数字图像处理与分析

Homework 3

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1. 证明如下的等式成立：

$$f(x,y)\exp\left[\frac{j2\pi(u_0x+v_0y)}{N}\right]\Leftrightarrow F(u-u_0,v-v_0)$$

$$f(x-x_0,y-y_0)\Leftrightarrow F(u,v)\exp\left[\frac{-j2\pi(ux_0+vy_0)}{N}\right]$$

答：首先有

$$\begin{aligned}F(u-u_0,v-v_0)&=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\exp\frac{-j2\pi\left[(u-u_0)x+(v-v_0)y\right]}{N}\\&=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\exp\frac{j2\pi(u_0x+v_0y)}{N}\exp\frac{-j2\pi(ux+vy)}{N}\\&=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f_1(x,y)\exp\frac{-j2\pi(ux+vy)}{N}\end{aligned}\tag{1}$$

其中 $f_1(x,y)=f(x,y)\exp\frac{j2\pi(u_0x+v_0y)}{N}$。同时又有

$$\begin{aligned}f(x-x_0,y-y_0)&=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(u,v)\exp\frac{j2\pi\left[u(x-x_0)+v(y-y_0)\right]}{N}\\&=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F(x,y)\exp\frac{-j2\pi(ux_0+vy_0)}{N}\exp\frac{j2\pi(ux+vy)}{N}\\&=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}F_1(x,y)\exp\frac{j2\pi(ux+vy)}{N}\end{aligned}\tag{2}$$

其中 $F_1(x,y)=F(x,y)\exp\frac{-j2\pi(ux_0+vy_0)}{N}$。

综上所述，原傅里叶变换对应关系成立。

2．证明$f(x)$的自相关函数的傅里叶变换就是$f(x)$的功率谱$|F(u)|^2$。

答：设$F(u) \Leftrightarrow f(x)$。先证明如下的引理：

$$F^*(u) = \int_{\infty}^{\infty} f^*(x)e^{j2\pi ux}dx$$

证明：可知

$$\begin{aligned} F(u) &= \int_{\infty}^{\infty} f(x)e^{-j2\pi ux}dx \\ &= \int_{\infty}^{\infty} \left[f_R(x) + jf_I(x)\right](\cos 2\pi ux - j\sin 2\pi ux)dx \\ &= \int_{\infty}^{\infty} \left[(f_R(x)\cos 2\pi ux + f_I(x)\sin 2\pi ux) + j(f_R(x)\sin 2\pi ux - f_I(x)\cos 2\pi ux)\right]dx \end{aligned} \tag{3}$$

所以

$$\begin{aligned} F^*(u) &= \int_{\infty}^{\infty} \left[(f_R(x)\cos 2\pi ux + f_I(x)\sin 2\pi ux) + j(f_I(x)\cos 2\pi ux - f_R(x)\sin 2\pi ux)\right]dx \\ &= \int_{\infty}^{\infty} \left[f_R(x) - jf_I(x)\right](\cos 2\pi ux + j\sin 2\pi ux)dx \\ &= \int_{\infty}^{\infty} f^*(x)e^{j2\pi ux}dx \end{aligned} \tag{4}$$

故引理得证。所以

$$\begin{aligned} |F(u)|^2 &= F(u)F^*(u) \\ &= \int_{\infty}^{\infty} f(x)e^{-j2\pi ux}dx \int_{\infty}^{\infty} f^*(y)e^{j2\pi uy}dy \\ &= \int_{\infty}^{\infty}\int_{\infty}^{\infty} f^*(x)f(y)e^{j2\pi u(x-y)}dxdy \end{aligned} \tag{5}$$

令 $\tau = y - x$，上式化为

$$\begin{aligned} |F(u)|^2 &= \int_{\infty}^{\infty}\int_{\infty}^{\infty} f^*(x)f(x+\tau)e^{-j2\pi u\tau}dxd\tau \\ &= \int_{\infty}^{\infty} \varphi(\tau)e^{-j2\pi u\tau}d\tau \end{aligned} \tag{6}$$

其中 $\varphi(\tau) = \int_{\infty}^{\infty} f^*(x)f(\tau + x)dx = f(x) * f(x)$。

所以 $|F(u)|^2 \Leftrightarrow f(x) * f(x)$，证毕。

3. 证明离散傅里叶变换和反变换都是周期函数（为简便可以用1-D函数为例）。

答：可知

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)e^{-j\frac{2\pi ux}{N}}$$

$$f(x)=\frac{1}{N}\sum_{u=0}^{N-1}F(u)e^{j\frac{2\pi ux}{N}}$$

所以

$$\begin{aligned}F(u+kN)&=\frac{1}{N}\sum_{x=0}^{N-1}f(x)e^{-j\frac{2\pi ux}{N}}e^{-j2\pi kx}\\&=\frac{1}{N}\sum_{x=0}^{N-1}f(x)e^{-j\frac{2\pi ux}{N}}(\cos 2\pi kx-\sin 2\pi kx)\\&=\frac{1}{N}\sum_{x=0}^{N-1}f(x)e^{-j\frac{2\pi ux}{N}}\\&=F(u),\quad k\in\mathbf{Z}\end{aligned}\tag{7}$$

$$\begin{aligned}f(x+kN)&=\frac{1}{N}\sum_{u=0}^{N-1}F(u)e^{j\frac{2\pi ux}{N}}e^{j2\pi kx}\\&=\frac{1}{N}\sum_{u=0}^{N-1}F(u)e^{j\frac{2\pi ux}{N}}(\cos 2\pi kx+\sin 2\pi kx)\\&=\frac{1}{N}\sum_{u=0}^{N-1}F(u)e^{j\frac{2\pi ux}{N}}\\&=f(x),\quad k\in\mathbf{Z}\end{aligned}\tag{8}$$

故离散傅里叶变换和反变换都是周期函数。证毕。