1 Syllabus Subsections

1. Systems of Linear Equations
2. RREF, Rank
3. Vectors, Dot, Cross Products
4. Inverses
5. Linear Transformations
6. Projection, Reflection, Rotation
7. Determinant and Cofactors
8. Eigenvalues, Eigenvectors
9. Diagonalization
10. Subspaces
11. Span
12. Linear Independence
13. Basis
14. Dimension
15. Orthogonality
16. Gram-Schmidt Algorithm
17. Orthogonal Diagonalization

Not all of these will be covered in this document, only the following sections:

- Gram-Schmidt
- Subspaces
- Span
- Orthogonality
- Basis

2 Subspaces

A set U is a subspace if:

- The zero vector is in U ($\vec{0} \in \mathbb{R}$)
- U is closed under vector addition
- U is closed under scalar multiplication

The **image space** of A $(im(A)) = \{A\vec{x} | \vec{x} \in \mathbb{R}^n\} = col(A)$
The **column space** of A is the set of all linear combinations of the columns of A, or the span of the columns of A.
We can verify that the image space is a subspace by checking it against the subspace conditions.

$$im(A) = 0$$

$\vec{x}, \vec{y} \in im(A) \Rightarrow A\vec{b} = \vec{x}, A\vec{c} = \vec{y}, A(x+y) = k \Rightarrow Ab+Ac = x+y$

Recall that we need to prove that there exists a vector d such that $A\vec{d} = x+y$. So, we factor to get $A(b+c) = \vec{x} + \vec{y}$

2.1 General Procedure to Prove something is a Vector Space or Linear Transformation etc.

Define the variables:

$$\lambda \in \mathbb{R}, x \in im(A)$$

Define the boundary conditions:

$$\vec{x} = A\vec{b} \Rightarrow \lambda\vec{x} = \lambda A\vec{b}$$

Find what we need to satisfy:

$$\lambda x = A(\lambda b)$$

Rearrange, algebraically: (just switch A, λ in boundary condition step).

2.2 Linear Independence

Linear Independence for a set of vectors V_1 to V_k occurs when $a_1V_1...a_kV_k = 0$ only if $a_1...a_k = 0$. That is, there is no vector in the set that is a linear combination of the others.

3 GSA

The purpose of GSA is to turn a set of bases into a set of orthogonal bases.
Say we have some basis vectors x_1, x_2, x_3 and we want to turn them into some f_1, f_2, f_3 such that all $f_i \cdot f_j = 0$ for all $f_i \neq f_j$.

We subtract the projection onto all previously orthogonal vectors onto each of the next vectors, iteratively - we project x_1 onto x_1 and get it back, we get $f_2 = x_2 - proj_{x_1} x_2$. Then, we take $f_3 = x_3 - proj_{x_1} x_3 - proj_{f_2} x_3$

A general formula for GSA of some basis vectors x_i into an orthogonal set f_i

$$f_n = x_n - \sum_{i}^{n} \text{proj}_{f_i} x_n$$

Do not try this on a set of vectors that is linearly dependent, as one of the vectors (redundant) end up going to zero.

To check, put the vectors as columns in a matrix A, reduce A to REF, and take the columns in A corresponding to the columns in A_{REF} with leading ones as the independent bases.

4 Orthogonal Diagonalization

Spectral Theorem: If A is a real symmetric matrix $A^T = A$, then A has an orthogonal diagonalization.

A general procedure for orthogonally diagonalizing a symmetric matrix A is as follows:

1. Find the eigenvalues λ_i of A by finding solutions to the characteristic polynomial $C_\lambda A = det(A - \lambda I)$.
2. Find the eigenvectors for each lambda by finding $null(A - \lambda I)$ for all λ_i.
3. For the eigenspaces $E_{\lambda_i} A$ with dimension greater than 1 (eg. the multiplicity of $\lambda i \geq 2$, there are two or more eigenvectors for λ_i), use GSA on the two vectors if they're not already orthogonal to make them orthogonal. Different eigenspaces will be orthogonal to each other already.
4. Write the diagonalizing matrix P with eigenvectors as cols and diagonal D (positioned with respective eigenvalues multiples of I for the eigenvectors in P), and normalize all values in P by dividing each eigenvector by its magnitude to get an orthogonal matrix for P.