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KOBAYASHI & NOMIZU

Foundations of Differential Geometry

Volume I

FOUNDATIONS OF DIFFERENTIAL GEOMETRY

VOLUME I

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PREFACE

Differential geometry has a long history as a field of mathematics and yet its rigorous foundation in the realm of contemporary mathematics is relatively new. We have written this book, the first of the two volumes of the *Foundations of Differential Geometry*, with the intention of providing a systematic introduction to differential geometry which will also serve as a reference book.

Our primary concern was to make it self-contained as much as possible and to give complete proofs of all standard results in the foundation. We hope that this purpose has been achieved with the following arrangements. In Chapter I we have given a brief survey of differentiable manifolds, Lie groups and fibre bundles. The readers who are unfamiliar with them may learn the subjects from the books of Chevalley, Montgomery-Zippin, Pontrjagin, and Steenrod, listed in the *Bibliography*, which are our standard references in Chapter I. We have also included a concise account of tensor algebras and tensor fields, the central theme of which is the notion of derivation of the algebra of tensor fields. In the *Appendices*, we have given some results from topology, Lie group theory and others which we need in the main text. With these preparations, the main text of the book is self-contained.

Chapter II contains the connection theory of Ehresmann and its later development. Results in this chapter are applied to linear and affine connections in Chapter III and to Riemannian connections in Chapter IV. Many basic results on normal coordinates, convex neighborhoods, distance, completeness and holonomy groups are proved here completely, including the de Rham decomposition theorem for Riemannian manifolds.

In Chapter V, we introduce the sectional curvature of a Riemannian manifold and the spaces of constant curvature. A more complete treatment of properties of Riemannian manifolds involving sectional curvature depends on calculus of variations and will be given in Volume II. We discuss flat affine and Riemannian connections in detail.

In Chapter VI, we first discuss transformations and infinitesimal transformations which preserve a given linear connection or a Riemannian metric. We include here various results concerning Ricci tensor, holonomy and infinitesimal isometries. We then treat the extension of local transformations and the so-called equivalence problem for affine and Riemannian connections. The results in this chapter are closely related to differential geometry of homogeneous spaces (in particular, symmetric spaces) which are planned for Volume II.

In all the chapters, we have tried to familiarize the readers with various techniques of computations which are currently in use in differential geometry. These are: (1) classical tensor calculus with indices; (2) exterior differential calculus of E. Cartan; and (3) formalism of covariant differentiation $\nabla_X Y$, which is the newest among the three. We have also illustrated, as we see fit, the methods of using a suitable bundle or working directly in the base space.

The *Notes* include some historical facts and supplementary results pertinent to the main content of the present volume. The *Bibliography* at the end contains only those books and papers which we quote throughout the book.

Theorems, propositions and corollaries are numbered for each section. For example, in each chapter, say, Chapter II, Theorem 3.1 is in Section 3. In the rest of the same chapter, it will be referred to simply as Theorem 3.1. For quotation in subsequent chapters, it is referred to as Theorem 3.1 of Chapter II.

We originally planned to write one volume which would include the content of the present volume as well as the following topics: submanifolds; variations of the length integral; differential geometry of complex and Kähler manifolds; differential geometry of homogeneous spaces; symmetric spaces; characteristic classes. The considerations of time and space have made it desirable to divide the book in two volumes. The topics mentioned above will therefore be included in Volume II.

In concluding the preface, we should like to thank Professor L. Bers, who invited us to undertake this project, and Interscience Publishers, a division of John Wiley and Sons, for their patience and kind cooperation. We are greatly indebted to Dr. A. J. Lohwater, Dr. H. Ozeki, Messrs. A. Howard and E. Ruh for their kind help which resulted in many improvements of both the content and the presentation. We also acknowledge the grants of the National Science Foundation which supported part of the work included in this book.

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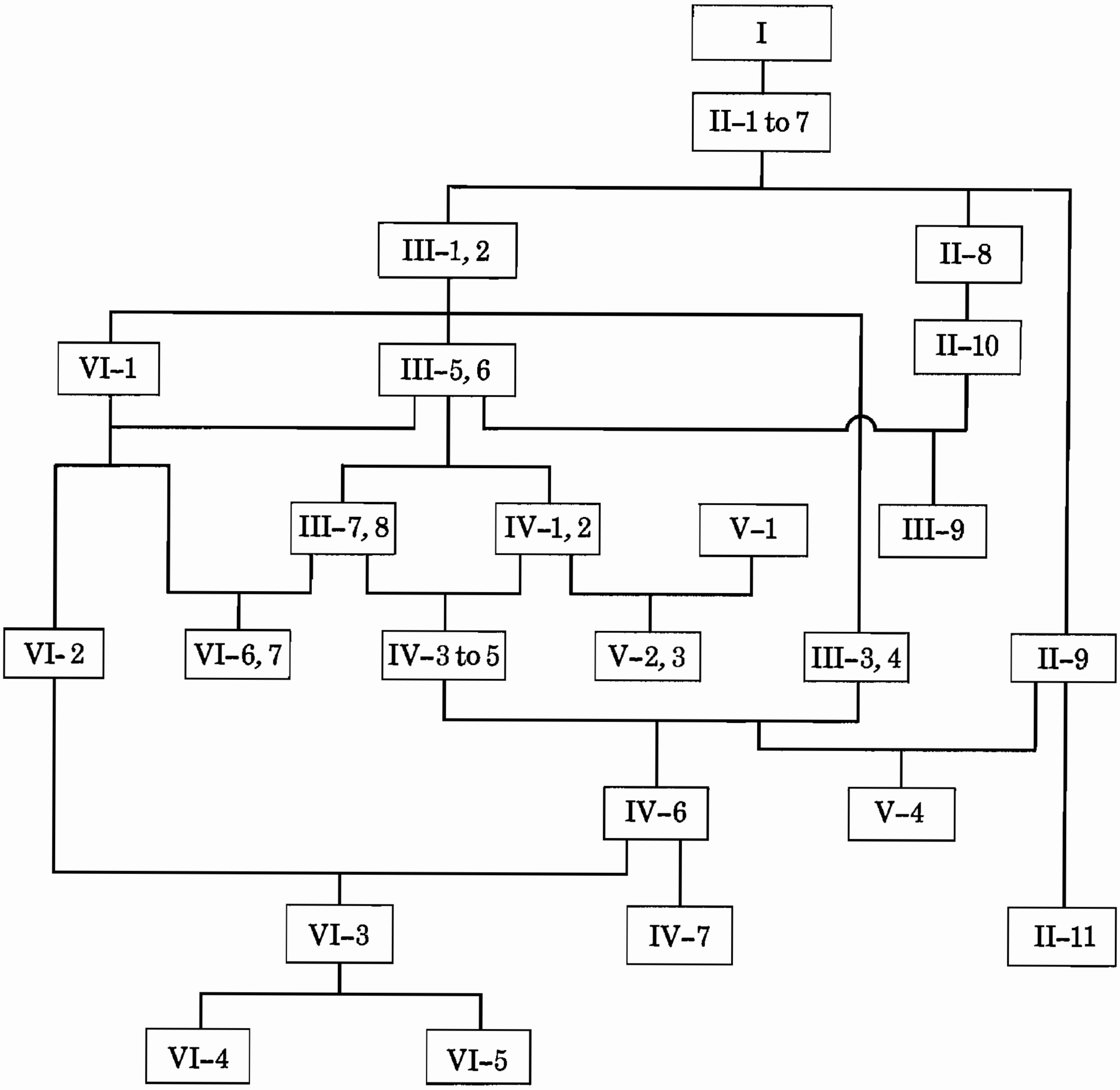

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CHAPTER I

Differentiable Manifolds

1. *Differentiable manifolds*

A *pseudogroup of transformations* on a topological space S is a set Γ of transformations satisfying the following axioms:

(1) Each $f \in \Gamma$ is a homeomorphism of an open set (called the domain of f) of S onto another open set (called the range of f) of S;

(2) If $f \in \Gamma$, then the restriction of f to an arbitrary open subset of the domain of f is in Γ;

(3) Let $U = \bigcup_i U_i$ where each U_i is an open set of S. A homeomorphism f of U onto an open set of S belongs to Γ if the restriction of f to U_i is in Γ for every i;

(4) For every open set U of S, the identity transformation of U is in Γ;

(5) If $f \in \Gamma$, then $f^{-1} \in \Gamma$;

(6) If $f \in \Gamma$ is a homeomorphism of U onto V and $f' \in \Gamma$ is a homeomorphism of U' onto V' and if $V \cap U'$ is non-empty, then the homeomorphism $f' \circ f$ of $f^{-1}(V \cap U')$ onto $f'(V \cap U')$ is in Γ.

We give a few examples of pseudogroups which are used in this book. Let $\mathbf{R}^n$ be the space of n-tuples of real numbers $(x^1, x^2, \ldots, x^n)$ with the usual topology. A mapping f of an open set of $\mathbf{R}^n$ into $\mathbf{R}^m$ is said to be of *class* C^r, $r = 1, 2, \ldots, \infty$, if f is continuously r times differentiable. By class C^0 we mean that f is continuous. By class C^ω we mean that f is real analytic. The *pseudogroup* $\Gamma^r(\mathbf{R}^n)$ *of transformations of class* C^r of $\mathbf{R}^n$ is the set of homeomorphisms f of an open set of $\mathbf{R}^n$ onto an open set of $\mathbf{R}^n$ such that both f and f^{-1} are of class C^r. Obviously $\Gamma^r(\mathbf{R}^n)$ is a pseudogroup of transformations of $\mathbf{R}^n$. If $r < s$, then $\Gamma^s(\mathbf{R}^n)$ is a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. If we consider only those $f \in \Gamma^r(\mathbf{R}^n)$ whose Jacobians are positive everywhere, we obtain a subpseudogroup of $\Gamma^r(\mathbf{R}^n)$. This subpseudogroup, denoted by $\Gamma^r_o(\mathbf{R}^n)$, is called the *pseudogroup of orientation-preserving transformations of class* C^r *of* $\mathbf{R}^n$. Let $\mathbf{C}^n$ be the space of n-tuples of complex numbers with the usual topology. The *pseudogroup of holomorphic* (i.e., complex analytic) *transformations of* $\mathbf{C}^n$ can be similarly defined and will be denoted by $\Gamma(\mathbf{C}^n)$. We shall identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, when necessary, by mapping $(z^1, \ldots, z^n) \in \mathbf{C}^n$ into $(x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbf{R}^{2n}$, where $z^j = x^j + iy^j$. Under this identification, $\Gamma(\mathbf{C}^n)$ is a subpseudogroup of $\Gamma^r_o(\mathbf{R}^{2n})$ for any r.

An *atlas* of a topological space M compatible with a pseudogroup Γ is a family of pairs (U_i, φ_i), called *charts*, such that

(a) Each U_i is an open set of M and $\bigcup_i U_i = M$;

(b) Each φ_i is a homeomorphism of U_i onto an open set of S;

(c) Whenever $U_i \cap U_j$ is non-empty, the mapping $\varphi_j \circ \varphi_i^{-1}$ of $\varphi_i(U_i \cap U_j)$ onto $\varphi_j(U_i \cap U_j)$ is an element of Γ.

A *complete* atlas of M compatible with Γ is an atlas of M compatible with Γ which is not contained in any other atlas of M compatible with Γ. Every atlas of M compatible with Γ is contained in a unique complete atlas of M compatible with Γ. In fact, given an atlas $A = \{(U_i, \varphi_i)\}$ of M compatible with Γ, let $\tilde{A}$ be the family of all pairs (U, φ) such that φ is a homeomorphism of an open set U of M onto an open set of S and that

$$\varphi_i \circ \varphi^{-1}: \varphi(U \cap U_i) \to \varphi_i(U \cap U_i)$$

is an element of Γ whenever $U \cap U_i$ is non-empty. Then $\tilde{A}$ is the complete atlas containing A.

If Γ' is a subpseudogroup of Γ, then an atlas of M compatible with Γ' is compatible with Γ.

A *differentiable manifold* of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma^r(\mathbf{R}^n)$. The integer n is called the dimension of the manifold. Any atlas of a Hausdorff space compatible with $\Gamma^r(\mathbf{R}^n)$, enlarged to a complete atlas, defines a differentiable structure of class C^r. Since $\Gamma^r(\mathbf{R}^n) \supset \Gamma^s(\mathbf{R}^n)$ for $r < s$, a differentiable structure of class C^s defines uniquely a differentiable structure of class C^r. A differentiable manifold of class C^ω is also called a *real analytic manifold.* (Throughout the book we shall mostly consider differentiable manifolds of class C^∞. By a *differentiable manifold* or, simply, *manifold*, we shall mean a differentiable manifold of class C^∞.) A *complex* (*analytic*) *manifold* of complex dimension n is a Hausdorff space with a fixed complete atlas compatible with $\Gamma(\mathbf{C}^n)$. An *oriented* differentiable manifold of class C^r is a Hausdorff space with a fixed complete atlas compatible with $\Gamma_o^r(\mathbf{R}^n)$. An oriented differentiable structure of class C^r gives rise to a differentiable structure of class C^r uniquely. Not every differentiable structure of class C^r is thus obtained; if it is obtained from an oriented one, it is called *orientable.* An orientable manifold of class C^r admits exactly two orientations if it is connected. Leaving the proof of this fact to the reader, we shall only indicate how to *reverse the orientation* of an oriented manifold. If a family of charts (U_i, φ_i) defines an oriented manifold, then the family of charts (U_i, ψ_i) defines the manifold with the reversed orientation where ψ_i is the composition of φ_i with the transformation $(x^1, x^2, \ldots, x^n) \to (-x^1, x^2, \ldots, x^n)$ of $\mathbf{R}^n$. Since $\Gamma(\mathbf{C}^n) \subset \Gamma_o^r(\mathbf{R}^{2n})$, every complex manifold is oriented as a manifold of class C^r.

For any structure under consideration (e.g., differentiable structure of class C^r), an *allowable* chart is a chart which belongs to the fixed complete atlas defining the structure. From now on, by a chart we shall mean an allowable chart. Given an allowable chart (U_i, φ_i) of an n-dimensional manifold M of class C^r, the system of functions $x^1 \circ \varphi_i, \ldots, x^n \circ \varphi_i$ defined on U_i is called a *local coordinate system* in U_i. We say then that U_i is a *coordinate neighborhood.* For every point p of M, it is possible to find a chart (U_i, φ_i) such that $\varphi_i(p)$ is the origin of $\mathbf{R}^n$ and φ_i is a homeomorphism of U_i onto an open set of $\mathbf{R}^n$ defined by $|x^1| < a, \ldots, |x^n| < a$ for some positive number a. U_i is then called a *cubic neighborhood* of p.

In a natural manner $\mathbf{R}^n$ is an oriented manifold of class C^r for any r; a chart consists of an element f of $\Gamma_o^r(\mathbf{R}^n)$ and the domain of f. Similarly, $\mathbf{C}^n$ is a complex manifold. Any open subset N of a manifold M of class C^r is a manifold of class C^r in a natural manner; a chart of N is given by $(U_i \cap N, \psi_i)$ where (U_i, φ_i) is a chart of M and ψ_i is the restriction of φ_i to $U_i \cap N$. Similarly, for complex manifolds.

Given two manifolds M and M' of class C^r, a mapping $f: M \to M'$ is said to be differentiable of class C^k, $k \leqq r$, if, for every chart (U_i, φ_i) of M and every chart (V_j, ψ_j) of M' such that

$f(U_i) \subset V_j$, the mapping $\psi_j \circ f \circ \varphi_i^{-1}$ of $\varphi_i(U_i)$ into $\psi_j(V_j)$ is differentiable of class C^k. If $u^1, \ldots, u^n$ is a local coordinate system in U_i and $v^1, \ldots, v^m$ is a local coordinate system in V_j, then f may be expressed by a set of differentiable functions of class C^k:

$$v^1 = f^1(u^1, \ldots, u^n), \ldots, v^m = f^m(u^1, \ldots, u^n).$$

By a *differentiable mapping* or simply, a *mapping*, we shall mean a mapping of class C^∞. A differentiable function of class C^k on M is a mapping of class C^k of M into $\mathbf{R}$. The definition of a *holomorphic* (or *complex analytic*) mapping or function is similar.

By a *differentiable curve* of class C^k in M, we shall mean a differentiable mapping of class C^k of a closed interval $[a, b]$ of $\mathbf{R}$ into M, namely, the restriction of a differentiable mapping of class C^k of an open interval containing $[a, b]$ into M. We shall now define a *tangent vector* (or simply a *vector*) at a point p of M. Let $\mathfrak{F}(p)$ be the algebra of differentiable functions of class C^1 defined in a neighborhood of p. Let $x(t)$ be a curve of class C^1, $a \leqq t \leqq b$, such that $x(t_0) = p$. The vector tangent to the curve $x(t)$ at p is a mapping X: $\mathfrak{F}(p) \to \mathbf{R}$ defined by

$$Xf = (df(x(t))/dt)_{t_0}.$$

In other words, Xf is the derivative of f in the direction of the curve $x(t)$ at $t = t_0$. The vector X satisfies the following conditions:

(1) X is a linear mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$;

(2) $X(fg) = (Xf)g(p) + f(p)(Xg) \qquad \text{for } f, g \in \mathfrak{F}(p)$.

The set of mappings X of $\mathfrak{F}(p)$ into $\mathbf{R}$ satisfying the preceding two conditions forms a real vector space. We shall show that the set of vectors at p is a vector subspace of dimension n, where n is the dimension of M. Let $u^1, \ldots, u^n$ be a local coordinate system in a coordinate neighborhood U of p. For each j, $(\partial/\partial u^j)_p$ is a mapping of $\mathfrak{F}(p)$ into $\mathbf{R}$ which satisfies conditions (1) and (2) above. We shall show that the set of vectors at p is the vector space with basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Given any curve $x(t)$ with $p = x(t_0)$, let $u^j = x^j(t)$, $j = 1, \ldots, n$, be its equations in terms of the local coordinate system $u^1, \ldots, u^n$. Then

$$(df(x(t))/dt)_{t_0} = \Sigma_j\, (\partial f/\partial u^j)_p \cdot (dx^j(t)/dt)_{t_0}\,^*,$$

* For the summation notation, see Summary of Basic Notations.

which proves that every vector at p is a linear combination of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$. Conversely, given a linear combination $\sum \xi^j(\partial/\partial u^j)_p$, consider the curve defined by

$$u^j = u^j(p) + \xi^j t, \qquad j = 1, \ldots, n.$$

Then the vector tangent to this curve at $t = 0$ is $\sum \xi^j(\partial/\partial u^j)_p$. To prove the linear independence of $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$, assume $\sum \xi^j(\partial/\partial u^j)_p = 0$. Then

$$0 = \sum \xi^j(\partial u^k/\partial u^j)_p = \xi^k \qquad \text{for } k = 1, \ldots, n.$$

This completes the proof of our assertion. The set of tangent vectors at p, denoted by $T_p(M)$ or T_p, is called the *tangent space* of M at p. The n-tuple of numbers $\xi^1, \ldots, \xi^n$ will be called the *components* of the vector $\sum \xi^j(\partial/\partial u^j)_p$ with respect to the local coordinate system $u^1, \ldots, u^n$.

Remark. It is known that if a manifold M is of class C^∞, then $T_p(M)$ coincides with the space of X: $\mathfrak{F}(p) \to \mathbf{R}$ satisfying conditions (1) and (2) above, where $\mathfrak{F}(p)$ now denotes the algebra of all C^∞ functions around p. From now on we shall consider mainly manifolds of class C^∞ and mappings of class C^∞.

A *vector field* X on a manifold M is an assignment of a vector X_p to each point p of M. If f is a differentiable function on M, then Xf is a function on M defined by $(Xf)(p) = X_p f$. A vector field X is called *differentiable* if Xf is differentiable for every differentiable function f. In terms of a local coordinate system $u^1, \ldots, u^n$, a vector field X may be expressed by $X = \sum \xi^j(\partial/\partial u^j)$, where ξ^j are functions defined in the coordinate neighborhood, called the *components* of X with respect to $u^1, \ldots, u^n$. X is differentiable if and only if its components ξ^j are differentiable.

Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on M. It is a real vector space under the natural addition and scalar multiplication. If X and Y are in $\mathfrak{X}(M)$, define the bracket $[X, Y]$ as a mapping from the ring of functions on M into itself by

$$[X, Y]f = X(Yf) - Y(Xf).$$

We shall show that $[X, Y]$ is a vector field. In terms of a local coordinate system $u^1, \ldots, u^n$, we write

$$X = \sum \xi^j(\partial/\partial u^j), \qquad Y = \sum \eta^j(\partial/\partial u^j).$$

Then

$$[X, Y]f = \Sigma_{j,k}(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))(\partial f/\partial u^j).$$

This means that $[X, Y]$ is a vector field whose components with respect to $u^1, \ldots, u^n$ are given by $\Sigma_k(\xi^k(\partial\eta^j/\partial u^k) - \eta^k(\partial\xi^j/\partial u^k))$, $j = 1, \ldots, n$. With respect to this bracket operation, $\mathfrak{X}(M)$ is a Lie algebra over the real number field (of infinite dimensions). In particular, we have Jacobi's identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$
$$\text{for } X, Y, Z \in \mathfrak{X}(M).$$

We may also regard $\mathfrak{X}(M)$ as a module over the algebra $\mathfrak{F}(M)$ of differentiable functions on M as follows. If f is a function and X is a vector field on M, then fX is a vector field on M defined by $(fX)_p = f(p)X_p$ for $p \in M$. Then

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$
$$f,g \in \mathfrak{F}(M), \qquad X,Y \in \mathfrak{X}(M).$$

For a point p of M, the dual vector space $T_p^*(M)$ of the tangent space $T_p(M)$ is called the space of *covectors* at p. An assignment of a covector at each point p is called a *1-form* (*differential form of degree* 1). For each function f on M, the *total differential* $(df)_p$ of f at p is defined by

$$\langle (df)_p, X\rangle = Xf \qquad \text{for } X \in T_p(M),$$

where $\langle\ ,\ \rangle$ denotes the value of the first entry on the second entry as a linear functional on $T_p(M)$. If $u^1, \ldots, u^n$ is a local coordinate system in a neighborhood of p, then the total differentials $(du^1)_p, \ldots, (du^n)_p$ form a basis for $T_p^*(M)$. In fact, they form the dual basis of the basis $(\partial/\partial u^1)_p, \ldots, (\partial/\partial u^n)_p$ for $T_p(M)$. In a neighborhood of p, every 1-form ω can be uniquely written as

$$\omega = \Sigma_j f_j\, du^j,$$

where f_j are functions defined in the neighborhood of p and are called the *components* of ω with respect to $u^1, \ldots, u^n$. The 1-form ω is called *differentiable* if f_j are differentiable (this condition is independent of the choice of a local coordinate system). We shall only consider differentiable 1-forms.

A 1-form ω can be defined also as an $\mathfrak{F}(M)$-linear mapping of the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$ into $\mathfrak{F}(M)$. The two definitions are related by (cf. Proposition 3.1)

$$(\omega(X))_p = \langle \omega_p, X_p \rangle, \qquad X \in \mathfrak{X}(M), \qquad p \in M.$$

Let $\Lambda T_p^*(M)$ be the exterior algebra over $T_p^*(M)$. An *r-form* ω is an assignment of an element of degree r in $\Lambda T_p^*(M)$ to each point p of M. In terms of a local coordinate system $u^1, \ldots, u^n$, ω can be expressed uniquely as

$$\omega = \Sigma_{i_1 < i_2 < \cdots < i_r} f_{i_1 \ldots i_r} \, du^{i_1} \wedge \cdots \wedge du^{i_r}.$$

The r-form ω is called *differentiable* if the components $f_{i_1 \cdots i_r}$ are all differentiable. By an *r-form* we shall mean a differentiable r-form. An r-form ω can be defined also as a skew-symmetric r-linear mapping over $\mathfrak{F}(M)$ of $\mathfrak{X}(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)$ (r times) into $\mathfrak{F}(M)$. The two definitions are related as follows. If $\omega_1, \ldots, \omega_r$ are 1-forms and $X_1, \ldots, X_r$ are vector fields, then $(\omega_1 \wedge \cdots \wedge \omega_r)(X_1, \ldots, X_r)$ is $1/r!$ times the determinant of the matrix $(\omega_j(X_k))_{j,k=1,\ldots,r}$ of degree r.

We denote by $\mathfrak{D}^r = \mathfrak{D}^r(M)$ the totality of (differentiable) r-forms on M for each $r = 0, 1, \ldots, n$. Then $\mathfrak{D}^0(M) = \mathfrak{F}(M)$. Each $\mathfrak{D}^r(M)$ is a real vector space and can be also considered as an $\mathfrak{F}(M)$-module: for $f \in \mathfrak{F}(M)$ and $\omega \in \mathfrak{D}^r(M)$, $f\omega$ is an r-form defined by $(f\omega)_p = f(p)\omega_p$, $p \in M$. We set $\mathfrak{D} = \mathfrak{D}(M) = \Sigma_{r=0}^n \mathfrak{D}^r(M)$. With respect to the exterior product, $\mathfrak{D}(M)$ forms an algebra over the real number field. *Exterior differentiation* d can be characterized as follows:

(1) d is an **R**-linear mapping of $\mathfrak{D}(M)$ into itself such that $d(\mathfrak{D}^r) \subset \mathfrak{D}^{r+1}$;

(2) For a function $f \in \mathfrak{D}^0$, df is the total differential;

(3) If $\omega \in \mathfrak{D}^r$ and $\pi \in \mathfrak{D}^s$, then

$$d(\omega \wedge \pi) = d\omega \wedge \pi + (-1)^r \omega \wedge d\pi;$$

(4) $d^2 = 0$.

In terms of a local coordinate system, if $\omega = \Sigma_{i_1 < \cdots < i_r} f_{i_1 \ldots i_r} \, du^{i_1} \wedge \cdots \wedge du^{i_r}$, then $d\omega = \Sigma_{i_1 < \cdots < i_r} df_{i_1 \cdots i_r} \wedge du^{i_1} \cdots \wedge du^{i_r}$.

It will be later necessary to consider differential forms with values in an arbitrary vector space. Let V be an m-dimensional real vector space. A V-valued r-form ω on M is an assignment to each point $p \in M$ a skew-symmetric r-linear mapping of $T_p(M) \times \cdots \times T_p(M)$ (r times) into V. If we take a basis $e_1, \ldots, e_m$ for V, we can write ω uniquely as $\omega = \Sigma_{j=1}^m \omega^j \cdot e_j$, where ω^j are usual r-forms on M. ω is *differentiable*, by definition, if ω^j are all differentiable. The exterior derivative $d\omega$ is defined to be $\Sigma_{j=1}^m d\omega^j \cdot e_j$, which is a V-valued $(r+1)$-form.

Given a mapping f of a manifold M into another manifold M', the *differential* at p of f is the linear mapping f_* of $T_p(M)$ into $T_{f(p)}(M')$ defined as follows. For each $X \in T_p(M)$, choose a curve $x(t)$ in M such that X is the vector tangent to $x(t)$ at $p = x(t_0)$. Then $f_*(X)$ is the vector tangent to the curve $f(x(t))$ at $f(p) = f(x(t_0))$. It follows immediately that if g is a function differentiable in a neighborhood of $f(p)$, then $(f_*(X))g = X(g \circ f)$. When it is necessary to specify the point p, we write $(f_*)_p$. When there is no danger of confusion, we may simply write f instead of f_*. The transpose of $(f_*)_p$ is a linear mapping of $T^*_{f(p)}(M')$ into $T^*_p(M)$. For any r-form ω' on M', we define an r-form $f^*\omega'$ on M by

$$(f^*\omega')(X_1, \ldots, X_r) = \omega'(f_*X_1, \ldots, f_*X_r), \qquad X_1, \ldots, X_r \in T_p(M).$$

The exterior differentiation d commutes with f^*: $d(f^*\omega') = f^*(d\omega')$.

A mapping f of M into M' is said to be of *rank* r at $p \in M$ if the dimension of $f_*(T_p(M))$ is r. If the rank of f at p is equal to $n = \dim M$, $(f_*)_p$ is injective and $\dim M \leqq \dim M'$. If the rank of f at p is equal to $n' = \dim M'$, $(f_*)_p$ is surjective and $\dim M \geqq \dim M'$. By the implicit function theorem, we have

Proposition 1.1. *Let f be a mapping of M into M' and p a point of M.*

(1) *If $(f_*)_p$ is injective, there exist a local coordinate system $u^1, \ldots u^n$ in a neighborhood U of p and a local coordinate system $v^1, \ldots, v^{n'}$ in a neighborhood of $f(p)$ such that*

$$v^i(f(q)) = u^i(q) \qquad \textit{for } q \in U \quad \textit{and} \quad i = 1, \ldots, n.$$

In particular, f is a homeomorphism of U onto $f(U)$.

(2) *If $(f_*)_p$ is surjective, there exist a local coordinate system $u^1, \ldots, u^n$ in a neighborhood U of p and a local coordinate system $v^1, \ldots, v^{n'}$ of $f(p)$ such that*

$$v^i(f(q)) = u^i(q) \qquad \text{for } q \in U \quad \text{and} \quad i = 1, \ldots, n'.$$

In particular, the mapping $f\colon U \to M'$ is open.

(3) *If $(f_*)_p$ is a linear isomorphism of $T_p(M)$ onto $T_{f(p)}(M')$, then f defines a homeomorphism of a neighborhood U of p onto a neighborhood V of $f(p)$ and its inverse $f^{-1}\colon V \to U$ is also differentiable.*

For the proof, see Chevalley [1, pp. 79–80].

A mapping f of M into M' is called an *immersion* if $(f_*)_p$ is injective for every point p of M. We say then that M is immersed in M' by f or that M is an immersed submanifold of M'. When an immersion f is injective, it is called an *imbedding* of M into M'. We say then that M (or the image $f(M)$) is an *imbedded submanifold* (or, simply, a *submanifold*) of M'. A submanifold may or may not be a closed subset of M'. The topology of a submanifold is in general finer than the relative topology induced from M'. An open subset M of a manifold M', considered as a submanifold of M' in a natural manner, is called an *open submanifold* of M'.

Example 1.1. Let f be a function defined on a manifold M'. Let M be the set of points $p \in M'$ such that $f(p) = 0$. If $(df)_p \neq 0$ at every point p of M, then it is possible to introduce the structure of a manifold in M so that M is a closed submanifold of M', called the *hypersurface defined by the equation* $f = 0$. More generally, let M be the set of common zeros of functions $f_1, \ldots, f_r$ defined on M'. If the dimension, say k, of the subspace of $T^*_p(M')$ spanned by $(df_1)_p, \ldots, (df_r)_p$ is independent of $p \in M$, then M is a closed submanifold of M' of dimension $\dim M' - k$.

A *diffeomorphism* of a manifold M onto another manifold M' is a homeomorphism φ such that both φ and φ^{-1} are differentiable. A diffeomorphism of M onto itself is called a *differentiable transformation* (or, simply, a *transformation*) of M. A transformation φ of M induces an automorphism φ^* of the algebra $\mathfrak{D}(M)$ of differential forms on M and, in particular, an automorphism of the algebra $\mathfrak{F}(M)$ of functions on M:

$$(\varphi^* f)(p) = f(\varphi(p)), \qquad f \in \mathfrak{F}(M), \qquad p \in M.$$

It induces also an automorphism φ_* of the Lie algebra $\mathfrak{X}(M)$ of vector fields by

$$(\varphi_* X)_p = (\varphi_*)_q(X_q),$$

where

$$\varphi(q) = p, \qquad X \in \mathfrak{X}(M).$$

They are related by

$$\varphi^*((\varphi_* X)f) = X(\varphi^* f) \qquad \text{for } X \in \mathfrak{X}(M) \quad \text{and} \quad f \in \mathfrak{F}(M).$$

Although any mapping φ of M into M' carries a differential form ω' on M' into a differential form $\varphi^*(\omega')$ on M, φ does not send a vector field on M into a vector field on M' in general. We say that a vector field X on M is *φ-related* to a vector field X' on M' if $(\varphi_*)_p X_p = X'_{\varphi(p)}$ for all $p \in M$. If X and Y are φ-related to X' and Y' respectively, then $[X, Y]$ is φ-related to $[X', Y']$.

A *distribution* S of dimension r on a manifold M is an assignment to each point p of M an r-dimensional subspace S_p of $T_p(M)$. It is called *differentiable* if every point p has a neighborhood U and r differentiable vector fields on U, say, $X_1, \ldots, X_r$, which form a basis of S_q at every $q \in U$. The set $X_1, \ldots, X_r$ is called a *local basis* for the distribution S in U. A vector field X is said to belong to S if $X_p \in S_p$ for all $p \in M$. Finally, S is called *involutive* if $[X, Y]$ belongs to S whenever two vector fields X and Y belong to S. By a distribution we shall always mean a differentiable distribution.

A connected submanifold N of M is called an *integral manifold* of the distribution S if $f_*(T_p(N)) = S_p$ for all $p \in N$, where f is the imbedding of N into M. If there is no other integral manifold of S which contains N, N is called a *maximal integral manifold* of S. The classical theorem of Frobenius can be formulated as follows.

PROPOSITION 1.2. *Let S be an involutive distribution on a manifold M. Through every point $p \in M$, there passes a unique maximal integral manifold $N(p)$ of S. Any integral manifold through p is an open submanifold of $N(p)$.*

For the proof, see Chevalley [1, p. 94]. We also state

PROPOSITION 1.3. *Let S be an involutive distribution on a manifold M. Let W be a submanifold of M whose connected components are all integral manifolds of S. Let f be a differentiable mapping of a manifold N*

into M such that $f(N) \subset W$. If W satisfies the second axiom of countability, then f is differentiable as a mapping of N into W.

For the proof, see Chevalley [1, p. 95, Proposition 1]. Replace analyticity there by differentiability throughout and observe that W need not be connected since the differentiability of f is a local matter.

We now define the product of two manifolds M and N of dimension m and n, respectively. If M is defined by an atlas $A = \{(U_i, \varphi_i)\}$ and N is defined by an atlas $B = \{(V_j, \psi_j)\}$, then the natural differentiable structure on the topological space $M \times N$ is defined by an atlas $\{(U_i \times V_j, \varphi_i \times \psi_j)\}$, where $\varphi_i \times \psi_j \colon U_i \times V_j \to \mathbf{R}^{m+n} = \mathbf{R}^m \times \mathbf{R}^n$ is defined in a natural manner. Note that this atlas is not complete even if A and B are complete. For every point (p, q) of $M \times N$, the tangent space $T_{(p,q)}(M \times N)$ can be identified with the direct sum $T_p(M) + T_q(N)$ in a natural manner. Namely, for $X \in T_p(M)$ and $Y \in T_q(N)$, choose curves $x(t)$ and $y(t)$ such that X is tangent to $x(t)$ at $p = x(t_0)$ and Y is tangent to $y(t)$ at $q = y(t_0)$. Then $(X, Y) \in T_p(M) + T_q(N)$ is identified with the vector $Z \in T_{(p,q)}(M \times N)$ which is tangent to the curve $z(t) = (x(t), y(t))$ at $(p, q) = (x(t_0), y(t_0))$. Let $\bar{X} \in T_{(p,q)}(M \times N)$ be the vector tangent to the curve $(x(t), q)$ in $M \times N$ at (p, q). Similarly, let $\bar{Y} \in T_{(p,q)}(M \times N)$ be the vector tangent to the curve $(p, y(t))$ in $M \times N$ at (p, q). In other words, $\bar{X}$ is the image of X by the mapping $M \to M \times N$ which sends $p' \in M$ into (p', q) and $\bar{Y}$ is the image of Y by the mapping $N \to M \times N$ which sends $q' \in N$ into (p, q'). Then $Z = \bar{X} + \bar{Y}$, because, for any function f on $M \times N$, $Zf = (df(x(t), y(t))/dt)_{t=t_0}$ is, by the chain rule, equal to

$$(df(x(t), y(t_0))/dt)_{t=t_0} + (df(x(t_0), y(t))/dt)_{t=t_0} = \bar{X}f + \bar{Y}f.$$

More generally:

Proposition 1.4 (Leibniz's formula). *Let φ be a mapping of the product manifold $M \times N$ into another manifold V. The differential φ_* at $(p, q) \in M \times N$ can be expressed as follows. If $Z \in T_{(p,q)}(M \times N)$ corresponds to $(X, Y) \in T_p(M) + T_q(N)$, then*

$$\varphi_*(Z) = \varphi_{1*}(X) + \varphi_{2*}(Y),$$

where $\varphi_1: M \to V$ *and* $\varphi_2: N \to V$ *are defined by*

$\varphi_1(p') = \varphi(p', q)$ *for* $p' \in M$ *and* $\varphi_2(q') = \varphi(p, q')$ *for* $q' \in N$.

Proof. From the definitions of $\bar{X}$, $\bar{Y}$, φ_1, and φ_2, it follows that $\varphi_*(\bar{X}) = \varphi_{1*}(X)$ and $\varphi_*(\bar{Y}) = \varphi_{2*}(Y)$. Hence, $\varphi_*(Z) = \varphi_*(\bar{X}) + \varphi_*(\bar{Y}) = \varphi_{1*}(X) + \varphi_{2*}(Y)$. QED.

Note that if $V = M \times N$ and φ is the identity transformation, then the preceding proposition reduces to the formula $Z = \bar{X} + \bar{Y}$.

Let X be a vector field on a manifold M. A curve $x(t)$ in M is called an *integral curve* of X if, for every parameter value t_0, the vector $X_{x(t_0)}$ is tangent to the curve $x(t)$ at $x(t_0)$. For any point p_0 of M, there is a unique integral curve $x(t)$ of X, defined for $|t| < \varepsilon$ for some $\varepsilon > 0$, such that $p_0 = x(0)$. In fact, let $u^1, \ldots, u^n$ be a local coordinate system in a neighborhood U of p_0 and let $X = \Sigma\, \xi^j(\partial/\partial u^j)$ in U. Then an integral curve of X is a solution of the following system of ordinary differential equations:

$$du^j/dt = \xi^j(u^1(t), \ldots, u^n(t)), \qquad j = 1, \ldots, n.$$

Our assertion follows from the fundamental theorem for systems of ordinary differential equations (see Appendix 1).

A *1-parameter group of (differentiable) transformations* of M is a mapping of $\mathbf{R} \times M$ into M, $(t, p) \in \mathbf{R} \times M \to \varphi_t(p) \in M$, which satisfies the following conditions:

(1) For each $t \in \mathbf{R}$, $\varphi_t: p \to \varphi_t(p)$ is a transformation of M;

(2) For all $t,s \in \mathbf{R}$ and $p \in M$, $\varphi_{t+s}(p) = \varphi_t(\varphi_s(p))$.

Each 1-parameter group of transformations φ_t induces a vector field X as follows. For every point $p \in M$, X_p is the vector tangent to the curve $x(t) = \varphi_t(p)$, called the *orbit* of p, at $p = \varphi_0(p)$. The orbit $\varphi_t(p)$ is an integral curve of X starting at p. A *local 1-parameter group of local transformations* can be defined in the same way, except that $\varphi_t(p)$ is defined only for t in a neighborhood of 0 and p in an open set of M. More precisely, let I_ε be an open interval $(-\varepsilon, \varepsilon)$ and U an open set of M. A local 1-parameter group of local transformations defined on $I_\varepsilon \times U$ is a mapping of $I_\varepsilon \times U$ into M which satisfies the following conditions:

(1′) For each $t \in I_\varepsilon$, $\varphi_t: p \to \varphi_t(p)$ is a diffeomorphism of U onto the open set $\varphi_t(U)$ of M;

(2′) If $t, s, t+s \in I_\varepsilon$ and if p, $\varphi_s(p) \in U$, then

$$\varphi_{t+s}(p) = \varphi_t(\varphi_s(p)).$$

As in the case of a 1-parameter group of transformations, φ_t induces a vector field X defined on U. We now prove the converse.

PROPOSITION 1.5. *Let X be a vector field on a manifold M. For each point p_0 of M, there exist a neighborhood U of p_0, a positive number ε and a local 1-parameter group of local transformations $\varphi_t\colon U \to M$, $t \in I_\varepsilon$, which induces the given X.*

We shall say that X generates a local 1-parameter group of local transformations φ_t in a neighborhood of p_0. If there exists a (global) 1-parameter group of transformations of M which induces X, then we say that X is *complete.* If $\varphi_t(p)$ is defined on $I_\varepsilon \times M$ for some ε, then X is complete.

Proof. Let $u^1, \dots, u^n$ be a local coordinate system in a neighborhood W of p_0 such that $u^1(p_0) = \cdots = u^n(p_0) = 0$. Let $X = \sum \xi^i(u^1, \dots, u^n)(\partial/\partial u^i)$ in W. Consider the following system of ordinary linear differential equations:

$$df^i/dt = \xi^i(f^1(t), \dots, f^n(t)), \qquad i = 1, \dots, n$$

with unknown functions $f^1(t), \dots, f^n(t)$. By the fundamental theorem for systems of ordinary differential equations (see Appendix 1), there exists a unique set of functions $f^1(t; u), \dots, f^n(t; u)$, defined for $u = (u^1, \dots, u^n)$ with $|u^j| < \delta_1$ and for $|t| < \varepsilon_1$, which form a solution of the differential equation for each fixed u and satisfy the initial conditions:

$$f^i(0; u) = u^i.$$

Set $\varphi_t(u) = (f^1(t; u), \dots, f^n(t; u))$ for $|t| < \varepsilon_1$ and u in $U_1 = \{u; |u^i| < \delta_1\}$. If $|t|$, $|s|$ and $|t+s|$ are all less than ε_1 and both u and $\varphi_s(u)$ are in U_1, then the functions $g^i(t) = f^i(t+s; u)$ are easily seen to be a solution of the differential equation for the initial conditions $g^i(0) = f^i(s; u)$. By the uniqueness of the solution, we have $g^i(t) = f^i(t; \varphi_s(u))$. This proves that $\varphi_t(\varphi_s(u)) = \varphi_{t+s}(u)$. Since φ_0 is the identity transformation of U_1, there exist $\delta > 0$ and $\varepsilon > 0$ such that, for $U = \{u; |u^i| < \delta\}$, $\varphi_t(U) \subset U_1$ if

$|t| < \varepsilon$. Hence, $\varphi_{-t}(\varphi_t(u)) = \varphi_t(\varphi_{-t}(u)) = \varphi_0(u) = u$ for every $u \in U$ and $|t| < \varepsilon$. This proves that φ_t is a diffeomorphism of U for $|t| < \varepsilon$. Thus, φ_t is a local 1-parameter group of local transformations defined on $I_\varepsilon \times U$. From the construction of φ_t, it is obvious that φ_t induces the given vector field X in U. QED.

Remark. In the course of the preceding proof, we showed also that if two local 1-parameter groups of local transformations φ_t and ψ_t defined on $I_\varepsilon \times U$ induce the same vector field on U, they coincide on U.

PROPOSITION 1.6. *On a compact manifold M, every vector field X is complete.*

Proof. For each point $p \in M$, let $U(p)$ be a neighborhood of p and $\varepsilon(p)$ a positive number such that the vector field X generates a local 1-parameter group of local transformations φ_t on $I_{\varepsilon(p)} \times U(p)$. Since M is compact, the open covering $\{U(p); p \in M\}$ has a finite subcovering $\{U(p_i); i = 1, \ldots, k\}$. Let $\varepsilon = \min\{\varepsilon(p_1), \ldots, \varepsilon(p_k)\}$. It is clear that $\varphi_t(p)$ is defined on $I_\varepsilon \times M$ and, hence, on $\mathbf{R} \times M$. QED.

In what follows, we shall not give explicitly the domain of definition for a given vector field X and the corresponding local 1-parameter group of local transformations φ_t. Each formula is valid whenever it makes sense, and it is easy to specify, if necessary, the domain of definition for vector fields or transformations involved.

PROPOSITION 1.7. *Let φ be a transformation of M. If a vector field X generates a local 1-parameter group of local transformations φ_t, then the vector field $\varphi_* X$ generates $\varphi \circ \varphi_t \circ \varphi^{-1}$.*

Proof. It is clear that $\varphi \circ \varphi_t \circ \varphi^{-1}$ is a local 1-parameter group of local transformations. To show that it induces the vector field $\varphi_* X$, let p be an arbitrary point of M and $q = \varphi^{-1}(p)$. Since φ_t induces X, the vector $X_q \in T_q(M)$ is tangent to the curve $x(t) = \varphi_t(q)$ at $q = x(0)$. It follows that the vector

$$(\varphi_* X)_p = \varphi_*(X_q) \in T_p(M)$$

is tangent to the curve $y(t) = \varphi \circ \varphi_t(q) = \varphi \circ \varphi_t \circ \varphi^{-1}(p)$. QED.

COROLLARY 1.8. *A vector field X is invariant by φ, that is, $\varphi_* X = X$, if and only if φ commutes with φ_t.*

We now give a geometric interpretation of the bracket $[X, Y]$ of two vector fields.

PROPOSITION 1.9. *Let X and Y be vector fields on M. If X generates a local 1-parameter group of local transformations φ_t, then*

$$[X, Y] = \lim_{t \to 0} \frac{1}{t} [Y - (\varphi_t)_* Y].$$

More precisely,

$$[X, Y]_p = \lim_{t \to 0} \frac{1}{t} [Y_p - ((\varphi_t)_* Y)_p], \qquad p \in M.$$

The limit on the right hand side is taken with respect to the natural topology of the tangent vector space $T_p(M)$. We first prove two lemmas.

LEMMA 1. *If $f(t, p)$ is a function on $I_\varepsilon \times M$, where I_ε is an open interval $(-\varepsilon, \varepsilon)$, such that $f(0, p) = 0$ for all $p \in M$, then there exists a function $g(t, p)$ on $I_\varepsilon \times M$ such that $f(t, p) = t \cdot g(t, p)$. Moreover, $g(0, p) = f'(0, p)$, where $f' = \partial f / \partial t$, for $p \in M$.*

Proof. It is sufficient to define

$$g(t, p) = \int_0^1 f'(ts, p)\, ds. \qquad \text{QED.}$$

LEMMA 2. *Let X generate φ_t. For any function f on M, there exists a function $g_t(p) = g(t, p)$ such that $f \circ \varphi_t = f + t \cdot g_t$ and $g_0 = Xf$ on M.*

The function $g(t, p)$ is defined, for each fixed $p \in M$, in $|t| < \varepsilon$ for some ε.

Proof. Consider $f(t, p) = f(\varphi_t(p)) - f(p)$ and apply Lemma 1. Then $f \circ \varphi_t = f + t \cdot g_t$. We have

$$\lim_{t \to 0} \frac{1}{t} [f(\varphi_t(p)) - f(p)] = \lim_{t \to 0} \frac{1}{t} f(t, p) = \lim_{t \to 0} g_t(p) = g_0(p).$$

QED.

Proof of Proposition 1.9. Given a function f on M, take a function g_t such that $f \circ \varphi_t = f + t \cdot g_t$ and $g_0 = Xf$ (Lemma 2). Set $p(t) = \varphi_t^{-1}(p)$. Then

$$((\varphi_t)_* Y)_p f = (Y(f \circ \varphi_t))_{p(t)} = (Yf)_{p(t)} + t \cdot (Yg_t)_{p(t)}$$

and

$$\lim_{t\to 0}\frac{1}{t}[Y-(\varphi_t)_*Y]_pf = \lim_{t\to 0}\frac{1}{t}[(Yf)_p-(Yf)_{p(t)}] - \lim_{t\to 0}(Yg_t)_{p(t)}$$
$$= X_p(Yf) - Y_pg_0 = [X,Y]_pf,$$

proving our assertion. QED.

Corollary 1.10. *With the same notations as in Proposition 1.9, we have more generally*

$$(\varphi_s)_*[X, Y] = \lim_{t\to 0}\frac{1}{t}[(\varphi_s)_*Y - (\varphi_{s+t})_*Y]$$

for any value of s.

Proof. For a fixed value of s, consider the vector field $(\varphi_s)_*Y$ and apply Proposition 1.9. Then we have

$$[X, (\varphi_s)_*Y] = \lim_{t\to 0}\frac{1}{t}[(\varphi_s)_*Y - (\varphi_t)_*\circ(\varphi_s)_*Y]$$
$$= \lim_{t\to 0}\frac{1}{t}[(\varphi_s)_*Y - (\varphi_{s+t})_*Y],$$

since $\varphi_s\circ\varphi_t = \varphi_{s+t}$. On the other hand, $(\varphi_s)_*X = X$ by Corollary 1.8. Since $(\varphi_s)_*$ preserves the bracket, we obtain

$$(\varphi_s)_*[X, Y] = [X, (\varphi_s)_*Y].$$

QED.

Remark. The conclusion of Corollary 1.10 can be written as

$$(d((\varphi_t)_*Y)/dt)_{t=s} = -(\varphi_s)_*[X, Y].$$

Corollary 1.11. *Suppose X and Y generate local 1-parameter groups φ_t and ψ_s, respectively. Then $\varphi_t\circ\psi_s = \psi_s\circ\varphi_t$ for every s and t if and only if $[X, Y] = 0$.*

Proof. If $\varphi_t\circ\psi_s = \psi_s\circ\varphi_t$ for every s and t, Y is invariant by every φ_t by Corollary 1.8. By Proposition 1.9, $[X, Y] = 0$. Conversely, if $[X, Y] = 0$, then $d((\varphi_t)_*Y)/dt = 0$ for every t by Corollary 1.10 (see Remark, above). Therefore, $(\varphi_t)_*Y$ is a constantvector at each point p so that Y is invariant by every φ_t. By Corollary 1.8, every ψ_s commutes with every φ_t. QED.

2. *Tensor algebras*

We fix a ground field $\mathbf{F}$ which will be the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$ in our applications. All vector spaces we consider are finite dimensional over $\mathbf{F}$ unless otherwise stated. We define the *tensor product* $U \otimes V$ of two vector spaces U and V as follows. Let $M(U, V)$ be the vector space which has the set $U \times V$ as a basis, i.e., the free vector space generated by the pairs (u, v) where $u \in U$ and $v \in V$. Let N be the vector subspace of $M(U, V)$ spanned by elements of the form

$$(u + u', v) - (u, v) - (u', v), \qquad (u, v + v') - (u, v) - (u, v'),$$
$$(ru, v) - r(u, v), \qquad (u, rv) - r(u, v),$$

where $u, u' \in U$, $v, v' \in V$ and $r \in \mathbf{F}$. We set $U \otimes V = M(U, V)/N$. For every pair (u, v) considered as an element of $M(U, V)$, its image by the natural projection $M(U, V) \to U \otimes V$ will be denoted by $u \otimes v$. Define the *canonical bilinear mapping* φ of $U \times V$ into $U \otimes V$ by

$$\varphi(u, v) = u \otimes v \qquad \text{for } (u, v) \in U \times V.$$

Let W be a vector space and ψ: $U \times V \to W$ a bilinear mapping. We say that the couple (W, ψ) has the *universal factorization property* for $U \times V$ if for every vector space S and every bilinear mapping f: $U \times V \to S$ there exists a unique linear mapping g: $W \to S$ such that $f = g \circ \psi$.

PROPOSITION 2.1. *The couple* $(U \otimes V, \varphi)$ *has the universal factorization property for* $U \times V$. *If a couple* (W, ψ) *has the universal factorization property for* $U \times V$, *then* $(U \otimes V, \varphi)$ *and* (W, ψ) *are isomorphic in the sense that there exists a linear isomorphism* σ: $U \otimes V \to W$ *such that* $\psi = \sigma \circ \varphi$.

Proof. Let S be any vector space and f: $U \times V \to S$ any bilinear mapping. Since $U \times V$ is a basis for $M(U, V)$, we can extend f to a unique linear mapping f': $M(U, V) \to S$. Since f is bilinear, f' vanishes on N. Therefore, f' induces a linear mapping g: $U \otimes V \to S$. Obviously, $f = g \circ \varphi$. The uniqueness of such a mapping g follows from the fact that $\varphi(U \times V)$ spans $U \otimes V$. Let (W, ψ) be a couple having the universal factorization property for $U \times V$. By the universal factorization property of $(U \otimes V, \varphi)$

(resp. of (W, ψ)), there exists a unique linear mapping $\sigma: U \otimes V \to W$ (resp. $\tau: W \to U \otimes V$) such that $\psi = \sigma \circ \varphi$ (resp. $\varphi = \tau \circ \varphi$). Hence, $\varphi = \tau \circ \sigma \circ \varphi$ and $\psi = \sigma \circ \tau \circ \psi$. Using the uniqueness of g in the definition of the universal factorization property, we conclude that $\tau \circ \sigma$ and $\sigma \circ \tau$ are the identity transformations of $U \times V$ and W respectively. QED.

Proposition 2.2. *There is a unique isomorphism of $U \otimes V$ onto $V \otimes U$ which sends $u \otimes v$ into $v \otimes u$ for all $u \in U$ and $v \in V$.*

Proof. Let $f: U \times V \to V \otimes U$ be the bilinear mapping defined by $f(u, v) = v \otimes u$. By Proposition 2.1, there is a unique linear mapping $g: U \otimes V \to V \otimes U$ such that $g(u \otimes v) = v \otimes u$. Similarly, there is a unique linear mapping $g': V \otimes U \to U \otimes V$ such that $g'(v \otimes u) = u \otimes v$. Evidently, $g' \circ g$ and $g \circ g'$ are the identity transformations of $U \otimes V$ and $V \otimes U$ respectively. Hence, g is the desired isomorphism. QED.

The proofs of the following two propositions are similar and hence omitted.

Proposition 2.3. *If we regard the ground field $\mathbf{F}$ as a 1-dimensional vector space over $\mathbf{F}$, there is a unique isomorphism of $\mathbf{F} \otimes U$ onto U which sends $r \otimes u$ into ru for all $r \in \mathbf{F}$ and $u \in U$. Similarly, for $U \otimes \mathbf{F}$ and U.*

Proposition 2.4. *There is a unique isomorphism of $(U \otimes V) \otimes W$ onto $U \otimes (V \otimes W)$ which sends $(u \otimes v) \otimes w$ into $u \otimes (v \otimes w)$ for all $u \in U$, $v \in V$, and $w \in W$.*

Therefore, it is meaningful to write $U \otimes V \otimes W$. Given vector spaces $U_1, \ldots, U_k$, the tensor product $U_1 \otimes \cdots \otimes U_k$ can be defined inductively. Let $\varphi: U_1 \times \cdots \times U_k \to U_1 \otimes \cdots \otimes U_k$ be the multilinear mapping which sends $(u_1, \ldots, u_k)$ into $u_1 \otimes \cdots \otimes u_k$. Then, as in Proposition 2.1, the couple $(U_1 \otimes \cdots \otimes U_k, \varphi)$ can be characterized by the universal factorization property for $U_1 \times \cdots \times U_k$.

Proposition 2.2 can be also generalized. For any permutation π of $(1, \ldots, k)$, there is a unique isomorphism of $U_1 \otimes \cdots \otimes U_k$ onto $U_{\pi(1)} \otimes \cdots \otimes U_{\pi(k)}$ which sends $u_1 \otimes \cdots \otimes u_k$ into $u_{\pi(1)} \otimes \cdots \otimes u_{\pi(k)}$.

Proposition 2.4.1. *Given linear mappings $f_j: U_j \to V_j$, $j = 1, 2$, there is a unique linear mapping $f: U_1 \otimes U_2 \to V_1 \otimes V_2$ such that $f(u_1 \otimes u_2) = f_1(u_1) \otimes f_2(u_2)$ for all $u_1 \in U_1$ and $u_2 \in U_2$.*

Proof. Consider the bilinear mapping $U_1 \times U_2 \rightarrow V_1 \otimes V_2$ which sends (u_1, u_2) into $f_1(u_1) \otimes f_2(u_2)$ and apply Proposition 2.1. QED.

The generalization of Proposition 2.4.1 to the case with more than two mappings is obvious. The mapping f just given will be denoted by $f_1 \otimes f_2$.

PROPOSITION 2.5. *If $U_1 + U_2$ denotes the direct sum of U_1 and U_2, then*

$$(U_1 + U_2) \otimes V = U_1 \otimes V + U_2 \otimes V.$$

Similarly,

$$U \otimes (V_1 + V_2) = U \otimes V_1 + U \otimes V_2.$$

Proof. Let $i_1: U_1 \rightarrow U_1 + U_2$ and $i_2: U_2 \rightarrow U_1 + U_2$ be the injections. Let $p_1: U_1 + U_2 \rightarrow U_1$ and $p_2: U_1 + U_2 \rightarrow U_2$ be the projections. Then $p_1 \circ i_1$ and $p_2 \circ i_2$ are the identity transformations of U_1 and U_2 respectively. Both $p_2 \circ i_1$ and $p_1 \circ i_2$ are the zero mappings. By Proposition 2.4.1, i_1 and the identity transformation of V induce a linear mapping $\bar{i}_1: U_1 \otimes V \rightarrow (U_1 + U_2) \otimes V$. Similarly, $\bar{i}_2$, $\bar{p}_1$, and $\bar{p}_2$ are defined. It follows that $\bar{p}_1 \circ \bar{i}_1$ and $\bar{p}_2 \circ \bar{i}_2$ are the identity transformations of $U_1 \otimes V$ and $U_2 \otimes V$ respectively and $\bar{p}_2 \circ \bar{i}_1$ and $\bar{p}_1 \circ \bar{i}_2$ are the zero mappings. This proves the first isomorphism. The proof for the second is similar. QED.

By the induction, we obtain

$$(U_1 + \cdots + U_k) \otimes V = U_1 \otimes V + \cdots + U_k \otimes V.$$

PROPOSITION 2.6. *If $u_1, \ldots, u_m$ is a basis for U and $v_1, \ldots, v_n$ is a basis for V, then $\{u_i \otimes v_j;\ i = 1, \ldots, m;\ j = 1, \ldots, n\}$ is a basis for $U \otimes V$. In particular,* $\dim U \otimes V = \dim U \dim V$.

Proof. Let U_i be the 1-dimensional subspace of U spanned by u_i and V_j the 1-dimensional subspace of V spanned by v_j. By Proposition 2.5,

$$U \otimes V = \Sigma_{i,j}\, U_i \otimes V_j.$$

By Proposition 2.3, each $U_i \otimes V_j$ is a 1-dimensional vector space spanned by $u_i \otimes v_j$. QED.

For a vector space U, we denote by U^* the dual vector space of U. For $u \in U$ and $u^* \in U^*$, $\langle u, u^* \rangle$ denotes the value of the linear functional u^* on u.

PROPOSITION 2.7. *Let $L(U^*, V)$ be the space of linear mappings of U^* into V. Then there is a unique isomorphism g of $U \otimes V$ onto $L(U^*, V)$ such that*

$$(g(u \otimes v))u^* = \langle u, u^* \rangle v \qquad \text{for all } u \in U, v \in V \text{ and } u^* \in U^*.$$

Proof. Consider the bilinear mapping $f: U \times V \to L(U^*, V)$ defined by $(f(u, v))u^* = \langle u, u^* \rangle v$ and apply Proposition 2.1. Then there is a unique linear mapping $g: U \otimes V \to L(U^*, V)$ such that $(g(u \otimes v))u^* = \langle u, u^* \rangle v$. To prove that g is an isomorphism, let $u_1, \ldots, u_m$ be a basis for U, $u_1^*, \ldots, u_m^*$ the dual basis for U^* and $v_1, \ldots, v_n$ a basis for V. We shall show that $\{g(u_i \otimes v_j); i = 1, \ldots, m; j = 1, \ldots, n\}$ is linearly independent. If $\Sigma\, a_{ij} g(u_i \otimes v_j) = 0$ where $a_{ij} \in \mathbf{F}$, then

$$0 = (\Sigma\, a_{ij} g(u_i \otimes v_j))u_k^* = \Sigma\, a_{kj} v_j$$

and, hence, all a_{ij} vanish. Since $\dim U \otimes V = \dim L(U^*, V)$, g is an isomorphism of $U \times V$ onto $L(U^*, V)$. QED.

PROPOSITION 2.8. *Given two vector spaces U and V, there is a unique isomorphism g of $U^* \otimes V^*$ onto $(U \otimes V)^*$ such that*

$$(g(u^* \otimes v^*))(u \otimes v) = \langle u, u^* \rangle \langle v, v^* \rangle$$

$$\text{for all } u \in U,\ u^* \in U^*,\ v \in V,\ v^* \in V^*.$$

Proof. Apply Proposition 2.1 to the bilinear mapping $f: U^* \times V^* \to (U \otimes V)^*$ defined by $(f(u^*, v^*))(u \otimes v) = \langle u, u^* \rangle \langle v, v^* \rangle$. To prove that g is an isomorphism, take bases for U, V, U^*, and V^* and proceed as in the proof of Proposition 2.7. QED.

We now define various tensor spaces over a fixed vector space V. For a positive integer r, we shall call $\mathbf{T}^r = V \otimes \cdots \otimes V$ (r times tensor product) the *contravariant tensor space of degree r*. An element of $\mathbf{T}^r$ will be called a *contravariant tensor of degree r*. If $r = 1$, $\mathbf{T}^1$ is nothing but V. By convention, we agree that $\mathbf{T}^0$ is the ground field $\mathbf{F}$ itself. Similarly, $\mathbf{T}_s = V^* \otimes \cdots \otimes V^*$ (s times tensor product) is called the *covariant tensor space of degree s* and its elements *covariant tensors of degree s*. Then $\mathbf{T}_1 = V^*$ and, by convention, $\mathbf{T}_0 = \mathbf{F}$.

We shall give the expressions for these tensors with respect to a basis of V. Let $e_1, \ldots, e_n$ be a basis for V and $e^1, \ldots, e^n$ the dual basis for V^*. By Proposition 2.6, $\{e_{i_1} \otimes \cdots \otimes e_{i_r}; 1 \leqq i_1, \ldots, i_r \leqq n\}$ is a basis for $\mathbf{T}_r$. Every contravariant tensor K of degree r can be expressed uniquely as a linear combination

$$K = \Sigma_{i_1, \ldots, i_r} K^{i_1 \cdots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r},$$

where $K^{i_1 \cdots i_r}$ are the *components* of K with respect to the basis $e_1, \ldots, e_n$ of V. Similarly, every covariant tensor L of degree s can be expressed uniquely as a linear combination

$$L = \Sigma_{j_1, \ldots, j_r} L_{j_1 \ldots j_r} e^{j_1} \otimes \cdots \otimes e^{j_r},$$

where $L_{j_1 \ldots j_r}$ are the components of L.

For a change of basis of V, the components of tensors are subject to the following transformations. Let $e_1, \ldots, e_n$ and $\bar{e}_1, \ldots, \bar{e}_n$ be two bases of V related by a linear transformation

$$\bar{e}_i = \Sigma_j A_i^j e_j, \qquad i = 1, \ldots, n.$$

The corresponding change of the dual bases in V^* is given by

$$\bar{e}^i = \Sigma_j B_j^i e^j, \qquad i = 1, \ldots, n,$$

where $B = (B_j^i)$ is the inverse matrix of the matrix $A = (A_j^i)$ so that

$$\Sigma_j A_j^i B_k^j = \delta_k^i.$$

If K is a contravariant tensor of degree r, its components $K^{i_1 \cdots i_r}$ and $\bar{K}^{i_1 \cdots i_r}$ with respect to $\{e_i\}$ and $\{\bar{e}_i\}$ respectively are related by

$$\bar{K}^{i_1 \cdots i_r} = \Sigma_{j_1, \ldots, j_r} A_{j_1}^{i_1} \cdots A_{j_r}^{i_r} K^{j_1 \cdots j_r}.$$

Similarly, the components of a covariant tensor L of degree s are related by

$$\bar{L}_{i_1 \ldots i_s} = \Sigma_{j_1, \ldots, j_s} B_{i_1}^{j_1} \cdots B_{i_s}^{j_s} L_{j_1 \ldots j_s}.$$

The verification of these formulas is left to the reader.

We define the (mixed) *tensor space of type* (r, s), or *tensor space of contravariant degree* r *and covariant degree* s, as the tensor product $\mathbf{T}_s^r = \mathbf{T}^r \otimes \mathbf{T}_s = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ (V: r times, V^*: s times). In particular, $\mathbf{T}_0^r = \mathbf{T}^r$, $\mathbf{T}_s^0 = \mathbf{T}_s$ and $\mathbf{T}_0^0 = \mathbf{F}$. An element of $\mathbf{T}_s^r$ is called a *tensor of type* (r, s), or *tensor of contravariant degree* r *and covariant degree* s. In terms of a basis $e_1, \ldots, e_n$ of V and the dual basis $e^1, \ldots, e^n$ of V^*, every tensor K of type (r, s) can be expressed uniquely as

$$K = \Sigma_{i_1,\ldots,i_r,j_1,\ldots,j_s} K^{i_1\cdots i_r}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$

where $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ are called the components of K with respect to the basis $e_1, \ldots, e_n$. For a change of basis $\bar{e}_i = \Sigma_j A^j_i e_j$, we have the following transformations of components:

$$\bar{K}^{i_1\cdots i_r}_{j_1\cdots j_s} = \Sigma A^{i_1}_{k_1} \cdots A^{i_r}_{k_r} B^{m_1}_{j_1} \cdots B^{m_s}_{j_s} K^{k_1\cdots k_r}_{m_1\cdots m_s}. \tag{2.1}$$

Set $\mathbf{T} = \Sigma^\infty_{r,s=0} \mathbf{T}^r_s$, so that an element of $\mathbf{T}$ is of the form $\Sigma^\infty_{r,s=0} K^r_s$, where $K^r_s \in \mathbf{T}^r_s$ are zero except for a finite number of them. We shall now make $\mathbf{T}$ into an associative algebra over $\mathbf{F}$ by defining the product of two tensors $K \in \mathbf{T}^r_s$ and $L \in \mathbf{T}^p_q$ as follows. From the universal factorization property of the tensor product, it follows that there exists a unique bilinear mapping of $\mathbf{T}^r_s \times \mathbf{T}^p_q$ into $\mathbf{T}^{r+p}_{s+q}$ which sends $(v_1 \otimes \cdots \otimes v_r \otimes v^*_1 \otimes \cdots \otimes v^*_s, w_1 \otimes \cdots \otimes w_p \otimes w^*_1 \otimes \cdots \otimes w^*_q) \in \mathbf{T}^r_s \times \mathbf{T}^p_q$ into $v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_p \otimes v^*_1 \otimes \cdots \otimes v^*_s \otimes w^*_1 \otimes \cdots \otimes w^*_q \in \mathbf{T}^{r+p}_{s+q}$. The image of $(K, L) \in \mathbf{T}^r_s \times \mathbf{T}^p_q$ by this bilinear mapping will be denoted by $K \otimes L$. In terms of components, if K is given by $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ and L is given by $L^{k_1\cdots k_p}_{m_1\cdots m_q}$, then

$$(K \otimes L)^{i_1\cdots i_{r+p}}_{j_1\cdots j_{s+q}} = K^{i_1\cdots i_r}_{j_1\cdots j_s} L^{i_{r+1}\cdots i_{r+p}}_{j_{s+1}\cdots j_{s+q}}.$$

We now define the notion of *contraction*. Let $r,s \geqq 1$. To each ordered pair of integers (i, j) such that $1 \leqq i \leqq r$ and $1 \leqq j \leqq s$, we associate a linear mapping, called the contraction and denoted by C, of $\mathbf{T}^r_s$ into $\mathbf{T}^{r-1}_{s-1}$ which maps $v_1 \otimes \cdots \otimes v_r \otimes v^*_1 \otimes \cdots \otimes v^*_s$ into

$$\langle v_i, v^*_j \rangle v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_r \otimes v^*_1 \otimes \cdots \otimes v^*_{j-1} \otimes v^*_{j+1} \otimes \cdots \otimes v^*_s,$$

where $v_1, \ldots, v_r \in V$ and $v^*_1, \ldots, v^*_s \in V^*$. The uniqueness and the existence of C follow from the universal factorization property of the tensor product. In terms of components, the contraction C maps a tensor $K \in \mathbf{T}^r_s$ with components $K^{i_1\cdots i_r}_{j_1\cdots j_s}$ into a tensor $CK \in \mathbf{T}^{r-1}_{s-1}$ whose components are given by

$$(CK)^{i_1\cdots i_{r-1}}_{j_1\cdots j_{s-1}} = \Sigma_k K^{i_1\cdots k \cdots i_r}_{j_1\cdots k \cdots j_s},$$

where the superscript k appears at the i-th position and the subscript k appears at the j-th position.

We shall now interpret tensors as multilinear mappings.

PROPOSITION 2.9. *$\mathbf{T}_r$ is isomorphic, in a natural way, to the vector space of all r-linear mappings of $V \times \cdots \times V$ into $\mathbf{F}$.*

PROPOSITION 2.10. *$\mathbf{T}^r$ is isomorphic, in a natural way, to the vector space of all r-linear mappings of $V^* \times \cdots \times V^*$ into $\mathbf{F}$.*

Proof. We prove only Proposition 2.9. By generalizing Proposition 2.8, we see that $\mathbf{T}_r = V^* \otimes \cdots \otimes V^*$ is the dual vector space of $\mathbf{T}^r = V \otimes \cdots \otimes V$. On the other hand, it follows from the universal factorization property of the tensor product that the space of linear mappings of $\mathbf{T}^r = V \otimes \cdots \otimes V$ into $\mathbf{F}$ is isomorphic to the space of r-linear mappings of $V \times \cdots \times V$ into $\mathbf{F}$. QED.

Following the interpretation in Proposition 2.9, we consider a tensor $K \in \mathbf{T}_r$ as an r-linear mapping $V \times \cdots \times V \to \mathbf{F}$ and write $K(v_1, \ldots, v_r) \in \mathbf{F}$ for $v_1, \ldots, v_r \in V$.

PROPOSITION 2.11. *$\mathbf{T}^1_r$ is isomorphic, in a natural way, to the vector space of all r-linear mappings of $V \times \cdots \times V$ into V.*

Proof. $\mathbf{T}^1_r$ is, by definition, $V \otimes \mathbf{T}_r$ which is canonically isomorphic with $\mathbf{T}_r \otimes V$ by Proposition 2.2. By Proposition 2.7, $\mathbf{T}_r \otimes V$ is isomorphic to the space of linear mappings of the dual space of $\mathbf{T}_r$, that is $\mathbf{T}^r$, into V. Again, by the universal factorization property of the tensor product, the space of linear mappings of $\mathbf{T}^r$ into V can be identified with the space of r-linear mappings of $V \times \cdots \times V$ into V. QED.

With this interpretation, any tensor K of type $(1, r)$ is an r-linear mapping of $V \times \cdots \times V$ into V which maps $(v_1, \ldots, v_r)$ into $K(v_1, \ldots, v_r) \in V$. If $e_1, \ldots, e_n$ is a basis for V, then $K = \Sigma K^i_{j_1 \ldots j_r} e_i \otimes e^{j_1} \otimes \cdots \otimes e^{j_r}$ corresponds to an r-linear mapping of $V \times \cdots \times V$ into V such that $K(e_{j_1}, \ldots, e_{j_r}) = \Sigma_i K^i_{j_1 \ldots j_r} e_i$.

Similar interpretation can be made for tensors of type (r, s) in general, but we shall not go into it.

Example 2.1. If $v \in V$ and $v^* \in V^*$, then $v \otimes v^*$ is a tensor of type $(1, 1)$. The contraction C: $\mathbf{T}^1_1 \to \mathbf{F}$ maps $v \otimes v^*$ into $\langle v, v^* \rangle$. In general, a tensor K of type $(1, 1)$ can be regarded as a linear endomorphism of V and the contraction CK of K is then the trace of the corresponding endomorphism. In fact, if $e_1, \ldots, e_n$ is a basis for V and K has components K^i_j with respect to this basis, then the endomorphism corresponding to K sends e_j into $\Sigma_i K^i_j e_i$. Clearly, the trace of K and the contraction CK of K are both equal to $\Sigma_i K^i_i$.

Example 2.2. An *inner product* g on a real vector space V is a covariant tensor of degree 2 which satisfies (1) $g(v, v) \geqq 0$ and $g(v, v) = 0$ if and only if $v = 0$ (positive definite) and (2) $g(v, v') = g(v', v)$ (symmetric).

Let $\mathbf{T}(U)$ and $\mathbf{T}(V)$ be the tensor algebras over vector spaces U and V. If A is a linear isomorphism of U onto V, then its transpose A^* is a linear isomorphism of V^* onto U^* and A^{*-1} is a linear isomorphism of U^* onto V^*. By Proposition 2.4, we obtain a linear isomorphism $A \otimes A^{*-1}$: $U \otimes U^* \rightarrow V \otimes V^*$. In general, we obtain a linear isomorphism of $\mathbf{T}(U)$ onto $\mathbf{T}(V)$ which maps $\mathbf{T}^r_s(U)$ onto $\mathbf{T}^r_s(V)$. This isomorphism, called the *extension* of A and denoted by the same letter A, is the unique algebra isomorphism $\mathbf{T}(U) \rightarrow \mathbf{T}(V)$ which extends A: $U \rightarrow V$; the uniqueness follows from the fact that $\mathbf{T}(U)$ is generated by $\mathbf{F}$, U and U^*. It is also easy to see that the extension of A commutes with every contraction C.

PROPOSITION 2.12. *There is a natural* $1:1$ *correspondence between the linear isomorphisms of a vector space U onto another vector space V and the algebra isomorphisms of* $\mathbf{T}(U)$ *onto* $\mathbf{T}(V)$ *which preserve type and commute with contractions.*

In particular, the group of automorphisms of V is isomorphic, in a natural way, with the group of automorphisms of the tensor algebra $\mathbf{T}(V)$ *which presserve type and commute with contractions.*

Proof. The only thing which has to be proved now is that every algebra isomorphism, say f, of $\mathbf{T}(U)$ onto $\mathbf{T}(V)$ is induced from an isomorphism A of U onto V, provided that f preserves type and commutes with contractions. Since f is type-preserving, it maps $\mathbf{T}^1_0(U) = U$ isomorphically onto $\mathbf{T}^1_0(V) = V$. Denote the restriction of f to U by A. Since f maps every element of the field $\mathbf{F} = \mathbf{T}^0_0$ into itself and commutes with every contraction C, we have, for all $u \in U$ and $u^* \in U^*$,

$$\begin{aligned}\langle Au, fu^* \rangle = \langle fu, fu^* \rangle = C(fu \otimes fu^*) &= C(f(u \otimes u^*)) \\ = f(C(u \otimes u^*)) &= f(\langle u, u^* \rangle) = \langle u, u^* \rangle.\end{aligned}$$

Hence, $fu^* = A^{*-1}u^*$. The extension of A and f agrees on $\mathbf{F}$, U and U^*. Since the tensor algebra $\mathbf{T}(U)$ is generated by $\mathbf{F}$, U and U^*, f coincides with the extension of A. QED.

Let $\mathbf{T}$ be the tensor algebra over a vector space V. A linear endomorphism D of $\mathbf{T}$ is called a *derivation* if it satisfies the following conditions:

(a) D is type-preserving, i.e., D maps $\mathbf{T}^r_s$ into itself;
(b) $D(K \otimes L) = DK \otimes L + K \otimes DL$ for all tensors K and L;
(c) D commutes with every contraction C.

The set of derivations of $\mathbf{T}$ forms a vector space. It forms a Lie algebra if we set $[D, D'] = DD' - D'D$ for derivations D and D'. Similarly, the set of linear endomorphisms of V forms a Lie algebra. Since a derivation D maps $\mathbf{T}^1_0 = V$ into itself by (a), it induces an endomorphism, say B, of V.

Proposition 2.13. *The Lie algebra of derivations of* $\mathbf{T}(V)$ *is isomorphic with the Lie algebra of endomorphisms of* V. *The isomorphism is given by assigning to each derivation its restriction to* V.

Proof. It is clear that $D \to B$ is a Lie algebra homomorphism. From (b) it follows easily that D maps every element of $\mathbf{F}$ into 0. Hence, for $v \in V$ and $v^* \in V^*$, we have

$$\begin{aligned} 0 = D(\langle v, v^*\rangle) &= D(C(v \otimes v^*)) = C(D(v \otimes v^*)) \\ &= C(Dv \otimes v^* + v \otimes Dv^*) = \langle Dv, v^*\rangle + \langle v, Dv^*\rangle. \end{aligned}$$

Since $Dv = Bv$, $Dv^* = -B^*v^*$ where B^* is the transpose of B. Since $\mathbf{T}$ is generated by $\mathbf{F}$, V and V^*, D is uniquely determined its restriction to $\mathbf{F}$, V and V^*. It follows that $D \to B$ is injective. Conversely, given an endomorphism B of V, we define $Da = 0$ for $a \in \mathbf{F}$, $Dv = Bv$ for $v \in V$ and $Dv^* = -B^*v^*$ for $v^* \in V^*$ and, then, extend D to a derivation of $\mathbf{T}$ by (b). The existence of D follows from the universal factorization property of the tensor product. QED.

Example 2.3. Let K be a tensor of type (1, 1) and consider it as an endomorphism of V. Then the automorphism of $\mathbf{T}(V)$ induced by an automorphism A of V maps the tensor K into the tensor AKA^{-1}. On the other hand, the derivation of $\mathbf{T}(V)$ induced by an endomorphism B of V maps K into $[B, K] = BK - KB$.

3. *Tensor fields*

Let $T_x = T_x(M)$ be the tangent space to a manifold M at a point x and $\mathbf{T}(x)$ the tensor algebra over T_x: $\mathbf{T}(x) = \Sigma\, \mathbf{T}^r_s(x)$, where $\mathbf{T}^r_s(x)$ is the tensor space of type (r, s) over T_x. A *tensor field of type* (r, s) on a subset N of M is an assignment of a tensor $K_x \in \mathbf{T}^r_s(x)$ to each point x of N. In a coordinate neighborhood U with a local coordinate system $x^1, \dots, x^n$, we take $X_i = \partial/\partial x^i$, $i = 1, \dots, n$, as a basis for each tangent space T_x, $x \in U$, and $\omega^i = dx^i$, $i = 1, \dots, n$, as the dual basis of T^*_x. A tensor field K of type (r, s) defined on U is then expressed by

$$K_x = \Sigma\, K^{i_1 \cdots i_r}_{j_1 \cdots j_s} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_s},$$

where $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are functions on U, called the *components* of K with respect to the local coordinate system $x^1, \dots, x^n$. We say that K is of class C^k if its components $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ are functions of class C^k; of course, it has to be verified that this notion is independent of a local coordinate system. This is easily done by means of the formula (2.1) where the matrix (A^i_j) is to be replaced by the Jacobian matrix between two local coordinate systems. From now on, we shall mean by a tensor field that of class C^∞ unless otherwise stated.

In section 5, we shall interpret a tensor field as a differentiable cross section of a certain fibre bundle over M. We shall give here another interpretation of tensor fields of type $(0, r)$ and $(1, r)$ from the viewpoint of Propositions 2.9 and 2.11. Let $\mathfrak{F}$ be the algebra of functions (of class C^∞) on M and $\mathfrak{X}$ the $\mathfrak{F}$-module of vector fields on M.

PROPOSITION 3.1. *A tensor field K of type $(0, r)$ (resp. type $(1, r)$) on M can be considered as an r-linear mapping of $\mathfrak{X} \times \cdots \times \mathfrak{X}$ into $\mathfrak{F}$ (resp. $\mathfrak{X}$) such that*

$$K(f_1 X_1, \dots, f_r X_r) = f_1 \cdots f_r K(X_1, \dots, X_r) \qquad \text{for } f_i \in \mathfrak{F} \quad \textit{and} \quad X_i \in \mathfrak{X}.$$

Conversely, any such mapping can be considered as a tensor field of type $(0, r)$ *(resp. type* $(1, r)$*).*

Proof. Given a tensor field K of type $(0, r)$ (resp. type $(1, r)$), K_x is an r-linear mapping of $T_x \times \cdots \times T_x$ into $\mathbf{R}$ (resp. T_x)

by Proposition 2.9 (resp. Proposition 2.11) and hence $(X_1, \ldots, X_r) \to (K(X_1, \ldots, X_r))_x = K_x((X_1)_x, \ldots, (X_r)_x)$ is an r-linear mapping of $\mathfrak{X} \times \cdots \times \mathfrak{X}$ into $\mathfrak{F}$ (resp. $\mathfrak{X}$) satisfying the preceding condition. Conversely, let $K: \mathfrak{X} \times \cdots \times \mathfrak{X} \to \mathfrak{F}$ (resp. $\mathfrak{X}$) be an r-linear mapping over $\mathfrak{F}$. The essential point of the proof is to show that the value of the function (resp. the vector field) $K(X_1, \ldots, X_r)$ at a point x depends only on the values of X_i at x. This will imply that K induces an r-linear mapping of $T_x(M) \times \cdots \times T_x(M)$ into $\mathbf{R}$ (resp. $T_x(M)$) for each x. We first observe that the mapping K can be localized. Namely, we have

Lemma. *If $X_i = Y_i$ in a neighborhood U of x for $i = 1, \ldots, r$, then we have*

$$K(X_1, \ldots, X_r) = K(Y_1, \ldots, Y_r) \quad \textit{in } U.$$

Proof of Lemma. It is sufficient to show that if $X_1 = 0$ in U, then $K(X_1, \ldots, X_r) = 0$ in U. For any $y \in U$, let f be a differentiable function on M such that $f(y) = 0$ and $f = 1$ outside U. Then $X_1 = fX_1$ and $K(X_1, \ldots, X_r) = f\,K(X_1, \ldots, X_r)$, which vanishes at y. This proves the lemma.

To complete the proof of Proposition 3.1, it is sufficient to show that if X_1 vanishes at a point x, so does $K(X_1, \ldots, X_r)$. Let $x^1, \ldots, x^n$ be a coordinate system around x so that $X_1 = \Sigma_i f^i\,(\partial/\partial x^i)$. We may take vector fields Y_i and differentiable functions g^i on M such that $g^i = f^i$ and $Y_i = (\partial/\partial x^i)$ for $i = 1, \ldots, n$ in some neighborhood U of x. Then $X_1 = \Sigma_i\, g^i Y_i$ in U. By the lemma, $K(X_1, \ldots, X_r) = \Sigma_i\, g^i \cdot K(Y_i, X_2, \ldots, X_r)$ in U. Since $g^i(x) = f^i(x) = 0$ for $i = 1, \ldots, n$, $K(X_1, \ldots, X_r)$ vanishes at x. QED.

Example 3.1. A (positive definite) *Riemannian metric* on M is a covariant tensor field g of degree 2 which satisfies (1) $g(X, X) \geqq 0$ for all $X \in \mathfrak{X}$, and $g(X, X) = 0$ if and only if $X = 0$ and (2) $g(Y, X) = g(X, Y)$ for all $X, Y \in \mathfrak{X}$. In other words, g assigns an inner product in each tangent space $T_x(M)$, $x \in M$ (cf. Example 2.2). In terms of a local coordinate system $x^1, \ldots, x^n$, the components of g are given by $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$. It has been customary to write $ds^2 = \Sigma\, g_{ij}\, dx^i\, dx^j$ for g.

Example 3.2. A differential form ω of degree r is nothing but a covariant tensor field of degree r which is skew-symmetric:

$$\omega(X_{\pi(1)}, \ldots, X_{\pi(r)}) = \varepsilon(\pi)\omega(X_1, \ldots, X_r),$$

where π is an arbitrary permutation of $(1, 2, \ldots, r)$ and $\varepsilon(\pi)$ is its sign. For any covariant tensor K at x or any covariant tensor field K on M, we define the *alternation* A as follows:

$$(AK)(X_1, \ldots, X_r) = \frac{1}{r!}\Sigma_\pi\, \varepsilon(\pi)\cdot K(X_{\pi(1)}, \ldots, X_{\pi(r)}),$$

where the summation is taken over all permutations π of $(1, 2, \ldots, r)$. It is easy to verify that AK is skew-symmetric for any K and that K is skew-symmetric if and only if $AK = K$. If ω and ω' are differential forms of degree r and s respectively, then $\omega \otimes \omega'$ is a covariant tensor field of degree $r + s$ and $\omega \wedge \omega' = A(\omega \otimes \omega')$.

Example 3.3. The *symmetrization* S can be defined as follows. If K is a covariant tensor or tensor field of degree r, then

$$(SK)(X_1, \ldots, X_r) = \frac{1}{r!}\Sigma_\pi\, K(X_{\pi(1)}, \ldots, X_{\pi(r)}).$$

For any K, SK is symmetric and $SK = K$ if and only if K is symmetric.

We now proceed to define the notion of Lie differentiation. Let $\mathfrak{T}^r_s(M)$ be the set of tensor fields of type (r, s) defined on M and set $\mathfrak{T}(M) = \Sigma^\infty_{r,s=0}\mathfrak{T}^r_s(M)$. Then $\mathfrak{T}(M)$ is an algebra over the real number field $\mathbf{R}$, the multiplication $\otimes$ being defined pointwise, i.e., if $K, L \in \mathfrak{T}(M)$ then $(K \otimes L)_x = K_x \otimes L_x$ for all $x \in M$. If φ is a transformation of M, its differential φ_* gives a linear isomorphism of the tangent space $T_{\varphi^{-1}(x)}(M)$ onto the tangent space $T_x(M)$. By Proposition 2.12, this linear isomorphism can be extended to an isomorphism of the tensor algebra $\mathbf{T}(\varphi^{-1}(x))$ onto the tensor algebra $\mathbf{T}(x)$, which we denote by $\tilde{\varphi}$. Given a tensor field K, we define a tensor field $\tilde{\varphi}K$ by

$$(\tilde{\varphi}K)_x = \tilde{\varphi}(K_{\varphi^{-1}(x)}), \qquad x \in M.$$

In this way, every transformation φ of M induces an algebra automorphism of $\mathfrak{T}(M)$ which preserves type and commutes with contractions.

Let X be a vector field on M and φ_t a local 1-parameter group of local transformations generated by X (cf. Proposition 1.5). We shall define the *Lie derivative* $L_X K$ *of a tensor field* K *with respect to a vector field* X as follows. For the sake of simplicity, we assume that φ_t is a global 1-parameter group of transformations of M; the reader will have no difficulty in modifying the definition when X is not complete. For each t, $\tilde{\varphi}_t$ is an automorphism of the algebra $\mathfrak{T}(M)$. For any tensor field K on M, we set

$$(L_X K)_x = \lim_{t \to 0} \frac{1}{t} [K_x - (\tilde{\varphi}_t K)_x].$$

The mapping L_X of $\mathfrak{T}(M)$ into itself which sends K into $L_X K$ is called the *Lie differentiation with respect to* X. We have

PROPOSITION 3.2. *Lie differentiation* L_X *with respect to a vector field* X *satisfies the following conditions:*

(a) L_X *is a derivation of* $\mathfrak{T}(M)$, *that is, it is linear and satisfies*

$$L_X(K \otimes K') = (L_X K) \otimes K' + K \otimes (L_X K') \qquad \text{for all } K, K' \in \mathfrak{T}(M);$$

(b) L_X *is type-preserving:* $L_X(\mathfrak{T}^r_s(M)) \subset \mathfrak{T}^r_s(M)$;

(c) L_X *commutes with every contraction of a tensor field;*

(d) $L_X f = Xf$ *for every function* f;

(e) $L_X Y = [X, Y]$ *for every vector field* Y.

Proof. It is clear that L_X is linear. Let φ_t be a local 1-parameter group of local transformations generated by X. Then

$$\begin{aligned} L_X(K \otimes K') &= \lim_{t \to 0} \frac{1}{t} [K \otimes K' - \tilde{\varphi}_t(K \otimes K')] \\ &= \lim_{t \to 0} \frac{1}{t} [K \otimes K' - (\tilde{\varphi}_t K) \otimes (\tilde{\varphi}_t K')] \\ &= \lim_{t \to 0} \frac{1}{t} [K \otimes K' - (\tilde{\varphi}_t K) \otimes K'] \\ &\qquad + \lim_{t \to 0} \frac{1}{t} [(\tilde{\varphi}_t K) \otimes K' - (\tilde{\varphi}_t K) \otimes (\tilde{\varphi}_t K')] \\ &= \left(\lim_{t \to 0} \frac{1}{t} [K - (\tilde{\varphi}_t K)]\right) \otimes K' \\ &\qquad + \lim_{t \to 0} (\tilde{\varphi}_t K) \otimes \left(\frac{1}{t} [K' - (\tilde{\varphi}_t K')]\right) \\ &= (L_X K) \otimes K' + K \otimes (L_X K'). \end{aligned}$$

Since $\tilde{\varphi}_t$ preserves type and commutes with contractions, so does L_X. If f is a function on M, then

$$(L_X f)(x) = \lim_{t \to 0} \frac{1}{t} [f(x) - f(\varphi_t^{-1} x)] = -\lim_{t \to 0} \frac{1}{t} [f(\varphi_t^{-1} x) - f(x)].$$

If we observe that $\varphi_t^{-1} = \varphi_{-t}$ is a local 1-parameter group of local transformations generated by $-X$, we see that $L_X f = -(-X)f = Xf$. Finally (e) is a restatement of Proposition 1.9. QED.

By a *derivation* of $\mathfrak{T}(M)$, we shall mean a mapping of $\mathfrak{T}(M)$ into itself which satisfies conditions (a), (b) and (c) of Proposition 3.2.

Let S be a tensor field of type (1, 1). For each $x \in M$, S_x is a linear endomorphism of the tangent space $T_x(M)$. By Proposition 2.13, S_x can be uniquely extended to a derivation of the tensor algebra $\mathbf{T}(x)$ over $T_x(M)$. For every tensor field K, define SK by $(SK)_x = S_x K_x$, $x \in M$. Then S is a derivation of $\mathfrak{T}(M)$. We have

PROPOSITION 3.3. *Every derivation D of $\mathfrak{T}(M)$ can be decomposed uniquely as follows:*

$$D = L_X + S,$$

where X is a vector field and S is a tensor field of type (1, 1).

Proof. Since D is type-preserving, it maps $\mathfrak{F}(M)$ into itself and satisfies $D(fg) = Df \cdot g + f \cdot Dg$ for $f, g \in \mathfrak{F}(M)$. It follows that there is a vector field X such that $Df = Xf$ for every $f \in \mathfrak{F}(M)$. Clearly, $D - L_X$ is a derivation of $\mathfrak{T}(M)$ which is zero on $\mathfrak{F}(M)$. We shall show that any derivation D which is zero on $\mathfrak{F}(M)$ is induced by a tensor field of type (1, 1). For any vector field Y, DY is a vector field and, for any function f, $D(fY) = Df \cdot Y + f \cdot DY = f \cdot DY$ since $Df = 0$ by assumption. By Proposition 3.1, there is a unique tensor field S of type (1, 1) such that $DY = SY$ for every vector field Y. To show that D coincides with the derivation induced by S, it is sufficient to prove the following

LEMMA. *Two derivations D_1 and D_2 of $\mathfrak{T}(M)$ coincide if they coincide on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$.*

Proof. We first observe that a derivation D can be localized, that is, if a tensor field K vanishes on an open set U, then DK vanishes on U. In fact, for each $x \in U$, let f be a function such that $f(x) = 0$ and $f = 1$ outside U. Then $K = f \cdot K$ and hence $DK = Df \cdot K + f \cdot DK$. Since K and f vanish at x, so does DK.

It follows that if two tensor fields K and K' coincide on an open set U, then DK and DK' coincide on U.

Set $D = D_1 - D_2$. Our problem is now to prove that if a derivation D is zero on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$, then it is zero on $\mathfrak{T}(M)$. Let K be a tensor field of type (r, s) and x an arbitrary point of M. To show that DK vanishes at x, let V be a coordinate neighborhood of x with a local coordinate system $x^1, \ldots, x^n$ and let

$$K = \Sigma K^{i_1 \cdots i_r}_{j_1 \cdots j_s} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_s},$$

where $X_i = \partial/\partial x^i$ and $\omega^j = dx^j$. We may extend $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, X_i and ω^j to M and assume that the equality holds in a smaller neighborhood U of x. Since D can be localized, it suffices to show that

$$D(K^{i_1 \cdots i_r}_{j_1 \cdots j_s} X_{i_1} \otimes \cdots \otimes X_{i_r} \otimes \omega^{j_1} \otimes \cdots \otimes \omega^{j_s}) = 0.$$

But this will follow at once if we show that $D\omega = 0$ for every 1-form ω on M. Let Y be any vector field and C: $\mathfrak{T}^1_1(M) \to \mathfrak{F}(M)$ the obvious contraction so that $C(Y \otimes \omega) = \omega(Y)$ is a function (cf. Example 2.1). Then we have

$$\begin{aligned} 0 = D(C(Y \otimes \omega)) &= C(D(Y \otimes \omega)) \\ &= C(DY \otimes \omega) + C(Y \otimes D\omega) = C(Y \otimes D\omega) = (D\omega)(Y). \end{aligned}$$

Since this holds for every vector field Y, we have $D\omega = 0$. QED.

The set of all derivations of $\mathfrak{T}(M)$ forms a Lie algebra over $\mathbf{R}$ (of infinite dimensions) with respect to the natural addition and multiplication and the bracket operation defined by $[D, D']K = D(D'K) - D'(DK)$. From Proposition 2.13, it follows that the set of all tensor fields S of type $(1, 1)$ forms a subalgebra of the Lie algebra of derivations of $\mathfrak{T}(M)$. In the proof of Proposition 3.3, we showed that a derivation of $\mathfrak{T}(M)$ is induced by a tensor field of type $(1, 1)$ if and only if it is zero on $\mathfrak{F}(M)$. It follows immediately that if D is a derivation of $\mathfrak{T}(M)$ and S is a tensor field of type $(1, 1)$, then $[D, S]$ is zero on $\mathfrak{F}(M)$ and, hence, is induced by a tensor field of type $(1, 1)$. In other words, *the set of tensor fields of type* $(1, 1)$ *is an ideal of the Lie algebra of derivations of* $\mathfrak{T}(M)$. On the other hand, *the set of Lie differentiations* L_X, $X \in \mathfrak{X}(M)$, *forms a subalgebra of the Lie algebra of derivations of* $\mathfrak{T}(M)$. This follows from the following

Proposition 3.4. *For any vector fields X and Y, we have*

$$L_{[X, Y]} = [L_X, L_Y].$$

Proof. By virtue of Lemma above, it is sufficient to show that $[L_X, L_Y]$ has the same effect as $L_{[X, Y]}$ on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$. For $f \in \mathfrak{F}(M)$, we have

$$[L_X, L_Y]f = XYf - YXf = [X, Y]f = L_{[X, Y]}f.$$

For $Z \in \mathfrak{X}(M)$, we have

$$[L_X, L_Y]Z = [X, [Y, Z]] - [Y, [X, Z]] = [[X, Y], Z]$$

by the Jacobi identity. QED.

Proposition 3.5. *Let K be a tensor field of type $(1, r)$ which we interpret as in Proposition 3.1. For any vector field X, we have then*

$$(L_X K)(Y_1, \ldots, Y_r) = [X, K(Y_1, \ldots, Y_r)] - \Sigma_{i=1}^r K(Y_1, \ldots, [X, Y_i], \ldots, Y_r).$$

Proof. We have

$$K(Y_1, \ldots, Y_r) = C_1 \cdots C_r(Y_1 \otimes \cdots \otimes Y_r \otimes K),$$

where $C_1, \ldots, C_r$ are obvious contractions. Using conditions (a) and (c) of Proposition 3.2, we have, for any derivation D of $\mathfrak{T}(M)$,

$$D(K(Y_1, \ldots, Y_r)) = (DK)(Y_1, \ldots, Y_r) + \Sigma_i K(Y_1, \ldots, DY_i, \ldots, Y_r).$$

If $D = L_X$, then (e) of Proposition 3.2 implies Proposition 3.5. QED.

Generalizing Corollary 1.10, we obtain

Proposition 3.6. *Let φ_t be a local 1-parameter group of local transformations generated by a vector field X. For any tensor field K, we have*

$$\tilde{\varphi}_s(L_X K) = -(d(\tilde{\varphi}_t K)/dt)_{t=s}.$$

Proof. By definition,

$$L_X K = \lim_{t \to 0} \frac{1}{t} [K - \tilde{\varphi}_t K].$$

Replacing K by $\tilde{\varphi}_s K$, we obtain

$$L_X(\tilde{\varphi}_s K) = \lim_{t\to 0} \frac{1}{t} [\tilde{\varphi}_s K - \tilde{\varphi}_{t+s} K] = -(d(\tilde{\varphi}_t K)/dt)_{t=s}.$$

Our problem is therefore to prove that $\tilde{\varphi}_s(L_X K) = L_X(\tilde{\varphi}_s K)$, i.e., $L_X K = \tilde{\varphi}_s^{-1} \circ L_X \circ \tilde{\varphi}_s(K)$ for all tensor fields K. It is a straightforward verification to see that $\tilde{\varphi}_s^{-1} \circ L_X \circ \tilde{\varphi}_s$ is a derivation of $\mathfrak{T}(M)$. By Lemma in the proof of Proposition 3.3, it is sufficient to prove that L_X and $\tilde{\varphi}_s^{-1} \circ L_X \circ \tilde{\varphi}_s$ coincide on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$. We already noted in the proof of Corollary 1.10 that they coincide on $\mathfrak{X}(M)$. The fact that they coincide on $\mathfrak{F}(M)$ follows from the following formulas (cf. §1, Chapter I):

$$\varphi^*((\varphi_* X)f) = X(\varphi^* f),$$
$$\tilde{\varphi}^{-1} f = \varphi^* f,$$

which hold for any transformation φ of M and from $(\varphi_s)_* X = X$ (cf. Corollary 1.8). QED.

COROLLARY 3.7. *A tensor field K is invariant by φ_t for every t if and only if $L_X K = 0$.*

Let $\mathfrak{D}^r(M)$ be the space of differential forms of degree r defined on M, i.e., skew-symmetric covariant tensor fields of degree r. With respect to the exterior product, $\mathfrak{D}(M) = \sum_{r=0}^n \mathfrak{D}^r(M)$ forms an algebra over $\mathbf{R}$. A *derivation* (resp. *skew-derivation*) of $\mathfrak{D}(M)$ is a linear mapping D of $\mathfrak{D}(M)$ into itself which satisfies

$$D(\omega \wedge \omega') = D\omega \wedge \omega' + \omega \wedge D\omega' \qquad \text{for } \omega, \omega' \in \mathfrak{D}(M)$$

$$(\text{resp.} = D\omega \wedge \omega' + (-1)^r \omega \wedge D\omega' \qquad \text{for } \omega \in \mathfrak{D}^r(M), \omega' \in \mathfrak{D}(M)).$$

A derivation or a skew-derivation D of $\mathfrak{D}(M)$ is said to be of degree k if it maps $\mathfrak{D}^r(M)$ into $\mathfrak{D}^{r+k}(M)$ for every r. The exterior differentiation d is a skew-derivation of degree 1. As a general result on derivations and skew-derivations of $\mathfrak{D}(M)$, we have

PROPOSITION 3.8. (a) *If D and D' are derivations of degree k and k' respectively, then $[D, D']$ is a derivation of degree $k + k'$.*

(b) *If D is a derivation of degree k and D' is a skew-derivation of degree k', then $[D, D']$ is a skew-derivation of degree $k + k'$.*

(c) *If D and D' are skew-derivations of degree k and k' respectively, then $DD' + D'D$ is a derivation of degree $k + k'$.*

(d) *A derivation or a skew-derivation is completely determined by its effect on $\mathfrak{D}^0(M) = \mathfrak{F}(M)$ and $\mathfrak{D}^1(M)$.*

Proof. The verification of (a), (b), and (c) is straightforward. The proof of (d) is similar to that of Lemma for Proposition 3.3. QED.

PROPOSITION 3.9. *For every vector field X, L_X is a derivation of degree 0 of $\mathfrak{D}(M)$ which commutes with the exterior differentiation d. Conversely, every derivation of degree 0 of $\mathfrak{D}(M)$ which commutes with d is equal to L_X for some vector field X.*

Proof. Observe first that L_X commutes with the alternation A defined in Example 3.2. This follows immediately from the following formula:

$$(L_X\omega)(Y_1, \ldots, Y_r) = X(\omega(Y_1, \ldots, Y_r)) - \Sigma_i\, \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_r),$$

whose proof is the same as that of Proposition 3.5. Hence, $L_X(\mathfrak{D}(M)) \subset \mathfrak{D}(M)$ and, for any $\omega, \omega' \in \mathfrak{D}(M)$, we have

$$\begin{aligned} L_X(\omega \wedge \omega') &= L_X(A(\omega \otimes \omega')) = A(L_X(\omega \otimes \omega')) \\ &= A(L_X\omega \otimes \omega') + A(\omega \otimes L_X\omega') \\ &= L_X\omega \wedge \omega' + \omega \wedge L_X\omega'. \end{aligned}$$

To prove that L_X commutes with d, we first observe that, for any transformation φ of M, $\tilde{\varphi}\omega = (\varphi^{-1})^*\omega$ and, hence, $\tilde{\varphi}$ commutes with d. Let φ_t be a local 1-parameter group of local transformations generated by X. From $\tilde{\varphi}_t(d\omega) = d(\tilde{\varphi}_t\omega)$ and the definition of $L_X\omega$ it follows that $L_X(d\omega) = d(L_X\omega)$ for every $\omega \in \mathfrak{D}(M)$. Conversely, let D be a derivation of degree 0 of $\mathfrak{D}(M)$ which commutes with d. Since D maps $\mathfrak{D}^0(M) = \mathfrak{F}(M)$ into itself, D is a derivation of $\mathfrak{F}(M)$ and there is a vector field X such that $Df = Xf$ for every $f \in \mathfrak{F}(M)$. Set $D' = D - L_X$. Then D' is a derivation of $\mathfrak{D}(M)$ such that $D'f = 0$ for every $f \in \mathfrak{F}(M)$. By virtue of (d) of Proposition 3.8, in order to prove $D' = 0$, it is sufficient to prove $D'\omega = 0$ for every 1-form ω. Just as in Lemma for Proposition 3.3, D' can be localized and it is sufficient to show that $D'\omega = 0$ when ω is of the form $f dg$ where $f, g \in \mathfrak{F}(M)$ (because

ω is locally of the form $\Sigma f_i\, dx^i$ with respect to a local coordinate system $x^1, \ldots, x^n$). Let $\omega = f dg$. From $D'f = 0$ and $D'(dg) = d(D'g) = 0$, we obtain

$$D'(\omega) = (D'f)\, dg + f \cdot D'(dg) = 0.$$

QED.

For each vector field X, we define a skew-derivation ι_X, called the *interior product* with respect to X, of degree -1 of $\mathfrak{D}(M)$ such that

(a) $\iota_X f = 0$ for every $f \in \mathfrak{D}^0(M)$;

(b) $\iota_X \omega = \omega(X)$ for every $\omega \in \mathfrak{D}^1(M)$.

By (d) of Proposition 3.8, such a skew-derivation is unique if it exists. To prove its existence, we consider, for each r, the contraction C: $\mathfrak{T}^1_r(M) \to \mathfrak{T}^0_{r-1}(M)$ associated with the pair (1, 1). Consider every r-form ω as an element of $\mathfrak{T}^0_r(M)$ and define $\iota_X \omega = C(X \otimes \omega)$. In other words,

$$(\iota_X\omega)(Y_1, \ldots, Y_{r-1}) = r \cdot \omega(X, Y_1, \ldots, Y_{r-1}) \qquad \text{for } Y_i \in \mathfrak{X}(M).$$

The verification that ι_X thus defined is a skew-derivation of $\mathfrak{D}(M)$ is left to the reader; $\iota_X(\omega \wedge \omega') = \iota_X\omega \wedge \omega' + (-1)^r \omega \wedge \iota_X\omega'$, where $\omega \in \mathfrak{D}^r(M)$ and $\omega' \in \mathfrak{D}^s(M)$, follows easily from the following formula:

$$(\omega \wedge \omega')(Y_1, Y_2, \ldots, Y_{r+s})$$
$$= \frac{1}{(r+s)!} \Sigma\, \varepsilon(j; k)\, \omega(Y_{j_1}, \ldots, Y_{j_r})\omega'(Y_{k_1}, \ldots, Y_{k_s}),$$

where the summation is taken over all possible partitions of $(1, \ldots, r+s)$ into $(j_1, \ldots, j_r)$ and $(k_1, \ldots, k_s)$ and $\varepsilon(j; k)$ stands for the sign of the permutation $(1, \ldots, r+s) \to (j_1, \ldots, j_r, k_1, \ldots, k_s)$.

Since $(\iota_X^2\omega)(Y_1, \ldots, Y_{r-2}) = r(r-1) \cdot \omega(X, X, Y_1, \ldots, Y_{r-2}) = 0$, we have

$$\iota_X^2 = 0.$$

As relations among d, L_X, and ι_X, we have

PROPOSITION 3.10. (a) *$L_X = d \circ \iota_X + \iota_X \circ d$ for every vector field X.* (b) *$[L_X, \iota_Y] = \iota_{[X, Y]}$ for any vector fields X and Y.*

Proof. By (c) of Proposition 3.8, $d \circ \iota_X + \iota_X \circ d$ is a derivation of degree 0. It commutes with d because $d^2 = 0$. By Proposition 3.9, it is equal to the Lie differentiation with respect to some vector field. To prove that it is actually equal to L_X, we have only to show that $L_X f = (d \circ \iota_X + \iota_X \circ d)f$ for every function f. But this is obvious since $L_X f = Xf$ and $(d \circ \iota_X + \iota_X \circ d)f = \iota_X(df) = (df)(X) = Xf$. To prove the second assertion (b), observe first that $[L_X, \iota_Y]$ is a skew-derivation of degree -1 and that both $[L_X, \iota_Y]$ and $\iota_{[X, Y]}$ are zero on $\mathfrak{F}(M)$. By (d) of Proposition 3.8, it is sufficient to show that they have the same effect on every 1-form ω. As we noted in the proof of Proposition 3.9, we have $(L_X\omega)(Y) = X(\omega(Y)) - \omega([X, Y])$ which can be proved in the same way as Proposition 3.5. Hence,

$$[L_X, \iota_Y]\omega = L_X(\omega(Y)) - \iota_Y(L_X\omega) = X(\omega(Y)) - (L_X\omega)(Y)$$
$$= \omega([X, Y]) = \iota_{[X, Y]}\omega.$$

QED.

As an application of Proposition 3.10 we shall prove

Proposition 3.11. *If ω is an r-form, then*

$$(d\omega)(X_0, X_1, \ldots, X_r)$$
$$= \frac{1}{r+1} \Sigma_{i=0}^{r} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r))$$
$$+ \frac{1}{r+1} \Sigma_{0 \leq i < j \leq r} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r),$$

where the symbol ^ means that the term is omitted. (The cases $r = 1$ and 2 are particularly useful.) If ω is a 1-form, then

$$(d\omega)(X, Y) = \tfrac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\},$$
$$X, Y \in \mathfrak{X}(M).$$

If ω is a 2-form, then

$$(d\omega)(X, Y, Z) = \tfrac{1}{3}\{X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y))$$
$$- \omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y)\},$$
$$X, Y, Z \in \mathfrak{X}(M).$$

Proof. The proof is by induction on r. If $r = 0$, then ω is a function and $d\omega(X_0) = X_0\omega$, which shows that the formula above is true for $r = 0$. Assume that the formula is true for $r - 1$. Let ω be an r-form and, to simplify the notation, set $X = X_0$. By (a) of Proposition 3.10,

$$\begin{aligned}(r+1)\, d\omega(X, X_1, \dots, X_r) &= (\iota_X \circ d\omega)(X_1, \dots, X_r) \\ &= (L_X\omega)(X_1, \dots, X_r) - (d \circ \iota_X\omega)(X_1, \dots, X_r).\end{aligned}$$

As we noted in the proof of Proposition 3.9,

$$\begin{aligned}(L_X\omega)(X_1, \dots, X_r) &= X(\omega(X_1, \dots, X_r)) \\ &\quad - \Sigma_{i=1}^r \omega(X_1, \dots, [X, X_i], \dots, X_r).\end{aligned}$$

Since $\iota_X\omega$ is an $(r-1)$-form, we have, by induction assumption,

$$\begin{aligned}(d \circ \iota_X\omega)(X_1, \dots, X_r) &= \frac{1}{r}\Sigma_{i=1}^r(-1)^{i-1} \\ &\qquad \times X_i(\iota_X\omega(X_1, \dots, \hat{X}_i, \dots, X_r)) \\ &+ \frac{1}{r}\Sigma_{1 \leqq i < j \leqq r}(-1)^{i+j}(\iota_X\omega)([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r) \\ &= \frac{1}{r}\Sigma_{i=1}^r(-1)^{i-1}X_i(\omega(X, X_1, \dots, \hat{X}_i, \dots, X_r)) \\ &- \frac{1}{r}\Sigma_{1 \leqq i < j \leqq r}(-1)^{i+j} \\ &\qquad \times \omega([X_i, X_j], X, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_r).\end{aligned}$$

Our Proposition follows immediately from these three formulas. QED.

Remark. Formulas in Proposition 3.11 are valid also for vector-space valued forms.

Various derivations allow us to construct new tensor fields from a given tensor field. We shall conclude this section by giving another way of constructing new tensor fields.

PROPOSITION 3.12. *Let A and B tensor fields of type* $(1, 1)$. *Set*

$$\begin{aligned}S(X, Y) &= [AX, BY] + [BX, AY] + AB[X, Y] + BA[X, Y] \\ &\quad - A[X, BY] - A[BX, Y] - B[X, AY] - B[AX, Y], \\ &\qquad X, Y \in \mathfrak{X}(M).\end{aligned}$$

Then the mapping $S: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *is a tensor field of type* $(1, 2)$ *and* $S(X, Y) = -S(Y, X)$.

Proof. By a straightforward calculation, we see that S is a bilinear mapping of the $\mathfrak{F}(M)$-module $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into the $\mathfrak{F}(M)$-module $\mathfrak{X}(M)$. By Proposition 3.1, S is a tensor field of type $(1, 2)$. The verification of $S(X, Y) = -S(Y, X)$ is easy.

QED.

We call S the *torsion* of A and B. The construction of S was discovered by Nijenhuis [1].

4. *Lie groups*

A *Lie group* G is a group which is at the same time a differentiable manifold such that the group operation $(a, b) \in G \times G \to ab^{-1} \in G$ is a differentiable mapping of $G \times G$ into G. Since G is locally connected, the connected component of the identity, denoted by G^0, is an open subgroup of G. G^0 is generated by any neighborhood of the identity e. In particular, it is the sum of at most countably many compact sets and satisfies the second axiom of countability. It follows that G satisfies the second axiom of countability if and only if the factor group G/G^0 consists of at most countably many elements.

We denote by L_a (resp. R_a) the left (resp. right) translation of G by an element $a \in G$: $L_a x = ax$ (resp. $R_a x = xa$) for every $x \in G$. For $a \in G$, $\operatorname{ad} a$ is the inner automorphism of G defined by $(\operatorname{ad} a)x = axa^{-1}$ for every $x \in G$.

A vector field X on G is called *left invariant* (resp. *right invariant*) if it is invariant by all left translations L_a (resp. right translations R_a), $a \in G$. A left or right invariant vector field is always differentiable. We define the *Lie algebra* $\mathfrak{g}$ of G to be the set of all left invariant vector fields on G with the usual addition, scalar multiplication and bracket operation. As a vector space, $\mathfrak{g}$ is isomorphic with the tangent space $T_e(G)$ at the identity, the isomorphism being given by the mapping which sends $X \in \mathfrak{g}$ into X_e, the value of X at e. Thus $\mathfrak{g}$ is a Lie subalgebra of dimension n ($n = \dim G$) of the Lie algebra of vector fields $\mathfrak{X}(G)$.

Every $A \in \mathfrak{g}$ generates a (global) 1-parameter group of transformations of G. Indeed, if φ_t is a local 1-parameter group of local transformations generated by A and $\varphi_t e$ is defined for $|t| < \varepsilon$, then $\varphi_t a$ can be defined for $|t| < \varepsilon$ for every $a \in G$ and is equal to $L_a(\varphi_t e)$ as φ_t commutes with every L_a by Corollary 1.8. Since $\varphi_t a$ is defined for $|t| < \varepsilon$ for every $a \in G$, $\varphi_t a$ is defined for $|t| < \infty$ for every $a \in G$. Set $a_t = \varphi_t e$. Then $a_{t+s} = a_t a_s$ for all $t,s \in \mathbf{R}$. We call a_t the *1-parameter subgroup of G generated by A*. Another characterization of a_t is that it is a unique curve in G such that its tangent vector $\dot{a}_t$ at a_t is equal to $L_{a_t} A_e$ and that $a_0 = e$. In other words, it is a unique solution of the differential equation $a_t^{-1}\dot{a}_t = A_e$ with initial condition $a_0 = e$. Denote $a_1 = \varphi_1 e$ by $\exp A$. It follows that $\exp tA = a_t$ for all t. The mapping $A \to \exp A$ of $\mathfrak{g}$ into G is called the *exponential mapping*.

Example 4.1. $GL(n; \mathbf{R})$ and $\mathfrak{gl}(n; \mathbf{R})$. Let $GL(n; \mathbf{R})$ be the group of all real $n \times n$ non-singular matrices $A = (a^i_j)$ (the matrix whose i-th row and j-th column entry is a^i_j); the multiplication is given by

$$(AB)^i_j = \Sigma_{k=1}^n a^i_k b^k_j \qquad \text{for } A = (a^i_j) \quad \text{and} \quad B = (b^i_j).$$

$GL(n; \mathbf{R})$ can be considered as an open subset and, hence, as an open submanifold of $\mathbf{R}^{n^2}$. With respect to this differentiable structure, $GL(n; \mathbf{R})$ is a Lie group. Its identity component consists of matrices of positive determinant. The set $\mathfrak{gl}(n; \mathbf{R})$ of all $n \times n$ real matrices forms an n^2-dimensional Lie algebra with bracket operation defined by $[A, B] = AB - BA$. It is known that the Lie algebra of $GL(n; \mathbf{R})$ can be identified with $\mathfrak{gl}(n; \mathbf{R})$ and the exponential mapping $\mathfrak{gl}(n; \mathbf{R}) \to GL(n; \mathbf{R})$ coincides with the usual exponential mapping $\exp A = \Sigma_{k=0}^\infty A^k/k!$

Example 4.2. $O(n)$ and $\mathfrak{o}(n)$. The group $O(n)$ of all $n \times n$ orthogonal matrices is a compact Lie group. Its identity component, consisting of elements of determinant 1, is denoted by $SO(n)$. The Lie algebra $\mathfrak{o}(n)$ of all skew-symmetric $n \times n$ matrices can be identified with the Lie algebra of $O(n)$ and the exponential mapping $\mathfrak{o}(n) \to O(n)$ is the usual one. The dimension of $O(n)$ is equal to $n(n - 1)/2$.

By a *Lie subgroup* of a Lie group G, we shall mean a subgroup H which is at the same time a submanifold of G such that H itself is a Lie group with respect to this differentiable structure. A left invariant vector field on H is determined by its value at e and this tangent vector at e of H determines a left invariant vector field on

G. It follows that the Lie algebra $\mathfrak{h}$ of H can be identified with a subalgebra of $\mathfrak{g}$. Conversely, every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the Lie algebra of a unique connected Lie subgroup H of G. This is proved roughly as follows. To each point x of G, we assign the space of all A_x, $A \in \mathfrak{h}$. Then this is an involutive distribution and the maximal integral submanifold through e of this distribution is the desired group H (cf. Chevalley [1; p. 109, Theorem 1]).

Thus there is a 1:1 correspondence between connected Lie subgroups of G and Lie subalgebras of the Lie algebra $\mathfrak{g}$. We make a few remarks about nonconnected Lie subgroups. Let H be an arbitrary subgroup of a Lie group G. By providing H with the discrete topology, we may regard H as a 0-dimensional Lie subgroup of G. This also means that a subgroup H of G can be regarded as a Lie subgroup of G possibly in many different ways (that is, with respect to different differentiable structures). To remedy this situation, *we impose the condition that H/H^0, where H^0 is the identity component of H with respect to its own topology, is countable, or in other words, H satisfies the second axiom of countability.* (A subgroup, with a discrete topology, of G is a Lie subgroup only if it is countable.) Under this condition, we have the uniqueness of Lie subgroup structure in the following sense. Let H be a subgroup of a Lie group G. Assume that H has two differentiable structures, denoted by H_1 and H_2, so that it is a Lie subgroup of G. If both H_1 and H_2 satisfy the second axiom of countability, the identity mapping of H onto itself is a diffeomorphism of H_1 onto H_2. Consider the identity mapping $f\colon H_1 \to H_2$. Since the identity component of H_2 is a maximal integral submanifold of the distribution defined by the Lie algebra of H_2, $f\colon H_1 \to H_2$ is differentiable by Proposition 1.3. Similary $f^{-1}\colon H_2 \to H_1$ is differentiable.

Every automorphism φ of a Lie group G induces an automorphism φ_* of its Lie algebra $\mathfrak{g}$; in fact, if $A \in \mathfrak{g}$, $\varphi_* A$ is again a left invariant vector field and $\varphi_*[A, B] = [\varphi_* A, \varphi_* B]$ for $A, B \in \mathfrak{g}$. In particular, for every $a \in G$, ad a which maps x into axa^{-1} induces an automorphism of $\mathfrak{g}$, denoted also by ad a. The representation $a \to \operatorname{ad} a$ of G is called the *adjoint representation* of G in $\mathfrak{g}$. For every $a \in G$ and $A \in \mathfrak{g}$, we have $(\operatorname{ad} a)A = (R_{a^{-1}})_* A$, because $axa^{-1} = L_a R_{a^{-1}} x = R_{a^{-1}} L_a x$ and A is left invariant. Let $A, B \in \mathfrak{g}$ and φ_t the 1-parameter group of transformations of G generated by A. Set $a_t = \exp tA = \varphi_t(e)$. Then $\varphi_t(x) = x a_t$ for

$x \in G$. By Proposition 1.9, we have

$$[B, A] = \lim_{t \to 0} \frac{1}{t} [(\varphi_t)_* B - B] = \lim_{t \to 0} \frac{1}{t} [(R_{a_t})_* B - B]$$
$$= \lim_{t \to 0} \frac{1}{t} [\operatorname{ad}(a_t^{-1})B - B].$$

It follows that if H is an invariant Lie subgroup of G, its Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, that is, $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$ imply $[B, A] \in \mathfrak{h}$. Conversely, the connected Lie subgroup H generated by an ideal $\mathfrak{h}$ of $\mathfrak{g}$ is an invariant subgroup of G.

A differential form ω on G is called left invariant if $(L_a)^*\omega = \omega$ for every $a \in G$. The vector space $\mathfrak{g}^*$ formed by all left invariant 1-forms is the dual space of the Lie algebra $\mathfrak{g}$: if $A \in \mathfrak{g}$ and $\omega \in \mathfrak{g}^*$, then the function $\omega(A)$ is constant on G. If ω is a left invariant form, then so is $d\omega$, because the exterior differentiation commutes with φ^*. From Proposition 3.11 we obtain the *equation of Maurer-Cartan*:

$$d\omega(A, B) = -\tfrac{1}{2}\omega([A, B]) \qquad \text{for } \omega \in \mathfrak{g}^* \quad \text{and} \quad A, B \in \mathfrak{g}.$$

The *canonical 1-form* θ on G is the left invariant $\mathfrak{g}$-valued 1-form uniquely determined by

$$\theta(A) = A \qquad \text{for } A \in \mathfrak{g}.$$

Let $E_1, \ldots, E_r$ be a basis for $\mathfrak{g}$ and set

$$\theta = \Sigma_{i=1}^r \theta^i E_i.$$

Then $\theta^1, \ldots, \theta^r$ form a basis for the space of left invariant real 1-forms on G. We set

$$[E_j, E_k] = \Sigma_{i=1}^r c^i_{jk} E_i,$$

where the c^i_{jk}'s are called the *structure constants* of $\mathfrak{g}$ with respect to the basis $E_1, \ldots, E_r$. It can be easily verified that the equation of Maurer-Cartan is given by

$$d\theta^i = -\tfrac{1}{2}\Sigma_{j,k=1}^r c^i_{jk} \theta^j \wedge \theta^k, \qquad i = 1, \ldots, r.$$

We now consider Lie transformation groups. We say that a Lie group G is a *Lie transformation group* on a manifold M or that G acts (differentiably) on M if the following conditions are satisfied:

(1) Every element a of G induces a transformation of M, denoted by $x \to xa$ where $x \in M$;

(2) $(a, x) \in G \times M \to xa \in M$ is a differentiable mapping;
(3) $x(ab) = (xa)b$ for all $a,b \in G$ and $x \in M$.

We also write $R_a x$ for xa and say that G acts on M on the right. If we write ax and assume $(ab)x = a(bx)$ instead of (3), we say that G acts on M on the left and we use the notation $L_a x$ for ax also. Note that $R_{ab} = R_b \circ R_a$ and $L_{ab} = L_a \circ L_b$. From (3) and from the fact that each R_a or L_a is one-to-one on M, it follows that R_e and L_e are the identity transformation of M.

We say that G acts *effectively* (resp. *freely*) on M if $R_a x = x$ for all $x \in M$ (resp. for some $x \in M$) implies that $a = e$.

If G acts on M on the right, we assign to each element $A \in \mathfrak{g}$ a vector field A^* on M as follows. The action of the 1-parameter subgroup $a_t = \exp tA$ on M induces a vector field on M, which will be denoted by A^* (cf. §1).

PROPOSITION 4.1. *Let a Lie group G act on M on the right. The mapping $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$ which sends A into A^* is a Lie algebra homomorphism. If G acts effectively on M, then σ is an isomorphism of $\mathfrak{g}$ into $\mathfrak{X}(M)$. If G acts freely on M, then, for each non-zero $A \in \mathfrak{g}$, $\sigma(A)$ never vanishes on M.*

Proof. First we observe that σ can be defined also in the following manner. For every $x \in M$, let σ_x be the mapping $a \in G \to xa \in M$. Then $(\sigma_x)_* A_e = (\sigma A)_x$. It follows that σ is a linear mapping of $\mathfrak{g}$ into $\mathfrak{X}(M)$. To prove that σ commutes with the bracket, let $A,B \in \mathfrak{g}$, $A^* = \sigma A$, $B^* = \sigma B$ and $a_t = \exp tA$. By Proposition 1.9, we have

$$[A^*, B^*] = \lim_{t\to 0} \frac{1}{t} [B^* - R_{a_t} B^*].$$

From the fact that $R_{a_t} \circ \sigma_{xa_t^{-1}}(c) = xa_t^{-1} c a_t$ for $c \in G$, we obtain (denoting the differential of a mapping by the same letter)

$$(R_{a_t} B^*)_x = R_{a_t} \circ \sigma_{xa_t^{-1}} B_e = \sigma_x(\mathrm{ad}\,(a_t^{-1}) B_e)$$

and hence

$$\begin{aligned}[A^*, B^*] &= \lim_{t\to 0} \frac{1}{t} [\sigma_x B_e - \sigma_x(\mathrm{ad}\,(a_t^{-1}) B_e)] \\ &= \sigma_x \left(\lim_{t\to 0} \frac{1}{t} [B_e - \mathrm{ad}\,(a_t^{-1}) B_e]\right) \\ &= \sigma_x([A, B]_e) = (\sigma[A, B])_x,\end{aligned}$$

by virtue of the formula for $[A, B]$ in $\mathfrak{g}$ in terms of ad G. We have thus proved that σ is a homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathfrak{X}(M)$. Suppose that $\sigma A = 0$ everywhere on M. This means that the 1-parameter group of transformations R_{a_t} is trivial, that is, R_{a_t} is the identity transformation of M for every t. If G is effective on M, this implies that $a_t = e$ for every t and hence $A = 0$. To prove the last assertion of our proposition, assume σA vanishes at some point x of M. Then R_{a_t} leaves x fixed for every t. If G acts freely on M, this implies that $a_t = e$ for every t and hence $A = 0$. QED.

Although we defined a Lie group as a group which is a differentiable manifold such that the group operation $(a, b) \to ab^{-1}$ is differentiable, we may replace differentiability by real analyticity without loss of generality for the following reason. The exponential mapping is one-to-one near the origin of $\mathfrak{g}$, that is, there is an open neighborhood N of 0 in $\mathfrak{g}$ such that exp is a diffeomorphism of N onto an open neighborhood U of e in G (cf. Chevalley [1; p. 118] or Pontrjagin [1; §39]). Consider the atlas of G which consists of charts (Ua, φ_a), $a \in G$, where φ_a: $Ua \to N$ is the inverse mapping of $R_a \circ \exp: N \to Ua$. (Here, Ua means $R_a(U)$ and N is considered as an open set of $\mathbf{R}^n$ by an identification of $\mathfrak{g}$ with $\mathbf{R}^n$.) With respect to this atlas, G is a real analytic manifold and the group operation $(a, b) \to ab^{-1}$ is real analytic (cf. Pontrjagin [1; p. 257]). We shall need later the following

Proposition 4.2. *Let G be a Lie group and H a closed subgroup of G. Then the quotient space G/H admits a structure of real analytic manifold in such a way that the action of G on G/H is real analytic, that is, the mapping $G \times G/H \to G/H$ which maps (a, bH) into abH is real analytic. In particular, the projection $G \to G/H$ is real analytic.*

For the proof, see Chevalley [1; pp. 109–111].

There is another important class of quotient spaces. Let G be an abstract group acting on a topological space M on the right as a group of homeomorphisms. The action of G is called *properly discontinuous* if it satisfies the following conditions:

(1) If two points x and x' of M are not congruent modulo G (i.e., $R_a x \neq x'$ for every $a \in G$), then x and x' have neighborhoods U and U' respectively such that $R_a(U) \cap U'$ is empty for all $a \in G$;

(2) For each $x \in G$, the isotropy group $G_x = \{a \in G; R_a x = x\}$ is finite;

(3) Each $x \in M$ has a neighborhood U, stable by G_x, such that $U \cap R_a(U)$ is empty for every $a \in G$ not contained in G_x.

Condition (1) implies that the quotient space M/G is Hausdorff. If the action of G is free, then condition (2) is automatically satisfied.

Proposition 4.3. *Let G be a properly discontinuous group of differentiable (resp. real analytic) transformations acting freely on a differentiable (resp. real analytic) manifold M. Then the quotient space M/G has a structure of differentiable (resp. real analytic) manifold such that the projection π: $M \to M/G$ is differentiable (resp. real analytic).*

Proof. Condition (3) implies that every point of M/G has a neighborhood V such that π is a homeomorphism of each connected component of $\pi^{-1}(V)$ onto V. Let U be a connected component of $\pi^{-1}(V)$. Choosing V sufficiently small, we may assume that there is an admissible chart (U, φ), where φ: $U \to \mathbf{R}^n$, for the manifold M. Introduce a differentiable (resp. real analytic) structure in M/G by taking (V, ψ), where ψ is the composite of π^{-1}: $V \to U$ and φ, as an admissible chart. The verification of details is left to the reader. QED.

Remark. A complex analytic analogue of Proposition 4.3 can be proved in the same way.

To give useful criteria for properly discontinuous groups, we define a weaker notion of discontinuous groups. The action of an abstract group G on a topological space M is called *discontinuous* if, for every $x \in M$ and every sequence of elements $\{a_n\}$ of G (where a_n are all mutually distinct), the sequence $\{R_{a_n} x\}$ does not converge to a point in M.

Proposition 4.4. *Every discontinuous group G of isometries of a metric space M is properly discontinuous.*

Proof. Observe first that, for each $x \in M$, the orbit $xG = \{R_a x;\ a \in G\}$ is closed in M. Given a point x' outside the orbit xG, let r be a positive number such that $2r$ is less than the distance between x' and the orbit xG. Let U and U' be the open spheres of radius r and centers x and x' respectively. Then $R_a(U) \cap U'$ is empty for all $a \in G$, thus proving condition (1). Condition (2) is always satisfied by a discontinuous action. To prove (3), for each $x \in M$, let r be a positive number such that $2r$ is less than the distance between x and the closed set $xG - \{x\}$. It suffices to take the open sphere of radius r and center x as U. QED.

Let G be a topological group and H a closed subgroup of G. Then G, hence, any subgroup of G acts on the quotient space G/H on the left.

PROPOSITION 4.5. *Let G be a topological group and H a compact subgroup of G. Then the action of every discrete subgroup D of G on G/H (on the left) is discontinuous.*

Proof. Assuming that the action of D is not discontinuous, let x and y be points of G/H and $\{d_n\}$ a sequence of distinct elements of D such that $d_n x$ converges to y. Let $p: G \to G/H$ be the projection and write $x = p(a)$ and $y = p(b)$ where $a, b \in G$. Let V be a neighborhood of the identity e of G such that $bVVV^{-1}V^{-1}b^{-1}$ contains no element of D other than e. Since $p(bV)$ is a neighborhood of y, there is an integer N such that $d_n x \in p(bV)$ for all $n > N$. Hence, $d_n aH = p^{-1}(d_n x) \subset p^{-1}(p(bV)) = bVH$ for $n > N$. For each $n > N$, there exist $v_n \in V$ and $h_n \in H$ such that $d_n a = bv_n h_n$. Since H is compact, we may assume (by taking a subsequence if necessary) that h_n converges to an element $h \in H$ and hence $h_n = u_n h$ for $n > N$, where $u_n \in V$. We have therefore $d_n = bv_n u_n ha^{-1}$ for $n > N$. Consequently, $d_i d_j^{-1}$ is in $bVVV^{-1}V^{-1}b^{-1}$ if $i, j > N$. This means $d_i = d_j$ if $i, j > N$, contradicting our assumption. QED.

In applying the theory of Lie transformation groups to differential geometry, it is important to show that a certain given group of differentiable transformations of a manifold can be made into a Lie transformation group by introducing a suitable differentiable structure in it. For the proof of the following theorem, we refer the reader to Montgomery-Zippin [1; p. 208 and p. 212].

THEOREM 4.6 *Let G be a locally compact effective transformation group of a connected manifold M of class C^k, $1 \leqq k \leqq \omega$, and let each transformation of G be of class C^1. Then G is a Lie group and the mapping $G \times M \to M$ is of class C^k.*

We shall prove the following result, essentially due to van Dantzig and van der Waerden [1].

THEOREM 4.7. *The group G of isometries of a connected, locally compact metric space M is locally compact with respect to the compact-open topology.*

Proof. We recall that the compact-open topology of G is defined as follows. For any finite number of pairs (K_i, U_i) of compact subsets K_i and open subsets U_i of M, let $W = W(K_1, \ldots, K_s; U_1, \ldots, U_s) = \{\varphi \in G; \varphi(K_i) \subset U_i \text{ for } i = 1, \ldots, s\}$. Then the sets W of this form are taken as a base for the open sets of G. Since M is regular and locally compact, the group multiplication $G \times G \to G$ and the group action $G \times M \to M$ are continuous (cf. Steenrod [1; p. 19]). The continuity of the mapping $G \to G$ which sends φ into φ^{-1} will be proved using the assumption in Theorem 4.7, although it follows from a weaker assumption (cf. Arens [1]).

Every connected, locally compact metric space satisfies the second axiom of countability (see Appendix 2). Since M is locally compact and satisfies the second axiom of countability, G satisfies the second axiom of countability. This justifies the use of sequences in proving the local compactness of G (cf. Kelley [1; p. 138]). The proof is divided into several lemmas.

LEMMA 1. *Let $a \in M$ and let $\varepsilon > 0$ be such that $U(a; \varepsilon) = \{x \in M; d(a, x) < \varepsilon\}$ has compact closure (where d is the distance). Denote by V_a the open neighborhood $U(a; \varepsilon/4)$ of a. Let φ_n be a sequence of isometries such that $\varphi_n(b)$ converges for some point $b \in V_a$. Then there exist a compact set K and an integer N such that $\varphi_n(V_a) \subset K$ for every $n > N$.*

Proof. Choose N such that $n > N$ implies $d(\varphi_n(b), \varphi_n(b)) < \varepsilon/4$. If $x \in V_a$ and $n > N$, then we have

$$\begin{aligned} d(\varphi_n(x), \varphi_N(a)) \leqq d(\varphi_n(x), \varphi_n(b)) + d(\varphi_n(b), \varphi_N(b)) \\ + d(\varphi_N(b), \varphi_N(a)) \\ = d(x, b) + d(\varphi_n(b), \varphi_N(b)) + d(b, a) < \varepsilon, \end{aligned}$$

using the fact that φ_n and φ_N are isometries. This means that $\varphi_n(V_a)$ is contained in $U(\varphi_N(a); \varepsilon)$. But $U(\varphi_N(a); \varepsilon) = \varphi_N(U(a; \varepsilon))$ since φ_N is an isometry. Thus the closure K of $U(\varphi_N(a); \varepsilon) = \varphi_N(U(a; \varepsilon))$ is compact and $\varphi_n(V_a) \subset K$ for $n > N$.

LEMMA 2. *In the notation of Lemma 1, assume again that $\varphi_n(b)$*

converges for some $b \in V_a$. *Then there is a subsequence* φ_{n_k} *of* φ_n *such that* $\varphi_{n_k}(x)$ *converges for each* $x \in V_a$.

Proof. Let $\{b_n\}$ be a countable set which is dense in V_a. (Such a $\{b_n\}$ exists since M is separable.) By Lemma 1, there is an N such that $\varphi_n(V_a)$ is in K for $n > N$. In particular, $\varphi_n(b_1)$ is in K. Choose a subsequence $\varphi_{1,k}$ such that $\varphi_{1,k}(b_1)$ converges. From this subsequence, we choose a subsequence $\varphi_{2,k}$ such that $\varphi_{2,k}(b_2)$ converges, and so on. The diagonal sequence $\varphi_{k,k}(b_n)$ converges for every $n = 1, 2, \ldots$. To prove that $\varphi_{k,k}(x)$ converges for every $x \in V_a$, we change the notation and may assume that $\varphi_n(b_i)$ converges for each $i = 1, 2, \ldots$. Let $x \in V_a$ and $\delta > 0$. Choose b_i such that $d(x, b_i) < \delta/4$. There is an N_1 such that $d(\varphi_n(b_i), \varphi_m(b_i)) < \delta/4$ for $n,m > N_1$. Then we have

$$\begin{aligned} d(\varphi_n(x), \varphi_m(x)) &\leqq d(\varphi_n(x), \varphi_n(b_i)) + d(\varphi_n(b_i), \varphi_m(b_i)) \\ &\qquad + d(\varphi_m(b_i), \varphi_m(x)) \\ &= 2d(x, b_i) + d(\varphi_n(b_i), \varphi_m(b_i)) < \delta. \end{aligned}$$

Thus $\varphi_n(x)$ is a Cauchy sequence. On the other hand, Lemma 1 says that $\varphi_n(x)$ is in a compact set K for all $n > N$. Thus $\varphi_n(x)$ converges.

LEMMA 3. *Let* φ_n *be a sequence of isometries such that* $\varphi_n(a)$ *converges for some point* $a \in M$. *Then there is a subsequence* φ_{n_k} *such that* $\varphi_{n_k}(x)$ *converges for each* $x \in M$. (The connectedness of M is essentially used here.)

Proof. For each $x \in M$, let $V_x = U(x; \varepsilon/4)$ such that $U(x; \varepsilon)$ has compact closure (this ε may vary from point to point, but we choose one such ε for each x). We define a chain as a finite sequence of open sets V_i such that (1) each V_i is of the form V_x for some x; (2) V_1 contains a; (3) V_i and V_{i+1} have a common point. We assert that every point y of M is in the last term of some chain. In fact, it is easy to see that the set of such points y is open and closed. M being connected, the set coincides with M.

This being said, choose a countable set $\{b_i\}$ which is dense in M. For b_1, let $V_1, V_2, \ldots, V_s$ be a chain with $b_1 \in V_s$. By assumption $\varphi_n(a)$ converges. By Lemma 2, we may choose a subsequence (which we may still denote by φ_n by changing the notation) such that $\varphi_n(x)$ converges for each $x \in V_1$. Since $V_1 \cap V_2$ is non-empty, Lemma 2 allows us to choose a subsequence which converges for each $x \in V_2$, and so on. Thus the original sequence φ_n has a subsequence $\varphi_{1,k}$ such that $\varphi_{1,k}(b_1)$ converges. From this subsequence, we may further choose a subsequence $\varphi_{2,k}$ such that $\varphi_{2,k}(b_2)$ converges. As in the proof of Lemma 2, we obtain the diagonal subsequence $\varphi_{k,k}$ such that $\varphi_{k,k}(b_n)$ converges for each n. Denote this diagonal subsequence by φ_n, by changing the notation. Thus $\varphi_n(b_i)$ converges for each b_i.

We now want to show that $\varphi_n(x)$ converges for each $x \in M$. In V_x, there is some b_i so that there exist an N and a compact set K such that $\varphi_n(V_x) \subset K$ for $n > N$ by Lemma 1. Proceeding as in the second half of the proof for Lemma 2, we can prove that $\varphi_n(x)$ is a Cauchy sequence. Since $\varphi_n(x) \in K$ for $n > N$, we conclude that $\varphi_n(x)$ converges.

Lemma 4. *Assume that φ_n is a sequence of isometries such that $\varphi_n(x)$ converges for each $x \in M$. Define $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ for each x. Then φ is an isometry.*

Proof. Clearly, $d(\varphi(x), \varphi(y)) = d(x,y)$ for any $x,y \in M$. For any $a \in M$, let $a' = \varphi(a)$. From $d(\varphi_n^{-1} \circ \varphi(a), a) = d(\varphi(a), \varphi_n(a))$, it follows that $\varphi_n^{-1}(a')$ converges to a. By Lemma 3, there is a subsequence φ_{n_k} such that $\varphi_{n_k}^{-1}(y)$ converges for every $y \in M$. Define a mapping ψ by $\psi(y) = \lim_{k\to\infty} \varphi_{n_k}^{-1}(y)$. Then ψ preserves distance, that is, $d(\psi(x), \psi(y)) = d(x,y)$ for any $x,y \in M$. From

$$\begin{aligned} d(\psi(\varphi(x)), x) &= d(\lim_{k\to\infty} \varphi_{n_k}^{-1}(\varphi(x)), x) = \lim_{k\to\infty} d(\varphi_{n_k}^{-1}(\varphi(x)), x) \\ &= \lim_{k\to\infty} d(\varphi(x), \varphi_{n_k}(x)) = d(\varphi(x), \varphi(x)) = 0, \end{aligned}$$

it follows that $\psi(\varphi(x)) = x$ for each $x \in M$. This means that φ maps M onto M. Since ψ preserves distance and maps M onto M, ψ^{-1} exists and is obviously equal to φ. Thus φ is an isometry.

Lemma 5. *Let φ_n be a sequence of isometries and φ an isometry. If $\varphi_n(x)$ converges to $\varphi(x)$ for every $x \in M$, then the convergence is uniform on every compact subset K of M.*

Proof. Let $\delta > 0$ be given. For each point $a \in K$, choose an integer N_a such that $n > N_a$ implies $d(\varphi_n(a), \varphi(a)) < \delta/4$. Let $W_a = U(a; \delta/4)$. Then for any $x \in W_a$ and $n > N_a$, we have

$$\begin{aligned} d(\varphi_n(x), \varphi(x)) &\leqq d(\varphi_n(x), \varphi_n(a)) + d(\varphi_n(a), \varphi(a)) + d(\varphi(a), \varphi(x)) \\ &< 2d(x, a) + \delta/4 < \delta. \end{aligned}$$

Now K can be covered by a finite number of W_a's, say $W_i = W_{a_i}$, $i = 1, \ldots, s$. It follows that if $n > \max_i \{N_{a_i}\}$, then

$$d(\varphi_n(x), \varphi(x)) < \delta \qquad \text{for each } x \in K.$$

LEMMA 6. *If $\varphi_n(x)$ converges to $\varphi(x)$ as in Lemma 5, then $\varphi_n^{-1}(x)$ converges to $\varphi^{-1}(x)$ for every $x \in M$.*

Proof. For any $x \in M$, let $y = \varphi^{-1}(x)$. Then

$$d(\varphi_n^{-1}(x), \varphi^{-1}(x)) = d(\varphi_n^{-1}(\varphi(y)), y) = d(\varphi(y), \varphi_n(y)) \to 0.$$

We shall now complete the proof of Theorem 4.7. First, observe that $\varphi_n \to \varphi$ with respect to the compact-open topology is equivalent to the uniform convergence of φ_n to φ on every compact subset of M. If $\varphi_n \to \varphi$ in G (with respect to the compact-open topology), then Lemma 6 implies that $\varphi_n^{-1}(x) \to \varphi^{-1}(x)$ for every $x \in M$, and the convergence is uniform on every compact subset by Lemma 5. Thus $\varphi_n^{-1} \to \varphi^{-1}$ in G. This means that the mapping $G \to G$ which maps φ into φ^{-1} is continuous.

To prove that G is locally compact, let $a \in M$ and U an open neighborhood of a with compact closure. We shall show that the neighborhood $W = W(a; U) = \{\varphi \in G; \varphi(a) \in U\}$ of the identity of G has compact closure. Let φ_n be a sequence of elements in W. Since $\varphi_n(a)$ is contained in the compact set $\bar{U}$, closure of U, we can choose, by Lemma 3, a subsequence φ_{n_k} such that $\varphi_{n_k}(x)$ converges for every $x \in M$. The mapping φ defined by $\varphi(x) = \lim \varphi_{n_k}(x)$ is an isometry of M by Lemma 4. By Lemma 5, $\varphi_{n_k} \to \varphi$ uniformly on every compact subset of M, that is, $\varphi_{n_k} \to \varphi$ in G, proving that W has compact closure. QED.

COROLLARY 4.8. *Under the assumption of Theorem 4.7, the isotropy subgroup $G_a = \{\varphi \in G; \varphi(a) = a\}$ of G at a is compact for every $a \in M$.*

Proof. Let φ_n be a sequence of elements of G_a. Since $\varphi_n(a) = a$ for every n, there is a subsequence φ_{n_k} which converges to an element φ of G_a by Lemmas 3, 4, and 5. QED.

COROLLARY 4.9. *If M is a locally compact metric space with a finite number of connected components, the group G of isometries of M is locally compact with respect to the compact-open topology.*

Proof. Decompose M into its connected components M_i, $M = \bigcup_{i=1}^{s} M_i$. Choose a point a_i in each M_i and an open neighborhood U_i of a_i in M_i with compact closure. Then $W(a_1, \ldots, a_s; U_1, \ldots, U_s) = \{\varphi \in G; \varphi(a_i) \in U_i \text{ for } i = 1, \ldots, s\}$ is a neighborhood of the identity of G with compact closure.

QED.

COROLLARY 4.10. *If M is compact in addition to the assumption of Corollary 4.9, then G is compact.*

Proof. Let $G^* = \{\varphi \in G; \varphi(M_i) = M_i \text{ for } i = 1, \ldots, s\}$. Then G^* is a subgroup of G of finite index. In the proof of Corollary 4.9, let $U_i = M_i$. Then G^* is compact. Hence, G is compact.

QED.

5. *Fibre bundles*

Let M be a manifold and G a Lie group. A (*differentiable*) *principal fibre bundle over M with group G* consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right: $(u, a) \in P \times G \to ua = R_a u \in P$;

(2) M is the quotient space of P by the equivalence relation induced by G, $M = P/G$, and the canonical projection $\pi: P \to M$ is differentiable;

(3) P is locally trivial, that is, every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$ in the sense that there is a diffeomorphism $\psi: \pi^{-1}(U) \to U \times G$ such that $\psi(u) = (\pi(u), \varphi(u))$ where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(ua) = (\varphi(u))a$ for all $u \in \pi^{-1}(U)$ and $a \in G$.

A principal fibre bundle will be denoted by $P(M, G, \pi)$, $P(M, G)$ or simply P. We call P the *total space* or the *bundle space*, M the *base space*, G the *structure group* and π the *projection*. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the *fibre* over x. If u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points ua, $a \in G$, and is called the fibre through u. Every fibre is diffeomorphic to G.

Given a Lie group G and a manifold M, G acts freely on $P = M \times G$ on the right as follows. For each $b \in G$, R_b maps $(x, a) \in M \times G$ into $(x, ab) \in M \times G$. The principal fibre bundle $P(M, G)$ thus obtained is called *trivial*.

From local triviality of $P(M, G)$ we see that if W is a submanifold of M then $\pi^{-1}(W)(W, G)$ is a principal fibre bundle.

We call it the *portion of P over W* or the *restriction of P to W* and denote it by $P|W$.

Given a principal fibre bundle $P(M, G)$, the action of G on P induces a homomorphism σ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{X}(P)$ of vector fields on P by Proposition 4.1. For each $A \in \mathfrak{g}$, $A^* = \sigma(A)$ is called the *fundamental vector field corresponding* to A. Since the action of G sends each fibre into itself, A_u^* is tangent to the fibre at each $u \in P$. As G acts freely on P, A^* never vanishes on P (if $A \neq 0$) by Proposition 4.1. The dimension of each fibre being equal to that of $\mathfrak{g}$, the mapping $A \to (A^*)_u$ of $\mathfrak{g}$ into $T_u(P)$ is a linear isomorphism of $\mathfrak{g}$ onto the tangent space at u of the fibre through u. We prove

Proposition 5.1. *Let A^* be the fundamental vector field corresponding to $A \in \mathfrak{g}$. For each $a \in G$, $(R_a)_*A^*$ is the fundamental vector field corresponding to* $(\mathrm{ad}\,(a^{-1}))A \in \mathfrak{g}$.

Proof. Since A^* is induced by the 1-parameter group of transformations R_{a_t} where $a_t = \exp tA$, the vector field $(R_a)_*A^*$ is induced by the 1-parameter group of transformations $R_aR_{a_t}R_{a^{-1}} = R_{a^{-1}a_ta}$ by Proposition 1.7. Our assertion follows from the fact that $a^{-1}a_ta$ is the 1-parameter group generated by $(\mathrm{ad}\,(a^{-1}))A \in \mathfrak{g}$. QED.

The concept of fundamental vector fields will prove to be useful in the theory of connections.

In order to relate our intrinsic definition of a principal fibre bundle to the definition and the construction by means of an open covering, we need the concept of transition functions. By (3) for a principal fibre bundle $P(M, G)$, it is possible to choose an open covering $\{U_\alpha\}$ of M, each $\pi^{-1}(U_\alpha)$ provided with a diffeomorphism $u \to (\pi(u), \varphi_\alpha(u))$ of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$ such that $\varphi_\alpha(ua) = (\varphi_\alpha(u))a$. If $u \in \pi^{-1}(U_\alpha \cap U_\beta)$, then $\varphi_\beta(ua)(\varphi_\alpha(ua))^{-1} = \varphi_\beta(u)(\varphi_\alpha(u))^{-1}$, which shows that $\varphi_\beta(u)(\varphi_\alpha(u))^{-1}$ depends only on $\pi(u)$ not on u. We can define a mapping $\psi_{\beta\alpha}\colon U_\alpha \cap U_\beta \to G$ by $\psi_{\beta\alpha}(\pi(u)) = \varphi_\beta(u)(\varphi_\alpha(u))^{-1}$. The family of mappings $\psi_{\beta\alpha}$ are called *transition functions* of the bundle $P(M, G)$ corresponding to the open covering $\{U_\alpha\}$ of M. It is easy to verify that

$$(*) \qquad \psi_{\gamma\alpha}(x) = \psi_{\gamma\beta}(x) \cdot \psi_{\beta\alpha}(x) \qquad \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma.$$

Conversely, we have

PROPOSITION 5.2. *Let M be a manifold, $\{U_\alpha\}$ an open covering of M and G a Lie group. Given a mapping $\psi_{\beta\alpha}$: $U_\alpha \cap U_\beta \to G$ for every non-empty $U_\alpha \cap U_\beta$, in such a way that the relations (*) are satisfied, we can construct a (differentiable) principal fibre bundle $P(M, G)$ with transition functions $\psi_{\beta\alpha}$.*

Proof. We first observe that the relations (*) imply $\psi_{\alpha\alpha}(x) = e$ for every $x \in U_\alpha$ and $\psi_{\alpha\beta}(x)\psi_{\beta\alpha}(x) = e$ for every $x \in U_\alpha \cap U_\beta$. Let $X_\alpha = U_\alpha \times G$ for each index α and let $X = \bigcup_\alpha X_\alpha$ be the topological sum of X_α; each element of X is a triple (α, x, a) where α is some index, $x \in U_\alpha$ and $a \in G$. Since each X_α is a differentiable manifold and X is a disjoint union of X_α, X is a differentiable manifold in a natural way. We introduce an equivalence relation ρ in X as follows. We say that $(\alpha, x, a) \in \{\alpha\} \times X_\alpha$ is equivalent to $(\beta, y, b) \in \{\beta\} \times X_\beta$ if and only if $x = y \in U_\alpha \cap U_\beta$ and $b = \psi_{\beta\alpha}(x)a$. We remark that (α, x, a) and (α, y, b) are equivalent if and only if $x = y$ and $a = b$. Let P be the quotient space of X by this equivalence relation ρ. We first show that G acts freely on P on the right and that $P/G = M$. By definition, each $c \in G$ maps the ρ-equivalence class of (α, x, a) into the ρ-equivalence class of (α, x, ac). It is easy to see that this definition is independent of the choice of representative (α, x, a) and that G acts freely on P on the right. The projection π: $P \to M$ maps, by definition, the ρ-equivalence class of (α, x, a) into x; the definition of π is independent of the choice of representative (α, x, a). For $u, v \in P$, $\pi(u) = \pi(v)$ if and only if $v = uc$ for some $c \in G$. In fact, let (α, x, a) and (β, y, b) be representatives for u and v respectively. If $v = uc$ for some $c \in G$, then $y = x$ and hence $\pi(v) = \pi(u)$. Conversely, if $\pi(u) = x = y = \pi(v) \in U_\alpha \cap U_\beta$, then $v = uc$ where $c = a^{-1}\psi_{\beta\alpha}(x)^{-1}b \in G$. In order to make P into a differentiable manifold, we first note that, by the natural mapping $X \to P = X/\rho$, each $X_\alpha = U_\alpha \times G$ is mapped 1:1 onto $\pi^{-1}(U_\alpha)$. We introduce a differentiable structure in P by requiring that $\pi^{-1}(U_\alpha)$ is an open submanifold of P and that the mapping $X \to P$ induces a diffeomorphism of $X_\alpha = U_\alpha \times G$ onto $\pi^{-1}(U_\alpha)$. This is possible since every point of P is contained in $\pi^{-1}(U_\alpha)$ for some α and the identification of (α, x, a) with $(\beta, x, \psi_{\beta\alpha}(x)a)$ is made by means of differentiable mappings. It is easy to check that the action of G on P is differentiable and $P(M, G, \pi)$ is a differentiable principal fibre bundle. Finally, the transition functions of P

corresponding to the covering $\{U_\alpha\}$ are precisely the given $\psi_{\beta\alpha}$ if we define ψ_α: $\pi^{-1}(U_\alpha) \to U_\alpha \times G$ by $\psi_\alpha(u) = (x, a)$, where $u \in \pi^{-1}(U)$ is the ρ-equivalence class of (α, x, a). QED.

A *homomorphism* f of a principal fibre bundle $P'(M', G')$ into another principal fibre bundle $P(M, G)$ consists of a mapping f': $P' \to P$ and a homomorphism f'': $G' \to G$ such that $f'(u'a') = f'(u')f''(a)$ for all $u' \in P'$ and $a' \in G'$. For the sake of simplicity, we shall denote f' and f'' by the same letter f. Every homomorphism f: $P' \to P$ maps each fibre of P' into a fibre of P and hence induces a mapping of M' into M, which will be also denoted by f. A homomorphism f: $P'(M', G') \to P(M, G)$ is called an *imbedding* or *injection* if f: $P' \to P$ is an imbedding and if f: $G' \to G$ is a monomorphism. If f: $P' \to P$ is an imbedding, then the induced mapping f: $M' \to M$ is also an imbedding. By identifying P' with $f(P')$, G' with $f(G')$ and M' with $f(M')$, we say that $P'(M', G')$ is a *subbundle* of $P(M, G)$. If, moreover, $M' = M$ and the induced mapping f: $M' \to M$ is the identity transformation of M, f: $P'(M', G') \to P(M, G)$ is called a *reduction* of the structure group G of $P(M, G)$ to G'. The subbundle $P'(M, G')$ is called a *reduced bundle*. Given $P(M, G)$ and a Lie subgroup G' of G, we say that the structure group G is reducible to G' if there is a reduced bundle $P'(M, G')$. Note that we do not require in general that G' is a closed subgroup of G. This generality is needed in the theory of connections.

PROPOSITION 5.3. *The structure group G of a principal fibre bundle $P(M, G)$ is reducible to a Lie subgroup G' if and only if there is an open covering $\{U_\alpha\}$ of M with a set of transition functions $\psi_{\beta\alpha}$ which take their values in G'.*

Proof. Suppose first that the structure group G is reducible to G' and let $P'(M, G')$ be a reduced bundle. Consider P' as a submanifold of P. Let $\{U_\alpha\}$ be an open covering of M such that each $\pi'^{-1}(U_\alpha)$ (π': the projection of P' onto M) is provided with an isomorphism $u \to (\pi'(u), \varphi'_\alpha(u))$ of $\pi'^{-1}(U_\alpha)$ onto $U_\alpha \times G'$. The corresponding transition functions take their values in G'. Now, for the same covering $\{U_\alpha\}$, we define an isomorphism of $\pi^{-1}(U_\alpha)$ (π: the projection of P onto M) onto $U_\alpha \times G$ by extending φ'_α as follows. Every $v \in \pi^{-1}(U_\alpha)$ may be represented in the form $v = ua$ for some $u \in \pi'^{-1}(U_\alpha)$ and $a \in G$ and we set $\varphi_\alpha(v) = \varphi'_\alpha(u)a$.

It is easy to see that $\varphi_\alpha(v)$ is independent of the choice of representation $v = ua$. We see then that $v \to (\pi(v), \varphi_\alpha(v))$ is an isomorphism of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$. The corresponding transition functions $\psi_{\beta\alpha}(x) = \varphi_\beta(v)(\varphi_\alpha(v))^{-1} = \varphi'_\beta(u)(\varphi'_\alpha(u))^{-1}$ take their values in G'.

Conversely, assume that there is a covering $\{U_\alpha\}$ of M with a set of transition functions $\psi_{\beta\alpha}$ all taking values in a Lie subgroup G' of G. For $U_\alpha \cap U_\beta \neq \phi$, $\psi_{\beta\alpha}$ is a differentiable mapping of $U_\alpha \cap U_\beta$ into a Lie group G such that $\psi_{\beta\alpha}(U_\alpha \cap U_\beta) \subset G'$. The crucial point is that $\psi_{\beta\alpha}$ is a differentiable mapping of $U_\alpha \cap U_\beta$ into G' with respect to the differentiable structure of G'. This follows from Proposition 1.3; note that a Lie subgroup satisfies the second axiom of countability by definition, cf. §4. By Proposition 5.2, we can construct a principal fibre bundle $P'(M, G')$ from $\{U_\alpha\}$ and $\{\psi_{\beta\alpha}\}$. Finally, we imbed P' into P as follows. Let f_α: $\pi'^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$ be the composite of the following three mappings:

$$\pi'^{-1}(U_\alpha) \to U_\alpha \times G' \to U_\alpha \times G \to \pi^{-1}(U_\alpha).$$

It is easy to see that $f_\alpha = f_\beta$ on $\pi'^{-1}(U_\alpha \cap U_\beta)$ and that the mapping f: $P' \to P$ thus defined by $\{f_\alpha\}$ is an injection. QED.

Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left: $(a, \xi) \in G \times F \to a\xi \in F$. We shall construct a fibre bundle $E(M, F, G, P)$ associated with P with standard fibre F. On the product manifold $P \times F$, we let G act on the right as follows: an element $a \in G$ maps $(u, \xi) \in P \times F$ into $(ua, a^{-1}\xi) \in P \times F$. The quotient space of $P \times F$ by this group action is denoted by $E = P \times_G F$. A differentiable structure will be introduced in E later and at this moment E is only a set. The mapping $P \times F \to M$ which maps (u, ξ) into $\pi(u)$ induces a mapping π_E, called the projection, of E onto M. For each $x \in M$, the set $\pi_E^{-1}(x)$ is called the fibre of E over x. Every point x of M has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic to $U \times G$. Identifying $\pi^{-1}(U)$ with $U \times G$, we see that the action of G on $\pi^{-1}(U) \times F$ on the right is given by

$$(x, a, \xi) \to (x, ab, b^{-1}\xi) \qquad \text{for } (x, a, \xi) \in U \times G \times F \quad \text{and} \quad b \in G.$$

It follows that the isomorphism $\pi^{-1}(U) \approx U \times G$ induces an isomorphism $\pi_E^{-1}(U) \approx U \times F$. We can therefore introduce a differentiable structure in E by the requirement that $\pi_E^{-1}(U)$ is an open submanifold of E which is diffeomorphic with $U \times F$ under the isomorphism $\pi_E^{-1}(U) \approx U \times F$. The projection π_E is then a differentiable mapping of E onto M. We call E or more precisely $E(M, F, G, P)$ the *fibre bundle over the base M, with (standard) fibre F and (structure) group G, which is associated with the principal fibre bundle P.*

Proposition 5.4. *Let $P(M, G)$ be a principal fibre bundle and F a manifold on which G acts on the left. Let $E(M, F, G, P)$ be the fibre bundle associated with P. For each $u \in P$ and each $\xi \in F$, denote by $u\xi$ the image of $(u, \xi) \in P \times F$ by the natural projection $P \times F \to E$. Then each $u \in P$ is a mapping of F onto $F_x = \pi_E^{-1}(x)$ where $x = \pi(u)$ and*

$$(ua)\xi = u(a\xi) \qquad \text{for } u \in P,\ a \in G,\ \xi \in F.$$

The proof is trivial and is left to the reader.

By an isomorphism of a fibre $F_x = \pi_E^{-1}(x)$, $x \in M$, onto another fibre F_y, $y \in M$, we mean a diffeomorphism which can be represented in the form $v \circ u^{-1}$, where $u \in \pi^{-1}(x)$ and $v \in \pi^{-1}(y)$ are considered as mappings of F onto F_x and F_y respectively. In particular, an automorphism of the fibre F_x is a mapping of the form $v \circ u^{-1}$ with $u,v \in \pi^{-1}(x)$. In this case, $v = ua$ for some $a \in G$ so that any automorphism of F_x can be expressed in the form $u \circ a \circ u^{-1}$ where u is an arbitrarily fixed point of $\pi^{-1}(x)$. The group of automorphisms of F_x is hence isomorphic with the structure group G.

Example 5.1. $G(G/H, H)$: Let G be a Lie group and H a closed subgroup of G. We let H act on G on the right as follows. Every $a \in H$ maps $u \in G$ into ua. We then obtain a differentiable principal fibre bundle $G(G/H, H)$ over the base manifold G/H with structure group H; the local triviality follows from the existence of a local cross section. It is proved in Chevalley [1; p. 110] that if π is the projection of G onto G/H and e is the identity of G, then there is a mapping σ of a neighborhood of $\pi(e)$ in G/H into G such that $\pi \circ \sigma$ is the identity transformation of the neighborhood. See also Steenrod [1; pp. 28–33].

Example 5.2. *Bundle of linear frames*: Let M be a manifold of dimension n. A *linear frame* u at a point $x \in M$ is an ordered basis $X_1, \ldots, X_n$ of the tangent space $T_x(M)$. Let $L(M)$ be the set of all linear frames u at all points of M and let π be the mapping of $L(M)$ onto M which maps a linear frame u at x into x. The general linear group $GL(n; \mathbf{R})$ acts on $L(M)$ on the right as follows. If $a = (a^i_j) \in GL(n; \mathbf{R})$ and $u = (X_1, \ldots, X_n)$ is a linear frame at x, then ua is, by definition, the linear frame $(Y_1, \ldots, Y_n)$ at x defined by $Y_i = \Sigma_j a^j_i X_j$. It is clear that $GL(n; \mathbf{R})$ acts freely on $L(M)$ and $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n; \mathbf{R})$. Now in order to introduce a differentiable structure in $L(M)$, let $(x^1, \ldots, x^n)$ be a local coordinate system in a coordinate neighborhood U in M. Every frame u at $x \in U$ can be expressed uniquely in the form $u = (X_1, \ldots, X_n)$ with $X_i = \Sigma_k X^k_i(\partial/\partial x^k)$, where (X^k_i) is a non-singular matrix. This shows that $\pi^{-1}(U)$ is in 1:1 correspondence with $U \times GL(n; \mathbf{R})$. We can make $L(M)$ into a differentiable manifold by taking (x^j) and (X^k_i) as a local coordinate system in $\pi^{-1}(U)$. It is now easy to verify that $L(M)(M, GL(n; \mathbf{R}))$ is a principal fibre bundle. We call $L(M)$ the *bundle of linear frames over* M. In view of Proposition 5.4, a linear frame u at $x \in M$ can be defined as a non-singular linear mapping of $\mathbf{R}^n$ onto $T_x(M)$. The two definitions are related to each other as follows. Let $e_1, \ldots, e_n$ be the natural basis for $\mathbf{R}^n$: $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. A linear frame $u = (X_1, \ldots, X_n)$ at x can be given as a linear mapping $u: \mathbf{R}^n \to T_x(M)$ such that $ue_i = X_i$ for $i = 1, \ldots, n$. The action of $GL(n; \mathbf{R})$ on $L(M)$ can be accordingly interpreted as follows. Consider $a = (a^i_j) \in GL(n; \mathbf{R})$ as a linear transformation of $\mathbf{R}^n$ which maps e_j into $\Sigma_i a^i_j e_i$. Then $ua: \mathbf{R}^n \to T_x(M)$ is the composite of the following two mappings:

$$\mathbf{R}^n \xrightarrow{a} \mathbf{R}^n \xrightarrow{u} T_x(M).$$

Example 5.3. *Tangent bundle*: Let $GL(n; \mathbf{R})$ act on $\mathbf{R}^n$ as above. The *tangent bundle* $T(M)$ *over* M is the bundle associated with $L(M)$ with standard fibre $\mathbf{R}^n$. It can be easily shown that the fibre of $T(M)$ over $x \in M$ may be considered as $T_x(M)$.

Example 5.4. *Tensor bundles*: Let $\mathbf{T}^r_s$ be the tensor space of type (r, s) over the vector space $\mathbf{R}^n$ as defined in §2. The group $GL(n: \mathbf{R})$ can be regarded as a group of linear transformations of the space $\mathbf{T}^r_s$ by Proposition 2.12. With this standard fibre $\mathbf{T}^r_s$, we obtain the *tensor bundle* $T^r_s(M)$ *of type* (r, s) *over* M which is associated with $L(M)$. It is easy to see that the fibre of $T^r_s(M)$ over $x \in M$ may be considered as the tensor space over $T_x(M)$ of type (r, s).

Returning to the general case, let $P(M, G)$ be a principal fibre bundle and H a closed subgroup of G. In a natural way, G acts on the quotient space G/H on the left. Let $E(M, G/H, G, P)$ be the associated bundle with standard fibre G/H. On the other hand, being a subgroup of G, H acts on P on the right. Let P/H be the quotient space of P by this action of H. Then we have

PROPOSITION 5.5. *The bundle $E = P \times_G (G/H)$ associated with P with standard fibre G/H can be identified with P/H as follows. An element of E represented by $(u, a\xi_0) \in P \times G/H$ is mapped into the element of P/H represented by $ua \in P$, where $a \in G$ and ξ_0 is the origin of G/H, i.e., the coset H.*

Consequently, $P(E, H)$ is a principal fibre bundle over the base $E = P/H$ with structure group H. The projection $P \to E$ maps $u \in P$ into $u\xi_0 \in E$, where u is considered as a mapping of the standard fibre G/H into a fibre of E.

Proof. The proof is straightforward, except the local triviality of the bundle $P(E, H)$. This follows from local triviality of $E(M, G/H, G, P)$ and $G(G/H, H)$ as follows. Let U be an open set of M such that $\pi_E^{-1}(U) \approx U \times G/H$ and let V be an open set of G/H such that $p^{-1}(V) \approx V \times H$, where $p\colon G \to G/H$ is the projection. Let W be the open set of $\pi_E^{-1}(U) \subset E$ which corresponds to $U \times V$ under the identification $\pi_E^{-1}(U) \approx U \times G/H$. If $\mu\colon P \to E = P/H$ is the projection, then $\mu^{-1}(W) \approx W \times H$. QED.

A *cross section* of a bundle $E(M, F, G, P)$ is a mapping $\sigma\colon M \to E$ such that $\pi_E \circ \sigma$ is the identity transformation of M. For $P(M, G)$ itself, a cross section $\sigma\colon M \to P$ exists if and only if P is the trivial bundle $M \times G$ (cf. Steenrod [1; p. 36]). More generally, we have

PROPOSITION 5.6. *The structure group G of $P(M, G)$ is reducible to a closed subgroup H if and only if the associated bundle $E(M, G/H, G, P)$ admits a cross section $\sigma\colon M \to E = P/H$.*

Proof. Suppose G is reducible to a closed subgroup H and let $Q(M, H)$ be a reduced bundle with injection $f\colon Q \to P$. Let $\mu\colon P \to E = P/H$ be the projection. If u and v are in the same fibre of Q, then $v = ua$ for some $a \in H$ and hence $\mu(f(v)) = \mu(f(u)a) = \mu(f(u))$. This means that $\mu \circ f$ is constant on each fibre of Q and induces a mapping $\sigma\colon M \to E$, $\sigma(x) = \mu(f(u))$

where $x = \pi(f(u))$. It is clear that σ is a section of E. Conversely, given a cross section $\sigma: M \to E$, let Q be the set of points $u \in P$ such that $\mu(u) = \sigma(\pi(u))$. In other words, Q is the inverse image of $\sigma(M)$ by the projection $\mu: P \to E = P/H$. For every $x \in M$, there is $u \in Q$ such that $\pi(u) = x$ because $\mu^{-1}(\sigma(x))$ is non-empty. Given u and v in the same fibre of P, if $u \in Q$ then $v \in Q$ when and only when $v = ua$ for some $a \in H$. This follows from the fact that $\mu(u) = \mu(v)$ if and only if $v = ua$ for some $a \in H$. It is now easy to verify that Q is a closed submanifold of P and that Q is a principal fibre bundle $Q(M, H)$ imbedded in $P(M, G)$. QED.

Remark. The correspondence between the sections $\sigma: M \to E = P/H$ and the submanifolds Q is 1:1.

We shall now consider the question of extending a cross section defined on a subset of the base manifold. A mapping f of a subset A of a manifold M into another manifold is called *differentiable on* A if for each point $x \in A$, there is a differentiable mapping f_x of an open neighborhood U_x of x in M into M' such that $f_x = f$ on $U_x \cap A$. If f is the restriction of a differentiable mapping of an open set W containing A into M', then f is clearly differentiable on A. Given a fibre bundle $E(M, F, G, P)$ and a subset A of M, by a cross section on A we mean a differentiable mapping σ of A into E such that $\pi_E \circ \sigma$ is the identity transformation of A.

THEOREM 5.7. *Let $E(M, F, G, P)$ be a fibre bundle such that the base manifold M is paracompact and the fibre F is diffeomorphic with a Euclidean space $\mathbf{R}^m$. Let A be a closed subset (possibly empty) of M. Then every cross section $\sigma: A \to E$ defined on A can be extended to a cross section defined on M. In the particular case where A is empty, there exists a cross section of E defined on M.*

Proof. By the very definition of a paracompact space, every open covering of M has a locally finite open refinement. Since M is normal, every locally finite open covering $\{U_i\}$ of M has an open refinement $\{V_i\}$ such that $\bar{V}_i \subset U_i$ for all i (see Appendix 3).

LEMMA 1. *A differentiable function defined on a closed set of $\mathbf{R}^n$ can be extended to a differentiable function on $\mathbf{R}^n$* (cf. Appendix 3).

LEMMA 2. *Every point of M has a neighborhood U such that every section of E defined on a closed subset contained in U can be extended to U.*

Proof. Given a point of M, it suffices to take a coordinate neighborhood U such that $\pi_E^{-1}(U)$ is trivial: $\pi_E^{-1}(U) \approx U \times F$. Since F is diffeomorphic with $\mathbf{R}^m$, a section on U can be identified with a set of m functions $f_1, \ldots, f_m$ defined on U. By Lemma 1, these functions can be extended to U.

Using Lemma 2, we shall prove Theorem 5.7. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of M such that each U_i has the property stated in Lemma 2. Let $\{V_i\}$ be an open refinement of $\{U_i\}$ such that $\bar{V}_i \subset U_i$ for all $i \in I$. For each subset J of the index set I, set $S_J = \bigcup_{i \in J} \bar{V}_i$. Let T be the set of pairs (τ, J) where $J \subset I$ and τ is a section of E defined on S_J such that $\tau = \sigma$ on $A \cap S_J$. The set T is non-empty; take U_i which meets A and extend the restriction of σ to $A \cap \bar{V}_i$ to a section on $\bar{V}_i$, which is possible by the property possessed by U_i. Introduce an order in T as follows: $(\tau', J') < (\tau'', J'')$ if $J' \subset J''$ and $\tau' = \tau''$ on $S_{J'}$. Let (τ, J) be a maximal element (by using Zorn's Lemma). Assume $J \neq I$ and let $i \in I - J$. On the closed set $(A \cup S_J) \cap \bar{V}_i$ contained in U_i, we have a well defined section σ_i: $\sigma_i = \sigma$ on $A \cap \bar{V}_i$ and $\sigma_i = \tau$ on $S_J \cap \bar{V}_i$. Extend σ_i to a section τ_i on $\bar{V}_i$, which is possible by the property possessed by U_i. Let $J' = J \cup \{i\}$ and τ' be the section on $S_{J'}$ defined by $\tau' = \tau$ on S_J and $\tau' = \tau_i$ on $\bar{V}_i$. Then $(\tau, J) < (\tau', J')$, which contradicts the maximality of (τ, J). Hence, $I = J$ and τ is the desired section. QED.

The proof given here was taken from Godement [1, p. 151].

Example 5.5. Let $L(M)$ be the bundle of linear frames over an n-dimensional manifold M. The homogeneous space $GL(n; \mathbf{R})/O(n)$ is known to be diffeomorphic with a Euclidean space of dimension $\frac{1}{2}n(n+1)$ by an argument similar to Chevalley [1, p. 16]. The fibre bundle $E = L(M)/O(n)$ with fibre $GL(n; \mathbf{R})/O(n)$, associated with $L(M)$, admits a cross section if M is paracompact (by Theorem 5.7). By Proposition 5.6, we see that the structure group of $L(M)$ can be reduced to the orthogonal group $O(n)$, provided that M is paracompact.

Example 5.6. More generally, let $P(M, G)$ be a principal fibre bundle over a paracompact manifold M with group G which is a connected Lie group. It is known that G is diffeomorphic with a direct product of any of its maximal compact subgroups H and a Euclidean space (cf. Iwasawa [1]). By the same reasoning as above, the structure group G can be reduced to H.

Example 5.7. Let $L(M)$ be the bundle of linear frames over a manifold M of dimension n. Let $(\ ,\ )$ be the natural inner product in $\mathbf{R}^n$ for which $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ are orthonormal and which is invariant by $O(n)$ by the very definition of $O(n)$. We shall show that each reduction of the structure group $GL(n; \mathbf{R})$ to $O(n)$ gives rise to a Riemannian metric g on M. Let $Q(M, O(n))$ be a reduced subbundle of $L(M)$. When we regard each $u \in L(M)$ as a linear isomorphism of $\mathbf{R}^n$ onto $T_x(M)$ where $x = \pi(u)$, each $u \in Q$ defines an inner product g in $T_x(M)$ by

$$g(X, Y) = (u^{-1}X, u^{-1}Y) \qquad \text{for } X, Y \in T_x(M).$$

The invariance of $(\ ,\ )$ by $O(n)$ implies that $g(X, Y)$ is independent of the choice of $u \in Q$. Conversely, if M is given a Riemannian metric g, let Q be the subset of $L(M)$ consisting of linear frames $u = (X_1, \ldots, X_n)$ which are orthonormal with respect to g. If we regard $u \in L(M)$ as a linear isomorphism of $\mathbf{R}^n$ onto $T_x(M)$, then u belongs to Q if and only if $(\xi, \xi') = g(u\xi, u\xi')$ for all $\xi, \xi' \in \mathbf{R}^n$. It is easy to verify that Q forms a reduced subbundle of $L(M)$ over M with structure group $O(n)$. The bundle Q will be called the *bundle of orthonormal frames over M* and will be denoted by $O(M)$. An element of $O(M)$ is an *orthonormal frame*. The result here combined with Example 5.5 implies that *every paracompact manifold M admits a Riemannian metric*. We shall see later that every Riemannian manifold is a metric space and hence paracompact.

To introduce the notion of induced bundle, we prove

PROPOSITION 5.8. *Given a principal fibre bundle $P(M, G)$ and a mapping f of a manifold N into M, there is a unique (of course, unique up to an isomorphism) principal fibre bundle $Q(N, G)$ with a homomorphism $f\colon Q \to P$ which induces $f\colon N \to M$ and which corresponds to the identity automorphism of G.*

The bundle $Q(N, G)$ is called the *bundle induced by f from $P(M, G)$* or simply the *induced bundle*; it is sometimes denoted by $f^{-1}P$.

Proof. In the direct product $N \times P$, consider the subset Q consisting of $(y, u) \in N \times P$ such that $f(y) = \pi(u)$. The group G acts on Q by $(y, u) \to (y, u)a = (y, ua)$ for $(y, u) \in Q$ and $a \in G$. It is easy to see that G acts freely on Q and that Q is a principal fibre bundle over N with group G and with projection π_Q given by $\pi_Q(y, u) = y$. Let Q' be another principal fibre bundle over N with group G and $f'\colon Q' \to P$ a homomorphism which induces $f\colon N \to M$ and which corresponds to the identity automorphism of G. Then it is easy to show that the mapping of Q' onto Q defined by $u' \to (\pi_{Q'}(u'), f'(u'))$, $u' \in Q'$, is an isomorphism of the bundle Q' onto Q which induces the identity transformation of N and which corresponds to the identity automorphism of G. QED.

We recall here some results on covering spaces which will be used later. Given a connected, locally arcwise connected topological space M, a connected space E is called a *covering space* over M with projection $p\colon E \to M$ if every point x of M has a connected open neighborhood U such that each connected component of $p^{-1}(U)$ is open in E and is mapped homeomorphically onto U by p. Two covering spaces $p\colon E \to M$ and $p'\colon E' \to M$ are *isomorphic* if there exists a homeomorphism $f\colon E \to E'$ such that $p' \circ f = p$. A covering space $p\colon E \to M$ is a *universal covering space* if E is simply connected. If M is a manifold, every covering space has a (unique) structure of manifold such that p is differentiable. From now on we shall only consider covering manifolds.

PROPOSITION 5.9. (1) *Given a connected manifold M, there is a unique* (*unique up to an isomorphism*) *universal covering manifold, which will be denoted by $\tilde{M}$.*

(2) *The universal covering manifold $\tilde{M}$ is a principal fibre bundle over M with group $\pi_1(M)$ and projection $p\colon \tilde{M} \to M$, where $\pi_1(M)$ is the first homotopy group of M.*

(3) *The isomorphism classes of the covering spaces over M are in a* 1:1 *correspondence with the conjugate classes of the subgroups of $\pi_1(M)$. The correspondence is given as follows. To each subgroup H of $\pi_1(M)$, we associate $E = \tilde{M}/H$. Then the covering manifold E corresponding to H is a fibre bundle over M with fibre $\pi_1(M)/H$ associated with the principal fibre bundle $\tilde{M}(M, \pi_1(M))$. If H is a normal subgroup of $\pi_1(M)$, $E = \tilde{M}/H$ is a principal fibre bundle with group $\pi_1(M)/H$ and is called a regular covering manifold of M.*

For the proof, see Steenrod [1, pp. 67–71] or Hu [1, pp. 89–97].

The action of $\pi_1(M)/H$ on a regular covering manifold $E = \tilde{M}/H$ is properly discontinuous. Conversely, if E is a connected manifold and G is a properly discontinuous group of transformations acting freely on E, then E is a regular covering manifold of

$M = E/G$ as follows immediately from the condition (3) in the definition of properly discontinuous action in §4.

Example 5.8. Consider $\mathbf{R}^n$ as an n-dimensional vector space and let $\xi_1, \ldots, \xi_n$ be any basis of $\mathbf{R}^n$. Let G be the subgroup of $\mathbf{R}^n$ generated by $\xi_1, \ldots, \xi_n$: $G = \{\sum m_i \xi_i;\ m_i \text{ integers}\}$. The action of G on $\mathbf{R}^n$ is properly discontinuous and $\mathbf{R}^n$ is the universal covering manifold of $\mathbf{R}^n/G$. The quotient manifold $\mathbf{R}^n/G$ is called an n-dimensional *torus*.

Example 5.9. Let S^n be the unit sphere in $\mathbf{R}^{n+1}$ with center at the origin: $S^n = \{(x^1, \ldots, x^{n+1}) \in \mathbf{R}^{n+1};\ \sum_i (x^i)^2 = 1\}$. Let G be the group consisting of the identity transformation of S^n and the transformation of S^n which maps $(x^1, \ldots, x^{n+1})$ into $(-x^1, \ldots, -x^{n+1})$. Then $S^n, n \geqq 2$, is the universal covering manifold of S^n/G. The quotient manifold S^n/G is called the n-dimensional *real projective space*.

CHAPTER II

Theory of Connections

1. *Connections in a principal fibre bundle*

Let $P(M, G)$ be a principal fibre bundle over a manifold M with group G. For each $u \in P$, let $T_u(P)$ be the tangent space of P at u and G_u the subspace of $T_u(P)$ consisting of vectors tangent to the fibre through u. A *connection* Γ in P is an assignment of a subspace Q_u of $T_u(P)$ to each $u \in P$ such that

(a) $T_u(P) = G_u + Q_u$ (direct sum);

(b) $Q_{ua} = (R_a)_* Q_u$ for every $u \in P$ and $a \in G$, where R_a is the transformation of P induced by $a \in G$, $R_a u = ua$;

(c) Q_u depends differentiably on u.

Condition (b) means that the distribution $u \to Q_u$ is invariant by G. We call G_u the *vertical subspace* and Q_u the *horizontal subspace* of $T_u(P)$. A vector $X \in T_u(P)$ is called *vertical* (resp. *horizontal*) if it lies in G_u (resp. Q_u). By (a), every vector $X \in T_u(P)$ can be uniquely written as

$$X = Y + Z \qquad \text{where } Y \in G_u \quad \text{and} \quad Z \in Q_u.$$

We call Y (resp. Z) the *vertical* (resp. *horizontal*) *component* of X and denote it by vX (resp. hX). Condition (c) means, by definition, that if X is a differentiable vector field on P so are vX and hX. (It can be easily verified that this is equivalent to saying that the distribution $u \to Q_u$ is differentiable.)

Given a connection Γ in P, we define a 1-form ω on P with values in the Lie algebra $\mathfrak{g}$ of G as follows. In §5 of Chapter I, we showed that every $A \in \mathfrak{g}$ induces a vector field A^* on P, called the fundamental vector field corresponding to A, and that $A \to (A^*)_u$ is a linear isomorphism of $\mathfrak{g}$ onto G_u for each $u \in P$. For each $X \in T_u(P)$, we define $\omega(X)$ to be the unique $A \in \mathfrak{g}$ such that

$(A^*)_u$ is equal to the vertical component of X. It is clear that $\omega(X) = 0$ if and only if X is horizontal. The form ω is called the *connection form* of the given connection Γ.

PROPOSITION 1.1. *The connection form ω of a connection satisfies the following conditions:*

(a′) $\omega(A^*) = A$ *for every* $A \in \mathfrak{g}$;

(b′) $(R_a)^*\omega = \text{ad}\,(a^{-1})\omega$, *that is*, $\omega((R_a)_*X) = \text{ad}\,(a^{-1}) \cdot \omega(X)$ *for every* $a \in G$ *and every vector field* X *on* P, *where* ad *denotes the adjoint representation of* G *in* $\mathfrak{g}$.

Conversely, given a $\mathfrak{g}$-*valued* 1-*form* ω *on* P *satisfying conditions* (a′) *and* (b′), *there is a unique connection* Γ *in* P *whose connection form is* ω.

Proof. Let ω be the connection form of a connection. The condition (a′) follows immediately from the definition of ω. Since every vector field of P can be decomposed into a horizontal vector field and a vertical vector field, it is sufficient to verify (b′) in the following two special cases: (1) X is horizontal and (2) X is vertical. If X is horizontal, so is $(R_a)_*X$ for every $a \in G$ by the condition (b) for a connection. Thus, both $\omega((R_a)_*X)$ and $\text{ad}\,(a^{-1}) \cdot \omega(X)$ vanish. In the case when X is vertical, we may further assume that X is a fundamental vector field A^*. Then $(R_a)_*X$ is the fundamental vector field corresponding to $\text{ad}\,(a^{-1})A$ by Proposition 5.1 of Chapter I. Thus we have

$$(R_a^*\omega)_u(X) = \omega_{ua}((R_a)_*X) = \text{ad}\,(a^{-1})A = \text{ad}\,(a^{-1})(\omega_u(X)).$$

Conversely, given a form ω satisfying (a′) and (b′), we define

$$Q_u = \{X \in T_u(P);\ \omega(X) = 0\}.$$

The verification that $u \to Q_u$ defines a connection whose connection form is ω is easy and is left to the reader. QED.

The projection $\pi\colon P \to M$ induces a linear mapping $\pi\colon T_u(P) \to T_x(M)$ for each $u \in P$, where $x = \pi(u)$. When a connection is given, π maps the horizontal subspace Q_u isomorphically onto $T_x(M)$.

The *horizontal lift* (or simply, the *lift*) of a vector field X on M is a unique vector field X^* on P which is horizontal and which projects onto X, that is, $\pi(X_u^*) = X_{\pi(u)}$ for every $u \in P$.

PROPOSITION 1.2. *Given a connection in P and a vector field X on M, there is a unique horizontal lift X^* of X. The lift X^* is invariant by R_a for every $a \in G$. Conversely, every horizontal vector field X^* on P invariant by G is the lift of a vector field X on M.*

Proof. The existence and uniqueness of X^* is clear from the fact that π gives a linear isomorphism of Q_u onto $T_{\pi(u)}(M)$. To prove that X^* is differentiable if X is differentiable, we take a neighborhood U of any given point x of M such that $\pi^{-1}(U) \approx U \times G$. Using this isomorphism, we first obtain a differentiable vector field Y on $\pi^{-1}(U)$ such that $\pi Y = X$. Then X^* is the horizontal component of Y and hence is differentiable. The invariance of X^* by G is clear from the invariance of the horizontal subspaces by G. Finally, let X^* be a horizontal vector field on P invariant by G. For every $x \in M$, take a point $u \in P$ such that $\pi(u) = x$ and define $X_x = \pi(X^*_u)$. The vector X_x is independent of the choice of u such that $\pi(u) = x$, since if $u' = ua$, then $\pi(X^*_{u'}) = \pi(R_a \cdot X^*_u) = \pi(X^*_u)$. It is obvious that X^* is then the lift of the vector field X. QED.

PROPOSITION 1.3. *Let X^* and Y^* be the horizontal lifts of X and Y respectively. Then*

(1) *$X^* + Y^*$ is the horizontal lift of $X + Y$;*

(2) *For every function f on M, $f^* \cdot X^*$ is the horizontal lift of fX where f^* is the function on P defined by $f^* = f \circ \pi$;*

(3) *The horizontal component of $[X^*, Y^*]$ is the horizontal lift of $[X, Y]$.*

Proof. The first two assertions are trivial. As for the third, we have

$$\pi(h[X^*, Y^*]) = \pi([X^*, Y^*]) = [X, Y].$$

QED.

Let $x^1, \ldots, x^n$ be a local coordinate system in a coordinate neighborhood U in M. Let X_i^* be the horizontal lift in $\pi^{-1}(U)$ of the vector field $X_i = \partial/\partial x^i$ in U for each i. Then $X_1^*, \ldots, X_n^*$ form a local basis for the distribution $u \to Q_u$ in $\pi^{-1}(U)$.

We shall now express a connection form ω on P by a family of forms each defined in an open subset of the base manifold M. Let $\{U_\alpha\}$ be an open covering of M with a family of isomorphisms $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$. For each α, let $\sigma_\alpha\colon U_\alpha \to P$ be the cross section on U_α defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$, $x \in U_\alpha$, where e is the identity of G. Let θ be the (left invariant $\mathfrak{g}$-valued) canonical 1-form on G defined in §4 of Chapter I (p. 41).

For each non-empty $U_\alpha \cap U_\beta$, define a $\mathfrak{g}$-valued 1-form $\theta_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ by

$$\theta_{\alpha\beta} = \psi_{\alpha\beta}^* \theta.$$

For each α, define a $\mathfrak{g}$-valued 1-form ω_α on U_α by

$$\omega_\alpha = \sigma_\alpha^* \omega.$$

Proposition 1.4. *The forms $\theta_{\alpha\beta}$ and ω_α are subject to the conditions:*

$$\omega_\beta = \mathrm{ad}\,(\psi_{\alpha\beta}^{-1})\,\omega_\alpha + \theta_{\alpha\beta} \qquad \text{on } U_\alpha \cap U_\beta.$$

Conversely, for every family of $\mathfrak{g}$-valued 1-forms $\{\omega_\alpha\}$ each defined on U_α and satisfying the preceding conditions, there is a unique connection form ω on P which gives rise to $\{\omega_\alpha\}$ in the described manner.

Proof. If $U_\alpha \cap U_\beta$ is non-empty, $\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$ for all $x \in U_\alpha \cap U_\beta$. Denote the differentials of σ_α, σ_β, and $\psi_{\alpha\beta}$ by the same letters. Then for every vector $X \in T_x(U_\alpha \cap U_\beta)$, the vector $\sigma_\beta(X) \in T_u(P)$, where $u = \sigma_\beta(x)$, is the image of $(\sigma_\alpha(X), \psi_{\alpha\beta}(X)) \in T_{u'}(P) + T_a(G)$, where $u' = \sigma_\alpha(x)$ and $a = \psi_{\alpha\beta}(x)$, under the mapping $P \times G \to P$. By Proposition 1.4 (Leibniz's formula) of Chapter I, we have

$$\sigma_\beta(X) = \sigma_\alpha(X)\psi_{\alpha\beta}(x) + \sigma_\alpha(x)\psi_{\alpha\beta}(X),$$

where $\sigma_\alpha(X)\psi_{\alpha\beta}(x)$ means $R_a(\sigma_\alpha(X))$ and $\sigma_\alpha(x)\psi_{\alpha\beta}(X)$ is the image of $\psi_{\alpha\beta}(X)$ by the differential of $\sigma_\alpha(x)$, $\sigma_\alpha(x)$ being considered as a mapping of G into P which maps $b \in G$ into $\sigma_\alpha(x)b$. Taking the values of ω on both sides of the equality, we obtain

$$\omega_\beta(X) = \mathrm{ad}\,(\psi_{\alpha\beta}(x)^{-1})\,\omega_\alpha(X) + \theta_{\alpha\beta}(X).$$

Indeed, if $A \in \mathfrak{g}$ is the left invariant vector field on G which is equal to $\psi_{\alpha\beta}(X)$ at $a = \psi_{\alpha\beta}(x)$ so that $\theta(\psi_{\alpha\beta}(X)) = A$, then $\sigma_\alpha(x)\psi_{\alpha\beta}(X)$ is the value of the fundamental vector field A^* at $u = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$ and hence $\omega(\sigma_\alpha(x)\psi_{\alpha\beta}(X)) = A$.

The converse can be verified by following back the process of obtaining $\{\omega_\alpha\}$ from ω. QED.

2. *Existence and extension of connections*

Let $P(M, G)$ be a principal fibre bundle and A a subset of M. We say that a connection is defined over A if, at every point $u \in P$ with $\pi(u) \in A$, a subspace Q_u of $T_u(P)$ is given in such a way that conditions (a) and (b) for connection (see §1) are satisfied and Q_u depends differentiably on u in the following sense. For every point $x \in A$, there exist an open neighborhood U and a connection in $P \mid U = \pi^{-1}(U)$ such that the horizontal subspace at every $u \in \pi^{-1}(A)$ is the given space Q_u.

THEOREM 2.1. *Let $P(M, G)$ be a principal fibre bundle and A a closed subset of M (A may be empty). If M is paracompact, every connection defined over A can be extended to a connection in P. In particular, P admits a connection if M is paracompact.*

Proof. The proof is a replica of that of Theorem 5.7 in Chapter I.

LEMMA 1. *A differentiable function defined on a closed subset of $\mathbf{R}^n$ can be always extended to a differentiable function on $\mathbf{R}^n$* (cf. Appendix 3).

LEMMA 2. *Every point of M has a neighborhood U such that every connection defined on a closed subset contained in U can be extended to a connection defined over U.*

Proof. Given a point of M, it suffices to take a coordinate neighborhood U such that $\pi^{-1}(U)$ is trivial: $\pi^{-1}(U) \approx U \times G$. On the trivial bundle $U \times G$, a connection form ω is completely determined by its behavior at the points of $U \times \{e\}$ (e: the identity of G) because of the property $R_a^*(\omega) = \mathrm{ad}\,(a^{-1})\omega$. Furthermore, if $\sigma\colon U \to U \times G$ is the natural cross section, that is, $\sigma(x) = (x, e)$ for $x \in U$, then ω is completely determined by the $\mathfrak{g}$-valued 1-form $\sigma^*\omega$ on U. Indeed, every vector $X \in T_{\sigma(x)}(U \times G)$ can be written uniquely in the form

$$X = Y + Z,$$

where Y is tangent to $U \times \{e\}$ and Z is vertical so that $Y = \sigma_*(\pi_* X)$. Hence we have

$$\omega(X) = \omega(\sigma_*(\pi_* X)) + \omega(Z) = (\sigma^*\omega)(\pi_* X) + A,$$

where A is a unique element of $\mathfrak{g}$ such that the corresponding fundamental vector field A^* is equal to Z at $\sigma(x)$. Since A depends only on Z, not on the connection, ω is completely determined by $\sigma^*\omega$. The equation above shows that, conversely, every $\mathfrak{g}$-valued 1-form on U determines uniquely a connection form on $U \times G$. Thus Lemma 2 is reduced to the extension problem for $\mathfrak{g}$-valued 1-forms on U. If $\{A_j\}$ is a basis for $\mathfrak{g}$, then $\omega = \Sigma\, \omega^j A_j$ where each ω^j is a usual 1-form. Thus it is sufficient to consider the extension problem of usual 1-forms on U. Let $x^1, \ldots, x^n$ be a local coordinate system in U. Then every 1-form on U is of the form $\Sigma f_i\, dx^i$ where each f_i is a function on U. Thus our problem is reduced to the extension problem of functions on U. Lemma 2 now follows from Lemma 1.

By means of Lemma 2, Theorem 2.1 can be proved exactly in the same way as Theorem 5.7 of Chapter I. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of M such that each U_i has the property stated in Lemma 2. Let $\{V_i\}$ be an open refinement of $\{U_i\}$ such that $\bar{V}_i \subset U_i$. For each subset J of I, set $S_J = \bigcup_{i \in J} \bar{V}_i$. Let T be the set of pairs (τ, J) where $J \subset I$ and τ is a connection defined over S_J which coincides with the given connection over $A \cap S_J$. Introduce an order in T as follows: $(\tau', J') < (\tau'', J'')$ if $J' \subset J''$ and $\tau' = \tau''$ on $S_{J'}$. Let (τ, J) be a maximal element of T. Then $J = I$ as in the proof of Theorem 5.7 of Chapter I and τ is a desired connection. QED.

Remark. It is possible to prove Theorem 2.1 using Lemma 2 and a partition of unity $\{f_i\}$ subordinate to $\{V_i\}$ (cf. Appendix 3). Let ω_i be a connection form on $\pi^{-1}(U_i)$ which extends the given connection over $A \cap \bar{V}_i$. Then $\omega = \Sigma_i\, g_i \omega_i$ is a desired connection form on P, where each g_i is the function on P defined by $g_i = f_i \circ \pi$.

3. *Parallelism*

Given a connection Γ in a principal fibre bundle $P(M, G)$, we shall define the concept of parallel displacement of fibres along any given curve τ in the base manifold M.

Let $\tau = x_t$, $a \leqq t \leqq b$, be a piecewise differentiable curve of class C^1 in M. A *horizontal lift* or simply a *lift* of τ is a horizontal curve $\tau^* = u_t$, $a \leqq t \leqq b$, in P such that $\pi(u_t) = x_t$ for $a \leqq t \leqq b$. Here a horizontal curve in P means a piecewise differentiable curve of class C^1 whose tangent vectors are all horizontal.

The notion of lift of a curve corresponds to the notion of lift of a vector field. Indeed, if X^* is the lift of a vector field X on M, then the integral curve of X^* through a point $u_0 \in P$ is a lift of the integral curve of X through the point $x_0 = \pi(u_0) \in M$. We now prove

Proposition 3.1. *Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a curve of class C^1 in M. For an arbitrary point u_0 of P with $\pi(u_0) = x_0$, there exists a unique lift $\tau^* = u_t$ of τ which starts from u_0.*

Proof. By local triviality of the bundle, there is a curve v_t of class C^1 in P such that $v_0 = u_0$ and $\pi(v_t) = x_t$ for $0 \leqq t \leqq 1$. A lift of τ, if it exists, must be of the form $u_t = v_t a_t$, where a_t is a curve in the structure group G such that $a_0 = e$. We shall now look for a curve a_t in G which makes $u_t = v_t a_t$ a horizontal curve. Just as in the proof of Proposition 1.4, we apply Leibniz's formula (Proposition 1.4 of Chapter I) to the mapping $P \times G \rightarrow P$ which maps (v, a) into va and obtain

$$\dot{u}_t = \dot{v}_t a_t + v_t \dot{a}_t,$$

where each dotted italic letter denotes the tangent vector at that point (e.g., $\dot{u}_t$ is the vector tangent to the curve $\tau^* = u_t$ at the point u_t). Let ω be the connection form of Γ. Then, as in the proof of Proposition 1.4, we have

$$\omega(\dot{u}_t) = \mathrm{ad}(a_t^{-1})\omega(\dot{v}_t) + a_t^{-1}\dot{a}_t,$$

where $a_t^{-1}\dot{a}_t$ is now a curve in the Lie algebra $\mathfrak{g} = T_e(G)$ of G. The curve u_t is horizontal if and only if $\dot{a}_t a_t^{-1} = -\omega(\dot{v}_t)$ for every t. The construction of u_t is thus reduced to the following

Lemma. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra identified with $T_e(G)$. Let Y_t, $0 \leqq t \leqq 1$, be a continuous curve in $T_e(G)$. Then there exists in G a unique curve a_t of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for $0 \leqq t \leqq 1$.*

Remark. In the case where $Y_t = A$ for all t, the curve a_t is nothing but the 1-parameter subgroup of G generated by A. Our differential equation $\dot{a}_t a_t^{-1} = Y_t$ is hence a generalization of the differential equation for 1-parameter subgroups.

Proof of Lemma. We may assume that Y_t is defined and continuous for all t, $-\infty < t < \infty$. We define a vector field X on

$G \times \mathbf{R}$ as follows. The value of X at $(a, t) \in G \times \mathbf{R}$ is, by definition, equal to $(Y_t a, (d/dz)_t) \in T_a(G) \times T_t(\mathbf{R})$, where z is the natural coordinate system in $\mathbf{R}$. It is clear that the integral curve of X starting from $(e, 0)$ is of the form (a_t, t) and a_t is the desired curve in G. The only thing we have to verify is that a_t is defined for all t, $0 \leqq t \leqq 1$. Let $\varphi_t = \exp tX$ be a local 1-parameter group of local transformations of $G \times \mathbf{R}$ generated by X. For each $(e, s) \in G \times \mathbf{R}$, there is a positive number δ_s such that $\varphi_t(e, r)$ is defined for $|r - s| < \delta_s$ and $|t| < \delta_s$ (Proposition 1.5 of Chapter I). Since the subset $\{e\} \times [0, 1]$ of $G \times \mathbf{R}$ is compact, we may choose $\delta > 0$ such that, for each $r \in [0, 1]$, $\varphi_t(e, r)$ is defined for $|t| < \delta$ (cf. Proof of Proposition 1.6 of Chapter I). Choose $s_0, s_1, \ldots, s_k$ such that $0 = s_0 < s_1 < \cdots < s_k = 1$ and $s_i - s_{i-1} < \delta$ for every i. Then $\varphi_t(e, 0) = (a_t, t)$ is defined for $0 \leqq t \leqq s_1$; $\varphi_u(e, s_1) = (b_u, u + s_1)$ is defined for $0 \leqq u \leqq s_2 - s_1$, where $\dot{b}_u b_u^{-1} = Y_{u+s_1}$, and we define $a_t = b_{t-s_1} a_{s_1}$ for $s_1 \leqq t \leqq s_2$; $\ldots$; $\varphi_u(e, s_{k-1}) = (c_u, s_{k-1} + u)$ is defined for $0 \leqq u \leqq s_k - s_{k-1}$, where $\dot{c}_u c_u^{-1} = Y_{u+s_{k-1}}$, and we define $a_t = c_{t-s_{k-1}} a_{s_{k-1}}$, thus completing the construction of a_t, $0 \leqq t \leqq 1$. QED.

Now using Proposition 3.1, we define the parallel displacement of fibres as follows. Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a differentiable curve of class C^1 on M. Let u_0 be an arbitrary point of P with $\pi(u_0) = x_0$. The unique lift τ^* of τ through u_0 has the end point u_1 such that $\pi(u_1) = x_1$. By varying u_0 in the fibre $\pi^{-1}(x_0)$, we obtain a mapping of the fibre $\pi^{-1}(x_0)$ onto the fibre $\pi^{-1}(x_1)$ which maps u_0 into u_1. We denote this mapping by the same letter τ and call it the *parallel displacement along the curve* τ. The fact that $\tau\colon \pi^{-1}(x_0) \to \pi^{-1}(x_1)$ is actually an isomorphism comes from the following

PROPOSITION 3.2. *The parallel displacement along any curve τ commutes with the action of G on P: $\tau \circ R_a = R_a \circ \tau$ for every $a \in G$.*

Proof. This follows from the fact that every horizontal curve is mapped into a horizontal curve by R_a. QED.

The parallel displacement along any piecewise differentiable curve of class C^1 can be defined in an obvious manner. It should be remarked that the parallel displacement along a curve τ is independent of a specific parametrization x_t used in the following sense. Consider two parametrized curves x_t, $a \leqq t \leqq b$, and y_s, $c \leqq s \leqq d$, in M. The parallel displacement along x_t and the one along y_s coincide if there is a homeomorphism φ of the interval $[a, b]$ onto $[c, d]$ such that (1) $\varphi(a) = c$ and $\varphi(b) = d$, (2) both φ and φ^{-1} are differentiable of class C^1 except at a finite number of parameter values, and (3) $y_{\varphi(t)} = x_t$ for all t, $a \leqq t \leqq b$.

If τ is the curve x_t, $a \leqq t \leqq b$, we denote by τ^{-1} the curve y_t, $a \leqq t \leqq b$, defined by $y_t = x_{a+b-t}$. The following proposition is evident.

PROPOSITION 3.3. (a) *If τ is a piecewise differentiable curve of class C^1 in M, then the parallel displacement along τ^{-1} is the inverse of the parallel displacement along τ.*

(b) *If τ is a curve from x to y in M and μ is a curve from y to z in M, the parallel displacement along the composite curve $\mu \cdot \tau$ is the composite of the parallel displacements τ and μ.*

4. *Holonomy groups*

Using the notion of parallel displacement, we now define the holonomy group of a given connection Γ in a principal fibre bundle $P(M, G)$. For the sake of simplicity we shall mean by a curve a piecewise differentiable curve of class C^k, $1 \leqq k \leqq \infty$ (k will be fixed throughout §4).

For each point x of M we denote by $C(x)$ the loop space at x, that is, the set of all closed curves starting and ending at x. If τ and μ are elements of $C(x)$, the composite curve $\mu \cdot \tau$ (τ followed by μ) is also an element of $C(x)$. As we proved in §3, for each $\tau \in C(x)$, the parallel displacement along τ is an isomorphism of the fibre $\pi^{-1}(x)$ onto itself. The set of all such isomorphisms of $\pi^{-1}(x)$ onto itself forms a group by virtue of Proposition 3.3. This group is called the *holonomy group of* Γ *with reference point* x. Let $C^0(x)$ be the subset of $C(x)$ consisting of loops which are homotopic to zero. The subgroup of the holonomy group consisting of the parallel displacements arising from all $\tau \in C^0(x)$ is called the *restricted holonomy group of* Γ *with reference point* x. The holonomy group and the restricted holonomy group of Γ with reference point x will be denoted by $\Phi(x)$ and $\Phi^0(x)$ respectively.

It is convenient to realize these groups as subgroups of the structure group G in the following way. Let u be an arbitrarily fixed point of the fibre $\pi^{-1}(x)$. Each $\tau \in C(x)$ determines an element, say, a, of G such that $\tau(u) = ua$. If a loop $\mu \in C(x)$ determines $b \in G$, then the composite $\mu \cdot \tau$ determines ba because $(\mu \cdot \tau)(u) = \mu(ua) = (\mu(u))a = uba$ by virtue of Proposition 3.2. The set of elements $a \in G$ determined by all $\tau \in C(x)$ forms a subgroup of G by Proposition 3.3. This subgroup, denoted by $\Phi(u)$, is called the *holonomy group of* Γ *with reference point* $u \in P$. The *restricted holonomy group* $\Phi^0(u)$ *of* Γ *with reference point* u can be defined accordingly. Note that $\Phi(x)$ is a group of isomorphisms of the fibre $\pi^{-1}(x)$ onto itself and $\Phi(u)$ is a subgroup of G. It is clear that there is a unique isomorphism of $\Phi(x)$ onto $\Phi(u)$ which makes the following diagram commutative:

$$\begin{tikzcd} & C(x) \arrow[dl] \arrow[dr] & \\ \Phi(x) \arrow[rr] & & \Phi(u). \end{tikzcd}$$

Another way of defining $\Phi(u)$ is the following: When two points u and v of P can be joined by a horizontal curve, we write $u \sim v$. This is clearly an equivalence relation. Then $\Phi(u)$ is equal to the set of $a \in G$ such that $u \sim ua$. Using the fact that $u \sim v$ implies $ua \sim va$ for any $u, v \in P$ and $a \in G$, it is easy to verify once more that this subset of G forms a subgroup of G.

PROPOSITION 4.1. (a) *If* $v = ua$, $a \in G$, *then* $\Phi(v) = \mathrm{ad}\,(a^{-1})(\Phi(u))$, *that is, the holonomy groups* $\Phi(v)$ *and* $\Phi(u)$ *are conjugate in G. Similarly,* $\Phi^0(v) = \mathrm{ad}\,(a^{-1})(\Phi^0(u))$.

(b) *If two points u and v of P can be joined by a horizontal curve, then* $\Phi(u) = \Phi(v)$ *and* $\Phi^0(u) = \Phi^0(v)$.

Proof. (a) Let $b \in \Phi(u)$ so that $u \sim ub$. Then $ua \sim (ub)a$ so that $v \sim (va^{-1})ba = va^{-1}ba$. Thus $\mathrm{ad}\,(a^{-1})(b) \in \Phi(v)$. It follows easily that $\Phi(v) = \mathrm{ad}\,(a^{-1})(\Phi(u))$. The proof for $\Phi^0(v) = \mathrm{ad}\,(a^{-1})(\Phi^0(u))$ is similar.

(b) The relation $u \sim v$ implies $ub \sim vb$ for every $b \in G$. Since the relation $\sim$ is transitive, $u \sim ub$ if and only if $v \sim vb$, that is, $b \in \Phi(u)$ if and only if $b \in \Phi(v)$. To prove $\Phi^0(u) = \Phi^0(v)$, let μ^* be a horizontal curve in P from u to v. If $b \in \Phi^0(u)$, then there is a horizontal curve τ^* in P from u to ub such that the curve $\pi(\tau^*)$ in M is a loop at $\pi(u)$ homotopic to zero. Then the composite

$(R_b\mu^*) \cdot \tau^* \cdot \mu^{*-1}$ is a horizontal curve in P from v to vb and its projection into M is a loop at $\pi(v)$ homotopic to zero. Thus $b \in \Phi^0(v)$. Similarly, if $b \in \Phi^0(v)$, then $b \in \Phi^0(u)$. QED.

If M is connected, then for every pair of points u and v of P, there is an element $a \in G$ such that $v \sim ua$. It follows from Proposition 4.1 that if M is connected, the holonomy groups $\Phi(u)$, $u \in P$, are all conjugate to each other in G and hence isomorphic with each other.

The rest of this section is devoted to the proof of the fact that the holonomy group is a Lie group.

THEOREM 4.2. *Let $P(M, G)$ be a principal fibre bundle whose base manifold M is connected and paracompact. Let $\Phi(u)$ and $\Phi^0(u)$, $u \in P$, be the holonomy group and the restricted holonomy group of a connection Γ with reference point u. Then*

(a) *$\Phi^0(u)$ is a connected Lie subgroup of G;*

(b) *$\Phi^0(u)$ is a normal subgroup of $\Phi(u)$ and $\Phi(u)/\Phi^0(u)$ is countable.*

By virtue of this theorem, $\Phi(u)$ is a Lie subgroup of G whose identity component is $\Phi^0(u)$.

Proof. We shall show that every element of $\Phi^0(u)$ can be joined to the identity element by a piecewise differentiable curve of class C^k in G which lies in $\Phi^0(u)$. By the theorem in Appendix 4, it follows then that $\Phi^0(u)$ is a connected Lie subgroup of G.

Let $a \in \Phi^0(u)$ be obtained by the parallel displacement along a piecewise differentiable loop τ of class C^k which is homotopic to 0. By the factorization lemma (Appendix 7), τ is (equivalent to) a product of small lassos of the form $\tau_1^{-1} \cdot \mu \cdot \tau_1$, where τ_1 is a piecewise differentiable curve of class C^k from $x = \pi(u)$ to a point, say, y, and μ is a differentiable loop at y which lies in a coordinate neighborhood of y. It is sufficient to show that the element of $\Phi^0(u)$ defined by each lasso $\tau_1^{-1} \cdot \mu \cdot \tau_1$ can be joined to the identity element. This element is obviously equal to the element of $\Phi^0(v)$ defined by the loop μ, where v is the point obtained by the parallel displacement of u along τ_1. It is therefore sufficient to show that the element $b \in \Phi^0(v)$ defined by the differentiable loop μ can be joined to the identity element in $\Phi^0(v)$ by a differentiable curve of G which lies in $\Phi^0(v)$.

Let $x^1, \ldots, x^n$ be a local coordinate system with origin at y

and let μ be defined by $x^i = x^i(t)$, $i = 1, \ldots, n$. Set $f^i(t, s) = s + (1 - s)x^i(t)$ for $i = 1, \ldots, n$ and $0 \leqq t, s \leqq 1$. Then $f(t, s) = (f^1(t, s), \ldots, f^n(t, s))$ is a differentiable mapping of class C^k of $I \times I$ into M (where $I = [0, 1]$) such that $f(t, 0)$ is the curve μ and $f(t, 1)$ is the trivial curve y. For each fixed s, let $b(s)$ be the element of $\Phi^0(v)$ obtained from the loop $f(t, s)$, $0 \leqq t \leqq 1$, so that $b(0) = b$ and $b(1) =$ identity. The fact that $b(s)$ is of class C^k in s (as a mapping of I into G) follows from the following

LEMMA. *Let $f: I \times I \to M$ be a differentiable mapping of class C^k and $u_0(s)$, $0 \leqq s \leqq 1$, a differentiable curve of class C^k in P such that $\pi(u_0(s)) = f(0, s)$. For each fixed s, let $u_1(s)$ be the point of P obtained by the parallel displacement of $u_0(s)$ along the curve $f(t, s)$, where $0 \leqq t \leqq 1$ and s is fixed. Then the curve $u_1(s)$, $0 \leqq s \leqq 1$, is differentiable of class C^k.*

Proof of Lemma. Let $F: I \times I \to P$ be a differentiable mapping of class C^k such that $\pi(F(t, s)) = f(t, s)$ for all $(t, s) \in I \times I$ and that $F(0, s) = u_0(s)$. The existence of such an F follows from local triviality of the bundle P. Set $v_t(s) = F(t, s)$. In the proof of Proposition 3.1, we saw that, for each fixed s, there is a curve $a_t(s)$, $0 \leqq t \leqq 1$, in G such that $a_0(s) = e$ and that the curve $v_t(s)a_t(s)$, $0 \leqq t \leqq 1$, is horizontal. Set $u_t(s) = v_t(s)a_t(s)$. To prove that $u_1(s)$, $0 \leqq s \leqq 1$, is a differentiable curve of class C^k, it is sufficient to show that $a_1(s)$, $0 \leqq s \leqq 1$, is a differentiable curve of class C^k in G. Let ω be the connection form of Γ. Set $Y_t(s) = -\omega(\dot{v}_t(s))$, where $\dot{v}_t(s)$ is the vector tangent to the curve described by $v_t(s)$, $0 \leqq t \leqq 1$, when s is fixed. Then as in the proof of Proposition 3.1, $a_t(s)$ is a solution of the equation $\dot{a}_t(s)a_t(s)^{-1} = Y_t(s)$. As in the proof of the lemma for Proposition 3.1, we define, for each fixed s, a vector field $X(s)$ on $G \times \mathbf{R}$ so that $(a_t(s), t)$ is the integral curve of the vector field $X(s)$ through the point $(e, 0) \in G \times \mathbf{R}$. The differentiability of $a_t(s)$ in s follows from the fact that each solution of an ordinary linear differential equation with parameter s is differentiable in s as many times as the equation is (cf. Appendix 1). This completes the proof of the lemma and hence the proof of (a) of Theorem 4.2.

We now prove (b). If τ and μ are two loops at x and if μ is homotopic to zero, the composite curve $\tau \cdot \mu \cdot \tau^{-1}$ is homotopic to zero. This implies that $\Phi^0(u)$ is a normal subgroup of $\Phi(u)$.

Let $\pi_1(M)$ be the first homotopy group of M with reference point x. We define a homomorphism f: $\pi_1(M) \rightarrow \Phi(u)/\Phi^0(u)$ as follows. For each element α of $\pi_1(M)$, let τ be a continuous loop at x which represents α. We may cover τ by a finite number of coordinate neighborhoods, modify τ within each neighborhood and obtain a piecewise differentiable loop τ_1 of class C^k at x which is homotopic to τ. If τ_1 and τ_2 are two such loops, then $\tau_1 \cdot \tau_2^{-1}$ is homotopic to zero and defines an element of $\Phi^0(u)$. Thus, τ_1 and τ_2 define the same element of $\Phi(u)/\Phi^0(u)$, which is denoted by $f(\alpha)$. Clearly, f is a homomorphism of $\pi_1(M)$ onto $\Phi(u)/\Phi^0(u)$. Since M is connected and paracompact, it satisfies the second axiom of countability (Appendix 3). It follows easily that $\pi_1(M)$ is countable. Hence, $\Phi(u)/\Phi^0(u)$ is also countable. QED.

Remark. In §3, we defined the parallel displacement along any piecewise differentiable curve of class C^1. In this section, we defined the holonomy group $\Phi(u)$ using piecewise differentiable curves of class C^k. If we denote by $\Phi_k(u)$ the holonomy group thus obtained from piecewise differentiable curves of class C^k, then we have obviously $\Phi_1(u) \supset \Phi_2(u) \supset \cdots \supset \Phi_\infty(u)$. We shall prove later in §7 that these holonomy groups coincide.

5. *Curvature form and structure equation*

Let $P(M, G)$ be a principal fibre bundle and ρ a representation of G on a finite dimensional vector space V; $\rho(a)$ is a linear transformation of V for each $a \in G$ and $\rho(ab) = \rho(a)\rho(b)$ for $a,b \in G$. A *pseudotensorial form of degree r on P of type* (ρ, V) is a V-valued r-form φ on P such that

$$R_a^*\varphi = \rho(a^{-1}) \cdot \varphi \qquad \text{for } a \in G.$$

Such a form φ is called a *tensorial form* if it is horizontal in the sense that $\varphi(X_1, \ldots, X_r) = 0$ whenever at least one of the tangent vectors X_i of P is vertical, i.e., tangent to a fibre.

Example 5.1. If ρ_0 is the trivial representation of G on V, that is, $\rho_0(a)$ is the identity transformation of V for each $a \in G$, then a tensorial form of degree r of type (ρ_0, V) is nothing but a form φ on P which can be expressed as $\varphi = \pi^*\varphi_M$ where φ_M is a V-valued r-form on the base M.

Example 5.2. Let ρ be a representation of G on V and E the bundle associated with P with standard fibre V on which G acts through ρ. A tensorial form φ of degree r of type (ρ, V) can be regarded as an assignment to each $x \in M$ a multilinear skew-symmetric mapping $\tilde{\varphi}_x$ of $T_x(M) \times \cdots \times T_x(M)$ (r times) into the vector space $\pi_E^{-1}(x)$ which is the fibre of E over x. Namely, we define

$$\tilde{\varphi}_x(X_1, \ldots, X_r) = u(\varphi(X_1^*, \ldots, X_r^*)), \qquad X_i \in T_x(M),$$

where u is any point of P with $\pi(u) = x$ and X_i^* is any vector at u such that $\pi(X_i^*) = X_i$ for each i. $\varphi(X_1^*, \ldots, X_r^*)$ is then an element of the standard fibre V and u is a linear mapping of V onto $\pi_E^{-1}(x)$ so that $u(\varphi(X_1^*, \cdots, X_r^*))$ is an element of $\pi_E^{-1}(x)$. It can be easily verified that this element is independent of the choice of u and X_i^*. Conversely, given a skew-symmetric multilinear mapping $\tilde{\varphi}_x$: $T_x(M) \times \cdots \times T_x(M) \to \pi_E^{-1}(x)$ for each $x \in M$, a tensorial form φ of degree r of type (ρ, V) on P can be defined by

$$\varphi(X_1^*, \ldots, X_r^*) = u^{-1}(\tilde{\varphi}_x(\pi(X_1^*), \ldots, \pi(X_r^*))), \qquad X_i^* \in T_u(P),$$

where $x = \pi(u)$. In particular, a tensorial 0-form of type (ρ, V), that is, a function f: $P \to V$ such that $f(ua) = \rho(a^{-1})f(u)$, can be identified with a cross section $M \to E$.

A few special cases of Example 5.2 will be used in Chapter III.

Let Γ be a connection in $P(M, G)$. Let G_u and Q_u be the vertical and the horizontal subspaces of $T_u(P)$, respectively. Let h: $T_u(P) \to Q_u$ be the projection.

Proposition 5.1. *If φ is a pseudotensorial r-form on P of type (ρ, V), then*

(a) *The form φh defined by $(\varphi h)(X_1, \ldots, X_r) = \varphi(hX_1, \ldots, hX_r)$, $X_i \in T_u(P)$, is a tensorial form of type (ρ, V);*

(b) *$d\varphi$ is a pseudotensorial $(r + 1)$-form of type (ρ, V);*

(c) *The $(r + 1)$-form $D\varphi$ defined by $D\varphi = (d\varphi)h$ is a tensorial form of type (ρ, V).*

Proof. From $R_a \circ h = h \circ R_a$, $a \in G$, it follows that φh is a pseudotensorial form of type (ρ, V), It is evident that

$$(\varphi h)(X_1, \ldots, X_r) = 0,$$

if one of X_i's is vertical. (b) follows from $R_a^* \circ d = d \circ R_a^*$, $a \in G$. (c) follows from (a) and (b). QED.

The form $D\varphi = (d\varphi)h$ is called the *exterior covariant derivative of* φ and D is called *exterior covariant differentiation.*

If ρ is the adjoint representation of G in the Lie algebra $\mathfrak{g}$, a (pseudo) tensorial form of type $(\rho, \mathfrak{g})$ is said to be *of type* ad G. The connection form ω is a pseudotensorial 1-form of type ad G. By Proposition 5.1, $D\omega$ is a tensorial 2-form of type ad G and is called the *curvature form* of ω.

THEOREM 5.2 (*Structure equation*). *Let ω be a connection form and Ω its curvature form. Then*

$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y) \qquad \textit{for } X, Y \in T_u(P), \quad u \in P.$$

Proof. Every vector of P is a sum of a vertical vector and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric in X and Y, it is sufficient to verify the equality in the following three special cases.

(1) X and Y are horizontal. In this case, $\omega(X) = \omega(Y) = 0$ and the equality reduces to the definition of Ω.

(2) X and Y are vertical. Let $X = A^*$ and $Y = B^*$ at u, where $A, B \in \mathfrak{g}$. Here A^* and B^* are the fundamental vector fields corresponding to A and B respectively. By Proposition 3.11 of Chapter I, we have

$$\begin{aligned} 2d\omega(A^*, B^*) &= A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega([A^*, B^*]) \\ &= -[A, B] = -[\omega(A^*), \omega(B^*)], \end{aligned}$$

since $\omega(A^*) = A$, $\omega(B^*) = B$ and $[A^*, B^*] = [A, B]^*$. On the other hand, $\Omega(A^*, B^*) = 0$.

(3) X is horizontal and Y is vertical. We extend X to a horizontal vector field on P, which will be also denoted by X. Let $Y = A^*$ at u, where $A \in \mathfrak{g}$. Since the right hand side of the equality vanishes, it is sufficient to show that $d\omega(X, A^*) = 0$. By Proposition 3.11 of Chapter I, we have

$$\begin{aligned} 2d\omega(X, A^*) &= X(\omega(A^*)) - A^*(\omega(X)) - \omega([X, A^*]) \\ &= -\omega([X, A^*]). \end{aligned}$$

Now it is sufficient to prove the following

LEMMA. *If A^* is the fundamental vector field corresponding to an element $A \in \mathfrak{g}$ and X is a horizontal vector field, then $[X, A^*]$ is horizontal.*

Proof of Lemma. The fundamental vector field A^* is induced by R_{a_t}, where a_t is the 1-parameter subgroup of G generated by $A \in \mathfrak{g}$. By Proposition 1.9 of Chapter I, we have

$$[X, A^*] = \lim_{t \to 0} \frac{1}{t} [R_{a_t}(X) - X].$$

If X is horizontal, so is $R_{a_t}(X)$. Thus $[X, A^*]$ is horizontal. QED.

COROLLARY 5.3. *If both X and Y are horizontal vector fields on P, then*

$$\omega([X, Y]) = -2\Omega(X, Y).$$

Proof. Apply Proposition 1.9 of Chapter I to the left hand side of the structure equation just proved. QED.

The structure equation (often called "the structure equation of E. Cartan") is sometimes written, for the sake of simplicity, as follows:

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega.$$

Let $e_1, \ldots, e_r$ be a basis for the Lie algebra $\mathfrak{g}$ and c^i_{jk}, $i, j, k = 1, \ldots, r$, the structure constants of $\mathfrak{g}$ with respect to $e_1, \ldots, e_r$, that is,

$$[e_j, e_k] = \Sigma_i \, c^i_{jk} e_i, \qquad j, k = 1, \ldots, r.$$

Let $\omega = \Sigma_i \, \omega^i e_i$ and $\Omega = \Sigma_i \, \Omega^i e_i$. Then the structure equation can be expressed as follows:

$$d\omega^i = -\tfrac{1}{2}\Sigma_{j,k} \, c^i_{jk} \omega^j \wedge \omega^k + \Omega^i, \qquad i = 1, \ldots, r.$$

THEOREM 5.4 (*Bianchi's identity*). $D\Omega = 0$.

Proof. By the definition of D, it suffices to prove that $d\Omega(X, Y, Z) = 0$ whenever X, Y, and Z are all horizontal vectors. We apply the exterior differentiation d to the structure equation. Then

$$0 = dd\omega^i = -\tfrac{1}{2}\Sigma \, c^i_{jk} \, d\omega^j \wedge \omega^k + \tfrac{1}{2}\Sigma \, c^i_{jk} \omega^j \wedge d\omega^k + d\Omega^i.$$

Since $\omega^i(X) = 0$ whenever X is horizontal, we have

$$d\Omega^i(X, Y, Z) = 0$$

whenever X, Y, and Z are all horizontal. QED.

Proposition 5.5. *Let ω be a connection form and φ a tensorial 1-form of type* ad G. *Then*

$$D\varphi(X, Y) = d\varphi(X, Y) + \tfrac{1}{2}[\varphi(X), \omega(Y)] + \tfrac{1}{2}[\omega(X), \varphi(Y)]$$
$$\text{for } X, Y \in T_u(P), \quad u \in P.$$

Proof. As in the proof of Theorem 5.2, it suffices to consider the three special cases. The only non-trivial case is the case where X is vertical and Y is horizontal. Let $X = A^*$ at u, where $A \in \mathfrak{g}$. We extend Y to a horizontal vector field on P, denoted also by Y, which is invariant by R_a, $a \in G$. (We first extend the vector πY to a vector field on M and then lift it to a horizontal vector field on P.) Then $[A^*, Y] = 0$. As A^* is vertical, $D\varphi(A^*, Y) = 0$. We shall show that the right hand side of the equality vanishes. By Proposition 3.11 of Chapter I, we have

$$d\varphi(A^*, Y) = \tfrac{1}{2}(A^*(\varphi(Y)) - Y(\varphi(A^*)) - \varphi([A^*, Y]) = \tfrac{1}{2}A^*(\varphi(Y)),$$

so that it suffices to show $A^*(\varphi(Y)) + [\omega(A^*), \varphi(Y)] = 0$ or $A^*(\varphi(Y)) = -[A, \varphi(Y)]$. If a_t denotes the 1-parameter subgroup of G generated by A, then

$$\begin{aligned} A^*_u(\varphi(Y)) &= \lim_{t\to 0} \frac{1}{t}[\varphi_{ua_t}(Y) - \varphi_u(Y)] = \lim_{t\to 0}\frac{1}{t}[(R^*_{a_t}\varphi)_u(Y) - \varphi_u(Y)] \\ &= \lim_{t\to 0}\frac{1}{t}[\mathrm{ad}\,(a_t^{-1})(\varphi_u(Y)) - \varphi_u(Y)] = -[A, \varphi_u(Y)], \end{aligned}$$

since Y is invariant by R_{a_t}. QED.

6. *Mappings of connections*

In §5 of Chapter I, we considered certain mappings of one principal fibre bundle into another such as a homomorphism, an injection, and a bundle map. We now study the effects of these mappings on connections.

Proposition 6.1. *Let f: $P'(M', G') \to P(M, G)$ be a homomorphism with the corresponding homomorphism f: $G' \to G$ such that the induced mapping f: $M' \to M$ is a diffeomorphism of M' onto M. Let Γ' be a connection in P', ω' the connection form and Ω' the curvature form of Γ'. Then*

(a) *There is a unique connection Γ in P such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by f.*

(b) *If* ω *and* Ω *are the connection form and the curvature form of* Γ *respectively, then* $f^*\omega = f \cdot \omega'$ *and* $f^*\Omega = f \cdot \Omega'$, *where* $f \cdot \omega'$ *or* $f \cdot \Omega'$ *means the* $\mathfrak{g}'$*-valued form on* P' *defined by* $(f \cdot \omega')(X') = f(\omega'(X'))$ *or* $(f \cdot \Omega')(X', Y') = f(\Omega'(X', Y'))$, *where* f *on the right hand side is the homomorphism* $\mathfrak{g}' \to \mathfrak{g}$ *induced by* f: $G' \to G$.

(c) *If* $u' \in P'$ *and* $u = f(u') \in P$, *then* f: $G' \to G$ *maps* $\Phi(u')$ *onto* $\Phi(u)$ *and* $\Phi^0(u')$ *onto* $\Phi^0(u)$, *where* $\Phi(u)$ *and* $\Phi^0(u)$ (*resp.* $\Phi(u')$ *and* $\Phi^0(u')$) *are the holonomy group and the restricted holonomy group of* Γ (*resp.* Γ') *with reference point* u (*resp.* u').

Proof. (a) Given a point $u \in P$, choose $u' \in P'$ and $a \in G$ such that $u = f(u')a$. We define the horizontal subspace Q_u of $T_u(P)$ by $Q_u = R_a \circ f(Q_{u'})$, where $Q_{u'}$ is the horizontal subspace of $T_{u'}(P')$ with respect to Γ'. We shall show that Q_u is independent of the choice of u' and a. If $u = f(v')b$, where $v' \in P'$ and $b \in G$, then $v' = u'c'$ for some $c' \in G'$. If we set $c = f(c')$, then $u = f(v')b = f(u'c')b = f(u')cb$ and hence $a = cb$. We have $R_b \circ f(Q_{v'}) = R_b \circ f(Q_{u'c'}) = R_b \circ f \circ R_{c'}(Q_{u'}) = R_b \circ R_c \circ f(Q_{u'}) = R_a \circ f(Q_{u'})$, which proves our assertion. We shall show that the distribution $u \to Q_u$ is a connection in P. If $u = f(u')a$, then $ub = f(u')ab$ and $Q_{ub} = R_{ab} \circ f(Q_{u'}) = R_b \circ R_a \circ f(Q_{u'}) = R_b(Q_u)$, thus proving the invariance of the distribution by G. We shall now prove $T_u(P) = Q_u + G_u$, where G_u is the tangent space to the fibre at u. By local triviality of P, it is sufficient to prove that the projection π: $P \to M$ induces a linear isomorphism π: $Q_u \to T_x(M)$, where $x = \pi(u)$. We may assume that $u = f(u')$ since the distribution $u \to Q_u$ is invariant by G. In the commutative diagram

$$\begin{array}{ccc} Q_{u'} & \xrightarrow{f} & Q_u \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ T_{x'}(M') & \xrightarrow{f} & T_x(M), \end{array}$$

the mappings π': $Q_{u'} \to T_{x'}(M')$ and f: $T_{x'}(M') \to T_x(M)$ are linear isomorphisms and hence the remaining two mappings must be also linear isomorphisms. The uniqueness of Γ is evident from its construction.

(b) The equality $f^*\omega = f \cdot \omega'$ can be rewritten as follows:

$$\omega(fX') = f(\omega'(X')) \qquad \text{for } X' \in T_{u'}(P'), \quad u' \in P'.$$

It is sufficient to verify the above equality in the two special cases: (1) X' is horizontal, and (2) X' is vertical. Since $f: P' \to P$ maps every horizontal vector into a horizontal vector, both sides of the equality vanish if X' is horizontal. If X' is vertical, $X' = A'^*$ at u', where $A' \in \mathfrak{g}'$. Set $A = f(A') \in \mathfrak{g}$. Since $f(u'a') = f(u')f(a')$ for every $a' \in G'$, we have $f(X') = A^*$ at $f(u')$. Thus

$$\omega(fX') = \omega(A^*) = A = f(A') = f(\omega'(A'^*)) = f(\omega'(X')).$$

From $f^*\omega = f \cdot \omega'$, we obtain $d(f^*\omega) = d(f \cdot \omega')$ and $f^*d\omega = f \cdot d\omega'$. By the structure equation (Theorem 5.2):

$$-\tfrac{1}{2}f^*([\omega, \omega]) + f^*\Omega = -\tfrac{1}{2}f([\omega', \omega']) + f \cdot \Omega',$$

we have

$$-\tfrac{1}{2}[f^*\omega, f^*\omega] + f^*\Omega = -\tfrac{1}{2}[f \cdot \omega', f \cdot \omega'] + f \cdot \Omega'.$$

This implies that $f^*\Omega = f \cdot \Omega'$.

(c) Let τ be a loop at $x = \pi(u)$. Set $\tau' = f^{-1}(\tau)$ so that τ' is a loop at $x' = \pi'(u')$. Let τ'^* be the horizontal lift of τ' starting from u'. Then $f(\tau'^*)$ is the horizontal lift of τ starting from u. The statement (c) is now evident. QED.

In the situation as in Proposition 6.1, we say that f maps the connection Γ' into the connection Γ. In particular, in the case where $P'(M', G')$ is a reduced subbundle of $P(M, G)$ with injection f so that $M' = M$ and $f: M' \to M$ is the identity transformation, we say that the connection Γ in P is *reducible* to the connection Γ' in P'. An automorphism f of the bundle $P(M, G)$ is called an *automorphism of a connection* Γ in P if it maps Γ into Γ, and in this case, Γ is said to be *invariant* by f.

Proposition 6.2. *Let $f: P'(M', G') \to P(M, G)$ be a homomorphism such that the corresponding homomorphism $f: G' \to G$ maps G' isomorphically onto G. Let Γ be a connection in P, ω the connection form and Ω the curvature form of Γ. Then*

(a) *There is a unique connection Γ' in P' such that the horizontal subspaces of Γ' are mapped into horizontal subspaces of Γ by f.*

(b) *If ω' and Ω' are the connection form and the curvature form of Γ' respectively, then $f^*\omega = f \cdot \omega'$ and $f^*\Omega = f \cdot \Omega'$.*

(c) *If $u' \in P'$ and $u = f(u') \in P$, then the isomorphism $f: G' \to G$ maps $\Phi(u')$ into $\Phi(u)$ and $\Phi^0(u')$ into $\Phi^0(u)$.*

Proof. We define Γ' by defining its connection form ω'. Set $\omega' = f^{-1} \cdot f^*\omega$, where f^{-1}: $\mathfrak{g} \to \mathfrak{g}'$ is the inverse of the isomorphism f: $\mathfrak{g}' \to \mathfrak{g}$ induced from f: $G' \to G$. Let $X' \in T_{u'}(P')$ and $a' \in G'$ and set $X = fX'$ and $a = f(a')$. Then we have

$$\begin{aligned}\omega'(R_{a'}X') &= f^{-1}(\omega(f(R_{a'}X'))) = f^{-1}(\omega(R_a X)) \\ &= f^{-1}(\mathrm{ad}\,(a^{-1})(\omega(X))) = \mathrm{ad}\,(a'^{-1})(f^{-1}(\omega(X))) \\ &= \mathrm{ad}\,(a'^{-1})(\omega(X')).\end{aligned}$$

Let $A' \in \mathfrak{g}'$ and set $A = f(A')$. Let A^* and A'^* denote the fundamental vector fields corresponding to A and A' respectively. Then we have

$$\omega'(A'^*) = f^{-1}(\omega(A^*)) = f^{-1}(A) = A'.$$

This proves that the form ω' defines a connection (Proposition 1.1). The verification of other statements is similar to the proof of Proposition 6.1 and is left to the reader. QED.

In the situation as in Proposition 6.2, we say that Γ' is *induced* by f from Γ. If f is a bundle map, that is, $G' = G$ and f: $G' \to G$ is the identity automorphism, then $\omega' = f^*\omega$. In particular, given a bundle $P(M, G)$ and a mapping f: $M' \to M$, every connection in P induces a connection in the induced bundle $f^{-1}P$.

For any principal fibre bundles $P(M, G)$ and $Q(M, H)$, $P \times Q$ is a principal fibre bundle over $M \times M$ with group $G \times H$. Let $P + Q$ be the restriction of $P \times Q$ to the diagonal ΔM of $M \times M$. Since ΔM and M are diffeomorphic with each other in a natural way, we consider $P + Q$ as a principal fibre bundle over M with group $G \times H$. The restriction of the projection $P \times Q \to P$ to $P + Q$, denoted by f_P, is a homomorphism with the corresponding natural homomorphism f_G: $G \times H \to G$. Similarly, for f_Q: $P + Q \to Q$ and f_H: $G \times H \to H$.

Proposition 6.3. *Let Γ_P and Γ_Q be connections in $P(M, G)$ and $Q(M, H)$ respectively. Then*

(a) *There is a unique connection Γ in $P + Q$ such that the homomorphisms f_P: $P + Q \to P$ and f_Q: $P + Q \to Q$ maps Γ into Γ_P and Γ_Q respectively.*

(b) *If ω, ω_P and ω_Q are the connection forms and Ω, Ω_P, and Ω_Q are the curvature forms of Γ, Γ_P, and Γ_Q respectively, then*

$$\omega = f_P^*\omega_P + f_Q^*\omega_Q, \qquad \Omega = f_P^*\Omega_P + f_Q^*\Omega_Q.$$

(c) *Let* $u \in P$, $v \in Q$, *and* $(u, v) \in P + Q$. *Then the holonomy group* $\Phi(u, v)$ *of* Γ (*resp. the restricted holonomy group* $\Phi^0(u, v)$ *of* Γ) *is a subgroup of* $\Phi(u) \times \Phi(v)$ (*resp.* $\Phi^0(u) \times \Phi^0(v)$). *The homomorphism* $f_G: G \times H \to G$ (*resp.* $f_H: G \times H \to H$) *maps* $\Phi(u, v)$ *onto* $\Phi(u)$ (*resp. onto* $\Phi(v)$) *and* $\Phi^0(u, v)$ *onto* $\Phi^0(u)$ (*resp. onto* $\Phi^0(v)$), *where* $\Phi(u)$ *and* $\Phi^0(u)$ (*resp.* $\Phi(v)$ *and* $\Phi^0(v)$) *are the holonomy group and the restricted holonomy group of* Γ_P (*resp.* Γ_Q).

The proof is similar to those of Propositions 6.1 and 6.2 and is left to the reader.

PROPOSITION 6.4. *Let* $Q(M, H)$ *be a subbundle of* $P(M, G)$, *where* H *is a Lie subgroup of* G. *Assume that the Lie algebra* $\mathfrak{g}$ *of* G *admits a subspace* $\mathfrak{m}$ *such that* $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (*direct sum*) *and* $\mathrm{ad}\,(H)(\mathfrak{m}) = \mathfrak{m}$, *where* $\mathfrak{h}$ *is the Lie algebra of* H. *For every connection form* ω *in* P, *the* $\mathfrak{h}$-*component* ω' *of* ω *restricted to* Q *is a connection form in* Q.

Proof. Let $A \in \mathfrak{h}$ and A^* the fundamental vector field corresponding to A. Then $\omega'(A^*)$ is the $\mathfrak{h}$-component of $\omega(A^*) = A$. Hence, $\omega'(A^*) = A$. Let φ be the $\mathfrak{m}$-component of ω restricted to Q. Let $X \in T_v(Q)$ and $a \in H$. Then

$$\omega(R_a X) = \omega'(R_a X) + \varphi(R_a X),$$
$$\mathrm{ad}\,(a^{-1})(\omega(X)) = \mathrm{ad}\,(a^{-1})(\omega'(X)) + \mathrm{ad}\,(a^{-1})(\varphi(X)).$$

The left-hand sides of the preceding two equalities coincide. Comparing the $\mathfrak{h}$-components of the right hand sides, we obtain $\omega'(R_a X) = \mathrm{ad}\,(a^{-1})(\omega'(X))$. Observe that we used the fact that $\mathrm{ad}\,(a^{-1})(\varphi(X))$ is in $\mathfrak{m}$. QED.

Remark. The connection defined by ω in P is reducible to a connection in the subbundle Q if and only if the restriction of ω to Q is $\mathfrak{h}$-valued. Under the assumption in Proposition 6.4, this means $\omega' = \omega$ on Q.

7. *Reduction theorem*

Unless otherwise stated, a curve will mean a piecewise differentiable curve of class C^∞. The holonomy group $\Phi_\infty(u_0)$ will be denoted by $\Phi(u_0)$.

We first establish

THEOREM 7.1 (*Reduction theorem*). *Let* $P(M, G)$ *be a principal fibre bundle with a connection* Γ, *where* M *is connected and paracompact. Let* u_0 *be an arbitrary point of* P. *Denote by* $P(u_0)$ *the set of points in* P

which can be joined to u_0 *by a horizontal curve. Then*

(1) $P(u_0)$ *is a reduced bundle with structure group* $\Phi(u_0)$.

(2) *The connection* Γ *is reducible to a connection in* $P(u_0)$.

Proof. (1) We first prove

LEMMA 1. *Let* Q *be a subset of* $P(M, G)$ *and* H *a Lie subgroup of* G. *Assume:* (1) *the projection* $\pi: P \to M$ *maps* Q *onto* M; (2) Q *is stable by* H, *i.e.,* $R_a(Q) = Q$ *for each* $a \in H$; (3) *if* $u, v \in Q$ *and* $\pi(u) = \pi(v)$, *then there is an element* $a \in H$ *such that* $v = ua$; *and* (4) *every point* x *of* M *has a neighborhood* U *and a cross section* $\sigma: U \to P$ *such that* $\sigma(U) \subset Q$. *Then* $Q(M, H)$ *is a reduced subbundle of* $P(M, G)$.

Proof of Lemma 1. For each $u \in \pi^{-1}(U)$, let $x = \pi(u)$ and $a \in G$ the element determined by $u = \sigma(x)a$. Define an isomorphism $\psi: \pi^{-1}(U) \to U \times G$ by setting $\psi(u) = (x, a)$. It is easy to see that ψ maps $Q \cap \pi^{-1}(U)$ 1:1 onto $U \times H$. Introduce a differentiable structure in Q in such a way that $\psi: Q \cap \pi^{-1}(U) \to U \times H$ becomes a diffeomorphism; using Proposition 1.3 of Chapter I as in the proof of Proposition 5.3 of Chapter I, we see that Q becomes a differentiable manifold. It is now evident that Q is a principal fibre bundle over M with group H and that Q is a subbundle of P.

Going back to the proof of the first assertion of Theorem 7.1, we see that, M being paracompact, the holonomy group $\Phi(u_0)$ is a Lie subgroup of G (Theorem 4.2) and that the subset $P(u_0)$ and the group $\Phi(u_0)$ satisfy conditions (1), (2), and (3) of Lemma 1 (cf. the second definition of $\Phi(u_0)$ given before Proposition 4.1 and also Proposition 4.1(b)). To verify condition (4) of Lemma 1, let $x^1, \ldots, x^n$ be a local coordinate system around x such that x is the origin $(0, \ldots, 0)$ with respect to this coordinate system. Let U be a cubical neighborhood of x defined by $|x^i| < \delta$. Given any point $y \in U$, let τ_y be the segment from x to y with respect to the coordinate system $x^1, \ldots, x^n$. Fix a point $u \in Q$ such that $\pi(u) = x$. Let $\sigma(y)$ be the point of P obtained by the parallel displacement of u along τ_y. Then $\sigma: U \to P$ is a cross section such that $\sigma(U) \subset Q$. Now (1) of Theorem 7.1 follows from Lemma 1.

(2) This is an immediate consequence of the following

LEMMA 2. *Let* $Q(M, H)$ *be a subbundle of* $P(M, G)$ *and* Γ *a connection in* P. *If, for every* $u \in Q$, *the horizontal subspace of* $T_u(P)$ *is tangent to* Q, *then* Γ *is reducible to a connection in* Q.

Proof of Lemma 2. We define a connection Γ' in Q as follows. The horizontal subspace of $T_u(Q)$, $u \in Q$, with respect to Γ' is by definition the horizontal subspace of $T_u(P)$ with respect to Γ. It is obvious that Γ is reducible to Γ'. QED.

We shall call $P(u)$ the *holonomy bundle* through u. It is evident that $P(u) = P(v)$ if and only if u and v can be joined by a horizontal curve. Since the relation $\sim$ introduced in §4 ($u \sim v$ if u and v can be joined by a horizontal curve) is an equivalence relation, we have, for every pair of points u and v of P, either $P(u) = P(v)$ or $P(u) \cap P(v) =$ empty. In other words, P is decomposed into the disjoint union of the holonomy bundles. Since every $a \in G$ maps each horizontal curve into a horizontal curve, $R_a(P(u)) = P(ua)$ and R_a: $P(u) \to P(ua)$ is an isomorphism with the corresponding isomorphism ad (a^{-1}): $\Phi(u) \to \Phi(ua)$ of the structure groups. It is easy to see that, given any u and v, there is an element $a \in G$ such that $P(v) = P(ua)$. Thus the holonomy bundles $P(u)$, $u \in P$, are all isomorphic with each other.

Using Theorem 7.1, we prove that the holonomy groups $\Phi_k(u)$, $1 \leqq k \leqq \infty$, coincide as was pointed out in Remark of § 4. This result is due to Nomizu and Ozeki [2].

THEOREM 7.2. *All the holonomy groups* $\Phi_k(u)$, $1 \leqq k \leqq \infty$, *coincide.*

Proof. It is sufficient to show that $\Phi_1(u) = \Phi_\infty(u)$. We denote $\Phi_\infty(u)$ by $\Phi(u)$ and the holonomy bundle through u by $P(u)$. We know by Theorem 7.1 that $P(u)$ is a subbundle of P with $\Phi(u)$ as its structure group. Define a distribution S on P by setting

$$S_u = T_u(P(u)) \qquad \text{for } u \in P.$$

Since the holonomy bundles have the same dimension, say k, S is a k-dimensional distribution. We first prove

LEMMA 1. (1) S is *differentiable and involutive.*

(2) *For each* $u \in P$, $P(u)$ *is the maximal integral manifold of* S *through* u.

Proof of Lemma 1. (1) We set

$$S_u = S'_u + S''_u, \qquad u \in P,$$

where S'_u is horizontal and S''_u is vertical. The distribution S' is differentiable by the very definition of a connection. To prove the differentiability of S, it suffices to show that of S''. For each $u \in P$, let U be a neighborhood of $x = \pi(u)$ with a cross section $\sigma: U \to P(u)$ such that $\sigma(x) = u$. (Such a cross section was constructed in the proof of Theorem 7.1.) Let $A_1, \ldots, A_r$ be a basis of the Lie algebra $\mathfrak{g}(u)$ of $\Phi(u)$. We shall define vector fields $\tilde{A}_1, \ldots, \tilde{A}_r$ on $\pi^{-1}(U)$ which form a basis of S'' at every point of $\pi^{-1}(U)$. Let $v \in \pi^{-1}(U)$. Then there is a unique $a \in G$ such that $v = \sigma(\pi(v))a$. Since $\mathrm{ad}\,(a^{-1})\colon \Phi(u) \to \Phi(v)$ is an isomorphism, $\mathrm{ad}\,(a^{-1})(A_i)$, $i = 1, \ldots, r$, are elements of $\mathfrak{g}(v)$ and form a basis for $\mathfrak{g}(v)$. We set

$$(\tilde{A}_i)_v = (\mathrm{ad}\,(a^{-1})(A_i))^*_v, \qquad i = 1, \ldots, r,$$

where $(\mathrm{ad}\,(a^{-1})(A_i))^*$ is the fundamental vector field on P corresponding to $\mathrm{ad}\,(a^{-1})(A_i) \in \mathfrak{g}(v) \subset \mathfrak{g}$, $i = 1, \ldots, r$. It is easy to see that $\tilde{A}_1, \ldots, \tilde{A}_r$ are differentiable and form a basis of S'' on $\pi^{-1}(U)$.

For each point u, $P(u)$ is an integral manifold of S, since for every $v \in P(u)$, we have $T_v(P(u)) = T_v(P(v)) = S_v$. This implies that S is involutive.

(2) Let $W(u)$ be the maximal integral manifold of S through u (cf. Proposition 1.2 of Chapter I). Then $P(u)$ is an open submanifold of $W(u)$. We prove that $P(u) = W(u)$. Let v be an arbitrary point of $W(u)$ and let $u(t)$, $0 \leqq t \leqq 1$, be a curve in $W(u)$ such that $u(0) = u$ and $u(1) = v$. Let t_1 be the supremum of t_0 such that $0 \leqq t \leqq t_0$ implies $u(t) \in P(u)$. Since $P(u)$ is open in $W(u)$, t_1 is positive. We show that $u(t_1)$ lies in $P(u)$; since $P(u)$ is open in $W(u)$, this will imply that $t_1 = 1$, proving that $u(1) = v$ lies in $P(u)$. The point $u(t_1)$ is in $P(u(t_1))$ and $P(u(t_1))$ is open in $W(u(t_1))$. There exists $\varepsilon > 0$ such that $t_1 - \varepsilon < t < t_1 + \varepsilon$ implies $u(t) \in P(u(t_1))$. Let t be any value such that $t_1 - \varepsilon < t < t_1$. By definition of t_1, we have $u(t) \in P(u)$. On the other hand, $u(t) \in P(u(t_1))$. This implies that $P(u) = P(u(t_1))$ so that $u(t_1) \in P(u)$ as we wanted to show. We have thereby proved that $P(u)$ is actually the maximal integral manifold of S through u.

LEMMA 2. *Let S be an involutive, C^∞-distribution on a C^∞-manifold. Suppose x_t, $0 \leqq t \leqq 1$, is a piecewise C^1-curve whose tangent vectors $\dot{x}_t$ belong to S. Then the entire curve x_t lies in the maximal integral manifold W of S through the point x_0.*

Proof of Lemma 2. We may assume that x_t is a C^1-curve. Take a local coordinate system $x^1, \ldots, x^n$ around the point x_0 such that

$\partial/\partial x^1, \ldots, \partial/\partial x^k$, $k = \dim S$, form a local basis for S (cf. Chevalley [1, p. 92]). For small values of t, say, $0 \leqq t \leqq \varepsilon$, x_t can be expressed by $x^i = x^i(t)$, $1 \leqq i \leqq n$, and its tangent vectors are given by $\Sigma_i\,(dx^i/dt)(\partial/\partial x^i)$. By assumption, we have $dx^i/dt = 0$ for $k+1 \leqq i \leqq n$. Thus, $x^i(t) = x^i(0)$ for $k+1 \leqq i \leqq n$ so that x_t, $0 \leqq t \leqq \varepsilon$, lies in the slice through x_0 and hence in W. The standard continuation argument concludes the proof of Lemma 2.

We are now in position to complete the proof of Theorem 7.2. Let a be any element of $\Phi_1(u)$. This means that u and ua can be joined by a piecewise C^1-horizontal curve u_t, $0 \leqq t \leqq 1$, in P. The tangent vector $\dot{u}_t$ at each point obviously lies in S_{u_t}. By Lemma 2, the entire curve u_t lies in the maximal integral manifold $W(u)$ of S through u. By Lemma 1, the entire curve u_t lies in $P(u)$. In particular, ua is a point of $P(u)$. Since $P(u)$ is a subbundle with structure group $\Phi(u)$, a belongs to $\Phi(u)$. QED.

COROLLARY 7.3. *The restricted holonomy groups* $\Phi^0_k(u)$, $1 \leqq k \leqq \infty$, *coincide.*

Proof. $\Phi^0_k(u)$ is the connected component of the identity of $\Phi_k(u)$ for every k (cf. Theorem 4.2 and its proof). Now, Corollary 7.3 follows from Theorem 7.2. QED.

Remark. In the case where $P(M, G)$ is a real analytic principal bundle with an analytic connection, we can still define the holonomy group $\Phi_\omega(u)$ by using only piecewise analytic horizontal curves. The argument used in proving Theorem 7.2 and Corollary 7.3 shows that $\Phi_\omega(u) = \Phi_1(u)$ and $\Phi^0_\omega(u) = \Phi^0_1(u)$.

Given a connection Γ in a principal fibre bundle $P(M, G)$, we shall define the notion of parallel displacement in the associated fibre bundle $E(M, F, G, P)$ with standard fibre F. For each $w \in E$, the *horizontal subspace* Q_w and the *vertical subspace* F_w of $T_w(E)$ are defined as follows. The vertical subspace F_w is by definition the tangent space to the fibre of E at w. To define Q_w, we recall that we have the natural projection $P \times F \to E = P \times_G F$. Choose a point $(u, \xi) \in P \times F$ which is mapped into w. We fix this $\xi \in F$ and consider the mapping $P \to E$ which maps $v \in P$ into $v\xi \in E$. Then the horizontal subspace Q_w is, by definition, the image of the horizontal subspace $Q_u \subset T_u(P)$ by this mapping $P \to E$. We see easily that Q_w is independent of the choice of

$(u, \xi) \in P \times F$. We leave to the reader the proof that $T_w(E) = F_w + Q_w$ (direct sum). A curve in E is *horizontal* if its tangent vector is horizontal at each point. Given a curve τ in M, a (*horizontal*) *lift* τ^* of τ is a horizontal curve in E such that $\pi_E(\tau^*) = \tau$. Given a curve $\tau = x_t$, $0 \leqq t \leqq 1$, and a point w_0 such that $\pi_E(w_0) = x_0$, there is a unique lift $\tau^* = w_t$ starting from w_0. To prove the existence of τ^*, we choose a point (u_0, ξ) in $P \times F$ such that $u_0\xi = w_0$. Let u_t be the lift of $\tau = x_t$ starting from u_0. Then $w_t = u_t\xi$ is a lift of τ starting from w_0. The uniqueness of τ^* reduces to the uniqueness of a solution of a system of ordinary linear differential equations satisfying a given initial condition just as in the case of a lift in a principal fibre bundle. A cross section σ of E defined on an open subset U of M is called *parallel* if the image of $T_x(M)$ by σ is horizontal for each $x \in U$, that is, for any curve $\tau = x_t$, $0 \leqq t \leqq 1$, the parallel displacement of $\sigma(x_0)$ along τ gives $\sigma(x_1)$.

PROPOSITION 7.4. *Let $P(M, G)$ be a principal fibre bundle and $E(M, G/H, G, P)$ the associated bundle with standard fibre G/H, where H is a closed subgroup of G. Let $\sigma\colon M \to E$ be a cross section and $Q(M, H)$ the reduced subbundle of $P(M, G)$ corresponding to σ (cf.* Proposition 5.6 *of* Chapter I). *Then a connection Γ in P is reducible to a connection Γ' in Q if and only if σ is parallel with respect to Γ.*

Proof. If we identify E with P/H (cf. Proposition 5.5 of Chapter I), then $\sigma(M)$ coincides with the image of Q by the natural projection $\mu\colon P \to E = P/H$; in other words, if $u \in Q$ and $x = \pi(u)$, then $\sigma(x) = \mu(u)$ (cf. Proposition 5.6 of Chapter I). Suppose Γ is reducible to a connection Γ' in Q. We note that if ξ is the origin (i.e., the coset H) of G/H, then $u\xi = \mu(u)$ for every $u \in P$ and hence if u_t, $0 \leqq t \leqq 1$, is horizontal in P, so is $\mu(u_t)$ in E. Given a curve x_t, $0 \leqq t \leqq 1$, in M, choose $u_0 \in Q$ with $\pi(u_0) = x_0$ so that $\sigma(x_0) = \mu(u_0)$. Let u_t be the lift to P of x_t starting from u_0 (with respect to Γ), so that $\mu(u_t)$ is the lift of x_t to E starting from $\sigma(x_0)$. Since Γ is reducible to Γ', we have $u_t \in Q$ and hence $\mu(u_t) = \sigma(x_t)$ for all t. Conversely, assume that σ is parallel (with respect to Γ). Given any curve x_t, $0 \leqq t \leqq 1$, in M and any point u_0 of Q with $\pi(u_0) = x_0$, let u_t be the lift of x_t to P starting from u_0. Since σ is parallel, $\mu(u_t) = \sigma(x_t)$ and hence $u_t \in Q$ for all t. This shows that every horizontal vector at $u_0 \in Q$ (with respect to Γ) is tangent to Q. By Proposition 7.2, Γ is reducible to a connection in Q. QED.

8. *Holonomy theorem*

We first prove the following result of Ambrose and Singer [1] by applying Theorem 7.1.

THEOREM 8.1. *Let $P(M, G)$ be a principal fibre bundle, where M is connected and paracompact. Let Γ be a connection in P, Ω the curvature form, $\Phi(u)$ the holonomy group with reference point $u \in P$ and $P(u)$ the holonomy bundle through u of Γ. Then the Lie algebra of $\Phi(u)$ is equal to the subspace of $\mathfrak{g}$, Lie algebra of G, spanned by all elements of the form $\Omega_v(X, Y)$, where $v \in P(u)$ and X and Y are arbitrary horizontal vectors at v.*

Proof. By virtue of Theorem 7.1, we may assume that $P(u) = P$, i.e., $\Phi(u) = G$. Let $\mathfrak{g}'$ be the subspace of $\mathfrak{g}$ spanned by all elements of the form $\Omega_v(X, Y)$, where $v \in P(u) = P$ and X and Y are arbitrary horizontal vectors at v. The subspace $\mathfrak{g}'$ is actually an ideal of $\mathfrak{g}$, because Ω is a tensorial form of type ad G (cf. §5) and hence $\mathfrak{g}'$ is invariant by ad G. We shall prove that $\mathfrak{g}' = \mathfrak{g}$.

At each point $v \in P$, let S_v be the subspace of $T_v(P)$ spanned by the horizontal subspace Q_v and by the subspace $\mathfrak{g}'_v = \{A^*_v; A \in \mathfrak{g}'\}$, where A^* is the fundamental vector field on P corresponding to A. The distribution S has dimension $n + r$, where $n = \dim M$ and $r = \dim \mathfrak{g}'$. We shall prove that S is differentiable and involutive. Let v be an arbitrary point of P and U a coordinate neighborhood of $y = \pi(v) \in M$ such that $\pi^{-1}(U)$ is isomorphic with $U \times G$. Let $X_1, \dots, X_n$ be differentiable vector fields on U which are linearly independent everywhere on U and $X^*_1, \dots, X^*_n$ the horizontal lifts of $X_1, \dots, X_n$. Let $A_1, \dots, A_r$ be a basis for $\mathfrak{g}'$ and $A^*_1, \dots, A^*_r$ the corresponding fundamental vector fields. It is clear that $X^*_1, \dots, X^*_n, A^*_1, \dots, A^*_r$ form a local basis for S. To prove that S is involutive, it suffices to verify that the bracket of any two of these vector fields belongs to S. This is clear for $[A^*_i, A^*_j]$, since $[A_i, A_j] \in \mathfrak{g}'$ and $[A_i, A_j]^* = [A^*_i, A^*_j]$. By the lemma for Theorem 5.2, $[A^*_i, X^*_j]$ is horizontal; actually, $[A^*_i, X^*_j] = 0$ as X^*_j is invariant by R_a for each $a \in G$. Finally, set $A = \omega([X^*_i, X^*_j]) \in \mathfrak{g}$, where ω is the connection form of Γ. By Corollary 5.3, $A = \omega([X^*_i, X^*_j]) = -2\Omega(X^*_i, X^*_j) \in \mathfrak{g}'$. Since the vertical component of $[X^*_i, X^*_j]$ at $v \in P$ is equal to $A^*_v \in S_v$,

$[X_i^*, X_j^*]$ belongs to S. This proves our assertion that S is involutive.

Let P_0 be the maximal integral manifold of S through u. By Lemma 2 in the proof of Theorem 7.2, we have $P_0 = P$. Therefore,

$$\dim \mathfrak{g} = \dim P - n = \dim P_0 - n = \dim \mathfrak{g}'.$$

This implies $\mathfrak{g} = \mathfrak{g}'$. QED.

Next we prove

THEOREM 8.2. *Let $P(M, G)$ be a principal fibre bundle, where P is connected and M is paracompact. If* $\dim M \geqq 2$, *there exists a connection in P such that all the holonomy bundles $P(u)$, $u \in P$, coincide with P.*

Proof. Let u_0 be an arbitrary point of P and $x^1, \ldots, x^n$ a local coordinate system with origin $x_0 = \pi(u_0)$. Let U and V be neighborhoods of x_0 defined by $|x^i| < \alpha$ and $|x^i| < \beta$ respectively, where $0 < \beta < \alpha$. Taking α sufficiently small, we may assume that $P \mid U = \pi^{-1}(U)$ is isomorphic with the trivial bundle $U \times G$. We shall construct a connection Γ' in $P \mid U$ such that the holonomy group of the bundle $P \mid V$ coincides with the identity component of G. We shall then extend Γ' to a connection Γ in P in such a way that Γ coincides with Γ' on $P \mid \bar{V}$ (cf. Theorem 2.1).

Let $A_1, \ldots, A_r$ be a basis for the Lie algebra $\mathfrak{g}$ of G. Choose real numbers $\alpha_1, \ldots, \alpha_r$ such that $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r < \beta$ and let $f_i(t)$, $i = 1, \ldots, r$, be differentiable functions in $-\alpha - \varepsilon < t < \alpha + \varepsilon$ such that $f_i(0) = 0$ for every i and $f_i(\alpha_j) = \delta_{ij}$ (Kronecker's symbol). On $\pi^{-1}(U) = U \times G$, we can define a connection form ω by requiring that

$$\omega_{(x,e)}(\partial/\partial x^1) = \sum_{j=1}^{r} f_j(x^2) A_j$$

and that

$$\omega_{(x,e)}(\partial/\partial x^i) = 0 \qquad \text{for } i = 2, 3, \ldots, n.$$

(Note that, by virtue of the property $R_a^*\omega = \operatorname{ad}(a^{-1})(\omega)$, the preceding conditions determine the values of ω at every point (x, a) of $U \times G$.)

Fixing t, $0 < t < \beta$, and α_k, $1 \leqq k \leqq r$, for the moment, consider the rectangle on the x^1x^2-plane in V formed by the line segments τ_1 from $(0, 0)$ to $(0, \alpha_k)$, τ_2 from $(0, \alpha_k)$ to (t, α_k), τ_3 from (t, α_k) to $(t, 0)$ and τ_4 from $(t, 0)$ to $(0, 0)$. (Here and in the following argument, the x^3 to x^n-coordinates of all the points remain 0 and are hence omitted.) In $\pi^{-1}(V) = V \times G$, we determine the horizontal lift of $\tau = \tau_4 \cdot \tau_3 \cdot \tau_2 \cdot \tau_1$ starting from the point $(0, 0; e)$. The lift τ_1^* of τ_1 starting from $(0, 0; e)$ is clearly $(0, s; e)$, $0 \leqq s \leqq \alpha_k$, since its tangent vectors $\partial/\partial x^2$ are horizontal. The lift τ_2^* of τ_2 starting from the end point $(0, \alpha_k; e)$ of τ_1^* is of the form $(s, \alpha_k; c_s)$, $0 \leqq s \leqq t$, where c_s is a suitable curve with $c_0 = e$ in G. Its tangent vector is of the form $(\partial/\partial x^1)_{(s,\alpha_k)} + \dot{c}_s$. By a similar computation to that for Proposition 3.1, we have

$$\begin{aligned}\omega((\partial/\partial x^1)_{(s,\alpha_k)} + \dot{c}_s) &= \text{ad}\,(c_s^{-1})\omega((\partial/\partial x^1))_{(s,\alpha_k;\,e)} + c_s^{-1} \cdot \dot{c}_s \\ &= \text{ad}\,(c_s^{-1})\left(\sum_{j=1}^{r} f_j(\alpha_k)A_j\right) + c_s^{-1} \cdot \dot{c}_s = \text{ad}\,(c_s^{-1})A_k + c_s^{-1} \cdot \dot{c}_s.\end{aligned}$$

Therefore we have $c_s \cdot c_s^{-1} = -A_k$, that is, $c_s = \exp(-sA_k)$. The end point of τ_2^* is hence $(t, \alpha_k; \exp(-tA_k))$. The lift τ_3^* of τ_3 starting from $(t, \alpha_k; \exp(-tA_k))$ is $(t, \alpha_k - s; \exp(-tA_k))$, $0 \leqq s \leqq \alpha_k$. Finally, the lift τ_4^* of τ_4 starting from the end point $(t, 0; \exp(-tA_k))$ of τ_3^* is $(t - s, 0; \exp(-tA_k))$, $0 \leqq s \leqq t$, since $\partial/\partial x^1$ is horizontal at the points with $x^2 = 0$. This shows that the end point of the lift τ^* of τ is $(0, 0; \exp(-tA_k))$, proving that $\exp(-tA_k)$ is an element of the holonomy group of $\pi^{-1}(V)$ with reference point $(0, 0; e)$. Since this is the case for every t, we see that A_k is in the Lie algebra of the holonomy group. The result being valid for any A_k, we see that the holonomy group of the connection in $\pi^{-1}(V)$ coincides with the identity component of G.

Let Γ be a connection in P which coincides with Γ' on $\pi^{-1}(\bar{V})$. Since the holonomy group $\Phi(u_0)$ of Γ obviously contains the identity component of G, the holonomy bundle $P(u_0)$ of Γ has the same dimension as P and hence is open in P. Since P is a disjoint union of holonomy bundles each of which is open, the connectedness of P implies that $P = P(u_0)$. QED.

COROLLARY 8.3. *Any connected Lie group G can be realized as the holonomy group of a certain connection in a trivial bundle $P = M \times G$, where M is an arbitrary differentiable manifold with* $\dim M \geqq 2$.

Theorem 8.2 was proved for linear connections by Hano and Ozeki [1] and then in the general case by Nomizu [5], both by making use of Theorem 8.1. The above proof which is more direct is due to E. Ruh (unpublished).

9. *Flat connections*

Let $P = M \times G$ be a trivial principal fibre bundle. For each $a \in G$, the set $M \times \{a\}$ is a submanifold of P. In particular, $M \times \{e\}$ is a subbundle of P, where e is the identity of G. The *canonical flat connection* in P is defined by taking the tangent space to $M \times \{a\}$ at $u = (x, a) \in M \times G$ as the horizontal subspace at u. In other words, a connection in P is the canonical flat connection if and only if it is reducible to a unique connection in $M \times \{e\}$. Let θ be the canonical 1-form on G (cf. §4 of Chapter I). Let $f: M \times G \to G$ be the natural projection and set

$$\omega = f^*\theta.$$

It is easy to verify that ω is the connection form of the canonical flat connection in P. The Maurer-Cartan equation of θ implies that the canonical flat connection has zero curvature:

$$\begin{aligned} d\omega = d(f^*\theta) &= f^*(d\theta) = f^*(-\tfrac{1}{2}[\theta, \theta]) \\ &= -\tfrac{1}{2}[f^*\theta, f^*\theta] = -\tfrac{1}{2}[\omega, \omega]. \end{aligned}$$

A connection in any principal fibre bundle $P(M, G)$ is called *flat* if every point x of M has a neighborhood U such that the induced connection in $P \mid U = \pi^{-1}(U)$ is isomorphic with the canonical flat connection in $U \times G$. More precisely, there is an isomorphism $\psi: \pi^{-1}(U) \to U \times G$ which maps the horizontal subspace at each $u \in \pi^{-1}(U)$ upon the horizontal subspace at $\psi(u)$ of the canonical flat connection in $U \times G$.

THEOREM 9.1. *A connection in $P(M, G)$ is flat if and only if the curvature form vanishes identically.*

Proof. The necessity is obvious. Assume that the curvature form vanishes identically. For each point x of M, let U be a simply connected open neighborhood of x and consider the induced connection in $P \mid U = \pi^{-1}(U)$. By Theorems 4.2 and 8.1, the holonomy group of the induced connection in $P \mid U$ consists of the identity only. Applying the Reduction Theorem (Theorem 7.1), we see that the induced connection in $P \mid U$ is isomorphic with the canonical flat connection in $U \times G$. QED.

COROLLARY 9.2. *Let Γ be a connection in $P(M, G)$ such that the curvature vanishes identically. If M is paracompact and simply connected,* *then P is isomorphic with the trivial bundle $M \times G$ and Γ is isomorphic with the canonical flat connection in $M \times G$.*

We shall study the case where M is not necessarily simply connected. Let Γ be a flat connection in $P(M, G)$, where M is connected and paracompact. Let $u_0 \in P$ and $M^* = P(u_0)$, the holonomy bundle through u_0; M^* is a principal fibre bundle over M whose structure group is the holonomy group $\Phi(u_0)$. Since $\Phi(u_0)$ is discrete by Theorems 4.2 and 8.1 and since M^* is connected, M^* is a covering space of M. Set $x_0 = \pi(u_0)$, $x_0 \in M$. Every closed curve of M starting from x_0 defines, by means of the parallel displacement along it, an element of $\Phi(u_0)$. Since the restricted holonomy group is trivial by Theorems 4.2 and 8.1, any two closed curves at x_0 representing the same element of the first homotopy group $\pi_1(M, x_0)$ give rise to the same element of $\Phi(u_0)$. Thus we obtain a homomorphism of $\pi_1(M, x_0)$ onto $\Phi(u_0)$. Let N be a normal subgroup of $\Phi(u_0)$ and set $M' = M^*/N$. Then M' is a principal fibre bundle over M with structure group $\Phi(u_0)/N$. In particular, M' is a covering space of M. Let $P'(M', G)$ be the principal fibre bundle induced from $P(M, G)$ by the covering projection $M' \to M$. Let f: $P' \to P$ be the natural homomorphism (cf. Proposition 5.8 of Chapter I).

PROPOSITION 9.3. *There exists a unique connection Γ' in $P'(M', G)$ which is mapped into Γ by the homomorphism f: $P' \to P$. The connection Γ' is flat. If u'_0 is a point of P' such that $f(u'_0) = u_0$, then the holonomy group $\Phi(u'_0)$ of Γ' with reference point u'_0 is isomorphically mapped onto N by f.*

Proof. The first statement is contained in Proposition 6.2. By the same proposition, the curvature form of Γ' vanishes identically and Γ' is flat. We recall that P' is the subset of $M' \times P$ defined as follows (cf. Proposition 5.8 of Chapter I):

$$P' = \{(x', u) \in M' \times P;\ \mu(x') = \pi(u)\},$$

where μ: $M' \to M$ is the covering projection. The projection π': $P' \to M'$ is given by $\pi'(x', u) = x'$ and the homomorphism f: $P' \to P$ is given by $f(x', u) = u$ so that the corresponding homomorphism f: $G \to G$ of the structure groups is the identity automorphism. To prove that f maps $\Phi(u'_0)$ isomorphically onto

N, it is therefore sufficient to prove $\Phi(u'_0) = N$. Write

$$u'_0 = (x'_0, u_0) \in P' \subset M' \times P.$$

Since $\mu(x'_0) = \pi(u_0)$, there exists an element $a \in \Phi(u_0)$ such that

$$x'_0 = \nu(u_0 a),$$

where $\nu\colon M^* = P(u_0) \to M' = P(u_0)/N$ is the covering projection. Let $\tau = u'_t$, $0 \leqq t \leqq 1$, be a horizontal curve in P' such that $\pi'(u'_0) = \pi'(u'_1)$. For each t, we set

$$u'_t = (x'_t, u_t) \in P' \subset M' \times P.$$

Then the curve u_t, $0 \leqq t \leqq 1$, is horizontal in P and hence is contained in $M^* = P(u_0)$. Since $\mu(x'_t) = \pi(u_t) = \mu \circ \nu(u_t)$ and $x'_0 = \nu(u_0 a)$, we have $x'_t = \nu(u_t a)$ for $0 \leqq t \leqq 1$. We have

$$\nu(u_1 a) = x'_1 = \pi'(u'_1) = \pi'(u'_0) = x'_0 = \nu(u_0 a)$$

and, consequently,

$$\nu(u_1) = \nu(u_0),$$

which means that $u_1 = u_0 b$ for some $b \in N$. This shows that $\Phi(u'_0) \subset N$. Conversely, let b be any element of N. Let u_t, $0 \leqq t \leqq 1$, be a horizontal curve in P such that $u_1 = u_0 b$. Define a horizontal curve u'_t, $0 \leqq t \leqq 1$, in P' by

$$u'_t = (x'_t, u_t),$$

where $x'_t = \nu(u_t a)$. Then $u'_1 = u'_0 b$, showing that $b \in \Phi(u'_0)$. QED.

10. *Local and infinitesimal holonomy groups*

Let Γ be a connection in a principal fibre bundle $P(M, G)$, where M is connected and paracompact. For every connected open subset U of M, let Γ_U be the connection in $P \mid U = \pi^{-1}(U)$ induced from Γ. For each $u \in \pi^{-1}(U)$, we denote by $\Phi^0(u, U)$ and $P(u, U)$ the restricted holonomy group with reference point u and the holonomy bundle through u of the connection Γ_U, respectively. $P(u, U)$ consists of points v of $\pi^{-1}(U)$ which can be joined to u by a horizontal curve in $\pi^{-1}(U)$.

The *local holonomy group* $\Phi^*(u)$ with reference point u of Γ is defined to be the intersection $\bigcap \Phi^0(u, U)$, where U runs through all connected open neighborhoods of the point $x = \pi(u)$. If $\{U_k\}$ is a sequence of connected open neighborhoods of x such that $U_k \supset \bar{U}_{k+1}$ and $\bigcap_{k=1}^{\infty} U_k = \{x\}$, then we have obviously $\Phi^0(u, U_1) \supset \Phi^0(u, U_2) \supset \cdots \supset \Phi^0(u, U_k) \supset \cdots$. Since, for every open neighborhood U of x, there exists an integer k such that $U_k \subset U$, we have $\Phi^*(u) = \bigcap_{k=1}^{\infty} \Phi^0(u, U_k)$. Since each group $\Phi^0(u, U_k)$ is a connected Lie subgroup of G (Theorem 4.2), it follows that $\dim \Phi^0(u, U_k)$ is constant for sufficiently large k and hence that $\Phi^*(u) = \Phi^0(u, U_k)$ for such k. The following proposition is now obvious.

Proposition 10.1. *The local holonomy groups have the following properties:*

(1) *$\Phi^*(u)$ is a connected Lie subgroup of G which is contained in the restricted holonomy group $\Phi^0(u)$;*

(2) *Every point $x = \pi(u)$ has a connected open neighborhood U such that $\Phi^*(u) = \Phi^0(u, V)$ for any connected open neighborhood V of x contained in U;*

(3) *If U is such a neighborhood of $x = \pi(u)$, then $\Phi^*(u) \supseteqq \Phi^*(v)$ for every $v \in P(u, U)$;*

(4) *For every $a \in G$, we have $\Phi^*(ua) = \mathrm{ad}\,(a^{-1})(\Phi^*(u))$;*

(5) *For every integer m, the set $\{\pi(u) \in M; \dim \Phi^*(u) \leqq m\}$ is open.*

As to (5), we remark that $\dim \Phi^*(u)$ is constant on each fibre of P by (4) and thus can be considered as an integer valued function on M. Then (5) means that this integer valued function is upper semicontinuous.

Theorem 10.2. *Let $\mathfrak{g}(u)$ and $\mathfrak{g}^*(u)$ be the Lie algebras of $\Phi^0(u)$ and $\Phi^*(u)$ respectively. Then $\Phi^0(u)$ is generated by all $\Phi^*(v)$, $v \in P(u)$, and $\mathfrak{g}(u)$ is spanned by all $\mathfrak{g}^*(v)$, $v \in P(u)$.*

Proof. If $v \in P(u)$, then $\Phi^0(u) = \Phi^0(v) \supset \Phi^*(v)$ and $\mathfrak{g}(u) = \mathfrak{g}(v) \supset \mathfrak{g}^*(v)$. By Theorem 8.1, $\mathfrak{g}(u)$ is spanned by all elements of the form $\Omega_v(X^*, Y^*)$ where $v \in P(u)$ and X^* and Y^* are horizontal vectors at v. Since $\Omega_v(X^*, Y^*)$ is contained in the Lie algebra of $\Phi^0(v, V)$ for every connected open neighborhood V of $\pi(v)$, it is contained in $\mathfrak{g}^*(v)$. Consequently, $\mathfrak{g}(u)$ is spanned by all $\mathfrak{g}^*(v)$

where $v \in P(u)$. The first assertion now follows from the following lemma.

Lemma. *If the Lie algebra* $\mathfrak{g}$ *of a connected Lie group* G *is generated by a family of subspaces* $\{\mathfrak{m}_\lambda\}$, *then every element of* G *can be written as a product* $\exp X_1 \cdot \exp X_2 \cdot \cdots \cdot \exp X_k$, *where each* X_i *is contained in some* $\mathfrak{m}_\lambda$.

Proof of Lemma. The set H of all elements of G of the above form is clearly a subgroup which is arcwise connected; indeed, every element of H can be joined to the identity by a differentiable curve which lies in H. By the theorem of Freudenthal-Kuranishi-Yamabe (proved in Appendix 4), H is a connected Lie subgroup of G. Its Lie algebra contains all $\mathfrak{m}_\lambda$ and thus coincides with $\mathfrak{g}$. Hence, $H = G$. QED.

Theorem 10.3. *If* $\dim \Phi^*(u)$ *is constant on* P, *then* $\Phi^0(u) = \Phi^*(u)$ *for every* u *in* P.

Proof. By (3) of Proposition 10.1, $x = \pi(u)$ has an open neighborhood U such that $\Phi^*(u) \supset \Phi^*(v)$ for each v in $P(u, U)$. Since $\dim \Phi^*(u) = \dim \Phi^*(v)$, we have $\Phi^*(u) = \Phi^*(v)$. By the standard continuation argument, we see that, if $v \in P(u)$, then $\Phi^*(u) = \Phi^*(v)$. By Theorem 10.2, we have $\Phi^0(u) = \Phi^*(u)$. QED.

We now define the infinitesimal holonomy group at each point u of P by means of the curvature form and study its relationship to the local holonomy group. We first define a series of subspaces $\mathfrak{m}_k(u)$ of $\mathfrak{g}$ by induction on k. Let $\mathfrak{m}_0(u)$ be the subspace of $\mathfrak{g}$ spanned by all elements of the form $\Omega_u(X, Y)$, where X and Y are horizontal vectors at u. We consider a $\mathfrak{g}$-valued function f on P of the form

$$(I_k) \qquad f = V_k \cdots V_1(\Omega(X, Y)),$$

where $X, Y, V_1, \ldots, V_k$ are arbitrary horizontal vector fields on P. Let $\mathfrak{m}_k(u)$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions f of the form (I_k). We then set $\mathfrak{g}'(u)$ to be the union of all $\mathfrak{m}_k(u)$, $k = 0, 1, 2, \ldots$.

Proposition 10.4. *The subspace* $\mathfrak{g}'(u)$ *of* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{g}^*(u)$.

The connected Lie subgroup $\Phi'(u)$ of G generated by $\mathfrak{g}'(u)$ is called the *infinitesimal holonomy group* at u.

Proof. We show that $\mathfrak{m}_k(u) \subset \mathfrak{g}^*(u)$ by induction on k. The case $k = 0$ is obvious. Assume that $\mathfrak{m}_{k-1}(u) \subset \mathfrak{g}^*(u)$ for every point u. It is sufficient to show that, for every horizontal vector field X and for every function f of the form (I_{k-1}), we have $X_u f \in \mathfrak{g}^*(u)$. Let u_t, $|t| < \varepsilon$ for some $\varepsilon > 0$, be the integral curve of X with $u_0 = u$. Since u_t is horizontal, we have $\mathfrak{g}^*(u_t) \subset \mathfrak{g}^*(u)$ by (3) of Proposition 10.1. Therefore, $f(u_t) \in \mathfrak{m}_{k-1}(u_t) \subset \mathfrak{g}^*(u_t) \subset \mathfrak{g}^*(u)$. On the other hand, $X_u f = \lim_{t\to 0} \frac{1}{t}[f(u_t) - f(u)]$ so that $X_u f$ is in $\mathfrak{g}^*(u)$. Consequently, $\mathfrak{g}'(u)$ is contained in $\mathfrak{g}^*(u)$.

To prove that $\mathfrak{g}'(u)$ is a subalgebra of $\mathfrak{g}$, we need the following two lemmas.

Lemma 1. *Let f be a $\mathfrak{g}$-valued function of type* ad G *on P. Then*

(1) *For any vector field X on P, we have $v(X)_u \cdot f = -[\omega_u(X), f(u)]$, where $v(X)$ denotes the vertical component of X.*

(2) *For any horizontal vector fields X and Y on P, we have*

$$v([X, Y]_u) \cdot f = 2[\Omega_u(X, Y), f(u)].$$

(3) *If X and Y are vector fields on P which are invariant by all R_a, $a \in G$, then $\Omega(X, Y)$ and Xf are functions of type* ad G.

Proof of Lemma 1. (1) Let $A = \omega_u(X) \in \mathfrak{g}$ and $a_t = \exp tA$. Then

$$\begin{aligned} v(X)_u \cdot f = A^*_u f &= \lim_{t\to 0} \frac{1}{t}[f(ua_t) - f(u)] \\ &= \lim_{t\to 0} \frac{1}{t}[\mathrm{ad}\,(a_t^{-1})(f(u)) - f(u)] \\ &= -[A, f(u)] = -[\omega_u(X), f(u)]. \end{aligned}$$

(2) By virtue of the structure equation (Theorem 5.2), we have

$$\begin{aligned} 2\Omega_u(X, Y) &= 2(d\omega)_u(X, Y) \\ &= X_u(\omega(Y)) - Y_u(\omega(X)) - \omega_u([X, Y]) \\ &= -\omega_u([X, Y]). \end{aligned}$$

Replacing X by $[X, Y]$ in (1), we obtain (2).

(3) Since Ω is of type ad G (cf. §5 of Chapter II), we have

$$\Omega_{ua}(R_a X, R_a Y) = \mathrm{ad}\,(a^{-1})(\Omega_u(X, Y)),$$

which shows that $\Omega(X, Y)$ is of type ad G, if $X = R_a X$ and $Y = R_a Y$. We have also

$$(Xf)_{ua} = X_{ua}f = (R_a X_u)f = X_u(f \circ R_a)$$
$$= \text{ad}\,(a^{-1})(X_u f) = \text{ad}\,(a^{-1})(Xf)_u,$$

if f is of type ad G and X is invariant by R_a. This completes the proof of Lemma 1.

Let $X_i = \partial/\partial x^i$, where $x^1, \ldots, x^n$ is a local coordinate system in a neighborhood U of $x = \pi(u)$. Let X_i^* be the horizontal lift of X_i. Consider a $\mathfrak{g}$-valued function f of the form

$$(II_k) \qquad f = X_{j_k}^* \cdots X_{j_1}^*(\Omega(X_i^*, X_l^*)),$$

where $i, l, j_1, \ldots, j_k$ are taken freely from $1, \ldots, n$.

LEMMA 2. *For each k, $\mathfrak{m}_k(u)$ is spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions f of the form (II_k).*

Proof of Lemma 2. The proof is by induction on k. The case $k = 0$ is obvious. Every horizontal vector field in $\pi^{-1}(U)$ is a linear combination of $X_1^*, \ldots, X_n^*$ with real valued functions as coefficients. It follows that every function f of the form (I_k) is a linear combination of functions of the form (II_s), $s \leqq k$, with real valued functions as coefficients, in a neighborhood of u. It is now clear that, if the assertion holds for $k - 1$, it holds for k.

We now prove that $\mathfrak{g}'(u)$ is a subalgebra of $\mathfrak{g}$ by establishing the relation $[\mathfrak{m}_k(u), \mathfrak{m}_s(u)] \subseteq \mathfrak{m}_{k+s+2}(u)$ for all pairs of integers k and s. In view of Lemma 2, it is sufficient to prove that, for every function f of the form (I_s) and every function g of the form (II_k), the function $[f, g](u) = [f(u), g(u)]$ is a linear combination of functions of the form (I_r), $r \leqq k + s + 2$, with real valued functions as coefficients. The proof is by induction on s.

Let $s = 0$ and let $f(u) = \Omega_u(X, Y)$, where X and Y are horizontal vector fields. Since g is of type ad G, we have, by (2) of Lemma 1,

$$2[\Omega_u(X, Y), g(u)] = v([X, Y])_u \cdot g.$$

On the other hand, we have

$$v([X, Y])_u \cdot g = [X, Y]_u \cdot g - h([X, Y])_u \cdot g$$
$$= X_u(Yg) - Y_u(Xg) - h([X, Y])_u \cdot g,$$

where $h[X, Y]$ denotes the horizontal component of $[X, Y]$. The functions $X(Yg)$ and $Y(Xg)$ are of the form (I_{k+2}) and the function $h([X, Y])g$ is of the form (I_{k+1}). This proves our assertion for $s = 0$ and for an arbitrary k.

Suppose now that our assertion holds for $s - 1$ and every k. Every function of the form (I_s) can be written as Xf, where f is a function of the form (I_{s-1}) and X is a horizontal vector field. Let g be an arbitrary function of the form (II_k). Then

$$[X_u f, g(u)] = X_u([f, g]) - [f(u), X_u g].$$

The function $[f, Xg]$ is a linear combination of functions of the form (I_r), $r \leqq k + s + 1$, by the inductive assumption. The function $X[f, g]$ is a linear combination of functions of the form (I_r), $r \leqq s + k + 2$, also by the inductive assumption. Thus, the function $[Xf, g]$ is a linear combination of functions of the form (I_r), $r \leqq s + k + 2$. QED.

PROPOSITION 10.5. *The infinitesimal holonomy groups have the following properties:*

(1) *$\Phi'(u)$ is a connected Lie subgroup of the local holonomy group $\Phi^*(u)$;*

(2) $\Phi'(ua) = \mathrm{ad}\,(a^{-1})(\Phi'(u))$ *and* $\mathfrak{g}'(ua) = \mathrm{ad}\,(a^{-1})(\mathfrak{g}'(u))$;

(3) *For each integer m, the set $\{\pi(u) \in M;\ \dim \Phi'(u) \geqq m\}$ is open;*

(4) *If $\Phi'(u) = \Phi^*(u)$ at a point u, then there exists a connected open neighborhood U of $x = \pi(u)$ such that $\Phi'(v) = \Phi^*(v) = \Phi'(u) = \Phi^*(u)$ for every $v \in P(u, U)$.*

Proof. (1) is evident from Proposition 10.4. (2) follows from

LEMMA 1. *For each k, we have* $\mathfrak{m}_k(ua) = \mathrm{ad}\,(a^{-1})(\mathfrak{m}_k(u))$.

Proof of Lemma 1. The proof is by induction on k. The case $k = 0$ is a consequence of the fact that Ω is of type ad G. Suppose the assertion holds for $k - 1$. By (3) of Lemma 1 for Proposition 10.4, every function of the form (II_k) is of type ad G. Our lemma now follows from Lemma 2 for Proposition 10.4.

(2) means that $\Phi'(u)$ can be considered as a function on M. (3) is a consequence of the fact that, if the values of a finite number of functions of the form (I_k) are linearly independent at a point u, then they are linearly independent at every point of a neighborhood of u. Note that (3) means that $\dim \Phi'(u)$, considered as a function on M, is lower semicontinuous. To prove (4), assume

$\Phi'(u) = \Phi^*(u)$ at a point u. Since $\dim \Phi'(u)$ is lower semicontinuous and $\dim \Phi^*(u)$ is upper semicontinuous [cf. (5) of Proposition 10.1], the point $x = \pi(u)$ has a neighborhood U such that

$$\dim \Phi'(v) \geqq \dim \Phi'(u) \quad \text{and} \quad \dim \Phi^*(v) \leqq \dim \Phi^*(u) \qquad \text{for } v \in \pi^{-1}(U).$$

On the other hand, $\Phi^*(v) \supset \Phi'(v)$ for every $v \in \pi^{-1}(U)$. Hence,

$$\dim \Phi^*(v) = \dim \Phi'(v) = \dim \Phi^*(u) = \dim \Phi'(u)$$

and, consequently, $\Phi^*(v) = \Phi'(v)$ for every $v \in \pi^{-1}(U)$. Applying Theorem 10.3 to $P \mid U$, we see that $\Phi^0(u, U) = \Phi^*(u)$ and $\Phi^0(v, U) = \Phi^*(v)$. If $v \in P(u, U)$, then $\Phi^0(u, U) = \Phi^0(v, U)$ so that $\Phi^*(u) = \Phi^*(v)$. QED.

THEOREM 10.6. *If* $\dim \Phi'(v)$ *is constant in a neighborhood of* u *in* P, *then* $\Phi'(u) = \Phi^*(u)$.

Proof. We first prove the existence of an open neighborhood U of $x = \pi(u)$ such that $\mathfrak{g}'(u) = \mathfrak{g}'(v)$ for every $v \in P(u, U)$. Let $f_1, \ldots, f_s$ be a finite number of functions of the form (II_k) such that $f_1(u), \ldots, f_s(u)$ form a basis of $\mathfrak{g}'(u)$. At every point v of a small neighborhood of u, $f_1(v), \ldots, f_s(v)$ are linearly independent and, by the assumption, they form a basis of $\mathfrak{g}'(v)$. Since $f_1, \ldots, f_s$ are of type ad G, $f_1(va), \ldots, f_1(va)$ form a basis of $\mathfrak{g}'(va) = \operatorname{ad}(a^{-1})(\mathfrak{g}'(v))$. This means that there exists a neighborhood U of $x = \pi(u)$ such that $f_1(v), \ldots, f_s(v)$ form a basis of $\mathfrak{g}'(v)$ for every point $v \in \pi^{-1}(U)$. Now, let v be an arbitrary point of $P(u, U)$ and let u_t, $0 \leqq t \leqq 1$, be a horizontal curve from u to v in $\pi^{-1}(U)$ so that $u = u_0$ and $v = u_1$. We may assume that u_t is differentiable; the case where u_t is piecewise differentiable follows easily. Set $g_i(t) = f_i(u_t)$, $i = 1, \ldots, s$, and $X = \dot{u}_t$. Since X is horizontal, we have

$$(dg_i/dt)_t = (Xf_i)(u_t) \in \mathfrak{g}'(u_t), \qquad i = 1, \ldots, s.$$

Since $g_1(t), \ldots, g_s(t)$ form a basis for $\mathfrak{g}'(u_t)$, dg_i/dt can be expressed by

$$(dg_i/dt)_t = \Sigma_{j=1}^s A_{ij}(t) g_j(t),$$

where $A_{ij}(t)$ are continuous functions of t. By the lemma for Proposition 3.1, there exists a unique curve $(a_{ij}(t))_{i,j=1,\ldots,s}$ in

$GL(s; \mathbf{R})$ such that

$$da_{ij}/dt = \Sigma_{k=1}^{s} A_{ik}a_{kj} \quad \text{and} \quad a_{ij}(0) = \delta_{ij}.$$

(Note that $(A_{ij}(t)) \in \mathfrak{gl}(s; \mathbf{R})$ corresponds to $Y_t \in T_e(G)$ in the lemma for Proposition 3.1.) Let $(b_{ij}(t))$ be the inverse matrix of $(a_{ij}(t))$ so that

$$db_{ij}/dt = -\Sigma_{k=1}^{s} b_{ik}A_{kj}.$$

Then

$$\frac{d}{dt}(\Sigma_{j=1}^{s} b_{ij}g_j) = \Sigma_{j=1}^{s}\left(\frac{db_{ij}}{dt}\right)g_j + \Sigma_{k=1}^{s} b_{ik}\left(\frac{dg_k}{dt}\right)$$

$$= \Sigma_{j=1}^{s}\left(\frac{db_{ij}}{dt} + \Sigma_{k=1}^{s} b_{ik}A_{kj}\right)g_j = 0.$$

Since $b_{ij}(0) = \delta_{ij}$, we have

$$\Sigma_{j=1}^{s} b_{ij}(t)g_j(t) = g_i(0).$$

This means that $\mathfrak{g}'(u_t) = \mathfrak{g}'(u)$ and, in particular, $\mathfrak{g}'(v) = \mathfrak{g}'(u)$.

Taking U sufficiently small, we may assume that

$$\mathfrak{g}^*(u) \supset \mathfrak{g}^*(v) \supset \mathfrak{g}'(v) \supset \mathfrak{m}_0(v) \qquad \text{for every } v \in P(u, U).$$

By Theorem 8.1, the Lie algebra of $\Phi^0(u, U)$ is spanned by all $\mathfrak{m}_0(v)$, $v \in P(u, U)$. *A fortiori*, $\mathfrak{g}^*(u)$ is spanned by all $\mathfrak{g}'(v)$, $v \in P(u, U)$. Since $\mathfrak{g}'(v) = \mathfrak{g}'(u)$ for every $v \in P(u, U)$ as we have just shown, we may conclude that $\mathfrak{g}^*(u) = \mathfrak{g}'(u)$ and $\Phi^*(u) = \Phi'(u)$. QED.

COROLLARY 10.7. *If* $\dim \Phi'(u)$ *is constant on* P, *then* $\Phi^0(u) = \Phi^*(u) = \Phi'(u)$.

Proof. This follows from Theorems 10.3 and 10.6. QED.

THEOREM 10.8. *For a real analytic connection in a real analytic principal fibre bundle* P, *we have* $\Phi^0(u) = \Phi^*(u) = \Phi'(u)$ *for every* $u \in P$.

Proof. We may assume that $P = P(u)$ and, in particular, P is connected. It suffices to show that $\dim \Phi'(u)$ is locally constant; it then follows that $\dim \Phi'(u)$ is constant on P and, by Corollary 10.8, that $\Phi^0(u) = \Phi^*(u) = \Phi'(u)$ for every $u \in P$. Let $x^1, \ldots, x^n$ be a real analytic local coordinate system with origin $x = \pi(u)$. Let U be a coordinate neighborhood of x given by $\Sigma_i (x^i)^2 < a^2$ for some $a > 0$. We want to show that $\dim \Phi'(u)$ is constant on $\pi^{-1}(U)$. Let $X_i = \partial/\partial x^i$ and let X_i^* be the horizontal lift of X_i.

For any set of numbers $(a^1, \ldots, a^n)$ with $\Sigma_i (a^i)^2 = 1$, consider the vector field $X = \Sigma_i a^i X_i$ on U. Let x_t be the ray given by $x^i(t) = a^i t$ and let u_t be the horizontal lift of x_t such that $u = u_0$. We prove that $\mathfrak{g}'(u) = \mathfrak{g}'(u_t)$ for every t with $|t| < a$.

Consider all the functions f of the form (II_k), $k \geqq 0$,

$$f = X^*_{j_k} \cdots X^*_{j_1}(\Omega(X^*_i, X^*_l))$$

defined on $\pi^{-1}(U)$. We set $h(t) = f(u_t)$. Then the functions $h(t)$ are analytic functions of t. For each t_0 with $|t_0| < a$, there exists $\delta > 0$ such that all the functions $h(t)$ can be expanded in a common neighborhood $|t - t_0| < \delta$ in the Taylor series:

$$h(t) = \Sigma_{m=0}^{\infty} \frac{1}{m!} (t - t_0)^m h^{(m)}(t_0)$$

$$h(t_0) = \Sigma_{m=0}^{\infty} \frac{1}{m!} (t_0 - t)^m h^{(m)}(t).$$

If X^* is the horizontal lift of X, then we can write $h'(t) = X^*_{u_t} f$, $h''(t) = X^*_{u_t}(X^* f)$ and so on. The fact that there exists such a δ common to all $h(t)$ follows from the lemma we prove below. Now, if $|t - t_0| < \delta$, then all $h^{(m)}(t)$ belong to $\mathfrak{g}'(u_{t_0})$. The first power series shows that $\mathfrak{g}'(u_t)$ is contained in $\mathfrak{g}'(u_{t_0})$. Similarly, the second power series shows that $\mathfrak{g}'(u_{t_0})$ is contained in $\mathfrak{g}'(u_t)$. This means that $\mathfrak{g}'(u_t) = \mathfrak{g}'(u_{t_0})$ for $|t - t_0| < \delta$. The standard continuation argument shows that $\mathfrak{g}'(u_t) = \mathfrak{g}'(u)$ for every t with $|t| < a$, proving our theorem.

LEMMA. *In a real analytic manifold, let x_t be the integral curve of a real analytic vector field X such that $x_0 = x$, where $X_x \neq 0$. For any real analytic function g and for a finite number of real analytic vector fields $X_1, \ldots, X_s$, consider all the functions of the form*

$$f(x) = (X_{j_k} \cdots X_{j_1} g)(x)$$

$$h(t) = f(x_t),$$

where $j_1, \ldots, j_k$ are taken freely from $1, 2, \ldots, s$. Then there exists $\delta > 0$ such that the functions $h(t)$ can be expanded into power series in a common neighborhood $|t| < \delta$ as follows: $h(t) = \Sigma_{m=0}^{\infty} \frac{t^m}{m!} h^{(m)}(0)$.

Proof. Since $X_x \neq 0$, we may take a local coordinate system $x^1, \ldots, x^n$ such that $X = \partial/\partial x^1$ and $x_t = (t, 0, \ldots, 0)$ in a neighborhood of x. The preceding expansions of $h(t)$ are nothing but the expansions of $f(x)$ into power series of x^1. Each X_i is of the form $X_i = \Sigma_j f_{ij} \cdot \partial/\partial x^j$. Since f and f_{ij} are all real analytic, they can be expanded into power series of $(x^1, \ldots, x^n)$ in a common neighborhood $|x^i| < a$ for some $a > 0$. Our lemma then follows from the fact that if f_1 and f_2 are real analytic functions which can be expanded into power series of $x^1, \ldots, x^n$ in a neighborhood $|x^i| < a$, then the functions $f_1 f_2$ and $\partial f_1/\partial x^j$ can be expanded into power series in the same neighborhood. QED.

The results in this section are due to Ozeki [1].

11. *Invariant connections*

Before we treat general invariant connections, we present an important special case.

THEOREM 11.1. *Let G be a connected Lie group and H a closed subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively.*

(1) *If there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum) and* $\mathrm{ad}\,(H)\mathfrak{m} = \mathfrak{m}$, *then the $\mathfrak{h}$-component ω of the canonical 1-form θ of G (cf. §4 of Chapter* I) *with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ defines a connection in the bundle $G(G/H, H)$ which is invariant by the left translations of G;*

(2) *Conversely, any connection in $G(G/H, H)$ invariant by the left translations of G (if it exists) determines such a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and is obtainable in the manner described in* (1);

(3) *The curvature form Ω of the invariant connection defined by ω in* (1) *is given by*

$$\Omega(X, Y) = -\tfrac{1}{2}[X, Y]_{\mathfrak{h}} \qquad (\mathfrak{h}\text{-}component\ of\ -\tfrac{1}{2}[X, Y] \in \mathfrak{g}),$$

where X and Y are arbitrary left invariant vector fields on G belonging to $\mathfrak{m}$;

(4) *Let $\mathfrak{g}(e)$ be the Lie algebra of the holonomy group $\Phi(e)$ with reference point e (identity element) of the invariant connection defined in* (1). *Then $\mathfrak{g}(e)$ is spanned by all elements of the form $[X, Y]_{\mathfrak{h}}$, $X, Y \in \mathfrak{m}$.*

Proof. (1) The proof is straightforward and is similar to that of Proposition 6.4. Under the identification $\mathfrak{g} \approx T_e(G)$, the subspace $\mathfrak{m}$ corresponds to the horizontal subspace at e.

(2) Let ω be a connection form on $G(G/H, H)$ invariant by the left translations of G. Let $\mathfrak{m}$ be the set of left invariant vector fields on G such that $\omega(X) = 0$. It is easy to verify that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a desired decomposition.

(3) A left invariant vector field is horizontal if and only if it is an element of $\mathfrak{m}$. Now, (3) follows from Corollary 5.3.

(4) Let $\mathfrak{g}_1$ be the subspace of $\mathfrak{g}$ spanned by the set $\{\Omega_e(X, Y); X, Y \in \mathfrak{m}\}$. Let $\mathfrak{g}_2$ be the subspace of $\mathfrak{g}$ spanned by the set $\{\Omega_u(X, Y); X, Y \in \mathfrak{m}$ and $u \in G\}$. By Theorem 8.1, we have $\mathfrak{g}_1 \subset \mathfrak{g}(e) \subset \mathfrak{g}_2$. On the other hand, we have $\mathfrak{g}_1 = \mathfrak{g}_2$ as $\Omega_u(X, Y) = \Omega_e(X, Y)$ for any $X, Y \in \mathfrak{m}$ and $u \in G$. Now, (4) follows from (3).

QED.

Remark. (1) can be considered as a particular case of Proposition 6.4. Let $P = (G/H) \times G$ be the trivial bundle over G/H with group G. We imbed the bundle $G(G/H, H)$ into P by the mapping f defined by

$$f(u) = (\pi(u), u), \qquad u \in G,$$

where $\pi\colon G \to G/H$ is the natural projection. Let φ be the form defining the canonical flat connection (cf. §9) of P. Its $\mathfrak{h}$-component, restricted to the subbundle $G(G/H, H)$, defines a connection (Proposition 6.4) and agrees with the form ω in (1).

Going back to the general case, we first prove the following proposition which is basic in many applications.

Proposition 11.2. *Let φ_t be a 1-parameter group of automorphisms of a principal fibre bundle $P(M, G)$ and X the vector field on P induced by φ_t. Let Γ be a connection in P invariant by φ_t. For an arbitrary point u_0 of P, we define curves u_t, x_t, v_t and a_t as follows:*

$$u_t = \varphi_t(u_0), \qquad x_t = \pi(u_t),$$

$$v_t = \text{the horizontal lift of } x_t \text{ such that } v_0 = u_0,$$

$$u_t = v_t a_t.$$

Then a_t is the 1-parameter subgroup of G generated by $A = \omega_{u_0}(X)$, where ω is the connection form of Γ.

Proof. As in the proof of Proposition 3.1, we have

$$\omega(\dot{u}_t) = (\mathrm{ad}\,(a_t^{-1}))\omega(\dot{v}_t) + a_t^{-1}\dot{a}_t.$$

Since v_t is horizontal, we have $\omega(\dot{u}_t) = a_t^{-1}\dot{a}_t$. On the other hand, we have $\dot{u}_t = \varphi_t(X_{u_0})$ and hence $\omega(\dot{u}_t) = \omega(X_{u_0}) = A$, since the connection form ω is invariant by φ_t. Thus we obtain $a_t^{-1}\dot{a}_t = A$. QED.

Let K be a Lie group acting on a principal fibre bundle $P(M, G)$ as a group of automorphisms. Let u_0 be an arbitrary point of P which we choose as a reference point. Every element of K induces a transformation of M in a natural manner. The set J of all elements of K which fix the point $x_0 = \pi(u_0)$ of M forms a closed subgroup of K, called the isotropy subgroup of K at x_0. We define a homomorphism $\lambda\colon J \to G$ as follows. For each $j \in J$, ju_0 is a point in the same fibre as u_0 and hence is of the form $ju_0 = u_0 a$ with some $a \in G$. We define $\lambda(j) = a$. If $j, j' \in J$, then

$$u_0\lambda(jj') = (jj')u_0 = j(u_0\lambda(j')) = (ju_0)\lambda(j')$$
$$= (u_0\lambda(j))\lambda(j') = u_0(\lambda(j)\lambda(j')).$$

Hence, $\lambda(jj') = \lambda(j)\lambda(j')$, which shows that $\lambda\colon J \to G$ is a homomorphism. It is also easy to check that λ is differentiable. The induced Lie algebra homomorphism $\mathfrak{j} \to \mathfrak{g}$ will be also denoted by the same λ. Note that λ depends on the choice of u_0; the reference point u_0 is chosen once for all and is fixed throughout this section.

Proposition 11.3. *Let K be a group of automorphisms of $P(M, G)$ and Γ a connection in P invariant by K. We define a linear mapping $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ by*

$$\Lambda(X) = \omega_{u_0}(\tilde{X}), \qquad X \in \mathfrak{k},$$

where $\tilde{X}$ is the vector field on P induced by X. Then

(1) $\Lambda(X) = \lambda(X)$ *for $X \in \mathfrak{j}$;*

(2) $\Lambda(\mathrm{ad}\,(j)(X)) = \mathrm{ad}\,(\lambda(j))(\Lambda(X))$ *for $j \in J$ and $X \in \mathfrak{k}$,*

where $\mathrm{ad}\,(j)$ *is the adjoint representation of J in $\mathfrak{k}$ and* $\mathrm{ad}\,(\lambda(j))$ *is that of G in $\mathfrak{g}$.*

Note that the geometric meaning of $\Lambda(X)$ is given by Proposition 11.2.

Proof. (1) We apply Proposition 11.2 to the 1-parameter subgroup φ_t of K generated by X. If $X \in \mathfrak{j}$, then the curve $x_t = \pi(\varphi_t(u_0))$ reduces to a single point $x_0 = \pi(u_0)$. Hence we have $\varphi_t(u_0) = u_0\lambda(\varphi_t)$. Comparing the tangent vectors of the orbits $\varphi_t(u_0)$ and $u_0\lambda(\varphi_t)$ at u_0, we obtain $\Lambda(X) = \lambda(X)$.

(2) Let $X \in \mathfrak{k}$ and $j \in J$. We set $Y = \mathrm{ad}\,(j)(X)$. Then Y generates the 1-parameter subgroup $j\varphi_t j^{-1}$ which maps u_0 into $j\varphi_t j^{-1}(u_0) = j\varphi_t(u_0\lambda(j^{-1})) = j(R_{\lambda(j^{-1})}\varphi_t u_0)$. It follows that $\tilde{Y}_{u_0} = j(R_{\lambda(j^{-1})}\tilde{X}_{u_0})$. Since the connection form ω is invariant by j, we have

$$\begin{aligned}\omega_{u_0}(\tilde{Y}) &= \omega_{u_0}(j(R_{\lambda(j^{-1})}\tilde{X}_{u_0})) = \omega_{j^{-1}u_0}(R_{\lambda(j^{-1})}\tilde{X}_{u_0}) \\ &= \mathrm{ad}\,(\lambda(j))(\omega_{u_0}(\tilde{X}_{u_0})) = \mathrm{ad}\,(\lambda(j))(\Lambda(X)).\end{aligned}$$

QED.

PROPOSITION 11.4. *With the notation of Proposition* 11.3, *the curvature form* Ω *of* Γ *satisfies the following condition:*

$$2\Omega_{u_0}(\tilde{X}, \tilde{Y}) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]) \qquad \textit{for } X, Y \in \mathfrak{k}.$$

Proof. From the structure equation (Theorem 5.2) and Proposition 3.11 of Chapter I, we obtain

$$\begin{aligned}2\Omega(\tilde{X}, \tilde{Y}) &= 2d\omega(\tilde{X}, \tilde{Y}) + [\omega(\tilde{X}), \omega(\tilde{Y})] \\ &= \tilde{X}(\omega(\tilde{Y})) - \tilde{Y}(\omega(\tilde{X})) - \omega([\tilde{X}, \tilde{Y}]) + [\omega(\tilde{X}), \omega(\tilde{Y})].\end{aligned}$$

Since ω is invariant by K, we have by (c) of Proposition 3.2 of Chapter I (cf. also Proposition 3.5 of Chapter I)

$$\begin{aligned}\tilde{X}(\omega(\tilde{Y})) - \omega([\tilde{X}, \tilde{Y}]) &= (L_{\tilde{X}}\omega)(\tilde{Y}) = 0, \\ \tilde{Y}(\omega(\tilde{X})) - \omega([\tilde{Y}, \tilde{X}]) &= (L_{\tilde{Y}}\omega)(\tilde{X}) = 0.\end{aligned}$$

On the other hand, $X \to \tilde{X}$ being a Lie algebra homomorphism, we have

$$\omega_{u_0}([\tilde{X}, \tilde{Y}]) = \Lambda([X, Y]).$$

Thus we obtain

$$\begin{aligned}2\Omega_{u_0}(\tilde{X}, \tilde{Y}) &= [\omega_{u_0}(\tilde{X}), \omega_{u_0}(\tilde{Y})] - \Lambda([X, Y]) \\ &= [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]).\end{aligned}$$

QED.

We say that K acts *fibre-transitively* on P if, for any two fibres of P, there is an element of K which maps one fibre into the other, that is, if the action of K on the base M is transitive. If J is the isotropy subgroup of K at $x_0 = \pi(u_0)$ as above, then M is the homogeneous space K/J.

The following result is due to Wang [1].

THEOREM 11.5. *If a connected Lie group K is a fibre-transitive automorphism group of a bundle $P(M, G)$ and if J is the isotropy subgroup of*

K at $x_0 = \pi(u_0)$, then there is a 1:1 correspondence between the set of K-invariant connections in P and the set of linear mappings $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ which satisfies the two conditions in Proposition 11.3; the correspondence is given by

$$\Lambda(X) = \omega_{u_0}(\tilde{X}) \qquad \textit{for } X \in \mathfrak{k},$$

where $\tilde{X}$ is the vector field on P induced by X.

Proof. In view of Proposition 11.3, it is sufficient to show that, for every $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ satisfying (1) and (2) of Proposition 11.3, there is a K-invariant connection form ω on P such that $\Lambda(X) = \omega_{u_0}(\tilde{X})$ for $X \in \mathfrak{k}$. Let $X^* \in T_u(P)$. Since K is fibre-transitive, we can write

$$u_0 = kua = k \circ R_a u$$

$$k \circ R_a X^* = \tilde{X}_{u_0} + A^*_{u_0},$$

where $k \in K$, $a \in G$, $X \in k$ and A^* is the fundamental vector field corresponding to $A \in \mathfrak{g}$. We then set

$$\omega(X^*) = \mathrm{ad}\,(a)(\Lambda(X) + A).$$

We first prove that $\omega(X^*)$ is independent of the choice of X and A. Let

$$\tilde{X}_{u_0} + A^*_{u_0} = \tilde{Y}_{u_0} + B^*_{u_0}, \qquad \text{where } Y \in \mathfrak{k} \quad \text{and} \quad B \in \mathfrak{g},$$

so that $\tilde{X}_{u_0} - \tilde{Y}_{u_0} = B^*_{u_0} - A^*_{u_0}$. From the definition of $\lambda\colon \mathfrak{j} \to \mathfrak{g}$, it follows that $\lambda(X - Y) = B - A$. By condition (1) of Proposition 11.3, we have $\lambda(X - Y) = \Lambda(X - Y) = \Lambda(X) - \Lambda(Y)$. Hence, $\Lambda(X) + A = \Lambda(Y) + B$.

We next prove that $\omega(X^*)$ is independent of the choice of k and a. Let

$$u_0 = kua = k_1ua_1 \qquad (k_1 \in K \quad \text{and} \quad a_1 \in G),$$

so that $k_1k^{-1}u_0 = u_0a_1^{-1}a$ and $k_1k^{-1} \in J$. We set $j = k_1k^{-1}$. Then $\lambda(j) = a_1^{-1}a$. We have

$$\begin{aligned} k_1 \circ R_{a_1}X^* &= jk \circ R_{a\lambda(j^{-1})}X^* \\ &= j \circ R_{\lambda(j^{-1})}(k \circ R_a X^*) = j \circ R_{\lambda(j^{-1})}(\tilde{X}_{u_0} + A^*_{u_0}). \end{aligned}$$

By Proposition 1.7 of Chapter I, we have

$$j \circ R_{\lambda(j^{-1})}(\tilde{X}_{u_0}) = j(\tilde{X}_{u_0\lambda(j^{-1})}) = \tilde{Z}_{u_0}, \qquad \text{where } Z = \mathrm{ad}\,(j)(X).$$

By Proposition 5.1 of Chapter I, we have

$$j \circ R_{\lambda(j^{-1})}(A^*_{u_0}) = R_{\lambda(j^{-1})}(jA^*_{u_0}) = R_{\lambda(j^{-1})}A^*_{ju_0} = R_{\lambda(j^{-1})}A^*_{u_0\lambda(j)} = C^*_{u_0},$$

where $C = \operatorname{ad}(\lambda(j))(A)$. Hence we have

$$k_1 \circ R_{a_1}X^* = \tilde{Z}_{u_0} + C^*_{u_0},$$

$$\begin{aligned} \operatorname{ad}(a_1)(\Lambda(Z) + C) &= \operatorname{ad}(a_1)(\Lambda(\operatorname{ad}(j)(X)) + \operatorname{ad}(\lambda(j))A) \\ &= \operatorname{ad}(a_1)[\operatorname{ad}(\lambda(j))(\Lambda(X) + A)] \\ &= \operatorname{ad}(a)(\Lambda(X) + A). \end{aligned}$$

This proves our assertion that $\omega(X^*)$ depends only on X^*.

We now prove that ω is a connection form. Let $X^* \in T_u(P)$ and $u_0 = kua$ as above. Let b be an arbitrary element of G. We set

$$Y^* = R_bX^* \in T_v(P), \qquad \text{where } v = ub,$$

so that $u_0 = kub(b^{-1}a) = kv(b^{-1}a)$. We then have

$$k \circ R_{b^{-1}a}Y^* = k \circ R_{b^{-1}a}R_bX^* = k \circ R_aX^* = (\tilde{X}_{u_0} + A^*_{u_0})$$

and hence

$$\omega(R_bX^*) = \omega(Y^*) = \operatorname{ad}(b^{-1}a)(\Lambda(X) + A) = \operatorname{ad}(b^{-1})(\omega(X^*)),$$

which shows that ω satisfies condition (b′) of Proposition 1.1. Now, let A be any element of $\mathfrak{g}$ and let $u_0 = kua$. Then

$$k \circ R_a(A^*_u) = R_a \circ k(A^*_u) = R_a(A^*_{ku}) = B^*_{u_0}, \quad \text{where } B = \operatorname{ad}(a^{-1})(A).$$

Hence we have

$$\omega(A^*_u) = \operatorname{ad}(a)(B) = A,$$

which shows that ω satisfies condition (a′) of Proposition 1.1.

To prove that ω is differentiable, let u_1 be an arbitrary point of P and let $u_0 = k_1u_1a_1$. Consider the fibre bundle $K(M, J)$, where $M = K/J$. Let $\sigma\colon U \to K$ be a local cross section of this bundle defined in a neighborhood U of $\pi(u_1)$ such that $\sigma(\pi(u_1)) = k_1$. For each $u \in \pi^{-1}(U)$, we define $k \in K$ and $a \in G$ by

$$k = \sigma(\pi(u)) \quad \text{and} \quad u_0 = kua.$$

Then both k and a depend differentiably on u. We decompose the vector space $\mathfrak{k}$ into a direct sum of subspaces: $\mathfrak{k} = \mathfrak{j} + \mathfrak{m}$. For an arbitrary $X^* \in T_u(P)$, we set

$$k \circ R_a(X^*) = \tilde{X}_{u_0} + A^*_{u_0}, \qquad \text{where } X \in \mathfrak{m}.$$

Then both X and A are uniquely determined and depend differentiably on X^*. Thus $\omega(X^*) = \text{ad}\,(a)(\Lambda(X) + A)$ depends differentiably on X^*.

Finally, we prove that ω is invariant by K. Let $X^* \in T_u(P)$ and $u_0 = kua$. Let k_1 be an arbitrary element of K. Then $k_1X^* \in T_{k_1u}(P)$ and $u_0 = kk_1^{-1}(k_1u)a$. Hence,

$$kk_1^{-1} \circ R_a(k_1X^*) = k \circ R_a(X^*).$$

From the construction of ω, we see immediately that $\omega(k_1X^*) = \omega(X^*)$. QED.

In the case where K is fibre-transitive on P, the curvature form Ω, which is a tensorial form of type ad G (cf. §5) and is invariant by K, is completely determined by the values $\Omega_{u_0}(\tilde{X}, \tilde{Y})$, $X, Y \in \mathfrak{k}$. Proposition 11.4 expresses $\Omega_{u_0}(\tilde{X}, \tilde{Y})$ in terms of Λ. As a consequence of Proposition 11.4 and Theorem 11.5, we obtain

COROLLARY 11.6. *The K-invariant connection in P defined by Λ is flat if and only if $\Lambda\colon \mathfrak{k} \to \mathfrak{g}$ is a Lie algebra homomorphism.*

Proof. A connection is flat if and only if its curvature form vanishes identically (Theorem 9.1). QED.

THEOREM 11.7. *Assume in Theorem 11.5 that $\mathfrak{k}$ admits a subspace $\mathfrak{m}$ such that $\mathfrak{k} = \mathfrak{j} + \mathfrak{m}$ (direct sum) and $\text{ad}\,(J)(\mathfrak{m}) = \mathfrak{m}$, where $\text{ad}\,(J)$ is the adjoint representation of J in $\mathfrak{k}$. Then*

(1) *There is a 1:1 correspondence between the set of K-invariant connections in P and the set of linear mappings $\Lambda_{\mathfrak{m}}\colon \mathfrak{m} \to \mathfrak{g}$ such that*

$$\Lambda_{\mathfrak{m}}(\text{ad}\,(j)(X)) = \text{ad}\,(\lambda(j))(\Lambda_{\mathfrak{m}}(X)) \qquad \text{for } X \in \mathfrak{m} \quad \text{and} \quad j \in J;$$

the correspondence is given via Theorem 11.5 by

$$\Lambda(X) = \begin{cases} \lambda(X) & \text{if } X \in \mathfrak{j}, \\ \Lambda_{\mathfrak{m}}(X) & \text{if } X \in \mathfrak{m}. \end{cases}$$

(2) *The curvature form Ω of the K-invariant connection defined by $\Lambda_{\mathfrak{m}}$ satisfies the following condition:*

$$2\Omega_{u_0}(\tilde{X}, \tilde{Y}) = [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X, Y]_{\mathfrak{m}}) - \lambda([X, Y]_{\mathfrak{j}}) \qquad \text{for } X, Y \in \mathfrak{m},$$

where $[X, Y]_{\mathfrak{m}}$ (*resp.* $[X, Y]_{\mathfrak{j}}$) *denotes the* $\mathfrak{m}$-*component* (*resp.* $\mathfrak{j}$-*component*) *of* $[X, Y] \in \mathfrak{k}$.

Proof. Let $\Lambda: \mathfrak{k} \to \mathfrak{g}$ be a linear mapping satisfying (1) and (2) of Proposition 11.3. Let $\Lambda_{\mathfrak{m}}$ be the restriction of Λ to $\mathfrak{m}$. It is easy to see that $\Lambda \to \Lambda_{\mathfrak{m}}$ gives a desired correspondence. The statement (2) is a consequence of Proposition 11.4. QED.

In Theorem 11.7, the K-invariant connection in P defined by $\Lambda_{\mathfrak{m}} = 0$ is called the *canonical connection* (with respect to the decomposition $\mathfrak{k} = \mathfrak{j} + \mathfrak{m}$).

Remark. (1) and (3) of Theorem 11.1 follow from Theorem 11.7 if we set $P(M, G) = G(G/H, H)$ and $K = G$; the invariant connection in Theorem 11.1 is the canonical connection just defined.

Finally, we determine the Lie algebra of the holonomy group of a K-invariant connection.

THEOREM 11.8. *With the same assumptions and notation as in Theorem* 11.5, *the Lie algebra* $\mathfrak{g}(u_0)$ *of the holonomy group* $\Phi(u_0)$ *of the K-invariant connection defined by* $\Lambda: \mathfrak{k} \to \mathfrak{g}$ *is given by*

$$\mathfrak{m}_0 + [\Lambda(\mathfrak{k}), \mathfrak{m}_0] + [\Lambda(\mathfrak{k}), [\Lambda(\mathfrak{k}), \mathfrak{m}_0]] + \cdots,$$

where $\mathfrak{m}_0$ *is the subspace of* $\mathfrak{g}$ *spanned by*

$$\{[\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]); X, Y \in \mathfrak{k}\}.$$

Proof. Since K is fibre-transitive on P, the restricted holonomy group $\Phi^0(u_0)$ coincides with the infinitesimal holonomy group $\Phi'(u_0)$ by virtue of Corollary 10.7. We define a series of subspaces $\mathfrak{m}_k$, $k = 0, 1, 2, \ldots$, of $\mathfrak{g}$ as follows:

$$\mathfrak{m}_1 = \mathfrak{m}_0 + [\Lambda(\mathfrak{k}), \mathfrak{m}_0],$$

$$\mathfrak{m}_2 = \mathfrak{m}_0 + [\Lambda(\mathfrak{k}), \mathfrak{m}_0] + [\Lambda(\mathfrak{k}), [\Lambda(\mathfrak{k}), \mathfrak{m}_0]]$$

and so on. We defined in §10 an increasing sequence of subspaces $\mathfrak{m}_k(u_0)$, $k = 0, 1, 2, \ldots$, of $\mathfrak{g}$. Since the union of these subspaces $\mathfrak{m}_k(u_0)$ is the Lie algebra $\mathfrak{g}'(u_0)$ of the infinitesimal holonomy group $\Phi'(u_0)$, it is sufficient to prove that $\mathfrak{m}_k = \mathfrak{m}_k(u_0)$ for $k = 0, 1, 2, \ldots$.

By Proposition 11.4, the subspace $\mathfrak{m}_0$ is spanned by $\{\Omega_{u_0}(\tilde{X}, \tilde{Y});$ $X, Y \in \mathfrak{k}\}$. Since $\Omega_{u_0}(\tilde{X}, \tilde{Y}) = \Omega_{u_0}(h\tilde{X}, h\tilde{Y})$, where $h\tilde{X}$ and $h\tilde{Y}$

denote the horizontal components of $\tilde{X}$ and $\tilde{Y}$ respectively, $\mathfrak{m}_0$ coincides with $\mathfrak{m}_0(u_0)$.

We need the following lemmas.

Lemma 1. *If Y is a horizontal vector field on P and $\tilde{X}$ is the vector field on P induced by an element X of $\mathfrak{k}$, then $[\tilde{X}, \tilde{Y}]$ is horizontal.*

Proof of Lemma 1. By (c) of Proposition 3.2 of Chapter I (cf. also Proposition 3.5 of Chapter I), we have

$$\tilde{X}(\omega(Y)) = (L_{\tilde{X}}\omega)(Y) + \omega([\tilde{X}, Y]).$$

Since $\omega(Y) = 0$ and $L_{\tilde{X}}\omega = 0$, we have $\omega([\tilde{X}, Y]) = 0$.

Lemma 2. *Let $V, W, Y_1, \ldots, Y_r$ be arbitrary horizontal vector fields on P and let $\tilde{X}$ be the vector field on P induced by an element X of $\mathfrak{k}$. Then*

$$\tilde{X}_{u_0}(Y_r \cdots Y_1(\Omega(V, W))) \in \mathfrak{m}_r(u_0).$$

Proof of Lemma 2. We have

$$\begin{aligned} &\tilde{X}_{u_0}(Y_r \cdots Y_1(\Omega(V, W))) \\ &\qquad \equiv (Y_r)_{u_0}(\tilde{X}Y_{r-1} \cdots Y_1(\Omega(V, W))) \qquad \mod \mathfrak{m}_r(u_0), \end{aligned}$$

since $[\tilde{X}, Y_r]$ is horizontal by Lemma 1 and $[\tilde{X}, Y_r]_{u_0}(Y_{r-1} \cdots Y_1(\Omega(V, W)))$ is in $\mathfrak{m}_r(u_0)$. Repeating this process, we obtain

$$\begin{aligned} &\tilde{X}_{u_0}(Y_r \cdots Y_1(\Omega(V, W))) \\ &\qquad \equiv (Y_r)_{u_0}(Y_{r-1} \cdots Y_1\tilde{X}(\Omega(V, W))) \qquad \mod \mathfrak{m}_r(u_0). \end{aligned}$$

By the same argument as in the proof of Lemma 1, we have

$$\tilde{X}(\Omega(V, W)) = (L_{\tilde{X}}\Omega)(V, W) + \Omega([\tilde{X}, V], W) + \Omega(V, [\tilde{X}, W]).$$

Since $L_{\tilde{X}}\Omega = 0$, we have

$$\begin{aligned} &(Y_r)_{u_0}(Y_{r-1} \cdots Y_1\tilde{X}(\Omega(V, W))) \\ &\qquad = (Y_r)_{u_0}(Y_{r-1} \cdots Y_1(\Omega([\tilde{X}, V], W))) \\ &\qquad\qquad + (Y_r)_{u_0}(Y_{r-1} \cdots Y_1(\Omega(V, [\tilde{X}, W]))). \end{aligned}$$

The two terms on the right hand side belong to $\mathfrak{m}_r(u_0)$ as $[\tilde{X}, V]$ and $[\tilde{X}, W]$ are horizontal by Lemma 1. This completes the proof of Lemma 2.

Let $X_i = \partial/\partial x^i$, where $x^1, \ldots, x^n$ is a local coordinate system in a neighborhood of $x_0 = \pi(u_0)$. Let X_i^* be the horizontal lift of

X_i. Let

$$f = X^*_{j_r} \cdots X^*_{j_1}(\Omega(X^*_i, X^*_l))$$

be a function of the form (II_r) as defined in §10. If $h\tilde{X}$ and $v\tilde{X}$ denote the horizontal and the vertical components of $\tilde{X}$ respectively, then Lemma 1 for Proposition 10.4 implies

$$(h\tilde{X})_{u_0}f = -(v\tilde{X})_{u_0}f + \tilde{X}_{u_0}f = [\omega_{u_0}(\tilde{X}), f(u_0)] + \tilde{X}_{u_0}f.$$

Since $\tilde{X}_{u_0}f \in \mathfrak{m}_r(u_0)$ by Lemma 2 and since $\omega_{u_0}(\tilde{X}) = \Lambda(X)$, we have

$$(h\tilde{X})_{u_0}f \equiv [\Lambda(X), f(u_0)] \qquad \text{mod } \mathfrak{m}_r(u_0).$$

Assuming that $\mathfrak{m}_r = \mathfrak{m}_r(u_0)$ for all $r < s$, we shall show that $\mathfrak{m}_s = \mathfrak{m}_s(u_0)$. Since K is fibre-transitive on P, every horizontal vector at u_0 is of the form $(h\tilde{X})_{u_0}$ for some $X \in \mathfrak{k}$. Hence, $\mathfrak{m}_s(u_0)$ is spanned by $\mathfrak{m}_{s-1}(u_0)$ and the set of all $(h\tilde{X})_{u_0}f$, where $X \in \mathfrak{k}$ and f is a function of the form (II_{s-1}). On the other hand, $\mathfrak{m}_s$ is spanned by $\mathfrak{m}_{s-1} = \mathfrak{m}_{s-1}(u_0)$ and by $[\Lambda(\mathfrak{k}), \mathfrak{m}_{s-1}] = [\Lambda(\mathfrak{k}), \mathfrak{m}_{s-1}(u_0)]$. In other words, $\mathfrak{m}_s$ is spanned by $\mathfrak{m}_{s-1} = \mathfrak{m}_{s-1}(u_0)$ and the set of all $[\Lambda(X), f(u_0)]$, where $X \in \mathfrak{k}$ and f is a function of the form (II_{s-1}). Our assertion $\mathfrak{m}_s = \mathfrak{m}_s(u_0)$ follows from the congruence $(h\tilde{X})_{u_0}f \equiv [\Lambda(X), f(u_0)]$ mod $\mathfrak{m}_{s-1}(u_0)$. QED.

Remark. (4) of Theorem 11.1 is a corollary to Theorem 11.8 (cf. Remark made after the proof of Theorem 11.7).

CHAPTER III

Linear and Affine Connections

1. *Connections in a vector bundle*

Let $\mathbf{F}$ be either the real number field $\mathbf{R}$ or the complex number field $\mathbf{C}$, $\mathbf{F}^m$ the vector space of all m-tuples of elements of $\mathbf{F}$ and $GL(m; \mathbf{F})$ the group of all $m \times m$ non-singular matrices with entries from $\mathbf{F}$. The group $GL(m; \mathbf{F})$ acts on $\mathbf{F}^m$ on the left in a natural manner; if $a = (a^i_j) \in GL(m; \mathbf{F})$ and $\xi = (\xi^1, \ldots, \xi^m) \in \mathbf{F}^m$, then $a\xi = (\Sigma_j a^1_j \xi^j, \ldots, \Sigma_j a^m_j \xi^j) \in \mathbf{F}^m$.

Let $P(M, G)$ be a principal fibre bundle and ρ a representation of G into $GL(m; \mathbf{F})$. Let $E(M, \mathbf{F}^m, G, P)$ be the associated bundle with standard fibre $\mathbf{F}^m$ on which G acts through ρ. We call E a real or complex *vector bundle* over M according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. Each fibre $\pi_E^{-1}(x)$, $x \in M$, of E has the structure of a vector space such that every $u \in P$ with $\pi(u) = x$, considered as a mapping of $\mathbf{F}^m$ onto $\pi_E^{-1}(x)$, is a linear isomorphism of $\mathbf{F}^m$ onto $\pi_E^{-1}(x)$. Let S be the set of cross sections φ: $M \to E$; it forms a vector space over $\mathbf{F}$ (of infinite dimensions if $m \geqq 1$) with addition and scalar multiplication defined by

$$\begin{aligned}
(\varphi + \psi)(x) &= \varphi(x) + \psi(x), \qquad && \varphi, \psi \in S, \qquad && x \in M,\\
(\lambda\varphi)(x) &= \lambda(\varphi(x)), && \varphi \in S,\ \lambda \in \mathbf{F}, && x \in M.
\end{aligned}$$

We may also consider S as a module over the algebra of $\mathbf{F}$-valued functions; if λ is an $\mathbf{F}$-valued function on M, then

$$(\lambda\varphi)(x) = \lambda(x) \cdot \varphi(x), \qquad \varphi \in S, \qquad x \in M.$$

Let Γ be a connection in P. We recall how Γ defined the notion of parallel displacement of fibres of E in §7 of Chapter II. If $\tau = x_t$, $a \leqq t \leqq b$, is a curve in M and $\tau^* = u_t$ is a horizontal lift of τ to P, then, for each fixed $\xi \in \mathbf{F}^m$, the curve $\tau' = u_t\xi$ is, by definition, a horizontal lift of τ to E.

Let φ be a section of E defined on $\tau = x_t$ so that $\pi_E \circ \varphi(x_t) = x_t$ for all t. Let $\dot{x}_t$ be the vector tangent to τ at x_t. Then, for each fixed t, the *covariant derivative* $\nabla_{\dot{x}_t}\varphi$ *of* φ *in the direction of* (or *with respect to*) $\dot{x}_t$ is defined by

$$\nabla_{\dot{x}_t}\varphi = \lim_{h\to 0} \frac{1}{h} [\tau_t^{t+h}(\varphi(x_{t+h})) - \varphi(x_t)],$$

where $\tau_t^{t+h}\colon \pi_E^{-1}(x_{t+h}) \to \pi_E^{-1}(x_t)$ denotes the parallel displacement of the fibre $\pi_E^{-1}(x_{t+h})$ along τ from x_{t+h} to x_t. Thus, $\nabla_{\dot{x}_t}\varphi \in \pi_E^{-1}(x_t)$ for every t and defines a cross section of E along τ. The cross section φ is parallel, that is, the curve $\varphi(x_t)$ in E is horizontal, if and only if $\nabla_{\dot{x}_t}\varphi = 0$ for all t. The following formulas are evident. If φ and ψ are cross sections of E defined on $\tau = x_t$, then

$$\nabla_{\dot{x}_t}(\varphi + \psi) = \nabla_{\dot{x}_t}\varphi + \nabla_{\dot{x}_t}\psi.$$

If λ is an $\mathbf{F}$-valued function defined on τ, then

$$\nabla_{\dot{x}_t}(\lambda\varphi) = \lambda(x_t) \cdot \nabla_{\dot{x}_t}\varphi + (\dot{x}_t\lambda) \cdot \varphi(x_t).$$

The last formula follows immediately from

$$\tau_t^{t+h}(\lambda(x_{t+h}) \cdot \varphi(x_{t+h})) = \lambda(x_{t+h}) \cdot \tau_t^{t+h}(\varphi(x_{t+h})).$$

Let $X \in T_x(M)$ and φ a cross section of E defined in a neighborhood of x. Then the *covariant derivative* $\nabla_X\varphi$ *of* φ *in the direction of* X is defined as follows. Let $\tau = x_t$, $-\varepsilon \leqq t \leqq \varepsilon$, be a curve such that $X = \dot{x}_0$. Then set

$$\nabla_X\varphi = \nabla_{\dot{x}_0}\varphi.$$

It is easy to see that $\nabla_X\varphi$ is independent of the choice of τ. A cross section φ of E defined on an open subset U of M is parallel if and only if $\nabla_X\varphi = 0$ for all $X \in T_x(U)$, $x \in U$.

PROPOSITION 1.1. *Let* $X, Y \in T_x(M)$ *and let* φ *and* ψ *be cross sections of* E *defined in a neighborhood of* x. *Then*

(1) $\nabla_{X+Y}\varphi = \nabla_X\varphi + \nabla_Y\varphi$;

(2) $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$;

(3) $\nabla_{\lambda X}\varphi = \lambda \cdot \nabla_X\varphi$, where $\lambda \in \mathbf{F}$;

(4) $\nabla_X(\lambda\varphi) = \lambda(x) \cdot \nabla_X\varphi + (X\lambda) \cdot \varphi(x)$, *where* λ *is an* $\mathbf{F}$*-valued function defined in a neighborhood of* x.

Proof. We proved (2) and (4). (3) is obvious. Finally, (1) will follow immediately from the following alternative definition of covariant differentiation.

Suppose that a cross section φ of E is defined on an open subset U of M. As in Example 5.2 of Chapter II, we associate with φ an $\mathbf{F}^m$-valued function f on $\pi^{-1}(U)$ as follows:

$$f(v) = v^{-1}(\varphi(\pi(v))), \qquad v \in \pi^{-1}(U).$$

Given $X \in T_x(M)$, let $X^* \in T_u(P)$ be a horizontal lift of X. Since f is an $\mathbf{F}^m$-valued function, X^*f is an element of $\mathbf{F}^m$ and $u(X^*f)$ is an element of the fibre $\pi_E^{-1}(x)$. We have

LEMMA. $\nabla_X\varphi = u(X^*f)$.

Proof of Lemma. Let $\tau = x_t$, $-\varepsilon \leq t \leq \varepsilon$, be a curve such that $X = \dot{x}_0$. Let $\tau^* = u_t$ be a horizontal lift of τ such that $u_0 = u$ so that $X^* = \dot{u}_0$. Then we have

$$X^*f = \lim_{h\to 0} \frac{1}{h}[f(u_h) - f(u)] = \lim_{h\to 0}\frac{1}{h}[u_h^{-1}(\varphi(x_h)) - u^{-1}(\varphi(x))]$$

and

$$u(X^*f) = \lim_{h\to 0}\frac{1}{h}[u \circ u_h^{-1}(\varphi(x_h)) - \varphi(x)].$$

In order to prove the lemma, it is sufficient to prove

$$\tau_0^h(\varphi(x_h)) = u \circ u_h^{-1}(\varphi(x_h)).$$

Set $\xi = u_h^{-1}(\varphi(x_h))$. Then $u_t\xi$ is a horizontal curve in E. Since $u_h\xi = \varphi(x_h)$, $\varphi(x_h)$ is the element of E obtained by the parallel displacement of $u_0\xi = u \circ u_h^{-1}(\varphi(x_h))$ along τ from x_0 to x_h. This implies $\tau_0^h(\varphi(x_h)) = u \circ u_h^{-1}(\varphi(x_h))$, thus completing the proof of the lemma.

Now, (1) of Proposition 1.1 follows from the lemma and the fact that, if $X, Y \in T_x(M)$ and $X^*, Y^* \in T_u(P)$ are horizontal lifts of X and Y respectively, then $X^* + Y^*$ is a horizontal lift of $X + Y$.

QED.

If φ is a cross section of E defined on M and X is a vector field on M, then the *covariant derivative* $\nabla_X\varphi$ *of* φ *in the direction of* (or *with respect to*) X is defined by

$$(\nabla_X\varphi)(x) = \nabla_{X_x}\varphi.$$

Then, as an immediate consequence of Proposition 1.1, we have

PROPOSITION 1.2. *Let X and Y be vector fields on M, φ and ψ cross sections of E on M and λ an F-valued function on M. Then*

(1) $\nabla_{X+Y}\varphi = \nabla_X\varphi + \nabla_Y\varphi$;
(2) $\nabla_X(\varphi + \psi) = \nabla_X\varphi + \nabla_X\psi$;
(3) $\nabla_{\lambda X}\varphi = \lambda \cdot \nabla_X\varphi$;
(4) $\nabla_X(\lambda\varphi) = \lambda \cdot \nabla_X\varphi + (X\lambda)\varphi$.

Let X be a vector field on M and X^* the horizontal lift of X to P. Then covariant differentiation ∇_X corresponds to Lie differentiation L_{X^*} in the following sense. In Example 5.2 of Chapter II, we saw that there is a 1:1 correspondence between the set of cross sections $\varphi: M \to E$ and the set of $\mathbf{F}^m$-valued functions f on P such that $f(ua) = a^{-1}(f(u))$, $a \in G$ (a^{-1} means $\rho(a^{-1}) \in GL(m; \mathbf{F})$). The correspondence is given by $f(u) = u^{-1}(\varphi(\pi(u)))$, $u \in P$. We then have

PROPOSITION 1.3. *If $\varphi: M \to E$ is a cross section and $f: P \to F^m$ is the corresponding function, then X^*f is the function corresponding to the cross section $\nabla_X\varphi$.*

Proof. This is an immediate consequence of the lemma for Proposition 1.1. QED.

A *fibre metric* g in a vector bundle E is an assignment, to each $x \in M$, of an inner product g_x in the fibre $\pi_E^{-1}(x)$, which is differentiable in x in the sense that, if φ and ψ are differentiable cross sections of E, then $g_x(\varphi(x), \psi(x))$ depends differentiably on x. When E is a complex vector bundle, the inner product is understood to be hermitian:

$$g_x(\Xi_1, \Xi_2) = \overline{g_x(\Xi_2, \Xi_1)} \qquad \text{for} \quad \Xi_1, \Xi_2 \in \pi_E^{-1}(x).$$

PROPOSITION 1.4. *If M is paracompact, every vector bundle E over M admits a fibre metric.*

Proof. This follows from Theorem 5.7 of Chapter I just as the existence of a Riemannian metric on a paracompact manifold. We shall give here another proof using a partition of unity. Let $\{U_i\}_{i \in I}$ be a locally finite open covering of M such that $\pi_E^{-1}(U_i)$ is isomorphic with $U_i \times \mathbf{F}^m$ for each i. Let $\{s_i\}$ be a partition of unity subordinate to $\{U_i\}$ (cf. Appendix 3). Let h^i be a fibre metric in $E \mid U_i = \pi_E^{-1}(U_i)$. Set $g = \Sigma_i s_i h^i$, that is,

$$g(\Xi_1, \Xi_2) = \Sigma_i s_i(x) h^i(\Xi_1, \Xi_2) \quad \text{for} \quad \Xi_1, \Xi_2 \in \pi_E^{-1}(x), \qquad x \in M.$$

Since $\{U_i\}$ is locally finite and s_i vanishes outside U_i, g is a well defined fibre metric. QED.

Given a fibre metric g in a vector bundle $E(M, \mathbf{F}^m, G, P)$, we construct a reduced subbundle $Q(M, H)$ of $P(M, G)$ as follows. In the standard fibre $\mathbf{F}^m$ of E, we consider the canonical inner product $(\ ,\ )$ defined by

$$(\xi, \eta) = \Sigma_i \xi^i \eta^i \quad \text{for} \quad \xi = (\xi^1, \ldots, \xi^m), \eta = (\eta^1, \ldots, \eta^m) \in \mathbf{R}^m,$$

$$(\xi, \eta) = \Sigma_i \xi^i \bar{\eta}^i \quad \text{for} \quad \xi = (\xi^1, \ldots, \xi^m), \eta = (\eta^1, \ldots, \eta^m) \in \mathbf{C}^m.$$

Let Q be the set of $u \in P$ such that $g(u(\xi), u(\eta)) = (\xi, \eta)$ for $\xi, \eta \in \mathbf{F}^m$. Then Q is a closed submanifold of P. It is easy to verify that Q is a reduced subbundle of P whose structure group H is given by

$$H = \{a \in G;\ \rho(a) \in O(m)\} \qquad \text{if } \mathbf{F} = \mathbf{R},$$

$$H = \{a \in G;\ \rho(a) \in U(m)\} \qquad \text{if } \mathbf{F} = \mathbf{C},$$

where ρ is the representation of G in $GL(m; \mathbf{F})$.

Given a fibre metric g in E, a connection in P is called a *metric connection* if the parallel displacement of fibres of E preserves the fibre metric g. More precisely, for every curve $\tau = x_t$, $0 \leqq t \leqq 1$, of M, the parallel displacement $\pi_E^{-1}(x_0) \to \pi_E^{-1}(x_1)$ along τ is isometric.

Proposition 1.5. *Let g be a fibre metric in a vector bundle $E(M, \mathbf{F}^m, G, P)$ and $Q(M, H)$ the reduced subbundle of $P(M, G)$ defined by g. A connection Γ in P is reducible to a connection Γ' in Q if and only if Γ is a metric connection.*

Proof. Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a curve in M. Let $\xi, \eta \in \mathbf{F}^m$ and $u_0 \in Q$ with $\pi(u_0) = x_0$. Let $\tau^* = u_t$ be the horizontal lift of τ to P starting from u_0 so that both $\tau' = u_t(\xi)$ and $\tau'' = u_t(\eta)$ are horizontal lifts of τ to E. If Γ is reducible to a connection Γ' in Q, then $u_t \in Q$ for all t. Hence,

$$g(u_0(\xi), u_0(\eta)) = (\xi, \eta) = g(u_t(\xi), u_t(\eta)),$$

proving that Γ is a metric connection. Conversely, if Γ is a metric connection, then

$$g(u_t(\xi), u_t(\eta)) = g(u_0(\xi), u_0(\eta)) = (\xi, \eta).$$

Hence, $u_t \in Q$. This means that Γ is reducible to a connection in Q by Proposition 7.2 of Chapter II. QED.

Proposition 1.5, together with Theorem 2.1 of Chapter II, implies that, given a fibre metric g in a vector bundle E over a paracompact manifold K, there is a metric connection in P.

Let $E(M, \mathbf{F}^m, G, P)$ be a vector bundle such that $G = GL(m; \mathbf{F})$. Let $E_i^j \in \mathfrak{gl}(m; \mathbf{F})$, Lie algebra of $GL(m; \mathbf{F})$, be the $m \times m$ matrix such that the entry at the j-th column and the i-th row is 1 and other entries are all zero. Then $\{E_i^j; i, j = 1, \ldots, m\}$ form a basis of the Lie algebra $\mathfrak{gl}(m; \mathbf{F})$. Let ω and Ω be the connection form and the curvature form of a connection Γ in P. Set

$$\omega = \Sigma_{i,j} \omega_j^i E_i^j, \quad \Omega = \Sigma_{i,j} \Omega_j^i E_i^j.$$

It is easy to verify that the structure equation of the connection Γ (cf. §5 of Chapter II) can be expressed by

$$d\omega_j^i = -\Sigma_k \omega_k^i \wedge \omega_j^k + \Omega_j^i, \quad i, j = 1, \ldots, m.$$

Let g be a fibre metric in E and Q the reduced subbundle of P defined by g. If Γ is a metric connection, then the restriction of ω to Q defines a connection in Q by Proposition 6.1 of Chapter II and Proposition 1.5. In particular, both ω and Ω, restricted to Q, take their values in the Lie algebra $\mathfrak{o}(m)$ or $\mathfrak{u}(m)$ according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. In other words, both (ω_j^i) and (Ω_j^i), restricted to Q, are skew-symmetric or skew-hermitian according as $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$.

2. *Linear connections*

Throughout this section, we shall denote the bundle of linear frames $L(M)$ by P and the general linear group $GL(n; \mathbf{R})$, $n = \dim M$, by G.

The *canonical form* θ of P is the $\mathbf{R}^n$-valued 1-form on P defined by

$$\theta(X) = u^{-1}(\pi(X)) \qquad \text{for } X \in T_u(P),$$

where u is considered as a linear mapping of $\mathbf{R}^n$ onto $T_{\pi(u)}(M)$ (cf. Example 5.2 of Chapter I).

PROPOSITION 2.1. *The canonical form θ of P is a tensorial 1-form of type $(GL(n; \mathbf{R}), \mathbf{R}^n)$. It corresponds to the identity transformation of the tangent space $T_x(M)$ at each $x \in M$ in the sense of Example 5.2 of Chapter II.*

Proof. If X is a vertical vector at $u \in P$, then $\pi(X) = 0$ and hence $\theta(X) = 0$. If X is any vector at $u \in P$ and a is any element of $G = GL(n; \mathbf{R})$, then $R_a X$ is a vector at $ua \in P$. Hence,

$$\begin{aligned}(R_a^* \theta)(X) = \theta(R_a X) &= (ua)^{-1}(\pi(R_a X)) \\ &= a^{-1}u^{-1}(\pi(X)) = a^{-1}(\theta(X)),\end{aligned}$$

thus proving our first assertion. The second assertion is clear. QED.

A connection in the bundle of linear frames P over M is called a *linear connection* of M. Given a linear connection Γ of M, we associate with each $\xi \in \mathbf{R}^n$ a horizontal vector field $B(\xi)$ on P as follows. For each $u \in P$, $(B(\xi))_u$ is the unique horizontal vector at u such that $\pi((B(\xi))_u) = u(\xi)$. We call $B(\xi)$ the *standard horizontal vector field corresponding to* ξ. Unlike the fundamental vector fields, the standard horizontal vector fields depend on the choice of connections.

PROPOSITION 2.2. *The standard horizontal vector fields have the following properties:*

(1) *If θ is the canonical form of P, then $\theta(B(\xi)) = \xi$ for $\xi \in \mathbf{R}^n$;*

(2) *$R_a(B(\xi)) = B(a^{-1}\xi)$ for $a \in G$ and $\xi \in \mathbf{R}^n$;*

(3) *If $\xi \neq 0$, then $B(\xi)$ never vanishes.*

Proof. (1) is obvious. (2) follows from the fact that if X is a horizontal vector at u, then $R_a(X)$ is a horizontal vector at ua and $\pi(R_a(X)) = \pi(X)$. To prove (3), assume that $(B(\xi))_u = 0$ at some point $u \in P$. Then $u(\xi) = \pi((B(\xi))_u) = 0$. Since $u\colon \mathbf{R}^n \to T_{\pi(u)}(M)$ is a linear isomorphism, $\xi = 0$. QED.

Remark. The conditions $\theta(B(\xi)) = \xi$ and $\omega(B(\xi)) = 0$ (where ω is the connection form) completely determine $B(\xi)$ for each $\xi \in \mathbf{R}^n$.

PROPOSITION 2.3. *If A^* is the fundamental vector field corresponding to $A \in \mathfrak{g}$ and if $B(\xi)$ is the standard horizontal vector field corresponding* *to* $\xi \in \mathbf{R}^n$, *then*

$$[A^*, B(\xi)] = B(A\xi),$$

where $A\xi$ *denotes the image of* ξ *by* $A \in \mathfrak{g} = \mathfrak{gl}(n; \mathbf{R})$ (*Lie algebra of all* $n \times n$ *matrices*) *which acts on* $\mathbf{R}^n$.

Proof. Let a_t be the 1-parameter subgroup of G generated by A, $a_t = \exp tA$. By Proposition 1.9 of Chapter I and (2) of Proposition 2.2,

$$[A^*, B(\xi)] = \lim_{t\to 0} \frac{1}{t}\,[B(\xi) - R_{a_t}(B(\xi))] = \lim_{t\to 0} \frac{1}{t}\,[B(\xi) - B(a_t^{-1}\xi)].$$

Since $\xi \to (B(\xi))_u$ is a linear isomorphism of $\mathbf{R}^n$ onto the horizontal subspace Q_u (cf. (*3*) of Proposition 2.2), we have

$$\lim_{t\to 0} \frac{1}{t}\,[B(\xi) - B(a_t^{-1}\xi)] = B\left(\lim_{t\to 0} \frac{1}{t}\,(\xi - a_t^{-1}\xi)\right) = B(A\xi).$$

QED.

We define the *torsion form* Θ of a linear connection Γ by

$$\Theta = D\theta \quad \text{(exterior covariant differential of } \theta).$$

By Proposition 5.1 of Chapter II and Proposition 2.1, Θ is a tensorial 2-form on P of type $(GL(n; \mathbf{R}), \mathbf{R}^n)$.

THEOREM 2.4 (*Structure equations*). *Let* ω, Θ, *and* Ω *be the connection form, the torsion form and the curvature form of a linear connection* Γ *of* M. *Then*

1st structure equation:

$$d\theta(X, Y) = -\tfrac{1}{2}(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)) + \Theta(X, Y),$$

2nd structure equation:

$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y),$$

where $X, Y \in T_u(P)$ *and* $u \in P$.

Proof. The second structure equation was proved in Theorem 5.2 of Chapter II (see also §1). The proof of the first structure equation is similar to that of Theorem 5.2 of Chapter II. There are three cases which have to be verified and the only non-trivial case is the one where X is vertical and Y is horizontal. Choose $A \in \mathfrak{g}$ and $\xi \in \mathbf{R}^n$ such that $X = A_u^*$ and $Y = B(\xi)_u$.

Then $\Theta(X, Y) = 0$, $\omega(Y) \cdot \theta(X) = 0$ and $\omega(X) \cdot \theta(Y) = \omega(A^*) \cdot \theta(B(\xi)) = A\xi$, since $\omega(A^*) = A$ and $\theta(B(\xi)) = \xi$. On the other hand, $2d\theta(X, Y) = A^*(\theta(B(\xi))) - B(\xi)(\theta(A^*)) - \theta([A^*, B(\xi)]) = -\theta([A^*, B(\xi)]) = -\theta(B(A\xi)) = -A\xi$ by Proposition 2.3. This proves the first structure equation. QED.

With respect to the natural basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$, we write

$$\theta = \Sigma_i \, \theta^i e_i, \quad \Theta = \Sigma_i \, \Theta^i e_i.$$

As in §1, with respect to the basis E_i^j of $\mathfrak{gl}(n; \mathbf{R})$, we write

$$\omega = \Sigma_{i,j} \omega_j^i E_i^j, \quad \Omega = \Sigma_{i,j} \Omega_j^i E_i^j.$$

Then the structure equations can be written as

$$(1) \quad d\theta^i = -\Sigma_j \, \omega_j^i \wedge \theta^j + \Theta^i, \qquad i = 1, \ldots, n,$$

$$(2) \quad d\omega_j^i = -\Sigma_k \, \omega_k^i \wedge \omega_j^k + \Omega_j^i, \qquad i, j = 1, \ldots, n.$$

Considering θ as a vector valued form and ω as a matrix valued form, we also write the structure equations in the following simplified form:

$$(1') \quad d\theta = -\omega \wedge \theta + \Theta$$

$$(2') \quad d\omega = -\omega \wedge \omega + \Omega.$$

In the next section, we shall give an interpretation of the torsion form and the first structure equation from the viewpoint of affine connections.

THEOREM 2.5 (*Bianchi's identities*). *For a linear connection, we have*

1st identity: $D\Theta = \Omega \wedge \theta$, *that is,*

$$3D\Theta(X, Y, Z) = \Omega(X, Y)\, \theta(Z) + \Omega(Y, Z)\, \theta(X) + \Omega(Z, X)\, \theta(Y),$$

where $X, Y, Z \in T_u(P)$.

2nd identity: $D\Omega = 0$.

Proof. The second identity was proved in Theorem 5.4 of Chapter II. The proof of the first identity is similar to that of Theorem 5.4. If we apply the exterior differentiation d to the first structure equation $d\theta = -\omega \wedge \theta + \Theta$, then we obtain

$$0 = -d\omega \wedge \theta + \omega \wedge d\theta + d\Theta.$$

Denote by hX the horizontal component of X. Then $\omega(hX) = 0$, $\theta(hX) = \theta(X)$ and $d\omega(hX, hY) = \Omega(X, Y)$. Hence,

$$\begin{aligned} D\Theta(X, Y, Z) &= d\Theta(hX, hY, hZ) \\ &= (d\omega \wedge \theta)(hX, hY, hZ) = (\Omega \wedge \theta)(X, Y, Z). \end{aligned}$$

QED.

Let $B_1, \ldots, B_n$ be the standard horizontal vector fields corresponding to the natural basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$ and $\{E_i^j{}^*\}$ the fundamental vector fields corresponding to the basis $\{E_i^j\}$ of $\mathfrak{gl}(n; \mathbf{R})$. It is easy to verify that $\{B_i, E_i^j{}^*\}$ and $\{\theta^i, \omega_j^i\}$ are dual to each other in the sense that

$$\theta^k(B_i) = \delta_i^k, \quad \theta^k(E_i^j{}^*) = 0,$$
$$\omega_l^k(B_i) = 0, \quad \omega_l^k(E_i^j{}^*) = \delta_i^k \delta_l^j.$$

Proposition 2.6. *The $n^2 + n$ vector fields $\{B_k, E_i^j{}^*;\ i, j, k = 1, \ldots, n\}$ define an absolute parallelism in P, that is, the $n^2 + n$ vectors $\{(B_k)_u, (E_i^j{}^*)_u\}$ form a basis of $T_u(P)$ for every $u \in P$.*

Proof. Since the dimension of P is $n^2 + n$, it is sufficient to prove that the above $n^2 + n$ vectors are linearly independent. Since $A \to A_u^*$ is a linear isomorphism of $\mathfrak{g}$ onto the vertical subspace of $T_u(P)$ (cf. §5 of Chapter I), $\{E_i^j{}^*\}$ are linearly independent at every point of P. By (3) of Proposition 2.2, $\{B_k\}$ are linearly independent at every point of P. Since $\{B_k\}$ are horizontal and $\{E_i^j{}^*\}$ are vertical, $\{B_k, E_i^j{}^*\}$ are linearly independent at every point of P.

QED.

Let $T_s^r(M)$ be the tensor bundle over M of type (r, s) (cf. Example 5.4 of Chapter I). It is a vector bundle with standard fiber $\mathbf{T}_s^r$ (tensor space over $\mathbf{R}^n$ of type (r, s)) associated with the bundle P of linear frames. A tensor field K of type (r, s) is a cross section of the tensor bundle $T_s^r(M)$. In §1, we defined covariant derivatives of a cross section in a vector bundle in general. As in §1, we can define covariant derivatives of K in the following three cases:

(1) $\nabla_{\dot{x}_t} K$, when K is defined along a curve $\tau = x_t$ of M;

(2) $\nabla_X K$, when $X \in T_x(M)$ and K is defined in a neighborhood of x;

(3) $\nabla_X K$, when X is a vector field on M and K is a tensor field on M.

For the sake of simplicity, we state the following proposition in case (3) only, although it is valid in cases (1) and (2) with obvious changes.

PROPOSITION 2.7. *Let $\mathfrak{T}(M)$ be the algebra of tensor fields on M. Let X and Y be vector fields on M. Then the covariant differentiation has the following properties:*

(1) *$\nabla_X: \mathfrak{T}(M) \to \mathfrak{T}(M)$ is a type preserving derivation;*
(2) *∇_X commutes with every contraction;*
(3) *$\nabla_X f = Xf$ for every function f on M;*
(4) *$\nabla_{X+Y} = \nabla_X + \nabla_Y$;*
(5) *$\nabla_{fX} K = f \cdot \nabla_X K$ for every function f on M and $K \in \mathfrak{T}(M)$.*

Proof. Let $\tau = x_t$, $0 \leq t \leq 1$, be a curve in M. Let $\mathbf{T}(x_t)$ be the tensor algebra over $T_{x_t}(M)$, $\mathbf{T}(x_t) = \Sigma \mathbf{T}^r_s(x_t)$ (cf. §3 of Chapter I). The parallel displacement along τ gives an isomorphism of the algebra $\mathbf{T}(x_0)$ onto the algebra $\mathbf{T}(x_1)$ which preserves type and commutes with every contraction. From the definition of covariant differentiation given in §1, we obtain (1) and (2) by an argument similar to the proof of Proposition 3.2 of Chapter I. (3), (4) and (5) were proved in Proposition 1.2. QED.

By the lemma for Proposition 3.3 of Chapter I, the operation of ∇_X on $\mathfrak{T}(M)$ is completely determined by its operation on the algebra of functions $\mathfrak{F}(M)$ and the module of vector fields $\mathfrak{X}(M)$. Since $\nabla_X f = Xf$ for every $f \in \mathfrak{F}(M)$, the operation of ∇_X on $\mathfrak{T}(M)$ is determined by its operation on $\mathfrak{X}(M)$. As an immediate corollary to Proposition 1.2, we have

PROPOSITION 2.8. *If X, Y and Z are vector fields on M, then*

(1) *$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$;*
(2) *$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$;*
(3) *$\nabla_{fX} Y = f \cdot \nabla_X Y$ for every $f \in \mathfrak{F}(M)$;*
(4) *$\nabla_X(fY) = f \cdot \nabla_X Y + (Xf) Y$ for every $f \in \mathfrak{F}(M)$.*

We shall prove later in §7 that any mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoted by $(X, Y) \to \nabla_X Y$, satisfying the four conditions above is actually the covariant derivative with respect to a certain linear connection.

The proof of the following proposition, due to Kostant [1], is similar to that of Proposition 3.3 of Chapter I and hence is left to the reader.

PROPOSITION 2.9. *Let M be a manifold with a linear connection. Every derivation D (preserving type and commuting with contractions) of the algebra $\mathfrak{T}(M)$ of tensor fields into the tensor algebra $\mathbf{T}(x)$ at $x \in M$ can be uniquely decomposed as follows:*

$$D = \nabla_X + S,$$

where $X \in T_x(M)$ and S is a linear endomorphism of $T_x(M)$.

Observe that, in contrast to Lie differentiation L_X with respect to a *vector field*, covariant differentiation ∇_X makes sense when X is a *vector at a point* of M.

Given a tensor field K of type (r, s), the *covariant differential* ∇K of K is a tensor field of type $(r, s + 1)$ defined as follows. As in Proposition 2.11 of Chapter I, we consider a tensor of type (r, s) at a point $x \in M$ as a multilinear mapping of $T_x(M) \times \cdots \times T_x(M)$ (s times product) into $\mathbf{T}_0^r(x)$ (space of contravariant tensors of degree r at x). We set

$$(\nabla K)(X_1, \ldots, X_s; X) = (\nabla_X K)(X_1, \ldots, X_s), \qquad X, X_i \in T_x(M).$$

PROPOSITION 2.10. *If K is a tensor field of type (r, s), then*

$$(\nabla K)(X_1, \ldots, X_s; X) = \nabla_X(K(X_1, \ldots, X_s)) - \Sigma_{i=1}^s K(X_1, \ldots, \nabla_X X_i, \ldots, X_s),$$

where $X, X_i \in \mathfrak{X}(M)$.

Proof. This follows from the fact that ∇_X is a derivation commuting with every contraction. The proof is similar to that of Proposition 3.5 of Chapter I and is left to the reader. QED.

A tensor field K on M, considered as a cross section of a tensor bundle, is parallel if and only if $\nabla_X K = 0$ for all $X \in T_x(M)$ and $x \in M$ (cf. §1). Hence we have

PROPOSITION 2.11. *A tensor field K on M is parallel if and only if $\nabla K = 0$.*

The second covariant differential $\nabla^2 K$ of a tensor field K of type (r, s) is defined to be $\nabla(\nabla K)$, which is a tensor field of type $(r, s + 2)$. We set

$$(\nabla^2 K)(;X; Y) = (\nabla_Y(\nabla K))(;X), \quad \text{where } X, Y \in T_x(M),$$

that is, if we regard K as a multilinear mapping of $T_x(M) \times \cdots \times T_x(M)$ (s times product) into $T_0^r(x)$, then

$$(\nabla^2 K)(X_1, \ldots, X_s; X; Y) = (\nabla_Y(\nabla K))(X_1, \ldots, X_s; X).$$

Similarly to Proposition 2.10, we have

Proposition 2.12. *For any tensor field K and for any vector fields X and Y, we have*

$$(\nabla^2 K)(;X; Y) = \nabla_Y(\nabla_X K) - \nabla_{\nabla_Y X} K.$$

In general, the m-th covariant differential $\nabla^m K$ is defined inductively to be $\nabla(\nabla^{m-1}K)$. We use the notation $(\nabla^m K)(;X_1; \ldots ; X_{m-1}; X_m)$ for $(\nabla_{X_m}(\nabla^{m-1}K))(;X_1; \ldots ; X_{m-1})$.

3. *Affine connections*

A linear connection of a manifold M defines, for each curve $\tau = x_t$, $0 \leqq t \leqq 1$, of M, the parallel displacement of the tangent space $T_{x_0}(M)$ onto the tangent space $T_{x_1}(M)$; these tangent spaces are regarded as vector spaces and the parallel displacement is a linear isomorphism between them. We shall now consider each tangent space $T_x(M)$ as an affine space, called the tangent affine space at x. From the viewpoint of fibre bundles, this means that we enlarge the bundle of linear frames to the bundle of affine frames, as we shall now explain.

Let $\mathbf{R}^n$ be the vector space of n-tuples of real numbers as before. When we regard $\mathbf{R}^n$ as an affine space, we denote it by A^n. Similarly, the tangent space of M at x, regarded as an affine space, will be denoted by $A_x(M)$ and will be called the *tangent affine space*. The group $A(n; \mathbf{R})$ of affine transformations of A^n is represented by the group of all matrices of the form

$$\tilde{a} = \begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix},$$

where $a = (a^i_j) \in GL(n; \mathbf{R})$ and $\xi = (\xi^i)$, $\xi \in \mathbf{R}^n$, is a column vector. The element $\tilde{a}$ maps a point η of A^n into $a\eta + \xi$. We have the following sequence:

$$0 \longrightarrow \mathbf{R}^n \xrightarrow{\alpha} A(n; \mathbf{R}) \xrightarrow{\beta} GL(n; \mathbf{R}) \longrightarrow 1,$$

where α is an isomorphism of the vector group $\mathbf{R}^n$ into $A(n; \mathbf{R})$ which maps $\xi \in \mathbf{R}^n$ into $\begin{pmatrix} I_n & \xi \\ 0 & 1 \end{pmatrix} \in A(n; \mathbf{R})$ (I_n = identity of $GL(n; \mathbf{R})$) and β is a homomorphism of $A(n; \mathbf{R})$ onto $GL(n; \mathbf{R})$ which maps $\begin{pmatrix} a & \xi \\ 0 & 1 \end{pmatrix} \in A(n; \mathbf{R})$ into $a \in GL(n; \mathbf{R})$. The sequence is exact in the sense that the kernel of each homomorphism is equal to the image of the preceding one. It is a splitting exact sequence in the sense that there is a homomorphism γ: $GL(n; \mathbf{R}) \to A(n; \mathbf{R})$ such that $\beta \circ \gamma$ is the identity automorphism of $GL(n; \mathbf{R})$; indeed, we define γ by $\gamma(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in A(n; \mathbf{R})$, $a \in GL(n; \mathbf{R})$. The group $A(n; \mathbf{R})$ is a semidirect product of $\mathbf{R}^n$ and $GL(n; \mathbf{R})$, that is, for every $\tilde{a} \in A(n; \mathbf{R})$, there is a unique pair $(a, \xi) \in GL(n; \mathbf{R}) \times \mathbf{R}^n$ such that $\tilde{a} = \alpha(\xi) \cdot \gamma(a)$.

An *affine frame* of a manifold M at x consists of a point $p \in A_x(M)$ and a linear frame $(X_1, \ldots, X_n)$ at x; it will be denoted by $(p; X_1, \ldots, X_n)$. Let o be the origin of $\mathbf{R}^n$ and $(e_1, \ldots, e_n)$ the natural basis for $\mathbf{R}^n$. We shall call $(o; e_1, \ldots, e_n)$ the canonical frame of A^n. Every affine frame $(p; X_1, \ldots, X_n)$ at x can be identified with an affine transformation $\tilde{u}$: $A^n \to A_x(M)$ which maps $(o; e_1, \ldots, e_n)$ into $(p; X_1, \ldots, X_n)$, because $(p; X_1, \ldots, X_n) \leftrightarrow \tilde{u}$ gives a 1:1 correspondence between the set of affine frames at x and the set of affine transformations of A^n onto $A_x(M)$. We denote by $A(M)$ the set of all affine frames of M and define the projection $\tilde{\pi}$: $A(M) \to M$ by setting $\tilde{\pi}(\tilde{u}) = x$ for every affine frame $\tilde{u}$ at x. We shall show that $A(M)$ is a principal fibre bundle over M with group $A(n; \mathbf{R})$ and shall call $A(M)$ the *bundle of affine frames* over M. We define an action of $A(n; \mathbf{R})$ on $A(M)$ by $(\tilde{u}, \tilde{a}) \to \tilde{u}\tilde{a}$, $\tilde{u} \in A(M)$ and $\tilde{a} \in A(n; \mathbf{R})$, where $\tilde{u}\tilde{a}$ is the composite of the affine transformations $\tilde{a}$: $A^n \to A^n$ and $\tilde{u}$: $A^n \to A_x(M)$. It can be proved easily (cf. Example 5.2 of Chapter I) that $A(n; \mathbf{R})$ acts freely on $A(M)$ on the right and that $A(M)$ is a principal fibre bundle over M with group $A(n; \mathbf{R})$.

Let $L(M)$ be the bundle of linear frames over M. Corresponding to the homomorphisms β: $A(n; \mathbf{R}) \to GL(n; \mathbf{R})$ and γ: $GL(n; \mathbf{R}) \to A(n; \mathbf{R})$, we have homomorphisms β: $A(M) \to L(M)$ and γ: $L(M) \to A(M)$. Namely, β: $A(M) \to L(M)$ maps $(p; X_1, \ldots, X_n)$ into $(X_1, \ldots, X_n)$ and γ: $L(M) \to A(M)$ maps

$(X_1, \ldots, X_n)$ into $(o_x; X_1, \ldots, X_n)$, where $o_x \in A_x(M)$ is the point corresponding to the origin of $T_x(M)$. In particular, $L(M)$ can be considered as a subbundle of $A(M)$. Evidently, $\beta \circ \gamma$ is the identity transformation of $L(M)$.

A *generalized affine connection* of M is a connection in the bundle $A(M)$ of affine frames over M. We shall study the relationship between generalized affine connections and linear connections. We denote by $\mathbf{R}^n$ the Lie algebra of the vector group $\mathbf{R}^n$. Corresponding to the splitting exact sequence $0 \to \mathbf{R}^n \to A(n; \mathbf{R}) \to GL(n; \mathbf{R}) \to 1$ of groups, we have the following splitting exact sequence of Lie algebras:

$$0 \to \mathbf{R}^n \to \mathfrak{a}(n; \mathbf{R}) \to \mathfrak{gl}(n; \mathbf{R}) \to 0.$$

Therefore,

$$\mathfrak{a}(n; \mathbf{R}) = \mathfrak{gl}(n; \mathbf{R}) + \mathbf{R}^n \quad \text{(semidirect sum)}.$$

Let $\tilde{\omega}$ be the connection form of a generalized affine connection of M. Then $\gamma^*\tilde{\omega}$ is an $\mathfrak{a}(n; \mathbf{R})$-valued 1-form on $L(M)$. Let

$$\gamma^*\tilde{\omega} = \omega + \varphi$$

be the decomposition corresponding to $\mathfrak{a}(n; \mathbf{R}) = \mathfrak{gl}(n; \mathbf{R}) + \mathbf{R}^n$, so that ω is a $\mathfrak{gl}(n; \mathbf{R})$-valued 1-form on $L(M)$ and φ is an $\mathbf{R}^n$-valued 1-form on $L(M)$. By Proposition 6.4 of Chapter II, ω defines a connection in $L(M)$. On the other hand, we see easily that φ is a tensorial 1-form on $L(M)$ of type $(GL(n; \mathbf{R}), \mathbf{R}^n)$ (cf. §5 of Chapter II) and hence corresponds to a tensor field of type (1, 1) of M as explained in Example 5.2 of Chapter II.

PROPOSITION 3.1. *Let $\tilde{\omega}$ be the connection form of a generalized affine connection $\tilde{\Gamma}$ of M and let*

$$\gamma^*\tilde{\omega} = \omega + \varphi,$$

where ω is $\mathfrak{gl}(n; \mathbf{R})$-valued and φ is $\mathbf{R}^n$-valued. Let Γ be the linear connection of M defined by ω and let K be the tensor field of type (1, 1) *of M defined by φ. Then*

(1) *The correspondence between the set of generalized affine connections of M and the set of pairs consisting of a linear connection of M and a tensor field of type* (1, 1) *of M given by $\tilde{\Gamma} \to (\Gamma, K)$ is* 1: 1.

(2) *The homomorphism $\beta\colon A(M) \to L(M)$ maps $\tilde{\Gamma}$ into Γ (cf.* §6 *of Chapter II).*

Proof. (1) It is sufficient to prove that, given a pair (Γ, K), there is $\tilde{\Gamma}$ which gives rise to (Γ, K). Let ω be the connection form of Γ and φ the tensorial 1-form on $L(M)$ of type $(GL(n; \mathbf{R}), \mathbf{R}^n)$ corresponding to K. Given a vector $\tilde{X} \in T_{\tilde{u}}(A(M))$, choose $X \in T_u(L(M))$ and $\tilde{a} \in A(n; \mathbf{R})$ such that $\tilde{u} = u\tilde{a}$ and $\tilde{X} - R_{\tilde{a}}(X)$ is vertical. There is an element $A \in \mathfrak{a}(n; \mathbf{R})$ such that

$$\tilde{X} = R_{\tilde{a}}(X) + A_u^*,$$

where A^* is the fundamental vector corresponding to A. We define $\tilde{\omega}$ by

$$\tilde{\omega}(\tilde{X}) = \text{ad}\,(\tilde{a}^{-1})(\omega(X) + \varphi(X)) + A.$$

It is straightforward to verify that $\tilde{\omega}$ defines the desired connection $\tilde{\Gamma}$.

(2) Let $\tilde{X} \in T_{\tilde{u}}(A(M))$. We set $u = \beta(\tilde{u})$ and $X = \beta(\tilde{X})$ so that $X \in T_u(L(M))$. Since $\beta\colon A(M) \to L(M)$ is the homomorphism associated with the homomorphism $\beta\colon A(n; \mathbf{R}) \to GL(n; \mathbf{R}) = A(n; \mathbf{R})/\mathbf{R}^n$, $L(M)$ can be identified with $A(M)/\mathbf{R}^n$ and $\beta\colon A(M) \to L(M)$ can be considered as the natural projection $A(M) \to A(M)/\mathbf{R}^n$. Since $X = \beta(X) = \beta(\tilde{X})$, there exist $\tilde{a} \in \mathbf{R}^n \subset A(n; \mathbf{R})$ and $A \in \mathbf{R}^n \subset \mathfrak{a}(n; \mathbf{R})$ such that $\tilde{u} = u\tilde{a}$ and $\tilde{X} = R_{\tilde{a}}(X) + A_u^*$. Assume that $\tilde{X}$ is horizontal with respect to $\tilde{\Gamma}$ so that $0 = \tilde{\omega}(\tilde{X}) = \tilde{\omega}(R_{\tilde{a}}(X)) + \tilde{\omega}(A_u^*) = \text{ad}\,(\tilde{a}^{-1})(\tilde{\omega}(X)) + A$. Hence, $\tilde{\omega}(X) = \text{ad}\,(\tilde{a})(A)$ and $\omega(X) + \varphi(X) = \text{ad}\,(\tilde{a})(A)$. Since both $\varphi(X)$ and $\text{ad}\,(\tilde{a})(A)$ are in $\mathbf{R}^n$ and $\omega(X)$ is in $\mathfrak{gl}(n; \mathbf{R})$, we have $\omega(X) = 0$. This proves that if $\tilde{X}$ is horizontal with respect to $\tilde{\Gamma}$, then $\beta(\tilde{X})$ is horizontal with respect to Γ. QED.

Proposition 3.2. *In Proposition* 3.1, *let* $\tilde{\Omega}$ *and* Ω *be the curvature forms of* $\tilde{\Gamma}$ *and* Γ *respectively. Then*

$$\gamma^*\tilde{\Omega} = \Omega + D\varphi,$$

where D is the exterior covariant differentiation with respect to Γ.

Proof. Let $X, Y \in T_u(L(M))$. To prove that $(\gamma^*\tilde{\Omega})(X, Y) = \Omega(X, Y) + D\varphi(X, Y)$, it is sufficient to consider the following two cases: (1) at least one of X and Y is vertical, (2) both X and Y are horizontal with respect to Γ. In the case (1), both sides vanish. In the case (2), $\omega(X) = \omega(Y) = 0$ and hence $\tilde{\omega}(X) = \varphi(X)$ and $\tilde{\omega}(Y) = \varphi(Y)$. From the structure equation of $\tilde{\Gamma}$, we have

$$d\tilde{\omega}(X, Y) = -\tfrac{1}{2}[\tilde{\omega}(X), \tilde{\omega}(Y)] + \tilde{\Omega}(X, Y)$$
$$= -\tfrac{1}{2}[\varphi(X), \varphi(Y)] + \tilde{\Omega}(X, Y).$$

(Here, considering $L(M)$ as a subbundle of $A(M)$, we identified $\gamma(X)$ with X.) On the other hand, $\gamma^* \, d\tilde{\omega} = d\omega + d\varphi$ and hence $d\tilde{\omega}(X, Y) = d\omega(X, Y) + d\varphi(X, Y)$. Since $\mathbf{R}^n$ is abelian, $[\varphi(X), \varphi(Y)] = 0$. Hence, $d\omega(X, Y) + d\varphi(X, Y) = \tilde{\Omega}(X, Y)$. Since both X and Y are horizontal, $D\omega(X, Y) + D\varphi(X, Y) = \tilde{\Omega}(X, Y)$. QED.

Consider again the structure equation of a generalized affine connection $\tilde{\Gamma}$:

$$d\tilde{\omega} = -\tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] + \tilde{\Omega}.$$

By restricting both sides of the equation to $L(M)$ and by comparing the $\mathfrak{gl}(n; \mathbf{R})$-components and the $\mathbf{R}^n$-components we obtain

$$d\varphi(X, Y) = -\tfrac{1}{2}([\omega(X), \varphi(Y)] - [\omega(Y), \varphi(X)]) + D\varphi(X, Y),$$
$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y), \qquad X, Y \in T_u(L(M)).$$

Just as in §2, we write

$$d\varphi = -\omega \wedge \varphi + D\varphi$$
$$d\omega = -\omega \wedge \omega + \Omega.$$

A generalized affine connection $\tilde{\Gamma}$ is called an *affine connection* if, with the notation of Proposition 3.1, the $\mathbf{R}^n$-valued 1-form φ is the canonical form θ defined in §2. In other words, $\tilde{\Gamma}$ is an affine connection if the tensor field K corresponding to φ is the field of identity transformations of tangent spaces of M. As an immediate consequence of Proposition 3.1, we have

THEOREM 3.3. *The homomorphism $\beta: A(M) \to L(M)$ maps every affine connection $\tilde{\Gamma}$ of M into a linear connection Γ of M. Moreover, $\tilde{\Gamma} \to \Gamma$ gives a 1:1 correspondence between the set of affine connections $\tilde{\Gamma}$ of M and the set of linear connections Γ of M.*

Traditionally, the words "linear connection" and "affine connection" have been used interchangeably. This is justified by Theorem 3.3. Although we shall not break with this tradition, we shall make a logical distinction between a linear connection and an affine connection whenever necessary; a linear connection of M is a connection in $L(M)$ and an affine connection is a connection in $A(M)$.

From Proposition 3.2, we obtain

PROPOSITION 3.4. *Let Θ and Ω be the torsion form and the curvature form of a linear connection Γ of M. Let $\tilde{\Omega}$ be the curvature form of the corresponding affine connection. Then*

$$\gamma^*\tilde{\Omega} = \Theta + \Omega,$$

where $\gamma\colon L(M) \to A(M)$ is the natural injection.

Replacing φ by the canonical form θ in the formulas:

$$d\varphi = -\omega \wedge \varphi + D\varphi, \quad d\omega = -\omega \wedge \omega + \Omega,$$

we rediscover the structure equations of a linear connection proved in Theorem 2.4.

Let $\Phi(\tilde{u})$ be the holonomy group of an affine connection $\tilde{\Gamma}$ of M with reference point $\tilde{u} \in A(M)$. Let $\Psi(u)$ be the holonomy group of the corresponding linear connection Γ of M with reference point $u = \beta(\tilde{u}) \in L(M)$. We shall call $\Phi(\tilde{u})$ the *affine holonomy group* of $\tilde{\Gamma}$ or Γ and $\Psi(u)$ the *linear holonomy group* (or *homogeneous holonomy group*) of $\tilde{\Gamma}$ or Γ. The *restricted affine* and *linear holonomy groups* $\Phi^0(\tilde{u})$ and $\Psi^0(u)$ are defined accordingly. From Proposition 6.1 of Chapter II, we obtain

PROPOSITION 3.5. *The homomorphism $\beta\colon A(n; \mathbf{R}) \to GL(n; \mathbf{R})$ maps $\Phi(\tilde{u})$ onto $\Psi(u)$ and $\Phi^0(\tilde{u})$ onto $\Psi^0(u)$.*

4. *Developments*

We shall study in this section the parallel displacement arising from an affine connection of a manifold M. Let $\tau = x_t$, $0 \leqq t \leqq 1$, be a curve in M. The affine parallel displacement along τ is an affine transformation of the affine tangent space at x_0 onto the affine tangent space at x_1 defined by the given connection in $A(M)$. It is a special case of the parallelism in an associated bundle which is, in our case, the affine tangent bundle whose fibres are $A_x(M)$, $x \in M$. We shall denote this affine parallelism by $\tilde{\tau}$.

The total space (i.e., the bundle space) of the affine .tangent bundle over M is naturally homeomorphic with that of the tangent (vector) bundle over M; the distinction between the two is that the affine tangent bundle is associated with $A(M)$ whereas the tangent (vector) bundle is associated with $L(M)$. A cross section of the affine tangent bundle is called a *point field*. There is a natural 1:1 correspondence between the set of point fields and the set of vector fields.

Let $\tilde{\tau}_s^t$ be the affine parallel displacement along the curve τ from x_t to x_s. In particular, $\tilde{\tau}_0^t$ is the parallel displacement $A_{x_t}(M) \to A_{x_0}(M)$ along τ (in the reversed direction) from x_t to x_0. Let p be a point field defined along τ so that p_{x_t} is an element of $A_{x_t}(M)$ for each t. Then $\tilde{\tau}_0^t(p_{x_t})$ describes a curve in $A_{x_0}(M)$. We identify the curve $\tau = x_t$ with the trivial point field along τ, that is, the point field corresponding to the zero vector field along τ. Then the *development* of the curve τ in M into the affine tangent space $A_{x_0}(M)$ is the curve $\tilde{\tau}_0^t(x_t)$ in $A_{x_0}(M)$.

The following proposition allows us to obtain the development of a curve by means of the linear parallel displacement, that is, the parallel displacement defined by the corresponding linear connection.

Proposition 4.1. *Given a curve $\tau = x_t$, $0 \leqq t \leqq 1$, in M, set $Y_t = \tau_0^t(\dot{x}_t)$, where τ_0^t is the linear parallel displacement along τ from x_t to x_0 and $\dot{x}_t$ is the vector tangent to τ at x_t. Let C_t, $0 \leqq t \leqq 1$, be the curve in $A_{x_0}(M)$ starting from the origin (that is, $C_0 = x_0$) such that C_t is parallel (in the affine space $A_{x_0}(M)$ in the usual sense) to Y_t for every t. Then C_t is the development of τ into $A_{x_0}(M)$.*

Proof. Let u_0 be any point in $L(M)$ such that $\pi(u_0) = x_0$ and u_t the horizontal lift of x_t in $L(M)$ with respect to the linear connection. Let $\tilde{u}_t$ be the horizontal lift of x_t in $A(M)$ with respect to the affine connection such that $\tilde{u}_0 = u_0$. Since the homomorphism β: $A(M) \to L(M) = A(M)/\mathbf{R}^n$ (cf. §3) maps $\tilde{u}_t$ into u_t, there is a curve $\tilde{a}_t$ in $\mathbf{R}^n \subset A(n; \mathbf{R})$ such that $\tilde{u}_t = u_t\tilde{a}_t$ and that $\tilde{a}_0$ is the identity. As in the proof of Proposition 3.1 of Chapter II, we shall find a necessary and sufficient condition for $\tilde{a}_t$ in order that $\tilde{u}_t$ be horizontal with respect to the affine connection. From

$$\dot{\tilde{u}}_t = \dot{u}_t\tilde{a}_t + u_t\dot{\tilde{a}}_t$$

which follows from Leibniz's formula as in the proof of Proposition 3.1 of Chapter II, we obtain

$$\begin{aligned}\tilde{\omega}(\dot{\tilde{u}}_t) &= \text{ad}\,(\tilde{a}_t^{-1})(\tilde{\omega}(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t \\ &= \text{ad}\,(\tilde{a}_t^{-1})(\omega(\dot{u}_t) + \theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t = \text{ad}\,(\tilde{a}_t^{-1})(\theta(\dot{u}_t)) + \tilde{a}_t^{-1}\dot{\tilde{a}}_t,\end{aligned}$$

where $\tilde{\omega}$ and ω are the connection forms of the affine and the linear connections respectively. Thus $\tilde{u}_t$ is horizontal if and only if $\theta(\dot{u}_t) = -\dot{\tilde{a}}_t\tilde{a}_t^{-1}$. Hence,

$$\begin{aligned}Y_t = \tau_0^t(\dot{x}_t) &= u_0(u_t^{-1}(\dot{x}_t)) = u_0(\theta(\dot{u}_t)) \\ &= -u_0(\dot{\tilde{a}}_t\tilde{a}_t^{-1}) = -u_0(d\tilde{a}_t/dt).\end{aligned}$$

On the other hand, we have

$$C_t = \tilde{\tau}_0^t(x_t) = u_0(\tilde{u}_t^{-1}(x_t)) = u_0(\tilde{a}_t^{-1}(u_t^{-1}(x_t))) = u_0(\tilde{a}_t^{-1}(0)).$$

Hence,

$$dC_t/dt = -u_0(d\tilde{a}_t/dt) = Y_t.$$

QED.

Corollary 4.2. *The development of a curve* $\tau = x_t$, $0 \leq t \leq 1$, *into* $A_{x_0}(M)$ *is a line segment if and only if the vector fields* $\dot{x}_t$ *along* $\tau = x_t$ *is parallel.*

Proof. In Proposition 4.1, C_t is a line segment if and only if Y_t is independent of t. On the other hand, Y_t is independent of t if and only if $\dot{x}_t$ is a parallel vector field along τ. QED.

5. *Curvature and torsion tensors*

We have already defined the torsion form Θ and the curvature form Ω of a linear connection. We now define the *torsion tensor field* (or simply, *torsion*) T and the *curvature tensor field* (or simply, *curvature*) R. We set

$$T(X, Y) = u(2\Theta(X^*, Y^*)) \qquad \text{for } X, Y \in T_x(M),$$

where u is any point of $L(M)$ with $\pi(u) = x$ and X^* and Y^* are vectors of $L(M)$ at u with $\pi(X^*) = X$ and $\pi(Y^*) = Y$. We already know that $T(X, Y)$ is independent of the choice of u, X^*, and Y^* (cf. Example 5.2 of Chapter II); this fact can be easily verified directly also. Thus, at every point x of M, T defines a skew symmetric bilinear mapping $T_x(M) \times T_x(M) \to T_x(M)$. In other words, T is a tensor field of type (1, 2) such that $T(X, Y) = -T(Y, X)$. We shall call $T(X, Y)$ the *torsion translation* in $T_x(M)$ determined by X and Y. Similarly, we set

$$R(X, Y)Z = u((2\Omega(X^*, Y^*))(u^{-1}Z) \qquad \text{for } X, Y, Z \in T_x(M),$$

where u, X^* and Y^* are chosen as above. Then $R(X, Y)Z$ depends only on X, Y and Z, not on u, X^* and Y^*. In the above definition, $(2\Omega(X^*, Y^*))(u^{-1}Z)$ denotes the image of $u^{-1}Z \in \mathbf{R}^n$ by the linear endomorphism $2\Omega(X^*, Y^*) \in \mathfrak{gl}(n; \mathbf{R})$ of $\mathbf{R}^n$. Thus, $R(X, Y)$ is an endomorphism of $T_x(M)$ and is called the *curvature transformation* of $T_x(M)$ determined by X and Y. It follows that R is a tensor field of type (1, 3) such that $R(X, Y) = -R(Y, X)$.

Theorem 5.1. *In terms of covariant differentiation, the torsion T and the curvature R can be expressed as follows:*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

where X, Y and Z are vector fields on M.

Proof. Let X^*, Y^* and Z^* be the horizontal lifts of X, Y and Z, respectively. We first prove

Lemma. $(\nabla_X Y)_x = u(X^*_u(\theta(Y^*)))$, *where* $\pi(u) = x$.

Proof of Lemma. In the lemma for Proposition 1.1, we proved that $(\nabla_X Y)_x = u(X^*_u f)$, where f is an $\mathbf{R}^n$-valued function defined by $f(u) = u^{-1}(Y_x)$. Hence, $f(u) = \theta(Y^*_u)$ for $u \in L(M)$. This completes the proof of the lemma.

We have therefore

$$\begin{aligned} T(X_x, Y_x) &= u(2\Theta(X^*_u, Y^*_u)) \\ &= u(X^*_u(\theta(Y^*)) - Y^*_u(\theta(X^*)) - \theta([X^*, Y^*]_u)) \\ &= (\nabla_X Y)_x - (\nabla_Y X)_x - [X, Y]_x, \end{aligned}$$

since $\pi([X^*, Y^*]) = [X, Y]$.

To prove the second equality, we set $f = \theta(Z^*)$ so that f is an $\mathbf{R}^n$-valued function on $L(M)$ of type $(GL(n; \mathbf{R}), \mathbf{R}^n)$. We have

then

$$([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)_x$$
$$= u(X_u^*(Y^*f) - Y_u^*(X^*f) - (h[X^*, Y^*])_u f) = u((v[X^*, Y^*])_u f),$$

where h (resp. v) denotes the horizontal (resp. vertical) component. Let A be an element of $\mathfrak{gl}(n; \mathbf{R})$ such that $A_u^* = (v[X^*, Y^*])_u$, where A^* is the fundamental vector field corresponding to A. Then by Corollary 5.3 of Chapter II, we have

$$2\Omega(X_u^*, Y_u^*) = -\omega([X^*, Y^*]_u) = -A.$$

On the other hand, if a_t is the 1-parameter subgroup of $GL(n; \mathbf{R})$ generated by A, then

$$\begin{aligned} A_u^* f &= \lim_{t\to 0} \frac{1}{t} [f(ua_t) - f(u)] \\ &= \lim_{t\to 0} \frac{1}{t} [a_t^{-1}(f(u)) - f(u)] \\ &= -A(f(u)), \end{aligned}$$

where $A(f(u))$ denotes the result of the linear transformation $A\colon \mathbf{R}^n \to \mathbf{R}^n$ applied to $f(u) \in \mathbf{R}^n$. Therefore, we have

$$([\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z)_x = u((v[X^*, Y^*])_u f) = u(-A(f(u)))$$
$$= u(2\Omega(X_u^*, Y_u^*)(f(u))) = u(2\Omega(X_u^*, Y_u^*)(u^{-1}Z)) = R(X, Y)Z.$$

QED.

Proposition 5.2. *Let $X,Y,Z,W \in T_x(M)$ and $u \in L(M)$ with $\pi(u) = x$. Let X^*, Y^*, Z^* and W^* be the standard horizontal vector fields on $L(M)$ corresponding to $u^{-1}X$, $u^{-1}Y$, $u^{-1}Z$ and $u^{-1}W$ respectively, so that $\pi(X_u^*) = X$, $\pi(Y_u^*) = Y$, $\pi(Z_u^*) = Z$ and $\pi(W_u^*) = W$. Then*

$$(\nabla_X T)(Y, Z) = u(X_u^*(2\Theta(Y^*, Z^*)))$$

and

$$((\nabla_X R)(Y, Z))W = u((X_u^*(2\Omega(Y^*, Z^*)))(u^{-1}W)).$$

Proof. We shall prove only the first formula. The proof of the second formula is similar to that of the first. We consider the torsion T as a cross section of the tensor bundle $T_2^1(M)$ whose standard fibre is the tensor space $\mathbf{T}_2^1$ of type (1, 2) over $\mathbf{R}^n$. Let f be the $\mathbf{T}_2^1$-valued function on $L(M)$ corresponding to the torsion T as in Example 5.2 of Chapter II so that, if we set $\eta = u^{-1}Y$ and $\zeta = u^{-1}Z$, then

$$f_u(\eta, \zeta) = u^{-1}(T(X, Z)) = 2\Theta(Y^*_u, Z^*_u).$$

By Proposition 1.3, $X^*_u f$ corresponds to $\nabla_X T$. Hence,

$$\begin{aligned} u^{-1}((\nabla_X T)(Y, Z)) &= (X^*_u f)(\eta, \zeta) \\ &= X^*_u(f(\eta, \zeta)) = X^*_u(2\Theta(Y^*, Z^*)) \end{aligned}$$

thus proving our assertion. QED.

Using Proposition 5.2, we shall express the Bianchi's identities (Theorem 2.5) in terms of T, R and their covariant derivatives.

THEOREM 5.3. *Let T and R be the torsion and the curvature of a linear connection of M. Then, for $X, Y, Z \in T_x(M)$, we have*

Bianchi's 1st identity:

$$\mathfrak{S}\{R(X, Y)Z\} = \mathfrak{S}\{T(T(X, Y), Z) + (\nabla_X T)(Y, Z)\};$$

Bianchi's 2nd identity:

$$\mathfrak{S}\{(\nabla_X R)(Y, Z) + R(T(X, Y), Z)\} = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z.

In particular, if $T = 0$, then

Bianchi's 1st identity: $\mathfrak{S}\{R(X, Y)Z\} = 0$;

Bianchi's 2nd identity: $\mathfrak{S}\{(\nabla_X R)(Y, Z)\} = 0$.

Proof. Let u be any point of $L(M)$ such that $\pi(u) = x$. We lift X to a horizontal vector at u and then extend it to a standard horizontal vector field X^* on $L(M)$ as in Proposition 5.2. Similarly, we define Y^* and Z^*. We shall derive the first identity from

$$D\Theta = \Omega \wedge \theta \quad \text{(Theorem 2.5)}.$$

We have

$$\begin{aligned} 6(\Omega \wedge \theta)(X^*_u, Y^*_u, Z^*_u) &= \mathfrak{S}\{2\Omega(X^*_u, Y^*_u)\theta(Z^*_u)\} \\ &= \mathfrak{S}\{u^{-1}(R(X, Y)Z)\}. \end{aligned}$$

On the other hand, by Proposition 3.11 of Chapter I, we have

$$\begin{aligned} 6D\Theta(X^*_u, Y^*_u, Z^*_u) &= 6d\Theta(X^*_u, Y^*_u, Z^*_u) \\ &= \mathfrak{S}\{X^*_u(2\Theta(Y^*, Z^*)) - 2\Theta([X^*, Y^*]_u, Z^*_u)\}. \end{aligned}$$

By Proposition 5.2, $X_u^*(2\Theta(Y^*, Z^*)) = u^{-1}((\nabla_X T)(Y, Z))$. It is therefore sufficient to prove that

$$-2\Theta([X^*, Y^*]_u, Z_u^*) = u^{-1}(T(T(X, Y), Z)).$$

We observe first that

$$\begin{aligned}\pi([X^*, Y^*]_u) = u(\theta[X^*, Y^*]_u)) &= -u(2d\theta(X_u^*, Y_u^*)) \\ = -u(2\Theta(X_u^*, Y_u^*)) &= -T(X, Y).\end{aligned}$$

Hence we have

$$\begin{aligned}-2\Theta([X^*, Y^*]_u, Z_u^*) &= -u^{-1}(T(\pi[X^*, Y^*]_u, Z)) \\ &= u^{-1}(T(T(X, Y), Z)).\end{aligned}$$

We shall derive the second identity from

$$D\Omega = 0 \qquad \text{(Theorem 2.5)}.$$

We have

$$\begin{aligned}0 &= 3D\Omega(X_u^*, Y_u^*, Z_u^*) \\ &= \mathfrak{S}\{X_u^*(\Omega(Y^*, Z^*)) - \Omega([X^*, Y^*]_u, Z_u^*)\}.\end{aligned}$$

On the other hand, by Proposition 5.2, we have

$$X_u^*(\Omega(Y^*, Z^*)) = \tfrac{1}{2}u^{-1}((\nabla_X R)(Y, Z)).$$

As in the proof of the first identity, we have

$$-\Omega([X^*, Y^*]_u, Z_u^*) = \tfrac{1}{2}u^{-1}(R(T(X, Y), Z)).$$

The second identity follows from these three formulas. QED.

Remark. Theorem 5.3 can be proved from the formulas in Theorem 5.1 (see, for instance, Nomizu [7, p. 611]).

PROPOSITION 5.4. *Let B and B' be arbitrary standard horizontal vector fields on $L(M)$. Then we have*

(1) *If $T = 0$, then $[B, B']$ is vertical;*

(2) *If $R = 0$, then $[B, B']$ is horizontal.*

Proof. (1) $\theta([B, B']) = -2d\theta(B, B') = -2\Theta(B, B') = 0$. Hence, $[B, B']$ is vertical. (2) $\omega([B, B']) = -2d\omega(B, B') = -2\Omega(B, B') = 0$. Hence, $[B, B']$ is horizontal. QED.

Let $P(u_0)$ be the holonomy subbundle of $L(M)$ through a point $u_0 \in L(M)$ and $\Psi(u_0)$ the linear holonomy group with reference point u_0. Let $A_1, \ldots, A_r$ be a basis of the Lie algebra of

$\Psi(u_0)$ and $A_1^*, \ldots, A_r^*$ the corresponding fundamental vector fields. Let $B_1, \ldots, B_n$ be the standard horizontal vector fields corresponding to the basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$. These vector fields $A_1^*, \ldots, A_r^*, B_1, \ldots, B_n$ (originally defined on $L(M)$), restricted to $P(u_0)$, define vector fields on $P(u_0)$. Just as in Proposition 2.6, they define an absolute parallelism on $P(u_0)$. We know that $[A_i^*, A_j^*]$ is the fundamental vector field corresponding to $[A_i, A_j]$ and hence is a linear combination of $A_1^*, \ldots, A_r^*$ with constant coefficients. By Proposition 2.3, $[A_i^*, B_j]$ is the standard horizontal vector field corresponding to $A_i e_j \in \mathbf{R}^n$. The following proposition gives some information about $[B_i, B_j]$.

Proposition 5.5. *Let $P(u_0)$ be the holonomy subbundle of $L(M)$ through u_0. Let B and B' be arbitrary standard horizontal vector fields. Then we have*

(1) *If $\nabla T = 0$, then the horizontal component of $[B, B']$ coincides with a standard horizontal vector field on $P(u_0)$.*

(2) *If $\nabla R = 0$, then the vertical component of $[B, B']$ coincides with the fundamental vector field A^* on $P(u_0)$, which corresponds to an element A of the Lie algebra of the linear holonomy group $\Psi(u_0)$.*

Proof. (1) Let X^* be any horizontal vector at $u \in L(M)$. Set $X = \pi(X^*)$, $Y = \pi(B_u)$ and $Z = \pi(B'_u)$. By Proposition 5.2, we have

$$X^*(2\Theta(B, B')) = u^{-1}((\nabla_X T)(Y, Z)) = 0.$$

This means that $\Theta(B, B')$ is a constant function (with values in $\mathbf{R}^n$) on $P(u_0)$. Since $\theta([B, B']) = -2\Theta(B, B')$, the horizontal component of $[B, B']$ coincides on $P(u_0)$ with the standard horizontal vector field corresponding to the element $-2\Theta(B, B')$ of $\mathbf{R}^n$.

(2) Again, by Proposition 5.2, $\nabla R = 0$ implies

$$X^*(\Omega(B, B')) = 0.$$

This means that $\Omega(B, B')$ is a constant function on $P(u_0)$ (with values in the Lie algebra of $\Psi(u_0)$). Since $\omega([B, B']) = -2\Omega(B, B')$, the vertical component of $[B, B']$ coincides on $P(u_0)$ with the fundamental vector field corresponding to the element $-2\Omega(B, B')$ of the Lie algebra of $\Psi(u_0)$. QED.

It follows that, if $\nabla T = 0$ and $\nabla R = 0$, then the restriction of $[B_i, B_j]$ to $P(u_0)$ coincides with a linear combination of $A_1^*, \ldots,$

$A_r^*, B_1, \ldots, B_n$ with constant coefficients on $P(u_0)$. Hence we have

COROLLARY 5.6. *Let* $\mathfrak{g}$ *be the set of all vector fields* X *on the holonomy bundle* $P(u_0)$ *such that* $\theta(X)$ *and* $\omega(X)$ *are constant functions on* $P(u_0)$ *(with values in* $\mathbf{R}^n$ *and in the Lie algebra of* $\Psi(u_0)$*, respectively). If* $\nabla T = 0$ *and* $\nabla R = 0$*, then* $\mathfrak{g}$ *forms a Lie algebra and* $\dim \mathfrak{g} = \dim P(u_0)$.

The vector fields $A_1^*, \ldots, A_r^*, B_1, \ldots, B_n$ defined above form a basis for $\mathfrak{g}$.

6. *Geodesics*

A curve $\tau = x_t$, $a < t < b$, where $-\infty \leqq a < b \leqq \infty$, of class C^1 in a manifold M with a linear connection is called a *geodesic* if the vector field $X = \dot{x}_t$ defined along τ is parallel along τ, that is, if $\nabla_X X$ exists and equals 0 for all t, where $\dot{x}_t$ denotes the vector tangent to τ at x_t. In this definition of geodesics, the parametrization of the curve in question is important.

PROPOSITION 6.1. *Let* τ *be a curve of class* C^1 *in* M. *A parametrization which makes* τ *into a geodesic, if any, is determined up to an affine transformation* $t \to s = \alpha t + \beta$*, where* $\alpha \neq 0$ *and* β *are constants.*

Proof. Let x_t and y_s be two parametrizations of a curve τ which make τ into a geodesic. Then s is a function of t, $s = s(t)$, and $y_{s(t)} = x_t$. The vector $\dot{y}_s$ is equal to $\dfrac{dt}{ds}\dot{x}_t$. Since the parallel displacement along τ is independent of parametrization (cf. §3 of Chapter II), $\dfrac{dt}{ds}$ must be a constant different from zero. Hence, $s = \alpha t + \beta$, where $\alpha \neq 0$. QED.

If τ is a geodesic, any parameter t which makes τ into a geodesic is called an *affine parameter*. In particular, let x be a point of a geodesic τ and $X \in T_x(M)$ a vector in the direction of τ. Then there is a unique affine parameter t for τ, $\tau = x_t$, such that $x_0 = x$ and $\dot{x}_0 = X$. The parameter t is called the affine parameter for τ determined by (x, X).

PROPOSITION 6.2. *A curve* τ *of class* C^1 *through* $x \in M$ *is a geodesic if and only if its development into* $T_x(M)$ *is (an open interval of) a straight line.*

Proof. This is an immediate consequence of Corollary 4.2. QED.

Another useful interpretation of geodesics is given in terms of the bundle of linear frames $L(M)$.

PROPOSITION 6.3. *The projection onto M of any integral curve of a standard horizontal vector field of $L(M)$ is a geodesic and, conversely, every geodesic is obtained in this way.*

Proof. Let B be the standard horizontal vector field on $L(M)$ which corresponds to an element $\xi \in \mathbf{R}^n$. Let b_t be an integral curve of B. We set $x_t = \pi(b_t)$. Then $\dot{x}_t = \pi(\dot{b}_t) = \pi(B_{b_t}) = b_t\xi$, where $b_t\xi$ denotes the image of ξ by the linear mapping $b_t\colon \mathbf{R}^n \to T_{x_t}(M)$. Since b_t is a horizontal lift of x_t and ξ is independent of t, $b_t\xi$ is parallel along the curve x_t (see §7, Chapter II, in particular, before Proposition 7.4).

Conversely, let x_t be a geodesic in M defined in some open interval containing 0. Let u_0 be any point of $L(M)$ such that $\pi(u_0) = x_0$. We set $\xi = u_0^{-1}\dot{x}_0 \in \mathbf{R}^n$. Let u_t be the horizontal lift of x_t through u_0. Since x_t is a geodesic, we have $\dot{x}_t = u_t\xi$. Since u_t is horizontal and since $\theta(\dot{u}_t) = u_t^{-1}(\pi(\dot{u}_t)) = u_t^{-1}\dot{x}_t = \xi$, u_t is an integral curve of the standard horizontal vector field B corresponding to ξ. QED.

As an application of Proposition 6.3, we obtain the following

THEOREM 6.4. *For any point $x \in M$ and for any vector $X \in T_x(M)$, there is a unique geodesic with the initial condition (x, X), that is, a unique geodesic x_t such that $x_0 = x$ and $\dot{x}_0 = X$.*

Another consequence of Proposition 6.3 is that *a geodesic, which is a curve of class C^1, is automatically of class C^∞* (provided that the linear connection is of class C^∞). In fact, every standard horizontal vector field is of class C^∞ and hence its integral curves are all of class C^∞. The projection onto M of a curve of class C^∞ in $L(M)$ is a curve of class C^∞ in M.

A linear connection of M is said to be *complete* if every geodesic can be extended to a geodesic $\tau = x_t$ defined for $-\infty < t < \infty$, where t is an affine parameter. In other words, for any $x \in M$ and $X \in T_x(M)$, the geodesic $\tau = x_t$ in Theorem 6.4 with the initial condition (x, X) is defined for all values of t, $-\infty < t < \infty$.

Immediate from Proposition 6.3 is the following

PROPOSITION 6.5. *A linear connection is complete if and only if every standard horizontal vector field on $L(M)$ is complete.*

We recall that a vector field on a manifold is said to be complete if it generates a global 1-parameter group of transformations of the manifold.

When the linear connection is complete, we can define the *exponential map* at each point $x \in M$ as follows. For each $X \in T_x(M)$, let $\tau = x_t$ be the geodesic with the initial condition (x, X) as in Theorem 6.4. We set

$$\exp X = x_1.$$

Thus we have a mapping of $T_x(M)$ into M for each x. We shall later (in §8) define the exponential map in the case where the linear connection is not necessarily complete and discuss its differentiability and other properties.

7. *Expressions in local coordinate systems*

In this section, we shall express a linear connection and related concepts in terms of local coordinate systems.

Let M be a manifold and U a coordinate neighborhood in M with a local coordinate system $x^1, \ldots, x^n$. We denote by X_i the vector field $\partial/\partial x^i$, $i = 1, \ldots, n$, defined in U. Every linear frame at a point x of U can be uniquely expressed by

$$(\Sigma_i X_1^i (X_i)_x, \ldots, \Sigma_i X_n^i (X_i)_x),$$

where $\det (X_i^j) \neq 0$. We take (x^i, X_k^j) as a local coordinate system in $\pi^{-1}(U) \subset L(M)$. (cf. Example 5.2 of Chapter I). Let (Y_k^j) be the inverse matrix of (X_k^j) so that $\Sigma_j X_i^j Y_j^k = \Sigma_j Y_i^j X_j^k = \delta_i^k$.

We shall express first the canonical form θ in terms of the local coordinate system (x^i, X_k^j). Let $e_1, \ldots, e_n$ be the natural basis for $\mathbf{R}^n$ and set

$$\theta = \Sigma_i \theta^i e_i.$$

PROPOSITION 7.1. *In terms of the local coordinate system (x^i, X_k^j), the canonical form $\theta = \Sigma_i \theta^i e_i$ can be expressed as follows:*

$$\theta^i = \Sigma_j Y_j^i \, dx^j.$$

Proof. Let u be a point of $L(M)$ with coordinates (x^i, X^j_k) so that u maps e_i into $\Sigma_j X^j_i(X_j)_x$, where $x = \pi(u)$. If $X^* \in T_u(L(M))$ and if

$$X^* = \Sigma_j \lambda^j \left(\frac{\partial}{\partial x^j}\right)_u + \Sigma_{j,k} \Lambda^j_k \left(\frac{\partial}{\partial X^j_k}\right)_u$$

so that $\pi(X^*) = \Sigma_j \lambda^j (X_j)_x$, then

$$\theta(X^*) = u^{-1}(\Sigma_j \lambda^j (X_j)_x) = \Sigma_{i,j}(Y^i_j \lambda^j)\, e_i.$$

QED.

Let ω be the connection form of a linear connection Γ of M. With respect to the basis $\{E^j_i\}$ of $\mathfrak{gl}(n; \mathbf{R})$, we write

$$\omega = \Sigma_{i,j}\, \omega^i_j E^j_i.$$

Let σ be the cross section of $L(M)$ over U which assigns to each $x \in U$ the linear frame $((X_1)_x, \ldots, (X_n)_x)$. We set

$$\omega_U = \sigma^* \omega.$$

Then ω_U is a $\mathfrak{gl}(n; \mathbf{R})$-valued 1-form defined on U. We define n^3 functions Γ^i_{jk}, $i, j, k = 1, \ldots, n$, on U by

$$\omega_U = \Sigma_{i,j,k}(\Gamma^i_{jk}\, dx^j) E^k_i.$$

These functions Γ^i_{jk} are called the *components* (or *Christoffel's symbols*) of the linear connection Γ with respect to the local coordinate system $x^1, \ldots, x^n$. It should be noted that they are not the components of a tensor field. In fact, these components are subject to the following transformation rule.

PROPOSITION 7.2. *Let Γ be a linear connection of M. Let Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ be the components of Γ with respect to local coordinate systems $x^1, \ldots, x^n$ and $\bar{x}^1, \ldots, \bar{x}^n$, respectively. In the intersection of the two coordinate neighborhoods, we have*

$$\bar{\Gamma}^\alpha_{\beta\gamma} = \sum_{i,j,k} \Gamma^i_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \Sigma_i \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i}.$$

Proof. We derive the above formula from Proposition 1.4 of Chapter II. Let V be the coordinate neighborhood where the coordinate system $\bar{x}^1, \ldots, \bar{x}^n$ is valid. Let $\bar{\sigma}$ be the cross section of $L(M)$ over V which assigns to each $x \in V$ the linear frame

$((\partial/\partial\bar{x}^1)_x, \ldots, (\partial/\partial\bar{x}^n)_x)$. We define a mapping $\psi_{UV}\colon U \cap V \to GL(n; \mathbf{R})$ by

$$\bar{\sigma}(x) = \sigma(x) \cdot \psi_{UV}(x) \qquad \text{for } x \in U \cap V.$$

Let φ be the (left invariant $\mathfrak{gl}(n; \mathbf{R})$-valued) canonical 1-form on $GL(n; \mathbf{R})$ defined in §4 of Chapter I; this form was denoted by θ in §4 of Chapter I and in §1 of Chapter II. If (s^i_j) is the natural coordinate system in $GL(n; \mathbf{R})$ and if (t^i_j) denotes the inverse matrix of (s^i_j), then

$$\varphi = \Sigma_{i,j,k}\, t^i_j\, ds^j_k\, E^k_i,$$

the proof being similar to that of Proposition 7.1. It is easy to verify that

$$\psi_{UV} = (\partial x^i/\partial\bar{x}^j)$$

and hence

$$\psi^*_{UV}\varphi = \Sigma_{\alpha,\beta}\left(\Sigma_i \frac{\partial\bar{x}^\alpha}{\partial x^i}\, d\left(\frac{\partial x^i}{\partial\bar{x}^\beta}\right)\right)E^\beta_\alpha = \Sigma_{\alpha,\beta}\left(\Sigma_{i,\gamma} \frac{\partial\bar{x}^\alpha}{\partial x^i}\, \frac{\partial^2 x^i}{\partial\bar{x}^\beta\partial\bar{x}^\gamma}\, d\bar{x}^\gamma\right)E^\beta_\alpha.$$

With our notation, the formula in Proposition 1.4 of Chapter II can be expressed as follows:

$$\omega_V = (\mathrm{ad}\,(\psi^{-1}_{UV}))\omega_U + \psi^*_{UV}\varphi.$$

By a simple calculation, we see that this formula is equivalent to the transformation rule of our proposition. QED.

From the components Γ^i_{jk} we can reconstruct the connection form ω.

PROPOSITION 7.3. *Assume that, for each local coordinate system* $x^1, \ldots, x^n$, *there is given a set of functions* Γ^i_{jk}, $i, j, k = 1, \ldots, n$, *in such a way that they satisfy the transformation rule of Proposition 7.2. Then there is a unique linear connection* Γ *whose components with respect to* $x^1, \ldots, x^n$ *are precisely the given functions* Γ^i_{jk}. *Moreover, the connection form* $\omega = \Sigma_{i,j}\, \omega^i_j E^j_i$ *is given in terms of the local coordinate system* (x^i, X^j_k) *by*

$$\omega^i_j = \Sigma_k\, Y^i_k(dX^k_j + \Sigma_{l,m}\, \Gamma^k_{ml} X^l_j\, dx^m), \qquad i, j = 1, \ldots, n.$$

Proof. It is easy to verify that the form ω defined by the above formula defines a connection in $L(M)$, that is, ω satisfies the conditions (a′) and (b′) of Proposition 1.1 of Chapter II. The fact that ω is independent of the local coordinate system used follows from the transformation rule of Γ^i_{jk}; this can be proved by reversing the process in the proof of Proposition 7.2. The cross section $\sigma: U \to L(M)$ used above to define ω_U is given in terms of the local coordinate systems (x^i) and (x^i, X^j_k) by $(x^i) \to (x^i, \delta^j_k)$. Hence, $\sigma^*\omega^i_j = \Sigma_m \Gamma^i_{mj}\, dx^m$. This shows that the components of the connection Γ defined by ω are exactly the functions Γ^i_{jk}. QED.

The components of a linear connection can be expressed also in terms of covariant derivatives.

PROPOSITION 7.4. *Let $x^1, \ldots, x^n$ be a local coordinate system in M with a linear connection Γ. Set $X_i = \partial/\partial x^i$, $i = 1, \ldots, n$. Then the components Γ^i_{jk} of Γ with respect to $x^1, \ldots, x^n$ are given by*

$$\nabla_{X_j} X_i = \Sigma_k \Gamma^k_{ji} X_k.$$

Proof. Let X^*_j be the horizontal lift of X_j. From Proposition 7.3, it follows that, in terms of the coordinate system (x^i, X^j_k), X^* is given by

$$X^*_j = (\partial/\partial x^j) - \Sigma_{i,k,l}\, \Gamma^i_{jk} X^k_l (\partial/\partial X^i_l).$$

To apply Proposition 1.3, let f be the $\mathbf{R}^n$-valued function on $\pi^{-1}(U) \subset L(M)$ which corresponds to X_i. Then

$$f = \Sigma_k Y^k_i e_k.$$

A simple calculation shows that

$$X^*_j f = \Sigma_{k,l} \Gamma^l_{ji} Y^k_l e_k.$$

By Proposition 1.3, $X^*_j f$ is the function corresponding to $\nabla_{X_j} X_i$ and hence

$$\nabla_{X_j} X_i = \Sigma_k \Gamma^k_{ji} X_k.$$

QED.

PROPOSITION 7.5. *Assume that a mapping $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, denoted by $(X, Y) \to \nabla_X Y$, is given so as to satisfy the conditions* (1), (2), (3) *and* (4) *of Proposition* 2.8. *Then there is a unique linear connection Γ of M such that $\nabla_X Y$ is the covariant derivative of Y in the direction of X with respect to Γ.*

Proof. Leaving the detail to the reader, we shall give here an outline of the proof. Let $x \in M$. If X, X', Y and Y' are vector fields on M and if $X = X'$ and $Y = Y'$ in a neighborhood of x, then $(\nabla_X Y)_x = (\nabla_{X'} Y')_x$. This implies that the given mapping

$\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ induces a mapping $\mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ satisfying the same conditions of Proposition 2.8 (where U is any open set of M). In particular, if U is a coordinate neighborhood with a local coordinate system $x^1, \ldots, x^n$, we define n^3 functions Γ^i_{jk} on U by the formula given in Proposition 7.4. Then these functions satisfy the transformation rule of Proposition 7.2. By Proposition 7.3, they define a linear connection, say, Γ. It is clear that $\nabla_X Y$ is the covariant derivative of Y in the direction of X with respect to Γ. QED.

Let η^i be the components of a vector field Y with respect to a local coordinate system $x^1, \ldots, x^n$, $Y = \Sigma_i \eta^i(\partial/\partial x^i)$. Let $\eta^i_{;j}$ be the components of the covariant differential ∇Y so that $\nabla_{X_j} Y = \Sigma_i \eta^i_{;j} X_i$, where $X_i = \partial/\partial x^i$. From Propositions 7.4 and 2.8, we obtain the following formula:

$$\eta^i_{;j} = \partial\eta^i/\partial x^j + \Sigma_k \Gamma^i_{jk}\eta^k.$$

If X is a vector field with components ξ^i, then the components of $\nabla_X Y$ are given by $\Sigma_j \eta^i_{;j}\xi^j$.

More generally, if K is a tensor field of type (r, s) with components $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, then the components of ∇K are given by

$$K^{i_1 \cdots i_r}_{j_1 \cdots j_s;k} = \partial K^{i_1 \cdots i_r}_{j_1 \cdots j_s}/\partial x^k + \Sigma^r_{\alpha=1} (\Sigma_l \Gamma^{i_\alpha}_{kl} K^{i_1 \cdots l \cdots i_r}_{j_1 \cdots j_s}) - \Sigma^s_{\beta=1}(\Sigma_m \Gamma^m_{kj_\beta} K^{i_1 \cdots i_r}_{j_1 \cdots m \cdots j_s}),$$

where l takes the place of i_α and m takes the place of j_β. The proof of this formula is the same as the one for a vector field, except that Proposition 2.7 has to be used in place of Proposition 2.8. If X is a vector field with components ξ^i, then the components of $\nabla_X K$ are given by

$$\Sigma_k K^{i_1 \cdots i_r}_{j_1 \cdots j_s;k}\xi^k.$$

The covariant derivatives of higher order can be defined similarly. For a tensor field K with components $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$, $\nabla^m K$ has components $K^{i_1 \cdots i_r}_{j_1 \cdots j_s; k_1; \cdots ;k_m}$.

The components T^i_{jk} of the torsion T and the components R^i_{jkl} of the curvature R are defined by

$$T(X_j, X_k) = \Sigma_i T^i_{jk} X_i, \quad R(X_k, X_l)X_j = \Sigma_i R^i_{jkl} X_i.$$

Then they can be expressed in terms of the components Γ^i_{jk} of the linear connection Γ as follows.

PROPOSITION 7.6. *We have*

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj};$$

$$R^i_{jkl} = (\partial\Gamma^i_{lj}/\partial x^k - \partial\Gamma^i_{kj}/\partial x^l) + \Sigma_m(\Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}).$$

Proof. These formulas follow immediately from Theorem 5.1 and Proposition 7.4. QED.

The proof of the following proposition is a straightforward calculation.

PROPOSITION 7.7. (1) *If f is a function defined on M, then*

$$f_{;k;j} - f_{;j;k} = \Sigma_i \, T^i_{kj} f_{;i}.$$

(2) *If X is a vector field on M with components ξ^i, then*

$$\xi^i_{;l;k} - \xi^i_{;k;l} = \Sigma_j \, R^i_{jkl}\xi^j + \Sigma_j \, T^j_{lk}\xi^i_{;j}.$$

Since $n^2 + n$ 1-forms θ^i, ω^j_k, $i, j, k = 1, \ldots, n$, define an absolute parallelism (Proposition 2.6), every differential form on $L(M)$ can be expressed in terms of these 1-forms and functions. Since the torsion form Θ and the curvature form Ω are tensorial forms, they can be expressed in terms of n 1-forms θ^i and functions. We define a set of functions $\tilde{T}^i_{jk}$ and $\tilde{R}^i_{jkl}$ on $L(M)$ by

$$\Theta^i = \Sigma_{j,k} \tfrac{1}{2}\tilde{T}^i_{jk}\theta^j \wedge \theta^k, \quad \tilde{T}^i_{jk} = -\tilde{T}^i_{kj},$$

$$\Omega^i_j = \Sigma_{k,l} \tfrac{1}{2}\tilde{R}^i_{jkl}\theta^k \wedge \theta^l, \quad \tilde{R}^i_{jkl} = -\tilde{R}^i_{jlk}.$$

These functions are related to the components of the torsion T and the curvature R as follows. Let $\sigma \colon U \to L(M)$ be the cross section over U defined at the beginning of this section. Then

$$\sigma^* \tilde{T}^i_{jk} = T^i_{jk}, \quad \sigma^*\tilde{R}^i_{jkl} = R^i_{jkl}.$$

These formulas follow immediately from Proposition 7.6 and from

$$\sigma^* \, d\theta^i = -\Sigma_j \, \sigma^*\omega^i_j \wedge \sigma^*\theta^j + \sigma^*\Theta^i,$$

$$\sigma^* \, d\omega^i_j = -\Sigma_k \, \sigma^*\omega^i_k \wedge \sigma^*\omega^k_j + \sigma^*\Omega^i_j,$$

$$\sigma^*\theta^i = dx^i \quad \text{and} \quad \sigma^*\omega^i_j = \Sigma_k \, \Gamma^i_{kj} \, dx^k.$$

PROPOSITION 7.8. *Let $x^i = x^i(t)$ be the equations of a curve*

$\tau = x_t$ *of class* C^2. *Then* τ *is a geodesic if and only if*

$$\frac{d^2x^i}{dt^2} + \Sigma_{j,k}\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i = 1, \ldots, n.$$

Proof. The components of the vector field $\dot{x}_t$ along τ are given by dx^i/dt. From the formula for the components of $\nabla_X Y$ given above, we see that, if we set $X = \dot{x}_t$, then $\nabla_X X = 0$ is equivalent to the above equations. QED.

We shall compare two or more linear connections by their components.

PROPOSITION 7.9. *Let* Γ *be a linear connection of* M *with components* Γ^i_{jk}. *For each fixed* t, $0 \leqq t \leqq 1$, *the set of functions* $\Gamma^{*i}_{jk} = t\Gamma^i_{jk} + (1-t)\Gamma^i_{kj}$ *defines a linear connection* Γ^* *which has the same geodesics as* Γ. *In particular,* $\Gamma^{*i}_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})$ *define a linear connection with vanishing torsion.*

Proof. Our proposition follows immediately from Propositions 7.3, 7.6 and 7.8. QED.

In general, given two linear connections Γ with components Γ^i_{jk} and Γ' with components Γ'^i_{jk}, the set of functions $t\Gamma^i_{jk} + (1-t)\Gamma'^i_{jk}$ define a linear connection for each t, $0 \leqq t \leqq 1$. Proposition 7.8 implies that Γ and Γ' have the same geodesics if $\Gamma^i_{jk} + \Gamma^i_{kj} = \Gamma'^i_{jk} + \Gamma'^i_{kj}$.

The following proposition follows from Proposition 7.2.

PROPOSITION 7.10. *If* Γ^i_{jk} *and* Γ'^i_{jk} *are the components of linear connections* Γ *and* Γ' *respectively, then* $S^i_{jk} = \Gamma'^i_{jk} - \Gamma^i_{jk}$ *are the components of a tenor field of type* $(1, 2)$. *Conversely, if* Γ^i_{jk} *are the components of a linear connection* Γ *and* S^i_{jk} *are the components of a tensor field* S *of type* $(1, 2)$, *then* $\Gamma'^i_{jk} = \Gamma^i_{jk} + S^i_{jk}$ *define a linear connection* Γ'. *In terms of covariant derivatives, they are related to each other as follows:*

$$\nabla'_X Y = \nabla_X Y + S(X, Y) \quad \textit{for any vector fields } X \textit{ and } Y \textit{ on } M,$$

where ∇ *and* ∇' *are the covariant differentiations with respect to* Γ *and* Γ' *respectively.*

8. *Normal coordinates*

In this section we shall prove the existence of normal coordinate systems and convex coordinate neighborhoods as well as the differentiability of the exponential map.

Let M be a manifold with a linear connection Γ. Given $X \in T_x(M)$, let $\tau = x_t$ be the geodesic with the initial condition (x, X) (cf. Theorem 6.3). We set

$$\exp tX = x_t.$$

As we have seen already in §6, exp tX is defined in some open interval $-\varepsilon_1 < t < \varepsilon_2$, where ε_1 and ε_2 are positive. If the connection is complete, the exponential map exp is defined on the whole of $T_x(M)$ for each $x \in M$. In general, exp is defined only on a subset of $T_x(M)$ for each $x \in M$.

PROPOSITION 8.1. *Identifying each $x \in M$ with the zero vector at x, we consider M as a submanifold of $T(M) = \bigcup_{x \in M} T_x(M)$. Then there is a neighborhood N of M in $T(M)$ such that the exponential map is defined on N. The exponential map $N \to M$ is differentiable of class C^∞, provided that the connection is of class C^∞.*

Proof. Let x_0 be any point of M and u_0 a point of $L(M)$ such that $\pi(u_0) = x_0$. For each $\xi \in \mathbf{R}^n$, we denote by $B(\xi)$ the corresponding standard horizontal vector field on $L(M)$ (cf. §2). By Proposition 1.5 of Chapter I, there exist a neighborhood U^* of u_0 and a positive number δ such that the local 1-parameter group of local transformations $\exp tB(\xi)\colon U^* \to L(M)$ is defined for $|t| < \delta$. Given a compact set K of $\mathbf{R}^n$, we can choose U^* and δ for all $\xi \in K$ simultaneously, because $B(\xi)$ depends differentiably on ξ. Therefore, there exist a neighborhood U^* of u_0 and a neighborhood V of 0 in $\mathbf{R}^n$ such that $\exp tB(\xi)\colon U^* \to L(M)$ is defined for $\xi \in V$ and $|t| \leqq 1$. Let U be a neighborhood of x_0 in M and σ a cross section of $L(M)$ over U such that $\sigma(x_0) = u_0$ and $\sigma(U) \subset U^*$. Given $x \in U$, let N_x be the set of $X \in T_x(M)$ such that $\sigma(x)^{-1}X \in V$ and set $N(x_0) = \bigcup_{x \in U} N_x$. Given $X \in N_x$, set $\xi = \sigma(x)^{-1}X$. Then $\pi((\exp tB(\xi)) \cdot \sigma(x))$ is the geodesic with the initial condition (x, X) and hence

$$\exp X = \pi((\exp B(\xi)) \cdot \sigma(x)).$$

It is now clear that $\exp\colon N(x_0) \to M$ is differentiable of class C^∞. Finally, we set $N = \bigcup_{x_0 \in M} N(x_0)$. QED.

PROPOSITION 8.2. *For every point $x \in M$, there is a neighborhood N_x of x (more precisely, the zero vector at x) in $T_x(M)$ which is mapped diffeomorphically onto a neighborhood U_x of x in M by the exponential map.*

Proof. From the definition of the exponential map, it is evident that the differential of the exponential map at x is non-singular. By the implicit function theorem, there is a neighborhood N_x of x in $T_x(M)$ which has the property stated above. QED.

Given a linear frame $u = (X_1, \ldots, X_n)$ at x, the linear isomorphism $u: \mathbf{R}^n \to T_x(M)$ defines a coordinate system in $T_x(M)$ in a natural manner. Therefore, the diffeomorphism $\exp: N_x \to U_x$ defines a local coordinate system in U_x in a natural manner. We call it the *normal coordinate system* determined by the frame u.

Proposition 8.3. *Let $x^1, \ldots, x^n$ be the normal coordinate system determined by a linear frame $u = (X^1, \ldots, X^n)$ at $x \in M$. Then the geodesic $\tau = x_t$ with the initial condition (x, X), where $X = \Sigma_i a^i X_i$, is expressed by*

$$x^i = a^i t, \quad i = 1, \ldots, n.$$

Conversely, a local coordinate system $x^1, \ldots, x^n$ with the above property is necessarily the normal coordinate system determined by $u = (X^1, \ldots, X^n)$.

Proof. The first assertion is an immediate consequence of the definition of a normal coordinate system. The second assertion follows from the fact that a geodesic is uniquely determined by the initial condition (x, X). QED.

Remark. In the above definition of a normal coordinate system, we did not specify the neighborhood in which the coordinate system is valid. This is because if $x^1, \ldots, x^n$ is the normal coordinate system valid in a neighborhood U of x and $y^1, \ldots, y^n$ is the normal coordinate system valid in a neighborhood V of x and if the both are determined by the frame $u = (X_1, \ldots, X_n)$, then they coincide in a neighborhood of x.

Proposition 8.4. *Given a linear connection Γ on M, let Γ^i_{jk} be its components with respect to a normal coordinate system with origin x_0. Then*

$$\Gamma^i_{jk} + \Gamma^i_{kj} = 0 \quad \text{at } x_0.$$

Consequently, if the torsion of Γ vanishes, then $\Gamma^i_{jk} = 0$ at x_0.

Proof. Let $x^1, \ldots, x^n$ be a normal coordinate system with origin x_0. For any $(a^1, \ldots, a^n) \in \mathbf{R}^n$, the curve defined by $x^i = a^i t$, $i = 1, \ldots, n$, is a geodesic and, hence, by Proposition 7.8,

$\Sigma_{j,k} \Gamma^i_{jk}(a^1t, \ldots, a^nt)a^ja^k = 0$. In particular,

$$\Sigma_{j,k} \Gamma^i_{jk}(x_0)a^ja^k = 0.$$

Since this holds for every $(a^1, \ldots, a^n)$, $\Gamma^i_{jk} + \Gamma^i_{kj} = 0$ at x_0. If the torsion vanishes, then $\Gamma^i_{jk} = 0$ at x_0 by Proposition 7.6. QED.

COROLLARY 8.5. *Let K be a tensor field on M with components $K^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ with respect to a normal coordinate system $x^1, \ldots, x^n$ with origin x_0. If the torsion vanishes, then the covariant derivative $K^{i_1 \cdots i_r}_{j_1 \cdots j_s; k}$ coincides with the partial derivative $\partial K^{i_1 \cdots i_r}_{j_1 \cdots j_s}/\partial x^k$ at x_0.*

Proof. This is immediate from Proposition 8.4 and the formula for the covariant differential of K in terms of Γ^i_{jk} given in §7. QED.

COROLLARY 8.6. *Let ω be any differential form on M. If the torsion vanishes, then*

$(-1)^r$

$$d\omega = A(\nabla\omega),$$

where $\nabla\omega$ is the covariant differential of ω and A is the alternation defined in Example 3.2 of Chapter I.

Proof. Let x_0 be an arbitrary point of M and $x^1, \ldots, x^n$ a normal coordinate system with origin x_0. By Corollary 8.5, $d\omega = A(\nabla\omega)$ at x_0. QED.

THEOREM 8.7. *Let $x^1, \ldots, x^n$ be a normal coordinate system with origin x_0. Let $U(x_0; \rho)$ be the neighborhood of x_0 defined by $\Sigma_i (x^i)^2 < \rho^2$. Then there is a positive number a such that if $0 < \rho < a$, then*

(1) $U(x_0; \rho)$ is convex in the sense that any two points of $U(x_0; \rho)$ can be joined by a geodesic which lies in $U(x_0; \rho)$.

(2) Each point of $U(x_0; \rho)$ has a normal coordinate neighborhood containing $U(x_0; \rho)$.

Proof. By Proposition 7.9, we may assume that the linear connection has no torsion.

LEMMA 1. *Let $S(x_0; \rho)$ denote the sphere defined by $\Sigma_i (x^i)^2 = \rho^2$. Then there exists a positive number c such that, if $0 < \rho < c$, then any geodesic which is tangent to $S(x_0; \rho)$ at a point, say y, of $S(x_0; \rho)$ lies outside $S(x_0; \rho)$ in a neighborhood of y.*

Proof of Lemma 1. Since the torsion vanishes by our assumption, the components Γ^i_{jk} of the linear connection vanish at x_0 by Proposition 8.4. Let $x^i = x^i(t)$ be the equations of a geodesic which is tangent to $S(x_0; \rho)$ at a point $y = (x^1(0), \dots, x^n(0))$ (ρ will be restricted later). Set

$$F(t) = \Sigma_i\,(x^i(t))^2.$$

Then

$$F(0) = \rho^2,$$

$$\left(\frac{dF}{dt}\right)_{t=0} = 2\Sigma_i\, x^i(0)\left(\frac{dx^i}{dt}\right)_{t=0} = 0,$$

$$\frac{d^2F}{dt^2} = 2\Sigma_i\left(\left(\frac{dx^i}{dt}\right)^2 + x^i(t)\,\frac{d^2x^i}{dt^2}\right).$$

Because of the equations of a geodesic given in Proposition 7.8, we have

$$\left(\frac{d^2F}{dt^2}\right)_{t=0} = \Sigma_{j,k}\left((\delta_{jk} - \Sigma_i\,\Gamma^i_{jk}x^i)\,\frac{dx^j}{dt}\,\frac{dx^k}{dt}\right)_{t=0}.$$

Since Γ^i_{jk} vanish at x_0, there exists a positive number c such that the quadratic form with coefficients $(\delta_{jk} - \Sigma_i\,\Gamma^i_{jk}x^i)$ is positive definite in $U(x_0; c)$. If $0 < \rho < c$, then $(d^2F/dt^2)_{t=0} > 0$ and hence $F(t) > \rho^2$ when $t \neq 0$ is in a neighborhood of 0. This completes the proof of the lemma.

LEMMA 2. *Choose a positive number c as in Lemma 1. Then there exists a positive number $a < c$ such that*

(1) *Any two points of $U(x_0; a)$ can be joined by a geodesic which lies in $U(x_0; c)$;*

(2) *Each point of $U(x_0; a)$ has a normal coordinate neighborhood containing $U(x_0; a)$.*

Proof of Lemma 2. We consider M as a submanifold of $T(M)$ in a natural manner. Set

$$\varphi(X) = (x, \exp X) \qquad \text{for } X \in T_x(M).$$

If the connection is complete, φ is a mapping of $T(M)$ into $M \times M$. In general, φ is defined only in a neighborhood of M in $T(M)$. Since the differential of φ at x_0 is nonsingular, there exist a neighborhood V of x_0 in $T(M)$ and a positive number $a < c$ such that $\varphi\colon V \to U(x_0; a) \times U(x_0; a)$ is a diffeomorphism. Taking V and a small, we may assume that $\exp tX \in U(x_0; c)$ for all $X \in V$ and $|t| \leq 1$. To verify condition (1), let x and y be points of $U(x_0; a)$. Let $X = \varphi^{-1}(x, y)$, $X \in V$. Then the geodesic with the initial condition (x, X) joins x and y in $U(x_0; c)$. To verify (2), let $V_x = V \cap T_x(M)$. Since $\exp\colon V_x \to U(x_0; a)$ is a diffeomorphism, condition (2) is satisfied.

To complete the proof of Theorem 8.7, let $0 < \rho < a$. Let x and y be any points of $U(x_0; \rho)$. Let $x^i = x^i(t)$, $0 \leqq t \leqq 1$, be the equations of a geodesic from x to y in $U(x_0; c)$ (see Lemma 2). We shall show that this geodesic lies in $U(x_0; \rho)$. Set

$$F(t) = \Sigma_i \, (x^i(t))^2 \qquad \text{for } 0 \leqq t \leqq 1.$$

Assume that $F(t) \geqq \rho^2$ for some t (that is, $x^i(t)$ lies outside $U(x_0; \rho)$ for some t). Let t_0, $0 < t_0 < 1$, be the value for which $F(t)$ attains the maximum. Then

$$0 = \left(\frac{dF}{dt}\right)_{t=t_0} = 2\Sigma_i \, x^i(t_0) \left(\frac{dx^i}{dt}\right)_{t=t_0}.$$

This means that the geodesic $x^i(t)$ is tangent to the sphere $S(x_0; \rho_0)$, where $\rho_0^2 = F(t_0)$, at the point $x^i(t_0)$. By the choice of t_0, the geodesic $x^i(t)$ lies inside the sphere $S(x_0; \rho_0)$, contradicting Lemma 1. This proves (1). (2) follows from (2) of Lemma 2.

QED.

The existence of convex neighborhoods is due to J. H. C. Whitehead [1].

9. *Linear infinitesimal holonomy groups*

Let Γ be a linear connection on a manifold M. For each point u of $L(M)$, the holonomy group $\Psi(u)$, the local holonomy group $\Psi^*(u)$ and the infinitesimal holonomy group $\Psi'(u)$ are defined as in §10 of Chapter II. These groups can be realized as groups of linear transformations of $T_x(M)$, $x = \pi(u)$, denoted by $\Psi(x)$, $\Psi^*(x)$ and $\Psi'(x)$ respectively (cf. §4 of Chapter II).

THEOREM 9.1. *The Lie algebra $\mathfrak{g}(x)$ of the holonomy group $\Psi(x)$ is equal to the subspace of linear endomorphisms of $T_x(M)$ spanned by all elements of the form $(\tau R)(X, Y) = \tau^{-1} \circ R(\tau X, \tau Y) \circ \tau$, where $X, Y \in T_x(M)$ and τ is the parallel displacement along an arbitrary piecewise differentiable curve τ starting from x.*

Proof. This follows immediately from Theorem 8.1 of Chapter II and from the relationship between the curvature form Ω on $L(M)$ and the curvature tensor field R (cf. §5 of Chapter III). QED.

It is easy to reformulate Proposition 10.1, Theorems 10.2 and 10.3 of Chapter II in terms of $\Psi'(x)$ and $\Psi'^*(x)$. We shall therefore proceed to the determination of the Lie algebra of $\Psi''(x)$.

THEOREM 9.2. *The Lie algebra* $\mathfrak{g}'(x)$ *of the infinitesimal holonomy group* $\Psi''(x)$ *is spanned by all linear endomorphisms of* $T_x(M)$ *of the form* $(\nabla^k R)(X, Y; V_1; \ldots; V_k)$, *where* $X, Y, V_1, \ldots, V_k \in T_x(M)$ *and* $0 \leqq k < \infty$.

Proof. The proof is achieved by the following two lemmas.

LEMMA 1. *By a tensor field of type* A_k *(resp.* B_k*), we mean a tensor field of type* $(1, 1)$ *of the form* $\nabla_{V_k} \cdots \nabla_{V_1}(R(X, Y))$ *(resp.* $(\nabla^k R)(X, Y; V_1; \ldots; V_k)$*), where* $X, Y, V_1, \ldots, V_k$ *are arbitrary vector fields on* M. *Then every tensor field of type* A_k *(resp.* B_k*) is a linear combination (with differentiable functions as coefficients) of a finite number of tensor fields of type* B_j *(resp.* A_j*),* $0 \leqq j \leqq k$.

Proof of Lemma 1. The proof is by induction on k. The case $k = 0$ is trivial. Assume that $\nabla_{V_{k-1}} \cdots \nabla_{V_1}(R(X, Y))$ is a sum of terms like

$$f(\nabla^j R)(U, V; W_1; \ldots; W_j), \quad 0 \leqq j \leqq k - 1,$$

where f is a function. Then we have

$$\begin{aligned}
\nabla_{V_k}&(f(\nabla^j R)(U, V; W_1, \ldots; W_j)) \\
&= (V_k f) \cdot (\nabla^j R)(U, V; W_1; \ldots; W_j) \\
&\quad + (\nabla^{j+1} R)(U, V; W_1; \ldots; W_j; V_k) \\
&\quad + (\nabla^j R)(\nabla_{V_k} U, V; W_1; \ldots; W_j) \\
&\quad + (\nabla^j R)(U, \nabla_{V_k} V; W_1, \ldots; W_j) \\
&\quad + \Sigma_{i=1}^{j} (\nabla^j R)(U, V; W_1; \ldots; \nabla_{V_k} W_i; \ldots; W_j).
\end{aligned}$$

This shows that every tensor field of type A_k is a linear combination of tensor fields of type B_j, $0 \leqq j \leqq k$.

Assume now that every tensor field of type B_{k-1} is a linear combination of tensor fields of type A_j, $0 \leqq j \leqq k-1$. We have

$$
\begin{aligned}
(\nabla^k R)(X, Y; V_1; \ldots; V_k) = \nabla_{V_k}((\nabla^{k-1}R)(X, Y; V_1; \ldots; V_{k-1}))& \\
- (\nabla^{k-1}R)(\nabla_{V_k}X, Y; V_1; \ldots; V_{k-1})& \\
- (\nabla^{k-1}R)(X, \nabla_{V_k}Y; V_1; \ldots; V_{k-1})& \\
- \Sigma_{i=1}^{k-1} (\nabla^{k-1}R)(X, Y; V_1; \ldots; \nabla_{V_k}V_i; \ldots; V_{k-1})&.
\end{aligned}
$$

The first term on the right hand side is a linear combination of tensor fields of type A_j, $0 \leqq j \leqq k$. The remaining terms on the right hand side are linear combinations of tensor fields of type A_j, $0 \leqq j \leqq k-1$. This completes the proof of Lemma 1.

By definition, $\mathfrak{g}'(u)$ is spanned by the values at u of all $\mathfrak{gl}(n; \mathbf{R})$-valued functions of the form (I_k), $k = 0, 1, 2, \ldots$ (cf. §10 of Chapter II). Theorem 9.2 will follow from Lemma 1 and the following lemma.

LEMMA 2. *If $X, Y, V_1, \ldots, V_k$ are vector fields on M and if $X^*, Y^*, V_1^*, \ldots, V_k^*$ are their horizontal lifts to $L(M)$, then we have*

$$
\begin{aligned}
(\nabla_{V_k} \cdots \nabla_{V_1}(R(X, Y)))_x Z& \\
= u \circ (V_k^* \cdots V_1^*(2\Omega(X^*, Y^*)))_u \circ u^{-1}(Z)& \\
\textit{for } Z \in T_x(M)&.
\end{aligned}
$$

Proof of Lemma 2. This follows immediately from Proposition 1.3 of Chapter III; we take $R(X, Y)$ and $2\Omega(X^*, Y^*)$ as φ and f in Proposition 1.3 of Chapter III. QED.

By Theorem 10.8 of Chapter II and Theorem 9.2, the restricted holonomy group $\Psi^0(x)$ of a real analytic linear connection is completely determined by the values of all successive covariant differentials $\nabla^k R$, $k = 0, 1, 2, \ldots$, at the point x.

The results in this section were obtained by Nijenhuis [2].

CHAPTER IV

Riemannian Connections

1. *Riemannian metrics*

Let M be an n-dimensional paracompact manifold. We know (cf. Examples 5.5, 5.7 of Chapter I and Proposition 1.4 of Chapter III) that M admits a Riemannian metric and that there is a 1:1 correspondence between the set of Riemannian metrics on M and the set of reductions of the bundle $L(M)$ of linear frames to a bundle $O(M)$ of orthonormal frames. Every Riemannian metric g defines a positive definite inner product in each tangent space $T_x(M)$; we write $g_x(X, Y)$ or, simply, $g(X, Y)$ for $X, Y \in T_x(M)$ (cf. Example 3.1 of Chapter I).

Example 1.1. The *Euclidean metric* g on $\mathbf{R}^n$ with the natural coordinate system $x^1, \dots, x^n$ is defined by

$$g(\partial/\partial x^i, \partial/\partial x^j) = \delta_{ij} \quad \text{(Kronecker's symbol)}.$$

Example 1.2. Let $f: N \to M$ be an immersion of a manifold N into a Riemannian manifold M with metric g. The *induced Riemannian metric* h on N is defined by $h(X, Y) = g(f_*X, f_*Y)$, $X, Y \in T_x(N)$.

Example 1.3. A homogeneous space G/H, where G is a Lie group and H is a compact subgroup, admits an *invariant metric*. Let $\tilde{H}$ be the linear isotropy group at the origin o (i.e., the point represented by the coset H) of G/H; $\tilde{H}$ is a group of linear transformations of the tangent space $T_o(G/H)$, each induced by an element of H which leaves the point o fixed. Since H is compact, so is $\tilde{H}$ and there is a positive definite inner product, say g_o, in $T_o(G/H)$ which is invariant by $\tilde{H}$. For each $x \in G/H$, we take an element $a \in G$ such that $a(o) = x$ and define an inner product g_x in $T_x(G/H)$ by $g_x(X, Y) = g_o(a^{-1}X, a^{-1}Y)$, $X, Y \in T_x(G/H)$. It is easy to verify that g_x is independent of the choice of $a \in G$ such that $a(o) = x$ and that the Riemannian metric g thus obtained is invariant by G. The homogeneous space G/H provided with an invariant Riemannian metric is called a *Riemannian homogeneous space*.

Example 1.4. Every compact Lie group G admits a Riemannian metric which is invariant by both right and left translations. In fact, the group $G \times G$ acts transitively on G by $(a, b) \cdot x = axb^{-1}$, for $(a, b) \in G \times G$ and $x \in G$. The isotropy subgroup of $G \times G$ at the identity e of G is the diagonal $D = \{(a, a); a \in G\}$, so that $G = (G \times G)/D$. By Example 1.3, G admits a Riemannian metric invariant by $G \times G$, thus proving our assertion. If G is compact and semisimple, then G admits the following *canonical invariant Riemannian metric*. In the Lie algebra $\mathfrak{g}$, identified with the tangent space $T_e(G)$, we have the Killing-Cartan form $\varphi(X, Y) = \text{trace}\,(\text{ad}\, X \circ \text{ad}\, Y)$, where $X, Y \in \mathfrak{g} = T_e(G)$. The form φ is bilinear, symmetric and invariant by ad G. When G is compact and semisimple, φ is negative definite. We define a positive definite inner product g_e in $T_e(G)$ by $g_e(X, Y) = -\varphi(X, Y)$. Since φ is invariant by ad G, g_e is invariant by the diagonal D. By Example 1.3, we obtain a Riemannian metric on G invariant by $G \times G$. We discuss this metric in detail in Volume II.

By a Riemannian metric, we shall always mean a positive definite symmetric covariant tensor field of degree 2. By an *indefinite Riemannian metric*, we shall mean a symmetric covariant tensor field g of degree 2 which is nondegenerate at each $x \in M$, that is, $g(X, Y) = 0$ for all $Y \in T_x(M)$ implies $X = 0$.

Example 1.5. An indefinite Riemannian metric on $\mathbf{R}^n$ with the coordinate system $x^1, \ldots, x^n$ can be given by

$$\Sigma_{i=1}^{p} (dx^i)^2 - \Sigma_{j=p+1}^{n} (dx^j)^2,$$

where $0 \leqq p \leqq n - 1$. Another example of an indefinite Riemannian metric is the *canonical metric* on a noncompact, semisimple Lie group G defined as follows. It is known that for such a group the Killing-Cartan form φ is indefinite and nondegenerate. The construction in Example 1.4 gives an indefinite Riemannian metric on G invariant by both right and left translations.

Let M be a manifold with a Riemannian metric or an indefinite Riemannian metric g. For each x, the inner product g_x defines a linear isomorphism ψ of $T_x(M)$ onto its dual $T_x^*(M)$ (space of covectors at x) as follows: To each $X \in T_x(M)$, we assign the covector $\alpha \in T_x^*(M)$ defined by

$$\langle Y, \alpha \rangle = g_x(X, Y) \qquad \text{for all } Y \in T_x(M).$$

The inner product g_x in $T_x(M)$ defines an inner product, denoted also by g_x, in the dual space $T_x^*(M)$ by means of the isomorphism ψ:

$$g_x(\alpha, \beta) = g_x(\psi^{-1}(\alpha), \psi^{-1}(\beta)) \qquad \text{for } \alpha, \beta \in T_x^*(M).$$

Let $x^1, \ldots, x^n$ be a local coordinate system in M. The components g_{ij} of g with respect to $x^1, \ldots, x^n$ are given by

$$g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), \qquad i, j = 1, \ldots, n.$$

The *contravariant components* g^{ij} of g are defined by

$$g^{ij} = g(dx^i, dx^j), \qquad i, j = 1, \ldots, n.$$

We have then

$$\Sigma_j\, g_{ij} g^{jk} = \delta_i^k.$$

In fact, define ψ_{ij} by $\psi(\partial/\partial x^i) = \Sigma_j \psi_{ij}\, dx^j$. Then we have

$$g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) = \langle \partial/\partial x^j, \psi(\partial/\partial x^i)\rangle = \psi_{ij}.$$

On the other hand, we have

$$\delta_i^k = \langle \partial/\partial x^i, dx^k \rangle = g(dx^k, \psi(\partial/\partial x^i)) = g(dx^k, \Sigma_j\, \psi_{ij}\, dx^j) = \Sigma_j\, \psi_{ij} g^{jk},$$

thus proving our assertion.

If ξ^i are the components of a vector or a vector field X with respect to $x^1, \ldots, x^n$, that is, $X = \Sigma_i\, \xi^i(\partial/\partial x^i)$, then the components ξ_i of the corresponding covector or the corresponding 1-form $\alpha = \psi(X)$ are related to ξ^i by

$$\xi^i = \Sigma_j\, g^{ij} \xi_j, \quad \xi_i = \Sigma_j\, g_{ij} \xi^j.$$

The inner product g in $T_x(M)$ and in $T_x^*(M)$ can be extended to an inner product, denoted also by g, in the tensor space $\mathbf{T}_s^r(x)$ at x for each type (r, s). If K and L are tensors at x of type (r, s) with components $K_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ and $L_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ (with respect to $x^1, \ldots, x^n$),

then

$$g(K, L) = \Sigma\, g_{i_1k_1}\dots g_{i_rk_r}g^{j_1l_1}\cdots g^{j_sl_s}K^{i_1\dots i_r}_{j_1\dots j_s}L^{k_1\dots k_r}_{l_1\dots l_s}.$$

The isomorphism $\psi\colon T_x(M) \to T^*_x(M)$ can be extended to tensors. Given a tensor $K \in \mathbf{T}^r_s(x)$ with components $K^{i_1\dots i_r}_{j_1\dots j_s}$, we obtain a tensor $K' \in \mathbf{T}^{r-1}_{s+1}(x)$ with components

$$K'^{i_1\dots i_{r-1}}_{j_1\dots j_{s+1}} = \Sigma_k\, g_{j_1k}K^{ki_1\dots i_{r-1}}_{j_2\dots j_{s+1}},$$

or $K'' \in \mathbf{T}^{r+1}_{s-1}(x)$ with components

$$K''^{i_1\dots i_{r+1}}_{j_1\dots j_{s-1}} = \Sigma_k\, g^{i_1k}K^{i_2\dots i_{r+1}}_{kj_1\dots j_{s-1}}.$$

Example 1.6. Let A and B be skew-symmetric endomorphisms of the tangent space $T_x(M)$, that is, tensors at x of type (1, 1) such that

$$g(AX, Y) = -g(AY, X) \quad\text{and}\quad g(BX, Y) = -g(BY, X) \qquad \text{for } X, Y \in T_x(M).$$

Then the inner product $g(A, B)$ is equal to $-\text{trace}\,(AB)$. In fact, take a local coordinate system $x^1, \dots, x^n$ such that $g_{ij} = \delta_{ij}$ at x and let a^i_j and b^i_j be the components of A and B respectively. Then

$$g(A, B) = \Sigma\, g_{ik}g^{jl}a^i_jb^k_l = \Sigma\, a^i_jb^i_j = -\Sigma\, a^i_jb^j_i = -\text{trace}\,(AB),$$

since B is skew-symmetric, i.e., $b^i_j = -b^j_i$.

On a Riemannian manifold M, the *arc length* of a differentiable curve $\tau = x_t$, $a \leqq t \leqq b$, of class C^1 is defined by

$$L = \int_a^b g(\dot{x}_t, \dot{x}_t)^{\frac{1}{2}}\, dt.$$

In terms of a local coordinate system $x^1, \dots, x^n$, L is given by

$$L = \int_a^b \left(\Sigma_{i,j}\, g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)^{\frac{1}{2}} dt.$$

This definition can be generalized to a piecewise differentiable curve of class C^1 in an obvious manner.

Given a Riemannian metric g on a connected manifold M, we define the *distance function* $d(x, y)$ on M as follows. The distance $d(x, y)$ between two points x and y is, by definition, the infinimum of the lengths of all piecewise differentiable curves of class C^1

joining x and y. Then we have

$$d(x, y) \geqq 0, \quad d(x, y) = d(y, x), \quad d(x, y) + d(y, z) \geqq d(x, z).$$

We shall see later (in §3) that $d(x, y) = 0$ only when $x = y$ and that the topology defined by the distance function (metric) d is the same as the manifold topology of M.

2. *Riemannian connections*

Although the results in this section are valid for manifolds with indefinite Riemannian metrics, we shall consider (positive definite) Riemannian metrics only for the sake of simplicity.

Let M be an n-dimensional Riemannian manifold with metric g and $O(M)$ the bundle of orthonormal frames over M. Every connection in $O(M)$ determines a connection in the bundle $L(M)$ of linear frames, that is, a linear connection of M by virtue of Proposition 6.1 of Chapter II. A linear connection of M is called a *metric connection* if it is thus determined by a connection in $O(M)$.

PROPOSITION 2.1. *A linear connection Γ of a Riemannian manifold M with metric g is a metric connection if and only if g is parallel with respect to Γ.*

Proof. Since g is a fibre metric (cf. §1 of Chapter III) in the tangent bundle $T(M)$, our proposition follows immediately from Proposition 1.5 of Chapter III. QED.

Among all possible metric connections, the most important is the *Riemannian connection* (sometimes called the *Levi-Civita connection*) which is given by the following theorem.

THEOREM 2.2. *Every Riemannian manifold admits a unique metric connection with vanishing torsion.*

We shall present here two proofs, one using the bundle $O(M)$ and the other using the formalism of covariant differentiation.

Proof (A). *Uniqueness.* Let θ be the canonical form of $L(M)$ restricted to $O(M)$. Let ω be the connection form on $O(M)$ definining a metric connection of M. With respect to the basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$ and the basis E_i^j, $i < j$, $i, j = 1, \ldots, n$, of the Lie algebra $\mathfrak{o}(n)$, we represent θ and ω by n forms θ^i, $i = 1, \ldots, n$, and a skew-symmetric matrix of differential forms ω^i respectively.

The proof of the following lemma is similar to that of Proposition 2.6 of Chapter III and hence is left to the reader.

LEMMA. *The n forms* θ^i, $i = 1, \ldots, n$, *and the* $\frac{1}{2}n(n-1)$ *forms* ω^j_k, $1 \leqq j < k \leqq n$, *define an absolute parallelism on* $O(M)$.

Let φ be the connection form defining another metric connection of M. Then $\varphi - \omega$ can be expressed in terms of θ^i and ω^j_k by the lemma. Since $\varphi - \omega$ annihilates the vertical vectors, we have

$$\varphi^i_j - \omega^i_j = \Sigma_k F^i_{jk}\theta^k,$$

where the F^i_{jk}'s are functions on $O(M)$. Assume that the connections defined by ω and φ have no torsion. Then, from the first structure equation of Theorem 2.4 of Chapter III, we obtain

$$0 = \Sigma_j (\varphi^i_j - \omega^i_j) \wedge \theta^j = \Sigma_{j,k} F^i_{jk}\, \theta^k \wedge \theta^j.$$

This implies that $F^i_{jk} = F^i_{kj}$. On the other hand, $F^i_{jk} = -F^j_{ik}$ since (ω^i_j) and (φ^i_j) are skew-symmetric. It follows that $F^i_{jk} = 0$, proving the uniqueness.

Existence. Let φ be an arbitrary metric connection form on $O(M)$ and Θ its torsion form on $O(M)$. We write

$$\Theta^i = \tfrac{1}{2}\Sigma_{j,k}\, \tilde{T}^i_{jk}\theta^j \wedge \theta^k, \quad \tilde{T}^i_{jk} = -\tilde{T}^i_{kj}$$

and set

$$\tau^i_j = \Sigma_k\, \tfrac{1}{2}(\tilde{T}^i_{jk} + \tilde{T}^j_{ki} + \tilde{T}^k_{ji})\, \theta^k$$

and

$$\omega^i_j = \varphi^i_j + \tau^i_j.$$

We shall show that $\omega = (\omega^i_j)$ defines the desired connection. Since both $(\tilde{T}^i_{jk} + \tilde{T}^j_{ki})$ and $\tilde{T}^k_{ji}$ are skew-symmetric in i and j, so is τ^i_j. Hence ω is $\mathfrak{o}(n)$-valued. Since θ annihilates the vertical vectors, so does $\tau = (\tau^i_j)$. It is easy to show that $R^*_a\tau = \mathrm{ad}\,(a^{-1})(\tau)$ for every $a \in O(n)$. Hence, ω is a connection form. Finally, we verify that the metric connection defined by ω has zero torsion. Since $(\tilde{T}^k_{ji} + \tilde{T}^j_{ki})$ is symmetric in j and k, we have

$$\Sigma_j\, \tau^i_j \wedge \theta^j = -\Theta^i,$$

and hence

$$d\theta^i = -\Sigma_j\, \varphi^i_j \wedge \theta^j + \Theta^i = -\Sigma_j\, \omega^i_j \wedge \theta^j,$$

proving our assertion.

Proof (B). *Existence.* Given vector fields X and Y on M, we define $\nabla_X Y$ by the following equation:

$$2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y]),$$

which should hold for every vector field Z on M. It is a straightforward verification that the mapping $(X, Y) \to \nabla_X Y$, satisfies the four conditions of Proposition 2.8 of Chapter III and hence determines a linear connection Γ of M by Proposition 7.5 of Chapter III. The fact that Γ has no torsion follows from the above definition of $\nabla_X Y$ and the formula $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ given in Theorem 5.1 of Chapter III. To show that Γ is a metric connection, that is, $\nabla g = 0$ (cf. Proposition 2.1), it is sufficient to prove

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \qquad \text{for all vector fields } X, Y \text{ and } Z,$$

by virtue of Proposition 2.10 of Chapter III. But this follows immediately from the definition of $\nabla_X Y$.

Uniqueness. It is a straightforward verification that if $\nabla_X Y$ satisfies $\nabla_X g = 0$ and $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, then it satisfies the equation which defined $\nabla_X Y$. QED.

In the course of the proof, we obtained the following

PROPOSITION 2.3. *With respect to the Riemannian connection, we have*

$$2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, Y])$$

for all vector fields X, Y and Z of M.

COROLLARY 2.4. *In terms of a local coordinate system $x^1, \dots, x^n$, the components Γ^i_{jk} of the Riemannian connection are given by*

$$\Sigma_l g_{lk} \, \Gamma^l_{ji} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k}\right).$$

Proof. Let $X = \partial/\partial x^j$, $Y = \partial/\partial x^i$ and $Z = \partial/\partial x^k$ in Proposition 2.3 and use Proposition 7.4 of Chapter III. QED.

Let M and M' be Riemannian manifolds with Riemannian metrics g and g' respectively. A mapping $f: M \to M'$ is called *isometric* at a point x of M if $g(X, Y) = g'(f_*X, f_*Y)$ for all $X, Y \in T_x(M)$. In this case, f_* is injective at x, because $f_*X = 0$ implies that $g(X, Y) = 0$ for all Y and hence $X = 0$. A mapping f which is isometric at every point of M is thus an immersion, which we call an *isometric immersion*. If, moreover, f is 1:1, then it is called an *isometric imbedding* of M into M'. If f maps M 1:1 *onto* M', then f is called an *isometry* of M onto M'.

PROPOSITION 2.5. *If f is an isometry of a Riemannian manifold M onto another Riemannian manifold M', then the differential of f commutes with the parallel displacement. More precisely, if τ is a curve from x to y in M, then the following diagram is commutative:*

$$\begin{array}{ccc} T_x(M) & \xrightarrow{\tau} & T_y(M) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ T_{x'}(M') & \xrightarrow{\tau'} & T_{y'}(M'), \end{array}$$

where $x' = f(x)$, $y' = f(y)$ and $\tau' = f(\tau)$.

Proof. This is a consequence of the uniqueness of the Riemannian connection in Theorem 2.2. Being a diffeomorphism between M and M', f defines a 1:1 correspondence between the set of vector fields on M and the set of vector fields on M'. From the Riemannian connection Γ' on M', we obtain a linear connection Γ on M by $\nabla_X Y = f^{-1}(\nabla_{fX}(fY))$, where X and Y are vector fields on M. It is easy to verify that Γ has no torsion and is metric with respect to g. Thus, Γ is the Riemannian connection of M. This means that $f(\nabla_X Y) = \nabla_{fX}(fY)$ with respect to the Riemannian connections of M and M'. This implies immediately our proposition. QED.

PROPOSITION 2.6. *If f is an isometric immersion of a Riemannian manifold M into another Riemannian manifold M' and if $f(M)$ is open in M', then the differential of f commutes with the parallel displacement.*

Proof. Since $f(M)$ is open in M', $\dim M = \dim M'$. Since f is an immersion, every point x of M has an open neighborhood U such that $f(U)$ is open in M' and $f: U \to f(U)$ is a diffeomorphism. Thus, f is an isometry of U onto $f(U)$. By Proposition 2.5, the differential of f commutes with the parallel displacement along any curve in U. Given an arbitrary curve τ from x to y in M, we can find a finite number of open neighborhoods in M with the above property which cover τ. It follows that the differential of f commutes with the parallel displacement along τ. QED.

Remark. It follows immediately that, under the assumption of Proposition 2.6, every geodesic of M is mapped by f into a geodesic of M'.

Example 2.1. Let M be a Riemannian manifold with metric g. Let M^* be a covering manifold of M with projection p. We can introduce a Riemannian metric g^* on M^* in such a way that $p: M^* \to M$ is an isometric immersion. Every geodesic of M^* projects on a geodesic of M. Conversely, given a geodesic τ from x to y in M and a point x^* of M^* with $p(x^*) = x$, there is a unique curve τ^* in M^* starting from x^* such that $p(\tau^*) = \tau$. Since p is a local isometry, τ^* is a geodesic of M^*. A similar argument, together with Proposition 2.6, shows that if $p(x^*) = x$, then the restricted linear holonomy group of M^* with reference point x^* is isomorphic by p to the restricted linear holonomy group of M with reference point x.

Proposition 2.5 and 2.6 were stated with respect to Riemannian connections which are special linear connections. Similar statements hold with respect to the corresponding affine connections. The statement concerning linear holonomy groups in Example 2.1 holds also for affine holonomy groups.

3. *Normal coordinates and convex neighborhoods*

Let M be a Riemannian manifold with metric g. The length of a vector X, i.e., $g(X, X)^{\frac{1}{2}}$, will be denoted by $\|X\|$.

Let $\tau = x_t$ be a geodesic in M. Since the tangent vectors $\dot{x}_t$ are parallel along τ and since the parallel displacement is isometric, the length of $\dot{x}_t$ is constant along τ. If $\|\dot{x}_t\| = 1$, then t is called the *canonical parameter* of the geodesic τ.

By a *normal coordinate system at x of a Riemannian manifold M*, we always mean a normal coordinate system $x^1, \dots, x^n$ at x such that $\partial/\partial x^1, \dots, \partial/\partial x^n$ form an orthonormal frame at x. However, $\partial/\partial x^1, \dots, \partial/\partial x^n$ may not be orthonormal at other points.

Let U be a normal coordinate neighborhood of x with a normal coordinate system $x^1, \dots, x^n$ at x. We define a cross section σ of

$O(M)$ over U as follows: Let u be the orthonormal frame at x given by $(\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x$. By the parallel displacement of u along the geodesics through x, we attach an orthonormal frame to every point of U. For the study of Riemannian manifolds, the cross section $\sigma: U \to O(M)$ thus defined is more useful than the cross section $U \to L(M)$ given by $\partial/\partial x^1, \ldots, \partial/\partial x^n$. Let $\theta = (\theta^i)$ and $\omega = (\omega^j_k)$ be the canonical form and the Riemannian connection form on $O(M)$ respectively. We set

$$\bar{\theta} = \sigma^*\theta = (\bar{\theta}^i) \quad \text{and} \quad \bar{\omega} = \sigma^*\omega = (\bar{\omega}^j_k),$$

where $\bar{\theta}^i$ and $\bar{\omega}^j_k$ are 1-forms on U. To compute these forms explicitly, we introduce the polar coordinate system $(p^1, \ldots, p^n; t)$ by

$$x^i = p^i t, \quad i = 1, \ldots, n; \quad \Sigma_i\, (p^i)^2 = 1.$$

Then, $\bar{\theta}^i$ and $\bar{\omega}^j_k$ are linear combinations of $dp^1, \ldots, dp^n$ and dt with functions of $p^1, \ldots, p^n, t$ as coefficients.

Proposition 3.1. (1) *$\bar{\theta}^i = p^i\, dt + \varphi^i$, where φ^i, $i = 1, \ldots, n$, do not involve dt;*

(2) $\bar{\omega}^j_k$ do not involve dt;

(3) $\varphi^i = 0$ and $\bar{\omega}^j_k = 0$ at $t = 0$ (i.e., at the origin x);

(4)
$$d\varphi^i = -(dp^i + \Sigma_j\, \bar{\omega}^i_j p^j) \wedge dt + \cdots,$$
$$d\bar{\omega}^i_j = -\Sigma_{k,l}\, \bar{R}^i_{jkl} p^k \varphi^l \wedge dt + \cdots,$$

where the dots $\cdots$ indicate terms not involving dt and $\bar{R}^i_{jkl}$ are the components of the curvature tensor field with respect to the frame field σ.

Proof. (1) For a fixed direction $(p^1, \ldots, p^n)$, let $\tau = x_t$ be the geodesic defined by $x^i = p^i t$, $i = 1, \ldots, n$. Set $u_t = \sigma(x_t)$. To prove that $\bar{\theta}^i - p^i\, dt$ do not involve dt, it is sufficient to prove that $\bar{\theta}^i(\dot{x}_t) = p^i$. From the definition of the canonical form θ, we have

$$\theta(\dot{u}_t) = \bar{\theta}(\dot{x}_t) = u_t^{-1}(\dot{x}_t).$$

Since both u_t and $\dot{x}_t$ are parallel along τ, $\bar{\theta}(\dot{x}_t)$ is independent of t. On the other hand, we have $\bar{\theta}^i(\dot{x}_0) = p^i$ and hence $\bar{\theta}^i(\dot{x}_t) = p^i$ for all t.

(2) Since u_t is horizontal by the construction of σ, we have

$$\bar{\omega}^j_k(\dot{x}_t) = \omega^j_k(\dot{u}_t) = 0.$$

This means that $\bar{\omega}^j_k$ do not involve dt.

(3) Given any unit vector X at x (i.e., the point where $t = 0$), let $\tau = x_t$ be the geodesic with the initial condition (x, X) so that $X = \dot{x}_0$. By (1) and (2), we have $\varphi^i(\dot{x}_0) = 0$ and $\bar{\omega}^j_k(\dot{x}_0) = 0$.

(4) From the structure equations, we obtain

$$d(p^i\, dt + \varphi^i) = -\Sigma_j\, \bar{\omega}^i_j \wedge (p^j\, dt + \varphi^j)$$

$$d\bar{\omega}^i_j = -\Sigma_k\, \bar{\omega}^i_k \wedge \bar{\omega}^k_j + \bar{\Omega}^i_j,$$

where

$$\bar{\Omega}^i_j = \Sigma_{k,l}\, \tfrac{1}{2}\bar{R}^i_{jkl}\bar{\theta}^k \wedge \bar{\theta}^l = \Sigma_{k,l}\, \tfrac{1}{2}R^i_{jkl}(p^k\, dt + \varphi^k) \wedge (p^l\, dt + \varphi^l)$$

(cf. §7 of Chapter III)

and hence (4). QED.

In terms of dt and φ^i, we can express the metric tensor g as follows (cf. the classical expression $ds^2 = \Sigma\, g_{ij}\, dx^i\, dx^j$ for g as explained in Example 3.1 of Chapter I).

Proposition 3.2. *The metric tensor g can be expressed by*

$$ds^2 = (dt)^2 + \Sigma_i\, (\varphi^i)^2.$$

Proof. Since $\bar{\theta}(X) = (\sigma(y))^{-1}(X)$ for every $X \in T_y(M)$, $y \in U$, and since $\sigma(y)$ is an isometric mapping of $\mathbf{R}^n$ onto $T_y(M)$, we have

$$g(X, Y) = \Sigma_i\, \bar{\theta}^i(X)\bar{\theta}^i(Y) \qquad \text{for } X, Y \in T_y(M) \quad \text{and} \quad y \in U.$$

In other words,

$$ds^2 = \Sigma_i\, (\bar{\theta}^i)^2.$$

By Proposition 3.1, we have

$$ds^2 = (dt)^2 + \Sigma_i\, (\varphi^i)^2 + 2\, \Sigma_i\, p^i\varphi^i\, dt.$$

Since $\varphi^i = 0$ at $t = 0$ by Proposition 3.1, we shall prove that $\Sigma_i\, p^i\varphi^i = 0$ by showing that $\Sigma_i\, p^i\varphi^i$ is independent of t. Since $\Sigma_i\, p^i\varphi^i$ does not involve dt by Proposition 3.1, it is sufficient to show that $d(\Sigma_i\, p^i\varphi^i)$ does not involve dt. We have, by Proposition 3.1,

$$d(\Sigma_i\, p^i\varphi^i) = -\Sigma_i\, p^i(dp^i + \Sigma_j\, \bar{\omega}^i_j p^j) \wedge dt + \cdots,$$

where the dots $\cdots$ indicate terms not involving dt.

From $\Sigma_i\, (p^i)^2 = 1$, we obtain

$$0 = d(\Sigma_i\, (p^i)^2) = 2\, \Sigma_i\, p^i\, dp^i.$$

On the other hand, $$\Sigma_{i,j}\, p^i \bar{\omega}^i_j p^j = 0,$$
because $(\bar{\omega}^i_j)$ is skew-symmetric. This proves that $d(\Sigma_i\, p^i \varphi^i)$ does not involve dt. QED.

From Proposition 3.2, we obtain

PROPOSITION 3.3. *Let $x^1, \ldots, x^n$ be a normal coordinate system at x. Then every geodesic $\tau = x_t$, $x^i = a^i t$ $(i = 1, \ldots, n)$, through x is perpendicular to the sphere $S(x; r)$ defined by $\Sigma_i\, (x^i)^2 = r^2$.*

For each small positive number r, we set

$N(x; r)$ = the neighborhood of 0 in $T_x(M)$ defined by $\|X\| < r$,

$U(x; r)$ = the neighborhood of x in M defined by $\Sigma_i\, (x^i)^2 < r^2$.

By the very definition of a normal coordinate system, the exponential map is a diffeomorphism of $N(x; r)$ onto $U(x; r)$.

PROPOSITION 3.4. *Let r be a positive number such that*
$$\exp\colon N(x; r) \to U(x; r)$$
is a diffeomorphism. Then we have

(1) *Every point y in $U(x; r)$ can be joined to x (origin of the coordinate system) by a geodesic lying in $U(x; r)$ and such a geodesic is unique;*

(2) *The length of the geodesic in* (1) *is equal to the distance $d(x, y)$;*

(3) *$U(x; r)$ is the set of points $y \in M$ such that $d(x, y) < r$.*

Proof. Every line in $N(x; r)$ through the origin 0 is mapped into a geodesic in $U(x; r)$ through x by the exponential map and *vice versa*. Now, (1) follows from the fact that $\exp\colon N(x; r) \to U(x; r)$ is a diffeomorphism. To prove (2), let $(a^1, \ldots, a^n; b)$ be the coordinates of y with respect to the polar coordinate system $(p^1, \ldots, p^n; t)$ introduced at the beginning of the section. Let $\tau = x_s$, $\alpha \leqq s \leqq \beta$, be any piecewise differential curve from x to y. We shall show that the length of τ is greater than or equal to b. Let
$$p^1 = p^1(s), \ldots, p^n = p^n(s), \quad t = t(s), \quad \alpha \leqq s \leqq \beta,$$
be the equation of the curve τ. If we denote by $L(\tau)$ the length of τ, then Proposition 3.2 implies the following inequalities:
$$L(\tau) \geqq \int_\alpha^\beta \left|\frac{dt}{ds}\right| ds \geqq \int_0^b dt = b.$$

We shall now prove (3). If y is in $U(x; r)$, then, clearly, $d(x, y) < r$. Conversely, let $d(x, y) < r$ and let τ be a curve from x to y such that $L(\tau) < r$. Suppose τ does not lie in $U(x; r)$. Let y' be the first point on τ which belongs to the closure of $U(x; r)$ but not to $U(x; r)$. Then, $d(x, y') = r$ by (1) and (2). The length of τ from x to y' is at least r. Hence, $L(\tau) \geqq r$, which is a contradiction. Thus τ lies entirely in $U(x; r)$ and hence y is in $U(x; r)$. QED.

PROPOSITION 3.5. *$d(x, y)$ is a distance function (i.e., metric) on M and defines the same topology as the manifold topology of M.*

Proof. As we remarked earlier (cf. the end of §1), we have

$$d(x, y) \geqq 0, \quad d(x, y) = d(y, x), \quad d(x, y) + d(y, z) \geqq d(x, z).$$

From Proposition 3.4, it follows that if $x \neq y$, then $d(x, y) > 0$. Thus d is a metric. The second assertion follows from (3) of Proposition 3.4. QED.

A geodesic joining two points x and y of a Riemannian manifold M is called *minimizing* if its length is equal to the distance $d(x, y)$. We now proceed to prove the existence of a *convex neighborhood* around each point of a Riemannian manifold in the following form.

THEOREM 3.6. *Let $x^1, \ldots, x^n$ be a normal coordinate system at x of a Riemannian manifold M. There exists a positive number a such that, if $0 < \rho < a$, then*

(1) Any two points of $U(x; \rho)$ can be joined by a unique minimizing geodesic; and it is the unique geodesic joining the two points and lying in $U(x; \rho)$;

(2) In $U(x; \rho)$, the square of the distance $d(y, z)$ is a differentiable function of y and z.

Proof. (1) Let a be the positive number given in Theorem 8.7 of Chapter III and let $0 < \rho < a$. If y and z are points of $U(x; \rho)$, they can be joined by a geodesic τ lying in $U(x; \rho)$ by the same theorem. Since $U(x; \rho)$ is contained in a normal coordinate neighborhood of y (cf. Theorem 8.7 of Chapter III), we see from Proposition 3.4 that τ is a unique geodesic joining y and z and lying in $U(x; \rho)$ and that the length of τ is equal to the distance, that is, τ is minimizing. It is clear that τ is the unique minimizing geodesic joining y and z in M.

(2) Identifying every point y of M with the zero vector at y, we consider y as a point of $T(M)$. For each y in $U(x; \rho)$, let N_y be the neighborhood of y in $T_y(M)$ such that $\exp: N_y \to U(x; \rho)$ is a diffeomorphism (cf. (2) of Theorem 8.7 of Chapter III). Set $V = \bigcup_{y \in U(x;\rho)} N_y$. Then the mapping $V \to U(x; \rho) \times U(x; \rho)$ which sends $Y \in N_y$ into $(y, \exp Y)$ is a diffeomorphism (cf. Proposition 8.1 of Chapter III). If $z = \exp Y$, then $d(y, z) = \|Y\|$. In other words, $\|Y\|$ is the function on V which corresponds to the distance function $d(y, z)$ under the diffeomorphism $V \to U(x; \rho) \times U(x; \rho)$. Since $\|Y\|^2$ is a differentiable function on V, $d(y, z)^2$ is a differentiable function on $U(x; \rho) \times U(x; \rho)$. QED.

As an application of Theorem 3.6, we obtain the following

THEOREM 3.7. *Let M be a paracompact differentiable manifold. Then every open covering $\{U_\alpha\}$ of M has an open refinement $\{V_i\}$ such that*

(1) *each V_i has compact closure;*

(2) *$\{V_i\}$ is locally finite in the sense that every point of M has a neighborhood which meets only a finite number of V_i's;*

(3) *any nonempty finite intersection of V_i's is diffeomorphic with an open cell of $\mathbf{R}^n$.*

Proof. By taking an open refinement if necessary, we may assume that $\{U_\alpha\}$ is locally finite and that each U_α has compact closure. Let $\{U'_\alpha\}$ be an open refinement of $\{U_\alpha\}$ (with the same index set) such that $\bar{U}'_\alpha \subset U_\alpha$ for all α (cf. Appendix 3). Take any Riemannian metric on M. For each $x \in M$, let W_x be a convex neighborhood of x (in the sense of Theorem 3.6) which is contained in some U'_α. For each α, let

$$\mathfrak{W}_\alpha = \{W_x;\ W_x \cap \bar{U}'_\alpha \text{ is non-empty}\}.$$

Since $\bar{U}'_\alpha$ is compact, there is a finite subfamily $\mathfrak{V}_\alpha$ of $\mathfrak{W}_\alpha$ which covers $\bar{U}'_\alpha$. Then the family $\mathfrak{V} = \bigcup_\alpha \mathfrak{V}_\alpha$ is a desired open refinement of $\{U_\alpha\}$. In fact, it is clear from the construction that $\mathfrak{V}$ satisfies (1) and (2). If $V_1, \dots, V_k$ are members of $\mathfrak{V}$ and if x and y are points of the intersection $V_1 \cap \cdots \cap V_k$, then there is a unique minimizing geodesic joining x and y in M. Since the geodesic lies in each V_i, $i = 1, \dots, k$, it lies in the intersection $V_1 \cap \cdots \cap V_k$. It follows that the intersection is diffeomorphic with an open cell of $\mathbf{R}^n$. QED.

Remark. A covering $\{V_i\}$ satisfying (1), (2) and (3) is called a *simple covering*. Its usefulness lies in the fact that the Čech cohomology of M can be computed by means of a simple covering of M (cf. Weil [1]).

In any metric space M, a *segment* is defined to be a continuous image $x(t)$ of a closed interval $a \leqq t \leqq b$ such that

$$d(x(t_1), x(t_2)) + d(x(t_2), x(t_3)) = d(x(t_1), x(t_3)) \quad \text{for } a \leqq t_1 \leqq t_2 \leqq t_3 \leqq b,$$

where d is the distance function. As an application of Theorem 3.6, we have

PROPOSITION 3.8. *Let M be a Riemannian manifold with metric g and d the distance function defined by g. Then every segment is a geodesic (as a point set).*

The parametrization of a segment may not be affine.

Proof. Let $x(t)$, $a \leqq t \leqq b$, be a segment in M. We first show that $x(t)$ is a geodesic for $a \leqq t \leqq a + \varepsilon$ for some positive ε. Let U be a convex neighborhood of $x(a)$ in the sense of Theorem 3.6. There exists $\varepsilon > 0$ such that $x(t) \in U$ for $a \leqq t \leqq a + \varepsilon$. Let τ be the minimizing geodesic from $x(a)$ to $x(a + \varepsilon)$. We shall show that τ and $x(t)$, $a \leqq t \leqq a + \varepsilon$, coincide as a point set. Suppose there is a number c, $a < c < a + \varepsilon$, such that $x(c)$ is not on τ. Then

$$d(x(a), x(a + \varepsilon)) < d(x(a), x(c)) + d(x(c), x(a + \varepsilon)),$$

contradicting the fact that $x(t)$, $a \leqq t \leqq a + \varepsilon$, is a segment. This shows that $x(t)$ is a geodesic for $a \leqq t \leqq a + \varepsilon$. By continuing this argument, we see that $x(t)$ is a geodesic for $a \leqq t \leqq b$. QED.

Remark. If x_t is a continuous curve such that $d(x_{t_1}, x_{t_2}) = |t_1 - t_2|$ for all t_1 and t_2, then x_t is a geodesic with arc length t as parameter.

COROLLARY 3.9. *Let $\tau = x_t$, $a \leqq t \leqq b$, be a piecewise differentiable curve of class C^1 from x to y such that its length $L(\tau)$ is equal to $d(x, y)$. Then τ is a geodesic as a point set. If, moreover, $\|\dot{x}_t\|$ is constant along τ, then τ is a geodesic including the parametrization.*

Proof. It suffices to show that τ is a segment. Let $a \leqq t_1 \leqq t_2 \leqq t_3 \leqq b$. Denoting the points x_{t_i} by x_i, $i = 1, 2, 3$, and the arcs into which τ is divided by these points by τ_1, τ_2, τ_3 and τ_4 respectively, we have

$$d(x, x_1) \leqq L(\tau_1), \quad d(x_1, x_2) \leqq L(\tau_2), \quad d(x_2, x_3) \leqq L(\tau_3),$$
$$d(x_3, y) \leqq L(\tau_4).$$

If we did not have the equality everywhere, we would have

$$d(x, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, y)$$
$$< L(\tau_1) + L(\tau_2) + L(\tau_3) + L(\tau_4) = L(\tau) = d(x, y),$$

which is a contradiction. Thus we have

$$d(x_1, x_2) = L(\tau_2), \quad d(x_2, x_3) = L(\tau_3).$$

Similarly, we see that

$$d(x_1, x_3) = L(\tau_2 + \tau_3).$$

Finally, we obtain

$$d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3).$$

QED.

Using Proposition 3.8, we shall show that the distance function determines the Riemannian metric.

THEOREM 3.10. *Let M and M' be Riemannian manifolds with Riemannian metrics g and g', respectively. Let d and d' be the distance functions of M and M' respectively. If f is a mapping (which is not assumed to be continuous or differentiable) of M onto M' such that $d(x, y) = d'(f(x), f(y))$ for all $x, y \in M$, then f is a diffeomorphism of M onto M' which maps the tensor field g into the tensor field g'.*

In particular, every mapping f of M onto itself which preserves d is an isometry, that is, preserves g.

Proof. Clearly, f is a homeomorphism. Let x be an arbitrary point of M and set $x' = f(x)$. For a normal coordinate neighborhood U' of x' let U be a normal coordinate neighborhood of x such that $f(U) \subset U'$. For any unit tangent vector X at x, let τ be a geodesic in U with the initial condition (x, X). Since τ is a segment with respect to d, $f(\tau)$ is a segment with respect to d' and hence is a geodesic in U' with origin x'. Since $\tau = x_s$ is parametrized by the arc length s and since $d'(f(x_{s_1}), f(x_{s_2})) = d(x_{s_1}, x_{s_2}) = |s_2 - s_1|$, $f(\tau) = f(x_s)$ is parametrized by the arc length s also. Let $F(X)$ be the unit vector tangent to $f(\tau)$ at x'. Thus, F is a mapping of the set of unit tangent vectors at x into the set of unit tangent vectors at x'. It can be extended to a mapping, denoted by the same F, of $T_x(M)$ into $T_{x'}(M')$ by proportion. Since f has an inverse which also preserves the distance functions, it is clear that F is a 1:1 mapping of $T_x(M)$ onto $T_{x'}(M')$. It is also clear that

$$f \circ \exp_x = \exp_{x'} \circ F \quad \text{and} \quad \|F(X)\| = \|X\| \quad \text{for } X \in T_x(M),$$

where $\exp_x$ (resp. $\exp_{x'}$) is the exponential map of a neighborhood of 0 in $T_x(M)$ (resp. $T_{x'}(M')$) onto U (resp. U'). Both $\exp_x$ and $\exp_{x'}$ are diffeomorphisms. To prove that f is a diffeomorphism of M onto M' which maps g into g', it is therefore sufficient to show that F is a linear isometric mapping of $T_x(M)$ onto $T_{x'}(M')$.

We first prove that $g(X, Y) = g'(F(X), F(Y))$ for all $X, Y \in T_x(M)$. Since $F(cX) = cF(X)$ for any $X \in T_x(M)$ and any constant c, we may assume that both X and Y are unit vectors. Then both $F(X)$ and $F(Y)$ are unit vectors at x'. Set

$$\cos \alpha = g(X, Y) \quad \text{and} \quad \cos \alpha' = g'(F(X), F(Y)).$$

Let x_s and y_s be the geodesics with the initial conditions (x, X) and (x, Y) respectively, both parametrized by their arc length from x. Set

$$x'_s = f(x_s) \quad \text{and} \quad y'_s = f(y_s).$$

Then x'_s and y'_s are the geodesics with the initial conditions $(x', F(X))$ and $(x', F(Y))$, respectively.

Lemma. $\sin \tfrac{1}{2}\alpha = \lim_{s\to 0} \dfrac{1}{2s}\, d(x_s, y_s)$ *and* $\sin \tfrac{1}{2}\alpha' = \lim_{s\to 0} \dfrac{1}{2s}\, d(x'_s, y'_s)$.

We shall give the proof of the lemma shortly. Assuming the lemma for the moment, we shall complete the proof of our theorem. Since f preserves distance, the lemma implies that

$$\sin \tfrac{1}{2}\alpha = \sin \tfrac{1}{2}\alpha'$$

and hence

$$\begin{aligned} g(X, Y) = \cos \alpha &= 1 - 2\sin^2 \tfrac{1}{2}\alpha \\ &= 1 - 2\sin^2 \tfrac{1}{2}\alpha' = \cos \alpha' = g'(F(X), F(Y)). \end{aligned}$$

We shall now prove that F is linear. We already observed that $F(cX) = cF(X)$ for any $X \in T_x(M)$ and for any constant c.

Let $X_1, \ldots, X_n$ be an orthonormal basis for $T_x(M)$. Then $X'_i = F(X_i)$, $i = 1, \ldots, n$, form an orthonormal basis for $T_{x'}(M')$ as we have just proved. Given X and Y in $T_x(M)$, we have

$$g'(F(X+Y), X'_i) = g(X+Y, X_i) = g(X, X_i) + g(Y, X_i)$$
$$= g'(F(X), X'_i) + g'(F(Y), X'_i) = g'(F(X) + F(Y), X'_i)$$

for every i, and hence

$$F(X+Y) = F(X) + F(Y).$$

Proof of Lemma. It is sufficient to prove the first formula. Let U be a coordinate neighborhood with a normal coordinate system $x^1, \ldots, x^n$ at x. Let h be the Riemannian metric in U given by $\Sigma_i (dx^i)^2$ and let $\delta(y, z)$ be the distance between y and z with respect to h. Supposing that

$$\overline{\lim_{s\to 0}} \frac{1}{2s} d(x_s, y_s) > \sin \tfrac{1}{2}\alpha,$$

we shall obtain a contradiction. (The case where the inequality is reversed can be treated in a similar manner.) Choose $c > 1$ such that

$$\overline{\lim_{s\to 0}} \frac{1}{2s} d(x_s, y_s) > c \sin \tfrac{1}{2}\alpha.$$

Taking U small, we may assume that $\frac{1}{c} h < g < ch$ on U in the sense that

$$\frac{1}{c} h(Z, Z) < g(Z, Z) < c\, h(Z, Z) \quad \text{for } Z \in T_z(M) \quad \text{and} \quad z \in U.$$

From the definition of the distances d and δ, we obtain

$$\frac{1}{c} \delta(y, z) < d(y, z) < c\, \delta(y, z).$$

Hence we have

$$\frac{c}{2s} \delta(x_s, y_s) > \frac{1}{2s} d(x_s, y_s) > c \sin \tfrac{1}{2}\alpha \qquad \text{for small } s.$$

On the other hand, h is a Euclidean metric and hence

$$\frac{1}{2s} \delta(x_s, y_s) = \sin \tfrac{1}{2}\alpha.$$

This is a contradiction. Hence,

$$\overline{\lim_{s\to 0}} \frac{1}{2s} d(x_s, y_s) = \sin \tfrac{1}{2}\alpha.$$

Similarly, we obtain

$$\underline{\lim_{s\to 0}} \frac{1}{2s} d(x_s, y_s) = \sin \tfrac{1}{2}\alpha.$$

QED.

Theorem 3.10 is due to Myers and Steenrod [1]; the proof is adopted from Palais [2].

4. *Completeness*

A Riemannian manifold M or a Riemannian metric g on M is said to be *complete* if the Riemannian connection is complete, that is, if every geodesic of M can be extended for arbitrarily large values of its canonical parameter (cf. §6 of Chapter III). We shall prove the following two important theorems.

THEOREM 4.1. *For a connected Riemannian manifold M, the following conditions are mutually equivalent:*

(1) *M is a complete Riemannian manifold;*

(2) *M is a complete metric space with respect to the distance function d;*

(3) *Every bounded subset of M (with respect to d) is relatively compact;*

(4) *For an arbitrary point x of M and for an arbitrary curve C in the tangent space $T_x(M)$ (or more precisely, the affine tangent space $A_x(M)$) starting from the origin, there is a curve τ in M starting from x which is developed upon the given curve C.*

THEOREM 4.2. *If M is a connected complete Riemannian manifold, then any two points x and y of M can be joined by a minimizing geodesic.*

Proof. We divide the proofs of these theorems into several steps.

(i) The implication (2) $\to$ (1). Let x_s, $0 \leqq s < L$, be a geodesic, where s is the canonical parameter. We show that this geodesic can be extended beyond L. Let $\{s_n\}$ be an infinite sequence such that $s_n \uparrow L$. Then

$$d(x_{s_m}, x_{s_n}) \leqq |s_m - s_n|,$$

so that $\{x_{s_n}\}$ is a Cauchy sequence in M with respect to d and hence converges to a point, say x. The limit point x is independent of the choice of a sequence $\{s_n\}$ converging to L. We set $x_L = x$. By using a normal coordinate system at x, we can extend the geodesic for the values of s such that $L \leqq s \leqq L + \varepsilon$ for some $\varepsilon > 0$.

(ii) Proof of Theorem 4.2. Let x be any point of M. For each $r > 0$, we set

$$S(r) = \{y \in M;\, d(x, y) \leqq r\}$$

and

$$E(r) = \{y \in S(r);\, y \text{ can be joined to } x \text{ by a minimizing geodesic}\}.$$

We are going to prove that $E(r)$ is compact and coincides with $S(r)$ for every $r > 0$. To prove the compactness of $E(r)$, let y_i, $i = 1, 2, \ldots$, be a sequence of points of $E(r)$ and, for each i, let τ_i be a minimizing geodesic from x to y_i. Let X_i be the unit vector tangent to τ_i at x. By taking a subsequence if necessary, we may assume that $\{X_i\}$ converges to a unit vector X_0 in $T_x(M)$. Since $d(x, y_i) \leqq r$ for all i, we may assume, again by taking a subsequence if necessary, that $d(x, y_i)$ converges to a non-negative number r_0. Since τ_i is minimizing, we have

$$y_i = \exp\, (d(x, y_i)X_i).$$

Since M is a complete Riemannian manifold, $\exp r_0 X_0$ is defined. We set

$$y_0 = \exp r_0 X_0.$$

It follows that $\{y_i\}$ converges to y_0 and hence that $d(x, y_0) = r_0$. This implies that the geodesic $\exp sX_0$, $0 \leqq s \leqq r_0$, is minimizing and that y_0 is in $E(r)$. This proves the compactness of $E(r)$.

Now we shall prove that $E(r) = S(r)$ for all $r > 0$. By the existence of a normal coordinate system and a convex neighborhood around x (cf. Theorem 3.6), we know that $E(r) = S(r)$ for $0 < r < \varepsilon$ for some $\varepsilon > 0$. Let r^* be the supremum of $r_0 > 0$ such that $E(r) = S(r)$ for $r < r_0$. To show that $r^* = \infty$, assume that $r^* < \infty$. We first prove that $E(r^*) = S(r^*)$. Let y be a point of $S(r^*)$ and let $\{y_i\}$ be a sequence of points with $d(x, y_i) < r^*$ which converges to y. (The existence of such a sequence $\{y_i\}$ follows from the fact that x and y can be joined by a curve whose length is as close to $d(x, y)$ as we wish.) Then each y_i belongs to some $E(r)$, where $r < r^*$, and hence each y_i belongs to $E(r^*)$.

Since $E(r^*)$ is compact, y belongs to $E(r^*)$. Hence $S(r^*) = E(r^*)$. Next we shall show that $S(r) = E(r)$ for $r < r^* + \delta$ for some $\delta > 0$, which contradicts the definition of r^*. We need the following

LEMMA. *On a Riemannian manifold M, there exists a positive continuous function $r(z)$, $z \in M$, such that any two points of $S_z(r(z)) = \{y \in M; d(z,y) \leqq r(z)\}$ can be joined by a minimizing geodesic.*

Proof of Lemma. For each $z \in M$, let $r(z)$ be the supremum of $r > 0$ such that any two points y and y' with $d(z,y) \leqq r$ and $d(z,y') \leqq r$ can be joined by a minimizing geodesic. The existence of a convex neighborhood (cf. Theorem 3.6) implies that $r(z) > 0$. If $r(z) = \infty$ for some point z, then $r(y) = \infty$ for every point y of M and any positive continuous function on M satisfies the condition of the lemma. Assume that $r(z) < \infty$ for every $z \in M$. We shall prove the continuity of $r(z)$ by showing that $|r(z) - r(y)| \leqq d(z,y)$. Without any loss of generality, we may assume that $r(z) > r(y)$. If $d(z,y) \geqq r(z)$, then obviously $|r(z) - r(y)| < d(z,y)$. If $d(z,y) < r(z)$, then $S_y(r') = \{y'; d(y,y') \leqq r'\}$ is contained in $S_z(r(z))$, where $r' = r(z) - d(z,y)$. Hence $r(y) \geqq r(z) - d(z,y)$, that is, $|r(z) - r(y)| \leqq d(z,y)$, completing the proof of the lemma.

Going back to the proof of Theorem 4.2, let $r(z)$ be the continuous function given in the lemma and let δ be the minimum of $r(z)$ on the compact set $E(r^*)$. To complete the proof of Theorem 4.2, we shall show that $S(r^* + \delta) = E(r^* + \delta)$. Let $y \in S(r^* + \delta)$ but $\notin S(r^*)$. We show first that there exists a point y' in $S(r^*)$ such that $d(x,y') = r^*$ and that $d(x,y) = d(x,y') + d(y',y)$. To this end, for every positive integer k, choose a curve τ_k from x to y such that $L(\tau_k) < d(x,y) + \frac{1}{k}$, where $L(\tau_k)$ is the length of τ_k. Let y_k be the last point on τ_k which belongs to $E(r^*) = S(r^*)$. Then $d(x,y_k) = r^*$ and $d(x,y_k) + d(y_k,y) \leqq L(\tau_k) < d(x,y) + \frac{1}{k}$.

Since $E(r^*)$ is compact, we may assume, by taking a subsequence if necessary, that $\{y_k\}$ converges to a point, say y', of $E(r^*)$. We have $d(x,y') = r^*$ and $d(x,y') + d(y',y) = d(x,y)$. Let τ' be a minimizing geodesic from x to y'. Since $d(y',y) \leqq \delta \leqq r(y')$, there is a minimizing geodesic τ'' from y' to y. Let τ be the join of τ' and

τ''. Then $L(\tau) = L(\tau') + L(\tau'') = d(x, y') + d(y', y) = d(x, y)$. By Corollary 3.9, τ is a geodesic, in fact, a minimizing geodesic from x to y. Hence $y \in E(r^* + \delta)$, completing the proof of Theorem 4.2.

Remark. To prove that $E(r) = S(r)$ is compact for every r, it is sufficient to assume that every geodesic issuing from the particular point x can be extended infinitely.

(iii) The implication (1) $\to$ (3) in Theorem 4.1. In (ii) we proved that (1) implies that $E(r) = S(r)$ is compact for every r. Every bounded subset of M is contained in $S(r)$ for some r, regardless of the point x we choose in the proof of (ii).

(iv) The implication (3) $\to$ (2) is evident.

(v) The implication (4) $\to$ (1). Since a geodesic is a curve in M which is developed upon a straight line (or a segment) in the tangent space, it is obvious that every geodesic can be extended infinitely.

(vi) The implication (1) $\to$ (4). Let C_t, $0 \leqq t \leqq a$, be an arbitrary curve in $T_x(M)$ starting from the origin. We know that there is $\varepsilon > 0$ such that C_t, $0 \leqq t \leqq \varepsilon$, is the development of a curve x_t, $0 \leqq t \leqq \varepsilon$, in M. Let b be the supremum of such $\varepsilon > 0$. We want to show that $b = a$. Assume that $b < a$. First we show that $\lim_{t \to b} x_t$ exists in M. Let $t_n \uparrow b$. Since the development preserves the arc length, the length of x_t, $t_n \leqq t \leqq t_m$, is equal to the length of C_t, $t_n \leqq t \leqq t_m$. On the other hand, the distance $d(x_{t_n}, x_{t_m})$ is less than or equal to the length of x_t, $t_n \leqq t \leqq t_m$. This implies that $\{x_{t_n}\}$ is a Cauchy sequence in M. Since we know the implication (1) $\to$ (3) by (iii) and (iv), we see that $\{x_{t_n}\}$ converges to a point, say y. It is easy to see that $\lim_{t \to b} x_t = y$. Let C'_t be the curve in $T_y(M)$ (or more precisely, in $A_y(M)$) obtained by the affine (not linear!) parallel displacement of the curve C_t along the curve x_t, $0 \leqq t \leqq b$. Then C'_b is the origin of $T_y(M)$. There exist $\delta > 0$ and a curve x_t, $b \leqq t \leqq b + \delta$, which is developed upon C'_t, $b \leqq t \leqq b + \delta$. Then the curve x_t, $0 \leqq t \leqq b + \delta$, is developed upon C_t, $0 \leqq t \leqq b + \delta$. This contradicts the definition of b. QED.

COROLLARY 4.3. *If all geodesics starting from any particular point x of a connected Riemannian manifold M are infinitely extendable, then M is complete.*

Proof. As we remarked at the end of (ii) in the proof of Theorem 4.2, $E(r) = S(r)$ is compact for every r. Every bounded subset of M is contained in $S(r)$ for some r and hence is relatively compact. QED.

Corollary 4.4. *Every compact Riemannian manifold is complete.*

Proof. This follows from the implication (3) $\rightarrow$ (1) in Theorem 4.1. QED.

A Riemannian manifold M is said to be *homogeneous* if the group of isometries, i.e., transformations preserving the metric tensor g, of M is transitive on M. (Cf. Example 1.3 and Theorem 3.4, Chapter VI.)

Theorem 4.5. *Every homogeneous Riemannian manifold is complete.*

Proof. Let x be a point of a homogeneous Riemannian manifold M. There exists $r > 0$ such that, for every unit vector X at x, the geodesic $\exp sX$ is defined for $|s| \leqq r$ (cf. Proposition 8.1 of Chapter III). Let $\tau = x_s$, $0 \leqq s \leqq a$, be any geodesic with canonical parameter s in M. We shall show that $\tau = x_s$ can be extended to a geodesic defined for $0 \leqq s \leqq a + r$. Let φ be an isometry of M which maps x into x_a. Then φ^{-1} maps the unit vector $\dot{x}_a$ at x_a into a unit vector X at x: $X = \varphi^{-1}(\dot{x}_a)$. Since $\exp sX$ is a geodesic through x, $\varphi(\exp sX)$ is a geodesic through x_a. We set

$$x_{a+s} = \varphi(\exp sX) \qquad \text{for } 0 \leqq s \leqq r.$$

Then $\tau = x_s$, $0 \leqq s \leqq a + r$, is a geodesic. QED.

Theorem 4.5 follows also from the general fact that every locally compact homogeneous metric space is complete.

Theorem 4.6. *Let M and M^* be connected Riemannian manifolds of the same dimension. Let p: $M^* \rightarrow M$ be an isometric immersion.*

(1) *If M^* is complete, then M^* is a covering space of M with projection p and M is also complete.*

(2) *Conversely, if p: $M^* \rightarrow M$ is a covering projection and if M is complete, then M^* is complete.*

Proof. The proof is divided into several steps.

(i) If M^* is complete so is M. Let $x^* \in M^*$ and set $x = p(x^*)$. Let X be any unit vector of M at x and choose a unit vector X^* at x^* such that $p(X^*) = X$. Then $\exp sX = p(\exp sX^*)$ is the geodesic in M with the initial condition (x, X). Since $\exp sX^*$ is defined for all s, $-\infty < s < \infty$, so is $\exp sX$. By Corollary 4.3, M is complete.

(ii) If M^* is complete, p maps M^* *onto* M. Let $x^* \in M^*$ and $x = p(x^*)$. Given a point y of M, let $\exp sX$, $0 \leqq s \leqq a$, be a geodesic from x to y, where X is a unit vector at x. Such a geodesic exists by Theorem 4.2 since M is complete by (i). Let X^* be the unit vector of M^* at x^* such that $p(X^*) = X$. Set $y^* = \exp aX^*$. Then $p(y^*) = \exp aX = y$.

(iii) If M^* is complete, then $p: M^* \to M$ is a covering projection. For a given $x \in M$ and for each positive number r, we set

$$U(x; r) = \{y \in M;\ d(x, y) < r\},\ N(x; r) = \{X \in T_x(M);\ \|X\| < r\}.$$

Similarly, we set, for $x^* \in M^*$,

$$U(x^*; r) = \{y^* \in M^*;\ d(x^*, y^*) < r\},$$

$$N(x^*; r) = \{X^* \in T_{x^*}(M^*);\ \|X\| < r\}.$$

Choose $r > 0$ such that $\exp: N(x; 2r) \to U(x; 2r)$ is a diffeomorphism. Let $\{x_1^*, x_2^*, \ldots\}$ be the set $p^{-1}(x)$. For each x_i^*, we have the following commutative diagram:

$$\begin{array}{ccc} N(x_i^*; 2r) & \xrightarrow{\exp} & U(x_i^*; 2r) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ N(x; 2r) & \xrightarrow{\exp} & U(x; 2r). \end{array}$$

It is sufficient to prove the following three statements:

(a) $p: U(x_i^*; r) \to U(x; r)$ is a diffeomorphism for every i;

(b) $p^{-1}(U(x; r)) = \bigcup_i U(x_i^*; r)$;

(c) $U(x_i^*; r) \cap U(x_j^*; r)$ is empty if $x_i^* \neq x_j^*$.

Now, (a) follows from the fact that both $p: N(x_i^*; 2r) \to N(x; 2r)$ and $\exp: N(x; 2r) \to U(x; 2r)$ are diffeomorphisms in the above diagram. To prove (b), let $y^* \in p^{-1}(U(x; r))$ and set $y = p(y^*)$. Let $\exp sY$, $0 \leqq s \leqq a$, be a minimizing geodesic from y to x, where Y is a unit vector at y. Let Y^* be the unit vector at y^* such that $p(Y^*) = Y$. Then $\exp sY^*$, $0 \leqq s \leqq a$, is a geodesic in M^* starting from y^* such that $p(\exp sY^*) = \exp sY$. In particular, $p(\exp aY^*) = x$ and hence $\exp aY^* = x_i^*$ for some x_i^*. Evidently, $y^* \in U(x_i^*; r)$, proving that $p^{-1}(U(x; r)) \subset \bigcup_i U(x_i^*; r)$. On the other hand, it is obvious that $p(U(x_i^*; r)) \subset U(x; r)$ for every i and hence $p^{-1}(U(x; r)) \supset \bigcup_i U(x_i^*; r)$. To prove (c), suppose $y^* \in U(x_i^*; r) \cap U(x_j^*; r)$. Then $x_i^* \in U(x_j^*; 2r)$. Using the above diagram, we have shown that $p: U(x_j^*; 2r) \to U(x; 2r)$ is a diffeomorphism. Since $p(x_i^*) = p(x_j^*)$, we must have $x_i^* = x_j^*$.

(iv) Proof of (2). Assume that $p: M^* \to M$ is a covering projection and that M is complete. Observe first that, given a curve x_t, $0 \leqq t \leqq a$, in M and given a point x_0^* in M^* such that $p(x_0^*) = x_0$, there is a unique curve x_t^*, $0 \leqq t \leqq a$, in M^* such that $p(x_t^*) = x_t$ for $0 \leqq t \leqq a$. Let $x^* \in M^*$ and let X^* be any unit vector at X^*. Set $X = p(X^*)$. Since M is complete, the geodesic $\exp sX$ is defined for $-\infty < s < \infty$. From the above observation, we see that there is a unique curve x_s^*, $-\infty < s < \infty$, in M^* such that $x_0^* = x^*$ and that $p(x_s^*) = \exp sX$. Evidently, $x_s^* = \exp sX^*$. This shows that M^* is complete. QED.

Corollary 4.7. *Let M and M^* be connected manifolds of the same dimension and let $p: M^* \to M$ be an immersion. If M^* is compact, so is M, and p is a covering projection.*

Proof. Take any Riemannian metric g on M. It is easy to see that there is a unique Riemannian metric g^* on M^* such that p is an isometric immersion. Since M^* is complete by Corollary 4.4, p is a covering projection by Theorem 4.6 and hence M is compact. QED.

Example 4.1. A Riemannian manifold is said to be *non-prolongeable* if it cannot be isometrically imbedded into another Riemannian manifold as a proper open submanifold. Theorem 5.6 shows that every complete Riemannian manifold is non-prolongeable. The converse is not true. For example, let M be the Euclidean plane with origin removed and M^* the universal covering space of M. As an open submanifold of the Euclidean plane, M has a natural Riemannian metric which is obviously not complete. With respect to the natural Riemannian metric on M^* (cf. Example 2.1), M^* is not complete by Theorem 4.6. It can be shown that M^* is non-prolongeable.

Corollary 4.8. *Let G be a group of isometries of a connected Riemannian manifold M. If the orbit $G(x)$ of a point x of M contains an* *open set of M, then the orbit $G(x)$ coincides with M, that is, M is homogeneous.*

Proof. It is easy to see that $G(x)$ is open in M. Let M^* be a connected component of $G(x)$. For any two points x^* and y^* of M^*, there is an element f of G such that $f(x^*) = y^*$. Since f maps every connected component of $G(x)$ onto a connected component of $G(x), f(M^*) = M^*$. Hence M^* is a homogeneous Riemannian manifold isometrically imbedded into M as an open submanifold. Hence, $M^* = M$. QED.

PROPOSITION 4.9. *Let M be a Riemannian manifold and M^* a submanifold of M which is locally closed in the sense that every point x of M has a neighborhood U such that every connected component of $U \cap M^*$ (with respect to the topology of M^*) is closed in U. If M is complete, so is M^* with respect to the induced metric.*

Proof. Let d be the distance function defined by the Riemannian metric of M and d^* the distance function defined by the induced Riemannian metric of M^*. Let x_s be a geodesic in M^* and let a be the supremum of s such that x_s is defined. To show that $a = \infty$, assume $a < \infty$. Let $s_n \uparrow a$. Since $d(x_{s_n}, x_{s_m}) \leqq d^*(x_{s_n}, x_{s_m}) \leqq |s_n - s_m|$, $\{x_{s_n}\}$ is a Cauchy sequence in M and hence converges to a point, say x, of M. Then $x = \lim_{s \to a} x_s$. Let U be a neighborhood of x in M with the property stated in Proposition. Then x_s, $b \leqq s < a$, lies in U for some b. Since the connected component of $M^* \cap U$ containing x_s, $b \leq s < a$, is closed in U, the point x belongs to M^*. Set $x_a = x$. Then x_s, $0 \leqq s \leqq a$, is a geodesic in M^*. Using a normal coordinate system at x_a, we see that this geodesic can be extended to a geodesic x_s, $0 \leqq s \leqq a + \delta$, for some $\delta > 0$. QED.

5. *Holonomy groups*

Throughout this section, let M be a connected Riemannian manifold with metric g and $\Psi(x)$ the linear or homogeneous holonomy group of the Riemannian connection with reference point $x \in M$ (cf. §4 of Chapter II and §3 of Chapter III). Then M is said to be *reducible* or *irreducible* according as $\Psi(x)$ is reducible or irreducible as a linear group acting on $T_x(M)$. In this section, we shall study $\Psi(x)$ and local structures of a reducible Riemannian manifold.

Assuming that M is reducible, let T'_x be a non-trivial subspace of $T_x(M)$ which is invariant by $\Psi(x)$. Given a point $y \in M$, let τ be a curve from x to y and T'_y the image of T'_x by the (linear) parallel displacement along τ. The subspace T'_y of $T_y(M)$ is independent of the choice of τ. In fact, if μ is any other curve from x to y, then $\mu^{-1} \cdot \tau$ is a closed curve at x and the subspace T'_x is invariant by the parallel displacement along $\mu^{-1} \cdot \tau$, that is, $\mu^{-1} \cdot \tau(T'_x) = T'_x$, and hence $\tau(T'_x) = \mu(T'_x)$. We thus obtain a distribution T' which assigns to each point y of M the subspace T'_y of $T_y(M)$.

A submanifold N of a Riemannian manifold (or more generally, a manifold with a linear connection) M is said to be *totally geodesic at a point* x of N if, for every $X \in T_x(N)$, the geodesic $\tau = x_t$ of M determined by (x, X) lies in N for small values of the parameter t. If N is totally geodesic at every point of N, it is called a *totally geodesic submanifold* of M.

PROPOSITION 5.1. (1) *The distribution T' is differentiable and involutive;*

(2) *Let M' be the maximal integral manifold of T' through a point of M. Then M' is a totally geodesic submanifold of M. If M is complete, so is M' with respect to the induced metric.*

Proof. (1) To prove that T' is differentiable, let y be any point of M and $x^1, \dots, x^n$ a normal coordinate system at y, valid in a neighborhood U of y. Let $X_1, \dots, X_k$ be a basis for T'_y. For each i, $1 \leq i \leq k$, we define a vector field X_i^* in U by

$$(X_i^*)_z = \tau X_i \qquad \text{for } z \in U,$$

where τ is the geodesic from y to z given by $x^j = a^j t$, $j = 1, \dots, n$, $(a^1, \dots, a^n)$ being the coordinates of z. Since the parallel displacement τ depends differentiably on $(a^1, \dots, a^n)$, we obtain a differentiable vector field X_i^* in U. It is clear that $X_1^*, \dots, X_k^*$ form a basis of T'_z for every point z of U.

To prove that T' is involutive, it is sufficient to prove that if X and Y are vector fields belonging to T', so are $\nabla_X Y$ and $\nabla_Y X$, because the Riemannian connection has no torsion and $[X, Y] = \nabla_X Y - \nabla_Y X$ (cf. Theorem 5.1 of Chapter III). Let x_t be the integral curve of X starting from an arbitrary point y. Let τ_0^t be the parallel displacement along this curve from the point x_t to the point $y = x_0$. Since Y_y and Y_{x_t} belong to T' for every t, $(\nabla_X Y) = \lim_{t \to 0} \frac{1}{t} (\tau_0^t Y_{x_t} - Y_y)$ belongs to T'_y.

(2) Let M' be a maximal integral manifold of T'. Let $\tau = x_t$ be a geodesic of M with the initial condition (y, X), where $y \in M$ and $X \in T_y(M') = T'_y$. Since the tangent vectors $\dot{x}_t$ are parallel along τ, we see that $\dot{x}_t$ belongs to T'_{x_t} for every t and hence τ lies in M' (cf. Lemma 2 for Theorem 7.2 of Chapter II). This proves that M' is a totally geodesic submanifold of M. From the following lemma, we may conclude that, if M is complete, so is M'.

Lemma. *Let N be a totally geodesic submanifold of a Riemannian manifold M. Every geodesic of N with respect to the induced Riemannian metric of N is a geodesic in M.*

Proof of Lemma. Let $x \in N$ and $X \in T_x(N)$. Let $\tau = x_t$, $0 \leqq t \leqq a$, be the geodesic of M with the initial condition (x, X). Since N is totally geodesic, τ lies in N. It now suffices to show that τ is a geodesic of N with respect to the induced Riemannian metric of N. Let d and d' be the distance functions of M and N respectively. Considering only small values of t, we may assume that τ is a minimizing geodesic from $x = x_0$ to x_a so that $d(x, x_a) = L(\tau)$, where $L(\tau)$ is the arc length of τ. The arc length of τ measured by the metric of M is the same as the one measured with respect to the induced metric of N. From the definition of the distance functions d and d', we obtain

$$d'(x, x_a) \geqq d(x, x_a) = L(\tau).$$

Hence, $d'(x, x_a) = L(\tau)$. By Corollary 3.9, τ is a geodesic with respect to the induced metric of N. QED.

Remark. The lemma is a consequence of the following two facts which will be proved in Volume II. (1) If M is a manifold with a linear connection whose torsion vanishes and if N is a totally geodesic submanifold of M, then N has a naturally induced linear connection such that every geodesic of N is a geodesic of M; (2) If N is a totally geodesic submanifold of a Riemannian manifold M, then the naturally induced linear connection of N is the Riemannian connection with respect to the induced metric of N. Note that Proposition 5.1 holds under the weaker assumption that M is a manifold with a linear connection whose torsion vanishes.

Let T' be a distribution defined as before. We now use the fact that the homogeneous holonomy group consists of orthogonal transformations of $T_x(M)$. Let T''_x be the orthogonal complement of T'_x in $T_x(M)$. Then $T_x(M)$ is the direct sum of two subspaces T'_x and T''_x which are invariant by $\Psi(x)$. From the subspace T''_x we obtain a distribution T'' just as we obtained T' from T'_x. The distributions T' and T'' are complementary and orthogonal to each other at every point of M.

Proposition 5.2. *Let y be any point of M. Let M' and M'' be the maximal integral manifolds of the distributions T' and T'' defined above. Then y has an open neighborhood V such that $V = V' \times V''$, where V' (resp. V'') is an open neighborhood of y in M' (resp. M''), and that the Riemannian metric in V is the direct product of the Riemannian metrics in V' and V''.*

Proof. We first prove the following

Lemma. *If T' and T'' are two involutive distributions on a manifold M which are complementary at every point of M, then, for each point y of M, there exists a local coordinate system $x^1, \ldots, x^n$ with origin at y such that $(\partial/\partial x^1, \ldots, \partial/\partial x^k)$ and $(\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n)$ form local bases for T' and T'' respectively. In other words, for any set of constants $(c^1, \ldots, c^k, c^{k+1}, \ldots, c^n)$, the equations $x^i = c^i$, $1 \leqq i \leqq k$, (resp. $x^j = c^j$, $k + 1 \leqq j \leqq n$) define an integral manifold of T'' (resp. T').*

Proof of Lemma. Since T' is involutive, there exists a local coordinate system $y^1, \ldots, y^k, x^{k+1}, \ldots, x^n$ with origin y such that $(\partial/\partial y^1, \ldots, \partial/\partial y^k)$ form a local basis for T'. In other words, the equations $x^j = c^j$, $k + 1 \leqq j \leqq n$, define an integral manifold of T'. Similarly, there exists a local coordinate system $x^1, \ldots, x^k, z^{k+1}, \ldots, z^n$ with origin y such that $(\partial/\partial z^{k+1}, \ldots, \partial/\partial z^n)$ form a local basis for T''. In other words, the equations $x^i = c^i$, $1 \leqq i \leqq k$, define an integral manifold of T''. It is easy to see that $x^1, \ldots, x^k, x^{k+1}, \ldots, x^n$ is a local coordinate system with the desired property.

Making use of the local coordinate system $x^1, \ldots, x^n$ thus obtained, we shall prove Proposition 5.2. Let V be the neighborhood of y defined by $|x^i| < c$, $1 \leqq i \leqq n$, where c is a sufficiently small positive number so that the coordinate system $x^1, \ldots, x^n$ gives a homeomorphism of V onto the cube $|x^i| < c$ in $\mathbf{R}^n$. Let V' (resp. V'') be the set of points in V defined by $|x^i| < c$, $1 \leqq i \leqq k$, and $x^j = 0$, $k + 1 \leqq j \leqq n$ (resp. $x^i = 0$, $1 \leqq i \leqq k$, and $|x^j| < c$, $k + 1 \leqq j \leqq n$). It is clear that V' (resp. V'') is an integral manifold of T' (resp. T'') through y and is a neighborhood of y in M' (resp. M'') and that $V = V' \times V''$. We set $X_i = \partial/\partial x^i$, $1 \leqq i \leqq n$. To prove that the Riemannian metric of V is the direct product of those in V' and V'', we show that $g_{ij} = g(X_i, X_j)$ are independent of $x^{k+1}, \ldots, x^n$ for $1 \leqq i, j \leqq k$, that $g_{ij} = g(X_i, X_j)$ are independent of $x^1, \ldots, x^k$ for $k + 1 \leqq i, j \leqq n$ and that $g_{ij} = g(X_i, X_j) = 0$ for $1 \leqq i \leqq k$ and $k + 1 \leqq j \leqq n$. The last assertion is obvious since X_i, $1 \leqq i \leqq k$, belong to T' and X_j, $k + 1 \leqq j \leqq n$, belong to T'' and since T' and T'' are orthogonal to each other at every point. We now prove the first assertion, and the proof of the second assertion is similar. Let $1 \leqq i \leqq k$ and $k + 1 \leqq m \leqq n$. As in the proof of (1) of Proposition 5.1, we see that $\nabla_{X_m} X_i$ belongs to T' and that $\nabla_{X_i} X_m$ belongs to T''. Since the torsion is zero and since $[X_i, X_m] = 0$, we have

$$\nabla_{X_i} X_m - \nabla_{X_m} X_i = \nabla_{X_i} X_m - \nabla_{X_m} X_i - [X_i, X_m] = 0.$$

Hence, $\nabla_{X_i} X_m = \nabla_{X_m} X_i = 0$. Since g is parallel, we have

$$\begin{aligned} X_m(g_{ij}) &= \nabla_{X_m}(g(X_i, X_j)) \\ &= g(\nabla_{X_m} X_i, X_j) + g(X_i, \nabla_{X_m} X_j) = 0, \quad 1 \leqq i, j \leqq k, \end{aligned}$$

thus proving our assertion. QED.

Proposition 5.3. *Let T' and T'' be the distributions on M used in Proposition 5.2. If M is simply connected, then the homogeneous holonomy group $\Psi(x)$ is decomposed into the direct product of two normal subgroups $\Psi'(x)$ and $\Psi''(x)$ such that $\Psi'(x)$ is trivial on T''_x and that $\Psi''(x)$ is trivial on T'_x.*

Proof. Given an element $a \in \Psi(x)$, let a_1 (resp. a_2) be the restriction of a to T'_x (resp. T''_x). Let a' (resp. a'') be the orthogonal transformation of $T_x(M)$ which coincides with a_1 on T'_x (resp. with a_2 on T''_x) and which is trivial on T''_x (resp. T'_x). If we take an orthonormal basis for $T_x(M)$ such that the first k vectors lie in T'_x and the remaining $n - k$ vectors lie in T''_x, then these linear transformations can be expressed by matrices as follows:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad a' = \begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a'' = \begin{pmatrix} 1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

We shall show that both a' and a'' are elements of $\Psi(x)$. Let τ be a closed curve at x such that the parallel displacement along τ is the given element $a \in \Psi(x)$. First we consider the special case where τ is a *small lasso* in the following sense. A closed curve τ at x is called a small lasso if it can be decomposed into three curves as follows: $\tau = \mu^{-1} \cdot \sigma \cdot \mu$, where μ is a curve from x to a point y (so that μ^{-1} is a curve from y to x going backward) and σ is a closed curve at y which is small enough to be contained in a neighborhood $V = V' \times V''$ of y given in Proposition 5.2. In this special case, we denote by σ' (resp. σ'') the image of σ by the natural projection $V \to V'$ (resp. $V \to V''$). We set

$$\tau' = \mu^{-1} \cdot \sigma' \cdot \mu, \quad \tau'' = \mu^{-1} \cdot \sigma'' \cdot \mu.$$

The parallel displacement along τ' (resp. τ'') is trivial on T''_y (resp. T'_y). The parallel displacement along σ is the product of those along σ' and σ''. Hence the parallel displacement along τ is the product of those along τ' and τ''. On the other hand, τ' (resp. τ'') is trivial on T''_x (resp. T'_x). It follows that a' (resp. a'') is the parallel displacement along τ' (resp. τ''), thus proving our assertion in the case where τ is a small lasso.

In the general case, we decompose τ into a product of small lassos as follows.

LEMMA. *If M is simply connected, then the parallel displacement along τ is the product of the parallel displacements along a finite number of small lassos at x.*

Proof of Lemma. This follows from the factorization lemma (cf. Appendix 7).

It is now clear that both a' and a'' belong to $\Psi(x)$ in the general case. We set

$$\Psi'(x) = \{a'; a \in \Psi(x)\}, \quad \Psi''(x) = \{a''; a \in \Psi(x)\}.$$

Then $\Psi(x) = \Psi'(x) \times \Psi''(x)$. QED.

We now proceed to define a most natural decomposition of $T_x(M)$ and derive its consequences. Let $T_x^{(0)}$ be the set of elements in $T_x(M)$ which are left fixed by $\Psi(x)$. It is the maximal linear subspace of $T_x(M)$ on which $\Psi(x)$ acts trivially. Let T'_x be the orthogonal complement of $T_x^{(0)}$ in $T_x(M)$. It is invariant by $\Psi(x)$ and can be decomposed into a direct sum $T'_x = \Sigma_{i=1}^k T_x^{(i)}$ of mutually orthogonal, invariant and irreducible subspaces. We shall call $T_x(M) = \Sigma_{i=0}^k T_x^{(i)}$ a *canonical decomposition* (or *de Rham decomposition*) of $T_x(M)$.

THEOREM 5.4. *Let M be a Riemannian manifold, $T_x(M) = \Sigma_{i=0}^k T_x^{(i)}$ a canonical decomposition of $T_x(M)$ and $T^{(i)}$ the involutive distribution on M obtained by parallel displacement of $T_x^{(i)}$ for each $i = 0, 1, \ldots, k$. Let y be a point of M and let, for each $i = 0, 1, \ldots, k$, M_i be the maximal integral manifold of $T^{(i)}$ through y. Then*

(1) *The point y has an open neighborhood V such that $V = V_0 \times V_1 \times \cdots \times V_k$ where each V_i is an open neighborhood of y in M_i and that the Riemannian metric in V is the direct product of the Riemannian metrics in the V_i's;*

(2) *The maximal integral manifold M_0 is locally Euclidean in the sense that every point of M_0 has a neighborhood which is isometric with an open set of an n_0-dimensional Euclidean space, where $n_0 = \dim M_0$;*

(3) *If M is simply connected, then the homogeneous holonomy group $\Psi(x)$ is the direct product $\Psi_0(x) \times \Psi_1(x) \times \cdots \times \Psi_k(x)$ of normal subgroups, where $\Psi_i(x)$ is trivial on $T_x^{(j)}$ if $i \neq j$ and is irreducible on $T_x^{(i)}$ for each $i = 1, \ldots, k$, and $\Psi_0(x)$ consists of the identity only;*

(4) *If M is simply connected, then a canonical decomposition $T_x(M) = \Sigma_{i=0}^k T_x^{(i)}$ is unique up to an order.*

Proof. (1) This is a generalization of Proposition 5.2.

(2) Since y is an arbitrary point of M, it is sufficient to prove that V_0 is isometric to an open subset of an n_0-dimensional Euclidean space. Since the homogeneous holonomy group of V_0 consists of the identity only, $T_y^{(0)}$ is the direct sum of n_0 1-dimensional subspaces. From the proof of Proposition 5.2, it follows that V_0 is a direct product of 1-dimensional submanifolds and that the Riemannian metric on V_0 is the direct product of the Riemannian metrics on these 1-dimensional submanifolds. On the other hand, on any 1-dimensional manifold with a local coordinate system x^1, every Riemannian metric is of the form $g_{11}\, dx^1\, dx^1$. If x^1 is a normal coordinate system, then the metric is of the form $dx^1\, dx^1$. Hence V_0 is isometric to an open set of a Euclidean space.

(3) This is clear from the definition of a canonical decomposition of $T_x(M)$ and from the proof of Proposition 5.3.

(4) First we prove

LEMMA. *Let S_x be any subspace of $T_x(M)$ invariant by $\Psi(x)$. Then, for each $i = 1, \ldots, k$, either S_x is orthogonal to $T_x^{(i)}$ or S_x contains $T_x^{(i)}$.*

Proof of Lemma. (i) Assume that all vectors of S_x are left fixed by $\Psi_i(x)$. Then S_x is orthogonal to $T_x^{(i)}$. In fact, let $X = \sum_{j=0}^k X_j$ be any element of S_x, where $X_j \in T_x^{(j)}$. For an arbitrary element a_i of $\Psi_i(x)$, we have

$$a_i(X) = X_0 + X_1 + \cdots + a_i(X_i) + \cdots + X_k$$

since a_i acts trivially on $T_x^{(j)}$ for $j \neq i$. If $a_i(X) = X$, then $a_i(X_i) = X_i$. Since this holds for every $a_i \in \Psi_i(x)$ and since $\Psi_i(x)$ is irreducible in $T_x^{(i)}$, we must have $X_i = 0$. This shows that X is orthogonal to $T_x^{(i)}$.

(ii) Assume that $a_i(X) \neq X$ for some $a_i \in \Psi_i(x)$ and for some $X \in S_x$. Let $X = \sum_{j=0}^k X_j$, where $X_j \in T_x^{(j)}$. Since each X_j, $j \neq i$, is left fixed by every element of $\Psi_i(x)$, $X - a_i(X) = X_i - a_i(X_i) \neq 0$ is a vector in $T_x^{(i)}$ as well as in S_x. The subset $\{b_i(X - a_i(X));\ b_i \in \Psi_i(x)\}$ is in $T_x^{(i)} \cap S_x$ and spans $T_x^{(i)}$, since $\Psi_i(x)$ is irreducible in $T_x^{(i)}$. This implies that $T_x^{(i)}$ is contained in S_x, thus proving the lemma.

Going back to the proof of (4), let $T_x(M) = \sum_{j=0}^l S_x^{(j)}$ be any other canonical decomposition. First it is clear that $T_x^{(0)} = S_x^{(0)}$. It is therefore sufficient to prove that each $S_x^{(j)}$, $1 \leqq j \leqq l$, coincides with some $T_x^{(i)}$. Consider, for example, $S_x^{(1)}$. By the lemma, either it is orthogonal to $T_x^{(i)}$ for every $i \geqq 1$ or it contains $T_x^{(i)}$ for some $i \geqq 1$. In the first case, it must be contained in the orthogonal complement $T_x^{(0)}$ of $\sum_{i=1}^k T_x^{(i)}$ in $T_x(M)$. This is obviously a contradiction. In the second case, the irreducibility of $S_x^{(1)}$ implies that $S_x^{(1)}$ actually coincides with $T_x^{(i)}$. QED.

The following result is due to Borel and Lichnerowicz [1].

THEOREM 5.5. *The restricted homogeneous holonomy group of a Riemannian manifold M is a closed subgroup of $SO(n)$, where $n = \dim M$.*

Proof. Since the homogeneous holonomy group of the universal covering space of M is isomorphic with the restricted homogeneous holonomy group of M (cf. Example 2.1), we may assume, without loss of generality, that M is simply connected. In view of (3) of Theorem 5.4, our assertion follows from the following result in the theory of Lie groups:

Let G be a connected Lie subgroup of $SO(n)$ which acts irreducibly on the n-dimensional vector space $\mathbf{R}^n$. Then G is closed in $SO(n)$.

The proof of this result is given in Appendix 5.

6. *The decomposition theorem of de Rham*

Let M be a connected, simply connected and complete Riemannian manifold. Assuming that M is reducible, let $T_x(M) = T'_x + T''_x$ be a decomposition into subspaces invariant by the linear holonomy group $\Psi(x)$ and let T' and T'' be the parallel distributions defined by T'_x and T''_x, respectively, as in the beginning of §5. We fix a point $o \in M$ and let M' and M'' be the maximal integral manifolds of T' and T'' through o, respectively. By Proposition 5.1, both M' and M'' are complete, totally geodesic submanifolds of M.

The purpose of this section is to prove

THEOREM 6.1. *M is isometric to the direct product $M' \times M''$.*

Proof. For any curve z_t, $0 \leqq t \leqq 1$, in M with $z_0 = o$, we shall define its projection on M' to be the curve x_t, $0 \leqq t \leqq 1$, with $x_0 = o$ which is obtained as follows. Let C_t be the development of z_t in the affine tangent space $T_o(M)$. (For the sake of simplicity we identify the affine tangent space with the tangent (vector) space.) Since $T_o(M)$ is the direct product of the two Euclidean spaces T'_o and T''_o, C_t may be represented by a pair (A_t, B_t), where A_t and B_t are curves in T'_o and T''_o respectively. By applying (4) of Theorem 4.1 to M', we see that there exists a unique curve x_t in M' which is developed upon the curve A_t. In view of Proposition 4.1 of Chapter III we may define the curve x_t as follows. For each t, let X_t be the result of the parallel displacement of the T'-component of $\dot{z}_t$ from z_t to $o = z_0$ (along the curve z_t). The curve x_t is a curve in M' with $x_0 = o$ such that the result of the parallel displacement of $\dot{x}_t$ along itself to o is equal to X_t for each t.

Before proceeding further, we shall indicate the main idea of the proof. We show that the end point x_1 of the projection x_t depends only on the end point z_1 of the curve z_t if M is simply connected.

Thus we obtain a projection $p': M \to M'$ and, similarly, a projection $p'': M \to M''$. The mapping $p = (p', p'')$ of M into $M' \times M''$ will be shown to be isometric at every point. Theorem 4.6 then implies that p is a covering projection of M onto $M' \times M''$. If h is a homotopy in M from a curve of M' to another curve of M', then $p'(h)$ is a homotopy between the two curves in M'. Thus, M' is simply connected. Similarly, M'' is simply connected. Thus p is an isometry of M onto $M' \times M''$. The detail now follows.

LEMMA 1. *Let $\tau = z_t$, $0 \leq t \leq 1$, be a curve in M with $z_0 = o$ and let a be any number with $0 \leqq a \leqq 1$. Let τ_1 be the curve z_t, $0 \leqq t \leqq a$, and let τ_2 be the curve z_t, $a \leqq t \leqq 1$.* Let *τ_2' be the projection of τ_2 in the maximal integral manifold $M'(z_a)$* of *T' through z_a. Then the projection of $\tau = \tau_2 \cdot \tau_1$ in M' coincides with the projection of $\tau' = \tau_2' \cdot \tau_1$.*

Proof of Lemma 1. This is obvious from the second definition of the projection by means of the (linear) parallel displacement of tangent vectors.

LEMMA 2. *Let $z \in M$ and let $V = V' \times V''$ be an open neighborhood of z in M, where V' and V'' are open neighborhoods of z in $M'(z)$ and $M''(z)$ respectively. For any curve z_t with $z_0 = z$ in V, the projection of z_t in $M'(z)$ is given by the natural projection of V onto V'.*

Proof of Lemma 2. For the existence of a neighborhood $V = V' \times V''$, see Proposition 5.2. Let z_t be given by the pair (x_t, y_t) where x_t (resp. y_t) is a curve in V' (resp. V'') with $x_0 = z$ (resp. $y_0 = z$). Since $V = V' \times V''$, the parallel displacement of the T'-component of $\dot{z}_t$ from z_t to $z_0 = z$ along the curve z_t is the same as the parallel displacement of $\dot{x}_t$ from x_t to $x_0 = z$ along the curve x_t. Thus x_t is the projection of the curve z_t in $M'(z)$.

We introduce the following terminologies. A (piecewise differentiable) curve z_t is called a T'-curve (resp. T''-curve) if $\dot{z}_t$ belongs to T'_{z_t} (resp. T''_{z_t}) for every t. Given a (piecewise differentiable) homotopy $z\colon [0, 1] \times [0, s_0] \to M$ which is denoted by $z(t, s) = z_t^s$, we shall denote by $z_t^{(s)}$ (resp. $z_{(t)}^s$) the curve with parameter t for the fixed value of s (resp. the curve with parameter s for the fixed value of t). Their tangent vectors will be denoted by $\dot{z}_t^{(s)}$ and $\dot{z}_{(t)}^s$, respectively. For any point $z \in M$, let d' (resp. d'') denote the distance function on the maximal integral manifold $M'(z)$ of T' (resp. $M''(z)$ of T'') through z. Let $U'(z; r)$ (resp. $U''(z; r)$) denote the set of points $w \in M'(z)$ (resp. $w \in M''(z)$) such that $d'(z, w) < r$ (resp. $d''(z, w) < r$).

LEMMA 3. *Let $\tau' = x_t$, $0 \leqq t \leqq 1$, be a T'-curve. Then there exist a number $r > 0$ and a family of isometries f_t, $0 \leqq t \leqq 1$, of $U''(x_0; r)$ onto $U''(x_t; r)$ with the following properties:*

(1) *The differential of f_t at x_0 coincides with the parallel displacement along the curve τ' from x_0 to x_t;*

(2) *For any curve $\tau'' = y^s$, $0 \leqq s \leqq s_0$, in $U''(x_0; r)$ with $y^0 = x_0$, set $z_t^s = f_t(y^s)$. Then*

(a) *For any $0 \leqq t_1 \leqq 1$ and $0 \leqq s_1 \leqq s_0$, the parallel displacement along the "parallelogram" formed by the curve x_t, $0 \leqq t \leqq t_1$, the curve $z_{(t_1)}^s$, $0 \leqq s \leqq s_1$, the inverse of the curve $z_t^{(s_1)}$, $0 \leqq t \leqq t_1$, and the inverse of the curve y^s, $0 \leqq s \leqq s_1$, is trivial;*

(b) *For any s and t, $\dot{z}_t^{(s)}$ is parallel to $\dot{x}_t$ along the curve $z_{(t)}^s$;*

(c) *For any s and t, $\dot{z}_{(t)}^s$ is parallel to $\dot{y}^s$ along the curve $z_t^{(s)}$.*

Proof of Lemma 3. Let V be a neighborhood of x_0 of the form $V = V' \times V''$ as in Proposition 5.2. Choose a number $r > 0$ sufficiently small so that $x_t \in V'$ and $U''(x_t; r) \subset \{x_t\} \times V''$ for $0 \leqq t \leqq r$. We define f_t by $f_t(x_0, y) = (x_t, y)$ for $y \in U''(x_0; r)$. It is clear that the family of isometries f_t, $0 \leqq t \leqq r$, has all the properties (1) and (2). The family f_t can be extended easily for $0 \leqq t \leqq 1$ and for a suitable $r > 0$ by covering the curve $\tau' = x_t$ by a finite number of neighborhoods of the form $V = V' \times V''$ and using the above argument for each neighborhood.

LEMMA 4. *Let $\tau' = x_t$, $0 \leqq t \leqq 1$, be a T'-curve and let $\tau'' = y^s$, $0 \leqq s \leqq s_0$, be a T''-geodesic with $y^0 = x_0$, where s is the arc length. Then there exists a homotopy z_t^s, $0 \leqq t \leqq 1$, $0 \leqq s \leqq s_0$, with the following properties:*

(1) *$z_t^{(0)} = x_t$ and $z_{(0)}^s = y^s$;*

(2) *z_t^s has properties* (a), (b) *and* (c) *of Lemma* 3.

The homotopy z_t^s is uniquely determined. In fact, if Y_t is the result of parallel displacement of the initial tangent vector $Y_0 = \dot{y}^0$ of the geodesic τ'' along the curve τ', then $z_t^s = \exp sY_t$.

Proof of Lemma 4. We first prove the uniqueness. By (a) and (c) and by the fact that τ'' is a geodesic, it follows that $\dot{z}_{(t)}^s$ is parallel to Y_t along the curve $z_{(t)}^s$. This means that $z_{(t)}^s$ is a geodesic with initial tangent vector Y_t. Thus, $z_t^s = \exp sY_t$, proving the uniqueness.

It remains therefore to prove that $z_t^s = \exp sY_t$ actually satisfies conditions (1) and (2). Condition (1) is obvious. To prove (2), we may assume that τ' is a differentiable curve so that z_t^s is differentiable in (t, s). Let f_t be the family of isometries as in Lemma 3. It is obvious that there exists a number $\delta > 0$ such that $z_t^s = f_t(y^s)$ for $0 \leqq t \leqq 1$ and $0 \leqq s \leqq \delta$. Thus, z_t^s satisfies condition (2) for $0 \leqq t \leqq 1$ and $0 \leqq s \leqq \delta$. Let a be the supremum of such δ. In order to prove $a = s_0$, assume $a < s_0$. First we show that z_t^s satisfies (2) for $0 \leqq t \leqq 1$ and $0 \leqq s \leqq a$. Since z_t^s is differentiable in (t, s), the parallel displacement along the curve $z_t^{(a)}$ is the limit of the parallel displacement along the curve $z_t^{(s)}$ as $s \uparrow a$ (cf. Lemma for Theorem 4.2 of Chapter II). Thus condition (a) is satisfied. We have also $\dot{z}_{(t)}^a = \lim_{s \uparrow a} \dot{z}_{(t)}^s$ and $\dot{z}_t^{(a)} = \lim_{s \uparrow a} \dot{z}_t^{(s)}$. Combined with the above limit argument, this gives conditions (b) and (c) for $0 \leqq t \leqq 1$ and $s = a$.

In order to show that z_t^s has property (2) beyond the value a, we apply Lemma 3 to the T'-curve $\tau^{(a)} = z_t^{(a)}$ and the T''-geodesic y^u, where $u = s - a$. We see then that there exist a number $r > 0$ and a homotopy w_t^u, $0 \leqq t \leqq 1$, $-r \leqq u \leqq r$, satisfying a condition similar to (2), such that $w_t^{(0)} = z_t^{(a)}$ and $w_0^u = y^s$. Since $\dot{w}_t^{(0)}$ is parallel to $\dot{y}^a$ along the curve $w_t^{(0)} = z_t^{(a)}$, it follows that $z_t^s = w_t^{s-a}$ for $0 \leqq t \leqq 1$ and $a - r \leqq s \leqq a + r$. This proves that z_s^t satisfies condition (2) for $0 \leqq t \leqq 1$ and $0 \leqq s \leqq a + r$, contradicting the assumption that $a < s_0$.

Lemma 5. *Keeping the notation of Lemma 4, the projection of the curve $\tau' \cdot \tau''^{-1}$ in $M'(y^{s_0})$ coincides with $\tau^{(s_0)} = z_t^{(s_0)}$, $0 \leqq t \leqq 1$.*

Proof of Lemma 5. Since τ''^{-1} is a T''-curve, its projection in $M'(y^{s_0})$ is trivial, that is, reduces to the point y^{s_0}. Conditions (a) and (b) imply that, for each t, the parallel displacement of $\dot{x}_t$ along $\tau'' \cdot \tau'^{-1}$ to y^{s_0} is the same as the parallel displacement of $\dot{z}_t^{(s_0)}$ along $z_t^{(s_0)}$ to y^{s_0}. This means that $\tau'' \cdot \tau'^{-1}$ projects on $\tau^{(s_0)}$.

We now come to the main step for the proof of Theorem 6.1.

Lemma 6. *If two curves τ_1 and τ_2 from o to a point z in M are homotopic to each other, then their projections in $M' = M'(o)$ have the same end point.*

Proof of Lemma 6. We first remark that τ_2 is obtained from τ_1 by a finite succession of small deformations. Here a small deformation of a curve z_t means that, for a certain small neighborhood V, we replace a portion z_t, $t_1 \leqq t \leqq t_2$, of the curve lying in V by a curve w_t, $t_1 \leqq t \leqq t_2$, with $w_{t_1} = z_{t_1}$ and $w_{t_2} = z_{t_2}$, lying in V. As a neighborhood V, we shall always take a neighborhood of the form $V' \times V''$ as in Lemma 2.

It suffices therefore to prove the following assertion. Let τ be a curve from o to z_1, μ a curve from z_1 to z_2 which lies in a small neighborhood $V = V' \times V''$ and κ a curve from z_2 to z. Let ν be another curve from z_1 to z_2 which lies in V. Then the projections of $\kappa \cdot \mu \cdot \tau$ and $\kappa \cdot \nu \cdot \tau$ in M' have the same end point.

To prove this, we may first replace the curve κ by its projection in $M'(z_2)$ by Lemma 1. Thus we shall assume that κ is a T'-curve. Let μ be represented by the pair (μ', μ'') in $V = V' \times V''$. By Lemma 2, the projection of μ in $M'(z_1)$ is μ'. Let μ^* be a T''-geodesic in V joining z_2 and the end point of μ'. The parallel displacement of T'-vectors at z_2 along μ^{-1} is the same as the parallel displacement along $\mu'^{-1} \cdot \mu^*$, because μ'' and μ^* give the same parallel displacement for T'-vectors. By Lemma 5, we see that the projection of $\kappa \cdot \mu$ in $M'(z_1)$ is the curve μ' followed by the curve κ' obtained by using the homotopy z_t^s constructed from the T''-geodesic μ^* and the T'-curve κ. The homotopy z_t^s depends only on μ^* and κ and not on μ. Thus if we replace μ by ν in the above argument, we see that the projection of $\kappa \cdot \nu$ is equal to ν' followed by κ', where $\nu = (\nu', \nu'')$ in $V = V' \times V''$.

We now divide τ into a finite number of arcs, say, $\tau_1, \tau_2, \ldots, \tau_k$, such that each τ_i lies in a small neighborhood V_i of the form $V_i' \times V_i''$. We show that the projections of the curves $\kappa' \cdot \mu' \cdot \tau_k$ and $\kappa' \cdot \nu' \cdot \tau_k$ have the same end point in the maximal integral manifold of T' through the initial point of τ_k. Again, let $\tau_k = (\tau_k', \tau_k'')$ in $V_k = V_k' \times V_k''$ and let τ_k^* be the geodesic in V_k joining the end point of τ_k to the end point of τ_k'. As before, the projection of $\kappa' \cdot \mu' \cdot \tau_k$ is the curve τ_k' followed by the curve obtained by the homotopy which is constructed from the T''-geodesic τ_k^* and the T'-curve $\kappa' \cdot \mu'$. Similarly for the projection of $\kappa' \cdot \nu' \cdot \tau_k$. Each homotopy was constructed by the parallel displacement of the initial tangent vector of the geodesic τ_k^* along $\kappa' \cdot \mu'$ or along

$\kappa' \cdot \nu'$. Since $\nu'^{-1} \cdot \mu'$ is a curve in V', the parallel displacement along $\nu'^{-1} \cdot \mu'$ is trivial for T''-vectors. This means that the parallel displacements of the initial tangent vector of τ_k^* along μ' and ν' are the same so that the two homotopies produce the curves μ_k and ν_k starting at the end point of τ_k' and ending at the same point, where a curve κ_k starts in such a way that $\kappa_k \cdot \mu_k \cdot \tau_k'$ and $\kappa_k \cdot \nu_k \cdot \tau_k'$ are the projections of $\kappa' \cdot \mu' \cdot \tau_k$ and $\kappa' \cdot \nu' \cdot \tau_k$ respectively. We also remark that the parallel displacements of every T''-vector along μ_k and ν_k are the same; this indeed follows from property (a) of the homotopy in Lemma 4.

We continue to the next stage of projecting the curves $\kappa_k \cdot \mu_k \cdot \tau_k' \cdot \tau_{k-1}$ and $\kappa_k \cdot \nu_k \cdot \tau_k' \cdot \tau_{k-1}$ by the same method. As a result of the above remark, we have two curves ending at the same point. Now it is obvious that this process can be continued, thus completing the proof of Lemma 6.

Now we are in position to complete the proof of Theorem 6.1.

Lemma 6 allows us to define a mapping p' of M into M'. Similarly, we define a mapping p'' of M into M''. These mappings are differentiable. As we indicated before Lemma 1, we have only to show that the mapping $p = (p', p'')$ of M into $M' \times M''$ is isometric at each point. Let z be any point of M and let τ be a curve from o to z. For any tangent vector $Z \in T_z(M)$, let $Z = X + Y$, where $X \in T_z'$ and $Y \in T_z''$. By definition of the projection, it is clear that $p'(Z)$ is the same as the vector obtained by the parallel displacement of X from z to o along τ and then from o to $p'(z)$ along $p'(\tau)$. Therefore, $p'(Z)$ and X have the same length. Similarly, $p''(Z)$ and Y have the same length. It follows that Z and $p(Z) = (p'(Z), p''(Z))$ have the same length, proving that p is isometric at z. QED.

Combining Theorem 5.4 and Theorem 6.1, we obtain the decomposition theorem of de Rham.

THEOREM 6.2. *A connected, simply connected and complete Riemannian manifold M is isometric to the direct product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Euclidean space (possibly of dimension 0) and $M_1, \ldots, M_k$ are all simply connected, complete, irreducible Riemannian manifolds. Such a decomposition is unique up to an order.*

Theorems 6.1 and 6.2 are due to de Rham [1]. The proof of Theorem 6.1 is new; it was inspired by the work of Reinhart [1].

7. *Affine holonomy groups*

Let M be a connected Riemannian manifold. Fixing a point x of M, we denote the affine holonomy group $\Phi(x)$ and the linear holonomy group $\Psi(x)$ simply by Φ and Ψ, respectively. We know (cf. Theorem 5.5) that the restricted linear holonomy group Ψ^0 is a closed subgroup of $SO(n)$, where $n = \dim M$. Φ is a group of Euclidean motions of the affine (or rather *Euclidean*) tangent space $T_x(M)$.

We first prove the following result.

THEOREM 7.1. *If Ψ^0 is irreducible, then either*

(1) Φ^0 *contains all translations of* $T_x(M)$,

or

(2) Φ^0 *fixes a point of* $T_x(M)$.

Proof. Let K be the kernel of the homomorphism of Φ^0 onto Ψ^0 (cf. Proposition 3.5 of Chapter III). Since K is a normal subgroup of Φ^0 and since every element a of Φ^0 is of the form $a = \xi \cdot \tilde{a}$ where $\tilde{a} \in \Psi^0$ and ξ is a pure translation, Ψ^0 normalizes K, that is, $\tilde{a}^{-1}K\tilde{a} = K$ for every $\tilde{a} \in \Psi^0$. Consider first the case where K is not discrete. Since Ψ^0 is connected, it normalizes the identity component K^0 of K. Let V be the orbit of the origin of $T_x(M)$ by K^0. It is a non-trivial linear subspace of $T_x(M)$ invariant by Ψ^0; the invariance by Ψ^0 is a consequence of the fact that Ψ^0 normalizes K^0. Since Ψ^0 is irreducible by assumption, we have $V = T_x(M)$. This means that Φ^0 contains all translations of $T_x(M)$. Consider next the case where K is discrete. Since Ψ^0 is connected, Ψ^0 commutes with K elementwise. Hence, for every $\xi \in K$, $\xi(0)$ is invariant by Ψ^0 (where 0 denotes the origin of $T_x(M)$). Since Ψ^0 is irreducible, $\xi(0) = 0$ for every $\xi \in K$. This means that K consists of the identity element only and hence that Φ^0 is isomorphic to Ψ^0 in a natural manner. In particular, Φ^0 is compact. On the other hand, any compact group of affine transformations of $T_x(M)$ has a fixed point. Although we shall prove a more general statement in Volume II, we shall give here a direct proof of this fact. Let f be the mapping from Φ^0 into $T_x(M)$ defined by

$$f(a) = a(0) \qquad \text{for } a \in \Phi^0.$$

Let da be a bi-invariant Haar measure on Φ^0 and define

$$X_0 = \int f(a)\, da.$$

It is easy to verify that X_0 is a fixed point of Φ^0. QED.

We now investigate the second case of Theorem 7.1 (without assuming the irreducibility of M).

THEOREM 7.2. *Let M be a connected, simply connected and complete Riemannian manifold. If the (restricted) affine holonomy group Φ^0 at a point x fixes a point of the Euclidean tangent space $T_x(M)$, then M is isometric to a Euclidean space.*

Proof. Assuming that $X_0 \in T_x(M)$ is a point fixed by Φ^0, let τ be the geodesic from x to a point y which is developed upon the line segment tX_0, $0 \leqq t \leqq 1$. We observe that the affine holonomy group $\Phi^0(y)$ at y fixes the origin of $T_y(M)$. In fact, for any closed curve μ at y, the affine parallel displacement along $\tau^{-1} \cdot \mu \cdot \tau$ maps X_0 into itself, that is, $(\tau^{-1} \cdot \mu \cdot \tau)X_0 = X_0$. Hence the origin of $T_y(M)$ given by $\tau(X_0)$ is left fixed by μ. This shows that we may assume that Φ^0 fixes the origin of $T_x(M)$. Since M is complete, the exponential mapping $T_x(M) \to M$ is surjective. We show that it is 1:1. Assume that two geodesics τ and μ issuing from x meet at a point $y \neq x$. The affine parallel displacement $\mu^{-1} \cdot \tau$ maps the origin 0_x of $T_x(M)$ into itself and hence we have

$$\tau^{-1}(0_y) = \mu^{-1}(0_y),$$

where 0_y denotes the origin of $T_y(M)$. Since $\tau^{-1}(0_y)$ and $\mu^{-1}(0_y)$ are the end points of the developments of τ and μ in $T_x(M)$ respectively, these developments which are line segments coincide with each other. Thus $\tau = \mu$, contradicting the assumption that $x \neq y$. This proves that the exponential mapping $T_x(M) \to M$ is 1:1.

Assume that $\exp_x$ is a diffeomorphism of $N(x; r) = \{X \in T_x(M); \|X\| < r\}$ onto $U(x; r) = \{y \in M; d(x, y) < r\}$, and let $x^1, \ldots, x^n$ be a normal coordinate system on $U(x; r)$.

We set $X = -\Sigma_{i=1}^n x^i\, (\partial/\partial x^i)$ and let p be the corresponding point field (cf. §4 of Chapter III). We show that p is a parallel point field. Since Φ^0 fixes the origin of $T_x(M)$, it is sufficient to prove that p is parallel along each geodesic through x. Our assertion follows therefore from

LEMMA 1. *Let* $\tau = x_t$, $0 \leqq t \leqq 1$, *be a curve in a Riemannian manifold* M *and let* $\tilde{\tau}_s^t$ *(resp.* τ_s^t*) denote the affine (resp. linear) parallel displacement along* τ *from* x_t *to* x_s. *Then*

$$\tilde{\tau}_0^t(Y) = \tau_0^t(Y) + C_t, \qquad Y \in T_{x_t}(M),$$

where C_t, $0 \leqq t \leqq 1$, *is the development of* $\tau = x_t$ *into* $T_{x_0}(M)$.

Proof of Lemma 1. Given $Y \in T_{x_t}(M)$, let p (resp. q) be the point field along τ defined by the affine parallel displacement of Y (resp. the origin of $T_{x_t}(M)$) and let Y^* be the vector field along τ defined by the linear parallel displacement of Y. Then $p = q + Y^*$ at each point of τ, that is, Y^* is the vector with initial point q and end point p at each point of τ. At the point x_0, this means precisely $\tilde{\tau}_0^t(Y) = \tau_0^t(Y) + C_t$.

Going back to the proof of Theorem 7.2, we assert that

$$\nabla_V X + V = 0 \qquad \text{for any vector field } V.$$

This follows from

LEMMA 2. *Let* p *be a point field along a curve* $\tau = x_t$, $0 \leqq t \leqq 1$, *in a Riemannian manifold* M *and let* X *be the corresponding vector field along* τ. *Then* p *is a parallel point field if and only if*

$$\nabla_{\dot{x}_t} X + \dot{x}_t = 0 \qquad \textit{for } 0 \leqq t \leqq 1.$$

Proof of Lemma 2. From Lemma 1, we obtain

$$\tilde{\tau}_t^{t+h}(p_{x_{t+h}}) = \tau_t^{t+h}(X_{x_{t+h}}) + C_{t,h},$$

where $C_{t,h}$ (for a fixed t and with parameter h) is the development of τ into $T_{x_t}(M)$. Since $\tilde{\tau}_t^{t+h}(p_{x_{t+h}})$ is independent of h (and depends only on t) if and only if p is parallel, we have

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{1}{h} [\tau_t^{t+h}(X_{x_{t+h}}) - X_{x_t}] + \lim_{h \to 0} \frac{1}{h} C_{t,h} \\ &= \nabla_{\dot{x}_t} X + \dot{x}_t \end{aligned}$$

for $0 \leqq t \leqq 1$ if and only if p is parallel, completing the proof of Lemma 2.

Let Y and Z be arbitrary vector fields on M. From $\nabla_Y X + Y = 0$ and $\nabla_Z X + Z = 0$, we obtain (cf. Theorem 5.1 of Chapter III)

$$\nabla_X Y = \nabla_Y X + [X, Y] = -Y + [X, Y]$$

and

$$\nabla_X Z = \nabla_Z X + [X, Z] = -Z + [X, Z].$$

Hence,

$$\begin{aligned} X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &= -2g(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]). \end{aligned}$$

Let $Y = \partial/\partial x^j$ and $Z = \partial/\partial x^k$ for any fixed j and k. Then we have

$$X \cdot g_{jk} = -2g_{jk} + g_{jk} + g_{jk} = 0.$$

This means that the functions g_{jk} are invariant by the local 1-parameter group of transformations φ_t generated by X. But φ_t is of the form

$$\varphi_t(x^1, \ldots, x^n) = (e^{-t}x^1, \ldots, e^{-t}x^n).$$

Thus the functions g_{jk} are constant along each geodesic through x. Hence,

$$g_{jk} = g_{jk}(x) = \delta_{jk} \quad \text{at every point of } U(x; r).$$

This shows that $\exp_x$ is an isometric mapping of $N(x; r)$ with Euclidean metric onto $U(x; r)$. Let r_0 be the supremum of $r > 0$ such that $\exp_x$ is a diffeomorphism of $N(x; r)$ onto $U(x; r)$. Since the differential $(\exp_x)_*$ is non-singular at every point of $N(x; r_0)$, $\exp_x$ is a diffeomorphism, hence an isometry, by the argument above, of $N(x; r_0)$ onto $U(x; r_0)$. If $r_0 < \infty$, it follows that $(\exp_x)_*$ is isometric at every point y of the boundary of $N(x; r_0)$ and hence nonsingular in a neighborhood of such y. Since the boundary of $N(x; r_0)$ is compact, we see that there exists $\varepsilon > 0$ such that $\exp_x$ is a diffeomorphism of $N(x; r_0 + \varepsilon)$ onto $U(x; r_0 + \varepsilon)$, contradicting the definition of r_0. This shows that $\exp_x$ is a diffeomorphism of $T_x(M)$ onto M. By choosing a normal coordinate system $x^1, \ldots, x^n$ on the whole M, we conclude that $g_{jk} = \delta_{jk}$ at every point of M, that is, M is a Euclidean space. QED.

As a consequence we obtain the following corollary due to Goto and Sasaki [1].

COROLLARY 7.3. *Let M be a connected and complete Riemannian manifold. If the restricted affine holonomy group $\Phi^0(x)$ fixes a point of the Euclidean tangent space $T_x(M)$ for some $x \in M$, then M is locally Euclidean (that is, every point of M has a neighborhood which is isometric to an open subset of a Euclidean space).*

Proof. Apply Theorem 7.2 to the universal covering space of M. QED.

COROLLARY 7.4. *If M is a complete Riemannian manifold of dimension > 1 and if the restricted linear holonomy group $\Psi^0(x)$ is irreducible, then the restricted affine holonomy group $\Phi^0(x)$ contains all translations of $T_x(M)$.*

Proof. Since $\Psi^0(x)$ is irreducible, M is not locally Euclidean. Our assertion now follows from Theorem 7.1 and Corollary 7.3. QED.

CHAPTER V

Curvature and Space Forms

1. *Algebraic preliminaries*

Let V be an n-dimensional real vector space and $R: V \times V \times V \times V \to \mathbf{R}$ a quadrilinear mapping with the following three properties:

(a) $R(v_1, v_2, v_3, v_4) = -R(v_2, v_1, v_3, v_4)$
(b) $R(v_1, v_2, v_3, v_4) = -R(v_1, v_2, v_4, v_3)$
(c) $R(v_1, v_2, v_3, v_4) + R(v_1, v_3, v_4, v_2) + R(v_1, v_4, v_2, v_3) = 0.$

Proposition 1.1. *If R possesses the above three properties, then it possesses also the following fourth property:*
(d) $R(v_1, v_2, v_3, v_4) = R(v_3, v_4, v_1, v_2)$.
Proof. We denote by $S(v_1, v_2, v_3, v_4)$ the left hand side of (c). By a straightforward computation, we obtain

$$\begin{aligned} 0 &= S(v_1, v_2, v_3, v_4) - S(v_2, v_3, v_4, v_1) - S(v_3, v_4, v_1, v_2) \\ &\qquad + S(v_4, v_1, v_2, v_3) \\ &= R(v_1, v_2, v_3, v_4) - R(v_2, v_1, v_3, v_4) - R(v_3, v_4, v_1, v_2) \\ &\qquad + R(v_4, v_3, v_1, v_2). \end{aligned}$$

By applying (a) and (b), we see that

$$2R(v_1, v_2, v_3, v_4) - 2R(v_3, v_4, v_1, v_2) = 0.$$

QED.

Proposition 1.2. *Let R and T be two quadrilinear mappings with the above properties* (a), (b) *and* (c). *If*

$$R(v_1, v_2, v_1, v_2) = T(v_1, v_2, v_1, v_2) \qquad \textit{for all } v_1, v_2 \in V,$$

then $R = T$.

Proof. We may assume that $T = 0$; consider $R - T$ and 0 instead of R and T. We assume therefore that $R(v_1, v_2, v_1, v_2) = 0$ for all $v_1, v_2 \in V$. We have

$$\begin{aligned} 0 &= R(v_1, v_2 + v_4, v_1, v_2 + v_4) \\ &= R(v_1, v_2, v_1, v_4) + R(v_1, v_4\, v_1, v_2) \\ &= 2R(v_1, v_2, v_1, v_4). \end{aligned}$$

Hence,

$$(1) \qquad R(v_1, v_2, v_1, v_4) = 0 \qquad \text{for all } v_1, v_2, v_4 \in V.$$

From (1) we obtain

$$\begin{aligned} 0 &= R(v_1 + v_3, v_2, v_1 + v_3, v_4) \\ &= R(v_1, v_2, v_3, v_4) + R(v_3, v_2, v_1, v_4). \end{aligned}$$

Now, by applying (d) and then (b), we obtain

$$\begin{aligned} 0 &= R(v_1, v_2, v_3, v_4) + R(v_1, v_4, v_3, v_2) \\ &= R(v_1, v_2, v_3, v_4) - R(v_1, v_4, v_2, v_3). \end{aligned}$$

Hence,

$$(2) \quad R(v_1, v_2, v_3, v_4) = R(v_1, v_4, v_2, v_3) \qquad \text{for all } v_1, v_2, v_3, v_4 \in V.$$

Replacing v_2, v_3, v_4 by v_3, v_4, v_2, respectively, we obtain

$$(3) \quad R(v_1, v_2, v_3, v_4) = R(v_1, v_3, v_4, v_2) \qquad \text{for all } v_1, v_2, v_3, v_4 \in V.$$

From (2) and (3), we obtain

$$\begin{aligned} 3R(v_1, v_2, v_3, v_4) = R(v_1, v_2, v_3, v_4) &+ R(v_1, v_3, v_4, v_2) \\ &+ R(v_1, v_4, v_2, v_3), \end{aligned}$$

where the right hand side vanishes by (c). Hence,

$$R(v_1, v_2, v_3, v_4) = 0 \qquad \text{for all } v_1, v_2, v_3, v_4 \in V.$$

QED.

Besides a quadrilinear mapping R, we consider an inner product (i.e., a positive definite symmetric bilinear form) on V, which will be denoted by (,). Let p be a plane, that is, a 2-dimensional subspace, in V and let v_1 and v_2 be an orthonormal basis for p. We set

$$K(p) = R(v_1, v_2, v_1, v_2).$$

As the notation suggests, $K(p)$ is independent of the choice of an orthonormal basis for p. In fact, if w_1 and w_2 form another orthonormal basis of p, then

$$w_1 = av_1 + bv_2, \quad w_2 = -bv_1 + av_2 \quad (\text{or } bv_1 - av_2),$$

where a and b are real numbers such that $a^2 + b^2 = 1$. Using (a) and (b), we easily obtain $R(v_1, v_2, v_1, v_2) = R(w_1, w_2, w_1, w_2)$.

PROPOSITION 1.3. *If v_1, v_2 is a basis (not necessarily orthonormal) of a plane p in V, then*

$$K(p) = \frac{R(v_1, v_2, v_1, v_2)}{(v_1, v_1)(v_2, v_2) - (v_1, v_2)^2}.$$

Proof. We obtain the formula making use of the following orthonormal basis for p:

$$\frac{v_1}{(v_1, v_1)^{\frac{1}{2}}}, \qquad \frac{1}{a}\,[(v_1, v_1)v_2 - (v_1, v_2)v_1]$$

where $a = [(v_1, v_1)((v_1, v_1)(v_2, v_2) - (v_1, v_2)^2)]^{\frac{1}{2}}$ QED.

We set

$$R_1(v_1, v_2, v_3, v_4) = (v_1, v_3)(v_2, v_4) - (v_2, v_3)(v_4, v_1)$$
$$\text{for } v_1, v_2, v_3, v_4 \in V.$$

It is a trivial matter to verify that R_1 is a quadrilinear mapping having the properties (a), (b) and (c) and that, for any plane p in V, we have

$$K_1(p) = R_1(v_1, v_2, v_1, v_2) = 1,$$

where v_1, v_2 is an orthonormal basis for p.

PROPOSITION 1.4. *Let R be a quadrilinear mapping with properties* (a), (b) *and* (c). *If $K(p) = c$ for all planes p, then $R = cR_1$.*

Proof. By Proposition 1.3, we have

$$R(v_1, v_2, v_1, v_2) = cR_1(v_1, v_2, v_1, v_2) \qquad \text{for all } v_1, v_2 \in V.$$

Applying Proposition 1.2 to R and cR_1, we conclude $R = cR_1$.
QED.

Let $e_1, \ldots, e_n$ be an orthonormal basis for V with respect to the inner product (,). To each quadrilinear mapping R having properties (a), (b) and (c), we associate a symmetric bilinear form S on V as follows:

$$S(v_1, v_2) = R(e_1, v_1, e_1, v_2) + R(e_2, v_1, e_2, v_2) + \cdots + R(e_n, v_1, e_n, v_2), \quad v_1, v_2 \in V.$$

It can be easily verified that S is independent of the choice of an orthonormal basis $e_1, \ldots, e_n$. From the definition of S, we obtain

PROPOSITION 1.5. *Let $v \in V$ be a unit vector and let $v, e_2, \ldots, e_n$ be an orthonormal basis for V. Then*

$$S(v, v) = K(p_2) + \cdots + K(p_n),$$

where p_i is the plane spanned by v and e_i.

2. *Sectional curvature*

Let M be an n-dimensional Riemannian manifold with metric tensor g. Let $R(X, Y)$ denote the curvature transformation of $T_x(M)$ determined by $X, Y \in T_x(M)$ (cf. §5 of Chapter III). The *Riemannian curvature tensor* (*field*) of M, denoted also by R, is the tensor field of covariant degree 4 defined by

$$R(X_1, X_2, X_3, X_4) = g(R(X_3, X_4)X_2, X_1), \quad X_i \in T_x(M), i = 1, \ldots, 4.$$

PROPOSITION 2.1. *The Riemannian curvature tensor, considered as a quadrilinear mapping $T_x(M) \times T_x(M) \times T_x(M) \times T_x(M) \to \mathbf{R}$ at each $x \in M$, possesses properties* (a), (b), (c) *and hence* (d) *of* §1.

Proof. Let u be any point of the bundle $O(M)$ of orthonormal frames such that $\pi(u) = x$. Let $X_3^*, X_4^* \in T_u(O(M))$ with $\pi(X_3^*) = X_3$ and $\pi(X_4^*) = X_4$. From the definition of the curvature transformation $R(X_3, X_4)$ given in §5 of Chapter III, we obtain

$$\begin{aligned} g(R(X_3, X_4)X_2, X_1) &= g(u[2\Omega(X_3^*, X_4^*)(u^{-1}X_2)], X_1) \\ &= ((2\Omega(X_3^*, X_4^*))(u^{-1}X_2), u^{-1}X_1), \end{aligned}$$

where (,) is the natural inner product in $\mathbf{R}^n$. Now we see that property (*a*) is a consequence of the fact that $\Omega(X_3^*, X_4^*) \in \mathfrak{o}(n)$ is a skew-symmetric matrix. (b) follows from $R(X_3, X_4) = -R(X_4, X_3)$. Finally, (c) is a consequence of Bianchi's first identity given in Theorem 5.3 of Chapter III. QED.

For each plane p in the tangent space $T_x(M)$, the *sectional curvature* $K(p)$ for p is defined by

$$K(p) = R(X_1, X_2, X_1, X_2) = g(R(X_1, X_2)X_2, X_1),$$

where X_1, X_2 is an orthonormal basis for p. As we saw in §1, $K(p)$ is independent of the choice of an orthonormal basis X_1, X_2. Proposition 1.2 implies that the set of values of $K(p)$ for all planes p in $T_x(M)$ determines the Riemannian curvature tensor at x.

If $K(p)$ is a constant for all planes p in $T_x(M)$ and for all points $x \in M$, then M is called a *space of constant curvature.* The following theorem is due to F. Schur [1].

THEOREM 2.2. *Let M be a connected Riemannian manifold of dimension $\geqq 3$. If the sectional curvature $K(p)$, where p is a plane in $T_x(M)$, depends only on x, then M is a space of constant curvature.*

Proof. We define a covariant tensor field R_1 of degree 4 as follows:

$$R_1(W, Z, X, Y) = g(W, X)g(Z, Y) - g(Z, X)g(Y, W), \qquad W, Z, X, Y \in T_x(M).$$

By Proposition 1.4, we have

$$R = kR_1,$$

where k is a function on M. Since g is parallel, so is R_1. Hence,

$$(\nabla_U R)(W, Z, X, Y) = (\nabla_U k)R_1(W, Z, X, Y) \qquad \text{for any } U \in T_x(M).$$

This means that, for any $X, Y, Z, U \in T_x(M)$, we have

$$[(\nabla_U R)(X, Y)]Z = (Uk)(g(Z, Y)X - g(Z, X)Y).$$

Consider the cyclic sum of the above identity with respect to (U, X, Y). The left hand side vanishes by Bianchi's second identity (Theorem 5.3 of Chapter III). Thus we have

$$\begin{aligned} 0 = {} & (Uk)(g(Z, Y)X - g(Z, X)Y) \\ & + (Xk)(g(Z, U)Y - g(Z, Y)U) \\ & + (Yk)(g(Z, X)U - g(Z, U)X). \end{aligned}$$

For an arbitrary X, we choose Y, Z and U in such a way that X, Y and Z are mutually orthogonal and that $U = Z$ with $g(Z, Z) = 1$. This is possible since $\dim M \geqq 3$. Then we obtain

$$(Xk)Y - (Yk)X = 0.$$

Since X and Y are linearly independent, we have $Xk = Yk = 0$. This shows that k is a constant. QED.

COROLLARY 2.3. *For a space of constant curvature k, we have*

$$R(X, Y)Z = k(g(Z, Y)X - g(Z, X)Y).$$

This was established in the course of proof for Theorem 2.2.

If k is a positive (resp. negative) constant, M is called a space of constant positive (resp. negative) curvature.

If R^i_{jkl} and g_{ij} are the components of the curvature tensor and the metric tensor with respect to a local coordinate system (cf. §7 of Chapter III), then the components R_{ijkl} of the Riemannian curvature tensor are given by

$$R_{ijkl} = \Sigma_m\, g_{im} R^m_{jkl}.$$

If M is a space of constant curvature with $K(p) = k$, then

$$R_{ijkl} = k(g_{ik}g_{jl} - g_{jk}g_{li}), \qquad R^i_{jkl} = k(\delta^i_k g_{jl} - g_{jk}\delta^i_l).$$

As in §7 of Chapter III, we define a set of functions $\tilde{R}^i_{jkl}$ on $L(M)$ by

$$\Omega^i_j = \Sigma_{k,l}\, \tfrac{1}{2}\tilde{R}^i_{jkl}\theta^k \wedge \theta^l,$$

where $\Omega = (\Omega^i_j)$ is the curvature form of the Riemannian connection. For an arbitrary point u of $O(M)$, we choose a local coordinate system $x^1, \ldots, x^n$ with origin $x = \pi(u)$ such that u is the frame given by $(\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x$. With respect to this coordinate system, we have

$$g_{ij} = \delta_{ij} \quad \text{at } x,$$

and hence

$$R^i_{jkl} = R_{ijkl} = k(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{li}) \quad \text{at } x.$$

Let σ be the local cross section of $L(M)$ given by the field of linear frames $\partial/\partial x^1, \ldots, \partial/\partial x^n$. As we have shown in §7 of

Chapter III, we have $\sigma^*\tilde{R}^i_{jkl} = R^i_{jkl}$. Hence,

$$\tilde{R}^i_{jkl} = k(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{li}) \quad \text{at } u,$$
$$\Omega^i_j = k\theta^i \wedge \theta^j \quad \text{at } u.$$

Since u is an arbitrary point of $O(M)$, we have

PROPOSITION 2.4. *If M is a space of constant curvature with sectional curvature k, then the curvature form $\Omega = (\Omega^i_j)$ is given by*

$$\Omega^i_j = k\theta^i \wedge \theta^j \quad \text{on } O(M),$$

where $\theta = (\theta^i)$ is the canonical form on $O(M)$.

3. *Spaces of constant curvature*

In this section, we shall construct, for each constant k, a simply connected, complete space of constant curvature with sectional curvature k. Namely, we prove

THEOREM 3.1. *Let $(x^1, \ldots, x^n, t)$ be the coordinate system of $\mathbf{R}^{n+1}$ and M the hypersurface of $\mathbf{R}^{n+1}$ defined by*

$$(x^1)^2 + \cdots + (x^n)^2 + rt^2 = r \quad (r\text{: a nonzero constant}).$$

Let g be the Riemannian metric of M obtained by restricting the following form to M:

$$(dx^1)^2 + \cdots + (dx^n)^2 + r\,dt^2.$$

Then

(1) *M is a space of constant curvature with sectional curvature $1/r$.*

(2) *The group G of linear transformations of $\mathbf{R}^{n+1}$ leaving the quadratic form $(x^1)^2 + \cdots + (x^n)^2 + rt^2$ invariant acts transitively on M as a group of isometries of M.*

(3) *If $r > 0$, then M is isometric to a sphere of a radius $r^{\frac{1}{2}}$. If $r < 0$, then M consists of two mutually isometric connected manifolds each of which is diffeomorphic with $\mathbf{R}^n$.*

Proof. First we observe that M is a closed submanifold of $\mathbf{R}^{n+1}$ (cf. Example 1.1 of Chapter I); we leave the verification to the reader.

We begin with the proof of (3). If $r > 0$, then we set $x^{n+1} = r^{\frac{1}{2}}t$. Then M is given by

$$(x^1)^2 + \cdots + (x^{n+1})^2 = r,$$

and the metric g is the restriction of $(dx^1)^2 + \cdots + (dx^{n+1})^2$ to M. This means that M is isometric with a sphere of radius $r^{\frac{1}{2}}$. If $r < 0$, then $t^2 \geqq 1$ at every point of M. Let M' (resp. M'') be the set of points of M with $t \geqq 1$ (resp. $t \leqq -1$). The mapping $(x^1, \ldots, x^n, t) \to (y^1, \ldots, y^n)$ defined by

$$y^i = x^i/t, \qquad i = 1, \ldots, n,$$

is a diffeomorphism of M' (and M'') onto the open subset of $\mathbf{R}^n$ given by

$$\Sigma_{i=1}^n (y^i)^2 + r < 0.$$

In fact, the inverse mapping is given by

$$x^i = y^i t, \qquad i = 1, \ldots, n,$$

$$t = \pm\left(\frac{r}{r + \Sigma_i (y^i)^2}\right)^{\frac{1}{2}}.$$

A straightforward computation shows that the metric g is expressed in terms of $y^1, \ldots, y^n$ as follows:

$$\frac{r[(r + \Sigma_i (y^i)^2)(\Sigma_i (dy^i)^2) - (\Sigma_i y^i\, dy^i)^2]}{(r + \Sigma_i (y^i)^2)^2}.$$

To prove (2), we first consider G as a group acting on $\mathbf{R}^{n+1}$. Since G is a linear group leaving $(x^1)^2 + \cdots + (x^n)^2 + rt^2$ invariant, it leaves the form $(dx^1)^2 + \cdots + (dx^n)^2 + r\, dt^2$ invariant. Thus, considered as a group acting on M, G is a group of isometries of the Riemannian manifold M. The transitivity of G on M is a consequence of Witt's theorem, which may be stated as follows.

Let Q be a nondegenerate quadratic form on a vector space V. If f is a linear mapping of a subspace U of V into V such that $Q(f(x)) = Q(x)$ for all $x \in U$, then f can be extended to a linear isomorphism of V onto itself such that $Q(f(x)) = Q(x)$ for all $x \in V$. In particular, if x_0 and x_1 are elements of V with $Q(x_0) = Q(x_1)$, there is a linear isomorphism f of V onto itself which leaves Q invariant and which maps x_0 into x_1.

For the proof of Witt's theorem, see, for example, Artin [1, p. 121].

Finally, we shall prove (1). Let H be the subgroup of G which consists of transformations leaving the point o with coordinates

$(0, \ldots, 0, 1)$ fixed. We define a mapping $f\colon G \to O(M)$ as follows. Let $u_0 \in O(M)$ be the frame at the point $o = (0, \ldots, 0, 1) \in M$ given by $(\partial/\partial x^1)_o, \ldots, (\partial/\partial x^n)_o$. Every element $a \in G$, being an isometric transformation of M, maps each orthonormal frame of M into an orthonormal frame. In particular, $a(u_0)$ is an orthonormal frame of M at the point $a(o)$. We define

$$f(a) = a(u_0), \qquad a \in G.$$

Lemma 1. *The mapping $f\colon G \to O(M)$ is an isomorphism of the principal fibre bundle $G(G/H, H)$ onto the bundle $O(M)(M, O(n))$.*

Proof of Lemma 1. If we consider G as a group of $(n + 1) \times (n + 1)$-matrices in a natural manner, then H is naturally isomorphic with $O(n)$:

$$H = \begin{pmatrix} O(n) & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to verify that $f\colon G \to O(M)$ commutes with the right translation R_a for every $a \in H = O(n)$:

$$f(ba) = f(b) \cdot a \qquad \text{for } b \in G \text{ and } a \in H = O(n).$$

The transitivity of G on M implies that the induced mapping $f\colon G/H \to M$ is a diffeomorphism and hence that $f\colon G \to O(M)$ is a bundle isomorphism.

The quadratic form defining M is given by the following $(n + 1) \times (n + 1)$-matrix:

$$Q = \begin{pmatrix} I_n & 0 \\ 0 & r \end{pmatrix}.$$

An $(n + 1) \times (n + 1)$-matrix a is an element of G if and only if ${}^t aQa = Q$, where ${}^t a$ is the transpose of a. Let

$$a = \begin{pmatrix} X & y \\ {}^t z & w \end{pmatrix},$$

where X is an $n \times n$-matrix, y and z are elements of $\mathbf{R}^n$ and w is a real number. Then the condition for a to be in G is expressed by

$${}^t XX + r \cdot z\, {}^t z = I_n, \quad {}^t Xy + r \cdot zw = 0, \quad {}^t yy + r \cdot w^2 = r.$$

It follows that the Lie algebra of G is formed by the matrices of the form

$$\begin{pmatrix} A & b \\ {}^tc & 0 \end{pmatrix},$$

where A is an $n \times n$-matrix with ${}^tA + A = 0$ and b and c are elements of $\mathbf{R}^n$ satisfying $b + rc = 0$. Let

$$\begin{pmatrix} \alpha_1^1 \ldots \alpha_n^1 & \beta^1 \\ \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots & \\ \alpha_1^n \ldots \alpha_n^n & \beta^n \\ \gamma_1 \ldots \gamma_n & 0 \end{pmatrix}$$

be the (left invariant) canonical 1-form on G (cf. §4 of Chapter I). We have

$$\alpha_j^i + \alpha_i^j = 0, \quad \beta^i + r\gamma_i = 0, \qquad i, j = 1, \ldots, n.$$

The Maurer-Cartan equation of G is expressed by

$$\begin{aligned} d\beta^i &= -\Sigma_k\, \alpha_k^i \wedge \beta^k, \\ d\alpha_j^i &= -\Sigma_k\, \alpha_k^i \wedge \alpha_j^k - \beta^i \wedge \gamma_j, \qquad i, j = 1, \ldots, n. \end{aligned}$$

LEMMA 2. *Let $\theta = (\theta^i)$ and $\omega = (\omega_j^i)$ be the canonical form and the connection form on $O(M)$. Then*

$$f^*\theta^i = \beta^i \quad \textit{and} \quad f^*\omega_j^i = \alpha_j^i, \qquad i, j = 1, \ldots, n.$$

Proof of Lemma 2. As we remarked earlier, every element $a \in G$ induces a transformation of $O(M)$; this transformation corresponds to the left translation by a in G under the isomorphism $f\colon G \rightarrow O(M)$. From the definition of θ, we see easily that $\theta = (\theta^i)$ is invariant by the transformation induced by each $a \in G$. On the other hand, (β^i) is invariant by the left translation by each $a \in G$. To prove $f^*\theta^i = \beta^i$, it is therefore sufficient to show that $(f^*\theta^i)(X^*) = \beta^i(X^*)$ for all $X^* \in T_e(G)$. Set $X_i = (\partial/\partial x^i)_o$ so that the frame u_0 is given by $(X_1, \ldots, X_n)$. The composite mapping $\pi \circ f\colon G \rightarrow O(M) \rightarrow M$ maps an element of $T_e(G)$ (identified with the Lie algebra of G) of the form

$$\begin{pmatrix} A & b \\ {}^tc & 0 \end{pmatrix}$$

into the vector $\Sigma_i b^i X_i \in T_o(M)$, where $b^1, \ldots, b^n$ are the components of b. Therefore, if $X^* \in T_e(G)$, then $\pi \circ f(X^*) = \Sigma_i \beta^i(X^*)X_i$ and hence

$$(f^*\theta^1(X^*), \ldots, f^*\theta^n(X^*)) = u_0^{-1}(\pi \circ f(X^*))$$
$$= (\beta^1(X^*), \ldots, \beta^n(X^*)),$$

which proves the first assertion of the lemma. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Let $\mathfrak{m}$ be the linear subspace of $\mathfrak{g}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & b \\ {}^t c & 0 \end{pmatrix}.$$

It is easy to verify that $\mathfrak{m}$ is stable under ad H, that is, ad $(a)(\mathfrak{m}) = \mathfrak{m}$ for every $a \in H$. Applying Theorem 11.1 of Chapter II, we see that (α^i_j) defines a connection in the bundle $G(G/H, H)$. Now, the second assertion of Lemma 2 follows from the following three facts: (1) (β^i) corresponds to (θ^i) under the isomorphism $f: G \to O(M)$; (2) the Riemannian connection form (ω^i_j) is characterized by the property that the torsion is zero (Theorem 2.2 of Chapter IV), that is, $d\theta^i = -\Sigma_k \omega^i_k \wedge \theta^k$; (3) the connection form (α^i_j) satisfies the equality: $d\beta^i = -\Sigma_k \alpha^i_k \wedge \beta^k$.

We shall now complete the proof of Theorem 3.1. Lemma 2, together with

$$d\alpha^i_j = -\Sigma_k \alpha^i_k \wedge \alpha^k_j - \beta^i \wedge \gamma_j$$

and

$$\beta^i + r\gamma_i = 0,$$

implies

$$d\alpha^i_j = -\Sigma_k \alpha^i_k \wedge \alpha^k_j + \frac{1}{r} \beta^i \wedge \beta^j,$$

showing that the curvature form of the Riemannian connection is given by $\dfrac{1}{r} \theta^i \wedge \theta^j$. By Proposition 2.4, M is a space of constant curvature with sectional curvature $1/r$. QED.

Remark. The group G is actually the group of all isometries of M. To see this, let $\mathfrak{I}(M)$ be the group of isometries of M and define a mapping $f: \mathfrak{I}(M) \to O(M)$ in the same way as we defined $f: G \to O(M)$. Then $G \subset \mathfrak{I}(M)$ and $f: \mathfrak{I}(M) \to O(M)$

is an extension of $f: G \to O(M)$. Since f maps $\mathfrak{I}(M)$ 1:1 into $O(M)$ and since $f(G) = O(M)$, we must have $G = \mathfrak{I}(M)$.

In the course of the proof of Theorem 3.1, we obtained

THEOREM 3.2. (1) *Let M be the sphere in $\mathbf{R}^{n+1}$ defined by*

$$(x^1)^2 + \cdots + (x^{n+1})^2 = a^2.$$

Let g be the restriction of $(dx^1)^2 + \cdots + (dx^{n+1})^2$ to M. Then, with respect to this Riemannian metric g, M is a space of constant curvature with sectional curvature $1/a^2$.

(2) *Let M be the open set in $\mathbf{R}^n$ defined by*

$$(x^1)^2 + \cdots + (x^n)^2 < a^2.$$

Then, with respect to the Riemannian metric given by

$$\frac{a^2[(a^2 - \Sigma_i (y^i)^2)(\Sigma_i (dy^i)^2) - (\Sigma_i y^i\, dy^i)^2]}{(a^2 - \Sigma_i (y^i)^2)^2},$$

M is a space of constant curvature with sectional curvature $-1/a^2$.

The spaces M constructed in Theorem 3.2 are all simply connected, homogeneous and hence complete by Theorem 4.5 of Chapter IV. The space $\mathbf{R}^n$ with the Euclidean metric $(dx^1)^2 + \cdots + (dx^n)^2$ gives a simply connected, complete space of zero curvature.

A Riemannian manifold of constant curvature is said to be *elliptic*, *hyperbolic* or *flat* (or *locally Euclidean*) according as the sectional curvature is positive, negative or zero. These spaces are also called *space forms* (cf. Theorem 7.10 of Chapter VI).

4. *Flat affine and Riemannian connections*

Throughout this section, M will be a connected, paracompact manifold of dimension n.

Let $A(M)$ be the bundle of affine frames over M; it is a principal fibre bundle with structure group $G = A(n; \mathbf{R})$ (cf. §3 of Chapter III). An affine connection of M is said to be *flat* if every point of M has an open neighborhood U and an isomorphism $\psi: A(M) \to U \times G$ which maps the horizontal space at each $u \in A(U)$ into the horizontal space at $\psi(u)$ of the canonical flat connection of $U \times G$. A manifold with a flat affine connection is said to be *locally affine*. A Riemannian manifold is *flat* (or *locally Euclidean*) if the Riemannian connection is a flat affine connection.

THEOREM 4.1. *For an affine connection of M, the following conditions are mutually equivalent:*

(1) *It is flat;*

(2) *The torsion and the curvature of the corresponding linear connection vanish identically;*

(3) *The affine holonomy group is discrete.*

Proof. By Theorem 9.1 of Chapter II, an affine connection is flat if and only if its curvature form $\tilde{\Omega}$ on $A(M)$ vanishes identically. The equivalence of (1) and (2) follows from Proposition 3.4 of Chapter III. The equivalence of (2) and (3) follows from Theorems 4.2 and 8.1 of Chapter II. QED.

Remark. Similarly, for a linear connection of M, the following conditions are mutually equivalent:

(1) It is flat, i.e., the connection in $L(M)$ is flat; (2) Its curvature vanishes identically; (3) The linear (or homogeneous) holonomy group is discrete.

When we say that the affine holonomy group and the linear holonomy group are discrete, we mean that they are 0-dimensional Lie groups. Later (cf. Theorem 4.2) we shall see that the affine holonomy group of a complete flat affine connection is discrete in the affine group $A(n; \mathbf{R})$. But the linear holonomy group need not be discrete in $GL(n; \mathbf{R})$ (cf. Example 4.3). It will be shown that the linear holonomy group of a compact flat Riemannian manifold is discrete in $O(n)$ (cf. the proof of (4) of Theorem 4.2 and the remark following Theorem 4.2).

Example 4.1. Let $\xi_1, \ldots, \xi_k$ be linearly independent elements of $\mathbf{R}^n$, $k \leqq n$. Let G be the subgroup of $\mathbf{R}^n$ generated by $\xi_1, \ldots, \xi_k$:

$$G = \{\Sigma\, m_i \xi_i;\ m_i \text{ integers}\}.$$

The action of G on $\mathbf{R}^n$ is properly discontinuous, and $\mathbf{R}^n$ is the universal covering manifold of $\mathbf{R}^n/G$. The Euclidean metric $(dx^1)^2 + \cdots + (dx^n)^2$ of $\mathbf{R}^n$ is invariant by G and hence induces a flat Riemannian metric on $\mathbf{R}^n/G$. The manifold $\mathbf{R}^n/G$ with the Riemannian metric thus defined will be called a *Euclidean cylinder*. It is called a *Euclidean torus* if $\xi_1, \ldots, \xi_k$ form a basis of $\mathbf{R}^n$, i.e., $k = n$. Every connected abelian Lie group with an invariant

Riemannian metric is a Euclidean cylinder, and if it is, moreover, compact, then it is a Euclidean torus. In fact, the universal covering group of such a Lie group is isomorphic with a vector group $\mathbf{R}^n$ and its invariant Riemannian metric is given by $(dx^1)^2 + \cdots + (dx^n)^2$ by a proper choice of basis in $\mathbf{R}^n$. Our assertion is now clear.

The following example shows that a torus can admit a flat affine connection which is not Riemannian. This was taken from Kuiper [1].

Example 4.2. The set G of transformations

$$(x,y) \to (x + ny + m, y + n),$$
$$n,m = 0, \pm 1, \pm 2, \ldots,$$

of $\mathbf{R}^2$ with coordinate sytem (x,y) forms a discrete subgroup of the group of affine transformations; it acts properly discontinuously on $\mathbf{R}^2$ and the quotient space $\mathbf{R}^2/G$ is diffeomorphic with a torus. The flat affine connection of $\mathbf{R}^2$ induces a flat affine connection on $\mathbf{R}^2/G$. This flat affine connection of $\mathbf{R}^2/G$ is not Riemannian. In fact, if it is Riemannian, the induced Riemannian metric on the universal covering space $\mathbf{R}^2$ must be of the form $a\,dx\,dx + 2b\,dx\,dy + c\,dy\,dy$, where a, b and c are constants, since the metric must be parallel. On the other hand, G is not a group of isometries of $\mathbf{R}^2$ with respect to this metric, thus proving our assertion.

Let M be locally affine and choose a linear frame $u_0 \in L(M) \subset A(M)$. Let M^* be the holonomy bundle through u_0 of the flat affine connection and M' the holonomy bundle through u_0 of the corresponding flat linear connection. Then M^* (resp. M') is a principal fibre bundle over M whose structure group is the affine holonomy group $\Phi(u_0)$ (resp. the linear holonomy group $\Psi(u_0)$). Since $\Phi(u_0)$ and $\Psi(u_0)$ are discrete, both M^* and M' are covering manifolds of M. The homomorphism $\beta\colon A(M) \to L(M)$ defined in §3 of Chapter III maps M^* onto M' (cf. Proposition 3.5 of Chapter III). Hence M^* is a covering manifold of M'.

THEOREM 4.2. *Let M be a manifold with a complete, flat affine connection. Let $u_0 \in L(M) \subset A(M)$. Let M^* be the holonomy bundle through u_0 of the flat affine connection and M' the holonomy bundle through u_0 of the corresponding flat linear connection. Then*

(1) *M^* is the universal covering space of M and, with respect to the flat affine connection induced on M^*, it is isomorphic to the ordinary affine space A^n.*

(2) *With respect to the flat affine connection induced on M', M' is a Euclidean cylinder, and the first homotopy group of M' is isomorphic to the kernel of the homomorphism $\Phi(u_0) \to \Psi(u_0)$.*

(3) *If M'' is a Euclidean cylinder and is a covering space of M, then it is a covering space of M'.*

(4) *M' is a Euclidean torus if and only if M is a compact flat Riemannian manifold.*

Proof. Let

$$d\theta^i = -\Sigma_j \, \omega^i_j \wedge \theta^j, \quad d\omega^i_j = -\Sigma_k \, \omega^i_k \wedge \omega^k_j, \quad i, j = 1, \ldots, n,$$

be the structure equations on $L(M')$ of the flat affine connection of M'. Let N be the kernel of the homomorphism $\Phi(u_0) \to \Psi(u_0)$. Since $M' = M^*/N$, the affine holonomy group of the flat affine connection on M' is naturally isomorphic with N (cf. Proposition 9.3 of Chapter II). The group N consists of pure translations only and the linear holonomy group of M' is trivial. Let $\sigma\colon M' \to L(M')$ be a globally defined parallel field of linear frames. Set

$$\bar{\theta}^i = \sigma^*\theta^i, \quad \bar{\omega}^i_j = \sigma^*\omega^i_j, \qquad i, j = 1, \ldots, n.$$

Since σ is horizontal, that is, $\sigma(M')$ is horizontal, we have $\bar{\omega}^i_j = 0$. The structure equations imply that $d\bar{\theta}^i = 0$. We assert that, for an arbitrarily chosen point o of M', there exists a unique abelian group structure on M' such that the point o is the identity element and that the forms $\bar{\theta}^i$ are invariant. Our assertion follows from the following three facts:

(a) $\bar{\theta}^1, \ldots, \bar{\theta}^n$ form a basis for the space of covectors at every point of M';

(b) $d\bar{\theta}^i = 0$ for $i = 1, \ldots, n$;

(c) Let X be a vector field on M' such that $\bar{\theta}^i(X) = c^i$ (c^i: constant) for $i = 1, \ldots, n$. Then X is complete in the sense that it generates a 1-parameter group of global transformations of M'.

The completeness of the connection implies (c) as follows. Let X^* be the horizontal vector field on $L(M')$ defined by $\theta^i(X^*) = c^i$, $i = 1, \ldots, n$. Under the diffeomorphism $\sigma\colon M' \to \sigma(M')$, X corresponds to X^*. Since X^* is complete (cf. Proposition 6.5 of

Chapter III), so is X. Note that (b) implies that the group is abelian.

It is clear that $\bar{\theta}^1\bar{\theta}^1 + \cdots + \bar{\theta}^n\bar{\theta}^n$ is an invariant Riemannian metric on the abelian Lie group M'. As we have seen in Example 4.1, M' is a Euclidean cylinder.

LEMMA 1. *Let* $\mathbf{R}^n/G$, $G = \{\sum_{i=1}^k m_i\xi_i;\ m_i$ *integers*$\}$, *be a Euclidean cylinder as defined in Example* 4.1. *Then the affine holonomy group of* $\mathbf{R}^n/G$ *is a group of translations isomorphic with* G.

Proof of Lemma 1. We identify the tangent space $T_a(\mathbf{R}^n)$ at each point $a \in \mathbf{R}^n$ with $\mathbf{R}^n$ by the following correspondence:

$$T_a(\mathbf{R}^n) \ni \textstyle\sum_{i=1}^n \lambda^i(\partial/\partial x^i)_a \leftrightarrow (\lambda^1, \ldots, \lambda^n) \in \mathbf{R}^n.$$

The linear parallel displacement from $0 \in R^n$ to $a \in R^n$ sends $(\lambda^1, \ldots, \lambda^n) \in T_0(\mathbf{R}^n)$ into the vector with the same components $(\lambda^1, \ldots, \lambda^n) \in T_a(\mathbf{R}^n)$. The affine parallel displacement from 0 to $a = (a^1, \ldots, a^n)$ sends $(\lambda^1, \ldots, \lambda^n)$, considered as an element of the tangent affine space $A_0(\mathbf{R}^n)$, into $(\lambda^1 + a^1, \ldots, \lambda^n + a^n) \in A_a(\mathbf{R}^n)$. Let $\tau^* = x_t^*$, $0 \leq t \leq 1$, be a line from 0 to $\sum_{i=1}^k m_i\xi_i \in G$ and let $\tau = x_t$, $0 \leq t \leq 1$, be the image of τ^* by the projection $\mathbf{R}^n \to \mathbf{R}^n/G$. Then τ is a closed curve in $\mathbf{R}^n/G$. Let

$$\textstyle\sum_{i=1}^k m_i\xi_i = (a^1, \ldots, a^n) \in \mathbf{R}^n.$$

Then the affine parallel displacement along τ yields the translation

$$(\lambda^1, \ldots, \lambda^n) \to (\lambda^1 + a^1, \ldots, \lambda^n + a^n).$$

This completes the proof of Lemma 1.

Being a covering space of M', M^* is also a Euclidean cylinder. By Proposition 9.3 of Chapter II, the affine holonomy group of M^* is trivial. By Lemma 1, M^* must be the ordinary affine space A^n, proving (1).

Since $M' = M^*/N$, the first homotopy group of M' is isomorphic with N. This completes the proof of (2).

Let M'' be a covering space of M. Since M^* is the universal covering space of M, we can write $M'' = M^*/H$, where H is a subgroup of $\Phi(u_0)$. The affine holonomy group of M'' is H by Proposition 9.3 of Chapter II. If M'' is a Euclidean cylinder, the affine holonomy group H consists of translations only (cf. Lemma 1) and hence is contained in the kernel N of the homomorphism

$\Phi(u_0) \to \Psi(u_0)$. Since $M' = M^*/N$, we may conclude that M'' is a covering space of M', thus proving (3).

Suppose M' is a Euclidean torus. It follows that M is compact and the linear holonomy group $\Phi(u_0)$ of M is a finite group. This implies that the flat affine connection of M is Riemannian. In fact, we choose an inner product in $T_{x_0}(M)$, $x_0 = \pi(u_0)$, invariant by the linear holonomy group with reference point x_0, and then we extend it to a Riemannian metric by parallel displacement. The flat affine connection of M is the Riemannian connection with respect to the Riemannian metric thus constructed.

Conversely, suppose M is a compact, connected, flat Riemannian manifold. By virtue of (1), identifying M^* with $\mathbf{R}^n$, we may write $M = \mathbf{R}^n/G$, where G is a discrete subgroup of the group of Euclidean motions acting on $\mathbf{R}^n$. Let N be the subgroup of G consisting of pure translations. In view of (2) and (3) our problem is to prove that $\mathbf{R}^n/N$ is a Euclidean torus. We first prove several lemmas.

Lemma 2. *Let A and B be unitary matrices of degree n such that A commutes with $ABA^{-1}B^{-1}$. If the characteristic roots of B have positive real parts, then A commutes with B.*

Proof of Lemma 2. Since $AABA^{-1}B^{-1} = ABA^{-1}B^{-1}A$, we have $ABA^{-1}B^{-1} = BA^{-1}B^{-1}A$. Without loss of generality, we may assume that B is diagonal with diagonal elements $b_k = \cos\beta_k + \sqrt{-1}\sin\beta_k$, $k = 1, \ldots, n$. Since $A^{-1} = {}^t\bar{A}$ and $B^{-1} = {}^t\bar{B} = \bar{B}$, we have

$$AB{}^t\bar{A}\bar{B} = ABA^{-1}B^{-1} = BA^{-1}B^{-1}A = B{}^t\bar{A}\bar{B}A.$$

Comparing the (i,i)-th entries, we have

$$\Sigma_{j=1}^n a_j^i b_j \bar{a}_j^i \bar{b}_i = \Sigma_{j=1}^n b_i \bar{a}_i^j \bar{b}_j a_i^j, \quad \text{where } A = (a_j^i).$$

Comparing the imaginary parts, we obtain

$$\Sigma_{j=1}^n (|a_j^i|^2 + |a_i^j|^2) \cdot \sin(\beta_j - \beta_i) = 0 \qquad \text{for } i = 1, \ldots, n.$$

We may also assume that $\beta_1 = \beta_2 = \cdots = \beta_{p_1} < \beta_{p_1+1} = \cdots = \beta_{p_1+p_2} < \cdots \leqq \beta_n < \beta_1 + \pi$. Since all the b_k's have positive real parts, we have

$$\sin(\beta_j - \beta_i) > 0 \qquad \text{for } i \leqq p_1 \text{ and } j > p_1.$$

Hence we must have

$$a^i_j = a^j_i = 0 \qquad \text{for } i \leqq p_1 \text{ and } j > p_1.$$

Similarly, we have

$$a^i_j = a^j_i = 0 \qquad \text{for } i \leqq p_1 + p_2 \text{ and } j > p_1 + p_2.$$

Continuing this argument we have

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & & \cdot \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & & \cdot \end{pmatrix},$$

$$B_1 = b_1 I_1, \quad B_2 = b_{p_1+1} I_2, \ldots,$$

where $A_1, A_2, \ldots$ are unitary matrices of degree $p_1, p_2, \ldots$, and $I_1, I_2, \ldots$ are the identity matrices of degree $p_1, p_2, \ldots$. This shows clearly that A and B commute.

For any matrix $A = (a^i_j)$ of type (r, s) we set

$$\varphi(A) = (\Sigma_{i,j} |a^i_j|^2)^{\frac{1}{2}}.$$

In other words, $\varphi(A)$ is the length of A when A is considered as a vector with rs components. We have

$$\varphi(A + B) \leqq \varphi(A) + \varphi(B),$$
$$\varphi(AB) \leqq \varphi(A) \cdot \varphi(B).$$

The latter follows from the inequality of Schwarz. If A is an orthogonal matrix, we have

$$\varphi(AB) = \varphi(B), \quad \varphi(CA) = \varphi(C).$$

Every *Euclidean motion* of $\mathbf{R}^n$ is given by

$$x \to Ax + p, \qquad x \in \mathbf{R}^n,$$

where A is an orthogonal matrix (called the rotation part of the motion) and p is an element of $\mathbf{R}^n$ (called the translation part of the motion). This motion will be denoted by (A, p).

LEMMA 3. *Given any two Euclidean motions (A, p) and (B, q), set*

$$(A_1, p_1) = (A, p)(B, q)(A, p)^{-1}(B, q)^{-1}.$$

Let I be the identity matrix of degree n. If $\varphi(A - I) < a$ and $\varphi(B - I) < b$, then we have

(1) $\varphi(A_1 - I) < 2ab$;

(2) $\varphi(p_1) < b \cdot \varphi(p) + a \cdot \varphi(q)$.

Proof of Lemma 3. We have

$$A_1 - I = ABA^{-1}B^{-1} - I = (AB - BA)A^{-1}B^{-1}$$
$$= ((A - I)(B - I) - (B - I)(A - I))A^{-1}B^{-1}.$$

Since $A^{-1}B^{-1}$ is an orthogonal matrix, we have

$$\varphi(A_1 - I) \leqq \varphi(A - I) \cdot \varphi(B - I) + \varphi(B - I) \cdot \varphi(A - I) < 2ab.$$

By a simple calculation, we obtain

$$p_1 = A(I - B)A^{-1}p + AB(I - A^{-1})B^{-1}q.$$

By the same reasoning as above, we obtain

$$\varphi(p_1) \leqq \varphi(I - B) \cdot \varphi(p) + \varphi(I - {}^tA) \cdot \varphi(q) < b\varphi(p) + a\varphi(q).$$

Lemma 4. *With the same notation as in Lemma 3, set*

$$(A_k, p_k) = (A, p)(A_{k-1}, p_{k-1})(A, p)^{-1}(A_{k-1}, p_{k-1})^{-1}, \quad k = 2, 3, \ldots.$$

Then, for $k = 1, 2, 3, \ldots$, we have

(1) $\varphi(A_k - I) < 2^k a^k b$;

(2) $\varphi(p_k) < (2^k - 1)a^{k-1}b \cdot \varphi(p) + a^k \cdot \varphi(q)$.

Proof of Lemma 4. A simple induction using Lemma 3 establishes the inequalities.

Lemma 5. *Let G be a discrete subgroup of the group of Euclidean motions of $\mathbf{R}^n$. Let $a < \frac{1}{2}$ and*

$$G(a) = \{(A, p) \in G;\ \varphi(A - I) < a\}.$$

Then any two elements (A, p) and (B, q) of $G(a)$ commute.

Proof of Lemma 5. By Lemma 4, $\varphi(A_k - I)$ and $\varphi(p_k)$ approach zero as k tends to infinity. Since G is discrete in $A(n, \mathbf{R})$, there exists an integer k such that $A_k = I$ and $p_k = 0$. We show that the characteristic roots $a_1, \ldots, a_n$ of an orthogonal matrix A with $\varphi(A - I) < \frac{1}{2}$ have positive real parts. If U is a unitary matrix such that UAU^{-1} is diagonal, then

$$\varphi(A - I) = \varphi(U(A - I)U^{-1}) = \varphi(UAU^{-1} - I)$$
$$= (|a_1 - 1|^2 + \cdots + |a_n - 1|^2)^{\frac{1}{2}} < \tfrac{1}{2},$$

which proves our assertion. By applying Lemma 2 to $A_k = AA_{k-1}A^{-1}A_{k-1}$, we see that $A_{k-1} = I$. Continuing this argument, we obtain $A_1 = I$. Thus A and B commute.
Hence,

$$\begin{aligned}
&p_1 = (I - B)p - (I - A)q, \quad p_2 = (A - I)p_1, \\
&p_3 = (A - I)p_2 = (A - I)^2 p_1, \\
&\cdots \\
&p_k = (A - I)p_{k-1} = (A - I)^{k-1}p_1.
\end{aligned}$$

Since $p_k = 0$, we have

$$(A - I)^{k-1}p_1 = 0.$$

Changing the roles of (A, p) and (B, q) and noting that

$$(B, q)(A, p)(B, q)^{-1}(A, p)^{-1} = (I, -p_1),$$

we obtain

$$(B - I)^{m-1}p_1 = 0 \qquad \text{for some integer } m.$$

Since A and B commute, there exists a unitary matrix U such that UAU^{-1} and UBU^{-1} are both diagonal. Set

$$UAU^{-1} = \begin{pmatrix} a_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & a_n \end{pmatrix}, \quad UBU^{-1} = \begin{pmatrix} b_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & b_n \end{pmatrix},$$

$$Up = \begin{pmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{pmatrix}, \quad Uq = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix}.$$

Then, from $(A - I)^{k-1}p_1 = (A - I)^{k-1}((I - B)p - (I - A)q) = 0$, we obtain

$$(a_i - 1)^{k-1}\{(1 - b_i)r_i - (1 - a_i)s_i\} = 0, \quad i = 1, \ldots, n,$$

Similarly, from

$$(B - I)^{m-1}p_1 = (B - I)^{m-1}\{(I - B)p - (I - A)q\} = 0,$$

we obtain

$$(b_i - 1)^{m-1}\{(1 - b_i)r_i - (1 - a_i)s_i\} = 0, \quad i = 1, \ldots, n.$$

Hence we have

$$(1 - b_i)r_i - (1 - a_i)s_i = 0, \quad i = 1, \ldots, n.$$

In other words, we have

$$p_1 = (I - B)p - (I - A)q = 0,$$

which completes the proof of Lemma 5.

If $(A, p) \in G(a)$ and $(B, q) \in G$, then $(B, q)(A, p)(B, q)^{-1} \in G(a)$. Indeed,

$$\varphi(BAB^{-1} - I) = \varphi(B(A - I)B^{-1}) = \varphi(A - I) < a.$$

This shows that the group generated by $G(a)$ is an invariant subgroup of G. By Lemma 5, it is moreover abelian if $a < \frac{1}{2}$.

A subset V of $\mathbf{R}^n$ is called a *Euclidean subspace* if there exist an element $x_0 \in \mathbf{R}^n$ and a vector subspace S of $\mathbf{R}^n$ such that $V = \{x + x_0; x \in S\}$. We say that a group G of Euclidean motions of $\mathbf{R}^n$ is *irreducible* if $\mathbf{R}^n$ is the only Euclidean subspace invariant by G.

Lemma 6. *If H is an abelian normal subgroup of an irreducible group G of Euclidean motions of $\mathbf{R}^n$, then H contains pure translations only.*

Proof of Lemma 6. Since H is abelian, we may assume, by applying an orthogonal change of basis of $\mathbf{R}^n$ if necessary, that the elements (A, p) of H are simultaneously reduced to the following form:

$$A = \begin{pmatrix} A_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & A_k & \\ 0 & & & & I_{n-2k} \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_k \\ p^* \end{pmatrix}, \quad A_i = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}, \quad (i = 1, \ldots, k)$$

where I_{n-2k} is the identity matrix of degree $n - 2k$, each p_i is a vector with 2 components and p^* is a vector with $n - 2k$ components. Moreover, for each i, there exists an element (A, p) of H such that A_i is different from the identity matrix I_2 so that $A_i - I_2$ is non-singular.

Our task is now to prove that $k = 0$, i.e., $A = I_n$ for all $(A, p) \in H$. Assuming $k \geqq 1$, we shall derive a contradiction.

For each i, choose $(A, p) \in H$ such that $A_i - I_2$ is non-singular and define a vector t_i with 2 components by

$$(A_i - I_2)t_i = p_i.$$

We shall show that

$$(B_i - I_2)t_i = q_i \qquad \text{for all } (B, q) \in H.$$

Since (A, p) and (B, q) commute, we have

$$A_i q_i + p_i = B_i p_i + q_i$$

or

$$(A_i - I_2)q_i = (B_i - I_2)p_i.$$

Hence we have

$$\begin{aligned}(B_i - I_2)t_i &= (B_i - I_2)(A_i - I_2)^{-1}p_i = (A_i - I_2)^{-1}(B_i - I_2)p_i \\ &= (A_i - I_2)^{-1}(A_i - I_2)q_i = q_i,\end{aligned}$$

thus proving our assertion. We define a vector $t \in \mathbf{R}^n$ by

$$t = \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_k \\ 0 \end{pmatrix}.$$

We have now

$$\begin{aligned}(I_n, t)(A, p)(I_n, t)^{-1} &= (I_n, t)(A, p)(I_n, -t) \\ &= (A, p - (A - I_n)t), \qquad (A, p) \in H,\end{aligned}$$

where

$$p - (A - I_n)t = \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_k \\ p^* \end{pmatrix} - \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ p^* \end{pmatrix}.$$

By translating the origin of $\mathbf{R}^n$ to t, we may now assume that the elements (A, p) of H are of the form

$$A = \begin{pmatrix} A_1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & A_k & \\ 0 & & & & I_{n-2k} \end{pmatrix}, \quad p = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ p^* \end{pmatrix}.$$

Let V be the vector subspace of $\mathbf{R}^n$ consisting of all vectors whose first $2k$ components are zero. Then V is invariant by all elements (A, p) of H. We shall show that V is also invariant by G. First we observe that V is precisely the set of all vectors which are left fixed by all A where $(A, p) \in H$. Let $(C, r) \in G$. Since H is a normal subgroup of G, for each $(A, p) \in H$, there exists an element $(B, q) \in H$ such that

$$(A, p)(C, r) = (C, r)(B, q).$$

If $v \in V$, then $ACv = CBv = Cv$. Since Cv is left fixed by all A, it lies in V. Hence C is of the form

$$C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix}$$

where C' and C'' are of degree $2k$ and $n - 2k$, respectively. To prove that the first $2k$ components of r are zero, write

$$r = \begin{pmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_k \\ r^* \end{pmatrix}.$$

For each i, let (A, p) be an element of H such that $A_i - I_2$ is non-singular. Applying the equality $(A, p)(C, r) = (C, r)(B, q)$ to the zero vector of $\mathbf{R}^n$ and comparing the $(2i - 1)$-th and $2i$-th components of the both sides, we have

$$A_i r_i = r_i.$$

Since $(A_i - I_2)$ is non-singular, we obtain $r_i = 0$. Thus every element (C, r) of G is of the form

$$C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ r^* \end{pmatrix}.$$

This shows that V is invariant by G, thus contradicting the irreducibility of G. This completes the proof of Lemma 6.

LEMMA 7. *A group G of Euclidean motions of $\mathbf{R}^n$ is irreducible if $\mathbf{R}^n/G$ is compact.*

Proof of Lemma 7. Assuming that G is not irreducible, let V be a proper Euclidean subspace of $\mathbf{R}^n$ which is invariant by G. Let x_0 be any point of V and let L be a line through x_0 perpendicular to V. Let $x_1, x_2, \ldots, x_m, \ldots$ be a sequence of points on L such that the distance between x_0 and x_m is equal to m. Let $G(x_0)$ denote the orbit of G through x_0. Since $G(x_0)$ is in V, the distance between $G(x_0)$ and x_m is at least m and, hence, is equal to m. Therefore the distance between the images of x_0 and x_m in $\mathbf{R}^n/G$ by the projection $\mathbf{R}^n \to \mathbf{R}^n/G$ is equal to m. This means that $\mathbf{R}^n/G$ is not compact.

We are now in position to complete the proof of (4). Let G and $G(a)$ be as in Lemma 5 and assume $a < \frac{1}{2}$. Let H be the group generated by $G(a)$; it is an abelian normal subgroup of G. Assume that $\mathbf{R}^n/G$ is compact. Lemmas 6 and 7 imply that H contains nothing but pure translations. On the other hand, since G is discrete, G/H is finite by construction of $G(a)$. Hence $\mathbf{R}^n/H$ is also compact and hence is a Euclidean torus. Let N be the subgroup of G consisting of all pure translations of G. Since $G(a)$ contains N, we have $N = H$. This proves that $\mathbf{R}^n/N$ is a Euclidean torus.

QED.

Remark. (4) means that the linear holonomy group of a compact flat Riemannian manifold $M = \mathbf{R}^n/G$ is isomorphic to G/N and hence is finite.

Although (1), (2) and (3) are essentially in Auslander-Markus [1], we laid emphasis on affine holonomy groups. (4) was originally proved by Bieberbach [1]. The proof given here was taken from Frobenius [1] and Zassenhaus [1].

Example 4.3. The linear holonomy group of a non-compact flat Riemannian manifold may not be finite. Indeed, fix an arbitrary irrational real number λ. For each integer m, we set

$$A(m) = \begin{pmatrix} \cos \lambda m\pi & \sin \lambda m\pi & 0 \\ -\sin \lambda m\pi & \cos \lambda m\pi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad p(m) = \begin{pmatrix} 0 \\ 0 \\ m \end{pmatrix}.$$

Then we set $G = \{(A(m), p(m))\,;\, m = 0, \pm 1, \pm 2, \ldots\}$. It is easy to see that G is a discrete subgroup of the group of Euclidean motions of $\mathbf{R}^3$ and acts freely on $\mathbf{R}^3$. The linear holonomy group of $\mathbf{R}^3/G$ is isomorphic to the group $\{A(m)\,;\, m = 0, \pm 1, \pm 2, \ldots\}$.

COROLLARY 4.3. *A manifold M with a flat affine connection admits a Euclidean torus as a covering space if and only if M is a compact flat Riemannian manifold.*

Proof. Let M'' be a Euclidean torus which is a covering space of M. By (3) of Theorem 4.2, M'' is a covering space of M'. Thus, M' is a compact, Euclidean cylinder and hence is a Euclidean torus. By (3) of Theorem 4.2, M is a compact flat Riemannian manifold. The converse is contained in (4) of Theorem 4.2. QED.

Example 4.4. In Example 4.2, set $M = \mathbf{R}^2/G$. Let N be the subgroup of G consisting of translations:

$$(x, y) \to (x + m, y), \quad m = 0, \pm 1, \pm 2, \ldots .$$

Then the covering space M' defined in Theorem 4.2 is given by $\mathbf{R}^2/N$ in this case. Clearly, M' is an ordinary cylinder, that is, the direct product of a circle with a line.

The determination of the 2-dimensional complete flat Riemannian manifolds is due to Killing [1,2], Klein [1,2] and H. Hopf [1]. We shall present here their results with an indication of the proof.

There are four types of two-dimensional complete flat Riemannian manifolds other than the Euclidean plane. We give the fundamental group (the first homotopy group) for each type, describing its action on the universal covering space $\mathbf{R}^2$ in terms of the Cartesian coordinate system (x, y).

(1) *Ordinary cylinder* (orientable)
$(x, y) \to (x + n, y)$,
$n = 0, \pm 1, \pm 2, \ldots$.

(2) *Ordinary torus* (orientable)
$(x, y) \to (x + ma + n, y + mb)$
$m, n = 0, \pm 1, \pm 2, \ldots,$
a, b: real numbers, $b \neq 0$.

(3) *Möbius band with infinite width or twisted cylinder* (non-orientable)
$(x, y) \to (x + n, (-1)^n y)$,
$n = 0, \pm 1, \pm 2, \ldots$.

(4) *Klein bottle or twisted torus* (non-orientable)
$(x, y) \to (x + n, (-1)^n y + bm)$,
$n, m = 0, \pm 1, \pm 2, \ldots,$
b: non-zero real number.

Any two-dimensional complete flat Riemannian manifold M is isometric, up to a constant factor, to one of the above four types of surfaces.

The proof goes roughly as follows. By Theorem 4.2, the problem reduces to the determination of the discrete groups of motions acting freely on $\mathbf{R}^2$. Let G be such a discrete group. We first prove that every element of G which preserves the orientation of $\mathbf{R}^2$ is necessarily a translation. Set $z = x + iy$. Then every orientation preserving motion of $\mathbf{R}^2$ is of the form

$$z \to \varepsilon z + w,$$

where ε is a complex number of absolute value 1 and w is a complex number. If we iterate the transformation $z \to \varepsilon z + w$ r times, then we obtain the transformation

$$z \to \varepsilon^r z + (\varepsilon^{r-1} + \varepsilon^{r-2} + \cdots + 1)w.$$

We see easily that, if $\varepsilon \neq 1$, then the point $w/(1 - \varepsilon)$ is left fixed by the transformation $z \to \varepsilon z + w$, in contradiction to the assumption that G acts freely on $\mathbf{R}^2$. Hence $\varepsilon = 1$, which proves our assertion. If f is an element of G which reverses the orientation of $\mathbf{R}^2$, then f^2 is an orientation preserving transformation and hence is a translation. We thus proved that every element of G is a transformation of the type

$$z \to z + w \quad \text{or} \quad z \to \bar{z} + w,$$

where $\bar{z}$ is the complex conjugate of z. It is now easy to conclude that M must be one of the above four types of surfaces. The detail is left to the reader.

CHAPTER VI

Transformations

1. *Affine mappings and affine transformations*

Let M and M' be manifolds provided with linear connections Γ and Γ' respectively. Throughout this section, we denote by $P(M, G)$ and $P'(M', G')$ the bundles of linear frames $L(M)$ and $L(M')$ of M and M', respectively, so that $G = GL(n; \mathbf{R})$ and $G' = GL(n'; \mathbf{R})$, where $n = \dim M$ and $n' = \dim M'$.

A differentiable mapping $f \colon M \to M'$ of class C^1 induces a continuous mapping $f \colon T(M) \to T(M')$, where $T(M)$ and $T(M')$ are the tangent bundles of M and M', respectively. We call $f \colon M \to M'$ an *affine mapping* if the induced mapping $f \colon T(M) \to T(M')$ maps every horizontal curve into a horizontal curve, that is, if f maps each parallel vector field along each curve τ of M into a parallel vector field along the curve $f(\tau)$.

Proposition 1.1. *An affine mapping $f \colon M \to M'$ maps every geodesic of M into a geodesic of M' (together with its affine parameter). Consequently, f commutes with the exponential mappings, that is,*

$$f \circ \exp X = \exp \circ f(X), \qquad X \in T_x(M).$$

Proof. This is obvious from the definition of an affine mapping. QED.

Proposition 1.1 implies that an affine mapping is necessarily of class C^∞ provided that the connections Γ and Γ' are of class C^∞.

We recall that a vector field X of M is *f-related to* a vector field X' of M' if $f(X_x) = X'_{f(x)}$ for all $x \in M$ (cf. §1 of Chapter I).

Proposition 1.2. *Let $f \colon M \to M'$ be an affine mapping. Let X, Y and Z be vector fields on M which are f-related to vector fields X', Y' and Z' on M', respectively. Then*

(1) *$\nabla_X Y$ is f-related to $\nabla_{X'} Y'$, where ∇ denotes covariant differentiation both in M and M';*

(2) *$T(X, Y)$ is f-related to $T'(X', Y')$, where T and T' are the torsion tensor fields of M and M', respectively;*

(3) *$R(X, Y)Z$ is f-related to $R'(X', Y')Z'$, where R and R' are the curvature tensor fields of M and M', respectively.*

Proof. (1) Let x_t be an integral curve of X such that $x = x_0$ and let τ_0^t be the parallel displacement along this curve from x_t to $x = x_0$. Then (cf. §1 of Chapter III)

$$(\nabla_X Y)_x = \lim_{t\to 0} \frac{1}{t}(\tau_0^t Y_{x_t} - Y_x).$$

Set $x'_t = f(x_t)$ and let τ'^t_0 be the parallel displacement along this image curve from x'_t to $x' = x'_0$. Since f commutes with parallel displacement, we have

$$\begin{aligned} f((\nabla_X Y)_x) &= \lim_{t\to 0} \frac{1}{t}[f(\tau_0^t Y_{x_t}) - f(Y_x)] \\ &= \lim_{t\to 0} \frac{1}{t}(\tau'^t_0 Y'_{x'_t} - Y'_{x'}) = (\nabla_{X'} Y')_{x'}. \end{aligned}$$

(2) and (3) follow from (1) and Theorem 5.1 of Chapter III.

QED.

A diffeomorphism f of M onto itself is called an *affine transformation* of M if it is an affine mapping. Any transformation f of M induces in a natural manner an automorphism $\tilde{f}$ of the bundle $P(M, G)$; $\tilde{f}$ maps a frame $u = (X_1, \dots, X_n)$ at $x \in M$ into the frame $\tilde{f}(u) = (fX_1, \dots, fX_n)$ at $f(x) \in M$. Since $\tilde{f}$ is an automorphism of the bundle P, it leaves every fundamental vector field of P invariant.

Proposition 1.3. (1) *For every transformation f of M, the induced automorphism $\tilde{f}$ of the bundle P of linear frames leaves the canonical form θ invariant. Conversely, every fibre-preserving transformation of P leaving θ invariant is induced by a transformation of M.*

(2) *If f is an affine transformation of M, then the induced automorphism $\tilde{f}$ of P leaves both the canonical form θ and the connection form ω invariant. Conversely, every fibre-preserving transformation of P leaving both θ and ω invariant is induced by an affine transformation of M.*

Proof. (1) Let $X^* \in T_u(P)$ and set $X = \pi(X^*)$ so that $X \in T_x(M)$, where $x = \pi(u)$. Then (cf. §2 of Chapter III)

$$\theta(X^*) = u^{-1}(X) \quad \text{and} \quad \theta(\tilde{f}X^*) = \tilde{f}(u)^{-1}(fX),$$

where the frames u and $\tilde{f}(u)$ are considered as linear mappings of $\mathbf{R}^n$ onto $T_x(M)$ and $T_{f(x)}(M)$, respectively. It follows from the definition of $\tilde{f}$ that the following diagram is commutative:

$$\begin{tikzcd}
& \mathbf{R}^n \arrow[dl, "u"'] \arrow[dr, "\tilde{f}(u)"] & \\
T_x(M) \arrow[rr, "f"'] & & T_{f(x)}(M).
\end{tikzcd}$$

Hence, $u^{-1}(X) = \tilde{f}(u)^{-1}(fX)$, thus proving that θ is invariant by $\tilde{f}$.

Conversely, let F be a fibre-preserving transformation of P leaving θ invariant. Let f be the transformation of the base M induced by F. We prove that $\tilde{f} = F$. We set $J = \tilde{f}^{-1} \circ F$. Then J is a fibre-preserving transformation of P leaving θ invariant. Moreover, J induces the identity transformation on the base M. Therefore, we have

$$u^{-1}(X) = \theta(X^*) = \theta(JX^*) = J(u)^{-1}(X) \qquad \text{for } X^* \in T_u(P).$$

This implies that $J(u) = u$, that is, $\tilde{f}(u) = F(u)$.

(2) Let f be an affine transformation of M. The automorphism $\tilde{f}$ of P maps the connection Γ into a connection, say, $\tilde{f}(\Gamma)$, and the form $\tilde{f}^*\omega$ is the connection form of $\tilde{f}(\Gamma)$ (cf. Proposition 6.1 of Chapter II). From the definition of an affine transformation, we see that $\tilde{f}$ maps, for each $u \in P$, the horizontal subspace of $T_u(P)$ onto the horizontal subspace of $T_{\tilde{f}(u)}(P)$. This means that $\tilde{f}(\Gamma) = \Gamma$ and hence $\tilde{f}^*\omega = \omega$.

Conversely, let F be a fibre-preserving transformation of P leaving θ and ω invariant. By (1), there exists a transformation f of M such that $F = \tilde{f}$. Since $\tilde{f}$ maps every horizontal curve of P into a horizontal curve of P, the transformation f: $T(M) \to T(M)$ maps every horizontal curve of $T(M)$ into a horizontal curve of $T(M)$. This means that f: $M \to M$ is an affine mapping, thus completing the proof. QED.

Remark. Assume that M is orientable. Then the bundle P consists of two principal fibre bundles, say $P^+(M, G^0)$ and

$P^-(M, G^0)$, where G^0 is the connected component of the identity of $G = GL(n; \mathbf{R})$. Then any transformation F of P^+ or P^- leaving θ invariant is fibre-preserving and hence is induced by a transformation f of the base M. In fact, every vertical vector X^* of P^+ or P^- is mapped into a vertical vector by F since $\theta(FX^*) = \theta(X^*) = 0$. Any curve in any fibre of P^+ or P^- is therefore mapped into a curve in a fibre by F. Since the fibres of P^+ or P^- are connected, F is fibre-preserving.

Proposition 1.4. *Let Γ be a linear connection on M. For a transformation f of M, the following conditions are mutually equivalent:*

(1) *f is an affine transformation of M;*

(2) *$\tilde{f}^*\omega = \omega$, where ω is the connection form of Γ and $\tilde{f}$ is the transformation of P induced by f;*

(3) *$\tilde{f}$ leaves every standard horizontal vector field $B(\xi)$ invariant;*

(4) *$f(\nabla_Y Z) = \nabla_{fY}(fZ)$ for any vector fields Y and Z on M.*

Proof. (i) The equivalence of (1) and (2) is contained in Proposition 1.3.

(ii) (2) $\rightarrow$ (3). By Proposition 1.3, we have

$$\xi = \theta(B(\xi)) = (\tilde{f}^*\theta)(B(\xi)) = \theta(\tilde{f}^{-1}B(\xi)).$$

Since $\omega(B(\xi)) = 0$, (2) implies

$$0 = \omega(B(\xi)) = (\tilde{f}^*\omega)(B(\xi)) = \omega(\tilde{f}^{-1}\cdot B(\xi)).$$

This means that $\tilde{f}^{-1}\cdot B(\xi) = B(\xi)$.

(iii) (3) $\rightarrow$ (2). The horizontal subspace at u is given by the set of $B(\xi)_u$. Hence (3) implies that $\tilde{f}$ maps every horizontal subspace into a horizontal subspace. This means that $\tilde{f}(\Gamma) = \Gamma$ and hence $\tilde{f}^*\omega = \omega$.

(iv) (1) $\rightarrow$ (4). This follows from Proposition 1.2.

(v) (4) $\rightarrow$ (1). Let Z be a parallel vector field along a curve $\tau = x_t$. Let Y be the vector field along τ tangent to τ, that is, $Y_{x_t} = \dot{x}_t$. We extend Y and Z to vector fields defined on M, which will be denoted by the same letters Y and Z respectively. (4) implies that fZ is parallel along $f(\tau)$. This means that f is an affine transformation. QED.

The set of affine transformations of M, denoted by $\mathfrak{A}(M)$ or $\mathfrak{A}(\Gamma)$, forms a group. The set of all fibre-preserving transformations of P leaving θ and ω invariant, denoted by $\mathfrak{A}(P)$, forms a group which is canonically isomorphic with $\mathfrak{A}(M)$. We prove that $\mathfrak{A}(M)$ is a Lie group by establishing that $\mathfrak{A}(P)$ is a Lie group with respect to the compact-open topology in P.

THEOREM 1.5. *Let Γ be a linear connection on a manifold M with a finite number of connected components. Then the group $\mathfrak{A}(M)$ of affine transformations of M is a Lie transformation group with respect to the compact-open topology in P.*

Proof. Let $\theta = (\theta^i)$ and $\omega = (\omega^j_k)$ be the canonical form and the connection form on P. We set

$$g(X^*, Y^*) = \Sigma_i\, \theta^i(X^*)\theta^i(Y^*) + \Sigma_{j,k}\, \omega^j_k(X^*)\omega^j_k(Y^*),$$
$$X^*, Y^* \in T_u(P).$$

Since the $n^2 + n$ 1-forms θ^i, ω^j_k, $i, j, k = 1, \ldots, n$, form a basis of the space of covectors at every point u of P (cf. Proposition 2.6 of Chapter III), g is a Riemannian metric on P which is invariant by $\mathfrak{A}(P)$ by Proposition 1.3. The group of isometries of P is a Lie transformation group of P with respect to the compact-open topology by Theorem 4.6 and Corollary 4.9 of Chapter I (cf. also Theorem 3.10 of Chapter IV). Since $\mathfrak{A}(P)$ is clearly a closed subgroup of the group of isometries of P, $\mathfrak{A}(P)$ is also a Lie transformation group of P. QED.

2. *Infinitesimal affine transformations*

Throughout this section, $P(M, G)$ denotes the bundle of linear frames over a manifold M so that $G = GL(n; \mathbf{R})$, where $n = \dim M$.

Every transformation φ of M induces a transformation of P in a natural manner. Correspondingly, every vector field X on M induces a vector field $\tilde{X}$ on P in a natural manner. More precisely, we prove

PROPOSITION 2.1. *For each vector field X on M, there exists a unique vector field $\tilde{X}$ on P such that*

(1) *$\tilde{X}$ is invariant by R_a for every $a \in G$;*
(2) *$L_{\tilde{X}}\theta = 0$;*
(3) *$\tilde{X}$ is π-related to X, that is, $\pi(\tilde{X}_u) = X_{\pi(u)}$ for every $u \in P$.*

Conversely, given a vector field $\tilde{X}$ *on* P *satisfying* (1) *and* (2), *there exists a unique vector field* X *on* M *satisfying* (3).

We shall call $\tilde{X}$ the *natural lift* of X.

Proof. Given a vector field X on M and a point $x \in M$, let φ_t be a local 1-parameter group of local transformations generated by X in a neighborhood U of x. For each t, φ_t induces a transformation $\tilde{\varphi}_t$ of $\pi^{-1}(U)$ onto $\pi^{-1}(\varphi_t(U))$ in a natural manner. Thus we obtain a local 1-parameter group of local transformations $\tilde{\varphi}_t\colon \pi^{-1}(U) \to P$ and hence the induced vector field on P, which will be denoted by $\tilde{X}$. Since $\tilde{\varphi}_t$ commutes with R_a for every $a \in G$, $\tilde{X}$ satisfies (1) (cf. Corollary 1.8 of Chapter 1). Since $\tilde{\varphi}_t$ preserves the form θ, $\tilde{X}$ satisfies (2). Finally, $\pi \circ \tilde{\varphi}_t = \tilde{\varphi}_t \circ \pi$ implies (3).

To prove the uniqueness of $\tilde{X}$, let $\tilde{X}_1$ be another vector field on P satisfying (1), (2) and (3). Let $\tilde{\psi}_t$ be a local 1-parameter group of local transformations generated by $\tilde{X}_1$. Then $\tilde{\psi}_t$ commutes with every R_a, $a \in G$, and preserves the canonical form θ. By (1) of Proposition 1.3, it follows that $\tilde{\psi}_t$ is induced by a local 1-parameter group of local transformations ψ_t of M. Because of (3), ψ_t induces the vector field X on M. Thus $\psi_t = \varphi_t$ and hence $\tilde{\psi}_t = \tilde{\varphi}_t$, which implies that $\tilde{X} = \tilde{X}_1$.

Conversely, let $\tilde{X}$ be a vector field on P satisfying (1) and (2). For each $x \in M$, choose a point $u \in P$ such that $\pi(u) = x$. We then set $X_x = \pi(\tilde{X}_u)$. Since $\tilde{X}$ satisfies (1), X_x is independent of the choice of u and thus we obtain a vector field X which satisfies (3). The uniqueness of X is evident. QED.

Let Γ be a linear connection on M. A vector field X on M is called an *infinitesimal affine transformation* of M if, for each $x \in M$, a local 1-parameter group of local transformations φ_t of a neighborhood U of x into M preserves the connection Γ, more precisely, if each $\varphi_t\colon U \to M$ is an affine mapping, where U is provided with the affine connection $\Gamma \mid U$ which is the restriction of Γ to U.

Proposition 2.2. *Let* Γ *be a linear connection on* M. *For a vector field* X *on* M, *the following conditions are mutually equivalent:*

(1) X *is an infinitesimal affine transformation of* M;

(2) $L_{\tilde{X}}\omega = 0$, *where* ω *is the connection form of* Γ *and* $\tilde{X}$ *is the natural lift of* X;

(3) $[\tilde{X}, B(\xi)] = 0$ *for every* $\xi \in \mathbf{R}^n$, *where* $B(\xi)$ *is the standard horizontal vector field corresponding to* ξ;

(4) $L_X \circ \nabla_Y - \nabla_Y \circ L_X = \nabla_{[X,Y]}$ *for every vector field* Y *on* M.

Proof. Let φ_t be a local 1-parameter group of local transformations of M generated by X and let, for each t, $\tilde{\varphi}_t$ be a local transformation of P induced by φ_t.

(i) (1) → (2). By Proposition 1.4, $\tilde{\varphi}_t$ preserves ω. Hence we have (2).

(ii) (2) → (3). For every vector field X, we have (Proposition 2.1)

$$0 = \tilde{X}(\theta(B(\xi))) = (L_{\tilde{X}}\theta)(B(\xi)) + \theta([\tilde{X}, B(\xi)]) = \theta([\tilde{X}, B(\xi)]),$$

which means that $[\tilde{X}, B(\xi)]$ is vertical. If $L_{\tilde{X}}\omega = 0$, then

$$0 = \tilde{X}(\omega(B(\xi))) = (L_{\tilde{X}}\omega)(B(\xi)) + \omega([\tilde{X}, B(\xi)]) = \omega([\tilde{X}, B(\xi)]),$$

which means that $[\tilde{X}, B(\xi)]$ is horizontal. Hence, $[\tilde{X}, B(\xi)] = 0$.

(iii) (3) → (1). If $[\tilde{X}, B(\xi)] = 0$, then $\tilde{\varphi}_t$ leaves $B(\xi)$ invariant and thus maps the horizontal subspace at u into the horizontal subspace at $\tilde{\varphi}_t(u)$, whenever $\tilde{\varphi}_t(u)$ is defined. Therefore $\tilde{\varphi}_t$ preserves the connection Γ and X is an infinitesimal affine transformation of M.

(iv) (1) → (4). By Proposition 1.4, we have

$$\varphi_t(\nabla_Y Z) = \nabla_{\varphi_t Y}(\varphi_t Z) \qquad \text{for any vector fields } Y \text{ and } Z \text{ on } M.$$

From the definition of Lie differentiation given in §3 of Chapter I, we obtain

$$\begin{aligned} L_X \circ \nabla_Y Z &= \lim_{t\to 0} \frac{1}{t}[\nabla_Y Z - \varphi_t(\nabla_Y Z)] \\ &= \lim_{t\to 0} \frac{1}{t}[\nabla_Y Z - \nabla_{\varphi_t Y} Z] + \lim_{t\to 0} \frac{1}{t}[\nabla_{\varphi_t Y} Z - \nabla_{\varphi_t Y}(\varphi_t Z)] \\ &= \nabla_{L_X Y} Z + \nabla_Y \circ L_X Z = \nabla_{[X,Y]} Z + \nabla_Y \circ L_X Z. \end{aligned}$$

We thus verified the formula:

$$L_X \circ \nabla_Y K - \nabla_Y \circ L_X K = \nabla_{[X,Y]} K,$$

when K is a vector field. If K is a function, the above formula is evidently true. By the lemma for Proposition 3.3 of Chapter I, the formula holds for any tensor field K.

(v) (4) → (1). We fix a point $x \in M$. We set

$$V(t) = (\varphi_t(\nabla_Y Z))_x \quad \text{and} \quad W(t) = (\nabla_{\varphi_t Y}(\varphi_t Z))_x.$$

For each t, both $V(t)$ and $W(t)$ are elements of $T_x(M)$. In view of Proposition 1.4, it is sufficient to prove that $V(t) = W(t)$. As in (iv), we obtain

$$dV(t)/dt = \varphi_t((L_X \circ \nabla_Y Z)_{\varphi_t^{-1}(x)}),$$
$$dW(t)/dt = \varphi_t((\nabla_{[X,Y]}Z + \nabla_Y \circ L_X Z)_{\varphi_t^{-1}(x)}).$$

From our assumption we obtain $dV(t)/dt = dW(t)/dt$. On the other hand, we have evidently $V(0) = W(0)$. Hence, $V(t) = W(t)$. QED.

Let $\mathfrak{a}(M)$ be the set of infinitesimal affine transformations of M. Then $\mathfrak{a}(M)$ forms a subalgebra of the Lie algebra $\mathfrak{X}(M)$ of all vector fields on M. In fact, the correspondence $X \to \tilde{X}$ defined in Proposition 2.1 is an isomorphism of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M into the Lie algebra $\mathfrak{X}(P)$ of vector fields on P. Let $\mathfrak{a}(P)$ be the set of vector fields X on P satisfying (1) and (2) of Proposition 2.1 and also (2) of Proposition 2.2. Since $L_{[X,X']} = L_X \circ L_{X'} - L_{X'} \circ L_X$ (cf. Proposition 3.4 of Chapter I), $\mathfrak{a}(P)$ forms a subalgebra of the Lie algebra $\mathfrak{X}(P)$. It follows that $\mathfrak{a}(M)$ is a subalgebra of $\mathfrak{X}(M)$ isomorphic with $\mathfrak{a}(P)$ under the correspondence $X \to \tilde{X}$ defined in Proposition 2.1.

THEOREM 2.3. *If M is a connected manifold with an affine connection Γ, the Lie algebra $\mathfrak{a}(M)$ of infinitesimal affine transformations of M is of dimension at most $n^2 + n$, where $n =$ dim M. If* dim $\mathfrak{a}(M) = n^2 + n$, *then Γ is flat, that is, both the torsion and the curvature of Γ vanish identically.*

Proof. To prove the first statement it is sufficient to show that $\mathfrak{a}(P)$ is of dimension at most $n^2 + n$, since $\mathfrak{a}(M)$ is isomorphic with $\mathfrak{a}(P)$. Let u be an arbitrary point of P. The following lemma implies that the linear mapping $f\colon \mathfrak{a}(P) \to T_u(P)$ defined by $f(\tilde{X}) = \tilde{X}_u$ is injective so that dim $\mathfrak{a}(P) \leq$ dim $T_u(P) = n^2 + n$.

LEMMA. *If an element $\tilde{X}$ of $\mathfrak{a}(P)$ vanishes at some point of P, then it vanishes identically on P.*

Proof of Lemma. If $\tilde{X}_u = 0$, then $\tilde{X}_{ua} = 0$ for every $a \in G$ as $\tilde{X}$ is invariant by R_a (cf. Proposition 2.1). Let F be the set of points $x = \pi(u) \in M$ such that $\tilde{X}_u = 0$. Then F is closed in M. Since M is connected, it suffices to show that F is open. Assume $\tilde{X}_u = 0$. Let b_t be a local 1-parameter group of local transformations generated by a standard horizontal vector field $B(\xi)$ in a neighborhood of u. Since $[\tilde{X}, B(\xi)] = 0$ by Proposition 2.2, $\tilde{X}$ is invariant by b_t and hence $\tilde{X}_{b_t u} = 0$. In the definition of a normal coordinate system (cf. §8 of Chapter III), we saw that the points of the form $\pi(b_t u)$ cover a neighborhood of $x = \pi(u)$ when ξ and t vary. This proves that F is open.

To prove the second statement, we assume that $\dim \mathfrak{a}(M) = \dim \mathfrak{a}(P) = n^2 + n$. Let u be an arbitrary point of P. Then the linear mapping f: $f(\tilde{X}) = \tilde{X}_u$, maps $\mathfrak{a}(P)$ onto $T_u(P)$. In particular, given any element $A \in \mathfrak{g}$, there exists a (unique) element $\tilde{X} \in \mathfrak{a}(P)$ such that $\tilde{X}_u = A^*_u$, where A^* denotes the fundamental vector field corresponding to A. Let $B = B(\xi)$ and $B' = B(\xi')$ be the standard horizontal vector fields corresponding to ξ and ξ', respectively. Then

$$\tilde{X}_u(\Theta(B, B')) = A^*_u(\Theta(B, B')).$$

We compute both sides of the equality separately. From $L_X\Theta = L_X(d\theta + \omega \wedge \theta) = 0$ and from (3) of Proposition 2.2, we obtain

$$X(\Theta(B, B')) = (L_X\Theta)(B, B') + \Theta([X, B], B') + \Theta(B, [X, B']) = 0.$$

To compute the right hand side, we first observe that the exterior differentiation d applied to the first structure equation yields

$$0 = -\Omega \wedge \theta + \omega \wedge \Theta + d\Theta.$$

Hence we have

$$\begin{aligned} L_{A^*}\Theta &= (d \circ \iota_{A^*} + \iota_{A^*} \circ d)\Theta \\ &= \iota_{A^*} \circ d\,\Theta = \iota_{A^*}(\Omega \wedge \theta - \omega \wedge \Theta) = -\omega(A^*) \cdot \Theta \end{aligned}$$

and

$$(L_{A^*}\Theta)(B, B') = -A \cdot \Theta(B, B').$$

Therefore,

$$A^*(\Theta(B, B')) = -A \cdot \Theta(B, B') + \Theta([A^*, B], B') + \Theta(B, [A^*, B']).$$

If we take as A the identity matrix of $\mathfrak{g} = \mathfrak{gl}(n; \mathbf{R})$, then, by Proposition 2.3 of Chapter III, we have

$$[A^*, B] = B \quad \text{and} \quad [A^*, B'] = B'.$$

Thus we have

$$\begin{aligned} 0 &= X_u(\Theta(B, B')) = A_u^*(\Theta(B, B')) \\ &= -\Theta_u(B, B') + \Theta_u(B, B') + \Theta_u(B, B') = \Theta_u(B, B'), \end{aligned}$$

showing that the torsion form vanishes.

Similarly, comparing the both sides of the equality:

$$X_u(\Omega(B, B')) = A_u^*(\Omega(B, B'))$$

and letting A equal the identity matrix of $\mathfrak{g} = \mathfrak{gl}(n; \mathbf{R})$, we see that the curvature form vanishes identically. QED.

We now prove the following result due to Kobayashi [2].

THEOREM 2.4. *Let Γ be a complete linear connection on M. Then every infinitesimal affine transformation X of M is complete, that is, generates a global* 1*-parameter group of affine transformations of M.*

Proof. It suffices to show that every element $\tilde{X}$ of $\mathfrak{a}(P)$ is complete under the assumption that M is connected. Let u_0 be an arbitrary point of P and let $\tilde{\varphi}_t\colon U \to P$, $|t| < \delta$, be a local 1-parameter group of local transformations generated by $\tilde{X}$ (cf. Proposition 1.5 of Chapter I). We shall prove that $\tilde{\varphi}_t(u)$ is defined for every $u \in P$ and $|t| < \delta$. Then it follows that $\tilde{X}$ is complete.

By Proposition 6.5 of Chapter III, every standard horizontal vector field $B(\xi)$ is complete since the connection is complete. Given any point u of P, there exist standard horizontal vector fields $B(\xi_1), \ldots, B(\xi_k)$ and an element $a \in G$ such that

$$u = (b_{t_1}^1 \circ b_{t_2}^2 \circ \cdots \circ b_{t_k}^k u_0)a,$$

where each b_t^i is the 1-parameter group of transformations of P generated by $B(\xi_i)$. In fact, the existence of normal coordinate neighborhoods (cf. Proposition 8.2 of Chapter III) and the connectedness of M imply that the point $x = \pi(u)$ can be joined to the point $x_0 = \pi(u_0)$ by a finite succession of geodesics. By Proposition 6.3 of Chapter III, every geodesic is the projection of an integral curve of a certain standard horizontal vector field. This means that by taking suitable $B(\xi_1), \ldots, B(\xi_k)$, we obtain a point $v = b_{t_1}^1 \circ b_{t_2}^2 \circ \cdots \circ b_{t_k}^k u_0$ which lies in the same fibre as u. Then $u = va$ for a suitable $a \in G$, thus proving our assertion. We then define $\tilde{\varphi}_t(u)$ by

$$\tilde{\varphi}_t(u) = (b_{t_1}^1 \circ b_{t_2}^2 \circ \cdots \circ b_{t_k}^k(\tilde{\varphi}_t(u_0)))a, \quad |t| < \delta.$$

The fact that $\tilde{\varphi}_t(u)$ is independent of the choice of $b^1_{t_1}, \ldots, b^k_{t_k}$, a and that $\tilde{\varphi}_t$ is generated by $\tilde{X}$ follows from (1) of Proposition 2.1 and (3) of Proposition 2.2; note that (3) of Proposition 2.2 implies that $b_s \circ \tilde{\varphi}_t(u) = \tilde{\varphi}_t \circ b_s(u)$ whenever they are both defined. QED.

In general, every element of the Lie algebra of the group $\mathfrak{A}(M)$ of affine transformations of M gives rise to an element of $\mathfrak{a}(M)$ which is complete, and conversely. In other words, the Lie algebra of $\mathfrak{A}(M)$ can be identified with the subalgebra of $\mathfrak{a}(M)$ consisting of complete vector fields. Theorem 2.4 means that if the connection is complete, then $\mathfrak{a}(M)$ can be considered as the Lie algebra of $\mathfrak{A}(M)$.

For any vector field X on M, the derivation $A_X = L_X - \nabla_X$ is induced by a tensor field of type (1, 1) because it is zero on the function algebra $\mathfrak{F}(M)$ (cf. the proof of Proposition 3.3 of Chapter I). This fact may be derived also from the following

Proposition 2.5. *For any vector fields X and Y on M, we have*

$$A_X Y = -\nabla_Y X - T(X, Y),$$

where T is the torsion.

Proof. By Theorem 5.1 of Chapter III, we have

$$\begin{aligned} A_X Y = L_X Y - \nabla_X Y &= [X, Y] - (\nabla_Y X + [X, Y] + T(X, Y)) \\ &= -\nabla_Y X - T(X, Y). \end{aligned}$$

QED.

We conclude this section by

Proposition 2.6. (1) *A vector field X on M is an infinitesimal affine transformation if and only if*

$$\nabla_Y (A_X) = R(X, Y) \qquad \textit{for every vector field } Y \textit{ on } M.$$

(2) *If both X and Y are infinitesimal affine transformations of M, then*

$$A_{[X,Y]} = [A_X, A_Y] + R(X, Y),$$

where R denotes the curvature.

Proof. (1) By Theorem 5.1 of Chapter III, we have

$$\begin{aligned} R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} &= [L_X - A_X, \nabla_Y] - \nabla_{[X,Y]} \\ &= [L_X, \nabla_Y] - \nabla_{[X,Y]} - [A_X, \nabla_Y]. \end{aligned}$$

By Proposition 2.2, X is an infinitesimal affine transformation if and only if $R(X, Y) = -[A_X, \nabla_Y]$ for every Y, that is, if and only if

$$R(X, Y)Z = \nabla_Y(A_X Z) - A_X(\nabla_Y Z) = (\nabla_Y(A_X))Z \quad \text{for all } Y \text{ and } Z.$$

(2) By Theorem 5.1 of Chapter III and Proposition 2.2, we have

$$\begin{aligned}[A_X, A_Y] - A_{[X,Y]} &= [L_X - \nabla_X, L_Y - \nabla_Y] - (L_{[X,Y]} - \nabla_{[X,Y]}) \\ &= [L_X, L_Y] - [\nabla_X, L_Y] - [L_X, \nabla_Y] \\ &\quad + [\nabla_X, \nabla_Y] - L_{[X,Y]} + \nabla_{[X,Y]} = R(X, Y).\end{aligned}$$

QED.

3. *Isometries and infinitesimal isometries*

Let M be a manifold with a Riemannian metric g and the corresponding Riemannian connection Γ. An isometry of M is a transformation of M which leaves the metric g invariant. We know from Proposition 2.5 of Chapter IV that an isometry of M is necessarily an affine transformation of M with respect to Γ.

Consider the bundle $O(M)$ of orthonormal frames over M which is a subbundle of the bundle $L(M)$ of linear frames over M. We have

PROPOSITION 3.1. (1) *A transformation f of M is an isometry if and only if the induced transformation $\tilde{f}$ of $L(M)$ maps $O(M)$ into itself;*

(2) *A fibre-preserving transformation F of $O(M)$ which leaves the canonical form θ on $O(M)$ invariant is induced by an isometry of M.*

Proof. (1) This follows from the fact that a transformation f of M is an isometry if and only if it maps each orthonormal frame at an arbitrary point x into an orthonormal frame at $f(x)$.

(2) Let f be the transformation of the base M induced by F. We set $J = \tilde{f}^{-1} \circ F$. Then J is a fibre-preserving mapping of $O(M)$ into $L(M)$ which preserves θ. Moreover, J induces the identity transformation on the base M, Therefore we have

$$u^{-1}(X) = \theta(X^*) = \theta(JX^*) = J(u)^{-1}(X), \qquad X^* \in T_u(O(M)), \quad X = \pi(X^*).$$

This implies that $J(u) = u$, that is, $\tilde{f}(u) = F(u)$. By (1), f is an isometry of M. QED.

A vector field X on M is called an *infinitesimal isometry* (or, a *Killing vector field*) if the local 1-parameter group of local transformations generated by X in a neighborhood of each point of M consists of local isometries. An infinitesimal isometry is necessarily an infinitesimal affine transformation.

PROPOSITION 3.2. *For a vector field X on a Riemannian manifold M, the following conditions are mutually equivalent:*

(1) *X is an infinitesimal isometry;*

(2) *The natural lift $\tilde{X}$ of X to $L(M)$ is tangent to $O(M)$ at every point of $O(M)$;*

(3) *$L_X g = 0$, where g is the metric tensor field of M;*

(4) *The tensor field $A_X = L_X - \nabla_X$ of type (1, 1) is skew-symmetric with respect to g everywhere on M, that is, $g(A_X Y, Z) = -g(A_X Z, Y)$ for arbitrary vector fields Y and Z.*

Proof. (i) To prove the equivalence of (1) and (2), let φ_t and $\tilde{\varphi}_t$ be the local 1-parameter groups of local transformations generated by X and $\tilde{X}$ respectively. If X is an infinitesimal isometry, then φ_t are local isometries and hence $\tilde{\varphi}_t$ map $O(M)$ into itself. Thus $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$. Conversely, if $\tilde{X}$ is tangent to $O(M)$ at every point of $O(M)$, the integral curve of $\tilde{X}$ through each point of $O(M)$ is contained in $O(M)$ and hence each $\tilde{\varphi}_t$ maps $O(M)$ into itself. This means, by Proposition 3.1, that each φ_t is a local isometry and hence X is an infinitesimal isometry.

(ii) The equivalence of (1) and (3) follows from Corollary 3.7 of Chapter I.

(iii) Since $\nabla_X g = 0$ for any vector field X, $L_X g = 0$ is equivalent to $A_X g = 0$. Since A_X is a derivation of the algebra of tensor fields, we have

$$A_X(g(Y, Z)) = (A_X g)(Y, Z) + g(A_X Y, Z) + g(Y, A_X Z) \qquad \text{for } Y, Z \in \mathfrak{X}(M).$$

Since A_X maps every function into zero, $A_X(g(Y, Z)) = 0$. Hence $A_X g = 0$ if and only if $g(A_X Y, Z) + g(Y, A_X Z) = 0$ for all Y and Z, thus proving the equivalence of (3) and (4). QED.

The set of all infinitesimal isometries of M, denoted by $\mathfrak{i}(M)$, forms a Lie algebra. In fact, if X and Y are infinitesimal isometries of M, then

$$L_{[X,Y]}g = L_X \circ L_Y g - L_Y \circ L_X g = 0$$

by Proposition 3.2. By the same proposition, $[X, Y]$ is an infinitesimal isometry of M.

THEOREM 3.3. *The Lie algebra* $\mathfrak{i}(M)$ *of infinitesimal isometries of a connected Riemannian manifold M is of dimension at most* $\frac{1}{2}n(n+1)$, *where* $n = \dim M$. *If* $\dim \mathfrak{i}(M) = \frac{1}{2}n(n+1)$, *then M is a space of constant curvature.*

Proof. To prove the first assertion, it is sufficient to show that, for any point u of $O(M)$, the linear mapping $X \to \tilde{X}_u$ maps $\mathfrak{i}(M)$ 1:1 into $T_u(O(M))$. By Proposition 3.2, $\tilde{X}_u$ is certainly an element of $T_u(O(M))$. If $\tilde{X}_u = 0$, then the proof of Theorem 2.3 shows that $X = 0$. We now prove the second assertion.

Let X, X' be an orthonormal basis of a plane p in $T_x(M)$ and let u be a point of $O(M)$ such that $\pi(u) = x$. We set $\xi = u^{-1}(X)$, $\xi' = u^{-1}(X')$, $B = B(\xi)$ and $B' = B(\xi')$, where $B(\xi)$ and $B(\xi')$ are the restrictions to $O(M)$ of the standard horizontal vector fields corresponding to ξ and ξ', respectively. From the definition of the curvature transformation given in §5 of Chapter III we see that the sectional curvature $K(p)$ (cf. §2 of Chapter V) is given by

$$K(p) = ((2\Omega(B_u, B'_u))\xi', \xi),$$

where $(\ , \ )$ denotes the natural inner product in $\mathbf{R}^n$. To prove that $K(p)$ is independent of p, let Y, Y' be an orthonormal basis of another plane q in $T_x(M)$ and set $\eta = u^{-1}(Y)$ and $\eta' = u^{-1}(Y')$. Let a be an element of $SO(n)$ such that $a\xi = \eta$ and $a\xi' = \eta'$. By Proposition 2.2 of Chapter III, we have

$$\begin{aligned}\Omega(B(\eta)_u, B(\eta')_u) &= \Omega(B(a\xi)_u, B(a\xi')_u) = \Omega(R_{a^{-1}}(B_{ua}), R_{a^{-1}}(B'_{ua})) \\ &= \mathrm{ad}\,(a)(\Omega(B_{ua}, B'_{ua})) = a \cdot \Omega(B_{ua}, B'_{ua}) \cdot a^{-1}.\end{aligned}$$

Hence the sectional curvature $K(q)$ is given by

$$\begin{aligned}K(q) &= ((2\Omega(B(\eta)_u, B(\eta')_u)\eta', \eta) \\ &= ((a \cdot 2\Omega(B_{ua}, B'_{ua}) \cdot a^{-1})a\xi', a\xi) \\ &= ((2\Omega(B_{ua}, B'_{ua}))\xi', \xi).\end{aligned}$$

To prove that $K(p) = K(q)$, it is sufficient to show that $\Omega(B_{ua}, B'_{ua}) = \Omega(B_u, B'_u)$. Given any vertical vector $X^* \in T_v(O(M))$ with $\pi(v) = x$, there exists an element $X \in \mathfrak{i}(M)$ such that $\tilde{X}_v = X^*$ if $\dim \mathfrak{i}(M) = \frac{1}{2}n(n+1)$, since the mapping $X \rightarrow \tilde{X}_v$ maps $\mathfrak{i}(M)$ onto $T_v(O(M))$. We have

$$\tilde{X}(\Omega(B, B')) = (L_{\tilde{X}}\Omega)(B, B') + \Omega([X, B], B') + \Omega(B, [X, B']) = 0.$$

This implies that $\Omega(B_u, B'_u) = \Omega(B_{ua}, B'_{ua})$ for every $a \in SO(n)$. We thus proved that $K(p)$ depends only on the point x. We prove that $K(p)$ does not depend even on x. Given any vector $Y^* \in T_u(O(M))$, let Y be an element of $\mathfrak{i}(M)$ such that $\tilde{Y}_u = Y^*$. We have again $\tilde{Y}(\Omega(B, B')) = 0$. Hence, for fixed ξ and ξ', the function $((2\Omega(B, B'))\xi', \xi)$ is constant in a neighborhood of u. This means that $K(p)$, considered as a function on M, is locally constant. Since it is continuous and M is connected, it must be constant on M. (If $\dim M \geqq 3$, the fact that $K(p)$ is independent of x follows also from Theorem 2.2 of Chapter V.) QED.

THEOREM 3.4. (1) *For a Riemannian manifold M with a finite number of connected components, the group $\mathfrak{I}(M)$ of isometries of M is a Lie transformation group with respect to the compact-open topology in M;*

(2) *The Lie algebra of $\mathfrak{I}(M)$ is naturally isomorphic with the Lie algebra of all complete infinitesimal isometries;*

(3) *The isotropy subgroup $\mathfrak{I}_x(M)$ of $\mathfrak{I}(M)$ at an arbitrary point x is compact;*

(4) *If M is complete, the Lie algebra of $\mathfrak{I}(M)$ is naturally isomorphic with the Lie algebra $\mathfrak{i}(M)$ of all infinitesimal isometries of M;*

(5) *If M is compact, then the group $\mathfrak{I}(M)$ is compact.*

Proof. (1) As we indicated in the proof of Theorem 1.5, this follows from Theorem 4.6 and Corollary 4.9 of Chapter I and Theorem 3.10 of Chapter IV.

(2) Every 1-parameter subgroup of $\mathfrak{I}(M)$ induces an infinitesimal isometry X which is complete on M and, conversely, every complete infinitesimal isometry X generates a 1-parameter subgroup of $\mathfrak{I}(M)$.

(3) This follows from Corollary 4.8 of Chapter I.

(4) This follows from (2) and Theorem 2.4.

(5) This follows from Corollary 4.10 of Chapter I. QED.

Clearly, $\mathfrak{I}(M)$ is a closed subgroup of $\mathfrak{A}(M)$. We shall see that, in many instances, the identity component $\mathfrak{I}^0(M)$ of $\mathfrak{I}(M)$ coincides with the identity component $\mathfrak{A}^0(M)$ of $\mathfrak{A}(M)$. We first prove a result by Hano [1].

THEOREM 3.5. *If $M = M_0 \times M_1 \times \cdots \times M_k$ is the de Rham decomposition of a complete, simply connected Riemannian manifold M, then*

$$\mathfrak{A}^0(M) \approx \mathfrak{A}^0(M_0) \times \mathfrak{A}^0(M_1) \times \cdots \times \mathfrak{A}^0(M_k),$$

$$\mathfrak{I}^0(M) \approx \mathfrak{I}^0(M_0) \times \mathfrak{I}^0(M_1) \times \cdots \times \mathfrak{I}^0(M_k).$$

Proof. We need the following two lemmas.

LEMMA 1. *Let $T_x(M) = \Sigma_{i=0}^k T_0^{(i)}$ be the canonical decomposition.*
(1) *If $\varphi \in A(M)$, then $\varphi(T_x^{(0)}) = T_{\varphi(x)}^{(0)}$ and for each i, $1 \leqq i \leqq k$, $\varphi(T_x^{(i)}) = T_{\varphi(x)}^{(j)}$ for some j, $1 \leqq j \leqq k$;*
(2) *If $\varphi \in A^0(M)$, then $\varphi(T_x^{(i)}) = T_{\varphi(x)}^{(i)}$ for every i, $0 \leqq i \leqq k$.*

Proof of Lemma 1. Let τ be any loop at x and set $\tau' = \varphi(\tau)$ so that τ' is a loop at $\varphi(x)$. If we denote by the same letter τ and τ' the parallel displacements along τ and τ' respectively, then

$$\varphi \circ \tau(X) = \tau' \circ \varphi(X) \qquad \text{for } X \in T_x(M).$$

It follows easily that $\varphi(T_x^{(0)})$ is invariant elementwise and every $\varphi(T_x^{(i)})$, $1 \leqq i \leqq k$, is irreducible by the linear holonomy group $\Psi(\varphi(x))$. Hence, $\varphi(T_x^{(0)}) \subset T_{\varphi(x)}^{(0)}$ and, their dimensions being the same, $\varphi(T_x^{(0)}) = T_{\varphi(x)}^{(0)}$. Thus we obtain the canonical decomposition $T_{\varphi(x)} = \Sigma_{i=0}^k \varphi(T_x^{(i)})$ which should coincide with the canonical decomposition $T_{\varphi(x)} = \Sigma_{i=0}^k T_{\varphi(x)}^{(i)}$ up to an order by (4) of Theorem 5.4 of Chapter IV. This means precisely the statement (1). Let φ_t be a 1-parameter subgroup of $\mathfrak{A}^0(M)$ and let X be a non-zero element of $T_x^{(i)}$. Let $\tau = x_t = \varphi_t(x)$. Since $g(\varphi_0(X), X) = g(X, X) \neq 0$, we have $g(\varphi_t(X), \tau_t^0 X) \neq 0$ for $|t| < \delta$ for some $\delta > 0$, where τ_t^0 denotes the parallel displacement from x_0 to x_t along τ. This means that $\varphi_t(T_x^{(i)}) = T_{\varphi_t(x)}^{(i)}$ for $|t| < \delta'$ for some positive number δ'; in fact, if $X_1, \ldots, X_r$ is a basis for $T_x^{(i)}$, then $g(\varphi_t(X_j), \tau_t^0 X_j) \neq 0$ for $1 \leqq j \leqq r$ and $|t| < \delta'$ for some positive number δ' and hence $\varphi_t(X_j) \in T_{\varphi_t(x)}^{(i)}$ for $|t| < \delta'$, which implies $\varphi_t(T_x^{(i)}) = T_{\varphi_t(x)}^{(i)}$ for $|t| < \delta'$ because of the linearity of φ_t. This concludes the proof of the statement (2), since $\mathfrak{A}^0(M)$ is generated by 1-parameter subgroups.

Lemma 1 is due to Nomizu [3].

LEMMA 2. *Let* φ_i *be an arbitrary transformation of* M_i *for every* i, $0 \leqq i \leqq k$. *Let* φ *be the transformation of* $M = M_0 \times M_1 \times \cdots \times M_k$ *defined by*

$$\varphi(x) = (\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_k(x_k)) \quad \text{for } x = (x_0, x_1, \ldots, x_k) \in M.$$

Then

(1) φ *is an affine transformation of* M *if and only if every* φ_i *is an affine transformation of* M_i.

(2) φ *is an isometry of* M *if and only if every* φ_i *is an isometry of* M_i.

The proof is trivial.

The correspondence $(\varphi_0, \varphi_1, \cdots, \varphi_k) \to \varphi$ defined in Lemma 2 maps $\mathfrak{A}(M_0) \times \mathfrak{A}(M_1) \times \cdots \times \mathfrak{A}(M_k)$ isomorphically into $\mathfrak{A}(M)$. To complete the proof of Theorem 3.5, it suffices to show that, for every $\varphi \in \mathfrak{A}^0(M)$, there exist transformations $\varphi_i\colon M_i \to M_i$, $0 \leqq i \leqq k$, such that

$$\varphi(x) = (\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_k(x_k)) \quad \text{for } x = (x_0, x_1, \ldots, x_k) \in M.$$

We prove that, if $p_i\colon M \to M_i$ denotes the natural projection, then $p_i(\varphi(x))$ depends only on $x_i = p_i(x)$. Given any point $y = (y_0, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_k)$, let, for each $j = 0, 1, \ldots, i-1, i+1, \ldots, k$, $\tau_j = x_j(t)$, $0 \leqq t \leqq 1$, be a curve from x_j to y_j in M_j so that $x_j(0) = x_j$ and $x_j(1) = y_j$. Let $\tau = x(t)$, $0 \leqq t \leqq 1$, be the curve from x to y in M defined by

$$x(t) = (x_0(t), x_1(t), \ldots, x_{i-1}(t), x_i, x_{i+1}(t), \ldots, x_k(t)), \quad 0 \leqq t \leqq 1.$$

For each t, the tangent vector $\dot{x}(t)$ to τ at $x(t)$ is in the distribution $T^{(0)} + \cdots + T^{(i-1)} + T^{(i+1)} + \cdots + T^{(k)}$. By Lemma 1, $\varphi(\dot{x}(t))$ lies also in the same distribution. Hence $p_i(\varphi(x(t)))$ is independent of t (cf. Lemma 2 for Theorem 7.2 of Chapter II). In particular, $p_i(\varphi(x)) = p_i(\varphi(y))$, thus proving our assertion. We then define a transformation $\varphi_i\colon M_i \to M_i$ by

$$\varphi_i(x_i) = p_i(\varphi(x)).$$

Clearly, we have

$$\varphi(x) = (\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_k(x_k)).$$

QED.

It is therefore important to study $\mathfrak{A}(M)$ when M is irreducible. The following result is due to Kobayashi [4].

THEOREM 3.6. *If M is a complete, irreducible Riemannian manifold, then $\mathfrak{A}(M) = \mathfrak{I}(M)$ except when M is a 1-dimensional Euclidean space.*

Proof. A transformation φ of a Riemannian manifold is said to be *homothetic* if there is a positive constant c such that $g(\varphi(X), \varphi(Y)) = c^2g(X, Y)$ for all $X, Y \in T_x(M)$ and $x \in M$. Consider the Riemannian metric g^* defined by $g^*(X, Y) = g(\varphi(X), \varphi(Y))$. From the proof (B) of Theorem 2.2 of Chapter III, we see that the Riemannian connection defined by g^* coincides with the one defined by g. This means that every homothetic transformation of a Riemannian manifold M is an affine transformation of M.

LEMMA 1. *If M is an irreducible Riemannian manifold, then every affine transformation φ of M is homothetic.*

Proof of Lemma 1. Since φ is an affine transformation, the two Riemannian metrics g and g^* (defined above) determine the same Riemannian connection, say Γ. Let $\Psi(x)$ be the linear holonomy group of Γ with reference point x. Since it is irreducible and leaves both g and g^* invariant, there exists a positive constant c_x such that $g^*(X, Y) = c_x^2 \cdot g(X, Y)$ for all $X, Y \in T_x(M)$, that is, $g_x^* = c_x^2 \cdot g_x$ (cf. Theorem 1 of Appendix 5). Since both g^* and g are parallel tensor fields with respect to Γ, c_x is constant.

LEMMA 2. *If M is a complete Riemannian manifold which is not locally Euclidean, then every homothetic transformation φ of M is an isometry.*

Proof of Lemma 2. Assume that φ is a non-isometric homothetic transformation of M. Considering the inverse transformation if necessary, we may assume that the constant c associated with φ is less than 1. Take an arbitrary point x of M. If the distance between x and $\varphi(x)$ is less than δ, then the distance between $\varphi^m(x)$ and $\varphi^{m+1}(x)$ is less than $c^m\delta$. It follows that $\{\varphi^m(x); m = 1, 2, \dots\}$ is a Cauchy sequence and hence converges to some point, say x^*, since M is complete. It is easy to see that the point x^* is left fixed by φ.

Let U be a neighborhood of x^* such that $\bar{U}$ is compact. Let K^* be a positive number such that $|g(R(Y_1, Y_2)Y_2, Y_1)| < K^*$ for any unit vectors Y_1 and Y_2 at $y \in U$, where R denotes the curvature tensor field. Let $z \in M$ and q any plane in $T_z(M)$. Let X, Y be an orthonormal basis for q. Since φ is an affine transformation, (3) of Proposition 1.2 implies that

$$R(\varphi^m X, \varphi^m Y)(\varphi^m Y) = \varphi^m(R(X, Y)Y).$$

Hence we have

$$\begin{aligned} g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y), \varphi^m X) &= g(\varphi^m(R(X, Y)Y), \varphi^m X) \\ &= c^{2m} g(R(X, Y)Y, X) = c^{2m} K(q). \end{aligned}$$

On the other hand, the distance between $x^* = \varphi^m(x^*)$ and $\varphi^m(z)$ approaches 0 as m tends to infinity. In other words, there exists an integer m_0 such that $\varphi^m(z) \in U$ for every $m \geqq m_0$. Since the lengths of the vectors $\varphi^m X$ and $\varphi^m Y$ are equal to c^m, we have

$$c^{4m} K^* \geqq |g(R(\varphi^m X, \varphi^m Y)(\varphi^m Y), \varphi^m X)| \qquad \text{for } m \geqq m_0.$$

Thus we obtain

$$c^{2m} K^* \geqq |K(q)| \qquad \text{for } m \geqq m_0.$$

Letting m tend to infinity, we have $K(q) = 0$. This shows that M is locally Euclidean. QED.

Let X be an infinitesimal affine transformation on a complete Riemannian manifold M. Using Theorems 3.5 and 3.6, we shall find a number of sufficient conditions for X to be an infinitesimal isometry. Assuming that M is connected, let $\tilde{M}$ be the universal covering manifold with the naturally induced Riemannian metric $\tilde{g} = p^*(g)$, where $p\colon \tilde{M} \rightarrow M$ is the natural projection. Let $\tilde{X}$ be the vector field on $\tilde{M}$ induced by X; $\tilde{X}$ is p-related to X. Then $\tilde{X}$ is an infinitesimal affine transformation of $\tilde{M}$. Clearly, $\tilde{X}$ is an infinitesimal isometry of $\tilde{M}$ if and only if X is an infinitesimal isometry of M. Let $\tilde{M} = M_0 \times M_1 \times \cdots \times M_k$ be the de Rham decomposition of the complete simply connected Riemannian manifold $\tilde{M}$. By Theorem 3.5, the Lie algebra $\mathfrak{a}(\tilde{M})$ is isomorphic with $\mathfrak{a}(M_0) + \mathfrak{a}(M_1) + \cdots + \mathfrak{a}(M_k)$. Let $(X_0, X_1, \ldots, X_k)$ be the element of $\mathfrak{a}(M_0) + \mathfrak{a}(M_1) + \cdots + \mathfrak{a}(M_k)$ corresponding to $\tilde{X} \in \mathfrak{a}(\tilde{M})$. Since $X_1, \ldots, X_k$ are all infinitesimal isometries by Theorem 3.6, X is an infinitesimal isometry if and only if X_0 is.

Corollary 3.7. *If M is a connected, complete Riemannian manifold whose restricted linear holonomy group $\Psi^0(x)$ leaves no non-zero vector at x fixed, then $\mathfrak{A}^0(M) = \mathfrak{I}^0(M)$.*

Proof. The linear holonomy group of M is naturally isomorphic with the restricted linear holonomy group $\Psi^0(x)$ of M (cf. Example 2.1 of Chapter IV). This means that M_0 reduces to a point and hence $X_0 = 0$ in the above notations. QED.

COROLLARY 3.8. *If X is an infinitesimal affine transformation of a complete Riemannian manifold and if the length of X is bounded, then X is an infinitesimal isometry.*

Proof. We may assume M to be connected. If the length of X is bounded on M, the length of X_0 is also bounded on M_0. Let $x^1, \ldots, x^r$ be the Euclidean coordinate system in M_0 and set

$$X_0 = \Sigma_{\alpha=1}^r \xi^\alpha(\partial/\partial x^\alpha).$$

Applying the formula $(L_{X_0} \circ \nabla_Y - \nabla_Y \circ L_{X_0})Z = \nabla_{[X_0,Y]}Z$ (cf. Proposition 2.2) to $Y = \partial/\partial x^\beta$ and $Z = \partial/\partial x^\gamma$, we see that

$$\frac{\partial^2 \xi^\alpha}{\partial x^\beta\, \partial x^\gamma} = 0.$$

This means that X_0 is of the form

$$\Sigma_{\alpha=1}^r \left(\Sigma_{\beta=1}^r a_\beta^\alpha x^\beta + b^\alpha\right)(\partial/\partial x^\alpha).$$

It is easy to see that length of X_0 is bounded on M_0 if and only if $a_\beta^\alpha = 0$ for $\alpha, \beta = 1, \ldots, r$. Thus if X_0 is of bounded length, then X_0 is an infinitesimal isometry of M_0. QED.

Corollary 3.8, obtained by Hano [1], implies the following result of Yano [1] which was originally proved by a completely different method.

COROLLARY 3.9. *On a compact Riemannian manifold M, we have* $\mathfrak{A}^0(M) = \mathfrak{I}^0(M)$.

Proof. On a compact manifold M, every vector field is of bounded length. By Corollary 3.8, every infinitesimal affine transformation X is an infinitesimal isometry. QED.

4. *Holonomy and infinitesiaml isometries*

Let M be a differentiable manifold with a linear connection Γ. For an infinitesimal affine transformation X of M, we give a geometric interpretation of the tensor field $A_X = L_X - \nabla_X$ introduced in §2.

Let x be an arbitrary point of M and let φ_t be a local 1-parameter group of affine transformations generated by X in a neighborhood of x. Let τ be the orbit $x_t = \varphi_t(x)$ of x. We denote by τ_t^s the parallel displacement along the curve τ from x_s to x_t. For each t, we consider a linear transformation $C_t = \tau_0^t \circ (\varphi_t)_*$ of $T_x(M)$.

PROPOSITION 4.1. *C_t is a local 1-parameter group of linear transformations of $T_x(M)$: $C_{t+s} = C_t \circ C_s$, and $C_t = \exp(-t(A_Z)_x)$.*

Proof. Since φ_t maps the portion of τ from x_0 to x_s into the portion of τ from x_t to x_{t+s} and since φ_t is compatible with parallel displacement, we have

$$\varphi_t \circ \tau_0^s = \tau_t^{t+s} \circ \varphi_t.$$

Hence

$$C_t \circ C_s = \tau_0^t \circ \varphi_t \circ \tau_0^s \circ \varphi_s = \tau_0^t \circ \tau_t^{t+s} \circ \varphi_t \circ \varphi_s = \tau_0^{t+s} \circ \varphi_{t+s} = C_{t+s}.$$

This proves the first assertion. Thus there is a linear endomorphism, say, A of $T_x(M)$ such that $C_t = \exp tA$. The second assertion says that $A = -(A_X)_x$. To prove this, we show that

$$\lim_{t\to 0} \frac{1}{t}(C_t Y_x - Y_x) = -(A_X)_x Y_x \qquad \text{for } Y_x \in T_x(M).$$

First, consider the case where $X_x \neq 0$. Then x has a coordinate neighborhood with local coordinate system $x^1, \dots, x^n$ such that the curve $\tau = x_t$ is given by $x^1 = t$, $x^2 = \cdots = x^n = 0$ for small values of t. We may therefore extend Y_x to a vector field Y on M in such a way that $\varphi_t(Y_x) = Y_{x_t}$ for small values of t. Evidently, $(L_X Y)_x = 0$. We have

$$\begin{aligned}
-(A_X)_x Y_x &= (\nabla_X Y)_x - (L_X Y)_x = (\nabla_X Y)_x \\
&= \lim_{t\to 0} \frac{1}{t}(\tau_0^t Y_{x_t} - Y_x) = \lim_{t\to 0} \frac{1}{t}(\tau_0^t \circ \varphi_t Y_x - Y_x) \\
&= \lim_{t\to 0} \frac{1}{t}(C_t Y_x - Y_x).
\end{aligned}$$

Second, consider the case where $X_x = 0$. In this case, φ_t is a local 1-parameter group of local transformations leaving x fixed and the parallel displacement τ_0^t reduces to the identity transformation of $T_x(M)$. Thus $(\nabla_X Y)_x = 0$. We have

$$-(A_X)_x Y_x = (\nabla_X Y)_x - (L_X Y)_x = -(L_X Y)_x$$
$$= -\lim_{t\to 0} \frac{1}{t}(Y_x - \varphi_t Y_x) = \lim_{t\to 0} \frac{1}{t}(C_t Y_x - Y_x).$$

This completes the proof of the second assertion. QED.

Remark. Proposition 4.1 is indeed a special case of Proposition 11.2 of Chapter II and can be derived from it.

PROPOSITION 4.2. *Let $N(\Psi(x))$ and $N(\Psi^0(x))$ be the normalizors of the linear holonomy group $\Psi(x)$ and the restricted linear holonomy group $\Psi^0(x)$ in the group of linear transformations of $T_x(M)$. Then C_t is contained in $N(\Psi(x))$ as well as in $N(\Psi^0(x))$.*

Proof. Let φ_t and τ_t^s be as before. For any loop μ at x, we set $\mu_t' = \varphi_t(\mu)$ so that μ_t' is a loop at $x_t = \varphi_t(x)$. We denote by μ and μ_t' the parallel displacements along μ and μ_t', respectively. Then $\varphi_t \circ \mu = \mu_t' \circ \varphi_t$. We have

$$C_t \circ \mu \circ C_t^{-1} = \tau_0^t \circ \varphi_t \circ \mu \circ \varphi_t^{-1} \circ \tau_t^0 = \tau_0{}' \circ \mu_t' \circ \varphi_t \circ \varphi_t^{-1} \circ \tau_t^0 = \tau_0^t \circ \mu_t' \circ \tau_t^0.$$

This shows that $C_t \circ \mu \circ C_t^{-1}$ is an element of $\Psi(x)$. It is in $\Psi^0(x)$ if μ is in $\Psi^0(x)$. (Note that $N(\Psi(x)) \subset N(\Psi^0(x))$ since $\Psi^0(x)$ is the identity component of $N(\Psi(x)$.) QED.

COROLLARY 4.3. *If X is an infinitesimal affine transformation of M, then, at each point $x \in M$, $(A_X)_x$ belongs to the normalizor $N(\mathfrak{g}(x))$ of the Lie algebra $\mathfrak{g}(x)$ of $\Psi(x)$ in the Lie algebra of endomorphisms of $T_x(M)$.*

We recall that $N(\mathfrak{g}(x))$ is by definition the set of linear endomorphisms A of $T_x(M)$ such that $[A, B] \in \mathfrak{g}(x)$ for every $B \in \mathfrak{g}(x)$.

If X is an infinitesimal isometry of a Riemannian manifold M, then A_X is skew-symmetric (cf. Proposition 3.2) and, for each t, C_t is an orthogonal transformation of $T_x(M)$. We have then

THEOREM 4.4. *Let M be a Riemannian manifold and $\mathfrak{g}(x)$ the Lie algebra of $\Psi(x)$. If X is an infinitesimal isometry of M, then, for each $x \in M$, $(A_X)_x$ is in the normalizor $N(\mathfrak{g}(x))$ of $\mathfrak{g}(x)$ in the Lie algebra $E(x)$ of skew-symmetric linear endomorphisms of $T_x(M)$.*

The following theorem is due to Kostant [1].

THEOREM 4.5. *If X is an infinitesimal isometry of a compact Riemannian manifold M, then, for each $x \in M$, $(A_X)_x$ belongs to the Lie algebra $\mathfrak{g}(x)$ of the linear holonomy group $\Psi(x)$.*

Proof. In the Lie algebra $E(x)$ of skew-symmetric endomorphisms of $T_x(M)$, we introduce a positive definite inner product $(\ ,\ )$ by setting

$$(A, B) = -\mathrm{trace}\,(AB).$$

Let $B(x)$ be the orthogonal complement of $\mathfrak{g}(x)$ in $E(x)$ with respect this inner product. For the given infinitesimal isometry X of M, we set

$$A_X = S_X + B_X,$$
$$\text{where } S_X \in \mathfrak{g}(x), \quad B_X \in B(x), \quad x \in M.$$

LEMMA. *The tensor field B_X of type $(1, 1)$ is parallel.*

Proof of Lemma. Let τ be an arbitrary curve from a point x to another point y. The parallel displacement τ gives an isomorphism of $E(x)$ onto $E(y)$ which maps $\mathfrak{g}(x)$ onto $g(y)$. Since the inner products in $E(x)$ and in $E(y)$ are preserved by τ, τ maps $B(x)$ onto $B(y)$. This means that, for any vector field Y on M, $\nabla_Y(S_X)$ is in $\mathfrak{g}(x)$ whereas $\nabla_Y(B_X)$ is in $B(x)$ at each point $x \in M$. On the other hand, the formula $\nabla_Y(A_X) = R(X, Y)$ (cf. Proposition 2.6) implies that $\nabla_Y(A_X)$ belongs to $\mathfrak{g}(x)$ at each $x \in M$ (cf. Theorem 9.1 of Chapter III). By comparing the $\mathfrak{g}(x)$-component and the $B(x)$-component of the equality $\nabla_Y(A_X) = \nabla_Y(B_X) + \nabla_Y(S_X)$, we see that $\nabla_Y(B_X)$ belongs to $\mathfrak{g}(x)$ also. Hence $\nabla_Y(B_X) = 0$, concluding the proof of the lemma.

We shall show that $B_X = 0$. We set $Y = B_X X$. By Green's theorem (cf. Appendix 6), we have (assuming that M is orientable for the moment)

$$\int_M \mathrm{div}\, Y\, dv = 0 \quad (dv\text{: the volume element}).$$

Since div Y is equal to the trace of the linear mapping $V \to \nabla_V Y$ at each point x, we have (Lemma and Proposition 2.5)

$$\begin{aligned} \mathrm{div}\, Y &= \mathrm{trace}\,(V \to \nabla_V(B_X X)) = \mathrm{trace}\,(V \to B_X(\nabla_V X)) \\ &= -\mathrm{trace}\,(B_X A_X) = -\mathrm{trace}\,(B_X B_X) - \mathrm{trace}\,(B_X S_X) \\ &= -\mathrm{trace}\,(B_X B_X) \geqq 0. \end{aligned}$$

Thus

$$\int_M \text{trace } (B_X B_X) \, dv = 0,$$

which implies trace $B_X B_X = 0$ and hence $B_X = 0$. If M is not orientable, we lift X to an infinitesimal isometry X^* of the two-fold orientable covering space M^* of M. Then $B_{X^*} = 0$ implies $B_X = 0$. QED.

As an application of Theorem 4.5, we prove a result of H. C. Wang [1].

THEOREM 4.6. *If M is a compact Riemannian manifold, then*

(1) *Every parallel tensor field K on M is invariant by the identity component $\mathfrak{I}^0(M)$ of the group of isometries of M;*

(2) *At each point x, the linear isotropy group of $\mathfrak{I}^0(M)$ is contained in the linear holonomy group $\Psi(x)$.*

Proof. (1) Let X be an arbitrary infinitesimal isometry of M. By Proposition 4.1 and Theorem 4.5, the 1-parameter group C_t of linear transformations of $T_x(M)$ is contained in $\Psi(x)$. When C_t is extended to a 1-parameter group of automorphisms of the tensor algebra over $T_x(M)$, it leaves K invariant. Thus $\varphi_t(K_x) = \tau_t^0 K_x = K_{x_t}$ for every t, where φ_t is the 1-parameter group of isometries generated by X. Since $\mathfrak{I}^0(M)$ is connected, it leaves K invariant.

(2) Let φ be any element of $\mathfrak{I}^0(M)$ such that $\varphi(x) = x$. Since $\mathfrak{I}^0(M)$ is a compact connected Lie group, there exists a 1-parameter subgroup φ_t such that $\varphi = \varphi_{t_0}$ for some t_0. In the proof of (1), we saw that C_t (obtained from φ_t) is in $\Psi(x)$. On the other hand, since $\varphi_{t_0}(x) = x$, $\tau_{t_0}^0$ is also in $\Psi(x)$. Hence $\varphi_{t_0} = \tau_{t_0}^0 \circ C_{t_0}$ belongs to $\Psi(x)$. QED.

5. *Ricci tensor and infinitesimal isometries*

Let M be a manifold with a linear connection Γ. *The Ricci tensor field S* is the covariant tensor field of degree 2 defined as follows:

$$S(X, Y) = \text{trace of the map } V \to R(V, X)Y \text{ of } T_x(M),$$

where $X, Y, V \in T_x(M)$. If M is a Riemannian manifold and if $V_1, \ldots, V_n$ is an orthonormal basis of $T_x(M)$, then

$$\begin{aligned} S(X, Y) &= \Sigma_{i=1}^{n} g(R(V_i, X)Y, V_i) \\ &= \Sigma_{i=1}^{n} R(V_i, Y, V_i, X), \quad X, Y \in T_x(M), \end{aligned}$$

where R in the last equation denotes the Riemannian curvature tensor (cf. §2 of Chapter V). Property (d) of the Riemannian curvature tensor (cf. §1 of Chapter V) implies $S(X, Y) = S(Y, X)$, that is, S is symmetric.

Proposition 5.1. *If X is an infinitesimal affine transformation of a Riemannian manifold M, then*

$$\operatorname{div}(A_X Y) = -S(X, Y) - \operatorname{trace}(A_X A_Y)$$

for every vector field Y on M. In particular,

$$\operatorname{div}(A_X X) = -S(X, X) - \operatorname{trace}(A_X A_X).$$

Proof. By Proposition 2.6, we have $R(V, X) = -R(X, V) = -\nabla_V(A_X)$ for any vector field V on M. Hence

$$\begin{aligned} R(V, X)Y &= -(\nabla_V(A_X))Y = -\nabla_V(A_X Y) + A_X(\nabla_V Y) \\ &= -\nabla_V(A_X Y) - A_X A_Y V. \end{aligned}$$

Our proposition follows from the fact that $S(X, Y)$ is the trace of $V \to R(V, X)Y$ and that $\operatorname{div}(A_X Y)$ is the trace of $V \to \nabla_V(A_X Y)$.
QED.

Proposition 5.2. *For an infinitesimal isometry X of a Riemannian manifold M, consider the function $f = \frac{1}{2}g(X, X)$ on M. Then*

(1) $Vf = g(V, A_X X)$ *for every tangent vector V;*

(2) $V^2 f = g(V, \nabla_V(A_X X))$ *for every vector field V such that $\nabla_V V = 0$;*

(3) $\operatorname{div}(A_X X) \geqq 0$ *at any point where f attains a relative minimum;* $\operatorname{div}(A_X X) \leqq 0$ *at any point where f attains a relative maximum.*

Proof. Since g is parallel, we have

$$Z(g(X, Y)) = \nabla_Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for arbitrary vector fields X, Y and Z on M. Applying this formula to the case where $X = Y$ and $Z = V$, we obtain

$$Vf = g(\nabla_V X, X) = -g(A_X V, X) = g(V, A_X X)$$

by virtue of Proposition 2.5 and the skew-symmetry of A_X (cf. Proposition 3.2). This proves (1). If V is a vector field such that $\nabla_V V = 0$, then

$$V^2 f = V(g(V, A_X X)) = g(\nabla_V V, A_X X) + g(V, \nabla_V(A_X X))$$
$$= g(V, \nabla_V(A_X X)),$$

proving (2). To prove (3), let $V_1, \ldots, V_n$ be an orthonormal basis for $T_x(M)$. For each i, let $\tau_i = x_i(t)$ be the geodesic with the initial condition (x, V_i) so that $V_i = \dot{x}_i(0)$. We extend each V_i to a vector field which coincides with $\dot{x}_i(t)$ at $x_i(t)$ for small values of t. Then we have

$$d^2 f(x_i(t))/dt^2 = V_i^2 f = g(\nabla_{V_i} V_i, A_X X) + g(V_i, \nabla_{V_i}(A_X X))$$
$$= g(V_i, \nabla_{V_i}(A_X X)).$$

Since div $(A_X X)$ is the trace of the linear mapping $V \to \nabla_V(A_X X)$, we have

$$\operatorname{div}(A_X X)_x = \Sigma_{i=1}^n (V_i^2 f)_x.$$

Now (3) follows from the fact that, if f attains a relative minimum (resp. maximum) at x, then $(V_i^2 f)_x \geqq 0$ (resp. $\leqq 0$). QED.

As an application of these two propositions, we prove the following result of Bochner [1].

THEOREM 5.3. *Let M be a connected Riemannian manifold whose Ricci tensor field S is negative definite everywhere on M. If the length of an infinitesimal isometry X attains a relative maximum at some point of M, then X vanishes identically on M.*

Proof. Assume the length of X attains a relative maximum at x. By Proposition 5.2, we have div $(A_X X) \leqq 0$ at x. By Proposition 5.1, we obtain $-S(X, X) -$ trace $(A_X A_X) \leqq 0$. But $S(X, X) \leqq 0$ by assumption, and trace $(A_X A_X) \leqq 0$ since A_X is skew-symmetric. Thus we have $S(X, X) = 0$ and $A_X = 0$ at x. Since S is negative definite, $X = 0$ at x. Since the length of X attains a relative maximum at x, X vanishes in a neighborhood x. If u is any point of $O(M)$ such that $\pi(u) = x$, then the natural lift $\tilde{X}$ of X vanishes in a neighborhood of u. As we have seen in the proof of Theorem 3.3, X vanishes identically on M. QED.

Corollary 5.4. *If M is a compact Riemannian manifold with negative definite Ricci tensor field, then the group $\mathfrak{I}(M)$ of isometries of M is finite.*

Proof. By Theorem 5.3, $\mathfrak{I}^0(M)$ reduces to the identity. Since $\mathfrak{I}(M)$ is compact (cf. Theorem 3.4), it is finite. QED.

Remark. Corollary 5.4 can be derived from Proposition 5.1 by means of Green's theorem in the following way.

We may assume that M is orientable; otherwise, we have only to consider the orientable twofold covering space of M. From Proposition 5.1 and Green's theorem, we obtain

$$\int_M [S(X, X) + \text{trace }(A_X A_X)]\, dv = 0.$$

Since $S(X, X) \leqq 0$ and trace $(A_X A_X) \leqq 0$, we must have $S(X, X) = 0$ and trace $(A_X A_X) = 0$ everywhere on M. Since S is negative definite, we have $X = 0$ everywhere on M. This proof gives also

Corollary 5.5. *If M is a compact Riemannian manifold with vanishing Ricci tensor field, then every infinitesimal isometry of M is a parallel vector field.*

Proof. By Proposition 2.5, we have $0 = A_X V = -\nabla_V X$ for every vector field V on M. QED.

From Corollary 5.5, we obtain the following result of Lichnerowicz [1].

Corollary 5.6. *If a connected compact homogeneous Riemannian manifold M has zero Ricci tensor, then M is a Euclidean torus.*

Proof. By Theorem 5.1 of Chapter III and Corollary 5.5, we have

$$[X, Y] = \nabla_X Y - \nabla_Y X = 0$$

for any infinitesimal isometries X, Y. Thus $\mathfrak{I}^0(M)$ is a compact abelian group. Since $\mathfrak{I}^0(M)$ acts effectively on M, the isotropy subgroup of $\mathfrak{I}^0(M)$ at every point M reduces to the identity element. As we have seen in Example 4.1 of Chapter V, M is a Euclidean torus. QED.

As another application of Proposition 5.2, we prove

PROPOSITION 5.7. *Let φ_t be the 1-parameter group of isometries generated by an infinitesimal isometry X of a Riemannian manifold M. If x is a critical point of the length function $g(X, X)^{\frac{1}{2}}$, then the orbit $\varphi_t(x)$ is a geodesic.*

Proof. If x is a critical point of $g(X, X)^{\frac{1}{2}}$, it is a critical point of the function $f = \frac{1}{2}g(X, X)$ also. By (1) of Proposition 5.2, we have $g(V, A_X X) = 0$ for every vector V at x. Hence $A_X X = 0$ at x, that is, $\nabla_X X = 0$ at x. Since $\varphi_t(X_x) = X_{\varphi_t(x)}$ by (1) of Proposition 1.2, we have $\nabla_X X = 0$ along the orbit $\varphi_t(x)$. This shows that the orbit $\varphi_t(x)$ is a geodesic. QED.

6. *Extension of local isomorphisms*

Let M be a real analytic manifold with an analytic linear connection Γ. The bundle $L(M)$ of linear frames is an analytic manifold and the connection form ω is analytic. The distribution Q which assigns the horizontal subspace Q_u to each point $u \in L(M)$ is analytic in the sense that each point u has a neighborhood and a local basis for the distribution Q consisting of analytic vector fields. The same is true for the distribution on the tangent bundle $T(M)$ which defines the notion of parallel displacement in the bundle $T(M)$ (for the notion of horizontal subspaces in an associated fibre bundle, see §7 of Chapter II).

The main object of this section is to prove the following theorem.

THEOREM 6.1. *Let M be a connected, simply connected analytic manifold with an analytic linear connection. Let M' be an analytic manifold with a complete analytic linear connection. Then every affine mapping f_U of a connected open subset U of M into M' can be uniquely extended to an affine mapping f of M into M'.*

The proof is preceded by several lemmas.

LEMMA 1. *Let f and g be analytic mappings of a connected analytic manifold M into an analytic manifold M'. If f and g coincide on a non-empty open subset of M, then they coincide on M.*

Proof of Lemma 1. Let x be any point of M and let $x^1, \dots, x^n$ be an analytic local coordinate system in a neighborhood of x. Let $y^1, \dots, y^m$ be an analytic local coordinate system in a neighborhood of the point $f(x)$. The mapping f can be expressed by a set of analytic functions

$$y^i = f^i(x^1, \ldots, x^n), \qquad i = 1, \ldots, m.$$

These functions can be expanded at x into convergent power series of $x^1, \ldots, x^n$. Similarly for the mapping g. Let N be the set of points $x \in M$ such that $f(x) = g(x)$ and that the power series expansions of f and g at x coincide. Then N is clearly a closed subset of M. From the well known properties of power series, it follows that N is open in M. Since M is connected, $N = M$.

Lemma 2. *Let S and S' be analytic distributions on analytic manifolds M and M'. Let f be an analytic mapping of M into M' such that*

$$(*) \qquad f(S_x) \subset S'_{f(x)}$$

or every point x of an open subset of M. If M is connected, then $(*)$ is satisfied at every point x of M.

Proof of Lemma 2. Let N be the set of all points $x \in M$ such that $(*)$ is satisfied in a neighborhood of x. Then N is clearly a non-empty open subset of M. Since M is connected, it suffices to show that N is closed. Let $x_k \in N$ and $x_k \to x_0$. Let $y^1, \ldots, y^m$ be an analytic local coordinate system in a neighborhood V of $f(x_0)$. Let $Z_1, \ldots, Z_h$ be a local basis for the distribution S' in V. From $\partial/\partial y^1, \ldots, \partial/\partial y^m$, choose $m - h$ vector fields, say, $Z_{h+1}, \ldots, Z_m$ such that $Z_1, \ldots, Z_h, Z_{h+1}, \ldots, Z_m$ are linearly independent at $f(x_0)$ and hence in a neighborhood V' of $f(x_0)$. Let U be a connected neighborhood of x_0 with an analytic local coordinate system $x^1, \ldots, x^n$ such that $f(U) \subset V'$ and that S has a local basis $X_1, \ldots, X_k$ consisting of analytic vector fields defined on U. Since f is analytic, we have

$$f(X_i)_x = \Sigma_{j=1}^m f_i^j(x) \cdot Z_j, \qquad i = 1, \ldots, k,$$

where $f_i^j(x)$ are analytic functions of $x^1, \ldots, x^n$. Since $x_k \in N$ and $x_k \to x_0$, there exists a neighborhood U_1 of some x_k such that $U_1 \subset U$ and that $(*)$ is satisfied at every point x of U_1. In other words, $f_i^j(x) = 0$ on U_1 for $1 \leqq i \leqq k$ and $h + 1 \leqq j \leqq m$. It follows that $f_i^j = 0$ on U for the same i and j. This proves that $(*)$ is satisfied at every point x of U.

Lemma 3. *Let M and M' be analytic manifolds with analytic linear connections and f an analytic mapping of M into M'. If the* *restriction of f to an open subset U of M is an affine mapping and if M is connected, then f is an affine mapping of M into M'.*

Proof of Lemma 3. Let F be the analytic mapping of the tangent bundle $T(M)$ into $T(M')$ induced by f. By assumption, F maps the horizontal subspace at each point of $\pi^{-1}(U)$ into a horizontal subspace in $T(M')$ (here, π denotes the projection of $T(M)$ onto M). Applying Lemma 2 to the mapping F, we see that f is an affine mapping of M into M'.

LEMMA 4. *Let M and M' be differentiable manifolds with linear connections and let f and g be affine mappings of M into M'. If $f(X) = g(X)$ for every $X \in T_x(M)$ at some point $x \in M$ and if M is connected, then f and g coincide on M.*

Proof of Lemma 4. Let N be the set of all points $x \in M$ such that $f(X) = g(X)$ for $X \in T_x(M)$. Then N is clearly a non-empty closed subset of M. Since f and g commute with the exponential mappings (Proposition 1.1), $x \in N$ implies that a normal coordinate neighborhood of x is in N. Thus N is open. Since M is connected, we have $N = M$.

We are now in position to prove Theorem 6.1. Under the assumptions in Theorem 6.1, let $x(t)$, $0 \leqq t \leqq 1$, be a curve in M such that $x(0) \in U$. An *analytic continuation* of f_U along the curve $x(t)$ is, by definition, a family of affine mappings f_t, $0 \leqq t \leqq 1$, satisfying the following conditions:

(1) For each t, f_t is an affine mapping of a neighborhood U_t of the point $x(t)$ into M';

(2) For each t, there exists a positive number δ such that if $|s - t| < \delta$, then $x(s) \in U_t$ and f_s coincides with f_t in a neighborhood of $x(s)$;

(3) $f_0 = f_U$.

It follows easily from Lemma 4 that an analytic continuation of f_U along the curve $x(t)$ is unique if it exists. We now show that it exists. Let t_0 be the supremum of $t_1 > 0$ such that an analytic continuation f_t exists for $0 \leqq t \leqq t_1$. Let W be a convex neighborhood of the point $x(t_0)$ as in Theorem 8.7 of Chapter III such that every point of W has a normal coordinate neighborhood containing W. Take t_1 such that $t_1 < t_0$ and that $x(t_1) \in W$. Let V be a normal coordinate neighborhood of $x(t_1)$ which contains W. Since there exists an analytic continuation f_t of f_U for $0 \leqq t \leqq t_1$, we have the affine mapping f_{t_1} of a neighborhood $x(t_1)$ into M'. We extend f_{t_1} to an analytic mapping, say g, of V into M' as follows. Since the exponential mapping gives a diffeomorphism of an open neighborhood V^* of the origin in $T_{x(t_1)}(M)$ onto V, each point $y \in V$ determines a unique element $X \in V^* \subset T_{x(t_1)}(M)$ such that $y = \exp X$. Set $X' = f_{t_1}(X)$ so that X' is a vector at $f_{t_1}(x(t_1))$. Since M' is complete, $\exp X'$ is well defined and we set $g(y) = \exp X'$. The extension g of f_{t_1} thus defined commutes with the exponential mappings. Since the exponential mappings are analytic, g is also analytic. By Lemma 3, g is an affine mapping of V into M'. We can easily define the continuation f_t beyond t_1 by using this affine mapping g. We have thus proved the existence of an analytic continuation f_t along the whole curve $x(t)$, $0 \leqq t \leqq 1$.

To complete the proof of Theorem 6.1, let x be an arbitrarily fixed point of U. For each point y of M, let $x(t)$, $0 \leqq t \leqq 1$, be a curve from x to y. The affine mapping f_U can be analytically continued along the curve $x(t)$ and gives rise to an affine mapping g of a neighborhood of y into M'. We show that $g(y)$ is independent of the choice of a curve from x to y. For this, it is sufficient to observe that if $x(t)$ is a closed curve, then the analytic continuation f_t of f_U along $x(t)$ gives rise to the affine mapping f_1 which coincides with f_U in a neighborhood x. Since M is simply connected, the curve $x(t)$ is homotopic to zero and our assertion follows readily from the factorization lemma (cf. Appendix 7) and from the uniqueness of an analytic continuation we have already proved. It follows that the given mapping f_U can be extended to an affine mapping f of M into M'. The uniqueness of f follows from Lemma 4. QED.

Corollary 6.2. *Let M and M' be connected and simply connected analytic manifolds with complete analytic linear connections. Then every affine isomorphism between connected open subsets of M and M' can be uniquely extended to an affine isomorphism between M and M'.*

We have the corresponding results for analytic Riemannian manifolds. The Riemannian connection of an analytic Riemannian metric is analytic; this follows from Corollary 2.4 of Chapter IV.

Theorem 6.3. *Let M and M' be analytic Riemannian manifolds. If M is connected and simply connected and if M' is complete, then every* *isometric immersion f_U of a connected open subset U of M into M' can be uniquely extended to an isometric immersion f of M into M'.*

Proof. The proof is quite similar to that of Theorem 6.1. We indicate only the necessary changes. Lemma 1 can be used without any change. Lemma 2 was necessary only to derive Lemma 3. In the present case, we prove the following Lemma 3′ directly.

LEMMA 3′. *Let M and M' be analytic manifolds with analytic Riemannian metrics g and g', respectively, and let f be an analytic mapping of M into M'. If the restriction of f to an open subset U of M is an isometric immersion and if M is connected, then f is an isometric immersion of M into M'.*

Proof of Lemma 3′. Compare g and $f^*(g')$. Since they coincide on U, the argument similar to the one used in the proof of Lemma 1 shows that they coincide on the whole of M.

In Lemma 4, we replace "affine mappings" by "isometric immersions." Since an isometric immersion maps every geodesic into a geodesic and hence commutes with the exponential mappings, the proof of Lemma 4 is still valid.

In the rest of the proof of Theorem 6.1, we replace "affine mapping" by "isometric immersion." Then the proof goes through without any other change. QED.

Remark. Since an isometric immersion $f: M \to M'$ is not necessarily an affine mapping, Theorem 6.3 does not follow from Theorem 6.1. If $\dim M = \dim M'$, then every isometric immersion $f: M \to M'$ is an affine mapping (cf. Proposition 2.6 of Chapter IV). Hence the following corollary follows from Corollary 6.2 as well as from Theorem 6.3.

COROLLARY 6.4. *Let M and M' be connected and simply connected, complete analytic Riemannian manifolds. Then every isometry between connected open subsets of M and M' can be uniquely extended to an isometry between M and M'.*

7. *Equivalence problem*

Let M be a manifold with a linear connection. Let $x^1, \ldots, x^n$ be a normal coordinate system at a point x_0 and let U be a neighborhood of x_0 given by $|x^i| < \delta$, $i = 1, \ldots, n$. Let u_0 be the linear frame at the origin x_0 given by $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. We define a cross section $\sigma: U \to L(M)$ as follows. If x is a point of U with coordinates $(a^1, \ldots, a^n)$, then $\sigma(x)$ is the frame obtained by the parallel displacement of u_0 along the geodesic given by $x^i = ta^i$, $0 \leqq t \leqq 1$. We call σ the *cross section adapted to the normal coordinate system* $x^1, \ldots, x^n$.

The first objective of this section is to prove the following theorem.

THEOREM 7.1. *Let M and M' be manifolds with linear connections. Let U (resp. V) be a normal coordinate neighborhood of a point $x_0 \in M$ (resp. $y_0 \in M'$) with a normal coordinate system $x^1, \ldots, x^n$ (resp. $y^1, \ldots, y^n$) and let $\sigma: U \to L(M)$ (resp. $\sigma': V \to L(M')$) be the cross section adapted to $x^1, \ldots, x^n$ (resp. $y^1, \ldots, y^n$). A diffeomorphism f of U onto V is an affine isomorphism if it satisfies the following two conditions:*

(1) f maps the frame $\sigma(x)$ into the frame $\sigma'(f(x))$ for each point $x \in U$;

(2) f preserves the torsion and curvature tensor fields.

Proof. Let $\theta = (\theta^i)$ and $\omega = (\omega^i_j)$ be the canonical form and the connection form on $L(M)$ respectively. We set

(1) $\bar{\theta}^i = \sigma^*\theta^i = \Sigma_j A^i_j \, dx^j, \qquad i = 1, \ldots, n,$

(2) $\bar{\omega}^i_j = \sigma^*\omega^i_j = \Sigma_k B^i_{jk} \, dx^k, \qquad i, j = 1, \ldots, n.$

LEMMA 1. *For any $(a^1, \ldots, a^n) \neq (0, \ldots, 0)$ with $|a^i| < \delta$, we have*

(3) $\Sigma_j A^i_j(ta)a^j = a^i, \qquad 0 \leqq t \leqq 1, \qquad i = 1, \ldots, n,$

(4) $\Sigma_k B^i_{jk}(ta)a^k = 0, \qquad 0 \leqq t \leqq 1, \qquad i, j = 1, \ldots, n,$

where ta stands for $(ta^1, \ldots, ta^n)$.

Proof of Lemma 1. For a fixed $a = (a^i)$, consider the geodesic x_t given by $x^i = ta^i$, $0 \leqq t \leqq 1$, $i = 1, \ldots, n$. Let $u_t = \sigma(x_t)$, which is the horizontal lift of x_t starting from u_0. Since the frames u_t are parallel along x_t, we have

$$\bar{\theta}^i(\dot{x}_t) = \theta^i(\dot{u}_t) = a^i.$$

On the other hand, we have

$$\dot{x}_t = \Sigma_j a^j \, (\partial/\partial x^j) \quad \text{and} \quad \bar{\theta}^i(\dot{x}_t) = \Sigma_j A^i_j(ta)a^j.$$

This proves (3). Similarly, (4) follows from the fact that $\omega(\dot{u}_t) = 0$.

We set (cf. §7 of Chapter III)

(5) $\bar{\Theta}^i = \sigma^*\Theta^i = \Sigma_{j,k} \frac{1}{2}\bar{T}^i_{jk}\, \bar{\theta}^j \wedge \bar{\theta}^k \quad (\bar{T}^i_{jk} = \sigma^*\tilde{T}^i_{jk})$,

(6) $\bar{\Omega}^i_j = \sigma^*\Omega^i_j = \Sigma_{k,l} \frac{1}{2}\bar{R}^i_{jkl}\bar{\theta}^k \wedge \bar{\theta}^l \quad (\bar{R}^i_{jkl} = \sigma^*\tilde{R}^i_{jkl})$.

Lemma 2. *For an arbitrarily fixed* $(a^1, \ldots, a^n)$, *we set*

$$\hat{A}^i_j(t) = tA^i_j(ta), \qquad \hat{B}^i_{jk}(t) = tB^i_{jk}(ta),$$
$$\hat{T}^i_{jk}(t) = \bar{T}^i_{jk}(ta), \qquad \hat{R}^i_{jkl}(t) = \bar{R}^i_{jkl}(ta).$$

Then the functions $\hat{A}^i_j(t)$ *and* $\hat{B}^i_{jk}(t)$ *satisfy the following system of ordinary linear differential equations:*

(7) $d\hat{A}^i_j(t)/dt = \delta^i_j + \Sigma_l\, \hat{B}^{\,i}_{lj}(t)a^l + \Sigma_{l,m}\, \hat{T}^{\,i}_{lm}(t)\hat{A}^m_j(t)a^l$,

(8) $d\hat{B}^i_{jk}(t)/dt = \Sigma_{l,m}\, \hat{R}^i_{jkl}(t)\hat{A}^m_k(t)a^l$.

with the initial conditions:

(9) $\hat{A}^i_j(0) = 0,\ \hat{B}^i_{jk}(0) = 0$.

Proof of Lemma 2. We consider the open set Q of $\mathbf{R}^{n+1}$ defined by $Q = \{(t, a^1, \ldots, a^n);\ |ta^i| < \delta \text{ for } i = 1, \ldots, n\}$. Let ρ be the mapping of Q into U defined by

$$\rho(t, a^1, \ldots, a^n) = (ta^1, \ldots, ta^n).$$

We set

$$\bar{\bar{\theta}}^i = \rho^*\bar{\theta}^i, \quad \bar{\bar{\omega}}^i_j = \rho^*\bar{\omega}^i_j, \quad \bar{\bar{\Theta}}^i = \rho^*\bar{\Theta}^i, \quad \bar{\bar{\Omega}}^i_j = \rho^*\bar{\Omega}^i_j.$$

From Lemma 1, we obtain

(10) $\bar{\bar{\theta}}^i = \Sigma_j\, tA^i_j(ta)\, da^j + a^i\, dt$,

(11) $\bar{\bar{\omega}}^i_j = \Sigma_k\, tB^i_{jk}(ta)\, da^k$.

From (5) and (6), we obtain

(12) $\bar{\bar{\Theta}}^i = \Sigma_{j,k}\, \frac{1}{2}\bar{T}^{\,i}_{jk}(ta)\, \bar{\bar{\theta}}^j \wedge \bar{\bar{\theta}}^k$,

(13) $\bar{\bar{\Omega}}^i_j = \Sigma_{k,l}\, \frac{1}{2}\bar{R}^i_{jkl}(ta)\, \bar{\bar{\theta}}^k \wedge \bar{\bar{\theta}}^l$.

From (10) and (11), we obtain

(14) $d\bar{\bar{\theta}}^i = -\Sigma_j \left[\frac{\partial}{\partial t}(tA^i_j(ta)) - \delta^i_j\right] da^j \wedge dt + \cdots$,

(15) $d\bar{\bar{\omega}}^i_j = -\Sigma_k \left[\frac{\partial}{\partial t}(tB^i_{jk}(ta))\right] da^k \wedge dt + \cdots$,

where the dots denote the terms not involving dt.

From (10), (11), (12) and (13), we obtain

$$(16)\quad -\Sigma_j\, \overline{\overline{\omega}}{}^i_j \wedge \overline{\overline{\theta}}{}^j + \overline{\overline{\Theta}}{}^i$$
$$= -\Sigma_j\, [\Sigma_l\, tB^i_{lj}(ta)a^l + \Sigma_l\, T^i_{lm}(ta)(tA^m_j(ta))a^l]\, da^j \wedge dt + \cdots,$$

$$(17)\quad -\Sigma_k\, \overline{\overline{\omega}}{}^i_k \wedge \overline{\overline{\omega}}{}^k_j + \overline{\overline{\Omega}}{}^i_j$$
$$= -\Sigma_k\, [\Sigma_{l,m}\, R^i_{jlm}(ta)(tA^m_k(ta))a^l]\, da^k \wedge dt + \cdots,$$

where the dots denote the terms not involving dt. Now (7) follows from (14), (16) and the first structure equation. Similarly, (8) follows from (15), (17) and the second structure equation. Finally, (9) is obvious from the definition of $\hat{A}^i_j(t)$ and $\hat{B}^i_{jk}(t)$. This proves Lemma 2.

From Lemma 2 and from the uniqueness theorem on systems of ordinary linear differential equations (cf. Appendix 1), it follows that the functions $\hat{A}^i_j(t)$ and $\hat{B}^i_{jk}(t)$ are uniquely determined by $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$. On the other hand, the functions $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$ are uniquely determined by the torsion tensor fields T and the curvature tensor fields R and also by the cross section (for each fixed $(a^1, \ldots, a^n)$). From (1) we see that the connection form ω is uniquely determined by T, R and σ. QED.

In the case of a real analytic linear connection, the torsion and curvature tensor fields and their successive covariant derivatives at a point determine the connection uniquely. More precisely, we have

THEOREM 7.2. *Let M and M' be analytic manifolds with analytic linear connections. Let T, R and ∇ (resp. T', R' and ∇') be the torsion, the curvature and the covariant differentiation of M (resp. M'). If a linear isomorphism F: $T_{x_0}(M) \to T_{y_0}(M')$ maps the tensors $(\nabla^m T)_{x_0}$ and $(\nabla^m R)_{x_0}$ into the tensors $(\nabla'^m T')_{y_0}$ and $(\nabla'^m R')_{y_0}$, respectively, for $m = 0, 1, 2, \ldots$, then there is an affine isomorphism f of a neighborhood U of x_0 onto a neighborhood V of y_0 such that $f(x_0) = y_0$ and that the differential of f at x_0 is F.*

Proof. Let $x^1, \ldots, x^n$, $|x^i| < \delta$, be a normal coordinate system in a neighborhood U of x_0. Let $y^1, \ldots, y^n$, $|y^i| < \delta$, be a normal coordinate system in a neighborhood V of y_0 such that $(\partial/\partial y^i)_{y_0} = F((\partial/\partial x^i)_{x_0})$, $i = 1, \ldots, n$; such a normal coordinate system exists and is unique. Let f be the analytic homeomorphism of U onto V defined by

$$y^i \circ f = x^i, \qquad i = 1, \ldots, n.$$

Clearly the differential of f at x_0 coincides with F. We shall show that f is an affine isomorphism of U onto V.

We use the same notation as in the proof of Theorem 7.1. It suffices to prove the following five statements. If the normal coordinate system $x^1, \ldots, x^n$ is fixed, then

(i) The tensors $(\nabla^m T)_{x_0}$, $m = 0, 1, 2, \ldots$, determine the functions $\hat{T}^i_{jk}(t)$, $0 \leqq t \leqq 1$;

(ii) The tensors $(\nabla^m R)_{x_0}$, $m = 0, 1, 2, \ldots$, determine the functions $\hat{R}^i_{jkl}(t)$, $0 \leqq t \leqq 1$;

(iii) The functions $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$ determine the forms $\bar{\theta}^i$ and $\bar{\omega}^i_j$;

(iv) The forms $\bar{\theta}^i$ determine the cross section σ;

(v) The cross section σ and the forms $\bar{\omega}^i_j$ determine the connection form ω.

To prove (i) and (ii) we need the following lemma.

LEMMA 1. *Let u_t, $0 \leqq t \leqq 1$, be a horizontal lift of a curve x_t, $0 \leqq t \leqq 1$, to $L(M)$. Let $\mathbf{T}^r_s$ be the tensor space of type (r, s) over $\mathbf{R}^n$. Given a tensor field K of type (r, s) along x_t, let $\tilde{K}$ be the $\mathbf{T}^r_s$-valued function defined along u_t by*

$$\tilde{K}(u_t) = u_t^{-1}(K_{x_t}), \qquad 0 \leqq t \leqq 1,$$

where u_t is considered as a linear mapping of $\mathbf{T}^r_s$ onto the tensor space $\mathbf{T}^r_s(x_t)$ at x_t of type (r, s). Then we have

$$\frac{d\tilde{K}(u_t)}{dt} = u_t^{-1}(\nabla_{\dot{x}_t} K), \quad 0 \leqq t \leqq 1.$$

Proof of Lemma 1. This is a special case of Proposition 1.3 of Chapter III. The tensor field K and the function $\tilde{K}$ here correspond to the cross section φ and the function f there. Although φ in Proposition 1.3 of Chapter III is defined on the whole of M, the proof goes through when φ is defined on a curve in M (cf. the lemma for Proposition 1.1 of Chapter III).

To prove (i), we apply Lemma 1 to the torsion T, the geodesic x_t given by $x^i = ta^i$, $i = 1, \ldots, n$, and the horizontal lift u_t of x_t

with $u_0 = ((\partial/\partial x^1)_{x_0}, \ldots, (\partial/\partial x^n)_{x_0})$. Then Lemma 1 (applied m times) implies that, for each t, $u_t^{-1}((\nabla_{\dot{x}_t})^m T)$ is the element of the tensor space $\mathbf{T}^1_2$ with components $d^m \hat{T}^i_{jk}(t)/dt^m$. In particular, setting $t = 0$, we see that, once the coordinate system $x^1, \ldots, x^n$ and $(a^1, \ldots, a^n)$ are fixed, $(d^m \hat{T}^i_{jk}/dt^m)_{t=0}$, $m = 0, 1, 2, \ldots$, are all determined by $(\nabla^m T)_{x_0}$. (Actually, it is not hard to see that

$$(d^m \hat{T}^i_{jk}/dt^m)_{t=0} = \Sigma_{l_1, \ldots, l_m} T^i_{jk;l_1;\ldots;l_m}(x_0) a^{l_1} \cdots a^{l_m},$$

where $T^i_{jk;l_1;\ldots;l_m}$ are the components of $\nabla^m T$ with respect to $x^1, \ldots, x^n$.) Since each $\hat{T}^i_{jk}(t)$ is an analytic function of t, it is determined by $(\nabla^m T)_{x_0}$, $m = 0, 1, 2, \ldots$. This proves (i). The proof of (ii) is similar.

Lemma 2 for Theorem 7.1 implies that the functions $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$ determine the functions $\hat{A}^i_j(t)$ and $\hat{B}^i_{jk}(t)$. Now (iii) follows from the formula (1) and (2) in the proof of Theorem 7.1.

(iv) follows from the following lemma.

Lemma 2. *Let σ and σ' be two cross sections of $L(M)$ over an open subset U of M. If $\sigma^*\theta = \sigma'^*\theta$ on U, then $\sigma = \sigma'$.*

Proof of Lemma 2. For each $X \in T_x(M)$, where $x \in U$, we have

$$(\sigma^*\theta)(X) = \theta(\sigma X) = \sigma(x)^{-1}(\pi(\sigma X)) = \sigma(x)^{-1}X,$$

where $\sigma(x) \in L(M)$ is considered as a linear isomorphism of $\mathbf{R}^n$ onto $T_x(M)$. Using the same equation for σ', we obtain

$$\sigma(x)^{-1}X = \sigma'(x)^{-1}X.$$

Since this holds for every X in $T_x(M)$, we obtain $\sigma(x) = \sigma'(x)$.

Finally, (v) is evident from the definition of $\bar{\omega}^i_j$. QED.

Corollary 7.3. *In Theorem 7.2, if M and M' are, moreover, connected, simply connected analytic manifolds with complete analytic linear connections, then there exists a unique affine isomorphism f of M onto M' whose differential at x_0 coincides with F.*

Proof. This is an immediate consequence of Corollary 6.2 and Theorem 7.2. QED.

Theorem 7.4. *Let M and M' be differentiable manifolds with linear connections. Let T, R and ∇ (resp. T', R' and ∇') be the torsion, the curvature and the covariant differentiation of M (resp. M'). Assume*

$\nabla T = 0$, $\nabla R = 0$, $\nabla' T' = 0$ *and* $\nabla' R' = 0$. *If F is a linear isomorphism of $T_{x_0}(M)$ onto $T_{y_0}(M')$ and maps the tensors T_{x_0} and R_{x_0} at x_0 into the tensors T'_{y_0} and R'_{y_0} at y_0 respectively, then there is an affine isomorphism f of a neighborhood U of x_0 onto a neighborhood V of y_0 such that $f(x_0) = y_0$ and that the differential of f at x_0 coincides with F.*

Proof. We follow the notation and the argument in the proof of Theorem 7.2. By Lemma 1 in the proof of Theorem 7.2, the functions $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$ are constant functions and hence are determined by T_{x_0} and R_{x_0} (and the coordinate system $x^1, \ldots, x^n$). Our theorem now follows from (iii), (iv) and (v) in the proof of Theorem 7.2. QED.

Corollary 7.5. *Let M be a differentiable manifold with a linear connection such that $\nabla T = 0$ and $\nabla R = 0$. Then, for any two points x and y of M, there exists an affine isomorphism of a neighborhood of x onto a neighborhood of y.*

Proof. Let τ be an arbitrary curve from x to y. Since $\nabla T = 0$ and $\nabla R = 0$, the parallel displacement $\tau \colon T_x(M) \to T_y(M)$ maps the tensors T_x and R_x at x into the tensors T_y and R_y at y. By Theorem 7.4, there exists a local affine isomorphism f such that $f(x) = y$ and that the differential of f at x coincides with τ. QED.

Let M be a manifold with a linear connection Γ. The connection Γ is said to be *invariant by parallelism* if, for arbitrary points x and y of M and for an arbitrary curve τ from x to y, there exists a (unique) local affine isomorphism f such that $f(x) = y$ and that the differential of f at x coincides with the parallel displacement $\tau \colon T_x(M) \to T_y(M)$. In the proof of Corollary 7.5, we saw that if $\nabla T = 0$ and $\nabla R = 0$, then the connection is invariant by parallelism. The converse is also true. Namely, we have

Corollary 7.6. *A linear connection is invariant by parallelism if and only if $\nabla T = 0$ and $\nabla R = 0$.*

Proof. Assuming that the connection is invariant by parallelism, let τ be an arbitrary curve from x to y. Let f be a local affine isomorphism such that $f(x) = y$ and that the differential of f at x coincides with the parallel displacement τ. Then f maps T_x and R_x into T_y and R_y respectively. Hence the parallel displacement τ maps T_x and R_x into T_y and R_y, respectively. This means that T and R are parallel tensor fields. QED.

THEOREM 7.7. *Let M be a differentiable manifold with a linear connection such that $\nabla T = 0$ and $\nabla R = 0$. With respect to the atlas consisting of normal coordinate systems, M is an analytic manifold and the connection is analytic.*

Proof. Let $x^1, \ldots, x^n$ be a normal coordinate system in an open set U. We introduce a coordinate system $(x^i, X^j_k)_{i,j,k=1,\ldots,n}$ in $\pi^{-1}(U) \subset L(M)$ in a natural way as in §7 of Chapter III (cf. Example 5.2 of Chapter I). If we denote by (U^j_k) the inverse matrix of (X^j_k), then the canonical form and the connection form can be expressed as follows (cf. Propositions 7.1 and 7.2 of Chapter III):

(18) $\theta^i = \Sigma_j\, U^i_j\, dx^j, \qquad i = 1, \ldots, n;$

(19) $\omega^i_j = \Sigma_k\, U^i_k(dX^k_j + \Sigma_{l,m}\, \Gamma^k_{ml}\, X^l_j\, dx^m), \qquad i,j = 1, \ldots, n.$

The forms θ^i are analytic with respect to (x^i, X^j_k). We show that the forms ω^i_j are also analytic with respect to (x^i, X^j_k). Clearly it is sufficient to show that the components Γ^i_{jk} of the connection are analytic in $x^1, \ldots, x^n$. We use the same notation as in the proof of Theorem 7.1. Since the functions $\hat{T}^i_{jk}(t)$ and $\hat{R}^i_{jkl}(t)$ are constants which do not depend on $(a^1, \ldots, a^n)$ by virtue of the assumption that $\nabla T = 0$ and $\nabla R = 0$, Lemma 2 in the proof of Theorem 7.1 implies (cf. Appendix 1) that the functions $\hat{A}^i_j(t)$ and $\hat{B}^i_{jkl}(t)$ are analytic in t and depend analytically on $(a^1, \ldots, a^n)$. Hence the functions $\hat{A}^i_j$ and $\hat{B}^i_{jk}$ are analytic in $x^1, \ldots, x^n$. From (1) in the proof of Theorem 7.1, we see that the cross section $\sigma\colon U \to L(M)$ is given by

(20) $U^i_j = A^i_j, \qquad i,j = 1, \ldots, n.$

Let (C^i_j) be the inverse matrix of (A^i_j). From (19) and (20), we obtain

(21) $\sigma^*\omega^i_j = \bar{\omega}^i_j = \Sigma_k\, A^i_k(dC^k_j + \Sigma_{l,m}\, \Gamma^k_{ml} C^l_j\, dx^m).$

By comparing (21) with (2) in the proof of Theorem 7.1, we obtain

(22) $B^i_{jm} = \Sigma_k\, A^i_k(\partial C^k_j/\partial x^m + \Sigma_l\, \Gamma^k_{ml} C^l_j).$

Transforming (22) we obtain

(23) $\Gamma^k_{ml} = \Sigma_j\, (\Sigma_i\, C^k_i B^i_{jm} - \partial C^k_j/\partial x^m) A^j_l,$

which shows that the components Γ^i_{jk} are analytic functions of $x^1, \ldots, x^n$.

Since the $n^2 + n$ 1-forms θ^i and ω^j_k are analytic with respect to (x^i, X^i_k) and define an absolute parallelism (Proposition 2.5 of Chapter III), the following lemma implies that $L(M)$ is an analytic manifold with respect to the atlas consisting of the coordinate system (x^i, X^i_j) induced from the normal coordinate systems $(x^1, \ldots, x^n)$ of M.

Lemma. *Let $\omega^1, \ldots, \omega^m$ be 1-forms defining an absolute parallelism on a manifold P of dimension m. Let $u^1, \ldots, u^m$ (resp. $v^1, \ldots, v^m$) be a local coordinate system valid in an open set U (resp. V). If the forms $\omega^1, \ldots, \omega^m$ are analytic with respect to both $u^1, \ldots, u^m$ and $v^1, \ldots, v^m$, then the functions*

$$v^i = f^i(u^i, \ldots, u^m), \qquad i = 1, \ldots, m,$$

which define the coordinate change are analytic.

Proof of Lemma. We write

$$\omega^i = \Sigma_j \, a^i_j(u) \, du^j = \Sigma_j \, b^i_j(v) \, dv^j,$$

where the functions $a^i_j(u)$ (resp. $b^i_j(v)$) are analytic in $u^1, \ldots, u^m$ (resp. $v^1, \ldots, v^m$). Let $(c^i_j(v))$ be the inverse matrix of $(b^i_j(v))$. Then the system of functions $v^i = f^i(u^1, \ldots, u^m)$, $i = 1, \ldots, m$, is a solution of the following system of linear partial differential equations:

$$\partial v^i / \partial u^j = \Sigma_k \, c^i_k(v) a^k_i(u), \qquad i,j = 1, \ldots, n.$$

Since the functions $c^i_k(v)$ and $a^k_i(u)$ are analytic in $v^1, \ldots, v^m$ and $u^1, \ldots, u^m$ respectively, the functions $f^i(u^1, \ldots, u^m)$ are analytic in $u^1, \ldots, u^m$ (cf. Appendix 1). This proves the lemma.

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ be two normal coordinate systems in M. Let (x^i, X^j_k) and (y^i, Y^j_k) be the local coordinate systems in $L(M)$ induced by these normal coordinate systems. By the lemma just proved, $y^1, \ldots, y^n$ are analytic functions of x^i and X^j_k. Since $y^1, \ldots, y^n$ are clearly independent of X^j_k, they are analytic functions of $x^1, \ldots, x^n$. This proves the first assertion of Theorem 7.7. Since we have already proved that the forms ω^i_j are analytic with respect to (x^i, X^j_k), the connection is analytic. QED.

As an application of Theorem 7.7 we have

THEOREM 7.8. *In Theorem 7.4, if M and M' are, moreover, connected, simply connected and complete then there exists a unique affine isomorphism f of M onto M' such that $f(x_0) = y_0$ and that the differential of f at x_0 coincides with F.*

Proof. This is an immediate consequence of Corollary 6.2, Theorem 7.4, and Theorem 7.7. QED.

COROLLARY 7.9. *Let M be a connected, simply connected manifold with a complete linear connection such that $\nabla T = 0$ and $\nabla R = 0$. If F is a linear isomorphism of $T_{x_0}(M)$ onto $T_{y_0}(M)$ which maps the tensors T_{x_0} and R_{x_0} into T_{y_0} and R_{y_0}, respectively, then there is a unique affine transformation f of M such that $f(x_0) = y_0$ and that the differential of f at x_0 is F.*

In particular, the group $\mathfrak{A}(M)$ of affine transformations of M is transitive on M.

Proof. The first assertion is clear. The second assertion follows from Corollary 7.5 and Theorem 7.8. QED.

In §3 of Chapter V, we constructed, for each real number k, a connected, simply connected complete Riemannian manifold of constant curvature k. Any connected, simply connected complete space of constant curvature k is isometric to the model we constructed. Namely, we have

THEOREM 7.10. *Any two connected, simply connected complete Riemannian manifolds of constant curvature k are isometric to each other.*

Proof. By Corollary 2.3 of Chapter V, for a space of constant curvature, we have $\nabla R = 0$. Our assertion now follows from Theorem 7.8 and from the fact that, if both M and M' have the same sectional curvature k, then any linear isomorphism $F: T_{x_0}(M) \to T_{y_0}(M')$ mapping the metric tensor g_{x_0} at x_0 into the metric tensor g'_{y_0} at y_0 necessarily maps the curvature tensor R_{x_0} at x_0 into the curvature tensor R'_{y_0} at y_0 (cf. Proposition 1.2 of Chapter V). QED.

APPENDIX 1

Ordinary linear differential equations

The purpose of this appendix is to state the fundamental theorem on ordinary linear differential equations in the form needed in the text. The proof will be found in various text books on differential equations.

For the sake of simplicity, we use the following abbreviated notation:

$$y = (y^1, \ldots, y^n), \quad \eta = (\eta^1, \ldots, \eta^n), \quad f = (f^1, \ldots, f^n),$$
$$\varphi = (\varphi^1, \ldots, \varphi^n), \quad s = (s^1, \ldots, s^m), \quad x = (x^1, \ldots, x^m).$$

Then we have

THEOREM. *Let $f(t, y, s)$ be a family of n functions defined in $|t| < \delta$ and $(y, s) \in D$, where D is an open set in $\mathbf{R}^{n+m}$. If $f(t, y, s)$ is continuous in t and differentiable of class C^1 in y, then there exists a unique family $\varphi(t, \eta, s)$ of n functions defined in $|t| < \delta'$ and $(\eta, s) \in D'$, where $0 < \delta' < \delta$ and D' is an open subset of D, such that*

(1) $\varphi(t, \eta, s)$ is differentiable of class C^1 in t and η;
(2) $\partial\varphi(t, \eta, s)/\partial t = f(t, \varphi(t, \eta, s), s)$;
(3) $\varphi(0, \eta, s) = \eta$.

If $f(t, y, s)$ is differentiable of class C^p, $0 \leqq p \leqq \omega$, in t and of class C^q, $1 \leqq q \leqq \omega$, in y and s, then $\varphi(t, \eta, s)$ is differentiable of class C^{p+1} in t and of class C^q in η and s.

Consider the system of differential equations:

$$dy/dt = f(t, y, s)$$

which depend on the parameters s. Then $y = \varphi(t, \eta, s)$ is called the solution satisfying the initial condition:

$$y = \eta \quad \text{when } t = 0.$$

Consider now a system of partial differential equations:

$$\partial y^i/\partial x^j = f^i_j(x, y), \qquad i = 1, \ldots, n; j = 1, \ldots, m.$$

It follows from the theorem that if the functions $f^i_j(x, y)$ are differentiable of class C^r, $0 \leqq r \leqq \omega$, then every solution $y = \psi(x)$ is differentiable of class C^{r+1}. This fact is used in the proof of Theorem 7.7 of Chapter VI.

A connected, locally compact metric space is separable

We recall that a topological space M is *separable* if there exists a dense subset D which contains at most countably many points. It is called *locally separable* if every point of M has a neighborhood which is separable. Note that, for a metric space, the separability is equivalent to the second axiom of countability (cf. Kelley [1, p. 120]). The proof of the statement in the title is divided into the following three lemmas.

LEMMA 1. *A compact metric space is separable.*

For the proof, see Kelley [1, p. 138].

LEMMA 2. *A locally compact metric space is locally separable.*

This is a trivial consequence of Lemma 1.
The following lemma is due to Sierpinski [1].

LEMMA 3. *A connected, locally separable metric space is separable.*
Proof of Lemma 3. Let d be the metric of a connected, locally separable metric space M. For every point $x \in M$ and every positive number r, let $U(x; r)$ be the interior of the sphere of center x and radius r, that is, $U(x; r) = \{y \in M; d(x, y) < r\}$. We say that two points x and y of M are R-related and write xRy, if there exist a separable $U(x; r)$ containing y and a separable $U(y; r')$ containing x. Evidently, xRx for every $x \in M$. We have also xRy if and only if yRx.

For every subset A of M, we denote by SA the set of points which are R-related to a point of A: $SA = \{y \in M; yRx \text{ for some } x \in A\}$. Set $S^n A = SS^{n-1}A$, $n = 2, 3, \ldots$. If $\{x\}$ is the set consisting of a single point x, we write Sx for $S\{x\}$. We see easily that $y \in S^n x$ if and only if $x \in S^n y$. We prove the following three statements:

(a) Sx is open for every $x \in M$;

(b) If A is separable, so is SA;

(c) Set $U(x) = \bigcup_{n=1}^{\infty} S^n x$ for each $x \in M$. Then, for any $x, y \in M$, either $U(x) \cap U(y)$ is empty or $U(x) = U(y)$.

Proof of (a). Let y be a point of Sx. Since xRy, there exist positive numbers r and r' such that $U(x; r)$ and $U(y; r')$ are separable and that $y \in U(x; r)$ and $x \in U(y; r')$. Since $d(x, y) < r'$, there is a positive number r_1 such that

$$d(x, y) < r_1 < r'.$$

Let r_0 be any positive number such that

$$r_0 < r' - r_1, \quad r_0 < r - d(x, y), \quad r_0 < r_1 - d(x, y).$$

It suffices to show that $U(y, r_0)$ is contained in Sx. If $z \in U(y; r_0)$, then

$$d(x, z) \leqq d(x, y) + d(y, z) < d(x, y) + r_0 < \min\{r, r_1\}.$$

Hence z is in $U(x; r)$ which is separable and x is in $U(z; r_1)$. To prove that $U(z; r_1)$ is separable, we shall show that $U(z; r_1)$ is contained in $U(y; r')$ which is separable. Let $w \in U(z; r_1)$ so that $d(z, w) < r_1$. Then

$$d(y, w) \leqq d(y, z) + d(z, w) < d(y, z) + r_1 < r_0 + r_1 < r'.$$

Hence $w \in U(y; r')$. This proves that zRx for every $z \in U(y; r_0)$, that is, $U(y; r_0) \subset Sx$.

Proof of (b). Let A be a separable subset of M and D a countable dense subset of A. It suffices to prove that every $x \in SA$ is contained in a separable sphere whose center is a point of D and whose radius is a rational number, because there are only countably many such spheres and the union of these spheres is separable. Let $x \in SA$. Then there is $y \in A$ such that xRy and there is a separable sphere $U(y; r)$ containing x. Let r_0 be a positive rational number such that $d(x, y) < r_0 < r$. Since D is dense in A, there is $z \in D$ such that

$$d(z, y) < \min\{r_0 - d(x, y), r - r_0\}.$$

It suffices to show that $U(z; r_0)$ contains x and is separable. From

$$d(x, z) \leqq d(x, y) + d(y, z) < r_0,$$

it follows that $x \in U(z; r_0)$. To prove that $U(z; r_0)$ is separable, we show that $U(z; r_0)$ is contained in $U(y; r)$ which is separable. If $w \in U(z; r_0)$, then

$$d(w, y) \leqq d(w, z) + d(z, y) < r_0 + d(z, y) < r,$$

and hence $w \in U(y; r)$.

Proof of (c). Assume that $U(x) \cap U(y)$ is non-empty and let $z \in U(x) \cap U(y)$. Then $z \in S^m x$ and $z \in S^n y$ for some m and n. From $z \in S^m x$, we obtain $x \in S^m z$. Hence $x \in S^m z \subset S^{m+n} y$. This implies $S^k x \subset S^{k+m+n} y$ for every k and hence $U(x) \subset U(y)$. Similarly, we have $U(y) \subset U(x)$, thus proving (c).

By (a), $SA = \bigcup_{x \in A} Sx$ is open for any subset A of M. Hence $U(x)$ is open for every $x \in M$. By (b), $S^n x$ is separable for every n. Hence $U(x)$ is separable. Since M is connected and since each $U(x)$ is open, (c) implies $M = U(x)$ for every $x \in M$. Hence M is separable, thus completing the proof of the statement in the title.

We are now in position to prove

THEOREM. *For a connected differentiable manifold M, the following conditions are mutually equivalent:*

(1) *There exists a Riemannian metric on M;*

(2) *M is metrizable;*

(3) *M satisfies the second axiom of countability;*

(4) *M is paracompact.*

Proof. The implication (1) → (2) was proved in Proposition 3.5 of Chapter IV. As we stated at the beginning, for a metric space, the second axiom of countability is equivalent to the separability. The implication (2) → (3) is therefore a consequence of the statement in the title. If (3) holds, then M is metrizable by Urysohn's metrization theorem (cf. Kelley [1, p. 125]) and, hence, M is paracompact (cf. Kelley [1, p. 156]). This shows that (3) implies (4). The implication (4) → (1) follows from Proposition 1.4 of Chapter III. QED.

APPENDIX 3

Partition of unity

Let $\{U_i\}_{i \in I}$ be a locally finite open covering of a differentiable manifold M, i.e., every point of M has a neighborhood which intersects only finitely many U_i's. A family of differentiable functions $\{f_i\}$ on M is called a *partition of unity subordinate to the covering* $\{U_i\}$, if the following conditions are satisfied:

(1) $0 \leqq f_i \leqq 1$ on M for every $i \in I$;

(2) The support of each f_i, i.e., the closure of the set $\{x \in M; f_i(x) \neq 0\}$, is contained in the corresponding U_i;

(3) $\Sigma_i f_i(x) = 1$.

Note that in (3), for each point $x \in M$, $f_i(x) = 0$ except for a finite number of i's so that $\Sigma_i f_i(x)$ is a finite sum for each x.

We first prove

THEOREM 1. *Let $\{U_i\}$ be a locally finite open covering of a paracompact manifold M such that each U_i has compact closure $\bar{U}_i$. Then there exists a partition of unity $\{f_i\}$ subordinate to $\{U_i\}$.*

Proof. We first prove the following three lemmas. The first two are valid without the assumption that M is paracompact whereas the third holds for any paracompact topological space.

LEMMA 1. *For each point $x \in M$ and for each neighborhood U of x, there exists a differentiable function f (of class C^∞) on M such that* (1) $0 \leqq f \leqq 1$ *on M;* (2) $f(x) = 1$*; and* (3) $f = 0$ *outside U.*

Proof of Lemma 1. This can be easily reduced to the case where $M = \mathbf{R}^n$, $x = 0$ and $U = \{(x^1, \ldots, x^n); |x^i| < a\}$. Then, for each $j, j = 1, \ldots, n$, we let $f_j(x^j)$ be a differentiable function such that $f_j(0) = 1$ and that $f_j(x^j) = 0$ for $|x^j| \geqq a$. We set $f(x^1, \ldots, x^n) = f_1(x^1) \cdots f_n(x^n)$. This proves Lemma 1.

LEMMA 2. *For every compact subset K of M and for every neighborhood U of K, there exists a differentiable function f on M such that* (1) $f \geqq 0$ *on M;* (2) $f > 0$ *on K; and* (3) $f = 0$ *outside U.*

Proof of Lemma 2. For each point x of K, let f_x be a differentiable function on M with the property of f in Lemma 1. Let V_x be the neighborhood of x defined by $f_x > \frac{1}{2}$. Since K is compact, there exist a finite number of points $x_1, \ldots, x_k$ of K such that $V_{x_1} \cup \cdots \cup V_{x_k} \supset K$. Then we set

$$f = f_{x_1} + \cdots + f_{x_k}.$$

This completes the proof of Lemma 2.

LEMMA 3. *Let $\{U_i\}$ be a locally finite open covering of M. Then there exists a locally finite open refinement $\{V_i\}$ (with the same index set) of $\{U_i\}$ such that $\bar{V}_i \subset U_i$ for every i.*

Proof of Lemma 3. For each point $x \in M$, let W_x be an open neighborhood of x such that $\bar{W}_x$ is contained in some U_i. Let $\{W'_\alpha\}$ be a locally finite refinement of $\{W_x; x \in M\}$. For each i, let V_i be the union of all W'_α whose closures are contained in U_i. Since $\{W'_\alpha\}$ is locally finite, we have $\bar{V}_i = \bigcup \bar{W}'_\alpha$, where the union is taken over all α such that $\bar{W}'_\alpha \subset U_i$. We thus obtained an open covering $\{V_i\}$ with the required property.

We are now in position to complete the proof of Theorem 1. Let $\{V_i\}$ be as in Lemma 3. For each i, let W_i be an open set such that $\bar{V}_i \subset W_i \subset \bar{W}_i \subset U_i$. By Lemma 2, there exists, for each i, a differentiable function g_i on M such that (1) $g_i \geqq 0$ on M; (2) $g_i > 0$ on $\bar{V}_i$; and (3) $g_i = 0$ outside W_i. Since the support of each g_i contains $\bar{V}_i$ and is contained in U_i and since $\{U_i\}$ is locally finite, the sum $g = \Sigma_i\, g_i$ is defined and differentiable on M. Since $\{V_i\}$ is an open covering of M, $g > 0$ on M. We set, for each i,

$$f_i = g_i/g.$$

Then $\{f_i\}$ is a partition of unity subordinate to $\{U_i\}$. QED.

Let f be a function defined on a subset F of a manifold M. We say that f is differentiable on F if, for each point $x \in F$, there exists a differentiable function f_x on an open neighborhood V_x of x such that $f = f_x$ on $F \cap V_x$.

THEOREM 2. *Let F be a closed subset of a paracompact manifold M. Then every differentiable function f defined on F can be extended to a differentiable function on M.*

Proof. For each $x \in F$, let f_x be a differentiable function on an open neighborhood V_x of x such that $f_x = f$ on $F \cap V_x$. Let U_i be a locally finite open refinement of the covering of M consisting of $M - F$ and V_x, $x \in F$. For each i, we define a differentiable function g_i on U_i as follows. If U_i is contained in some V_x, we choose such a V_x and set

$$g_i = \text{restriction of } f_x \text{ to } U_i.$$

If there is no V_x which contains U_i, then we set

$$g_i = 0.$$

Let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$. We define

$$g = \Sigma_i f_i g_i.$$

Since $\{U_i\}$ is locally finite, every point of M has a neighborhood in which $\Sigma_i f_i g_i$ is really a finite sum. Thus g is differentiable on M. It is easy to see that g is an extension of f. QED.

In the terminologies of the sheaf theory, Theorem 2 means that the sheaf of germs of differentiable functions on a paracompact manifold M is soft ("*mou*" in Godement [1]).

APPENDIX 4

On arcwise connected subgroups of a Lie group

Kuranishi and Yamabe proved that every arcwise connected subgroup of a Lie group is a Lie subgroup (see Yamabe [1]). We shall prove here the following weaker theorem, which is sufficient for our purpose (cf. Theorem 4.2 of Chapter II). This result is essentially due to Freudenthal [1].

THEOREM. *Let G be a Lie group and H a subgroup of G such that every element of H can be joined to the identity e by a piecewise differentiable curve of class C^1 which is contained in H. Then H is a Lie subgroup of G.*

Proof. Let S be the set of vectors $X \in T_e(G)$ which are tangent to differentiable curves of class C^1 contained in H. We identify $T_e(G)$ with the Lie algebra $\mathfrak{g}$ of G. Then

LEMMA. *S is a subalgebra of* $\mathfrak{g}$.

Proof of Lemma. Given a curve x_t in G, we denote by $\dot{x}_t$ the vector tangent to the curve at the point x_t. Let r be any real number and set $z_t = x_{rt}$. Then $\dot{z}_0 = r\,\dot{x}_0$. This shows that if $X \in S$, then $rX \in S$. Let x_t and y_t be curves in G such that $x_0 = y_0 = e$. If we set $v_t = x_t y_t$, then $\dot{v}_0 = \dot{x}_0 + \dot{y}_0$ (cf. Chevalley [1, pp. 120–122]). This shows that if $X, Y \in S$, then $X + Y \in S$. There exists a curve w_t such that $w_{t^2} = x_t y_t x_t^{-1} y_t^{-1}$ and we have $\dot{w}_0 = [\dot{x}_0, \dot{y}_0]$ (cf. Chevalley [1, pp. 120–122] or Pontrjagin [1, p. 238]). This shows that if $X, Y \in S$, then $[X, Y] \in S$, thus completing the proof of the lemma.

Since $S \subset T_e(G) = \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$, the distribution $x \to L_x S$, $x \in G$, is involutive (where L_x is the left translation by x) and its maximal integral manifold through e, denoted by K, is the Lie subgroup of G corresponding to the subalgebra S. We shall show that $H = K$.

We first prove that $K \supset H$. Let a be any point of H and $\tau = x_t$, $0 \leqq t \leqq 1$, a curve from e to a so that $e = x_0$ and $a = x_1$. We claim that the vector $\dot{x}_t$ is in $L_{x_t}S$ for all t. In fact, for each fixed t, $L_{x_t}^{-1}(\dot{x}_t)$ is the vector tangent to the curve $L_{x_t}^{-1}(\tau)$ at e and hence lies in S, thus proving our assertion. Since $\dot{x}_t \in L_{x_t}S$ for all t and $x_0 = e$, the curve x_t lies in the maximal integral manifold K of the distribution $x \to L_x S$ (cf. Lemma 2 for Theorem 7.2 of Chapter II). Hence $a \in K$, showing that $K \supset H$.

To prove that $H \supset K$, let $e_1, \ldots, e_k$ be a basis for S and $x_t^1 \cdots x_t^k$, $0 \leqq t \leqq 1$, be curves in H such that $x_0^i = e$ and $\dot{x}_0^i = e_i$ for $i = 1, \ldots, k$. Consider the mapping f of a neighborhood U of the origin in R^k into K defined by $f(t_1, \ldots, t_k) = x_{t_1}^1 \cdots x_{t_k}^k$, $(t_1, \ldots, t_k) \in U$. Since $\dot{x}_0^1, \ldots, \dot{x}_0^k$ form a basis for S, the differential of f: $U \to K$ at the origin is non-singular. Taking U sufficiently small, we may assume that f is a diffeomorphism of U onto an open subset $f(U)$ of K. From the definition of f, we have $f(U) \subset H$. This shows that a neighborhood of e in K is contained in H. Since K is connected, $K \subset H$. QED.

APPENDIX 5

Irreducible subgroups of $O(n)$

We prove the following two theorems.

THEOREM 1. *Let G be a subgroup of $O(n)$ which acts irreducibly on the n-dimensional real vector space $\mathbf{R}^n$. Then every symmetric bilinear form on $\mathbf{R}^n$ which is invariant by G is a multiple of the standard inner product $(x, y) = \sum_{i=1}^{n} x^i y^i$.*

THEOREM 2. *Let G be a connected Lie subgroup of $SO(n)$ which acts irreducibly on $\mathbf{R}^n$. Then G is closed in $SO(n)$.*

We begin with the following lemmas.

LEMMA 1. *Let G be a subgroup of $GL(n; \mathbf{R})$ which acts irreducibly on $\mathbf{R}^n$. Let A be a linear transformation of $\mathbf{R}^n$ which commutes with every element of G. Then*

(1) *If A is nilpotent, then $A = 0$.*

(2) *The minimal polynomial of A is irreducible over $\mathbf{R}$.*

(3) *Either $A = aI_n$ (a: real number, I_n: the identity transformation of $\mathbf{R}^n$), or $A = aI_n + bJ$, where a and b are real numbers, $b \neq 0$, J is a linear transformation such that $J^2 = -I_n$, and n is even.*

Proof. (1) Let k be the smallest integer such that $A^k = 0$. Assuming that $k \geqq 2$, we derive a contradiction. Let $W = \{x \in \mathbf{R}^n; Ax = 0\}$. Since W is invariant by G, we have either $W = \mathbf{R}^n$ or $W = (0)$. In the first case, $A = 0$. In the second case, A is non-singular and $A^{k-1} = A^{-1} \cdot A^k = 0$.

(2) If the minimal polynomial $f(x)$ of A is a product $f_1(x) \cdot f_2(x)$ with $(f_1, f_2) = 1$, then $\mathbf{R}^n = W_1 + W_2$ (direct sum), where $W_i = \{x \in \mathbf{R}^n; f_i(A)x = 0\}$. Since every element of G commutes with A and hence with $f_i(A)$, it follows that W_i are both invariant by G, contradicting the assumption of irreducibility. Thus

$f(x) = g(x)^k$, where $g(x)$ is irreducible. Applying (1) to $g(A)$, we see that $f(A) = g(A)^k = 0$ implies $g(A) = 0$. Thus $f = g$.

(3) By (2), the minimal polynomial $f(x)$ of A is either $(x - a)$ or $(x - a)^2 + b^2$ with $b \neq 0$. In the first case, $A = aI_n$. In the second case, let $J = (A - aI_n)/b$. Then $J^2 = -I_n$ and $A = aI_n + bJ$. We have $(-1)^n = \det J^2 = (\det J)^2 > 0$ so that n is even.

LEMMA 2. *Let G be a subgroup of $O(n)$ which acts irreducibly on $\mathbf{R}^n$. Let $A, B, \ldots$ be linear transformations of $\mathbf{R}^n$ which commute with G.*

(1) *If A is symmetric, i.e., $(Ax, y) = (x, Ay)$, then $A = aI_n$.*

(2) *If A is skew-symmetric, i.e., $(Ax, y) + (x, Ay) = 0$, then $A = 0$ or $A = bJ$, where $J^2 = -I_n$ and $n = 2m$.*

(3) *If $A \neq 0$ and B are skew-symmetric and $AB = BA$, then $B = cA$.*

Proof. (1) By (3) of Lemma 1, $A = aI_n + bJ$, possibly with $b = 0$. If A is symmetric, so is bJ. If $b \neq 0$, J is symmetric so that $(Jx, Jx) = (x, J^2x) = -(x, x)$, which is a contradiction for $x \neq 0$.

(2) Since the eigenvalues of skew-symmetric A are 0 or purely imaginary, the minimal polynomial of A is either x or $x^2 + b^2$, $b \neq 0$. In the first case, $A = 0$. In the second case, $A = -bJ$ with $J^2 = -I_n$.

(3) Let $A = bJ$ and $B = b'K$, where $J^2 = K^2 = -I_n$. We have $JK = KJ$. We show that $\mathbf{R}^n = W_1 + W_2$ (direct sum), where $W_1 = \{x \in \mathbf{R}^n; Jx = Kx\}$ and $W_2 = \{x \in \mathbf{R}^n; Jx = -Kx\}$. Clearly, $W_1 \cap W_2 = (0)$. Every $x \in \mathbf{R}^n$ is of the form $y + z$ with $y \in W_1$ and $z \in W_2$, as we see by setting $y = (x - JKx)/2$ and $z = (x + JKx)/2$. W_1 and W_2 are invariant by G, because J and K commute with every element of G. Since G is irreducible, we have either $W_1 = \mathbf{R}^n$ or $W_2 = \mathbf{R}^n$, that is, either $K = J$ or $K = -J$. This means that $B = cA$ for some c.

Proof of Theorem 1. For any symmetric bilinear form $f(x, y)$, there is a symmetric linear transformation A such that $f(x, y) = (Ax, y)$. If f is invariant by G, then A commutes with every element of G. By (1) of Lemma 2, $A = aI_n$ and hence $f(x, y) = a(x, y)$.

Proof of Theorem 2. We first show that the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{g}$ of G is at most 1-dimensional. Let $A \neq 0$ and $B \in \mathfrak{z}$. Since A, B are skew-symmetric linear transformations which commute with every element of G, (3) of Lemma 2 implies that $B = cA$ for some c. Thus $\dim \mathfrak{z} \leqq 1$. If $\dim \mathfrak{z} = 1$, then $\mathfrak{z} = \{cJ; c \text{ real}\}$, where

J is a certain skew-symmetric linear transformation with $J^2 = -I_n$. Now J is representable by a matrix which is a block form, each block being $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with respect to a certain orthonormal basis of $\mathbf{R}^n$. The 1-parameter subgroup $\exp tJ$ consists of matrices of a block form, each block being $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, and hence is isomorphic with the circle group.

Since $\mathfrak{g}$ is the subalgebra of the Lie algebra of all skew-symmetric matrices, $\mathfrak{g}$ has a positive definite inner product $(A, B) = -\operatorname{trace}(AB)$ which is invariant by $\operatorname{ad}(G)$. It follows that the orthogonal complement $\mathfrak{s}$ of the center $\mathfrak{z}$ in $\mathfrak{g}$ with respect to this inner product is an ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{z} + \mathfrak{s}$ is the direct sum. If $\mathfrak{s}$ contains a proper ideal, say, $\mathfrak{s}_1$, then the orthogonal complement $\mathfrak{s}'$ of $\mathfrak{s}_1$ in $\mathfrak{s}$ is an ideal of $\mathfrak{s}$ (in fact of $\mathfrak{g}$) and $\mathfrak{s} = \mathfrak{s}_1 + \mathfrak{s}'$. Thus we see that $\mathfrak{s}$ is a direct sum of simple ideals: $\mathfrak{s} = \mathfrak{s}_1 + \cdots + \mathfrak{s}_k$. We have already seen that the connected Lie subgroup generated by $\mathfrak{z}$ is closed in $SO(n)$. We now show that the connected Lie subgroup generated by $\mathfrak{s}$ is closed in $SO(n)$. This will finish the proof of Theorem 2.

We first remark that Yosida [1] proved the following result. Every connected semisimple Lie subgroup G of $GL(n; \mathbf{C})$ is closed in $GL(n; \mathbf{C})$. His proof, based on a theorem of Weyl that any representation of a semisimple Lie algebra is completely reducible, also works when we replace $GL(n; \mathbf{C})$ by $GL(n; \mathbf{R})$. In the case of a subgroup G of $SO(n)$, we need not use the Weyl theorem. We now prove the following result by the same method as Yosida's.

A connected semisimple Lie subgroup G of $SO(n)$ is closed in $SO(n)$.

Proof. Since $\mathfrak{g}$ is a direct sum of simple ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ of dimension > 1 and since $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$ for each i, it follows that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Now consider $SO(n)$ and hence its subgroup G as acting on the complex vector space $\mathbf{C}^n$ with standard hermitian inner product which is left invariant by $SO(n)$. Then $\mathbf{C}^n$ is the direct sum of complex subspaces $V_1, \ldots, V_r$ which are invariant and irreducible by G. Assuming that G is not closed in $SO(n)$, let $\bar{G}$ be its closure. Since $\bar{G}$ is a connected closed subgroup of $SO(n)$, it is a Lie subgroup. Let $\bar{\mathfrak{g}}$ be its Lie algebra. Obviously, $\mathfrak{g} \subset \bar{\mathfrak{g}}$. Since $\operatorname{ad}(G)\mathfrak{g} \subset \mathfrak{g}$, we have $\operatorname{ad}(\bar{G})\mathfrak{g} \subset \mathfrak{g}$, which implies that $\mathfrak{g}$ is an ideal of $\bar{\mathfrak{g}}$. Since the Lie algebra of $SO(n)$ has a positive definite inner product invariant by ad $(SO(n))$ as we already noted, it follows that $\bar{\mathfrak{g}}$ is the direct sum of $\mathfrak{g}$ and the orthogonal complement $\mathfrak{u}$ of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$. Each summand V_i of $\mathbf{C}^n$ is also invariant by $\bar{G}$ and hence by $\bar{\mathfrak{g}}$ acting on $\mathbf{C}^n$. For any $A \in \bar{\mathfrak{g}}$, denote by A_i its action on V_i for each i. For any $A, B \in \mathfrak{g}$, we have obviously trace $[A_i, B_i] = 0$. Since $A \rightarrow A_i$ is a representation of $\mathfrak{g}$ on V_i and since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we have trace $A_i = 0$ for every $A \in \mathfrak{g}$. Thus the restriction of $a \in G$ on each V_i has determinant 1 (cf. Corollary 1 of Chevalley [1; p. 6]). By continuity, the restriction of $a \in \bar{G}$ on each V_i has determinant 1. This means that trace $A_i = 0$ for every $A \in \mathfrak{g}$ and for each i. Now let $B \in \mathfrak{u}$. Its action B_i on V_i commutes with the actions of $\{A_i; A \in \mathfrak{g}\}$. By Schur's Lemma (which is an obvious consequence of Lemma 1, (2), which is valid for any field instead of $\mathbf{R}$), we have $B_i = b_i I$, where I is the identity transformation of V_i. Since trace $B_i = 0$, it follows that $b_i = 0$, that is, $B_i = 0$. This being the case for each i, we have $B = 0$. This means that $\mathfrak{u} = (0)$ and $\bar{\mathfrak{g}} = \mathfrak{g}$. This proves that $\bar{G} = G$, that is, G is closed in $SO(n)$.

APPENDIX 6

Green's theorem

Let M be an oriented n-dimensional differentiable manifold. An n-form ω on M is called a *volume element*, if $\omega(\partial/\partial x^1, \ldots, \partial/\partial x^n) > 0$ for each oriented local coordinate system $x^1, \ldots, x^n$. With a fixed volume element ω (which will be also denoted by a more intuitive notation dv), the integral $\int_M f\,dv$ of any continuous function f with compact support can be defined (cf. Chevalley [1, pp. 161–167]).

For each vector field X on M with a fixed volume element ω, the *divergence of* X, denoted by div X, is a function on M defined by

$$(\operatorname{div} X)\,\omega = L_X\omega,$$

where L_X is the Lie differentiation in the direction of X.

GREEN'S THEOREM. *Let M be an oriented compact manifold with a fixed volume element $\omega = dv$. For every vector field X on M, we have*

$$\int_M \operatorname{div} X\,dv = 0.$$

Proof. Let φ_t be the 1-parameter group of transformations generated by X (cf. Proposition 1.6 of Chapter I). Since we have (cf. Chevalley [1, p. 165])

$$\int_M \varphi_t^{-1*}\omega = \int_M \omega,$$

$\int_M \varphi_t^{-1*}\omega$, considered as a function of t, is a constant. By definition of L_X, we have $\left[\frac{d}{dt}(\varphi_t^{-1*}\omega)\right]_{t=0} = -L_X\omega$. Hence

$$0 = \left[\frac{d}{dt}\int_M \varphi_t^{-1*}\omega\right]_{t=0} = \int_M \left[\frac{d}{dt}(\varphi_t^{-1*}\omega)\right]_{t=0}$$
$$= -\int_M L_X\omega = -\int_M \operatorname{div} X \, dv.$$

QED.

Remark 1. The above formula is valid for a non-compact manifold M as long as X has a compact support.

Remark 2. The above formula follows also from Stokes' formula. In fact, since $d\omega = 0$, we have $L_X\omega = d \circ \iota_X \omega + \iota_X \circ d\omega = d \circ \iota_X\omega$. We then have

$$\int_M L_X\omega = \int_{\partial M} \iota_X\omega = 0.$$

PROPOSITION. *Let M be an oriented manifold with a fixed volume element $\omega = dv$. If Γ is an affine connection with no torsion on M such that ω is parallel with respect to Γ, then, for every vector field X on M, we have*

$$(\operatorname{div} X)_x = \textit{trace of the endomorphism } V \to \nabla_V X, \quad V \in T_x(M).$$

Proof. Let A_X be the tensor field of type (1, 1) defined by $A_X = L_X - \nabla_X$ as in §2 of Chapter VI. Let $X_1, \ldots, X_n$ be a basis of $T_x(M)$. Since $\nabla_X\omega = 0$ and since A_X, as a derivation, maps every function into zero, we have

$$\begin{aligned}
(L_X\omega)(X_1, \ldots, X_n) &= (A_X\omega)(X_1, \ldots, X_n) \\
&= A_X(\omega(X_1, \ldots, X_n)) - \Sigma_i\, \omega(X_1, \ldots, A_X X_i, \ldots, X_n) \\
&= -\Sigma_i\, \omega(X_1, \ldots, A_X X_i, \ldots, X_n) \\
&= -(\operatorname{trace} A_X)_x \omega(X_1, \ldots, X_n).
\end{aligned}$$

This shows that

$$\operatorname{div} X = -\operatorname{trace} A_X.$$

Our assertion follows from the formula (cf. Proposition 2.5 of Chapter VI):

$$A_X Y = -\nabla_Y X - T(X, Y)$$

and from the assumption that $T = 0$. QED.

Remark 3. The formula div $X = -\text{trace } A_X$ holds without the assumption $T = 0$.

Let M be an oriented Riemannian manifold. We define a natural volume element dv on M. At an arbitrary point x of M, let $X_1, \ldots, X_n$ be an orthonormal basis of $T_x(M)$ compatible with the orientation of M. We define an n-form dv by

$$dv(X_1, \ldots, X_n) = 1.$$

It is easy to verify that dv is defined independently of the frame $X_1, \ldots, X_n$ chosen. In terms of an allowable local coordinate system $x^1, \ldots, x^n$ and the components g_{ij} of the metric tensor g, we have

$$dv = \sqrt{G}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \text{where} \quad G = \det(g_{ij}).$$

In fact, let $(\partial/\partial x^i)_x = \Sigma_k C_i^k X_k$ so that $g_{ij} = \Sigma_k C_i^k C_j^k$ and $G = \det(C_i^k)^2$ at x. Since $\partial/\partial x^1, \ldots, \partial/\partial x^n$ and $X_1, \ldots, X_n$ have the same orientation, we have $\det(C_i^k) = \sqrt{G} > 0$. Hence, at x, we have

$$\begin{aligned} dv(\partial/\partial x^1, \ldots, \partial/\partial x^n) &= \Sigma_{i_1, \ldots, i_n} \varepsilon\, C_1^{i_1} \cdots C_n^{i_n}\, dv(X_{i_1}, \ldots, X_{i_n}) \\ &= \det(C_i^k) = \sqrt{G}, \end{aligned}$$

where ε is 1 or -1 according as $(i_1, \ldots, i_n)$ is an even or odd permutation of $(1, \ldots, n)$.

Since the parallel displacement along any curve τ of M maps every orthonormal frame into an orthonormal frame and preserves the orientation, the volume element dv is parallel. Thus the proposition as well as Green's theorem is valid for the volume element dv of a Riemannian manifold.

APPENDIX 7

Factorization lemma

Let M be a differentiable manifold. Two continuous curves $x(t)$ and $y(t)$ defined on the unit interval $I = [0, 1]$ with $x(0) = y(0)$ and $x(1) = y(1)$ are said to be *homotopic* to each other if there exists a continuous mapping f: $(t, s) \in I \times I \to f(t, s) \in M$ such that $f(t, 0) = x(t), f(t, 1) = y(t), f(0, s) = x(0) = y(0)$ and $f(1, s) = x(1) = y(1)$ for every t and s in I. When $x(t)$ and $y(t)$ are piecewise differentiable curves of class C^k (briefly, piecewise C^k-curves), they are *piecewise C^k-homotopic*, if the mapping f can be chosen in such a way that it is piecewise C^k on $I \times I$, that is, for a certain subdivision $I = \sum_{i=1}^{r} I_i$, f is a differentiable mapping of class C^k of $I_i \times I_j$ into M for each (i, j).

LEMMA. *If two piecewise C^k-curves $x(t)$ and $y(t)$ are homotopic to each other, then they are piecewise C^k-homotopic.*

Proof. We can take a suitable subdivision $I = \sum_{i=1}^{r} I_i$ so that $f(I_i \times I_j)$ is contained in some coordinate neighborhood for each pair (i, j). By modifying the mapping f in the small squares $I_i \times I_j$ we can obtain a piecewise C^k-homotopy between $x(t)$ and $y(t)$.

Now let $\mathfrak{U}$ be an arbitrary open covering. We shall say that a closed curve τ at a point x is a $\mathfrak{U}$-*lasso* if it can be decomposed into three curves $\tau = \mu^{-1} \cdot \sigma \cdot \mu$, where μ is a curve from x to a point y and σ is a closed curve at y which is contained in an open set of $\mathfrak{U}$. Two curves τ and τ' are said to be equivalent, if τ' can be obtained from τ by replacing a finite number of times a portion of the curve of the form $\mu^{-1} \cdot \mu$ by a trivial curve consisting of a single point or vice versa. With these definitions, we prove

Factorization Lemma. *Let $\mathfrak{U}$ be an arbitrary open covering of M.*

(a) *Any closed curve which is homotopic to zero is equivalent to a product of a finite number of $\mathfrak{U}$-lassos.*

(b) *If the curve is moreover piecewise C^k, then each $\mathfrak{U}$-lasso in the product can be chosen to be of the form $\mu^{-1} \cdot \sigma \cdot \mu$, where μ is a piecewise C^k-curve and σ is a C^k-curve.*

Proof. (a) Let $\tau = x(t)$, $0 \leqq t \leqq 1$, so that $x = x(0) = x(1)$. Let f be a homotopy $I \times I \to M$ such that $f(t, 0) = x(t), f(t, 1) = x, f(0, s) = f(1, s) = x$ for every t and s in I. We divide the square $I \times I$ into m^2 equal squares so that the image of each small square by f lies in some open set of the covering $\mathfrak{U}$. For each pair of integers (i, j), $1 \leqq i, j \leqq m$, let $\lambda(i, j)$ be the closed curve in the square $I \times I$ consisting of line segments joining lattice points in the following order:

$$\begin{aligned}&(0, 0) \to (0, j/m) \to ((i-1)/m, j/m) \to \\ &\quad ((i-1)/m, (j-1)/m) \to (i/m, (j-1)/m) \to (i/m, j/m) \to \\ &\quad ((i-1)/m, j/m) \to (0, j/m) \to (0, 0).\end{aligned}$$

Geometrically, $\lambda(i, j)$ looks like a lasso. Let $\tau(i, j)$ be the image of $\lambda(i, j)$ by the mapping f. Then τ is equivalent to the product of $\mathfrak{U}$-lassos

$$\tau(m, m) \cdots \tau(1, m) \cdots \tau(m, 2) \cdots \tau(1, 2) \cdot \tau(m, 1) \cdots \tau(1, 1).$$

(b) By the preceding lemma, we may assume that the homotopy mapping f is piecewise C^k. By choosing m larger if necessary, we may also assume that f is C^k on each of the m^2 small squares. Then each lasso $\tau(i, j)$ has the required property. QED.

The factorization lemma is taken from Lichnerowicz [2, p. 51].

Note 1. *Connections and holonomy groups*

1. Although differential geometry of surfaces in the 3-dimensional Euclidean space goes back to Gauss, the notion of a Riemannian space originates with Riemann's Habilitationsschrift [1] in 1854. The Christoffel symbols were introduced by Christoffel [1] in 1869. Tensor calculus, founded and developed in a series of papers by Ricci, was given a systematic account in Levi-Civita and Ricci [1] in 1901. Covariant differentiation which was formally introduced in this tensor calculus was given a geometric interpretation by Levi-Civita [1] who introduced in 1917 the notion of parallel displacement for the surfaces. This discovery led Weyl [1, 2] and E. Cartan [1, 2, 4, 5, 8, 9] to the introduction of affine, projective and conformal connections. Although the approach of Cartan is the most natural one and reveals best the geometric nature of the connections, it was not until 1950 that Ehresmann [2] clarified the general notion of connections from the point of view of contemporary mathematics. His paper was followed by Chern [1, 2], Ambrose-Singer [1], Kobayashi [6], Nomizu [7], Lichnerowicz [2] and others.

Ehresmann [2] defined, for the first time, a connection in an arbitrary fibre bundle as a field of horizontal subspaces and proved the existence of connections in any bundle. He introduced also a connection form ω and defined the curvature form Ω by means of the structure equation. The definition of Ω given in this book is due to Ambrose and Singer [1] who proved also the structure equation (Theorem 5.2 of Chapter II). Chern [1, 2] defined a connection by means of a set of differential forms ω_α on U_α with values in the Lie algebra of the structure group, where $\{U_\alpha\}$ is an open covering of the base manifold (see Proposition 1.4 of Chapter II).

Ehresmann [2] also defined the notion of a Cartan connection, whose examples include affine, projective and conformal connections. See also Kobayashi [6] and Takizawa [1]. We have given in the text a detailed account of the relationship between linear and affine connections.

2. The notion of holonomy group is due to E. Cartan [1, 6]. The fact that the holonomy group is a Lie group was taken for granted even for a Riemannian connection until Borel and Lichnerowicz [1] proved it explicitly. The holonomy theorem (Theorem 8.1 of Chapter II) of E. Cartan was rigorously proved first by Ambrose-Singer [1]. The proof was simplified by Nomizu [7] and Kobayashi [6] by first proving the reduction theorem (Theorem 7.1 of Chapter II), which is essentially due to Cartan and Ehresmann. Kobayashi [6] showed that Theorem 8.1 is essentially equivalent to the following fact. For a principal fibre bundle $P(M, G)$, consider the principal fibre bundle $T(P)$ over $T(M)$ with group $T(G)$, where $T(\ )$ denotes the tangent bundle. For any connection Γ in P, there is a naturally induced connection $T(\Gamma)$ in $T(P)$ whose holonomy group is $T(\Phi)$, where Φ is the holonomy group of Γ.

The result of Hano and Ozeki [1] and Nomizu [5] (Theorem 8.2 of Chapter II) to the effect that the structure group G of $P(M, G)$ can be reduced to a subgroup H if and only if there exists a connection in P whose holonomy group is exactly H means that the holonomy group by itself does not give any information other than those obtainable by topological methods. However, combined with other conditions (such as a "*torsion-free* linear connection"), the holonomy group is of considerable interest.

3. Chern [3] defined the notion of a G-structure on a differentiable manifold M, where G is a certain Lie subgroup of $GL(n; \mathbf{R})$ with $n = \dim M$. In our terminologies, a *G-structure* on M is a reduction of the bundle of linear frames $L(M)$ to the subgroup G. For $G = O(n)$, a G-structure is nothing but a Riemannian metric given on M (see Example 5.7 of Chapter I). For a general theory of G-structures, see Chern [3], Bernard [1] and Fujimoto [1]. We mention some other special cases.

Weyl [3] and E. Cartan [3] proved the following. *For a closed subgroup G of $GL(n; \mathbf{R})$, $n \geqq 3$, the following two conditions are equivalent:*

(1) *G is the group of all matrices which preserve a certain non-degenerate quadratic form of any signature:*

(2) *For every n-dimensional manifold M and for every reduced subbundle P of $L(M)$ with group G, there is a unique torsion-free connection in P.*

The implication (1) $\rightarrow$ (2) is clear from Theorem 2.2 of Chapter IV (in which g can be an indefinite Riemannian metric); in fact, if G is such a group, any G-structure on M corresponds to an indefinite Riemannian metric on M in a similar way to Example 5.7 of Chapter I. The implication (2) $\rightarrow$ (1) is non-trivial. See also Klingenberg [1].

Let G be the subgroup of $GL(n; \mathbf{R})$ consisting of all matrices which leave the r-dimensional subspace $\mathbf{R}^r$ of $\mathbf{R}^n$ invariant. A G-structure on an n-dimensional manifold M is nothing but an r-dimensional distribution. Walker [3] proved that an r-dimensional distribution is parallel with respect to a certain torsion-free linear connection if and only if the distribution is integrable. See also Willmore [1, 2].

Let G be $GL(n; \mathbf{C})$ regarded as a subgroup of $GL(2n; \mathbf{R})$ in a natural manner. A G-structure on a $2n$-dimensional manifold M is nothing but an almost complex structure on M. This structure will be treated in Volume II.

4. The notions of local and infinitesimal holonomy groups were introduced systematically by Nijenhuis [2]. The results in §10 of Chapter II were obtained by him in the case of a linear connection (§9 of Chapter III). Nijenhuis' results were generalized by Ozeki [1] to the general case as presented in §10 of Chapter II. See also Nijenhuis [3]. Chevalley also obtained Corollary 10.7 of Chapter II in the case of a linear connection (unpublished) and his result was used by Nomizu [2] who discussed invariant linear connections on homogeneous spaces. His results were generalized by Wang [1] as in §11 of Chapter II.

5. By making use of a connection, one can define characteristic classes of any principal fibre bundle. This will be treated in Volume II. See Chern [2], H. Cartan [2, 3]. We shall here state a result of Narasimhan and Ramanan [1] which is closely related to the notion of a universal bundle (cf. Steenrod [1, p. 101]).

Theorem. *Given a compact Lie group G and a positive integer n, there exists a principal bundle $E(N, G)$ and a connection Γ_0 on E such that any connection Γ in any principal bundle $P(M, G)$,* dim $M \leqq n$, *can be obtained as the inverse image of Γ_0 by a certain homomorphism of P into E*

(*that is,* $\omega = f^*\omega_0$, *where* ω *and* ω_0 *are the connection forms of* Γ *and* Γ_0, *respectively, see* Proposition 6.2 *of* Chapter II).

The connection Γ_0 is therefore called a *universal connection* for G (and n). For example, the canonical connection in a Stiefel manifold with structure group $O(k)$ is universal for $O(k)$. For the canonical connections in the Stiefel manifolds, see also Kobayashi [5] who gave an interpretation of the Riemannian connections of manifolds imbedded in Euclidean spaces (see Volume II).

6. The holonomy groups of linear and Riemannian connections were studied in detail by Berger [1]. By a careful examination of the curvature tensor, he obtained a list of groups which can be restricted linear holonomy groups of irreducible Riemannian manifolds with non-parallel curvature tensor. His list coincides with the list of connected orthogonal groups acting transitively on spheres. Simons [1] proved directly that the linear holonomy group of an irreducible Riemannian manifold with non-parallel curvature tensor is transitive on the unit sphere in the tangent space. See Note 7 (symmetric spaces).

7. The local decomposition of a Riemannian manifold (Proposition 5.2 of Chapter IV) has been treated by a number of authors. The global decomposition (Theorem 6.2 of Chapter IV) was proved by de Rham [1]; the same problem was also treated by Walker [2]. A more general situation than the direct product has been studied by Reinhart [1], Nagano [2] and Hermann [1].

It is worthwhile noting that even the local decomposition is a strongly metric property. Ozeki gave an example of a torsion-free linear connection with the following property. The linear holonomy group is completely reducible (that is, the tangent space is the direct sum of invariant irreducible subspaces) but the linear connection is not a direct product even locally. His example is as follows: On $\mathbf{R}^2$ with coordinates (x^1, x^2), take the linear connection given by the Christoffel symbols $\Gamma^1_{11}(x^1, x^2) = x^2$ and other $\Gamma^i_{jk} = 0$. The holonomy group is $\left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a > 0\right\}$.

8. The restricted linear holonomy group of an arbitrary Riemannian manifold is a closed subgroup of the orthogonal group. Hano and Ozeki [1] gave an example of a torsion-free linear connection whose restricted linear holonomy group is not

closed in the general linear group. The linear holonomy group of an arbitrary Riemannian manifold is not in general compact, as Example 4.3 of Chapter V shows. For a compact flat Riemannian manifold, it is compact (Theorem 4.2 of Chapter V). Recently, Wolf [6] proved that this is also the case for a compact locally symmetric Riemannian manifold.

Note 2. Complete affine and Riemannian connections

Hopf and Rinow [1] proved Theorem 4.1 (the equivalence of (1), (2) and (3)), Theorem 4.2 and Theorem 4.4 of Chapter IV. Theorem 4.2 goes back to Hilbert [1]; his proof can be also found in E. Cartan's book [8]. In §4 of Chapter IV, we followed the appendix of de Rham [1]. Condition (4) of Theorem 4.1 of Chapter IV was given as the definition of completeness by Ehresmann [1, 2].

For a complete affine connection, it does not hold in general that every pair of points can be joined by a geodesic. To construct counterexamples, consider an affine connection on a connected Lie group G such that the geodesics emanating from the identity are precisely the 1-parameter groups of G. Such connections will be studied in Volume II. For our present purpose, it suffices to consider the affine connection which makes every left invariant vector field parallel; the existence and the uniqueness of such a connection is easy to see. Then the question is whether every element of G is on a 1-parameter subgroup. The answer is yes, if G is compact (well known) or if G is nilpotent (cf. Matsushima [1]). For a solvable group G, this is no longer true in general; Saito [1] gave a necessary and sufficient condition in terms of the Lie algebra of G for the answer to be affirmative when G is a simply connected solvable group. For some linear real algebraic groups, this question was studied by Sibuya [1]. Even for a simple group, the answer is not affirmative in general. For instance, a direct computation shows that an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

of $SL(2; \mathbf{R})$ lies on some 1-parameter subgroup if and only if either $a + d > -2$ or $a = d = -1$ and $b = c = 0$. This means that, for every element A of $SL(2; \mathbf{R})$, either A or $-A$ (possibly both) lies on a 1-parameter subgroup. Thus the answer to our question is negative for $SL(2; \mathbf{R})$ and is affirmative for $SL(2; \mathbf{R})$ modulo its center. Smith [1] also constructed a Lorentz metric, i.e., an indefinite Riemannian metric, on a 2-dimensional manifold such that the (Riemannian) connection is complete and that not every pair of points can be connected by a geodesic. It is not known whether every pair of points of a compact, connected manifold with a complete affine connection can be joined by a geodesic.

An affine connection on a compact manifold is not necessarily complete as the following example of Auslander and Markus [1] shows. Consider the Riemannian connection on the real line $\mathbf{R}^1$ defined by the metric $ds^2 = e^x\, dx^2$, where x is the natural coordinate system in $\mathbf{R}^1$; it is flat. It is not complete as the length of the geodesic from $x = 0$ to $x = -\infty$ is equal to 2. The translation $x \to x + 1$ is an affine transformation as it sends the metric $e^x\, dx^2$ into $e\, e^x\, dx^2$. Thus the real line modulo 1, i.e., a circle, has a non-complete flat affine connection. This furnishes a non-complete, compact, homogeneous affinely connected manifold. An example of a non-complete affine connection on a simply connected compact manifold is obtained by defining the above affine connection on the equator of a sphere and extending it on the whole sphere so that the equator is a geodesic.

It is known that every metrizable space admits a complete uniform structure (compatible with the topology) (Dieudonné [1]). Nomizu and Ozeki [1] proved that, given a Riemannian metric g on a manifold M, there exists a positive function f on M such that $f \cdot g$ is a complete Riemannian metric.

Note 3. *Ricci tensor and scalar curvature*

Analogous to the theorem of Schur (Theorem 2.2 of Chapter V), we have the following classical result.

THEOREM 1. *Let M be a connected Riemannian manifold with metric tensor g and Ricci tensor S. If $S = \lambda g$, where λ is a function on M, then λ is necessarily a constant provided that $n = \dim M \geq 3$.*

Proof. The simplest proof is probably by means of the classical tensor calculus. Let g_{ij}, R_{ijkl} and R_{ij} be the components of the metric tensor g, the Riemannian curvature tensor R and the Ricci tensor S, respectively, with respect to a local coordinate system $x^1, \ldots, x^n$. Then Bianchi's second identity (Theorem 5.3 of Chapter III) is expressed by

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0.$$

Multiplying by g^{ik} and g^{jl}, summing with respect to i, j, k and l and finally using the following formulas

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}, \quad \Sigma_{i,k}\, g^{ik} R_{ijkl} = R_{jl} = \lambda g_{jl},$$

we obtain

$$(n-2)\,\lambda_{;m} = 0.$$

Hence λ is a constant. QED.

A Riemannian manifold is called an *Einstein manifold* if $S = \lambda g$, where λ is a constant.

The following proposition is due to Schouten and Struik [1].

Proposition 2. *If M is a 3-dimensional Einstein manifold, then i is a space of constant curvature.*

Proof. Let p be any plane in $T_x(M)$ and let X_1, X_2, X_3 be an orthonormal basis for $T_x(M)$ such that p is spanned by X_1, X_2. Let p_{ij} be the plane spanned by X_i and X_j $(i \neq j)$ so that $p_{ij} = p_{ji}$. Then

$$\begin{aligned} S(X_1, X_1) &= \phantom{K(p_{21}) +{}} K(p_{12}) + K(p_{13}) \\ S(X_2, X_2) &= K(p_{21}) + \phantom{K(p_{12}) +{}} K(p_{23}) \\ S(X_3, X_3) &= K(p_{31}) + K(p_{32}), \end{aligned}$$

where $K(p_{ij})$ denotes the sectional curvature determined by the plane p_{ij}. Hence we have

$$S(X_1, X_1) + S(X_2, X_2) - S(X_3, X_3) = 2K(p_{12}) = 2K(p).$$

Since $S(X_i, X_i) = \lambda$, we have $K(p) = \frac{1}{2}\lambda$. QED.

Remark. The above formula implies also that, if $0 < c < S(X, X) < 2c$ for all unit vectors $X \in T_x(M)$, then $K(p) > 0$ for all planes p in $T_x(M)$. Similarly, if $2c < S(X, X) < c < 0$ for all unit vectors $X \in T_x(M)$, then $K(p) < 0$ for all planes p in $T_x(M)$.

Going back to the general case where $n = \dim M$ is arbitrary, let $X_1, \ldots, X_n$ be an orthonormal basis for $T_x(M)$. Then $S(X_1, X_1) + \cdots + S(X_n, X_n)$ is independent of the choice of orthonormal basis and is called the *scalar curvature* at x. In terms of the components R_{ij} and g_{ij} of S and g, respectively, the scalar curvature is given by $\Sigma_{i,j}\, g^{ij} R_{ij}$.

Note 4. Spaces of constant positive curvature

Let M be an n-dimensional, connected, complete Riemannian manifold of constant curvature $1/a^2$. Then, by Theorem 3.2 of Chapter V and Theorem 7.10 of Chapter VI, the universal covering manifold of M is isometric to the sphere S^n of radius a in $\mathbf{R}^{n+1}$ given by $(x^1)^2 + \cdots + (x^{n+1})^2 = a^2$, that is, $M = S^n/G$, where G is a finite subgroup of $O(n+1)$ which acts freely on S^n.

In the case where n is even, the determination of these groups G is extremely simple. Let $\chi(M)$ denote the Euler number of M. Then we have (cf. Hu [1; p. 277])

$$2 = \chi(S^n) = \chi(M) \times \text{order of } G \quad (\text{if } n \text{ is even}).$$

Hence, G consists of either the identity I only or I and another element A of $O(n+1)$ such that $A^2 = I$. Clearly, the eigen-values of A are ± 1. Since A can not have any fixed point on S^n, the eigen-values of A are all equal to -1. Hence, $A = -I$. We thus obtained

THEOREM 1. *Every connected, complete Riemannian manifold M of even dimension n with constant curvature $1/a^2$ is isometric either to the sphere S^n of radius a or to the real projective space $S^n/\{\pm I\}$.*

The case where n is odd has not been solved completely. The most general result in this direction is due to Zassenhaus [2].

THEOREM 2. *Let G be a finite subgroup of $O(n+1)$ which acts freely on S^n. Then, any subgroup of G of order pq (where p and q are prime numbers, not necessarily distinct) is cyclic.*

Proof. It suffices to prove that if G is order pq, then G is cyclic. First, consider the case G is of order p^2. Then, G is either cyclic or a direct product of two cyclic groups G_1 and G_2 of order p (cf. Hall [1, p. 49]). Assuming the latter, let A and B be generators of

G_1 and G_2, respectively. Since every element $T \neq I$ of G is of order p, we have $T(\sum_{i=0}^{p-1} T^i y) = \sum_{i=0}^{p-1} T^i y$ for each $y \in \mathbf{R}^{n+1}$. Since T has no fixed point on S^n, we have

$$\textstyle\sum_{i=0}^{p-1} T^i y = 0 \qquad \text{for } y \in \mathbf{R}^{n+1}.$$

By setting $T = A^i B$ and $y = x$, we obtain

$$\textstyle\sum_{j=0}^{p-1} (A^i B)^j x = 0 \qquad \text{for } x \in \mathbf{R}^{n+1} \quad \text{and} \quad i = 0, 1, \dots p - 1,$$

and hence

$$0 = \textstyle\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (A^i B)^j x = \sum_{j=0}^{p-1} \sum_{i=0}^{p-1} A^{ij} B^j x \qquad \text{for } x \in \mathbf{R}^{n+1}.$$

On the other hand, by setting $T = A^j$ and $y = B^j x$, we obtain

$$\textstyle\sum_{i=0}^{p-1} A^{ij} B^j x = 0 \qquad \text{for } x \in \mathbf{R}^{n+1} \quad \text{and} \quad j = 1, 2, \dots, p - 1.$$

Hence, we have

$$0 = \textstyle\sum_{j=0}^{p-1} \sum_{i=0}^{p-1} A^{ij} B^j x = \sum_{i=0}^{p-1} A^0 B^0 x = px \qquad \text{for } x \in \mathbf{R}^{n+1},$$

which is obviously a contradiction. Thus, G must be cyclic.

Second, consider the case where $p < q$. Then G is either cyclic or non-abelian. Assuming that G is non-abelian, let S and A be elements of order p and q, respectively. Then, we have (cf. Hall [1, p. 51])

$$SAS^{-1} = A^t,$$

where $1 < t < q$ and $t^p \equiv 1 \mod q$, and every element of G can be written uniquely as $A^i S^k$, where $0 \leqq i \leqq q - 1$ and $0 \leqq k \leqq p - 1$. For each integer k, define an integer $f(k)$ by $f(k) = 1 + t + t^2 + \cdots + t^{k-1}$. We then have

(a) $f(p) \equiv 0 \mod q$;
(b) $f(k) \equiv 1 \mod q$, if $k \equiv 1 \mod p$;
(c) $(A^i S)^k = A^{i \cdot f(k)} S^k$.

Indeed, (a) follows from $t^p \equiv 1 \mod q$, and (c) follows from $SAS^{-1} = A^t$. For each i, $0 \leqq i \leqq q - 1$, let G_i be the cyclic subgroup of G generated by $A^i S$. Since $(A^i S)^p = A^{i \cdot f(p)} S^p = I$, G_i is of order p. Hence we have either $G_i \cap G_j = \{I\}$ or $G_i = G_j$ for $0 \leqq i, j \leqq q - 1$. We prove that $G_i \cap G_j = \{I\}$ if $i \neq j$. If $G_i = G_j$, there exists an integer k such that $(A^i S)^k = A^j S$. By (c), we have $A^{i \cdot f(k)} S^k = A^j S$ and, hence, $S^k = S$. This implies $k \equiv 1 \mod p$ and $f(k) \equiv 1 \mod q$. Hence, we have $A^i S^k = A^j S$, which implies $i = j$. Let N be the normal subgroup of G generated by A. Since N is of order q and since G_i is of order p, we have $G_i \cap N = \{I\}$ for each i, $0 \leqq i \leqq q-1$. By counting the orders of $N, G_0, G_1, \ldots, G_{q-1}$, we see that G is a disjoint union of N, $G_0 - \{I\}$, $G_1 - \{I\}, \ldots, G_{q-1} - \{I\}$. Therefore we have

$$\Sigma_{T \in N} Tx + \Sigma_{T \in G_0} Tx + \cdots + \Sigma_{T \in G_{q-1}} Tx = \Sigma_{T \in G} Tx + qx \qquad \text{for } x \in \mathbf{R}^{n+1}.$$

On the other hand, for every $T_0 \in N$, we have

$$T_0(\Sigma_{T \in N} Tx) = \Sigma_{T \in N} T_0 Tx = \Sigma_{T \in N} Tx \qquad \text{for } x \in \mathbf{R}^{n+1}.$$

Since G acts freely on S^n, we have $\Sigma_{T \in N} Tx = 0$. By the same reasoning, we have $\Sigma_{T \in G_i} Tx = 0$ for $i = 0, 1, \ldots, q-1$ and $\Sigma_{T \in G} Tx = 0$. Hence, we have $qx = 0$ for each $x \in \mathbf{R}^{n+1}$, which is obviously a contradiction. QED.

Recently, Wolf [1] classified the homogeneous Riemannian manifolds of constant curvature $1/a^2$. His result may be stated as follows.

THEOREM 3. *Let $M = S^n/G$ be a homogeneous Riemannian manifold of constant curvature $1/a^2$.*

(1) *If $n + 1 = 2m$ (but not divisible by 4), then*

$$S^n = \{(z^1, \ldots, z^m) \in \mathbf{C}^m;\ |z^1|^2 + \cdots + |z^m|^2 = a^2\},$$

and G is a finite group of matrices of the form λI_m, where $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ and I_m is the $m \times m$ identity matrix;

(2) *If $n + 1 = 4m$, then*

$$S^n = \{(q^1, \ldots, q^m) \in \mathbf{Q}^m;\ |q^1|^2 + \cdots + |q^m|^2 = a^2\}$$

(where $\mathbf{Q}$ is the field of quaternions), and G is a finite group of matrices of the form ρI_m, where $\rho \in \mathbf{Q}$ with $|\rho| = 1$.

Conversely, if G is a finite group of the type described in (1) or (2), then $M = S^n/G$ is homogeneous.

In view of Theorem 1, we do not have to consider the case where n is even.

The reader interested in the classification problem of elliptic spaces, i.e., spaces of constant positive curvature, is referred to the following papers: Vincent [1], Wolf [5]; for $n = 3$, H. Hopf [1] and Seifert and Threlfall [1]. Milnor [1] partially generalized

Theorem 2 to the case where G is a group of homeomorphisms acting freely on S^n. Calabi and Markus [1] and Wolf [3, 4] studied Lorentz manifolds of constant positive curvature. See also Helgason [1]. For the study of spaces covered by a homogeneous Riemannian manifold, see Wolf [2].

Note 5. *Flat Riemannian manifolds*

Let $M = \mathbf{R}^n/G$ be a compact flat Riemannian manifold, where G is a discrete subgroup of the group of Euclidean motions of $\mathbf{R}^n$. Let N be the subgroup of G consisting of pure translations. Then

(1) N is an abelian normal subgroup of G and is free on n generators;

(2) N is a maximal abelian subgroup of G;

(3) G/N is finite;

(4) G has no finite subgroup.

Indeed, (1) and (3) have been proved in (4) of Theorem 4.2 of Chapter V. To prove (2), let K be any abelian subgroup of G containing N. Since G/K is also finite by (2), $\mathbf{R}^n/K$ is a compact flat Riemannian manifold. Since K is an abelian normal subgroup of K, K contains nothing but translations by Lemma 6 for Theorem 4.2 of Chapter V. Hence $K = N$. Finally, (4) follows from the fact that G acts freely on $\mathbf{R}^n$. In fact, any finite group of Euclidean motions has a fixed point (cf. the proof of Theorem 7.1 of Chapter IV) and hence G has no finite subgroup.

Auslander and Kuranishi [1] proved the converse:

Let G be a group with a subgroup N satisfying the above conditions (1), (2), (3) *and* (4). *Then G can be realized as a group of Euclidean motions of $\mathbf{R}^n$ such that $\mathbf{R}^n/G$ is a compact flat Riemannian manifold.*

Let $\mathbf{R}^n/G$ and $\mathbf{R}^n/G'$ be two compact flat Riemannian manifolds. We say that they are equivalent, if there exists an affine transformation φ such that $\varphi G \varphi^{-1} = G'$, that is, if G and G' are conjugate in the group of affine transformations of $\mathbf{R}^n$. In addition to (4) of Theorem 4.2 of Chapter V, Bieberbach [1] obtained the following results:

(a) *If G and G' are isomorphic as abstract group, then $\mathbf{R}^n/G$ and $\mathbf{R}^n/G'$ are equivalent.*

(b) *For each n, there are only a finite number of equivalence classes of compact flat Riemannian manifolds $\mathbf{R}^n/G$.*

We shall sketch here an outline of the proof. We denote by (A, p) an affine transformation of $\mathbf{R}^n$ with linear part A and translation part p. Let N be the subgroup of G consisting of pure translations and let $(I, t_1), \ldots, (I, t_n)$ be a basis of N, where I is the identity matrix and $t_i \in \mathbf{R}^n$. Since $(A, p)(I, t_i)(A, p)^{-1} = (I, At_i) \in N$ for any $(A, p) \in G$, we can write $At_i = \sum_{j=1}^n a_i^j t_j$, where each a_i^j is an integer. Let T be an $(n \times n)$-matrix whose i-th column is given by t_i, that is, $T = (t_1 \ldots t_n)$. Then $(a_i^j) = T^{-1}AT$ is unimodular. (A matrix is called unimodular if it is non-singular and integral together with its inverse.)

To prove (a), let $(A', p') \in G'$ be the element corresponding to $(A, p) \in G$ by the isomorphism $G' \approx G$. Let N' be the subgroup of G' corresponding to N by the isomorphism $G' \approx G$. Then N' is normal and maximal abelian in G'. Hence N' is the subgroup of G' consisting of pure translations. Let (I, t_i') correspond to (I, t_i). Since $(A', p')(I, t_i')(A', p')^{-1} = (I, A't_i')$, $(I, A't_i')$ corresponds to (I, At_i). Hence we have $A't_i' = \sum_{j=1}^n a_i^j t_j'$. In other words, if we set $T' = (t_1' \ldots t_n')$, then $T'^{-1}A'T' = T^{-1}AT$. Set

$$G^* = \{(T^{-1}AT,\ T^{-1}p - T'^{-1}p');\quad (A, p) \in G\}.$$

Then G^* is a group which contains no pure translations and hence is finite. Let $u \in \mathbf{R}^n$ be a point left fixed by G^*. Then we have

$$(T, Tu)^{-1}(A, p)(T, Tu) = (T', 0)^{-1}(A', p')(T, 0) \qquad \text{for all } (A, p) \in G.$$

This completes the proof of (a).

To prove (b), it suffices to show that there are only a finite number of mutually non-isomorphic groups G such that $\mathbf{R}^n/G$ are compact flat Riemannian manifolds. Each G determines a group extension

$$0 \to N \to G \to K \to 1,$$

where the finite group $K = G/N$ acts linearly on N when N is considered as a subgroup of $\mathbf{R}^n$. Given such a finite group K, the set of group extensions $0 \to N \to G \to K \to 1$ is given by $H^2(K, N)$. Since K is finite and N is finitely generated, $H^2(K, N)$ is finite. As we have seen in the proof of (a), if we identify N with the integral lattice points of $\mathbf{R}^n$, then $K = G/N$ is given by unimodular matrices. Let K and K' be two finite groups of unimodular matrices of degree n which are conjugate in the group $GL(n; \mathbf{Z})$ of all unimodular matrices so that $SKS^{-1} = K'$ for some $S \in GL(n; \mathbf{Z})$. The mapping which sends $t \in N$ into $St \in N$ is an automorphism of N. Hence S induces an isomorphism $H^2(K, N) \approx H^2(K', N)$, and if $0 \to N \to G' \to K' \to 1$ is the element of $H^2(K', N)$ corresponding to an element $0 \to N \to G \to K \to 1$ of $H^2(K, N)$, then G and G' are isomorphic. Thus our problem is reduced to the following theorem of Jordan:

There are only a finite number of conjugate classes of finite subgroups of $GL(n; \mathbf{Z})$.

This theorem of Jordan follows from the theory of Minkowski-Siegel. Let H_n be the space of all real symmetric positive definite matrices of degree n. Then $GL(n; \mathbf{Z})$ acts properly discontinuously on H_n as follows:

$$X \to {}^tSXS \qquad \text{for } X \in H_n \quad \text{and} \quad S \in GL(n; \mathbf{Z}).$$

Let R be the subset of H_n consisting of reduced matrices in the sense of Minkowski. Denote ${}^tSXS$ by $S[X]$. Then

(i) $\bigcup_{S \in GL(n;\mathbf{Z})} S[R] = H_n$;

(ii) The set F defined by $F = \{S \in GL(n; \mathbf{Z});\ S[R] \cap R = \text{non-empty}\}$ is a finite set.

The first property of R implies that any finite subgroup K of $GL(n; \mathbf{Z})$ is conjugate to a subgroup of $GL(n; \mathbf{Z})$ contained in F. Indeed, let $X_0 \in H_n$ be a fixed point of K (for instance, set $X_0 = \Sigma_{A \in K}\, {}^tAA$). Let $S \in GL(n; \mathbf{Z})$ be such that $S[X_0] \in R$. Then $S^{-1}KS \subset F$. Since F is finite, there are only a finite number of conjugate classes of finite subgroups of $GL(n; \mathbf{Z})$. QED.

As references we mention Minkowski [1], Bieberbach [2], Bieberbach and Schur [1] and Siegel [1].

Note that (a) implies that two compact flat Riemannian manifolds are equivalent if and only if they are homeomorphic to each other. Although (b) does not hold for non-compact flat Riemannian manifolds, there are only a finite number of homeomorphism classes of complete flat Riemannian manifolds for each dimension (Bieberbach [3]).

For the classification of 3-dimensional complete flat Riemannian manifolds, see Hantzche and Wendt [1] and Nowacki [1].

Most of the results for flat Riemannian manifolds cannot be generalized to flat affine connections, see, for example, Auslander [1].

Note 6. Parallel displacement of curvature

Let M and M' be Riemannian manifolds and $\varphi\colon M \to M'$ a diffeomorphism which preserves the curvature tensor fields. In general, this does not imply the existence of an isometry of M onto M'. For instance, let M be a compact Riemannian manifold obtained by attaching a unit hemisphere to each end of the right circular cylinder $S^1 \times [0, 1]$, where S^1 is the unit circle, and then smoothing out the corners. Similarly, let M' be a compact Riemannian manifold obtained by attaching a unit hemisphere to each end of the right circular cylinder $S^1 \times [0, 2]$ and then smoothing out the corners in the same way. Let $\varphi\colon M \to M'$ be a diffeomorphism which induces an isometry on the attached hemispheres and their neighborhoods. Since the cylinder parts of M and M' are flat, φ preserves the curvature tensor fields. However, M and M' cannot be isometric with each other.

Ambrose [1] obtained the following result, which generalizes Theorem 7.4 of Chapter VI in the Riemannian case.

Let M and M' be complete, simply connected Riemannian manifolds, x an arbitrarily fixed point of M and x' an arbitrarily fixed point of M'. Let $f\colon T_x(M) \to T_{x'}(M')$ be a fixed orthogonal transformation. Let τ be a simply broken geodesic of M from x to a point y and τ' the corresponding simply broken geodesic of M' from x' to a point y', the correspondence being given by f through parallel displacement. Let p (resp. p') be a plane in $T_x(M)$ (resp. $T_{x'}(M')$) and q (resp. q') the plane in $T_y(M)$ (resp. $T_{y'}(M')$) obtained from p (resp. p') by parallel displacement along τ (resp. τ'). Assume that p' corresponds to p by f. If the sectional curvature $K(q)$ is equal to the sectional curvature $K'(q')$ for all simply broken geodesics τ and all planes p in $T_x(M)$, then there exists a unique isometry $F\colon M \to M'$ whose differential at x coincides with f.

Hicks [1] obtained a similar result in the case of affine connection; his result generalizes Theorem 7.4 of Chapter VI.

Note 7. Symmetric spaces

Although the theory of symmetric spaces, in particular, Riemannian symmetric spaces, will be taken up in detail in Volume II, we shall give here its definition and basic properties.

Let G be a connected Lie group with an involutive automorphism σ ($\sigma^2 = 1$, $\sigma \neq 1$). Let H be a closed subgroup which lies between the (closed) subgroup of all fixed points of σ and its identity component. We shall then say that G/H is a *symmetric homogeneous space* (defined by σ). Denoting by the same letter σ the involutive automorphism of the Lie algebra $\mathfrak{g}$ induced by σ, we have $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (direct sum), where $\mathfrak{h} = \{X \in \mathfrak{g};\ X^\sigma = X\}$ coincides with the subalgebra corresponding to H and $\mathfrak{m} = \{X \in \mathfrak{g};\ X^\sigma = -X\}$. We have obviously $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

The automorphism σ of G also induces an involutive diffeomorphism σ_o of G/H such that $\sigma_o(\pi x) = \pi(x^\sigma)$ for every $x \in G$, where π is the canonical projection of G onto G/H. The origin $o = \pi(e)$ of G/H is then an isolated fixed point of σ_o. We call σ_o the *symmetry* around o.

By Theorem 11.1 of Chapter II, the bundle $G(G/H, H)$ admits an invariant connection Γ determined by the subspace $\mathfrak{m}$. We call this connection the *canonical connection* in $G(G/H, H)$.

THEOREM 1. *For a symmetric space G/H, the canonical connection Γ in $G(G/H, H)$ has the following properties:*

(1) *Γ is invariant by the automorphism σ of G (which is a bundle automorphism of $G(G/H, H)$);*

(2) *The curvature form is given by $\Omega(X, Y) = -(1/2)[X, Y] \in \mathfrak{h}$, where X and Y are arbitrary left invariant vector fields belonging to $\mathfrak{m}$;*

(3) *For any $X \in \mathfrak{m}$, let $a_t = \exp tX$ and let $x_t = \pi(a_t) = a_t(o)$. The parallel displacement of the fibre H along the curve x_t coincides with the left translation $h \to a_t h$, $h \in H$.*

Proof. (1) follows easily from $\mathfrak{m}^\sigma = \mathfrak{m}$. (2) is contained in Theorem 11.1 of Chapter II. (3) follows from the fact that $a_t h$ for any fixed $h \in H$ is the horizontal lift through h of the curve x_t. QED.

The projection π gives a linear isomorphism of the horizontal subspace $\mathfrak{m}$ at e of Γ onto the tangent space $T_o(G/H)$ at the origin o. If $h \in H$, then ad (h) on $\mathfrak{m}$ corresponds by this isomorphism to the linear isotropy $\tilde{h}$, i.e., the linear transformation of $T_o(G/H)$ induced by the transformation h of G/H which fixes o.

Now, denoting G/H by M, we define a mapping f of G into the bundle of frames $L(M)$ over M as follows. Let u_0 be an arbitrarily fixed frame $X_1, \ldots, X_n$ at o, which can be identified with a certain basis of $\mathfrak{m}$. For any $a \in G$, $f(a)$ is the frame at $a(o)$ consisting of the images of X_i by the differential of a. In particular, for $h \in H$, $f(h) = h \cdot u_0 = u_0 \cdot \varphi(h)$, where $\varphi(h) \in GL(n; \mathbf{R})$ is the matrix which represents the linear transformation of $T_o(M)$ induced by h with respect to the basis u_0. It is easy to see that f is a bundle homomorphism of G into $L(M)$ corresponding to the homomorphism φ of H into $GL(n; \mathbf{R})$. If G is effective on G/H (or equivalently, if H contains no non-trivial invariant subgroup of G), then f and φ are isomorphisms.

By Proposition 6.1, of Chapter II, the canonical connection Γ in $G(M, H)$ induces a connection in $L(M)$, which we shall call the *canonical linear connection* on G/H and denote still by Γ.

THEOREM 2. *The canonical linear connection on a symmetric space G/H has the following properties:*

(1) *Γ is invariant by G as well as the symmetry σ_o around o;*

(2) *The restricted homogeneous holonomy group of Γ at o is contained in the linear isotropy group $\tilde{H}$;*

(3) *For any $X \in \mathfrak{m}$, let $a_t = \exp tX$ and $x_t = \pi(a_t) = a_t(o)$. The parallel displacement of vectors along x_t is the same as the transformation by a_t. In particular, x_t is a geodesic;*

(4) *The torsion tensor field is 0;*

(5) *Every G-invariant tensor field on G/H is parallel with respect to Γ. In particular, the curvature tensor field R is parallel, i.e., $\nabla R = 0$.*

Proof. (1), (2) and (3) follow from the corresponding properties in Theorem 1. (4) follows from (1); since the torsion tensor field T is invariant by σ_o, we have $T(X, Y) = (T(X^{\sigma_o}, Y^{\sigma_o}))^{\sigma_o} = -T(-X, -Y) = -T(X, Y)$ and hence $T(X, Y) = 0$ for any X and Y in $T_o(M)$. Thus $T = 0$ at o and hence everywhere. (5) follows from (3). In fact, if K is a G-invariant tensor field, then $\nabla_{X_0} K = 0$ for any $X_0 \in T_o(M)$, since there exists $X \in \mathfrak{m}$ such that x_t in (3) has the initial tangent vector X_0. QED.

Remark. Γ is the unique linear connection on G/H which has property (1). This justifies the name of canonical linear connection.

Let G/H be a symmetric space with compact H. There exists a G-invariant Riemannian metric on G/H. For any such metric g, the Riemannian connection coincides with Γ. In fact, the metric tensor field g is parallel with respect to Γ by (5). Since Γ has zero torsion, it is the Riemannian connection by the uniqueness (Theorem 2.2, Chapter II).

Example. In $G = SO(n+1)$, let σ be the involutive automorphism $A \in SO(n+1) \to SAS^{-1} \in SO(n+1)$ where S is the matrix of the form $\begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$ with identity matrix I_n of degree n. The identity component H^0 of the subgroup H of fixed points of σ consists of all matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$, where $B \in SO(n)$. We shall write $SO(n)$ for H^0 with this understanding. The symmetric homogeneous space $SO(n+1)/SO(n)$ is naturally diffeomorphic with the unit sphere S^n in $\mathbf{R}^{n+1}$. In fact, let $e_0, e_1, \ldots, e_n$ be the standard orthonormal basis in $\mathbf{R}^{n+1}$. The mapping $A \in SO(n+1) \to Ae_0 \in S^n$ induces a diffeomorphism of $SO(n+1)/SO(n)$ onto S^n. The set of vectors $Ae_1, \ldots, Ae_n$ can be considered as an orthonormal frame at the point Ae_0 of S^n. This gives an isomorphism of the bundle $SO(n+1)$ over $SO(n+1)/SO(n)$ onto the bundle of orthonormal frames over S^n. The canonical linear connection on $SO(n+1)/SO(n)$ coincides with the Riemannian connection of S^n with respect to the Riemannian metric of S^n as imbedded submanifold of $\mathbf{R}^{n+1}$.

A linear connection Γ on a differentiable manifold M is said to be *locally symmetric* at $x \in M$, if there exists an involutive affine transformation of an open neighborhood U of x which has x as an isolated fixed point. This local symmetry at x, if it exists, must be of the form $(x^i) \to (-x^i)$ with respect to any normal coordinate system with origin x, since it induces the linear transformation $X \to -X$ in $T_x(M)$. We say that Γ is *locally symmetric*, if it is locally symmetric at every point x of M.

THEOREM 3. *A linear connection Γ on M is locally symmetric if and only if $T = 0$ and $\nabla R = 0$.*

Proof. If Γ is locally symmetric, then any tensor field of type (r, s) with odd $r + s$ which is invariant by the local symmetry at x is 0 at x. Hence $T = 0$ and $\nabla R = 0$ on M. The converse follows from Theorem 7.4 of Chapter VI. QED.

THEOREM 4. *Let Γ be a locally symmetric linear connection on M. If M is connected, simply connected and complete, then the group $\mathfrak{A}(M)$ of all affine transformations is transitive on M. Let $G = \mathfrak{A}^0(M)$. Then $M = G/H$ is a symmetric space for which Γ is the canonical linear connection.*

Proof. The first assertion follows from Corollary 7.9 of Chapter VI. Let σ_o be the local symmetry at a point o of M. By Corollary 6.2, σ_o can be extended to an affine transformation of M onto itself which is involutive. Define an involutive automorphism of G by $a^\sigma = \sigma_o \circ a \circ \sigma_o$. Then H lies between the subgroup of all fixed elements of σ and its identity component. QED.

The Riemannian versions of Theorems 3 and 4 are obvious.

The Riemannian symmetric spaces were introduced and studied extensively by Cartan [7]. For the canonical linear connection on symmetric G/H, see Nomizu [2] and Kobayashi [3]. Nomizu [4, 6] proved the converse of (2) of Theorem 2 that if the restricted linear holonomy group of a complete Riemannian manifold M is contained in the linear isotropy group at every point, then M is locally symmetric. Simons [1] has a similar theorem.

Nomizu and Ozeki [3] proved that, for any complete Riemannian manifold M, the condition $\nabla^m R = 0$ for some $m > 1$ implies $\nabla R = 0$. (This was known by Lichnerowicz [3, p. 4] when M is compact.) They remarked later that the assumption of completeness is not necessary.

Note 8. Linear connections with recurrent curvature

Let M be an n-dimensional manifold with a linear connection Γ. A non-zero tensor field K of type (r, s) on M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$. The following result is due to Wong [1].

THEOREM 1. *In the notation of* §5 *of Chapter* III, *let f: $L(M) \to \mathbf{T}^r_s(\mathbf{R}^n)$ be the mapping which corresponds to a given tensor field K of type (r, s). Then K is recurrent if and only if, for the holonomy bundle $P(u_0)$ through any $u_0 \in L(M)$, there exists a differentiable function $\varphi(u)$ with no zero on $P(u_0)$ such that*

$$f(u) = \varphi(u) \cdot f(u_0) \qquad \textit{for } u \in P(u_0).$$

As a special case, K is parallel if and only if $f(u)$ is constant on $P(u_0)$.

Using this result and the holonomy theorem (Theorem 8.1 of Chapter II), Wong obtained

THEOREM 2. *Let Γ be a linear connection on M with recurrent curvature tensor R. Then the Lie algebra of its linear holonomy group $\Psi(u_0)$ is spanned by all elements of the form $\Omega_{u_0}(X, Y)$, where Ω is the curvature form and X and Y are horizontal vectors at u_0. In particular, we have*

$$\dim \Psi(u_0) \leqq \tfrac{1}{2}n(n-1).$$

As an application of Theorem 1, we shall sketch the proof of the following

THEOREM 3. *For a Riemannian manifold M with recurrent curvature tensor whose restricted linear holonomy group is irreducible, the curvature tensor is necessarily parallel provided that* $\dim M \geqq 3$.

Proof. Let R^i_{jkl} be the components of the $\mathbf{T}^1_3(\mathbf{R}^n)$-valued function on $O(M)$ which corresponds to the curvature tensor field R. We apply Theorem 1 to R. Since $\Sigma_{i,j,k,l}\,(R^i_{jkl})^2$ is constant on each fibre of $O(M)$, φ^2 is constant on each fibre of $P(u_0)$. Since φ never vanishes on $P(u_0)$, it is either always positive or always negative. Hence φ itself is constant on each fibre of $P(u_0)$. Let λ be the function on M defined by $\lambda(x) = 1/\varphi(u)$, where $x = \pi(u) \in M$. Then λR is a parallel tensor field. If we denote by S the Ricci tensor field, then λS is also parallel. The irreducibility of M implies that $\lambda S = c \cdot g$, where c is a constant and g is the metric tensor (cf. Theorem 1 of Appendix 5). If $\dim M \geqq 3$ and if the Ricci tensor S is non-trivial, then λ is a constant function by Theorem 1 of Note 3. Since λR is parallel and since λ is a constant, R is parallel.

Next we shall consider the case where the Ricci tensor S vanishes identically. Let $\nabla R = R \otimes \alpha$ and let R^i_{jkl} and α_m be the components of R and α with respect to a local coordinate system $x^1, \ldots, x^n$. By Bianchi's second identity (Theorem 5.3 of Chapter III; see also Note 3), we have

$$R^i_{jkl}\alpha_m + R^i_{jlm}\alpha_k + R^i_{jmk}\alpha_l = 0.$$

Multiply by g^{jm} and sum with respect to j and m. Since the Ricci tensor vanishes identically, we have $\Sigma_{j,m}g^{jm}R^i_{jlm} = \Sigma_{j,m}g^{jm}R^i_{jmk} = 0$. Hence,

$$\Sigma_j R^i_{jkl}\alpha^j = 0, \quad \text{where} \quad \alpha^j = \Sigma_m g^{jm}\alpha_m.$$

This equation has the following geometric implication. Let x be an arbitrarily fixed point of M and let X and Y be any vectors at x. If we denote by V the vector at x with components $\alpha^j(x)$, then the linear transformation $R(X, Y)\colon T_x(M) \to T_x(M)$ maps V into the zero vector. On the other hand, by the Holonomy Theorem (Theorem 8.1 of Chapter II) and Theorem 1 of this Note (see also Wong [1]), the Lie algebra of the linear holonomy group $\Psi(x)$ is spanned by the set of all endomorphisms of $T_x(M)$ given by $R(X, Y)$ with $X, Y \in T_x(M)$. It follows that V is invariant by $\Psi(x)$ and hence is zero by the irreducibility of $\Psi(x)$. Consequently, ∇R vanishes at x. Since x is an arbitrary point of M, R is parallel. QED.

On the other hand, every non-flat 2-dimensional Riemannian manifold is of recurrent curvature if the sectional curvature does not vanish anywhere.

Corollary. *If M is a complete Riemannian manifold with recurrent curvature tensor, then the universal covering manifold $\tilde{M}$ of M is either a symmetric space or a direct product of the Euclidean space $\mathbf{R}^{n-2}$ and a 2-dimensional Riemannian manifold.*

Proof. Use the decomposition theorem of de Rham (Theorem 6.2 of Chapter IV) and Theorem 3 above together with the following fact which can be verified easily. Let M and M' be manifolds with linear connections and let R and R' be their curvature tensors, respectively. If the curvature tensor of $M \times M'$ is recurrent, then there are only three possibilities: (1) $\nabla R = 0$ and $\nabla R' = 0$; (2) $R = 0$ and $\nabla R' \neq 0$; (3) $\nabla R \neq 0$ and $R' = 0$. (See also Walker [1].) QED.

Note 9. The automorphism group of a geometric structure

Given a differentiable manifold M, the group of all differentiable transformations of M is a very large group. However, the group of differentiable transformations of M leaving a certain geometric structure is often a Lie group. The first result of this nature was given by H. Cartan [1] who proved that the group of all complex analytic transformations of a bounded domain in $\mathbf{C}^n$ is a Lie group. Myers and Steenrod [1] proved that the group of all isometries of a

Riemannian manifold is a Lie group. Bochner and Montgomery [1, 2] proved that the group of all complex analytic transformations of a compact complex manifold is a complex Lie group; they made use of a general theorem concerning a locally compact group of differentiable transformations which is now known to be valid in the form of Theorem 4.6, Chapter I. The theorem that the group of all affine transformations of an affinely connected manifold is a Lie group was first proved by Nomizu [1] under the assumption of completeness; this assumption was later removed by Hano and Morimoto [1]. Kobayashi [1, 6] proved that the group of all automorphisms of an absolute parallelism is a Lie group by imbedding it into the manifold. This method can be applied to the absolute parallelism of the bundle of frames $L(M)$ of an affinely connected manifold M (cf. Proposition 2.6 of Chapter III and Theorem 1.5 of Chapter VI).

Automorphisms of a complex structure and a Kählerian structure will be discussed in Volume II.

A global theory of Lie transformation groups was studied in Palais [1]. We shall here state one theorem which has a direct bearing on us. Let G be a certain group of differentiable transformations acting on a differentiable manifold M. Let $\mathfrak{g}'$ be the set of all vector fields X on M which generate a global 1-parameter group of transformations which belong to the given group G. Let $\mathfrak{g}$ be the Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$ generated by $\mathfrak{g}'$.

THEOREM. *If* $\mathfrak{g}$ *is finite-dimensional, then* G *admits a Lie group structure (such that the mapping* $G \times M \rightarrow M$ *is differentiable) and* $\mathfrak{g} = \mathfrak{g}'$. *The Lie algebra of* G *is naturally isomorphic with* $\mathfrak{g}$.

We have the following applications of this result. If G is the group of all affine transformations (resp. isometries) of an affinely connected (resp. Riemannian) manifold M, then $\mathfrak{g}'$ is the set of all infinitesimal affine transformations (resp. infinitesimal isometries) which are globally integrable (note that if M is complete, these infinitesimal transformations are always globally integrable by Theorem 2.4 of Chapter VI). By virtue of Theorem 2.3 (resp. Theorem 3.3) of Chapter VI, it follows that $\mathfrak{g}$ is finite-dimensional. By the theorem above, G is a Lie group.

The Lie algebra $\mathfrak{i}(M)$ of all infinitesimal isometries of a Riemannian manifold M was studied in detail by Nomizu [8, 9]. At each point x of M, a certain Lie algebra $\mathfrak{i}(x)$ is constructed by using the curvature tensor field and its covariant differentials. If M is simply connected and analytic together with the metric, then $\mathfrak{i}(M)$ is naturally isomorphic with $\mathfrak{i}(x)$, where x is an arbitrary point.

Note 10. Groups of isometries and affine transformations with maximum dimensions

In Theorem 3.3 of Chapter VI, we proved that the group $\mathfrak{I}(M)$ of isometries of a connected, n-dimensional Riemannian manifold M is of dimension at most $\frac{1}{2}n(n+1)$ and that if $\dim \mathfrak{I}(M) = \frac{1}{2}n(n+1)$, then M is a space of constant curvature. We shall outline the proof of the following theorem.

THEOREM 1. *Let M be a connected, n-dimensional Riemannian manifold. If* $\dim \mathfrak{I}(M) = \frac{1}{2}n(n+1)$, *then M is isometric to one of the following spaces of constant curvature:*

(a) *An n-dimensional Euclidean space $\mathbf{R}^n$;*
(b) *An n-dimensional sphere S^n;*
(c) *An n-dimensional real projective space $S^n/\{\pm I\}$;*
(d) *An n-dimensional, simply connected hyperbolic space.*

Proof. From the proof of Theorem 3.3 of Chapter VI, we see that M is homogeneous and hence is complete. The universal covering space $\tilde{M}$ of M is isometric to one of (a), (b) and (d) above (cf. Theorem 7.10 of Chapter VI). Every infinitesimal isometry X of M induces an infinitesimal isometry $\tilde{X}$ of $\tilde{M}$. Hence, $\frac{1}{2}n(n+1) = \dim \mathfrak{I}(M) \leq \dim \mathfrak{I}(\tilde{M}) \leq \frac{1}{2}n(n+1)$, which implies that every infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ is induced by an infinitesimal isometry X of M. If $\tilde{M}$ is isometric to (a) or (d), then there exists an infinitesimal isometry $\tilde{X}$ of $\tilde{M}$ which vanishes only at a single point of $\tilde{M}$. Hence, M is simply connected in case the curvature is nonpositive. If $\tilde{M}$ is isometric to a sphere S^n for any antipodal points x and x', there exists an infinitesimal isometry $\tilde{X}$ of $\tilde{M} = S^n$ which vanishes only at x and x'. This implies that $M = S^n$ or $M = S^n/\{\pm I\}$. We see easily that if M is isometric to the projective space $S^n/\{\pm I\}$, then $\mathfrak{I}(M)$ is isomorphic to $O(n+1)$ modulo its center and hence of dimension $\frac{1}{2}n(n+1)$.
QED.

In Theorem 2.3 of Chapter VI, we proved that the group

$\mathfrak{A}(M)$ of affine transformations of a connected, n-dimensional manifold M with an affine connection is of dimension at most $n^2 + n$ and that if $\dim \mathfrak{A}(M) = n^2 + n$, then the connection is flat. We prove

THEOREM 2. *If* $\dim \mathfrak{A}(M) = n^2 + n$, *then M is an ordinary affine space with the natural flat affine connection.*

Proof. Every element of $\mathfrak{A}(M)$ induces a transformation of $L(M)$ leaving the canonical form and the connection form invariant (cf. §1 of Chapter VI). From the fact that $\mathfrak{A}(M)$ acts freely on $L(M)$ and from the assumption that $\dim \mathfrak{A}(M) = n^2 + n = \dim L(M)$, it follows that $\mathfrak{A}^0(M)$ is transitive on each connected component of $L(M)$. This implies that every standard horizontal vector field on $L(M)$ is complete; the proof is similar to that of Theorem 2.4 of Chapter VI. In other words, the connection is complete. By Theorem 4.2 of Chapter V or by Theorem 7.8 of Chapter VI, the universal covering space $\tilde{M}$ of M is an ordinary affine space. Finally, the fact that $\tilde{M} = M$ can be proved in the same way as Theorem 1 above. QED.

Theorems 2.3 and 3.3 are classical (see, for instance, Eisenhart [1]).

Riemannian manifolds and affine connections admitting very large groups of automorphisms have been studied by Egorov, Wang, Yano and others. The reader will find references on the subject in the book of Yano [2].

Note 11. *Conformal transformations of a Riemannian manifold*

Let M be a Riemannian manifold with metric tensor g. A transformation φ of M is said to be *conformal* if $\varphi^*g = \rho g$, where ρ is a positive function on M. If ρ is a constant function, φ is a homothetic transformation. If ρ is identically equal to 1, φ is nothing but an isometry. An infinitesimal transformation X of M is said to be *conformal* if $L_X g = \sigma g$, where σ is a function on M. It is homothetic if σ is a constant function, and it is isometric if $\sigma = 0$. The local 1-parameter group of local transformations generated by an infinitesimal transformation X is conformal if and only if X is conformal.

THEOREM 1. *The group of conformal transformations of a connected, n-dimensional Riemannian manifold M is a Lie transformation group of dimension at most* $\frac{1}{2}(n+1)(n+2)$, *provided* $n \geqq 3$.

This can be proved along the following line. The integrability conditions of $L_X g = \sigma g$ imply that the Lie algebra of infinitesimal conformal transformation X is of dimension at most $\frac{1}{2}(n+1)(n+2)$ (cf. for instance, Eisenhart [1, p. 285]). By the theorem of Palais cited in Note 9, the group of conformal transformations is a Lie transformation group.

In §3 of Chapter VI we showed that, for almost all Riemannian manifolds M, the largest connected group $\mathfrak{A}^0(M)$ of affine transformations of M coincides with the largest connected group $\mathfrak{I}_0(M)$ of isometries of M. For the largest connected group $\mathfrak{C}^0(M)$ of conformal transformations of M, we have the following several results in the same direction.

THEOREM 2. *Let M be a connected n-dimensional Riemannian manifold for which* $\mathfrak{C}^0(M) \neq \mathfrak{I}^0(M)$. *Then,*

(1) *If M is compact, there is no harmonic p-form of constant length for* $1 \leqq p < n$ (Goldberg and Kobayashi [1]);

(2) *If M is compact and homogeneous, then M is isometric to a sphere provided* $n > 3$ (Goldberg and Kobayashi [2]);

(3) *If M is a complete Riemannian manifold of dimension $n \geqq 3$ with parallel Ricci tensor, then M is isometric to a sphere* (Nagano [1]);

(4) *M cannot be a compact Riemannian manifold with constant non-positive scalar curvature* (Yano [2; p. 279] and Lichnerowicz [3; p. 134]).

(3) is an improvement of the result of Nagano and Yano [1] to the effect that if M is a complete Einstein space of dimension $\geqq 3$ for which $\mathfrak{C}^0(M) \neq \mathfrak{I}^0(M)$, then M is isometric to a sphere. Nagano [1] made use of a result of Tanaka [1].

On the other hand, it is easy to construct Riemannian manifolds (other than spheres) for which $\mathfrak{C}^0(M) \neq \mathfrak{I}^0(M)$. Indeed, let M be a Riemannian manifold with metric tensor g which admits a 1-parameter group of isometries. Let ρ be a positive function on M which is not invariant by this 1-parameter group of isometries. Then, with respect to the new metric ρg, this group is a 1-parameter group of non-isometric, conformal transformations.

To show that $\dim \mathfrak{C}^0(M) = \frac{1}{2}(n+1)(n+2)$ for a sphere M of dimension n, we imbed M into the real projective space of dimension $n+1$. Let $x^0, x^1, \ldots, x^{n+1}$ be a homogeneous coordinate system of the real projective space P_{n+1} of dimension $n+1$. Let M be the n-dimensional sphere in $\mathbf{R}^{n+1}$ defined by $(y^1)^2 + \cdots + (y^{n+1})^2 = 1$. We imbed M into P_{n+1} by means of the mapping defined by

$$x_0 = \frac{1}{\sqrt{2}}(1 + y^{n+1}), \quad x^1 = y^1, \ldots, x^n = y^n, \quad x^{n+1} = \frac{1}{\sqrt{2}}(1 - y^{n+1}).$$

The image of M in P_{n+1} is given by

$$(x^1)^2 + \cdots + (x^n)^2 - 2x^0x^{n+1} = 0.$$

Let h be the Riemannian metric on P_{n+1} given by

$$p^*h = 2\,\frac{(\Sigma_{i=0}^{n+1}(x^i)^2)(\Sigma_{i=0}^{n+1}(dx^i)^2) - (\Sigma_{i=0}^{n+1} x^i\, dx^i)^2}{(\Sigma_{i=0}^{n+1}(x^i)^2)^2},$$

where p is the natural projection from $\mathbf{R}^{n+2} - \{0\}$ onto P_{n+1}. Then the imbedding $M \to P_{n+1}$ is isometric. Let G be the group of linear transformations of $\mathbf{R}^{n+2}$ leaving the quadratic form $(x^1)^2 + \cdots + (x^n)^2 - 2x^0x^{n+1}$ invariant. Then G maps the image of M in P_{n+1} onto itself. It is easy to verify that, considered as a transformation group acting on M, G is a group of conformal transformations of dimension $\frac{1}{2}(n+1)(n+2)$.

The case $n = 2$ is exceptional in most of the problems concerning conformal transformations for the following reason. Let M be a complex manifold of complex dimension 1 with a local coordinate system $z = x + iy$. Let g be a Riemannian metric on M which is of the form

$$f(dx^2 + dy^2) = f\,dz\,d\bar{z},$$

where f is a positive function on M. Then every complex analytic transformation of M is conformal.

SUMMARY OF BASIC NOTATIONS

We summarize only those basic notations which are used most frequently throughout the book.

1. Σ_i, $\Sigma_{i,j,\dots}$, etc., stand for the summation taken over i or $i, j, \dots$, where the range of indices is generally clear from the context.

2. **R** and **C** denote the real and complex number fields, respectively.

$\mathbf{R}^n$: vector space of n-tuples of real numbers $(x^1, \dots, x^n)$
$\mathbf{C}^n$: vector space of n-tuples of complex numbers $(z^1, \dots, z^n)$
(x, y): standard inner product $\Sigma_i x^i y^i$ in $\mathbf{R}^n$ ($\Sigma_i x^i \bar{y}^i$ in $\mathbf{C}^n$)
$GL(n; \mathbf{R})$: general linear group acting on $\mathbf{R}^n$
$\mathfrak{gl}(n; \mathbf{R})$: Lie algebra of $GL(n; \mathbf{R})$
$GL(n; \mathbf{C})$: general linear group acting on $\mathbf{C}^n$
$\mathfrak{gl}(n; \mathbf{C})$: Lie algebra of $GL(n; \mathbf{C})$
$O(n)$: orthogonal group
$\mathfrak{o}(n)$: Lie algebra of $O(n)$
$U(n)$: unitary group
$\mathfrak{u}(n)$: Lie algebra of $U(n)$
$\mathbf{T}^r_s(V)$: tensor space of type (r, s) over a vector space V
$\mathbf{T}(V)$: tensor algebra over V
A^n: space $\mathbf{R}^n$ regarded as an affine space
$A(n; \mathbf{R})$: group of affine transformations of A^n
$\mathfrak{a}(n; \mathbf{R})$: Lie algebra of $A(n; \mathbf{R})$

3. M denotes an n-dimensional differentiable manifold.

$T_x(M)$: tangent space of M at x
$\mathfrak{F}(M)$: algebra of differentiable functions on M
$\mathfrak{X}(M)$: Lie algebra of vector fields on M
$\mathfrak{T}(M)$: algebra of tensor fields on M
$\mathfrak{D}(M)$: algebra of differential forms on M
$T(M)$: tangent bundle of M
$L(M)$: bundle of linear frames of M

$O(M)$: bundle of orthonormal frames of M (with respect to a given Riemannian metric)
$\theta = (\theta^i)$: canonical 1-form on $L(M)$ or $O(M)$
$A(M)$: bundle of affine frames of M
$T^r_s(M)$: tensor bundle of type (r, s) of M
f_*: differential of a differentiable mapping f
$f^*\omega$: the transform of a differential form ω by f
$\dot{x}_t$: tangent vector of a curve x_t, $0 \leqq t \leqq 1$, at the point x_t
L_X: Lie differentiation with respect to a vector field X

4. For a Lie group G, G^0 denotes the identity component and $\mathfrak{g}$ the Lie algebra of G.
L_a: left translation by $a \in G$
R_a: right translation by $a \in G$
ad a: inner automorphism by $a \in G$; also adjoint representation in $\mathfrak{g}$
$P(M, G)$: principal fibre bundle over M with structure group G
A^*: fundamental vector field corresponding to $A \in \mathfrak{g}$
$\omega = (\omega^i_j)$: connection form
$\Omega = (\Omega^i_j)$: curvature form
$E(M, F, G, P)$: bundle associated to $P(M, G)$ with fibre F

5. For an affine (linear) connection Γ on M,
$\Theta = (\Theta^i_j)$: torsion form
Γ^i_{jk}: Christoffel's symbols
$\Psi(x)$: linear holonomy group at $x \in M$
$\Phi(x)$: affine holonomy group at $x \in M$
∇_X: covariant differentiation with respect to a vector (field) X
R: curvature tensor field (with components R^i_{jkl})
T: torsion tensor field (with components T^i_{jk})
S: Ricci tensor field (with components R_{ij})
$\mathfrak{A}(M)$: group of all affine transformations
$\mathfrak{a}(M)$: Lie algebra of all infinitesimal affine transformations
$\mathfrak{I}(M)$: group of all isometries
$\mathfrak{i}(M)$: Lie algebra of all infinitesimal isometries

BIBLIOGRAPHY

AMBROSE, W.

[1] Parallel translation of Riemannian curvature, Ann. of Math. 64 (1956), 337–363.

AMBROSE, W. and SINGER, I. M.

[1] A theorem on holonomy, Trans. Amer. Math. Soc. 75 (1953), 428–443.

ARTIN, E.

[1] Geometric Algebra, Interscience Tracts #3. Interscience, New York, 1957.

ARENS, R.

[1] Topologies for homeomorphism groups, Ann. of Math. 68 (1946), 593–610.

AUSLANDER, L.

[1] Examples of locally affine spaces, Ann. of Math. 64 (1956), 255–259.

AUSLANDER, L. and KURANISHI, M.

[1] On the holonomy group of locally Euclidean spaces, Ann. of Math. 65 (1957), 411–415.

AUSLANDER, L. and MARKUS, L.

[1] Holonomy of flat affinely connected manifolds, Ann. of Math. 62 (1955), 139–151.

BERGER, M.

[1] Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math, France 83 (1953), 279–330.

BERNARD, D.

[1] Sur la géométrie différentielle des G-structures, Ann. Inst. Fourier (Grenoble) 10 (1960), 151–270.

BIEBERBACH, L.

[1] Über die Bewegungsgruppen der Euklidischen Räume I, Math. Ann. 70 (1911), 297–336; II. 72 (1912), 400–412.

[2] Über die Minkowskische Reduktion der positiven quadratischen Formen und die endliche Gruppen linearer ganzzahlige Substitutionen, Göttingen Nachr. (1912), 207–216.

BIEBERBACH, L. and SCHUR, I.

[1] Über die Minkowskische Reduktiontheorie der positiven quadratischen Formen, Sitzungsberichte d. Preuss. Akad. Wiss. Berlin 33 (1928), 510–535.

BOCHNER, S.

[1] Vector fields and Ricci curvature, Bull. Amer. Math. Soc. 52 (1946), 776–797.

BOCHNER, S. and MONTGOMERY, D.
[1] Locally compact groups of differentiable transformations, Ann. of Math. 47 (1946), 639–653.
[2] Groups on analytic manifolds, Ann. of Math. 48 (1947), 659–669.
BOREL, A. and LICHNEROWICZ, A.
[1] Groupes d'holonomie des variétés riemanniennes, C. R. Acad. Sci. Paris 234 (1952), 1835–1837.
CALABI, E. and MARKUS, L.
[1] Relativistic space forms, Ann. of Math. 75 (1962), 63–76.
CARTAN, E.
[1] Sur les variétés à connexion affine et la théorie de la relativité généralisée, Ann. Ec. Norm. Sup. 40 (1923), 325–412; 41 (1924), 1–25; 42 (1925), 17–88.
[2] Les espaces à connexion conforme, Ann. Soc. Pol. Math. 2 (1923), 171–221.
[3] Sur un théorème fondamental de M. H. Weyl, J. Math. Pures Appl. 2 (1923), 167–192.
[4] Sur les variétés à connexion projective, Bull. Soc. Math. France, 52 (1924), 205–241.
[5] La Géométrie des Espaces de Riemann, Mémorial des Sciences Math., Vol. 9, 1925.
[6] Les groupes d'holonomie des espaces généralises, Acta Math. 48 (1926), 1–42.
[7] Sur une class remarquable d'espaces de Riemann, Bull. Soc. Math. France, 54 (1926), 214–264; 55 (1927), 114–134.
[8] Leçons sur la Géométrie des Espaces de Riemann, Gauthiers-Villars, Paris, 1928; 2nd edition, 1946.
[9] La Méthode du Repère Mobile, la Théorie des Groupes Continus et les Espaces Généralisés, Actualités Sci. et Ind., #194, Hermann, Paris, 1935.
CARTAN, H.
[1] Sur les groupes de transformations analytiques, Actualités Sci. et Ind., #198, Hermann, Paris, 1935.
[2] Notion d'algèbre différentielle; applications aux groupes de Lie et aux variétés où opère un groupe de Lie, Colloque de Topologie, Bruxelles (1950), 15–27.
[3] La transgression dans un groupe de Lie et dans un espace fibré principal, Colloque de Topologie, Bruxelles (1950), 57–71.
CHERN, S. S.
[1] Differential geometry of fiber bundles, Proc. Int. Congress (1950), Vol. II, 397–411.
[2] Topics in Differential Geometry, Inst. for Adv. Study, Princeton, 1951.
[3] Pseudo-groupes continus infinis, Colloque de Géométrie Différentielle, Strasbourg (1953), 119–136.

CHEVALLEY, C.
[1] Theory of Lie Groups, Princeton University Press, 1946.
CHRISTOFFEL, E. B.
[1] Über die Transformation der homogenen Differential-ausdrücke zweiten Grades, J. Reine Angew. Math. 70 (1869), 46–70.
VAN DANTZIG, D. and VAN DER WAERDEN, B. L.
[1] Über metrisch homogenen Räume, Abh. Math. Sem. Hamburg 6 (1928), 374–376.
DIEUDONNÉ, J.
[1] Sur les espaces uniformes complets, Ann. Ec. Norm. Sup. 56 (1939), 277–291.
EHRESMANN, C.
[1] Sur la notion d'espace complet en géométrie différentielle, C. R. Acad. Sci. Paris 202 (1936), 2033.
[2] Les connexions infinitésimales dans un espace fibré différentiable. Colloque de Topologie, Bruxelles (1950), 29–55.
EISENHART, L. P.
[1] Riemannian Geometry, 2nd edition, Princeton University Press, 1949.
FREUDENTHAL, H.
[1] Die Topologie der Lieschen Gruppen als algebraisches Phänomen I, Ann. of Math. 42 (1941), 1051–1074.
FROBENIUS, G.
[1] Über die unzerlegbaren diskreten Bewegungsgruppen, Sitzungsberichte d. König. Preuss. Akad. Wiss. Berlin 29 (1911), 654–665.
FUJIMOTO, A.
[1] On the structure tensor of G-structure, Mem. Coll. Sci. Univ. Kyoto Ser. A, 18 (1960), 157–169.
GODEMENT, R.
[1] Théorie des Faisceaux: Actualités Sci. et Ind., #1252, Hermann, Paris, 1958.
GOLDBERG, S. I. and KOBAYASHI, S.
[1] The conformal transformation group of a compact Riemannian manifold, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), 25–26; Amer. J. Math. 84 (1962), 170–174.
[2] The conformal transformation group of a compact homogeneous Riemannian manifold, Bull. Amer. Math. Soc. 68 (1962), 378–381.
GOTO, M. and SASAKI, S.
[1] Some theorems on holonomy groups of Riemannian manifolds, Trans. Amer. Math. Soc. 80 (1955), 148–158.
HALL, M.
[1] The Theory of Groups, Macmillan, New York, 1959.
HANO, J.
[1] On affine transformations of a Riemannian manifold, Nagoya Math. J. 9 (1955), 99–109.

HANO, J. and MOMORIMOTO, A.
[1] Note on the group of affine transformations of an affinely connected manifold, Nagoya Math. J., 8 (1955), 71–81.
HANO, J. and OZEKI, H.
[1] On the holonomy groups of linear connections, Nagoya Math J. 10 (1956), 97–100.
HANTZSCHE, W. and WENDT, W.
[1] Drei dimensionale euklidische Raumformen, Math Ann., 110 (1934), 593–611.
HELGASON, S.
[1] Some remarks on the exponential mapping for an affine connection, Math. Scand. 9 (1961), 129–146.
HERMANN, R.
[1] On the differential geometry of foliations, Ann. of Math. 72 (1960), 445–457.
HICKS, N.
[1] A theorem on affine connexions, Illinois J. Math. 3 (1959), 242–254.
HILBERT, D.
[1] Über das Dirichletsche Prinzip, J. Reine Angew. Math. 129 (1905), 63–67.
HOPF, H.
[1] Zum Clifford-Kleinschen Raumproblem, Math. Ann. 95 (1926), 313–339.
HOPF, H. and RINOW, W.
[1] Über den Begriff der vollständigen differentialgeometrischen Flächen. Comm. Math. Helv. 3 (1931), 209–225.
HU, S. T.
[1] Homotopy Theory, Academic Press, New York, 1959.
IWASAWA, K.
[1] On some types of topological groups, Ann. of Math. 50 (1949), 507–558.
KELLEY, J. L.
[1] General Topology, Van Nostrand, Princeton, 1955.
KILLING, W.
[1] Die nicht-euklidischen Raumformen in analytische Behandlung, Teubner, Leibzig, 1885.
[2] Über die Clifford-Kleinschen Raumformen, Math. Ann. 39 (1891), 257–278.
KLEIN, F.
[1] Vergleichende Betrachtungen über neuere geometrische Forchungen, Math. Ann. 43 (1893), 63–100.
[2] Vorlesungen über nicht-euklidische Geometrie, Springer, Berlin, 1928.

Klingenberg, W.

[1] Eine Kennzeichnung der Riemannschen sowie der Hermitschen Mannigfaltigkeiten, Math. Z. 70 (1959), 300–309.

Kobayashi, S.

[1] Le groupe de transformations qui laissent invariant un parallélisme, Colloque de Topologie, Strasbourg (1954).

[2] Espaces à connexion de Cartan complète, Proc. Japan Acad. Sci. 30 (1954), 709–710.

[3] Espaces à connexions affines et riemanniennes symétriques, Nagoya Math. J. 9 (1955), 25–37.

[4] A theorem on the affine transformation group of a Riemannian manifold, Nagoya Math. J. 9 (1955), 39–41.

[5] Induced connections and imbedded Riemannian spaces, Nagoya Math. J. 10 (1956), 15–25.

[6] Theory of connections, Annali di Mat. 43 (1957), 119–194.

Kostant, B.

[1] Holonomy and the Lie algebra in infinitesimal motions of a Riemannian manifold, Trans. Amer. Math. Soc. 80 (1955), 528–542.

Kuiper, N. H.

[1] Sur les surfaces localement affines, Colloque de Géométrie Différentielle, Strasbourg (1953), 79–88.

Levi-Civita, T.

[1] Nozione di parallelismo in una varieta qualunque e consequente specificazione geometrica della curvature Riemanniana, Rend. Palermo 42 (1917), 73–205.

Levi-Civita, T. and Ricci, G.

[1] Méthodes de calcul différentiel absolu et leur applications, Math. Ann. 54 (1901), 125–201.

Lichnerowicz, A.

[1] Espaces homogènes kählériens, Colloque de Géométrie Différentielle, Strasbourg (1953), 171–201.

[2] Théorie Globale des Connexions et des Groupes d'Holonomie, Ed. Cremonese, Rome, 1955.

[3] Géométrie des Groupes de Transformations, Dunod, Paris, 1958.

Matsushima, Y.

[1] On the discrete subgroups and homogeneous spaces of nilpotent Lie groups, Nagoya Math. J. 2 (1951), 95–110.

Milnor, J.

[1] Groups which act on S^n without fixed points, Amer. J. Math. 79 (1957), 623–630.

Minkowski, H.

[1] Diskontinuitätsbereich für arithmetische Äquivalenz, J. Reine Angew. Math. 129 (1905), 220–274.

Montgomery, D. and Zippin, L.

[1] Transformation Groups, Interscience Tracts #1, Interscience, New York, 1955.

Myers, S. B. and Steenrod, N.

[1] The group of isometries of a Riemannian manifold, Ann. of Math. 40 (1939), 400–416.

Nagano, T.

[1] The conformal transformation on a space with parallel Ricci tensor, J. Math. Soc. Japan 11 (1959), 10–14.

[2] On fibered Riemannian manifolds, Sci. Papers, College General Educ. Univ. of Tokyo 10 (1960), 17–27.

Nagano, T. and Yano, K.

[1] Einstein spaces admitting a one-parameter group of conformal transformations, Ann. of Math. 69 (1959), 451–461.

Narasimhan, M. S. and Ramanan, S.

[1] Existence of universal connections, Amer. J. Math. 83 (1961), 563–572.

Nijenhuis, A.

[1] X_{n-1}-forming sets of eigenvectors, Indag. Math. 13 (1951), 200–212.

[2] On the holonomy group of linear connections, Indag. Math. 15 (1953), 233–249; 16 (1954), 17–25.

[3] A note on infinitesimal holonomy groups, Nagoya Math. J. 15 (1957), 145–146.

Nomizu, K.

[1] On the group of affine transformations of an affinely connected manifold, Proc. Amer. Math. Soc., 4 (1953), 816–823.

[2] Invariant affine connections on homogeneous spaces, Amer. J. Math., 76 (1954), 33–65.

[3] Studies on Riemannian homogeneous spaces, Nagoya Math. J., 9 (1955), 43–56.

[4] Reduction theorem for connections and its application to the problem of isotropy and holonomy groups of a Riemannian manifold, Nagoya Math. J. 9 (1955), 57–66.

[5] Un théorème sur les groupes d'holonomie, Nagoya Math. J. 10 (1956), 101–103.

[6] On infinitesimal holonomy and isotropy groups, Nagoya Math. J. 11 (1957), 111–114.

[7] Lie Groups and Differential Geometry, Publ. Math. Soc. Japan, #2, 1956.

[8] On local and global existence of Killing vector fields, Ann. of Math. 72 (1960), 105–120.

[9] Sur les algèbres de Lie de générateurs de Killing et l'homogénéité d'une variété riemannienne, Osaka Math. J. 14 (1962), 45–51.

NOMIZU, K. and OZEKI, H.

[1] The existence of complete Riemannian metrics, Proc. Amer. Math. Soc. 12 (1961), 889–891.

[2] On the degree of differentiability of curves used in the definition of the holonomy group, Bull. Amer. Math. Soc. 68 (1962), 74–75.

[3] A theorem on curvature tensor fields, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), 206–207.

NOWACKI, W.

[1] Die euklidischen, dreidimensionalen geschlossenen und offenen Raumformen, Comm. Math. Helv. 7 (1934), 81–93.

OZEKI, H.

[1] Infinitesimal holonomy groups of bundle connections, Nagoya Math. J. 10 (1956), 105–123.

PALAIS, R. S.

[1] A Global Formulation of the Lie Theory of Transformation Groups. Mem. Amer. Math. Soc., #22, 1957.

[2] On the differentiability of isometries, Proc. Amer. Math. Soc. 8 (1957), 805–807.

PONTRJAGIN, L.

[1] Topological Groups, Princeton University Press, 1956.

REINHART, B.

[1] Foliated manifolds with bundle-like metrics, Ann. of Math. 69 (1959), 119–132.

DE RHAM, G.

[1] Sur la réductibilité d'un espace de Riemann, Comm. Math. Helv. 26 (1952), 328–344.

RIEMANN, B.

[1] Über die Hypothesen, welche de Geometrie zugrunde liegen, Ges. Werke.

SAITO, M.

[1] Sur certains groupes de Lie résolubles I, Sci. Papers College General Educ. Univ. of Tokyo 7 (1957), 1–11; II, *ibid.*, 157–168.

SCHOUTEN. J. A. and STRUIK, D. J.

[1] On some properties of general manifolds relating to Einstein's theory of gravitation, Amer. J. Math. 43 (1921), 213–216.

SCHUR, F.

[1] Über den Zusammenhang der Räume konstanter Krümmungsmasses mit den projektiven Räumen, Math Ann. 27 (1886), 537–567.

SEIFERT, H. and THRELFALL, W.

[1] Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidmensionalen sphärischen Räumes, Math. Ann. 104 (1930), 1–70; 107 (1933), 543–586.

SIBUYA, Y.

[1] Note on real matrices and linear dynamical systems with periodic coefficients, J. Math. Anal. Appl. 1 (1960), 363–372.

SIEGEL, C. L.

[1] Einheiten quadratische Formen, Abh. Math. Sem. Hamburg 13 (1940), 209–239.

SIERPINSKI, W.

[1] Sur les espaces métriques localement separables, Fund. Math. 21 (1933), 107–113.

SIMONS, J.

[1] On transitivity of holonomy systems, Ann. of Math. 76 (1962), 213–234.

SMITH, J. W.

[1] Lorentz structures on the plane, Trans. Amer. Math. Soc. 95 (1960), 226–237.

STEENROD, N.

[1] The Topology of Fibre Bundles, Princeton University Press, 1951.

TAKIZAWA, N.

[1] On Cartan connexions and their torsions, Mem. Coll. Sci. Univ. Kyoto, ser A, 29 (1955), 199–217.

TANAKA, N.

[1] Conformal connections and conformal transformations, Trans. Amer. Math. Soc. 92 (1959), 168–190.

VINCENT, G.

[1] Les groupes linéaires finis sans points fixes, Comm. Math. Helv. 20 (1947), 117–171.

WALKER, A. G.

[1] On Ruse's spaces of recurrent curvature, Proc. London Math. Soc. (2) 52 (1950), 36–64.

[2] The fibring of Riemannian manifolds, Proc. London Math. Soc. (3) 3 (1953), 1–19.

[3] Connexions for parallel distribution in the large, Quart. J. Math. (Oxford) (2) 6 (1955), 301–308; II, 9 (1958), 221–231.

WANG, H. C.

[1] On invariant connections over a principal fibre bundle. Nagoya Math. J. 13 (1958), 1–19.

WEIL, A.

[1] Sur les théorèmes de de Rham, Comm. Math. Helv. 26 (1952), 119–145.

WEYL, H.

[1] Reine Infinitesimalgeometrie, Math. Z. 2 (1918), 384–411.

[2] Zur Infinitesimalgeometrie; Einornung der projektiven und konformen Auffassung, Göttingen Nachrichten (1912), 99–112.

[3] Raum, Zeit, Materie, Springer, Berlin, 1918.

WHITEHEAD, J. H. C.

[1] Convex regions in the geometry of paths, Quart. J. Math. 3 (1932), 33–42.

WILLMORE, T. J.

[1] Parallel distributions on manifolds, Proc. London Math. Soc. (3) 6 (1956), 191–204.

[2] Connexions for systems of parallel distributions, Quart. J. Math. (2) 7 (1956), 269–276.

WOLF, J. A.

[1] Sur la classification des variétés riemanniennes homogènes à courbure constante, C. R. Acad. Sci. Paris 250 (1960), 3443–3445.

[2] The manifolds covered by a Riemannian homogeneous manifold, Amer. J. Math. 82 (1960), 661–688.

[3] Homogeneous manifolds of constant curvature, Comm. Math. Helv. 36 (1961), 112–147.

[4] The Clifford-Klein space forms of indefinite metric, Ann. of Math. 75 (1962), 77–80.

[5] Vincent's conjecture of Clifford translations of the sphere, Comm. Math. Helv. 36 (1961), 33–41.

[6] Discrete groups, symmetric spaces, and global holonomy, Amer. J. Math. 84 (1962), 527–542.

WONG, Y. C.

[1] Recurrent tensors on a linearly connected differentiable manifold, Trans. Amer. Math. Soc. 99 (1961), 325–341.

YANO, K.

[1] On harmonic and Killing vectors, Ann. of Math. 55 (1952), 38–45.

[2] The Theory of Lie Derivatives and Its Applications, North Holland Publishing Co., Amsterdam, 1957.

YAMABE, H.

[1] On an arcwise connected subgroup of a Lie group, Osaka Math. J. 2 (1950), 13–14.

YOSIDA, K.

[1] A theorem concerning the semi-simple Lie groups, Tohoku Math. J., 43 (Part II) (1937), 81–84.

ZASSENHAUS, H.

[1] Beweis eines Satzes über diskrete Gruppen, Abh. Math. Sem. Hamburg. 12 (1938), 289–312.

[2] Über endliche Fastkörper, Abh. Math. Sem. Hamburg 11 (1936), 187–220.

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