What you learned in Calc 1

2019 Fall

Main Ideas of Calculus

The Derivative

- "Instantaneous rate of change"
- "Slope of a tangent line"
- Limit of the secant line, as the end-points get arbitrarily close

The Integral

- "Area under a curve"
  - Approximate by summing up rectangles
  - Approximate better by using infinitely many rectangles

FTC (Fundamental Theorem of Calculus)

Slogan: "Differentiation and Integration are inverse operations"

Main Ideas of Calculus (Mathematically)

The Derivative

$$f'(a) := \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \text{ or } f'(a) := \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

The Integral

$$\int_a^b f(x)dx := \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \cdot \frac{b-a}{N}$$

FTC (Fundamental Theorem of Calculus)

- Part 1: $\frac{d}{dx} \int_a^x f(t)dt = f(x)$
- Part 2: $\int_a^b f(x)dx = F(b) - F(a)$

Limits

We want to be to evaluate limits. They are just a calculation.

- Start with direct substition
- If that fails, try:
  - Algebraic technique
  - L'Hopital's:
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
    - The LHS must be of type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$ for this to hold

Indeterminate forms

- $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \cdot \infty$
- 1^{∞}, 0^0, ∞^0
  - Note that taking ln() of these reduces them to the type $0 \cdot \infty$

Derivative Rules

Basic Derivative rules

- Product rule: $(fg)' = f'g + fg'$
- Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
- Chain rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$
- Power rule: $(x^n)' = nx^{n-1}$
  - Do not confuse with: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Trig Derivatives

- $\sin \to \cos$, $\cos \to -\sin$
- $\tan \to \sec^2$, $\cot \to -\csc^2$
- $\sec(x) \to \sec(x)\tan(x)$, $\csc(x) \to -\csc(x)\cot(x)$

More Derivative Rules

Log and Exponential Derivatives

- Special
  - $\ln x \to \frac{1}{x}$
  - $e^x \to e^x$
- General
  - $\log_b x \to \frac{1}{\ln b}\frac{1}{x}$
  - $b^x \to (\ln b)b^x$

Inverse Trig Derivatives

- $\sin^{-1} x \to \frac{1}{\sqrt{1-x^2}}$, $\cos^{-1} x \to -\frac{1}{\sqrt{1-x^2}}$
- $\tan^{-1} x \to \frac{1}{x^2+1}$, $\cot^{-1} x \to -\frac{1}{x^2+1}$
- $\sec^{-1} x \to \frac{1}{|x|\sqrt{x^2-1}}$, $\csc^{-1} x \to -\frac{1}{|x|\sqrt{x^2-1}}$

Logarithmic differentiation

What is it

Taking ln() of both sides before differentiating (and remembering to undo this).

Why use it

- ln() will turn multiplication/division into addition/subtraction, so that you can avoid product rule and quotient rule.
- If x occurs in both the base and the power, then this technique is necessary (e.g. $y = x^x$).

Implicit differentiation

- Using $y \to y'$ in addition to ordinary derivative rules.
- Why?
  - Sometimes both x and y are on the same side of the equation, and the equation cannot be solved for y fully.
  - However, we can still think about slopes of tangent lines for such curves, so we want to be able to compute the derivative
  - E.g. A circle. $x^2 + y^2 = 1$

Related rates are an example of this

Linearization

- The idea behind it: Use the derivative as a linear approximation for your function.
- To construct an $L(x)$ that approximates $f(x)$ near $x = a$:

$$L(x) = f(a) + f'(a)(x - a)$$

Curve sketching

- f' denotes increasing/decreasing
  - positive: increasing
  - negative: decreasing
  - zero or undefined: critical point
- f'' denotes concavity
  - positive: concave up
  - negative: concave down
  - zero: inflection point

Optimization

Overall: Want to find critical points, and classify as local min/max.

To classify:

1. First derivative test: Evaluate $f(x)$ on either side of the critical point
   - $+|-$: max
   - $-|+$: min
   - otherwise : neither
2. Second derivative test: Evaluate $f''(p)$ at the critical point
   - $f''(p) > 0$: Concave up, so local min
   - $f''(p) < 0$: Concave down, so local max
   - $f''(p) = 0$: Inconclusive

Making the case that a local min/max is a global min/max requires an extra condition:

1. First derivative test: $f(x)$ has no other critical points
2. Second derivative test: $f'' > 0$ always or $f'' < 0$ always.

Integral techniques

- Spotting the antiderivative by inspection
- U-sub

Area between curves

"Upper function − lower function"

Volume of revolution

Disk / Washer method

- Disk:

$$V = \pi \int_a^b [r(x)]^2 \, dx$$

- Washer:

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2\right) \, dx$$

Shell method

$$V = 2\pi \int_a^b r(x) \cdot h(x) \, dx$$