Exam 2 Prep

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Contents

1 Material outline:

- Previous material:
  - Unit circle
  - Product rule
  - Quotient rule
  - Chain rule
  - Trig derivatives
- Derivatives of log and exponential:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\log_b x = \frac{1}{\ln b}\frac{1}{x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}a^x = (\ln a)a^x$$

- Derivatives of inverse trig functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \frac{1}{x^2+1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cot^{-1}x = -\frac{1}{x^2+1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Logarithmic differentiation
- Implicit differentiation
- Related rates
- Linearization
- Differentials
- Mean value theorem?
- Finding / identifying local and global extrema on a graph
- The process of finding absolute min/max on a closed domain
- First and second derivative tests
- Calculating $\lim\limits_{x\to\infty}$ and $\lim\limits_{x\to-\infty}$
- Curve sketching
- Reading data off of a graph
  - Intervals where increasing/decreasing
  - Local mins and maxes
  - Intervals where concave up/down

2 Past exam problems

The following problems all have written solutions in my exam archive at http://www.math.ksu.edu/~winstonc/exams/calc1.

Note, I have rephrased some of the problems here, to ease typing. Unless I've made a mistake, the exam listed should be where the problem was originally found.

2.1 Take Derivatives

1. (2018 Fall Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x \cdot \arctan\left(3x^2\right)\right)$
2. (2018 Fall Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x \cdot \dfrac{2^x - \ln x}{e^x + 1}\right)$
3. (2018 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\tan x}{\ln x + 3}\right)$
4. (2018 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^2\right) \cdot \arctan(x)$
5. (2017 Fall Exam 2) $w(x) = \dfrac{\tan(e^x)}{1 + x^3}$
6. (2017 Fall Exam 2) $h(x) = \sin^{-1}(\ln(x))$
7. (2017 Summer Exam 2) $f(x) = \tan\left(\ln\left(6x^4 + x^2\right)\right)$
8. (2017 Summer Exam 2) $y = e^{\sin x}$
9. (2017 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(2^x - \dfrac{5}{x^2} + \ln(7)\right)$
10. (2017 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x} \cdot \tan x\right)$
11. (2017 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\arctan\left(2t^2 - 3\right)$
12. (2017 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{6\ln x - 2x^3}{e^x + 3}\right)$
13. (2016 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x \cdot 2^x$
14. (2016 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\sqrt{x}}{\cos x}$
15. (2016 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}w}\arctan\left(5w^2 + 3\right)$
16. (2016 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{e^x}\right)$
17. (2016 Spring Exam 2) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta \cdot \ln\theta$
18. (2016 Fall Exam 2) $w(x) = x^3 e^{1/x}$
19. (2016 Fall Exam 2) $h(x) = \tan^{-1}(x^2)$

2.1.1 Using Logarithmetic Differentiation

Question: If you weren't prompted, would you know that these problems involve logarithmic diff just by looking?

1. (2018 Fall Exam 2) Find the derivative of $h(x) = x^{5x^2}$
2. (2018 Fall Final) Find the derivative of $f(x) = x^{3x}$
3. (2017 Fall Exam 2) $y = x^{5x}(2 + 3x^2)^4$
4. (2017 Summer Exam 2) Find the derivative of $y = \dfrac{(x+2)^2}{(x+5)(3x-4)}$.
5. (2017 Spring Exam 2) Find the derivative of $h(x) = e^x \cdot x^{5\cos(x)}$
6. (2016 Spring Exam 2) Find the derivative of $h(x) = \dfrac{x^x}{(3x^2+4)^5}$
7. (2016 Spring Exam 2) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $y = x^{2x}(1-x)^7$.

2.1.2 Using Implicit Differentiation

1. (2018 Fall Exam 2) Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $x^2y - e^y = x + 1$
2. (2018 Fall Final) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $x^3 + xy + y^4 = 5$
3. (2017 Fall Exam 2) Find an equation for the tangent line to the curve $x^2y + y^2 = x^3 - 3$ at the point $(2, 1)$.
4. (2017 Summer Exam 2) Compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $3xy^2 = y^3 - \cos x$
5. (2017 Spring Exam 2) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $\sin\left(xy^2\right) = x^2$.
6. (2016 Spring Exam 2) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $x^2y^3 = e^x - y^2$.
7. (2016 Fall Exam 2) Find an equation for the tangent line to the curve $xy + 7 = x^3 + y^3$ at the point $(2, 1)$.
8. Find an equation of the tangent line to the curve $x^2y^3 - x^3y^2 = 4$ at the point $(1, 2)$.

2.2 Linearize and approximate

1. (2018 Fall Exam 2) Find the linearization of $g(x) = \sqrt{x}$ at $x = 25$. Use this to estimate $\sqrt{26}$.

2. (2017 Fall Exam 2) Find the linear approximation of $f(x) = \cos(x)$ near $x = \dfrac{\pi}{4}$. Use this to estimate $\cos\left(\dfrac{\pi}{4} + \dfrac{1}{10}\right)$.

3. (2018 Fall Final) Use the linearization of $u(x) = \ln x$ at $x = 1$ to approximate $\ln(0.9)$.

4. (2017 Spring Exam 3) Find linearization for $g(x) = \ln(x)$ at $x = 1$. Use it to approximate $\ln(1.15)$.

5. (2016 Spring Exam 3) Find the linearization of $w(x) = \sqrt{x}$ at $x = 9$. Use it to estimate $\sqrt{9.6}$.

6. (2016 Fall Exam 2) Find the linear approximation of $f(x) = \sqrt[3]{x}$ near $x = 8$. Use it to estimate $\sqrt[3]{8.1}$.

7. (2015 Fall Exam 2) Find an equation of the tangent line to the curve $y = \sqrt[5]{x}$ at $x = 32$, and use this to approximate $\sqrt[5]{33}$.

2.3 Differentials

1. (2018 Fall Exam 2) Let V denote the volume of a cube of side length x. Find the differential dV in terms of x and dx.

2. (2017 Fall Exam 2) The volume of a cone of height 9 ft is given by $V = 3\pi r^2$. Estimate the change in volume using the differentials dV and dr, if $r = 5$ ft and increased by $\dfrac{1}{10}$ ft.

3. (2017 Spring Exam 3) The volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$. Find the differential dV.

4. (2016 Fall Exam 2) The volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$. Estimate the change in volume by calculating dV, given that $r = 3$ inches and $dr = \dfrac{1}{12\pi}$ inches.

5. (2016 Spring Exam 3) Find dy if $y = \cos\left(4x^2\right)$.

2.4 Find absolute max/min on a closed interval

1. (2018 Fall Exam 2) $w(x) = x + \sin x$ on $[0, 2\pi]$.
2. (2018 Fall Final) $w(x) = x - \sqrt{x}$ on $[0, 4]$.
3. (2017 Fall Exam 2) Determine the absolute minimum and absolute maximum value of the function $f(x) = x^4 - 2x^2$ over the interval $[0, 2]$.
4. (2017 Spring Exam 3) Find the absolute min and max of $w(x) = 2x^3 - 9x^2 + 3$ on $[-1, 1]$.
5. (2016 Spring Exam 3) Find absolute min/max of $g(x) = x^3 - 3x^2 + 4$ on $[-1, 1]$.
6. (2016 Fall Exam 2) Find absolute min/max value of $f(x) = x^3 - 3x$ over the interval $[0, 2]$.
7. (2015 Fall Exam 2) Find the x and y coordinates of the absolute maximum of the function $y = x^3 - x^2 - x$ on the closed interval $0 \leq x \leq 2$

2.5 Horizontal Asymptotes

1. (2018 Fall Exam 2) Find $\displaystyle\lim_{x\to-\infty} \frac{3x^9 - 7x + 3}{2 + 5x + 6x^9}$.
2. (2017 Fall Exam 2) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{7 - 3x^5}{2x^5 - 15x^3}$.
3. (2016 Fall Exam 2) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{x + 3}{\sqrt{x^2 - 1}}$
4. (2015 Spring Exam 1) Find horizontal asymptote(s) for $y = \dfrac{\sqrt{9x^2 + 5}}{2x - 7}$

2.6 Related Rates

Always include units!

1. (2018 Fall Exam 2) A hot air balloon rising vertically is tracked by an observer located 2 miles from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{4}$, and it is changing at a rate of $\frac{1}{10}$ radians/minute. How fast is the balloon rising at this moment?
2. (2018 Fall Final) A 5-foot ladder rests against a wall. The bottom of the ladder slides away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 ft from the wall?

3. (2017 Fall Exam 2) A rocket is launched vertically upward from a point P. At the same time, a car is driving on a straight line away from the point P. Use related rates to determine the rate that the distance between the rocket and the car is increasing at the instant when the rocket is 3 miles up and travelling 500 miles per hour, and the car is 4 miles away from P and travelling 30 miles per hour. (Assume the ground is flat.)
4. (2017 Summer Exam 2) A plane is flying away from you at 500 mph at a height of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?
5. (2017 Spring Exam 2) Boyle's Law says that when a sample of gas is compressed at a constant temperature, the pressure P and the volume V satisfy the equation $PV = C$, where C is a constant. Suppose that at a certain instant, the volume is $300\,\mathrm{cm}^3$, the pressure is 100 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume changing at this instant?
6. (2017 Spring Exam 2) The length of a rectangle is increasing at a rate of 2 ft/s, and its width is increasing at a rate of 3 ft/s. At what rate is the area of the rectangle increasing when the length is 6 ft and the width is 7 ft?

2.7 Curve sketching

By "the full process for curve sketching" I mean:

1. Finding intercepts
2. Finding asymptotes (HA, VA)
3. Finding critical numbers and sign chart for f'
4. Finding inflection points and sign chart for f''
5. Sketching the curve using the above data.

1. (2018 Fall Exam 2) Sketch $f(x) = x^2(x + 3)$, which has derivatives $f'(x) = 3x(x + 2)$, $f''(x) = 6(x + 1)$. Do the entire process.
2. (2017 Fall Exam 2) For $g(x) = 20x^3 - 3x^5$, draw the number line for $g'(x)$ and $g''(x)$. Classify the critical points.
3. (2017 Fall Exam 2) Let $f(x) = \dfrac{2 - x}{x^2 - 1}$. Then $f'(x) = \dfrac{x^2 - 4x + 1}{(x^2 - 1)^2}$. Find asymptotes, intercepts, critical points. Classify the critical points.
4. (2017 Summer Exam 2) Let $f(x) = x^3 - 2x^2 + x - 1$. Find when f is concave up/down, increasing/decreasing, the inflection points, the critical points, and the local minima and maxima.

5. (2017 Spring Exam 3) Sketch $f(x) = \frac{x^2}{x^2+3}$. It has $f'(x) = \frac{6x}{(x^2+3)^2}$ and $f''(x) = \frac{-18(x^2-1)}{(x^2+3)^3}$. Do the entire process.

6. (2016 Spring Exam 3) Sketch $f(x) = \frac{x^2-1}{x^2+3}$. It has $f'(x) = \frac{8x}{(x^2+3)^2}$ and $f''(x) = \frac{-24(x^2-1)}{(x^2+3)^3}$. Do the entire process.

7. (2016 Fall Exam 2) Let $g(x) = x^5 - \frac{20}{3}x^3$. Draw the number line for g' and g''. Classify the critical points. State when $g(x)$ is concave up.

8. (2016 Fall Exam 2) Let $f(x) = \frac{2x^2-6x}{4-x^2}$. Then $f'(x) = \frac{-2(3x^2-8x+12)}{(4-x^2)^2}$. Sketch $f(x)$ (skip using f'' or inflection points).

9. (2015 Fall Exam 3) Let $f(x) = \left(\frac{x-2}{x}\right)^2$. Then $f'(x) = \frac{4(x-2)}{x^3}$ and $f''(x) = \frac{-8(x-3)}{x^4}$. Do the entire process.