Final Exam Prep

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Contents

1 Material outline:

- Previous material:
  - Limits
    - Algebraically
    - End behavior

        - L'Hôpital
    - Derivatives
        - Using limit definition
        - All derivative rules
        - Implicit diff
        - Related rates
        - Reading from a graph ($f(x)$ or $f'(x)$) the intervals where a function is increasing/decreasing, concave up/down
        - Finding the linearization, and using it to approximate
        - Finding absolute max and min on a closed interval
        - Applied optimization
            - Arguing that something is an *absolute* max/min using 1st derivative, 2nd derivative, or closed interval method
    - Integrals
        - FTC part 1
        - Finding a particular antiderivative given boundary conditions.
        - U-sub technique
        - Computing displacement given a velocity function and endpoints.
        - Graphically, as areas under curves.
        - Approximating area using Left/Right/Mid

- New material
    - Area between two curves
    - Volumes of revolution
        - Disk / Washer method
            - Disk:
$$V = \pi \int_a^b \big(r(x)\big)^2 \, \mathrm{d}x$$
            - Washer:
$$V = \pi \int_a^b \Big(\big(R(x)\big)^2 - \big(r(x)\big)^2\Big) \, \mathrm{d}x$$
        - Shell method
$$V = 2\pi \int_a^b r(x) \cdot h(x) \, \mathrm{d}x$$

2 Past exam problems

The following problems all have written solutions in my exam archive at http://www.math.ksu.edu/~winstonc/exams/calc1.

Note, I have rephrased some of the problems here, to ease typing. Unless I've made a mistake, the exam listed should be where the problem was originally found.

2.1 Limits

Potentially a mixed bag. These may or may not require the usage of L'Hôpital

1. (2018 Fall Final) $\lim_{x \to 1} \frac{\ln x}{x-1}$
2. (2017 Fall Final) $\lim_{x \to 2} \frac{x^3-4x}{2-x}$
3. (2017 Fall Final) $\lim_{h \to 0} \frac{\sin(3h)}{\sin(2h)} + \frac{\cos(3h)}{\cos(2h)}$
4. (2017 Fall Final) $\lim_{x \to \infty} (x^2+5)^{1/x}$
5. (2017 Spring Final) $\lim_{x \to 0} \frac{e^x}{x+1}$
6. (2017 Spring Final) $\lim_{\theta \to 0} \frac{\sin(\theta^2)}{\theta}$
7. (2016 Fall Final) $\lim_{x \to 3} \frac{x-3}{9x-x^3}$
8. (2016 Fall Final) $\lim_{h \to 0} \frac{\tan(2h)}{\sin(5h)}$
9. (2016 Fall Final) $\lim_{x \to \infty} (5+x)^{1/x}$
10. (2016 Spring Final) $\lim_{x \to \pi} \frac{\cos x}{x}$
11. (2016 Spring Final) $\lim_{x \to \infty} \frac{23+x-5x^5-3x^9}{4x^9-3x-2}$

2.2 Derivatives

2.2.1 Mixed bag of problems

Some of these are just a straightforward application of derivative rules, while others may use implicit differentiation or logarithmic differentiation.

1. (2018 Fall Final) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{\tan x}{\ln x + 3}$

2. (2018 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin\left(x^2\right)\cdot\arctan(x)$
3. (2018 Fall Final) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if $x^3 + xy + y^4 = 5$
4. (2018 Fall Final) $f'(x)$ if $f(x) = x^{3x}$
5. (2017 Fall Final) $f'(t)$ where $f(t) = \sin^5(\ln t)$
6. (2017 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}e^{3x}\tan^{-1}(x)$
7. (2017 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x+\tan x}{1-x^2}$
8. (2017 Spring Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{\tan x}{e^x+5}$
9. (2017 Spring Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos\left(x^3\right)\cdot\arctan(x)$
10. (2017 Spring Final) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $x^2 - xy + y^3 = 5$
11. (2017 Spring Final) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $y = x^{7x}$
12. (2016 Fall Final) $f'(t)$ where $f(t) = \cos^2(2t+1)$
13. (2016 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}x\ln\left(x^2+2\right)$
14. (2016 Fall Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{e^{5x}}{x^2+1}$
15. (2016 Spring Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{e^x}{\sqrt{x}}$
16. (2016 Spring Final) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(x)\cdot\sin\left(x^2\right)$
17. (2016 Spring Final) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $x^3 + y^3 = 5xy$
18. (2016 Spring Final) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $x^{\cos x}$

2.2.2 Using limit definition

For these problems, the limit definition **must** be used.

1. (2018 Fall Final) Find $f'(2)$ if $f(x) = x^2$.
2. (2017 Fall Final) $f'(x)$ for $f(x) = x^2 + x$.
3. (2017 Spring Final) Find $f'(2)$ if $f(x) = 3x^2$
4. (2016 Fall Final) Find $f'(x)$ for $f(x) = 3x^2 - x$
5. (2016 Spring Final) Find $f'(2)$ if $f(x) = x^2 + x$.

2.3 Integrals

Some of these will require u-sub.

1. (2018 Fall Final) $\displaystyle\int \left(\frac{2}{x} - \sqrt{x}\right) \mathrm{d}x$
2. (2018 Fall Final) $\displaystyle\int \frac{\cos(\ln x)}{x} \mathrm{d}x$
3. (2017 Fall Final) $\displaystyle\int e^{5x} - \frac{1}{\sqrt{4-x^2}} \mathrm{d}x$
4. (2017 Fall Final) $\displaystyle\int \sin^5(2x)\cos(2x)\,\mathrm{d}x$
5. (2017 Fall Final) $\displaystyle\int_1^e \frac{(\ln x)^2}{x} \mathrm{d}x$
6. (2017 Spring Final) $\displaystyle\int (\sqrt{x} + \cos x - 5)\,\mathrm{d}x$
7. (2017 Spring Final) $\displaystyle\int \frac{\sqrt{\ln x}}{x} \mathrm{d}x$
8. (2017 Spring Final) $\displaystyle\int_0^{\pi/2} \sin^4\theta\cos\theta\,\mathrm{d}\theta$
9. (2016 Fall Final) $\displaystyle\int \sin\bigl(\pi x/2\bigr) + 2^x - \frac{1}{\sqrt{1-x^2}} \mathrm{d}x$
10. (2016 Fall Final) $\displaystyle\int \tan^3(2x)\sec^2(2x)\,\mathrm{d}x$
11. (2016 Fall Final) $\displaystyle\int_0^1 \frac{x+2}{x^2+4x+1} \mathrm{d}x$

12. (2016 Spring Final) $\int \frac{7\,\mathrm{d}x}{1+x^2}$
13. (2016 Spring Final) $\int t\sqrt{t^2+3}\,\mathrm{d}t$
14. (2016 Spring Final) $\int_0^{\pi/2} \sin^3\theta\cos\theta\,\mathrm{d}\theta$

2.3.1 Initial value problems

1. (2017 Fall Final) $f'(t) = 4t^3 - \sin t,\ f(0) = 1$.
2. (2016 Fall Final) $f'(t) = \sqrt{t},\ f(1) = 2$.

2.3.2 Approximating integrals

1. (2017 Fall Final) Estimate the area below the curve $y = x^2 + 2$ over the interval $[0, 3]$ using R_3. Make a sketch of the curve with the rectangles used.
2. (2016 Fall Final) Estimate the area below the curve $y = x^2$ over the interval $[0, 2]$ using L_4. Make a sketch of the curve with the rectangles used.

2.4 FTC

1. (2018 Fall Final) $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^5 e^{\sin t}\,\mathrm{d}t$
2. (2017 Spring Final) $\frac{\mathrm{d}}{\mathrm{d}x}\int_3^x t\cdot\sin\left(t^3\right)\mathrm{d}t$
3. (2016 Spring Final) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{\cos t}\,\mathrm{d}t$

2.5 Applications of derivatives

2.5.1 Related rates

1. (2018 Fall Final) A 5-foot ladder rests against a wall. The bottom of the ladder slides away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 ft from the wall?
2. (2017 Fall Final) Consider a rectangle with edges of length x, y. If x is increasing at a rate of 5 m/sec and y is decreasing at a rate of 2 m/sec, at what rate is the area A of the rectangle changing when $x = 3$ m and $y = 4$ m?
3. (2017 Spring Final) Suppose that an oil spill from a ruptured tanker spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 2 ft/sec, how fast is the area of the spill increasing when the radius is 10 ft?

4. (2016 Fall Final) Consider a right triangle with edges of length x, y, z, with z the hypotenuse. If x is increasing at a rate of $5\,\mathrm{m/sec}$ and z is increasing at a rate of $7\,\mathrm{m/sec}$, at what rate is y increasing when $x = 3\,\mathrm{m}$ and $z = 5\,\mathrm{m}$?

5. (2016 Spring Final) A hot air balloon rising vertically is tracked by an observer located 5 miles from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{4}$, and it is changing at a rate of $\frac{1}{10}$ radians/minute. How fast is the balloon rising at this moment?

2.5.2 Finding tangent lines to curves

1. (2017 Fall Final) Find the equation of the tangent line to the curve $xy + y^2 = 2x - 1$ at $(2, 1)$.

2. (2016 Fall Final) Find the equation of the tangent line to the curve $x^3+y^2 = 5y+4$ at $(2, 1)$.

2.5.3 Linearization

1. (2018 Fall Final) Use a linearization of $u(x) = \ln x$ at $x = 1$ to approximate $\ln(0.9)$.

2. (2017 Fall Final) Find the linear approximation of $f(x) = \sqrt{x}$ near $x = 9$. Use this to estimate $\sqrt{8.9}$.

3. (2017 Spring Final) Use a linearization of $u(x) = \sqrt{x}$ at $x = 9$ to approximate $\sqrt{9.6}$.

4. (2016 Fall Final) Find the linear approximation of $f(x) = \sqrt{x}$ near $x = 4$. Use this to estimate $\sqrt{4.1}$.

5. (2016 Spring Final) Use linearization to approximate $\sin(0.01)$.

2.5.4 Gathering data using f' and f"

1. (2017 Fall Final) Let $f(x)$ be a function with $f'(x) = x^2(x^2-4)(x-7)$. Find the critical points of $f(x)$, and the intervals where $f(x)$ is increasing / decreasing. Classify each critical point as a local min, local max, or neither.

2. (2017 Fall Final) Let $g(x) = 2x^6 - 5x^4$. Determine the intervals where $g(x)$ is concave up / concave down. Determine all inflection points.

3. (2017 Spring Final) Given $w''(x) = \dfrac{3-x}{x^2+7}$, find the intervals where $w(x)$ is concave up / concave down. Determine inflection points.

4. (2016 Fall Final) Let $g(x) = 3x^5 + 20x^3$. Determine intervals where $g(x)$ is concave up / concave down. Determine all inflection points.

5. (2016 Fall Final) Let $f(x) = x^2(x-4)^3$. Given $f'(x) = x(x-4)^2(5x-8)$. Find the critical points of $f(x)$. Find the intervals where $f(x)$ is increasing / decreasing, and classify the critical points (local min / local max / neither).

6. (2016 Spring Final) Given $w''(x) = \dfrac{2(x-1)}{x^2+3}$, find the intervals where $w(x)$ is concave up / concave down. Determine inflection points.

2.5.5 Find absolute min / max on a closed interval

1. (2018 Fall Final) $w(x) = x - \sqrt{x}$ on $[0, 4]$
2. (2017 Spring Final) $v(x) = x^3 + 3x^2 + 1$ on $[-1, 1]$
3. (2016 Spring Final) $v(x) = x^3 - 3x + 1$ on $[0, 2]$

2.5.6 Optimization

These problems require some justification that the calculated answer is an absolute max/min.

1. (2018 Fall Final) When a company charges x dollars per chair, it makes a total profit $P(x) = -2x^2 + 200x - 50$ dollars. If the company wants to maximize total profit, what should it charge per chair?

2. (2017 Fall Final) A rectangular fence consists of three sides costing \$2 per meter and one side costing \$1 per meter. If the area of the rectangle is 12 square meters, find the dimensions that minimize the cost of the fence.

3. (2017 Spring Final) A farmer has 20 feet of fencing and wants to fence off a rectangular region that borders a straight river. The farmer needs no fencing along the river. What dimensions will maximize the fenced-in area?

4. (2016 Fall Final) Find the dimensions of a cylinder with total surface area 6π square meters, including top and bottom, that maximizes its volume.

5. (2016 Spring Final) Suppose you want to enclose a $25\,\text{ft}^2$ rectangular area with fencing. What is the minimum length of fencing needed?

2.6 Applications of Integrals

2.6.1 Displacement

1. (2018 Fall Final) Suppose a particle has position $s(t)$ feet at time t seconds and a velocity function $s'(t) = t \cdot \sin\left(\pi t^2\right)$ ft/s. Find the displacement from time $t = 0$ to time $t = 1$.

2.6.2 Area between curves

1. (2018 Fall Final) Find the area between the curves $y = 4$ and $y = x^2$
2. (2017 Fall Final) Calculate the area of the region with $x \geq 0$ bounded by the y-axis, the parabola $y = x^2 - 2x$ and the line $y = 6 - x$.
3. (2017 Spring Final) Find the area bounded between $y = 4$ and $y = x^2$.
4. (2016 Fall Final) Calculate the area between $y = 8 - x^2$ and $y = x + 2$.
5. (2016 Spring Final) Find the area bounded between $y = 2x^2$ and $y = 3 - x^2$.

2.6.3 Volume of revolution

1. (2018 Fall Final) Rotate the region bounded by $y = x^3$, $x = 0$, and $y = 1$ around the x-axis. Find the volume.
2. (2018 Fall Final) Rotate the region bounded by $y = x^2$ and $y = x$ around the y-axis. Find the volume.
3. (2017 Fall Final) Rotate the region bounded by the y-axis, $y = x$, and $y = 3 + \frac{1}{2}x$ about the y-axis. Set up the integral for the volume.
4. (2017 Fall Final) Rotate the region bounded by the y-axis, $y = x$, and $y = 3 + \frac{1}{2}x$ about the x-axis. Set up the integral for the volume.
5. (2017 Spring Final) Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ around the x-axis.
6. (2016 Fall Final) Take the region bounded between the curves $y = 4x - x^3$ and $y = x^2$, with $x \geq 0$, and rotate around the x-axis. Set up the integral for the volume. (The curves don't intersect at a nice x-value.)
7. (2016 Fall Final) Take the region bounded between the curves $y = 4x - x^3$ and $y = x^2$, with $x \geq 0$, and rotate around the y-axis. Set up the integral for the volume. (The curves don't intersect at a nice x-value.)
8. (2016 Spring Final) Find the volume of the solid obtained by rotating the region bounded by $y = 4$ and $y = x^2$ around the x-axis.