Problem Sheet III

Introduction to Random Matrices, IST Austria
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1 Limiting density for XX^t random matrices

Let $X = (x_{ij})_{i\in[M],j\in[N]}$ be a (non-symmetric) random matrices with $M = M(N) \le N$, and independent, identically distributed entries

$$x_{ij} \in \mathbb{R}, \qquad \mathbf{E}\, x_{ij} = 0, \qquad \mathbf{E}\, x_{ij}^2 = 1, \qquad |x_{ij}| \le C.$$

The symmetric random matrix $H = N^{-1}XX^t \in \mathbb{R}^{M\times M}$ then has real eigenvalues $\lambda_1 \le \cdots \le \lambda_M$. The goal of this exercise is to show that in the asymptotic scaling $M/N = \lambda + \mathcal{O}\left(N^{-1}\right)$ with $\lambda \in (0,1]$ the empirical spectral density $\mu_N = M^{-1}\sum_i \delta_{\lambda_i}$ of H converges weakly in probability to the absolutely continues measure $\mathrm{d}\mu = \rho\,\mathrm{d}x$ with density

$$\rho(x) = \frac{1}{2\lambda\pi x}\sqrt{(b-x)_+(x-a)_+}, \qquad a = \left(1-\sqrt{\lambda}\right)^2, \quad b = \left(1+\sqrt{\lambda}\right)^2. \tag{1}$$

Problem 1. *We define the empirical moments to be*

$$m_{k,N} := \int x^k\,\mathrm{d}\mu_N(x) = \frac{1}{MN^k}\sum_{\substack{i_1,\dots,i_k\in[M]\\ j_1,\dots,j_k\in[N]}} x_{i_1j_1}x_{i_2j_1}x_{i_2j_2}x_{i_3j_2}\dots x_{i_kj_k}x_{i_1j_k}.$$

(i) Use similar arguments to those in the moment computation for the Wigner semicircle law to show that

$$\mathbf{E}\, m_{k,N} = m_k + \mathcal{O}\left(N^{-1}\right), \qquad m_k = \sum_{2k-\textit{Dyck paths}} \lambda^{u_{even}},$$

where u_{even} counts the number of upstrokes in even steps of the Dyck path.

(ii) Define the auxiliary quantity

$$m_k' := \sum_{2k-\textit{Dyck paths}} \lambda^{u_{odd}}.$$

and prove the recursion

$$m_k = \sum_{j=1}^k m_{j-1}'m_{k-j}, \qquad m_k' = \lambda\sum_{j=1}^k m_{j-1}m_{k-j}',\ k\ge 1, \qquad m_0 = m_0' = 1$$

to conclude that the generating function $f(x) = \sum_{k\ge 0} m_k x^k$ of m_k satisfies the equation

$$f(x) = 1 + (1-\lambda)xf(x) + \lambda x f(x)^2.$$

(iii) Relate the Stieltjes transform of μ to f to conclude that the density of μ is indeed given by (1).

(iv) Show that $\mathbf{Var}\, m_{k,N} = \mathcal{O}\left(N^{-2}\right)$ *and conclude that $m_{k,N}$ converges almost surely to m_k for each k.* ***Hint.*** *The Borel-Cantelli Lemma might be helpful.*

(v) Conclude that almost surely μ_N converges weakly to μ. ***Hint.*** *You may assume that μ is uniquely determined by its moments and therefore convergence of moments implies weak convergence.*

□

Problem 2. *Use Problem 1 to show that for square X as in Problem 1, the singular value distribution of $N^{-1/2}X$ converges weakly to a quarter-circle distribution with density*

$$\frac{\sqrt{(4-s^2)_+}}{\pi}\mathbf{1}_{s\geq 0}\,\mathrm{d}s.$$

□

2 Operator norm of Wigner matrices

In the lecture it was proved that the empirical spectral density $\mu_N = N^{-1}\sum_i \delta_{\lambda_i}$ concentrated in the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ of a Wigner matrix H converges weakly in probability to a semi-circular distribution $\mathrm{d}\mu_{sc} = \rho_{sc}(x)\,\mathrm{d}x$ with density

$$\rho_{sc}(x) = \frac{\sqrt{(4-x^2)_+}}{2\pi}.$$

This might suggest that λ_N converges to 2 in probability. This is indeed the case but requires an extra argument. The lower bound, however, follows directly from the weak convergence in probability:

Problem 3. *Prove that for each $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbf{P}(\lambda_N < 2-\epsilon) = 0. \tag{2}$$

□

For the upper bound we impose additional assumptions on the growth of moments. Specifically, we will assume that

$$\sup_{ij} \mathbf{E}\left|\sqrt{N}h_{ij}\right|^k \leq \mu_k \leq k^{Ck} \tag{3}$$

for some constant C.

Problem 4. *Prove that for each $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbf{P}(\lambda_N > 2+\epsilon) = 0. \tag{4}$$

Hint. *In the lecture it was proved that for each fixed k, the empirical moments $m_{k,N} = N^{-1}\,\mathbf{E}\,\mathrm{Tr}\,H^k$ converge to $C_{k/2}$ for even k and 0 for odd k. This was achieved by counting the so called back-tracking graphs which form the leading term in the expansion*

$$N^{-1}\,\mathbf{E}\,\mathrm{Tr}\,H^k = N^{-1}\sum_{i_1,\ldots,i_k\in[n]} \mathbf{E}\, h_{i_1 i_2}\ldots h_{i_k i_1}. \tag{5}$$

Using a more careful counting of the remaining graphs (i.e., all graphs in which each edge occurs at least twice which are not backtracking trees) this can be improved to also cover slowly growing k. One can show[1] that the number $N_{k,j}$ of cycles $i_1,\ldots,i_k,i_1$ which visit j unique vertices and pass along each edge at least twice is bounded by

$$N_{k,j} \leq 2^k k^{3(k-2j+2)} N^j, \quad j < \frac{k}{2}+1, \qquad N_{k,\lfloor k/2+1\rfloor} \leq C_{k/2} N^{\lfloor k/2+1\rfloor} \leq 2^k N^{\lfloor k/2+1\rfloor}. \tag{6}$$

Use the combinatorial bound (6) in (5) to prove that $N^{-1}\,\mathbf{E}\,\mathrm{Tr}\,H^k \leq C' 2^k$ for $k \sim \log^2 N$.

□

[1] For those interested this combinatorial argument with a slightly less precise bound is spelled out in the proof of Theorem 12 of `https://terrytao.wordpress.com/2010/01/09/254a-notes-3-the-operator-norm-of-a-random-matrix`