Problem Sheet II

Introduction to Random Matrices, IST Austria
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LÁSZLÓ ERDŐS, DOMINIK SCHRÖDER

1 Bound on largest eigenvalue

The goal of this exercise is to obtain a direct proof of the fact that the operator norm of random matrices with independent entries of size $1/\sqrt{N}$ is bounded. Because the argument is based on simple concentration estimates it is only applicable to the case of uniformly sub-gaussian matrix entries.

Definition. A centered random variable X is called subgaussian if there exists $a > 0$ such that for all $t \in \mathbb{R}$ it holds that $\mathbf{E}\, e^{tX} \le e^{at^2/2}$. A family of random variables is called uniformly subgaussian if the same constant a can be chosen for all random variables.

Problem 1. *Prove that the following three assertions about a centered random variable X are equivalent:*

(i) *X is subgaussian,*

(ii) *there exists $b > 0$ such that for all $\lambda > 0$, $\mathbf{P}(|X| \ge \lambda) \le 2e^{-b\lambda^2}$,*

(iii) *there exits $c > 0$ such that $\mathbf{E}\, e^{cX^2} \le 2$.*

Let H be a real $N \times N$ random matrix with independent entries $(h_{ij})_{i,j\in[N]}$ of zero mean such that the normalized random variables $\sqrt{N} h_{ij}$ are uniformly subgaussian. The goal of this exercise is to prove that there exists a constant $c > 0$ such that

$$\mathbf{P}(\|H\| > C) \le e^{-cNC^2} \tag{1}$$

for all C large enough.

Problem 2. *Let x be a fixed vector of (Euclidean) length $\|x\| = 1$. Show that there exists a constant $c > 0$ such that*

$$\mathbf{P}(\|Hx\| > C) \le e^{-cNC^2} \tag{2}$$

for all C large enough.

Problem 3. *Let $P \subset S := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$ be a maximal $1/2$-separated set, i.e., a set for which for any $x \ne y \in P$ we have that $\|x - y\| \ge 1/2$ and for each $z \in S \setminus P$ there exist $x \in P$ such that $\|x - z\| < 1/2$. Prove that*

$$\mathbf{P}\left(\|H\| > C\right) \le \mathbf{P}\left(\|Hx\| > C/2 \text{ for some } x \in P\right). \tag{3}$$

Problem 4. *Combine the statements of problems 2 and 3 to prove* (1).

2 Interlacing eigenvalues

Let A be a Hermitian $N \times N$ matrix and let B be the $(N-1) \times (N-1)$ principal submatrix, i.e.,

$$A = \begin{pmatrix} B & a \\ a^* & b \end{pmatrix}$$

for some vector $a \in \mathbb{C}^{N-1}$ and scalar $b \in \mathbb{R}$ and suppose that A and B have disjoint, simple spectra. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ and $\mu_1 < \cdots < \mu_{N-1}$ denote the ordered eigenvalues of A and B. The goal of this exercise is to prove that the eigenvalues of B interlace the eigenvalues of A, i.e., that

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{N-1} < \mu_{N-1} < \lambda_N. \tag{4}$$

We begin with an equivalent characterisation of interlacing.

Problem 5. *Let f, g be polynomials with distinct simple real roots and leading positive coefficients of degree N and $N-1$. Prove that the roots interlace if and only if for each $\lambda \in [0,1]$ the polynomial $\lambda f + (1-\lambda) g$ has only real roots.*

It turns out that Problem 5 provides a useful characterisation of interlacing for proving (4).

Problem 6. *Find a Hermitian matrix $C = C(\lambda)$ such that*

$$\det(x - C) = \det(x - A) + \frac{1-\lambda}{\lambda} \det(x - B)$$

and conclude from Problem 5 that the eigenvalues of B interlace those of A, i.e., (4)*.*

3 Resolvent identities

The goal of this exercise is to prove two identities relating resolvent elements to those of the resolvent of certain minors. Let H be a Hermitian $N \times N$ matrix. For $i \in [N]$ let $H^{(i)}$ denote the matrix with the i-th row and column set to zero, i.e., $H^{(i)}_{kl} = H_{kl} \mathbf{1}_{i \neq k} \mathbf{1}_{i \neq l}$. The entries of the resolvent $G^{(i)}(z) := (H^{(i)} - z)^{-1}$ of such a minor satisfy the *first resolvent decoupling identity* for $i, j \neq k$

$$G_{ij} = G^{(k)}_{ij} + \frac{G_{ik} G_{kj}}{G_{kk}} \tag{5a}$$

as well as the *second resolvent decoupling identity* for $i \neq j$

$$G_{ij} = -G_{ii} \sum_{k \neq i} h_{ik} G^{(i)}_{kj} = -G_{jj} \sum_{k \neq j} G^{(j)}_{ik} h_{kj}. \tag{5b}$$

Before proving (5a)–(5b) we recall the standard resolvent expansion formula.

Problem 7. *Show that for matrices A, B it holds that*

$$(A - B)^{-1} = A^{-1} + A^{-1} B (A - B)^{-1} = A^{-1} + (A - B)^{-1} B A^{-1}, \tag{6}$$

provided that all inverses exist.

Problem 8. *Use* (6) *to first prove* (5b) *and then use* (6) *and* (5b) *to prove* (5a)*.*