1 Subspaces II

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1.1 Finding Bases

With any mxn matrix A, we can row reduce it.

$$A_{(mxn)} \to R_{REF,(mxn)}$$

And we find that they have certain relationships and similarities. These are because the elementary operations preserve some things. Eg:

- The row spaces are equal: $row(A) = row(R)$. Recall, the **row space** is equal to the span of the rows of A. If

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$row(A) = span \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

When we have some matrix in row echelon form, it becomes very easy to check linear independence.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$row(A) = span \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can put this into an equation, giving each row a parameter:

$$a \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} a \\ 2a \\ -a+b \\ 2b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can take a = 0 from the first line, and calculate for the rest of the rows we know that all the coefficients a,b, and c must be zero. Thus, $a = b = c = 0$ is the only solution, so it is linearly independent.

This tells us that **For any row echelon R, the set of *NONZERO* rows is linearly independent.**

EX.
If we want to find a basis for

$$spanof \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$$

We can write them as row vectors and row reduce it.

$$\begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 1 & 3 & -1 & 0 \end{bmatrix} \text{is equivalent to} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This provides the basis $\begin{bmatrix} 1 & 2 & -1 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T$. However, these are not in the original spanning set.

To find vectors that are a bit closer to the original matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ -1 & -2 & -1 \\ 1 & 2 & 0 \end{bmatrix} \text{equiv to} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The leading ones designate the columns (R1, R3) of A from which we take the basis, and provide us a basis **composed** of a subset of the originals. Thus, the basis is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}$$

.
Say we had a 2x4 matrix A. If we multiply $A\vec{x} = \vec{y}$, we know that $\vec{x}$ has to have 4 elements, and that $\vec{y}$ has 2 elements. ($\vec{x} \in \mathbb{R}^2$, $\vec{y} \in \mathbb{R}^4$).

We can conclude that $dim(row(A)) = 0, 1, 2$ but definitely not 3, so our row space can have at most 2 vectors.
Also, the dimension of the column space is equal to the dimension of the row space, and this is the value we call the rank.

$$dim(rowA) = dim(colA)$$

We also know that $im(A) = col(A)$. This is because the span of the columns of in A:

$col(A) = span\{\vec{c_1}, \vec{c_2}...\vec{c_n}\}$
Also

$$im(A) = \{A\vec{x}|\vec{x} \in \mathbb{R}^n\}$$

But

$$A\vec{x} = [\vec{c_1}, \vec{c_2}...\vec{c_n}] \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \vec{c_1}x_1...\vec{c_n}x_n$$

which is equal to our column space! Just interpret x_i as some arbitrary constant.

2 Summary

2.1 Subspaces

For any matrix A:

- We have 4 names for 3 subspaces (im(A) = col(A))
- row(A) = span (rows)
- col(A) = span(cols) = im(A)
- null(A) = solutions to $A\vec{x} = 0$

2.2 Connections

- dim(row(A) = dim(col(A))). In the REF form, the number of vectors is the rank, or the number of leading ones. These are in the same dimension, though in different spaces.
- For every column, we either have a leading one or a non-leading variable. The nonleading variables are the parameters, which give us solutions to the null space. Thus $dim(nullA) = n - rank(A) = \text{number of columns} - \text{number of leading variables}$. This is known as the **Rank + Nullity Theorem** which tell us that

  $$rank(A) + dim(null(A)) = n$$

  or: (number of cols with leading variables) + (number of cols with nonleading variables) = (total number of columns)

Thus we can conclude some things about a matrix without knowing anything except for the size. So, for a 2x4 matrix of A:

- dim(row(A)) = 0, 1, or 2
- dim(col(A)) = 0, 1, or 2
- dim(null(A)) = 4, 3, or 2 from the previous two points