\circ\sigma} G \tag{5}$$

are the identity morphisms of the functors F and G respectively.

It turns out that the existence of the adjunction morphisms is equivalent to the fact that F and G are mutually adjoint. Namely, let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two functors and let morphisms of functors (4) be given such that the conditions (5) are satisfied. Then the functors F and G are mutually adjoint; the isomorphism α is $\alpha(u) = \sigma_Y \circ F(u)$ and the inverse isomorphism α^{-1} is $\alpha^{-1}(v) = G(v) \circ \tau_X$.

1.25. Examples. a. Several constructions in algebra and in topology can be interpreted as the application of functors adjoint to the some standard functors (like the forgetful functors), see Table 1.

Table 1

	The functor $F\colon \mathcal{C} \to \mathcal{D}$	The left adjoint $G\colon \mathcal{C} \to \mathcal{D}$
1	**Gr** $\to$ **Set**	the free group with the given set of generators
2	**Ab** $\to$ **Set**	the free abelian group with the given set of generators
3	R-mod $\to$ **Ab** (R is a fixed ring)	$A \to R \otimes_{\mathbb{Z}} A$
4	k-alg (associative algebras over a field k) $\to$ **Vect** (vector spaces over k)	the tensor algebra of a space V
5	**Top** $\to$ **Set**	a set with the discrete topology
6	**Commet** (complete metric spaces) $\to$**Met** (metric spaces)	the completion of a metric space
7	k-alg$\to$Lie$_k$ (Lie algebras over k); algebra A becomes the Lie algebra with the bracket $[a, b] = ab - ba$	the universal enveloping algebra

b. Let R, S be two rings, $M = {}_RM_S$ be a (R, S)-bimodule (that is, a left R-module and a right S-module). Then the functor $X \to M \otimes_S X$ from the category S-mod to the category R-mod is left adjoint to the functor $Y \to \mathrm{Hom}_R(M, Y)$.

c. The functor $\lim\operatorname{proj} F$ is left adjoint and the functor $\lim\operatorname{ind} F$ is right adjoint to the diagonal functor $\Delta : \mathcal{C} \to \mathbf{Funct}(J, \mathcal{C})$.

d. Let F be the functor from the category **SCat** of small categories ($\operatorname{Ob}\mathcal{C}$ is a set) to the category **Set**, which associates to each $\mathcal{C}$ the set $\operatorname{Ob}\mathcal{C}$. Then F has the left adjoint functor G_l, which associates to a set X the discrete category $\mathcal{C}_X$ ($\operatorname{Ob}\mathcal{C}_X = X$, $\operatorname{Hom}_{\mathcal{C}_X}(x,x) = \mathrm{id}_x$, $\operatorname{Hom}_{\mathcal{C}_X}(x,y) = \emptyset$ if $x \neq y$), and the right adjoint functor G_r, which associated to X the category $\widetilde{\mathcal{C}_X}$ with $\operatorname{Ob}\widetilde{\mathcal{C}_X} = X$, $\operatorname{Hom}_{\widetilde{\mathcal{C}_X}(x,x)}$ is an one-element set for all x, y. In turn, G_l has the left adjoint functor, which associates to a small category $\mathcal{C}$ the set of its connected components (i.e. the set of equivalence classes of the set $\operatorname{Ob}\mathcal{C}$ by the relation $x \approx y$ if and only if $\operatorname{Hom}_{\mathcal{C}}(x,y)$ is non-empty).

§ 2. Additive and Abelian Categories

2.1. Homological Algebra and Categories. From the point of view of category theory, homological algebra studies properties of the functors with values in so called abelian categories. The notion of an abelian category axiomatizes the main properties of the following categories:

A. The category of abelian groups.

B. The category of modules over a fixed ring.

C. The category of coefficient systems (see Chap. 1, 2.4) and the category of presheaves of abelian groups over a fixed space.

D. The category of sheaves of abelian groups over a fixed space.

We formulate the axioms A1–A4 of an abelian category $\mathcal{C}$ one by one (defining all the relevant notions) and comment why these axioms are violated for some non-abelian categories such that the category of topological abelian groups and the category of abelian groups with filtrations.

2.2. Axiom A1. *Each set* $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ *is endowed with the structure of an abelian group (we will use the additive notation); the composition of morphisms is bi-additive with respect to these structures.*

In the other words, $\operatorname{Hom}_{\mathcal{C}}$ is a functor $\mathcal{C}^\circ \times \mathcal{C} \to \mathbf{Ab}$.

2.3. Axiom A2. *There exists a zero object* $0 \in \operatorname{Ob}\mathcal{C}$ *such that* $\operatorname{Hom}_{\mathcal{C}}(0,0)$ *is the zero group. We will denote zero morphisms also by* 0.

2.4. Axiom A3. *For any pair of objects* X_1, X_2 *there exist an object* Y *and morphisms* p_1, p_2, i_1, i_2

$$X_1 \xrightarrow{i_1} Y \xleftarrow{i_2} X_2, \qquad X_1 \xleftarrow{p_1} Y \xrightarrow{p_2} X_2 \tag{1}$$

with the following properties:

$$p_1 i_1 = \mathrm{id}_{X_1}, \quad p_2 i_2 = \mathrm{id}_{X_2}, \quad p_2 i_1 = p_1 i_2 = 0, \quad i_1 p_1 + i_2 p_2 = \mathrm{id}_Y\,.$$

This axiom can be reformulated by saying that two squares below are respectively cartesian and cocartesian (see 1.17 and 1.18), so that the object Y is both the direct product and the direct sum of X_1 and X_2:

$$\begin{tikzcd} Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X_1 \arrow[d] \\ X_2 \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} Y & X_1 \arrow[l, "i_1"'] \\ X_2 \arrow[u, "i_2"] & 0 \arrow[l] \arrow[u] \end{tikzcd}$$

In particular, for given X_1 and X_2 any two diagrams of the form (1) are canonically isomorphic.

Now we begin the categorical analysis of the least trivial property of the categories A–D from 2.1, namely the existence of exact sequences.

2.5. Kernel. Let a category $\mathcal{C}$ satisfies the axioms $A1$ and $A2$, and let $\varphi : X \to Y$ be a morphism in $\mathcal{C}$. Consider the following functor $\operatorname{Ker}\varphi : \mathcal{C}^\circ \to \mathbf{Ab}$:

$$(\operatorname{Ker}\varphi)(Z) = \operatorname{Ker}(h_X(Z) \to h_Y(Z)),$$
$$(\operatorname{Ker}\varphi)(f) \text{ is the restriction of } h_X(f) \text{ to } (\operatorname{Ker}\varphi)(Z).$$

(see 1.15). The embedding $(\operatorname{Ker}\varphi)(Z) \subset h_X(Z)$ defines the morphism of functors $\operatorname{Ker}\varphi \to h_X$. Let us assume that the functor $\operatorname{Ker}\varphi$ is represented by an object K. This object is defined together with a morphism $k : K \to X$ such that $\varphi \circ k = 0$. The diagram $K \xrightarrow{k} X \xrightarrow{\varphi} Y$ satisfies the following universality condition: for any morphism $k' : K' \to X$ such that $\varphi \circ k' = 0$ there exists a unique morphism $h : K' \to K$ such that $k' = k \circ h$.

We call the morphism k or the pair (K, k) the *kernel* of φ; sometimes by the kernel we mean also the object K.

One can easily verify that if the kernel (K, k) of a morphism φ exists it is defined uniquely.

In the categories from 2.1.A–D we have the set-theoretical construction of the kernel: $\varphi^{-1}(0)$ for groups and for modules; the family $\varphi_x^{-1}(0)$ for coefficient systems; the family $\varphi_U^{-1}(0)$ for a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ represented by the family $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open sets U.

2.5.1. Lemma. a. *For each of the categories* 2.1.A–D *the set-theoretic kernel of a morphism* $\varphi : X \to Y$ *is an object of the same category.*

b. *The canonical embedding of this object* K *into* X *is the kernel in the sense of* 2.5.

2.6. Cokernel. The naive definition of the cokernel $\operatorname{Coker}\varphi$ as the object representing the functor $Z \to \operatorname{Coker}(h_X(Z) \to h_Y(Z))$ is false. For example, even in the category of abelian groups it is not isomorphic to the functor represented by the set-theoretic cokernel. For example, let $X = Y = \mathbb{Z}$,

φ be the multiplication by an integer $n > 1$, $Z = \mathbb{Z}/n\mathbb{Z}$. Then $h_X(Z) = h_Y(Z) = 0$, so that we have $\operatorname{Coker}(h_X(Z) \to h_Y(Z)) = 0$. On the other hand, $\operatorname{Hom}(\mathbb{Z}/n\mathbb{Z}, \operatorname{Coker}\varphi) \neq 0$ (here $\operatorname{Coker}\varphi = \mathbb{Z}/n\mathbb{Z}$ is the set-theoretic cokernel of φ). We leave to the reader to verify that in this example the functor $Z \mapsto \operatorname{Coker}(h_X(Z) \to h_Y(Z))$ is even non-representable.

The appropriate definition of $\operatorname{Coker}\varphi$ (if this functor is representable) requires the double dualization:

$$\operatorname{Coker}\varphi = (\operatorname{Ker}\varphi^\circ)^\circ,$$

where $^\circ$ denotes the passage to the dual category (see 1.5.b). This definition is equivalent to each of the following two definitions.

a. The cokernel of a morphism $\varphi : X \to Y$ is a morphism $c : Y \to K'$ such that for any $Z \in \operatorname{Ob}\mathcal{C}$ the sequence of groups

$$0 \to \operatorname{Hom}_{\mathcal{C}}(K', Z) \to \operatorname{Hom}_{\mathcal{C}}(Y, Z) \to \operatorname{Hom}_{\mathcal{C}}(X, Z)$$

is exact (this just means that $(K')^\circ$ represents $\operatorname{Ker}\varphi^\circ$).

b. The cokernel of a morphism $\varphi : X \to Y$ is a morphism $c : Y \to K'$ such that $c \circ \varphi = 0$ and for any morphism $c_1 : Y \to K'_1$ with $c_1 \circ \varphi = 0$ there exists a unique morphism $h : K' \to K'_1$ with $c_1 = h \circ c$.

Similarly to the kernel, the cokernel of $c : Y \to K'$, if it exists, is unique up to a unique isomorphism.

In each of the categories 2.1.A–C there exists a set-theoretic definition of the cokernel, and an analog of Lemma 2.5.1 holds.

The situation in the category **SAb** of sheaves of abelian groups is more complicated. The reason is that even if $\varphi : X \to Y$ is a morphism of sheaves, the family $\{K'(U) = \operatorname{Coker}\varphi_U\}$ is a presheaf, but sometimes not a sheaf. One can verify that $\{K'(U)\}$ is the cokernel in the category of presheaves.

Below we will show how to define the cokernel in the category of sheaves. But first we formulate the last axiom.

2.7. Axiom A4. *For any morphism $\varphi : X \to Y$ there exists a sequence*

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K' \tag{2}$$

with the following properties:

a. $j \circ i = \varphi$;

b. *K is the kernel of φ, K' is the cokernel of φ;*

c. *I is the kernel of k and the cokernel of c.*

Such sequence is called the *canonical decomposition* of φ.

2.8. Definition. A category satisfying the axioms A1–A3 is called an *additive category.* A category satisfying the axioms A1–A4 is called an *abelian category.*

All categories 2.1.A–D are additive. Moreover, all these categories are abelian. The existence of the canonical decomposition of a morphism φ :

$G \to H$ in the category **Ab** (i.e., the isomorphism $\operatorname{Im}\varphi \cong H/\operatorname{Ker}\varphi$) is guaranteed by the homomorphism theorem for abelian groups; similarly one proves that the category of modules over a fixed ring is abelian. The canonical decomposition in categories from 2.1.C. is constructed componentwise. For example, the canonical decomposition of a morphism $\varphi : X \to Y$ in the category **PAb** of presheaves of abelian groups is obtained from the canonical decompositions of morphisms $\varphi_U : X(U) \to Y(U)$ in **Ab**:

$$K(U) \xrightarrow{k_U} X(U) \xrightarrow{i_U} I(U) \xrightarrow{j_U} Y(U) \xrightarrow{c_U} K'(U).$$

One can easily verify that these decompositions are compatible with the restriction homomorphisms.

2.9. Comments to the Axiom A4. a. All canonical decompositions of a given morphism φ are isomorphic, and these isomorphisms are uniquely determined.

b. If, instead of A4, we require only the existence of kernels and of cokernels, then for any morphism φ we can construct two halves of the diagram (2):

$$K \xrightarrow{k} X \xrightarrow{i} I, \quad I' \xrightarrow{j} Y \xrightarrow{c} K',$$

where $k = \operatorname{Ker}\varphi$, $i = \operatorname{Coker} k$, $c = \operatorname{Coker}\varphi$, $j = \operatorname{Ker} c$. The additional requirement in the axiom A4 is that I and I' must "coincide", i.e. that there should exist an isomorphism $I \xrightarrow{l} I'$ such that $\varphi = j \circ l \circ i$. One can verify that a morphism l with this property can be constructed canonically, so that the axiom A4 requires essentially that this morphism must be an isomorphism. The pair (I', j) is called sometimes the *image* of φ, and (I, i) is called the *coimage* of φ.

c. The axiom A4 is selfdual in the following sense. Let us consider the diagram (2) in the dual category $\mathcal{C}^\circ$:

$$K'^\circ \xrightarrow{c^\circ} Y^\circ \xrightarrow{j^\circ} I^\circ \xrightarrow{i^\circ} X^\circ \xrightarrow{k^\circ} K^\circ. \tag{3}$$

If the diagram (2) is the canonical decomposition of φ in the category $\mathcal{C}$, then the diagram (3) is the canonical decomposition of φ° in the category $\mathcal{C}^\circ$.

A similar autoduality property holds for axioms A1–A3 as well. Hence, assuming that $\operatorname{Hom}_{\mathcal{C}}(X, Y) = \operatorname{Hom}_{\mathcal{C}^\circ}(X^\circ, Y^\circ)$ as an abelian group, we see that the dual to an additive category is additive and the dual to an abelian category is abelian.

d. If $\mathcal{C}$ is an abelian category, then any morphism φ in $\mathcal{C}$ such that $\operatorname{Ker}\varphi = 0$ and $\operatorname{Coker}\varphi = 0$ is an isomorphism. In fact, the cokernel of a (unique) morphism $0 \to X$ is isomorphic to $\mathrm{id} : X \to X$, and the cokernel of a (unique) morphism $Y \to 0$ is isomorphic to $\mathrm{id} : Y \to Y$. The axiom A4 shows that morphisms i and j are isomorphisms, hence so is $\varphi = j \circ i$.

e. Morphisms φ with $\operatorname{Ker}\varphi = 0$ are called *monomorphisms*; morphisms φ with $\operatorname{Coker}\varphi = 0$ are called *epimorphisms*.

2.10. Sheaves and Presheaves. Let **SAb** (resp. **PAb**) be the category of sheaves (resp. of presheaves) of abelian groups on a fixed topological space M. A sheaf is a presheaf with some additional properties; hence there exists the embedding functor $\iota : \mathbf{SAb} \to \mathbf{PAb}$. In proving that **SAb** is an abelian category, the important role is played by the functor $s : \mathbf{PAb} \to \mathbf{SAb}$, which is left adjoint to ι, and by the corresponding isomorphism of bifunctors

$$\mathrm{Hom}_{\mathbf{SAb}}(sX, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{PAb}}(X, \iota Y).$$

After proving that s exists, we can define the cokernel of a morphism $\varphi : X \to Y$ of sheaves as $s(\widetilde{K})$, where $\widetilde{K}$ is the cokernel of φ in the category **PAb**, and prove the existence of the canonical decomposition.

2.10.1. Proposition. *The functor* $\iota : \mathbf{SAb} \to \mathbf{PAb}$ *admits the left adjoint functor.*

2.11. Proposition. *Let* $\varphi : X \to Y$ *be a morphism of sheaves in* **SAb** *and let*

$$K \xrightarrow{k} \iota X \xrightarrow{i} I \xrightarrow{j} \iota Y \xrightarrow{c} K'$$

be the canonical decomposition of the morphism $\iota\varphi$ *in the abelian category* **PAb**. *Then the diagram*

$$sK \xrightarrow{s(k)} X = s\iota X \xrightarrow{s(i)} sI \xrightarrow{s(j)} Y = s\iota Y \xrightarrow{s(c)} sK'$$

is the canonical decomposition of the morphism φ *in the category* **SAb**. *In particular,* **SAb** *is an abelian category.*

2.12. Examples of Additive Non-Abelian Categories. In the categories below the axiom A4 is not satisfied.

a. *Filtered abelian groups.* An object of the category **AbF** is an abelian group X with a sequence of subgroups $\cdots \subset F^iX \subset F^{i+1}X \subset \cdots \subset X$. Denote

$$\mathrm{Hom}_{\mathbf{AbF}}(X, Y) = \{\varphi \in \mathrm{Hom}_{\mathbf{Ab}}(X, Y) \mid \varphi(F^iX) \subset F^iY \text{ for all } i\}.$$

Denote by $F^i\varphi : F^iX \to F^iY$ the restriction of φ to F^iX.

The kernel of a morphism φ in **AbF** as a group coincides with the kernel of φ in **Ab**; $\mathrm{Ker}\,\varphi$ is endowed with the filtration by subgroups $\mathrm{Ker}_{\mathbf{Ab}}(F^i\varphi)$. As a group, the cokernel of a morphism φ in **AbF** coincides with the cokernel of φ in **Ab**; the filtration is

$$F^i\,\mathrm{Coker}_{\mathbf{AbF}}\,\varphi = F^iY/F^iY \cap \varphi(X).$$

The following construction gives morphisms with zero kernel and zero cokernel that are not isomorphisms. Let X be a group with two filtrations F_1^i and F_2^i such that $F_1^iX \subset F_2^iX$ for all i, and let φ be the identity morphism. If $F_1^iX \neq F_2^iX$ for at least one i, this morphism is not an isomorphism. By 2.9.d, this implies that **AbF** is not an abelian category.

For a general morphism $\varphi : X \to Y$ in **AbF** we have, in the notation of 2.9.b, that $I = X/\operatorname{Ker}\varphi$, $I' = \varphi(X)$ as groups with filtrations

$$F^i I = F^i X/\operatorname{Ker} F^i\varphi, \quad F^i I' = F^i Y \cap \varphi(X).$$

The canonical morphism $I \to I'$ is induced by ν; it is an isomorphism in **Ab**. However, the filtrations $\varphi(F^i X)$ and $F^i Y \cap \varphi(X)$ in Y can be distinct, as in the above example, and if this is the case, φ does not possess the canonical decomposition.

b. *Topological abelian groups.* Objects of the category **AbT** are Hausdorff topological abelian groups, morphisms are continuous homomorphisms of groups. Any morphism in this category has the kernel and the cokernel: for $\varphi : X \to Y$, $\operatorname{Ker}\varphi$ is the kernel of φ in **Ab** with the induced topology, and $\operatorname{Coker}\varphi$ is $Y/\overline{\varphi(X)}$, where $\overline{\varphi(X)}$ is the closure of the group-theoretic image of φ in the topology induced from Y.

In the notations of 2.9.b, $I = X/\operatorname{Ker}\varphi$, while $I' = \varphi(X)$ with the topology induced from that on Y. The canonical morphism $I \to I'$ is not continuous if $\varphi(X)$ is not closed. It can happen also that $\varphi(X)$ is closed, but Y induces on it a weaker topology. For example, the identity mapping of $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the ordinary topology has zero kernel and zero cokernel, but is not an isomorphism.

2.13. Several Definitions. Most definitions from Chap. 1 can be literally generalized to an arbitrary abelian category. We repeat some of these definitions.

a. A (cochain) *complex* in an additive category C is a sequence of objects and morphisms

$$X^\cdot : \ldots \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \ldots$$

with the property $d^n \circ d^{n-1} = 0$ for all n.

b. Let the category C be abelian. The *n-dimensional cohomology* of a complex $X^\cdot$ in C is defined as the object

$$H^n(X^\cdot) = \operatorname{Coker} a^n = \operatorname{Ker} b^{n+1} \qquad (4)$$

defined from the following commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Coker} d^n \arrow[dr, dashed, "b^{n+1}"] & \\
X^n \arrow[r, "d^n"] \arrow[dr, dashed, "a^n"'] & X^{n+1} \arrow[u] \arrow[r, "d^{n+1}"] & X^{n+2} \\
 & \operatorname{Ker} d^{n+1} \arrow[u] &
\end{tikzcd}$$

(The second equality in (4) means the canonical isomorphism.)

c. A complex $X^\cdot$ in an abelian category C is said to be *acyclic at* X^n if $H^n(X^\cdot) = 0$.

d. A complex $X^{\cdot}$ in an abelian category $\mathcal{C}$ is said to be *exact* (or an *exact sequence*) if it is acyclic at all terms.

In a similar way one can generalize other notions from § 1 of Chap. 1, and the definition of spectral sequence from § 3 of Chap. 1 to an arbitrary abelian category. The analogs of Theorem 1.5.1 and of Theorem 3.3 from Chap. 1 remain true.

2.14. About Proofs in Abelian Categories. The main thesis in working with abelian categories is that if a statement involving a finite number of objects and a finite number of morphisms is true in the category of modules over a ring, the it remains true in an arbitrary abelian category. The thesis is based on the following embedding theorem.

2.14.1. Theorem. *Let $\mathcal{A}$ be an abelian category whose objects form a set. Then there exists a ring R and an exact functor $F : \mathcal{A} \longrightarrow R$-*mod*, which is an embedding on objects and an isomorphism on* Hom*'s.*

This theorem (or some similar method, for example, one based on the notion of an element of an object in an abelian category) is used to verify some properties of objects or (more often) of morphisms constructed using some universality properties. For example, this theorem enables us to claim that a sequence is exact, or that a morphism is an isomorphism, if this property holds in the module categories.

The following two results are often used in homological algebra.

2.14.2. The Five-Lemma. *Let we are given a commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & X_3 \arrow[r] \arrow[d, "f_3"'] & X_4 \arrow[r] \arrow[d, "f_4"'] & X_5 \arrow[d, "f_5"'] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5
\end{tikzcd}$$

with exact rows, in which f_1 is an epimorphism, f_5 is a monomorphism, f_2 and f_4 are isomorphisms. Then f_3 is also an isomorphism.

2.14.3. The Snake Lemma. *Let*

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & X_2 \arrow[r, "g_2"] \arrow[d, "f_2"'] & X_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & Y_1 \arrow[r, "h_1"] & Y_2 \arrow[r, "h_2"] & Y_3 \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows.

a. *The sequences*

$$0 \longrightarrow \operatorname{Ker} f_1 \xrightarrow{a_1} \operatorname{Ker} f_2 \xrightarrow{a_2} \operatorname{Ker} f_3,$$

$$\operatorname{Coker} f_1 \xrightarrow{b_1} \operatorname{Coker} f_2 \xrightarrow{b_2} \operatorname{Coker} f_3 \longrightarrow 0,$$

where a_1, a_2 (resp. b_1, b_2) are induced by morphisms g_1, g_2 (resp. h_1, h_2), are exact.

b. *There exists a natural morhism* $\delta : \operatorname{Ker} f_3 \to \operatorname{Coker} f_1$ *that glue these two exact sequences into one long exact sequence.*

§ 3. Functors in Abelian Categories

3.1. Definition. Let $\mathcal{C}$, $\mathcal{C}'$ be additive categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be *additive* if all mappings

$$F : \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{C}'}(F(X), F(Y)), \quad X, Y \in \operatorname{Ob}\mathcal{C},$$

are homomorphisms of abelian groups.

One of the main characteristics of additive functors is to what extend they preserve kernels and cokernels.

3.2. Definition. Let $\mathcal{C}$, $\mathcal{C}'$ be additive categories and $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. It is said to be *exact* if for any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$ the sequence

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0 \tag{1}$$

is exact in $\mathcal{C}'$. The functor F is said to be *left exact* if the sequence (1) is exact everywhere except possibly at the term $F(Z)$, and *right exact* if the sequence (1) is exact everywhere except possibly at the term $F(X)$.

First we present the main facts about the exactness of the most important functors.

3.3. Proposition. *Let* $\mathcal{C}$ *be an abelian category. The functors*

$$\mathcal{C} \to \mathbf{Ab}, \quad X \mapsto \operatorname{Hom}_{\mathcal{C}}(Y, X)$$

(for fixed Y*) and*

$$\mathcal{C}^{\circ} \to \mathbf{Ab}, \quad X \mapsto \operatorname{Hom}_{\mathcal{C}}(X, Y)$$

(for fixed Y*) are left exact.*

3.4. Proposition. *Let* A-mod *(resp.* mod-A*) be the abelian category of left (resp. right) modules over a fixed ring* A*. The functors*

$$A\text{-mod} \to \mathbf{Ab}, \quad X \mapsto Y \otimes_A X,$$

where Y *is a fixed object from* mod-A*, and*

$$\text{mod-}A \to \mathbf{Ab}, \quad X \mapsto X \otimes_A Y,$$

where Y *is a fixed object from* A-mod*, are right exact.*

3.5. Proposition. *Let X be a topological space, $U \subset X$ be an open set,* **SAb** *the category of sheaves of abelian groups on X. The functor*

$$\Gamma(U,\cdot) : \mathbf{SAb} \to \mathbf{Ab}, \quad \mathcal{F} \to \Gamma(U,\mathcal{F}),$$

is left exact.

Proof. Let **PAb** be the category of presheaves of abelian groups on X, ι be the natural embedding functor (see 2.10). Let us prove first that ι is left exact. Let

$$0 \to \mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}'' \to 0$$

be an exact sequence of sheaves. We can assume that $(\mathcal{F}', f) = \operatorname{Ker} g$ (the kernel in **SAb**). Next, $\operatorname{Ker}(\iota g : \iota\mathcal{F} \to \iota\mathcal{F}'') = (\iota\mathcal{F}, \iota f)$. Therefore the sequence

$$0 \to \iota\mathcal{F}' \xrightarrow{\iota f} \iota\mathcal{F} \xrightarrow{\iota g} \iota\mathcal{F}'$$

is exact in **PAb**.

The kernel and the cokernel of a morphism in **PAb** are defined on each open set separately. Hence the functor

$$\Gamma(U,\cdot) : \mathbf{PAb} \to \mathbf{Ab}, \quad \mathcal{F} \to \Gamma(U,\mathcal{F}),$$

is exact, so that the sequence

$$0 \to \Gamma(U,\iota\mathcal{F}') \to \Gamma(U,\mathcal{F}) \to \Gamma(U,\iota\mathcal{F}'')$$

is also exact. For any sheaf $\mathcal{F}$ we have clearly $\Gamma(U,\iota\mathcal{F}) = \Gamma(U,\mathcal{F})$. Hence the functor $\mathcal{F} \to \Gamma(U,\mathcal{F})$ is left exact on the category **SAb**.

Let us remark also that the functor $\mathcal{F} \to \Gamma(Y,\mathcal{F})$ on the category **PAb** is exact for an arbitrary (not necessarily open) subset $Y \subset X$.

3.6. Definition. a. An object Y of an abelian category is said to be *projective* if the functor $X \to \operatorname{Hom}(Y,X)$ is exact.

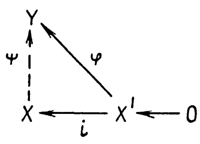
b. An object Y of an abelian category is said to be *injective* if the functor $X \to \operatorname{Hom}(Y,X)$ is exact.

c. A left A-module X (resp. a right A-module Y) is said to be *flat* if the functor $Y \to Y \otimes_A X$ (resp. the functor $X \to Y \otimes_A X$) is exact.)

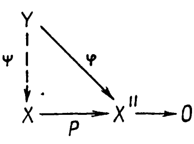
Part **c** of this definition can be applied also to sheaves of modules.

Let us discuss various aspects of this definition.

3.7. Injectivity, Projectivity, and Extensions of Morphisms. Let us consider the following two diagram in an abelian category:



Injectivity diagram



Projectivity diagram

We will think of these diagrams as encoding the following properties of the object Y:

a. for any epimorphism $X \to X''$ and any morphism $Y \to X''$ there exists a morphism $X \to Y$ that makes the projectivity diagram commutative.

b. for any monomorphism $X' \to X$ and any morphism $X' \to Y$ there exists a morphism $Y \to X$ that makes the injectivity diagram commutative.

We claim that the the statement **a** (resp. the statement **b**) is equivalent to the projectivity (resp. to the injectivity) of the object Y.

3.8. Projective Modules and Free Modules. The results of the previous subsection yield the following description of projective objects in the category of modules (left or right): a module is projective if and only if it is a direct summand of a free module.

The category of sheaves of modules over a ringed space usually contains few projective modules. In some constructions projective sheaves can be replaced by locally free sheaves (i.e. by sheaves that become free after being restricted to elements of an appropriate covering).

3.9. Flatness and Relations. Let us consider a left A-module X and a finite sequence of elements $(x_1, \ldots, x_n) \in X^n$. A sequence $(a_1, \ldots, a_n) \in A^n$ is said to be a *relation* between $x_1, \ldots, x_n$ if $\sum_{i=1}^n a_i x_i = 0$.

More generally, a sequence of elements $(y_1, \ldots, y_n) \in Y^n$, where Y is a right A-module, is said to be a relation between $x_1, \ldots, x_n$ with coefficients in Y if $\sum_{i=1}^n y_i \otimes x_i = 0$ in $Y \otimes_A X$.

Relations with coefficients in a module can be obtained from relations with coefficients in a ring. More explicitly, let $(a_1^{(j)}, \ldots, a_n^{(j)}) \in A^n$, $j = 1, \ldots, m$, be a family of relations between $x_1, \ldots, x_n$, and $y^{(j)}$ be a family of elements from Y. Then $\left(y_i = \sum_{j=1}^n y^{(j)} a_i^{(j)}\right) \in Y^n$ is a relation. Indeed,

$$\sum_{i=1}^n y_i \otimes x_i = \sum_{i=1}^n \sum_{j=1}^m y^{(j)} a_i^{(j)} \otimes x_i = \sum_{i=1}^n \sum_{j=1}^m y^{(j)} \otimes a_i^{(j)} x_i$$
$$= \sum_{j=1}^n y^{(j)} \otimes \sum_{i=1}^n a_i^{(j)} x_i = 0.$$

Now we can characterize flat right A-modules Y by the following property: for any left A-module X and any finite sequence of elements $x_1, \ldots, x_n$ of X any relation between x_i with coefficients in Y is a consequence of relations between x_i with coefficients in A.

3.10. Flatness and Projectivity. The following properties of flat modules are almost obvious.

a. Free modules are flat.

b. Direct summands of flat modules are flat.

c. Inductive limits of families of flat modules are flat. (The proof uses the fact that inductive limits commute with tensor products and preserve exactness. See some details in the next section.)

Properties **a** and **b** imply that projective modules are flat, and property **c** implies that inductive limit of projective limits are flat.

The theorem proved independently by A.Lazard and V.E.Govorov asserts that the converse is also true: any flat A-module is isomorphic to the inductive limit of free modules of finite type over a directed family of indices.

The last functors whose exactness properties we will discuss are direct and inverse images of sheaves.

3.11. Definition. Let $f : M \to N$ be a continuous mapping of topological spaces, $\mathcal{F}$ be a sheaf of sets on M. Its *direct image* is the sheaf $f_.(\mathcal{F})$ on N whose sections over an open subset $U \subset N$ are defined by the formula

$$f_.(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U)),$$

and the restriction from U to $V \subset U$ is induced by the restriction from $f^{-1}(U)$ to $f^{-1}(V)$.

The fact that $f_.(\mathcal{F})$ is a presheaf is clear; the fact that it is a sheaf can be proved directly.

3.12. Properties of the Direct Image. a. The functor $f_.$ preserves all additional structures the sheaf $\mathcal{F}$ can possess: the direct image of a sheaf of groups is a sheaf of groups, etc. If $\Phi = (f, \varphi)$ is a morphism of ringed spaces, and $\mathcal{F}$ is a sheaf of $\mathcal{O}_M$-modules, then $f_.(\mathcal{F})$ is a sheaf of $\mathcal{O}_N$-modules. In fact, we must define the action of $\mathcal{O}_N(U)$ on $f_.(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. But the structure of a morphism of ringed spaces includes a family of ring homomorphisms $\varphi_U : \mathcal{O}_N(U) \to \mathcal{O}_M(f^{-1}(U))$ compatible with restrictions, and $\mathcal{F}(f^{-1}(U))$ is an $\mathcal{O}_M(f^{-1}(U))$-module. Hence $\mathcal{F}(f^{-1}(U))$ can be considered as an $\mathcal{O}_N(U)$ -module. The same arguments show that $\{\varphi_U\}$ can be viewed as a morphism of sheaves of rings $\mathcal{O}_N \to f_.(\mathcal{O}_M)$.

b. Let $f : M \to \{\cdot\}$ is the morphism to a point. Then $f_.(\mathcal{F}) = \Gamma(M, \mathcal{F})$ (global sections).

More generally, let $f : M \to N$ be a locally trivial fibration. Then $f_.(\mathcal{F})$ is "the sheaf of sections of $\mathcal{F}$ along the fibers of f".

Let $i : M \to N$ be a closed embedding. The sheaf $i_.(\mathcal{F})$ is sometimes called "the extension of $\mathcal{F}$ by zero": if $U = N \setminus M$ is an open set then $i_.(\mathcal{F})(U)$ is an one-point set (if $\mathcal{F}$ is a sheaf of sets) or $i_.(\mathcal{F})(U) = \{0\}$ (if $\mathcal{F}$ is a sheaf of abelian groups).

The usage of the same name for non-closed embedding is safe if we will remember that the stalks of $i_.(\mathcal{F})$ over points of the boundary of M can be non-trivial.

c. Any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on M induces (in an obvious way) the morphism of sheaves $f_.(\varphi) : f_.(\mathcal{F}) \to f_.(\mathcal{G})$ on N. Hence, $f_.$ is a functor from the category of sheaves on M to the category of sheaves on N.

The same holds for categories of sheaves of abelian groups, of modules over ringed spaces, etc.

Among the functoriality properties of $f_{.}$ with respect to f we mention the following:

$$(fg)_{.} = f_{.}g_{.}, \qquad \mathrm{id}_{.} = \mathrm{Id}.$$

3.13. The Inverse Image. Let again $f : M \to N$ be a continuous mapping of topological spaces, $\mathcal{F}$ be a sheaf of sets on N. The main property of the *direct image* $f^{\cdot}(N)$ is that the functor $f^{\cdot}$ is left adjoint to the functor $f_{.}$ (see 1.23). Hence we give the definition of the functor $f^{\cdot}$ as the existence theorem. One of the justifications for such approach is that in passing, say, to sheaves of modules over ringed spaces the construction of $f^{\cdot}(\mathcal{F})$ changes, while the adjointness property remains valid.

3.13.1. Proposition-Definition. *There exists a functor* $f^{\cdot} : \mathbf{SSet}_N \to \mathbf{SSet}_M$ *between the categories of sheaves of sets on* N *and on* M*, and the isomorphism of bifuctors*

$$\mathrm{Hom}(f^{\cdot}(\mathcal{F}), \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}, f_{.}(\mathcal{G})).$$

3.14. Properties of the Inverse Image. a. The sheaf $f^{\cdot}(\mathcal{F})$ can be constructed explicitly as the sheaf on M associated to the presheaf

$$U \mapsto \mathcal{F}(f(U)), \qquad U \subset M.$$

Since in general the set $f(U)$ is not open in N, this definition of $F^{\cdot}(\mathcal{F})$ requires two "passages to the limit": one to construct $\mathcal{F}(f(U))$, and another one to define the sheaf associated to a presheaf.

One can easily verify that $f^{\cdot}(\mathcal{F})_X = \mathcal{F}_{f(X)}$ for any point $x \in M$.

b. The construction of $f^{\cdot}$ preserves the internal structures: if $\mathcal{F}$ is a sheaf of abelian groups, rings, etc., then $f^{\cdot}(\mathcal{F})$ belongs to a similar category. However, if $\Phi = (f, \varphi)$ is a morphism of ringed spaces and $\mathcal{F}$ is a sheaf of $\mathcal{O}_N$-modules, then $f^{\cdot}(\mathcal{F})$ does not possess, in general, a natural structure of a sheaf of $\mathcal{O}_M$-modules. In fact, the morphism Φ defines a morphism of sheaves of rings $f^{\cdot}(\mathcal{O}_N) \to \mathcal{O}_M$, but now $\mathcal{F}$ is a sheaf of modules not over the second, as for the direct limit, but over the first of these two sheaves. Algebraic wisdom tells us that the base change in this case is performed using the tensor product, so that the natural definition is

$$f^{*}(\mathcal{F}) = \mathcal{O}_M \underset{f^{\cdot}(\mathcal{O}_N)}{\otimes} f^{\cdot}(\mathcal{F}).$$

One can verify that

$$\mathrm{Hom}_{\mathcal{O}_M}(f^{*}(\mathcal{F}), \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_N}(\mathcal{F}, f_{.}(\mathcal{G})),$$

so that $f^{*}(\mathcal{F})$ is an appropriate definition of the inverse image for sheaves of modules.

c. If $i : M \to N$ is an embedding, then $i^{\cdot}(\mathcal{F})$ is the restriction of the sheaf $\mathcal{F}$ to M.

d. Similarly to the case of the direct image, we have

$$(fg)^{\cdot} = g^{\cdot} f^{\cdot}, \quad \mathrm{id}^{\cdot} = \mathrm{Id}.$$

The exactness properties of the functors $f_{.}$, $f^{\cdot}$, f^{*} are summarized in the following proposition.

3.14.1. Proposition. a. *On the categories of sheaves of abelian groups the functor $f_{.}$ is left exact and the functor $f^{\cdot}$ is exact.*

b. *On the categories of sheaves of modules over ringed spaces the functor $f_{.}$ is left exact and the functor f^{*} is right exact.*

The proof uses the following exactness properties of adjoint functors.

3.15. Exactness of Adjoint Functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories, $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be two additive functors. Assume we are given an isomorphism of bifunctors

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

so that F is left adjoint to G and G is right adjoint to F. Then F is right exact and G is left exact.

§ 4. Classical Derived Functors

4.1. Introduction. To a large extend the appearence of homological algebra is due to the fact that the standard functors originated in algebra, geometry and topology are usually only exact from one side (from the right or from the left). Hence, for example, a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories (such as Hom or Γ) maps an exact sequence

$$0 \to A' \xrightarrow{\varphi} A \xrightarrow{\psi} A'' \to 0 \tag{1}$$

in $\mathcal{A}$ to the exact sequence

$$0 \to F(A') \xrightarrow{F(\varphi)} F(A) \xrightarrow{F(\psi)} F(A'') \tag{2}$$

in $\mathcal{B}$, and the morphism $F(\psi)$ is not, in general, an epimorphism. The classical derived functors R^iF (in our case, the right derived functors) to some extent control the non-exactness of $F(\psi)$. Namely, one of the most important properties of derived functors $R^iF : \mathcal{A} \to \mathcal{B}$ is that for any short exact sequence (1) there exists a long exact sequence which extends (2) to the right:

$$\begin{gathered} 0 \to F(A') \xrightarrow{F(\varphi)} F(A) \xrightarrow{F(\psi)} F(A'') \xrightarrow{\delta^0} R^1F(A') \xrightarrow{R^1F(\varphi)} \dots \\ \to R^{n-1}F(A'') \xrightarrow{\delta^{n-1}} R^nF(A') \xrightarrow{R^nF(\varphi)} R^nF(A) \xrightarrow{R^nF(\psi)} R^nF(A'') \to \dots \end{gathered} \tag{3}$$

and depends functorially on (1).

In this section we give the precise definition and the main properties of derived functors.

4.2. Definition. a. An *injective resolution* of an object A from an abelian category $\mathcal{A}$ is an exact sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \to I^1 \to I^2 \to \dots$$

in $\mathcal{A}$, with all I^i, $i \geq 0$, injective.

b. A *projective resolution* of an object A from an abelian category $\mathcal{A}$ is an exact sequence

$$\dots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{\varepsilon} A \to 0$$

in $\mathcal{A}$, with all P^{-i}, $i \geq 0$, projective.

Sometimes resolutions are denoted $A \xrightarrow{\varepsilon} I^\cdot$, $P^\cdot \xrightarrow{\varepsilon} A$; ε is called the augmentation morphism.

4.3. Properties of Resolutions. a. A neccesary and sufficient condition for any object from $\mathcal{A}$ to have an injective resolution is that any object from $\mathcal{A}$ is a subobject of an injective object of $\mathcal{A}$.

In this case we say that $\mathcal{A}$ has *sufficiently many injective objects.*

b. Let $\varphi : A \to A'$ be a morphism in $\mathcal{A}$, $A \xrightarrow{\varepsilon} I^\cdot$ and $A' \xrightarrow{\varepsilon'} I'^\cdot$ be injective resolutions of A and A'. Then there exists a morphism of complexes $\varphi^\cdot : I^\cdot \to I'^\cdot$ that extends φ (that is, $\varepsilon'\varphi = \varphi^\cdot\varepsilon$). Any two such extensions are homotopic. In particular, any two injective resolutions of the object A are homotopically equivalent.

Similar results hold for projective resolutions (in **a** we must require that any object from $\mathcal{A}$ is a quotient of a projective object).

4.4. Derived Functors. We begin with the definition of right derived functors.

4.4.1. Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Assume that $\mathcal{A}$ has sufficiently many injective objects. The *right derived functors* $R^iF : \mathcal{A} \to \mathcal{B}$, $i \geq 0$ are defined as follows:

$$\text{on objects:} \quad R^iF(A) = H^i(F(I^\cdot)) \ \text{ for } A \in \operatorname{Ob}\mathcal{A},$$

where $A \xrightarrow{\varepsilon} I^\cdot$ is an injective resolution of A, $F(I^\cdot)$ is the complex obtained by termwise application of F to $I^\cdot$;

$$\text{on morphisms:} \quad R^iF(\varphi) = H^i(F(\varphi^\cdot)) \ \text{ for } \varphi : A \to A',$$

where $\varphi^\cdot : I^\cdot \to I'^\cdot$ is an extension of φ to resolutions (as in 4.3.a), $F(\varphi^\cdot) : F(I^\cdot) \to F(I'^\cdot)$ is the corresponding morphism of resolutions.

4.4.2. Proposition. a. *$R^iF(A)$ and $R^iF(\varphi)$ are well defined (more explicitely, the objects $R^iF(A)$ constructed from different injective resolutions of*

A are canonically isomorphic; morphisms $R^iF(\varphi)$ constructed from different extensions of φ on resolutions coincide).

b. *R^iF are additive functors from $\mathcal{A}$ to $\mathcal{B}$.*

c. *R^0F is canonically isomorphic to F.*

d. *From any short exact sequence* (1) *in $\mathcal{A}$ one can construct a long exact sequence* (3) *in $\mathcal{B}$ functorially depending on* (1).

e. *If F is exact, then $R^iF = 0$ for $i > 0$.*

The proofs of all these statements essentially follows from 4.3. In particular, to prove **d**, we must construct from (1) the exact sequence of resolutions

$$0 \to I'^{\cdot} \xrightarrow{\varphi^{\cdot}} I^{\cdot} \xrightarrow{\psi^{\cdot}} I''^{\cdot} \to 0$$

and use Theorem 1.5.1 from Chap. 1.

4.4.3. Left Derived Functors. If $\mathcal{A}$ contains sufficiently many projective objects, the left derived functors L_iG of a right exact functor $G : \mathcal{A} \to \mathcal{B}$ are defined (on objects from $\mathcal{A}$) by the formula $L_iG = H^{-i}(G(P^{\cdot}))$, where $P^{\cdot} \xrightarrow{\varepsilon} A$ is a projective resolution of A. Analogs of statements a–e of Proposition 4.4.2 hold for left derived functors.

4.5. Derived Functors of the Composition. The Grothendieck Spectral Sequence. Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be two left exact functors. One can easily verify that $G \circ F : \mathcal{A} \to \mathcal{C}$ is also left exact. Hence, assuming that $\mathcal{A}$ and $\mathcal{B}$ contain sufficiently many injective objects, we can define right derived functors R^iF, R^iG, $R^i(G \circ F)$. The relations among these derived functors are expressed by the Grothendieck spectral sequence.

4.5.1. Definition. An object $A \in \operatorname{Ob}\mathcal{A}$ is said to be acyclic with respect to a left exact functor $F : \mathcal{A} \to \mathcal{B}$ (or F-acyclic) if $R^iF(A) = 0$ for all $i > 0$.

Clearly, any injective object is F-acyclic for any left exact F. However, for some functors F there may exist other F-acyclic objects (see examples in Chap. 4, §5).

4.5.2. Theorem. *Let $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be two left exact functors. Assume that $\mathcal{A}$ and $\mathcal{B}$ contain sufficiently many injective objects, and for any injective $I \in \operatorname{Ob}\mathcal{A}$ the object $F(I)$ is injective in $\mathcal{B}$. Then for any $A \in \operatorname{Ob}\mathcal{A}$ there exists a spectral sequence* (see 3.1 in Chap. 1) *with*

$$E_2^{p,q} = (R^pG)(R^qF(A)),$$

converging to $R^n(G \circ F)(X)$. It is functorial in X.

4.5.3. Plan of the Proof. Objects $RF(A)$ are the cohomology objects of the complex $F(I^{\cdot})$, where $A \xrightarrow{\varepsilon} I^{\cdot}$ is an injective resolution of A. Similarly, objects $R^n(G \circ F)(A)$ are the cohomology objects of the complex $(G \circ F)(I^{\cdot})$.

On the other hand, to compute $(R^pG)(R^qF(A))$ we must apply G term by term to injective resolutions $R^qF(A) \to J^{\cdot,q}$ of objects $R^qF(A)$.

The relation between $(G \circ F)(I^{\cdot})$ and $G(J^{\cdot,q})$ is expresses by the Cartan-Eilenberg resolution of the complex $K^{\cdot} = F(I^{\cdot})$. This resolution consists of the following data.

a. A *double complex* L^{ij} with differentials d_{I}, d_{II} of bidegree $(1,0)$ and $(0,1)$ respectively; it satisfies the conditions $L^{ij} = 0$ for $j < 0$ or $i \leq 0$, L^{ij} are injective (the main definitions concerning double complexes see in 3.5 from Chap. 1).

b. A *morphism of complexes* $\varepsilon^{\cdot} : K^{\cdot} \to L^{\cdot,0}$.

To formulate the conditions imposed in the above data we note that (L^{ij}) yields the following complexes:

$$\begin{aligned}
0 \to & K^i \xrightarrow{\varepsilon^i} L^{i,0} \to L^{i,1} \to \dots, \\
0 \to & B^i(K^{\cdot}) \to B^i_{\mathrm{I}}(L^{\cdot,0}) \to B^i_{\mathrm{I}}(L^{\cdot,1}) \to \dots, \\
0 \to & Z^i(K^{\cdot}) \to Z^i_{\mathrm{I}}(L^{\cdot,0}) \to Z^i_{\mathrm{I}}(L^{\cdot,1}) \to \dots, \\
0 \to & H^i(K^{\cdot}) \to H^i_{\mathrm{I}}(L^{\cdot,0}) \to H^i_{\mathrm{I}}(L^{\cdot,1}) \to \dots
\end{aligned} \tag{4}$$

(here B and Z denote respectively the boundaries and the cycles). The following two conditions should be satisfied.

c. All these complexes are acyclic.

d. Exact triples

$$\begin{aligned}
0 \to & B^i_{\mathrm{I}}(L^{\cdot,j}) \to Z^i_{\mathrm{I}}(L^{\cdot,j}) \to H^i_{\mathrm{I}}(L^{\cdot,j}) \to 0, \\
0 \to & Z^i_{\mathrm{I}}(L^{\cdot,j}) \to L^{ij} \to B^{i+1}_{\mathrm{I}}(L^{\cdot,J}) \to 0
\end{aligned}$$

split.

Under these conditions all objects B_{I}, Z_{I}, H_{I} are injective, so that complexes (4) are injective resolutions of K^i, $B^i(K^{\cdot})$, $Z^i(K^{\cdot})$, $H^i(K^{\cdot})$ respectively.

In particular, $R^pG(R^qF(A))$ can be computed as the p-th cohomology of the complex $G(H^q_{\mathrm{I}}(L^{\cdot\cdot})$. On the other hand, the cohomology of the complex $K^{\cdot} = F(I^{\cdot}_X)$ is isomorphic to the cohomology of the diagonal complex $SL^{\cdot}$ of $L^{\cdot}$ (see 3.5 in Chap. 1), so that $R^n(G \circ F)(A) = H^n(G(F(I^{\cdot})))$ is isomorphic to $H^n(SL^{\cdot})$. Moreover, L has a filtration

$$(F^pSL^{\cdot})^n = \bigoplus_{\substack{i+j=n \\ j \geq p}} L^{ij}$$

and the term E_2 of the spectral sequence corresponding to this filtration is isomorphic to $(R^pG)(R^qF(A)) = H^p(G(H^q_{\mathrm{I}}(L^{\cdot\cdot})))$.

Hence, the above spectral sequence is a special case of the spectral sequence associated to a filtered complex, see 3.3 in Chap. 1.

4.6. Standard Derived Functors. Some most commonly used derived functors from algebra and topology have standard names. We mention some of these functors.

4.6.1. Functors Ext. Let $\mathcal{A}$ be an abelian category. The abelian group $\mathrm{Hom}_{\mathcal{A}}(M,N)$ for $M,N \in \mathrm{Ob}\,\mathcal{A}$ can be interpreted as a functor in three different ways.

a. as a functor $\mathrm{Hom}_{\mathcal{A}}(M,\cdot) : \mathcal{A} \to \mathbf{Ab}$ for a fixed M;

b. as a functor $\mathrm{Hom}_{\mathcal{A}}(\cdot,N) : \mathcal{A}^{\circ} \to \mathbf{Ab}$ for a fixed N;

c. as a functor $\mathrm{Hom}_{\mathcal{A}}(\cdot,\cdot) : \mathcal{A}^{\circ} \times \mathcal{A} \to \mathbf{Ab}$.

All these functors are left exact. The corresponding right derived functors are denoted $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$; one can show that the groups $\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ do not depend on the choice in the interpretation of $\mathrm{Hom}_{\mathcal{A}}$ as a functor. See also 3.2 in Chap. 3.

4.6.2. Functors Tor. For a fixed ring R and two R-modules M (right) and N (left) the tensor product $M \otimes_R N$ is an abelian group. Similarly to 4.6.2, one can define three functors

$$M \otimes_R \cdot : R\text{-mod} \to \mathbf{Ab}, \quad \cdot \otimes_R N : \text{mod-}R \to \mathbf{Ab},$$
$$\cdot \otimes_R \cdot : \text{mod-}R \times R\text{-mod} \to \mathbf{Ab}.$$

All three functors are right exact. Their left derived functors are denoted $\mathrm{Tor}_i^R(M,N)$, $i \geq 0$. Again, these functors do not depend on the choice in the definition of the functor $\otimes_R$.

4.6.3. Cohomology with Coefficients in a Sheaf. The functor $\Gamma : \mathbf{SAb}_X \to \mathbf{Ab}$ which associates to each sheaf $\mathcal{F}$ of abelian groups on X the group of its global sections $\Gamma(X,\mathcal{F})$ is left exact. Its right derived functors are called *cohomologies of* X *with coefficients in* $\mathcal{F}$ and are denoted $H^i(X,\mathcal{F})$, $i \geq 0$. See also 5.3 in Chap. 4.

4.6.4. Group Cohomology. Let G be a group, G-mod be the category of G-modules (see 2.7 in Chap. 1). The functor from G-mod to $\mathbf{Ab}$ which associated to a G-module A the group A^G of G-invariant elements of A is left exact. Its right derived functors are called *cohomologies* $H^i(G,A)$ *of the group* G *with coefficients in* A.

4.6.5. Lie Algebra Cohomology. Similarly, let $\mathfrak{g}$ be a Lie algebra over a field k, $\mathfrak{g}$-mod be the category of $\mathfrak{g}$-modules. The *cohomology* $H^i(\mathfrak{g},M)$ *of the Lie algebra* $\mathfrak{g}$ *with coefficients in a* $\mathfrak{g}$*-module* M is the i-th right derived functor of the functor from $\mathfrak{g}$-mod to $\mathbf{Vect}_k$ which associated to M the space $M^{\mathfrak{g}} = \{m \in M \mid Xm = 0 \text{ for all } X \in \mathfrak{g}\}$.

4.6.6. The Zuckerman Functors. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{h}$ its subalgebra, $U(\mathfrak{g})$, $U(\mathfrak{h})$ the corresponding enveloping algebras. Assume that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is completely reducible. An element m from a $\mathfrak{g}$-module M is said to be $\mathfrak{h}$-finite if $\dim_k U(\mathfrak{h})m < \infty$. One can easily verify that $\mathfrak{h}$-finite elements of M form a $\mathfrak{g}$-submodule $M^{(\mathfrak{g})}$ of M. The functor $M \to M^{(\mathfrak{h})}$ from the category $\mathfrak{g}$-mod to itself is left exact. Its left derived functors are called the *Zuckerman functors*. They are very important in representation theory (especially in the case when both $\mathfrak{g}$ and $\mathfrak{h}$ are reductive), being analogs of derived functors of the functor "holomorphic sections of an analytical vector bundle on G/H", where G and H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

Bibliographic Hints

In the first two sections of this chapter we presented a collection of standard facts of category theory; more detailed information can be obtained from the corresponding parts of (Gelfand, Manin 1988; Faith 1973; Grothendieck 1957; MacLane 1971). In particular, Theorem 1.16 is one of Freyd's theorems about the general characterization of representable functors; see (MacLane 1971). About the embedding Theorem 2.13.1 see (Bass 1968). Lemmas 2.13.2 and 2.13.3 form the basis of a majority of proofs which use the so called "diagram chase", see the discussion in the book (MacLane 1971).

Results presented in Sect. 3 are concentrated around the notion of an exact functor. These results are mostly classical, and their proofs may be found in textbooks on homological algebra (Cartan, Eilenberg 1956; MacLane 1963) (for 3.3.–3.10) and on sheaf theory (Bredon 1967; Iversen 1986). See also (Gelfand, Manin 1988).

In Sect. 4 we present the theory of derived functors as it was understood at the beginning of fifties and at the end of sixties; this is how this theory is presented in (Cartan, Eilenberg 1956; Grothendieck 1957; MacLane 1963). The proofs of these results can be found also in (Gelfand, Manin 1988). Proposition 4.4.2 is proved in (Hartshorne 1966), Theorem 4.5.2 in (Grothendieck 1957).

Chapter 3
Homology Groups in Algebra and in Geometry

§ 1. Small Dimensions

1.1. Dimension 0. Zero-dimensional (co)homology groups often appear as the main objects of interest in classical problems: they are spaces of functions with prescribed singularities, or invariants of group actions, etc. There is not much to say about them in purely homological terms.

1.2. Dimension 1. One-dimensional classes can be especially interesting for certain coefficient groups.

1.2.1. Sheaves. a. Let X be a complex manifold (or space), $\mathcal{O}_X$ the sheaf of holomorphic functions on it, $\mathcal{M}_X$ the sheaf of meromorphic functions, $\mathcal{P}_X = \mathcal{M}_X/\mathcal{O}_X$ the sheaf of (additive) principal parts. The classical (additive) Cousin problem is that of finding a meromorphic function on X with prescribed principal parts. It is always solvable if and only if the map $H^0(\mathcal{M}_X) \to H^0(\mathcal{P}_X)$ is surjective. Therefore the cohomology classes lying in the image of the boundary map $\delta : H^0(\mathcal{P}_X) \to H^1(\mathcal{O}_X)$ are obstructions to the solvability of the *Cousin problem.*

b. In the same setting, denote by $\mathcal{O}^*$ (resp. $\mathcal{M}^*$) the sheaf of holomorphic functions with holomorphic inverse (resp. meromorphic functions with meromorphic inverse). The sheaf of multiplicative principal parts $\mathcal{P}^* = \mathcal{M}^*/\mathcal{O}^*$ is called the *sheaf of Cartier divisors.* Obstructions for existence of a meromorphic function with a prescribed Cartier divisor lie in the group $H^1(\mathcal{O}^*)$.

The same group classifies invertible sheaves on X, i.e. locally free rank one $\mathcal{O}_X$-modules. In order to construct a Čech cocycle corresponding to such a sheaf $\mathcal{L}$, we choose a covering (U_α) over elements of which $\mathcal{L}$ is freely generated by sections t_α and put $g_{\alpha_0\alpha_1} = t_{\alpha_0}t_{\alpha_1}^{-1} \in \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{O}_X^*)$. The class of this cocycle in $H^1(\mathcal{O}^*)$ is well defined. Two such classes coincide if and only if the corresponding sheaves are isomorphic. Any class is defined by an invertible sheaf. Product of classes corresponds to the tensor product of sheaves.

So interpreted, $H^1(\mathcal{O}^*)$ is called the *Picard group* of the space X. It is usually denoted $\operatorname{Pic} X$ and admits a similar description in various other categories of locally ringed spaces, e.g. schemes. If A is a ring of integers of a field of algebraic numbers K, $X = \operatorname{Spec}(A)$, then $\operatorname{Pic} X$ is the classical ideal class group of K.

c. Consider a submersion $\pi : X \to S$ of complex manifolds as a family of its fibers. Let $\mathcal{T}X$, $\mathcal{T}S$ be the tangent sheaves and $\mathcal{T}X/S$ the sheaf of vertical tangent fields. The exact sequence on X

$$0 \to \mathcal{T}X/S \to \mathcal{T}X \to \pi^*(\mathcal{T}S) \to 0$$

induces the boundary map

$$\mathcal{T}S = \pi_*(\pi^*\mathcal{T}S) \to R^1\pi_*\mathcal{T}X/S,$$

which is called the *Kodaira-Spencer map.* Considering it pointwise, we see that it associates with a tangent vector at a point s_0 in the deformation space S of the fiber $X_{s_0} = \pi^{-1}(s_0)$ a certain cohomology class in $H^1(X_{s_0}, \mathcal{T}X_{s_0})$. Kodaira and Spencer proved that if X_{s_0} is compact and its second cohomology group with coefficients in the tangent sheaf vanishes, there is a local deformation of this fiber for which the Kodaira-Spencer map is an isomorphism. Moreover, any other deformation is locally induced by this one.

1.2.2. Groups. Let G be a group, M a G-module with trivial action. Then $B^1(G,M) = 0$, $H^1(G,M) = Z^1(G,M) = \operatorname{Hom}(G,M)$ (homomorphisms of groups). Therefore, general 1-cocycles are also called crossed homomorphisms. In particular, if G is a finite (or profinite) p-group, we have $H^1(G,\mathbb{Z}_p) = \operatorname{Hom}(G,\mathbb{Z}_p) = \operatorname{Hom}(G/[G,G],\mathbb{Z}_p)$. The dimension of this linear space over $\mathbb{Z}_p$ coincides with the minimal number of (topological) generators of G.

1.2.3. Lie Algebras. Let $\mathfrak{g}$ be a Lie algebra over a field k. Consider k as $\mathfrak{g}$-module with trivial action. Then $H_1(\mathfrak{g},k) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, $H^1(\mathfrak{g},k) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$.

Consider now $\mathfrak{g}$ as $\mathfrak{g}$-module with adjoint action. Then $H^1(\mathfrak{g}, \mathfrak{g}) = \operatorname{Der}(\mathfrak{g})/\operatorname{Int}(\mathfrak{g})$, where $\operatorname{Der}(\mathfrak{g})$ is the space of derivations, $\operatorname{Int}(\mathfrak{g})$ the space of inner derivations.

1.2.4. Extension Classes. If X, Y are objects of an abelian category with sufficiently many projectives or injectives, $\operatorname{Ext}^1(X, Y)$ can be described as the group of classes of exact triples $0 \to Y \to E \to X \to 0$ with respect to the equivalence relation defined by the existence of a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] & E' \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

In order to describe the element of $\operatorname{Ext}^1(X, Y)$ corresponding to such a triple, consider the first terms of an injective resolution $0 \to Y \to I^0 \to I^1 \to I^2$. In view of the extension property, there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "c"'] & 0 \arrow[d] \\
0 \arrow[r] & Y \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2
\end{tikzcd}.$$

The morphism $c : X \to I^1$ defines a cocycle in $\operatorname{Hom}(X, I^1)$, whose class is our element of Ext^1. The sum of two elements of $\operatorname{Ext}^1(X, Y)$ defined by the triples $0 \to Y \to E' \to X \to 0$ and $0 \to Y \to E'' \to X \to 0$ is the triple $0 \to Y \to E \to X \to 0$, which is defined by the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \oplus Y \arrow[r] & E' \oplus E'' \arrow[r] & X \oplus X \arrow[r] & 0 \\
0 \arrow[r] & Y \oplus Y \arrow[r] \arrow[u, equal] \arrow[d, "\nabla"'] & \widetilde{E}' \arrow[r] \arrow[u] \arrow[d] & X \arrow[r] \arrow[u, leftarrow, "\Delta"] \arrow[d, equal] & 0, \\
0 \arrow[r] & Y \arrow[r] & E \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

where Δ is the diagonal morphism, ∇ is the sum morphism, 1 and 2 are cartesian and cocartesian squares, respectively (cf. Chap. 2, 1.17, 1.18).

1.3. Dimension 2. A series of classical construction can be interpreted in terms of two-dimensional cohomology groups.

1.3.1. Sheaves. Two-dimensional sheaf cohomology arises in various extension problems; see some details in the next section.

1.3.2. Groups. a. For any G-module M, elements of $H^2(G, M)$ classify group extensions of the type

$$1 \to M \to E \to G \to 1,$$

where the adjoint action of G on M coincides with the given one. Two extensions are called equivalent if they fit into a diagram like in 1.2.4. Given a cocycle $a \in Z^2(G,M)$, E is defined as the set $M \times G$ with the multiplication law

$$(m,g)(n,h) = (m + gn + a(g,h), gh).$$

In particular, if the action of G on M is trivial, $H^2(G,M)$ classifies central extensions. It follows that if G is free, $H^2(G,M) = 0$.

b. Let G be a finite p-group, as in 1.2.2. Consider a surjective morphism $f : F \to G$, where F is a free group. Let $R = \mathrm{Ker}(f)$. From the Hochschild-Serre spectral sequence we find the exact sequence

$$0 \to H^1(G,\mathbb{Z}_p) \to H^1(F,\mathbb{Z}_p) \to H^1(R,\mathbb{Z}_p)^G \to H^2(G,\mathbb{Z}) \to 0.$$

The dimension of $H^1(R,\mathbb{Z})^G$ can be identified with the minimal number of elements generating R as a normal subgroup in G. If the numbers of generators of F and G coincide, then $H^1(R,\mathbb{Z})^G$ is isomorphic to $H^2(G,\mathbb{Z})$. Hence the dimension of this last group can be interpreted as the minimal number of generating relations between a minimal set of generators of G.

c. Let G be the Galois group of a separable field extension K/k. Then the *Brauer group* $H^2(G,K^*) = \mathrm{Br}(K/k)$ classifies finite-dimensional simple central algebras over k split over K. As is well known, any such algebra is isomorphic to $\mathrm{Mat}(n,A)$, where A is a division algebra over k with center k. Class of such an algebra in the Brauer group is by definition the class of A up to an isomorphism. Given a cocycle $a \in Z^2(G,K^*)$, the corresponding algebra can be described as a cross-product. Namely, let G be finite. Then we can construct the algebra $\oplus_{g\in G} K[g]$, where $[g]$ are symbols with multiplication table

$$[g][h] = a(g,h)[gh], \quad [g]b = (gb)[g]; \qquad g,h \in G, \quad b \in K.$$

Composition in $H^2(G,K^*)$ corresponds to the tensor product of algebras.

1.3.3. Lie Algebras. a. For any $\mathfrak{g}$-module M, elements of $H^2(\mathfrak{g},M)$ classify Lie algebra extensions

$$0 \to M \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0,$$

such that M is an abelian ideal, and the adjoint action of $\mathfrak{g}$ on M coincides with the given one. Given a cocycle $a \in Z^2(\mathfrak{g},M)$, $\tilde{\mathfrak{g}}$ can be described as $M \oplus \mathfrak{g}$ with the bracket

$$[(m,X),(n,Y)] = (Xn - Ym + a(X,Y), [X,Y]).$$

For a trivial $\mathfrak{g}$-module M, $H^2(\mathfrak{g},M)$ classifies central extensions.

Recently central extensions of certain infinite-dimensional Lie algebras were actively studied in connection with theoretical physics. As a characteristic example, we mention here the *Virasoro algebra* – the central extension of the algebra of Laurent polynomial vector fields on a line defined with the help of the cocycle

$$a(f(t)\,d/dt, g(t)\,d/dt) = \mathrm{res}(f\,dg/dt).$$

b. Elements of $H^2(\mathfrak{g}, \mathfrak{g})$ classify infinitesimal deformations of $\mathfrak{g}$, that is, Lie algebras $\tilde{\mathfrak{g}}$ over dual numbers $k[\varepsilon]$, $\varepsilon^2 = 0$, free as $k[\varepsilon]$-module and endowed with an isomorphism $\tilde{\mathfrak{g}}/(\varepsilon\tilde{\mathfrak{g}}) = \mathfrak{g}$.

§ 2. Obstructions, Torsors, Characteristic Classes

2.1. From Local to Global. In various geometric situations, (co)homology classes appear as global obstructions to the solution of a locally solvable problem. The local solvability often can be interpreted as a statement that a certain morphism of sheaves $\mathcal{G} \to \mathcal{H}$ on a topological space X is surjective: such was the case of the Cousin problems described in 1.2.1. The kernel of such a morphism then determines the set of local solutions (if $\mathcal{G}$, $\mathcal{H}$ are sheaves of abelian groups).

The standard exact sequence of the cohomology groups

$$\dots \to H^i(X, \mathcal{K}) \to H^i(X, \mathcal{G}) \to H^i(X, \mathcal{H}) \xrightarrow{\delta} H^{i+1}(X, \mathcal{K}) \to \dots$$

corresponding to the exact triple of sheaves

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0$$

can also be viewed as follows:

a. Let $c \in H^i(X, \mathcal{H})$. Suppose we want to lift c to a cohomology class $c' \in H^i(X, \mathcal{G})$. This problem is solvable if and only if the obstruction $\delta(c) \in H^{i+1}(X, \mathcal{K})$ vanishes. In particular, c' always exists if $H^{i+1}(X, \mathcal{G}) = 0$.

b. When $\delta(c) = 0$, the set of all liftings is a homogeneous space over the group $H^i(X, \mathcal{K})$. In particular, the lifting is unique if $H^i(X, \mathcal{K}) = 0$.

In many classical problems $i = 0$, i.e. we want to lift sections. On the other hand, $\mathcal{G}$ and $\mathcal{H}$ are not necessarily sheaves of abelian groups, so that this simplest scheme does not apply literally and needs certain modifications.

We shall consider now some examples.

2.2. Map Extensions: a Topological Situation. Let $i : Y \to Y'$ be a pair (topological space, subspace), and $f : Y \to X$ a continuous map. We want to construct a continuous map $f' : Y' \to X$ coinciding with f on Y. Assume that Y is a local deformation retract of Y'. Then locally on Y an extension f' does exist. This means that the restriction map of sheaves of sets on Y'

$$\mathrm{Map}(Y', X) \to i_* \mathrm{Map}(Y, X)$$

is surjective. Generally we cannot construct an obstruction as in 2.1, but in the category of, say, CW-complexes we can abelianize the problem as follows. Suppose that $Y = Y_{n-1}$, $Y' = Y_n$ are two consecutive skeleta of Y. We shall assume that either $n > 3$, or $n = 2$ and $\pi_1(X)$ is an abelian group. Given f, construct a cochain $c^n(f) \in C^n(Y, \pi_{n-1}(X))$ whose value on a cell e_n of Y is the homotopy class of the map $S^{n-1} \xrightarrow{h} Y \xrightarrow{f} X$, where h is the boundary of the characteristic map of e_n.

2.2.1. Theorem. a. *$c^n(f)$ is a cocycle. Its cohomology class vanishes if and only if f admits an extension $f : Y \to X$.*

b. *Let f', f'' be two extensions of f. Then we have $c^{n+1}(f') - c^{n+1}(f'') \in B^{n+1}(Y, \pi_n(X))$, and varying f'' we obtain the entire group $B^{n+1}(Y, \pi_n(X))$.*

Corollary. a. *With the same notation, the restriction of f to Y_{n-2} can be extended to a map $Y_n \to X$ if and only if the cohomology class $[c^n(f)] \in H^n(Y, p_{n-1}(X))$ vanishes.*

b. *The set of homotopy classes of maps $f' : Y_{n-1} \to X$ coinciding with f on Y_{n-2} is a homogeneous space over $B^{n+1}(Y, \pi_n(X))$.*

Thus, working with skeleta and homotopy groups, we achieve a (partial) abelianization of the map extension problem. In 2.5 we shall explain a different construction adapted to the category of complex spaces.

Using similar considerations, one can prove the following important result on the representability of the cohomology functors (in the CW category).

2.2.2. Theorem. a. *For every pair π, n, where n is a natural number, π is a group (abelian if $n > 2$) there exists a CW-complex $K(\pi, n)$ such that $H^n(X, \pi) = [X, K(\pi, n)]$ as functors on the homotopy category of CW-complexes.*

b. *The object $K(\pi, n)$ of this category can be uniquely characterized by the following properties: $\pi_i(K(\pi, n)) = 0$ for $i \neq n$, $\pi_n(K(\pi, n)) = \pi$.*

2.3. Torsors. If G is a non-commutative group one can define a *set* $H^1(X, G)$. Properties of this set constitute the most simple and useful part of the still fragmentary noncommutative cohomology theory.

a. Let G be a topological group, X a topological space. A principal G-fibration, or simply *G-torsor* on X consists of the following data: a continuous map $\pi : P \to X$; a continuous action $G \times P \to P$, $(g, p) \to gp$, such that $\pi(p) = \pi(gp)$ and locally π is isomorphic to a projection $G \times U \to U$ with the fiberwise action $g(h, p) = (gh, p)$. Denote by $T(X, G)$ the set of G-torsors up to an isomorphism.

b. Let P be a G-torsor endowed with a set of local trivializations over an open covering $X = \bigcap_i U_i$. Denote by $e_i : U_i \to P$ the image of $\{\mathrm{id}\} \times U_i$ under this trivialization. Define a continuous map $g_{ij} : U_i \cap U_j \to G$ by the condition $e_i = g_{ij} e_j$ on $U_i \cap U_j$. Clearly, g_{ij} satisfies the *non-commutative 1-cocycle* conditions:

$$g_{ij} g_{ji} = 1, \quad g_{jk} g_{ki} g_{ij} = 1.$$

Denote by $Z^1((U_i), G)$ the set of all 1-cocycles of this kind.

A change of trivialization $e_i' = h_i e_i$ replaces the cocycle g_{ij} by a *cohomological* cocycle

$$g_{ij}' = h_i g_{ij} h_j^{-1}.$$

Denote by $H^1((U_i), G$ the set of equivalence classes of cocycles $Z^1((U_i), G)$. The sets $Z^1((U_i), G$ and $H^1((U_i), G)$ obviously form inductive systems with respect to refinement of coverings. Put $H^1(X, G) = \lim\operatorname{ind} H^1((U_i), G)$: this is the set of 1-cohomology classes of X in the sheaf of local continuous maps of X to G.

2.3.1. Theorem. *The natural map $T(X, G) \to H^1(X, G)$ is a bijection.*

One can show that in various homotopy categories $T(X, G)$, as a functor of X, is representable. The corresponding object BG is called a *classifying space* of G. In view of 2.3.1, this fact can be considered as a non-commutative analog of Theorem 2.2.4. For example:

$$G = O(n), \quad BO(n) = \lim\operatorname{ind}_N \mathbf{Gr}(n, \mathbb{R}^N),$$
$$G = U(n), \quad BU(n) = \lim\operatorname{ind}_N \mathbf{Gr}(n, \mathbb{C}^N),$$

where $\mathbf{Gr}$ denotes the corresponding Grassmannian. These Grassmannians are classifying spaces for $G = O(n)$, resp. $U(n)$, in the homotopy category of Hausdorff compact spaces X, i.e. $H^1(X, G) = [X, BG]$.

2.4. Characteristic Classes. Suppose that a G-torsor $P \in H^1(X, G)$ corresponds to a map $f : X \to BG$. The induced map $f^* : H^*(BG) \to H^*(X)$ is well defined, that is, independent on the choice of f in the homotopy class corresponding to P (cohomology theory H can be chosen more or less arbitrarily, e.g. as singular cohomology with coefficients $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$). For classical Lie groups G (and various other groups) cohomology rings $H^*(BG)$ were calculated explicitly and have a distinguished generator set, say, $\{c_i(G),\ i \in I\}$. In this case classes $f^*c_i(G) \in H^*(X)$ are called the *characteristic classes* of the torsor P.

2.5. Analytic Spaces. Another version of the ideology explained in 2.2 is useful in the complex analytic geometry. Let Y' be an analytic space, $Y \subset Y'$ a closed analytic subspace, defined by the sheaf of ideals $J \subset \mathcal{O}_Y$. We shall consider the case when Y and Y' have a common support, that is, J consists of nilpotents. In this case Y' is called an infinitesimal extension of Y. Similarly to the topological case, we can consider several extension problems:

(i) Extension to Y' of a morphism of analytical spaces $Y \to X$.

(ii) Extension to Y' of a locally free sheaf $\mathcal{E}$, or of a G-torsor, where G is a complex Lie group.

(iii) Extension to $(Y', \mathcal{E}')$ of a cohomology class $c \in H^i(Y, \mathcal{E})$, where $\mathcal{E}$ is an extension of $\mathcal{E}'$ to Y'.

The last problem can be solved in the same way as in 2.1, with the help of the exact sequence

$$\dots \to H^i(Y', J\mathcal{E}') \to H^i(Y', \mathcal{E}') \to H^i(Y, \mathcal{E}) \to H^{i+1}(Y, J\mathcal{E}') \to \dots$$

The first two problems can be tackled in a similar way by considering successive extensions. The simplest one-step infinitesimal extension verifies two conditions:

a. $J^2 = 0$;

b. The sequence of $\mathcal{O}_Y$-sheaves

$$0 \to J \overset{d\otimes 1}{\to} \Omega^1 X \underset{\mathcal{O}_{Y'}}{\otimes} \mathcal{O}_Y \to \Omega^1 Y \to 0.$$

is exact.

We shall give the simplest statement that can be proved in this situation.

2.5.1. Theorem. a. *In the above notation, let $\mathcal{E}$ be a locally free sheaf on Y. It can be extended to a locally free sheaf $\mathcal{E}'$ on Y' if and only if a certain (explicitly constructed) cohomology class*

$$\omega(\mathcal{E}) \in H^2(Y, \operatorname{End} \mathcal{E} \otimes J)$$

vanishes.

b. *If $\omega(\mathcal{E}) = 0$, then the set of all classes of extension $\mathcal{E}$ is endowed with a transitive action of the cohomology group $H^1(Y, \operatorname{End} \mathcal{E} \otimes J)$. This action is effective if there exists an extension $\mathcal{E}'$ such that every section of* $\operatorname{End} \mathcal{E}$ *extends to a section of* $\operatorname{End} \mathcal{E}'$.

For a proof, one must consider the surjection of sheaves $GL(n, \mathcal{O}_{Y'}) \to GL(n, \mathcal{O}_Y)$ and a segment of an exact sequence that can be constructed for non-commutative 1-cohomology. This theorem can also be generalized to other complex Lie groups instead of GL and torsors of this groups.

2.6. The Atiyah Class and Characteristic Classes in the Hodge Cohomology. Now let us apply Theorem 2.5.1 in the following situation: Y is a complex manifold, Y' is the first infinitesimal neighbourhood of the diagonal in $Y \times Y$. There is a canonical homomorphism $J = \Omega^1 Y$. Put $\mathcal{E}' = p_1^*(\mathcal{E})$, $\mathcal{E}'' = p_2^*(\mathcal{E})$, where $p_i : Y' \to Y$ are induced by the two projections. Since $\mathcal{E}'$, $\mathcal{E}''$ have the same restriction to Y, in view of Theorem 2.5.1 we can define their *difference cohomology class*

$$a(\mathcal{E}) \in H^1(Y, \operatorname{End} \mathcal{E} \otimes \Omega^1 Y),$$

called the *Atiyah class.* We can also define the multiplication maps

$$H^p(Y, \operatorname{End} \mathcal{E} \otimes \Omega^q Y) \times H^{p'}(Y, \operatorname{End} \mathcal{E} \otimes \Omega^{q'} Y) \to H^{p+p'}(Y, \operatorname{End} \mathcal{E} \otimes \Omega^{q+q'} Y),$$

and the trace maps

$$\operatorname{tr} : H^p(Y, \operatorname{End} \mathcal{E} \otimes \Omega^q Y) \to H^p(Y, \Omega^q Y).$$

Put

$$e_p(\mathcal{E}) = \operatorname{tr} a(\mathcal{E})^p \in H^p(Y, \Omega^p Y).$$

These characteristic classes of $\mathcal{E}$ take values in the Hodge cohomology ring $\oplus_{p,q} H^p(Y, \Omega^q Y)$.

§ 3. Cyclic (Co)Homology

3.1. Cyclic Objects. Recall that a simplicial object of a category $\mathcal{C}$ is a functor $\Delta^\circ \to \mathcal{C}$, where Δ is the category of sets $[n] = \{0, 1, \dots, n\}$ with non-decreasing maps as morphisms. A. Connes defined in a similar way a cyclic object as a functor $\Lambda^\circ \to \mathcal{C}$, where the cyclic category Λ consists of objects

$$\{n\} = \text{roots of unity of degree } n+1.$$

Morphisms between these objects can be defined by one of the following equivalent ways. Let us denote the root $e^{2\pi i k/(n+1)}$ by k. Introduce on $\{n\}$ the cyclic order $0 < 1 < \cdots < n < 0$.

Variant 1. $\mathrm{Hom}(\{n\}, \{m\})$ is the set of homotopy classes of continuous cyclically non-decreasing maps $\varphi : S^1 \to S^1$ of degree 1 (here $S^1 = \{z \in \mathbb{C}, |z| = 1\}$) such that $\varphi(\{n\}) \subset \{m\}$ (homotopy must preserve this property).

Variant 2. A morphism $\{n\} \to \{m\}$ is a pair consisting of a map $f : \{n\} \to \{m\}$ and of a set σ of total orders on all fibers $f^{-1}(i)$, $i \in \{m\}$, such that the cyclic order on $\{n\}$ is induced by the cyclic order on $\{m\}$ and σ coincides with the initial cyclic order on $\{n\}$. The composition rule is $(g, \tau)(f, \sigma) = (gf, \tau\sigma)$, where $i < j$ with respect to $\tau\sigma$ if either $f(i) < f(j)$ with respect to τ, or $f(i) = f(j)$ and $i < j$ with respect to σ.

Let us note that a given $f : \{n\} \to \{m\}$ extends to a morphism (f, σ) if and only if f is cyclically non-decreasing, that is, $i < j < k < i$ in $\{n\}$ implies $f(i) < f(j) < f(k) < f(i)$ in $\{m\}$. Such an extension is unique unless f is constant, and for a constant f, σ can be chosen in $n+1$ ways.

Variant 3. Morphisms in Λ are described by the set of generators

$$\partial_n^i : \{n-1\} \to \{n\}, \quad \sigma_i^n : \{n+1\} \to \{n\}, \quad \tau_n : \{n\} \to \{n\}$$

satisfying the relations

$$\begin{aligned}
\partial_n^j \partial_{n-1}^i &= \partial_n^i \partial_{n-1}^{j-1} \quad \text{for} \quad i < j;\\
\sigma_n^j \sigma_{n+1}^i &= \sigma_n^i \sigma_{n+1}^{j+1} \quad \text{for} \quad i \le j;\\
\sigma_n^j \partial_{n+1}^i &= \begin{cases} \partial_n^i \sigma_{n-1}^{j-1} & \text{for } i < j,\\ \mathrm{Id} & \text{for } i = j, j+1,\\ \partial_n^{i-1} \sigma_{n-1}^j & \text{for } i > j+1; \end{cases}\\
\tau_n \partial_n^i &= \partial_n^{i-1} \tau_{n-1}, \quad i = 1, \dots, n;\\
\tau_n \sigma_n^i &= \sigma_n^{i-1} \tau_{n+1}, \quad i = 1, \dots, n;\\
(\tau_n)^{n+1} &= \mathrm{id}.
\end{aligned}$$

In terms of (f, σ), ∂_n^i omits i, σ_n^i covers i twice, $\tau(j) = j+1$. Finally, for σ_0^0 the fiber is ordered by $0 < 1$.

3.2. Autoduality of Λ. Given a morphism $(f,\sigma):\{n\}\to\{m\}$, we define $(f,\sigma)^*=(g,\tau):\{m\}\to\{n\}$ by the following prescription: $g(i)$ is the σ-minimal element of $f^{-1}(j)$, where j is the maximal (cyclically) element of $f(\{n\})$ preceding i. The order τ must be defined only when f is constant. In this case the image of f is, by definition, the τ-minimal element of $\{n\}$.

3.2.1. Lemma. *The map $\{n\}\to\{n\}^\circ$, $(f,\sigma)\mapsto(f,\sigma)^{\circ *}$, is an autodual isomorphism of Λ with Λ°.*

3.3. Cyclic Complex. Consider now a cyclic object of an abelian category $\mathcal{C}$: $E=(E_n,d_i^n,s_i^n,t_n)$, where d,s,t correspond to ∂,σ,τ. Put

$$d^n=\sum_{i=0}^{n}(-1)^i d_i^n : E_n\to E_{n-1};$$

$$d'^n=\sum_{i=0}^{n-1}(-1)^i d_i^n : E_n\to E_{n-1};$$

$$t=(-1)^n t_n : E_n\to E_n;$$

$$N=\sum_{i=0}^{n} t_i.$$

Denote by $E_.$ the complex (E_n,d^n) and consider the complex $(E_n/(1-t)E_n, d^n \bmod (1-t))$. It is well defined because $d(1-t)=(1-t)d'$. If the multiplication by $n+1$ is an automorphism of E_n for every n, we put, as in 2.11 of Chap. 1.

$$H_.^\lambda(E)=HC_.(E)=H_.(E/(1-t)E).$$

In the general case, denote by $C_.E$ the complex associated with the bicomplex

$$\begin{tikzcd}
{} \arrow[d, "d"'] & {} \arrow[d, "-d'"'] & {} \arrow[d, "d"'] & \\
E_2 \arrow[d, "d"'] & E_2 \arrow[l, "1-y"'] \arrow[d, "-d'"'] & E_2 \arrow[l, "N"'] \arrow[d, "d"'] & \cdots \arrow[l, "1-t"'] \\
E_1 \arrow[d, "d"'] & E_1 \arrow[l, "1-y"'] \arrow[d, "-d'"'] & E_1 \arrow[l, "N"'] \arrow[d, "d"'] & \cdots \arrow[l, "1-t"'] \\
E_0 & E_0 \arrow[l, "1-y"'] & E_0 \arrow[l, "N"'] & \cdots \arrow[l, "1-t"']
\end{tikzcd}$$

and put $HC_.(E)=H(C_.E)$. In the situation of the preceding paragraph, lines of this bicomplex are exact everywhere except of the leftmost place, so that the new definition reduces to the old one.

A cocyclic object is a functor $E:\Lambda\to\mathcal{C}$. The dual version of the previous constructions lead to a bicomplex of the same structure but with reversed arrows, the associated complex $C^{\cdot}E$ and the cyclic cohomology $HC^{\cdot}(E)$.

3.4. Cyclic Cohomology as a Derived Functor. Consider the category $\mathcal{C}$ of abelian groups or modules over a commutative ring k. It has internal Hom's and an object $\mathbb{1}$ sich that $\mathrm{Hom}(\mathbb{1}, A) = A$ for all A (for brevity, we make no distinction in notation between the usual Hom and the internal one). Namely, in the category of abelian groups, $\mathbb{1} = \mathbb{Z}$, and in the category of k-modules, $\mathbb{1} = k$.

The category $\Lambda\mathcal{C}$ of cocyclic objects of $\mathcal{C}$ is an abelian category. Denote by $\Lambda\mathbb{1}$ the object of $\Lambda\mathcal{C}$ with all components being equal to $\mathbb{1}$ and all structure morphisms being identical.

3.4.1. Theorem. *For any object $E \in \Lambda\mathcal{C}$ we have*

$$HC^k(E) = \mathrm{Ext}^k_{\Lambda\mathcal{C}}(\Lambda\mathbb{1}, E).$$

Sketch of the proof. Denote by $L_{ik} \in \mathrm{Ob}(\mathcal{C})$ the direct sum of objects $\mathbb{1}$ indexed by the set $\mathrm{Hom}(\{i\}, \{k\})$. With i fixed, we can consider L_i with components L_{ik} as an object of $\Lambda\mathcal{C}$. In fact, a morphism $\psi : \{k\} \to \{l\}$ induces a map $\mathrm{Hom}_\Lambda(\{i\}, \{k\}) \to \mathrm{Hom}_\Lambda(\{i\}, \{l\})$ on the indices of summands of L_{ik}, which we replace by identical morphisms on the summands themselves.

Similarly, a morphism $\varphi : \{i\} \to \{j\}$ in Λ induces a morphism $L(\varphi) : L_j \to L_i$ of the cocyclic complexes. Therefore, we can construct a bicomplex similar to $C_. E$ above, with E_i, d, d', t, N replaced by $L_i, L(d), L(d'), L(t), L(N)$ respectively.

$$\begin{tikzcd}
{} \arrow[d, "L(d)"'] & {} \arrow[d, "-L(d')"'] & {} \arrow[d, "L(d)"'] & \\
L_1 \arrow[d, "L(d)"'] & L_1 \arrow[l, "L(1-t)"'] \arrow[d, "-L(d')"'] & L_1 \arrow[l, "L(N)"'] \arrow[d, "L(d)"'] & \cdots \arrow[l] \\
L_0 & L_0 \arrow[l, "L(1-t)"'] & L_0 \arrow[l, "L(N)"'] & \cdots \arrow[l]
\end{tikzcd}$$

The complex associated to this bicomplex is a resolution of $\Lambda\mathbb{1}$ in $\Lambda\mathcal{C}$. Since for any cocyclic object E we have $\mathrm{Hom}_{\Lambda\mathcal{C}}(L_i, E) = E_i$, this resolution consists of projective objects.

Calculating $\mathrm{Ext}^k_{\Lambda\mathcal{C}}(\Lambda\mathbb{1}, E)$ using this resolution, we get the complex whose cohomology is, by definition, $HC^k(E)$.

3.4.2. Remarks. a. Of course, Theorem 3.4.1 is valid for any abelian category with internal Hom's and an object $\mathbb{1}$.

b. One can give a similar interpretation of cyclic homology. Namely, in the category k-mod, $HC_.(E) = \mathrm{Tor}_i^{k[\Lambda^\circ]}(\Lambda\mathbb{1}, E)$, where $k[\Lambda^\circ]$ is a semiring generated by morphisms of Λ° over k, and $\Lambda\mathbb{1}$ (resp. E) is considered as a right (resp. left) module over this semiring.

3.5. Connection with Hochschild Homology. Let A be a k-algebra, $A \supset k$. Define a cyclic k-module A^c whose i-th component is $A^{\otimes(i+1)}$, and structure morphisms are

$$
\begin{aligned}
d_i^n(a_0 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \quad 0 \le i < n, \\
d_n^n(a_0 \otimes \ldots \otimes a_n) &= a_n a_0 \otimes a_1 \ldots \otimes a_{n-1}, \\
s_i^n(a_0 \otimes \ldots \otimes a_n) &= a_0 \otimes \ldots \otimes a_i \otimes \mathbb{1} \otimes a_{i+1} \otimes \ldots \otimes a_n, \\
t_n(a_0 \otimes \ldots \otimes a_n) &= a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}.
\end{aligned}
$$

The border column of the associated bicomplex (see 3.4) is the Hochschild complex $C_.(A, A)$ (cf. 2.10 in Chap. 1). Let us denote by L the entire bicomplex, by S its endomorphism of the shift two columns to the right, and, finally, by K the bicomplex coinciding with L in the first two columns and containing zero elements elsewhere. Clearly, we have the exact sequence

$$0 \to K \to L \to L/K \to 0$$

and two isomorphisms $H_.(\Delta K) = H_.(A, A)$ and $S : L \to L/K$. Passing to the ordinary complexes we get the following result.

3.5.1. Theorem. *The cyclic homology and the Hochschild homology fit into the exact sequence*

$$\cdots \to H_n(A, A) \to HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{\delta} H_{n-1}(A, A) \to \ldots$$

We write $HC_n(A)$ instead of $HC_n(A^c)$ to conform with the notation of 2.10 in Chap. 1.

In the same way we obtain an exact sequence for cyclic cohomology:

$$\cdots \to H^n(A, A^*) \to HC^{n-1}(A) \to HC^{n+1}(A) \to H_{n+1}(A, A^*) \to \ldots$$

3.6. Relative Cyclic Homology of Algebras. Let k be a field of characteristic zero. A *differential graded algebra* (dga) is a k-algebra $R = \oplus_{n \in \mathbb{Z}} R_n$ with $R_m R_n \subset R_{m+n}$ endowed with a differential $\partial : R_n \to R_{n-1}$, $\partial^2 = 0$, satisfying the graded Leibniz identity. Morphisms of DGA are algebra morphisms of degree zero commuting with ∂.

A DGA R is called *free* if it is isomorphic to the tensor algebra of a graded linear space over k (without restrictions on the differential). More generally, consider a DGA morphism $R_1 \to R_2$. We say that R_2 is free over R_1 if this morphism is isomorphic to a canonical morphism $R_1 \to R_1 * S$, where S is free and $*$ denotes the amalgamated product.

The category of associative k-algebras is embedded into the category of those DGA that are concentrated in degree zero and have $\partial = 0$.

Consider a k-algebra homomorphism $f : A \to B$. A resolution of B over A is a commutative diagram

$$
\begin{tikzcd}
 & R \arrow[d, "\pi"] \\
A \arrow[ur, "i"] \arrow[r, "f"'] & B
\end{tikzcd}
$$

where R is a free DGA over A, and p is a surjective morphism of DGA and quasiisomorphism of complexes. Every morphism admits a resolution.

Consider the resolution R as a complex. From the Leibniz formula, it follows that $[R, R] + i(A)$ is a subcomplex (here $[R, R]$ means the linear span of supercommutators $[r, s] = rs - (-1)^{\deg r \deg s} sr$). Put

$$HC_n(A \to B) = H_{n+1}(R/([R, R] + i(A))).$$

3.6.1. Theorem. a. *This homology does not depend on the choice of a resolution and defines a covariant functor on the category of A-algebra morphisms.*

b. *Every commutative triangle*

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur] \arrow[r] & C \end{tikzcd}$$

of DGA *morphisms determines an exact sequence*

$$\cdots \to HC_n(A \to B) \to HC_n(A \to C) \to$$
$$\to HC(B \to C) \to HC_{n-1}(A \to B) \to \ldots$$

c. *There is a canonical isomorphism*

$$HC_n(A \to 0) = HC_n(A).$$

3.7. Relation with the de Rham Cohomology. Let k be a field of characteristic zero, A a finitely generated commutative k-algebra. By definition, the A-module of 1-differentials $\Omega^1 A$ (or, more precisely, $\Omega^1 A/k$) is the universal A-module fitting into a diagram of k-linear spaces $d : A \to \Omega^1 A$, where d satisfies the Leibniz formula. The algebraic de Rham complex of A is $\Lambda^{\cdot}_A(\Omega^1 A)$ with d extended by the Leibniz rule and $d = 0$. We shall denote it $\Omega^{\cdot} A$.

If $k = \mathbb{C}$, A is a ring of polynomial functions on a non-singular affine variety V, the cohomology of $\Omega^{\cdot} A$ can be identified with $H^*(V, \mathbb{C})$. In the general case this is wrong, and in order to calculate $H^*(V, \mathbb{C})$ algebraically one must utilise also the de Rham complex of an ambient affine space.

To be more precise, let B a polynomial algebra over k, $B \to A$ a surjection with the kernel I which defines V as a closed subscheme in $\operatorname{Spec} B$. Consider a filtration of $\Omega^{\cdot}$ consisting of the following subcomplexes:

$$F^n \Omega^j B = \begin{cases} \Omega^j B & \text{for } n \leq j, \\ I^{n-j} \Omega^j B & \text{for } n > j. \end{cases}$$

Put

$$H^{\cdot}_{\text{crys}}(A; n) = H^{\cdot}(\Omega^{\cdot} B / F^{n+1} \Omega^{\cdot} B).$$

These groups do not depend on the choice of the morphism $B \to A$.

A. Grothendieck proved that $H^{\cdot}(\operatorname{lim\,ind} \Omega^{\cdot} B / F^{n+1} \Omega^{\cdot} B)$ are canonically isomorphic to $H^{\cdot}(\operatorname{Spec} A, \mathbb{C})$. On the other hand, the cyclic homology of A is connected with the crystal cohomology groups themselves.

3.7.1. Theorem. a. *For all n, i with $0 \le 2i \le n$ there exist functorial maps*

$$\chi_{n,i} : HC_n(A) \to H^{n-2i}_{\text{crys}}(A; n-i)$$

for which the following diagrams are commutative

$$\begin{array}{ccc} HC_n(A) & \xrightarrow{\chi_{n,i}} & H^{n-2i}_{\text{crys}}(A; n-i) \\ S \big\downarrow & & \big\downarrow \text{reduction} \\ HC_{n-2}(A) & \xrightarrow{\chi_{n-2,i-1}} & H^{n-2i}_{\text{crys}}(A; n-1-i) \end{array}$$

(S is defined in 3.5).

b. *Suppose that for a n appropriate surjection $B \to A$ the ideal of equations I is locally generated by a regular sequence. Then the map*

$$\oplus\chi_{n,i} : HC_n(A) \to \bigoplus_{0 \le 2i \le n} H^{n-2i}_{\text{crys}}(A; n-i)$$

is an isomorphism.

3.7.2. The Case of Non-Singular Spectrum. If $V = \operatorname{Spec} A$ is a reduced smooth scheme, we have

$$H^{n-2i}_{\text{crys}}(A; m) = \begin{cases} H^n(\Omega^{\cdot} A) = H_{DR}(A) & \text{for } n < m, \\ \Omega^n(A)d/\Omega^{n-1}(A) & \text{for } n = m. \end{cases}$$

Therefore,

$$HC_n(A) \simeq \Omega^n A/d\Omega^{n-1} A \oplus \Big(\bigoplus_{i \ge 1} H^{n-2i}_{DR}(V)\Big).$$

3.8. Lie Algebra of Infinite Matrices and Its Homology. For an arbitrary Lie algebra $\mathfrak{g}$ over a field k, its homology space $H_{\cdot}(\mathfrak{g}, k)$ has a canonical comultiplication $\Delta : H_{\cdot}(\mathfrak{g}, k) \to H_{\cdot}(\mathfrak{g}, k) \otimes H_{\cdot}(\mathfrak{g}, k)$. It is induced by the comultiplication of the chain algebra $\Lambda^{\cdot}(\mathfrak{g}) \to \Lambda^{\cdot}(\mathfrak{g}) \otimes \Lambda^{\cdot}(\mathfrak{g})$ given on 1-chains by $\xi \to 1 \otimes \xi + \xi \otimes 1$. This comultiplication is coassociative and supercommutative with respect to the natural $\mathbb{Z}_2$-grading.

Consider now an associative algebra A and denote by $\mathrm{gl}(A)$ the Lie algebra of infinite finitely supported matrices over A of the format a_{ij}, $i, j \ge 1$. There is a "direct sum" operation

$$(a_{ij}) \oplus (b_{ij}) = (c_{ij}), \quad c_{ij} = \begin{cases} a_{i/2,j/2} & \text{if } i, j \equiv 1 \mod 2, \\ b_{(i+1)/2,(j+1)/2} & \text{if } i, j \equiv 1 \mod 2, \\ 0 & \text{otherwise.} \end{cases}$$

It induces on $H_{\cdot}(\mathfrak{g}, k)$ a supercommutative associative multiplication compatible with the comultiplication described above, so that $H_{\bullet}(\mathfrak{g}, k)$ becomes a Hopf algebra.

Let us recall that an element $x \in H_{\cdot}(\mathfrak{g}, k)$ of a bialgebra is called *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

In what follows we assume that the characteristic of k is zero.

3.8.1. Theorem. *A graded Hopf algebra with supercommutative multiplication and comultiplication is isomorphic to the symmetric (super) algebra generated by its primitive elements.*

3.8.2. Theorem. *Denote by* $\operatorname{Prim} H_n(\mathrm{gl}(A), k)$ *the space of primitive elements of degree* n. *There is a canonical isomorphism*

$$HC_{n-1}(A) = \operatorname{Prim} H_n(\mathrm{gl}(A), k).$$

In the same vein, $H^{\cdot}(\mathrm{gl}(A), k)$ can be endowed with a structure of a Hopf algebra, so that the following dual statement is valid.

3.8.3. Theorem. *There is a canonical isomorphism*

$$HC^{n-1} = \operatorname{Prim} H^n(\mathrm{gl}(A), k).$$

3.8.4. Sketch of Proof of Theorem 3.8.1. The proof consists of several steps.

a. The homology of the chain complex $C_{.}(\mathrm{gl}(n, A), k)$ coincides with the homology of the complex of the coinvariants $C'_{.}(\mathrm{gl}(n, A), k)$ with respect to the adjoint action of $\mathrm{gl}(n, k)$ because the latter is trivial on the homology, and completely reducible on the chains.

b. One can calculate $C'_{.}(\mathrm{gl}(n, A), k)$ using a classical result due to H. Weyl. For primitive elements, the answer can be phrased as follows. Consider the map

$$(\otimes_1^m)/\operatorname{Im}(1-t) \to C_{.}(\mathrm{gl}(n, A)), \qquad n \geq m$$

transforming the class of $r_1 \otimes \cdots \otimes r_m$ into $E_{12}r_1 \wedge E_{23}r_2 \wedge \cdots \wedge E_{m1}r_m$. It is well defined and induces an isomorphism

$$(\otimes_1^m)/\operatorname{Im}(1-t) \to \operatorname{Prim} C_{.}(\mathrm{gl}(n, A)).$$

c. The isomorphism defined above commutes with the differential which, in particular, transforms the primitive elements into the primitive ones.

d. Every primitive homology class in $H_{.}(\mathrm{gl}(A), k)$ is represented by a primitive cycle in $C'_{.}(\mathrm{gl}(n, A), k)$ for an appropriate n.

3.9. Dihedral Homology. Theorem 3.8.2 admits an extension to the stable orthogonal and symplectic Lie algebras, in which case, however, the cyclic homology should be replaced by the dihedral one. We shall only state the barest minimum of facts.

Let A be a coalgebra endowed with an involution $a \to a^*$ which is k-linear and reverses the multiplication.

$$(a^*)^* = a; \quad (\lambda a)^* = \lambda a^* \quad \text{for} \quad \lambda \in k, \quad a \in A;$$
$$(ab)^* = b^* a^*; \quad (a+b) = a^* + b^*.$$

We extend its action on the tensor algebra of A putting

$$h(a_0 \otimes \cdots \otimes a_n) = (-1)^{(n+1)(n+2)/2} a_n^* \otimes \cdots \otimes a_1^* \otimes a_0^*.$$

The action of t and h on $\otimes_{n+1}A$ generates the dihedral group. The h-invariant part of the cyclic complex $C^{\lambda}_{\cdot}(A)$ is a subcomplex whose homology is denoted $HD_n(A)$.

3.10. Orthogonal and Symplectic Lie Algebras. Let $\varepsilon = \pm 1$, t is the transposition, $*$ acts on matrices coefficientwise. The involution on $\mathrm{gl}(2n, A)$ defined in the block notation by

$$\begin{pmatrix} X & Y \\ Z & U \end{pmatrix}^T = \begin{pmatrix} U^{t*} & \varepsilon Y^{t*} \\ \varepsilon Z^{t*} & X^{t*} \end{pmatrix}$$

satisfies the condition $[B, C]^T = -[B^T, C^T]$. Therefore,

$$O_\varepsilon(n, A) = \{B \in \mathrm{gl}(2n, A) \mid B^T + B = 0\}$$

is a Lie subalgebra of $\mathrm{gl}(2n, A)$. Passing to the limit as $n \to \infty$, we obtain the stable algebras $O_\varepsilon(A)$.

3.10.1. Theorem. *There are canonical isomorphisms*

$$HD_{n-1}(A) = \mathrm{Prim}\, H_n(O_\varepsilon(A)).$$

§ 4. Non-Commutative Differential Geometry

4.1. Cycles. Let X be a compact oriented differentiable n-dimensional manifold, $\Omega^{\cdot}(X)$ its de Rham complex, $\int : \Omega^n(X) \to \mathbb{R}$ the linear operator "integration of volume forms". The algebraic properties of the triple $(\Omega^{\cdot}, d, \int)$ encode an important part of the topological properties of X. Completing the picture by taking into consideration also the functorial properties, one can obtain even more information about the topology and geometry.

A. Connes suggested to consider a non-commutative version of such a triple as a basic object of the "non-commutative differential geometry" and discovered the central role of cyclic (co)homology in this theory. We explain below some of his results.

4.1.1 Definition. An *n-dimensional cycle* is a triple $(\Omega, d, \int)$, where $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ is a graded $\mathbb{C}$-algebra, d its graded differential of degree 1 with $d^2 = 0$, and $\int : \Omega^n \to \mathbb{C}$ is a linear functional with the following properties:

$$\int d\omega = 0, \qquad \omega \in \Omega^{n-1},$$

$$\int \omega\omega' = (-1)^{\deg\omega \cdot \deg\omega'} \int \omega'\omega.$$

In other words, $\int$ is a closed graded trace.

4.1.2. Examples. a. Let A be an arbitrary $\mathbb{C}$-algebra, $\mathrm{tr} : A \to \mathbb{C}$ a linear functional with $\mathrm{tr}([a,b]) = 0$. Put $\Omega^0 = A$, $W^i = 0$ for $i > 0$, $d = 0$, $\int = \mathrm{tr}$. This is a 0-cycle.

b. Let X be a n-dimensional compact differentiable manifold, C a closed q-current on X, $q < n$. Put $\Omega = \bigoplus_{i=0}^{q} \Omega^i(X)$, $\int \omega = \langle C, \omega \rangle$ for $\omega \in \Omega^q$. This gives a q-cycle.

c. For a cycle $(\Omega, d, \int)$, multiplying $\int$ by -1 we obtain the cycle with reversed orientation. The direct sum of n-cycles is defined in an obvious way.

The following example shows how cycles emerge in functional analysis.

4.2. Fredholm Modules. Let H be a separable Hilbert space, $\mathcal{L}(H)$ its bounded operator algebra, $\mathcal{L}^\infty(H)$ the ideal of compact operators. For $T \in \mathcal{L}^\infty(H)$, denote by $\mu_n(T)$ the n-th singular number of T, that is, the n-th eigenvalue of $|T| = (T^*T)^{1/2}$. For $1 < p < \infty$ put

$$\mathcal{L}^p(H) = \left\{ T \in \mathcal{L}^\infty(H) \quad \text{such that } \sum_n \mu_n(T) < \infty \right\}.$$

These two-sided ideals in $\mathcal{L}(H)$ are called the *Schatten ideals*. They can be alternatively defined as

$$\mathcal{L}^p(H) = \{T \in \mathcal{L}(H) \quad \text{such that } \mathrm{tr}\,|T|^p < \infty\},$$

where $\mathrm{tr}\, T = \sum \langle T\xi_n, \xi_n \rangle$, $\{\xi_n\}$ is an orthonormal basis of H. We shall topologize $\mathcal{L}^p(H)$ using the norm

$$\|T\| = \left(\sum \mu_n(T)^p \right)^{1/p}.$$

4.2.1. Definition. Let A be a $\mathbb{C}$-algebra (not necessarily commutative). An *n-summable Fredholm A-module* is a $\mathbb{Z}_2$-graded left A-module $H = H_0 \oplus H_1$ endowed with an odd $\mathbb{C}$-linear map $F : H \to H$ with the following properties.

a. H is a Hilbert space, and multiplication by an arbitrary element of A is a bounded operator;

b. F is bounded, $F^2 = 1$, and $[F, a] \stackrel{\mathrm{df}}{=} Fa - aF \in \mathcal{L}^n(H)$ for all $a \in A$.

4.3. Cycle Associated to a Fredholm Module. Let H be a summable Fredholm module. Put $\Omega^0 = A$. Denote by Ω^q the closed span in $\mathcal{L}^{n/q}(H)$ of the operators $(a^0 + \lambda \cdot 1)[F, a][F, a^2] \dots [F, a^q]$, where $a^i \in A$, $\lambda \in \mathbb{C}$. Define $d : \Omega^0 \to \Omega^1$ by $da = i[F, a]$. One can check that d can be uniquely extended to a graded continuous differential on $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ (with the multiplication $\Omega^j \times \Omega^h \to \Omega^{j+k}$ coinciding with operator composition). The extension is given by the same formula $d\omega = i[F, \omega]$. Finally, for $\omega \in \Omega^n$ put

$$\int_\omega = (-1)^{\deg \omega}\, \mathrm{tr}\, \omega.$$

This construction has the following abstract algebraic version.

4.4. Universal Algebra of Differential Forms. Let A be an associative $\mathbb{C}$-algebra, with or without identity. Consider the category of ring homomorphisms $f : A \to \Omega^{\cdot}$, where $\Omega^{\cdot} == (\bigoplus_{i\geq 0} \Omega^i, d)$ a differential graded algebra with identity, $d^2 = 0$. We do not require that $f(1) = 1$ even if 1 exists.

In the category of such homomorphisms there is a universal (initial) object $\Omega^{\cdot}(A)$. Here is its direct construction.

a. $\Omega^0 = \widetilde{A} = A \oplus \mathbb{C}1\!\!I$ (extension of A by $1\!\!I$).

b. $\Omega^n = \widetilde{A} \otimes A^{\otimes n}$ (tensor product over $\mathbb{C}$).

c. $d((a^0 + \lambda 1\!\!I) \otimes a^1 \otimes \cdots \otimes a^n) = 1\!\!I \otimes a^0 \otimes a^1 \otimes \cdots \otimes a^n$. In particular, $da = 1\!\!I \otimes a$ for $a \in A$.

d. The multiplication $\Omega^m \otimes \Omega^n \to \Omega^{m+n}$ is uniquely defined by the Leibniz formula and the condition that the left multiplication on $\Omega^0 = \widetilde{A}$ is the standard one. For example, put $\widetilde{a^0} = a^0 + \lambda 1\!\!I$, $\widetilde{b^0} = b^0 + \mu 1\!\!I$. Then

$$
\begin{aligned}
(\widetilde{a^0} \otimes a^1)(\widetilde{b^0} \otimes b^1) &= (\widetilde{a^0} da^1)(\widetilde{b^0} db^1) = \widetilde{a^0}(da\widetilde{b^0})db^1 \\
&= \widetilde{a^0}[d(a^1\widetilde{b^1} - a^1 d\widetilde{b^0}]db^1 = \widetilde{a^0}(a^1\widetilde{b^0})db^1 - (\widetilde{a^0}a^1)db^0 db^1 \\
&= \widetilde{a^0} \otimes a^1\widetilde{b^0} \otimes b^1 - \widetilde{a^0}a^1 \otimes b^0 \otimes b^1.
\end{aligned}
$$

Here is the general formula for the right multiplication by A:

$$
(\widetilde{a^0} \otimes a^1 \otimes \ldots \otimes a^n)b = \sum_{j=0}^{n} (-1)^{n-j} \widetilde{a^0} \otimes \ldots \otimes a^j a^{j+1} \otimes \ldots \otimes b.
$$

4.5. Cycles $(\Omega^{\leq q(A)}, d, \int)$. Let $\Omega^{\cdot}(A)$ be the algebra constructed in 4.4. Put $\Omega^{\leq q}(A) = \bigoplus_{j=0}^{q} \Omega^j(A)$. Given a linear functional $\int : \Omega^q(A) \to \mathbb{C}$ with the properties 4.1.1, define $\tau : A^{\otimes q} \to A^*$ by the formula

$$
\tau(a^1 \otimes \ldots \otimes a^q)(a^0) = \int a^0 da^1 \ldots da^q.
$$

4.5.1. Theorem. *This construction defines a bijection between closed graded traces on $\Omega^q(A)$ and cyclic q-cocycles of A* (cf. Chap. 1, 2.11).

More generally, a q-cycle over an algebra A is a pair consisting of a cycle $\Omega, d, \int)$ and a homomorphism $\rho : A \to \Omega^0$. The functional τ defined by

$$
\tau(a^1, \ldots, a^q)(a^0) = \int \rho(a^0)\, d(\rho(a^1)) \ldots d(\rho(a^q)),
$$

is called the *character* of this cycle. From the universality property of $\Omega^{\cdot}(A)$, it follows that any character τ is induced by a cyclic cocycle.

We can now describe non-commutative versions of two differential geometric notions: cobordism and connection.

4.6. Cobordism. An $(n+1)$-chain is a triple $(\Omega, \partial\Omega, \int)$, where

a. $\Omega = \overset{n-1}{\underset{i=0}{\oplus}} \Omega^i$, $\partial\Omega = \overset{n-1}{\underset{i=0}{\oplus}} (\partial\Omega)^i$, are graded differential algebras endowed with a surjective morphism $r : \Omega \to \partial\Omega$ of degree zero.

b. $\int : \Omega^{n+1} \to \mathbb{C}$ is a trace satisfying the condition

$$\int d\omega = 0 \quad \text{if} \quad r(\omega) = 0.$$

The *boundary* of this chain is, by definition, the n-cycle $(\partial\Omega, d, \int')$ where $\int' \omega' = \int d\omega$ if $r(\omega) = \omega'$.

4.6.1. Definition. Cycles Ω', Ω'' are said to be *cobordant* if there is a chain with the boundary $\Omega' \oplus \widetilde{\Omega''}$, where tilda means the reversed orientation.

One can define in a similar way the relative cobordism relation over an algebra A: the chain itself and homomorphisms should be defined over A.

4.6.2. Lemma. *Cobordism is an equivalence relation.*

4.6.3. Theorem. *Let Ω', Ω'' be two n-cycles over A, τ', τ'' their characters. Let $B : H^{n+1}(A, A^*) \to HC^n(A)$ be the morphism described in* 3.5. *Then Ω', Ω'' are cobordant if and only if the difference of their cyclic cohomology classes belongs to* $\mathrm{Im}(B)$.

4.7. Connections. Let $\rho : A \to \Omega$ be a cycle over A, E a projective right A-module of finite rank. A *connection* on E is, by definition, a $\mathbb{C}$-linear map $\nabla : E \to E \underset{A}{\otimes} \Omega^1$ with the following property:

$$\nabla(ea) = (\nabla e)a + a \otimes d(\rho(a)), \qquad e \in E, \quad a \in A.$$

Assume now that A has an identity.

Put $\widetilde{E} = E \underset{A}{\otimes} \Omega$. Extend ∇ to $\widetilde{E}$ by the formula $\nabla(e \otimes \omega) = (\nabla e) \otimes \omega + e \otimes d\omega$. Consider the graded $\mathrm{End}_A(E)$-algebra $\mathrm{End}_\Omega(\widetilde{E})$. For an element T of this algebra put

$$\delta(T) = \nabla T - (-1)^{\deg T} T \nabla.$$

Define the trace functional on $\mathrm{End}_\Omega(\widetilde{E})^n$ as the compsition of the matrix trace and $\int : \Omega^n \to \mathbb{C}$. The triple $(\mathrm{End}_\Omega(\widetilde{E}), \delta, \int)$ fails to be an $\mathrm{End}_A(E)$-cycle only because $\delta^2 \neq 0$. More precisely, ∇^2 on $\widetilde{E}$ is the multiplication by a non-commutative curvature form θ, and one can check that

$$\delta^2 T = \theta T - T\theta$$

for all $T \in \mathrm{End}_\Omega(\widetilde{E})$.

The next construction axiomatizes this situation and simultaneously shows how to correct our triple, making it a cycle.

Consider an object $(\Xi, \delta, \theta, \int)$ consisting of a graded algebra $\Xi = \overset{n}{\underset{i=0}{\oplus}} \Xi^i$ with a derivation δ of degree 1, a closed trace $\int$, and a form $\theta \in \Xi^2$ such that $\delta\theta \int = 0$ and $\delta T = [\theta, T]$ for all $T \in \Xi$.

Adjoin to Ξ an element X of degree 1 subject to the relations $X^2 = 0$, $\omega_1 X \omega_2 = 0$ for all $\omega_1, \omega_2 \in \Xi$.

Let $\Omega' = \Xi[X]$. Define d' and $\int'$ setting $d'X = 0$ and

$$d'\omega = \delta(\omega) + X\omega - (-1)^{\deg \omega}\omega X \qquad \text{for } \omega \in \Xi;$$

$$\int' (\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X) = \int \omega_{11} - (-1)^{\deg \omega_{11}} \int \omega_{22}\theta,$$

where $\deg(\omega_{11}) = n = \deg(\omega_{12}) + 1 = \deg(\omega_{21}) + 1 = \deg(\omega_{22}) + 2$.

4.7.1. Lemma. *$(\Omega', d', \int')$ is a cycle.*

Now we apply this construction to $(\mathrm{End}_{\Omega}(\widetilde{E}), \delta, \theta, \int)$. In this way, starting with a pair (projective A-module E, cycle on A) we obtain a cycle on $\mathrm{End}_A(E)$ depending on the connection. A. Connes proves that its character depends only on the class of E in $K_0(A)$ and the character of the initial cycle. Moreover, $H_\lambda^{\cdot}(\mathrm{End}_A(E))$ is canonically isomorphic to $H_\lambda^{\cdot}(A)$.

4.7.2. Theorem. *This construction determines a biadditive multiplication $K_0(A) \times H_\lambda^{\cdot}(A) \to H_\lambda^{\cdot}(A)$. When A is commutative this makes $H_\lambda^{\cdot}(A)$ a $K_0(A)$-module.*

§ 5. (Co)Homology of Discrete Groups

5.1. Topological Definition of the Group Cohomology. Given a group G acting upon a topological space X, denote by Y the orbit space G with induced topology and by $\pi : X \to Y$ the canonical projection.

5.1.1. Theorem. *Assume that X is contractible, π is a fibration, and G acts freely upon X. Then for any abelian group A we have $H^n(G, A) = H^n(Y, A)$, $H_n(G, A) = H_n(Y, A)$, where A is considered as a trivial G-module and equalities denote functorial isomorphisms.*

A proof is based on the fact that in this situation the singular chain complex of X

$$\cdots \to S_1(X) \to S_0(X) \to \mathbb{Z} \to 0$$

is a resolution of $\mathbb{Z}$ consisting of free $\mathbb{Z}[G]$-modules, and the G-invariant chains $S_n(X)$ can be identified with $S_n(Y)$.

We must add that $\pi_1(Y) = G$, $\pi_i(Y) = 0$ for $i \geq 2$, so that Y is a $K(G,1)$-space. Since all these spaces are homotopically equivalent and, in particular, have the same (co)homology, the Theorem 5.1.1 furnishes topological tools for calculating $H^*(G, A)$ as soon as we have a concrete construction of $K(G,1)$, e.g., the classifying space BG (Chap. 1, 2.7). An economical resolution can be sometimes obtained using CW-complexes.

Let X be a CW-complex on which G acts by a permutation of cells (preserving orientation). Clearly, this action extends to chains.

5.1.2. Proposition. *If the action of G upon cells of X is free and X is contractible then the chain complex*

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X)$$

is a free resolution of $\mathbb{Z}[G]$-module Z (the augmentation map $\varepsilon : C_0(X) \to \mathbb{Z}$ takes value 1 on each 0-cell).

Since for a G-module F we have

$$H^n(G,F) = \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},F), \quad H_n(G,F) = \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},F),$$

this resolution can be used to calculate (co)homology of G-modules. We shall give below some examples.

5.2. Free Groups. Bouquet $B(I)$ of a family of circles indexed by a set I is $K(F(I),1)$, where $F(I)$ is a free group generated by I. Hence

$$H_n(F(I),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0, \\ F(I)/[F(I),F(I)] & \text{for } n=1, \\ 0 & \text{for } n>1. \end{cases}$$

5.3. Groups with One Relation. Let $F(I)$ be a free group as above, $r \in F(I)$, N the minimal normal subgroup containing r, $G = F(I)/N$. The element r defines a map $\rho : S^1 \to B(I)$ for which r is the class $\rho(S^1)$ in $\pi_1(B(I)) = F(I)$. Denote by Y the result of glueing a 2-cell to $B(I)$ with boundary r.

5.3.1. Theorem. *If r is primitive in $F(I)$ (i.e. not a power of another element), then Y is $K(G,1)$.*

It follows that with this condition we have

$$H_n(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0, \\ F(I)/[F(I),F(I)] & \text{for } n=1, \\ 0 & \text{for } n\geq 3 \end{cases}$$

and

$$H_2(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r \in [F(I),F(I)], \\ 0 & \text{otherwise.} \end{cases}$$

This result is applicable to the fundamental group of a compact oriented surface of genus $g \geq 1$ for which $I = \{1,\dots,2g\}$, $r = A_1B_1A_1^{-1}B_1^{-1}\dots A_gB_g A_g^{-1}B_g^{-1}$.

5.4. Abelian Groups. a. For $G = \mathbb{Z}$ acting on $\mathbb{R}^n = E$ by translations we obtain n-torus as the $K(G,1)$-space. Therefore

$$H_.(G,\mathbb{Z}) = \Lambda^{\cdot}_{\mathbb{Z}}(\mathbb{Z}^n).$$

b. For $G = \mathbb{Z}_2$ acting by the reflection upon $S^\infty = \bigcup_{n=0}^\infty S^n$ we have $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$ and $H_{2i+1}(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}_2$, $H_0(\mathbb{Z}_2, \mathbb{Z}) = \mathbb{Z}$, $H_{2i}(\mathbb{Z}_2, \mathbb{Z}) = 0$ for $i \geq 1$.

To extend this to general cyclic groups, look first at the action of $\mathbb{Z}_n = \{t^i\}$ upon the circle divided into n equal parts. This gies an acyclic complex of $\mathbb{Z}[\mathbb{Z}_n]$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}_n] \to \mathbb{Z} \to 0,$$

where the second arrow maps 1 to $N = \sum t^i$. Now, without trying to visualize the classifying space, we can simply glue these complexes together getting an infinite resolution of $\mathbb{Z}$

$$\ldots \to \mathbb{Z}[\mathbb{Z}_n] \xrightarrow{N} \mathbb{Z}[\mathbb{Z}_n] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}_n] \to \ldots,$$

which shows that

$$H_i(\mathbb{Z}_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_n & \text{for } i \equiv 1 \mod 2, \\ 0 & \text{for } i \equiv 0 \mod 2, \quad i \geq 2. \end{cases}$$

5.5. Amalgamation and the Mayer-Vietoris Sequence. Consider a diagram of the group homomorphisms $G_1 \overset{\alpha_1}{\leftarrow} A \overset{\alpha_2}{\rightarrow} G_2$. In the category of groups, it can be extended to a cocartesian square defining the amalgam $G_1 \underset{A}{*} G_2$.

5.5.1. Theorem. *Every cocartesian square of groups in which α_1, α_2 are inclusions, is isomorphic to the diagram of fundamental groups of a square of $K(\cdot, 1)$-spaces of the form*

$$\begin{tikzcd} X_1 \cap X_2 = Y \arrow[r, hook] \arrow[d, hook] & X_2 \arrow[d, hook] \\ X_1 \arrow[r, hook] & X_1 \underset{Y}{\cup} X_2 \end{tikzcd}$$

5.5.2. Corollary. *From the Mayer-Vietoris sequence for this square of $K(\cdot, 1)$-spaces we get the following exact sequence for amalgams:*

$$\cdots \to H_n(A, \mathbb{Z}) \to H_n(C_1, \mathbb{Z}) \oplus H_n(G_2, \mathbb{Z}) \to$$
$$\to H_n(G_1 * G_2, \mathbb{Z}) \to H_{n-1}(A, \mathbb{Z}) \to \ldots$$

5.5.3. Example. $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$. This presentation corresponds to the generators $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Using the Mayer-Vietoris sequence we obtain

$$H_i(PSL(2, \mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_3 & \text{for } i \equiv 1 \mod 2, \\ 0 & \text{for } i \equiv 0 \mod 2; \quad i \geq 2. \end{cases}$$

The same presentation shows that $SL(2,\mathbb{Z}) = \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6$, and

$$H_i(SL(2,\mathbb{Z}),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i=0, \\ \mathbb{Z}_{12} & \text{for } i \equiv 1 \mod 2, \\ 0 & \text{for } i \equiv 0 \mod 2; \quad i \geq 2. \end{cases}$$

5.6. Discrete Subgroups of Lie Groups. (Co)homology of $SL(2,\mathbb{Z})$ can also be calculated by another method which can be considerably generalized. It is based on the following result.

5.6.1. Theorem. *Let G be a connected Lie group. Then its maximal compact connex subgroups $K \subset G$ are pairwise conjugate, and the homogeneous space $E = G/K$ is diffeomorphic to $\mathbb{R}^d$.*

5.6.2. Corollary. *Let $\Gamma \subset G$ be a discrete subgroup without torsion. Then $\Gamma \setminus E = \Gamma \setminus G/K$ is a $K(G,1)$-space so that*

$$H.(\Gamma,\mathbb{Z}) = H.(\Gamma \setminus E).$$

The absence of torsion guarantees that Γ acts upon E freely; in general this is not so. For this reason, we cannot apply this method directly to $SL(n,\mathbb{Z})$. However, it can be applied to congruence subgroups of finite index

$$\Gamma(N) = \{g \in SL(n,\mathbb{Z}) \mid g \equiv 1 \mod N\}, \quad N \geq 3.$$

In order to state some qualitative results we shall now define two finiteness properties of a group Γ.

5.7. Finiteness Conditions. a. The *cohomological dimension* of an abstract group Γ is

$$\operatorname{cd} \Gamma = \operatorname{projdim}_{\mathbb{Z}[\Gamma]} \mathbb{Z} = \inf\{n \mid H^i(\Gamma, M) = 0 \quad \text{for all } i > n \text{ and all } M\}.$$

b. Γ is said to be of *FL-type* if $\mathbb{Z}$ has a resolution of finite length consisting of free $\mathbb{Z}[\Gamma]$-modules of finite rank.

Clearly, $\operatorname{cd} \Gamma \leq d$ if Γ acts freely on a contractible d-dimensional space. Let us apply this result to a discrete torsion-free subgroup $\Gamma \subset SL(n,\mathbb{R})$. The space denoted by E in the Theorem 5.6.1 is now

$$E = SL(n,\mathbb{R})/SO(n,\mathbb{R}) = \{\text{positive quadratic forms of rank } n\}/\mathbb{R}^*_+.$$

Therefore,

$$\operatorname{cd} \Gamma \leq n(n+1)/2 - 1.$$

Actually, one can prove a more precise result.

5.7.1. Theorem. *Let $\Gamma \subset SL(n,\mathbb{Z})$ be a torsion-free subgroup of finite index. Then it is of FL-type, and*

$$\operatorname{cd} \Gamma = n(n-1)/2.$$

A proof is on the fact that E/Γ is homotopically equivalent to a $n(n-1)/2$-dimensional finite CW-complex which can be constructed explicitly.

5.8. Duality. Let Γ be an abstract group. A right Γ-module D is called *dualizing* (in dimension n) if there are functorial isomorphisms compatible with the canonical (co)homology exact sequences

$$H^i(\Gamma, M) \xrightarrow{\sim} H_{n-i}(\Gamma, D \underset{\mathbb{Z}}{\otimes} M),$$

where $g(d \otimes m) = dg^{-1} \otimes gm$ for all $g \in \Gamma$, $d \in D$, $m \in M$.

5.8.1. Theorem. *Assume that $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}[\Gamma]$ consisting of projective modules of finite type. Then Γ has a dualizing module in dimension n if and only if $H^i(\Gamma, \mathbb{Z}[\Gamma]) = 0$ for $i \neq n$ and $H^n(\Gamma, \mathbb{Z}[\Gamma])$ is a free abelian group. In this case $n = \mathrm{cd}(\Gamma)$ and $D = H^n(\Gamma, \mathbb{Z}[\Gamma])$.*

5.8.2. Theorem. *Suppose that a $K(\Gamma, 1)$-space Y is compact d-dimensional manifold. Let $\widetilde{Y}$ be its universal covering. Put $\varepsilon(g) = 1$ if $g \in \Gamma$ preserves an orientation of $\widetilde{Y}$ and $\varepsilon(g) = -1$ otherwise. Then the Γ-module $D = \mathbb{Z}$ with the action $g \cdot 1 = \varepsilon(g)$ is dualizing in dimension d.*

5.9. Euler Characteristic. Let Γ be a torsion-free group such that $\mathrm{cd}\,\Gamma < \infty$ and $\mathrm{rk}_{\mathbb{Z}} H_i(\Gamma, \mathbb{Z}) < \infty$. Put

$$\chi(\Gamma) = \sum_i (-1)^i \,\mathrm{rk}_{\mathbb{Z}} H_i(\Gamma, \mathbb{Z}).$$

More generally, if a group Γ' contains a subgroup of finite idex Γ with such properties, put

$$\chi(\Gamma') = \frac{1}{[\Gamma' : \Gamma]} \chi(\Gamma).$$

5.9.1. Theorem. *$\chi(\Gamma')$ is well defined, i.e. does not depend on the choice of Γ in Γ'.*

5.9.2. Examples. a. $\chi(F(I)) = 1 - |I|$, where $F(I)$ is the free group generated by a finite set I.

b. $\chi(SL(2, \mathbb{Z}) = -1/12$.

c. $\chi(\Gamma) = 1/|\Gamma|$ if Γ is finite.

d. If $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ is exact and Γ has a subgroup with Euler characteristic then the same is true for Γ',Γ'', and $\chi(\Gamma) = \chi(\Gamma')\chi(\Gamma'')$.

e. If $\Gamma = \Gamma \underset{A}{*} \Gamma_2$ and Γ has Euler characteristic then $\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2) - \chi(A)$.

One can prove the following topological formula for the calculation of Euler characteristic. Assume that Γ acts on a CW-complex X in such a way that: a) stabilizers Γ_e of all cells e have Euler characteristics; b) the number of orbits for the action Γ on cells is finite. Put

$$\chi_\Gamma(X) = \sum_e (-1)^{\dim e} \chi(\Gamma_e),$$

where summation is taken over a set of representatives of all orbits.

5.9.3. Theorem. *If in the described situation X is contractible and Γ has a torsion-free subgroup of finite index, then $\chi(\Gamma)$ is defined and coincides with $\chi_\Gamma(X)$.*

5.10. Euler Characteristic of Arithmetical Groups. Applying Theorem 5.9.3 to the case when Γ acts freely on X we get $\chi(\Gamma) = \chi(X/\Gamma)$. Suppose that $X/\Gamma = Y$ is a compact differentiable manifold. Any Riemannian metric on Y determines a volume form $d\mu$ polynomial in curvature tensor (Gauss-Bonnet form) such that

$$\chi(Y) = \int_Y d\mu = \mu(Y).$$

This result usually cannot be applied directly to the discrete subgroups Γ of a Lie group G because $\Gamma \setminus G/K$ is not compact. Harder has proved that it may be still applicable if this space has a finite invariant volume.

5.10.1. Theorem. *Let G be a semisimple algebraic group over $\mathbb{Q}$, $\Gamma \subset G(\mathbb{Q})$ a torsion-free aritmetical subgroup, $K \subset G(\mathbb{R})$ a maximal compact subgroup, $X = G(\mathbb{R})/K$, $d\mu$ the Gauss-Bonnet volume form of a $G(\mathbb{R})$-invariant metric. Then*

$$\chi(\Gamma) = \mu(\Gamma \setminus X).$$

The measure in the right hand side of this formula can be interpreted as a volume of a fundamental domain. It was explicitly calculated, e.g. for all Chevalley groups. We formulate two simple results.

5.10.2. Examples. a. We have

$$\chi(SL(n,\mathbb{Z})) = \prod_{k=2}^{n} \zeta(1-k) = \begin{cases} -1/12 & \text{for } n=2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

b. We have

$$\chi(Sp(2n,\mathbb{Z})) = \prod_{k=1}^{n} \zeta(1-2k).$$

§ 6. Generalities on Lie Algebra Cohomology

6.1. Calculation Methods. Let $\mathfrak{g}$ be a Lie algebra, A a $\mathfrak{g}$-module. In this section we shall described several tools for calculating $H(\mathfrak{g}, A)$ and some results of calculations for finite-dimensional Lie algebras.

We start with a situation typical for the semi-simple algebras. Let $\mathfrak{h} \subset \mathfrak{g}$ be an abelian subalgebra such that $\mathfrak{g}$ (resp. A) admits a decomposition into the direct sum of root (resp. weight) subspaces with respect to $\mathfrak{h}$: $\mathfrak{g} = \bigoplus_{\gamma\in\mathfrak{h}^*} \mathfrak{g}_\gamma$, $A = \bigoplus_{\mu\in\mathfrak{h}^*} A_\mu$. Then the cochain complex $C^\cdot(\mathfrak{g}, A)$ admits a similar decomposition. Denote by $C^\cdot_{(0)}(\mathfrak{g}, A)$ its invariant part.

6.1.1. Proposition. *The embedding $C^\cdot_{(0)}(\mathfrak{g}, A) \to C^\cdot(\mathfrak{g}, A)$ is a quasi-isomorphism.*

Similarly one can use the decomposition with respect to the characters of the center of $U(\mathfrak{g})$.

Given an invariant scalar product on $\mathfrak{g}$ (and on A, when A is unitary) one can use the following trick invented by Hodge. Let $(K^\cdot, d)$ be a complex of finite-dimensional Hilbert spaces (over $\mathbb{R}$ or $\mathbb{C}$). Let $(K^\cdot, d^*)$ be the dual complex, $\Delta^i = dd^* + d^*d : K^i \to K^i$ the Laplace operators.

6.1.2. Proposition. *The embedding* $(\operatorname{Ker} \Delta^\cdot, 0) \to (K^\cdot, d)$ *is a quasi-isomorphism. In particular,* $H^i(K') = \operatorname{Ker} \Delta^i$.

In general, given a Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{h}$, one can construct the *Hochschild-Serre spectral sequence.*

6.1.3. Theorem. *There is a spectral sequence with* $E_1^{pq} = H^q(\mathfrak{h}, \operatorname{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), A)$ *converging to* $H^n(\mathfrak{g}, A)$. *If* $\mathfrak{h}$ *is an ideal, then* $E_2^{pq} = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, A))$.

6.2. Semisimple Algebras. Let $\mathfrak{g}$ be a semisimple algebra over $k = \mathbb{R}$ or $\mathbb{C}$, A a finite-dimensional $\mathfrak{g}$-module. Considering the action of the center of $U(\mathfrak{g})$, one can obtain the following result in the vein of 6.1.1.

6.2.1. Theorem. $H^q(\mathfrak{g}, A) = H^q(\mathfrak{g}, A^{\mathfrak{g}}) = H^q(\mathfrak{g}, k) \otimes A^{\mathfrak{g}}$. *A similar result is true for the homology* $H_q(\mathfrak{g}, A)$.

Recall that in view of the Cartan theorem

$$H^\cdot(\mathfrak{g}, k) = H^\cdot_{\text{top}}(G, k),$$

where G is a connected simply connected compact Lie group with the Lie algebra $\mathfrak{g}$, and $H^\cdot_{\text{top}}$ is the cohomology of the topological space G.

In order to calculate $H^\cdot_{\text{top}}$ one can use a general theorem due to Hopf: it is a finite-dimensional supercommutative Hopf superalgebra, which is therefore freely generated by a finite set of odd-dimensional generators. The dimensions of these generators are known for all (semi)simple groups.

§ 7. Continuous Cohomology of Lie Groups

7.1. Continuous Homological Algebra. Working with topological groups and modules, one usually utilizes complexes and resolutions compatible with the topological structure. The categorical formalism recedes into backstage serving rather as a model for choosing definitions and making calculations.

7.2. G-modules and Complexes of G-modules. Let G be a separable locally compact group.

a. A *continuous G-module* is locally convex separable topological vector space (LCSS) E endowed with a continuous linear representation of G. A G-morphism of two such modules is a continuous linear operator $\varphi : E \to F$ compatible with the G-structures. The set of G-morphisms is denoted $\mathrm{Hom}_G(E, F)$. Continuous G-modules form a category, which is additive but not abelian.

b. Continuous linear injective map $\varphi : E \to F$ is called *strong* if it has a continuous left inverse map (E, F are assumed to be LCSS). An arbitrary continuous linear map $\varphi : E \to F$ is called *strong* if $\mathrm{Ker}\,\varphi \to E$ and $E/\mathrm{Ker}\,\varphi \to F$ are strong. In this case, $\mathrm{Im}\,\varphi$ is closed in F, the map $E/\mathrm{Ker}\,\varphi \to \mathrm{Im}\,\varphi$ is bicontinuous, and $\mathrm{Ker}\,\varphi$, $\mathrm{Im}\,\varphi$ have topological complements in E, resp. in F.

c. A (cochain) complex of LCSS

$$E^{\cdot} = \{\ldots \to E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \to \ldots\}$$

is called *strong* if all d^i are strong morphisms. We impose upon $\mathrm{Ker}\,d^n$ and $\mathrm{Im}\,d^{n-1}$ the topology induced by E^n , and consider on $H^n = \mathrm{Ker}\,d^n/\,\mathrm{Im}\,d^{n-1}$ the quotient topology. Then H^n are separable.

A complex $0 \to E \to E^0 \to E^1 \to \ldots$ is called a *resolution* of E if it is exact.

d. A G-morphism of two continuous G-modules is called *strong* if it is strong as a linear operator. Similarly, a complex of G-modules is *strong* if it is strong in the sense of c).

e. A G-module F is called relatively injective if it satisfies the following extension condition for G-morphisms: for any G-monomorphism (in the set-theoretical sense) $\varphi : E_1 \to E_2$ and any G-morphism $u : E_1 \to F$ there exists a G-morphism $w : E_2 \to F$ extending u.

The next proposition essentially shows that homological constructions can be based upon these definitions.

7.2.1. Proposition. *Suppose that the following data are given: two G-modules E, F, a strong resolution*

$$0 \to E \to E^0 \to E^1 \to \ldots,$$

and a complex of G-modules

$$0 \to F \to F^0 \to F^1 \to \ldots,$$

in which F^i are relatively injective. Then an arbitrary G-morphism $\varphi : E \to F$ extends to a G-morphism of complexes $\varphi^{\cdot} : E^{\cdot} \to F^{\cdot}$, and two such extensions differ by a G-homotopy.

7.3. Functors Ext^n_G. A *strong injective resolution* of a G-module F is a strong resolution of F consisting of relatively injective G-modules:

$$0 \to F \to F^0 \to F^1 \to F^2 \to \ldots .$$

7.3.1. Definition. $\mathrm{Ext}^n_G(E,F)$ is the n-th cohomology group of the complex

$$0 \to \mathrm{Hom}_G(E,F^0) \to \mathrm{Hom}_G(E,F^1) \to \dots$$

7.3.2. Proposition. *Every G-module admits a strong injective resolution; any two resolutions are connected by a continuous homotopy. Therefore topological spaces $\mathrm{Ext}^n_G(E,F)$ are well defined and functorial in E, F.*

Put $H^n_{\mathrm{cont}}(G,E) = \mathrm{Ext}^n_G(\mathbb{C},E)$, where G acts trivially on $\mathbb{C}$.

7.4. Cochains. Consider a LCSS E. For any separable locally compact group G, denote by $C^n_{\mathrm{cont}}(G,E)$ the space of continuous maps $G^{n+1} \to E$ with the topology of uniform convergence on compacts. G acts on $C^n_{\mathrm{cont}}(G,E)$ in a usual way:

$$(gf)(g_0,\dots,g^n) = g\cdot f(g^{-1}g_0,\dots,g^{-1}g_n),$$

The differential and the augmentation map are the standard ones:

$$(df)(g_0,\dots,g_{n+1}) = \sum_{i=0}^{n+1}(-1)^i f(g_0,\dots,\widehat{g}_i,\dots,g_{n+1})$$

$$\varepsilon : E \to C^0(G,E), \quad \varepsilon(e)g = e.$$

If G is a Lie group, we can construct in a similar way the spaces of C^∞-cochains $C^n_\infty(G,E)$ and L^p_{loc}-cochains $L^p_{\mathrm{loc}}C^n(G,E)$. The action of G, d and ε are again well defined.

7.4.1. Proposition. *The complexes $C^\cdot_{\mathrm{cont}}(G,E)$; $C^\cdot_\infty(G,E)$, for quasi-complete E, and $L^p_{\mathrm{loc}}C^\cdot(G,E)$ for complete E, are strong relatively injective resolutions of G-module E.*

Therefore, $H^n_{\mathrm{cont}}(G,E)$ can be calculated with the help of continuous, smooth or locally summable cochains with usual properties.

7.4.2. Proposition. *Every strong exact triple of G-modules $0 \to E_1 \to E_2 \to E_3 \to 0$ induces the usual infinite sequence of the cohomolgy groups $H^n(G,E)$, which is algebraically exact.*

7.4.3. Some Properties of Continuous Cohomology. a. $H^0(G,E) = E^G$.

b. If E is a Frechet space and if $H^n(G,E)$ is separable, then $H^n(G,E)$ is also Frechet.

c. If E is a barrel space, there is topological isomorphism between $\mathrm{Ext}^n_G(E,F)$ and $H^n_{\mathrm{cont}}(G,\mathrm{Hom}(E,F))$.

d. If G is compact, then every quasi-complete G-module F is relatively injective. In particular, $\mathrm{Ext}^n_G(E,F) = 0$ for $n \ge 1$.

e. Let $\widehat{G}$ be the space of classes of unitary irreducible representations of G. If the trivial representation is isolated in $\widehat{G}$, then $H^1_{\mathrm{cont}}(G,E) = 0$ for all unitary G-modules E.

f. Let $z \in G$ be a central element such that $z - \mathrm{id}$ is an automorphism of a G-module E. Then $H^n_{\mathrm{cont}}(G, E) = 0$ for all $n \geq 0$. In particular, if the center acts upon E via a non-trivial character, we have $H^{\cdot}_{\mathrm{cont}}(G, E) = 0$, and if the center acts upon E, F via two different characters, we have $\mathrm{Ext}^{\cdot}_G(E, F) = 0$.

g. Similarly, let μ be a measure with compact support on G, which is central in the algebra of these measures. If $\mu(1) = 1$ and the action of $\mu - \mathrm{id}$ on E is invertible, then $H^n_{\mathrm{cont}}(G, E) = 0$. In particular, if all conjugacy classes in G are relatively compact, then for every irreducible unitary G-module $E \neq \mathbb{C}$ we have $H^{\cdot}_{\mathrm{cont}}(G, E) = 0$.

7.5. Induced Modules. Let $H \subset G$ be a closed subgroup, E an H-module. Denote by $\mathrm{Ind}_{\mathrm{cont}}\, E$ the space of continuous maps $f : G \to E$ satisfying the condition $f(gh) = h^{-1}f(g)$, endowed with the G-action $(gf)(g') = f(g^{-1}g')$.

7.5.1. Proposition. *If E is quasi-complete, the spaces $H^{\cdot}_{\mathrm{cont}}(G, \mathrm{Ind}_{\mathrm{cont}}\, E)$ and $H^{\cdot}_{\mathrm{cont}}(G, E)$ are topologically isomorphic.*

For a Lie group G, one can similarly define $\mathrm{Ind}_\infty E$ and $L^p_{\mathrm{loc}} \mathrm{Ind}\, E$ and prove a statement similar to 7.5.1.

7.6. The Lyndon-Serre-Hochschild Spectral Sequence. Let H be a closed normal subgroup in G such that the factorization morphism $G \to G/H$ admits local continuous sections.

Consider a quasi-complete G-module E. Suppose that E is a Frechet space and that all $H^n_{\mathrm{cont}}(H, E)$ are Hausdorff. Define the action of G/H upon $H^n_{\mathrm{cont}}(H, E)$ by the following formula which should be applied to a representative cocycle $f \in Z^n_{\mathrm{cont}}(H, E)$:

$$(gf)(h_0, \ldots, h_n) = g(f(g^{-1}h_0g, \ldots, g^{-1}h_ng)).$$

7.6.1. Theorem. *In these conditions, there exists a spectral sequence with $E_2^{pq} = H^p_{\mathrm{cont}}(G/H, H^q_{\mathrm{cont}}(H, E))$ converging to $H^n_{\mathrm{cont}}(G, E)$.*

7.7. Van Est Theorems. In this subsection G denotes a connected Lie group, $K \subset G$ is its maximal connected compact subgroup, $M = G/K$. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K represented by right-invariant vector fields.

Let E be a G-module. An element $e \in E$ is called differentiable (or C^∞) if the map $\widetilde{e} : G \to E$: $\widetilde{e}(g) = g \cdot e$ is C^∞. Denote by E^∞ the set of such elements. It is a dense G-invariant subspace of E. The map $E^\infty \to C^\infty(G, E)$: $e \mapsto e$ is injective and compatible with the action of G by the right shifts on $C^\infty(G, E)$. Consider the induced topology on E^∞. Call E a G-module if the same topology is induced by the injection $E^\infty \to E$.

On a C^∞-module E one can define an action of $\mathfrak{g}$ by the formula

$$Xe = \lim_{t \to 0} t^{-1}(\exp(tX)e - e); \qquad X \in \mathfrak{g}, \quad e \in E.$$

This action is continuous and can be extended to $U(\mathfrak{g})$. Van Est proved that one can calculate $H^{\cdot}_{\mathrm{cont}}(G, E)$ in terms of the pair $(\mathfrak{g}, \mathfrak{k})$.

7.7.1. $(\mathfrak{g}, \mathfrak{k})$-cohomology. Consider the complex

$$C^\cdot(\mathfrak{g}, \mathfrak{k}, E) = \mathrm{Hom}_K(\Lambda^\cdot(\mathfrak{g}/\mathfrak{k}), E),$$

where K acts upon $\Lambda^\cdot(\mathfrak{g}/\mathfrak{k})$ via the adjoint representation and the differential is defined by the usual formula

$$\begin{aligned}(df)(\overline{X_0}, \dots, \overline{X_n}) = &\sum_{i=0}^{n} (-1)^i X_i f(\overline{X_0}, \dots, \widehat{\overline{X_i}}, \dots, \overline{X_n}) \\ &+ \sum_{i<j} (-1)^{i+j} f(\overline{[X_i, X_j]}, \overline{X_1}, \dots, \widehat{\overline{X_i}}, \dots, \widehat{\overline{X_j}}, \dots, \overline{X_n}),\end{aligned}$$

where $X_i \in \mathfrak{g}$, $\overline{X_i} = X_i \mod \mathfrak{k}$. Denote by $H^n(\mathfrak{g}, \mathfrak{k}, E)$ the cohomology of this complex.

7.7.2. Theorem. *For a C^∞ G-module E, there are canonical isomorphisms*

$$H^\cdot_{\mathrm{cont}}(G, E) = H^\cdot(\mathfrak{g}, \mathfrak{k}, E).$$

This theorem is used together with the following fact.

7.7.3. Proposition. *For any complete G-module E*

$$H^\cdot_{\mathrm{cont}}(G, E) = H^\cdot_{\mathrm{cont}}(G, E^\infty).$$

§ 8. Cohomology of Infinite-Dimensional Lie Algebras

8.1. Vector Fields and Current Algebras. In this section we consider Lie algebras over $k = \mathbb{R}$ or $\mathbb{C}$ of the following types.

a. W_n-Lie algebra of formal vector fields, that is,of the derivations of the formal series ring $k[[x_1, \dots, x_n]]$. Put $L_k = L_k(n) = \{\sum f_i \partial/\partial x_i \mid f_i \in (x_1, \dots, x_n)^{k+1}\}$, $k = -1, 0, 1, \dots$. We have $W_n = L_{-1}(n) \supset L_0(n) \supset L_1(n) \supset \dots$; $[L_a, L_b] \subset L_{a+b}$; $\mathrm{gl}(n) = L_0(n)/L_1(n)$.

b. $\widehat{S}_n = \{X = \sum f_i \frac{\partial}{\partial x_i} \mid \sum \frac{\partial f_i}{\partial x_i} = c_X \in k\} \subset W_n$, $S_n = \{X \mid c_X = 0\}$

c. $\widehat{H}_n = \{X = \sum f_i \frac{\partial}{\partial x_i} \mid \mathrm{Lie}_X \omega = c_X \omega,\ c_X \in k\} \subset W_{2n}$, where $\omega = dx_1 \wedge dx_{n+1} + \dots + dx_n \wedge dx_{2n}$ (Hamiltonian vector fields); $H_n = \{X \mid c_X = 0\} \subset \widehat{H}_n$.

d. $K_n = \{X = \sum f_i \frac{\partial}{\partial x_i} \mid \mathrm{Lie}_X \nu = g\nu\} \supset W_{2n+1}$, where $\nu = x_1 dx_{n+1} + \dots + x_n dx_{2n} + dx_{2n+1}$ (contact vector fields).

e. $T(M)$: Lie algebra of vector fields on a C^∞-manifold M.

f. $\mathfrak{g}^M$: smooth functions on a C^∞-manifold M with values in a Lie algebra $\mathfrak{g}$ (current algebras).

We shall be mostly concerned with the case $M = S^1$.

All these Lie algebras have a natural topology: linear in the cases **a**–**d**, C^∞ in the remaining cases. We shall consider only topological modules and the cohomology defined with the help of continuous cochains. We omit this condition in notation like $H^q(W, A)$.

8.2. Finiteness Theorem. Let $k = \mathbb{R}$, and assume that a Lie subalgebra $\mathfrak{g} \subset W_n$ contains an element $\sum c_i x_i \frac{\partial}{\partial x_i}$, with $c_i > 0$. Then for every finite-dimensional $\mathfrak{g}$-module A the graded cohomology space $H^{\cdot}(\mathfrak{g}, A)$ is finite-dimensional.

8.2.1. Corollary. *If* $\dim A < \infty$, *then* $H^{\cdot}(\mathfrak{g}, A)$ *is finite-dimensional for* $\mathfrak{g} = W_n$, $\widehat{S}_n$, $\widehat{H}_n$, K_n *(for* $k = \mathbb{R}$ *or* $\mathbb{C}$*).*

A proof is based on the principle similar to that of 6.2.1.One should consider a weight decomposition with respect to the element $\sum x_i \frac{\partial}{\partial x_i}$ or $\sum_{i=1}^n x_i \frac{\partial}{\partial x_i} + 2x_{2n+1} \frac{\partial}{\partial x_{2n+1}}$ (for K_n).

8.3. The Ring $H^{\cdot}(W_n, k)$. Denote by X_n the CW-complex which can be obtained by restricting to the $2n$-skeleton of the infinite Grassmannian $\mathbf{Gr}(n, \mathbb{C}^\infty)$ the principal $U(n)$-fibration over this Grassmannian.

8.3.1. Theorem. *There is a ring isomorphism* $H^{\cdot}(W_n, k) \cong H^{\cdot}(X_n, k)$. *Any product of positive-dimensional elements in these rings vanishes.*

A proof is based on the direct construction of an isomorphism between E_2-parts of the following spectral sequences:
a. The Serre-Hochschild spectral sequence corresponding to the subalgebra $\mathrm{gl}(n, k) \subset W$.
b. The Leray spectral sequence of the fibre space $X_n \to \mathrm{sk}_{2n}\mathbf{Gr}(n, \mathbb{C}^\infty)$.

8.4. Cohomology of W_n with Other Coefficients. We formulate the following result.

8.4.1. Theorem. *Let* $\Omega_n^{\cdot}$ *be the* W_n*-module of exterior forms of* $k[[x_1, \dots, x_n]]$. *The bigraded algebra* $H^{\cdot}(W_n, \Omega_n^{\cdot})$ *is generated by certain elements*

$$\lambda_i \in H^{2i-1}(W_n, \Omega_n^0), \qquad 1 \le i \le n,$$
$$\mu_j \in H^j(W_n, \Omega_n^j), \qquad 1 \le j \le n,$$

which satisfy the supercommutativity relations with respect to the total degree and also relations

$$\mu_j \dots \mu_{j_s} = 0 \quad \textit{for} \quad j_1 + \dots + j_s > 0.$$

A proof is based on the existence of an isomorphism $H^{\cdot}(W_n, \Omega_n^{\cdot}) = H^{\cdot}(L_0, \Lambda^{\cdot} V)$, where $V = \{\sum a_i dx_i \mid a_i \in k\}$, and L_0 acts on $\Lambda^{\cdot} V$ via $L_0 \to L_0/L_1 = \mathrm{gl}(n, k)$. Similarly one can prove the following result.

8.4.2. Theorem. *Let $\mathcal{A}$ be the W_n-module of formal tensor fields associated with a tensor* $\mathrm{gl}(n)$*-module A. Then there is an isomorphism*

$$H^{\cdot}(W_n, \mathcal{A}) \cong H^{\cdot}(\mathrm{gl}(n)) \otimes H^{\cdot}(L_1(n), k) \otimes A)^{\mathrm{gl}(n)}.$$

8.4.3. Theorem. *Denote by* W_n' *the* W_n*-module of continuous linear functionals on* W_n. *There exists a canonical isomorphism*

$$H^q(W'_n; W'_n) \cong H^{2n+1}(W_n) \otimes H^{q-2n}(\mathrm{gl}(n); k).$$

A proof uses the Serre-Hochschild spectral sequence corresponding to the subalgebra $\mathrm{gl}(n, k)$.

For an arbitrary Lie algebra $\mathfrak{g}$, there is a morphism of cochain complexes

$$C^{q+1}(\mathfrak{g}, k) \to C^q(\mathfrak{g}, \mathfrak{g}') : f \to \widetilde{f},$$
$$\widetilde{f}(x_1, \dots, x_q)(x) = f(x_1, \dots, x_q, x).$$

Denote by $\mathrm{var} : H^{q+1}(\mathfrak{g}, k) \to H^q(\mathfrak{g}, \mathfrak{g}')$ the induced cohomology morphism.

8.4.4. Theorem. a. $\mathrm{var} : H^{2n+1}(W_n, k) \to H^{2n}(W_n, W'_n)$ *is an isomorphism.*

b. *The sequence*

$$H^{q+1}(W_{n+1}) \xrightarrow{i} H^{q+1}(W_n) \xrightarrow{\mathrm{var}} H^q(W_n, W'_n)$$

is exact (here i is induced by the embedding $W_n \subset W_{n+1}$).

8.5. Other Formal Vector Fields Algebras. The following analogs of Theorem 8.3.1. are known:

8.5.1. Theorem. a. *Let Y be the inverse image of the $2n$-skeleton of the infinite Grassmannian in the principal $SU(n)$-bundle. There is a ring isomorphism*

$$H^{\cdot}(\widehat{S}_n, k) = H^{\cdot}(S^1 \times Y_n, k).$$

b. *Let Z_n be the inverse image of the $(4n+2)$-skeleton of the infinite Grassmannian in the principal $S^1 \times Sp(2n)$-bundle. There is a ring isomorphism*

$$H^{\cdot}(K_n, k) = H^{\cdot}(Z_n, k).$$

For $\widehat{H}_n$, H_n, S_n analogs of these results are seemingly unknown. Only the stable cohomology of these algebras is calculated, "stability" refers here to the dimensions ($\leq n$ or $\leq$ a linear function of n).

8.6. Cohomology of Lie Algebras of Vector Fields. Let M be a n-dimensional C^∞-manifold. Denote by $x(M)$ the fiber space with the base M, structure group $U(n)$, and fiber X_n described in 8.3, which is associated with the complexified tangent bundle to M. Denote by $\mathrm{Sec}\, x(M)$ the functional space of sections of $x(M)$. We need actually only its cohomolgy. It can be defined, for example, with the help of singular chains: by definition, a singular q-symplex of $\mathrm{Sec}\, x(M)$ is a continuous map $\Delta_q \times M \to x(M)$, commuting with the projection onto M.

8.6.1. Theorem. *There is a natural graded ring isomorphism*

$$H^{\cdot}(T(M), k) \cong H^{\cdot}(\mathrm{Sec}\, x(M), k).$$

A proof starts with a special case of this theorem referring to $M = \mathbb{R}^n$:

8.6.2. Proposition. $H^{\cdot}(T(\mathbb{R}^n), k) = H^{\cdot}(k \otimes W_n)$.

The isomorphism here is induced by the homomorphism $T(\mathbb{R}^n) \to W_n$, which maps a vector field to its ∞-jet at the origin.

The cohomology of $T(M)$ is then calculated with the help of Čech cochains, and Theorem 8.3.1 is used.

One can calculate in a similar way the cohomology with coefficients in the $T(M)$-module $C^\infty(M)$.

Denote by $u(M)$ the principal $\mathrm{GL}(n, \mathbb{C})$-bundle associated with the complexified tangent bundle of M. We may and will assume that $u(M)$ is a subbundle of $x(M)$.

8.6.3. Theorem. *There is a natural isomorphism*

$$H^{\cdot}(T(M), C^\infty(M)) \cong H^{\cdot}(Y(M), \mathbb{R}),$$
$$Y(M) = \{(y, s) \in M \times \operatorname{Sec} x(M) \mid s(y) \in u(M)\}.$$

For $M = S^1$, one can obtain more explicit results for a wider class of coefficients.

8.6.4. Theorem. *$H_{\cdot}(T(S^1), C^\infty(S^1))$ is a supercommutative $\mathbb{R}$-algebra freely generated by the following cycles of degrees* 1,1,2 *(φ is a cyclic coordinate on S^1):*

$$f(\varphi)d/d\varphi \to f(\varphi), \quad f(\varphi)d/d\varphi \to f'(\varphi),$$
$$(f(\varphi)d/d\varphi, g(\varphi)d/d\varphi) \to \int_{S^1} \begin{vmatrix} f'(\varphi) & g'(\varphi) \\ f''(\varphi) & g''(\varphi) \end{vmatrix} d\varphi.$$

8.6.5. Theorem. *$H^{\cdot}(T(S^1), \Omega^1(S^1))$ is a free $H^{\cdot}(T(S^1), C^\infty(S^1))$-module with one* 1*-dimensional generator represented by the cocycle*

$$f(\varphi)d/d\varphi \to f'(\varphi)d\varphi.$$

More generally:

8.6.6. Theorem. a. *$H^{\cdot}(T(S^1), \Omega^1(S^1)^{\otimes^s}) = 0$ if $s \neq (3r^2 \pm r)/2$, $r =$* $1, 2. \ldots.$

b. *$H^{\cdot}(T(S^1), \Omega(S^1)^{\otimes^s})$ is a free $H^{\cdot}(T(S^1), C^\infty(S^1))$-module with one generator of degree r for $s = (3r^2 \pm r)/2$.*

A proof of this theorem takes of several steps. First, the cohomology of $T(S^1)$ is reduced to the cohomology of W_1. Second, Theorem 8.4.2 is used. Third, the cohomology of $L_1(1)$ is calculated (this can be done by various methods).

8.7. Cohomology of Current Algebras. Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra.

8.7.1. Theorem. *The canonical homomorphism* $H^\cdot(\mathfrak{g}^{S^1}, \mathbb{R}) \to H^\cdot(G^{S^1}, \mathbb{R})$ (*cohomology of the functional loop space*) *is an isomorphism.*

More generally, let $\mathfrak{g}$ be a semi-simple real Lie algebra and let G be a compact Lie group such that $\mathfrak{g} \otimes \mathbb{C} = \mathrm{Lie}G \otimes \mathbb{C}$.

8.7.2. Theorem. $H^\cdot(\mathfrak{g}^{S^1}, \mathbb{R}) \simeq H^\cdot(G^{S^1}, \mathbb{R})$.

Applying this result to $\mathrm{sl}(n, \mathbb{R})$ and $SU(n)$ we see that $H^\cdot(\mathrm{sl}(n, \mathbb{R})^{S^1}, \mathbb{R}) \cong H^\cdot(SU(n)^{S^1}, \mathbb{R}) \cong H^\cdot(SU(n) \otimes \Omega SU(n), \mathbb{R})$ is a supercommutative algebra freely generated by cycles of degrees $2, \ldots, 2n-1$. There are algebraic versions of current algebras, the so called Kac-Moody Lie algebras. In the corresponding theorems $\mathfrak{g}$ should be replaced by those Kac-Moody algebras that correspond to the symmetrizable generalized Cartan algebras, and loop spaces should be replaced by the infinite-dimensional algebraic groups introduced by Shafarevich.

Bibliographic Hints

The interpretation of various constructions in algebra, topology and geometry in terms of low-dimensional (co)homology classes belongs to the basics of classical homological algebra. One can find a more detailed information in the following sources: sheaf cohomology in (Golovin 1986; Bredon 1967; Godement 1958; Iversen 1986; Wells 1973), group (co)homology in (Brown 1982; Wall 1979), Lie algebra (co)homology in (Fuchs 1984; Feigin, Fuchs 1988). One-dimensional cocycles $Z^1(G, K^*)$ in the case when G is the Galois group of a finite extension K/k were considered by Hilbert who proved that all such cocycles are coboundaries ("Theorem 90", cf. (Serre 1965)). The group of extension classes, now denoted by Ext^1, was invented and investigated by Baer. The interpretation of $H^i(G, \mathbb{Z}_p)$ in terms of generators ($i = 1$) and relations ($i = 2$) was suggested by Tate (cf. (Serre 1965)). The Brauer group was discovered in the classification theory of the simple central division algebras (cf. (Weil 1967)). We refer to (Fuchs 1984; Feigin, Fuchs 1988; Kac 1983) for information about central extensions of infinite-dimensional Lie algebras (Kac-Moody, Virasoro and alike).

The homotopy obstruction theory of Sect. 2 is explained in many textbooks of algebraic topology, e.g. (Fuchs *et al.* 1969). One can find there also H. Cartan's theory of $K(\Pi, n)$-spaces (see (Cartan 1954)). The torsor formalism in topology was elaborated by Grothendieck (1955); cf. the group-theoretic version in (Serre 1965), where Theorem 2.3.1 is also proved. For Theorem 2.5, see (Grothendieck 1958). One can find the Kodaira-Spencer theory in (Kodaira, Spencer 1958) or in the textbook by Wells (1973).

Cyclic homology was introduced from different viewpoints by Feigin, Tsygan (1983, 1987) and by A. Connes (1986); cf. also (Karoubi 1983). In particular, one can find proofs of Theorems 3.4.1, 3.5.1, and 3.7.1 in (Feigin, Tsygan 1987), and of Theorems 3.8.1 and 3.8.2 in (Loday, Quillen 1986). See (Loday 1985) for the results presented in 3.8, 3.9; cf. also the report by Cartier (1985). The noncommutative differential geometry was created and actively pursued by A. Connes; see his fundamental paper (Connes 1986), where the proofs of results from Sect. 4 can be found.

Main results on the group cohomology, in particular, all proofs for Sect. 5, are described in (Brown 1982). We had to omit here the Galois cohomology which is now an essential part of the class field theory; it will be explained in the volumes devoted to algebraic number theory. One can find basics of the Galois cohomology in Serre's lectures (1965).

One can also recommend to the reader the collection (Wall 1979) and various papers from collected papers of Serre (1986) and Cartan (1979).

The book by Fuchs (1984) and the report by Feigin, Fuchs (1988) are devoted to the Lie algebra cohomology (including infinite-dimensional Lie algebras). They contain, in particular, detailed calculations of the cohomology of certain infinite-dimensional Lie algebras due to I. M. Gelfand, D. B. Fuchs and others. See also (Gelfand 1970).

Continuous cohomology of Lie groups is described in the book by Guichardet (1980), see also (Feigin, Fuchs 1988) and (Borel, Wallach 1980). A different approach to continuous cohomology is discussed in (Helemskii 1986).

Chapter 4
Derived Categories and Derived Functors

§ 1. Definition of the Derived Category

1.1. Definition. A morphism $f : K^\cdot \to L^\cdot$ of complexes in an abelian category $\mathcal{A}$ is said to be a *quasi-isomorphism* if the corresponding homology morphism $H^n(f) : H^n(K^\cdot) \to H^n(L^\cdot)$ is an isomorphism for any n.

Earlier we have encountered following the examples of quasi-isomorphisms:

a. There exists a quasi-isomorphism between any two projective (injective) resolutions of one object (Theorem 3.a and Lemma 2.c).

b. Any object X of an abelian category a can be considered as a complex $\dots \to 0 \to 0 \to X \to 0 \to 0 \to \dots$ (with X at the 0-th place). This complex is acyclic outside zero, and its 0-th cohomology is isomorphic to X; such a complex will be called a 0-complex. The augmentation ε_X of a left resolution $P^\cdot \xrightarrow{\varepsilon_X} X$ determines a quasi-isomorphism of complexes

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow{\scriptstyle \varepsilon_X} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Hence the notion of a resolution is a special case of the notion of a quasi-isomorphism.

c. A morphism of the zero complex $0^\cdot \to K^\cdot$ (and $K^\cdot \to 0^\cdot$) is a quasi-isomorphism if and only if $K^\cdot$ is acyclic.

1.2. The Idea of the Derived Category. The ideology of the derived category, as we understand it today, can be formulated as follows.

a. An object X of an abelian category should be identified with all its resolutions.

b. The main reason for such an identification is that some most important functors, such as Hom, tensor products, Γ, should be redefined. Their "naive" definitions should be applied only to some special objects that are acyclic with respect to this functor. If, say, X is a flat module and Y is an arbitrary one, then $X \otimes Y$ is a good definition of the tensor product. But in the general case to get a correct definition one must replace $X \otimes Y$ by $P^\cdot \otimes Y$, where $P^\cdot$ is a flat resolution of the module X. Similarly, to get a correct definition of the group $\Gamma(\mathcal{F})$ of section of a sheaf $\mathcal{F}$ one must take the complex $\Gamma(\mathcal{I}^\cdot)$, where $\mathcal{F} \to \mathcal{I}^\cdot$ is an injective resolution of $\mathcal{F}$.

c. To pursue this point of view we must consider from the very beginning not only objects of an abelian category and their resolutions, but arbitrary complexes. One of the reasons why we have to do this is that $P^\cdot \otimes Y$ and $\Gamma(\mathcal{I})$ in the above examples usually have nontrivial cohomology not only in the degree 0, but in other degrees as well. (Recall that the corresponding cohomology groups are called derived functors $\mathrm{Tor}_i(X, Y)$ and $H^i(\mathcal{F})$ respectively.) Hence the relation between an object and its resolution that enables us to identify them should be generalized to arbitrary complexes. The appropriate generalization is the notion of a quasi-isomorphism.

d. The equivalence relation between complexes generated by quasi-isomorphisms is rather complicated, and what happens after the factorization by this equivalence relation is difficult to trace. The corresponding technique is the core of the theory of derived categories, and the main part of the present chapter is devoted to this technique.

e. The new definition of such functors as $\otimes$, Γ and others (see **b.**) makes semiexact functors in some sense "exact". However, the very notion of exactness in a derived category is by no means obvious, see the discussion in 4.2. In classical homological algebra this notion is based on the exact sequence for higher derived functor, which is invariant under a change of a resolution.

1.3. Definition-Theorem. *Let $\mathcal{A}$ be an abelian category,* $\mathrm{Kom}(\mathcal{A})$ *be the category of complexes over $\mathcal{A}$. There exists a category $D(\mathcal{A})$ and a functor $Q : \mathrm{Kom}(\mathcal{A}) \to D(\mathcal{A})$ with the following properties:*

a. *$Q(f)$ is an isomorphism for any quasi-isomorphism f.*

b. *Any functor $F : \mathrm{Kom}(\mathcal{A}) \to D$ transforming quasi-isomorphisms into isomorphisms can be uniquely factorized through $D(\mathcal{A})$, i.e., there exists a unique functor $G : D(\mathcal{A}) \to D$ with $F = G \circ Q$.*

The category $D(\mathcal{A})$ is called the derived category *of the abelian category $\mathcal{A}$.*

1.4. A Simple Existence Proof: Localization of a Category. Let $\mathcal{B}$ be an arbitrary category and S be an arbitrary class of morphisms in $\mathcal{B}$. We will show that there exists a universal functor transforming elements of S into isomorphisms. More precisely, we will construct a category $\mathcal{B}[S^{-1}]$ and a "localization by S" functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ with the universality property similar to that in 1.3.b.

To do this we first set $\mathrm{Ob}\,\mathcal{B}[S^{-1}] = \mathrm{Ob}\,\mathcal{B}$ and define Q to be the identity on objects.

Morphisms in $\mathcal{B}[S^{-1}]$ are constructed in several steps.

a. Introduce variables x_s, one for every morphism $s \in S$.

b. Construct an oriented graph Γ as follows:

vertices of Γ = objects of $\mathcal{B}$;

edges of Γ = {morphisms in $\mathcal{B}$} $\cup$ $\{x_s,\ s \in S\}$;

the edge $X \to Y$ is oriented from X to Y;

the edge x_s has the same vertices as the edge s,

but the opposite orientation.

c. A *path* in Γ is a finite sequence of edges such that the end of any edge coincides with the beginning of the next edge.

d. A *morphism* in $\mathcal{B}[S^{-1}]$ is an equivalence class of paths in Γ with the common beginning and the common end. Two paths are said to be *equivalent* if they can be connected by a chain of elementary equivalencies of the following type:

two consecutive arrows in a path can be replaced

by their composition;

arrows $X \xrightarrow{s} Y \xrightarrow{x_s} X$ (resp. $Y \xrightarrow{x_s} x \xrightarrow{s} Y$) can be replaced by

$X \xrightarrow{\mathrm{id}} X$ (resp. $Y \xrightarrow{\mathrm{id}} Y$).

Finally, the composition of two morphisms is induced by the conjunction of paths and the functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ maps a morphism $X \to Y$ into the class of the corresponding path (of length 1). For any $s \in S$ the morphism $Q(s)$ is clearly an isomorphism in $\mathcal{B}[S^{-1}]$, the inverse morphism being the class of the path x_s.

For any other functor $\mathcal{B} \to \mathcal{B}'$ transforming morphisms from S into quasi-isomorphisms the functor $G : \mathcal{B}[S^{-1}] \to \mathcal{B}'$ with the condition $F = G \circ Q$ is constructed as follows:

$$G(X) = F(X), \quad X \in \operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{B}[S^{-1}];$$
$$G(f) = F(f), \quad f \in \operatorname{Mor}\mathcal{B};$$
$$G(\text{class of } x_s) = F(s)^{-1}, \quad s \in S.$$

The reader can easily verify that all definitions are unambiguous and that the functor G is unique.

1.5. Splitting and Derived Categories. First insight into the structure of the derived category can be obtained from the following construction. A complex $K^\cdot$ is said to be *cyclic* if all its differentials are zero (so that all chains are cycles). Cyclic complexes form a full subcategory $\mathrm{Kom}_0(\mathcal{A}) \subset \mathrm{Kom}(\mathcal{A})$. The structure of $\mathrm{Kom}_0(\mathcal{A})$ is obvious: it is isomorphic to the category $\prod_{n=-\infty}^{\infty} \mathcal{A}[n]$, where $\mathcal{A}[n]$ is the "n-th copy of $\mathcal{A}$". Let i be the inclusion functor $\mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}_0(\mathcal{A})$ and h be the cohomology functor

$$h : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}_0(\mathcal{A}),$$
$$h((K^n, d^n)) = (H^n(K^\cdot), 0), \qquad h(f : K^\cdot \to L^\cdot) = (H^n(f)).$$

Since h transforms quasi-isomorphisms into isomorphisms, it can be factored through $D(\mathcal{A})$, so that for any $\mathcal{A}$ we have a functor

$$k : D(\mathcal{A}) \to \mathrm{Kom}_0(\mathcal{A}).$$

An abelian category $\mathcal{A}$ is said to be *semisimple* if any exact triple in $\mathcal{A}$ splits, i.e. is isomorphic to a triple of the form $0 \to X \xrightarrow{(\mathrm{id},0)} X \oplus Y \xrightarrow{(0,\mathrm{id})} Y \to 0$. For example, the category of finite-dimensional linear spaces over a field or the category of finite-dimensional linear representations of a finite group over a field of characteristic zero is semisimple. The category of abelian groups is, of course, not semisimple: the sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ does not split.

1.5.1. Proposition. *If an abelian category $\mathcal{A}$ is semisimple, then the functor $D(\mathcal{A}) \to \mathrm{Kom}_0(\mathcal{A})$ is an equivalence of categories.*

1.6. Variants. In applications it is often useful to consider complexes with various finiteness conditions. In particular, let

$$\begin{aligned}
&\mathrm{Kom}^+(\mathcal{A}) : K^i = 0 \quad \text{for } i \le i_0(K^\cdot);\\
&\mathrm{Kom}^-(\mathcal{A}) : K^i = 0 \quad \text{for } i \ge i_0(K^\cdot);\\
&\mathrm{Kom}^b(\mathcal{A}) = \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A}).
\end{aligned}$$

These categories are full subcategories in $\mathrm{Kom}(\mathcal{A})$, and it is often useful to consider the corresponding derived categories $D^+(\mathcal{A})$, $D^-(\mathcal{A})$, $D^b(\mathcal{A})$. For example, left projective resolutions belong to $\mathrm{Kom}^-(\mathcal{A})$, while right injective resolutions belong to $\mathrm{Kom}^+(\mathcal{A})$.

Now we see that we have two definitions of, say, $D^+(\mathcal{A})$: it could be defined either as the localization of $\mathrm{Kom}^+(\mathcal{A})$ by quasi-isomorphisms or as the full subcategory of $D^+(\mathcal{A})$ consisting of complexes $K^\cdot$ bounded from the left. We would like to be confident that these two constructions coincide. However, to prove this we do not have enough technique.

The problem is that morphisms in $\mathcal{B}[S^{-1}]$ constructed in 1.4 are formal expressions of the form

$$f_1 \circ s_1^{-1} \circ f_2 \circ s_2^{-1} \circ \cdots \circ s_k^{-1} \circ f_k, \qquad \text{where } f_i \in \mathrm{Mor}\,\mathcal{B},\ s_i \in S. \tag{1}$$

and to deal with such expressions we need some algebraic identities like "finding the common denominator". For example, in special cases in is rather hard even to find out whether or not $\mathcal{B}[S^{-1}]$ is equivalent to the trivial category with one object and one morphism.

All the required algebraic identities are provided by the following definition.

1.7. Definition. A class of morphisms $S \subset \mathrm{Mor}\,\mathcal{B}$ is said to be *localizing* if the following conditions are satisfied:

a. S is closed under multiplication: $\mathrm{id}_X \in S$ for any $X \in \mathrm{Ob}\,\mathcal{B}$ and $s \circ t \in S$ for any $s, t \in S$ whenever the composition is defined.

b. Extension conditions: for any $f \in \mathrm{Mor}\,\mathcal{B}$, $s \in S$ there exist $g \in \mathrm{Mor}\,\mathcal{B}$, $t \in S$ such that the following squares

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d, dashed, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} W & Z \arrow[l, dashed, "g"'] \\ X \arrow[u, dashed, "t"] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd} \tag{2}$$

are commutative.

c. Let f, g be two morphisms from X to Y; the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ with $ft = gt$.

1.8. Remarks. a. Let us consider the paths $x_s f$ and $g x_t$ from X to Z in the left square in (2). We claim that they represent the same morphism $X \to Z$ in $\mathcal{B}[S^{-1}]$. Indeed, the commutativity of the square means that $ft = sg$ in $\mathrm{Mor}\,\mathcal{B}$, which implies the equivalence of paths $x_s f t x_t$ and $x_s s g x_t$ and the equivalence of paths $x_s f$ and $g x_t$.

Hence in $\mathcal{B}[S^{-1}]$ we have (in obvious notations) $s^{-1} f = g t^{-1}$, so that if S satisfies conditions **a** and **b** in 1.7, we can move all denominators in (1) to the right. Similarly, the second square in (2) enables us to move all denominators to the left. These two properties make the study of the localized categories much simpler.

The condition 1.7.c defines the equivalence between the moving to the right and to the left.

b. Unfortunately, in general quasi-isomorphisms in $\mathrm{Kom}\,\mathcal{A}$ do not form a localizing class. To bypass this obstacle we proceed as follows: first we construct the category $K(\mathcal{A})$ of complexes modulo homotopic equivalence and then verify that quasi-isomorphisms in this new category already form a localizing class of morphisms. This will be done in the next section.

1.9. Lemma. *Let S be a localizing class of morphisms in a category $\mathcal{B}$. Then $\mathcal{B}[S^{-1}]$ can be described as follows:* $\mathrm{Ob}\,\mathcal{B}[S^{-1}] = \mathrm{Ob}\,\mathcal{B}$, *and then*

a. *One morphism $X \to Y$ in $\mathcal{B}[S^{-1}]$ is a class of "roofs", i.e. of diagrams (s, f) in $\mathcal{B}$ of the form*

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad , \quad s \in S,\ f \in \mathrm{Mor}\,\mathcal{B} \tag{3}$$

two roofs are equivalent, $(s, f) \sim (t, g)$, and only if there exists a third roof forming into a commutative diagram of the form

$$\begin{tikzcd}
& X''' \arrow[dl, "r"'] \arrow[dr, "h"] & \\
X' \arrow[d, "s"'] \arrow[drr, "f" near end] & & X'' \arrow[dll, "t"' near end] \arrow[d, "g"] \\
X & & Y
\end{tikzcd}$$

The identity morphism $\mathrm{id} : X \to X$ *is a class of the roof* $(\mathrm{id}_X, \mathrm{id}_X)$.

b. *The composition of morphisms represented by the roofs* (s, f) *and* (t, g) *is a class of the roof* (st', gf') *obtained using the first square in* (2):

$$\begin{tikzcd}
& & X'' \arrow[dl, dashed, "t'"'] \arrow[dr, dashed, "f''"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

1.9.1. Remark. A diagram of the form (3) is called a *left S-roof.* There exists a variant of Lemma 1.8 which uses *right S-roofs*

$$\begin{tikzcd}
& X' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"']
\end{tikzcd}$$

instead of left S-roofs. The composition of morphisms represented by right S-roofs is constructed using the second square in (2).

According to these two possibilities one can introduce the notions of left- and right-localizing classes of morphisms satisfying only half of the conditions 1.6.b and 1.6.c.

1.10. Proposition. *Let* $\mathcal{C}$ *be a category,* S *be a localizing system of morphisms in* $\mathcal{C}$ *and* $\mathcal{B} \subset \mathcal{C}$ *be a full subcategory. Let* **a** *and either* $\mathbf{b}_1$ *or* $\mathbf{b}_2$ *be satisfied, where* **a**, $\mathbf{b}_1$, $\mathbf{b}_2$ *are the following conditions:*

a. $S_{\mathcal{B}} \cap \mathrm{Mor}\, \mathcal{B}$ *is a localizing system in* $\mathcal{B}$.

$\mathbf{b}_1$. *For any* $s : X' \to X$ *with* $s \in S$, $X \in \mathrm{Ob}\, \mathcal{B}$ *there exists* $f : X'' \to X'$, *such that* $sf \in S$, $X'' \in \mathrm{Ob}\, \mathcal{B}$.

$\mathbf{b}_2$. *The same as* $\mathbf{b}_1$ *with all arrows reversed.*

Then $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ *is a full subcategory in* $\mathcal{C}[S^{-1}]$. *More precisely, the canonical functor* $I : \mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{C}[S^{-1}]$ *is fully faithful.*

Using this proposition one can prove that all functors in the diagram

$$\begin{tikzcd}
 & D^+(\mathcal{A}) \arrow[dr] & \\
D^b(\mathcal{A}) \arrow[ur] \arrow[dr] & & D(\mathcal{A}) \\
 & D^-(\mathcal{A}) \arrow[ur] &
\end{tikzcd}$$

are imbeddings of full subcategories.

§ 2. Derived Category as the Localization of Homotopic Category

2.1. The Plan. First we introduce certain diagrams in derived categories – called *distinguished triangles* – that replace and generalize exact triples in abelian categories. The definition of such diagrams is not at all obvious. First of all, we do not even know that the category $D(\mathcal{A})$ is additive: to add two morphisms we have, in a sense, to find their "common denominator". Next, although $D(\mathcal{A})$ will happen to be additive, it is will almost never be abelian. Therefore we can not apply to $D(\mathcal{A})$ the standard definition of exactness.

However, although $D(\mathcal{A})$ is not abelian, distinguished triangles in $D(\mathcal{A})$ form a remarkable structure that reflects the main homological properties of the initial abelian category.

2.2. Translation, Cylinder, Cone Functors. Let $\mathcal{A}$ be an abelian category.

a. Fix an integer n and for any complex $K^\cdot = (K^i, d_K^i)$ define a new complex $K^\cdot[n]$ by $(K[n])^i = K^{n+i}$, $d_{K[n]} = (-1)^n d_K$. For a morphism of complexes $f : K^\cdot \to L^\cdot$ define $f[n] : K^\cdot[n] \to L^\cdot[n]$ as a morphism of complexes that coincides with f on components of $K^\cdot[n]$ (=components of $K^\cdot$).

It is clear that $T^n : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$, where $T^n(K^\cdot) = K^\cdot[n]$, $T^n(f) = f[n]$ is a functor, called the *translation functor*; it is an autoequivalence of $\mathrm{Kom}(\mathcal{A})$. This functor induces an autoequivalence of any of categories $\mathrm{Kom}^+(\mathcal{A})$, $\mathrm{Kom}^-(\mathcal{A})$, $\mathrm{Kom}^b(\mathcal{A})$ and of the corresponding derived categories.

b. Let $f : K^\cdot \to L^\cdot$ be a morphism of complexes. The *cone* of f is the following complex $C(f)$:

$$C(f)^i = K[1]^i \oplus L^i, \qquad d_{C(f)}(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i).$$

It is convenient to write elements of $C(f)$ as columns of height 2 and morphisms as matrices, so that

$$d_{C(f)} = \begin{pmatrix} d_K[1] & 0 \\ f[1] & d_L \end{pmatrix}.$$

One easily verifies that $d_{C(f)}^2 = 0$.

If f is a morphism of 0-complexes (see 1.1.b) then $C(f)$ is a complex $\dots \to 0 \to K^0 \xrightarrow{f} L^0 \to 0 \to \dots$, where K^0 sits in degree -1, and L^0 sits in degree 0. In particular,

$$H^{-1}(C(f)) = \operatorname{Ker} f, \quad H^0(C(f)) = \operatorname{Coker} f.$$

c. In the same notation the *cylinder* $\mathrm{Cyl}(f)$ of a morphism f is the following complex:

$$\mathrm{Cyl}(f) = K^{\cdot} \oplus K^{\cdot}[1] \oplus L^{\cdot},$$
$$d^i_{\mathrm{Cyl}(f)}(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i).$$

Let us remark that if both complexes $K^{\cdot}$, $L^{\cdot}$ are bounded either from the left or from the right, or from both sides, then $C(f)$ and $\mathrm{Cyl}(f)$ are also bounded from the same side. If, moreover, $K^{\cdot}$ and $L^{\cdot}$ lie in $\mathrm{Kom}(\mathcal{B})$, where $\mathcal{B} \subset \mathcal{A}$ is some additive subcategory, then $C(f)$, $\mathrm{Cyl}(f) \in \mathrm{Kom}(\mathcal{B})$ as well.

Main properties of the cone and of the cylinder are summarized in the following lemma.

2.2.1. Lemma. *For any morphism $f : K^{\cdot} \to L^{\cdot}$ there exists the following commutative diagram in* $\mathrm{Kom}(\mathcal{A})$ *with exact rows which is functorial in f:*

$$\begin{tikzcd}
 & 0 \arrow[r] & L^{\cdot} \arrow[r, "\overline{\pi}"] \arrow[d, "\alpha"] & C(f) \arrow[r, "\delta=\delta(f)"] \arrow[d, equal] & K^{\cdot}[1] \arrow[r] & 0 \\
0 \arrow[r] & K^{\cdot} \arrow[r, "\overline{f}"] \arrow[d, equal] & \mathrm{Cyl}(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & C(f) \arrow[r] & 0 & \\
 & K^{\cdot} \arrow[r, "f"] & L^{\cdot} & & &
\end{tikzcd} \tag{1}$$

It has the following properties:

- *α and β are quasi-isomorphisms; moreover $\beta\alpha = \mathrm{id}_L$ and $\alpha\beta$ is homotopic to $\mathrm{id}_{\mathrm{Cyl}(f)}$, so that $L^{\cdot}$ and $\mathrm{Cyl}(f)$ are canonically isomorphic in the derived category.*

2.3. Definition. a. A *triangle* in a category of complexes $(\mathrm{Kom}, K, D, D^+, D^-, \dots)$ is a diagram of the form

$$K^{\cdot} \xrightarrow{u} L^{\cdot} \xrightarrow{v} M^{\cdot} \xrightarrow{w} K^{\cdot}[1].$$

b. A *morphism of trianges* is a commutative diagram of the form

$$\begin{tikzcd}
K^{\cdot} \arrow[r, "u"] \arrow[d, "f"] & L^{\cdot} \arrow[r, "v"] \arrow[d, "g"] & M^{\cdot} \arrow[r, "w"] \arrow[d, "h"] & K^{\cdot}[1] \arrow[d, "f[1]"] \\
K_1^{\cdot} \arrow[r, "u_1"] & L_1^{\cdot} \arrow[r, "v_1"] & M_1^{\cdot} \arrow[r, "w_1"] & K_1^{\cdot}[1].
\end{tikzcd}$$

Such a morphism is said to be an *isomorphism* if f, g, h are isomorphisms in the corresponding category.

c. A triangle is said to be *distinguished* if it is isomorphic to the middle part

$$K^{\cdot} \xrightarrow{\overline{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} K^{\cdot}[1]$$

of some diagram of the form (1).

The next proposition shows that any exact triple can be completed to a distinguished triangle.

2.4. Proposition. *Any exact triple of complexes in* Kom($\mathcal{A}$) *is quasi-isomorphic to the middle row of an appropriate diagram of the form* (1).

Proof. Let $0 \to K^{\cdot} \xrightarrow{f} L^{\cdot} \xrightarrow{g} M^{\cdot} \to 0$ be an exact triple. The required quasi-isomorphism is of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K^{\cdot} & \longrightarrow & L^{\cdot} & \longrightarrow & M^{\cdot} & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle g} & & \\ 0 & \longrightarrow & K^{\cdot} & \longrightarrow & \mathrm{Cyl}(f) & \longrightarrow & C(f) & \longrightarrow & 0 \end{array}$$

where β is taken from (1) and γ is defined by $\gamma(k^{i+1}, l^i) = g(l^i)$.

The next theorem shows that cohomological properties of distinguished triangles in $D(\mathcal{A})$ (or in $D^+(\mathcal{A})$,...) are quite similar to those of exact triples of complexes.

Theorem 2.5. *Let*

$$K^{\cdot} \xrightarrow{\ u\ } L^{\cdot} \xrightarrow{\ v\ } M^{\cdot} \xrightarrow{\ w\ } K^{\cdot}[1]. \tag{2}$$

be an exact sequence in $D(\mathcal{A})$. *Then the sequence*

$$\ldots \to H^i(K^{\cdot}) \xrightarrow{H^i(u)} H^i(L^{\cdot}) \xrightarrow{H^i(v)} H^i(M^{\cdot}) \xrightarrow{H^i(w)}$$
$$\xrightarrow{H^i(w)} H^i(K^{\cdot}[1]) = H^{i+1}(K^{\cdot}) \to \ldots$$

is exact.

It is sufficient to prove the theorem for the distinguished triangle

$$K^{\cdot} \xrightarrow{\ \overline{u}\ } \mathrm{Cyl}(u) \xrightarrow{\ \pi\ } C(f) \xrightarrow{\ \delta\ } K^{\cdot}[1]$$

from (1), which is quasi-isomorphic to (2). This can be done by direct computations.

2.6. Definition. Let $\mathcal{A}$ be an abelian category. The *homotopic category* $K(\mathcal{A})$ is defined as follows:

$$\mathrm{Ob}\, K(\mathcal{A}) = \mathrm{Ob}\, \mathrm{Kom}(\mathcal{A}),$$
$$\mathrm{Mor}\, K(\mathcal{A}) = \mathrm{Mor}\, \mathrm{Kom}(\mathcal{A}) \quad \text{modulo homotopic equivalence.}$$

By $K^+(\mathcal{A})$, $K^-(\mathcal{A})$, $K^b(\mathcal{A})$ we denote full subcategories of $K(\mathcal{A})$ formed by complexes satisfying the corresponding boundness conditions.

It is clear that $K(\mathcal{A})$ is an additive category on which the functors H^i are well defined. Hence the definition of a quasi-isomorphism from 1.1 can be literally applied to $K(\mathcal{A})$.

2.7. Theorem. a. *The class of quasi-isomorphisms in $K(\mathcal{A})$ is localizing.*

b. *The localization of $K(\mathcal{A})$ by quasi-isomorphisms is canonically isomorphic to the derived category $D(\mathcal{A})$.*

The same is true for $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$, where $* = +, -$, or b.

The proof of this theorem is standard, although somewhat cumbersome. The main technique is the application of Lemma 2.2.1.

2.8. Additivity of $D(\mathcal{A})$. To be able to define the sum of morphisms in $D(\mathcal{A})$ we will show that for any two morphisms $\varphi, \varphi' : X \to Y$ in $D(\mathcal{A})$ one can find "a common denominator", i.e. represent them by roofs

$$\begin{tikzcd} & U \arrow[dl, "t"'] \arrow[dr] & \\ X & & Y \end{tikzcd}, \quad \begin{tikzcd} & U \arrow[dl, "t"'] \arrow[dr, "g'"] & \\ X & & Y \end{tikzcd}$$

with the common top and the common left side.

Indeed, let φ, φ' be represented by roofs

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}, \quad \begin{tikzcd} & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

Let us extend the diagram

$$\begin{tikzcd} U \arrow[r, dashed, "r'"] \arrow[d, dashed, "r"'] & Z' \arrow[d, "s'"] \\ Z \arrow[r, "s"] & X \end{tikzcd}$$

to a commutative square in $K(\mathcal{A})$. Since s, s', r are quasi-isomorphisms, r' is also a quasi-isomorphism. Hence φ and φ' can be represented by roofs of the form (3) with $t = sr = s'r'$, $g = fr$, $g' = f'r'$.

Now we define the sum $\varphi + \varphi'$ in $D(\mathcal{A})$ to be the class of the roof $X \xleftarrow{t} U \xrightarrow{g+g'} Y$.

2.9. Morphisms in $D(\mathcal{A})$. Let $\mathcal{A}$ be an abelian category, $f : K^\cdot \to L^\cdot$ be a morphism in $\mathrm{Kom}(\mathcal{A})$. By the definition of morphisms in $D(\mathcal{A})$, it is clear that $f = 0$ in $D(\mathcal{A})$ if and only if there exists a quasi-isomorphism $t : L^\cdot \to M^\cdot$ such that sf is homotopic to the zero morphism (equivalently, there exists a quasi-isomorphism $t : N^\cdot \to K^\cdot$ such that ft is homotopic to the zero morphism).

If $f = 0$ in $D(\mathcal{A})$, then clearly $H^i(f) = 0$ for all i. Considering the morphism of complexes of length 2 (for $\mathcal{A} = \mathbf{Ab}$)

$$\begin{array}{ccccccc}
\dots 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{a} & \mathbb{Z} & \longrightarrow & 0\dots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} & & \\
\dots 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{c} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0\dots
\end{array}$$

where b and c map the generator of the corresponding group $\mathbb{Z}$ to the generator, and a and d multiply the generator by 2, one can see that the converse is wrong. Therefore, in the sequence of implications $\{f = 0 \text{ in } \mathrm{Kom}(\mathcal{A})\} \Rightarrow \{f = 0 \text{ in } K(\mathcal{A})\} \Rightarrow \{f = 0 \text{ in } D(\mathcal{A})\} \Rightarrow \{H^i(f) = 0 \text{ for all } i\}$ neither arrow is an equivalence.

2.10. Truncation Functors. For a complex $K^\cdot \in \mathrm{Ob}\,\mathrm{Kom}\,\mathcal{A}$ and for an integer i define the complexes $\tau_{\le i}K^\cdot$, $\tau_{\ge i}K^\cdot$ (cf. 3.3 from Chap. 1) as follows:

$$(\tau_{\le i}K^\cdot)^n = \begin{cases} K^n & \text{for } n < i, \\ \mathrm{Ker}\, d^i & \text{for } n = i, \\ 0 & \text{for } n > i. \end{cases}$$

$$(\tau_{geqi}K^\cdot)^n = \begin{cases} K^n & \text{for } n < i-1, \\ K^{i-1}/\,\mathrm{Ker}\, d^i & \text{for } n = i-1, \\ 0 & \text{for } n \ge i. \end{cases}$$

It is clear that

$$H^n(\tau_{\le i}K^\cdot) = \begin{cases} H^n(K^\cdot) & \text{for } n \le i, \\ 0 & \text{for } n > i. \end{cases}$$

$$H^n(\tau_{\ge i}K^\cdot) = \begin{cases} 0 & \text{for } n < i, \\ H^n(K^\cdot) & \text{for } n \ge i. \end{cases}$$

2.10.1. Proposition. a. *Natural morphisms of complexes $\alpha : \tau_{\ge i}K^\cdot \to K^\cdot$ (resp. $\beta : K^\cdot \to \tau_{\ge i}K^\cdot$) induce the functor $\tau_{\le i}$ (resp. $\tau_{\ge i}$) from the category* $\mathrm{Kom}\,\mathcal{A}$ *into itself.*

b. *The functors $\tau_{\le i}$, $\tau_{\ge i}$ map quasi-isomorphisms into quasi-isomorphisms. Hence they induce the functors (denoted again by $\tau_{\le i}$, $\tau_{\ge i}$) from the category $D(\mathcal{A})$ into itself.*

c. *There exists a distinguished triangle*

$$\tau_{\le i}K^\cdot \xrightarrow{\alpha} K^\cdot \xrightarrow{\beta} \tau_{\ge i+1}K^\cdot \longrightarrow \tau_{\le i}K^\cdot[1].$$

in $D(\mathcal{A})$, which depends functorially in $K^\cdot \in \mathrm{Ob}\, D(\mathcal{A})$.

The proof of all these statements can be easily obtained using Proposition 2.4. More about the properties and generalizations of truncation functors $\tau_{\le i}$, $\tau_{\ge i}$ see 3.4 in Chap. 5.

§ 3. Structure of the Derived Category

3.1. Complexes-Objects. By a H^0-complex we mean a complex $K^\cdot$ such that $H^i(K^\cdot) = 0$ for $i \neq 0$. In this definition $K^\cdot$ can be considered as an object of any of the categories of complexes, since the functor H^i transforms quasi-isomorphisms into isomorphisms, so it is defined not only on $\mathrm{Kom}^*(\mathcal{A})$, but also on $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ for $* = \emptyset, +, -, b$.

3.1.1. Proposition. *The functor $Q : \mathcal{A} \to D^*(\mathcal{A})$ establishes an equivalence of $\mathcal{A}$ with the full subcategory of $D^*(\mathcal{A})$ consisting of H^0-complexes.*

3.2. Extensions and the Functors Ext. Instead of 0-complexes and H^0-complexes we can consider i-complexes and H^i-complexes for any i. The structure of morphisms between such complexes in $D(\mathcal{A}^*)$ is the information of the next level of complexity about the structure of the derived category.

We shall denote the i-complex with an object $X \in \mathrm{Ob}\,\mathcal{A}$ at the i-th place by $X[-i]$; in the previous subsection we used the notation X instead of $X[0]$.

3.2.1. Definition. $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{D^*(\mathcal{A})}(X[0], Y[i])$.

Later we shall show that this definition coincides with the one given in § 4 of Chap. 2.

3.3. Remarks. a. It does not matter in the above definition what is the index $*$, since i-complexes are bounded, and all embeddings of derived categories are fully faithful functors.

b. Using the functor T^k we can identify $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ with $\mathrm{Hom}_{D(\mathcal{A})}(X[k], Y[i + k])$ as well. The composition of morphisms in the derived category enables us to define the multiplication on Ext's:

$$\begin{array}{ccc}
\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \quad \times \quad \mathrm{Ext}^j_{\mathcal{A}}(Y,Z) & & \xrightarrow{\alpha} \\
\Big\| & & \\
\mathrm{Hom}_{D(\mathcal{A})}(X[k], Y[i+k]) \times \mathrm{Hom}_{D(\mathcal{A})}(Y[i+k], Z[i+j+k]) & & \xrightarrow{\beta} \\
\xrightarrow{\alpha} & \mathrm{Ext}^{i+j}_{\mathcal{A}}(X,Z) & \\
& \Big\| & \\
\xrightarrow{\beta} & \mathrm{Hom}_{D(\mathcal{A})}(X[k], Z[i+j+k]) &
\end{array}$$

This composition on Ext's does depend on the choice of k in the lower line.

c. Since the category $D(\mathcal{A})$ is additive, $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ are abelian groups. The multiplication on Ext's is biadditive. Moreover, $\mathrm{Ext}^i_{\mathcal{A}}$ determines a bifunctor $\mathcal{A}^\circ \times \mathcal{A} \to \mathbf{Ab}$.

Viewing an exact triple $0 \to X' \to X \to X'' \to 0$ (resp. $0 \to Y' \to Y \to Y'' \to 0$) in $\mathcal{A}$ as a distinguished triangle, we obtain from 2.5 the following exact sequence:

$$\cdots \to \mathrm{Ext}^i_{\mathcal{A}}(X'',Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X,Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X',Y) \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X'',Y) \to \dots$$

(resp.

$$\cdots \to \mathrm{Ext}^i_{\mathcal{A}}(X,Y') \to \mathrm{Ext}^i_{\mathcal{A}}(X,Y) \to \mathrm{Ext}^i_{\mathcal{A}}(X,Y'') \to \mathrm{Ext}^{i+1}_{\mathcal{A}}(X,Y') \to \dots).$$

d. Following Yoneda, we consider the following construction of elements from $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$, $i > 0$. Let $K^\cdot$ be an acyclic complex of the form

$$K^\cdot : \cdots \to 0 \to K^{-i} = Y \to K^{-i+1} \to \cdots \to K^0 \to X = K^1 \to 0 \to \dots \quad (1)$$

It determines the following left roof:

$$\begin{tikzcd} & \widetilde{K} \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X[0] & & Y[i] \end{tikzcd}$$

where $\widetilde{K}^l = K^l$ for $l \neq 1$, $\widetilde{K}^1 = 0$, $s^0 = d^0_K$, $f^{-i} = \mathrm{id}_Y$. Denote by $y(K^\cdot)$ the morphism $X[0] \to Y[i]$ in the derived category corresponding to this roof.

Let we have now two finite acyclic complexes $K^\cdot$, $L^\cdot$ such that the extreme left term of $K^\cdot$ (i.e. Y in (1)) coincides with the extreme right term of $L^\cdot$. Then we can form the third finite acyclic complex, "gluing" $K^\cdot$ and $L^\cdot$ together:

$$\begin{aligned} L^\cdot \circ K^\cdot : \cdots \to 0 \to Z = L^{-j} \to \cdots \to L^{-1} \xrightarrow{d_L} L^0 \xrightarrow{f} K^{-i+1} \\ \cdots \to K^0 \to X = K^1 \to 0 \to \dots, \end{aligned}$$

where f is the composition

$$f : L^0 \xrightarrow{d_L} L^1 = Y = K^{-i} \xrightarrow{d_K} K^{-i+1}$$

(one can easily prove that $L^\cdot \circ K^\cdot$ is an acyclic complex).

3.4. Theorem. a. $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) = 0$ *for* $i < 0$.

b. $\mathrm{Ext}^0_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(X,Y)$.

c. *Any element of* $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ *is of the form* $y(K^\cdot)$ *for some complex* $K^\cdot$ *of the form* (1), *and for* $y(K^\cdot) \in \mathrm{Ext}^i_{\mathcal{A}}(X,Y)$, $y(L^\cdot) \in \mathrm{Ext}^i_{\mathcal{A}}(Y,Z)$ *we have*

$$y(L^\cdot \circ K^\cdot) = y(L^\cdot)y(K^\cdot).$$

3.5. Homological Dimension. The previous theorem shows that the complexity of the derived category can be roughly measured by the following parameter.

The *homological dimension* $\mathrm{dh}(\mathcal{A})$ of the category $\mathcal{A}$ is the maximal number p such that there exist objects X, Y from $\mathcal{A}$ with $\mathrm{Ext}^i_{\mathcal{A}}(X,Y) \neq 0$, or ∞, if such p does not exist.

Clearly, $\mathrm{dh}(\mathcal{A}) \geq 0$. Categories of dimension 0 are especially simple.

3.5.1. Proposition. *The following conditions are equivalent:*

a. $\mathrm{dh}(\mathcal{A}) = 0$.

b. $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)=0$ *for all* $X,Y\in \mathrm{Ob}\,\mathcal{A}$.
c. *The category* $\mathcal{A}$ *is semisimple* (see 1.5).

3.6. One-Dimensional Categories. Examples of categories of homological dimension 1 are: a) the category **Ab**, b) the category $K[x]$-mod, where K is a field.

One can easily prove that for each of these categories we have $\mathrm{dh}(\mathcal{A})\geq 1$. Namely, the following exact triples are indecomposable:

$$\begin{aligned}&0\to\mathbb{Z}\xrightarrow{m}\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}\to 0,\quad m>1, && \text{in } \mathbf{Ab},\\ &0\to K[x]\xrightarrow{x}K[x]\to K\to 0, && \text{in } K[x]\text{-mod}.\end{aligned}$$

To prove that the dimension of **Ab** is exactly 1 is also rather easy. To prove that the dimension of $K[x]$-mod equals 1, one can use a technique that we shall now describe.

3.7. Homological Dimension of an Object. Denote, for $X\in\mathrm{Ob}\,\mathcal{A}$,

$$\begin{aligned}\mathrm{dhp}\,X&=\sup\{n: \text{there exists } Y\in\mathrm{Ob}\,\mathcal{A}\ \text{ with } \mathrm{Ext}^n_{\mathcal{A}}(X,Y)\neq 0\},\\ \mathrm{dhi}\,X&=\sup\{n: \text{there exists } Y\in\mathrm{Ob}\,\mathcal{A}\ \text{ with } \mathrm{Ext}^n_{\mathcal{A}}(Y,X)\neq 0\}.\end{aligned}$$

Letters p and i stay for shortened projective and injective. These notations are justified by the following lemma.

3.7.1. Lemma. *The following conditions on* $X\in\mathrm{Ob}\,\mathcal{A}$ *are equivalent:*
$\mathbf{a}_p$. $\mathrm{dhp}\,X=0$,
$\mathbf{b}_p$. $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)=0$ *for all* $Y\in\mathrm{Ob}\,\mathcal{A}$,
$\mathbf{c}_p$. X *is projective.*
Similarly, the following conditions on X *are equivalent:*
$\mathbf{a}_i$. $\mathrm{dhi}\,X=0$,
$\mathbf{b}_i$. $\mathrm{Ext}^1_{\mathcal{A}}(Y,X)=0$ *for all* $Y\in\mathrm{Ob}\,\mathcal{A}$,
$\mathbf{c}_i$. X *is injective.*

3.7.2. Proposition. a. *Let a complex*

$$\dots\to 0\to X'\to P^{-k}\to\dots\to P^{-1}\to P^0\to 0\to\dots$$

be acyclic and objects P^{-i} *be projective. Then*

$$\mathrm{dhp}\,X'=\max(\mathrm{dhp}\,X-k-1,0).$$

b. *Let a complex*

$$\dots\to 0\to X\to I^0\to I^1\to\dots\to I^k\to X'\to 0\to\dots$$

be acyclic and objects I^i *be injective. Then*

$$\mathrm{dhi}\,X'=\max(\mathrm{dhi}\,X-k-1,0).$$

Corollary. a. *If the category* $\mathcal{A}$ *has sufficiently many projective objects* (*any object is isomorphic to a quotient of a projective one*) *then the condition*

$\operatorname{dhp} X \leq k$ *is equivalence to the existence of a projective resolution of* X *of length* $\leq k+1$.

b. *If the category* $\mathcal{A}$ *has sufficiently many injective objects (any object is isomorphic to a subobject of an projective one) then the condition* $\operatorname{dhi} X \leq k$ *is equivalent to the existence of an injective resolution of* X *of length* $\leq k+1$.

3.8. The Hilbert Theorem. In § 4 of Chap. 2 we have formulated the Hilbert theorem about the syzygies for modules over the polynomial ring $k[t_1, \ldots, t_n]$. The next theorem gives a categorical version of this theorem. Being applied to modules over $k[t_1, \ldots, t_n]$, this theorem gives a somewhat weaker result than the classical Hilbert theorem, since we shall obtain a bound for the length of projective resolutions, and not of free ones.

(Actually, any projective module of finite type over the ring of polynomials with coefficients in a field is free. This highly non-trivial theorem was conjectured by Serre and proved independently by Suslin (1976) and Quillen (1976).)

Let $\mathcal{A}$ be an abelian category. Denote by $\mathcal{A}[T]$ the following category:

$$\operatorname{Ob} \mathcal{A}[T] = \{\text{pairs } (X, t), \quad \text{where } X \in \operatorname{Ob} \mathcal{A}, \quad t \in \operatorname{Hom}_{\mathcal{A}}(X, X)\}.$$

A morphism $(X, t) \to (X', t')$ in $\mathcal{A}[T]$ is a morphism $f : X \to X'$ in $\mathcal{A}$ such that $t' \circ f = t \circ f$.

3.9. Theorem. a. *The category* $\mathcal{A}[T]$ *is abelian.*

b. *Let us assume that the category* $\mathcal{A}$ *has sufficiently many projective objects and possesses infinite direct sums. Then for any* $(X, t) \in \operatorname{Ob} \mathcal{A}[T]$ *we have*

$$\operatorname{dhp}_{\mathcal{A}[T]}(X, t) \leq \operatorname{dhp}_{\mathcal{A}} X + 1.$$

c. *Under the same conditions*

$$\operatorname{dhp}_{\mathcal{A}[T]}(X, 0) = \operatorname{dhp}_{\mathcal{A}} X + 1.$$

This theorem clearly implies that $\operatorname{dh}(\mathcal{A}[T]) = 1$.

3.10. Derived Categories and Resolutions. In this subsection we give a description of the derived category using injective resolutions. Let $\mathcal{I}$ be the full subcategory of an abelian category $\mathcal{A}$ formed by all injective objects. Let us consider the category $K^+(\mathcal{I})$ consisting of all left bounded complexes of injective objects and morphisms of complexes modulo homotopic equivalence. Let $K^+(\mathcal{I}) \to D^+(\mathcal{A})$ be a natural functor.

3.10.1. Theorem. a. *The above functor establishes an equivalence of* $K^+(\mathcal{I})$ *with a full subcategory of* $D^+(\mathcal{A})$.

b. *If* $\mathcal{A}$ *has sufficiently many injective objects, then this functor is an equivalence of categories* $K^+(\mathcal{I})$ *and* $D^+(\mathcal{A})$.

3.10.2. Plan of the Proof. First of all, one must verify that the pair of categories $K^+(\mathcal{I}) \subset K^+(\mathcal{A})$ and the system S of quasi-isomorphisms in $K^+(\mathcal{I})$

satisfies the assumptions of Proposition 1.10. This requires the following verifications:

A. Quasi-isomorphisms in $K^+(\mathcal{I})$ form a localizing systems.

B. The condition $\mathbf{b}_2$ of Proposition 1.10 is satisfied.

Applying Proposition 1.10 we obtain that $K^+(\mathcal{I})[S_{\mathcal{I}}^{-1}]$ is a full subcategory in $K^+(\mathcal{A})[S^{-1}] = D^+(\mathcal{A})$. (The last equality is Theorem 2.7.b.)

C. $S_{\mathcal{I}}$ consists of quasi-isomorphisms. Therefore, $K^+(\mathcal{I})[S_{\mathcal{I}}^{-1}] = K^+(\mathcal{I})$.

D. If $\mathcal{A}$ has sufficiently many injective objects then any object from $D^+(\mathcal{A})$ is isomorphic to an objects from $K^+(\mathcal{I})$.

Let us remark also that in this case the composition of functors $D^+(\mathcal{A}) \xrightarrow{\sim} K^+(\mathcal{I}) \to K^+(\mathcal{A})$ is the right adjoint to the localization $K^+(\mathcal{A}) \to D^+(\mathcal{A})$.

3.11. Ext **and Resolutions.** In proving statements B) and C) from the previous subsection the following generalization of the condition $\mathbf{b}_1$ of Proposition 1.10 turns out to be very useful:

(*) Let $s : I^\cdot \to K^\cdot$ be a quasi-isomorphism of an object from $K^+(\mathcal{I})$ with an object from $K^+(\mathcal{A})$. Then there exists a morphism of complexes $t : K^\cdot \to I^\cdot$ such that $t \circ s$ is homotopic to $\mathrm{id}_{I^\cdot}$.

The statement (*), together with the dual statement for complexes of projective objects imply that the natural homomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(X^\cdot, Y^\cdot) \to \mathrm{Hom}_{D(\mathcal{A})}(X^\cdot, Y^\cdot)$$

is an isomorphism in each of the following cases:

(i) $Y^\cdot \in \mathrm{Ob}\,\mathrm{Kom}^+(\mathcal{I})$;

(ii) $X^\cdot \in \mathrm{Ob}\,\mathrm{Kom}^-(\mathcal{P})$ ($\mathcal{P}$ is the class of projective objects in $\mathcal{A}$).

Statements (i) and (ii) imply that the definition of groups $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ from 3.2.1. coincides with the definition given in § 4 of Chap. 2: $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is the n-th cohomology group of the complex

$$0 \to \mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^{-1}, Y) \to \mathrm{Hom}(P^{-2}, Y) \to \dots$$

(resp. of the complex

$$0 \to \mathrm{Hom}(X, I^0) \to \mathrm{Hom}(X, I^1) \to \mathrm{Hom}(X, I^2) \to \dots)$$

where $\dots \to P^{-1} \to P^0 \to X \to 0$ is a projective resolution of X (resp. $0 \to Y \to I^0 \to I^1 \to \dots$ is an injective resolution of Y).

3.12. Equivalence of K^b and D^b. Let $\mathcal{A}$ be an abelian category, $X \in \mathrm{Ob}\,\mathcal{A}$. Denote by $\mathrm{add}\,X$ the full subcategory of $\mathcal{A}$ consisting of finite direct sums of direct summands of X. It is clear that $\mathrm{add}\,X$ is an additive subcategory of $\mathcal{A}$. There exists a natural functor $\varphi_X : K^b(\mathrm{add}\,X) \to D^b(\mathcal{A})$ (composition of the inclusion $K^b(\mathrm{add}\,X) \to K^b(\mathcal{A})$ with the localization $Q : K^b(\mathcal{A}) \to D^b(\mathcal{A})$).

One can verify the following generalizations of 3.11(i), (ii): if $\mathrm{Ext}^i(X, X) = 0$ for all $i > 0$, then φ_X is a fully faithful functor. Moreover, if any object Y of $\mathcal{A}$ has a finite resolution by objects from $\mathrm{add}\,X$ (i.e. if there exists a complex from $\mathrm{Kom}^b(\mathrm{add}\, X)$ quasi-isomorphic to the 0-complex Y), then φ_X is an equivalence of categories.

§ 4. Derived Functors

4.1. Motivations. We have already mentioned in 1.2 that some of the most important additive functors F in abelian categories, such as Hom, $\otimes$, Γ, are not exact, and to restore their exactness we must redefine them. More explicitly, let $F : \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor between abelian categories. In this section we define and study its extension $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ (resp. $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$) which will be called *right* (resp. *left*) *derived functor of* F.

The functors RF (resp. LF) will be *exact* in the following sense: *they map distinguished triangles into distinguished ones.*

In particular, if we define classical derived functors by

$$R^iF = H^i(RF), \quad L^iF = H^i(LF),$$

then to any exact triple $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ we can associate a long exact cohomology sequence

$$\dots \to R^iF(A) \to R^iF(B) \to R^iF(C) \to R^{i+1}F(A) \to \dots$$

and similarly for L.

But what should be an appropriate way to extend F to complexes? The first and most obvious idea is to make F to act on complexes componentwise. In any case, such an extension transforms homotopic morphisms into homotopic ones, so that we obtain functors

$$K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{B}), \quad * = \emptyset, +, -, b.$$

4.2. Proposition. *Assume that F is exact.*

a. *$K^*(F)$ transforms quasi-isomorphisms into quasi-isomorphisms, hence induces a functor $D^*(F) : D^*(\mathcal{A}) \to D^*(\mathcal{B})$.*

b. *$D^*(F)$ is an exact functor, i.e. it transforms distinguished triangles into distinguished triangles.*

Indeed, one can easily verify that $K^*(F)$ maps acyclic (i.e. quasi-isomorphic to 0) complexes $K^\cdot$ into acyclic ones.

For a morphism of complexes $f : K^\cdot \to L^\cdot$ there exists a canonical isomorphism of $F(C(f))$ onto $C(F(f))$. Since f is a quasi-isomorphism if and only if $C(f)$ is acyclic, $F(C(f))$ is an acyclic complex, so that $F(f)$ is a quasi-isomorphism.

Looking over the definitions in 2.2 and 2.3 we see that any additive functor F maps the cylinder of f into the cylinder of $F(f)$, and the main diagram of Lemma 2.3 into a similar diagram. Hence any additive functor F maps a triangle of the form

$$K^{\cdot} \xrightarrow{\bar{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} C(f) \xrightarrow{\delta} K^{\cdot}[1]$$

into a triangle of the same form. If F is exact, then $K^*(F)$ maps distinguished (i.e. quasi-isomorphic to the above) triangles into distinguished triangles.

4.3. Adapted Classes of Objects. The main idea in the construction of derived functors RF and LF in the general case is that we have to apply F componentwise not to an arbitrary complex, but to some appropriately selected representatives in equivalence classes of quasi-isomorphic complexes. For example, as we will see later, to compute $R\Gamma$ we must apply Γ to (bounded from the left) complexes of injective sheaves, and to compute $M \otimes \cdot$ we must use (bounded from the right) complexes of flat modules.

We axiomize the general situation as follows. A class of objects $R \subset \mathcal{A}$ is said to be *adapted* to a (left or right) exact functor F if it is stable under finite direct sums and satisfies the following two conditions:

a. *If F is left exact, it maps any acyclic complex from* $\mathrm{Kom}^+(\mathcal{R})$ *into an acyclic complex.*

If F is right exact, it maps any acyclic complex from $\mathrm{Kom}^-(\mathcal{R})$ *into an acyclic complex.*

b. *If F is left exact, then any object from $\mathcal{A}$ is a subobject of an object from $\mathcal{R}$.*

If F is right exact, then any object from $\mathcal{A}$ is a quotient of an object from $\mathcal{R}$.

(Whenever this condition is satisfied, we say that $\mathcal{R}$ is sufficiently large, or that there are sufficiently many elements in $\mathcal{R}$.)

Let us remark that if F is exact then the first part of Proposition 4.2 implies that any class $\mathcal{R}$ satisfying the condition b) (in particular, the class of all objects of $\mathcal{A}$) is adapted to F.

4.4. Proposition. *Let $\mathcal{R}$ be a class of objects adapted to a left exact functor $F : \mathcal{A} \to \mathcal{B}$ and $S_{\mathcal{R}}$ be a class of quasi-isomorphisms in $K^{\pm}(\mathcal{R})$. (Here and below K^+ is taken for left exact F's, and K^- is taken for right exact F's.) Then $S_{\mathcal{R}}$ is a localizing class of morphisms in $K^{\pm}(\mathcal{R})$ and the canonical functor*

$$K^{\pm}(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to D^{\pm}(\mathcal{A})$$

is an equivalence of categories.

4.5. The Construction of the Derived Functor. Under the assumptions of Proposition 4.4 we define the derived functor RF of a left exact functor F on objects of the category $K^{\pm}(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ term by term:

$$RF(K^{\cdot})^i = F(K^i) \quad \text{for } K^{\cdot} \in K^+(\mathcal{R}),$$
$$LF(K^{\cdot})^i = F(K^i) \quad \text{for } K^{\cdot} \in K^-(\mathcal{R}).$$

Since the componentwise application of F maps acyclic objects from $K^{\pm}(\mathcal{R})$ into acyclic ones, arguments similar to those used in the proof on Proposition 4.2 show that quasi-isomorphisms in $K^{\pm}(\mathcal{R})$ are mapped into quasi-isomorphisms. So we can consider RF as a functor from $K^{\pm}(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ to $D^{\pm}(\mathcal{B})$.

Now we have to choose an equivalence of categories $\Phi : D^{\pm}(\mathcal{A}) \to K^{\pm}(\mathcal{R})[S_{\mathcal{R}}^{-1}]$ that is left inverse to the natural embedding, and define RF and LF by

$$RF(K^{\cdot}) = RF(\Phi(K^{\cdot})), \quad LF(K^{\cdot}) = LF(\Phi(K^{\cdot})).$$

This construction contains some ambiguity: the choice of Φ and, more important, the choice of $\mathcal{R}$ are non-unique. It is rather clear in what sense the construction does not depend on Φ. To state and to prove the independence on the choice of $\mathcal{R}$ we need a formal definition of the derived functor via some universal property.

4.6. Definition. The *derived functor* of an additive left exact functor $F : \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $D^+(F) : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and a morphism of functors $\varepsilon_F : Q_{\mathcal{B}} \circ K^+(F) \to K^+(F) \circ Q_{\mathcal{A}}$:

$$\begin{tikzcd} & D^+(\mathcal{A}) \arrow[dr, "D^+(F)"] & \\ K^+(\mathcal{A}) \arrow[ur, "Q_{\mathcal{A}}"] \arrow[dr, "K^+(F)"'] & & D^+(\mathcal{B}) \\ & K^+(\mathcal{B}) \arrow[ur, "Q_{\mathcal{B}}"'] & \end{tikzcd}$$

satisfying the following universal property: for any exact functor $G : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and any morphism of functor $\varepsilon : Q_{\mathcal{B}} \circ K^+(F) \to G \circ Q_{\mathcal{A}}$ there exists a unique morphism of functors $\eta : D^+(F) \to G$ making the following diagram

$$\begin{tikzcd} & G \circ Q_{\mathcal{A}} \\ Q_{\mathcal{B}} \circ K^+(F) \arrow[ur, "\varepsilon"] \arrow[dr, "\varepsilon_F"'] & \\ & D^+(F) \circ Q_{\mathcal{A}} \arrow[uu, "\eta \circ Q_{\mathcal{A}}"'] \end{tikzcd} \tag{1}$$

commutative.

Similarly, the derived functor of a right exact functor $F : \mathcal{A} \to \mathcal{B}$ is a pair consisting of an exact functor $D^-(F) : D^-(\mathcal{A}) \to D^-(\mathcal{B})$ and a morphism of functors $\varepsilon_F : D^-(F) \circ Q_{\mathcal{A}} \to Q_{\mathcal{B}} \circ K^-(F)$ satisfying the universal property similar to (1) (with a morphism of functors $\eta : G \to D^-(F)$).

Let us remark that if F is exact then, by Proposition 4.2.b $D^*(F)$ coincides with the term by term application of F to complexes.

4.7. Uniqueness of the Derived Functor. Let $(D^*(F), \varepsilon_F)$, $(\widetilde{D}^*(F), \widetilde{\varepsilon}_F)$ be two derived functors for F. By definition, there exist unique morphisms of functors $D^*(F) \xrightarrow{\eta} \widetilde{D}^*(F)$, $\widetilde{D}^*(F) \xrightarrow{\widetilde{\eta}} D^*(F)$ with the required commutativity properties. These commutativity properties imply that $\widetilde{\eta} \circ \eta$ and $\eta \circ \widetilde{\eta}$ are automorphisms of functors $D^*(F)$ and $\widetilde{D}^*(F)$ respectively. Hence, by the uniqueness, they are identity isomorphisms, so that η and $\widetilde{\eta}$ are mutually inverse isomorphisms of functors, which, moreover, are uniquely defined.

4.8. Theorem. *Assume that a left exact functor F admits an adapted class of objects $\mathcal{R}$. Then the derived functor $D^+(F)$ exists and can be defined by the construction from* 4.5 (*i.e.* $D^+(F) = RF$, $D^-(F) = LF$).

The proof of this theorem consists in accurate formalization of the arguments from 4.5.

In applications it is sometimes useful to know that some classes of objects are adapted to all functors that are exact from a fixed side.

4.9. Theorem. *If $\mathcal{A}$ contains sufficiently many injective* (*resp. projective*) *objects then the class of all these objects is adapted to any left exact* (*resp. right exact*) *functor F.*

Proof. Let, say, F be left exact, and let $\mathcal{I}$ be the class of all injective objects. By definition, we have to verify that F maps acyclic complexes from $\mathrm{Kom}^+(\mathcal{I})$ into acyclic ones. Let $I^\cdot$ be such a complex. The zero morphism $0 : I^\cdot \to I^\cdot$ is a quasi-isomorphism. By (*) from 3.12, it is homotopic to $\mathrm{id}_{I^\cdot}$. Hence the zero morphism of $F(I^\cdot)$ is homotopic to $\mathrm{id}_{F(I^\cdot)}$, so that $F(I^\cdot)$ is acyclic.

Up to now we have derived the existence of the derived functor from the existence of an adapted class. A partial inversion of this result sounds as follows.

Let us assume that $D^+(F)$ exists. An object X is said to be F-*acyclic* if $D^iF(X) = 0$ for all $i \neq 0$.

4.10. Theorem. a. *A class of objects adapted to F exists if and only if the class $\mathcal{Z}$ of all F-acyclic objects is sufficiently large, i.e. if any object is a subobject of an F-acyclic one.*

b. *If $\mathcal{Z}$ is sufficiently large then it contains any adapted to F class and any sufficiently large subclass of $\mathcal{Z}$ is adapted.*

c. *If $\mathcal{Z}$ is sufficiently large then it contains all injective objects.*

Proof. Let there exists a class of objects $\mathcal{R}$ adapted to F. By the construction from 4.5, $DF(X[0])$ is quasi-isomorphic to $F(X)[0]$ for any $X \in \mathcal{R}$. Hence $\mathcal{R} \subset \mathcal{Z}$ and $\mathcal{Z}$ is sufficiently large.

Conversely, let $\mathcal{R}$ be a sufficiently large subclass in $\mathcal{Z}$. To prove that it is adapted to F we have to prove that F maps acyclic complexes from $\mathrm{Kom}^+(\mathcal{R})$ into acyclic ones. If this acyclic complex is a triple $0 \to K^0 \to K^1 \to K^2 \to 0$ then the exactness of $0 \to F(K^0) \to F(K^1) \to F(K^2) \to 0$ follows from the equality $D^1F(K^0) = 0$. In the general case we can successively split

$$0 \to K^0 \to K^1 \to X^1 \to 0,$$
$$0 \to K^1 \to K^2 \to X^2 \to 0,$$

and so on. We have $X^{i+1} \in \mathcal{Z}$ because $X^i, K^{i+1} \in \mathcal{Z}$. Hence the triples $0 \to F(X^i) \to F(K^{i+1}) \to F(X^{i+1}) \to 0$ are exact and $F(K^\cdot)$ is acyclic.

Let, finally, $\mathcal{Z}$ be sufficiently large and F is still left exact. Embed an injective object I into an acyclic object X. By the injectivity diagram

$$\begin{tikzcd}
I & & \\
X \arrow[u, dashed, "\varphi"] & I \arrow[l] \arrow[ul, "id"'] & 0 \arrow[l]
\end{tikzcd}$$

we see that φ splits off I as a direct summand of X. Since F is an additive functor, $D^iF(I)$ is a direct summand of $D^iF(X) = 0$ for $i \neq 0$.

4.11. Classical Derived Functors. a. A functor H from a derived (or homotopic) category to an abelian category is said to be cohomological if for any distinguished triangle $X \to Y \to Z \to X[1]$ the sequence

$$\cdots \to H(T^iX) \to H(T^iY) \to H(T^iZ) \to H(T^{i+1}X) \to \ldots$$

is exact. For example, $H = H^0$ is a cohomological functor (see 2.4 and 1.5.1 of Chap. 1). Another example: $H = \mathrm{Hom}(U, \cdot)$ (see 1.3 of Chap. 5).

b. Let F be a left exact (resp. right exact) functor between two abelian categories. Then $R^iF = H^0(T^i(D^+F)) = H^i(D^+F)$ (resp. $L^iF = H^i(D^-F)$) is called *the classical i-th derived functor* for F. One can easily see that $R^iF = 0$ for $i < 0$, $R^0F = F$ (resp. $L^iF = F$ for $i > 0$, $L^0F = F$).

Examples see in Chap. 2, § 4.

4.12. Weak Derived Functors. There exists another construction of derived functors. It guarantees the existence of, say, RF in a much more general situation; however, now RF will take values not in the derived category, but in its extension.

4.12.1. Category of Coindices. A category I is said to be a category of *coindices* if it is small, non-empty, and satisfies the following conditions:

a. I is connected, i.e. any two objects can be joint by a sequence of morphisms (directions are irrelevant).

b. Any pair of morphisms $j' \leftarrow i \to j$ can be embedded into a commutative square

$$\begin{tikzcd}
i \arrow[r] \arrow[d] & j \arrow[d, dashed] \\
j' \arrow[r, dashed] & k
\end{tikzcd}$$

c. For any pair of morphisms $u, v : i \to j$ there exists a morphism $w : j \to k$ such that $w \circ u = w \circ v$.

A category J is said to be a category of *indices* if J° is a category of coindices.

4.12.2. Inductive Limits. Let I be a category of coindices and $F : I \to \mathcal{C}$, $i \mapsto X_i$ be a functor. It determines the functor $\widehat{F} : I \to \widehat{\mathcal{C}} = \mathbf{Funct}(\mathcal{C}^\circ, \mathbf{Set})$ by

$$\widehat{F}(i) = \mathrm{Hom}_{\mathcal{C}}(Y, F(i)), \qquad Y \in \mathrm{Ob}\,\mathcal{C}^\circ = \mathrm{Ob}\,\mathcal{C}, \quad i \in \mathrm{Ob}\,I.$$

Define an object L of the category $\widehat{\mathcal{C}}$ as the inductive limit $L = \lim\mathrm{ind}\,\widehat{F}$ (see Chap. 2, 1.21), so that for any $Y \in \mathrm{Ob}\,\mathcal{C}^\circ$

$$L(Y) = \lim\mathrm{ind}\,F_Y,$$

where $F_Y : I \to \mathbf{Set}$, $F_Y(i) = \mathrm{Hom}_{\mathcal{C}}(Y, F(i))$. The existence of L follows from the existence, for any Y, of the inductive limit $L = \lim\mathrm{ind}\,F_Y$ in the category $\mathbf{Set}$.

Such functors L (for all possible I and F) for a full subcategory $\mathrm{Ind}\,\mathcal{C}$ of $\mathcal{C}$, which is called the *category of inductive limits* in $\mathcal{C}$.

Define the *category of projective limits* $\mathrm{Pro}\,\mathcal{C}$ in $\mathcal{C}$ as $(\mathrm{Ind}\,\mathcal{C}^\circ)^\circ$. This is a subcategory of $(\widehat{\mathcal{C}^\circ})^\circ$. In $\mathrm{Pro}\,\mathcal{C}$ one can take limits of functors of the form $J \to \mathcal{C}$, where J is a category of indices.

4.12.3. Inductive Limits and Localization. Let S be a localizing class of morphisms in a category $\mathcal{C}$. It is said to be *saturated* if any morphism which is both a right divisor of some morphism in S and a left divisor of some morphism in S itself belongs to S.

For any object $X \in \mathrm{Ob}\,\mathcal{C}$ the category I_X of morphisms $s : X \to X'$, $s \in S$, is a category of coindices, and the category J_X of morphisms $s : X' \to X$, $s \in S$, is a category of indices. Let

$$X^+ = \lim_{I_X}\mathrm{ind} X', \quad X^- = \lim_{J_X}\mathrm{proj} X'.$$

Assume that S is saturated. Then the mappings $X \to X^+$, $X \to X^-$ can be extended to functors $\mathcal{C} \to \mathrm{Ind}\,\mathcal{C}$ and $\mathcal{C} \to \mathrm{Pro}\,\mathcal{C}$, which map morphisms from S into isomorphisms. Hence they determine canonical functors $\mathcal{C}[S^{-1}] \to \mathrm{Ind}\,\mathcal{C}$ and $\mathcal{C}[S^{-1}] \to \mathrm{Pro}\,\mathcal{C}$.

4.12.4. Definition. Let $\mathcal{A}$, $\mathcal{B}$ be two abelian categories, $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Denote also by F the term by term extension of F to a functor from $K^*(\mathcal{A})$ to $K^*(\mathcal{B})$. Let $S_{\mathcal{A}}$ (resp. $S_{\mathcal{B}}$) be the class of quasi-isomorphisms in $K^*(\mathcal{A})$ (resp. $K^*(\mathcal{B})$). Define the *weak right derived functor*

$$R_w F : K^*(\mathcal{A})[S_{\mathcal{A}}^{-1}] = D^*(\mathcal{A}) \to \mathrm{Ind}\,K^*(\mathcal{B})[S_{\mathcal{B}}^{-1}] = \mathrm{Ind}\,D^*(\mathcal{B})$$

as a unique functor for which the diagram

$$\begin{tikzcd}
K^*(\mathcal{A}) \arrow[r, "X \mapsto F(X^+)"] \arrow[d] & \mathrm{Ind}\,K^*(\mathcal{B}) \arrow[d] \\
K^*(\mathcal{A})[S_{\mathcal{A}}^{-1}] \arrow[r, "R_w F"] & \mathrm{Ind}\,K^*(\mathcal{B})[S_{\mathcal{B}}^{-1}]
\end{tikzcd}$$

is commutative (here $F(X^+) = \lim\limits_{I_X} \operatorname{ind} F(X'))$.

If R_w takes values in the subcategory of $\operatorname{Ind} D^*(\mathcal{B})$ formed by representable objects then it "coincides" with RF.

An object $X \in KK^*(\mathcal{A})$ is said to be *F-acyclic from the right* if the canonical morphism $F(X) \to R_wF(X)$ is an isomorphism.

The *weak left derived functor* is defined similarly using the diagram

$$\begin{array}{ccc} K^*(\mathcal{A}) & \xrightarrow{X \mapsto F(X^-)} & \operatorname{Pro} K^*(\mathcal{B}) \\ \downarrow & & \downarrow \\ K^*(\mathcal{A})[S_{\mathcal{A}}^{-1}] & \xrightarrow{L_wF} & \operatorname{Pro} K^*(\mathcal{B})[S_{\mathcal{B}}^{-1}] \end{array}$$

4.12.5. Relation to Derived Functors. Keeping the assumptions of the previous subsection, let us assume also that any object of $\mathcal{A}$ is a quotient (resp. a subobject) of some left (resp. right) F-acyclic object. Then the functor L_wF on $D^-(\mathcal{A})$ (resp. the functor R_wF on $D^+(\mathcal{B})$) takes values in $D^-(\mathcal{B})$ (resp. in $D^+(\mathcal{B})$) and, therefore, "coincides" with LF (resp. with RF).

4.13. Exactness of the Functor lim proj. Let $\mathcal{A}$ be an abelian category with countable direct products, $\mathcal{C}(\mathbb{Z}^+)$ be the category of the ordered set of positive integers (see 1.23 in Chap. 2). The following results hold:

a. The category $\mathcal{A}^{\mathbb{Z}^+} = \mathbf{Funct}(\mathcal{C}(\mathbb{Z}^+), \mathcal{A}^\circ)$ is abelian.

b. The functor $\lim\operatorname{proj} : \mathcal{A}^{\mathbb{Z}^+} \to \mathcal{A}$ is right exact.

c. An object $X = (X_i, p_{ij}) \in \mathcal{A}^{\mathbb{Z}^+}$ ia said to satisfy the *condition* ML (Mittag-Leffler) if for any i there exists $j > i$ such that $p_{ij} : X_i \to X_j$ is an epimorphism. Then the class of all objects X satisfying the condition ML is adapted to the functor lim proj.

d. If $0 \to X \to S \to Y \to 0$ is an exact sequence in $\mathcal{A}^{\mathbb{Z}^+}$ and S satisfies the condition ML, then Y also satisfies this condition. This implies that the right derived functors $R^i \lim\operatorname{proj}$ vanish for $i \geq 2$.

4.14. Unbounded Complexes. Below we present some results of Spaltenstein (1988) that make it possible to work with unbounded complexes in derived categories.

By a *left projective resolution* of a complex $A^\cdot \in \operatorname{Ob}\operatorname{Kom}(\mathcal{A})$ we mean a quasi-isomorphism $P^\cdot \to A^\cdot$, where all P^i's are projective in $\mathcal{A}$. We define a *right injective resolution* similarly.

Theorem 3.10.1 (and its analog for projective resolutions) claims that any object $A^\cdot \in \operatorname{Ob}\operatorname{Kom}^+(\mathcal{A})$ (resp. $A^\cdot \in \operatorname{Ob}\operatorname{Kom}^-(\mathcal{A})$) has a right injective (resp. left projective) resolution, which is unique up to a homotopic equivalence.

Without the boundness assumptions the uniqueness may fail: for $\mathcal{A} = (\mathbb{Z}/4)$-mod the complex

$$P^\cdot : \ldots \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \ldots$$

is acyclic and consists of free $\mathbb{Z}/4$-modules. Hence it is a left projective resolution of the zero complex. However, the morphism $P^\cdot \to 0^\cdot$ is not a homotopic equivalence: tensoring with $\mathbb{Z}/2$ we obtain the complex

$$P^\cdot \otimes \mathbb{Z}/2 : \ldots \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \ldots,$$

which has nonzero cohomology and, therefore, is not homotopic to $0^\cdot$.

4.14.1. Definition. A complex $P^\cdot$ is said to be K-*projective* if for any acyclic complex $A^\cdot$ the complex of abelian groups $\mathrm{Hom}^\cdot(P^\cdot, A^\cdot)$ is acyclic. Similarly one can define K-*injective* complexes $I^\cdot$ (using $\mathrm{Hom}^\cdot(A^\cdot, I^\cdot)$).

4.14.2. Properties of K-projective Complexes. a. Let $A^\cdot$ be a 0-complex (i.e. $A^i = 0$ for $i \neq 0$). Then $A^\cdot$ is K-projective if and only if A^0 is projective in $\mathcal{A}$.

b. If any two vertices of a distinguished triangle in $D(\mathcal{A})$ are K-projective, so is the third.

c. For $P^\cdot \in \mathrm{Kom}(\mathcal{A})$ the following conditions are equivalent:

(i) $P^\cdot$ is K-projective.

(ii) For any $A^\cdot \in \mathrm{Ob}\,\mathrm{Kom}(\mathcal{A})$ the natural homomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\cdot, A^\cdot) \to \mathrm{Hom}_{D(\mathcal{A})}(P^\cdot, A^\cdot)$$

is an isomorphism.

(iii) Any quasi-isomorphism $s : A^\cdot \to P^\cdot$ admits a right inverse $t : P^\cdot \to A^\cdot$ in $K(\mathcal{A})$.

d. By a K-projective (left) resolution of a complex $A^\cdot$ we mean a quasi-isomorphism $P^\cdot \to A^\cdot$ with a K-projective $P^\cdot$. Such a K-projective resolution (if it exists) is unique up to a homotopic equivalence. If $A^\cdot \in \mathrm{Ob}\,\mathrm{Kom}^-(\mathcal{A})$ and $\mathcal{A}$ has sufficiently many projective objects, then a K-projective resolution is a left projective resolution of $A^\cdot$.

Similar results hold for K-injective (right) resolutions.

4.14.3. By the property 4.14.3.d above we can use K-projective and K-injective resolutions to compute values of derived functors on unbounded complexes. In particular, to compute $R\,\mathrm{Hom}(A^\cdot, B^\cdot)$ we can use either a K-projective resolution of $A^\cdot$ or a K-injective resolution of $B^\cdot$.

As to the existence of K-resolutions, the following results are proved in Spaltenstein (1988).

a. Let R be an associative ring with unity and $\mathcal{A} = R$-mod. Then any complex $A^\cdot \in \mathrm{Ob}\,\mathrm{Kom}(\mathcal{A})$ admits a K-projective and a K-injective resolutions.

b. Let $\mathcal{O}$ be a sheaf of rings on a topological space X and $\mathcal{A}$ be the category of sheaves of $\mathcal{O}$-modules. Then any complex $A^\cdot \in \mathrm{Ob}\,\mathrm{Kom}(\mathcal{A})$ admits a K-injective resolution.

4.14.4. Similarly one can define and prove the existence of K-flat resolutions (used to compute the derived functors for tensor product), K-soft resolutions (used to compute $Rf_!$, see the next section), and so on.

4.15. Derived Functor of the Composition and the Spectral Sequence. The theorem about the Grothendieck spectral sequence for the composition of two left exact functors has the following interpretation in terms of derived categories and derived functors.

4.15.1. Theorem. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be three abelian categories, $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be two additive left exact functors. Let $\mathcal{R}_\mathcal{A} \subset \mathrm{Ob}\,\mathcal{A}$ (resp. $\mathcal{R}_\mathcal{B} \subset \mathrm{Ob}\,\mathcal{B}$) be a class of objects adapted to F (resp. to G). Assume moreover that $F(\mathcal{R}_\mathcal{A}) \subset \mathcal{R}_\mathcal{B}$. Then the derived functors RF, RG, $R(G \circ F) : D^+(\cdot) \to D^+(\cdot)$ exist and the natural morphism of functors $R(G \circ F) \to RG \circ RF$ is an isomorphism.*

Proof. The definition of an adapted class in 4.3 and the conditions of the theorem imply that $\mathcal{R}_\mathcal{A}$ is adapted not only to F, but also to $G \circ F$. Hence, RF, RG and $R(G \circ F)$ exist and to compute them we can use the construction from 4.6.

Since RF and RG are exact functors, the composition $R(G \circ F)$ is also exact and the morphism $E : R(G \circ F) \to RG \circ RF$ is defined by the universality property from 4.6.

For $K^\cdot \in \mathrm{Ob}\,\mathrm{Kom}^+(\mathcal{R}_\mathcal{A})$ the morphism $E(K^\cdot) : R(G \circ F)(K^\cdot) \to RG \circ RF(K^\cdot)$ is an isomorphism. Since any object of $D^+(\mathcal{A})$ is isomorphic to such an object $K^\cdot$, E is an isomorphism of functors.

A similar result holds for right exact functors.

§ 5. Sheaf Cohomology

5.1. Proposition. *Let X be a topological space, $\mathcal{R}$ be a sheaf of rings with unity on X. Then any sheaf of $\mathcal{R}$-modules can be embedded into an injective sheaf of $\mathcal{R}$-modules.*

The proof is based on the following standard method of Godement. Let $\mathcal{F}$ be a sheaf of (say, left) $\mathcal{R}$-modules. For any point $x \in X$ we can construct a monomorphism $\mathcal{F}_x \hookrightarrow I(x)$ of $\mathcal{R}_x$-modules, where $I(x)$ is injective over $\mathcal{R}_x$. Let us define now a sheaf of $\mathcal{R}$-modules $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I(x), \qquad U \text{ open in } X$$

(with obvious restriction mappings). We have a canonical embedding $\mathcal{F} \to \mathcal{I}$ and one can easily verify that $\mathcal{I}$ is injective.

5.2. Direct Images and Cohomology. Proposition 5.1 enables us to construct the derived functor for the direct image in the following situation. Let $(f, \varphi) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a morphism of ringed spaces, so that $\varphi : \mathcal{R}_Y \to f_\cdot(\mathcal{R}_X)$ is a morphism of sheaves of modules. Then for any $\mathcal{F} \in \mathcal{R}_X\text{-mod}$, the sheaf $f_\cdot(\mathcal{F})$ inherits, via φ, a natural structure of an

$\mathcal{R}_Y$-module and the functor $f_\cdot : \mathcal{R}_X\text{-mod} \to \mathcal{R}_Y\text{-mod}$ is left exact. Hence we can construct the right derived functor

$$Rf_\cdot : D^+(\mathcal{R}_X\text{-mod}) \to D^+(\mathcal{R}_Y\text{-mod})$$

In particular, when Y is a point, so that $\mathcal{R}_Y = \mathbb{Z}$, we obtain the derived functor $R\Gamma : D^+(\mathcal{R}_X\text{-mod}) \to D^+(\mathbf{Ab})$ and $R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$ is *the cohomology of X with coefficients in $\mathcal{F}$.*

5.3. Theorem. a. *Let $\Phi : \mathcal{R}_X\text{-mod} \to \mathbf{SAb}$ be the forgetful functor (of the structure of $\mathcal{R}_X$-module). Then the functors $R\Gamma$ and $R(\Gamma \circ \Phi)$ are naturally isomorphic. In the other words, in computing $H^i(X, \mathcal{F})$ it does not matter whether we consider $\mathcal{F}$ as a $\mathcal{R}_X$-module or just a sheaf of abelian groups.*

b. *Let $X = \bigcup U_i$ be an open covering with $H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$ for all $q > 0$, $p \geq 1$. Then $H^i(X, \mathcal{F})$ coincides with the i-dimensional cohomology of the Čech complex of this covering* (see 2.6 in Chap. 1).

c. $H^i(X, \mathcal{F}) = \mathrm{Ext}_{\mathcal{R}_X\text{-mod}}(\mathcal{R}_X, \mathcal{F})$.

d. *Let $f : X \to Y$ be a mapping of topological spaces. Then $R^q f_\cdot(\mathcal{F})$ is naturally isomorphic to the sheaf associated to the presheaf $U \to H^q(f^{-1}(U), \mathcal{F})$.*

e. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be three topological spaces and two mappings, $\mathcal{F}$ be a sheaf of $\mathcal{R}_X$-modules. There exists a spectral sequence with $E_2^{pq} = R^p g_\cdot(R^q f_\cdot(\mathcal{F}))$ converging to $R^{p+q}(gf)_\cdot(\mathcal{F})$; it is functorial in $\mathcal{F}$.*

5.4. Proof of Theorem 5.3. We will comment here on the proofs of parts **a** and **e**.

5.4.1. Flabby Sheaves. Proof of 5.3.a. It is clear that $\Gamma = \Gamma \circ \Phi$ (as functors from $\mathcal{R}_X$-mod to $\mathbf{Ab}$), so that $R\Gamma = R(\Gamma \circ \Phi)$. We will show that we can apply Theorem 4.15.1 and get the natural isomorphism $R(\Gamma \circ \Phi) = R\Gamma \circ R\Phi$. The required statement would then follow since, by the exactness of Φ, $R\Phi$ coincides with the componentwise application of Φ.

To apply Theorem 4.15.1 we must show that there exists a class of sheaves of $\mathcal{R}_X$-modules that is adapted to Φ and is transformed by Φ into a class of sheaves adapted to Γ. As the first class we take injective $\mathcal{R}_X$-modules, and as the second class we take flabby sheaves of abelian groups. Let us recall that a sheaf $\mathcal{F}$ is said to be *flabby* if the restriction maps $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ are surjective for all open $U \subset X$. We list here all the facts that should be verified.

a. Any sheaf of abelian groups is a subsheaf of a flabby sheaf. Any injective sheaf of $\mathcal{R}_X$-modules is flabby.

b. Let

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0 \tag{1}$$

be an exact sequence of sheaves of abelian groups with $\mathcal{F}$ flabby. Then the sequence

$$0 \to \Gamma(X, \mathcal{F}) \xrightarrow{\Gamma(\varphi)} \Gamma(X, \mathcal{G}) \xrightarrow{\Gamma(\psi)} \Gamma(X, \mathcal{H}) \to 0 \tag{2}$$

is exact.

c. If in (1) the sheaves $\mathcal{F}$ and $\mathcal{G}$ are flabby, then $\mathcal{H}$ is also flabby.

d. Γ transforms a left bounded acyclic complex of flabby sheaves into an acyclic complex of abelian groups.

5.4.2. Proof of Theorem 5.3.e. Part **a** of the Theorem shows that it suffices to prove our statement for the category **SAb** of sheaves of abelian groups. We apply Theorem 4.5.2 of Chap. 2 to the pair of functors $F = f_.$, $G = g_.$. This is possible because $f_.$ maps injective sheaves on X into injective sheaves on Y. Indeed, injective sheaves $\mathcal{F}$ can be characterized by the property that $\mathrm{Hom}(\mathcal{G},\mathcal{F}) \to \mathrm{Hom}(\mathcal{G}',\mathcal{F})$ is an epimorphism for any monomorphism of sheaves $\mathcal{G}' \to \mathcal{G}$. But by 3.13.1 from Chap. 3, $\mathrm{Hom}(\mathcal{G}, f_.\mathcal{F}) = \mathrm{Hom}(f^{\cdot}\mathcal{G},\mathcal{F})$ and, by 3.14.1.a from Chap. 3, $F^{\cdot}$ is an exact functor. So $f^{\cdot}$ maps monomorphisms into monomorphisms, and $f_.\mathcal{F}$ is an injective sheaf whenever $\mathcal{F}$ is.

5.5. Functor $\Gamma_{[Z]}$. Canonical Decomposition. Let $i : Z \to X$ be a closed embedding, $j : U \to X$ be the embedding of the complementary open set. Consider the functor $\Gamma_{[Z]} : \mathbf{SAb}_X \to \mathbf{SAb}_X$ defined by $\Gamma_{[Z]}(\mathcal{F})(V) = \xi \in \mathcal{F}(V)$, $\operatorname{supp}\xi \subset V\}$. This functor is left exact and for any sheaf $\mathcal{F}$ we have the following exact sequence

$$0 \to \Gamma_{[Z]}(\mathcal{F}) \to \mathcal{F} \xrightarrow{\alpha} j_.j^{\cdot}(\mathcal{F}).$$

For an injective sheaf $\mathcal{F}$ the morphism α is an epimorphism. Hence we have the following distinguished triangle in $D^b(\mathbf{SAb}_X)$:

$$R\Gamma_{[Z]}(\mathcal{F}) \to \mathcal{F} \to Rj_.j^{\cdot}(\mathcal{F}) \to R\Gamma_{[Z]}(\mathcal{F})[1].$$

This triangle is called the *canonical decomposition of the sheaf* $\mathcal{F}$ with respect to the pair (U, Z).

5.6. Tensor Products and Flat Sheaves. Let $\mathcal{R}$ be a sheaf of rings with unit on a topological space X. For any sheaf of left $\mathcal{R}$-modules $\mathcal{N}$ we can define a functor

$$\cdot \otimes \mathcal{N} : \text{mod-}\mathcal{R} \to \mathbf{SAb}_X, \quad \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{R}} \mathcal{N}.$$

Similarly to the corresponding statement for modules over a ring, one can prove that this functor is right exact.

A sheaf $\mathcal{N}$ is said to be *flat* (over $\mathcal{R}$) if the functor $\cdot \otimes \mathcal{N}$ is exact. The reader can easily verify that $\mathcal{N}$ is flat if and only if its stalk $\mathcal{N}_X$ at any point $x \in X$ is a flat module over the ring $\mathcal{R}_X$.

5.6.1. Proposition. a. *Any sheaf of left $\mathcal{R}$-modules is a quotient of a flat sheaf.*

b. *The class of all flat sheaves is adapted to the functor $\mathcal{M} \otimes \cdot : \mathcal{R}$-mod $\to$* **SAb** *of tensoring with an arbitrary sheaf $\mathcal{M}$ of right $\mathcal{R}$-modules.*

5.7. Inverse Images and Tensor Products. By the previous proposition, we can construct the left derived functor

$$\mathcal{M} \overset{L}{\otimes} \cdot : D^-(\mathcal{R}\text{-mod}) \to D^-(\mathbf{SAb}).$$

Its cohomology sheaves are denoted Tor:

$$\mathrm{Tor}_i(\mathcal{M}, \mathcal{N}) = H^i(\mathcal{M} \overset{L}{\otimes} \mathcal{N}).$$

We can construct also the functor

$$\mathcal{M}^\cdot \overset{L}{\otimes} \cdot : D^-(\mathcal{R}\text{-mod}) \to D^-(\mathbf{SAb})$$

for $\mathcal{M}^\cdot \in D^-(\mathcal{R}\text{-mod})$.

In the same way one can define the functors

$$\cdot \otimes^L \mathcal{N}, \quad \cdot \otimes^L \mathcal{N}^\cdot : D^-(\mathcal{R}\text{-mod}) \to D^-(\mathbf{SAb})$$

for $\mathcal{N} \in \mathcal{R}$-mod, $\mathcal{N}^\cdot \in D^-(\mathcal{R}\text{-mod})$, and the bifunctor

$$\cdot \overset{L}{\otimes} \cdot : D^-(\mathcal{R}\text{-mod}) \times D^-(\mathcal{R}\text{-mod}) \to D^-(\mathbf{SAb}).$$

Similarly to modules over a ring, the sheaf $\mathrm{Tor}_i(\mathcal{M}, \mathcal{N})$ does not depend on whether we define it using $\mathcal{M} \overset{L}{\otimes} \cdot$, or $\cdot \overset{L}{\otimes} \mathcal{N}$, or $\cdot \overset{L}{\otimes} \cdot$. If $\mathcal{R}$ is a sheaf of commutative (or supercommutative) rings then left modules can be identified with right ones and $\mathcal{M} \otimes \mathcal{N}$ has the structure of an $\mathcal{R}$-module, so that $\mathcal{M} \overset{L}{\otimes} \cdot$ takes values in $D^-(\mathcal{R}\text{-mod})$.

Let now $(f, \varphi) : (X, \mathcal{R}_X) \to (Y, \mathcal{R}_Y)$ be a morphism of (super)commutatively ringed spaces. For any sheaf of $\mathcal{R}_X$-modules $\mathcal{F}$ we can define a sheaf of $\mathcal{R}_Y$-modules

$$f^*(\mathcal{F}) = \mathcal{R}_X \underset{f^\cdot(\mathcal{R})}{\otimes} f^\cdot(\mathcal{F}).$$

The corresponding left derived functor

$$Lf^*(\mathcal{F}) = \mathcal{R}_X \underset{f^\cdot(\mathcal{R})}{\overset{L}{\otimes}} f^\cdot(\mathcal{F})$$

determines higher inverse image functors:

$$L_i f^*(\mathcal{F}) = H^{-i}\Big(\mathcal{R}_X \underset{f^\cdot(\mathcal{R})}{\overset{L}{\otimes}} f^\cdot(\mathcal{F})\Big).$$

A morphism (f, φ) is said to be *flat* if $\mathcal{R}_X$ is a flat $f^\cdot(\mathcal{R}_Y)$-module. This property is one of the weakest and, at the same time, one of the most useful algebraic analogies of the geometrical notion of a "locally trivial fibration". It is widely used in algebraic and analytic geometry.

5.8. Higher Direct Images with Compact Support. Up to the end of this section we will consider only sheaves of abelian groups on locally compact topological spaces. Later we will impose also some finite-dimensionality conditions which are not, however, too restrictive; in particular, they hold for all topological manifolds. In this situation to any mapping $f : X \to Y$ we will associate functors $Rf_!$ and $f^!$ between appropriate derived categories.

5.8.1. Lemma-Definition. Let $f : X \to Y$ be a morphism of locally compact topological spaces and $\mathcal{F}$ be a sheaf on X. For any open $U \subset X$ let

$$f_!(\mathcal{F})(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \quad \text{such that } \operatorname{supp}(s) \xrightarrow{f} U \text{ is proper}\}.$$

(Recall that a morphism is said to be *proper* if the preimage of any compact set is compact.) Then

a. $f_!(\mathcal{F})$ is a subsheaf of $f_.(\mathcal{F})$.

b. The mapping $\mathcal{F} \to f_!(\mathcal{F})$ can be extended to a left exact functor.

5.9. Sections with Compact Support. An important special case of the above situation occurs when $f : X \to pt$ is the mapping to a point. In this case $f_!(\mathcal{F})$ is an abelian group formed by all sections $s \in \Gamma(X, \mathcal{F})$ such that $\operatorname{supp}(s)$ is a compact subset in X. This group is called the *group of sections of $\mathcal{F}$ with compact support* and is denoted by $\Gamma_c(X, \mathcal{F})$.

For an arbitrary $f : X \to Y$ the sheaf $f_!(\mathcal{F})$ can be, in some sense, recovered from the groups of compactly supported sections of $\mathcal{F}$ over various subsets of Y. More explicitly, we have the following result.

5.9.1. Proposition. *The stalk of the sheaf $f_!(\mathcal{F})$ at a point $y \in Y$ is isomorphic to $\Gamma_c(f^{-1}(Y), \mathcal{F}\mid_{f^{-1}(Y)})$.*

5.10. Sheaves Adapted to $f_!$. A sheaf $\mathcal{F}$ on X is said to be *soft* if for any closed $K \subset X$ the restriction $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is surjective.

Since any injective sheaf is flabby (see 5.4.a), and any flabby sheaf is, by obvious reasons, soft, the class of soft sheaves is sufficiently large.

5.10.1. Proposition. *The class of soft sheaves is adapted to the functor $f_!$.*

The proof is essentially similar to the proof of the part **a** of Proposition 5.3. The main step of this proof is the following statement.

(*) For an exact sequence of soft sheaves (1) the mapping $\Gamma_c(X, \mathcal{G}) \to \Gamma_c(X, \mathcal{H})$ is surjective.

Let us prove this statement. For $s \in \Gamma_c(X, \mathcal{H})$ let K be a compact set containing $\operatorname{supp} s$. Let us cover K by a finite number of compact sets $K_1, \dots, K_n$ such that $s\mid_{K_i}$ can be lifted to some section $t_i \in \Gamma(K_i, \mathcal{G})$. Denote $L_i = K_1 \cup \dots \cup K_i$ and prove by induction in i that there exists a section $r_i \in \Gamma(L_i, \mathcal{G})$ with $\psi(r_i) = s\mid_{L_i}$. Let us assume that r_{i-1} is already constructed. Denote $v = r_{i-1}\mid_{L_{i-1}\cap K_i} - t_i\mid_{L_{i-1}\cap K_i}$. We have $\psi(v) = 0$, so that $v = \varphi(v')$ for some $v' \in \Gamma(L_{i-1} \cap K_i, \mathcal{F})$. Extend v' to a section v'' of the sheaf $\mathcal{F}$ over K_i (using the softness of $\mathcal{F}$). and set $t_i' = t_i + \varphi(v'')$. Then the restrictions of t_i' and of r_{i-1} to $L_{i-1} \cap K_i$ coincide, so that they can be glued together to the required section r of the sheaf $\mathcal{G}$ over $L_i = L_{i-1} \cup K_i$.

Hence we have constructed a section $r \in \Gamma(K, \mathcal{G})$ with $\psi(r) = s$. Let M be the boundary of K. Then $\psi(r\mid_M) = 0$ so that $r\mid_M = \varphi(u)$ for some $u \in \Gamma(M, \mathcal{F})$. Since $\mathcal{F}$ is a soft sheaf, u can be extended to a section $u' \in \Gamma(K, \mathcal{F})$. Then $r' = r - \varphi(u')\mid_M$ so that we can extend r' by zero outside K, obtaining a section $s' \in \Gamma_c(X, \mathcal{F})$ with $\psi(s') = s$.

Let us remark that in the proof we have used only that $\mathcal{F}$ is a soft sheaf.

5.11. Higher Direct Images with Compact Support. The previous proposition enables us to define the right derived functor

$$Rf_! : D^+(\mathbf{SAb}_X) \to D^+(\mathbf{SAb}_Y).$$

Its cohomology sheaves are called *higher direct images with compact support* and denoted by $R^i f_!(\mathcal{F}) \in \mathbf{SAb}_Y$ (for $\mathcal{F} \in \mathbf{SAb}_X$).

In particular, for $f : X \to$ point we obtain the functor

$$R\Gamma_c : D^+(\mathbf{SAb}_X) \to D^+(\mathbf{Ab})$$

and its cohomology functors $H_c^i(X, \mathcal{F})$ (*cohomology of* $\mathcal{F}$ *with compact support*).

Let us list several properties of $Rf_!$.

a. The stalk of $R^i f_!(\mathcal{F})$ at a point $y \in Y$ is canonically isomorphic to $H_c^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$. This follows from Proposition 5.9 and from the softness of the restriction of a soft sheaf to $f^{-1}(y)$.

b. For continuous mappings $f : X \to Y$, $g : Y \to Z$ we have

$$R(gf)_! = Rg_! \circ Rf_! \tag{3}$$

(this follows from the fact that $f_!$ maps soft sheaves on X to soft sheaves on Y).

Using the results of Chap. 2, § 4, we can rewrite (3) as a spectral sequence connecting $R^p f_!$, $R^q g_!$ and $R^{p+q}(gf)_!$.

5.12. Inverse Image with Compact Support. By the general ideology (see Chap. 2, 3.13), for any mapping $f : X \to Y$ the inverse image $f^!$ with compact support should be defined as a functor from the category of sheaves on Y to the category of sheaves on X that is adjoint to $f_! : \mathbf{SAb}_X \to \mathbf{SAb}_Y$. However, for a general f the functor $f_!$ does not have the right adjoint functor, and to define $f^!$ we must pass to derived categories. Moreover, we have to assume X and Y to be finite-dimensional.

5.12.1. Theorem. *Let $f : X \to Y$ be a continuous mapping of locally compact finite-dimensional topological spaces. There exists a functor*

$$f^! : D^+(\mathbf{SAb}_Y) \to D^+(\mathbf{SAb}_X)$$

and a functorial in $\mathcal{F}^\cdot \in (D^+(\mathbf{SAb}_X))^\circ$, $\mathcal{G}^\cdot \in D^+(\mathbf{SAb}_Y)$ isomorphism

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\cdot, \mathcal{G}^\cdot) \xrightarrow{\sim} R\operatorname{Hom}(\mathcal{F}^\cdot, f^!\mathcal{G}^\cdot)$$

in $D^+(\mathbf{Ab})$.

5.12.2. Corollary. *The functor $f^!$ is right adjoint to $Rf_!$.*

5.13. Comments to Theorem 12.1 and the Plan of the Proof. For a sheaf $\mathcal{F}$ on X and an open set $U \subset X$ denote by $\mathcal{F}_U$ the extension of $\mathcal{F}$ by zero outside U (using the functors $j_!$, $j^{\cdot}$ where $j : U \to X$ is the embedding, we can write $\mathcal{F}_U = j_! j^{\cdot} \mathcal{F}$). For two open sets $U \subset V$ there exists a natural morphism of sheaves $\mathcal{F}_V \to \mathcal{F}_U$ which induces, for each sheaf $\mathcal{G}$ on Y, a homomorphism

$$\operatorname{Hom}(f_! \mathcal{F}_U, \mathcal{G}) \to \operatorname{Hom}(f_! \mathcal{F}_V, \mathcal{G}). \tag{4}$$

It is clear that the correspondence

$$U \mapsto \operatorname{Hom}(f_! \mathcal{F}_U, \mathcal{G}), \tag{5}$$

together with restriction mappings (4), defines a presheaf on X. If this presheaf were a sheaf, everything would be fine: denoting the sheaf $U \mapsto \operatorname{Hom}(f_! \mathbb{Z}_U, \mathcal{G})$ (where $\mathbb{Z}$ is the constant sheaf on X) by $f^? \mathcal{G}$, we would have

$$\operatorname{Hom}(f_! \mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{F}_U, f^? \mathcal{G})$$

so that $f_!$ would possess the right adjoint functor. However, (5) is a sheaf only in very special cases. More precisely,

$$U \mapsto \operatorname{Hom}(f_!(\mathcal{F}_U \otimes \mathcal{L}), \mathcal{G})$$

is a sheaf if $\mathcal{L}$ is a flat soft sheaf on X and in this case

$$\operatorname{Hom}(f_!(\mathcal{L} \otimes \mathcal{F}), \mathcal{G}) = \operatorname{Hom}(\mathcal{F}, f^!(\mathcal{L}, \mathcal{G})).$$

Therefore, we must pass to derived categories and to replace the constant sheaf $\mathbb{Z}$ on X by its flat resolution $\mathcal{L}^{\cdot}$

$$0 \to \mathbb{Z} \to \mathcal{L}^0 \to \cdots \to \mathcal{L}^{n-1} \to \mathcal{L}^n \to 0.$$

This being done, we can define $f^!(\mathcal{G}^{\cdot})$ as the diagonal complex associated to the bicomplex

$$A^{ij} = f^!(\mathcal{L}^{-i}, \mathcal{G}^j),$$

where $\mathcal{G}^{\cdot} \to \mathcal{I}^{\cdot}$ is a quasi-isomorphism of $\mathcal{G}^{\cdot}$ with a complex of injective sheaves on Y.

5.14. Representable Functors in Sheaf Categories. An important step in the proof of Theorem 5.13.1 is the following theorem (which is often applied in situations when one must construct a sheaf with prescribe properties).

5.14.1. Theorem. *A functor $F : \mathbf{SAb}_X \to (\mathbf{Ab})^{\circ}$ is representable if and only if it transforms inductive limits in $\mathbf{SAb}_X$ into projective limits in $\mathbf{Ab}$.*

Proof. The "only if" part of the theorem is true in the general situation: for an arbitrary abelian category $\mathcal{A}$ a functor of the form $X \mapsto \operatorname{Hom}_{\mathcal{A}}(X, Y)$ transforms inductive limits in $\mathcal{A}$ into projective limits in $\mathbf{Ab}$. To prove the "if" part let us note that

$$U \mapsto F(\mathbb{Z}_U)$$

together with restriction mappings $F(\mathbb{Z}_U) \to F(\mathbb{Z}_V)$ for $U \subset V$ induced by embeddings $\varphi_{UV} : \mathbb{Z}_U \to \mathbb{Z}_V$, determines a presheaf $\mathcal{G}$ of abelian groups on X, so that $\Gamma(U, \mathcal{G}) = F(\mathbb{Z}_U)$. Now one can use that F commutes with limits to prove that $\mathcal{G}$ is a sheaf.

5.15. Properties of $f^!$. a. The construction of $f^!$ and Theorem 5.14.1 can be generalized to the case when the category **SAb** is replaced to the category of sheaves of R-modules, where R is a noetherian ring, in particular, a field.

b. For two continuous mappings $f : X \to Y$, $g : Y \to Z$ we have $(gf)^! = g^! \circ f^!$.

c. If $f : X \to Y$ is an embedding of an open or of a closed subset, then the right adjoint functor to $f_!$ exists already on the level of the category of sheaves (without passing to derived categories). Namely, if $f : U \to Y$ is an open embedding, then the right adjoint to $f_! : \mathbf{SAb}_U \to \mathbf{SAb}_X$ is the restriction $f^\cdot : \mathbf{SAb}_X \to \mathbf{SAb}_U$.

If $f : X \to Y$ is a closed embedding, then the right adjoint to $f_! = f_\cdot :$ $\mathbf{SAb}_X \to \mathbf{SAb}_Y$ is the functor Γ_X (see 5.5).

Let us consider now the situation opposite to c), when $f : X \to pt$ is the mapping to the point.

5.16. The Dualizing Complex. The Duality. Assume that $Y = pt$ is a point. Then sheaves of Y are just abelian groups, and by $\mathbb{Z} \in D^+(\mathbf{SAb}_Y) = D^+(\mathbf{Ab})$ we will denote the 0-complex with the zero component $\mathbb{Z}$. For any (finite-dimensional, locally compact) topological space X let $\mathbf{D}^\cdot_X = f^!(\mathbb{Z})$, where $f : X \to pt$. The complex $\mathbf{D}^\cdot_X$ is called *the dualizing complex* on X. In this case Theorem 5.12.1 takes the following form:

$$R\operatorname{Hom}(R\Gamma_C(X, \mathcal{F}^\cdot), \mathbb{Z}) \tilde{\to} R\operatorname{Hom}(\mathcal{F}^\cdot, \mathbf{D}^\cdot_X). \tag{6}$$

There exists a more explicit construction of the dualizing complex $\mathbf{D}^\cdot_X$. For any open $U \subset X$ denote by $K_\cdot(U)$ the complex of relative integral chains $K_\cdot(U) = C_\cdot(X, X \setminus U; \mathbb{Z})$. It is clear that for $U \subset V$ the embedding of chains determines a morphism of complexes $K_\cdot(U) \to K_\cdot(V)$, and $U \mapsto K_\cdot(U)$ is a complex of presheaves on X. Now the dualizing complex $\mathbf{D}^\cdot_X$ is the complex of sheaves associated to the complex of presheaves $K_\cdot$.

Another formulation of this statement is as follows. Denote by $Hom(\mathcal{F}, \mathcal{G})$ the sheaf of morphisms of a sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ and by $RHom$ the corresponding derived functor, and define the duality functor

$$\mathfrak{D}_X : D^b(\mathbf{SAb}_X)^\circ \to D^b(\mathbf{SAb}_X)$$

by the formula

$$\mathfrak{D}_X(\mathcal{F}^\cdot) = Hom(\mathcal{F}^\cdot, \mathbf{D}^\cdot_X).$$

The complex $\mathfrak{D}_X(\mathcal{F}^\cdot)$ is called the *Verdier dual* to $\mathcal{F}^\cdot$.

Next, for any two sheaves $\mathcal{F}$ and $\mathcal{G}$ on X there exists a natural morphism

$$\mathcal{F} \mapsto Hom(Hom(\mathcal{F}, \mathcal{G})).$$

An easy verification shows that this morphism can be extended to derived categories. In particular, we have a morphism of functors $\alpha : \mathrm{Id} \to \mathfrak{D}_X\mathfrak{D}_X$ from $D^b(\mathbf{SAb}_X)$ into itself.

On the entire category $D^b(\mathbf{SAb}_X)$ the functor α is not, in general, an isomorphism, so that $\mathfrak{D}_X$ is not an autoequivalence of categories. But one can prove that α is an isomorphism when restricted to an important full subcategory consisting of complexes with constructive cohomology. Namely, let X be a stratified space whose strata are non-singular topological spaces. Then the cohomology sheaves of $\mathbf{D}^{\cdot}_X$ are constructible with respect to this stratification, the functor $\mathfrak{D}_X$ preserves the subcategory $\mathrm{Constr}_X \subset D^b(\mathbf{SAb}_X)$ consisting of complexes with constructive cohomology, and α is an isomorphism when restricted to this subcategory (see Chap. 7 for more details).

In the case when $R = k$ is a field, there exists yet another description of the duality functor $\mathfrak{D}_X$. Let $\mathcal{F}$ be a sheaf. Define the presheaf $\mathcal{F}^{*\mathrm{naive}}$ by the formula

$$\mathcal{F}^{*\mathrm{naive}}(U) = \Gamma_c(U, \mathcal{F})^*.$$

This presheaf is always flabby (any section over an open set can be extended to a section over the whole X), and becomes a sheaf if $\mathcal{F}$ is a soft sheaf. Let us consider the functor $\mathbf{Sh}_k \to (\mathbf{Sh}_k)^\circ$ (here $\mathbf{Sh}_k$ is the category of sheaves of k-modules), which maps $\mathcal{F}$ to the sheaf associated to the presheaf $\mathcal{F}^{*\mathrm{naive}}$. This functor is left exact and its right derived functor is $\mathfrak{D}_X$. In the other words, $\mathfrak{D}_X(\mathcal{F}) = (S^{\cdot})^{*\mathrm{naive}}$, where $S^{\cdot} \to \mathcal{F} \to 0$ is a soft resolution of $\mathcal{F}$.

Particularly simple is the structure of the dualizing complex in the case when X is non-singular.

5.16.1. Corollary. *Let X be an n-dimensional topological manifold with boundary. Then $\mathbf{D}^{\cdot}_X = \omega_X[n]$, where the sheaf ω_X is defined by*

$$\Gamma(U, \omega_X) = \mathrm{Hom}_{\mathbf{Ab}}(H^n_c(U, \mathbb{Z}), \mathbb{Z})$$

for any open $U \subset X$.

5.17. Remarks. Replacing $\mathbb{Z}$ by an arbitrary noetherian ring R, we can obtain analogs of the above corollary. In particular, if $R = k$ is a field and X is a topological manifold without boundary then $\omega_X = \tau_X$ is the sheaf of k-orientations of X (the constant sheaf k if X is oriented or if $\mathrm{char}\, k = 2$). If X is a manifold with boundary δX then $\omega_X = i_! \tau$, where τ is the sheaf of k-orientations of $X - \delta X$ and $i : X - \delta X \to X$ is the inclusion.

Taking cohomology of both sides of (6) we can express the Poincaré-Verdier duality in a more standard form (if k is a field) by saying that for any sheaf of k-modules $\mathcal{F}$ on X there exists a canonical isomorphism

$$\mathrm{Hom}_k(H^i_c(X, \omega_X), k) \xrightarrow{\sim} \mathrm{Ext}^{n-1}(\mathcal{F}, \omega_X). \tag{7}$$

Denoting by $\int_X : H^n_c(X, \omega_X) \to k$ the fundamental class of X, i.e. the preimage of $1 \in \mathrm{Hom}(\omega_X, \omega_X)$ under the isomorphism (13) for $i = n = \dim X$, $\mathcal{F} = \omega_X$ we can represent (13) as the composition of the canonical pairing

$$\mathrm{Ext}^{n-1}(\mathcal{F}, \omega_X) \times H^i_c(X, \mathcal{F}) \to H^n_c(X, \omega_X)$$

with $\int_X$.

5.18. Relations Among Functors in $D(\mathbf{SAb}_X)$. Here we present some relations among various functors in derived categories of sheaves on topological spaces. All spaces are assumed to be locally compact, paracompact and finite-dimensional, and all mappings are assumed to be continuous. The equalities between objects of (derived) categories are functorial isomorphisms.

a. *$f^{\cdot}$ and $\overset{L}{\otimes}$.* For $f : X \to T$, $\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot} \in D^{-}(\mathbf{SAb}_Y)$, we have

$$f^{\cdot}(\mathcal{F}^{\cdot} \overset{L}{\otimes} \mathcal{G}^{\cdot}) = f^{\cdot}\mathcal{F}^{\cdot} \overset{L}{\otimes} f^{\cdot}\mathcal{G}^{\cdot}.$$

For a proof replace $\mathcal{F}^{\cdot}$ and $\mathcal{G}^{\cdot}$ by their flat resolutions (quasi-isomorphic complexes of flat sheaves), and use the formula $f^{\cdot}(\mathcal{F} \otimes \mathcal{G}) = f^{\cdot}\mathcal{F} \otimes f^{\cdot}\mathcal{G}$ for $\mathcal{F}, \mathcal{G} \in D^{-}(\mathbf{SAb}_Y)$ which follows from $(\mathcal{F} \otimes \mathcal{G})_Y = \mathcal{F}_Y \otimes \mathcal{G}_Y$.

b. *RHom and $\overset{L}{\otimes}$.* For $\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot} \in D^{-}(\mathbf{SAb}_X)$, $\mathcal{H}^{\cdot} \in D^{+}(\mathbf{SAb}_X)$ we have

$$RHom(\mathcal{F}^{\cdot} \otimes \mathcal{G}^{\cdot}, \mathcal{H}^{\cdot}) = RHom(\mathcal{F}^{\cdot}, RHom(\mathcal{G}^{\cdot}, \mathcal{H}^{\cdot})).$$

For the proof verify the corresponding statement for sheaves of abelian groups on X, and then replace $\mathcal{G}^{\cdot}$ by a flat resolution and $\mathcal{H}^{\cdot}$ by an injective resolution.

c. *$Rf_.$ and RHom.* For $f : X \to Y$, $\mathcal{F}^{\cdot} \in D^{-}(\mathbf{SAb}_Y)$, $\mathcal{G}^{\cdot} \in D^{+}(\mathbf{SAb}_X)$ we have

$$Rf_. RHom(f^{\cdot}\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}) = RHom(\mathcal{F}^{\cdot}, Rf_.\mathcal{G}^{\cdot}).$$

Replacing $\mathcal{G}^{\cdot}$ by an injective resolution we see that $f_.\mathcal{G}^{\cdot}$ (with the componentwise action) is formed by injective sheaves, and $Hom(f^{\cdot}\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ is formed by soft sheaves. After that the required isomorphism follows from the isomorphism

$$f_. RHom(f^{\cdot}\mathcal{F}, \mathcal{G}) = RHom(\mathcal{F}, Rf_.\mathcal{G})$$

in $\mathbf{SAb}_Y$ which, in turn, follows from the adjointness of $f^{\cdot}$ and $f_.$.

d. *The base change formula.* Let

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

be a commutative diagram of spaces and continuous mappings. Then in $D^{+}(\mathbf{SAb}_Y)$ we have

$$q^{\cdot}Rf_! = Rg_! p^{\cdot}\mathcal{F}^{\cdot},$$
$$Rg_. p^{!}\mathcal{F}^{\cdot} = q^{!}Rf_!\mathcal{F}^{\cdot}, \quad \mathcal{F}^{\cdot} \in D^{+}(\mathbf{SAb}_X).$$

To prove the first formula one must verify the equality $q^{\cdot}f_!\mathcal{F} = g_! p^{\cdot}\mathcal{F}$, $\mathcal{F} \in \mathbf{SAb}_X$, by computing the stalks of both sheaves at a point $y' \in Y'$ using Proposition 5.9.1, and then replace $\mathcal{F}^{\cdot}$ by its soft resolution. To prove the second formula, one must interchange X and Y', apply the first formula and use the adjointness of the functors in the following pairs: $(f^{\cdot}, Rf_.)$, $(g^{\cdot}, Rg_.)$, $(Rq_!, q^{!})$, $(Rp_!, p^{!})$.

e. *The projection formula.* For $f : X \to Y$, $\mathcal{F}^{\cdot} \in D^{-}(\mathbf{SAb}_X)$, $\mathcal{G}^{\cdot} \in D^{-}(\mathbf{SAb}_Y)$ we have

$$Rf_!(\mathcal{F}^\cdot \overset{L}{\otimes} f^\cdot \mathcal{G}^\cdot) = Rf_!\mathcal{F}^\cdot \overset{L}{\otimes} \mathcal{G}^\cdot.$$

The proof follows from the corresponding formula in $\mathbf{SAb}_Y$ if we replace $\mathcal{F}^\cdot$ by a soft resolution, and $\mathcal{G}^\cdot$ by an injective resolution.

f. *$f^!$ and RHom.* For $f : X \to Y$, $\mathcal{F}^\cdot, \mathcal{G}^\cdot \in D^+(\mathbf{SAb}_Y)$ we have

$$f^! RHom(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = RHom(f^\cdot \mathcal{F}^\cdot, f^! \mathcal{G}^\cdot).$$

To get the proof it suffices to replace $\mathcal{F}^\cdot$ by a soft resolution, $\mathcal{G}^\cdot$ by an injective resolution, and to use the explicit construction of $f^!$.

g. *Functors and Duality.* First of all,

$$f^! \mathbf{D}^\cdot_X = \mathbf{D}^\cdot_Y.$$

Next,

$$\begin{aligned} f^! \mathbf{D}_Y \mathcal{F}^\cdot &= \mathbf{D}_X f^\cdot \mathcal{F}^\cdot, \quad && \mathcal{F}^\cdot \in D^b(\mathbf{SAb}_X), \\ \mathbf{D}_Y Rf_! \mathcal{G}^\cdot &= Rf_. \mathbf{D}_X \mathcal{F}^\cdot, && \mathcal{G}^\cdot \in D^b(\mathbf{SAb}_Y). \end{aligned}$$

Bibliographic Hints

In the first four sections of this chapter we have presented the theory of derived categories and the corresponding version of the theory of derived functors. The primary sources here are the notes of Hartshorne (1966) and in (Verdier 1977); see also (Gelfand, Manin 1988; Happel 1988; Iversen 1986). Complete proof can be found in (Gelfand, Manin 1988) and (Hartshorne 1966). The localization conditions in 1.7 are the classical Ore conditions which appear in the study of fractions in skew fields (see, e.g., (Herstein 1968)). Lemma 2.2.1 is taken from (Bourbaki 1980). The results from 3.1 2 (and their deep generalizations) see in (Happel 1988). The results from 4.12 are due to Deligne (Appendix to (Cattani, Kaplan 1982)), those from 4.13 to Roos (1961). The Grothendieck spectral sequence from 4.15 illustrates the nature of many spectral sequences in algebra, topology, and algebraic geometry, see (Grothendieck 1957).

Sheaf cohomology presented in Sect. 5 are explained in several books; we mention (Golovin 1986; Bredon 1967; Godement 1958; Iversen 1986). The proof of Proposition 5.1 see in (Godement 1968), and that of Theorem 5.3 in (Godement 1968) and in (Gelfand, Manin 1988). The results from 5.8–5.11 see in (Iversen 1968) and in the papers of Borel (1984) and Grivel (1984). Theorems 5.13.1 and 5.14.1 are due to Verdier (1963), about proofs see (Gelfand, Manin 1988) and (Iversen 1986). Short proof of the results from 5.18 can be found in (Borel 1984).

Chapter 5
Triangulated Categories

§ 1. Main Notions

1.1. Axioms. Let $\mathcal{D}$ be an additive category. The structure of a *triangulated category* on $\mathcal{D}$ is given by the following data **a, b** that must satisfy the axioms TR1–TR4 below.

a. An additive automorphism $T : \mathcal{D} \to \mathcal{D}$ called the *translation functor*.

Similarly to 2.2 in Chap. 4, we will write $X[n]$ instead of $T^n(X)$ and $f[n]$ instead of $T^n(f)$ (for a morphism f in $\mathcal{D}$). Now we can literally repeat parts a) and b) of Definition 2.3. from Chap. 4, introducing in $\mathcal{D}$ *triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and their *morphisms*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

The last part of Definition 2.3 from Chap. 4 is introduced axiomatically; among triangles in $\mathcal{D}$ one must distinguish

b. The class of distinguished triangles.

As we will gradually see, the following axioms give a satisfactory description of working properties of the construction from 2.3.c in Chapter 4.

TR1. a. For any $X \in \mathrm{Ob}\,\mathcal{D}$ the triangle

$$X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$$

is distinguished.

b. A triangle isomorphic to a distinguished one, is itself distinguished.

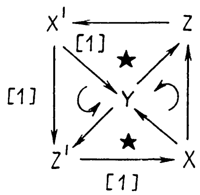
c. Any morphism $X \xrightarrow{u} Y$ can be completed to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.

TR2. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is distinguished.

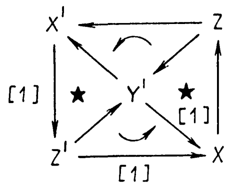
TR3. Assume we are given a diagram consisting of two distinguished triangles and two morphisms f and g between first two objects of these triangles. This diagram can be completed (not necessarily uniquely) to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

The last axiom deal with a rather complicated "octahedron diagram". One of the methods to draw this diagram is to represent in the form of two "caps" of the octahedron with common "rim":



upper cap



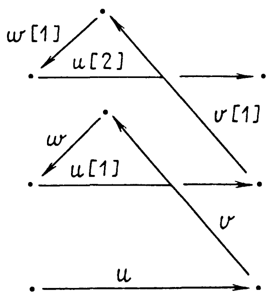
lower cap

In these diagrams X, Y, etc, are objects from $\mathcal{D}$, the arrow of the form $X' \overset{[1]}{\to} Y'$ means a morphism $f : X' \to Y'[1]$, triangles marked * are distinguished, triangles marked ↻ are commutative. The last part of the definition of an octahedron diagram requires that the two composite morphisms $Y \to Y'$, through Z and through Z', coincide, and the two composite morphisms $Y' \to Y[1]$, through $X[1]$ and through X', coincide.

After giving the definition of an octahedron diagram we can formulate the last axiom.

TR4. Any diagram of the form "upper cap" can be completed to an octahedron diagram.

1.2. Remarks About the Formal Structure of Axioms. a. Axiom TR2 implies that each distinguished triangle can be embedded into a "helix" in which any three consecutive morphisms form a distinguished triangle:



Hence a morphism of distinguished triangles generates a "double helix", i.e. two helixes chained together by horizontal arrows. Axioms TR3 and TR2 imply that given two neighboring arrows forming a commutative square, they can be completed to a morphism of helixes.

b. Any diagram of the form "upper cap" can be considered as a morphism of distinguished triangles with the middle morphism being the identity. Embedding this morphism into a double helix one can see that this double helix contains diagrams of the form "lower cap". Hence, axiom TR4 can be reformulated in the following equivalent (modulo axioms TR1–TR3) form:

TR4′. Any lower cap can be completed to an octahedron.

c. In TR1.c any two completions of $X \xrightarrow{u} Y$ to a distinguished triangle are isomorphic: the morphism h, whose existence is guaranteed by TR3 for $f = \mathrm{id}_X$, $g = \mathrm{id}_Y$, is an isomorphism.

Hence, an "upper cap" diagram can be uniquely up to an isomorphism recovered by one commutative face $X \to Y \to Z$, completing $X \to Y$ and $Y \to Z$ to distinguished triangles.

From axioms TR1–TR3 one can deduce the following property of distinguished triangles (cf. 2.5 in Chap. 4).

1.3. Proposition. *Let $\mathcal{D}$ be a triangulated category, $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a distinguished triangle. For any object U from $\mathcal{D}$ the following sequences are exact:*

$$\ldots \to \mathrm{Hom}(U, X[i]) \xrightarrow{u_*[i]} \mathrm{Hom}(U, Y[i]) \xrightarrow{v_*[i]} \mathrm{Hom}(U, Z[i]) \xrightarrow{w_*[i]}$$
$$\xrightarrow{w_*[i]} \mathrm{Hom}(U, X[i+1]) \to \ldots$$

$$\ldots \to \mathrm{Hom}(X[i+1], U) \xrightarrow{w^*[i]} \mathrm{Hom}(Z[i], U) \xrightarrow{v^*[i]} \mathrm{Hom}(Y[i], U) \xrightarrow{u^*[i]}$$
$$\xrightarrow{u^*[i]} \mathrm{Hom}(X[i], U) \to \ldots$$

Proof. To illustrate the methods of working with triangulated categories, let us prove the exactness of the first sequence. By the remark 1.2.a, it suffices to prove the exactness at the term $\mathrm{Hom}(U, Y[i])$. First we verify that $vu = 0$ (hence the composition of any two consecutive morphisms in a distinguished triangle equals zero). Axiom TR3 applied to the distinguished triangle

$$X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$$

and to the initial distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ gives

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}$$

The only possible morphism h is zero, and the commutativity implies $vu = 0$.

Let now $f : U \to Y$ be a morphism satisfying $vf = 0$. We want to prove that $f = ug$ for some $f : U \to X$. We take g from the following morphism of distinguished triangles:

$$\begin{array}{ccccccc}
U & \xrightarrow{\mathrm{id}} & U & \longrightarrow & 0 & \longrightarrow & U[1] \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g[1]} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}$$

This morphism is constructed by using TR2 and TR3: first TR3 is applied to

$$\begin{array}{ccccccc} U & \longrightarrow & 0 & \longrightarrow & U[1] & \xrightarrow{-\mathrm{id}} & U[1] \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g[1]} & & \downarrow{\scriptstyle f[1]} \\ Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] & \xrightarrow{-u[1]} & Y[1] \end{array}$$

and then g is recovered from $g[1]$.

1.4. Corollary. a. *If in the diagram* TR3 f *and* g *are isomorphisms, then* h *is also an isomorphism.*

b. *In axiom* TR1.c *a distinguished triangle completing the given morphism is unique up to an isomorphism.*

1.5. Corollary. *If* $v'gu = 0$ *then* g *can be embedded into a morphism of distinguished triangles. If, moreover,* $\mathrm{Hom}(X, Z'[-1]) = 0$, *this morphism is determined uniquely.*

1.6. Cohomological Functors. A functor $H : \mathcal{D} \to \mathcal{A}$ from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{A}$ is said to be *cohomological* if for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in $\mathcal{D}$ the sequence

$$H(X) \xrightarrow{H(u)} H(Y) \xrightarrow{H(v)} H(Z)$$

is exact in $\mathcal{A}$ (cf. Chap. 4, 4.11.a).

Axiom TR2 implies that if H is a cohomological functor, then any distinguished triangle in $\mathcal{D}$ yields the following long exact sequence in $\mathcal{A}$:

$$\ldots \to H(X[i]) \xrightarrow{H(u[i])} H(Y[i]) \xrightarrow{H(v[i])} H(Z[i]) \xrightarrow{H(w[i])} H(X[i+1]) \to \ldots$$

The main example of a cohomological functor is the mapping of a complex to its zero cohomology, $C^\cdot \to H^0(C^\cdot)$, considered as a functor from $K(\mathcal{A})$ or from $D(\mathcal{A})$ to $\mathcal{A}$.

Another example is the functor $\mathrm{Hom}_{\mathcal{D}}(U, \cdot) : \mathcal{D} \to \mathbf{Ab}$ for a fixed $U \in \mathrm{Ob}\,\mathcal{D}$.

1.7. A Cone of a Morphism. By axioms TR1–TR3 each morphism $X \xrightarrow{u} Y$ from a triangulated category $\mathcal{D}$ defines an object $C(u)$ as the third object in the distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z = C(u) \xrightarrow{w} X[1]$. This object is unique up to a *non-canonical* isomorphism, and is called a *cone* of the morphism u. (The indefinite article here indicates the non-uniqueness of the choice.) A cone is defined together with morphisms $Y \to C(u) \to X[1]$.

The choice of the name can be motivated as follows.

One of the main classes of triangulated categories form the categories $K(\mathcal{A})$ and $D(\mathcal{A})$ for various abelian categories $\mathcal{A}$. Proposition 2.4 from Chap. 4

easily implies that for any distinguished triangle $X^{\cdot} \xrightarrow{u} Y^{\cdot} \to Z^{\cdot} \to X^{\cdot}[1]$ in these categories the complex $Z^{\cdot}$ is isomorphic (although non-canonically) to the cone of the morphism u.

Let us return to a general triangulated category. If we would like to make C a functor, we might proceed as follows: a) define on objects $C : \mathrm{Ob}(\mathbf{Mor}\mathcal{D}) \to \mathrm{Ob}\,\mathcal{D}$ using the axiom of choice (here $\mathbf{Mor}\mathcal{D}$ is the category of morphisms of the category $\mathcal{D}$); b) Define on morphisms $C : \mathrm{Mor}(\mathbf{Mor}\mathcal{D}) \to \mathrm{Mor}\,\mathcal{D}$ using TR3 and the axiom of choice. After that we see that there is no reason at all to expect that $C(u \circ v) = C(u) \circ C(v)$. This "non-functoriality of a cone" is the first symptom that something is wrong with axioms of triangulated categories. Unfortunately, now we do not have a satisfactory way to fix the axioms. One of the possible approaches to improve thesituation in this and similar cases can be described as follows. In trying to make some construction (for example, the cone) canonical, we must define another triangulated category whose objects are "homotopical classes of constructions" in the initial category, and to associate to each object of this new category some object of the initial category. For example, this scheme can be successfully implemented in the case when the initial category has the structure of a filtered triangulated category.

Now we can ask the following question. In some cases we meet with categories $\mathcal{B}$ supplied with the structure of an "abstract cone" mapping $C : \mathbf{Mor}\mathcal{B} \to \mathcal{B}$. What might be good axioms of such mappings? These axioms must deal with at least two structures: a) The functorial properties of C with respect to morphisms in $\mathbf{Mor}\mathcal{B}$; b) the behavior of C under the composition of morphisms. A particular answer to this question is given by the octahedron axiom. Namely, let $X \xrightarrow{u} Y, Y \xrightarrow{v} Z$ be two morphisms in a triangulated category $\mathcal{D}$. Complete these morphisms to distinguished triangles with third objects $C(u)$, $C(v)$. The composition of morphism $C(v) \to Y[1]$ (from the second triangle) and $Y[1] \to C(u)[1]$ (from the first triangle) yields the morphism $w : C(v) \to C(u)[1]$. A part of the octahedron axiom can be expressed by the formula $C(v \circ u) = C(C(v) \xrightarrow{w} C(u)[1])[-1]$. Indeed, let us consider the upper cap of the octahedron constructed from the commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "v"] & & Z \\ & X \arrow[ul, "u"] \arrow[ur, "vu"'] & \end{tikzcd}$$

In notations of 1.1 we have

$$Z' = C(u), \quad X' = C(v), \quad Y' = C(v \circ u).$$

(the right triangle of the lower cap). On the other hand, the left triangle of the lower cap gives $Y' = C(w)[-1]$.

The remaining part of the octahedron axiom describes the arrows coming from and to $C(v \circ u)$.

1.8. $K(\mathcal{A})$ is a Triangulated Category. As we have already mentioned, the main examples of triangulated categories are the categories $K(\mathcal{A})$ and $D(\mathcal{A})$. The proof of the following theorem is reduced to the direct verification of axioms.

1.8.1. Theorem. *Let $\mathcal{A}$ be an abelian category. Then the category $K(\mathcal{A})$ with the standard translation functor and the distinguished triangles described in* 2.3 *of Chap.* 4, *is a triangulated category.*

The same is true for categories $K^+(\mathcal{A})$ and $K^b(\mathcal{A})$.

1.9. Localizing Classes of Morphisms in Triangulated Categories. To prove that $D(\mathcal{A})$ is a triangulated category, we have to establish that the structure of a triangulated category on $K(\mathcal{A})$ is compatible with the localization by quasi-isomorphisms.

In a more general setup, the structure of a triangulated category can be transferred to the localized category if the class of morphisms S satisfies the following properties of compatibility with triangulization:

a. $s \in S$ if and only if $T(s) \in S$.

b. If in a diagram TR3 (see 1.1) the morphisms f and g belong to S then we can choose a completing morphism h also belonging to S.

1.9.1. Theorem. *Let $\mathcal{D}$ be a triangulated category, S be a localizing class of morphisms in $\mathcal{D}$ compatible with the triangulization. Define the translation functor $T_S : \mathcal{D}_S \to \mathcal{D}_S$ tautologically* ($\operatorname{Ob} \mathcal{D}_S = \operatorname{Ob} \mathcal{D}$, $T_s = T$ *on objects*). *Define distinguished triangles in $\mathcal{D}_S$ as triangles isomorphic to images of distinguisged triangles in $\mathcal{D}$ under the localization functor $\mathcal{D}_S \to \mathcal{D}$. Then $\mathcal{D}_S$ with these structures is a triangulated category.*

To prove this theorem we define T_S on morphisms (roofs) by the formula

$$T\left[\begin{array}{ccc} & Z & \\ {}^{s}\swarrow & & \searrow^{u} \\ X & & Y \end{array}\right] = \begin{array}{ccc} & T(Z) & \\ {}^{T(s)}\swarrow & & \searrow^{T(u)} \\ T(X) & & T(Y) \end{array}$$

After that the axioms are verified directly.

1.9.2. Corollary. *The derived categies $D^*(\mathcal{A})$ are triangulated.*

1.10. Thick Subcategories. There exists an equivalent method to define the localization procedure in a triangulated category. In this method we define a family of objects that become isomorphic to the zero object in the localized category (instead of the class of morphisms that become isomorphisms).

1.10.1. Definition. A full triangulated subcategory $\mathcal{C}$ of a triangulated category $\mathcal{D}$ is said to be *thick* (*epaissé* in French) if the following condition is satisfied:

(T) Let a morphism $f : X \to Y$ in $\mathcal{D}$ can be factored through an object from $\mathcal{C}$ (i.e. can be represented in the form $X \to V \to Y$ with $V \in \operatorname{Ob}\mathcal{C}$) and can be embedded into a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ with $Z \in \operatorname{Ob}\mathcal{C}$. Then $X, Y \in \operatorname{Ob}\mathcal{C}$.

A standard example of a thick subcategory is the category of all acyclic objects in $K(\mathcal{A})$. In fact, the first condition in (T) means that $H^{\cdot}(f)$ is the zero morphism, and the second condition means that $H^{\cdot}(f)$ is an isomorphism. Hence X and Y are acyclic complexes.

The relation between thick subcategories in $\mathcal{D}$ and localizing classes of morphisms can be described as follows. A localizing class S in $\mathcal{D}$ is said to be *saturated* if the condition $s \in S$ is equivalent to the existence of morphisms f, f' in $\mathcal{D}$ such that $f \circ s \in S$, $s \circ f' \in S$.

1.10.2. Theorem. *Let $\mathcal{D}$ be a triangulated category. The mapping*

$$\mathcal{C} \mapsto \varphi(\mathcal{C}) = \{s \in \operatorname{Mor}\mathcal{C},\ \textit{such that } s \textit{ is contained in a distinguished triangle } X \xrightarrow{s} Y \to Z \to X[1] \textit{ with } s \in S\}$$

yields a one-to-one correspondence between the set of thick subcategories in $\mathcal{D}$ and the set of saturated localizing classes of morphisms in $\mathcal{D}$ compatible with triangulization.

The inverse mapping associates to a class $S \subset \operatorname{Mor}\mathcal{D}$ the full subcategory $\psi(S)$ generated by objects $Z \in \operatorname{Ob}\mathcal{D}$ contained in distinguished triangles $X \xrightarrow{s} Y \to Z \to X[1]$ with $s \in S$.

Below (in 3.9 of Chap. 5) we will need the following notion. By a *exact triple of triangulated categories* we mean a triple $\mathcal{C} \xrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$ where $\mathcal{C}$ is a thick subcategory of a triangulated category $\mathcal{D}$, $\mathcal{E} = \mathcal{D}[\varphi(\mathcal{C})^{-1}]$ is the localization of $\mathcal{D}$ by the class $S = \varphi(\mathcal{C})$ $P : \mathcal{C} \to \mathcal{D}$, and $Q : \mathcal{D} \to \mathcal{E}$ are the embedding and localization functors respectively.

1.11. Distinguished Triangles, Cones, Octahedrons in $D(\mathcal{A})$. Distinguished triangles in $D(\mathcal{A})$ are described by the following proposition (which shows that they are similar to exact triples in abelian categories).

1.11.1. Proposition. *Any exact triple of complexes $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ in $K(\mathcal{A})$ can be completed to a distinguished triangle in $D(\mathcal{A})$ by an appropriate morphism $Z \xrightarrow{w} X[1]$, and any distinguished triangle in $D(\mathcal{A})$ is isomorphic to a triangle of such a form.*

Proposition 1.11.1 enables us to give the following interpretation of a cone and of the octahedron axiom in $D(\mathcal{A})$.

If a morphism $X \xrightarrow{u} Y$ in $D(\mathcal{A})$ is represented by an embedding of complexes, then a cone is represented by the quotient complex $Y/u(X)$, and the corresponding morphism $Y \to C(u)$ is the factorization.

Let us consider now some octahedron diagram with morphisms $X \to Y$ and $Y \to Z$ (hence also $X \to Z$) represented by embeddings of complexes. We claim that the octahedron diagram yields the natural isomorphism

$$Z/Y \xrightarrow{\sim} (Z/X)/(Y/X).$$

In fact, $X' = Z/Y$, $Z' = Y/X$. Next, from the lower cap we have $Y' = Z/X$ and the natural morphism $Z' \to Y'$ is represented by the natural embedding $Y/X \to Z/X$. Finally, the third vertex X' of the left triangle in the lower cap is represented by the quotient $Y'/Z' = (Z/X)/(Y/X)$, and in the upper cap this vertex is Z/Y.

§ 2. Examples

2.1. The Grassmann Algebra. Let k be a field of characteristic $\neq 2$, E be a linear space over k of dimension $n+1$. Denote by Λ the $\mathbb{Z}$-graded exterior algebra of E:

$$\Lambda = \Lambda(E) = \bigoplus_{i=0}^{n+1} \wedge^i(E).$$

Let $\mathcal{M}(\Lambda)$ be the category of left unitary $\mathbb{Z}$-graded Λ-modules with morphisms of degree 0 as homomorphisms, and $\mathcal{M}^b(\Lambda)$ be the full subcategory of $\mathcal{M}(\Lambda)$ formed by finitely generated (=finite-dimensional over k) $\mathbb{Z}$-modules.

2.2. Operations over Λ-modules. a. Let V be a Λ-module. For $m \in \mathbb{Z}$ denote

$$V(m)^i = V^{i-m}, \quad V(m) = \bigoplus_{i\in\mathbb{Z}} V(m)^i.$$

Leaving the multiplication by elements of Λ unchanged, we see that the the shift of the grading $V \mapsto V(m)$, together with the identity mapping on morphisms, $f \mapsto f(m) = f$, is a functor from $\mathcal{M}(\Lambda)$ to itself.

b. For two Λ-modules V and V' denote

$$V \otimes V' = \bigoplus_l (V \otimes V')^l, \quad (V \otimes V')^l = \bigoplus_{i+j=l} (V^i \otimes_k V'^j),$$

$$e(v \otimes v') = ev \otimes v' + (-1)^{\deg v} v \otimes ev', \qquad v \in V, \quad v' \in V', \quad e \in E.$$

Let us note that in this section the tensor product is always taken *over k, and not over Λ.*

c. Any left Λ-module V carries a canonical structure of a right Λ-module V_r with the multiplication

$$v\lambda = (-1)^{\deg v \cdot \deg \lambda} \lambda v, \qquad v \in V, \quad \lambda \in \Lambda.$$

The mapping $V \mapsto V_r$ can be extended in an obvious manner to a functor which establishes an equivalence between the categories of left and of right (graded) modules.

d. Endow the space $V^* = \mathrm{Hom}_k(V, k)$ with the multiplication by elements from Λ by the formula $(\lambda\varphi)(v) = (-1)^{\deg v \cdot \deg \lambda}\varphi(\lambda v)$. Together with the grading $(V^*)^i = \mathrm{Hom}_k(V^{-i}, k)$, this determines on V^* the structure of a Λ-module.

2.3. The Category $\mathcal{M}^b(\Lambda)/\mathcal{F}$. Denote by $\mathcal{F} \subset \mathcal{M}^b(\Lambda)$ the full subcategory of free graded Λ-modules. A morphism $f : V \to V'$ in $\mathcal{M}^b(\Lambda)$ is said to be *equivalent to zero* if it can be represented as the composition $V \to F \to V'$, where $F \in \mathrm{Ob}\,\mathcal{F}$. Morphisms equivalent to zero clearly form a two-sided ideal I in $\mathrm{Mor}\,\mathcal{M}^b(\Lambda)$. Denote

$$\mathrm{Ob}(\mathcal{M}^b(\Lambda)/\mathcal{F}) = \mathrm{Ob}\,\mathcal{M}^b(\Lambda), \quad \mathrm{Mor}(\mathcal{M}^b(\Lambda)/\mathcal{F}) = \mathrm{Mor}\,\mathcal{M}^b(\Lambda)/I.$$

2.3.1. Theorem. *The category $\mathcal{M}^b(\Lambda)/\mathcal{F}$ carries a natural structure of a triangulated category.*

The most unusual in this structure (which will be described below) is the form of the translation functor T:

$$T(V) = (\Lambda(-n) \otimes_k V)/i(V)(-n), \quad n = \dim E - 1, \tag{1}$$

where $i(V) = \Lambda^{n+1}(E) \otimes_k V \subset \Lambda \otimes_k V$.

2.4. The Plan of the Proof. Any Λ-module V determines a family of complexes of vector spaces numbered by elements $e \in E$:

$$L_e(V) : \ldots \to V^{j-1} \xrightarrow{d^{j-1}(e)} V^j \xrightarrow{d^j(e)} V^{j+1} \to \ldots,$$

where $d^j(e)v = ev$. Clearly $L_e(V)$ and $L_{ce}(V)$ for $c \in k$, $c \neq 0$, are canonically isomorphic, so that the family $L_e(V)$ for $e \neq 0$ is essentially parameterized by points of the projective space $\mathbb{P}(E)$ of lines in E. Framing this remark in the algebraic geometry manner, we call a rigid complex a complex of quasicoherent sheaves on $\mathbb{P}(E)$ isomorphic to the complex of the form

$$L : \cdots \to V^j \otimes \mathcal{O}(j) \to V^{j+1} \otimes \mathcal{O}(j+1) \to \ldots$$

whose differentials are morphisms of $\mathcal{O}$-modules. A rigid complex L is said to be *finite* if it is bounded and all spaces V^j are finite-dimensional.

Construct from a rigid complex L the following graded Λ-module $V(L)$:

$$V(L) = \bigoplus_j \Gamma(L^j(-j)) = \bigoplus_j V(L)^j.$$

Consider the mapping

$$a : \Gamma(d^j(-j)) : V(L)^j \to V(L)^{j+1} \otimes E^*,$$

where we identified canonically $\Gamma(\mathcal{O}(1))$ with E^*. For any element $e \in E$ and any $v \in V(L)^j$ define

$$ev = (-1)^j(\mathrm{id} \otimes s_e)a(v),$$

where $s_e : E^* \to k$ is the convolution with e. The equality $d^{j+1} \circ d^j = 0$ implies that $e^2 v = 0$, so that $V(L)$ becomes a graded Λ-module.

One can easily see that this construction yields a functor from the category **CRig** of rigid complexes to the category $\mathcal{M}(\Lambda)$ which is an equivalence of categories. The inverse functor was essentially described earlier: a linear in e family of mappings $d^j(e) : V^j \to V^{j+1}$ is nothing but a linear mapping $V^j \to V^{j+1} \otimes E^*$ which determines a morphism of sheaves $V^j \otimes \mathcal{O} \to V^{j+1} \otimes \mathcal{O}(1)$ on $\mathbb{P}(E)$. Its tensor product with $\mathrm{id}_{\mathcal{O}(j)}$ yields the differential $d^j : V^j \otimes \mathcal{O}(j) \to V^{j+1} \otimes \mathcal{O}(j+1)$.

Objects from $\mathcal{M}^b(\Lambda)$ correspond to bounded complexes with finite-dimensional components.

Consider now the composition of functors

$$\Phi : \mathcal{M}^b(\Lambda) \to \mathbf{CRig}^b \to \mathcal{D}^b,$$

where by $\mathcal{D}^b$ we denoted the derived category of bounded complexes of quasi-coherent sheaves on $\mathbb{P}(E)$.

Theorem 2.3.1 is a corollary of the following results, which are based essentially on the properties of sheaves $\mathcal{O}(i)$ on $\mathbb{P}(E)$.

a. The essential image of the functor Φ (i.e. the full subcategory of $\mathcal{D}^b$ formed by all objects isomorphic to the objects of the form $\varphi(V)$) is closed under the translation functor T.

b. There exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{M}^b(\Lambda)}(V, W) \ \mathrm{mod}\ I \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}^b}(\Phi(V), \Phi(W)).$$

c. The constructions used in the verification of axioms TR1.c and TR4 (see 1.1), being applied to diagrams from the essential image of Φ, lead to diagrams from the essential image of Φ.

After that we can induce the structure of a triangulated category from $\mathcal{D}^b$ to the essential image of Φ and then to the equivalent category $\mathcal{M}^b(\Lambda)/\mathcal{F}$. (One can prove that the essential image of Φ coincides with $\mathcal{D}^b$.)

Now we describe one general method to construct triangulated categories; Theorem 2.3.1 can be thought of as a special case of this construction.

2.5. Exact Categories. a. Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ be its full additive subcategory. Let us assume that $\mathcal{B}$ is closed under extensions; by definition, this means that for any exact triple $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ with $X', X'' \in \mathrm{Ob}\,\mathcal{B}$ the object X is isomorphic to an object from $\mathcal{B}$. A pair $(\mathcal{B}, \mathcal{E})$, where $\mathcal{E}$ is the class of triples $X' \xrightarrow{i} X \xrightarrow{p} X''$ in $\mathcal{B}$ that become short exact triples in $\mathcal{A}$, is called an *exact category*. In particular, each abelian category $\mathcal{A}$ is an exact category ($\mathcal{E}$ is the class of all exact triples in $\mathcal{A}$).

Exact categories $(\mathcal{B}, \mathcal{E})$ can be defined using a system of axioms which does not appeal to the ambient category $\mathcal{A}$ (see, e.g., Quillen (1973)). There exists a canonical way to represent $\mathcal{B}$ as a full subcategory of an abelian category (namely, of the category of additive functors $F : \mathcal{B}^\circ \to \mathbf{Ab}$ such that for any triple $(X \to Y \to Z) \in \mathcal{E}$ the triple $0 \to F(Z) \to F(Y) \to F(X)$ of abelian groups is exact).

Each additive category can be made an exact category in at least one way (taking for $\mathcal{E}$ the class of all split triples $X \to X \oplus Y \to Y$).

b. *$\mathcal{E}$-injective* and *$\mathcal{E}$-projective objects.* Let $(\mathcal{B}, \mathcal{E})$ be an exact category. An object $I \in \operatorname{Ob} \mathcal{E}$ is said to be *$\mathcal{E}$-injective* if any triple $(I \to Y \to Z) \in \mathcal{E}$ splits. The class of all $\mathcal{E}$-injective objects will be denoted $\mathcal{I}_{\mathcal{E}}$. Similarly, $P \in \operatorname{Ob} \mathcal{E}$ is said to be *$\mathcal{E}$-projective* if any triple $(X \to Y \to P) \in \mathcal{E}$ splits. The class of all $\mathcal{E}$-projective objects will be denoted $\mathcal{P}_{\mathcal{E}}$.

$\mathcal{E}$-injective objects satisfy the following property: if

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[r] \arrow[d, dashed, "g"] & Z \\ & I & \end{tikzcd}$$

is a diagram in $\mathcal{B}$ with $(X \to Y \to Z) \in \mathcal{E}$ and $I \in \mathcal{I}_{\mathcal{E}}$, then there exists a morphism $g : Y \to I$ which makes it commutative.

$\mathcal{E}$-projective objects satisfy a similar property.

2.6. Frobenius Categories. a. An exact category $(\mathcal{B}, \mathcal{E})$ is said to be a *Frobenius category* if $\mathcal{I}_{\mathcal{E}} = \mathcal{P}_{\mathcal{E}}$ and for any $X \in \operatorname{Ob} \mathcal{E}$ there exist triples $Y \to I \to X$ and $X \to I' \to Y'$ in $\mathcal{E}$ with $I, I' \in \mathcal{I}_{\mathcal{E}}$ (roughly speaking, projective and injective objects coincide and there are sufficiently many of them in $\mathcal{B}$: each object is a subobject and a quotient of such an object).

In **b–d** examples of Frobenius categories are given.

b. The abelian category $\mathcal{M}^b(\Lambda)$ of finite-dimensional graded Λ-modules is a Frobenius category.

c. The abelian category of finite-dimensional modules over the group algebra $k[G]$ of a finite group G is a Frobenius category. More generally, the category of finite-dimensional modules over any Frobenius k-algebra (for the definition see (Curtis, Reiner 1982)) is a Frobenius category.

d. Let $\mathcal{B}'$ be an additive category with split idempotents (i.e. any morphism $\alpha : X \to X$ in $\mathcal{B}'$ with $\alpha^2 = \alpha$ is the projection onto a direct summand). Let $\mathcal{B} = \operatorname{Kom}^b(\mathcal{B}')$ and define $\mathcal{E}$ as the class of all triples $X^{\cdot} \to Y^{\cdot} \to Z^{\cdot}$ such that for any i the sequence $X^i \to Y^i \to Z^i$ splits. Then $\mathcal{E}$ is a Frobenius category and $\mathcal{E}$-projective ($=\mathcal{E}$-injective) complexes are finite direct sums of complexes of the form $\dots 0 \to X \xrightarrow{\mathrm{id}} X \to 0 \dots$ with $X \in \operatorname{Ob} \mathcal{B}'$.

2.6.1. Definition. a. Let $\mathcal{B}$ be a Frobenius category. For $X, Y \in \operatorname{Ob} \mathcal{B}'$ denote by $I(X, Y)$ the set of all morphisms $f : X \to Y$ in $\mathcal{B}$ that can be factored through an object from $\mathcal{I}_{\mathcal{E}}$.

b. Define the *stable category* $\mathcal{B}_0$ by $\operatorname{Ob} \mathcal{B}_0 = \operatorname{Ob} \mathcal{B}$, $\operatorname{Hom}_{\mathcal{B}_0}(X, Y) = \operatorname{Hom}_{\mathcal{B}}(X, Y)/I(X, Y)$.

One can easily verify that the composition in $\mathcal{B}_0$ is well defined and that $\mathcal{B}_0$ is an additive category.

2.7. The Suspension. a. Let $\mathcal{B}$ be a Frobenius category and

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d, dashed, "u"] & Y \arrow[d, dashed, "v"] \\ X \arrow[r] & I' \arrow[r] & Y' \end{tikzcd}$$

be a diagram whose rows are triples from $\mathcal{E}$ and $I, I' \in \mathcal{I}_{\mathcal{E}}$. Then there exist morphisms $u : I \to I'$, $v : Y \to Y'$ making this diagram commutative. Moreover, for two such extensions (u, v), $(\widetilde{u}, \widetilde{v})$ the images of v and $\widetilde{v}$ in $\mathrm{Hom}_{\mathcal{B}_0}(Y, Y')$ (Y,Y') coincide. Hence for any such pair (u, v) the image of v in $\mathrm{Hom}_{\mathcal{B}_0}(Y, Y')$ is an isomorphism. The existence of such canonical extension $v$in $\mathrm{Hom}_{\mathcal{B}_0}(Y, Y')$ enables us to define the *suspension functor* $T : \mathcal{B}_0 \to \mathcal{B}_0$ such that for any $X \in \mathrm{Ob}\,\mathcal{B}_0 = \mathrm{Ob}\,\mathcal{B}$ there exists a triple $(X \to I \to TX) \in \mathcal{E}$ with $I \in \mathcal{I}_{\mathcal{E}}$.

b. The equality $\mathcal{I}_{\mathcal{E}} = \mathcal{P}_{\mathcal{E}}$ implies that T is a self-equivalence of the category $\mathcal{B}_0$.

2.8. Distinguished Triangles. Let now $X, Y \in \mathrm{Ob}\,\mathcal{B}$, $u : X \to Y$ be an arbitrary morphism in $\mathcal{B}$ and $X \xrightarrow{i} I \xrightarrow{p} TX$ be a triple from $\mathcal{E}$ with $I \in \mathcal{I}_{\mathcal{E}}$. Then in the diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "u"] & I \arrow[r] \arrow[d, "y"] & TX \arrow[d, equal] \\ Y \arrow[r, "v"] & C \arrow[r, "w"] & TX \end{tikzcd}$$

in which the left square is cocartesian, there exists a unique morphism w that makes it commutative.

Triangles $X \xrightarrow{u} Y \xrightarrow{v} C \xrightarrow{w} TX$ in $\mathcal{B}$ that can be embedded into such a diagram, as well as their images in $\mathcal{B}_0$, will be called *standard triangles.* Any triangle isomorphic to a standard one will be called *a distinguished triangle.*

2.9. Theorem. *Let $\mathcal{B}$ be a Frobenius category, $\mathcal{B}_0$ be the corresponding stable category. Let us assume that the suspension functor is an automorphism of $\mathcal{B}_0$. Then the category $\mathcal{B}_0$ with T as the translation functor and distinguished triangles defined as in the previous subsection is triangulated* (*this gives, of course, another proof of Theorem* 3.5).

2.9.1. One of the approaches to the proof of Theorem 2.9 can be described as follows. Let $\mathcal{E}(\mathcal{I})$ be the category whose objects are acyclic complexes of objects from $\mathcal{I}$ (without any boundness conditions) and whose morphisms are homotopy classes of morphisms of complexes. A remarkable result is that the categories $\mathcal{E}(\mathcal{I})$ and $\mathcal{B}_0$ are equivalent. The functor $\alpha : \mathcal{E}(\mathcal{I}) \to \mathcal{B}_0$ which establishes this equivalence can be constructed as follows. Let $X^{\cdot} \in \mathrm{Ob}\,\mathcal{E}(\mathcal{I})$. Then

$$\alpha(X^{\cdot}) = \mathrm{Ker}(d^0 : X^0 \to X^1) = \mathrm{Im}(d^{-1} : X^{-1} \to X^0)$$

(as an element of $\operatorname{Ob}\mathcal{B}_0 = \operatorname{Ob}\mathcal{B}$). The quasi-inverse functor $\beta : \mathcal{B}_0 \to \mathcal{E}(\mathcal{I})$ associates to an object $X \in \operatorname{Ob}\mathcal{B}_0$ the complex

$$\dots \to P^{-1} \to P^0 \xrightarrow{\varepsilon \circ \varepsilon'} I^0 \to I^1 \to \dots,$$

where $P^\cdot \xrightarrow{\varepsilon'} X \to 0$ and $0 \to X \xrightarrow{\varepsilon} I^\cdot$ are left and right resolutions of X formed by projective (=injective) objects. Next, $\mathcal{E}(\mathcal{I})$ has a natural structure of a triangulated category. Theorem 2.9 is proved by verifying that α maps the translation functor and the distinguished triangles in $\mathcal{E}(\mathcal{I})$ into the translation functor and the distinguished triangles in $\mathcal{B}_0$.

2.9.2. Tate Cohomology. Let G be a finite group, k be either a finite field F_q or the ring $\mathbb{Z}/n\mathbb{Z}$, and $A = k[G]$ be the group ring of G over k. Then the category $\mathcal{B} = A$-mod of finitely generated A-module is a Frobenius category (see (Curtis, Reiner 1982)). Let B_0 be the corresponding stable category. For any A-module M the group $\operatorname{Hom}_{\mathcal{B}_0}(k, M[i])$ (where k is the trivial one-dimensional A-module) coincides with the i-th Tate cohomology group (see Chap. 4 in (Cassels, Fröhlich 1967)) of the group G with coefficients in M.

§ 3. Cores

3.1. What is the Problem? An important discovery in the homological algebra in the last few years was the fact that the same triangulated category can be can be represented as the derived category of two absolutely different abelian categories. In this section we describe an axiomatic approach to a technique that allows us to see various abelian subcategories inside a given triangulated category. This technique is called the formalism of t-structures.

The axioms of a t-structure formalize the following situation. Let $\mathcal{A}$ be an abelian category and $\mathcal{D} = D^*(\mathcal{A})$ be its derived category. Denote by $\mathcal{D}^{\geq n}$ (resp. $\mathcal{D}^{\leq n}$) the full subcategory of $\mathcal{D}$ formed by complexes $K^\cdot$ with $H^i(K^\cdot) = 0$ for $i < n$ (resp. for $i > n$).

By Proposition 3.1.1 from Chap. 4, the full subcategory $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ coincides with $\mathcal{A}$; more precisely, the functor $\mathcal{A} \to \{\text{category of } H^0\text{-complexes}\} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is an equivalence of categories.

To prove that the intersection $\mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ is abelian we need only the following formal properties.

3.2. Definition. A *t-structure* on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the conditions a)-c) below. Denote $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$, $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

a. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

b. $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \operatorname{Ob}\mathcal{D}^{\leq 0}$, $Y \in \operatorname{Ob}\mathcal{D}^{\geq 1}$.

c. For any $X \in \operatorname{Ob}\mathcal{D}$ there exists a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \operatorname{Ob}\mathcal{D}^{\leq 0}$, $B \in \operatorname{Ob}\mathcal{D}^{\geq 1}$.

The *core* of the t-structure is the full additive subcategory $\mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$.

This definition is motivated by the following proposition.

3.3. Proposition. *If $\mathcal{D} = D^*(\mathcal{A})$ is the derived category of an abelian category $\mathcal{A}$ then the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ described in 3.1 is a t-structure with the core $\mathcal{A}$.*

The main property of cores in triangulated categories is expressed by the following theorem.

3.4. Theorem. *The core $\mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ of any t-structure in $\mathcal{D}$ is an abelian category.*

The plan of the proof. First of all, we must construct the *truncation functors* τ corresponding to a given t-structure (see Chap. 4, 2.10).

3.4.1. Lemma. a. *There exist the functors $\tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$ (resp. $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$) that are right (resp. left) adjoint to the corresponding embedding functors.*

b. *For any $X \in \operatorname{Ob} \mathcal{D}$ there exists a distinguished triangle of the form*

$$\tau_{\leq 0} X \to X \to \tau_{\geq 1} X \xrightarrow{d} \tau_{\leq 0} X[1] \tag{1}$$

and any two distinguished triangles $A \to X \to B \to A[1]$ with $A \in \operatorname{Ob} \mathcal{D}^{\leq 0}$, $B \in \operatorname{Ob} \mathcal{D}^{\geq 1}$ are canonically isomorphic.

Proof. Let us prove the existence of $\tau_{\leq 0}$ and $\tau_{\geq 1}$. Other cases are considered similarly.

For any X let us choose a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \operatorname{Ob} \mathcal{D}^{\leq 0}$, $B \in \operatorname{Ob} \mathcal{D}^{\geq 1}$ and define $\tau_{\leq 0}$ and $\tau_{\geq 1}$ on objects by $\tau_{\leq 0} X = A$ and $\tau_{\geq 1} X = B$. Let $f : X \to Y$ be a morphism in $\mathcal{D}$ and $A' \to X \to B' \to A'[1]$ be the triangle corresponding to the object Y. Let us show that the composition $A \to X \xrightarrow{f} Y$ can be uniquely factored through A'. We have the exact sequence

$$\operatorname{Hom}(A, B'[1]) \to \operatorname{Hom}(A, A') \to \operatorname{Hom}(A, Y) \to \operatorname{Hom}(A, B').$$

By 3.2.b and 3.2.a the left and the right groups in this sequence vanish. Hence $f : X \to Y$ yields a unique morphism $\tau_{\leq 0}(f) : A \to A'$ and the family of these morphisms for all f's complete $\tau_{\leq 0}$ to a functor. Similarly one establishes the functoriality of $\tau_{\geq 1}$ and the uniqueness of triangles $A \to X \to B \to A[1]$.

To prove that $\tau_{\leq 0}$ is adjoint to the embedding $\mathcal{D}^{\leq 0} \to \mathcal{D}$ one must use the isomorphism of functors (in Y)

$$\operatorname{Hom}_{\mathcal{D}^{\leq 0}}(A, \tau_{\leq 0} Y \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(A, Y), \quad A \in \operatorname{Ob} \mathcal{D}^{\leq 0}.$$

constructed above. Similarly one deals with the functor $\tau_{\geq 1}$.

3.4.2. Relations Among Truncation Functors. The functors τ satisfy the following properties (immediate in the case $\mathcal{D} = D^*(\mathcal{A})$).

a. $\tau_{\leq n}X = 0$ if and only if $X \to \tau_{\geq n+1}X$ is an isomorphism.

b. For $m \leq n$ there exist natural isomorphisms $\tau_{\leq m}X \to \tau_{\leq m}\tau_{\leq n}X$ and $\tau_{\geq n}X \to \tau_{\geq n}\tau_{\geq m}X$.

c. For $m \leq n$ there exists a natural isomorphism $\tau_{\geq m}\tau_{\leq n}X \to \tau_{\leq n}\tau_{\geq m}X$ $(\overset{\text{df}}{=} \tau_{[m,n]}X)$.

3.4.3. Construction of Kernels and Cokernels in $\mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$. Let now $f : X \to Y$ be a morphism in $\mathcal{A}$. Denote by Z a cone of f and set

$$K = \tau_{\leq -1}Z, \quad C = \tau_{\geq 0}Z.$$

Define k and c as compositions $k : \tau_{\leq -1}Z \to Z \to X[1]$, $c : Y \to Z \to \tau_{\geq 0}Z$. Using the properties **a–c** above, one can easily verify that $(K[-1], k[-1])$ and (C, c) are respectively the kernel and the cokernel of f, and that the kernel of $k[-1] : K[-1] \to X$ is isomorphic to the cokernel of $c : Y \to C$.

3.5. The Cohomology Functors. Let $\mathcal{D}$ be a triangulated category, $\mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ be the core of some t-structure in $\mathcal{D}$. Define

$$H^0 = \tau_{[0,0]} : \mathcal{D} \to \mathcal{A}, \quad H^i = H^0(X[i]) : \mathcal{D} \to \mathcal{A}.$$

In the case when $\mathcal{D} = D^*(\mathcal{A})$ with the t-structure from 3.1, H^i is usual cohomology of a complex.

3.5.1. Theorem. a. *H^0 is a cohomology functor.*

Assume, in addition, that $\cap_n \operatorname{Ob}\mathcal{D}^{\leq n} = \cap_n \operatorname{Ob}\mathcal{D}^{\geq n} = \{0\}$ (such a structure is called non-degenerate). Then

b. *A morphism $f : X \to Y$ in $\mathcal{D}$ is an isomorphism if and only if all $H^i(f)$ are isomorphisms in $\mathcal{A}$.*

c. $\operatorname{Ob}\mathcal{D}^{\leq n} = \{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \textit{for all } i > n\}$. *Similarly,* $\operatorname{Ob}\mathcal{D}^{\geq n} = \{X \in \operatorname{Ob}\mathcal{D} \mid H^i(X) = 0 \quad \textit{for all } i < n\}$.

3.6. t-exact Functors. Let $\mathcal{D}$, $\widetilde{\mathcal{D}}$ be two triangulated categories, each endowed with a t-structure, and let $F : \mathcal{D} \to \mathcal{D}$ be an exact functor (so that F commutes with translations and maps distinguished triangles into distinguished triangles). The functor F is said to be *left t-exact* if $F(\mathcal{D}^{\geq 0}) \subset \widetilde{\mathcal{D}^{\geq o}}$, *right t-exact* if $F(\mathcal{D}^{\leq 0}) \subset \widetilde{\mathcal{D}^{\leq 0}}$, and *$t$-exact* if it is both left t-exact and right t-exact.

This definition models, of course, the situation when $\mathcal{D} = D(\mathcal{A})$, $\widetilde{\mathcal{D}} = D(\widetilde{\mathcal{A}})$ for two abelian catefories $\mathcal{A}$, $\widetilde{\mathcal{A}}$, F is the derived functor (left or right) of a functor $\varphi : \mathcal{A} \to \widetilde{\mathcal{A}}$ (which is respectively left exact, right exact, or exact).

Let us remark that to recover φ from F using the t-strcuture language one can proceed as follows. Let $\mathcal{D}$, $\widetilde{\mathcal{D}}$ be two triangulated categories, each endowed with a t-structure, $\mathcal{A}$, $\widetilde{\mathcal{A}}$ be the corresponding cores. Define the functor $\varphi : \mathcal{A} \to \widetilde{\mathcal{A}}$ by the formula

$$\varphi(X) = H^0(F(X)), \quad X \in \operatorname{Ob}\mathcal{A} \subset \operatorname{Ob}\mathcal{D},$$

where H^0 is the cohomology functor in $\widetilde{\mathcal{D}}$. Then φ is the additive functor between abelian categories which is left exact, right exact or exact, if F was respectively left exact, right exact or exact (in the last case we can omit the functor H^0 since $F(\mathcal{A}) \subset \widetilde{\mathcal{D}}^{\leq 0} \cap \widetilde{\mathcal{D}}^{\geq 0} = \widetilde{\mathcal{A}}$).

3.7. The Derived Category of the Core. Let $\mathcal{D}$ be a triangulated category with a t-structure, $\mathcal{A}$ be its core. In the general case we can say nothing about the relation of $\mathcal{D}$ to the deribed category $D(\mathcal{A})$ of $\mathcal{A}$. Moreover, we cannot even relate $\mathcal{D}$ with the category of complexes over $\mathcal{A}$. The reason for this is the non-uniqueness of a cone $C(f)$ of a morphism f in $\mathcal{D}$: to construct a functor $\mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ we have to be able to associate objects of $\mathcal{D}$ to complexes over $\mathcal{A}$. To a complex formed by one object X from $\mathcal{A}$ in degree n we associate, of course, the object $X[n]$ from $\mathcal{D}$. But for complexes of length 2 we already meet with some difficulties: the only natural candidate to the role of the object from $\mathcal{D}$ corresponding to the complex of the form, $\ldots \to A \xrightarrow{f} B \to 0 \ldots$ is the third vertex $C(f)$ of the triangle $A \xrightarrow{f} B \to C \to A[1]$ (recall that $\mathcal{A}$ is a full subcategory of $\mathcal{D}$). However, this definition is not functorial since C is determined only up to a non-canonical isomorphism.

However, there exists situations when the existence of auxiliary structures enables us to construct an exact functor $\mathcal{D} \to D(\mathcal{A})$ (where $\mathcal{A}$ is the core of a t-structure in $\mathcal{D}$). Below we will study when such a functor is an equivalence of categories.

3.7.1. Definition. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is said to be *bounded* if it non-degenerate (see Theorem 3.5.1) and for any $X \in \mathrm{Ob}\,\mathcal{D}$ only a finite number of objects $H^i(X) \in \mathrm{Ob}\,\mathcal{A}$ does not vanish.

Clearly the standard t-structure in $D^b(\mathcal{A})$ is bounded, while the standard t-structure in $D(\mathcal{A})$ is not.

3.7.2. Definition. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure, $\mathcal{A}$ be its core. For $X, Y \in \mathrm{Ob}\,\mathcal{A}$ define the groups $\mathrm{Ext}^i_{\mathcal{D}}(X, Y)$ by the formula

$$\mathrm{Ext}^i_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{D}}(X, Y[n]).$$

For $\mathcal{D} = D^b(\mathcal{A})$ the groups $\mathrm{Ext}^i_{\mathcal{D}}(X, Y)$ coincide, clearly, with the groups $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$ (see Definition 3.2.1 from Chap. 4). It is clear also that one can define the composition

$$\mathrm{Ext}^i_{\mathcal{D}}(X, Y) \times \mathrm{Ext}^j_{\mathcal{D}}(Y, Z) \to \mathrm{Ext}^{i+j}_{\mathcal{D}}(X, Z)$$

(similarly to Chap. 4, 3.3.a). The following theorem shows that the difference between $\mathcal{D}$ and $D^b(\mathcal{A})$ is controlled by the difference between $\mathrm{Ext}^i_{\mathcal{D}}(X, Y)$ and $\mathrm{Ext}^i_{\mathcal{A}}(X, Y)$.

3.7.3. Theorem. *Let $\mathcal{A}$ be the core of a bounded t-structure in a triangulated categoty $\mathcal{D}$, $F : D^b(\mathcal{A}) \to \mathcal{D}$ be a t-exact functor. The functor F is an equivalence of categories if and only if* $\mathrm{Ext}^*_{\mathcal{D}}$ *is generated by* $\mathrm{Ext}^1_{\mathcal{D}}$ *(that* *is, any element $\alpha \in \mathrm{Ext}^i_{\mathcal{D}}(X,Y)$, $X,Y \in \mathrm{Ob}\,\mathcal{A}$, is a linear combination of monomials $\beta_1\beta_2\dots\beta_i$, $\beta_i \in \mathrm{Ext}^j_{\mathcal{D}}(X_j, X_{j+1})$ with $X_1 = X$, $X_{i+1} = Y$).*

Let as remark that the interpretation of $\mathrm{Ext}^i_{\mathcal{A}}(X,Y)$ by Yoneda (see Theorem 3.4.c from Chap. 4) shows that $\mathrm{Ext}^*_{\mathcal{A}}$ is generated by $\mathrm{Ext}^1_{\mathcal{A}}$, so that the above condition is clearly necessary.

3.8. Glueing t-structures. An important method to construct new t-structures is the glueing theorem which enables one to relate t-structures on a triangulated category $\mathcal{D}$ with t-structures on its thick subcategory $\mathcal{C}$ and on the quotient category $\mathcal{E}$ (see 1.10).

Let

$$\mathcal{C} \xrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E} \tag{2}$$

be an exact triple of triangulated categories (see 1.10.2). Assume we are given t-structures on $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$. These t-structures are said to be *compatible* (or the triple (2) is said to be *t-exact*) if P and Q are t-exact functors.

First of all, it is clear that a t-structure on $\mathcal{D}$ determines unique compatible t-structures on $\mathcal{C}$ and on $\mathcal{E}$ (namely, $\mathcal{C}^{\le 0} = \mathcal{C} \cap \mathcal{D}^{\le 0}$, $\mathcal{E}^{\le 0} = Q\mathcal{D}^{\le 0}$ and similarly for $\mathcal{C}^{\ge 0}$, $\mathcal{E}^{\ge 0}$).

Conversly, for two t-structures on $\mathcal{C}$ and $\mathcal{E}$ there exists at most one compatible t-structure on $\mathcal{D}$. More precisely, the following theorem holds.

3.8.1. Theorem. *Let $\mathcal{C} \xrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$ be a t-exact triple of triangulated categories. Denote*

$$\begin{aligned} {}^{\perp}(P\mathcal{C}^{>0}) &= \{X \in \mathrm{Ob}\,\mathcal{D} \mid \mathrm{Hom}(X,Y) = 0 \quad \text{for all } \; Y \in P\mathcal{C}^{>0}\}, \\ (P\mathcal{C}^{<0})^{\perp} &= \{X \in \mathrm{Ob}\,\mathcal{D} \mid \mathrm{Hom}(Y,X) = 0 \quad \text{for all } \; Y \in P\mathcal{C}^{<0}\}. \end{aligned}$$

Then the t-structure on $\mathcal{D}$ is determined by t-structures on $\mathcal{C}$ and $\mathcal{E}$ as follows:

$$\begin{aligned} \mathcal{D}^{\le 0} &= Q(\mathcal{E}^{\le 0}) \cap {}^{\perp}(P\mathcal{C}^{>0}), \\ \mathcal{D}^{\ge 0} &= Q(\mathcal{E}^{\ge 0}) \cap (P\mathcal{C}^{<0})^{\perp}. \end{aligned}$$

It might happen that for given t-structures on $\mathcal{C}$ and $\mathcal{E}$ there exists no compatible t-structure on $\mathcal{D}$. However, there is a class of important cases when the existence of a compatible t-structure on $\mathcal{D}$ can be guaranteed.

3.8.2. Theorem. *Let $\mathcal{C} \xrightarrow{P} \mathcal{D} \xrightarrow{Q} \mathcal{E}$ be an exact triple of triangulated categories. Assume that P has left and right adjoint functors (this is equivalent to the existence of left and right adjoint functors for Q). Then for any t-structures on $\mathcal{C}$ and $\mathcal{E}$ there exists a compatible t-structure on $\mathcal{D}$.*

3.9. Examples. Non-trivial example of t-structures in derived categories of sheaves on topological spaces are related to the theory of so called perverse sheaves (see Chap. 7).

3.9.1. Relations Between the Categories of Sheaves on a Topological Space and on Its Subspaces. Below we shall describe an axiomatic approach to the following situation. Let X be a topological space, $U \subset X$ an open subspace, $Y = X - U$ be the complement, $i : Y \to X$, $j : U \to X$ be the inclusions. Denote by $\mathcal{A}_X$, $\mathcal{A}_Y$, $\mathcal{A}_U$ the categories of sheaves of abelian groups on X, Y, U and by $\mathcal{D}_X$, $\mathcal{D}_Y$, $\mathcal{D}_U$ the corresponding derived categories (say, bounded).

Consider the following 6 functors

$$Rj_{.}, j_! : \mathcal{D}_U \to \mathcal{D}_X, \quad i^{\cdot} : \mathcal{D}_X \to \mathcal{D}_U,$$
$$i_{.} : \mathcal{D}_Y \to \mathcal{D}_X, \quad i^{\cdot}, i^! : \mathcal{D}_X \to \mathcal{D}_Y$$

(see Chap. 4, Sect. 5; the functor $i^!$ can be defined as the right adjoint to the left exact funcor "sections with the support in Y", see Chap. 4, 5.5). These functors satisfy the following properties.

a. They are exact functors between corresponding triangulated categories.

b. $i^{\cdot}$ and $i^!$ are respectively the right and the left adjoint to $i_{.}$.

c. $j_!$ and $Rj_{.}$ are respectively the right and the left adjoint to $j^{\cdot}$.

d. $j^{\cdot}i_{.} = O$. By adjointness, this implies $i^{\cdot}j_! = 0$, $i^!Rj_{.} = 0$ and

$$\mathrm{Hom}_{\mathcal{D}_X}(j_!\mathcal{H}^{\cdot}, i_{.}\mathcal{G}^{\cdot}) = 0, \quad \mathrm{Hom}_{\mathcal{D}_X}(i_{.}\mathcal{G}^{\cdot}, Rj_{.}\mathcal{H}^{\cdot}) = 0$$

for $\mathcal{G}^{\cdot} \in \mathrm{Ob}\,\mathcal{D}_Y$, $\mathcal{H}^{\cdot} \in \mathrm{Ob}\,\mathcal{D}_U$.

e. There exist (functorial in $\mathcal{F}^{\cdot} \in \mathrm{Ob}\,\mathcal{D}_X$) morphisms

$$w : i_{.}i^{\cdot}\mathcal{F}^{\cdot} \to j_!j^{\cdot}\mathcal{F}^{\cdot}[1], \quad w' : Rj_{.}j^{\cdot}\mathcal{F}^{\cdot} \to i_{.}i^!\mathcal{F}^{\cdot}[1]$$

such that the triangles

$$j_!j^{\cdot}\mathcal{F}^{\cdot} \xrightarrow{u} \mathcal{F}^{\cdot} \xrightarrow{v} i_{.}i^{\cdot}\mathcal{F}^{\cdot} \xrightarrow{w} j_!j^{\cdot}\mathcal{F}^{\cdot}[1],$$
$$i_{.}i^!\mathcal{F}^{\cdot} \xrightarrow{u'} \mathcal{F}^{\cdot} \xrightarrow{v'} Rj_{.}j^{\cdot}\mathcal{F}^{\cdot}i \xrightarrow{w'} i_{.}i^!\mathcal{F}^{\cdot}[1]$$

are distinguished (here u, u', v, v' are the adjunction morphisms corresponding to functors from **b** and **c**. By **d** and Corollary 1.5, w and w' are determined uniquely.

f. The adjunction morphisms

$$i^{\cdot}i_{.}\mathcal{G}^{\cdot} \to \mathcal{G}^{\cdot} \to i^!i_{.}\mathcal{G}^{\cdot}, \quad \mathcal{G}^{\cdot} \in \mathrm{Ob}\,\mathcal{D}_Y,$$
$$j^{\cdot}Rj_{.}\mathcal{H}^{\cdot} \to \mathcal{H}^{\cdot} \to j^{\cdot}j_!\mathcal{H}^{\cdot}, \quad \mathcal{H}^{\cdot} \in \mathrm{Ob}\,\mathcal{D}_U,$$

are isomorphisms.

3.9.2. Glueing. A family consisting of three triangulated categories $\mathcal{D}_X$, $\mathcal{D}_Y$, $\mathcal{D}_U$ (not necesarily related to sheaf categories) and of six exact functors satisfying the conditions **a**–**f** is called the *glueing data*. One of examples was described in the previous subsection. Glueing data can be obtained if we define appropriate functors in categories of coherent sheaves (when X, Y, U are algebraic varieties or schemes) or in categories of sheaves in etale topologies.

Let us assume now that we have glueing data. Then it is easy to verify that $\mathcal{D}_Y \xrightarrow{i_\cdot} \mathcal{D}_X \xrightarrow{j^\cdot} \mathcal{D}_U$ is an exact triple of triangulated categories. Since $i_\cdot$ has both the left and the right adjoint functors, Theorems 3.8.1 and 3.8.2 show that any pair of t-structures $(\mathcal{D}_Y{}^{\leq 0}, \mathcal{D}_Y{}^{\geq 0})$ on $\mathcal{D}_Y$ and $(\mathcal{D}_U{}^{\leq 0}, \mathcal{D}_U{}^{\geq 0})$ on $\mathcal{D}_U$ determines a unique compatible t-structure on $\mathcal{D}_X$. Moreover, $\mathcal{D}_X{}^{\leq 0}$ and $\mathcal{D}_X{}^{\geq 0}$ for this structure are given by

$$\begin{aligned}\mathcal{D}_X{}^{\leq 0} &= \{\mathcal{F} \in \operatorname{Ob}\mathcal{D}_X \mid j^\cdot\mathcal{F} \in \operatorname{Ob}\mathcal{D}_U{}^{\leq 0}, \quad i^\cdot\mathcal{F} \in \operatorname{Ob}\mathcal{D}_Y{}^{\leq 0}\},\\ \mathcal{D}_X{}^{\geq 0} &= \{\mathcal{F} \in \operatorname{Ob}\mathcal{D}_X \mid j^\cdot\mathcal{F} \in \operatorname{Ob}\mathcal{D}_U{}^{\geq 0}, \quad i^\cdot\mathcal{F} \in \operatorname{Ob}\mathcal{D}_Y{}^{\geq 0}\}.\end{aligned}$$

This result about the glueing of t-structures was proved by Beilinson, Bernstein, Deligne (1982), Sect. 1.4. They used it to construct the category of perverse sheaves. Glueing together shifted t-structures on $\mathcal{D}_U$ and on $\mathcal{D}_Y$, one can also construct some non-standard t-structures on $\mathcal{D}_X$.

Bibliographic Hints

The notion of a triangulated catogory discussed in Sect. 1 appeared in the attempts to axiomatize the main properties of complexes up to quasi-isomorphism without mentioning the original abelian category. Original exposition see in the notes of the Hartshorne's seminar (Hartshorne 1966) and in the resume of Verdier's thesis (Verdier 1963); see also (Happel 1988; Iversen 1986). The proofs of the results from Sect. 1 can be found in (Hartshorne 1966) and in (Gelfand, Manin 1988).

In Sect. 2 we give examples of non-standard triangulated categories whose relationship to derived categories is no immediately clear. The first example (2.1–2.4) illustrates the so-called "S-Λ duality"; this duality enables us to give a description of coherent algebraic sheaves on projective spaces which is dual with respect to the Serre theorem. The proof of Theorem 4.3 and further generalizations see in (Beilinson 1978; Bernstein, Gelfand, Gelfand 1978; S.Gelfand 1984; Gelfand, Manin 1988; Kapranov 1988; Rudakov, Gorodentsev 1987).

One of the observations essentially underlying the S-Λ duality from (Bernstein, Gelfand, Gelfand 1978) is that the derived categories $D^*(\mathcal{A})$ and $D^*(\mathcal{B})$ can be equivalent even if the original abelian categories $\mathcal{A}$ and $\mathcal{B}$ were non-equivalent. This situation occurs, for example, in the theory of so-called tilting modules which play an important role in the study of the module categories over finite-dimensional algebras. The detailed exposition of the theory of tilting modules is contained in (Happel 1988). The same theory of tilting modules is related to the second method of constructing triangulated categories presented in Sect. 2 (2.5–2.9). The proof of the corresponding results (in particular, of Theorem 2.9) can be found in (Happel 1988).

The theory of cores in triangulated categories (Sect. 3) solves a problem which is in a sense converse to that from Chap. 4: how to recognize the abelian category $\mathcal{A}$ inside the triangulated category $D(\mathcal{A})$. This problem, which arose in relation to the theory of perverse sheaves (see Chap. 7) was discussed in (Beilinson, Bernstein, Deligne 1982), where the reader can find the proofs of the results from 3.1.

Chapter 6
Mixed Hodge Structures

§ 0. Introduction

Any (co)homology theory can be considered as a tool for linearization of certain nonlinear problems. It maps a geometrical universe (a category of topological spaces, a site, a topos, etc.) into a certain algebraic universe (modules, modules with some structure, complexes, etc.) that preserves sufficiently many properties of the category of linear spaces.

The main geometrical universe in this chapter is the category of complex algebraic varieties, possibly non-compact and/or singular, and the main algebraic universe is the category of Hodge structures. The functor from the first category to the second one is a modern version of the classical theory of periods of differential forms.

The subject of the classical Hodge theory is the cohomology (with constant coefficients) of smooth projective varieties. Its principal results are the following: the construction of the (p,q)-decomposition, the hard Lefschetz theorem (decomposition by primitive cycles), the theorem about the index of the intersection form (polarization on primitive cycles). The tools are (global) theory of harmonic forms and (local) Kähler identities.

In mid-sixties a (very important for the future development of the theory) analogies between (p,q)-decompositions and representations of the Galois group in etale cohomology was understood. The formal side of this similarity is quite simple: (p,q)-decomposition is the action of the torus $\mathbb{C}^*$ (see Sect. 1), while the class field theory identifies the multiplicative group of a (non-Archimedian) local field with the maximal abelian quotient of its Galois group, so that the (p,q)-decomposition is the Archimedian analog of the Galois representation. On the other hand, the non-formal part of this similarity is misterious: Galois representations result from the Galois symmetries in the etale topologe, while the roots of the Hodge structures are lost in darkness: hidden "Hodge symmetries" generating these structure are still unknown. In any case, this similarity forced us to suggest the existence of a natural Hodge structure on the cohomology of an arbitrary (not necessaraly smooth or compact) algebraic variety. An important difference with the standard Hodge theory is that one cohomology group carries (p,q)-components corresponding to *different* weights (this is illustrated by an example of a curve, see Sect. 2). The corresponding linear algebra structure, the so-called mixed Hodge structure, was suggested by Deligne (see Sect. 1), who also constructed a mixed Hodge structure in the cohomology (with constant coefficients) of an arbitrary algebraic variety. His construction is pure algebraic: it reduces a general case to the case of a smooth projective variety by using the Hironaka desingularization theorem. One must remark that a "natural" (not using resolution and compactification) construction is still unknown; in particular, it is not known what kind of analysis lies behind the notion of a mixed Hodge structure (see, however, papers by Varchenko (1981, 1983) about mixed Hodge structures related to singularities of functions).

Next, the above similarity requires the existence of an appropriate functorial category of "Hodge sheaves" (that is, of Hodge structures on constructible sheaves). Attempts to construct such category go back to Griffiths' papers on variations of Hodge structures (in the sixties); however, it seems that only recently such a theory was constructed by Saito (1986).

Example. Let X_s be a family of smooth projective varieties depending on a parameter $s \in S$. Then S carries a local system formed by cohomology of a fiber $\mathcal{H}_s^{\cdot} = H^{\cdot}(X_s, Z)$; $\mathcal{H}_s^{\cdot}$ is endowed with a Hodge structure depending continuously on the parameter s. Consider the maximal constant local subsystem $\mathcal{H}^f \subset \mathcal{H}^{\cdot}$ (its fibers are invariants of the monodromy). Then $\mathcal{H}_s^f$ is a Hodge substructure of $\mathcal{H}_s$, and it does not depend on s. A similar statement in the arithmetic situation is clear (just because of the existence of Galois symmetries); in the Hodge situation its proof is quite non-trivial.

We must remark that a majority of results about the topology of complex algebraic varieties (beginning with the Grothendieck theorem about the quasi-unipotence of the local monodromy) proved by arithmetic methods; after Deligne proved the Weil conjectures and developed a convenient functorial technique of perverse sheaves in finite characteristics, arithmetic methods soon became beyond comparison. Now, due to Saito, these results can be reproved in the framework of the Hodge theory. However, although the congruity between Hodge theory and arithmetic can now be considered as an experimently established fact, its explanation today is as misterious as it was some 20 years ago.

In a wide sense the main problem in the theory of Hodge structure can be formulated as follows: how close is the Hodge cohomology to the universal, "motivic", cohomology whose definition was suggested by Grothendieck? In its simplest form a motif is an object of a certain extension of the category of algebraic varieties with correspondences as morphisms; this category can be obtained by formally adding to algebraic varieties kernels and cokernels of projections. Correspondences are considered modulo some equivalence relation (rational, integral, etc.) and the choice of this equivalence relation determines the algebraic geometry subtleries accounted to by a particular motivic cohomology.

Further development of the theory of motives is complicated by the fact that the so-called "standard conjectures" are still unproved, and from this point of view several main problems and results of Hodge theory in few last years fall into the following directions.

a. The reconstruction of a variety by a Hodge structure on its cohomology: Torelli-type problems. We mention also similar problems for families of varieties. In certain special situations, a variation of Hodge structures yields, via the Griffiths transversality theorem, more information than the value at a point. These variations can be studies infinitisemally; this was done in a series of papers by Griffiths and his students.

b. The recognition of morphisms of varieties and, more generally, of algebraic correspondences, among generic morphisms of Hodge structures: Hodge-type conjectures about the characterization of algebraic cycles.

Together, problems of these two types try to find out to what extend the algebraic geometry information is preserved after passing to Hodge structures.

c. Description of those variations of Hodge structures that have the algebraic geometry origin, and the definition of Hodge structures in the cohomology with coefficients in sheaves of Hodge structures.

d. The construction of Hodge structures in other topologival invariants of a variety, such as homotopy groups, Deligne-Goresky-MacPherson cohomology, L_2-cohomology, etc.

Mixed Hodge structures form an abelian category $\mathcal{H}$, and the idea to consider the corresponding derived category is quite natural. Following the general line of this book we emphasize homological properties of $\mathcal{H}$ and $D(\mathcal{H})$, ingnoring somehow geometrical properties of the Hodge cohomology functor, which is also studied in other volumes of this Encyclopaedia.

§ 1. The Category of Hodge Structures

1.1. Definition. Let $A \subset \mathbb{R}$ be a Noetherian ring such that $A \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is a field, M_A be an A-module of the finite type. A *pure Hodge structure of the weight n*, $n \in \mathbb{Z}$, on M is any of the following collection of data:

a. A finite descending filtration on $M_\mathbb{C} = \mathbb{C} \underset{A}{\otimes} M_A$, $\ldots \supset F^p M_\mathbb{C} \supset F^{p+1} M_\mathbb{C} \supset \ldots$, with

$$F^p M_\mathbb{C} \oplus \overline{F^p M_\mathbb{C}} = M_\mathbb{C} \quad \text{for all } p + q = n + 1.$$

b. An action of $\mathbb{C}^*$ on $M_\mathbb{C}$ arising from a real action of the algebraic two-dimensional torus such that a real number $a \in \mathbb{R}^* \subset \mathbb{C}^*$ acts as the multiplication by a^n.

c. A double grading on $M_\mathbb{C}$:

$$M_\mathbb{C} = \bigoplus_{p+q=n} M^{p,q}$$

with the property $\overline{M^{p,q}} = M^{q,p}$ (the complex conjugation is taken with respect to $M_\mathbb{R} = \mathbb{R} \underset{A}{\otimes} M_A$).

The equivalence of these data is established as follows:

a. $\Rightarrow$ c.: $M^{p,q} = F^p M_\mathbb{C} \oplus \overline{F^q M_\mathbb{C}}$;

c. $\Rightarrow$ a.: $F^p M_\mathbb{C} = \underset{i>p}{\oplus} M^{i,j}$;

b. $\Leftrightarrow$ c.: $M^{p,q} = \{m \in M_{\mathbb{C}} \mid z \in \mathbb{C}^*$ acts on m as the multiplication by $z^p \overline{z}^q\}$.

1.2. Definition. In the same setup, a *mixed Hodge structure* on a Noetherian A-module M_A is the following collection of data:

A finite descending filtration $F^p M_{\mathbb{C}}$ (Hodge filtration) and a finite ascending filtration $W_i(M_A \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ (weight filtration) satisfying the following condition: for each $n \in \mathbb{Z}$ the module $\mathbf{Gr}_n^W(M_A \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ together with the filtration induced on it by F form a pure Hodge structure of weight n.

A Hodge structure on an A-module is called also a Hodge A-structure.

1.3. Morphisms. A *morphism of mixed Hodge A-structures* $(M_A, W, F) \to (N_A, W, F)$ is a morphism $f : M_A \to N_A$ of A-modules that induces morphisms compatible with filtrations F, W (so also with $\overline{F}$). An essential point is that this condition automaticaly implies the strong compatibility with filtrations: $f(M_{\mathbb{C}}) \cap F^p N_{\mathbb{C}} = F^p M_{\mathbb{C}}$, etc. This can be established together with the proof of the following result:

1.4. Theorem. *Mixed Hodge A-structures form an abelian category $\mathcal{H}_A$. Kernels and cokernels of morphisms in this category are kernels and cokernels of morphisms of A-modules with induced filtrations.*

1.5. The Tensor Algebra. Let $M = (M_A, W, F)$, $N = (N_A, W, F)$ be two mixed Hodge structures. Define

$$(M \otimes N)_A = M_A \underset{A}{\otimes} N_A,$$

$$W_i((M \otimes N)_A \otimes \mathbb{Q}) = \operatorname{Im}\Big(\sum_{k+l=i} W_k(M_A \otimes \mathbb{Q}) \otimes W_l(N_A \otimes \mathbb{Q}) \Big),$$

$$F^p(M \otimes N)_{\mathbb{C}} = \operatorname{Im}\Big(\sum_{k+l=p} F^k(M_{\mathbb{C}}) \otimes F^l(N_{\mathbb{C}}) \Big).$$

These data form a mixed Hodge structure called the *tensor product* of initial structures.

In the same way one defines inner **Hom**:

$$\mathbf{Hom}(M, N)_A = \operatorname{Hom}_A(M_A, N_A),$$
$$F^i\,\mathbf{Hom}(M, N) = \{f : M_A \to N_A \mid f(M_{\mathbb{C}}) \subset F^{n+i} N_{\mathbb{C}} \text{ for all } n\},$$

and similarly for W.

The ring A, viewed as an A-module with filtration $F^0 A = A$, $F^1 A = \{0\}$, $W_{-1}(A \otimes \mathbb{Q}) = \{0\}$, $W_0(A \otimes \mathbb{Q}) = A \otimes \mathbb{Q}$, is the identity element in this tensor category.

The dual Hodge structure is defined by $M^\vee = \mathbf{Hom}(M, A)$.

1.6. The Category of Real Hodge Structures as a Category of Representations. Certain general results (Tannaka-Krein type theorems) guarantee that a linear category with tensor products is equivalent to some category of representations.

For real mixed Hodge structures this category is the category of representations of a proalgebraic group which we will now describe.

Consider a free nilpotent Lie algebra L over $\mathbb{C}$ with generators $T^{i,j}$ $(i,j > 0)$. Introduce on L a grading by defining $\deg T^{i,j} = i+j$. Denote by W_nL the ideal in L generated by elements of degree $\leq n$. Let U_n be the simply connected Lie group with the Lie algebra L/W_nL, and $U = \operatorname{lim\,proj} U_n$. Define the action of $\mathbf{G}_m \times \mathbf{G}_m$ on L by the formula $(\lambda,\mu)T^{i,j} = \lambda^{-i}\mu^{-j}T^{i,j}$. This action can be transfered to U. Denote by G the semi-direct product $G = (\mathbf{G}_m \times \mathbf{G}_m) \ltimes L$ and introduce on G a real structure induced by the complex conjufation $(\lambda,\mu) \to (\overline{\lambda},\overline{\mu})$ on $\mathbf{G}_m \times \mathbf{G}_m$ and $T^{i,j} \to -T^{i,j}$ on L .

Let V be a real vector space endowed with a real action of G, that is, a representation of G in $V_{\mathbb{C}}$ compatible with the complex conjugation. This representation defines on $V_{\mathbb{C}}$ the following structures.

a. A double grading $V_{\mathbb{C}} = \oplus V^{p,q}$ which is defined as the decomposition by characters of $\mathbf{G}_m \times \mathbf{G}_m$. Since the action is real, $\overline{V^{p,q}} = V^{q,p}$.

b. An automorphism $t = \exp\left(\sum_{r,s} T^{r,s}\right)$. Since the action is real, $\overline{t} = t^{-1}$. Moreover

$$(t-1)V^{p,q} = \bigoplus_{\substack{i<p,\\ j<q}} V^{i,j}.$$

Conversely, double grading and t as above define a real representation of G (the action of $T^{r,s}$ is reconstructed from t and the double grading $V^{p,q}$). Denote the category of such structures by $\mathcal{G}_{\mathbb{R}}$.

The next proposition can be considered as a generalization of the characterizations 1.1.a and 1.1.b of pure Hodge structures to mixed Hodge structure (an analog of the characterization 1.1.a is Definition 1.2).

1.7. Proposition. *The category of real mixed Hodge structures is equivalent to the category of real finite-dimensional representations of the group G defined in* 1.6.

The sketch of the proof. The functor $\mathcal{G}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ maps $(V, V_{\mathbb{C}}^{p,q}, t)$ to (V, W, F), where

$$W_i = \bigoplus_{p+q<i} V_{\mathbb{C}}^{p,q}, \quad F^j = \Big(\bigoplus_{p>j} V_{\mathbb{C}}^{p,q}\Big).$$

One can easily verify that it defines an equivalence of categories. A quasi-inverse functor associates to $(M_{\mathbb{R}}, W, F)$ the space $V = \mathbf{Gr}^W(M_{\mathbb{R}})$; the double grading on $V_{\mathbb{C}}$ arises from the definiton of the mixed Hodge structure.

To construct t we introduce the following notations:

$$M_F^{p,q} = (W_{p+q} \cap F^p) \cap \Big((W_{p+q} \cap \overline{F^q}) + \sum_{i \geq 0} (W_{p+q-i} \cap \overline{F^{q-i+1}}) \Big),$$

$$M_{\overline{F}}^{p,q} = (W_{p+q} \cap \overline{F^q}) \cap \Big((W_{p+q} \cap F^p) + \sum_{i \geq 0} (W_{p+q-i} \cap F^{p-i+1}) \Big).$$

One can prove that for $p+q=n$ the projection of $W_n \subset M_{\mathbb{R}}$ onto V_W^n induces isomorphisms $M_F^{p,q} \xrightarrow{\sim} V_{\mathbb{C}}^{p,q}$ and $M_{\overline{F}}^{p,q} \xrightarrow{\sim} V_{\mathbb{C}}^{p,q}$. Denote by a_F and $a_{\overline{F}}$ the sums of these isomorphisms. Then $t = a_F a_{\overline{F}}^{-1}$ is an automorphism of $V_{\mathbb{C}}$ satisfying the condition 1.6.b.

1.8. Hodge-Tate Structures. A *Hodge-Tate A-structure* $A(1)$ is, by definition, a pure Hodge structure of the weight -2 on the module $M_A = 2\pi i A$ concentrated in the bidegree $(-1,-1)$.

Denote $A(n) = A(1)^{\otimes n}$, so that $M_A = (2\pi i)^n A$, the weight equals $-2n$, the bidegree equals $(-n,-n)$. For an arbitrary Hodge structure M denote $M(n) = M \otimes A(n)$.

§ 2. Mixed Hodge Structures on Cohomology with Constant Coefficients

The first fundamental result by Deligne after defining mixed Hodge structures was to construct a mixed Hodge structure on cohomology of an arbitrary (possibly singular and/or non-compact) separable algebraic variety X over $\mathbb{C}$ with values in $\mathcal{H}_{\mathbb{Z}}$. In this section we formulate the Deligne theorem and give some comments about its proof.

2.1. Theorem. *For each n the cohomology $H^n(X,\mathbb{C})$ carries a functorial in X mixed Hodge structure satisfying the following properties:*

a. *If X is smooth and complete, then this structure is pure of weight n. The Hodge filtration on $H^n(X,\mathbb{C})$ is determined by hypercohomology groups of the truncated holomorphic de Rham complex $\mathbb{H}^n(\Omega^i \to \Omega^{i+1} \to \dots) = F^i H^n(X,\mathbb{C})$. Since the spectral sequence $E_1^{pq} = H^p(X,\Omega^q) \Rightarrow H^{p+q}(X,\mathbb{C})$ degenerates, these hypercohomology groups are subgroups of $H^n(X,\mathbb{C})$.*

b. *The Kunneth isomorphism*

$$H^\cdot(X \times Y, \mathbb{Q}) \simeq H^\cdot(X,\mathbb{Q}) \otimes H^\cdot(Y,\mathbb{Q})$$

is an isomorphism of mixed Hodge structures.

c. *The multiplication in cohomology*

$$H^\cdot(X,\mathbb{Z}) \otimes H^\cdot(X,\mathbb{Z}) \to H^\cdot(X,\mathbb{Z})$$

is a morphism of mixed Hodge structures.

Below we give some comments to the proof.

2.2. Smooth Manifolds. Let U be a smooth variety represented in the form $U = X \setminus Y$, where X is a smooth compact variety and Y is a divisor with normal crossings (i.e. a divisor that is locally isomorphic to the union of several coordinate hyperplanes). By the Hironaka desingularization theorem any smooth algebraic variety U can be represented in such form.

Denote by $\Omega^p_Y(\log Y)$ the sheaf of those meromorphic p-forms on X that are locally generated by holomorphic forms and the forms dz/z, where $z = 0$ is the local equation of some branch of Y. The direct sum of all sheaves $\Omega^p_X(\log Y)$ is the logarithmic de Rham complex $\Omega^{\cdot}_X(\log Y)$. The definition of the mixed Hodge structure on $H^{\cdot}(U, \mathbb{C})$ is done in several steps.

a. There exists a canonical isomorphism

$$H^{\cdot}(U, \mathbb{C}) = H^{\cdot}(X, \Omega^p_X(\log Y))$$

(with hypercohomology at the right-hand side). This is done using the Leray spectral sequence for the embedding $j : U \to X$, which yields an isomorphism

$$H^{\cdot}(U, \mathbb{C}) \xrightarrow{\sim} H^{\cdot}(X, j_* \Omega^{\cdot}_U),$$

and the verification of the fact that the embedding $\Omega^p_X(\log Y) \hookrightarrow j_* \Omega^{\cdot}_U$ is a quasi-isomorphism of complexes of sheaves.

b. The weight filtration on $H^{\cdot}(U, \mathbb{C})$ is induced by any of two filtrations $\widetilde{W}$ or τ on $\Omega^{\cdot}_X(\log Y)$:

$$W_n H^m(X, \mathbb{C}) = H^m(X, \widetilde{W}_{n-m} \Omega^{\cdot}_Y(\log Y)) = H^m(X, \tau_{<n-m} \Omega^{\cdot}_X(\log Y)),$$

where

$$\widetilde{W}_n(\Omega^{\cdot}_Y(\log Y)) = \text{linear combination of forms } \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_m}{z_m}$$
$$\text{with holomorphic coefficients, } m \le n,\ z_i \text{ are local equations of } Y.$$

$$\tau_{<n}(\Omega^p_X(\log Y)) = \begin{cases} \omega^p_X(\log Y) & \text{for } p > n, \\ Z^n(\Omega^{\cdot}_X(\log Y)) & \text{for } p = n, \\ 0 & \text{for } p > n. \end{cases}$$

c. The Hodge filtration on $H^{\cdot}(U, \mathbb{C})$ is induced by the filtration

$$\sigma_{>n}(\Omega^p_X(\log Y)) = \begin{cases} \Omega^p_X(\log Y) & \text{for } p \ge n, \\ 0 & \text{for } p < n. \end{cases}$$

d. The fact that the weight filtration on $H^{\cdot}(U, \mathbb{C})$ is induced by a filtration on $H^{\cdot}(U, \mathbb{Q})$ is established by the verification that the hypercohomology spectral sequence for the filtered complex $\Omega^{\cdot}_X(\log Y)$ (with the filtration $\widetilde{W}$) coincides with the Leray spectral sequence for j_*, and this last sequence is the complexification of some sequence of $\mathbb{Q}$-spaces.

The proof is completed by the verification of the axioms of a Hodge structure, and of the independence on compactification.

2.3. Singular Varieties. This case involves a more sofisticated homological technique, namely the use of simplicial schemes at one side, and the introduction of the technical notion of a Hodge structure on a complex at the other side. We describe the corresponding methods in Sect. 4 and 5.

In conclusion we formulate a theorem about relative cohomology.

2.4. Theorem. *Relative cohomology of an algebraic variety over $\mathbb{C}$ carries a functorial mixed Hodge structure such that the exact sequence of relative cohomology is an exact sequence of mixed Hodge structures.*

2.5. Hodge Structure on the Cohomology of a Curve. We describe the Hodge structure on the cohomology group $H^1(X,\mathbb{C})$, where X is a curve.

2.5.1. *X is a smooth non-compact curve.* Let $j : X \to \overline{X}$ be the completion of X, $D = \overline{X} \setminus X$. The Leray spectral sequence for j yields the following exact sequence:

$$0 \to H^1(\overline{X}, j_*\mathbb{C}_X) \to H^1(X,\mathbb{C}) \to H^1(\overline{X}, R^1j_*\mathbb{C}_X) \to H^2(\overline{X}, j_*\mathbb{C}_X),$$

where $\mathbb{C}_X$ is the constant sheaf on X with the fiber $\mathbb{C}$. Since $j_*\mathbb{C}_X = \mathbb{C}_{\overline{X}}$, $R^1j_*\mathbb{C}_X = \mathbb{C}_D$, this exact sequence takes the form

$$0 \to H^1(\overline{X},\mathbb{C}) \xrightarrow{\alpha} H^1(X,\mathbb{C}) \to \mathbb{C}_D \to H^2(\overline{X},\mathbb{C})$$

Introduce the weight filtration W on $H^1(X,\mathbb{C})$ by setting $W^1H^1(X,\mathbb{C}) = \operatorname{Im}\alpha$, $W^2H^1(X,\mathbb{C}) = H^1(X,\mathbb{C})$; the Hodge filtration is defined by the pure Hodge structure of weights 1 and 2 on $H^1(\overline{X},\mathbb{C})$ and $\mathbb{C}^D$ (where $\mathbb{C}^D$ is naturally interpreted as the two-dimensional cohomology group $H^2(\overline{X}, X; \mathbb{C})$, so that the only nonzero space has $(p,q) = (1,1)$).

2.5.2. *X is a compact curve.* Let $\pi : Y \to X$ be the normalization of X. The exact sequence of sheaves on X

$$0 \to \mathbb{C}_X \to \pi_*\mathbb{C}_Y \to \mathcal{L} \to 0$$

(where $\mathcal{L}$ is a sheaf concentrated in singular points of X; the dimension of its fiber at $x \in X$ equals the number of local branches in x) yields the exact sequence

$$H^0(X,\mathcal{L}) \to H^1(X,\mathbb{C}_X) \to H^1(X,\pi_*\mathbb{C}_Y) \to 0.$$

We set $W^0H^1(X,\mathbb{C}) = \operatorname{Im}\delta$, $W^1H^1(X,\mathbb{C}) = H^1(X,\mathbb{C})$. The Hodge filtration on $H^1(X,\mathbb{C})$ is induced by pure Hodge structure of the weight 0 on $H^0(X,\mathcal{L})$ and of weight 1 on $H^1(X,\pi_*\mathbb{C}_Y) = H^1(Y,\mathbb{C})$.

2.5.3. *The general case.* Denote by Y the normalization of X and by $\overline{Y}$ the smooth compactification of Y. Similarly to 2.5.1, 2.5.2, we obtain the following three-term filtration on $H^1(X,\mathbb{C})$:

$$0 \subset \operatorname{Im}\delta \leq \theta^{-1}(H^1(\overline{Y},\mathbb{C})) \subset H^1(X,\mathbb{C}),$$

and the quotients of this filtration carry the natural Hodge structures of the weights 0, 1, 2 respectively.

§ 3. Hodge Structures on Homotopic Invariants

3.1. Malcev Completion. Let k be a field of characteristic zero, G be a group. Denote by $\varepsilon : k[G] \to k$ the linear functinal determined by the condition $\varepsilon(s) = 1$ for $s \in G$, and by J the kernel of ε. Let also $\widehat{k[G]}$ be the J-adic completion of $k[G]$, and $\widehat{J}$ the corresponding ideal in $\widehat{k[G]}$. The algebra $\widehat{k[G]}$ inherits from $k[G]$ a comultiplication $\widehat{\Delta} : \widehat{k[G]} \to \widehat{k[G]} \otimes \widehat{k[G]}$. Denote by $\mathfrak{g}$ the Lie algebra of primitive elements of $\widehat{k[G]}$:

$$\mathfrak{g} = \{x \in \widehat{J} \mid \widehat{\Delta}(x) = 1 \otimes x + x \otimes 1\},$$

and by $\widehat{G}$ the group of multiplicative elements:

$$\widehat{G} = \{s \in 1 + \widehat{J} \mid \widehat{\Delta}(s) = s \otimes s\}.$$

There exist natural mappings

$$G \xrightarrow{\theta} \widehat{G} \xrightarrow{\log} \mathfrak{g} \xrightarrow{\exp} \widehat{G}.$$

The mapping θ is the universal homomorphism of G into a prounipotent proalgebraic group over k. The mappins log and exp are bijections. The homomorphism θ, called the Malcev completion of G, satisfies the following properties:

The lower central series of the group $\widehat{G}$ coincides with $\widehat{G}^r = \widehat{G} \cap (1 + \widehat{J}^r)$. Let $G = \Gamma^{(1)} \supset \Gamma^{(2)} \supset \dots$ be the lower central series of the group G. Then θ induces the isomorphism $\widehat{G}/\widehat{G}^{r+1}$ with $(\widehat{G/\Gamma^{(r+1)}})$. If G is finitely generated and $k = \mathbb{R}$, we have $\widehat{G}^r/\widehat{G}^{r+1} = \mathfrak{g}/\mathfrak{g}^{r+1}$, where $\mathfrak{g}^r = \mathfrak{g} \cap \widehat{J}^{r+1}$.

3.2. The Homotopic Lie Algebra. Let (X, x) be a topological space with a base point. Define:

$$\mathfrak{g}_0(X, x) = \text{the Lie algebra of the Malcev competion of } \pi_1(X, x);$$

$$\mathfrak{g}_k(X, x) = \begin{cases} \pi_{k+1}(X, x) \otimes \mathbb{Q} & \text{if } (X, x) \text{ is a nilpotent space,} \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathfrak{g}_.(X, x) = \bigoplus_{k>0} \mathfrak{g}_k(X, x).$$

The Whitehead bracket defines on $\mathfrak{g}_.(X, x)$ the structure of a $\mathbb{Z}$-graded Lie superalgebra (with the $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}$-grading). It will be called the *homotopic Lie algebra.*

3.3. Theorem. *Let X be an algebraic variety over $\mathbb{C}$. Then the homotopic Lie algebra $\mathfrak{g}_.(X, x)$ carries a mixed Hodge $\mathbb{Q}$-structure, such that the Whitehead bracket and the Hurewicz homomorphism*

$$\mathfrak{g}_.(X, x) \to H_.(X, x)$$

are morphisms of mixed Hodge structures. All these structures are functorial in (X, x).

3.4. Remarks. a. The space $\mathfrak{g}_{\cdot}(X,x)$ is, in general, infinite-dimensional. Generalizing the definition of the mixed Hodge structure to this case we require the filtration W to be bounded from above ($W_N = 0$ for $N \gg 0$); however, it is allowed to be infinite. The filtration F also is allowed to be infinite, but the induced filtration on each $\mathbf{Gr}_n^W$ must be a finite filtration satisfying the condition of Definition 1.1.a.

b. Not only the algebra $\mathfrak{g}_0(X,x)$, but the completed group ring $\mathbb{Q}\widehat{[\pi_1(X,x)]}$ admits a mixed Hodge structure such that the multiplication and the comultiplication are morphisms of mixed Hodge structures. Next, $\mathbb{Z}[\pi_1(X,x)]/J^r$ admits a mixed Hodge $\mathbb{Z}$-structure.

c. There exists an analog of Theorem 3.3 for relative homotopic groups.

Let $f : X \to Y$ be a morphism of algebraic varieties. Denote by $E_f(y)$ the homotopic fiber of f over a point $y \in Y$.

3.5. Theorem. *If $E_f(y)$ is connected and the action of the group $\pi_1(Y,y)$ on $H^{\cdot}(E_f(y),\mathbb{Q})$ is unipotent, then both the cohomology and the homotopic Lie algebra of $E_f(y)$ carry mixed Hodge structures such that the following statements hold:*

a. *The natural morphisms*

$$H^{\cdot}(E_f(y),\mathbb{Q}) \to H^{\cdot}(X_y,\mathbb{Q}),$$
$$H^{\cdot}(E_f(y),\mathbb{Q}) \otimes \mathfrak{g}_0(Y,y) \to H^{\cdot}(E_f(y),\mathbb{Q})$$

are morphisms of mixed Hodge structures.

b. *If both X and Y are simply connected, then the homotopic exact sequence of f is an exact sequence of mixed Hodge structures.*

3.6. The Plan of the Proof. Let P_xX be the space of loops of (X,x), and $\widetilde{x} \in P_xX$ be the constant loop. It is well known that

$$\pi_{k+1}(X,x) \cong \pi_k(P_xX,\widetilde{x}),$$
$$H^0(P_xX,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}([\pi_k(X,x)],\mathbb{Z}).$$

Since for a simply connected X the space P_xX is an H-space, we have for such X

$$\mathrm{Hom}(\pi_{k+1}(X,x),\mathbb{Q}) \simeq QH^k(P_xX,\mathbb{Q}),$$

where QH^k is the space of indecomposable elements for $H^k(P_xX,\mathbb{Q})$ (i.e. the quotient I/I^2, where I is the ideal of the augmentation "the value at the point x").

To compute the cohomology algebra $H^{\cdot}(P_xX,\mathbb{Q})$ one can use either the iterated integrals introduced by Chen, or using the algebraic bar construction applied to the de Rham complex. Below (in 3.7 and 3.8) we describe both these methods.

3.7. The Bar Construction. Let $A^{\cdot}$ be a (super)commutative differential graded algebra over $\mathbb{C}$. This means that A is endowed with a grading $A^{\cdot} =$

$\underset{i>0}{\oplus} A^i$ such that $A^iA^j \subset A^{i+j}$, $ab = (-1)^{\deg a \deg b}ba$ for homogeneous a and b, and with a differential $d : A^\cdot \to A^\cdot$ such that $dA^i \subset A^{i+1}$, $d(ab) = da \cdot b + (-1)^{\deg a} a \cdot db$. We consider also the augmentation of $A^\cdot$, i.e. a homomorphism of algebras $\varepsilon : A^\cdot \to \mathbb{C}$ such that $\varepsilon(A^i) = 0$ for $i > 0$. The ideal $IA^\cdot$ of the augmentation is defined as the kernel of ε. In our case $A^\cdot$ is the algebra of differential forms on a variety or a subalgebra of this algebra.

A *bar construction* on $A^\cdot$ is, by definition, the complex $B^\cdot(A^\cdot)$ associated to the following bicomplex $B^{\cdot\cdot}(A^\cdot)$:

$$B^{-s,t} = [\otimes^s IA^\cdot]^t$$

(elements of degree t in the s-th tensor power). A standard notation for an element of $B^{-s,t}$ is $[a_1 \mid \cdots \mid a_s]$, $a_i \in IA^\cdot$. Differentials d' and d'' are defined by the following formulas:

$$d'[a_1 \mid \cdots \mid a_s] = \sum_{i=1}^{s} (-1)^i [Ja_1 \mid \cdots \mid Ja_{i-1} \mid da_i \mid a_{i+1} \mid \cdots \mid a_s],$$

$$d''[a_1 \mid \cdots \mid a_s] = \sum_{i=1}^{s} (-1)^{i+1} [Ja_1 \mid \cdots \mid Ja_{i-1} \mid Ja_i \cdot a_{i+1} \mid a_{i+2} \mid \cdots \mid a_s],$$

where $J : IA^\cdot \to IA^\cdot$ is a linear mapping defined on homogeneous elements a by the formula

$$Ja = (-1)^{\deg a} a.$$

Define the diagonal mapping Δ and the product $\wedge$ in $B(A^\cdot)$ as follows:

$$\Delta : B^\cdot(A^\cdot) \to B^\cdot(A^\cdot) \otimes B^\cdot(A^\cdot),$$

$$\Delta[a_1 \mid \cdots \mid a_s] = \sum_{i=0}^{s} [a_1 \mid \cdots \mid a_i] \otimes [a_{i+1} \mid \cdots \mid a_s],$$

$$\wedge : B^\cdot(A^\cdot) \otimes B^\cdot(A^\cdot) \to B^\cdot(A^\cdot),$$

$$[a_1 \mid \cdots \mid a_r] \wedge [a_{r+1} \mid \cdots \mid a_{r+s}] = \sum_{\sigma \in \Pi_{r,s}} \varepsilon(\sigma, a_1, \ldots, a_{r+s}) [a_{\sigma(1)} \mid \cdots \mid a_{\sigma(r+s)}]$$

In the last formula the summation is performed over the set $\Pi_{r,s}$ of all (r,s)-shuffles, i.e. over all elements of the symmetric group S_{r+s} such that $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(r+s)$; the sign $\varepsilon(\sigma, a_1, \ldots, a_{r+s})$ is determined by the number of inversions in σ between those indices i that $\deg a$ is *even*.

Examples:

$$a_1 \otimes a_2 = [a_1 \mid a_2] + (-1)^{(\deg a_1 + 1)(\deg a_2 + 1)} [a_2 \mid a_1],$$

$$\begin{aligned} a_1 \otimes [a_2 \mid a_3] = {} & [a_1 \mid a_2 \mid a_3] + (-1)^{(\deg a_1 + 1)(\deg a_2 + 1)} [a_2 \mid a_1 \mid a_3] \\ & + (-1)^{(\deg a_1 + 1)(\deg a_2 + \deg a_3)} [a_2 \mid a_3 \mid a_1]. \end{aligned}$$

3.7.1. Theorem. *The bar construction $B^\cdot(A^\cdot)$, together with the above diagonal mapping Δ and the product $\wedge$, is a differential graded Hopf algebra.*

Let us remark that the differential $d_{B^{\cdot}(A^{\cdot})}$ is related to the operations in $B^{\cdot}(A^{\cdot})$ as follows: both Δ and $\wedge$ are morphisms of complexes $B^{\cdot}(A^{\cdot})$ and $B^{\cdot}(A^{\cdot}) \otimes B^{\cdot}(A^{\cdot})$ (for $\wedge$ this is expressed by the (super)Leibniz rule).

3.8. Iterated Integrals. Let M be a smooth manifold, PM be the space of continuous piecewise smooth paths $\gamma : [0,1] \to M$ in M. This space is not, of course, a manifold. However, one can generalize to it a lot of constructions from differential geometry (PM is a differential space in the sense of Chen (1977)). In particular, a mapping $f : X \to PM$ (where X is a smooth manifold) is said to be smooth if it is continuous and if the corresponding mapping

$$\varphi_f : [0,1] \times X \to M, \quad \varphi_f(t,x) = f(x)(t)$$

satisfies the following condition: for some decomposition $0 = t_0 < t_1 \cdots < t_i = 1$ the restriction of each φ_f to $[t_j, t_{j+1}] \times X$ is smooth.

A differential form ω of degree n on PM is a collection of n-forms $f^*\omega$ on X, one for each smooth $f : X \to PM$, such that the following compatibility condition holds: if $g : Y \to X$ is a smooth mapping of manifolds, then

$$g^*(f^*\omega) = (fg)^*\omega,$$

where g^* is the usual inverse image of forms.

The exterior product and the differential of forms on PM is defined in a natural way using the corresponding operations on X. Let $\Omega^{\cdot}(PM)$ be the space of all forms on PM.

One can easily see that 0-forms on PM are smooth functions.

Let now $\omega_1, \ldots, \omega_r$ be differential forms on PM. Their *iterated integral* $\int \omega_1 \ldots \omega_r$ is the differential form of degree $\sum(\deg \omega_i - 1)$ on PM, which is defined for each smooth $f : X \to PM$ as follows. Express the form $\varphi_f \omega_j$ on $X \times [0,1]$ in the form:

$$\varphi_f \omega_j = \omega_j'(t,x) + dt \wedge \omega_j''(t,x),$$

where ω_j' and ω_j'' do not contain dt. Then

$$f^*\Big(\int \omega_1 \ldots \omega_r\Big) = \iint\limits_{0<t_1<\cdots<t_r<1} \omega_1'(t,x) \wedge \cdots \wedge \omega_r'(t,x) dt_1 \ldots dt_r.$$

Denote by $\int \Omega^{\cdot}$ the subspace of $\Omega^{\cdot}(PM)$ generated by all iterated integrals. One can easily see that $\int \Omega^{\cdot}$ is invariant under the exterior product and under the differential in $\Omega^{\cdot}(PM)$, and that it is a differential graded algebra.

Take a point $x \in M$, restrict each iterated integral to the subspace $P_x M \subset PM$, and denote the obtained algebra by $\int_x \Omega^{\cdot}$. The composition of loops enables us to define the comultiplication in $\int_x \Omega^{\cdot}$, and one can easily see that $\int \Omega^{\cdot}$ becomes a differential graded Hopf algebra.

3.8.1. Theorem. *Let $x \in M$; define the augmentation $\varepsilon_x : \Omega^{\cdot} M \to \mathbb{C}$ by the formula $\varepsilon_x(f) = f(x)$ for $f \in \Omega^{\cdot} M$. Let $B_x^{\cdot} = B_x^{\cdot}(\Omega^{\cdot} M)$ be the bar construction corresponding to this augmentation. Then the linear mapping*

$$B_x^{\cdot} \to \int_x \Omega^{\cdot}, \quad \omega_1 \otimes \ldots \otimes \omega_r \mapsto \int \omega_1 \ldots \omega_r$$

is a homomorphism of differential graded Hopf algebras which induces an isomorphism on cohomology.

The relation between $\int_x \Omega^{\cdot}$ and $H^{\cdot}(P_x M, \mathbb{Q})$ is given by the following Chen-Adams theorem.

3.8.2. Theorem. *Let M be simply connected. Then the integration mapping $H^{\cdot}\left(\int_x \Omega^{\cdot}\right) \to H^{\cdot}(P_x M, \mathbb{Q})$ is an isomorphism of Hopf algebras.*

3.9. Example. Let X be a smooth algebraic variety, $\overline{X}$ be its smooth completion by a divisor with normal crossings. Assume that $H^0(\overline{X}, \Omega^1_{\overline{X}}) = 0$. Then the mixed Hodge structure on $\Pi^r_{\mathbb{Q}} = \mathbb{Q}[\pi_1(X,x)]/J^r$ can be described as follows.

a. *The structure of an algebra over* $\mathbb{C}$. Choose some bases $w_1, \ldots, w_m$ in the space $W = H^0(\Omega^1_{\overline{X}}(\log D))$ and $z_1, \ldots z_n$ in the space $H^0(\Omega^2_{\overline{X}}(\log D))$. Let

$$w_i \wedge w_j = \sum_{k=1}^{m} a^k_{ij} w_k.$$

Denote by $\{w^j\}$ the dual basis in the dual space W^*. Consider the elements

$$R^k = \sum a^k_{ij}(w^i \otimes w^j - w^j \otimes w^i) \in T(W^*),$$

where $T(W^*)$ is the tensor algebra of the space W^* over $\mathbb{C}$, and define

$$\widetilde{\Pi}^r_{\mathbb{C}} = T(W^*)/[(R^k) + J^r],$$

where J is the kernel of the augmentation in $T(W^*)$. Then there exists a family of natural isomorphisms of rings $\widetilde{\Pi}^r_{\mathbb{C}} \simeq \Pi^r_{\mathbb{Q}} \otimes \mathbb{C}$ (they are defined using iterated integrals).

b. *The filtration over* $\mathbb{C}$. We have canonically $H^1(X, \mathbb{C}) \cong W$, so that W is a $\mathbb{C}$-component of a pure Hodge structure of weight 2 and of type $(1,1)$, and W^* is a component of the type $(-1,-1)$. Since the relations R^k are homogeneous, $\widetilde{\Pi}^r_{\mathbb{C}}$ possesses a double grading, and using this double grading the filtrations over $\mathbb{C}$ can be defined as follows:

$$F^p(\widetilde{\Pi}^r_{\mathbb{C}}) = \bigoplus_{u>p} (\widetilde{\Pi}^r_{\mathbb{C}})^{u,\cdot}, \quad W_l(\widetilde{\Pi}^r_{\mathbb{C}}) = \bigoplus_{p+q=l} (\widetilde{\Pi}^r_{\mathbb{C}})^{p,q}.$$

c. *The* $\mathbb{Q}$*-structure.* It is defined by the image of $\mathbb{Q}[\pi_1(X,x)]$ in $\widetilde{\Pi}^r_{\mathbb{C}}$. One can easily see that the W-filtration is defined over $\mathbb{Q}$.

§ 4. Hodge-Deligne Complexes

4.1. How to Introduce a Hodge Structure in Cohomology of Singular Varieties. Following Deligne (1971), the corresponding construction consists of the following steps.

a. A general (possibly, singular and/or non-compact) variety S is replaced by its "simplicial resolution" $U_{.}$. Such a resolution $U_{.}$ is a simplicial scheme formed by smooth varieties, together with an augmentation $a : U_{.} \to S$, which can be considered as an analog of a simplicial covering for the category of sheaves of abelian groups on S. Moreover, we assume that $U_{.}$ can be embedded into a smooth proper simplicial scheme $X_{.}$ such that $X_{.} \setminus U_{.}$ is a divisor with normal crossings.

b. Cohomology $H^{\cdot}(S, \mathbb{C})$ is computed using a generalized Čech resolution constructed from $H^{\cdot}(U_{.}, \mathbb{C})$, and on elements of this resolution the Hodge structure is constructed as in Sect. 2.

c. Intermediate objects in this construction are complexes of modules with filtrations appear whose properties generalize those of Hodge structures.

In this section we define such complexes following Deligne (1971); we call them Hodge-Deligne complexes. In Sect. 5 we use these complexes to describe mixed Hodge structures on cohomology of simplicial schemes. In Sect. 6 we give a modified definition of Hodge complexes following Beilinson and describe their relation to the derived category of Hodge structures.

We must warn the reader that from the point of view of algebraic geometry these notions play somewhat technical and auxiliary role. We have chosen to emphasize these notions here since they are rather typical both from the categorical and homological points of view.

4.2. The Filtered Derived Category. Let $\mathcal{A}$ be an abelian category. Denote by $K^+F\mathcal{A}$ the category of bounded from below filtered complexes over $\mathcal{A}$ modulo homotopies compatible with filtrations. A filtered quasi-isomorphism $f : (K, F) \to (K', F')$ is a morphism of complexes which is compatible with filtrations such that $\mathbf{Gr}_F(f)$ is a quasi-isomorphism. Denote by $D^+F\mathcal{A}$ the category obtained from $K^+F\mathcal{A}$ by inverting all quasi-isomorphisms.

The category $K^+F\mathcal{A}$ is convenient when we want to introduce a pure Hodge structure on cohomology. Mixed Hodge structures can be similarly obtained from the category $K^+F_2\mathcal{A}$ of complexes (K, W, F) with double filtrations and the corresponding bifiltered derived category $D^+F_2\mathcal{A}$ (f is a bifiltered quasi-isomorphism if it induces a quasi-isomorphism on $\mathbf{Gr}_F\mathbf{Gr}_W f$).

4.3. Pure Hodge-Deligne Complexes. In the setup of 1.1 a pure Hodge-Deligne A-complex of weight n is a complex $K_A \in \operatorname{Ob} D^+(A\text{-mod})$ with Noetherian cohomology $H^{\cdot}(K_A)$, endowed with the following structures:

a. A filtration F on $K_A \otimes \mathbb{C}$, which is understood as a triple $(K_{\mathbb{C}}, F, \alpha)$ where $(K_{\mathbb{C}}, F) \in \operatorname{Ob} D^+(\mathbb{C}\text{-mod})$ and $\alpha : K_{\mathbb{C}} \to K_A \otimes \mathbb{C}$ is a quasi-isomorphism. The differential in $K_{\mathbb{C}}$ is strongly compatible with F.

b. For any k the induced filtration on $H^k(K_A) \otimes \mathbb{C} = H^k(K_\mathbb{C})$ is a pure Hodge structure of the weight $n + k$.

A generalization of this notion is the notion of a pure Deligne-Hodge A-complex of the weight n over a topological space X. In the above definition we must replace A-modules and $\mathbb{C}$-modules by sheaves of A-modules and sheaves of $\mathbb{C}$-modules on X and to impose the following condition:

a′. The data $(R\Gamma(K_\mathbb{C}), R\Gamma(F), R\Gamma(\alpha))$ forms a pure Hodge A-complex of the weight n.

As a basic example we reformulate in this Hodge theorem. Let X be a smooth projective variety, $K_\mathbb{Z}$ be the sheaf $\mathbb{Z}$ (viewed as a complex concentrated at degree 0), $K_\mathbb{C}$ be the holomorphic de Rham complex F be the stupid filtration; the natural morphism $a : K_\mathbb{Z} \otimes \mathbb{C} \to K_\mathbb{C}$ is a quasi-isomorphism (by the de Rham lemma). Then $(K_\mathbb{Z}, (K_\mathbb{C}, \alpha))$ is a pure $\mathbb{Z}$-complex on X of the weight 0.

4.4. Mixed Hodge-Deligne Complexes. A mixed Hodge-Deligne A-complex consists of complexes

$$\begin{aligned} &K_A \in \operatorname{Ob} D^+(A\text{-mod}) \quad \text{with Noetherian cohomology,}\\ &(K_{A\otimes\mathbb{Q}}, W_.) \in \operatorname{Ob} D^+F(A \otimes \mathbb{Q}\text{-mod}),\\ &(K_\mathbb{C}, W_., F^\cdot) \in \operatorname{Ob} D^+F_2(\mathbb{C}\text{-mod}), \end{aligned}$$

and isomorphisms

$$\begin{aligned} \alpha_\mathbb{Q} : K_{A\otimes\mathbb{Q}} &\to K_A \otimes \mathbb{Q} && \text{in } D^+(A \otimes \mathbb{Q}\text{-mod}),\\ \alpha : (K_\mathbb{C}, W) &\to (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C} && \text{in } D^+F(\mathbb{C}\text{-mod}). \end{aligned}$$

These data must satisfy the following condition.

For any n the triple

$$(\mathbf{Gr}_n^W(K_{A\otimes\mathbb{Q}}), (\mathbf{Gr}_n^W(K_\mathbb{C}, F), (\mathbf{Gr}_n^W(\alpha))$$

is a pure Hodge $A \otimes \mathbb{Q}$-complex of weight n.

In this language the construction of the mixed Hodge structure from Sect. 2 can be expressed as follows.

Let X be a smooth proper variety, $Y \subset X$ be a divisor with normal crossings, $U = X \setminus Y$, $j : U \to X$ be the embedding. Denote

$$\begin{gathered} K_\mathbb{Z} = Rj_*\mathbb{Z},\\ K_\mathbb{Q} = Rj_*\mathbb{Q}, \quad W_n(K_\mathbb{Q}) = \tau_{<n}(Rj_*\mathbb{Q}),\\ K_\mathbb{C} = \Omega^\cdot_X(\log Y), \quad W_n = \tau_{<n}(\cdot), \quad F^p = \sigma_{>p}(\cdot). \end{gathered}$$

The construction of the isomorphism α was briefly discussed in Sect. 2. These data form a mixed Hodge-Deligne $\mathbb{Z}$-complex on X. Applying to this complex the functor $R\Gamma$, we obtain the mixed Hodge-Deligne $\mathbb{Z}$-complex, whose cohomology is the mixed Hodge structure $H^\cdot(U, \mathbb{Z})$.

§ 5. Hodge-Deligne Complexes for Singular and Simplicial Varieties

5.1. Simplicial Resolution of a Variety. Let S be an arbitrary complex algebraic variety. Its *simplicial resolution* is the following collection of data:

a. A simplicial variety $X_.$ over $\mathbb{C}$ with all X_n smooth and complete.

b. A simplicial subvariety $D_. \subset X_.$ such that $D_n \subset X_n$ is a divisor with normal crossings.

c. The augmentation morphism $U_. = X_. \setminus D_. \xrightarrow{d} S$, which makes $U_.$ a proper hypercovering of S in the category of topological spaces (with the ordinary $\mathbb{C}$-topology).

The formal definition of a hypercovering will be given in the next subsection. Properties of a hypercovering generalize those of an ordinary covering $S = \cup S_i$ if we formulate these last properties in terms of the simplicial space $U_.$ such that $U_0 = \bigsqcup S_i$, $U = U_0 \underset{S}{\times} \dots \underset{S}{\times} U_0$ ($n+1$ times). In particular, the value of the functor $R\Gamma$ at a sheaf on S can be computed from the values of $R\Gamma$ at its preimage on U_0.

The main algebraic geometry theorem used here construction (its proof is based on the Hironaka resolution of singularities) claims that

- a simplicial resolution exists for any S;
- any two simplicial resolutions can be covered by a third one;
- a morphism of varieties can be extended to a morphism of appropriate simplicial resolutions.

These properties ensures existence, uniqueness and functioriality of Hodge structures that will be constructed below.

5.2. Hypercoverings. a. Let $f : X \to Y$ be a morphism of topological spaces, $\mathcal{F}$ be a sheaf on X, $\mathcal{G}$ be a sheaf on Y. By an *f-morphisms* $\mathcal{G} \to \mathcal{F}$ we mean elements of the set

$$\mathrm{Hom}_f(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}(f^{\cdot}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, f_.\mathcal{F}).$$

For $X \xrightarrow{f} Y \xrightarrow{g} Z$ and three sheaves $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ one can define the composition of a g-morphism and a f-morphism; this composition is a $(g \circ f)$-morphism $\mathcal{H} \to \mathcal{F}$.

b. Let $U_.$ be a simplicial topological space. By a *sheaf* $\mathcal{F}$ on $U_.$ we mean a family of sheaves $\mathcal{F}^n$ on U_n connected by $U_.(f)$-morphisms $\mathcal{F}^{\cdot}(f)$ for any non-descreasing $f : [m] \to [n]$ such that the condition $\mathcal{F}^{\cdot}(f \circ g) = \mathcal{F}^{\cdot}(f) \circ \mathcal{F}^{\cdot}(g)$ holds.

The definition of a morphism of sheaves $\mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$ is clear.

If $S_.$ is a constant simplicial space, so that $S_n = S$ for all n, $S(f) = \mathrm{id}$ for all f, then a sheaf $\mathcal{F}^{\cdot}$ on S is just a cosimplicial sheaf on S.

c. Let $a : U_. \to S$ be an augmented simplicial space represented by morphisms $a_n : U_n \to S$ that equilize all $U_.(f)$. For any sheaf $\mathcal{F}$ on S the family

$a^\cdot(\mathcal{F}) = \{a^\cdot_n(\mathcal{F})\}$ is a sheaf on $U_.$. The functor $\mathcal{F} \to a^\cdot\mathcal{F}$ admits a right adjoint functor

$$a_. : \mathcal{G} \mapsto \operatorname{Ker}\left(d_1^1, d_1^0 : \mathcal{G}^0 \to \mathcal{G}^1\right).$$

so that there exists a morphism of functors $\mathrm{Id} \to a_.a^\cdot$.

An augmentation a is said to be a *morphism of cohomological descent* if for any sheaf f $\mathcal{F}$ abelian groups on S we have $\mathcal{F} \cong a_.a^\cdot\mathcal{F}$, $R^ia_.a^\cdot\mathcal{F} = 0$ for $i > 0$.

A continuous mapping $a : U_. \to S$ of topological spaces is said to be a *morphism of cohomological descent* if the augmentation $a : U_. \to S$, where $U_n = U \underset{S}{\times} \ldots \underset{S}{\times} U$ $(n+1$ times) is such a morphism.

A continuous mapping $a : U \to S$ is said to be a *morphism of universal cohomological descent* if for any $S' \to S$ the mapping $U \underset{S}{\times} S' \to S'$ is a morphism of cohomological descent. Any proper mapping has this property, and in our applications it suffices to consider only proper morphisms.

d. Let again $a : U_. \to S$ be an augmented sinplicial space. It is said to be a *hypercovering* of S if all canonical mappings

$$(\varphi_n)_{n+1} : U_{n+1} \to (\operatorname{cosk}\, \operatorname{sk}_n U_.)_{n+1}$$

are morphisms of universal cohomological descent (in the main constructions all these mappings will be proper).

Any hypercovering is itself a morphism of universal cohomological descent. In particular, for any complex $K \in \operatorname{Ob} D^+(S)$ the canonical mapping

$$R\Gamma(S, K) \to R\Gamma(S, Ra_.a^\cdot K) \sim R\Gamma(U_., a^\cdot K)$$

is an isomorphism (the functor $\Gamma(U_., \mathcal{F}^\cdot)$ is $\operatorname{Ker}(\Gamma(U_0, \mathcal{F}^0) \rightrightarrows \Gamma(U_1, \mathcal{F}^1)))$.

It is this isomorphism which is used to construct the Hodge-Deligne complex which computes the cohomology of S, since, by Sect. 2, we know the corresponding complexes on elements of the resolution $\{U_n\}$.

5.3. The Construction of the Hodge-Deligne Complex. In the notations of 4.4 we define the mixed Hodge-Deligne complex which computes cohomology of S from the simplicial resolution $X_. \xleftarrow{j} U_. \xrightarrow{a} S$ as consisting of the following data:

$$K_\mathbb{Z} = R\Gamma(U_., \mathbb{Z}), \quad K_\mathbb{Q} = R\Gamma(U_., \mathbb{Q}), \quad K_\mathbb{C} = R\Gamma(U_., \mathbb{C}).$$

As before, the weight filtration and the Hodge filtration on $K_\mathbb{C}$ are introduced via the identification

$$R\Gamma(U_., \mathbb{C}) \cong R\Gamma Rj_*\mathbb{C} \cong sR\Gamma^\cdot \Omega^\cdot_X(\log D).$$

Here the functor $\Gamma^\cdot$ associates to a sheaf $\mathcal{F}$ on $U_.$ the cosimplicial group $\{\Gamma(U_n, \mathcal{F}^n)\}$, and to a complex of sheaves the corresponding complex of cosimplisial groups. Hence the functor $R\Gamma^\cdot$ maps a complex of sheaves to a complex of cosimplicial groups, the alternating sum of faces makes it a bicomplex, and $sR\Gamma^\cdot$ is the corresponding diagonal complex.

Finally,

$$\begin{aligned} W_n[sR\Gamma^{\cdot}(\Omega^{\cdot}_X(\log D))]_m &= W_n\left[\bigoplus_{p+q=m}(R\Gamma(\Omega^{\cdot}_{X_p}(\log D_p))^q\right] \\ &= \bigoplus_{p+q=m} W_{n+p}[(R\Gamma^{\cdot}\Omega^{\cdot}_{X_p}(\log D_p))^q], \\ F^n[sR\Gamma^{\cdot}(\Omega^{\cdot}_X(\log D))]_m &= \bigoplus_{p+q=m} F^n[R\Gamma^{\cdot}(\Omega^{\cdot}_{X_p}(\log D_p))^q]. \end{aligned}$$

The fact that W comes from some filtration on $K_{\mathbb{Q}}$ is verified using the same arguments as in the smooth case.

§ 6. Hodge-Beilinson Complexes and Derived Categories of Hodge Structures

6.1. The Derived Category. In the setup of Sect. 1 denote by $\mathcal{H}$ the abelian category of mixed Hodge A-structures. Denote also by $\Gamma_{\mathcal{H}} : \mathcal{H} \to A$-mod the functor $\Gamma_{\mathcal{H}}(M) = \mathrm{Hom}_{\mathcal{H}}(A(0), M)$. One can easily see that this functor is left exact and $\mathrm{Hom}_{\mathcal{H}}(M, N) = \Gamma_{\mathcal{H}}\,\mathbf{Hom}(M, N)$. We can define the corresponding derived functors

$$R\,\mathbf{Hom} : D^-(\mathcal{H})^0 \times D^+(\mathcal{H}) \to D^+(\mathcal{H}),$$
$$\overset{L}{\otimes} : D^-(\mathcal{H}) \times D^-(\mathcal{H}) \to D^-(\mathcal{H}).$$

6.2. Proposition. *Let $M^{\cdot}$, $N^{\cdot}$ be two complexes of mixed Hodge structures. The canonical mappings*

$$(R\,\mathbf{Hom}(M^{\cdot}, N^{\cdot}))_A \to R\,\mathrm{Hom}_{A\text{-mod}}(M^{\cdot}_A, N^{\cdot}_A),$$
$$(M^{\cdot} \overset{L}{\otimes} N^{\cdot})_A \to M^{\cdot}_A \overset{L}{\underset{A}{\otimes}} N^{\cdot}_A$$

are quasi-isomorphisms. In particular, for $M, N \in \mathrm{Ob}\,\mathcal{H}$, $i > 0$,

$$(R\,\mathbf{Hom}(M, N))_A = \mathrm{Ext}^i_{A\text{-mod}}(M_A, N_A)$$

is a periodic A-module.

6.3. How to Compute $R\,\mathbf{Hom}$. Consider a diagram of complexes of A-modules of the form

$$\begin{tikzcd}[column sep=small] & & B_1 & & & \cdots & & B_n & \\ D: & A_1 \arrow[ur, "f_1"] & & A_2 \arrow[ul, "g_1"'] \arrow[urr, "f_2"] & & \cdots & A_n \arrow[ur, "f_n"] & & A_{n+1} \arrow[ul, "g_n"'] \end{tikzcd}$$

Set

$$f = \sum f_i - \sum g_i : \oplus A_i = \widetilde{\Gamma}^0(D) \to \oplus B_i = \widetilde{\Gamma}^1(D),$$

and further

$$\Gamma(D) = \operatorname{Ker} f, \quad \Gamma^1(D) = \operatorname{Coker} f, \quad \widetilde{\Gamma}(D) = \operatorname{Cone}(f).$$

Let $M^{\cdot}$ be a complex of Hodge structures. It generates a diagram of the above form:

$$D_{\mathcal{H}}(M^{\cdot}):\quad \begin{tikzcd}[column sep=small] & M^{\cdot}_{\mathbb{Q}} & & W_{\mathbb{C},0}(M^{\cdot}) & \\ M^{\cdot}_A \arrow[ur] & & W_{\mathbb{Q},0}(M^{\cdot}) \arrow[ul] \arrow[ur] & & (F^0 \cap W_{\mathbb{C},0})(M^{\cdot}) \arrow[ul] \end{tikzcd}$$

Denote $\widetilde{\Gamma}_{\mathcal{H}}(M^{\cdot}) = \widetilde{\Gamma}(D_{\mathcal{H}}(M^{\cdot}))$, $\Gamma_{\mathcal{H}}(M^{\cdot}) = \Gamma(D_{\mathcal{H}}(M^{\cdot}))$, etc. This new definition of $\Gamma_{\mathcal{H}}(M^{\cdot})$ coincides with the previous one when $M^{\cdot}$ is a 0-complex. The functor $\widetilde{\Gamma}_{\mathcal{H}}(M^{\cdot})$ maps quasi-isomorphisms into quasi-isomorphisms, hence induces a functor between derived categories.

6.3.1. Proposition. a. *The canonical mapping*

$$R\operatorname{Hom}^{\cdot}(M^{\cdot}, N^{\cdot}) \to \widetilde{\Gamma}_{\mathcal{H}} R\,\mathbf{Hom}(M^{\cdot}, N^{\cdot}))$$

is an isomorphism.

b. *For any Hodge structures M, N and any $i > 1$ we have*

$$\operatorname{Ext}^i_{\mathcal{H}}(M_A, N_A) = \operatorname{Ext}_{A\text{-mod}}(M_A, N_A).$$

In particular, if A is a regular ring of dimension ≤ 1 (for example, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$), then $\operatorname{Ext}^i_{\mathcal{H}} = 0$ for $i > 1$, so that any complex $M^{\cdot}$ is quasi-isomorphic to $\oplus H^i(M^{\cdot})[-i]$ and the corresponding quasi-isomorphism induces the identity morphism on cohomology.

6.4. Hodge-Beilinson Complexes. A *Hodge-Beilinson complex* is the following generalization of the diagram $D_{\mathcal{H}}(M^{\cdot})$:

$$\mathcal{M}:\quad \begin{tikzcd}[column sep=small] & \widetilde{M}^{\cdot}_{\mathbb{Q}} & & (\widetilde{M}^{\cdot}_{\mathbb{C}}, \widetilde{W}_{\mathbb{C}\cdot}) & \\ M^{\cdot}_A \arrow[ur, "\alpha_1"] & & (M^{\cdot}_{\mathbb{Q}}, W_{\mathbb{Q}\cdot}) \arrow[ul, "\alpha_2"'] \arrow[ur, "\alpha_3"] & & (M^{\cdot}_{\mathbb{C}}, W_{\mathbb{C}\cdot}, F^{\cdot}) \arrow[ul, "\alpha_4"'] \end{tikzcd}$$

In this diagram $M^{\cdot}_A$, $\widetilde{M}^{\cdot}_{\mathbb{Q}}$, etc., are complexes of A-modules, of $\mathbb{Q}$-modules, etc.; W and F are their filtrations, mappings

$$\alpha_{1,\mathbb{Q}} : M^{\cdot}_A \underset{A}{\otimes} \mathbb{Q} \to \widetilde{M}^{\cdot}_{\mathbb{Q}}, \quad \alpha_2 : M^{\cdot}_{\mathbb{Q}} \to \widetilde{M}^{\cdot}_{\mathbb{Q}},$$

$$\alpha_{3,\mathbb{C}} : (M^{\cdot}_{\mathbb{Q}}, W_{\mathbb{Q}\cdot}) \underset{A}{\otimes} \mathbb{C} \to (\widetilde{M}^{\cdot}_{\mathbb{C}}, \widetilde{W}_{\mathbb{C}\cdot}),$$

$$\alpha_4 : (M_{\mathbb{C}}, W_{\mathbb{C}\cdot}) \to (\widetilde{M}_{\mathbb{C}}, \widetilde{W}_{\mathbb{C}\cdot})$$

are (filtered) quasi-isomorphisms. Also, the following conditions should be satistied:

a. $M^{\cdot}_A$ has Noetherian cohomlogy.

b. For any $n \in \mathbb{Z}$ the differential of the filtered complex $\mathbf{Gr}^W_n(M^{\cdot}_{\mathbb{C}})$, $\mathbf{Gr}^W_n(F^{\cdot})$ is strongly compatible with the filtration.

c. This filtration can be pushed down to $H^{\cdot}(\mathbf{Gr}^W_n M^{\cdot}_{\mathbb{Q}})$ and defines on this cohomology a pure Hodge $A \otimes \mathbb{Q}$-structure of the weight n.

One can verify that under these conditions the spectral sequences of the complexes $M^{\cdot}_{\mathbb{Q}}$ and $M^{\cdot}_{\mathbb{C}}$ with respect to the filtrations degenerate in the term E_1, so that we can define a Hodge A-structure on $H^{\cdot}(M^{\cdot}_A)$. Denote this structure by $H^{\cdot}(M^{\cdot})$.

A morphism $f : M^{\cdot}_1 \to M^{\cdot}_2$ of Hodge-Beilinson complexes is defined as a morphism of corresponding diagrams. One can define a homotopy and the cone in this category. Let $K^*_{\mathcal{H}}$ (with $* = \emptyset, +, -, b$) be the corresponding triangulated category. The cohomology functor $\mathbb{H} : K^*_{\mathcal{H}} \to \mathcal{H}$ is defined, and a morphism in $K^*_{\mathcal{H}}$ is said to be a *quasi-isomorphism* if it induces an isomorphism on each $\mathbb{H}$. Let $D^*_{\mathcal{H}} = K^*_{\mathcal{H}}[\mathrm{Qis}^{-1}]$.

For any complex of Hodge structures $M^{\cdot}$ the diagram $D_{\mathcal{H}}(M^{\cdot})$ is a Hodge-Beilinson complex. Hence a functor $D^*(\mathcal{H}) \to D^*_{\mathcal{H}}$ is defined.

6.4.1. Theorem. *The functor $D^b(\mathcal{H}) \to D^b_{\mathcal{H}}$ is an equivalence of categories.*

This theorem, together with Deligne constructions, yields the following result.

6.5. Theorem. *There exists a functor $R\Gamma(\cdot, A)$ from the category of algebraic varieties over $\mathbb{C}$ to the derived category $D^b(\mathcal{H}_A)$ such that $R\Gamma(X, A)_A$ is the complex of singular cochains with coefficients in the constant sheaf A_X.*

The direct construction of this functor associates to a manifold X an appropriate Hodge-Beilinson complex as an object of $D^b_{\mathcal{H}}$.

§ 7. Variations of Hodge Structures

7.1. Main Problems. Let $f : X \to S$ be an morphism of algebraic varieties which can be naturally considered as a family of fibers $X_s f^{-1}(s)$ parameterized by points $s \in S$ (various regularity conditions may be imposed on f). We obtain in such a way the family of Hodge structures $H^{\cdot}(X_s)$; any such family is said to be geometric (or possessing a geometrical realization).The main properties of such families were established on the early stage of the development of Hodge theory. We mention the following properties (see the survey paper (Griffiths 1970)).

a. *The Griffiths transversality theorem*: the covariant derivative along any holomorphic vector field with respect to the canonical flat connection on $H^{\cdot}(X_s)$ maps F^p to F^{p-1}.

b. *The monodromy theorem*: let the family $f : X \to S$ be embedded into a family $\overline{f} : \overline{X} \to \overline{S}$, where $\overline{X} \setminus X$ and $\overline{S} \setminus S$ are divisors with normal crossings. Then the monodromy T around a branch of S is a quasi-unipotent operator: $(T^r - 1)^q = 0$ for appropriate r and q.

c. *The semisiplicity theorem*: the global action of $\pi_1(S, s_0)$ on $H^*(X_{s_0}, \mathbb{C})$ is semisimple.

d. *The regularity theorem*: singular points of the canonical connection over a one-dimensional base are regular in the sense of Fuchs.

e. *Schmid's SL_2-theorem.* This theorem gives explicit information about the asymptotic behavior of Hodge structures near singular points.

In the course of the further development of the theory the notion of a "variation of Hodge structures" was made more precise, and some properties of geometrical Hodge structures was included into the definition. One of the goals of the theory was to find such definition of a variation of Hodge structures that would be invariant under all natural geometrical and cohomological constructions, in particular, cohomology with values in a Hodge structure must itself carry a natural Hodge structure.

Not pretending to be complete, we list here some recent results about variations of Hodge structures.

7.2. Definition. *A variation of (pure) Hodge A-structures* of the weight n over a complex variety n is the following collection of data:

a. A locally constant sheaf of Noetherian A-modules $\mathcal{M}_A$ on S.

b. A finite descending filtration $\{F^p\}$ of the locally constant sheaf $\mathcal{M} = \mathcal{O}_S \underset{A}{\otimes} \mathcal{M}_A$ such that each F^p is a locally constant locally direct summand of $\mathcal{M}$, and in each fiber $\mathcal{M}(s)$, $s \in S$, this filtration induces a pure Hodge structure of the weight n.

c. F^p must satisfy the following transversality condition: let ∇ be a holomorphic connection on $\mathcal{M}$ such that $\mathcal{M}_A$ is horizontal with respect to ∇. Then $\nabla F^p\mathcal{M} \subset \Omega^1_S \underset{\mathcal{O}_S}{\otimes} F^{p-1}\mathcal{M}$ for each p.

7.3. Definition. *A polarization* of a Hodge A-structure of the weight n on S as in 7.2. is a non-degenerate flat bilinear pairing $\beta : \mathcal{M}_A \times \mathcal{M}_A \to A$ that is $(-1)^n$-symmetric and satisfies the following property: the Hermitian form $\beta_s(C_s m, \overline{n})$ is positive definite (here C_s is the multiplication by i^{p-q} on $\mathcal{M}^{p,q}(s)$).

The variation of Hodge structures is said to be *polarizable* if it admits a polarization. In the geometric situation a polarization comes from the Kähler metric and the primitive decomposition.

7.4. Definition. *A variation of mixed Hodge A-structures* over a complex variety S is the following collection of data.

a. A locally constant sheaf $\mathcal{M}_A$ of Noetherian A-modules on S.

b. A filtration W of the sheaf $\mathcal{M} = \mathcal{O}_S \underset{A}{\otimes} \mathcal{M}_A$ by locally constant subsheaves.

c. A finite descending filtration F of the locally constant sheaf $\mathcal{M} = \mathcal{O}_S \underset{A}{\otimes} \mathcal{M}_A$ such that each F^p is a locally constant locally direct summand of $\mathcal{M}$, and F satisfies the transversality condition as in 7.2.c.

On each fiber this data must define a mixed Hodge structure. Then, for each m, $\mathbf{Gr}_m^W \mathcal{M}_A$ will be the variation of mixed Hodge structures.

A variation of mixed Hodge structures is said to be *polarizable* if all variations $\mathbf{Gr}_m^W \mathcal{M}_A$ are polarizable.

7.5. Example. Let $f : X \to S$ be a topologically locally trivial family of algebraic varieties. Then the sheaf $\mathcal{M}_{\mathbb{Z}} = R^q f_* \mathbb{Z}$ carries a natural structure of a variation of Hodge structures.

7.6. Example. Let S be an arbitrary smooth algebraic variety. Consider on S a local system Π^r whose fiber at $\mathbb{Z}[\pi_1(S,s)]/J^r$. By Sect. 3, it carries a fiberwise mixed Hodge $\mathbb{Z}$-structure. There exists a unique variation of Hodge structures inducing these fiberwise structures. This variation is called tautological.

This example is quite important, since its generalization enables us to give a description of an important subcategory of variations of Hodge structures in terms of the representation theory.

To give such a description, let us remark first that a locally constant sheaf $\mathcal{M}_A$ on a connected base S is uniquely determined by the representation of the group $\pi_1(S,s)$ in the fiber $\mathcal{M}_{A,s}$ (where s is an arbitrary point of S).

7.7. Definition. A variation of Hodge structures $(\mathcal{M}_A, F, W)$ is said to be *unipotent* if any of the following equivalent condition is satisfied:

a. The representation of $\pi_1(S,s)$ in $\mathcal{M}_{A,s}$ is unipotent.

b. This representation is induced from a representation of $\mathbb{Z}[\pi_1(S,s)]/J^r$ (in this case we say that the unipotence index does not exceed $r-1$).

c. The variations $\mathbf{Gr}_m^W \mathcal{M}_A$ are constant for all m.

7.8. Behavior at Infinity. Let $S = \overline{S} \setminus D$, where $\overline{S}$ is a compact variety, D is a divisor with normal crossings. Let also N be a locally constant sheaf of complex vector spaces on S, $\mathcal{N} = \mathcal{O}_S \underset{\mathbb{C}}{\otimes} N$. Introduce, following Deligne, the canonical extension $\widetilde{\mathcal{N}}$ as the subsheaf of $j_*(\mathcal{N})$ (where $j : S \to \overline{S}$) defined in the following way. Choose a neighborhood of some point of D which is of the form $(\Delta^*)^r \times \Delta^{n-r}$, where Δ is the unit disk, $\Delta^* = \Delta \setminus \{0\}$ is the punctured unit disk. Let $(z_1, \ldots, z_r)$ be the coordinates in $(\Delta^*)^r$, and $N_i = r^{-1} \log T_i^r$, where T_i is the monodromy around zero in the i-factor Δ_i^* in $(\Delta^*)^r$ (T_i is unipotent). Write $z_j = \exp(2\pi i t_j)$, $\operatorname{Im} t_j > 0$. As generators of $\widetilde{\mathcal{N}}$ we choose the sections $\widetilde{n} = \exp(-\sum t_j N_j) n$, where n runs over (multi-valued) local sections of $\mathcal{N}$.

7.9. Definition. A unipotent variation of Hodge structures $(\mathcal{M}_A, F, W)$ is said to be *good* if

a. Terms of the Hodge filtration $F^p\mathcal{M}$ can be extended to subsheaves $\widetilde{F}^p = F^p\widetilde{\mathcal{M}}$ of the canonical extension in such a way that the sheaves $\mathbf{Gr}^W \mathbf{Gr}_F$ are locally free.

b. Any nilpotent logarithm N_i of the local monodromy operator satisfies the condition: $N_i W_m \subset W_{m-2}$ for any m.

7.10. Hodge Representations of π_1. A unipotent representation of $\pi_1(S, s)$ in an A-module M_A endowed with a Hodge structure is said to be a *Hodge representation* if the corresponding mapping

$$A[\pi_1(S, s)]/J^r \to \operatorname{End} M_A$$

is a morphism of Hodge structures.

7.11. Theorem. *If $(\mathcal{M}_A, F, W)$ is a good unipotent variation of Hodge structures, the the representation of $\pi_1(S, s)$ in its fiber is a Hodge representation.*

7.12. Theorem. *The monodromy functor defines an equivalence of categories*:

$$\left\{\begin{array}{l}\text{good unipotent variations}\\ \text{of Hodge structures}\\ \text{with the unipotence index} \le r-1\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Hodge representations}\\ \text{of the group } \pi_1(S, s) \text{ with}\\ \text{the unipotence index} \le r-1\end{array}\right\}$$

7.13. Properties of Extensions of Variations. Conditions 7.9.a and 7.9.b on a variation of Hodge structures can be formulated without the unipotence requirement. We say that a variation is *preadmissible* is it satisfies this weak conditions, and, moreover, is polarizable. A variation is said to be admissible if for any morphism of the unit disk D into S the induced variation of Hodge structures is preadmissible.

Kashiwara (1986) proved that if a variation of Hodge structures is admissible on a complement to a subset of codimension 2, it is admissible everywhere under the additional condition that the base can be compactified by a divisor with normal crossings.

Bibliographic Hints

For excellent expositions of the classical Hodge theory see review papers (Cornalba, Griffiths 1975; Griffiths 1970; Griffiths, Schmid 1973). The theory of mixed Hodge structures was developed by Deligne (1971, 1972), see also his talks (Deligne 1970a, 1974). Applications of mixed Hodge structures to the study of singularities of differentiable mappings and to the behavior of ocsillating integrals see (Varchenko 1981, 1983). Various versions of the Torelli-type theorems were considered in many papers on algebraic geometry; see, for example (Griffiths 1984), as well as other volumes of this Encyclopaedia. The review of results on the characterization of algebraic cohomology classes (Hodge conjectures) can be found in (Shioda 1983); see also (Grothendieck 1979). Algebraic geometry aspects of variations of Hodge structures see in the series of papers (Carlson *et al.* 1983) and in (Schmid 1973). About the problems that are not considered in this volume, we mention the p-adic Hodge structures, see (Tate 1967; Fontaine 1982; Bloch, Kato 1986; Faltings 1988).

Let us describe now the content of the sections in this chapter. The categorical approach to theory of mixed Hodge structures, and, in particular, Theorem 1.4, can be found in (Deligne 1971). About Proposition 1.7 see the Appendix of Deligne to (Cattani, Kaplan 1982b). Theorem 2.1 is proved in (Deligne 1971); about the example from 2.5 see Sect. 10.3 in (Deligne 1972). Theory of mixed Hodge structures on homotopic invariants of algebraic varieties was developed by Morgan (1978) and further generalized by Hain (1986, 1987). Our exposition in Sect. 3 follows (Hain 1987), where onw can find the proofs of Theorems 3.3, 3.5, 3.7.1, 3.8.1, 3.8.2. The theory of interated integrals was developed by Chen (1977), and, independently, by Parshin (1966). The exposition in Sect. 4,5 follows (Deligne 1972), that in Sect. 6 follows (Beilinson 1986). About the contemporary approach to the variations of mixed Hodge structures see (Cattani, Kaplan 1982a; Cattani, Kaplan, Schmid 1986; Hain, Zucker 1987; Kashiwara 1985, 1986).

Chapter 7 Perverse Sheaves

§ 1. Perverse Sheaves

1.1. Stratifications. Let X be a topological space. A finite decomposition of X into nonempty disjoint locally closed subsets (strata) is called a *stratification* if the closure of any stratum is a union of strata.

A complex analytic or an algebraic variety X admits a stratification whose strata are non-singular and satisfy the following equisingularity condition: for any two points p, q of one stratum there exists a diffeomorphism of X that preserves all strata and maps p to q.

Let $\mathcal{S}$ be a stratification. A *perversity* is a function $p : \mathcal{S} \to \mathbb{Z}$.

1.2. t-Structure Defined by p and $\mathcal{S}$. Let $D(X)$ be the derived category of sheaves of abelian groups on X. For any stratum $S \in \mathcal{S}$ denote by i_S the embedding of S into X.

Denote

$$^pD^{<0}(X) = \text{the full subcategory of complexes } K^{\cdot} \in \operatorname{Ob} D(X) \text{ such that } H^n i_S^{\cdot}(K^{\cdot}) = 0 \text{ for } n > p(S) \text{ for all } S; \tag{1}$$

$$^pD^{>0}(X) = \text{the full subcategory of complexes } K^{\cdot} \in \operatorname{Ob} D(X) \text{ such that } H^n i_S^{!}(K^{\cdot}) = 0 \text{ for } n < p(S) \text{ for all } S. \tag{2}$$

1.2.1. Proposition. *$({}^pD^{<0}(X), {}^pD^{>0}(X))$ is a t-structure on $D(X)$; it induces t-structures on $D^+(X)$ and $D^b(X)$.*

The proof is by induction in the number of strata and uses the gluing theorem (see Chap. 5, 3.7.3).

1.3. Definition. The core of this t-structure, i.e., the category $\mathcal{M}(p,X) = {}^pD^{<0}(X) \cap {}^pD^{>0}(X)$ is called the category of *p-perverse sheaves.*

The general theory of cores (Chap. 5, Theorem 3.4) implies that $\mathcal{M}(p,X)$ is an abelian category. Sheaves of abelian groups can be replaced by sheaves of $\mathcal{O}$-modules on X, where $\mathcal{O}$ is a fixed sheaf of modules. The corresponding category of perverse sheaves are denoted $\mathcal{M}(p,X,\mathcal{O})$ or again by $\mathcal{M}(p,X)$.

For $p = 0$ we obtain the ordinary category of sheaves.

1.4. Example. Let X be an n-dimensional complex variety stratified by non-singular complex subvarieties S, and $C_.(X)$ be the complex of piecewise linear chains (with respect to some piecewise linear structure on X) with coefficients in a field. Denote by $IC_.(X)$ the subcomplex of $C_.(X)$ consisting of all chains c satisfying the following conditions:

$$\begin{gathered}\dim(\operatorname{supp} c \cap S) < \dim \operatorname{supp} c - \operatorname{codim}_{\mathbb{C}} S,\\ \dim(\operatorname{supp} \partial c \cap S) < \dim \operatorname{supp} c - 1 - \operatorname{codim}_{\mathbb{C}} S,\end{gathered}$$

for all strata S of positive dimension.

Define similarly $IC_.(U)$ for an arbitrary open $U \subset X$ with the induced stratification and denote by $IC_.^{\mathrm{cl}}(U)$ the subgroup of chains with closed support.

Define now $\mathbf{IC}^i(U) = [IC_{2n-i}^{\mathrm{cl}}(U)]^*$. Then $\mathbf{IC}^{\cdot}$ is a p-perverse sheaf in the category $\mathcal{M}(p_{1/2}, X, A)$, where A is the constant sheaf corresponding to the coefficient group A, and the self-dual perversity $p_{1/2}$ is given by $p_{1/2}(S) = -\dim_{\mathbb{C}} S$.

Intuitively speaking, a perverse sheaf $\mathbf{IC}^{\cdot}$ must be thought of as an appropriate analog of the constant sheaf A for a singular variety. One can prove that as an object of the derived category $\mathbf{IC}^{\cdot}$ can be obtained by some natural extension procedure applied to the constant sheaf on the open dense stratum.

This construction can be applied also to non-constant local systems and to any perversity. We describe this construction in some details.

1.5. The Construction of the Extension. We assume that the perversity p is monotone: $p(S) \geq p(T)$ for $S \subset \overline{T}$. Denote $F_n = \bigcup_{p(S)\geq n} S$, $U_n = \bigcup_{p(S)<n} S$, $j_n : U_{n-1} \hookrightarrow U_n$. Sets U_n are open, and F_n's are closed.

Let A be a perverse sheaf on U_k, $a \geq k$ be an integer satisfying $p(S) \leq a$ for all S and $j = j_k$. Define

$$j_{!*}A = \tau_{<a-1}R(j_a)_{.} \ldots \tau_{<k}R(j_{k+1})_{.}(A),$$

where $\tau_{<i}$ be an ordinary truncation functor in $D(X)$ (see Chap. 4, 2.10) and $Rj_.$ is the direct image functor (see Chap. 4, 5.2).

1.5.1 Proposition. *The extension $j_{!*}A$ of the perverse sheaf A from U_k to X can be characterized as a unique extension of A to an object $P \in \operatorname{Ob} D(X)$ such that*

$$H^n i_S^{\cdot}(P) = 0 \quad \text{for } n \geq p(S),$$
$$H^n i_S^!(P) = 0 \quad \text{for } n \leq p(S)$$

for all $S \subset X \setminus U_k$.

More generally, a similar theorem about existence and uniqueness of the extension holds in the following situation: U_k is replaced by an arbitrary open union of strata and p is not necessarily monotone. However, the described direct construction can fail.

To generalize this construction to a general case we must introduce the direct image functors ${}^p j_.$, ${}^p j_!$ associated to a given perversity p.

Denote by U an arbitrary locally closed union of strata and by $j : U \to X$ the inclusion. Then the functors $Rj_., j_! : D(U) \to D(X)$ (see 3.7.1 in Chap. 5) are defined. One can verify that the functor $Rj_.$ is left exact and the functor $j_!$ is right exact with respect to the t-structure defined in 1.2. This means that for $P \in \mathcal{M}(p, U)$ we have $Rj_.(P) \in {}^pD^{>0}(X)$, $j_!(P) \in {}^pD_{<0}(X)$. Define the functors ${}^p j_., {}^p j_! : \mathcal{M}(p, U) \to \mathcal{M}(p, X)$ by the following formulas:

$${}^p j_.(P) = {}^p\tau_{<0} Rj_.(P),$$
$${}^p j_!(P) = {}^p\tau_{>0} j_!(P),$$

where ${}^p\tau$ denotes the truncation functor with respect to the t-structure from 1.2 (see Lemma 3.4 in Chap. 5).

The natural morphism of functors $j_! \to j_.$ can be extended from sheaves to derived categories and then to the morphism ${}^p j_! \to {}^p j_.$. Finally, define

$$j_{!*}(P) : \operatorname{Im}({}^p j_!(P) \to {}^p j_.(P)).$$

1.5.2 Proposition. *Let $B \in \mathcal{M}(p, U, \mathcal{O})$, where U is an open union of strata. Then $P = j_{!*}(B)$ is a unique extension of B to $D(X, \mathcal{O})$ with the property as in Proposition* 1.5.1.

1.6. Constructible Complexes. Later in this chapter we will assume the following conditions to be satisfied.

a. $\mathcal{O}$ is a constant sheaf whose fiber is a Noetherian ring R (most often, $\mathbb{Z}$ or a field).

b. X admits a locally finite triangulation; each stratum of the stratification $\mathcal{S}$ is an equidimensional topological manifold and can be obtained as a union of open simplices; moreover, $\dim S < \dim \overline{T}$ if $S \subset \overline{T}$, $S \neq T$.

c. For any S the functor i_S has a finite cohomological dimension.

A standard example is a Whitney stratification of a real algebraic variety, and, in particular, a stratification of a complex variety by non-singular complex subvarieties (in this case all strata are of even dimension).

1.6.1. Definition. a. A sheaf $\mathcal{F}$ on X is said to be *constructible* with respect to $\mathcal{S}$ if for any $S \in \mathcal{S}$ the sheaf $i_S^{\cdot}(\mathcal{F})$ is locally constant.

b. A complex of sheaves $\mathcal{F}^{\cdot}$ on X is said to be *cohomologically constructible* with respect to $\mathcal{S}$ if all sheaves $H^i(\mathcal{F}^{\cdot})$ are constructible with finitely generated (over R) fibers.

Denote by $D_{\mathcal{S}}(X,R)$ the full subcategory of the category $D(\mathcal{S}(R\text{-mod}))$ formed by sheaves of R-modules that are cohomologically constructible with respect to $\mathcal{S}$, and by $D_c(X,R)$ the union of the categories $D_{\mathcal{S}}(X,R)$ for all $\mathcal{S}$ satisfying conditions **a**–**c** above. One can easily verify that $D_c(X,R)$, $D_{\mathcal{S}}(X,R)$ are triangulated subcategories of $D(\mathcal{S}(R\text{-mod}))$.

Let us remark that if the conditions bf b and **c** are satisfied for any $S \in \mathcal{S}$ and for any locally constant sheaf $\mathcal{F}$ on S, then the sheaf $(i_S)_{.}(\mathcal{F})$ is constructible with respect to $\mathcal{S}$.

The main reason why we must introduce $D_C(X,R)$ is the following proposition.

1.6.2. Proposition. a. *The dualizing complex* $\mathbf{D}_X$ *(see* 5.16 *in Chap.* 4*) belongs to* $D_c(X,R)$ *(in fact, it belongs to* $D_{\mathcal{S}}(X,R)$ *for any stratification* $\mathcal{S}$ *satisfying* **b, c** *above).*

b. (Poincaré-Verdier duality) *Let* $R=k$ *be a field. The functor* $\mathfrak{D}_X(\mathcal{F}^{\cdot}) = RHom(\mathcal{F}^{\cdot},\mathbf{D}_X)$ *define an equivalence of categories* $D_c(X,k) \to D_c(X,k)^0$.

1.7. Duality for Perverse Sheaves. Let p be a perversity on a stratification $\mathcal{S}$ of a space X. Define the dual perversity p^* by the formula

$$p^*(S) = -p(S) - \dim S.$$

If all strata are of even dimension, the *self-dual perversity* $p_{1/2}$ is defined by

$$p_{1/2}(S) = -(1/2)\dim S$$

Using the formulas from 5.18.g in Chap. 4, we can rewrite conditions (1) and (2) in the definition of perverse sheaves as follows: for any stratum $S \in \mathcal{S}$ we have

$$H^n i_S^{\cdot}(\mathcal{F}^{\cdot}) = 0 \qquad \text{for } n > p(S), \tag{3}$$
$$H^n i_S^{\cdot}(\mathfrak{D}_X\mathcal{F}^{\cdot}) = 0 \qquad \text{for } n < p(S). \tag{4}$$

Conditions (3), (4) are, of course, dual to each other. Hence the duality functor $\mathfrak{D}_X$ maps $\mathcal{M}(p,X)$ to $\mathcal{M}(p^*,X)^{\circ}$. Moreover, imposing on $\mathcal{F}^{\cdot}$ the selfduality condition with respect to $\mathfrak{D}_X$, we can consider only one of two conditions (3) or (4). Hence, Proposition 1.5.2 can be rewritten as follows.

1.7.1. Proposition. *Let all strata of the stratification* $\mathcal{S}$ *are even-dimensional, and* $p = p_{1/2}$ *is the self-dual perversity. Let* $j: U \to X$ *be the embedding of* *the union of strata, and $\mathcal{A}$ be a p-perverse self-dual (with respect to $\mathfrak{D}_U$) sheaf in U. Then $j_{!*}(\mathcal{A})$ is a unique self-dual (with respect to $\mathfrak{D}_X$) extension P of the complex $\mathcal{A}$ in $D_{\mathcal{C}}(X,k)$ such that for any stratum $S \in \mathcal{S}$ we have $H^i i_S^{\cdot}(\mathcal{F}^{\cdot}) = 0$ for $i > p_{1/2}(S) = -(1/2)\dim S$.*

1.8. Subdivision of Stratifications. Let a stratification $\mathcal{T}$ be a subdivision of a stratification $\mathcal{S}$ (this means that each stratum of $\mathcal{S}$ is the union of several strata of $\mathcal{T}$), so that we have the embedding of categories $I : D_{\mathcal{S}}(X,R) \to D_{\mathcal{T}}(X,R)$. Let p and q be perversities on $\mathcal{S}$ and $\mathcal{T}$ respectively such that

$$p(S) \le q(T) \le p(S) + \dim S - \dim T$$

whenever $S \in \mathcal{S}$, $T \in \mathcal{T}$, $T \subset S$.

1.8.1. Proposition. *Under the above assumptions the t-structure of perversity p on $D_{\mathcal{T}}(X,R)$ induces the t-structure of perversity q on $D_{\mathcal{S}}(X,R)$ (under the embedding of the categories I). In particular, any p-perverse sheaf is also q-perverse, I induces an exact embedding of categories $I_{p,q} : \mathcal{M}(p,X) \to \mathcal{M}(q,X)$, and for any embedding $j : U \to X$ of a union of strata of $\mathcal{S}$ the restrictions of functors ${}^q j_., {}^q j_!, {}^q j^{\cdot}, {}^q j^!, {}^q j_{!*}$ to $\mathcal{M}(p,U)$ and $\mathcal{M}(p,X)$ coincide with the corresponding functors with the index p.*

1.9. Complex Varieties. Consider the case when X is a complex variety, $\mathcal{S}$ is a stratification of X by non-singular subvarieties, and the perversity p is such that $p(S)$ depends only on the dimension of $S \in \mathcal{S}$ (i.e., $p(S) = p(\dim_{\mathbb{R}} S) = p(2\dim_{\mathbb{C}} S)$). We will assume also that $p(n)$ satisfies the condition

$$0 \le p(n) - p(m) \le m - n \quad \text{for } n \le m. \tag{5}$$

By Proposition 1.8.1, under these assumptions the subdivision of stratifications is compatible with t-structures, so that the passage to the inductive limit defines the t-structure of perversity p:

$$(D_c^{b,<p}(X,\mathbb{C}), \quad D_c^{b,\ge p}(X,\mathbb{C}))$$

on the triangulated category $D_c^b(X,\mathbb{C})$ of bounded complexes of sheaves of vector spaces on X with constructible (with respect to some stratification of X by complex varieties) cohomology.

1.9.1. Proposition. *For $\mathcal{F}^{\cdot} \in D_c^b(X,\mathbb{C})$ the following conditions are equivalent:*

a. *$\mathcal{F}^{\cdot}$ is a perverse sheaf.*

b. *Any irreducible submanifold $S \subset X$ contains a Zariski open $U \subset S$ such that*

$$H^i(Rj^{\cdot}\mathcal{F}^{\cdot}) = 0 \quad \text{for } i > p(\dim_{\mathbb{R}} S);$$
$$H^i(Rj^{!}\mathcal{F}^{\cdot}) = 0 \quad \text{for } i < p(\dim_{\mathbb{R}} S).$$

1.10. Simple Objects. In the case when $R = k$ is a field and the perversity p depends only on $\dim S$ and satisfies the condition (5), simple objects of the abelian category $\mathcal{M}_{\mathcal{S}}(p,X) = \mathcal{M}(p,X) \cap D_{\mathcal{S}}(X,k)$ admit the following description.

1.10.1. Proposition. *The category $\mathcal{M}_{\mathcal{S}}(p,X)$ is Artinian. Its simple objects are of the form $L(S,\mathcal{E}) = {}^p(i_s)_{!*}[p(S)]$, where $S \in \mathcal{S}$ and $\mathcal{E}$ is an irreducible locally constant sheaf of vector spaces on S. In particular, if all strata are simply connected, simple objects of $\mathcal{M}_{\mathcal{S}}(p,X)$ are in one-to-one correspondence with strata $S \in \mathcal{S}$.*

If we do not fix a stratification, then the category of perverse sheaves of a given perversity p is only noetherian, but not artinian. If, however, $p = p_{1/2}$ is the self-dual perversity, then, by the Poincaré-Verdier duality the fact that the category is noetherian implies that it is also artinian. In particular, if X is a complex variety, we obtain the following proposition.

1.10.2. Proposition. *The category $\mathcal{M}(p_{1/2},X)$ of $p_{1/2}$-perverse sheaves on X with the cohomology that are constructible with respect to some stratification of X by complex non-singular varieties* (see 1.9) *is artinian. Its simple objects are of the form $L(S,\mathcal{E})$* (see 1.10.1), *where $S \subset X$ is a non-singular irreducible subvariety, $\mathcal{E}$ is an irredicuble locally constant sheaf in S. Moreover, $L(S,\mathcal{E}) \cong L(S',\mathcal{E}')$ if and only if $S \cap S'$ is dense in S and in S', and $\mathcal{E}\mid_{S\cap S'} = \mathcal{E}'\mid_{S\cap S'}$.*

§ 2. Glueing

2.1. What is Glueing. In this section we discuss the following problem. Let X be a topological space, $U \subset X$ be an open set, $Y = X \setminus U$ be its complements. Denote by $j : U \to X$, $i : Y \to X$ the embeddings. We are interested in relations between perverse sheaves on X and perverse sheaves on U and on Y. All results about glueing can be roughly divided into two large classes that can be called glueing of t-structures and glueing of cores. Typical representatives of these two classes are Theorem 2.2.1 and Theorem 2.5.1.

2.2. Glueing t-structures. Let us assume that X is endowed with a stratification $\mathcal{S}$ and U is an open union of strata (so that $Y = X \setminus U$ also is a union of strata). Let p be a perversity on $\mathcal{S}$; denote by the same letter the induced perversities on U and on Y. Let us recall that in Chap. 5, 3.7.3, we have described how we can construct a t-structure on $D(\mathcal{S}\mathbf{Ab}_X)$ from given t-structures on $D(\mathcal{S}\mathbf{Ab}_U)$ and on $D(\mathcal{S}\mathbf{Ab}_Y)$ using six functors $i^{\cdot}$, $i^!$, $i_{\cdot}$, $Rj_{\cdot}$, $j_!$, $j^{\cdot}$ satisfying some compatibility conditions (see 3.7.1 in Chap. 5).

2.2.1. Theorem. *The t-structures of perversity p on $D(\mathcal{S}\mathbf{Ab}_X)$, $D(\mathcal{S}\mathbf{Ab}_U)$ and $D(\mathcal{S}\mathbf{Ab}_Y)$ are related to each other by the construction from* 3.7.3 *in Chap.* 5.

This theorem enables us to use the general technique of t-structures to the study of perverse sheaves. In particular, the description of simple perverse sheaves from 1.10 can be generalized as follows. Taking the composition of the functor H^0 (with respect to the t-structure of perversity p) with $j_!$, we obtain the functors

$$^p j_., {}^p j_! : \mathcal{M}(p, U, \mathcal{O}) \to \mathcal{M}(p, X, \mathcal{O})$$

and the morphism of functors

$$F : {}^p j_. \to {}^p j_!.$$

After that $^p j_{*!} : \mathcal{M}(p, U, \mathcal{O}) \to \mathcal{M}(p, X, \mathcal{O})$ is defined as the functor that associates to $\mathcal{F}^\cdot \in \operatorname{Ob} \mathcal{M}(p, U, \mathcal{O})$ the object $\operatorname{Im} F_{\mathcal{F}^\cdot}$ of the category $\mathcal{M}(p, X, \mathcal{O})$.

2.2.2. Proposition. *Any simple object of $\mathcal{M}(p, X, \mathcal{O})$ is isomorphic either to the object $^p i_. \mathcal{F}^\cdot$, where $\mathcal{F}^\cdot$ is a simple object of the category $\mathcal{M}(p, Y, \mathcal{O})$, or to the object $^p j_{*!} \mathcal{F}^\cdot$, where $\mathcal{F}^\cdot$ is a simple object of the category $\mathcal{M}(p, U, \mathcal{O})$.*

Let us mention two characterizations of the object $\mathcal{G}^\cdot = {}^p j_{*!} \mathcal{F}^\cdot$ for $\mathcal{F}^\cdot \in \mathcal{M}(p, X, \mathcal{O})$. First, $\mathcal{G}^\cdot$ is a unique object of $D^b(\mathcal{S}\mathbf{Ab}_X)$ which is an extension of $\mathcal{F}^\cdot$ to X (i.e. $j^\cdot \mathcal{G}^\cdot = \mathcal{F}^\cdot$) such that

$$i^\cdot \mathcal{G}^\cdot \in D^{\leq -1}(p, Y, \mathcal{O}), \quad i^! \mathcal{G}^\cdot \in D^{>1}(p, Y, \mathcal{O}).$$

Second, $\mathcal{G}^\cdot$ is a unique extension of $\mathcal{F}^\cdot$ to X that has neither subobjects, nor quotient objects supported on Y (i.e., ofthe form $^p i_. \mathcal{H}^\cdot$ with $\mathcal{H}^\cdot \in \mathcal{M}(p, Y, \mathcal{O})$). In this sense $^p j_{*!} \mathcal{F}^\cdot$ can be thought of as the *minimal* extension of $\mathcal{F}^\cdot$ to X.

2.3. Glueing of Perverse Sheaves. The second group of results deals with the following situation. Let $\mathcal{F}^\cdot$ and $\mathcal{H}^\cdot$ be perverse sheaves on U and on Y respectively. What additional data one needs to construct from $\mathcal{F}^\cdot$ and $\mathcal{H}^\cdot$ a perverse sheaf on X? Answers to this questions (expressed in different languages, but essentially equivalent) were obtained by several authors. In the purely topological setup the most general result belongs to MacPherson and Vilonen (1986). Here we present the approach of Beilinson (1987) and Verdier (1985).

2.3.1. Assumptions. We fix the notations and the assumptions that will be used in the remainder of this section.We assume that X is a smooth complex algebraic variety, $Y \subset X$ is a subvariety defined in X by one equation $f = 0$ for some function $f \in \Gamma(X, \mathcal{O}_X)$. We will consider the self-dual perversity $p_{1/2}(S) = -(1/2) \dim_{\mathbb{R}} S$; the categories $D^b_C(X, \mathbb{C})$ and $\mathcal{M}(p_{1/2}, X, \mathbb{C})$ (see 1.10.2) will be denoted simply D_X and $\mathcal{M}_X$; objects of $\mathcal{M}_X$ will be called perverse sheaves on X. Similar notations will be used for perverse sheaves on Y and on $U = X - Y$.

2.4. Vanishing Cycles Functors and Their Properties. Following (Deligne 1973b) we define two vanishing cycle functors related to the function f. Denote $B = \mathbb{C}^1$, $B^* = \mathbb{C}^1 \setminus \{0\}$, $\widetilde{B}^*$ the universal covering of B^*, $p : \widetilde{B}^* \to B^*$

the covering map (one can take $\widetilde{B}^* = \mathbb{C}$, $p(z) = e^z$). We have the following commutative diagram of spaces and mappings:

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[d, "f"'] & X \arrow[d, "f"] & U \arrow[l, hook', "j"'] \arrow[d, "f"] \\ \{0\} \arrow[r, hook] & B & B^* \arrow[l, hook'] \end{tikzcd}$$

Denote by $\widetilde{X}^*$ the fiber product of X and $\widetilde{B}^*$ over B (with respect to the pair of mappings f, p) and by $\pi : \widetilde{X}^* \to X$ the natural projection.

2.4.1 Definition. (of the *nearby cycles functor*). For $\mathcal{F}^{\cdot} \in D_X$ define

$$\psi_f(\mathcal{F}^{\cdot}) = i^{\cdot} R\pi_{.}\pi^{\cdot}\mathcal{F}^{\cdot}. \tag{1}$$

2.4.2. Proposition. a. *Formula* (1) *defines a functor* $\psi_f : D_X \to D_Y$.
b. *If* $\mathcal{F}^{\cdot} \in \mathcal{M}_X$, *then* $\psi_f\mathcal{F}^{\cdot} \in \mathcal{M}_Y$.

Since $\pi(\widetilde{X}^*) = U \subset X$, $\psi_f\mathcal{F}^{\cdot}$ depends only on $j^{\cdot}\mathcal{F}^{\cdot}$. Hence ψ_f defines a functor (also denoted ψ_f) from D_U to D_Y. Next, the natural morphism $\mathcal{F}^{\cdot} \to R\pi_{.}\pi^{\cdot}\mathcal{F}^{\cdot}$ (the adjunction morphism for functors $R\pi_{.}$ and $\pi^{\cdot}$) defines a morphism of functors $\theta : i^{\cdot} \to \psi_F$.

2.4.3. Definition (of the *vanishing cycles functor*). For $\mathcal{F}^{\cdot} \in D_X$ denote

$$\varphi_f = \mathrm{cone}(\theta_{\mathcal{F}^{\cdot}} : i^{\cdot}\mathcal{F}^{\cdot} \to \psi_f\mathcal{F}^{\cdot}). \tag{2}$$

(i.e., $\varphi_f(\mathcal{F}^{\cdot})$ is the third vertex of the distinguished triangle containing $\theta_{\mathcal{F}}$).

2.4.4. Proposition. a. *Formula* (2) *defines a functor* $\varphi_f : D_X \to D_Y$.
b. *If* $\mathcal{F}^{\cdot} \in \mathcal{M}_X$, *then* $\varphi_f\mathcal{F}^{\cdot} \in \mathcal{M}_Y$.

To prove **a** we must interpret $\varphi_f\mathcal{F}^{\cdot}$ as the ordinary complex associated to the bicomplex

$$\theta_{\mathcal{F}^{\cdot}} : i^{\cdot}\mathcal{F}^{\cdot} \to \psi_f\mathcal{F}^{\cdot}.$$

2.4.5. The Monodromy. The "complete turn" mapping $t : \widetilde{B}^* \to \widetilde{B}^*$ defined by $t(z) = z + 2\pi i$ commutes with p, $p \circ t = p$. Hence, it determines a mapping $\tau : \widetilde{X}^* \to \widetilde{X}^*$ with $\pi \circ \tau = \pi$. Therefore the morphism of functors λ : $\mathrm{Id} \to R\tau_{.}\tau^{\cdot}$ determines a morphism $R\pi_{.} \circ \lambda \circ \pi^{\cdot} : R\pi_{.}\pi^{\cdot} \to R\pi_{.}\pi^{\cdot}$, hence a morphism of functors $T : \psi_f \to \psi_f$. This morphism T, called the *monodromy*, commutes with θ, $T \circ \theta = \theta$, hence determines a morphism of functors T : $\varphi_f \to \varphi_f$ (with the same name and notation). Since t is an isomorphism, T is an automorphism of functors (in both cases).

2.4.6. Morphisms can **and** var. By can and var we denote the following morphisms of functors:

$$\mathrm{can} : \varphi_f \to \psi_f$$

is the morphism from the distinguished triangle defining φ_f, and

$$\mathrm{var}: \psi_f \to \varphi_f$$

is defined from the following morphism of distinguished triangles:

$$\begin{tikzcd} i^{\cdot}\mathcal{F}^{\cdot} \arrow[r] \arrow[d] & \psi_f(\mathcal{F}^{\cdot}) \arrow[r, "\mathrm{can}_{\mathcal{F}^{\cdot}}"] \arrow[d, "T_{\mathcal{F}^{\cdot}}-\mathrm{id}"] & \varphi_f(\mathcal{F}^{\cdot}) \arrow[r] \arrow[d, "\mathrm{var}_{\mathcal{F}^{\cdot}}"] & i^{\cdot}\mathcal{F}^{\cdot}[1] \arrow[d] \\ 0 \arrow[r] & \psi_f(\mathcal{F}^{\cdot}) \arrow[r] & \psi_f(\mathcal{F}^{\cdot}) \arrow[r] & 0 \end{tikzcd}$$

so that $\mathrm{var} \circ \mathrm{can} = T - \mathrm{id}$.

2.5. Glueing Data. Define the category $\mathrm{Glue}(T, U)$ as follows. Objects of $\mathrm{Glue}(T, U)$ are quadruples $(\mathcal{G}^{\cdot}, \mathcal{H}^{\cdot}, a, b)$, where $\mathcal{G}^{\cdot} \in \mathrm{Ob}\,\mathcal{M}_U$, $\mathcal{H}^{\cdot} \in \mathrm{Ob}\,\mathcal{M}_Y$, $a : \psi_f(\mathcal{G}^{\cdot}) \to \mathcal{H}^{\cdot}$, $b : \mathcal{H}^{\cdot} \to \psi_f(\mathcal{G}^{\cdot})$ are morphisms in $\mathcal{M}_Y$ such that $b \circ a = T_{\mathcal{G}^{\cdot}} - \mathrm{Id}$. Morphisms of $\mathrm{Glue}(T, U)$ are defined in a natural way.

2.5.1. Theorem. *Categories $\mathcal{M}_X$ and* $\mathrm{Glue}(T, U)$ *are equivalent.*

The functor $\mathcal{M}_X \to \mathrm{Glue}(T, U)$ that establishes this equivalence, maps $\mathcal{F}^{\cdot} \in \mathrm{Ob}\,\mathcal{M}_X$ to the quadruple

$$(j^{\cdot}\mathcal{F}^{\cdot}, \varphi_f\mathcal{F}^{\cdot}, \mathrm{can}_{\mathcal{F}^{\cdot}}, \mathrm{var}_{\mathcal{F}^{\cdot}}).$$

2.6. Example. The simplest example corresponds to the case $X = B$, $Y = \{0\}$, $U = B^*$, $f(z) = z$. Denote by $\mathcal{M}_{X,\mathcal{S}}$ the category of perverse sheaves on X that are smooth with respect to the stratification $X = U \cup Y$ of X, and by $\mathcal{M}_{U,\mathcal{S}}$ the corresponding category of perverse sheaves on U. It is clear that $\mathcal{M}_{U,\mathcal{S}}$ is the category of locally constant sheaves $\mathcal{F}$ (local systems) on B^*. Since $\pi_1(B^*) = \mathbb{Z}$, $\mathcal{M}_{U,\mathcal{S}}$ is equivalent to the category of pairs (V, A), where V is a finite-dimensional vector space (the fiber of $\mathcal{F}$ over a point of B^*) and $A : V \to V$ is an invertible linear operator (geometric monodromy operator corresponding to the path around zero in the positive direction). Next, the category $\mathcal{M}_Y$ is equivalent to the category of finite-dimensional vector spaces. The functor $\psi_f : \mathcal{M}_{U,\mathcal{S}} \to \mathcal{M}_Y$ maps (V, A) to A and the monodromy operator T equals $A - \mathrm{Id}$. Hence, the category $\mathcal{M}_{X,\mathcal{S}}$ is equivalent to the following "linear algebra" (or quiver) category $\mathcal{A}$:

$$\mathrm{Ob}\,\mathcal{A} = \{(V, W, E, F)\},$$

where V, W are finite-dimensional linear spaces, $E : V \to W$, $F : W \to V$ are linear operators such that $\mathrm{Id} + FE$ is invertible, morphisms in $\mathcal{A}$ are defined in a natural way.

2.7. Another Example: the Coordinate Stratification. Applying Theorem 2.5.1 several times we can obtain the "linear algebra (quiver) description" of the category $\mathcal{M}_{X,\mathcal{S}}$ for more complicated stratifications $\mathcal{S}$. The next example illustrates the type of answers we can obtain. Let $X = \mathbb{C}^n$, $\mathcal{S}$ be the stratification of X by subsets X_I, $I \subset \{1, 2, \dots, n\}$ defined as follows: $X_I = \{(x_1, \dots, x_n)$ such that $x_i = 0$ for $i \in I, x_i \neq 0$ for $i \notin I\}$. The category of perverse sheaves on X whose cohomology is constructible with respect to this stratification is equivalent to the following category $\mathcal{A}_{(n)}$:

One object of $\mathcal{A}_{(n)}$ is a family of 2^n finite-dimensional vector spaces V_I, $I \subset \{1, 2, \dots, n\}$, and linear mappings $E_{I,i} : V_I \to V_{I \cup \{i\}}$, $F_{I,i} : V_{I\cup\{i\}} \to V_I$, $i \notin I$, satisfying the following conditions:

$$\begin{aligned} E_{I\cup\{j\},i}E_{I,j} &= E_{I\cup\{i\},j}E_{I,i}, \\ F_{I,j}F_{I\cup\{j\},i} &= F_{I,i}F_{I\cup\{i\},j} \quad \text{for } i,j \notin I, \quad i \neq j, \\ E_{I\setminus\{j\},i}F_{I\setminus\{j\},j} &= F_{I\cup\{i\}\setminus\{j\},j}E_{I,i} \quad \text{for } i \notin I, \quad j \in I. \\ F_{I,i}E_{I,I} &+ \mathrm{Id} \quad \text{is invertible.} \end{aligned}$$

Morphisms in $\mathcal{A}_{(n)}$ are families of mappings $f_I : V_I \to V'_I$ commuting with all $E_{I,i}$ and $F_{I,i}$.

2.7.1. Theorem. *The categories $\mathcal{M}_{X,\mathcal{S}}$ and $\mathcal{A}_{(n)}$ are equivalent.*

Bibliographic Hints

The notion of a perverse sheaf in the form it is presented in this chapter was first introduced by Beilinson, Bernstein, Deligne, Gabber (see (Beilinson, Bernstein, Deligne 1982)). The original idea of their paper was to find a relationship between modules over rings of algebraic differential operators and complexes of constructible sheaves in the framework of the Riemann–Hilbert correspondence (see the next chapter). A class of extremely important examples of perverse sheaves is supplied by the so-called intersection (co)homology sheaves which were introduced (following Deligne's suggestion) by Goresky, MacPherson (1983) in the continuation of their study of the Poincaré duality for singular varieties. A detailed exposition of these results, together with all necessary notions of sheaf theory, see in (Borel 1984).

The subsequent analysis showed that perverse sheaves are in many respects not worse (and in some cases even better) than ordinary sheaves suited for the study of analytical and topological problems on singular varieties. In this review we omit the relation of perverse sheaves to analysis (L_2-cohomology, see (Cheeger, Goresky, MacPherson 1982)), representation theory (Ginzburg 1986; Joseph 1983; Springer 1982; Vogan 1981), Hodge structures (Saito 1986) topology of singular varieties (Borel 1984), concentrating mostly on the homological aspects of the theory of perverse sheaves. A good review of the above relations of the theory of perverse sheaves is contained in MacPherson (1984). A rather complete bibliography see in (Goresky, MacPherson 1984).

Proofs of the majority of the results from § 1 can be found in (Beilinson, Bernstein, Deligne 1982). Proposition 1.6.2 is proved in (Goresky, MacPherson 1983) and in (Borel 1984). Theorem 2.2.1 and Proposition 2.2.2 are proved in (Beilinson, Bernstein, Deligne 1982). The functors similar to the vanishing cycle functors were introduced (in the context of derived categories) by Deligne (1973b), who generalized the topological construction by Milnor. The idea to use these functors for glueing perverse sheaves is due to several authors: in addition to topological approaches by MacPherson, Vilonen (1986), Deligne, Verdier (see (Verdier 1985)) and Beilinson (1987a, 1987b), there exists an (equivalent) approach by Kashiwara (1983), which uses the language of D-modules and the Riemann–Hilbert correspondence (see the next chapter). The description of certain special categories of perverse sheaves by these approaches see (MacPherson, Vilonen 1988; Maisonobe 1987; Narvaes-Macarro 1988). In particular, example in 2.7 is borrowed from (Maisonobe 1988).

Chapter 8
$\mathcal{D}$-Modules

§ 0. Introduction

0.1. Linear Differential Equations and $\mathcal{D}$-modules. Consider a system E of linear equations

$$\sum_{j=1}^{p} R_{ij} u_j = 0, \qquad i = 1, \ldots, q, \tag{1}$$

where u_j are unknown functions in variables $x_1, \ldots, x_n$, R_{ij} are linear differential operators (with variable coefficients). In the classical theory of differential equations we are interested in solutions of such systems.

To reformulate this problem in algebraic language, denote by A a ring of functions that contains coefficients of the system; it might be the ring of smooth, real-analytic, complex-analytic, or polynomial functions in some region U. Next, denote by $\mathcal{D}$ the ring of linear differential operators with coefficients in A. Construct from the system (1) a left $\mathcal{D}$-module M as the cokernel of the morphism of left $\mathcal{D}$-modules $R : \mathcal{D}^p \to \mathcal{D}^q$ defined as the right multiplication by the matrix operator (R_{ij})

$$\mathcal{D}^p \xrightarrow{R} \mathcal{D}^q \to M \to 0. \tag{2}$$

Of course, we are not obliged to look for solutions of the system (1) in A; for example, we can be interested in distribution solutions. The only essential thing is that we can differentiate these solutions and multiplicate them by functions. Therefore, it makes sense to consider another left $\mathcal{D}$-modules N and to define $\mathrm{Sol}(E, N)$ as the set of solutions of the system E in N. Clearly,

$$\mathrm{Sol}(E, N) = \mathrm{Hom}_{\mathcal{D}}(M, N). \tag{3}$$

This equality shows that the algebraic counterpart of the theory of equations (1) is the study of the category of left $\mathcal{D}$-modules.

0.2. Characteristic Variety. Let $p = q = 1$ and let $r(x, \xi) = 0$ be the highest symbol of the operator R. The equation $r(x, \xi) = 0$ defines a subvariety in the cotangent bundle to the region U, which is called the characteristic variety of the equation $Ru = 0$. The role of characteristic directions is well known: the corresponding hypersurfaces are the constant phase surfaces of short wave solutions, and the projection of the characteristic variety to U is the support of possible singularities of solutions.

To define the characteristic variety of a general system (1) we must consider points (x, ξ) where the rank of the matrix of highest symbols of coefficients in (1) is less than the number of unknown functions. However, this naive definition becomes true only after we add to the system (1) all its differential consequences, and it is much more convenient to use the following equivalent definition in terms of the corresponding (see (2)) $\mathcal{D}$-module M. Let $\mathcal{D}_i \subset \mathcal{D}$ be the space of operators of order $\leq i$, $M_i = \mathcal{D}_i M_0$, where M_0 is a finitely generated A-submodule of M that generates M over $\mathcal{D}$. Then $\operatorname{gr} M = \bigoplus_{i=1}^{\infty} M_i/M_{i+1}$ is a graded module over the commutative ring $\operatorname{gr} \mathcal{D} = \bigoplus_{i=1}^{\infty} \mathcal{D}_i/\mathcal{D}_{i+1}$. This ring is the ring of functions on the cotangent bundle T^*U that are polynomial on each fiber, and the module $\operatorname{gr} M$ is the module of sections of a certain sheaf on T^*U. The support of this sheaf is the characteristic variety of the system (1).

One of the first results of the homological theory of $\mathcal{D}$-modules relates the dimension of the characteristic variety of M with the smallest j such that $\operatorname{Ext}^j_{\mathcal{D}}(M, A) = 0$ (see Chap. 1, and also 6.1 and 2.7 in this chapter).

0.3. Localization. Varying the region U inside some manifold X we come to the necessity to adopt the sheaf-theoretic point of view: the ring A is replaced by the structure sheaf $\mathcal{O}_X$, the ring $\mathcal{D}$ becomes the sheaf of rings $\mathcal{D}_X$ of differential operators on X, M is replaced by a sheaf of $\mathcal{D}_X$-modules. This generalization brings a new aspect into the theory: namely, we must study homological invariants of $\mathcal{D}_X$-modules on a manifold.

In this chapter we will mainly consider the case when $(X, \mathcal{O}_X)$ is a complex analytic manifold. In Sect. 3–5 we present the formalism of $\mathcal{D}_X$-modules. Whenever it becomes necessary, we introduce the corresponding derived category. From the classical point of view this can be justified by the possibility to formulate a far reaching generalization of the Riemann–Hilbert program about the description of a system of equations by the singularities of solutions.

0.4. Holonomicity and Regularity. The dimension of the characteristic variety of the system (1) on an n-dimensional region U is not less that n. If this dimension equals n, both the system (1) and the corresponding module M are called *holonomic*. In classical theory such systems were called maximally overdetermined systems. These systems are the closest to systems of ordinary differential equations in the respect that their solutions depend on a finite number of constants. This is why these systems are most suitable for the study by algebraic methods. On the other hand, the classical partial derivative equations, such as the Laplace equation or the wave equations, are far from being holonomic.

General properties of holonomic modules are discussed in Sect. 5.

The importance of the notion of regularity (introduced by Fuchs) in the theory of ordinary differential equations with analytic coefficients is well known. This property guranteed a good behavior of solutions near a singular point x_0, and, in particular, the possibility to expand a solution in monomials of the form $(x - x_0)^\alpha \log(x - x_0)^\beta$, and the possibility to reconstruct locally the equation near a singular point from its monodromy, i.e. from the branching properties of solutions.

The notion of regularity can be introduced also for holonomic modules, and the above properties can be generalized, see Sect. 6,7.

0.5. Riemann-Hilbert Program. One of the most simple holonomic systems is the system of the form

$$\frac{\partial u_j}{\partial x_i} = 0; \qquad i = 1, \ldots, n, \quad j = 1, \ldots, p.$$

The solutions of this system the constant vectors $u_j = c_j$. A holonomic system is called smooth if it can be transformed to such form near any point of U by a linear invertible change of (u_i) with analytic coefficients. The same terminology applies to a $\mathcal{D}_X$-module M.

The category of holonomic smooth $\mathcal{D}$-modules on a complex analytic manifold X_{an} is equivalent to the category of local systems of finite-dimensional vector spaces on X_{an}. The equivalence is given by the functor Sol:

$$M \to Hom_{\mathcal{D}_X}(M, \mathcal{O}_X) \quad \text{(the sheaf of homomorphisms)}.$$

If we drop the smoothness condition, this theorem has the following deep generalization, which can be considered as one of the central results of the homological theory of $\mathcal{D}_X$-modules.

Let $D^b_{hr}(\mathcal{D}_X\text{-mod})$ be the derived category of holonomic regular $\mathcal{D}_X$-modules, and $D^b_c(X)$ be the triangulated category of complexes of sheaves of vector spaces with constructible cohomology sheaves (see 1.6.1 in Chap. 7). Then the functor $RHom_{\mathcal{D}_X}(\cdot, \mathcal{O}_X)$ determines an antiequivalence of categories

$$D^b_{hr}(\mathcal{D}_X\text{-mod}) \xrightarrow{\sim} D^b_c(X).$$

§ 1. The Weyl Algebra

1.1. Definition. The *Weyl algebra* $A(\mathbb{C})_n = A_n$ is the algebra with $2n$ generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ satisfying the following commutation relations: $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = 0$ for $i \neq j$, $[\partial_i, x_i] = 1$.

It is clear that an arbitrary element $P \in A_n$ can be uniquely written in the form

$$P = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha \partial^\beta, \tag{1}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial^\beta = \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$, $a_{\alpha\beta} \in \mathbb{C}$, the sum in (1) is finite. Hence, A_n can be thought of as the algebra of differential operators with polynomial coefficients.

Unless specifically mentioned, by an A_n-module we mean a left unitary A_n-module with a finite number of generators.

Two important examples:

a. $M = A_n/A_n(\partial_1, \dots, \partial_n)$. One can easily see that M can be represented as $M = \mathbb{C}[x_1, \dots, x_n]$, $\partial_j(F) = \frac{\partial F}{\partial x_i}$ (formal derivative), $x_j(F) = x_j F$ (multiplication).

b. $M = A_n/A_n(x_1, \dots, x_n)$. It is convenient to think about this module M as the module of distributions (functionals on polynomials) supported at the origin.

1.2. Properties of A_n. **a.** A_n is a simple algebra;

b. A_n is left and right Noetherian;

c. Any A_n-module of finite length is generated by one element;

d. The map $x_j \to \partial_j$, $\partial_j \to -x_j$ defines an automorphism of the ring A_n (the formal Fourier transform).

Let us prove the property **a**. Let $I \subset A_n$ be a nonzero two-sided ideal, $0 \neq P = \sum a_{\alpha\beta} x^\alpha \partial^\beta \in I$ Let $m > 0$ be the maximal degree of x_j in P. Then $[\partial_j, P] \neq 0$ and the maximal degree of x_j in $[\partial_j, P]$ is $m - 1$. Commuting m times ∂_j with P we obtain a nonzero element of I, which does not contain x_j. Commuting our element an appropriate number of times with each ∂_j and x_j we see that $1 \in I$, so that $I = A_n$.

Property **b** follows from the Noether property of the ring of polynomials in $2n$ variables, which is the associated graded ring for some filtration in A_n (see below).

Property **c** holds for any algebra A that has the infinite length as the left A-module (Stafford 1985).

Property **d** is straightforward. Let us remark that the formal Fourier transform interchanges modules from **a** and **b** in 1.1.

1.3. Homological Properties. One of the main properties of the algebra A_n from the point of view of homological algebra is the following result.

1.3.1. Theorem. *The homological dimension of A_n equals n; in the other words, for any A_n-modules M, N we have*

$$\operatorname{Ext}^j(M, N) = 0 \quad \text{for} \quad j > n.$$

In the next few subsection we sketch the proof of this theorem, as well as of certain more precise results.

1.4. Filtrations. We endow A_n with the following two increasing filtrations by linear subspaces:

a. *Bernstein filtration:*

$$B_j = \{a_{\alpha\beta} x^\alpha \partial^\beta : \quad |\alpha| + |\beta| \le j\}.$$

b. *Standard filtration* (by the order of a differential operator):

$$\Sigma_j = \{a_{\alpha\beta} x^\alpha \partial^\beta : \quad |\beta| \le j\}.$$

Denoting by $\{\mathcal{F}_j\}$ any of these filtrations, we have, of course,

$$\mathcal{F}_i \mathcal{F}_j \subset \mathcal{F}_{i+j}, \quad \bigcup \mathcal{F}_j = A_n.$$

Denote

$$\operatorname{gr}^{\mathcal{F}} A_n = \bigoplus_{j=0}^{\infty} \mathcal{F}_j / \mathcal{F}_{j-1}$$

(we assume that $\mathcal{F}_{-1} = \{0\}$). One can easily verify that for each of the filtrations $\{B_j\}$, $\{\Sigma_j\}$, the algebra $\operatorname{gr}^{\mathcal{F}} A_n$ is the algebra of polynomials in $2n$ variables $\overline{x_j}, \overline{\partial_j} \in \mathcal{F}_1 / \mathcal{F}_0$.

A *filtration of an A_n-module M compatible with $\mathcal{F}$* is an increasing sequence $\{0\} \subset \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \subset \cdots \subset M$ of linear subspaces such that

$$\mathcal{F}_i \Gamma_j \subset \Gamma_{i+j}, \quad \bigcup \Gamma_j = M,$$

$\mathcal{F}_0$-modules Γ_j are finitely generated.

The corresponding graded space

$$\operatorname{gr}^{\Gamma} M = \bigoplus_{j=0}^{\infty} \Gamma_j / \Gamma_{j-1}$$

has a natural structure of a $\operatorname{gr}^{\mathcal{F}} A_n$-module.

If $v \in \Gamma_j$, $v_j \notin \Gamma_{j-1}$, the image of v in Γ_j / Γ_{j-1} is called the *symbol* of v and denoted $\sigma(v)$.

A filtration $\{\Gamma_j\}$ of the module M is said to be *good* if there exists j_0 such that

$$\mathcal{F}_i \Gamma_j = \Gamma_{i+j} \quad \text{for all } i \ge 0,\ j \ge j_0.$$

1.4.1. Proposition. a. *Any finitely generated A_n-module M has a good filtration.*

b. *A filtration $\{\Gamma_j\}$ of a module M is good if and only if $\operatorname{gr}^{\Gamma} M$ is finitely generated over $\operatorname{gr}^{\mathcal{F}} A_n$. In this case M is necesserally finitely generated over A_n.*

c. *Let $\{\Gamma_j\}$, $\{\Gamma'_j\}$ be two good filtrations of a module M. There exists j_0 such that $\Gamma_{j-j_0} \subset \Gamma'_j \subset \Gamma_{j+j_0}$ for all j.*

1.5. Characteristic Variety of an A_n-module. Let M be a finitely generated A_n-module, $\{\Gamma_j\}$ be a good filtration of M with respect to the standard filtration $\{\Sigma_j\}$ on A_n. Let $I = \operatorname{Ann} \operatorname{gr}^{\Gamma} M \subset \operatorname{gr}^{\Sigma} A_n$ (i.e. I is the ideal of $P \in \operatorname{gr}^{\Sigma} A_n$ such that $Pu = 0$ for some $0 \ne u \in \operatorname{gr}^{\Gamma} M$) and let $J(M) = \sqrt{I}$

be the radical of I. Viewing $\operatorname{gr}^\Sigma A_n$ as the ring of polynomials in $2n$ variables $\overline{x_j}$, $\overline{\partial_j}$, denote by $\operatorname{ch} M \subset \mathbb{C}^{2n}$ the variety of zeros of the ideal $J(M)$. By 1.4.1.c, $J(M)$ and $\operatorname{ch} M$ (but not I) depend only on M, and not on $\{\Gamma_j\}$. The variety $\operatorname{ch} M$ is called the *characteristic variety* of the A_n-module M.

1.6. Relations Between $\operatorname{ch} M$ **and Cohomological Properties of** M. Associate to any A_n-module M the following numbers:

$$\delta(M) = \dim \operatorname{ch} M \ (= \text{the Krull dimension of } \operatorname{gr}^\Sigma A_n/J(M)),$$
$$j(M) = \min\{j : \operatorname{Ext}^j_{A_N}(M, A_n) \neq 0\}.$$

1.6.1. Theorem. *We have* $j(M) + \delta(M) = 2n$.

1.7. The Plan of the Proof of Theorem 1.6.1. a. Denote for brevity $S = \operatorname{gr}^\Sigma A_n$. Let Γ be a good (with respect to Σ) filtration on M. First of all, dimension theory of graded finitely generated modules over commutative regular algebras shows that the statement similar to Theorem 1.6.1 holds for the S-module $\operatorname{gr} M = N$. Namely, denoting

$$\delta'(N) = \text{the Krull dimension of } \operatorname{Ann}_S N,$$
$$j'(N) = \min\{j : \operatorname{Ext}^j_S(N, S) \neq 0\}$$

(Ext is taken in the category of graded S-modules), we have $j'(N) + \delta'(N) = 2n$.

b. It is clear from the definition that $\delta'(\operatorname{gr} M) = \delta(M)$. Therefore, to prove the theorem it suffices to verify that $j'(\operatorname{gr} M) = j(M)$. To do this we construct a spectral sequence that converges to $\operatorname{Ext}^j(M, A_n)$.

c. *Spectral sequence.* We claim that there exists a spectral sequence with

$$E_1^{pq} = \operatorname{Ext}_S^{p+q}(\operatorname{gr} M, S[-p])$$

(here $S[-p]$ is the free S-module with one generator of degree p) converging to

$$E_\infty^{p+q} = \operatorname{Ext}^{p+q}_{A_n}(M, A_n).$$

The existence of this spectral sequence immediately implies the desired equality $j'(\operatorname{gr} M) = j(M)$. To construct such a sequence we proceed as follows.

There exists a free resolution of M by filtered A_n-modules

$$\cdots \to L^{-2} \to L^{-1} \to L^0 \to M \to 0$$

such that

$$\cdots \to \operatorname{gr} L^{-2} \to \operatorname{gr} L^{-1} \to \operatorname{gr} L^0 \to \operatorname{gr} M \to 0$$

is a resolution of $\operatorname{gr} M$ by free graded S-modules and the S-rank of $\operatorname{gr} L^{-i}$ equals to the A_n-rank of L^{-i}.

Define the descending filtration on $K^i = \operatorname{Hom}_{A_n}(L^{-i}, A_n)$ by the formula $F^p K^i = \{\varphi : L^{-i} \to A_n, \varphi(\Gamma_k(L^{-i})) \subset \mathcal{F}_{k-p}\}$. One can easily verify that the differentials $d : L^{-i} \to L^{-i+1}$ induces on $\{K^i\}$ the structure of a filtered complex, and the desired spectral sequence is the spectral sequence associated to this filtered complex.

d. In fact, the analysis of the just constructed spectral sequence enables us not only to prove that $j'(\operatorname{gr} M) = j(M)$, but also to establish the following statement (using a similar result for S-modules):

$$\delta(\operatorname{Ext}^j_{A_n}(M, A_n)) \leq 2n - 1 \quad \text{for all} \quad j,$$

with the equality for $j = j'(\operatorname{gr} M) = j(M)$; here $\operatorname{Ext}^j_{A_n}(M, A_n)$ is considered as a right A_n-module with the structure induced by the right action of A_n on itself.

1.8. Remark. Theorem 1.6.1 and the statement from 1.7.d hold in a more general situation when A_n is replaced by a filtered ring R such that $\operatorname{gr} R$ is a regular commutative Noetherian ring of pure dimension s, and M is replaced by a finitely generated R-module with a good filtration (and $2n$ must be replaced by s).

In particular, these results hold when we consider the ring A_n with the filtration $\{B_j\}$ instead of $\{\Sigma_j\}$. Denoting the corresponding Krull dimension by $d(M)$, we obtain the equality $d(M) + j(M) = 2n$ (since $j(M)$ does not depend on filtration), so that, in particular, $d(M) = \delta(M)$. On the other hand, for $d(M)$ we can prove an important inequality called Bernstein inequality (see 1.10).

1.9. Hilbert Polynomial. In the next two subsections we will consider the Bernstein filtration $\{B_j\}$ on A_n. For this filtration we have

$$\dim B_j = j^{2n}/(2n)! + \text{terms of lower order in } j.$$

Let M be a finitely generated A_n-module with a good filtration $\{\Gamma_j\}$, so that $\operatorname{gr}^\Gamma M$ is a finitely generated module over the graded ring $S = \operatorname{gr}^B A_n = \mathbb{C}[\overline{x_1}, \dots, \overline{x_n}, \overline{\partial_1}, \dots, \overline{\partial_n}]$ (with $\deg \overline{x_j} = \deg \overline{\partial_j} = 1$). According to the dimension theory, there exists a polynomial

$$\chi(M, \Gamma, t) = \frac{m}{d!} t^d + O(t^{d-1}), \quad m, d \in \mathbb{Z}_+,$$

such that

$$\dim \Gamma_j = \sum_{i=0}^{j} \dim \Gamma_j / \Gamma_{j-1} = \chi(M, \Gamma, j) \quad \text{for} \quad j \gg 0.$$

Moreover, $d = d(M)$ (Krull dimension of the S-module $\operatorname{Ann}_S \operatorname{gr}^\Gamma M$) and m do not depend on the filtration $\{\Gamma_j\}$ (they are called respectively the dimension and the multiplicity of M).

1.10. Bernstein Inequality. This surprising property of A_n was proved by Bernstein (1971). It can be formulated as follows.

Let $M \neq \{0\}$ be a finitely generated A_n-module. Then $d(M) \geq n$.

In particular, not every S-module is of the form $\operatorname{gr}^\Gamma M$ for some A_n-module M.

We present here the proof of the Bernstein inequality given by A. Joseph.

Let $\{\Gamma_j\}$ be a good filtration of M. We can, of course, assume that $\Gamma_0 \neq \{0\}$. Let us prove, first of all, that the $\mathbb{C}$-linear mapping

$$B_i \to \operatorname{Hom}_{\mathbb{C}}(\Gamma_i, \Gamma_{2i}), \quad P \mapsto (f \to Pf), \qquad P \in A_n,\ f \in M,$$

is an embedding. We use the induction in i. For $i = 0$ the statement follows from $\Gamma_0 \neq \{0\}$.

Let $0 \neq P \in B_i$. We must prove that $P\Gamma_i \neq \{0\}$. Let us assume, on the contrary, that $P\Gamma_i = \{0\}$. Then P cannot be a constant, hence the expression

$$P = \sum a_{\alpha\beta} x^\alpha \partial^\beta$$

contains either some x_j, or some ∂_j. In the first case, $[P, \partial_j] \neq 0$, $[P, \partial_j] \in B_{i-1}$, and $[P, \partial_j]M = \{0\}$, which contradicts the induction hypothesis for $i - 1$. In the second case we must take ∂_j instead of x_j. Let now $\chi(t) = \chi(M, \Gamma, t)$ be the Hilbert polynomials, so that

$$\chi(M, \Gamma, i) = \dim \Gamma_i \quad \text{for} \quad i \gg 0.$$

Then for $i \gg 0$ we have

$$\frac{1}{(2n)!} i^{2n} + O(i^{2n-1}) = \dim B_i \leq \dim \operatorname{Hom}_{\mathbb{C}}(\Gamma_i, \Gamma_{2i}) = \chi(i)\chi(2i)$$

hence $\deg \chi \geq n$, so that $d(M) \geq n$.

1.10.1. Involutivity. Since, by 1.8, $d(M) = \delta(M)$, Bernstein inequality implies

$$\delta(M) = \dim \operatorname{ch} M \geq n \quad \text{for} \quad M \neq \{0\}.$$

This inequality has a far reaching (and much more difficult) generalization (Sternberg-Guillemin-Malgrange-Gabber, see (Björk 1979; Gabber 1981; Malgrange 1979)):

$$\operatorname{ch} M \subset \operatorname{spec} \mathbb{C}[\overline{x_1}, \ldots, \overline{x_n}, \overline{\partial_1}, \ldots, \overline{\partial_n}] = \mathbb{C}^{2n}$$

is an involutive subvariety with respect to the symplectic structure defined by the form $\omega = \sum d\overline{x_j} \wedge d\overline{\partial_j}$.

1.11. Homological Dimension of A_n Equals n. All the above can be equally applied to right A_n-modules. For any left A_n-module M the space $\operatorname{Ext}^j_{A_n}(M, A_n)$ is naturally endowed with the structure of a right A_n-module. Formula (1) and the Bernstein inequality for right A_n-modules imply that

$$\operatorname{Ext}^j_{A-N}(M, A_n) = 0 \quad \text{for} \quad j > n.$$

Next, using a free resolution of an arbitrary A_n-module N, one can prove that

$$\operatorname{Ext}^j_{A_n}(M, N) = 0 \quad \text{for} \quad j > n$$

for any two left (or right) A_n-modules with a finitely generated M. Finally,

$$\operatorname{dh} A_n \leq n.$$

Results of the next subsection will imply that dh A_n equals exactly n.

1.12. Holonomic Modules. By 1.6.1 and 1.11, for an arbitrary A_n-module M the groups $\operatorname{Ext}^j_{A_n}(M, A_n)$ can be different from 0 only if $2n - d(M) \leq j \leq n$. Those modules for which this interval is reduced to only one value $j = n$ are called *holonomic*. Hence, an A_n-module M is said to be holonomic if either $M = \{0\}$, or $d(M) = n$.

Modules in examples **a** and **b** in 1.1 are clearly holonomic.

Let us list some properties and examples of holonomic modules.

a. Submodules and factormodules of holonomic modules are holonomic.

b. Let M be an arbitrary A_n-module and $\{\Gamma_j\}$ be a filtration of M that is compatible with $\{B_j\}$ and satisfies $\dim \Gamma_j \leq \frac{c}{n!} j^n + c_1 j^{n-1}$. Then M is holonomic and $m(M) \leq c$. In particular, M is finitely generated (for the proof it suffices to verify that the length of the strictly increasing chain of submodules of M does not exceed n).

c. The length of a holonomic module M does not exceed $m(M)$ (in particular, M is a cyclic module).

Further examples of holonomic modules see below in 1.14.

1.13. Duality for Holonomic Modules. For any left (resp. right) holonomic A_n-module M set

$$M^* = \operatorname{Ext}^n_{A_N}(M, A_n).$$

This is a right (resp. left) A_n-module which is called dual of M. Its properties:

a. M^* is a holonomic A_n-module.

b. $M \to M^*$ is an exact functor from the category of left (resp. right) holonomic A_n-modules to the category of right (resp. left) holonomic A_n-modules which establishes an equivalence of these categories.

Property **a** follows essentially from (1), property **b** follows from the long exact sequence of Ext's.

1.14. More Examples and Properties of Holonomic Modules.

a. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $M = \mathbb{C}[x_1, \ldots, x_n, f^{-1}]$ with the action of A_n given by the formal differentiation of rational functions. Then M is holonomic (for the proof set $\Gamma_j = \{q(x)f(x)^{-j} \deg q \leq j(\deg f + 1)\}$ and apply **b**). The property $\cup \Gamma_j = M$ follows from $qf^{-1} \in \Gamma_{j+\deg q}$.

b. More generally, let M be a holonomic module. Then $M[f^{-1}] \stackrel{\text{def}}{=} M \underset{\mathbb{C}[x_1,\ldots,x_n]}{\otimes} \mathbb{C}[x_1, \ldots, x_n, f^{-1}]$ is also holonomic.

c. Let us remark that, although in some sense holonimic A_n-modules are "the smallest" A_n-modules, there exist simple nonholonomic A_n-modules. Two types of examples are constructed by Stafford (1985) and by Bernstein and Lunts (1988). In both cases $n = 2$ and $M = A_2/A_2P$ (so that $d(M) = 3$).

In the first paper $P = x_1 + x_2 + \partial_1 + x_1\partial_1\partial_2$, in the second paper P is a generic homogeneous (in $x_1, x_2, \partial_1, \partial_2$) element of degree $k \geq 4$.

1.15. Bernstein Polynomial. Let $f \in \mathbb{C}[x_1, \dots, x_n]$, and λ be a transcendental variable over $\mathbb{C}[x_1, \dots, x_n]$. Consider the Weyl algebra $A_n(\mathbb{C}(\lambda))$ over the field of rational functions in λ. Let $N = \mathbb{C}(\lambda)[x_1, \dots, x_n, f^{-1}]f^\lambda$ be the module over $A_n(\mathbb{C}(\lambda))$ generated over $\mathbb{C}(\lambda)[x_1, \dots, x_n, f^{-1}]$ by one generator f^λ, on which the elements ∂_j act by formal differentiation: $\partial_j f^\lambda = \lambda f^{-1}\frac{\partial f}{\partial x_j} f^\lambda$. Introducing in N a filtration similarly to 1.14.d, we see that N is a holonomic $A_n(\mathbb{C}(\lambda))$-module.

Let $N_j \subset N$, $j = 0, 1, 2, \dots,$, be an $A_n(\mathbb{C}(\lambda))$-submodule generated by $f^j f^\lambda$. Since N has a finite length, $N_k = N_{k+1}$ for some k, so that

$$f^k f^\lambda = P_1(\lambda)^{k+1} f^\lambda, \quad P_1(\lambda) \in A_n(\mathbb{C}(\lambda)).$$

replacing λ by $\lambda - k$ (λ is transcendental!) we obtain

$$f^\lambda = P_1(\lambda - k) f^{\lambda+1}.$$

Let $B(\lambda) \in \mathbb{C}[\lambda]$ be the common denominator of all coefficients (they belong to $\mathbb{C}(\lambda)$) of the operator $P_1(\lambda - k)$ and $P(\lambda) = B(\lambda)P_1(\lambda - k)$. Then

$$P(\lambda) f^{\lambda+1} = B_1(\lambda) f^\lambda, \qquad P(\lambda) \in A_n(\mathbb{C}[\lambda]), \quad B(\lambda) \in \mathbb{C}[\lambda]. \tag{2}$$

The existence of elements $P(\lambda)$ and $B(\lambda)$ is the Bernstein theorem. It is clear that all possible $B(\lambda)$ (for a given f) form an ideal in $\mathbb{C}[\lambda]$. The generator $b(\lambda)$ of this ideal is called the *Bernstein polynomial* for f. Let us remark that (2) implies $N = N_0$, so that N is generated by f^λ over $A_n(\mathbb{C})$.

1.15.1. Theorem. *Let $f \in \mathbb{C}[x_1, \dots, x_n]$. Then*

$$M = A_n(\mathbb{C})[\lambda] f^\lambda / A_n(\mathbb{C})[\lambda] f \cdot f^\lambda$$

is a holonomic $A_n(\mathbb{C})$-module, and the minimal polynomial of multiplication by λ is b.

§ 2. Algebraic $\mathcal{D}$-Modules

Everywhere below (unless mentioned otherwise) by X we mean a smooth complex algebraic variety.

2.1. The Sheaf $\mathcal{D}_x$. Let $U \subset X$ be an open set. A *differential operator of order $\leq n$ on U* is a linear mapping $P : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ satisfying the condition

$$[[[P, f_1], f_2], \dots, f_{n+1}] = 0$$

for any $f_1, \dots, f_{n+1} \in \mathcal{O}_X(U)$.

Denote by $\mathcal{D}_X(U)(n)$ the set of all differential operators of order $\leq n$ on U. The union $\mathcal{D}_X(U) = \bigcup_n \mathcal{D}_X(U)(n)$ is a ring, called the *ring of linear algebraic differential operators on U*. The increasing chain of subspaces

$$\{0\} = \mathcal{D}_X(U)(-1) \subset \mathcal{D}_X(U)(0) \subset \dots$$

determines on $\mathcal{D}_X(U)$ the structure of a filtered ring.

We have $\mathcal{D}_X(U)(0) = \mathcal{O}_X(U)$ (operators of the multiplication by a function). Next, an arbitrary element of $\mathcal{D}_X(U)(1)$ can be represented (uniquely) in the form $P = \theta + f$, where $f \in \mathcal{D}_X(U)(0)$ and θ is a vector field on X, i.e. a $\mathbb{C}$-differentiation of the ring $\mathcal{O}_X(U)$.

If U is an affine open subset of X, $\mathcal{D}_X(U)$ is generated by $\mathcal{D}_X(U)(1)$. If $U \subset V$ and both U, V are affine, we have

$$\mathcal{D}_X(V) = \mathcal{O}_X(V) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{D}_X(U).$$

Therefore there exists a unique $\mathcal{O}_X$-quasicoherent sheaf of rings $\mathcal{D}_X$ whose sections over any open affine $U \subset X$ coincide with $\mathcal{D}_X(U)$. This sheaf is called the *sheaf of germs of algebraic linear differential operators on* X. In the same way one defines the subsheaves $\mathcal{D}_X(n)$, which determine the structure of a sheaf of filtered rings on $\mathcal{D}_X$. We have $\mathcal{D}_X(0) = \mathcal{O}_X$ and each $\mathcal{D}_X(n)$ is a coherent $\mathcal{O}_X$-module.

If $X = \mathbb{C}^n$ is the n-dimensional affine space, $\mathcal{D}_X(X)$ coincides with the Weyl algebra $A_n(\mathbb{C})$ (∂_i is the vector field $\partial/\partial x_i$). In particular, $\mathcal{D}_X(X)$ is generated by $\mathcal{O}_X(X)$ and n pairwise commuting everywhere linearly independent vector fields. One can show that these facts remain true locally in Zariski topology for any smooth X.

2.2. $\mathcal{D}_X$-modules. Denote by $\mathcal{M}_X$ the abelian category of sheaves of (left) modules over $\mathcal{D}_X$.

Examples of $\mathcal{D}_X$-modules:

a. $M = \mathcal{O}_X$.

b. $M = \mathcal{D}_X$.

c. *Flat connections.* A connection ∇ on a quasicoherent $\mathcal{O}_X$-module M is a mapping of the $\mathcal{O}_X$-module Θ_X of vector fields on X to $\mathrm{Hom}_{\mathbb{C}}(M, M)$, $\xi \mapsto \nabla_\xi$, which is $\mathcal{O}_X$-linear:

$$\nabla_{f\xi+g\eta} = f\nabla_\xi + g\nabla_\eta, \qquad f, g \in \mathcal{O}_X, \quad \xi, \eta \in \Theta_X,$$

and satisfies the Leibniz rule:

$$\nabla_\xi(fm) = (\xi f)m + f\nabla_\xi m, \qquad m \in M, \quad \xi \in \Theta_X, \quad f \in \mathcal{O}_X.$$

A connection ∇ is said to be *flat* if

$$[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi,\eta]}.$$

One can easily see that the structure of a $\mathcal{D}_X$-module on M is equivalent to the structure of a flat connection ∇ on M.

Usually a flat connection is defined on a locally free $\mathcal{O}_X$-module of finite rank (i.e. on a sheaf of germs of sections of an algebraic vector bundle). The next proposition characterizes such sheaves.

2.3. Proposition. *A $\mathcal{D}_X$-module is $\mathcal{O}_X$-coherent if and only if it is locally free of finite rank over $\mathcal{O}_X$.*

Proof. This is a local problem. Let $x \in X$, $s_1, \dots, s_p$ be sections of M over some neighborhood of x such that their images in the finite-dimensional vector space $\overline{M_x} = M_x/m_x M_x$ (here m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at the point x) form a basis of $\overline{M_x}$. By the Nakayama lemma, $s_1, \dots, s_p$ generate M over $\mathcal{O}_X$ in a neighborhood of the point x. We must only verify that $s_1, \dots, s_p$ are linearly independent over $\mathcal{O}_X$. If

$$\sum \varphi_i s_i = 0, \quad \varphi_i \in \mathcal{O}_X, \tag{1}$$

is a linear relation among s_i, then necessarily $\varphi_i(x) = 0$ for all i. Let us choose among all relations (1) the one with the minimal value of $\nu = \min_i\{$the order of zero of φ_i at $x\}$. Let in this relation $\nu = \operatorname{ord} \varphi_1$. One can easily see that there exists a local vector field ∂ on X such that $\operatorname{ord}(\partial\varphi_i) = \nu - 1$. Then

$$0 = \partial\Big(\sum \varphi_i s_i\Big) = (\partial\varphi_1)s_1 + \sum_{i=2}^{p}(\partial\varphi_i)s_i + \sum \varphi_i(\partial s_i).$$

Expressing ∂s_i in terms of s_i, we obtain a relation of the form

$$\sum \psi_i s_i = 0$$

with $\operatorname{ord}\psi_1 = \nu - 1$, contradicting the choice of n.

2.4. The Sheaf Ω_X. Left and Right $\mathcal{D}_X$-modules. Denote by $\mathcal{M}_X^R$ the category of right $\mathcal{D}_X$-modules that are quasicoherent as $\mathcal{O}_X$-modules. A natural method to pass from $\mathcal{M}_X$ to $\mathcal{M}_X^R$ and back can be described as follows.

Let Ω_X be the sheaf of germs of algebraic differential forms on X of the highest ($= \dim X$) order. This is a locally $\mathcal{O}_X$-free sheaf of rank 1. A vector field ξ on X acts on Ω_X as the Lie derivative L_ξ. We have

$$\begin{aligned} [L_\xi, L_\eta] &= L_{[\xi,\eta]}, \\ L_{f\xi}\omega &= fL_\xi\omega - L_\xi(f\omega). \end{aligned} \tag{2}$$

These formulas show that Ω_X is a right $\mathcal{D}_X$-module with respect to the action

$$\omega \circ \xi = -L_\xi\omega, \quad \omega \circ f = f\omega.$$

For any $\mathcal{F} \in \mathcal{M}_X$ set

$$\Omega(\mathcal{F}) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \Omega_X,$$

and define the right action of $\mathcal{D}_X$ on $\Omega(\mathcal{F})$ as follows:

$$\begin{aligned} (u \otimes \omega)f &= fu \otimes \omega = u \otimes f\omega, \quad f \in \mathcal{O}_X, \\ (u \otimes \omega)\xi &= -\xi u \otimes \omega - u \otimes L_\xi\omega, \quad \xi \in \Theta_X. \end{aligned}$$

Formulas (2) show that $\Omega(\mathcal{F}) \in \mathcal{M}_X^R$.

One can easily verify that Ω determines an equivalence between $\mathcal{M}_X$ and $\mathcal{M}_X^R$. A quasi-inverse functor is

$$\mathcal{G} \to \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{G}) = \mathcal{G} \underset{\mathcal{O}_X}{\otimes} \Omega_X^{-1}.$$

2.5. Coherent $\mathcal{D}_X$-modules. According the a general definition, a $\mathcal{D}_X$-module is said to be *coherent* if it can be represented locally as the cokernel of a morphism of free $\mathcal{D}_X$-modules of finite rank. Denote by Coh_X the full subcategory of $\mathcal{M}_X$ formed by coherent modules.

2.5.1. Theorem. a. *The sheaf of rings $\mathcal{D}_X$ is coherent and locally Noetherian.*

b. *The homological dimension of the stalk $\mathcal{D}_{X,x}$ at any point $x \in X$ equals* $\dim X$*; the homological dimension of $\mathcal{D}_X$ does not exceed* $2\dim X$.

Part **a** follows from general properties of filtered rings (and sheaves of rings). The first statement in **b** follows from a similar result for the Weyl algebra A_n (see 1.11); the second part follows from the Serre-Grothendieck theorem about cohomology of X and from the spectral sequence of the composition of functors $\mathrm{Hom} = \Gamma \circ \mathcal{H}om$.

2.6. Characteristic Variety. The graded sheaf of rings $\mathrm{gr}\,\mathcal{D}_X$ associated to the filtration $\{\mathcal{D}_X(n)\}$ on $\mathcal{D}_X$ is clearly isomorphic to the sheaf $\pi_\cdot(\mathcal{O}_{T^*X}^{\mathrm{pol}})$, where T^*X is the cotangent bundle of X, $\pi : T^*X \to X$ is the projection, $\mathcal{O}_{T^*X}^{\mathrm{pol}}$ is the sheaf of germs of regular functions on T^*X that are polynomial along the fibers of π.

Let $M \in \mathrm{Coh}_X$ and let x be a point in X. Similarly to 1.4.1.a, we can assert that for some affine neighborhood U of x in X, $M(U)$ has a filtration that is good with respect to the filtration $\{\mathcal{D}_X(n)(U)\}$ on $\mathcal{D}_X(U)$. The corresponding graded module $\mathrm{gr}\,M(U)$ is finitely generated over $\mathrm{gr}\,\mathcal{D}_X(U) = \pi_\cdot(\mathcal{O}_{T^*X}^{\mathrm{pol}})(U)$. The subvariety $(\mathrm{ch}\,M)\,|_U \subset \pi^{-1}(U)$ corresponding to the ideal $J = \sqrt{\mathrm{Ann}\,\mathrm{gr}\,M(U)}$ does not depend on the choice of a good filtration (similarly to 1.5). Therefore the subvarieties $(\mathrm{ch}\,M)\,|_U$ for various $U \subset X$ can be glued together to a subvariety $\mathrm{ch}\,U \subset T^*X$, called the *characteristic variety* of the coherent $\mathcal{D}_X$-module M.

For $x \in X$ denote $d_x(M) = \inf\limits_{U, x \in X} \dim(\mathrm{ch}\,M \cap \pi^{-1}(U))$ and let $j_x(M)$ be the smallest j such that $\mathrm{Ext}^j_{\mathcal{D}_{X,x}}(M_x, \mathcal{D}_{X,x}) \neq 0$.

2.7. Theorem. a. $\mathrm{ch}\,M$ *is a conical subvariety of* T^*X *(i.e.* $\mathrm{ch}\,M$ *is invariant under dilations in fibers $\pi^{-1}(x)$, $x \in X$).*

b. $j_x(M) + d_x(M) = 2\dim X$.

c. *If $M_x \neq 0$ then* $d_x(M) \geq \dim X$.

d. $\mathrm{ch}\,M$ *is an involutive subvariety (i.e., the tangent space to* $\mathrm{ch}\,M$ *at each smooth point contains its orthogonal complement with respect to the standard symplectic form in* T^*X*).*

Part **a** is clear. Parts **b** and **c** follow from the corresponding statements for the Weyl algebra. Part **d** is a difficult theorem by Stenberg-Guillemin-Malgrange-Gabber-Sato-Kashiwara-Kawai. Known proofs of this theorem include the proof by Sato, Kashiwara, Kawai (1973), the proof by Malgrange (1979) (see also (Björk 1979)), where the action of generic symplectic transformations of T^*X on the ring $\mathcal{D}_X$ is analysed, and the interpretation of this proof in algebraic language by Gabber (1981).

2.8. Examples. a. $M = \mathcal{D}_X$, $\operatorname{ch} M = T^*X$.

b. $M = \mathcal{O}_X$, $\operatorname{ch} M = X \subset T^*X$ (the zero section).

c. M is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank with a locally flat connection, $\operatorname{ch} M \subset X$ is the support of $\mathcal{E}$ (equal to X if X is irreducible).

d. $M = \mathcal{D}_x/m_x\mathcal{D}_x$, $x \in X$ (distributions supported at the point x, cf. 1.1.b), $\operatorname{ch} M = \pi^{-1}(x)$ (the fiber over x).

2.9. Derived Categories. Most constructions and results of the theory of $\mathcal{D}$-modules can be most naturally formulated in the context of derived categories. Denote by $D^b(\mathcal{D}_X\text{-mod})$, $D^b(\mathcal{M}_X)$, $D^b(\mathrm{Coh}_X)$ the bounded derived categories of all (left) $\mathcal{D}_X$-modules, $\mathcal{O}_X$-quasicoherent $\mathcal{D}_X$-modules, and coherent $\mathcal{D}_X$-modules respectively. It is clear that the embeddings $\mathrm{Coh}_X \to \mathcal{M}_X \to \mathcal{D}_X$-mod determine the functors

$$D^b(\mathrm{Coh}_X) \to D^b(\mathcal{M}_X) \to D^b(\mathcal{D}_X\text{-mod}). \tag{3}$$

Denote by $D^b_{qc}(\mathcal{D}_X\text{-mod})$ and $D^b_{\mathrm{Coh}}(\mathcal{D}_X\text{-mod})$ the full subcategories of the category $D^b(\mathcal{D}_X\text{-mod})$ formed by complexes with cohomology in $\mathcal{M}_X$ and Coh_X respectively.

2.10. Theorem. *The functors* $\alpha : D^b(\mathrm{Coh}_X) \to D^b(\mathcal{D}_X\text{-mod})$ *and* $\beta : D^b(\mathcal{M}_X) \to D^b(\mathcal{D}_X\text{-mod})$ *determine equivalencies of triangulated categories*

$$D^b(\mathrm{Coh}_X) \to D^b_{\mathrm{Coh}}(\mathcal{D}_X\text{-mod}),$$
$$D^b(\mathcal{M}_X) \to D^b_{qc}(\mathcal{D}_X\text{-mod}).$$

The proof of the second statement was obtained by Bernstein. Its main steps are as follows.

One can easily see that $\mathcal{O}_X$-quasicoherent $\mathcal{D}_X$-modules generate the category $D^b_{qc}(\mathcal{D}_X\text{-mod})$, so that we must only prove that β is a fully faithful functor, i.e. that

$$\mathrm{Hom}_{D^b(\mathcal{M}_X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = \mathrm{Hom}_{D^b(\mathcal{D}_X\text{-mod})}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

for any $\mathcal{F}^\cdot, \mathcal{G}^\cdot \in \mathrm{Ob}\, D^b(\mathcal{M}_X)$. In fact, a similar equality holds even when we replace of Hom by $R\,\mathrm{Hom}$. The verification is first reduced to the case when X is affine. Next, using the appropriate resolutions of $\mathcal{F}^\cdot$ and $\mathcal{G}^\cdot$ everything is reduced to the case when $\mathcal{F}^\cdot$ and $\mathcal{G}^\cdot$ are complexes supported in one degree with $\mathcal{F}$ free and $\mathcal{G}$ injective. In this case the statement follows from the Serre theorem about the cohomology of quasicoherent sheaves on affine varieties.

The proof of the first statement follows from the fact that each object in $\mathcal{M}_X$ is the direct limit of subobjects belonging to Coh_X.

2.11. Resolutions. To compute various functors acting in derived categories of $\mathcal{D}_X$-modules it is useful to have conditions that guarantee the existence of particular resolutions of $\mathcal{D}_X$-modules. The results we will need later can be summarized as follows.

a. Any object $\mathcal{F} \in \mathcal{M}_X$ has a right injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^m \to 0$$

with $m \leq 2\dim X + 1$.

b. Let us assume that X is quasi-projective and $m \geq \dim X$. Then any object $\mathcal{F} \in \mathcal{M}_X$ has a left resolution

$$0 \to \mathcal{P}^{-m} \to \mathcal{P}^{-m+1} \to \cdots \to \mathcal{P}^0 \to \mathcal{F} \to 0$$

with $\mathcal{P}^{-i}$ locally free for $0 \leq i \leq m-1$ and $\mathcal{P}^{-m}$ locally projective.

Standard arguments show that any $\mathcal{F} \in \mathrm{Ob}\, D^b(\mathcal{M}_X)$ admits a bounded right injective resolution, a bounded left projective resolution, and a bounded left flat (over $\mathcal{D}_X$) resolution.

2.11.1. The de Rham Complex. For any $\mathcal{D}_X$-module M the corresponding *de Rham complex* $\Omega^{\cdot}(M)$ is the complex

$$\Omega^{\cdot}(M) : 0 \to M \to \Omega^1_X \underset{\mathcal{O}_X}{\otimes} M \overset{d^1}{\to} \cdots \to \Omega^{\dim X}_X \underset{\mathcal{O}_X}{\otimes} M \to 0,$$

where Ω^1_X is the sheaf of germs of algebraic i-forms on X. In local coordinates the differential d^i is defined by the formula

$$d^i(\omega \otimes m) = d\omega \otimes m + \sum_l (dx \wedge \omega) \otimes \frac{\partial}{\partial x_l} m.$$

One easily verify that d^i does not depend on the choice of local coordinates and that $d^{i+1}d^i = 0$.

Considering, in particular, the de Rham complex $\Omega^{\cdot}(\mathcal{D}_X)$ we obtain (after the shift by $\dim X$) a left resolution $\Omega^{\cdot}(\mathcal{D}_X)[\dim X]$ of $\Omega_X = \Omega_X^{\dim X}$ formed by locally free right $\mathcal{D}_X$-modules. This resolution is called the *de Rham resolution.*

Concluding this section we describe an analog of the notion of an affine variety in the category of $\mathcal{D}_X$-modules.

2.12. Definition. The variety X is said to be *$\mathcal{D}$-affine* if the functor $\mathcal{F} \to \Gamma(X, \mathcal{F})$ is exact on the the category $\mathcal{M}_X$.

Formal arguments (similar to those used in algebraic geometry for the description of $\mathcal{O}_X$-modules on affine schemes) show that for a $\mathcal{D}$-affine variety X the following results hold:

(i) Any sheaf $\mathcal{F} \in \mathcal{M}_X$ is generated by its global sections.
(ii) $\mathcal{D}_X$ is a projective generator of $\mathcal{M}_X$.

It is clear that any affine variety is also $\mathcal{D}$-affine. However, there exist important $\mathcal{D}$-affine varieties that are not affine.

2.12.1. Theorem. *Let Y be a (possibly degenerate) flag variety of a complex semisimple Lie group G (that is, $Y = G/P$, where $P \subset G$ is a parabolic subgroup), and Z is an arbitrary affine variety. Then $X = T \times Z$ is $\mathcal{D}$-affine. In particular, the projective space $P^n(\mathbb{C})$ is $\mathcal{D}$-affine.*

2.12.2. Relations to Representation Theory. Let again $Y = G/P$ be a flag variety of a semisimple complex Lie group G. The action of the group G on Y yields a homomorphism τ of the Lie algebra $\mathfrak{g}$ of G to the Lie algebra Θ of global vector fields on Y. This homomorphism is naturally extended to a homomorphism $\overline{\tau} : U(\mathfrak{g}) \to \mathcal{D}_Y(Y)$ of the universal enveloping algebra $U(\mathfrak{g})$ to the algebra of global differential operator on Y. One can easily prove (either directly or together with the proof of Theorem 2.12.1) that $\overline{\tau}$ is an epimorphism, so that global differential operator on Y are generated by constants and $\tau(\mathfrak{g})$. Moreover, it is rather easy to describe the kernel of $\overline{\tau}$.

Together with Theorem 2.12.1, this result enables us to reduce, at least to some extent, the study of $U(\mathfrak{g})$-modules (i.e. of representations of G) with kernels containing the kernel of $\overline{\tau}$ to the study of $\mathcal{D}_Y$-modules.

§ 3. Inverse Image

3.1. About Notations. Below we will work with direct and inverse images in various categories of sheaves (of abelian groups, or $\mathcal{O}$-modules, or $\mathcal{D}$-modules).

Let $f : X \to Y$ be a morphism of algebraic varieties. The categories of sheaves on X and on Y are related by the following functors:

a. Sheaves of abelian groups (in complex analytic topology):

$f_\cdot : \mathbf{SAb}_X \to \mathbf{SAb}_Y$ (the direct image),

$f^\cdot : \mathbf{SAb}_Y \to \mathbf{SAb}_X$ (the inverse image),

$f_! : \mathbf{SAb}_X \to \mathbf{SAb}_Y$ (the direct image with compact support),

$f^! : D^b(\mathbf{SAb}_Y) \to D^b(\mathbf{SAb}_X)$ (the extraordinary inverse image).

By $Rf_\cdot$, $Rf_!$, $f^\cdot$ we denote the corresponding functors between derived categories (let us recall that $f_\cdot$ and $f_!$ are left exact and $f^\cdot$ is exact). All four functors preserve the subcategories of constructible sheaves.

b. The sheaves of $\mathcal{O}$-modules:

$$f_* : \mathcal{O}_X\text{-mod} \to \mathcal{O}_Y\text{-mod} \quad \text{(the direct image)},$$
$$f^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod} \quad \text{(the inverse image)}.$$

Here $f_* = f_\cdot$ is left exact, f^* is right exact, and the corresponding derived functors are denoted by Rf_* and Lf^*.

3.2. The Action of Vector Fields on $f^*(M)$. Let $M \in \mathcal{M}_Y$. Viewing M as an $\mathcal{O}_X$-module, we can define $f^*(M)$. Our goal in this subsection is to make $f^*(M)$ a $\mathcal{D}_X$-module. To do this it suffices to define the action of vector fields on X on the sheaf $f^*(M)$. We define such an action locally in Y. Let $x \in X$, $y = f(y)$. In some neighborhood of y there exists $d_Y = \dim Y$ functions y_i and d_Y pairwise commuting vector fields ∂_i such that the differentials of y_i at y are linearly independent (so that y_i are local coordinates in some analytic neighborhood of y) and $[\partial_i, x_j] = \delta_{ij}$ (so that $\partial_i = \frac{\partial}{\partial y_i}$).

For a vector field ξ on Y and a section $\gamma = \varphi \otimes f^{\cdot} m$ of the sheaf $f^*(M) = \mathcal{O}_X \underset{f^{\cdot}\mathcal{O}_Y}{\otimes} f^{\cdot} M$ we set

$$\xi\gamma = \xi\varphi \otimes f^{\cdot} m + \sum_i \varphi \xi(y_i \circ f) \otimes f^{\cdot}(\partial_i m). \tag{1}$$

One can easily see that $\xi\gamma$ is well defined: $\xi(\varphi f^{\cdot}(\psi) \otimes f^{\cdot} m)) = \xi(\varphi \otimes f^{\cdot}(\psi m))$, $\psi \in \mathcal{O}_Y$, and that $\xi\gamma$ does not depend on the choice of (y_i, ∂_i). To verify the last assertion it suffices to use the coordinate change formula (locally near y) and the Leibniz rule.

Let us remark that formula (1) expresses a simple fact that if ξ is tangent to fibers of f, then $\xi(1 \otimes f^{\cdot} m) = 0$.

3.3. Lemma. (i) *The above action of vector fields makes $f^*(M)$ a $\mathcal{D}_X$-module; denote this module by $f^{\nabla}(M)$.*

(ii) *$M \to f^{\nabla}(M)$ extends to a right exact functor $f^{\nabla} : \mathcal{M}_Y \to \mathcal{M}_X$.*

(iii) *If $g : Y \to Z$ is another morphism of algebraic varieties, we have $(gf)^{\nabla} = f^{\nabla} g^{\nabla}$; moreover, $\mathrm{id}^{\nabla} = \mathrm{id}$.*

The proof reduces to easy verifications.

3.4. The Sheaf $\mathcal{D}_{X\to Y}$. Define

$$\mathcal{D}_{X\to Y} = f^{\nabla}(\mathcal{D}_Y) = \mathcal{O}_X \underset{f^{\cdot}(\mathcal{O}_Y)}{\otimes} f^{\cdot}(\mathcal{D}_Y).$$

This sheaf is a left $\mathcal{D}_X$-module and a right $f^{\cdot}(\mathcal{D}_Y)$-module. Using $\mathcal{D}_{X\to Y}$, we can define $f^{\nabla}(M)$ as follows:

$$f^{\nabla}(M) = \mathcal{D}_{X\to Y} \underset{f^{\cdot}(\mathcal{D}_Y)}{\otimes} f^{\cdot}(M).$$

The property (ii) of Lemma 3.3 is now equivalent to the equality

$$\mathcal{D}_{X\to Z} = \mathcal{D}_{X\to Y} \underset{f^{\cdot}(\mathcal{D}_Y)}{\otimes} f^{\cdot}(\mathcal{D}_{Y\to Z}). \tag{2}$$

3.5. The Functor f^D. Passing to derived categories we obtain the functor $Lf^{\nabla} : D^b(\mathcal{M}_Y) \to D^b(\mathcal{M}_X)$ defined by the formula

$$LF^{\nabla}(M^{\cdot}) = \mathcal{D}_{X\to Y} \underset{f^{\cdot}(\mathcal{D}_Y)}{\overset{L}{\otimes}} f^{\cdot}(M^{\cdot})$$

(since $f^{\cdot}$ is exact). However, there exists a lot of reasons to shift the degree (and to change the notation) and to consider the functor $f^D : D^b(\mathcal{M}_Y) \to D^b(\mathcal{M}_X)$ defined by the formula

$$f^D(M^{\cdot}) = \mathcal{D}_{X\to Y} \overset{L}{\underset{f^{\cdot}(\mathcal{D}(Y))}{\otimes}} f^{\cdot}(M^{\cdot})[d_{X,Y}]$$

(here $d_{X,Y} = \dim X - \dim Y$). For $M \in \mathcal{M}_Y$ the cohomology of the complex $f^D(M)$ can be computed using any of the following two formulas:

$$H^i(f^D(M)) = \mathrm{Tor}_{i-d_{X,Y}}^{f^{\cdot}(\mathcal{D}_Y)}(\mathcal{D}_{X\to Y}, f^{\cdot}(M)),$$
$$H^i(f^D(M)) = \mathrm{Tor}_{i-d_{X,Y}}^{f^{\cdot}(\mathcal{O}_Y)}(\mathcal{O}_X, f^{\cdot}(M)).$$

3.6. Proposition. *In the setup of Lemma* 3.3(iii) *we have* $(gf)^D = f^D g^D$, $\mathrm{id}^D = \mathrm{id}$.

The proof uses Lemma 3.3(iii) and the computation of the corresponding derived functors using flat resolutions.

3.7. Examples. a. $f : X \to Y$ is an open embedding. One can easily verify that in this case $f^{\nabla}(M) = f^{\cdot}(M) = M$ (as a module over $\mathcal{D}_X = \mathcal{D}_Y|_X$), so that f^{∇} is an exact functor and $f^D = Lf^{\nabla} = f^{\nabla}$. The functor f^D preserves coherence and holonomicity of $\mathcal{D}$-modules.

b. Let $X = Y \times Z$, $f : X \to Y$ be the projection to the first factor. Denote by $g : X \to Y$ the projection to the second factor. We have

$$\mathcal{D}_X = f^{\cdot}\mathcal{D}_Y \underset{\mathbb{C}}{\otimes} g^{\cdot}\mathcal{D}_Z. \tag{3}$$

One can easily verify that $f^{\nabla}(M) = M \underset{\mathbb{C}}{\otimes} g^{\cdot}\mathcal{O}_Z$ with the structure of an $\mathcal{D}_X$-module defined by the decomposition (3). Therefore the functor f^{∇} is exact. Hence, f^D coincides with f^{∇} up to the translation by $d_Z = \dim Z$, that is

$$H^i(f^D(M^{\cdot})) = f^{\nabla}(H^{i+d_Z}M^{\cdot})), \qquad M^{\cdot} \in D^b(\mathcal{M}_X).$$

Again one can easily verify that f^{∇} preserves coherence and holonomicity.

c. The most interesting case of a closed embedding will be discussed at the end of the next section (Kashiwara theorem).

§ 4. Direct Image

4.1. Direct Image of Right $\mathcal{D}$-modules. Let $f : X \to Y$ be a morphism of algebraic varieties. The main with the definition of the direct image of a $\mathcal{D}$-module is that in general f does not yield a morphism of ringed spaces $(X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ (contrary to what we have had for $\mathcal{O}$-modules). However, f determines a morphism of ringed spaces

$$\widetilde{f} : (X, f^{\cdot}\mathcal{D}_X) \to (Y, \mathcal{D}_Y)$$

(this is true for an arbitrary sheaf of rings on Y), which will be used to define the direct image. First we give a more natural definition of the direct image of a right $\mathcal{D}$-module (about left modules see 4.5). In doing so we must work from the very beginning with derived categories.

4.2. Definition. For $N^{\cdot} \in D^b(\mathcal{M}_X^R)$ set

$$f_+^{(R)} N^{\cdot} = R\widetilde{f}_{\cdot}(N^{\cdot} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X\to Y}).$$

Here $\mathcal{D}_{X\to Y}$ is a $(\mathcal{D}_X, f^{\cdot}\mathcal{D}_Y)$-bimodule defined in 3.5.

4.3. Properties of $f_+^{(R)}$. a. $f_+^{(R)}$ is an exact functor between triangulated categories $D^b(\mathcal{M}_X^R) \to D^b(\mathcal{M}_Y^R)$.

For the proof we must verify that $f_+^{(R)}$ has $\mathcal{D}$-quasicoherent cohomology. This is first done in the case when both X and Y are affine (using a free left resolution of $N^{\cdot}$); to pass to the general case we use the Čech spectral sequence for the affine covering of $f^{-1}(U)$ for an affine $U \subset Y$.

Other statements in **a** are obvious.

b. In the situation $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have

$$(g \circ f)_+^{(R)} = g_+^{(R)} \circ f_+^{(R)}.$$

The proof uses the projection formula relating $R\widetilde{f}$ and $f^{\cdot}$, the isomorphism (2) from Sect. 3, and the formula

$$R(\widetilde{g \circ f})_{\cdot} = R\widetilde{g}_{\cdot} \circ R\widetilde{f}_{\cdot}.$$

4.4. Remark. The functor $f_+^{(R)}$ is the composition of the left derived functor $\cdot \overset{L}{\otimes} \mathcal{D}_{X\to Y}$ and the right derived functor $R\widetilde{f}_{\cdot}$. In particular, $f_+^{(R)}$ is not, in general, the derived functor neither of its zero cohomology $H^0(f_+^{(R)})$, nor of the functor $N \to \widetilde{f}_{\cdot}(N^{\cdot} \otimes \mathcal{D}_{X\to Y})$ (and these two functors are, in general, distinct). Moreover, neither of these two functors possesses a property similar to 4.3.b.

4.5. Direct Image of Left $\mathcal{D}$-modules. The passage from the right $\mathcal{D}$-modules to the left ones is done using the construction from 2.5. To do this it is useful to introduce a certain sheaf $\mathcal{D}_{Y\leftarrow X}$ on X, which is a left $f^{\cdot}\mathcal{D}_Y$-module and a right $\mathcal{D}_X$-module. Namely, define $\widetilde{\mathcal{D}_Y} = \mathcal{D}_Y^r \underset{\mathcal{O}_Y}{\otimes} \Omega_Y^{-1}$, where $\mathcal{D}_Y^r$ is the sheaf $\mathcal{D}_Y$ considered as the right $\mathcal{D}_Y$-module under the adjoint action. By 2.5, $\widetilde{\mathcal{D}_Y}$ has two commuting structures of a left $\mathcal{D}_Y$-module: one arising from 2.5 (structure 1) and the other arising from the left adjoint action on $\mathcal{D}_Y$ (structure 2). Next, let

$$\mathcal{D}_{Y \leftarrow X} = \Omega_X \underset{f^{\cdot}\mathcal{O}_Y}{\otimes} f^{\cdot}\widetilde{\mathcal{D}_Y},$$

where the tensor product is taken with respect to structure 2.

The structure of the left $f^{\cdot}\mathcal{D}_Y$-module on $\mathcal{D}_{Y \leftarrow X}$ arises from the structure 1 on $\widetilde{\mathcal{D}_Y}$. The structure of the right $\mathcal{D}_X$-module arises from the structure of the left $\mathcal{D}_X$-module on

$$f^{\nabla}(\mathcal{D}_Y) = \mathcal{O}_X \underset{f^{\cdot}\mathcal{O}_Y}{\otimes} f^{\cdot}\mathcal{D}_Y$$

corresponding to structure 2, and further application of 2.5.

Now for $M^{\cdot} \in D^b(\mathcal{M}_X)$ we set

$$f_+ M^{\cdot} = R\widetilde{f_{\cdot}}\big(\mathcal{D}_{Y \leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} M^{\cdot}\big).$$

One can easily verify that this definition of the direct image for left $\mathcal{D}$-modules agrees with the Definition 4.2 for right $\mathcal{D}$-modules. Moreover, f_+ is a functor $D^b(\mathcal{M}_X) \to D^b(\mathcal{M}_Y)$ and for $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have

$$(g \circ f)_+ = g_+ \circ f_+. \tag{1}$$

4.6. Direct Image for Open Embeddings. Let $f : X \to Y$ be an open embedding. Then $\Omega_X = \Omega_Y|_X$, $\mathcal{D}_X = \mathcal{D}_Y|_X$ and $\mathcal{D}_{Y \leftarrow X} = \mathcal{D}_X$ as a $(\mathcal{D}_Y, \mathcal{D}_X)$-bimodule, so that f defines a morphism of ringed spaces $f : (X, \mathcal{D}_Y) \to (Y, \mathcal{D}_Y)$. Hence f_+ is given by the formula

$$f_+ = Rf_{\cdot} = Rf_*,$$

so that f_+ is the right derived functor of the left exact functor $f_{\cdot}$. This fact, together with 3.6, implies that f_+ is left adjoint to f^D, so that there exists a canonical morphism of functors $\mathrm{Id} \to f_+ f^D$.

Next, for $M \in \mathcal{M}_X$ we have

$$H^i f_+(M) = 0 \quad \text{for } \ i < 0,$$
$$H^i f_+(M)_x = \varinjlim_{U,\, x \in U} H^i(X \cap U, M), \quad x \in X, \qquad \text{for } \ i \geq 0,$$

where U run over the system of affine neighborhoods of x.

If X is affine, then f_* is an exact functor, so that $f_+ = Rf_* = f_*$ and

$$H^i f_+(M) = \begin{cases} M & \text{for } \ i = 0, \\ 0 & \text{for } \ i \neq 0. \end{cases}$$

Let us remark here, that for a generic open embedding f the functor f_+ does not preserve the coherence over $\mathcal{D}$ (an example: X is the complement to a hyperplane in Y); see, however, 4.10.c and 5.6.1.

4.7. Direct Image for a Smooth Morphism. For a smooth variety X denote by DR_X the de Rham complex

$$\Omega_X^0 \xrightarrow{d} \Omega_X^1 \to \cdots \to \Omega_X^{\dim X},$$

where Ω_X^i is the sheaf of germs of algebraic i-forms on X.

For an arbitrary $\mathcal{D}_X$-module M denote by $DR_X(M)$ the de Rham complex of M, namely $DR_X(M) = DR_X \otimes M$.

Next, for a smooth morphism $f : X \to Y$ denote by $\Omega^i_{X/Y}$, $0 \leq i \leq d_{X,Y} = \dim X - \dim Y$ the sheaf of germs of relative i-forms, by $DR^{\cdot}_{X/Y}$ the relative de Rham complex, and by $DR_{X/Y}(M)$, $M \in \mathcal{D}_X$-mod, the complex

$$DR^{\cdot}_{X/Y} \underset{\mathcal{O}_X}{\otimes} M.$$

It is clear that $DR_{X/Y}(\mathcal{D}_X)$ is a complex of locally free right $\mathcal{D}_X$-modules.

Since $\mathcal{D}_{Y \leftarrow X} = \mathcal{D}_{X \to Y} \underset{\mathcal{O}_X}{\otimes} \Omega^{d_{X,Y}}_{X,Y}$, there exists a natural morphism $\varepsilon : \Omega^{d_{X,Y}}_{X,Y} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \mathcal{D}_{Y \leftarrow X}$, which commutes with the right action of $\mathcal{D}_X$.

4.8. Lemma. *ε defines a left resolution*

$$DR_{X/Y}(\mathcal{D}_X)[d_{X,Y}] \to \mathcal{D}_{Y \leftarrow X}$$

of the module $\mathcal{D}_{Y \leftarrow X}$ by locally free right $\mathcal{D}_X$-modules.

The proof can be easily obtained using computations in local coordinates on X such that f is the projection of the direct product to one of the factors.

The lemma implies that for a smooth morphism $f : X \to Y$ the direct image f_+M, $M \in \mathcal{M}_X$, is given by the formula

$$f_+M = R\tilde{f}_{\cdot}(DR_{X/Y}(M))[d_{X,Y}].$$

Let us remark, however, that this formula defines f_+M as a complex of $\mathcal{O}_X$-modules. The action of vector fields on f_+M is given by rather complicated formulas. These formulas become somewhat simpler in the case when f is a projection to the direct factor (using the action of lifted vector fields on M).

4.9. Direct and Inverse Image for Closed Embeddings. Let $f : X \to Y$ be a closed embedding. Define the following functors:

$$f_\circ : \mathcal{M}_X \to \mathcal{M}_Y, \quad f_\circ(M) = f_{\cdot}(\mathcal{D}_{Y \leftarrow X} \underset{\mathcal{D}_X}{\otimes} M),$$
$$f^\circ : \mathcal{M}_Y \to \mathcal{M}_X, \quad f^\circ(M) = \mathrm{Hom}_{f^{\cdot}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, f^{\cdot}M).$$

The relations between these functors and the relations of these functors with the the earlier functors of the direct and inverse image is as follows.

a. $f_\circ$ is exact, f° is left exact, $f_\circ$ is left adjoint to f°.

b. $Rf_\circ = f_+$, $Rf^\circ = f^D$.

The following very important theorem of Kashiwara shows that $f_\circ$ and f° establishes an equivalence between $\mathcal{M}_X$ and some subcategory of $\mathcal{M}_Y$. Namely, for a closed subvariety $Z \subset Y$ denote by $\mathcal{M}_Y(Z)$ the full subcategory of $\mathcal{M}_Y$ formed by modules with support in Z.

4.9.1. Theorem. *Let $f: X \to Y$ be a closed embedding. Then the functors $f_\circ : \mathcal{M}_X \to \mathcal{M}_Y(X)$, $f^\circ : \mathcal{M}_Y(X) \to \mathcal{M}_X$ are quasi-inverse to each other, so that they establish an equivalence of categories.*

Sketch of the proof. The statement is local, so that we can assume that Y is affine and X is given by equations $\varphi_1 = \cdots = \varphi_d = 0$. The induction by d reduces the verification to the case when x is a hypersurface in Y defined by one equation $\varphi = 0$.

Choose a vector field θ on Y such that $[\theta, \varphi] = 1$ (locally this is always possible) and define $I = \varphi\theta$.

Let $\mathcal{F} \in \mathcal{M}_Y(X)$. Since $\mathcal{F}$ is quasicoherent, any section ξ satisfies $\varphi^k \xi = 0$ for a sufficiently large k.

Denote $\mathcal{F}_i = \{\xi \mid I\xi = i\xi\}$. Then $\varphi : \mathcal{F}_i \to \mathcal{F}_{i+1}$, $\theta : \mathcal{F}_{i+1} \to \mathcal{F}_i$, and one can easily verify that these morphisms are isomorphisms for $i < -1$. Next, the induction by k, together with the formula $\theta\varphi - \varphi\theta = 1$, shows that if $\varphi^k \xi = 0$ then $\varphi \in \mathcal{F}_{-1} \oplus \cdots \oplus \mathcal{F}_{-k}$. Hence, $\mathcal{F} = \bigoplus_{i=1}^{\infty} \mathcal{F}_{-1} = \mathbb{C}[\theta] \otimes \mathcal{F}_{-1}$ and $\ker \varphi \mid \mathcal{F} = \mathcal{F}_{-1}$. Therefore, the mappings $\mathcal{F} \to \mathcal{F}_{-1}$ and $\mathcal{G} \to \mathbb{C}[\theta] \otimes \mathcal{G}$ are mutually inverse isomorphisms between $\mathcal{M}_Y(X)$ and $\mathcal{M}_X$. One can easily verify that these morphisms coincide with f° and $f_\circ$.

4.10. Comments to the Kashiwara Theorem. a. This theorem illustrates the following essential difference between $\mathcal{D}$-modules and $\mathcal{O}$-modules. In the category of $\mathcal{O}$-modules a sheaf $\mathcal{F}$ supported on a closed subvariety $X \subset Y$ has an important invariant: the smallest degree of the ideal J_X of X in Y which annihilates $\mathcal{F}$ (so that, for example, all sheaves $\mathcal{O}_Y/J_X^k$, $k = 1, 2, \ldots$, are distinct, although they have a similar behavior along X). In the category of $\mathcal{D}$-modules differentiations in the directions transversal to X make this degree equal ∞ (from all sheaves $\mathcal{O}_Y/J_X^k$ one can form only one sheaf on which $\mathcal{D}_Y$ acts, namely the sheaf $\varinjlim \mathcal{O}_Y/J_X^k$).

b. The Kashiwara theorem yields simple proofs of statements from 2.4 and 2.15.

c. Let $f: X \to Y$ be a proper morphism. Then $f_+(D^b(\mathrm{Coh}_X)) \subset D^b(\mathrm{Coh}_Y)$ Indeed, any proper morphism is a composition of a closed embedding and a proper smooth morphism. Therefore our statement follows from (1), the Kashiwara theorem, and the description 4.7 of f_+ for a smooth proper morphism.

d. Let $f: X \to Y$ be a closed embedding. Then

$$\mathrm{ch}\, f_\circ(M) = \{(y, \eta) \mid y \in X;\ (y, \mathrm{Pr}_{T^*Y \to T^*X}\eta) \in \mathrm{ch}\, M\}.$$

In particular, $f_\circ(M)$ is holonomic if and only if M is holonomic. Since $f_\circ$ is exact and $f_+ = f_\circ$, the complex $f_+(M^\cdot)$ is holonomic if and only if M is holonomic.

4.11. The Functor $\Gamma_{[Z]}$. Canonical Decomposition. Let $i: Z \to Y$ be a closed embedding, $j: U \to Y$ be the embedding of the complementary open set. In 5.5 of Chap. 4 we have defined the functor $\Gamma_{[Z]} : \mathbf{SAb}_Y \to \mathbf{SAb}_X$. Recalling 3.7.a, we obtain a distinguished triangle

$$R\Gamma_{[Z]}M^{\cdot} \to M^{\cdot} \to j_+ j^D M^{\cdot}, \tag{2}$$

which is functorial in $M^{\cdot} \in D^b(\mathcal{M}_Y)$. It is called the *canonical decomposition* of $M^{\cdot}$ with respect to (Z, U).

4.11.1. Lemma. *We have*

$$R\Gamma_{[Z]} = i^D i_+$$

and the morphism $R\Gamma_{[Z]}M^{\cdot} \to M^{\cdot}$ *in* (2) *coincides with the adjunction morphism* $i^D i_+ \to \mathrm{Id}$ (*see* 4.6).

The proof is based on a straightforward, although cumbersome, verification of the formula $\Gamma_{[Z]} = i^D i_+$.

4.12. The Base Change Theorem. *Let*

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{p} & S \end{array}$$

be a commutative diagram of morphisms of algebraic varieties. Then the functors $p^{\nabla} f_+$ *and* $g_+ q^{\nabla}$ *from* $D^b(\mathcal{M}_Y)$ *to* $D^b(\mathcal{M}_X)$ *are naturally isomorphic. In particular, if* $Z = \emptyset$, *i.e.* $p(X) \cap f(Y) = \emptyset$, *then* $p^{\nabla} f_+ = 0$.

The proof reduces to the analysis of the following two cases: a) $f : T \times S \to S$ is the projection, b) f is a closed embedding. The first case is studied directly. In the second case one must use the base change theorem in the category **SAb** and the Kashiwara theorem.

§ 5. Holonomic Modules

5.1. Definition. a. A module $M \in \mathcal{M}_X$ is said to be *holonomic* if either $M = \{0\}$ or $\dim \mathrm{ch}\, M = \dim X$ (cf. 1.12).

b. A complex $M^{\cdot} \in D^b(\mathcal{D}\text{-mod})$ is said to be *holonomic* if all $H^i(M^{\cdot})$ are holonomic modules.

The category of holonomic modules is denoted $\mathbf{Hol}_X$; the category of holonomic complexes (the full subcategory of $D^b(\mathcal{D}_X\text{-mod})$) is denoted $D^b_{\mathbf{Hol}}(\mathcal{D}_X)$.

5.2. Properties of Holonomic Modules and Complexes. a. In an exact sequence $0 \to M' \to M \to M'' \to 0$ the module M is holonomic if and only if M' and M'' are holonomic. Therefore, the category $\mathbf{Hol}_X$ is abelian.

b. The embedding $\mathbf{Hol}_X \to \mathcal{D}_X\text{-mod}$ gives the functor $D^b(\mathbf{Hol}_X) \to D^b_{\mathbf{Hol}}(\mathcal{D}_X)$. A difficult theorem of Beilinson (see (Beilinson 1987b)) claims that this functor is an equivalence of categories (the proof uses results from Sect. 2 of Chap. 7).

5.3. Duality. The duality functor is naturally defined on the category $D^b_{\mathrm{Coh}}(\mathcal{D}_X)$ of complexes with coherent cohomology. Let us recall that in 4.5 we have defined the sheaf $\widetilde{\mathcal{D}_X}$ with two structures of the left $\mathcal{D}_X$-module.

5.3.1. Definition. For $M^\cdot \in D^b(\mathcal{D}_X\text{-mod})$ define

$$\Delta_X M^\cdot = RHom_{\mathcal{D}_X}(M^\cdot, \widetilde{\mathcal{D}_X})[\dim X],$$

where $RHom_{\mathcal{D}_X}$ is taken with respect to structure 1 on $\widetilde{\mathcal{D}_X}$.

Structure 2 enables us to consider $\Delta_X M^\cdot$ as an object of $D^b(\mathcal{D}_X\text{-mod})$. Therefore, to compute $\Delta_X M^\cdot$ we must replace $M^\cdot$ by a quasi-isomorphic complex $P^\cdot = \{\ldots \to P_{-1} \to P_0 \to P_1 \to \ldots\}$ consisting of locally projective (or locally free) $\mathcal{D}_X$-modules, and set $\Delta_X M^\cdot = \{\ldots \to Q_{-1} \to Q_0 \to Q_1 \to \ldots\}$, where

$$Q_i = Hom_{\mathcal{D}_X}(P_{-i-\dim X}, \widetilde{\mathcal{D}_X}).$$

It is clear that $\Delta_X M^\cdot$ defines a functor

$$\Delta_X : D^b(\mathcal{D}_X\text{-mod}) \to D^b(\mathcal{D}_X\text{-mod}).$$

This functor satisfies $\Delta_X \circ \Delta_X = \mathrm{Id}$.

5.4. Duality for Coherent Modules. Considering $M \in \mathrm{Coh}_X$ as a $\mathcal{O}_X$-module, we have

$$H^i(\Delta_X M) = Ext^{i+\dim X}_{\mathcal{D}_X}(M, \widetilde{\mathcal{D}_X})$$

(here $Ext^i_{\mathcal{D}_X}$ is the i-th derived functor of $RHom$). The study of duality for holonomic modules is based on the following theorem by J.-E. Roos.

5.4.1. Theorem. *Let* $M \in \mathrm{Coh}_X$. *Then*

a. $\dim \mathrm{ch}(Ext^i_{\mathcal{D}_X}(M, \mathcal{D}_X)) \le 2\dim X - i$.

b. $Ext^i_{\mathcal{D}_X}(M, \mathcal{D}_X) = 0$ *for* $i < 2\dim X - \dim \mathrm{ch}\, M$.

Since the statements are local in X, we can assume that X is affine and M possesses a good filtration. Using this filtration, we reduce the statements of the theorem to the corresponding statements for the graded module $\mathrm{gr}\, M$ over the graded ring $\mathrm{gr}\, \mathcal{D}_X = \mathcal{O}^{\mathrm{pol}}_{T^*X}$ (see 2.8), where they follow from standard dimension theory.

5.5. Corollary. a. $\Delta_X(D^b_{\mathrm{Coh}}(\mathcal{M}_X)) \subset D^b_{\mathrm{Coh}}(\mathcal{M}_X)$.

b. *Let* $M \in \mathrm{Coh}_X$. *Then the complex* $\Delta_X \mu$ *is supported in dimensions between* $-\dim X$ *and* 0, *i.e.* $H^i(\Delta_X M) = 0$ *for* $i < \dim X$ *or* $i > 0$.

c. M *is holonomic if and only if* $H^i(\Delta_X M) = 0$ *for* $i \ne 0$.

d. Δ_X *yields an autoequivalence* $\Delta_X : \mathbf{Hol}^\circ_X \to \mathbf{Hol}_X$, *i.e.* $\Delta_X \circ \Delta_X = \mathrm{Id}_{\mathbf{Hol}_X}$. *In particular, the functor* Δ_X *is exact on* $\mathbf{Hol}_X$.

The first statement follows from the standard properties of $Ext_{\mathcal{D}_X}$. Let us remark that if M is not coherent over $\mathcal{D}_X$, then the cohomology of $\Delta_X M$ is not even quasicoherent.

The second and third statements follow from the Roos theorem in 5.4.1.

The fourth statement follows immediately from the third and from the remark at the end of 5.3.1.

Let us also point out that part b) of the theorem implies the existence of a locally projective resolution of the length $\leq \dim X$ for an arbitrary coherent $\mathcal{D}_X$-module.

5.6. Holonomic Modules and Functors. The next theorem asserts that the holonomicity is preserved under the direct and the inverse image (Sect. 3,4). Namely,

5.6.1. Theorem. *For* $f : X \to Y$ *we have*
a. $f^D(D^b_{\mathbf{Hol}}(\mathcal{D}_Y)) \subset D^b_{\mathbf{Hol}}(\mathcal{D}_X)$;
b. $f_+(D^b_{\mathbf{Hol}}(\mathcal{D}_X)) \subset D^b_{\mathbf{Hol}}(\mathcal{D}_Y)$.

The proof is based on the following Key Lemma.

5.6.2. Lemma. *If* $i \colon X \to Y$ *is a locally closed embedding, then* $i_+(D^b_{\mathbf{Hol}}(\mathcal{D}_X)) \subset D^b_{\mathbf{Hol}}(\mathcal{D}_Y)$.

Below we indicate the main steps of the proof of Theorem 5.6.1, postponing the discussion of the proof of the main lemma and of the related notion of b-function until 5.10.

5.7. The Proof of 5.6.1.a. Each morphism f of algebraic varieties can be represented as a composition of a projection and a closed embedding, so it is sufficient to consider these two cases. If f is a projection, then, by 3.6.b, f^∇ is an exact functor preserving holonomicity. Since f^D differs from f^∇ only by a shift of degree, we have $f^D(D^b_{\mathbf{Hol}}(\mathcal{D}_Y)) \subset D^b_{\mathbf{Hol}}(\mathcal{D}_X)$.

Let now $f : X \to Y$ be a closed embedding. Consider the canonical decomposition of $M^{\cdot} \in D^b_{Hol}(\mathcal{D}_Y)$,

$$f_+F^DM^{\cdot} \to M^{\cdot} \to j_+j^DM^{\cdot},$$

where $j : U \to Y$ is the open embedding of the complement of X. Since the complex $j^DM^{\cdot} = M^{\cdot}\,|_U$ is holonomic, $j_+J^DM^{\cdot}$ is holonomic by the Key Lemma 5.6.2. Hence $f_+F^DM^{\cdot}$ is also holonomic, and by 4.10.d, $f^DM^{\cdot}$ is holonomic.

5.8. The Criterion for Holonomicity. This criterion asserts that a complex $M^{\cdot} \in D^b_{\mathrm{Coh}}(\mathcal{D}_X)$ is holonomic if and only if for any point $x \in X$ its stalk $i_x^D M^{\cdot}$ at x (where $i_x : x \to X$ is the embedding) has finite-dimensional cohomology.

The necessity of this condition follows from 5.6.1.a, since for $\mathcal{D}$-modules over a point holonomicity is equivalent to the finite-dimensionality.

To establish the sufficiency we must prove the following lemma.

5.8.1. Lemma. *Let $M \in \mathrm{Coh}_X$. There exists an open dense $U \subset X$ such that $M\mid_U$ is a locally free $\mathcal{D}$-module.*

The verification of the lemma can be reduced (since the statement is local) to the case when M is endowed with a good filtration. In this case $\operatorname{gr} M = \oplus M_i/M_{i-1}$ is a finitely generated module over the finitely generated $\mathcal{O}_X$-algebra $\operatorname{gr}\mathcal{D}_X$. Hence $\operatorname{gr} M$ is free over $\mathcal{O}_U$ for some open dense $U \subset X$. Therefore all M_i/M_{i-1} are projective over $\mathcal{O}_U$, so that $M \cong \operatorname{gr} M$ as an $\mathcal{O}_U$-module, so it is free over $\mathcal{O}_U$.

Returning to the verification of the sufficiency of the holonomicity criterion, we use the induction in $\dim S$, where $S = \operatorname{supp} M^\cdot$. By Lemma 5.8.1, there exists an open $Y \subset S$ such that $\dim(S \setminus Y) < \dim S$, and all cohomologies of $M^\cdot\mid_Y$ are free over $\mathcal{O}_Y$. Since the stalks of cohomology sheaves over points of Y are finite-dimensional, the cohomology of $M^\cdot\mid_Y$ are $\mathcal{O}_Y$-coherent, so that $M^\cdot\mid_Y \in D^b_{\mathbf{Hol}}(\mathcal{D}_Y)$. By the Key Lemma 5.6.2, $j_+M^\cdot\mid_Y$ is also holonomic (here j is the embedding $Y \to S$). Now the canonical decomposition of $M^\cdot\mid_S$ with respect to $(Y, S \setminus Y)$ reduces the verification of the holonomicity of $M^\cdot$ to complexes with the support inside the set $S \setminus Y$, whose dimension is less than that of S.

Let us remark that together with the proof of the criterion we have proved the following statement.

A complex $M^\cdot$ of $\mathcal{D}_X$-modules is holonomic if and only if there exists a smooth stratification $X = \cup S_\alpha$ of X such that all complexes $M^\cdot_\alpha = j^D_\alpha M^\cdot$ (where $j_\alpha : S_\alpha \to X$ is an embedding) are $\mathcal{O}_{S_\alpha}$-coherent (i.e., their cohomologies are $\mathcal{O}_{S_\alpha}$-coherent).

5.9. The Proof of 5.6.1.b. Since the case of a closed embedding is covered by the Key Lemma, it suffices to consider the case when f is a projection. In this case the verification reduces to the application of the criterion from 5.8 and the Base Change Theorem 4.12 for the diagram

$$\begin{tikzcd} T_y \arrow[r, hook] \arrow[d] & T \times Y \arrow[d] \\ y \arrow[r, hook] & Y \end{tikzcd}$$

5.10. The Extension Lemma. In the proof of the Key Lemma 5.6.2 the following extension lemma is used.

Let $U \subset X$ be an open subset, $N \in \mathcal{D}_X$-mod, M be a holonomic $\mathcal{D}_U$-submodule of $N\mid_U$. Then N contains a holonomic submodule N' such that $N'\mid_U = M$.

In the proof we can assume that $M = N\mid_U$. After that we must set $N' = \Delta_X(H^0(\Delta_X N))$ and use the Roos theorem 5.4.1.

5.11. The Proof of the Key Lemma. The case of a closed embedding is considered in 4.10.d. Hence, we can assume that $f : X \to Y$ is an open embedding. Next, we can assume that Y is affine and M is a holonomic $\mathcal{D}_X$-module generated by one section ξ. Using the Čech resolution associated with an affine covering $\{X_\alpha\}$ of X we came to the following situation:

Y is affine, φ is a function on Y, $X = \{y, \varphi(y) \neq 0\}$, $i : X \to Y$ is the embedding, M is a holonomic $\mathcal{D}_X$-module generated by one section ξ. We must prove that i_+M is holonomic (note that in this case $i_+ = i_\bullet$ is an exact functor).

An essential part of the proof is the verification of the coherence of i_+M. It is clear that as a $\mathcal{D}_Y$-module, i_+M is generated by the sections $\varphi^{-n}\xi$, $n \geq 0$. Hence the coherence of i_+M follows from the existence of n_0 such that

$$\varphi^{-n-1}\xi \in \mathcal{D}(Y)(\varphi^{-n}\xi)$$

for any $n > n_0$ (here $\mathcal{D}(Y)$ denotes the algebra of global sections of $\mathcal{D}_Y$ on Y).

This statement is similar to Theorem 1.9, and its proof is also similar. We must extend the scalars to the field $\mathbb{C}(\lambda)$ of rational functions in one variable, consider the extended modules $\widehat{M} = M \otimes \mathbb{C}(\lambda)$ over $\mathcal{D}(X) \otimes \mathbb{C}(\lambda)$ and $\widehat{N} = (i_+M)[\varphi^\lambda]$ over $\mathcal{D}(Y) \otimes \mathbb{C}(\lambda)$, and use the extension lemma 5.10.

Similarly to Theorem 1.9, in the proof of the coherence we obtain simultaneously that $\widehat{N}$ is holonomic and is generated over $\mathcal{D}(X) \otimes \mathbb{C}(\lambda)$ by one section $\varphi^\lambda\xi$, while i_+M is generated by $\varphi^{-n}\xi$ for any $n > n_0$. Next, by the holonomicity of $\widehat{N}$ over $\mathcal{D}(X) \otimes \mathbb{C}(\lambda)$, we can (substituting $\lambda = -n$ with $n > n_0$) find differential operators $P_1^{(n)}$ such that $P_1^{(n)}(\varphi^n) = 0$ and the set common zeros of its symbols is of dimension $\leq \dim X$. Hence i_+M is holonomic.

5.12. Functors f_D, f^+ and Their Properties. For an arbitrary morphism $f : X \to Y$ define the functors $f_D : D^b_{\mathbf{Hol}}(\mathcal{D}_X) \to D^b_{\mathbf{Hol}}(\mathcal{D}_Y)$, $f^+ : D^b_{\mathbf{Hol}}(\mathcal{D}_Y) \to D^b_{\mathbf{Hol}}(\mathcal{D}_X)$ by the formulas

$$f_D = \Delta_Y f_+ \Delta_X, \quad f^+ = \Delta_X f^D \Delta_Y.$$

By Theorem 5.6.1, these definitions make sense.

Hence, to any $f : X \to Y$ we have associated 4 functors between categories of holonomic complexes of $\mathcal{D}$-modules: 2 direct images f_+, f_D and 2 inverse images f^+, f^D.

The next theorem describes some properties of these functors.

5.12.1. Theorem. a. *There exists a canonical morphism of functors $f_D \to f_+$, which is an isomorphism for proper f.*

b. $f^D = f^+[2(\dim X - \dim Y)]$ *for a smooth morphism f.*

c. f_D *is left adjoint to f^D, f^+ is left adjoint to f_+.*

Let us make several comments about the proof of this theorem.

Assume first that $f = j$ is an open embedding. In this case $j^D = j^+$ is the restriction to an open set $X \subset Y$. Hence $j^+ = j^D$ is left adjoint to j_+ (see 4.6) so that $j_D = \Delta_Y j_+ \Delta_X$ is left adjoint to $j^D = \Delta_X j^+ \Delta_Y$.

Next, $j^+ \circ j_+ = \mathrm{Id}$, whence formally (using $\Delta_X \circ \Delta_X = \mathrm{Id}$) $j^+ \circ j_D = \mathrm{Id}$, i.e. the restriction of $j_D M^\cdot$ to X equals $M^\cdot$. This gives a natural morphism $j_D M^\cdot \to j_+ M^\cdot$, which is the identity on X.

So, it remains to prove parts **a** and **c** for proper f and part **b** for smooth proper f. (Let us remark also that the second statement in **c** follows from the first one). However, all these statements hold in a more general situation of coherent (and not only holonomic) complexes, since f_+ for a proper f and f^D for a smooth f map coherent complexes into coherent ones (see 4.10.c and 3.7.b).

Part **b** follows from the formula $RHom_{\mathcal{D}_X}(\mathcal{D}_{X\to Y}, \mathcal{D}_X) = \mathcal{D}_{Y\leftarrow X}[\dim Y - \dim X]$ valid for an arbitrary proper $f : X \to Y$.

The proof of parts **a** and **c** for proper morphisms uses the following duality theorem 5.13.

5.13. Theorem. *Let $f : X \to Y$ be a proper morphism. Then*

a. $\Delta_Y f_+ = f_D \Delta_X$;

b. f_+ *is left adjoint to* f^D.

The proof is based on the following proposition.

5.13.1. Proposition. *Let p be the projection of X to a point. Then for $M^\cdot \in D^b_{\mathrm{Coh}}(\mathcal{M}_X)$, $N^\cdot \in D^b(\mathcal{M}_X)$ we have*

$$p_+\big(\Delta_X M^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} N^\cdot\big) = RHom_{\mathcal{D}_X}(M^\cdot, N^\cdot)[\dim X]$$

(*the action of $\mathcal{D}_X$ in the tensor product over $\mathcal{O}_X$ is given by the Leibniz rule*).

5.14. Inner Hom and Tensor Troduct of $\mathcal{D}$-modules. Denote the functors "$\mathcal{D}$-tensor product"

$$\overset{D}{\otimes} : D^b(\mathcal{M}_X) \times D^b(\mathcal{M}_X) \to D^b(\mathcal{M}_X)$$

and "$\mathcal{D}$-inner *Hom*"

$$I\,\mathrm{Hom}^D : D^b_{\mathrm{Coh}}(\mathcal{D}_X)^\circ \times D^b_{\mathrm{Coh}}(\mathcal{D}_X) \to D^b(\mathcal{M}_X)$$

by the formulas

$$M^\cdot \overset{D}{\otimes} N^\cdot = (\mathrm{diag})^D(M^\cdot \boxtimes N^\cdot),$$

$$I\,\mathrm{Hom}^D(M^\cdot, N^\cdot) = \Delta_X(M^\cdot \overset{D}{\otimes} N^\cdot),$$

where $\mathrm{diag} : X \to X \times X$ is the diagonal embedding. Due to the equality

$$M^\cdot \overset{D}{\otimes} N^\cdot = M^\cdot \overset{L}{\underset{\mathcal{O}_X}{\otimes}} N^\cdot[\dim X],$$

Proposition 5.13.1 can be interpreted as the formula

$$R\operatorname{Hom}_{\mathcal{D}_X}(M^{\cdot}, N^{\cdot}) = p_+(I\operatorname{Hom}^D(M^{\cdot}, N^{\cdot})),$$

where p is the morphism to a point.

5.15. Irreducible Holonomic $\mathcal{D}$-modules. Let $f : X \to Y$ be a locally closed embedding. Define the morphism $f_{D+} : \mathbf{Hol}_X \to \mathbf{Hol}_Y$ by the formula

$$f_{D+}(M) = \operatorname{Im}\bigl(H^0(f_D(M)) \to H^0(f_+(M))\bigr).$$

If f is affine (in particular, if X is affine), then f_+ (hence f_D on **Hol**) is an exact functor, so that

$$f_{D+}(M) = \operatorname{Im}\bigl(f_D(M) \to f_+(M)\bigr).$$

5.15.1. Theorem. a. *Let $f : X \to Y$ be an affine embedding with irreducible X, and E be an irreducible $\mathcal{O}_X$-coherent $\mathcal{D}$-module. Then $f_{D+}(E)$ is an irreducible holonomic module. It can be characterized as a unique irreducible submodule of $f_+(E)$, or as a unique irreducible quotient of $f_D(E)$, or as a unique irreducible subquotient of $f_+(E)$ (or of $f_D(E)$) whose restriction to X is different from zero.*

b. *Any irreducible holonomic $\mathcal{D}_Y$-module M is of the form $f_{D+}(E)$ for some affine embedding $f : X \to Y$ with an irreducible X and some irreducible $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module E.*

Denote such a module by $L(X, E)$.

c. *$L(X, E)$ is isomorphic to $L(X', E')$ if and only if the closures of X and X' in Y coincide and $E\mid_U = E'\mid_U$ for some open dense $U \subset X \cap X'$.*

Let us sketch the proof of part **a** of the theorem. First of all, $f_+(E)$ and $f_D(E)$ are holonomic, so that they have a finite length. Let N be an irreducible submodule of $f_+(E)$. Then

$$\operatorname{Hom}_{\mathcal{D}_Y}(N, f_+(E)) = \operatorname{Hom}_{\mathcal{D}_X}(f^D(N), E) \neq 0,$$

and both $\mathcal{D}_X$-modules $f^D(N)$, E are irreducible, so that $f^D(N) = E$. Next, $f^D f_+(E) = E$, so that $f_+(E)$ has only one irreducible subquotient N such that $f^D(N) \neq 0$, hence only one irreducible submodule N.

Similarly, $f_D(E)$ has only one irreducible quotient N'. Finally,

$$\operatorname{Hom}(f_D(E), f_+(E)) = \operatorname{Hom}(E, f^D f_+(E)) = \mathbb{C}$$

and

$$\operatorname{Hom}(f_D(E), N) = \mathbb{C}.$$

Hence, $M = \operatorname{Im}\bigl(f_D(E) \to f_+(E)\bigr)$. The statement about N' is proved similarly.

The second and the third parts of Theorem 5.15.1 are proved using Lemma 5.8.1.

§ 6. Regular Connections

6.1. Integrable Connections. The most simple (from the geometrical point of view) $\mathcal{D}_X$-modules are $\mathcal{O}_X$-coherent modules, i.e. (see 2.2.c) integrable connections. Let us recall that a *connection* ∇ in a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ (the sheaf of germs of an algebraic vector bundle $\mathcal{F}$ on X) is a $\mathbb{C}$-linear mapping

$$\nabla : \mathcal{F} \to \Omega^1_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$$

satisfying the condition $\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f$ for sections $\varphi \in \mathcal{O}_X(U)$, $f \in \mathcal{F}(U)$. From ∇ one can naturally define the mappings

$$\nabla^{(p)} : \Omega^p_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F} \to \Omega^{p+1}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}, \quad \nabla^{(0)} = \nabla,$$

satisfying the condition $\nabla^{(p)}(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla^{(0)} f$. A connection is said to be *integrable* if $\nabla^{(p+1)}\nabla^{(p)} = 0$ for all p. Interpreting ∇ as a family of mappings $\nabla_\Phi : \mathcal{F} \to \mathcal{F}$, $\Phi \in \mathrm{Vect}_X$, given by $\nabla_\Phi = \langle \nabla f, \Phi \rangle$, one can easily see that the above definition of integrability is equivalent to the condition $\nabla_{[\Phi,\Psi]} = \nabla_\Phi \nabla_\Psi - \nabla_\Psi \nabla_\Phi$.

6.2. Analytic Manifolds. Similarly one can define analytic integrable connections in locally free sheaves on analytic manifolds. Such connections admit the following description.

6.2.1. Theorem. *Let $\widetilde{X}$ be a connected analytic nonsingular manifold. The following categories are equivalent:*

a. *The category* $\mathrm{Conn}(\widetilde{X})$ *of integrable connections in locally free sheaves of $\mathcal{O}_{\widetilde{X}}$-modules of finite rank.*

b. *The category* $\mathcal{LC}(\widetilde{X})$ *of locally constant sheaves of finite-dimensional vector spaces on $\widetilde{X}$.*

c. *The category* $\pi_1(\widetilde{X})$-mod *of finite-dimensional representations of the fundamental group $\pi_1(\widetilde{X})$ of the manifold $\widetilde{X}$.*

The functors establishing these equivalences can be described as follows: to $(\nabla, \mathcal{F})$ one associates the sheaf $\mathcal{E}$ of germs of flat sections of ∇: $\mathcal{E}(U) = \{f \in \mathcal{F}(U), \nabla f = 0\}$; this sheaf is locally constant due to the existence and uniqueness theorem for analytic solutions of analytic linear differential equations of the first order. Next, to $\mathcal{E}$ one associates the monodromy representation of $\pi_1(\widetilde{X})$ in the fiber of $\mathcal{E}$ over some point $x \in \widetilde{X}$, defined by continuation of sections of $\mathcal{E}$ along paths in $\widetilde{X}$.

6.3. Algebraic Connections. For any algebraic complex manifold X denote by X^{an} the corresponding analytic manifold. From a connection ∇ in a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ one can naturally construct a connection ∇^{an} in the corresponding locally free $\mathcal{O}_{X^{\mathrm{an}}}$-module $\mathcal{F}^{\mathrm{an}}$. If X is a projective variety, GAGA-type theorems say that the mapping $(\nabla, \mathcal{F}) \to (\nabla^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ establishes an equivalence between the categories of algebraic connections on X and analytic connections on X^{an}. Using Theorem 6.2.1, we obtain the following result.

6.3.1. Theorem. *For a projective algebraic variety X the category of integrable connections on X is equivalent to each of the categories $\mathcal{LC}(X^{an})$ and $\pi_1(X^{an})$-mod from Theorem 6.2.1.*

The problem of the generalization of this result to non-compact varieties goes back to Riemann (who considered complex curves with some points removed) and Hilbert, forming the content of the 21-th Hilbert problem. A detailed discussion of various aspects of this problem see (Katz 1976; Malgrange 1987). Here we only present a simplest example which shows that a straightforward generalizatioin of Theorem 6.3.1 to non-compact manifolds X is wrong.

Take for X the projective line $\mathbb{P}^1$ with one point removed. Introduce the coordinate z on $\mathbb{P}^1$ so that the removed point is $z = 0$. For any polynomial P the connection $\nabla_P = dz \otimes \frac{d\varphi}{dz} + dz \otimes \frac{1}{z^2} p\left(\frac{1}{z}\right)$ in the trivial one-dimensional vector bundle on X is non-singular on X (including the point $x = \infty$). All these connection have the trivial monodromy (since $\pi_1(X^{an}) = \{e\}$). It is clear, however, that from the algebraic point of view all these connections are non-equivalent (flat sections of ∇_P are of the form $\varphi_p(z) = c \exp\left\{ -\int^z \frac{1}{\zeta^2} P\left(\frac{1}{\zeta}\right) d\zeta \right\}$, and for different P they have algebraically non-equivalent singularities at the point $z = 0$.

To improve the situation we have to impose on the connections the so-called regularity conditions at infinity.

6.4. Connections with Regular Singularities on a Curve. First consider connections on one-dimensional manifolds. Let C be a non-singular curve, $i : C \to \overline{C}$ be its embedding into a non-singular complete curve as a dense subset. For a point $p \in \overline{C} \setminus C$ choose a local parameter z at p and denote by $\mathcal{D}^p_{\overline{C}}$ the subsheaf of the sheaf $\mathcal{D}_{\overline{C}}$ generated (as a sheaf of algebras) by $\mathcal{O}_{\overline{C}}$ and the operator $\frac{d}{dz}$. It is clear that $\mathcal{D}^p_{\overline{C}}$ does not depend on the choice of a local parameter at p.

6.4.1. Definition. Let ∇ be a connection in a locally free sheaf of $\mathcal{O}_C$-modules $\mathcal{F}$, i.e. an $\mathcal{O}_C$-coherent $\mathcal{D}_C$-module. We say that ∇ has a *regular singularity at p* if the direct image $i_.\mathcal{F}$ near p is the union of $\mathcal{O}_{\overline{C}}$-coherent $\mathcal{D}^p_{\overline{C}}$-modules. We say that ∇ has *regular singularities on C* (or is a regular connection on C) if ∇ has a regular singularity at any point $p \in \overline{C} \setminus C$.

One can easily verify that the property to have regular singularities does not depend on the choice of the completion $\overline{C}$ of the curve C. In the example in 6.3 the only connection with a regular singular at $0 \in \overline{C} \setminus C$ is the connection ∇_P corresponding to $P = 0$. Indeed, sections of $i_.\mathcal{F}$ in a neighborhood U of the point $z = 0$ are meromorphic functions in z, and the action of $d = z\frac{d}{dz}$ on $Q(z) \in i_{\cdot}\mathcal{F}(U)$ is given by the formula

$$dQ = z\frac{dQ}{dz} + \frac{1}{z}P\left(\frac{1}{z}\right)Q(z).$$

Therefore, for $P \neq 0$ any $\mathcal{D}_{\overline{C}}^{z=0}$-invariant submodule contains meromorphic functions with poles of an arbitrary high order at $z = 0$, so it can not be generated over $\mathcal{O}_{\overline{C}}(U) = \mathbb{C}[z]$ by a finite set of generators.

A general criterion of regularity of a connection ∇ at a point $p \in \overline{C} \setminus C$ can be formulated as follows. For some neighborhood $U \subset \overline{C}$ of the point p choose a basis $(e_1, \dots, e_n)$ of the space of sections of the vector bundle F corresponding to $\mathcal{F}$. We can write the action of ∇ on a section $\sum \varphi_i(z)e_i$ in the following form:

$$\nabla\varphi = dz \otimes \frac{d\varphi}{dz} + dz \otimes A(z)\varphi,$$

where $\varphi = (\varphi_1, \dots, \varphi_n)$, $A(z)$ is the matrix of the connection ∇ consisting of functions meromorphic in U with poles at $p = \{z = 0\}$. A connection ∇ represented in such form has a regular singularity at p if and only if the order of the pole of $A(z)$ at $z = 0$ does not exceed 1.

Yet another characterization of connections with regular singularities (useful in analytical problems) can be formulated as follows. A connection has regular singularity at $z = 0$ if and only is any flat section is given by a multi-valued analytic function with polynomial (in $1/|z|$) growth at $z \to 0$.

6.5. Connections with Regular Singularities – the General Case. Let now X be an arbitrary smooth algebraic manifold, ∇ be an integrable connection in a locally free sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$. For an embedding $j : Y \to X$ of a smooth submanifold the inverse image $j^{\cdot}\nabla$ (as a $\mathcal{D}$-module) is given by a connection $\nabla\mid_Y$ in a locally free sheaf $\mathcal{F}\mid_Y$ of $\mathcal{O}_Y$-modules.

6.5.1. Definition. We say that ∇ has *regular singularities on X* (or is a *regular connection on X*) if for any embedding $j : C \to X$ of a smooth curve C into X the connection $j^{\cdot}\nabla$ has regular singularities on C.

A standard method to study connections with regular singularities on X is to represent X as a complement $X = \overline{X} \setminus D$ to a divisor with normal crossings D in a complete manifold $\overline{X}$. Using such embedding one can give a definition of a connection with regular singularities which does not use curves (even two such definitions, one generalizing Definition 6.4.1, and another expressed in terms of the growth and the ramification of ∇-flat sections of $\mathcal{F}$ near D).

Let us remark also that in Definition 6.5.1 it suffices to consider not all curves $C \subset X$ but only a dense subset (in the space of curves). In particular, representing X as $X = \overline{X} \setminus D$ as above, it suffice to consider only those curves which pass only through non-singular points of D (i.e. those $C \subset X$ that the closure $\overline{C}$ of C in X intersects D at non-singular points).

6.6. Deligne Theorem. Consider the category of regular connections on X by $\mathrm{Conn}_r(X)$. Its objects of $\mathrm{Conn}_r(X)$ are pairs $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, ∇ is a connection with regular singularity in $\mathcal{F}$; morphisms in $\mathrm{Conn}_r(X)$ are mappings of sheaves with connections, so that $\mathrm{Conn}_r(X)$ is a full subcategory of the category $\mathcal{M}_X$.

6.6.1. Theorem. *Let X be a smooth connected algebraic variety, X^{an} be the corresponding analytic variety. The category* $\mathrm{Conn}_r(X)$ *is equivalent to each of the categories $\mathcal{LC}(X^{\mathrm{an}})$, $\pi_1(X^{\mathrm{an}})$*-mod *from Theorem* 6.2.1.

6.7. The de Rham Functor. Let ∇ be a on X as a $\mathcal{D}_X$-module. Consider it as $\mathcal{D}_X$-module and define the de Rham functor by the formula $DR(\nabla) = \mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \nabla^{\mathrm{an}})$. One can easily see that $DR(\nabla)$ is a locally constant sheaf on X^{an}, and $DR : \mathrm{Conn}_r(X^{\mathrm{an}}) \to \mathcal{LC}(X^{\mathrm{an}})$ establishes an equivalence of categories from Theorem 6.6.1 (or Theorem 6.2.1): the image of $1 \in \mathcal{O}_{X^{\mathrm{an}}}$ is a ∇^{an}-flat section of $\mathcal{F}^{\mathrm{an}}$.

The Riemann-Hilbert correspondence, which will be explained in the next section, generalizes this result so that it enables us to get a similar description of an arbitrary (and not only $\mathcal{O}_X$-coherent) $\mathcal{D}_X$-module. Roughly speaking, we again must use a de Rham-type functor. By technical reasons, instead of the functor $\mathcal{H}om_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{O}_{X^{\mathrm{an}}}, \cdot)$ it is more convenient to use the tensor product with Ω^{an} (the sheaf of forms of the highest degree on X^{an}). An important difference with the special case of $\mathcal{O}_X$-coherent modules is that we must pass to the derived category, obtaining for each $M^{\cdot} \in D^b_{\mathbf{Hol}}(\mathcal{D}_X)$ a complex $DR(M^{\cdot}) = \Omega_{X^{\mathrm{an}}} \overset{L}{\underset{\mathcal{D}_{X^{\mathrm{an}}}}{\otimes}} M^{\cdot}$. However, contrary to Theorem 6.6.1, we obtain a complex of sheaves whose cohomology is not locally constant, but only constructible sheaf of vector spaces. The main result, proved independently by Kashiwara (1984) and Mebkhout (1984a), (1984b), says that

a. the functor $\Omega_{X^{\mathrm{an}}} \overset{L}{\underset{\mathcal{D}_{X^{\mathrm{an}}}}{\otimes}} \cdot$ establishes an equivalence between a certain derived category of $\mathcal{D}_X$-modules (holonomic modules with regular singularities) and the derived category of complexes of sheaves of vector spaces whose cohomology is constructible with respect to some algebraic stratification;

b. under this equivalence holonomic modules with regular singularities (i.e. H^0-complexes in the derived category) correspond to perverse sheaves on X^{an} (of middle perversity).

§ 7. $\mathcal{D}$-Modules with Regular Singularities

7.1. $\mathcal{D}$-modules on a Curve. Let C be a smooth curve and M be a holonomic $\mathcal{D}_C$-module. By 5.8.1, there exists a Zariski open dense subset $C' \subset C$ such that the restriction of M to C' is given by an integrable connection (i.e. is an $\mathcal{O}_{C'}$-coherent $\mathcal{D}'_C$-module). We say that M is a *regular holonomic* (r.h.)

$\mathcal{D}_C$*-module* if $M_{C'}$ is a connection with regular singularities. We say that $M^\cdot \in \mathcal{D}^b_{\mathrm{Coh}}(\mathcal{D}_C)$ is a *regular holonomic complex* if all $H^i(M^\cdot)$ are r.h. modules. Denote by $RH(\mathcal{D}_C)$ the full subcategory of $\mathcal{D}_C$-mod consisting of r.h. modules, and by $D^b_{rh}(\mathcal{D}_C)$ the full subcategory of $D^b(\mathcal{D}_C\text{-mod})$ consisting of r.h. complexes.

7.1.1. Main Properties. a. Any holonomic $\mathcal{D}_C$-modules with zero-dimensional support belongs to $RH(\mathcal{D}_C)$.

b. The category $RH(\mathcal{D}_C)$ is closed under extensions and subquotients; in particular, $RH(\mathcal{D}_C)$ is an abelian category.

c. If two vertices of a distinguished triangles in $D^b(\mathcal{D}_C\text{-mod})$ belong to $D^b_{rh}(\mathcal{D}_C)$, then the third one also belongs to $D^b_{rh}(\mathcal{D}_C)$; in particular, $D^b_{rh}(\mathcal{D}_C)$ is a triangulated category.

7.2. The General Case. The definition of modules and complexes with regular singularities on an arbitrary manifold X can be reduced to the case of curves as in 6.5.

7.2.1. Definition. A complex $M^\cdot \in D^b(\mathcal{D}_x\text{-mod})$ is said to be *regural holonomic* (r.h.) if

a. $M^\cdot \in D^b_{\mathbf{Hol}}(\mathcal{D}_X)$.

b. For any morphism $\varphi : C \to X$ of a smooth curve C in X a complex $\varphi^D M^\cdot$ has regular singularities on C.

Applying this definition to a 0-complex of $\mathcal{D}_X$-modules we obtain the notion of a $\mathcal{D}_X$*-module with regular singularities.*

Denote by $RH(\mathcal{D}_X)$ the category of r.h. modules on X and by $D^b_{rh}(\mathcal{D}_X)$ the category of (bounded) r.h. complexes.

7.3. Properties of Regular Holonomic Complexes. a. If φ in 7.2.1.b is a constant morphism (the mapping to a point) and $M^\cdot \in D^b_{\mathbf{Hol}}(\mathcal{D}_X)$, then $\varphi^D M^\cdot$ has $\mathcal{O}_C$-free $\mathcal{O}_C$-coherent cohomology, so it is regular.

b. If X is a curve, we have two definitions of $D^b_{rh}(\mathcal{D}_X)$: internal, as in 7.1, and external, as in 7.2.1. One can verify that these two definitions are equivalent. To do this one must use the property **a** above together with the following statement. Let $\varphi : \widetilde{C} \to C$ be a dominant morphism of curves. Then $M^\cdot \in D^b(\mathcal{D}_{C'})$ is an r.h. complex if and only if $\varphi^D M^\cdot$ is an r.h. complex.

c. If two vertices of a distinguished triangles in $D^b(\mathcal{D}_X\text{-mod})$ belong to $D^b_{rh}(\mathcal{D}_X)$, then the third one also belongs to $D^b_{rh}(\mathcal{D}_X)$.

7.4. Regularity is Preserved Under Operations with $\mathcal{D}$-modules. In this subsection we formulate certain statements showing that regularity and holonomicity properties are preserved under the main operations with $\mathcal{D}$-modules.

Let $f : X \to Y$ be a morphism of algebraic varieties.

a. If $M^\cdot \in D^b_{rh}(\mathcal{D}_Y)$, then $f^D M^\cdot \in D^b_{rh}(\mathcal{D}_X)$. This follows immediately from Definition 7.2.1, since $\varphi^D \circ f^D = (f \circ \varphi)^D$.

b. If $M^{\cdot} \in D^b_{rh}(\mathcal{D}_X)$, then $f_+M^{\cdot} \in D^b_{rh}(\mathcal{D}_Y)$. Since any morphism f can be represented as a composition of an embedding and a projection, it suffices to consider these two cases separately. The case of an embedding is rather easy. The case of a projection is much more difficult. The main technical problems causes a special case when $f : C \times \mathbf{A}^1 \to C$ is the projection of the product of a smooth affine curve and the affine line $\mathbf{A}^1$ to the first factor. The proof of the statement **b** in this case is reduced to the explicit computation of f_+M for the so called standard modules, which will be defined in 7.4.c below.

Let $i : X \to Y$ be an embedding of a non-singular affine locally closed submanifold and $\mathcal{F}$ is a connection with regular singularities on Y. A module of the form $i_+\mathcal{F}$ on X is said to be a *standard* $\mathcal{D}_X$*-module* (since Y is affine, $i_+\mathcal{F}$ sits at degree 0). By 7.4.b for embeddings, any standard module is holonomic regular. Moreover, *standard modules generate* $D^b_{rh}(\mathcal{D}_X)$ (i.e. the smallest triangulated subcategory of $D^b_{rh}(\mathcal{D}_X)$ containing all standard $\mathcal{D}_X$-modules, coincides with $M^{\cdot} \in D^b_{rh}(\mathcal{D}_X)$). Therefore in some cases it is sufficient to verify a statement about objects of $D^b_{rh}(\mathcal{D}_X)$ for standard modules only (in particular, this is true for the statement from 7.4.b about the projection $f : C \times \mathbf{A}^1 \to C$).

d. The category $D^b_{rh}(\mathcal{D}_X)$ is closed under the duality functor Δ_X.

e. Let $M^{\cdot} \in D^b_{rh}(\mathcal{D}_X)$. Then any irreducible subquotient of any cohomology module $H^i(M^{\cdot})$ belongs to $RH(\mathcal{D}_X)$.

Both these statements are proved by formal arguments involving the induction by the dimension of the support of $M^{\cdot}$ and 7.4.c.

f. Categories of r.h. complexes are preserved under the functors f_D and f^+. This follows from 7.4.a, b, d.

7.5. Remark. Another approach to the definition of r.h. complexes is that first we define irreducible (or standard) r.h. modules from connections with regular singularities similarly to 7.4.c, and then to adopt 7.4.e as the definition of a general r.h. complex. With such definition we must prove 7.4.a, b, d, and the technical problems here are more or less equivalent to those in the approach described here.

7.6. *G*-equivariant $\mathcal{D}$-modules. Let an algebraic complex group G act on an algebraic manifold X. Such an action is determined by a morphism $\nu : G \times X \to X$ ($\nu(g, x) = gx$) satisfying the usual associativity conditions. Denote also by $p : G \times X \to X$ the projection.

7.6.1. Definition. A *G-equivariant $\mathcal{D}$-module* is a pair (M, s) consisting of a $\mathcal{D}$-module M and an isomorphism of $\mathcal{D}_{G\times X}$-modules $s : \nu^D M \to p^D M$. A morphism $\Phi : (M_1, s_1) \to (M_2, s_2)$ of G-equivariant $\mathcal{D}$-modules is a morphism of $\mathcal{D}_X$-modules $\varphi : M_1 \to M_2$ such that $p^D(\varphi)s_1 = s_2\nu^D(\varphi)$.

Denote by $\mathcal{M}_{G,\mathrm{Coh}}(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-coherent G-equivariant $\mathcal{D}_X$-modules. The main result, which, together with Theorem 2.12.1, enables us to use the theory of $\mathcal{D}$-modules in representation theory, can be formulated as follows.

7.6.2. Theorem. *Let us assume X consists of a finite number of G-orbits. Then any module from $\mathcal{M}_{G,\mathrm{Coh}}(\mathcal{D}_X)$ is a regular holonomic $\mathcal{D}_X$-module.*

The proof of this theorem is based on the following property of the inverse image of a $\mathcal{D}$-module (which is a partial converse of 7.4.a).

7.6.3. Proposition. *Let $f: X \to Y$ be a smooth surjective morphism and M a coherent $\mathcal{D}_Y$-module such that $f^D M$ is an r.h. complex. Then M is an r.h. module.*

7.6.4. Remark. Definition 7.6.1 deals with G-equivariant $\mathcal{D}$-modules, and not with G-equivariant objects of the appropriate derived category. The attempts to generalize Definition 7.6.1 to the derived category turns out to be unsuccessful, so that to give an appropriate (from the point of view of representation theory) definition of G-equivariant complexes of $\mathcal{D}$-modules is a difficult problem.

§ 8. Equivalence of Categories (Riemann-Hilbert Correspondence)

8.1. The de Rham Functor. Let us recall that for any complex algebraic variety X by X^{an} we denote the corresponding analytic variety and by $\mathcal{O}_{X^{\mathrm{an}}}$, $\Omega^i_{X^{\mathrm{an}}}$, $\mathcal{D}_{X^{\mathrm{an}}}$ the natural sheaves (of analytic functions, of holomorphic i-forms, of holomorphic differential operators) on it. In particular, $\Omega_{X^{\mathrm{an}}} = \Omega^{\dim X}_{X^{\mathrm{an}}}$ is (similarly to 2.4) a sheaf of right $\mathcal{D}_{X^{\mathrm{an}}}$-modules. Let also $\mathbf{Sh}_{X^{\mathrm{an}}}$ be the category of sheaves of vector spaces on X^{an}.

8.1.1. Definition. The *de Rham functor* $DR: D^b(\mathcal{D}_X\text{-mod}) \to D^b(\mathbf{Sh}_{X^{\mathrm{an}}})$ is defined by the formula

$$DR(M^{\cdot}) = \Omega_{X^{\mathrm{an}}} \overset{L}{\underset{\mathcal{O}_{X^{\mathrm{an}}}}{\otimes}} M^{\cdot}[\dim X].$$

To compute $\overset{L}{\underset{\mathcal{O}_{X^{\mathrm{an}}}}{\otimes}}$ we can use the left locally free resolution of $\Omega_{X^{\mathrm{an}}}$ similar to 2.11.1:

$$0 \to \mathcal{O}_{X^{\mathrm{an}}} \to \Omega^1_{X^{\mathrm{an}}} \underset{\mathcal{O}_{X^{\mathrm{an}}}}{\otimes} \mathcal{D}_{X^{\mathrm{an}}} \to \cdots \to \Omega_{X^{\mathrm{an}}} \underset{\mathcal{O}_{X^{\mathrm{an}}}}{\otimes} \mathcal{D}_{X^{\mathrm{an}}} \to \Omega_{X^{\mathrm{an}}} \to 0$$

(the right action of $\mathcal{D}_{X^{\mathrm{an}}}$ on elements of the complex is given by the multiplication in $\mathcal{D}_{X^{\mathrm{an}}}$). Therefore, $DR(M^{\cdot})$ is the complex of sheaves associated to the bicomplex

$$C^{ij} = \Omega^{i-\dim X}_{X^{\mathrm{an}}} \underset{\mathcal{O}_{X^{\mathrm{an}}}}{\otimes} M^j.$$

8.2. Theorem. *The functor DR satisfies the following properties:*

a. *It establishes an equivalence between the categories $D^b_{rh}(\mathcal{D}_X)$ and $D^b_c(X^{\mathrm{an}}, \mathbb{C})$ (the subcategory of $D^b(\mathbf{Sh}_{X^{\mathrm{an}}})$ consisting of complexes with cohomology constructible with respect to some stratification of X^{an} by algebraic varieties, see* Sect. 1 *of Chap.* 7).

b. *It commutes with direct and inverse images, with duality, and with tensor product.*

c. *It establishes an equivalence between the category $RH(\mathcal{D}_X) \subset D^b_{rh}(\mathcal{D}_X)$ of regular holonomic modules and the category $\mathcal{M}(p_{1/2}, X^{\mathrm{an}}, \mathbb{C})$ of perverse sheaves (of middle perversity) on X^{an}.*

8.3. Plan of the Proof. Theorem 8.2 follows from a series of statements **a–h** below, some of which hold not only for r.h., but for any holonomic complexes. Let us remark also that statements **a, b, d, e** make part **b** of Theorem 8.2 more precise.

Let $f : X \to Y$ be a morphism of algebraic varieties, $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the corresponding morphism of analytic varieties.

a. $DR \circ f_+ = Rf^{\mathrm{an}}_{\cdot} \circ DR$.

b. $DR(M^{\cdot} \boxtimes N^{\cdot}) = DR(M^{\cdot}) \overset{\mathrm{an}}{\boxtimes} DR(N^{\cdot})$, where $M^{\cdot}$, $N^{\cdot}$ are complexes of $\mathcal{D}$-modules on varieties X_1, X_2, $M^{\cdot} \boxtimes N^{\cdot}$ is their exterior tensor product (it is a complex on $X_1 \times X_2$), $\overset{\mathrm{an}}{\boxtimes}$ is the exterior product of complexes of sheaves of vector spaces, the complex $M^{\cdot}$ is holonomic, the complex $N^{\cdot}$ is coherent (over $\mathcal{D}_{X_2}$).

c. If $M^{\cdot} \in D^b_{\mathbf{Hol}}(\mathcal{D}_X)$ then $DR(M^{\cdot}) \in D^b_c(X^{\mathrm{an}}, \mathbb{C})$.

d. $DR \circ f^D = (f^{\mathrm{an}})^! \circ DR$.

e. $DR \circ \Delta_X = \mathfrak{D}_{X^{\mathrm{an}}} \circ DR$ (here Δ_X is the duality of $\mathcal{D}$-modules, see 5.3.1, $\mathfrak{D}_{X^{\mathrm{an}}}$ is the Verdier duality, see 5.16 in Chap. 4).

f. DR is a fully faithful functor (i.e. the corresponding morphism of Hom's is an isomorphism).

g. DR is surjective on classes of objects.

h. $M^{\cdot}$ is a H^0-complex if and only if $DR(M^{\cdot})$ is a $p_{1/2}$-perverse sheaf.

We do not give here proofs of these statements. Let us indicate only that in proving **a, b, d, e** one constructs first a functorial morphism of the left-hand side of the required equality to the right-hand side, and then verifies that it is an isomorphism. This verification must be done only for some set of generators of the category D^b_{rh} (for example, for standard modules, see 7.4.c). Statements **c** and **f** are also verified for standard generators. Statement **g** is proved for simple objects of the category $RH(\mathcal{D}_X)$, for which it follows from the Deligne theorem 6.6.1. To prove **h** one uses the following characterization of holonomic modules: an object $M^{\cdot} \in D^b_{\mathbf{Hol}}(\mathcal{D}_X)$ is an H^0-complex if and only if both $M^{\cdot}$ and $\Delta_X M^{\cdot}$ satisfy the following condition;

(*) For each locally closed subvariety Z in X (with the embedding $i_Z : Z \to X$) there exists a Zariski open dense subset $U \subset Z$ such that the cohomology $H^k((i_Z^D M^\cdot)|_U)$ vanishes for $k < 0$ and is an $\mathcal{O}_Z$-coherent module for $k \geq 0$.

8.4. Remarks. a. *Solution functor.* Define the functor Sol : $D^b(\mathcal{D}_X\text{-mod})^\circ \to D^b(\mathbf{Sh}_{X^{\mathrm{an}}})$ by the formula $\mathrm{Sol}(M^\cdot) = RHom_{\mathcal{D}_{X^{\mathrm{an}}}}(M^\cdot, \mathcal{O}_{X^{\mathrm{an}}})$. One can easily see that $\mathrm{Sol} = DR \circ \Delta_X$, so that Sol define an autoequivalence between the categories $D^b_{rh}(\mathcal{D}_X)$ and $D^b_c(X^{\mathrm{an}}, \mathbb{C})$. It is this functor that was used by Kashiwara (1984).

b. It follows from 8.3.a, b, e that DR transforms f^+ into $(f^{\mathrm{an}})^\cdot$ and f_D into $Rf_!^{\mathrm{an}}$.

c. By 8.3.a, for any holonomic complex $M^\cdot$ there exists a unique regular holonomic complex $\widetilde{M}^\cdot$ such that $DR(M^\cdot) = DR(\widetilde{M}^\cdot)$ (hence $\mathrm{Sol}(M^\cdot) = \mathrm{Sol}(\widetilde{M}^\cdot)$).

Bibliographic Hints

A rather complete exposition of the theory of D-modules can be found in the books by Björk (1979) and Pham (1979) and in three papers by Borel in (Borel et al. 1987), which are based on the unpublished lectures by Bernstein. A close theory of modules over the rings of microdifferential operators can be found in (Schapira 1985) and in the review paper (Kashiwara, Kawai 1983), which contains also further references.

Results about the Weyl algebras from Sect. 1 are proved in (Björk 1979) and (Ehlers 1987). A majority of the results from Sect. 2–5 can be found in (Borel et al. 1987). The notion of a D-affine manifolds from 2.12, and Theorem 2.12.1, which form a starting point of applications of D-modules to representations of semisimple Lie groups is due to Beilinson, Bernstein (1981). The realtionship between D-modules and representation theory, which is almost omitted here, drastically changed representation theory; the review of these results see in (Ginzburg 1986; Joseph (1983) and in the corresponding volumes of this Encyclopaedia.

Connections with regular singularities on a general variety were introduced by Deligne (1973b), who generalized the notion of a Fuchsian singular point of an ordinary differential equation. See the reviews by Katz (1976) and Malgrange (1987), where most of the results from Sect. 6 are proved.

Proofs of properties of D-modules with regular singularities from Sect. 7 can be found in (Borel et al. 1987). The Riemann–Hilbert correspondence (Theorem 8.12) is proved (in a somewhat different form) by Kashiwara (1984) and Mebkhout (1984a, 1984b). Our exposition follows (Borel et al. 1987).

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