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Laminar Flow of Newtonian Liquids in Ducts of Rectangular Cross-Section an Interesting Model for Both Physics and Mathematics

Franck Delplace

ESI Group Scientific Committee, 100 Av. De Suffren, Paris, France

Abstract In this paper, we considered the laminar fully developed flow, of a Newtonian fluid, in ducts of rectangular cross-section. Poisson's partial differential equation Saint-Venant solution was used, to calculate Poiseuille number values whatever is rectangles aspect ratio. From these results, we considered limit cases of square duct and plane Poiseuille flow (infinite parallel plates). We showed there exists a rectangle equivalent to a circular cross-section for energy dissipation through viscous friction. Finally, we gave some mathematical consequences of this approach for odd integers zeta function calculations and Catalan's constant.

Keywords Rectangular ducts, Poisson's equation, Saint-Venant solution, Viscous friction, Zeta function, Catalan's constant

1. Introduction

Pipes used in most applications always have a circular cross-section. That is why Poiseuille law/equation is used to calculate the pressure drop produced by a liquid flowing in a pipe in the laminar flow regime. Poiseuille famous equation tells us that pressure drop is proportional to liquid flow-rate $Q(m^3.s^{-1})$. In Engineering, this relationship is expressed using dimensionless numbers: The Fanning friction factor $(f/2)$ and the Reynolds number (Re):

$$\frac{f}{2} = \frac{8}{Re} \tag{1}$$

With

$$\frac{f}{2} = \frac{\tau_w}{\rho \bar{v}^2} \tag{2}$$

And

$$Re = \frac{\rho \bar{v} D}{\eta} \tag{3}$$

In equation (2), $D(m)$ is the pipe diameter, $\tau_w(Pa)$ is the wall shear stress due to liquid friction on pipe wall. In the case of the perfectly symmetric circular cross-section, its value is identical whatever is the position along the perimeter giving the local value τ_w equal to the mean value $\bar{\tau}_w$. $\rho(kg.m^{-3})$ is the liquid density and $\bar{v}(m.s^{-1})$ its mean velocity calculated from the flow-rate $Q(m^3.s^{-1})$ measurement using $\bar{v} = Q/S$ where $S(m^2) = \pi R^2$ is the cross-section area. From a balance between pressure drop $\Delta P(Pa)$ and viscous friction on pipe wall, it is possible to obtain a simple relationship between τ_w and ΔP:

$$\tau_w = \frac{\Delta P D}{4L} \tag{4}$$

In this equation, $D(m)$ is the pipe diameter and $L(m)$ is the pipe length where pressure drop ΔP is measured by use of a pressure sensor. The last parameter involved in equation (3) is well-known Newtonian liquid dynamic viscosity $\eta(Pa.s)$.

Finally, in equation (2), terms τ_w and $\rho \bar{v}^2$ are both energy concentrations, respectively energy dissipated by viscous friction and kinetic energy introduced in the liquid by the pumping system. From these considerations, dimensionless number $f/2$ represents the percentage of energy concentration dissipated by the liquid at pipe wall. Of course, this mechanical energy is converted into heat.

From equation (1), we can form the product:

$$f/2\ Re = Po \tag{5}$$

The new dimensionless quantity Po is called Poiseuille number in honour of important Poiseuille work on liquids laminar flow. In the simple case of pipe flow, we have $Po = 8$ (you sometimes find $Po = 16$ or $Po = 64$ depending on how you define Fanning friction factor).

Now, the question is what is the situation when a duct has a non-circular cross-section?

As reported in famous Shah & London [1] source book entitled "Laminar forced convection in ducts", and experimentally or numerically verified by numerous authors

* Corresponding author:
fr.delplace@gmail.com (Franck Delplace)
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[2], we generally have $Po \neq 8$ (we will explain why we say generally in the following of this paper).

An interesting and important geometry to investigate is rectangular ducts, from square cross-section exhibiting high symmetry properties (regular compact convex shape) to all rectangles of aspect ratio we called b/a. In fluid mechanics, we consider a limit case for rectangular geometries: the often called "Plane Poiseuille flow" corresponds to a rectangle such as $b \gg a$ giving $b/a \to +\infty$. This ideal type of flow is highly symmetric like the flow in a pipe because small side length has no influence on the velocity field which remains the same along large side length. The following figure 1 illustrates rectangular geometries considered in fluid mechanics.

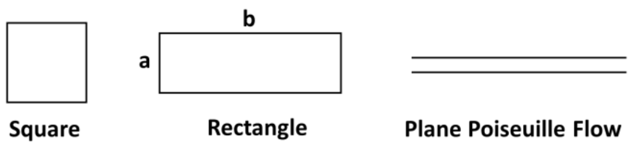

Figure 1. Rectangular cross-sections considered in fluid mechanics.

It is well-known that ideal plane Poiseuille flow gives a theoretical value $Po = 12$.

Moreover, as recently showed by Delplace [2], Po values could explain critical Reynolds number values for the change in the flow regime from laminar to transition and turbulent.

The objective of this paper is then to recall how Po values are obtained from Poisson partial differential equation (PDE) and also to try to explain why these results could be very important in both Physics and Mathematics.

2. Theory of Laminar Flow in Rectangular Ducts

Considering cartesian coordinates (x, y, z) with origin at the centre of the duct of rectangular cross-section, the fully established laminar flow of a Newtonian liquid is described by the following well-known Poisson equation:

$$\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} = -\frac{\Delta P}{\eta L} \tag{6}$$

Solutions of this PDE depends on the boundary conditions (Dirichlet problem) and the general case of rectangles with aspect ratio b/a can be solved by use of Saint-Venant method [3] giving the velocity field: $v_z(x, y)$:

$$v_z(x, y) = \frac{16\Delta P a^2}{\eta L \pi^3} \sum_{n=1,3,5,\dots}^{+\infty} \frac{(-1)^{(n-1)/2}}{n^3} \left(1 - \frac{ch\left(\frac{n\pi}{2a}y\right)}{ch\left(\frac{n\pi}{2a}b\right)}\right) cos\left(\frac{n\pi}{2a}x\right) \tag{7}$$

This equation allows components of wall shear-rate:

$$\left(\frac{\partial v_z(x,y)}{\partial x}\right)_w \text{ and } \left(\frac{\partial v_z(x,y)}{\partial y}\right)_w \tag{8}$$

To be calculated and then components of wall shear stress $\tau_w(x)$ and $\tau_w(y)$ by use of the rheological equation of state:

$$\tau_w(\blacksquare) = \eta \left(\frac{\partial v_z(x,y)}{\partial(\blacksquare)}\right)_w \tag{9}$$

The average wall shear stress can then be calculated using classical integral mean value:

$$\bar{\tau}_w = \frac{1}{a+b}\left(\int_0^a \tau_w(x)dx + \int_0^b \tau_w(y)dy\right) \tag{10}$$

We can also calculate from equation (7) the velocity mean value:

$$\bar{v} = \frac{1}{a\,b} \int_0^a \int_0^b v_z(x, y)dxdy \tag{11}$$

To finally obtain:

$$\frac{f}{2} = \frac{\bar{\tau}_w}{\rho \bar{v}^2} = \frac{Po}{Re} \tag{12}$$

Which is the analogous of relation (1) for the case of rectangular ducts. From knowledge of $\bar{\tau}$ and $\bar{v}$, we obtain:

$$Po = \frac{\pi^4 b^2}{8(a+b)^2 \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^4}\left(1 - \frac{2a}{n\pi b} th\left(\frac{n\pi b}{2a}\right)\right)} \tag{13}$$

It is now interesting to evaluate this last result for different aspect ratios b/a. The first elementary case is of course the square cross-section giving $b = a$.

Equation (13) reduces to:

$$Po = \frac{\pi^4}{32 \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^4}\left(1 - \frac{2}{n\pi} th\left(\frac{n\pi}{2}\right)\right)} \tag{14}$$

Considering now, the well-known mathematical result coming from Euler-Riemann zeta function knowledge:

$$\sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^4} = \frac{\pi^4}{96} \tag{15}$$

We obtain:

$$Po = \frac{\pi^4}{\frac{\pi^4}{3} - \frac{64}{\pi} \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^5} th\left(n\frac{\pi}{2}\right)} \tag{16}$$

The series in equation (16) can easily be evaluated numerically giving:

$$\sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^5} th\left(n\frac{\pi}{2}\right) \cong 0.92167516 \dots \tag{17}$$

Finally, we obtain the Po value for a duct of square cross-section shape:

$$Po = \frac{\pi^4}{\frac{\pi^4}{3} - \frac{64}{\pi} 0.92167516} = 7.11353554 \dots \tag{18}$$

This purely theoretical result is in perfect agreement with experimental results obtained by many authors [2] and of course with the value reported in Shah & London [1] source book.

Let us now consider the other limit case described above i.e. the plane Poiseuille flow obtained for infinite parallel plates. As previously reported, this highly symmetric case gives a well-known value of Poiseuille number: $Po = 12$.

If we consider $b \gg a$ in equation (13), we obtain:

$$Po = \frac{\pi^4}{8 \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^4}} = \frac{\pi^4}{8 \frac{\pi^4}{96}} = 12 \tag{19}$$

Remarkably, this result is in perfect agreement with both experimental and theoretical results reported above. Moreover, it shows that Saint-Venant solution of Poisson PDE, established for elasticity theory [3], is of great importance for the study of laminar flow in rectangular ducts. Equation (13) established for aspect ratios b/a varying in the range 1 (square) to $+\infty$ (infinite parallel plates) is then of major interest for all these geometrical shapes.

We know from experiments that for these shapes, we have $7.1135 ... \leq Po \leq 12$ and this result is in perfect agreement with equation (13). We can then write the following theorem:

Theorem 1: For $b/a \in [1;+\infty[$ *we have, according to equation (12):* $7.1135 ... \leq Po \leq 12$.

This fundamental result clearly demonstrates that every value of Poiseuille number are possible between 7.1135… and 12. Particularly, it exists an aspect ratio b/a which gives $Po = 8$ like in pipes i.e. for a circular cross-section shape. Numerical calculations performed with equation (13) gave:

$$\frac{b}{a} = 2.26930413446618 ... \qquad (20)$$

This result signifies that, in fluid mechanics, it exists a rectangle having the same property than a circle for mechanical energy dissipation through viscous friction and this rectangular duct has an aspect ratio $b/a = 2.2693 ...$

3. Discussion on Mathematical Consequences

Of course, these results give an equivalence between rectangular and circular geometries in terms of energy dissipation and we can write the following theorem:

Theorem 2: Considering energy dissipation by viscous friction during the fully established laminar flow of a Newtonian fluid, the equivalent geometry for a pipe of circular cross-section is a rectangular duct having an aspect ratio $b/a = 2.2693 ....$

This result could be extended to others geometries like triangles. We know that for an equilateral triangle, $Po = 20/3$ and stretching of this triangle giving isosceles triangles increases Po values until it also reaches $Po = 12$ for an infinite triangle comparable to infinite parallel plates [2]. In that sense, there also exists a triangle for which $Po = 8$ meaning a triangle equivalent to a circle. We can then propose the following conjecture:

Conjecture 1: For any compact convex shape, there exists a non-regular geometry giving $Po = 8$ *and then giving an equivalence with circular geometry in terms of mechanical energy degradation by viscous friction.*

If this conjecture was true, signification of Poiseuille number values could be very important in Physics and Mathematics. Considering well-known membrane deformation problem giving Poisson's PDE, equation (6) is clearly its analogous for the laminar flow of a Newtonian liquid in a duct of arbitrary cross-section shape. The Saint-Venant solution given by equation (7) gives the velocity field shape which depends on the boundary conditions i.e. the shape of the duct cross-section perimeter. In the case of a pipe with circular cross-section, the high symmetry allows simple calculations and velocity field has a parabolic shape according to Poiseuille law. But for polygonal geometries like rectangles or triangles, shape is much more complicated. But, at the end, calculation of $\bar{\tau}_w$ and $\bar{v}$ allows a simple dimensionless equation of the same form (equation (12)) to be obtained and this equation involves Po numbers varying in the range 20/3 to 12.

Another consequence of the rectangular approach is the close relation between Po numbers and the Euler-Riemann zeta function. The problem of $\zeta(s)$ values for odd integer remains unsolved because at this time, we have no idea of a closed form for $s = 3, 5, 7, ...$ [4] Equation (13) gives interesting properties which could help approaching a closed form for $\zeta(5)$.

By considering well-known properties of hyperbolic tangent function $th(x)$, this function reaches very rapidly asymptotic value of 1 when x reach sufficiently large values (greater than 10). We can then consider that for sufficiently high values of ratio b/a in equation (13), the quantity $th(n\pi b/2a) = 1$ giving the following relationship for the sum over odd integers of $1/n^5$:

$$\sum_{n=1,3,5,...}^{+\infty} \frac{1}{n^5} = \frac{b}{192\,a}\,\pi^5 - \frac{b^3}{16\,a\,(a+b)^2\,Po}\,\pi^5 \qquad (21)$$

Of course, $b, a\ and\ Po$ values are linked together (for example you have $Po = 11$ for $b/a = 14.84241923166$) but equation (21) is surely an interesting result for understanding of $\zeta(5)$ behaviour even if the sum only concerns odd values of n.

Complex calculations in rectangular ducts also give others surprising and interesting results in numbers theory. For example, it is possible to calculate the wall shear stress along the side length a:

$$\tau_w(x) = \eta\left(\frac{\partial v_z}{\partial y}\right)_{y=b} = \frac{8\Delta Pa}{\pi^2 L}\sum_{n=1,3,5,...}^{+\infty}\frac{1}{n^2}(-1)^{n+1/2}th\left(\frac{n\pi b}{2a}\right)cos\left(\frac{n\pi x}{2a}\right) \qquad (22)$$

The maximum value of $\tau_w(x)$ is obtained for $x = 0$, giving:

$$\tau_w^{max} = \frac{8\Delta Pa}{\pi^2 L}\sum_{n=1,3,5,...}^{+\infty}\frac{1}{n^2}(-1)^{n+1/2}th\left(\frac{n\pi b}{2a}\right) \qquad (23)$$

If we consider the limit case of infinite parallel plates giving $b \gg a$, we obtain:

$$\tau_w^{max} = \frac{8\Delta Pa}{\pi^2 L}\sum_{n=1,3,5,...}^{+\infty}\frac{1}{n^2}(-1)^{n+1/2} \qquad (24)$$

The series can be written as followed:

$$\sum_{n=1,3,5,...}^{+\infty}\frac{1}{n^2}(-1)^{n+1/2} = -\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2} = G \qquad (25)$$

Where G is the well-known Catalan's constant. Until now, we ignore if this number is irrational even if it is conjectured. What we obtained from equation (24) gives interesting information about this number. Moreover, if we calculate the mean value of wall shear-stress along the same side, we obtain:

$$\bar{\tau}_w = \frac{1}{a}\int_0^a \tau_w(x)dx = \frac{16\Delta Pa}{\pi^3 L}\sum_{n=1,3,5,...}^{+\infty}\frac{1}{n^3}(-1)^n th\left(\frac{n\pi b}{2a}\right) \qquad (26)$$

Considering now $b \gg a$ gives:

$$\bar{\tau}_w = \frac{16\Delta Pa}{\pi^3 L} \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^3}(-1)^n = \frac{16\Delta Pa}{\pi^3 L}\frac{\pi^3}{32} = \frac{\Delta Pa}{2L} \quad (27)$$

Reporting this result in equation (24) gives:

$$\frac{\tau_w^{max}}{\bar{\tau}_w} = \frac{16\,G}{\pi^2} \cong 1.484907491\,\dots \quad (28)$$

This result clearly shows that Catalan's constant is proportional to π^2 which is an irrational number and then it could be considered as a proof of Catalan's constant irrationality.

We can then write the following theorem:

Theorem 3: Catalan's constant is proportional to π^2 and then is an irrational number.

Finally, it is also possible to calculate the ratio $v_{max}/\bar{v}$ of the maximum velocity at the centre of the rectangular duct and the average velocity. For maximum velocity, we obtain:

$$v_{max} = \frac{\Delta Pa^2}{2\eta L} - \frac{16\Delta Pa^2}{\pi^3 \eta L} \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^3}(-1)^{n-1/2} \frac{1}{ch\left(\frac{n\pi b}{2a}\right)} \quad (29)$$

For mean velocity, we obtain from equation (11) and Fubini theorem:

$$\bar{v} = \frac{32\Delta Pa^2}{\pi^4 \eta L} \sum_{n=1,3,5,\dots}^{+\infty} \frac{1}{n^4}\left(1 - \frac{2a}{n\pi b}\, th\left(\frac{n\pi b}{2a}\right)\right) \quad (30)$$

Giving,

$$\frac{v_{max}}{\bar{v}} = \frac{\pi^4}{64\sum_{n=1,3,5,\dots}^{+\infty}\frac{1}{n^4}\left(1-\frac{2a}{n\pi b}\, th\left(\frac{n\pi b}{2a}\right)\right)} - \frac{\pi\sum_{n=1,3,5,\dots}^{+\infty}\frac{1}{n^3}(-1)^{n-1/2}\frac{1}{ch\left(\frac{n\pi b}{2a}\right)}}{2\sum_{n=1,3,5,\dots}^{+\infty}\frac{1}{n^4}\left(1-\frac{2a}{n\pi b}\, th\left(\frac{n\pi b}{2a}\right)\right)} \quad (31)$$

It is then easy to consider the limit case of infinite parallel plates by taking $b \gg a$:

$$\frac{v_{max}}{\bar{v}} = \frac{\pi^4}{64\frac{\pi^4}{96}} = \frac{96}{64} = \frac{3}{2} \quad (32)$$

Remarkably, this last result is well-known in fluid mechanics for the case of infinite parallel plates. For pipes of circular cross-section, we have: $v_{max}/\bar{v} = 2$.

Equation (31) allows this ratio to be calculated whatever is the rectangle aspect ratio b/a.

4. Conclusions

In this paper, we deeply investigated the Poisson's PDE describing the fully established laminar flow of a Newtonian fluid in a duct of rectangular cross-section. We used the Saint-Venant solution, established for torsion of prismatical bars to obtain the velocity field whatever is the rectangle aspect ratio b/a.

From this equation, we showed how Poiseuille number values can be calculated giving a simple theorem for evolution of $7.1135\,\dots \leq Po \leq 12$ when $b/a \in [1, +\infty[$. This result allowed the rectangle, giving the same value than the circular cross-section, to be defined with an aspect ratio $b/a = 2.2693\,\dots$

We tried to give some mathematical consequences of this approach. Among them, we conjecture that for any convex shape of non-circular cross-section, there always exists one having a Poiseuille number value equal to the circle value i.e. $Po = 8$.

We also showed, from Poiseuille number equation for rectangular ducts, that Euler-Riemann zeta function $\zeta(s)$ for odd integer $s = 5$, for summation over odd integers $n = 1,3,5,\dots$ can be calculated as proportional to π^5.

From calculation of both, maximum wall shear stress, and average wall shear stress, we showed that famous Catalan's constant G is proportional to π^2 which could be a proof of its irrationality.

Finally, by integrating the velocity field, we found an expression for the ratio $v_{max}/\bar{v}$ which gave, for the limit case of plane Poiseuille flow, a value of $3/2$ in perfect agreement with fluid mechanics results.

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