STACK it up: Automated Assessment in Higher Mathematics

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Contents

Declaration

This piece of work is a result of my own work, and I have complied with the Department's guidance on multiple submission and on use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

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Chapter 1

Introduction

For mathematics higher education in the UK, closed book written exams have always been the most common method of assessment [1, p. 1048]. However, since the Covid-19 pandemic in 2019, e-assessments have become more widely used as an assessment format in higher educational institutions (HEI), and researchers started to focus more on investigating the application of e-assessments in HEIs [2, p. 2046].

The study in [2] focused on the students' perspective of e-assessments in mathematics higher education. The research was conducted using data collected from undergraduate students taking the courses Mathematics 1 and Mathematics 2 [2, p. 2050]. The findings of the research suggested that students generally hold positive views towards e-assessments, and they find that **formative e-assessments are useful in preparation for summative assessments**, such as the final exams [2, p. 2060]. It is worth noting that the study in the paper is focused on maths education conducted in a fully online environment, where the students also have their summative exams conducted online [2, p. 2058]. Our research focus will be slightly different since we aim to discuss formative e-assessments aimed at helping students revise for the final exams, which could be conducted written or online. However, we believe that well-designed formative e-assessments can still help students prepare for a written final exam. This is suggested from the paper [3, p. 2] that formative e-assessments can help students "monitor their progress, encourage further studies" and most importantly, **"increase their learning and understanding"**. Besides, based on the paper [4, p. 6-7] which organises the mathematics assessment tasks in an assessment component taxonomy, it suggests that several components such as *technical* and *problem solving* require the accurate use of mathematical methods, while the *conceptual* and *consolidation* components assess conceptual understanding. In our research, we will therefore design an e-assessment focused on enhancing students' **conceptual understanding** and practice of **mathematical methods**, which are crucial skills for written exams.

We will use the online assessment system STACK [5] for writing an e-assessment question in the learning management system Moodle, where students will be able to access the e-assessment questions. Our research goal is to design effective formative e-assessments for students. The research focus will be divided into two parts: **technical aspects** and **pedagogical analysis**.

In the first part, we will discuss the technical aspects of STACK, including the fundamental structure of a STACK question, how to write a simple **fill-in-the-blank** question of STACK, and finally move on to the more advanced question types of STACK, namely **multiple-choice** and **interactive graph** questions, where we will explore the more advanced technical features that we have implemented in each question type. The focus is on the technical set-up and functionality of the features.

In the second part, we will present three case studies from our designed e-assessment questions in the mathematics third year module: **Mathematical Biology**, where we will discuss the pedagogical aspect of STACK questions, including the pedagogical function of each question type and how the associated technical features help in achieving the pedagogical goals of the question. Then, we will identify and analyse the **common student errors** (CSE) from selected sub-questions. These identified errors benefit both the educators and students since educators can utilise the data to provide enhanced feedback to students and improve their curriculum design [6, p. 207], while students can reflect on the errors they have made and identify areas for improvements in preparation for the exams. Finally, we will move on to discuss potential future work that can be done to enhance an e-assessment, from the perspectives of common student errors and feedback.

1.1 What is STACK E-assessment?

STACK is an online assessment system for mathematics and STEM [5]. It is used worldwide by many educational institutions and developers, with more than 2000 sites as of February 2025 [7], in over 25 countries [5]. For educators, it is an e-assessment platform that can be used to assess the students and help them understand the subject materials better. STACK supports a lot of question and input types, with many built-in technical features that make the e-assessment format more accessible by the students and better assess the students skills. We will start by discussing the structure of a STACK question in the following section.

1.1.1 Overview: Structure of a STACK question set-up in Moodle

STACK e-assessment questions can be implemented in Moodle [5], a learning management system. Depending on the Moodle version, there are mainly four important components in the set-up of a STACK question:

1. Question variables,
2. Question text,
3. Input,
4. Potential Response Trees (PRT).

Question Variables

The **question variables** component, in addition to defining question variables, is where we set up teacher's answers (`ta`) and in particular, options for a multiple-choice question. Question variables are coded in the computer algebra system, Maxima. It supports simple calculations, as well as more complex operations such as differentiation and integration. It also supports many mathematical object types, such as matrices, lists and sets [8]. We can set up randomised question variables to increase randomness in our assessment for further attempts and practice by students. In addition, we can also define the model answer variables in the question variables component, which will then be utilised in the corresponding input settings and PRTs.

Question Text

The question text is written to display the question content to the students in the actual generated question. It is written in HTML and it supports the inclusion of mathematical expressions using LaTeX. Besides, we can apply the question variables defined in Maxima earlier in the question text. This displays to the students different numbers or versions of the question in each generated question variant due to the randomisation of question variables.

Inputs

Inputs are where student will provide their answers to our questions. Each input will be labelled and supplemented with **input settings**, where educators can choose the **type of input**, e.g. algebraic input, checkbox and matrix. The **model answer** will also be defined in the input settings, which will be displayed to the students when they have submitted the e-assessment. There are also more advanced settings such as provision of a syntax hint and forbidding floats.

Potential Response Tree (PRT)

Potential Response Tree is where we check students' answers, score them points and provide them with feedback. The PRT component consists of two parts: feedback variables, and True or False nodes. **Feedback variables** are variables set up to be used after collecting the student' response, usually for answer checking and scoring purposes. Figure 1.1 shows the other part of the PRT which consists of many True or False Nodes for performing answer tests and scoring. Marks allocated in each node are fully customised by the teacher. Besides, it is worthy to note that all answers will only be marked, and all marks will only be scored from the nodes of the PRTs.
Feedback can be provided to the students in two ways, namely **specific feedback** or **general feedback**. Specific feedback can be written in the True or False nodes of the PRT component, as shown in Figure 1.1, where the feedback is given to tell the student specifically where they have done wrong. On the other hand, general feedback is written in a separate general feedback section, which can be used for providing students with the fully worked solution after they have submitted the question.

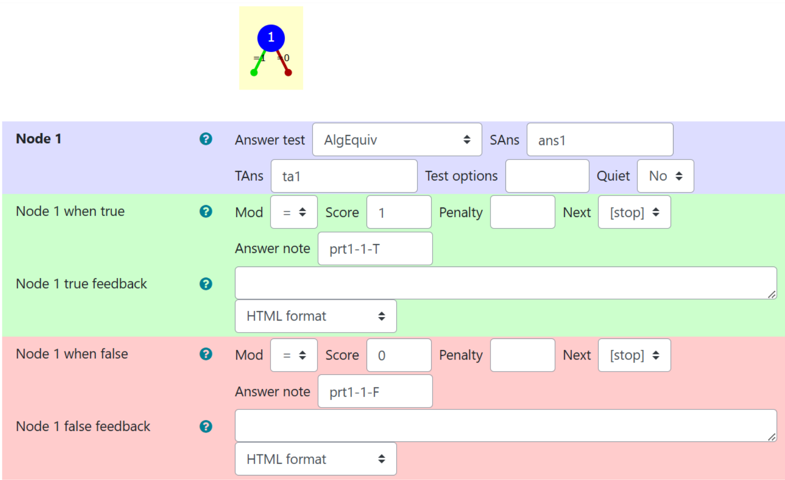


Figure 1.1: The PRT component of a STACK question consisting of True or False Nodes

1.1.2 Overview: Question Types of STACK

In this paper, we will explore **three** question types of STACK, namely **fill-in-the-blank**, **multiple-choice** and **interactive graph questions**. Fill-in-the-blank question is the simplest question type in STACK. Therefore, we will explore how to write a fill-in-the-blank question to demonstrate the function of each component of a STACK question, as well as the implementation of fundamental technical features of STACK in Section 1.2. Then, we will introduce the more advanced question types: multiple-choice and interactive graph questions in Sections 2.1 and 2.2, where we will explore more advanced technical features specific to the two question types.

1.2 Creating an algebraic Fill-in-the-Blank Question: Question Variables and Question Text

Fill-in-the-blank question is the fundamental question type of STACK. It is also a good starting point as a question type for new users of STACK to explore. In the following section, we will demonstrate the set-up of a fill-in-the-blank question: an algebraic input question from our designed Mathematical Biology e-assessment. We will provide an overview of writing the question using the **question variables** component of STACK, as introduced in Section 1.1.1, and include a general introduction of how to incorporate the question variables into the question text component of a STACK question. While setting up the question needs to use all four components of a STACK question, we will only focus on these two components to demonstrate the set-up and application of the **fundamental feature** of STACK: **randomisation**.

Example 1.2.1. An algebraic input fill-in-the-blank question from Question 2 (b) of our designed e-assessment.
As shown in Figure 1.2, students are given a modified biological system with equations modelling the population change of two biological species. They are then asked to nondimensionalise the system. In brief, the process of nondimensionalisation is writing the variables in terms of the dimensional and nondimensional parts, in order to eliminate some coefficients of the equations. In this example, it is writing $x = \hat{x}X$, $y = \hat{y}Y$ and $t = \hat{t}T$, where $\hat{x}, \hat{y}, \hat{t}$ are nondimensional and x, y and t are dimensional parts (with units). To make sure the answers are unique, students are also provided the nondimensionalised system of equations for them to compare the coefficients and write down the unknown variables: K, T, Y, X in terms of the known variables: a, b, c (and d).

Question Variables (written in Maxima)

To allow students to reattempt the question, we have created **three variants** to the equations of the system, as shown in Figure 1.3, such that the generated differential equations are not always the same. While it is possible to create a separate question for each variant and select a random question into the e-assessment, we will instead create a **randomised variable** called `variant` in the question variables component of a single STACK question, which allows us to better organise the question variants and enables easier editing, such that any future changes to a question can be reflected in all question variants.

We first create variable `variant` using STACK's `rand(lst)` function, which randomly picks an item from the list `lst`. It takes a randomised number from $\{1, 2, 3\}$:

```
/* Create 3 variants using STACK's rand function. */
variant: rand([1,2,3]);
```

Create a list containing the three variants for the terms corresponding to $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Then extract the pair corresponding to the randomly selected variant:

```
/* Create 3 pairs of equations depending on each variant.*/
terms_list: [[a*x-a*x*y/(x+b),-c*y+a*x*y/(x+b)],[a*x-b*x*y,-c*y+d*x*y],
            [a*x*(1-x)-b*x*y, -c*y*(1-y)+d*x*y]];

/* Given the variant, extract the pair from the list. */
[dxterm, dyterm]: terms_list[variant];
```

Similarly, we will also provide three versions of the nondimensionalised system of equations given each variant. Figure 1.2 shows the system of equations for `variant = 3`.

```
/* Create 3 different forms of nondimensionalised equations depending on each
    question variant. */
terms_nondim_list: [[x-x*y/(x+1),-K*y+x*y/(x+1)],[x-x*y,K*(-y+x*y)],
                    [x*(1-x)-x*y, K*(-y*(1-y)+x*y)]];

/* Given the variant, extract the pair from the list. */
[dxterm_nondim, dyterm_nondim]: terms_nondim_list[variant];
```

Consider a modified biological system between species X and Y, with (scaled) population x and y respectively.

It is given that

$$\frac{dx}{dt} = a \cdot (1-x) \cdot x - b \cdot x \cdot y$$

$$\frac{dy}{dt} = d \cdot x \cdot y - c \cdot (1-y) \cdot y.$$

(b) To perform nondimensionalisation, we can write $x = \hat{x}X, y = \hat{y}Y$ and $t = \hat{t}T$ and substitute in the system of equations. Finally, by treating $\hat{x}, \hat{y}$ and $\hat{t}$ as x, y and t respectively, we obtain the nondimensionalised system of equations:

$$\frac{dx}{dt} = (1-x) \cdot x - x \cdot y$$

$$\frac{dy}{dt} = K \cdot (x \cdot y - (1-y) \cdot y).$$

Express K, T, Y and X in terms of variables a, b, c and d.

Figure 1.2: A randomly generated question variant (`variant = 3`) of a fill-in-the-blank question: the figure at the top shows the given equations; the figure at the bottom shows the nondimensionalised equations for students to compare coefficients

It is given that

$$\frac{dx}{dt} = a \cdot x - \frac{a \cdot x \cdot y}{x+b}$$

$$\frac{dy}{dt} = \frac{a \cdot x \cdot y}{x+b} - c \cdot y.$$

(a) `variant = 1`

It is given that

$$\frac{dx}{dt} = a \cdot x - b \cdot x \cdot y$$

$$\frac{dy}{dt} = d \cdot x \cdot y - c \cdot y.$$

(b) `variant = 2`

Figure 1.3: Two other variants of the given equations

Note: It is also possible to randomise the coefficients a, b, c and d, but we have not done so because this question is testing students the procedure and let them practise with different systems of equations, instead of practising on substituting different numbers in the same system.

Express K, T, Y and X in terms of variables a, b, c and d.

$K =$

$T =$

$Y =$

$X =$

Figure 1.4: Fill-in-the-blank inputs for Example 1.2.1

Finally, we will define the model answer variables in the question variables component. Since some of the model answers are the same for all variants, it is quite simple to define the model answer variables and we can simply present them in `if/else` statements if we do not want to create a list:

```
/* ta2-5 correspond to the inputs for K, T, Y and X respectively. */
ta2: c/a;
ta3: 1/a;

/* Same value for variant 2 and 3. */
ta4: if variant = 1 then b else a/b ;
ta5: if variant = 1 then b else c/d ;
```

After setting up the question variables, we need to type in our randomised question variables in the **Question Note**. In our example, we can type in `Variant = {@variant@}` for our randomised variable `variant`. (You may refer to the following Question Text subsection for explanation of the code.) It helps STACK distinguish different question variants based on the equivalence in the Question Note.

Question Text (written in HTML)

We will omit the introduction of writing HTML code for the question text here since it is quite straightforward and not our main focus, but there are two important notes for users while writing the question text. Firstly, if we want to write equations or text formatted in LaTeX, we can use `\(...\)` for inline maths and `\[...\]` for displayed maths, which will be displayed in a new line [9, p. 13].

Secondly, we need to display the corresponding locations of the inputs and PRTs in the question text. For example, we would write `[[input:ans1]] [[validation:ans1]][[feedback:prt1]]` at the specified location in the question text which will be displayed to the student the corresponding input box `ans1` when they load in the question, and the corresponding PRT `prt1` when the answer is submitted. (Refer to Section 1.3 for the introduction of answer validation.)

The key is how we incorporate the defined question variables into the question text. Note that the question text is written in HTML, which is different from the question variables, but we can still include our question variables from Maxima using `{@var@}`, which will display the variable formatted in LaTeX.

Difference between `{#var#}` and `{@var@}`:
We can extract the question variables from Maxima using `{#var#}` or `{@var@}`, but they serve different purposes. Based on [9, p. 13], `{#var#}` extracts the unformatted value of the variable in CASText syntax, which is the same text type for the question text in STACK. This enables the value of the variable to be directly utilised. On the other hand, `{@var@}` extracts the variable and format it using LaTeX. It is usually used for displaying the variable to the students in the question text. These methods enable randomised variables to be applicable for displaying to the students or further use, e.g. in JSXGraph (Refer to Section 2.2.1).

1.3 Creating a numerical Fill-in-the-Blank Question: Inputs and Potential Response Tree

We have now discussed the set-up of an algebraic input question through the first two key components of STACK and introduced the randomisation feature. We would like to move on to another example that demonstrates the fundamental features in the **inputs** and the **PRT** components.

Example 1.3.1. A *numerical* input fill-in-the-blank Question from Question 1 (a) of our designed e-assessment.

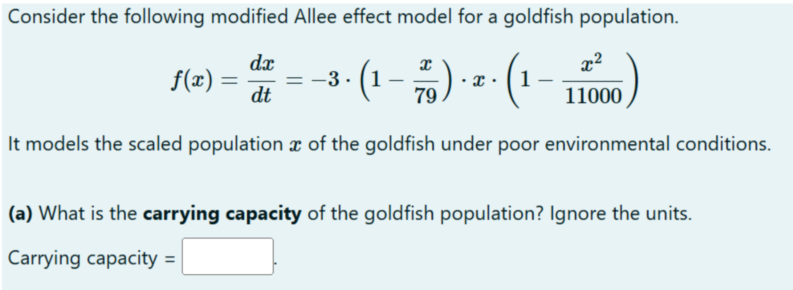


Figure 1.5: A fill-in-the-blank question with numerical input

Figure 1.5 shows a randomly generated question variant of a fill-in-the-blank question. The question has 3 variants with different equations for $f(x)$. In addition, students are provided with randomised variables of a, K and A where $f(x) = \frac{dx}{dt} = -ax\left(1 - \frac{x^2}{100K}\right)\left(1 - \frac{x}{A}\right)$ in this variant. The carrying capacity is the maximum root for the equation $\frac{dx}{dt} = 0$, that is $x = 10\sqrt{K} = 10\sqrt{110}$ in this example, which is irrational. Since the question does not specify that the student need to provide an exact answer, we should also expect the student's answers could be rounded off. Therefore, we will use numerical inputs in this case.

Inputs

In the input settings, we can change the input type from the default *algebraic input* to *numerical input*, which **requires students to type a numerical value**. It will reject any input with variables, which is very suitable for this example because students are only expected to provide a numerical answer. This also helps introducing the problem that sometimes students are confused about what form of the answer is expected, such as whether they should provide variables, numbers or strings. Therefore, STACK has provided a fundamental feature: **answer validation**. This function is enabled by default in the input settings, which allows students to verify what the system reads from **the expression they have provided**. It also displays the **variable list** which should only contain the variables that are intended. This prevents students from providing unexpected input types or making typos. The location of the validation is usually set up right after the input box, as introduced in Section 1.2.1. Fill-in-the-blank questions are particularly prone to typos, hence answer validation should always be shown to the students together with the variable lists for them to check their answers before submission. However, for some other question types such as MCQs, the validation can be turned off, since there is no need to validate the answers if the options are already provided for the student to select.

Potential Response Tree (PRT):

After setting up the input types, we will check the student's answer in the PRT. As we have mentioned that the model answer is irrational, we should allow the student's answer to be marked as correct within an acceptable range of error. This can be done by setting up appropriate answer tests in the True/False nodes of the PRT.

Answer Tests:
There are several types of answer tests in the PRTs. The default answer test is `algebraic equivalence`, where the expression provided in the input will be checked with the solutions in the PRT, whether they are exactly algebraically equivalent. However, for questions that only require students to provide an answer approximation, such as our example, we should change the type of answer tests in the PRT. Figure 1.6 illustrates the use of the `NumAbsolute` answer test of STACK, which checks that the following condition between the student's answer (`sa`) and the teacher's answer (`ta`) is satisfied: `|sa - ta| < opt`, where `opt` is the test option variable which accounts for the acceptable range of error between the two answers. If the student's answer is within the range, then it will be marked as correct, and vice versa. This ensures that the student's answer is at least rounded off to the nearest integer, in our example.

Besides, we can also provide feedback to the student to let them know where they could have gone wrong, such as the specific feedback provided in Figure 1.6. By predicting common student errors, we can set up several True/False nodes that provide students with different feedback based on their mistake. Common student errors will be discussed in Chapter 3 while feedback will be further discussed in Chapter 4.

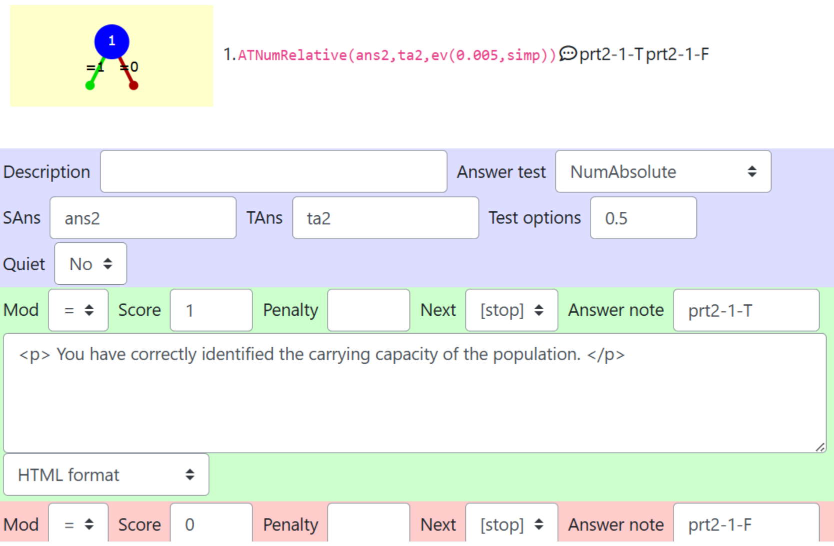


Figure 1.6: Set-up of the PRT with `NumAbsolute` answer test, where `opt = 0.5`

Chapter 2

Advanced Technical Features of 2 STACK Question Types

In the previous chapter, we have provided an overview of how to set up a fill-in-the-blank question in STACK, and introduced the fundamental features of STACK, in particular randomisation and answer tests. These two features are very useful and important in STACK e-assessments. They can also be implemented in other question types of STACK. In this chapter, we will explore more advanced technical features which are **specific** to two other question types of STACK.

2.1 Multiple-Choice Question (MCQ)

Multiple-choice question (MCQ) is a built-in question type well supported in STACK e-assessment. In STACK, there are three main types of input for MCQs. These include:

1. Radio,
2. Dropdown,
3. Checkbox.

The radio and dropdown menus return a single value. Hence, they only allow for one correct option in the question. The main difference between them is about how they present the mathematical expressions: Dropdown menus present expressions in Unicode text, while radio buttons support LaTeX formatting, which can present expressions more neatly and support more complicated notations.

Checkbox returns a list of values, e.g. `[A,B]`, i.e. choosing the list of options `A` and `B`. Hence, they allow for more than one correct options in the question. This enables partial scoring with respect to the number of correct options a student has selected, which will be discussed later in the partial scoring section in 2.1.1.

To set up an MCQ in STACK, we first create the question variables, which defines the multiple-choice options. Then, we will change the input type to checkbox, drop down or radio in the input settings. Finally, we set up the answers, scoring and feedback in the corresponding PRTs.

2.1.1 Main features of Multiple-Choice Questions

Randomisation and option shuffling in MCQs

Overview:
In a typical multiple-choice question, the position of options are fixed. Since students may reattempt the e-assessment, it is better to add some randomness to the question. STACK has internal functions that support the **shuffling of options**. We can make use of the `random_permutation` function, as implemented in Step 4 of Example 2.1.1, which shuffles the items in a given list. With shuffling, the correct options for the MCQ will not always appear in the same position in each question variant generated.

Besides, in order to increase the difficulty of the question, STACK supports the **random addition of distractor options** to a question. We will make use of the `random_selection` function of STACK, as implemented in Step 2 of Example 2.1.1, which randomly selects a specified number of items from a given list. This presents students with a different subset of the distractor options in each generated question variant, enhancing their understanding through evaluation of all options. Additionally, we can use the same STACK function to **randomly present students with a different subset of the correct options** in each question variant. This allows students to reattempt the question without encountering the same correct options each time.

Based on the STACK documentation page [10], which provides examples on many technical features of STACK MCQs, we have implemented and extended some features to our designed e-assessment questions, and we will conclude the steps for doing so.

Example 2.1.1. As shown in Figure 2.1, this is part (a) of the same question from Example 1.2.1. Students are first provided with a system of differential equations. Then they are asked to analyse the biological interpretation of the system with a checkbox type of multiple-choice question.

Note: As mentioned in Example 1.2.1, we have created three variants of the differential equations, each with different terms that lead to different correct options for this MCQ. In order to focus on explaining the advanced technical features of MCQ, we will assume the generated equation variant is `variant = 1`. However, the same method in Example 1.2.1 can be used to implement these features on all three variants.

Code set-up in the question variables, written in Maxima:

Step 1: Create two sets of options for the MCQ:

- A list of correct options (we name it `tat1`)
- A list of incorrect options, i.e. distractors (we name it `taf1`).

```
/* Set up a list of CORRECT answers*/
tat1: ["It is a Predator-prey model where X is prey and Y is predator.",
      "Y gets full so Y cannot consume X indefinitely.",
      "Both X and Y are following exponential growth or decay if we ignore
          the interaction terms."];

/* Set up a list of INCORRECT answers*/
taf1: ["It is a competitive Lotka-Volterra model.",
      "It is a Predator-prey model where X is predator and Y is prey.",
      "X gets full so X cannot consume Y indefinitely.",
      "Both X and Y are following logistic growth or decay if we ignore
          the interaction terms."];
```

Consider a modified biological system between species X and Y, with (scaled) population x and y respectively.

It is given that

$$\frac{dx}{dt} = a \cdot x - \frac{a \cdot x \cdot y}{x+b}$$

$$\frac{dy}{dt} = \frac{a \cdot x \cdot y}{x+b} - c \cdot y.$$

(a) Which of the following is/are the appropriate biological interpretation for the system above?

- ☐ Both X and Y are following logistic growth or decay if we ignore the interaction terms.
- ☐ It is a competitive Lotka-Voltera model.
- ☐ X gets full so X cannot consume Y indefinitely.
- ☐ It is a Predator-prey model where X is prey and Y is predator.
- ☐ It is a Predator-prey model where X is predator and Y is prey.
- ☐ None of the above

Figure 2.1: A checkbox type of multiple-choice question

Step 2: Use STACK's function `random_selection(taf1,n)` to randomly select `n` distractors from `taf1`. We choose `n = rand([2,3,4])`, where the function `rand(list)` generates a random item from `list`. The minimum value of `n` is 2, ensuring there are at least two distractors. Similarly, we can randomly select a specified number of correct options from `tat1`, where we choose `n = rand([0,1,2,3])`. Here, if `n = 0`, then the correct solution will be "None of the above" (refer to step 5).

```
/* Use STACK's built-in function random_selection to randomly choose 2
    elements from the list of incorrect answers */
taf1: rand_selection(taf1, rand([2,3,4]));
tat1: rand_selection(tat1, rand([0,1,2,3]));
```

Step 3: Apply `lambda` functions, which tag "true" to each element of `tat1` and tag "false" to each selected distractor from `taf1`. This creates the lists in the format:
`tat1_mapped: [[option1, true],[option2, true],...]` and
`taf1_mapped: [[option3, false],[option4, false],...]`.
Combine them using the `append` function to create the full list of options for the question.

```
/* Apply lambda function on each element of tat1, and tag true to them. */
tat1_mapped: maplist(lambda([ex],[ex, true]), tat1);

/* Apply lambda function on each element of taf1, and tag false to them. */
taf1_mapped: maplist(lambda([ex],[ex, false]), taf1);

/* Append the incorrect list to the correct list to produce the full
    question set ta1. */
ta1: append(tat1_mapped,taf1_mapped);
```

Step 4: Shuffle the options using STACK's `random_permutation` function.

```
/* Shuffle the options */
ta1:random_permutation(ta1);
```

Step 5: If applicable, add in the option “None of the above” at the bottom, which is conditional on whether the list `tat1` is empty.

```
/* Add in None of the above. */
ta1: if length(tat1)=0 then append(ta1,[["None of the above",true]])
     else append(ta1,[["None of the above",false]]);
```

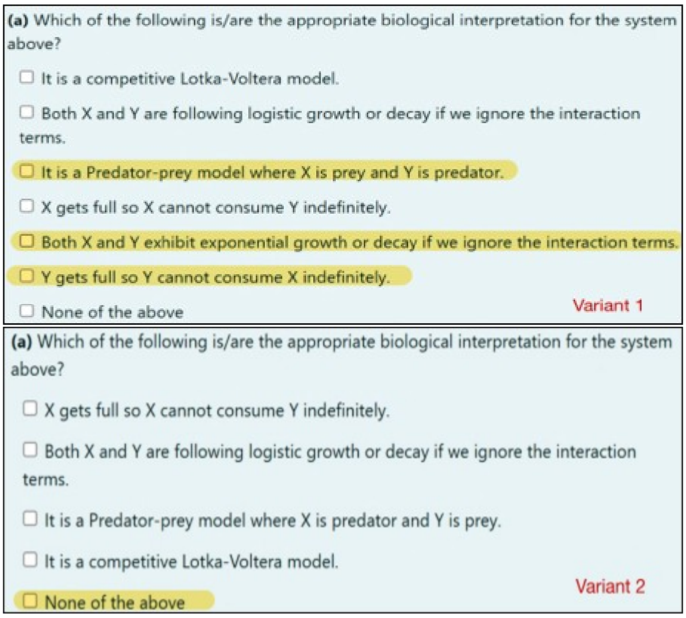


Figure 2.2: The resulting question: two randomly generated question variants

The resulting question variants randomly generated are shown in Figure 2.2. The correct options in each variant are highlighted. Excluding the option “None of the above”, Variant 1 contains 3 correct options (highlighted) and 3 distractors, while Variant 2 contains 0 correct option (hence “None of the above” is highlighted) and 4 distractors.

Partial scoring in checkbox type of multiple-choice questions

In radio and dropdown types of MCQs, there is only one correct option so a student will either be scored full mark or zero mark for the correct or incorrect option selected. Based on [11, p. 159], if this scoring method is also used in the checkbox type of MCQs, where there could be more than one correct answer, this is also known as **Dichotomous scoring**, which is paraphrased as: “one point is awarded if all correct options are selected and no wrong answer choices are identified. No partial credit is awarded.” Based on [11, p. 160], it requires the selection of all correct options yet applies a very strict penalty for any selection of irrelevant information. For the checkbox type of MCQs, especially for the cases there are many correct options, using the Dichotomous scoring method will be too harsh to only give the student zero mark for just one missing option while the other correct options selected. Hence, we want to apply more lenient partial scoring methods to the checkbox type of MCQ in our e-assessment questions.

There are many partial scoring algorithms for MCQs, as suggested in the paper [11, p. 159]. It compares several multiple choice scoring algorithms, such as Dichotomous, Subset, Plus/Minus (PM), and on p. 171, it concludes that **PM method is the most promising method**.

The Plus/Minus (PM) method algorithm is described as follows from [11, p.166]:
"This method scores any selection of a correct option as +1 point and any selection of a distractor option as −1 point. The total score on the item is then the sum of the response set, that is, adding the +1s and the −1s. If the sum is negative, the score is set to 0."
In other words, mathematically we can express the score as follows: Let N be the number of correct options, C and I be the number of correct and incorrect inputs a student has provided. Let S be the student's score for the question. Then we have

$$S = \begin{cases} \frac{C-I}{N} & \text{if } C > I \\ 0 & \text{if } C \leq I, \end{cases}$$

where we have normalised the score through division by N. This is to ensure that a student gets the score in the range $[0, 1]$.
Evaluating this method, it ensures students get fairly rewarded and penalised for the correct and incorrect options they have checked.
Note: It requires $N \geq 1$ so that there is no division by zero. Therefore, we should add in the option "None of the above" if there is no other correct option.

Example 2.1.2. To implement this partial scoring method in our STACK question, we need to change the set up for the marks awarded in the potential response tree (PRT).
STACK documentation provides the example for setting up the Subset partial scoring method [10]. We would like to **modify the code to the PM method**. In the following, we have defined that `ans1` is the list of answers submitted by the student, `ta1` is the list of the full question options. It is also worth noting about STACK's `mcq_correct` and `mcq_incorrect` functions, which select the options tagged as "true" or "false", according to [10].

Feedback variables in the question's corresponding PRT with the **Subset partial scoring method** provided by STACK documentation:

```
/* Find the incorrect inputs from the student. */
incorrectInputs : sublist(ans1, lambda([x], member(x, mcq_incorrect(ta1))));

/* If no incorrect option is selected, the student will be awarded the
    proportion of correct options they have selected.*/
 mark : if emptyp(incorrectInputs) then length(ans1) / length(mcq_correct(ta1))
        else 0;
```

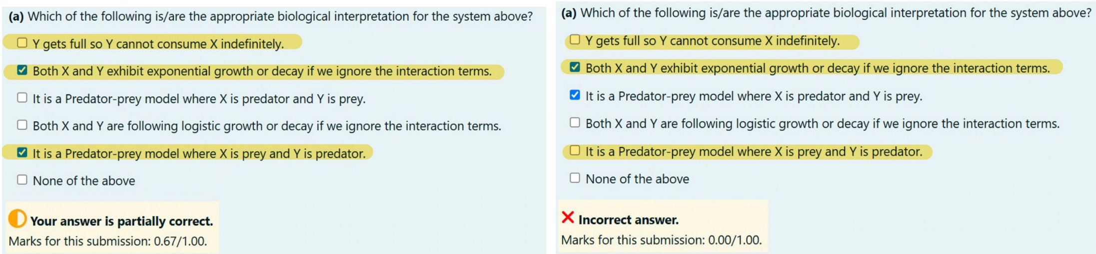
We will modify it to the **Plus/Minus method** using the following code:

```
/* Find the incorrect inputs (I) from the student. */
incorrectInputs : sublist(ans1, lambda([x], member(x, mcq_incorrect(ta1))));

C: length(ans1)-length(incorrectInputs);
I: length(incorrectInputs);
N: length(mcq_correct(ta1));

mark: if C>I then (C-I)/N else 0;
```

We can apply partial scoring to Example 2.1.1. We first use the code above in the feedback variables in the PRT. Then, we put `mark` as the score S the student awards for this question.



(a) $C = 2, I = 0, N = 3$ (b) $C = 1, I = 1, N = 3$

Figure 2.3: Implementation of partial scoring on 2 different inputs from the same question variant

In Figure 2.3, the two figures (a) and (b) are the same question variant but with different inputs, which could be submitted by two different students or by two separate attempts of a student. The correct options are highlighted.

In attempt (a), the student got 2 options correctly selected and 0 option incorrectly selected. Hence, since $C > I$, the student's score is $S = \frac{2-0}{3} \approx 0.67$.
In attempt (b), the student got 1 option correctly selected and one option incorrectly selected. Hence, since $C = I$, the student's score is $S = 0$.

2.2 Interactive Graph Question

Interactive graphs allow students to consolidate their learning through engagement with visualisation of graphs. They can be applied in STACK questions in many ways, e.g. moving objects, constructing graphs, visualising change of parameters and transformations.

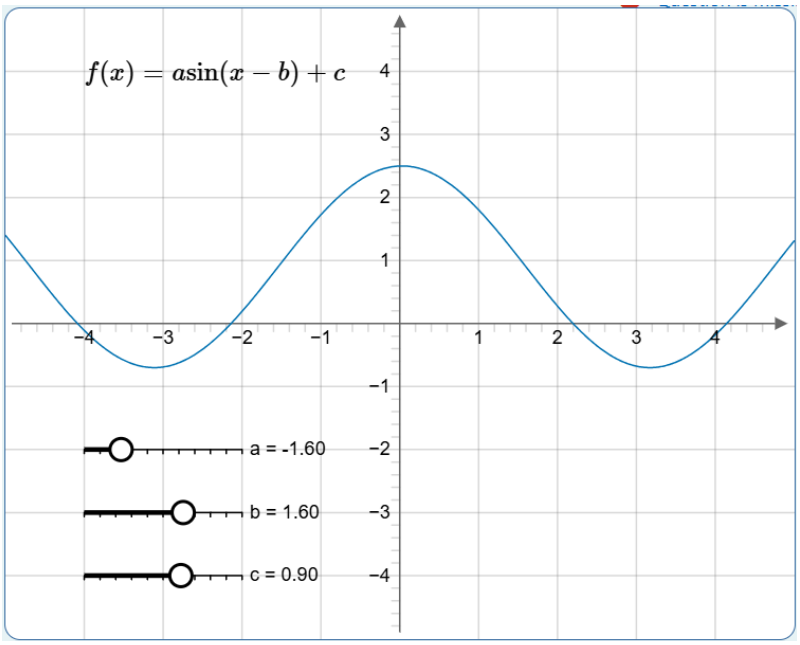


Figure 2.4: A simple example of visualising change of parameters with JSXGraph

STACK supports basic plotting using Maxima's `plot2d` function. However, to include dynamic graphs in STACK questions, which involves more advanced functions such as interactivity with students' inputs, we will make use of JSXGraph. According to JSXGraph webpage [12], "JSXGraph is a cross-browser JavaScript library for interactive geometry, function plotting, charting, and data visualization in the web browser". Building interactive graphs using JSXGraph requires more advanced coding due to the technical complexity of applying advanced JavaScript settings and functions. It is therefore recommended for users to first fully grasp the idea of fill-in-the-blank and multiple-choice questions set-up before delving into JSXGraph implementation.

2.2.1 Main features of Interactive Graph Questions

Moving points to the required positions

In some maths questions, students may need to locate a point in a graph, such as finding the local and global maxima, and points of inflexion of a function. This can be done in STACK questions with the implementation of JSXGraph. Students will be asked to move a point to the required position in the graph. The coordinates of the point will be stored in the system and updated instantaneously as students drag the point, and the final coordinates of the point submitted can be checked with the teacher's answer, and will be marked as correct if the coordinates are within an acceptable range of error, mainly due to manual point dragging.

To create such a question, we will need to set up the code in both the question text and the corresponding PRTs. We will first implement JSXGraph in the question text. The code should be embedded inside the blocks `[[jsxgraph]]` and `[[/jsxgraph]]`. We start by creating a board and set its dimensions and attributes:

```
var board = JXG.JSXGraph.initBoard(divid, {boundingbox: [left, top, right,
    bottom], axis: true, showCopyright: false});
```

Then we can create other objects such as the functions and the points. If we have defined randomised parameters in the question variables of STACK, we might want to utilise them in our JSXGraph.
As introduced in Section 1.2, we can extract the question variables from Maxima using `{#var#}` or `{@var@}`. Here, we will use `{#var#}` since it enables the variable to be directly utilised in the code, and further formatted by the function `board.jc.snippet` into JavaScript code.

The most complex aspect of coding with JSXGraph is **making the objects interactive**, such as constantly **updating** the coordinates of a point and **storing** its coordinates in the system for checking with the solution. In addition, if a student saves and reloads the page, we need to make sure that the state of the graph is saved so that returning to the graph will show the previous state, e.g. where they left their point at, how much they zoomed in the graph. To achieve these, STACK provides several methods to store these changes. Based on STACK documentation [10], we will first introduce the **general method**, which stores the input type as a **"string"**. The alternative method is more specific: the input binding functions of STACK, which will be discussed later in the buttons section in 2.2.1, but we will first introduce "string" type storage which works for most types of object.

Step 1: We first need to define where we store the state of our point. We will link it to an input, e.g. `stateStore` by changing the JSXGraph block to

`[[jsxgraph input-ref-stateStore = "stateRef"]]`, where `stateRef` is the element identifier for the input element `stateStore`.

Step 2: Set up a variable `state` inside JSXGraph which defines the default coordinates of a point.

```
/* State represented as a JS-object, first define default, e.g. (0,0). */
var state = {'x':0, 'y':0};
```

Step 3: Load the stored coordinates of the point if exist, from the reference `stateRef` and convert them into JavaScript object.

```
var stateInput = document.getElementById(stateRef);
if (stateInput.value && stateInput.value != '') {
    state = JSON.parse(stateInput.value)};
```

Note: it is essential that Step 3 is done after Step 2 because we want to make sure that any stored coordinates, if exist, will be loaded replacing the default. These stored coordinates come from previous attempts of the question, if exist.

Step 4: Create the point and make it "interactive", which updates the state with the coordinates of the point whenever a student drags it, so that the student can save the page and return back to the question with the point located at where it was left.

```
var point = board.create('point',[state['x'],state['y']]);
point.on('drag', function() {
    var newState = {'x':point.X(), 'y':point.Y()};
    /* Encode the state as JSON for storage and store it. */
    stateInput.value = JSON.stringify(newState);
    /* Dispatch an event to notify about the changed value. */
    stateInput.dispatchEvent(new Event('change'));
});
```

Example 2.2.1. Figure 2.5 illustrates a Mathematical Biology question where students are provided with an interactive graph displaying the graph of $f(x)$ against x. Again, we have set up several variants on $f(x)$, but we will focus on `variant = 1` in the following for clearer explanation, where $f(x) = \frac{dx}{dt} = -ax\left(1 - \frac{x}{K}\right)\left(1 - \frac{x}{A}\right)$. It is given that a, A and K are randomised variables where $a > 0$ and $0 < A < K$. Students are also given 2 movable points and 3 buttons (which will be disucussed in the buttons section in 2.2.1). The graph describes the biological phase plot of x, where x represents a population. Students are asked to **move the blue and red points to the stable and unstable equilibria of the system respectively**.

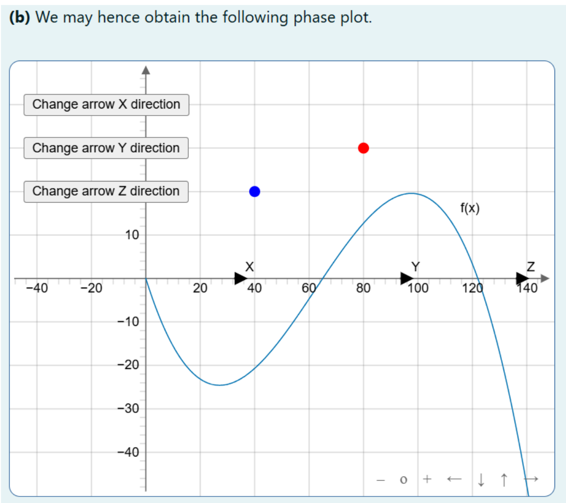


Figure 2.5: Example of an interactive graph question asking students to drag the points to the correct positions

The JSXGraph code for this question is as follows:

```
[[jsxgraph input-ref-stateStore="stateRef" input-ref-stateStore2="stateRef2"]]
  /* Store the references for the two inputs (blue point and red point) in
      "stateRef" and "stateRef2".*/

  /* Create a normal board. */
  /* boundingbox:[left, top, right, bottom]. */
  var board = JXG.JSXGraph.initBoard(divid, {boundingbox: [-50, 50, 150, -50],
      axis: true, showCopyright: false});

  var f = board.jc.snippet('{#fx#}', true, 'x', true);
  board.create('functiongraph', [f,0,150],{name:'f(x)', withLabel:true});

var state = {'x':40, 'y':20};
  var stateInput = document.getElementById(stateRef);
  if (stateInput.value && stateInput.value != '') {
    state = JSON.parse(stateInput.value);
  };

var state2 = {'x':80, 'y':30};
  var stateInput2 = document.getElementById(stateRef2);
  if (stateInput2.value && stateInput2.value != '') {
    state2 = JSON.parse(stateInput2.value);
  };

  /* Then make the graph represent the state. */
  var p = board.create('point',[state['x'],state['y']], {Color : 'blue', name :
      ' ' });
  var p2 = board.create('point',[state2['x'],state2['y']], {Color : 'red', name
      : ' ' });

  /* Update the stored state when things change. */
  p.on('drag', function() {
    var newState = {'x':p.X(), 'y':p.Y()};
    stateInput.value = JSON.stringify(newState);
    stateInput.dispatchEvent(new Event('change'));
  });

  p2.on('drag', function() {
    var newState2 = {'x':p2.X(), 'y':p2.Y()};
    stateInput2.value = JSON.stringify(newState2);
    stateInput2.dispatchEvent(new Event('change'));
  });

[[/jsxgraph]]
```

It generates a graph in a drawing board, with two movable points, red and blue, at the default positions (40, 20) and (80, 30) respectively. Their coordinates are stored in the system and updated instantaneously when dragged.

Then, we want to set up the corresponding potential response trees (PRT) to check the answers provided by a student. Note that there are only a maximum of three equilibria in the system since the function is cubic. The stable equilibria are at $(0, 0)$ and $(K, 0)$, and the unstable equilibrium is at $(A, 0)$.

We will first set up the feedback variables for answer checking.
Feedback variables in the corresponding PRT:

```
tmp:stackjson_parse(stateStore);
sx:stackmap_get(tmp,"x");
sy:stackmap_get(tmp,"y");
isongraph:numabsolutep(at(fx,[x=sx]),sy,5);
```

Through the first three feedback variables, we have obtained the final stored coordinates of the variable referenced by `stateStore`, in this case, it is the blue point (the stable equilibrium). For the last variable, we have applied the STACK function `numabsolutep(sa, ta, tol)`, which checks whether the first argument (`sa`) is within Euclidean distance of the third argument (`tol`) from the second argument (`ta`), and returns either `True` or `False`. The variable `isongraph` defined using this function checks that the student's answer (the blue point's coordinates) are on the graph within an acceptable range of error (5 units).

In the PRTs, we will also check whether the following conditions are satisfied by the student's answer given by the coordinates of the blue/red point. The structure of the corresponding PRT is as follows, consisting of several True/False nodes, as shown in Figure 2.6.

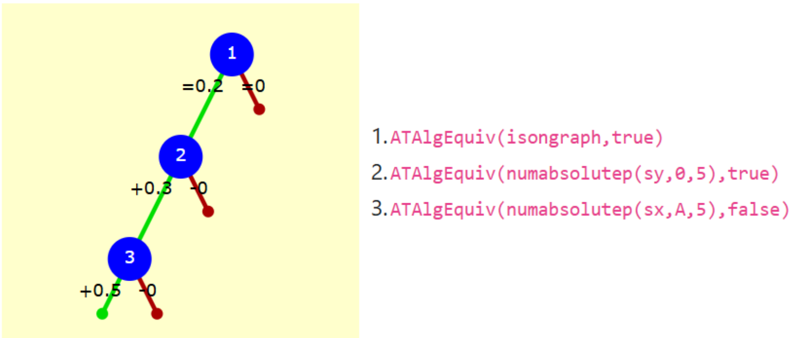


Figure 2.6: Corresponding Potential Response Tree with partial scoring for Example 2.2.1

Node 1: Is the blue point on the functioning graph?
This can be checked by the `isongraph` Boolean variable that we defined earlier, which checks whether the blue point is on the graph, and returns `TRUE` or `FALSE`. If the node is marked as `TRUE`, student will be awarded 0.2 marks and move on to Node 2, else `STOP`.

Node 2: Is the y-coordinate of the blue point zero?
We need the equilibrium to be on the x-axis, meaning that $\frac{dx}{dt} = 0$. This can be verified using `numabsolutep(sy, 0, 5)`, which checks whether the y-coordinate of the blue point is within $[-5, 5]$. If the node is marked as `TRUE`, student will be further awarded 0.3 marks and move on to Node 3, else `STOP`.

Node 3: Is the blue point a stable equilibrium?
This can be checked using the function `numabsolutep(sx,A,5)`, which checks whether the student's answer `sx` is within $[A-5, A+5]$. As mentioned earlier, there are only 3 equilibria of the system, and the stable equilibria are located at $(0,0)$ and $(K,0)$. Therefore, to make it simple, we can set the teacher's answer to be `FALSE` in response to the function `numabsolutep(sx,A,5)`. This excludes any possibility of choosing an unstable equilibrium, and so the student's answer have to be one of the two stable equilibria. If the node is marked as `TRUE`, the student will be further awarded 0.5 marks and `STOP`.

We will apply similar approach in the PRT for the red point (unstable equilibrium), which will not be repeatedly discussed here.

Buttons

Example 2.2.2. Buttons can be implemented in Example 2.2.1 to guide the students through classifying the equilibria positions. In Figure 2.5, the three buttons: `change arrow X/Y/Z direction` control the directions of arrow X, Y and Z in the graph respectively. These arrows should point at the direction of change in population at the point where the arrow tail is located: $\left(\frac{A}{2}, 0\right), \left(\frac{A+K}{2}, 0\right)$ and $\left(\frac{K+150}{2}, 0\right)$ respectively. These arrows are positioned such that they won't overlap with the equilibria, where the change in population will be zero. In a typical phase plot, the arrow heads should point at the stable equilibria and the tails should point at the unstable equilibria. We can perform this in JSXGraph by creating buttons that alternate the directions of the arrows, so that they will point at opposite directions.

Firstly, we will create the starting points (Xs, Ys, Zs) and ending points (Xe, Ye, Ze) of the arrows X, Y, Z (to be used later), and then set up the arrows. The following JavaScript code snippet does the above:

```
var Xs = board.create("point", [{#A#}/2, 0],{size:0 ,name:
    "X",color:"blue",visible: true,fixed:true});
var Xe = board.create("point", [{#A#}/2+5, 0],{visible: false,fixed:true});
 var Ys = board.create("point", [({#A#}+{#K#})/2, 0],{size:0,name:
     "Y",color:"blue",visible: true, fixed: true});
var Ye = board.create("point", [({#A#}+{#K#})/2+5, 0],{visible:
    false,fixed:true});
var Zs = board.create("point", [(150+{#K#})/2, 0],{size:0,name:
    "Z",color:"blue",visible: true,fixed:true});
var Ze = board.create("point", [(150+{#K#})/2+5, 0],{visible: false,fixed:true});

var X = board.create('arrow', [Xs, Xe],{name: "X", color:"black",size: 50});
var Y = board.create('arrow', [Ys, Ye],{name: "Y", color:"black",size: 50});
var Z = board.create('arrow', [Zs, Ze],{name: "Z", color:"black",size: 50});
```

Note that we have changed some attributes of the objects so that they will be **fixed**, i.e. cannot be manually dragged by the student.

We then create buttons that alter the directions of arrows X, Y and Z.

```
var button1 = board.create('button', [-45, 40, 'Change arrow X direction',
    function(){
     var Xs_temp = [Xs.X(), Xs.Y()];
     Xs.moveTo([Xe.X(), Xe.Y()], 10);
     Xe.moveTo(Xs_temp, 10);}]);
var button2 = board.create('button', [-45, 30, 'Change arrow Y direction',
    function(){
     var Ys_temp = [Ys.X(), Ys.Y()];
     Ys.moveTo([Ye.X(), Ye.Y()], 10);
     Ye.moveTo(Ys_temp, 10);}]);
var button3 = board.create('button', [-45, 20, 'Change arrow Z direction',
    function(){
     var Zs_temp = [Zs.X(), Zs.Y()];
     Zs.moveTo([Ze.X(), Ze.Y()], 10);
     Ze.moveTo(Zs_temp, 10);}]);
```

For each arrow, we have set up a button with a function that acts on upon pressing the button, in which the starting point and the ending point of each arrow will be switched.

This is further acted on by the functions `pt.moveTo(newpt, duration)`. Its first argument (`newpt`) controls the coordinates where the point (`pt`) will be moved to, while the second argument (`duration`) controls the time it takes for the movement, here we have chosen 10 milliseconds. Pressing the button once simply switches the arrows from pointing left to right, and vice versa.

Storing arrow directions and checking the solutions with one-way binding:
We could use the general method from STACK as discussed earlier to store the state of the arrows in the graph for further checking. However, STACK also introduces **convenient binding functions** that can particularly handle simple **points and sliders**. These binding functions can bind the inputs to the interactive elements in the graph. There are two methods in doing so.

One-way binding:
In this method, changing the elements in a graph can update the inputs, but not the other way round. For example, if the input (`ans1`) is bound to the coordinates of a point on the graph, then moving the point will update the coordinates in `ans1`. However the student cannot update the graph by changing the input `ans1`. This can be implemented by setting the input field `ans1` to be invisible by the students.

Two-way binding:
This method allow the changes in **both ways**. Changing the elements in a graph can update the inputs, and the same opposite way round. This binding method can be applicable for people who have **difficulty in manually controlling the attributes in the graph accurately**, potentially due to equipment difficulties or disabilities.

Moreover, the greatest advantage of this convenient binding method is that the coordinates of the points will be stored as algebraic inputs instead of strings, which greatly streamlines the process of setting up the code.

For this example of implementing buttons for controlling arrow directions, we will apply the function `stack_jxg.bind_point_dual(inputRef, point1, point2)` which stores the coordinates of the starting and ending points of the arrows and bind their coordinates to the corresponding inputs as an array, through their element identifiers. For example, the input `ans12` referenced by the element identifier `ans12Ref` is bound to the coordinates of the points Xs and Xe by the binding function.

```
stack_jxg.bind_point_dual(ans12Ref, Xs, Xe);
stack_jxg.bind_point_dual(ans13Ref, Ys, Ye);
stack_jxg.bind_point_dual(ans14Ref, Zs, Ze);
```

Whenever a change is made to the starting or ending points of an arrow, e.g. when a button is pressed, the input will display an **array**, such as $[[p, q], [r, s]]$, showing the instantaneous coordinates of the starting and ending points of arrow X, where $Xs = (p, q)$ and $Xe = (r, s)$.

However, by default settings of STACK, **before** the button is pressed, no coordinates will be shown in the input. In other words, if a student does not press the button, no answer will be submitted even though the arrow in the graph points to the right by default.

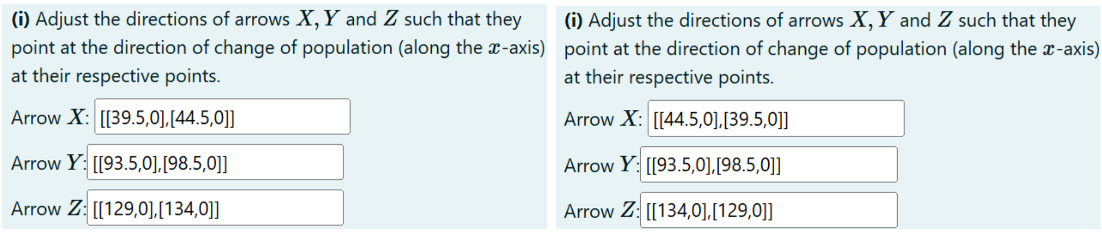
Therefore, we will **update the inputs** for the three arrows **before setting up the buttons**. This enables the inputs display the default coordinates of the arrows before any button is pressed. For example, we set up the following code for `ans12`:

```
var ans12Input = document.getElementById(ans12Ref);
/* Only update the input with the default arrow direction if no previous state
    are found in the input. */
if (!ans12Input.value || ans12Input.value == "") {
    ans12Input.value = "[["+Xs.X()+","+Xs.Y()+"],["+ Xe.X()+","+Xe.Y()+"]]"
  }
  /* Dispatch an event to notify about the changed value. */
ans12Input.dispatchEvent(new Event('change'));
```

We will also do the same for `ans13` and `ans14`.

Note: This approach is specific to our example because usually we expect the student to provide an input, e.g. by moving a point. However, in this example, the student could have not pressed the button at all. This is simply about whether to alternate the arrow directions or not.

In a randomly generated question variant, the three arrays for arrows X, Y and Z are as shown in Figure 2.7.



(a) Default arrows coordinates

(b) Teacher's solutions

Figure 2.7: Input fields with the implementation of the `stack_jxg.bind_point_dual` functions

Figure 2.7 (a) shows the three arrays corresponding to the arrows pointing to the right by default. After pressing the buttons for arrows X and Z (and keeping arrow Y unchanged), we have that the starting and ending points inside the array swapped for arrows X and Z, which changes the directions of arrows X and Z to the left, giving the teacher's solutions in (b).

*Note: In this question, we have shown the three arrays just **for demonstration purposes.** However, when we display the question to the students, we would like to implement **one-way binding** by **hiding the input fields**. This is because we want the students to control the arrow directions only by pressing the buttons, but not modifying the inputs from the arrays since they may not understand the meaning of the arrays. Hence, the inputs that store the arrays for the three arrows should be embedded inside the html span tag, with the style set to be `"display:none"` to hide them.*

In the corresponding PRTs, we can simply check the algebraic equivalence of the arrays with the actual solutions of the arrays corresponding to the arrows pointing left or right. This does not require an acceptable range of error, unlike the example in the movable points 2.2.1. This is because the coordinates of the points are only controlled by the buttons that can precisely change the coordinates in the input arrays.

Sliders

JSXGraph has extensive object types and attributes to explore, one of which is sliders, which are very useful for students to visualise the effect of parameter changes in the graphs. They will be able to experiment with different inputs by adjusting the sliders.

Example 2.2.3. The following Mathematical Biology question is designed to help students understand bifurcation phenomena through bifurcations: variations in model parameter values such that the stability or number of equilibria changes.

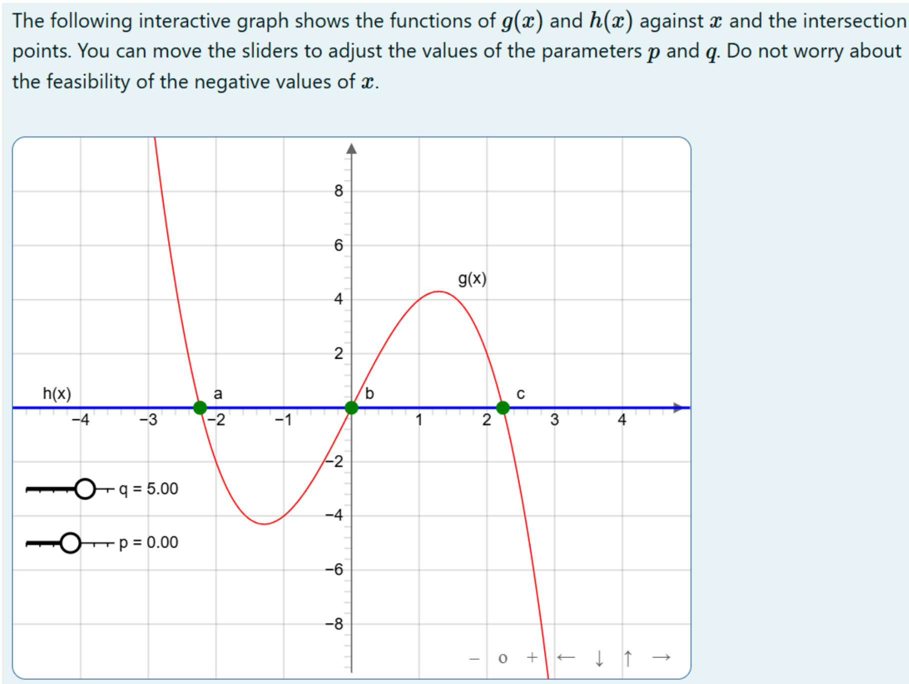


Figure 2.8: Implementation of sliders in a question to modify parameter values for understanding bifurcation phenomena

In the example in Figure 2.8, students are given two graphs $g(x) = -x^3+qx$ and $h(x) = -p$ with unknown biological parameters p and q, which can be adjusted by the sliders, as shown in Figure 2.9. For the values of p and q such that the graph $g(x)$ has **two turning points**, $g(x)$ is divided into three sections, separated at the two turning points, where the three sections are labelled as $g_0(x)$, $g_1(x)$ and $g_2(x)$, from left to right. The intersection points of the graph $h(x)$ with $g(x)$ will also be shown in the graph. Each intersection point will be labelled as a, b or c given which **section** of $g(x)$ intersects $h(x)$, rather than representing the actual coordinates. This means that all intersection points that intersect section $g_0(x)$ will be labelled as a on the graph, and all points that intersect section $g_1(x)$ will be labelled as b, and the same applies to the third section. The labelling of the intersection points has specific meaning in Mathematical Biology but is not important here.

At the values of p and q where the graph has **no turning points**, we will define the section in the left half plane as $g_0(x)$, the section in the right half plane as $g_2(x)$ and the origin itself as section $g_1(x)$.

Students will also be provided with an interactive graph for each part of the question where they are asked to adjust the two sliders for p and q to the correct positions. As shown in Figure 2.9, when the two sliders are adjusted, the shapes of the graph and the intersection points change. This helps students deduce relationships between the variables and the graph, helping them visualise bifurcation concepts.

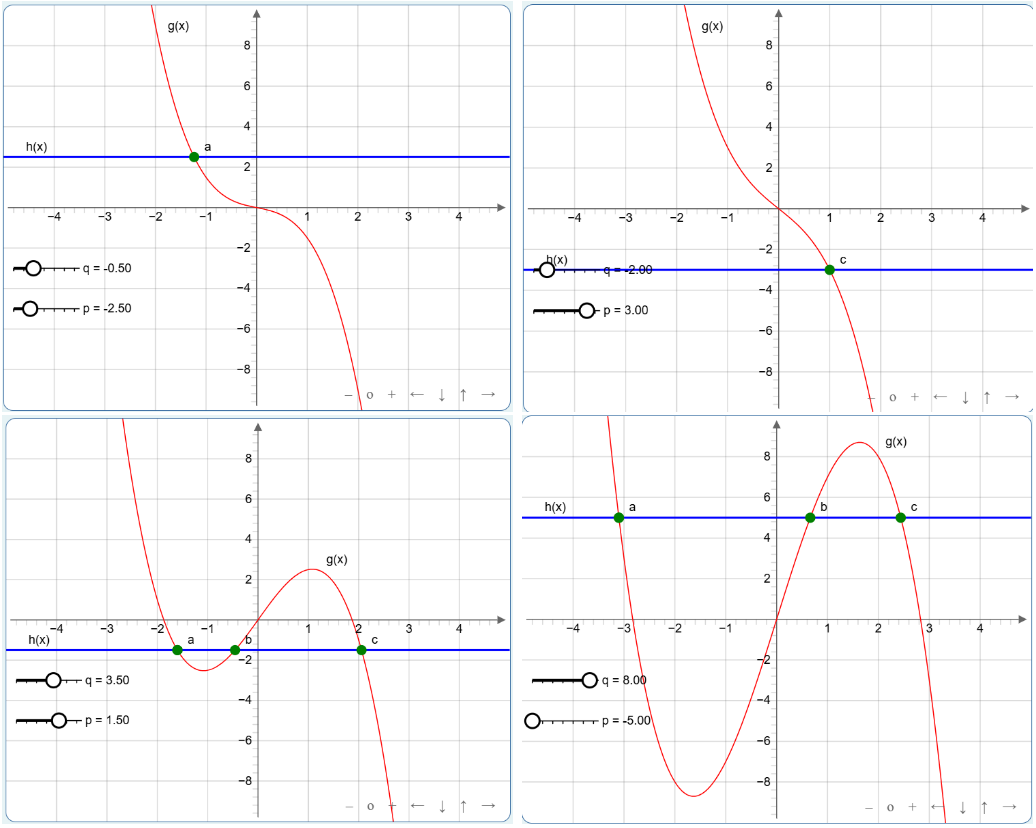


Figure 2.9: Example of graphs generated by moving the two sliders to different values

Creating sliders:

```
var p = board.create('slider', [[-4.8, -5], [-3.5,-5], [-5, 0, 5]],{name:'p',
    snapWidth: 0.5});

var q = board.create('slider', [[-4.8, -3], [-3.5,-3], [-5, 5, 10]],{name:'q',
    snapWidth: 0.5});
```

In the example code above, we have 2 parameters p and q presented as slider objects. The `board.create` function for a slider has 3 arguments. The first argument simply defines the object type is "slider". The second argument is an array with 3 lists. The first two lists defines the starting and ending positions of the slider. The final list has 3 arguments, defining the range of values and the default starting value.

For example, in our variable p, it specifies that the position of the slider will begin at coordinates $[-4.8, -3]$ and end at $[-3.5, -5]$ in the graph. $[-5, 0, 5]$ implies that the default starting value of the slider p is at $p = 0$, with the range of values from -5 to 10 adjustable by the slider. Finally the last argument specifies further properties, such as setting the name to be "p", while `snapWidth` controls the values returned by the slider as integer multiples of 0.5. Hence, the slider can only be adjusted to values such as $\{-1.5, 0, 0.5, 4.5\}$ within the specified range.

Next, it moves on to the problem that we want to adjust the shapes of the graphs $g(x, q)$ and $h(x, p)$ subject to the instantaneous changes in parameter values adjusted by the sliders. We need to use the control `(var).Value` to obtain the value of a variable `var`. However, in order to make sure it always gives the most updated value adjusted by the sliders, we need to always embed it within a function, e.g. `function() {return p.Value();}`. This allows the graph to always adjust itself with the updated parameter values.

For example, when we set up the variable for the graph $g(x)$, we will use the following code snippet:

```
var gx = board.create('functiongraph', [function(x) {return -Math.pow(x, 3) +
    q.Value() * x;
}, -5, 5], {strokeColor: "red", name: "g(x)", withLabel:true});
```

"Intersection" object type in STACK

Another point to note is that JSXGraph mainly supports finding intersection points between circlines (circles and lines or between two circles), by implementing the `intersection` object type in `board.create` function.

Intersection points between a functioning graph and a line are achievable, but it can only find a maximum of **2 distinct intersection points**, by using the positive and negative square roots. However, in our example, we have 3 intersection points for some values of p and q, since $g(x)$ is a cubic polynomial. Therefore, in order to find 3 intersection points in our example, we have divided the graph $g(x)$ into three parts: $g_i(x)$, $i = 0, 1, 2$, separated at the turning points $\pm\sqrt{\frac{q}{3}}$, as introduced earlier. Then we create three objects of type: `intersection` between each of $g_i(x)$ and $h(x)$:

```
var intersection1 = board.create('intersection', [gx0, hx, 0], {name: 'a',color:
    'green'});
var intersection2 = board.create('intersection', [gx1, hx, 0], {name: 'b',color:
    'green'});
var intersection3 = board.create('intersection', [gx2, hx, 0], {name: 'c',color:
    'green'});
```

Finally, the intersection points should display different names ('a', 'b', 'c') due to their intersection with the three different parts of the graph. This can be easily done since we have divided the graph into three parts. However, notice that for the variable $q < 0$, the graph will be shown as in the second figure in Figure 2.9. In this case, the turning points $\pm\sqrt{\frac{q}{3}}$ will have imaginary values, which will not be displayed by the system. To make it clear, we will further set up a variable `intersection4`, which basically defines the name of the intersection point as follows:

$$\text{name(intersection4)} = \begin{cases} \text{“ a ”} & \text{if } p < 0 \text{ and } q < 0 \\ \text{“ b ”} & \text{if } p = 0 \text{ and } q < 0 \\ \text{“ c ”} & \text{if } p > 0 \text{ and } q < 0 \\ \text{“ ”} & \text{otherwise .} \end{cases}$$

In other words, if there is **no turning point**, the intersection point will be named "a" if it has negative x-coordinate, "b" if it is at the origin, and "c" if it has positive x-coordinate:

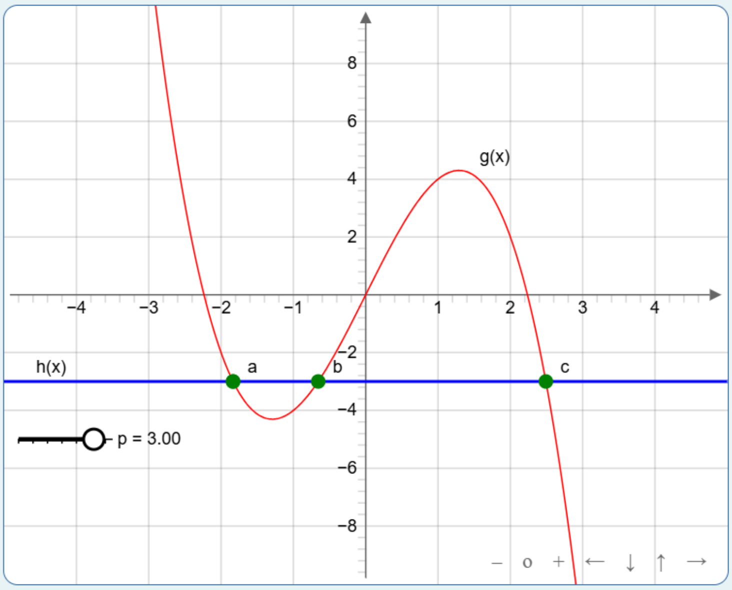
```
var intersection4 = board.create('intersection', [gx, hx, 0], {name: function(){
if (q.Value()<0 && p.Value()<0){return 'a'}
else if (q.Value()<0 && p.Value()==0){return 'b'}
else if (q.Value()<0 && p.Value()>0){return 'c'}
else {return ' '}}, color: 'green'});
```

Input binding for sliders: two-way binding

Similar to the input binding functions for the arrows mentioned in Section 2.2.2, STACK also provides a convenient slider binding function: `stack_jxg.bind_slider(ansXRef, p)`, which binds the value of a slider `p` to the specialised input `ansX`. The implementation code is very similar to the arrows binding method introduced in Section 2.2.2.

As discussed earlier in 2.2.3, the students are provided with an interactive graph for each part of the question that requires them to move the slider to the required position. For example, part (b)(i) of the question is shown in Figure 2.10. To demonstrate bifurcation

(b) (i) Consider the interactive graph below.



In the graph, q is fixed at 5.

Assume initially $p = 0$, $x = a$. What value should p be moved to such that x will eventually shift to c, given that the change in p is minimal in magnitude? **(*)**

$p =$ 3.

Figure 2.10: Part (b)(i) of an interactive graph question with slider binding

phenomena in Mathematical Biology, we have also fixed one of the parameters q at 5, and ask the student to move p to a required value. This can be easily done by modifying the code:

1. Keep p as a slider but changing q back to a constant: `var q = 5`.
2. Change `q.Value()` back to `q`.
3. Bind the slider p to the input `ans5` using the slider binding function:

   `stack_jxg.bind_slider(ans5Ref, p)`

It is a **two-way binding** method which means that unlike the arrows binding example, we will display the input blank to the student so that it allows the student to either adjust the value of the slider in the graph or manually type the value in the input blank, which can then be reflected in the graph. This provides students with **two methods of providing the inputs**. It also benefits students with **difficulty in controlling the sliders**, especially when the `snapWidth` of the slider is set very small, it will be more difficult to precisely adjust the values. In this case, students can choose to type the values instead.

Answer tests:
The question asks the student to find the value of p **approximately**. This is because the model answer for p is irrational, and also there could be manual errors in dragging the sliders or typing the exact answer only by observing the graph. Therefore, we can implement the `NumAbsolute` answer test as introduced in Section 1.3.1, where we set the student's input `ans5` to be marked as correct if `|ans5 - ta5| < 0.5`. Here `ta5` is the model answer and 0.5 is chosen to be the test option because it matches with the slider having `snapWidth = 0.5`.

Chapter 3

Pedagogical Analysis Presented with 3 Case Studies

In the previous chapter, we discussed the technical features of three question types of STACK in detail, namely fill-in-the-blank, multiple-choice and interactive graph questions. From a pedagogical perspective, why have we chosen to implement these features in our questions?

We believe that each technical feature could serve a different purpose in enhancing the question design, and each question type also serves a different pedagogical function. In the following chapter, we will shift our focus from technical to pedagogical perspective on maths e-assessments. We will present **three case studies** on the three questions from our proposed e-assessment. We will start by explain the purpose of each case study and the concepts or skills analysed in each question. Then we will examine the benefits of implementing the technical features in the specific questions, and finally analyse the expected **common student errors** of selected questions. These errors are classified according to a **model of taxonomy** and then related back to the concepts the students may have misunderstood. Since the e-assessment questions have not been released to the students, we do not have student response data for analysis of common student errors. Hence, we will identify the errors in the following case studies based on our own interpretation of potential mistakes that students who take the module would commonly make. In addition, the specifics of the question design, such as MCQ distractors, could also help us in identifying these errors, which can reflect the students' area of misunderstanding in a concept or misuse of a method in the question.

The three questions in our case studies are designed to support students in revising the materials covered in the Michaelmas term of the module: **Mathematical Biology III**. However, students are recommended to attempt the questions progressively, since their focus are different and they are designed to support students in revision based on their understanding and revision progress.

3.1 Model of Common Student Errors (CSE)

Before we delve into our case studies, we will first introduce a model of taxonomy which classifies common student errors in mathematics.

In 2018, an academic paper published from Cardiff University provided a model of taxonomy for classifying common student errors from undergraduate mathematics students [13]. Based on [13, p. 37], there is research on the common student errors made by pupils in primary or secondary education. However, these errors are usually more simple, such as simple algebraic or arithmetic errors, which we do not expect to be made by students in higher education of mathematics. Therefore, they have provided a taxonomy for classifying common student errors particularly for **undergraduate** mathematics students.

Later, researchers from UWE Bristol provided a **modified version** of the model which selects only the CSEs relevant to **e-assessments**, where they eliminated some CSEs related to the proof type of questions (P) or communication of solutions (C) which are not assessed by their e-assessment [14, p. 60]. They then used the model to classify CSEs from their first-year Engineering Mathematics students in their e-assessment. We have analysed the modified model of taxonomy and we believe it is highly applicable to our case studies, since both their Engineering Mathematics and our Mathematical Biology courses are in e-assessment contexts and the questions analysed are also maths-based.

Table 3.1: Modified Taxonomy of Common Student Errors

Main Category	**Code**	**Error Type**	**Examples**
Slip of action	S1	Copying error	Incorrect copying of the question
			Mistake copying/submitting answer into e-assessment
			Incorrect interpretation of the question
	S2	Careless errors on simple calculations	Overlooking negative signs
			Omission of denominator
	S3	Incorrect algebraic manipulation	Incorrect division of two complex numbers
			Sum of product is split as a product of two sums
			Incorrect handling of powers
Errors of understanding	U1	Confusing different mathematical structures	Confusing the structure of completing the square and the quadratic equation
			Stating that a unit step function is a number
	U2	Incorrect argument	Incorrectly assuming the derivative of the product of two functions is equal to the product of the individual derivatives
			Taking the integration of the product of two functions as the product of individual integrals
	U3	Lack of consideration of potential indeterminate forms	Taking the square of a negative number to be negative
	U4	Proposed solution is not viable	Angle is not within the given range
	U5	Definition/method/theorem not recalled correctly	Method of completing the square is not recalled correctly
			Definition of waveform properties not recalled correctly
			Method of differentiating a standard function is not recalled correctly
			Method of solving trigonometry equation is not recalled correctly

			Chain rule is not recalled correctly
			Method of Partial differentiation not recalled correctly
			Method of differentiating implicit functions is not recalled correctly
	U6	Partial solution	Correct workings but unfinished solution
	U7	Incorrect assumptions	Incorrect assumptions on the mean value theorem
Errors in choice of method	CM1	Applying an inappropriate formula/ method/ theorem	Uses a method which is not relevant in the situation
			Uses a formula which is not relevant in the situation
Errors in use of method	UM2	Error in use of an appropriate definition/ method/ theorem	Error in the use of the chain rule
			Error in use of partial differentiation method
			Incorrect units applied
			Method finding the volume of revolution is incorrectly followed

In Table 3.1, we have adopted UWE Bristol's modified taxonomy of CSEs [14, p. 60]. The CSEs are classified into different main categories. Within each category, there are several types of errors with the associated code. The examples are selected from both the original model [13] and the modified model [14].

Among the four main categories, *error of understanding* (U) is one of the most important categories to focus on. This category is where students have misunderstood some concepts in our question topics and hence made these common mistakes, which they would like to reflect on during revision. Through identifying the topics where errors of understanding are found, educators can also adjust their curriculum or focus of teaching to better cope with the areas where students struggle with the concepts. Most importantly, they can also provide more effective feedback particularly in response to these CSEs [14, p. 55].

Besides, *error in choice of method* (CM) and *error in use of method* (UM) are also significant since many mathematical methods assessed in an e-assessment are routine procedures that likely appear in the final exams. Students should then focus on practising the method for the questions where these types of errors are identified. In addition, we believe errors related to methods (UM, CM) may also indirectly reflect students' error of understanding (U), especially when the method is linked to a key mathematical concept. Therefore, we will focus on errors in the categories (U), (UM) and (CM) in the case studies.

Slip of action (S) is where lots of students make careless mistakes on. These CSEs are the most undesirable ones because students actually understand the concepts. Educators may want to improve their e-assessment question settings to reduce the chance of students making these errors, such as typing a fraction instead of a decimal, forgetting the denominator or negative sign (S2), which will be discussed in Section 4.1.3.

Our goal is to identify the common student errors in selected questions of our e-assessment and classify them using the model of taxonomy 3.1 in the following case studies.

3.2 Case Study 1: Learning foundational concepts and methods — studying the Allee effect model

(The full question can be found in A.1, A.2.)
Overview: In Question 1, students will study the **Allee effect model**, which is a one-dimensional population model for a population x. In our context, Allee effect is the correlation between population size and population growth or decay rate. It was found that animal aggregation can increase their chances of survival, which was proven in experiments with a small goldfish population by Allee and Bowen in 1932 [15]. An example of the model is given in [16, p. 379], which is also used in the lecture notes:

$$\frac{dx}{dt} = -ax\left(1 - \frac{x}{K}\right)\left(1 - \frac{x}{A}\right) \tag{3.1}$$

where $a > 0$ and $0 < A < K$. We will explore variations of this model in our question. We have decided to delve into the Allee effect model because students already have a basic understanding of this population model. The concepts introduced in this question will not be too difficult, which is why this question is very suitable for students who just started revision to recall the definitions and delve into this simple population model of Mathematical Biology. At the same time, they will also be tested on some newer concepts and applications of this model in new situations.

Aim: This question aims to help students achieve two goals:

- **Recall** some definitions and concepts of population models.
- Familiarise with an important **graphical procedure** that is widely used in Mathematical Biology.

Structure:

1. Fill-in-the-blank (parts (a), (c));
2. Multiple-choice (part (d));
3. Interactive graph (part (b))

The question consists of 4 parts. Parts (a) and (c) focus on easier understanding of the content that students will be able to answer if they have briefly understood the material. These parts are set to consolidate students' foundational knowledge and guide them through to the more difficult parts of the question. Part (b) demonstrates a routine but important graphical procedure. Part (d) builds up the skills in part (b) and (c), and requires higher-order understanding since it is a new situation not discussed in lectures.

Analysis of concepts and skills tested in each sub-question:

(a) Recalling definitions
This is a *fill-in-the-blank* question where students are asked to find the carrying capacity of the population. If the student recalls and understands the definition, it can be directly read out, or at least as simple as taking a square root from a term in the differential equation, depending on the question variant given.

(b) Learning about phase portraits
This is an *interactive graph* question which aims to help students understand the procedure of locating the stable and unstable equilibria in a phase portrait. The graph for $\frac{dx}{dt}$ with respect to x is provided. Students interact with the graph to achieve the following:

- Change arrows X, Y, Z to the correct directions depending on the sign of $\frac{dx}{dt}$ at that point.
- Move the blue and red points to the stable and unstable equilibria respectively, using the arrows.

This is a **routine graphical procedure** which is fundamental in Mathematical Biology. Although the procedure can also be done mathematically, the graphical method greatly streamlines the process for more mathematically complex biological systems.

(c) Conceptual understanding
This is a *fill-in-the-blank* question that aims to test students for the concept of threshold (or critical) population, which is the initial population below which the species will go extinct. Students can refer to the graph in part (b), where they should identify the threshold population by noticing arrow X should point from the threshold population (unstable equilibrium) towards the origin (stable equilibrium) directly, implying extinction. This question also demonstrates the meaning of Allee effect: under certain environmental circumstances, smaller population decreases survival rates.

(d) Application of phase portraits and system analysis in a new context
This is an unseen *multiple-choice* question where students are given three population models to

- (i) determine whether they have a threshold population,
- (ii) classify whether they exhibit strong, weak or no Allee effects.

Part (i) requires the student to already know how to draw a phase portrait (skills built up from part (b)) in order to find the stable and unstable equilibria, which determines whether there is a threshold population (skills built up in part (c)).

Part (ii) requires the student to understand the unseen classification of Allee effects, and the most difficult part is to notice that the third model has a weak Allee effect because the x^2 term in $\frac{dx}{dt} = \frac{ax^2}{K}\left(1 - \frac{x}{K}\right)$ makes the population growth at small population much slower. This requires **higher level of understanding** of how the mathematical terms in the differential equation affect the shape of the graph.

3.2.1 Technical features implemented in Question 1 and their intended goal

1. **Randomisation**

   Implementation:
   At the start of the question, students are given a randomly generated differential equation from 3 variants based on the equation 3.1:

   - `variant = 1`: $\frac{dx}{dt} = -ax(1 - \frac{x}{K})(1 - \frac{x}{A})$.
   - `variant = 2`: $\frac{dx}{dt} = -20ax\left(1 - \frac{25\log(1+x)}{K}\right)\left(1 - \frac{19\log(1+x)}{A}\right)$.
   - `variant = 3`: $\frac{dx}{dt} = -ax\left(1 - \frac{x^2}{100K}\right)(1 - \frac{x}{A})$.

   In addition, the coefficients a, A and K are randomised numbers. The equations are carefully scaled such that their graphs fit well in the JSXGraph in part (b).

Intended purpose/goal of implementation:
We implement randomisation to the question for the following two goals:

- **Allow students for repeated practice:**
We have randomised the question with 3 equation variants and randomised coefficients in the equations. The answer for the carrying capacity in part (a) and the shape of the interactive graph provided in part (b) will differ slightly depending on the variant and randomised numbers given, allowing students to practise on the question repeatedly with different numbers and system of equations.

- **Ensure fairness:**
Although the e-assessment is purely formative, it can still be used for gathering the statistics of the students performance in the future, which will be discussed in Section 4.1.1. To obtain more accurate statistics, we want to discourage students from cheating. Based on [17, p. 2], randomised mathematical problems ensure fairness, preventing students from copying from their peers directly. Specifically, according to the paper [18, p. 100], randomisation by **using different numbers** or **selecting a random question** from a larger pool can discourage cheating in online assessments, which is exactly what we have implemented in this question.

2. **Movable points** and **Buttons**
Implementation:
In part (b), as discussed in Example 2.2.1 and 2.2.2, students are given two movable points: blue and red, as well as three buttons controlling arrows X, Y and Z.

Why have we chosen interactive graph questions in the first place?

There are not many academic papers discussing pedagogical functions of JSXGraph, in particular. However, similar studies have been conducted on digital geometry softwares (DGS) and interactive diagrams (ID), such as GeoGebra and Cinderella, which share similar features as what JSXGraph can provide.

Supporting studies:
According to [19, p.27], JSXGraph is also considered a dynamic geometry system , since it can "import various file formats of Dynamic Geometry Systems" [19, p.26]. Based on the paper [20, p.29], it has conducted a lot of studies on the use of DGS on junior and high schools as well as univerisites in improving students' mathematical skills, such as mathematical critical thinking, representation and reasoning abilities [20, p.35]. It has found that the use of DGS in maths education overall boosts students' maths abilities. In addition, it is "more effective on colleges than junior high school". Therefore, we believe that the implementation of JSXGraph in higher maths education is highly beneficial for the students.

Purpose/Goal of implementation:

- **Promote active learning through interaction with graphs:**
We believe that implementing JSXGraph features allow students to interact with the graph since they need to think about their mathematical meaning as they identify the equilibria or change the arrow directions which correspond to population change. In addition, instead of just implementing the traditional constructed-response questions in maths, we present an interactive graph question which aims to arouse students' interest.

- **Encourage the graphical method to the problem:**
  Part (b) aims to help students identify the stable and unstable equilibria of the system. This can be done with both mathematical and graphical methods, and we have encouraged the student to approach the problem graphically with phase plot provided in the interactive graph, which is more intuitive than the mathematical approach. We also aim to prevent students from just reciting the mathematical definition and not try to understand the underlying concepts. We believe the graphical approach strengthens their concepts better than the mathematical approach since it clearly indicates the flow of the population in the phase plot.

3.2.2 Common Student Errors in Question 1

The following section identifies several common student errors (CSEs) expected from students in selected parts of the question. Each CSE will be labelled and numbered in ascending order for reference: CSE 1, CSE 2, ... These errors will be classified using the taxonomy in Table 3.1, and will be supplemented with explanations for making these errors, in relation to the level of understanding in the concepts and skills mentioned in 3.2.

Errors of understanding (U) and Methods (CM, UM)

We will analyse the common student errors for parts (b) and (d) of this question. Note that the identified types of common student errors are very similar regardless of the generated question variant.

1. (b) **Interactive graph**

For this question, we will discuss the student errors by considering both the positions of the points and the arrow directions together. There are 2 possible correct answers to this question:

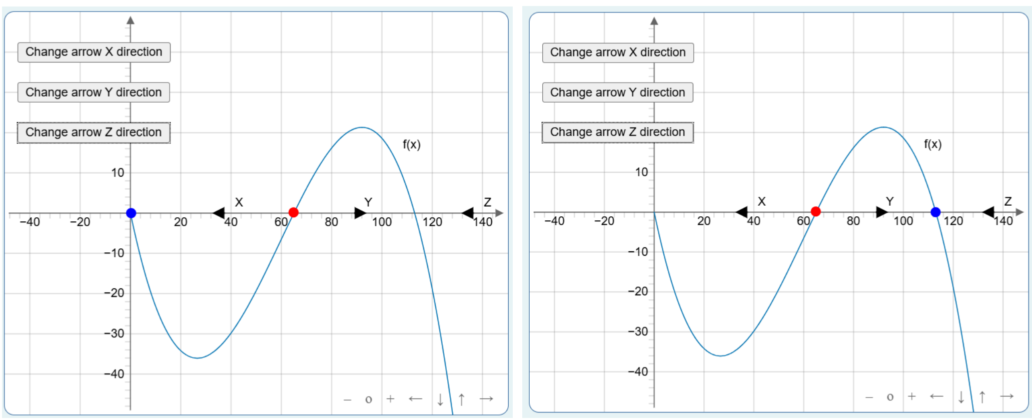
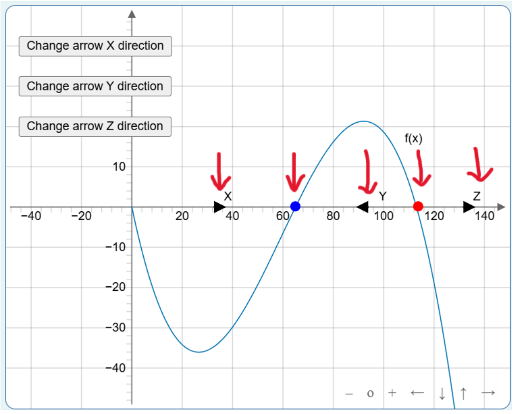


Figure 3.1: Two possible correct answers

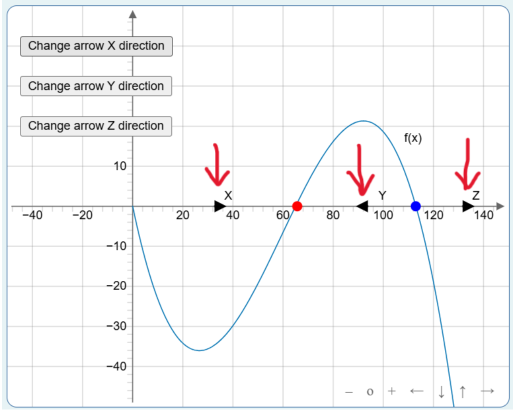
Here, we can identify 3 common student errors CSE 1-3, as shown in Figure 3.2.

CSE 1 is classified as *(U5) Definition/method/theorem not recalled correctly.*
The student has incorrectly assigned the arrow directions. However, the stable (blue) and unstable equilibrium (red) are misclassified as well. This is **consistent** with the arrow directions, which should point at where the stable equilibrium (blue point) is. This implies that the student **has understood the procedure**, including how the arrow directions affect the flow of the population and hence the equilibrium stability. However, the student



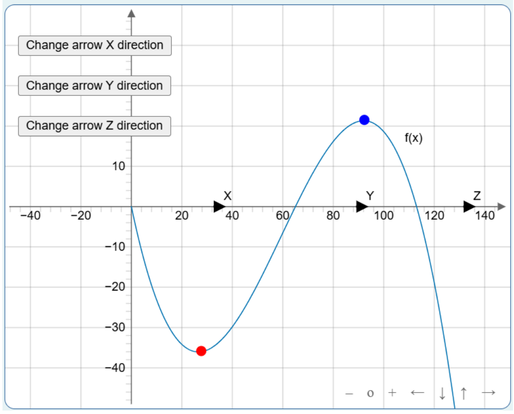
CSE 1

Classification: U5



CSE 2

Classification: UM2/CM1



CSE 3

Classification: U1

Figure 3.2: Common student errors (CSE 1-3)

has not noticed how the sign of $\frac{dx}{dt}$ determines the direction of change of population, which could be recalled from the definition.

CSE 2 is classified as *(UM2) Error in use of an appropriate definition/method/theorem* or *(CM1) Applying an inappropriate formula/method/ theorem.*
The student has incorrectly assigned the arrow directions, but the equilibria are correctly classified, which is not consistent with the method of using arrow directions to locate the stable equilibria. This implies that the student does not understand the procedure and **the method is incorrectly followed**. Otherwise, they may have computed the stable and unstable equilibria mathematically, instead of using the graph and the arrows provided, which can be considered *(CM1) Applying an inappropriate formula/method/ theorem* since they have not approached the question using the arrows in the graph.

CSE 3 is classified as *(U1) Confusing different mathematical structures.*
The student has found the local maximum and minimum points instead of the equilibria. This indicates a more serious error of fundamental understanding in this context, and the student may need more guidance on this topic.

1. (d) **Multiple-choice**

This question consists of 2 sub-parts (i) and (ii), each consisting of 3 multiple-choice questions for each population model, with the equations provided. We will consider the student's response from all the three models together.

(i) Any **threshold population** for the following models?

Sub-questions	**Teacher's solutions**	**Student's answers (CSE 4)**
Logistic equation	No	Yes
Goldfish model	Yes	Yes
Modified model	No	Yes

Table 3.2: Correct solutions and CSEs for (d)(i)

CSE 4 is classified as *(UM2) Error in use of an appropriate definition/method/theorem.* For this question, the student needs to apply the method of drawing a phase portrait and identifying the equilibria as introduced in part (b). The error: *incorrectly stating that the logistic and modified model has threshold population*, is likely made because they have not correctly classified $x = 0$ as an unstable equilibrium for the first and third models, which implies that these two models **do not have a threshold population** below which leads to extinction. Hence, this is mainly due to errors within the use of the phase portrait method.

(ii) Any **Allee effect** for the following models? (Strong/ Weak/ No Allee effect)

Sub-questions	**Teacher's solutions**	**Student's answers (CSE 5)**	**Student's answers (CSE 6)**
Logistic equation	No	Weak	No
Goldfish model	Strong	Strong	Strong
Modified model	Weak	Weak	No

Table 3.3: Correct solutions and CSEs for (d)(ii)

CSE 5 is classified as *(U2) Incorrect argument.*
Students who make this error: *incorrectly stating that the logistic equation and the modified model exhibit weak Allee effects, but correctly identified the Goldfish model as Strong Allee effect*, have noticed in the Goldfish model that x has negative growth rate when the population is small. However, they argue that since both the logistic and modified model have positive growth rate when the population is small, both of them exhibit weak Allee effects (reduced growth rate). However, this is not true. For the logistic model, the population actually starts at an almost linear growth rate, which is not reduced at all compared to modified model with the x^2 as the coefficient.

CSE 6 is classified as *(U7) Incorrect assumptions.*
The student may not have understood or was misled by part (i) of the question which asks about whether the model has a threshold population. They may assume that not having a threshold population would imply having no Allee effect, which is incorrect since it only excludes the possibility of having strong Allee effect but it tells nothing about whether the model has weak or no Allee effect.

3.2.3 Case Study 1 Reflections

We would now like to reflect on the findings in this case study by considering the following questions:

What is the pedagogical significance of this question?
This question assesses **fundamental concepts and methods** in Mathematical Biology. It also aims to help students with final exam revision through **enhancing their recall** of definitions and simple concepts in Mathematical Biology. This is also why we have tested students on recalling definitions of the carrying and threshold capacities, and included an interactive graph question in part (b) to help them recall the useful graphical method.

How did the technical features help in achieving the pedagogical goal of this question?
Not only does **randomisation** discourage cheating, but also it helps **reveal errors or gaps in the students' understanding** of fundamental concepts and methods. It provides several question variants and randomised coefficients, which test students on adapting to different mathematical models, instead of just asking them to solve a fixed problem. For example, even if a student has correctly answered part (b) due to luck or partial understanding, he may still get it wrong in another attempt with a different question variant, which could indicate an error in methods (UM2), (CM1) or concepts (U1), (U5).

However, an important consideration is that when we create the question variants, we should always make sure that they are not significantly different, so that the type of common student errors should be very similar. This is because they are essentially still the same question, with the goal of assessing the **same concept/method** but in a **slightly different mathematical context**.

The JSXGraph with interactive features can potentially mitigate student errors such as (U5), where they cannot recall the definition or method. The interactive graph provide students with a **visual image**. According to [21, p. 154], a visual image, such as the function of a graph, is a conceptualised image, which is a "pictorial representation of concepts". Besides, according to [21, p.155], "human minds are inclined to visualise facts" and we "think more easily with images". These suggest that solving the problem with a graphical approach allow students to better understand concepts than to merely memorise the mathematical definitions without processing the meaning. Through clarifying the concepts with a visual image and utilising the buttons and draggable points, we believe that students **recall the graphical method** with more practice.

Should the question be practised repeatedly by the students, and when should they move on to the next question?
Randomisation has created several variants to the question, and it allows students to familiarise with the basic concepts and a graphical procedure in Mathematical Biology through **repeated practice**. Since the question mainly covers fundamental concepts, students should aim to get full marks before moving on to the next question, as it provides a good foundation for the more advanced topics in Mathematical Biology.

3.3 Case Study 2: Performing advanced system computations and analysis — Studying the predator-prey model

(The full question can be found in A.3, A.4.)
Overview: In Question 2, students will study a two-dimensional biological system: a modified predator-prey model, which has always been a crucial topic in Mathematical Biology. The details of the model, e.g. predator-prey, the role of each species and other characteristics are not explicitly stated, which will be left for students to analyse in the question. This question is set to be a step up from the previous question. Students will be tested on the more computation mathematical procedures, as well as more advanced conceptual understanding and interpretation of the system.

To start with, we would like to introduce the classical **Lotka-Volterra** system [22, p. 199]:

$$\frac{dx}{dt} = x - xy \tag{3.2}$$

$$\frac{dy}{dt} = \gamma(-y + xy), \tag{3.3}$$

which describes a predator-prey model between two species: the prey and the predator, with population x and y respectively. The terms involving both x and y, e.g. $-xy$ and γxy are also called the **interaction terms** of the system, since they represent the change to the two populations due to the interaction between the two species.

Based on this model, we have created several variants to the system of equations by adding coefficients and modifying the terms in the equations to create a modified predator-prey system to the students, which are displayed at the start of the question. These equations illustrate slightly different biological meaning to test students' **conceptual understanding**.

- `variant = 1`:

$$\begin{aligned}\frac{dx}{dt} &= ax - \frac{axy}{x+b} \\ \frac{dy}{dt} &= -cy + \frac{axy}{x+b}.\end{aligned}$$

- `variant = 2`:

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy\end{aligned}$$

- `variant = 3`:

$$\begin{aligned}\frac{dx}{dt} &= ax(1-x) - bxy \\ \frac{dy}{dt} &= -cy(1-y) + dxy\end{aligned}$$

where a, b, c (and d) are some positive constants, but not randomised.

Aim: This question aims to help students achieve the following goals:

- Be able to interpret the mathematical and biological meanings of a population model.
- Practise important computational methods and analyse stability of the model.

Structure:

1. Fill-in-the-blank (parts (b), (c));
2. Multiple-choice (parts (a), (d))

The question consists of 4 parts. Part (a) is a checkbox type of MCQ which requires the most conceptual understanding, while parts (b) to (d) focus more on computational skills. They are designed to be **exam-type of questions**.

Analysis of concepts and skills tested in each sub-question:

(a) Biological interpretation of a system
This is a *multiple-choice question* which tests students on **biological interpretations** of the system. These include determining the type of population model, the role of each species, the type of growth under isolation, and saturation effect of the predator.

To answer this question, students need to be **familiar with the definition of a predator-prey (Lotka-Volterra) model**, and hence the **biological implication of the terms**. They need to notice the details, e.g. why should a term be positive or negative, and also **deduce the indeterminate forms of the terms, e.g. limits**. For example, in order to distinguish the predator and the prey, students need to consider the interaction term, i.e. the term with both x and y involved. Mathematically, the term should be positive to the predator, since they consume the prey, leading to increase in their population. It should be negative to the prey, since they are eaten by the predator. In addition, students need to notice that under isolation, the interaction terms vanish leading to the differential equations corresponding to exponential growth (or logistic growth) of each species.

Finally, **saturation effect** is the hardest to notice. Looking at the interaction terms in `variant = 1`, as the population x (the prey) gets very large, since the term $\frac{axy}{x+b}$ will approach ay, the interaction term will only depend on y but not x. This means the change in population y will eventually only be limited by itself, even if x gets infinitely large. Hence, y cannot consume x indefinitely.

(b) Performing routine numerical operations: nondimensionalisation
This is a *fill-in-the-blank* question asking students to perform nondimensionalisation of the system of equations in part (a). The process of nondimensionalisation means **eliminating some coefficients of the equations**, by writing them in terms of the others, to give a more interpretable form with fewer parameters.

This is a **routine numerical operation** that is fundamental and crucial for many models of Mathematical Biology, as this simplifies the model and allows for further analysis. Students need to write the parameters in the form provided and substitute into the equations. Then, they will compare the coefficients of the terms with the system of nondimensionalised equations provided, by writing down the unknown variables in terms of known variables, which greatly reduces the number of unknown variables to just K.

(c) Computational methods: solving system of equations for equilibria, finding the derivatives for the Jacobian
This is a two-part *fill-in-the-blank* question asking students to find the equilibria and the Jacobian matrix of the system. In order to locate the equilibrium apart from the origin, students need to recall that an equilibrium is the point where there is no change in population. However, this is now a two-dimensional system, hence two equations need to be set up and simultaneously satisfied: $\frac{dx}{dt} = \frac{dy}{dt} = 0$. This method is the **generalisation** of question 1 (b) **to two-dimensional space** but it should be pretty straight forward to follow once the concept and method is understood.
For the Jacobian matrix J, students are simply asked to recall the formula and find the derivatives in terms of the variables x, y and K. Let $\frac{dx}{dt} = F(x, y)$ and $\frac{dy}{dt} = G(x, y)$. Then $J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$. Part (d) then requires the students to find the

eigenvalues of J, which can determine the stability of the system with linear stability analysis.

(d) Linear stability analysis
This is a *multiple-choice question* which requires students to analyse the stability of the two equilibria found in part (c) depending on the range of values of K, one of which is the origin. The stability of the system is determined by the perturbation $\boldsymbol{x_1}$ around the equilibrium point $\boldsymbol{x_0}$, where $\boldsymbol{x_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ Ce^{\lambda_1 t} + De^{\lambda_2 t} \end{bmatrix}$, with constants A, B, C and D. Hence, the equilibrium $\boldsymbol{x_0}$ is only stable if $\boldsymbol{x_1} \to 0$ as $t \to \infty$, i.e. $\boldsymbol{x_0}$ is

$$\begin{cases} \text{stable} & \text{if } \mathrm{Re}(\lambda_i) < 0 \ \forall i \\ \text{unstable} & \text{if } \exists i \text{ such that } \mathrm{Re}(\lambda_i) > 0 \\ \text{degenerate} & \text{if } \exists i \text{ such that } \mathrm{Re}(\lambda_i) = 0 \text{ and } \mathrm{Re}(\lambda_j) \leq 0 \ \forall j \neq i. \end{cases}$$

Students need to **compute the eigenvalues** of the Jacobian matrix J and classify the stability of each equilibrium based on these conditions given in the lecture notes, which requires them to deal with **complex eigenvalues and inequalities**.

3.3.1 Technical features implemented in Question 2 and their intended goal

1. **Randomisation**

Implementation:
First of all, we have provided 3 variants of the system of equations at the start of the question, as mentioned earlier in 3.3. This was also implemented in Case Study 1, so we will not discuss it again here. However, we have further implemented randomisation features in the MCQ in part (a), including shuffling, random subset of correct options and random addition of distractors.

Intended purpose/goal of implementation:
To understand what we want to achieve with these MCQ features, let us first introduce the background of multiple-choice questions in maths education. Based on [1, p. 1], for mathematics higher education in the UK, the traditional assessment format is closed-book examination, while the most traditional question type has always been constructed-response questions [23, p. 1], which are open-ended questions where students need to construct their arguments. Provided response questions, such as multiple-choice questions (MCQ) are considered an "innovative" assessment form [1, p. 5].

In fact, based on the research results from [1, p. 20] conducted to investigate students' preferences for the method of assessment in higher mathematics, it suggests that students do not prefer MCQ because they believe it is "unfair", since it is subject to random guessing. A student was interviewed during the research and claimed: "they could have just ticked that because they didn't know what the answer was, so they just chose 'b' just randomly.", which suggests that some students believe they can guess the correct answer of an MCQ randomly, hence it does not reflect they have understood the underlying concepts [1, p. 21].

Given the concerns regarding whether MCQs are suitable as a question type in e-assessments, we still decided to implement them in our e-assessments due to several reasons. Firstly, it is **easily applicable** in e-assessment platforms such as STACK. Secondly, it also effectively assesses students in different **mathematical skills and understanding**.

Supporting studies:
According to [4, p. 15], multiple-choice question is in fact proven to be applicable in certain areas (assessment components) of mathematics, namely technical, conceptual, modelling and problem solving assessment components. In our multiple-choice question, we aim to focus on testing the **conceptual** component.

Regarding the concern from students about random guessing, firstly, the checkbox type of MCQ can have more than one correct option, which increases the **variety of possible responses** provided by the students, reducing the chance that all the correct options are selected by random guessing. Besides, we have implemented several technical features to the MCQ. We have further **enhanced the randomness** of the question through random subsetting of correct options and shuffling. Besides, we have **increased the questions' difficulty** through random addition of distractors, requiring students to fully understand the underlying concepts in order to identify the correct options.

2. **Partial Scoring**

   Implementation:
   We have implemented partial scoring in part (a) of the question because this is the only question that could have partially correct answers. We have implemented the Plus/Minus scoring algorithm, which was discussed in Section 2.1.1.

   Intended purpose/goal of implementation:
   We have implemented the Plus/Minus scoring method to prevent the extreme scoring such as the all-or-nothing Dichotomous method. The Plus/Minus method also prevents students from selecting all the options provided because any selection of distractor will reduce their marks. Through providing partial credits, we aim to provide a score that **accurately reflects the student's level of understanding to the question**, which allows them to improve on during revision.

3.3.2 Common Student Errors in Question 2

We have chosen to analyse the common student errors for parts (a) and (b) of this question.

Errors of understanding (U) and Methods (CM, UM)

2. (a) **Multiple-choice question**

Three variants of the system of equations as well as randomisation features of MCQ have been implemented to this question. Although the correct options for each question variant are different, our analysis will be focused on the conceptual error that leads to a wrong option being chosen. We will first **group** all the options that are **linked together to a particular concept**, then we will analyse the common student errors according to the group.

Grouping of the options based on the concept being tested:

- Options related to the **type of model**:
  - *"It is a Predator-prey model where X is prey and Y is predator."*
  - *"It is a Predator-prey model where X is predator and Y is prey."*
  - *"It is a competitive Lotka-Volterra model."*
- Options related to the **type of model** when the two species are **separated**:
  - *"Both X and Y exhibit exponential growth or decay if we ignore the interaction terms."*
  - *"Both X and Y are following logistic growth or decay if we ignore the interaction terms."*
- Options related to the **roles of X and Y** in the system:
  - *"It is a Predator-prey model where X is predator and Y is prey."*
  - *"X gets full so X cannot consume Y indefinitely."*
  - *"Y gets full so Y cannot consume X indefinitely."*
  - *"It is a Predator-prey model where X is prey and Y is predator."*
- Options related to the **saturation effect** of the predator:
  - *"X gets full so X cannot consume Y indefinitely."*
  - *"Y gets full so Y cannot consume X indefinitely."*

First of all, when considering the **type of model**, all the variants are **predator-prey models**, because clearly the interaction terms are negative to the prey and positive to the predator.

CSE 7 is identified as follows: If a student has instead chosen the option: *"It is a competitive Lotka-Volterra model."*, then they have mixed up the predator-prey model with the competitive Lotka-Volterra model, which by definition, the interaction terms are negative for both species since competition exists between them. We will classify this as *(U5) Definition/method/theorem not recalled correctly.*

Secondly, considering the the **type of model** when the two species are **separated**, we will neglect the interaction terms in the differential equations, left with the equations: $\frac{dx}{dt} = ax$ and $\frac{dy}{dt} = -cy$ for `variant = 1 or 2`, which are following **exponential growth** (or decay) for both species. On the other hand, the **logistic growth** (or decay) model has scaled equations: $\frac{dx}{dt} = ax(1-x)$ and $\frac{dy}{dt} = -cy(1-y)$, which are the equations for `variant = 3` instead.

CSE 8 is identified as follows: If a student has chosen the option that does not correspond to the type of model in the equation variant, then they have mixed up the definition of logistic growth and exponential growth equations. This should be again classified as *(U5) Definition/method/theorem not recalled correctly.*

Thirdly, we consider the options related to the **roles of X and Y** in the system. This is a bit different to the previous groups of errors which are primarily due to not recalling the definitions. Consider the incorrect statement *"It is a Predator-prey model where X is predator and Y is prey."*, where the roles of X and Y are reversed. It is composed of two parts.

CSE 9 is identified as follows: If a student has selected this option, they agree that the model is predator-prey, which implies that they can recall the equations or at least the structure for a predator-prey model, but they have also reversed the roles of X and Y in the system, which is a bigger problem because they cannot relate the signs of the interaction terms to the biological meaning to each species. We will therefore classify this error as *(U2) Incorrect argument.*

It is worth noting that the options involving the roles of X and Y have appeared in multiple groups of options, this is because we regard it as a significant error in this category. These options have been linked to the other categories so that it can be reflected in the final score.

Finally, we consider the two options related to the **saturation effect** of the predator. Again if a student selected *"X gets full so X cannot consume Y indefinitely."*, it should also be considered an error in reversing the roles of the predator and prey, as in CSE 9. Under the three variants, only `variant = 1` has saturation effect of the predator. The reason has been given earlier in the analysis of concepts in Section 3.3, mainly due to the interaction term $\frac{axy}{x+b}$ will approach ay as x gets large, and hence does not depend on x, the prey.

CSE 10 is identified as follows: If a student does not select the option *"Y gets full so Y cannot consume X indefinitely."*, then they have not considered the indeterminate forms (i.e. the limit) of the interaction term, and hence should be classified as *(U3) Lack of consideration of potential indeterminate forms.*

2. (b) **Fill-in-the-blank question**

For this question, since there are 3 variants to the system of equations, the method of nondimensionalisation is slightly different for each variant. We will demonstrate `variant = 1` as an example. In the future, we can also classify the common student errors for the other two variants.

The **model solution's method** goes as follows:
Substitute $x = \hat{x}X$, $y = \hat{y}Y$ and $t = \hat{t}T$ into the two equations provided in `variant = 1`. Then we have

$$\begin{cases} \dfrac{d\hat{x}}{d\hat{t}}\dfrac{X}{T} &= \quad a\hat{x}X - \dfrac{aXY\hat{x}\hat{y}}{\hat{x}X+b}, \\ \dfrac{d\hat{y}}{d\hat{t}}\dfrac{Y}{T} &= \quad -c\hat{y}Y + \dfrac{aXY\hat{x}\hat{y}}{\hat{x}X+b} \end{cases} \tag{3.4}$$

Multiplying both sides by $\frac{T}{X}$ and $\frac{T}{Y}$ respectively, we have

$$\begin{cases} \dfrac{d\hat{x}}{d\hat{t}} &= \quad aT\hat{x} - \dfrac{aYT\hat{x}\hat{y}}{\hat{x}X+b}, \\ \dfrac{d\hat{y}}{d\hat{t}} &= \quad -cT\hat{y} + \dfrac{aXT\hat{x}\hat{y}}{\hat{x}X+b} \end{cases} \tag{3.5}$$

By scaling the fractions, we have

$$\implies \begin{cases} \dfrac{d\hat{x}}{d\hat{t}} &= \quad aT\hat{x} - \dfrac{\frac{aYT}{X}\hat{x}\hat{y}}{\hat{x}+\frac{b}{X}}, \\ \dfrac{d\hat{y}}{d\hat{t}} &= \quad -cT\hat{y} + \dfrac{aT\hat{x}\hat{y}}{\hat{x}+\frac{b}{X}} \end{cases} \tag{3.6}$$

By comparing the coefficients with the nondimensionalised equations provided, we have:

$$\begin{cases} aT = & 1 \\ \dfrac{aYT}{X} = & 1 \\ \dfrac{b}{X} = & 1 \\ K = & cT \end{cases} \tag{3.7}$$

$$\implies \begin{cases} K = & \dfrac{c}{a} \\ T = & \dfrac{1}{a} \\ Y = & b \\ X = & b \end{cases} \tag{3.8}$$

The final solution in equation 3.8 provides the **model answer** to this question.
Due to a great number of computations in this question, there can be plenty of computational mistakes that lead to different kinds of errors in each step. We will focus on two possible common student errors in the use of the nondimensionalisation method, assuming other numerical operations are done correctly.

CSE 11 is identified as follows: The student has **partial substituted the scaled variables**, for example, not substituting t: The student has partially substituted the scaled variables $x = \hat{x}X$, $y = \hat{y}Y$ but not $t = \hat{t}T$. They have simply changed t with $\hat{t}$. Then they obtain

$$\begin{cases} \dfrac{d\hat{x}}{d\hat{t}}X = & a\hat{x}X - \dfrac{aXY\hat{x}\hat{y}}{\hat{x}X + b}, \\ \dfrac{d\hat{y}}{d\hat{t}}Y = & -c\hat{y}Y + \dfrac{aXY\hat{x}\hat{y}}{\hat{x}X + b} \end{cases} \tag{3.9}$$

instead of equation 3.4. Hence, they have obtained solutions:

$$\begin{cases} K = c \\ Y = b \\ X = b \end{cases} \tag{3.10}$$

where they have assumed $a = 1$, and they have not found a solution for the variable T.

Another error **CSE 12** is identified as follows: The student has **not scaled the fraction** in the interaction term.
After arriving at equations 3.5, they have not scaled the fraction as in equations 3.6. They have simply compared the coefficients of the numerator and the denominator, but this is not possible unless $b = 1$. Hence, they have assumed $b = 1$ and obtained the solutions:

$$\begin{cases} K = & \dfrac{c}{a} \\ T = & \dfrac{1}{a} \\ Y = & 1 \\ X = & 1 \end{cases} \tag{3.11}$$

CSE 11 and **CSE 12** both demonstrate **procedural errors** in nondimensionalisation, such as partial substitution and not scaling the fraction. Besides, in both cases, solutions should not be found unless the student has taken some variables to be 1, which is an incorrect argument because these variables are fixed and unknown. Hence, CSE 11 and CSE 12 are both classified as *(U2) Incorrect argument* and *(UM2) Error in use of an appropriate definition/ method/ theorem.*

3.3.3 Case Study 2 Reflections

We would now like to reflect on the findings in this case study by considering the following questions:

What is the pedagogical significance of this question?
This question serves a different purpose than question 1. Many parts of the question are designed to be **exam-type questions** with **routine computations** which could be repeatedly practised by the students. They should aim to fully familiarise with all the computational procedures since they are likely to appear in the exam. Besides, the question also help students **develop conceptual understanding** from **interpreting biological systems**, which is crucial for studying any biological model since students need to understand both the mathematical and biological implications of the model.

How did the technical features help in achieving the pedagogical goal of this question?

Randomisation features:
Firstly, looking back at part (a) of the question, we have included at least 1 distractor in each group of concepts which requires students to critically evaluate whether the statements are correct. This assesses their **conceptual understanding** and strengthens their recall of definitions related to the concepts. For example, we have been able to identify common student errors in (U2), (U3) and (U5), which belong to different types of errors in **understanding**. Besides, the three question variants as well as the random subsetting of correct options provide different model answers to the question. Hence, even if the students reattempt the question, they will not be immediately revealed the correct options due to the fact that they have attempted the question before. For example, some variants are models with logistic growth while the others with exponential growth, requiring students to differentiate between the two systems of equations when answering the MCQ.

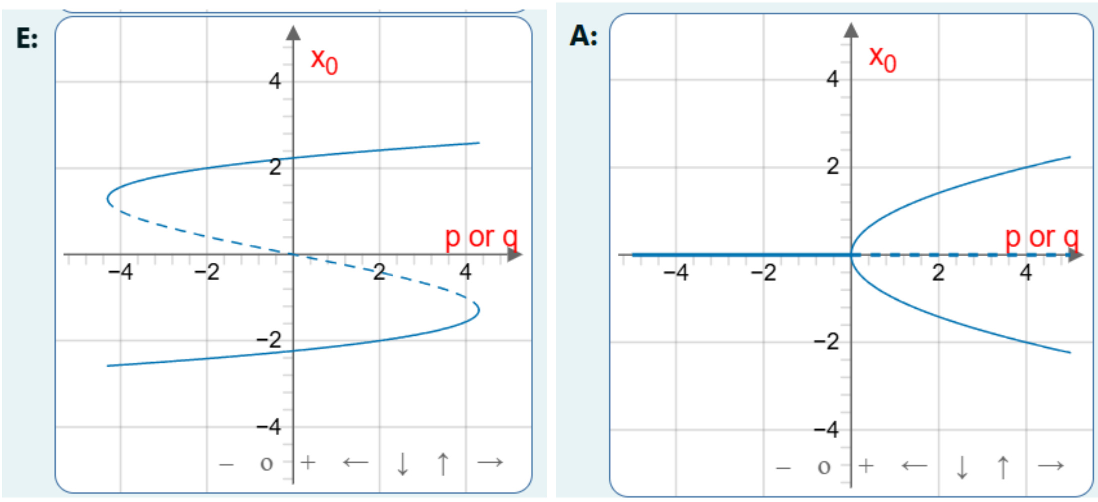
Partial scoring:
Partial scoring of MCQs does not aim to help students improve mathematical skills, but it provides a score which accurately reflects students' level of understanding to the topic. It allows for students' self-reflection during revision to better focus on their weaker concepts in preparation for the final exams.

When should the student move on to the next question?
Students should aim to practise the question until they score full mark and have understood all the concepts and procedures before moving on to the last question.

3.4 Case Study 3: Exploring complex concepts through interactive graphs — Studying bifurcation phenomena

(The full question can be found in A.5, A.6, A.7, A.8.)
Overview: In Question 3, we will study two **bifurcation phenomena**: *hysteresis* and *pitchfork bifurcation* in Mathematical Biology. There are many definitions of **bifurcation**. Generally, it refers to the point where a system has changes in behaviour as a parameter is changed. In our context, we are mostly referring to the point where the stability or the number of equilibria changes due to the change in parameter values. We will only explore the **mathematical aspects** of the two bifurcation phenomena, as in the lectures. The application in a biological context is currently not our focus. The concept of **hysteresis** is rather advanced but it is very useful for more advanced topics in Mathematical Biology. It refers to the process that the parameters have been changed such that the system

cannot return to its initial state even if the parameters return to their original values. The bifurcation diagram is shown in Figure 3.3 (a). **Pitchfork bifurcation** is a less complex concept describing bifurcations such that the shape of the bifurcation diagram looks like a pitchfork, as shown in Figure 3.3 (b).



(a) Hysteresis by p (b) Pitchfork bifurcation by q

Figure 3.3: Bifurcation diagrams for each bifurcation phenomenon displayed in a multiple-choice question

Students should only delve into this topic when they have fully understood the foundational content in Mathematical Biology and completed the previous two questions, since it is conceptually difficult and is intended for high-achievers in the exam. Besides, the design of this question is substantially different from the previous two. This question focuses on **explaining** the two phenomena through visualised concepts using interactive graphs.

At the start of the question, students are provided an interactive graph of $g(x) = -x^3 + qx$ and $h(x) = -p$ against x, where $\frac{dx}{dt} = g(x) - h(x)$, p and q are parameters controlled by the sliders. They are also given the intersection points of $g(x)$ and $h(x)$, with the labelling described in Example 2.2.3. The labels a, b and c have specific meaning in Mathematical Biology. However, since we only focus on the mathematical aspect of the concepts, they will only be considered the intersection points at different regions of the graph. The interactive graph is shown in Figure 2.8 in Section 2.2.3. Students should not worry about the feasibility of the negative values of x in the graph in a biological context.

Aim: Help students visualise the concepts of hysteresis and pitchfork bifurcations.

Structure:

1. Multiple-choice (parts (a), (c));
2. Interactive graph (part (b))

The question consists of 3 parts. Part (a) consists of two simple multiple-choice questions which help students understand the mathematical meaning of the graph provided. Part (b) consists of 3 sub-parts, each applying an interactive graph, which guide students to understand the two bifurcation phenomena. Finally, part (c) is a multiple-choice question that sums up the question by testing students on whether they have understood the two phenomena.

Analysis of concepts and skills tested in each sub-question:

(a) Interpreting the mathematical meaning of a graph
This question consists of two *multiple-choice questions* of the type of radio buttons which **aims to help students understand the graph** for answering the latter parts of the question. Students are asked to interpret the given graph in relation to the usual phase plot that they are familiar with, including how to find the equilibrium points from the graph and the mathematical meaning of parameter p. The interpretation of the graph is quite **simple**, where the equilibrium points are just the intersections of the graphs of $g(x)$ and $h(x)$, and p implies the shifting of the graph $y = g(x)$ upwards by p in the phase plot. Note that we have shown them the graphs of $g(x)$ and $h(x)$ instead of the usual phase plot of $\frac{dx}{dt}$ against x because this will help them understand the bifurcation concepts better and present them another way of analysing the system's stability.

(b) Understanding the bifurcation phenomena through interactive graphs
This question consists of three sub-parts of *interactive graph* questions with slider bindings. In sub-parts (i) and (ii), students will be **given the scenario in hysteresis** where the parameter q is fixed while the parameter p has been changed such that the system cannot be returned to the original state even if the change in p is restored. While for sub-part (iii), it demonstrates pitchfork bifurcations through parameter q. All sub-parts of the question ask students to adjust the parameter values in the graph (or through the input blank) to the required state of the system. It aims to help them understand the two phenomena through students' **experimentation of slider inputs**.

(c) Identifying the bifurcation diagram for each phenomenon
This is a *multiple-choice question* to sum up the question by asking the student to **identify** the correct **bifurcation diagram** for each bifurcation phenomenon: hysteresis and pitchfork bifurcations. It is difficult to draw the bifurcation diagrams intuitively, but it helps to provide students the graphs and ask them to identify the correct one. Four distractor options have been implemented to further **test their understanding of each phenomenon**.

3.4.1 Technical features implemented in Question 3 and their intended goal

We mainly focus on implementing **sliders** in our question. We have provided a general interactive graph at the start of the question, where the parameters p and q are both adjustable by the sliders. In part (b), students are given a specific interactive graph for each sub-part of the question, where one of the parameters p or q is fixed, for explaining the concept of bifurcations induced by the changes in one parameter. In these sub-parts, we have also implemented **dual slider bindings**, allowing students to adjust the slider values in two ways: from the graph or from the input blanks.

Intended purpose/goal of implementation:
In Case Study 1, we discussed why we would like to include interactive graph questions in our e-assessment. The reason is that we believe the implementation of JSXGraph in higher maths education is highly beneficial for the students in boosting their mathematical abilities. Now we would like to specifically discuss for this question how the implementation of JSXGraph with sliders can boost students' conceptual understanding in the bifurcation phenomena.

- **Visualise the concepts with interactive graphs:**
The concepts of the two bifurcation phenomena, especially hysteresis is quite complex

to understand intuitively through the definition. We would like to demonstrate the two phenomena through an example with interactive graphs visualising the concepts to the students.

- **Guide students to explore the concepts through experimentation:** Sliders allow students to **experiment with different inputs** through adjusting the parameter values in the graph which are constrained, e.g. one of the parameters is fixed to a value while the other parameter is adjustable, the range of values of the parameters are constrained, which provides guidance for the exploration of the concept. It is also user-friendly given the two-way binding which allows them to adjust the values from the graph or the input blanks.

Supporting studies:
Based on several studies cited in [24, p. 465], technological tools that involve interactions helps shape one's mind and mathematical thinking. This implies that JSXGraph sliders, which often requires students to interact with different inputs, can also help students enhance their conceptual understanding.
The paper [24, p. 466] also discussed several **pedagogical functions** of interactive diagrams (ID), which work similarly as what JSXGraph with sliders can provide, such as the **presentational** and **organisation** functions of IDs in particular.

We interpret the **presentational function** of an ID as what it aims to present to the students. It can be classified according to the type of example provided, namely *specific*, *random* and *generic*. In our question, we have provided a **generic** example of ID, which presents a specific graph of the functions $g(x)$ and $h(x)$ but it is **representative** of a bigger picture of concepts. It is aimed to help students "become aware of the representativeness of the example" to the bifurcation phenomenon being discussed.

Based on [24, p. 466-467], we interpret the **organisational function** of an ID as how it organises the **interactivity** of the diagram in supporting students' understanding of the concept. It can be classified as *illustrating*, *elaborating* and *guiding* IDs. In our question, we exactly match the approach of a **guiding ID** because the graph is designed to "call for an action" (input) through the experimentation of different slider inputs that "supports the construction" of the idea in the phenomenon.

Therefore, we believe that the implementation of sliders in JSXGraph in our question has fulfilled the presentational and organisation functions of an interactive diagram, which could guide students to experiment with different slider inputs, aiming to help them understand the bigger concepts in the bifurcation phenomena.

3.4.2 Common Student Errors in Question 3

Parts (a) and (b) are aimed at explaining the phenomena by guiding students through answering the question, so there are not as many student errors identified. We have chosen to analyse the common student errors for only part (c) of this question since it is aimed at testing whether the student has understood the two bifurcation phenomena.

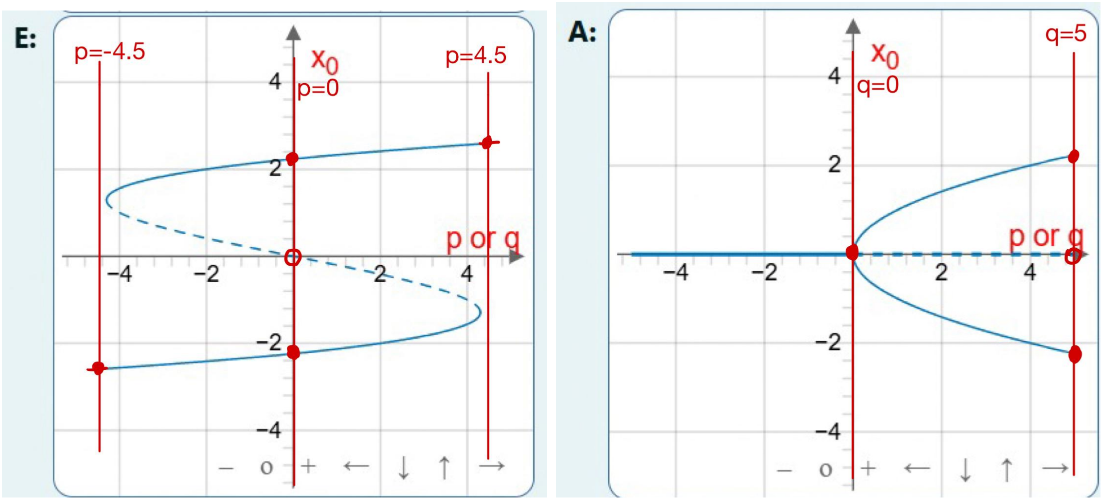
Errors of understanding (U) and Methods (CM, UM)

3. (c) **Multiple-choice question**

We will first provide the model answers and suggested method below. Note that *solid* and *dotted* lines in the bifurcation diagrams correspond to the *stable* and *unstable* equilibria respectively. Similarly, *filled* and *open* circles represent *stable* and *unstable* equilibria.

Bifurcation diagram of p:
The model answer is option E, which is shown in Figure 3.3 (a).
Bifurcation diagram of q:
The model answer is option A, which is shown in Figure 3.3 (b).

Suggested method when considering the bifurcation diagrams:
When considering the bifurcation diagrams, students should always **draw a vertical line** in the graph. The intersection points of the vertical line and the graph will then show the values of the equilibrium points on the vertical line at a given value of the parameter (p or q). They can then compare the values of the parameters obtained in part (b)(i) to (iii) of the question to choose the correct bifurcation diagrams.



(a) Hysteresis by p

(b) Pitchfork bifurcation by q

Figure 3.4: Annotated bifurcation diagrams for parameters p and q

For example, in Figure 3.4 (a), we have drawn three vertical lines corresponding to $p = -4.5$, $p = 0$ and $p = 4.5$. The default starting value in part (b) (i) is $p = 0$, where we have found 3 equilibria in the graph shown. We then adjusted p to 4.5, which is the bifurcation point where the number of equilibrium changes to 1, at $x = c$. Finally, in part (b) (ii), we adjusted p to -4.5, where the number of equilibrium is 1, at $x = a$. These can all be reflected in the bifurcation diagram in Figure 3.4 (a). Hence, the correct answer is option E.

Similarly, we can do the same analysis for the bifurcation diagram of q, as shown in Figure 3.4 (b), and obtain that the solution is option A.
The suggested method with the annotated bifurcation diagrams in Figure 3.4 will be displayed to the students in the *general feedback*.

For the bifurcation diagrams of p and q, we have identified two common student errors: CSE 13-14.

CSE 13 is identified as follows:
For the bifurcation diagram of p, which should be a hysteresis diagram, the student has chosen option A or B, which clearly shows a pitchfork bifurcation diagram. At the same time, for the bifurcation diagram of q, which should be a pitchfork bifurcation diagram, the student has chosen the options corresponding to hysteresis. This means that the student has not understood the bifurcation phenomena at all, leading to the type of bifurcation diagrams being incorrectly chosen. This error is primarily due to the student **mixing up**

the two bifurcation phenomena, or **mixing up the functions of the parameters** p and q in the graph, leading to an incorrect argument for each parameter. Therefore, we have classified CSE 13 as *(U2) Incorrect argument.*

CSE 14 is identified as follows:
The student has chosen the option C/F for the bifurcation diagram of p, and chosen the option B for the diagram of q, as shown in Figure 3.5. This means that the student has not found that the middle section of the diagrams of both p and q are unstable, i.e. the equilibrium $x = b$ is unstable. They have found the stability to be the other way round, where the middle section is a solid line, as shown in Figure 3.5. This is primarily due to an **error in the method of classifying the stability** of an equilibrium point, which is a skill developed back in Question 1 (b). Hence, we will classify CSE 14 as *(UM2) Error in use of an appropriate definition/method/theorem.*

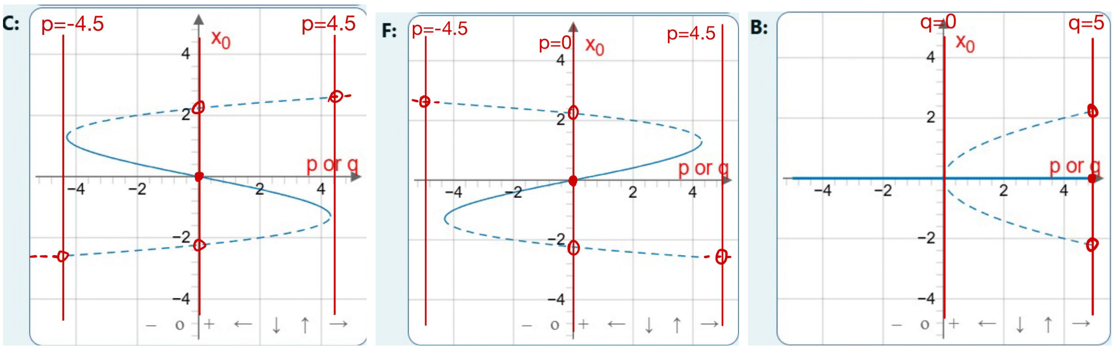


Figure 3.5: Annotated bifurcation diagrams for the incorrect options corresponding to CSE 14: Choosing options C/F for p and option B for q

3.4.3 Case Study 3 Reflections

We would now like to reflect on the findings in this case study by considering the following questions:

What is the pedagogical significance of this question?
As we mentioned before, the question is aimed at helping students **understand the complex concepts** of hysteresis and pitchfork bifurcations with interactive graphs. Students are not advised to practise on this question repeatedly because it is not designed to be an exam-type of question but instead to visualise concepts. They should take it as a learning material that helps them clarify the complex concepts especially for high-achievers who have mastered the foundational content in Mathematical Biology.

How did the technical features help in achieving the pedagogical goal of this question?
We have discussed the purpose of implementation of **sliders** with relevant supporting studies that showed how interactive diagrams can help students improve their understanding to the concept. However, to truly evaluate whether the student has fully understood the two bifurcation phenomena, they will be assessed in the MCQ in part (c) to **apply** the concept in bifurcation diagrams, which has provided several distractors to increase the question's difficulty. We have identified errors in (U2) and (UM2) which reflect errors in different areas of understanding or methods related to the concept.

3.5 Summary of Common Student Errors for each question type

We have shown 14 common student errors (CSE 1 - CSE 14) identified in the three case studies based on questions that we have selected on the three question types, from 1 fill-in-the-blank question, 3 multiple-choice questions and 1 interactive graph question. We can summarise the common student errors in Table 3.4, which are classified according to the question type and the classification codes.

Table 3.4: Classified common student errors

Code	Fill-in-the-Blank	Multiple-choice	Interactive graph
U1			CSE 3
U2	CSE 11, 12	CSE 5, 9, 13	
U3		CSE 10	
U4			
U5		CSE 7, 8	CSE 1
U6			
U7		CSE 6	
CM1			CSE 2
UM2	CSE 11, 12	CSE 4, 14	CSE 2

Note that since we have only analysed a very small portion of the questions selected for each question type, the summary of implications from the results may **not be very representative** of the actual performance of all the question types. In the future, more analysis of common student errors can be done on more questions to obtain a more accurate and representative result.
Just by analysing Table 3.4, we first divide the errors into two groups:

- Group U: the errors in **understanding** (U1-U7);
- Group M: the errors in **method** (CM1, UM2).

Note that for both *fill-in-the-blank* and *interactive graph questions*, we have 2 errors from Group U and 2 errors from Group M. Hence, they can test students in both **conceptual understanding** and **mathematical methods** equally. For *multiple-choice questions*, there are 7 errors from Group U and 2 errors from Group M, which implies that it can test more **conceptual understanding** than mathematical methods.

Does the common student errors match the expected pedagogical function of each question type?

As we have discussed in Chapters 1 and 2, **fill-in-the-blank question** is the simplest question type in STACK, which supports algebraic inputs. We do expect that a lot of questions can be designed to ask students to perform a long computational procedure and provide the final answer in the blank. For example, we have implemented several fill-in-the-blanks in questions 2 (b) and (c) aimed at helping students **practise the routine computational procedures**, where we identified errors in Group M and U. The errors in Group U are (U2) incorrect arguments, which indicates that some errors in computational procedures could also be **linked to conceptual misunderstandings**.

For **multiple-choice questions**, we have discussed in Section 3.3.1 that they can be applicable in testing the conceptual understanding component of mathematics. In questions 2 (a) and 3 (c), we have implemented multiple-choice questions aimed at assessing interpretation of biological models and concepts of bifurcation phenomena, where we identified a lot of errors in Group U. Therefore, it is no surprise that a lot more errors in Group U have been identified compared to Group M.

Finally, for **interactive graph** questions, we have discussed that they can provide visual images that represent concepts and help demonstrate and recall a graphical method, such as identifying equilibria using the phase plot in question 1 (b), where we identified errors in both Group U and Group M. For the interactive tools such as sliders in question 3 (b), we discussed that they can further shape one's mathematical thinking and enhance their conceptual understanding. Therefore, the common student errors that we identified match the pedagogical function of each question type.

How to improve the design of an e-assessment from the choice of question types?

Once we have confirmed the pedagogical function of each question type, we can optimise an e-assessment by selecting the question type based on its function to best achieve the pedagogical goal of the questions. For example, using our current result of the function of each question type, if we have designed a question to assess students in conceptual understanding, we may prefer using a multiple-choice question to assess the concept. Similarly, if we have a question focused on performing a graphical method, an interactive graph may be the most suitable option that visualises the concept and assesses students' ability to perform the graphical procedure.

Chapter 4

Potential Areas for Future Work

In the previous chapter, we included three case studies where we analysed the common student errors and used them for deducing the pedagogical function of each question type of STACK. We would like to discuss potential areas for future work that could enhance an e-assessment from the perspectives of **common student errors** and **feedback**.

4.1 Common Student Errors

4.1.1 Identifying common student errors

First of all, we have mentioned that the common student errors we have analysed are only based on our own interpretations and the specifics of the question design, which may not be reflective of the actual errors that students would make in common when they attempt our e-assessment questions. The analysis of common student errors can be overhauled in the future if we can release our questions to the students who take the module.

According to [14, p. 59] from UWE Bristol, they have gathered the common student errors of an e-assessed examination using the data from the **e-assessment system** as well as the **students' rough work scripts**, which were given to the students for working through the question before submission to the system. The second gathering method may not be feasible, since our e-assessments are designed for revision purpose but not for examinations. The first method is worth considering. We can gather the common student errors from the e-assessment system when we have collected large enough data from the students who have attempted our questions. We can then obtain the **statistics** for the most common student errors instead of identifying them from our own interpretations.

However, since students can reattempt the questions for revision, they may perform better in the subsequent attempts, so we may want to **calculate the statistics of common student errors only based on the first attempt or an average of the first few attempts**. Although the formative e-assessment is designed for revision purposes, we can also make it compulsory for students to treat it as a regular formative assignment so that we can obtain more accurate student performance statistics.

4.1.2 Utilising common student errors in question design

Educators can utilise the identified common student errors to refine their teaching materials and curriculum design. For example, they can allocate more lecture time on the topics where they have found many students with errors of conceptual understanding. They can also improve the question design of the e-assessments. As suggested in [6, p. 214], we can

create distractors of a multiple-choice question using the identified common student errors. This further enhances students' conceptual understanding.

4.1.3 Differentiating between errors of slips of action (S) and errors of understanding and methods (U/UM/CM)

In the three case studies that we discussed, we have analysed common student errors according to Table 3.1 in the categories of errors in understanding (U) and methods (UM)/(CM). However, we have not covered the errors in slip of action (S). This is because we only focused on whether the question can achieve the goal: to help students understand or assess them on a certain concept or method. We have ignored errors in slip of action which are the most undesirable since they do not reflect a student's misunderstanding to the concept.

In fact, some common student errors that we have classified could also come from slip of actions, especially for **multiple-choice questions**, which have way fewer possibilities of student answers than fill-in-the-blanks, since there are only a few possible options in MCQs. Therefore, we only assumed the errors did not come from an unintentional mistake, but in reality, there are a lot of students who will make careless mistakes, such as (S1-S3), reflecting copying errors, e.g. neglecting the negative sign, copying the wrong numbers, which we do not know whether they are just slips of action or conceptual misunderstandings. The errors in slips of action can hardly be reduced technically from the e-assessment system, except for improving the design of the question to minimise these errors as many as possible. For example, hinting the answer is negative by providing a negative sign before an input box, but this could potentially lower the difficulty for some other questions when students are expected to construct their answers fully by themselves but were given too many hints. Potential research can be done to detect the errors in slip of action from the other two.

There are also **technical errors** due to the e-assessment setting which we do not expect from traditional written assignments, such as providing the incorrect input type, pressing the wrong button [6, p. 207]. The first error does not always apply because STACK has a built-in function of input validation as introduced in Section 1.3.1, which checks the student's input type before submission. The second error could be due to the fact that the student is not used to the e-assessment platform at the beginning, but it is expected to reduce in the future as e-assessment becomes a more regular assessment format.

4.2 Feedback

4.2.1 Specific feedback

We aim to provide specific feedback which directly points out the common student error identified from the student and **provide hints or suggestions on how to address the error**. Firstly, to detect the common student errors, we can create several True or False nodes based on the identified CSE in the PRT component.

For example, we can construct the specific feedback for CSE 5 as follows. You may refer back to Section 3.2.2 for CSE 5. In the feedback variables of the corresponding PRT, we first set up the variable `CSE5` as a list of the student's answers for the three inputs corresponding to the Logistic, Goldfish and modified models respectively. This ensures that we can **check the 3 inputs simultaneously** to identify CSE 5:

```
CSE5: [ans6, ans7, ans8];
```

We then create a new node under one of the inputs, e.g. the incorrect input `ans6` for the Logistic equation's Allee effect. Then we complete the Node by providing the specific feedback as in Figure 4.1. Note that we have labelled option A as "Strong Allee effect", B as "Weak Allee effect" and C as "No Allee effect" in the question variables component.

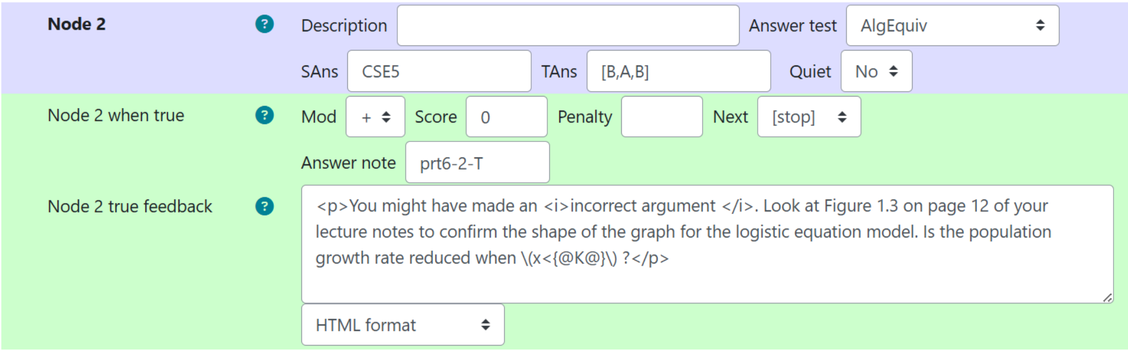


Figure 4.1: True or False Node for testing CSE 5 and the corresponding feedback

When we write the specific feedback, we should aim to provide feedback tailored to each error. In addition, we can enhance the feedback by providing more details to the students for self-reflection, such as indicating the type of error and the explanation, and provide reference to the relevant theorems or methods in the lecture notes. An example of tailored feedback for CSE 5 is demonstrated in Figure 4.1.

In some questions, we can also demonstrate that the student's answer is incorrect by **substituting their answers into the system**. For example, in Question 2(c)(i), students are asked to find the equilibrium of the system. We can substitute the student's answer into the differential equations and check the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Then we can demonstrate that if they are not both equal to zero, the student's answer is incorrect by definition. Similarly, we can also analyse the students' incorrect answers by providing a **graphical explanation**. For example, we analysed the incorrect options from CSE 14 in Question 3 (c) by annotating the bifurcation diagrams in Figure 3.5. They can be displayed to the students in the specific feedback to visually demonstrate why they are incorrect.

If the statistics for students' performance in the e-assessment in previous years are available, we can also provide them with the percentage of students who have answered the question correctly. If the percentage is high and the student has got it wrong, it might reflect a more serious error of understanding to the simple problem and they might need to put more attention to it, or it could simply be an error of slip of action. This allows students to **monitor their progress of revision**, which is what a formative e-assessment should be able to achieve, based on [3, p. 2].

4.2.2 General feedback

In addition to providing the fully worked solution to the students in the general feedback, we can provide an analysis of key concepts and skills tested in each sub-question, similar to what we presented in each of our case studies. It allows students to link each sub-question to the relevant concepts or mathematical methods for them to reflect and focus on their weaker areas during revision.

Chapter 5

Conclusion

To conclude, our research goal is to create effective formative e-assessments for revision purpose, allowing students to practise and prepare for their final exams. Therefore, we explored the technical and pedagogical aspects of setting up a STACK e-assessment through three e-assessment questions that we designed for the module: Mathematical Biology.

In the **technical** chapters, we introduced three question types of STACK, where we demonstrated the structure of a STACK question and how to implement the fundamental features of STACK in a fill-in-the-blank question, such as *randomisation* and *answer tests*. We then explored the more advanced technical features of multiple-choice questions and interactive graph questions. In multiple-choice questions, we explored the *advanced randomisation features* and *partial scoring* algorithms. For interactive graphs, we explored the diverse object types that JSXGraph can provide, including *movable points*, *buttons* and *sliders*, and we also implemented several methods of *input bindings* for answer checking in the potential response trees.

In the **pedagogical** chapter, we analysed three case studies for each e-assessment question that we have designed and identified the common student errors of selected sub-questions. Each case study focused on a distinct pedagogical focus and implemented different technical features introduced in the technical chapter. We discussed the intended purpose of implementing the technical features. Some technical features can support students' learning, while some can help in increasing the question's difficulty and target key concepts. We also summarised the common student errors and the pedagogical function of each question type: fill-in-the-blank and interactive graph questions can test students in both **conceptual understanding** and the practice of **mathematical methods**, while multiple-choice questions can test more of **conceptual understanding**. Through enhancing the students' mathematical skills in these two areas, we believe the formative e-assessment could be an effective learning material for students' revision.

Finally, we discussed potential areas for future work from the perspectives of **common student errors** and **feedback**. We believe that releasing the e-assessment to students taking the module Mathematical Biology will be highly beneficial for both the students and the educators. It allows educators to obtain more accurate statistics in student performance in order to refine the question design, identify more representative common student errors, as well as providing more effective feedback which allows students to reflect on during revision.

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Appendix A

Full Questions

A.1 Question 1 (randomly generated)

STACK question dashboarc

Consider the following modified Allee effect model for a goldfish population.

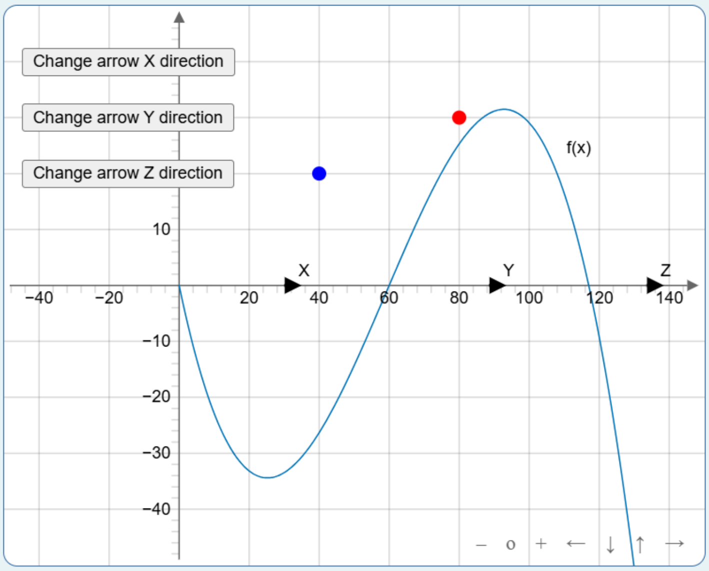
$$f(x) = \frac{dx}{dt} = -3 \cdot \left(1 - \frac{x}{60}\right) \cdot \left(1 - \frac{x}{117}\right) \cdot x$$

It models the scaled population x of the goldfish under poor environmental conditions.

(a) What is the **carrying capacity** of the goldfish population? Ignore the units.

Carrying capacity = ______.

(b) We may hence obtain the following phase plot.



(i) Adjust the directions of arrows X, Y and Z such that they point at the direction of change of population (along the x-axis) at their respective points.

(ii) Move the blue point to any stable equilibrium of the graph.

(iii) Move the red point to any unstable equilibrium of the graph.

A.2 Question 1 (continued)

It is claimed that the initial conditions of a system may be crucial in the long term population of a species.

(c) What happens to the goldfish population x as $t \to \infty$ if initially $x < 60$?

$x \to$ [ ]

This value of x: 60 is also called the critical/threshold capacity (*) of the population.

(d) Allee effect can be classified as strong Allee effect if the population has a negative growth rate when the population size is small, and weak Allee effect if the population has reduced but positive growth rate. **(i)** Determine whether the following population models has a threshold capacity (*), **(ii)** and whether they exhibit strong, weak or no Allee effect.

The logistic equation:

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{117}\right)$$

Any threshold population? (Clear my choice)

Any Allee effect? (Clear my choice)

The goldfish population model:

$$\frac{dx}{dt} = -3 \cdot \left(1 - \frac{x}{60}\right) \cdot \left(1 - \frac{x}{117}\right) \cdot x$$

Any threshold population? (Clear my choice)

Any Allee effect? (Clear my choice)

A modified population model:

$$\frac{dx}{dt} = \frac{3x^2}{117}\left(1 - \frac{x}{117}\right)$$

Any threshold population? (Clear my choice)

Any Allee effect? (Clear my choice)

A.3 Question 2 (randomly generated)

STACK question dashboard

Consider a modified biological system between species X and Y, with (scaled) population x and y respectively.

It is given that

$$\frac{dx}{dt} = a \cdot x - \frac{a \cdot x \cdot y}{x+b}$$

$$\frac{dy}{dt} = \frac{a \cdot x \cdot y}{x+b} - c \cdot y,$$

where a, b, c (and d) are positive constants.

(a) Which of the following is/are the appropriate biological interpretation for the system above?

- ☐ Both X and Y are following logistic growth or decay if we ignore the interaction terms.
- ☐ It is a competitive Lotka-Voltera model.
- ☐ X gets full so X cannot consume Y indefinitely.
- ☐ It is a Predator-prey model where X is prey and Y is predator.
- ☐ It is a Predator-prey model where X is predator and Y is prey.
- ☐ None of the above

(b) To perform nondimensionalisation, we can write $x = \hat{x}X, y = \hat{y}Y$ and $t = \hat{t}T$ and substitute in the system of equations. Finally, by treating $\hat{x}, \hat{y}$ and $\hat{t}$ as x, y and t respectively, we obtain the nondimensionalised system of equations:

$$\frac{dx}{dt} = x - \frac{x \cdot y}{x+1}$$

$$\frac{dy}{dt} = \frac{x \cdot y}{x+1} - K \cdot y.$$

Express K, T, Y and X in terms of variables a, b, c (and d).

$K =$ ______

$T =$ ______

$Y =$ ______

$X =$ ______

A.4 Question 2 (continued)

(c) (i) You are given the system of equations:

$$\frac{dx}{dt} = x - \frac{xy}{x+1}$$

$$\frac{dy}{dt} = -Ky + \frac{xy}{x+1}.$$

Find the equilibrium of the system **apart from the origin**. Leave your answer in term of K.

$[x, y] = \begin{bmatrix} \boxed{} & \boxed{} \end{bmatrix}$

(ii) Hence, find the Jacobian matrix J for the system. Express in terms of the variables x, y and K.

$J = \begin{bmatrix} \boxed{} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}$

(d) Stability Analysis.

The stability of an equilibrium may be dependent on parameter values.

Consider the equilibrium $(0, 0)$.

If $K < 0$, then it is (Clear my choice) ⬍ .

If $K = 0$, then it is (Clear my choice) ⬍ .

If $K > 0$, then it is (Clear my choice) ⬍ .

Consider the other equilibrium we found in (c)(i).

If $K < 0$, then it is (Clear my choice) ⬍ .

If $K = 0$, then it is (Clear my choice) ⬍ .

If $0 < K < 1$, then it is (Clear my choice) ⬍ .

If $K > 1$, then it is (Clear my choice) ⬍ .

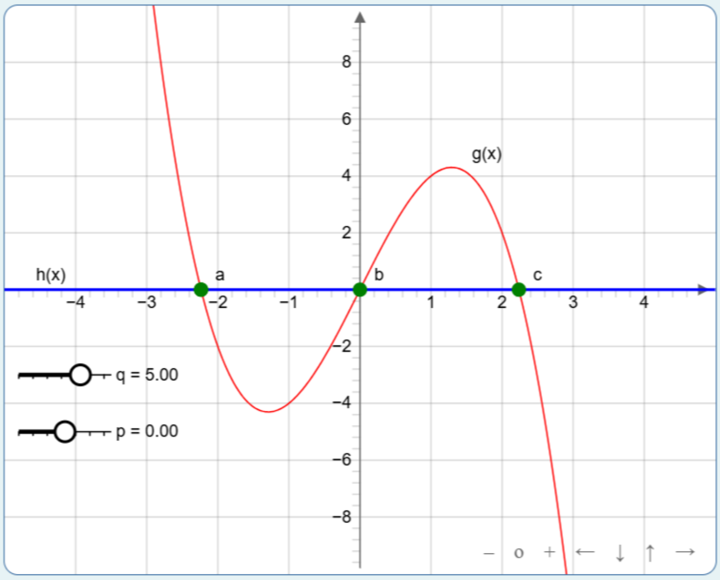
A.5 Question 3

STACK question dashboard

Let $g(x) = -x^3 + qx$ and $h(x) = -p$, where p and q are some biological parameters. Consider the two-parameter model $\frac{dx}{dt} = g(x) - h(x)$.

Let a, b, c be the points of intersection of $g(x)$ and $h(x)$ at the three sections (from left to right) separated by the turning points of $g(x)$, as introduced in lectures. At the values of p and q where the graph has **no turning points**, we will define a as the intersection point in the left half plane, c as the intersection point in the right half plane and b if the intersection point is on the y-axis.

The following interactive graph shows the functions of $g(x)$ and $h(x)$ against x and the intersection points. You can move the sliders to adjust the values of the parameters p and q. Do not worry about the feasibility of the negative values of x.



(a) (i) Where are the equilibria positions of the system?

- ○ (Clear my choice)

- ○ x-intercepts of the curve $y = g(x)$.
- ○ Intersection of $g(x)$ and $h(x)$.
- ○ Intersection of $g'(x)$ and $h'(x)$.

A.6 Question 3 (continued)

(ii) What does the parameter p also represent in the phase portrait $\frac{dx}{dt}$ against x?

○ (Clear my choice)

○ Shifting of the graph $y = g(x)$ downwards by p.

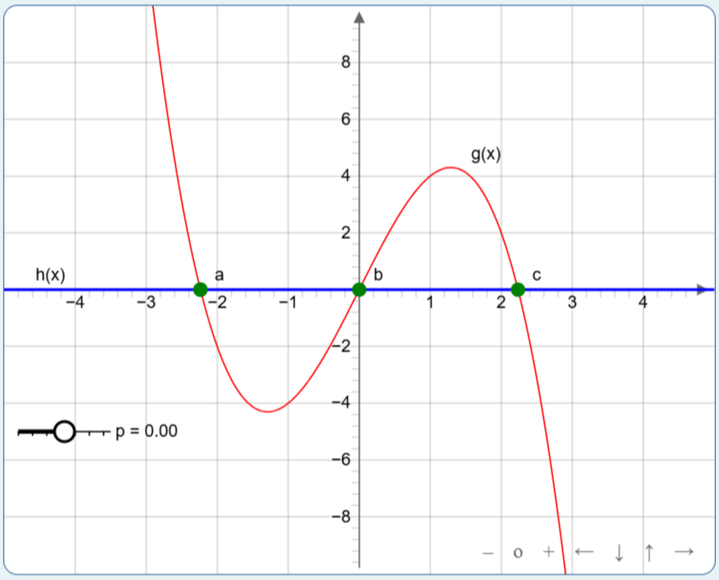
○ Shifting of the graph $y = g(x)$ upwards by p.

○ Shifting of the graph $y = g(x)$ to the left by p.

○ Shifting of the graph $y = g(x)$ to the right by p.

For questions (b)(i), (ii) and (iii), use the graph specified by each question to move the sliders p and q **approximately** to the correct values required by the question.

(b) (i) Consider the interactive graph below.



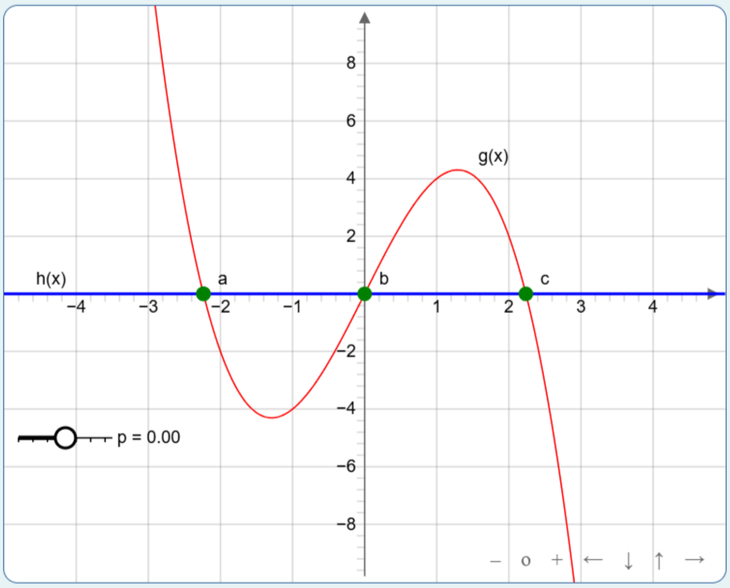
In the graph, q is fixed at 5.

Assume initially $p = 0$, $x = a$. What value should p be moved to such that x will eventually shift to c, given that the change in p is minimal in magnitude?

$p =$ ______.

A.7 Question 3 (continued)

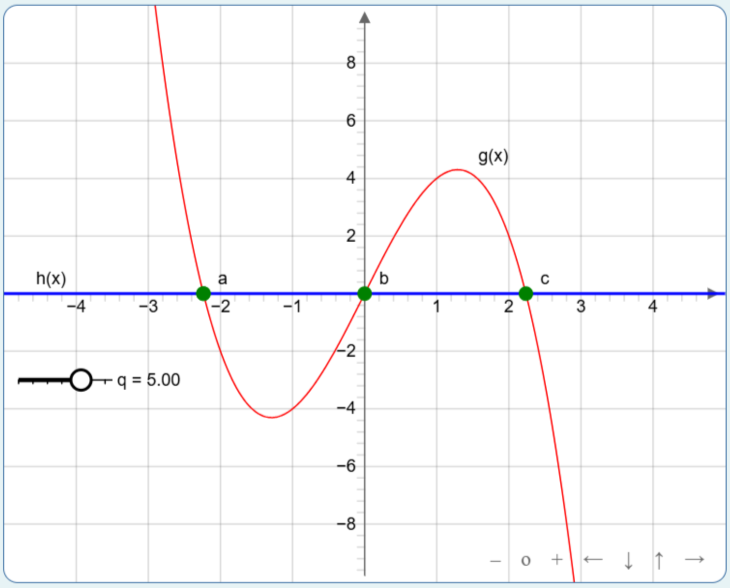
(ii) Consider the interactive graph below:



Hysteresis **has occurred** due to changes in p in (c)(i). Even if we restore the value of p to the original, x will still remain at c. Given the occurence of the hysteresis and that the change in p is minimal in magnitude, what value should p be moved to such that x will eventually return back to its original value: $x = a$?

$p =$ ______.

(iii) Consider the interactive graph below:



In the graph, p is fixed at 0.

Assume initially $q = 5$, $x = c$. Where should q be moved to such that x will eventaully shift to b, given that the change in q is minimal in magnitude?

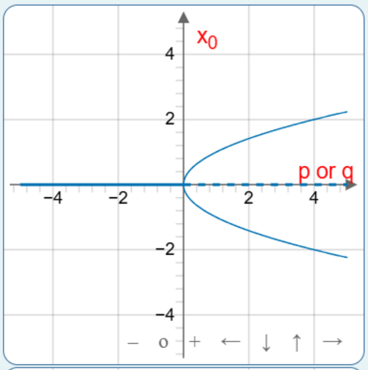
$q =$ ______.

A.8 Question 3 (continued)

(c) Choose the bifurcation diagrams (x_0 against (p or q)) for the parameters p (if we keep $q = 5$) and q (if we keep $p = 0$), where x_0 refers to the value of the equilibrium point(s). Note that solid line and dotted line imply stable and unstable equilibria respectively in the graphs.

p : (Clear my choice) , q : (Clear my choice) .

A:



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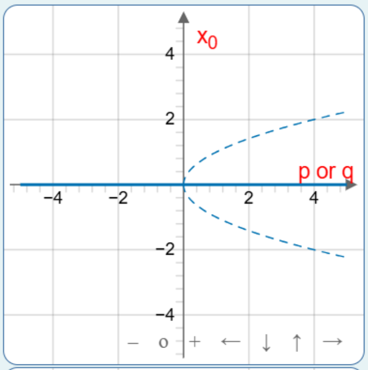
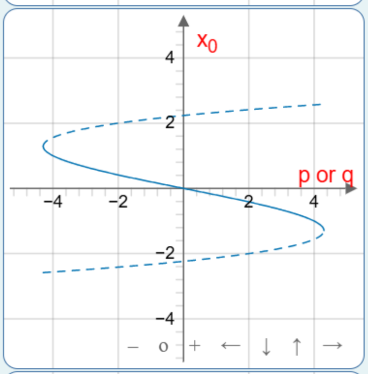
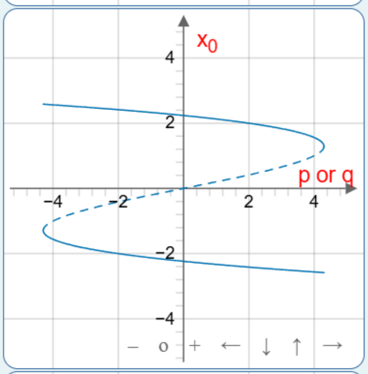


C:



D:



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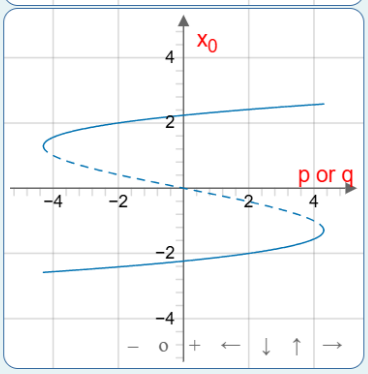

F: