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Phase Transition in the Density of States of Quantum Spin Glasses

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Declaration of Authorship

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Abstract

We prove that the empirical density of states of quantum spin glasses on arbitrary graphs converges to a normal distribution as long as the maximal degree is negligible compared with the total number of edges. This extends the recent results of [10] that were proved for graphs with bounded chromatic number and with symmetric coupling distribution. Furthermore, we generalise the result to arbitrary hypergraphs. We test the optimality of our condition on the maximal degree for p-uniform hypergraphs that correspond to p-spin glass Hamiltonians acting on n indistinguishable spin-$1/2$ particles. At the critical threshold $p = n^{1/2}$ we find a sharp classical-quantum phase transition between the normal distribution and the Wigner semicircle law. The former is characteristic to classical systems with commuting variables, while the latter is a signature of noncommutative random matrix theory. The main results of this thesis are also summarised in [5].

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Chapter 1

Introduction

1.1 Universal Probability Distributions

In Probability Theory the classical central limit theorem can be understood as a statement about the universality of the normal distribution.

Theorem 1.1 (Classical Central Limit Theorem). *Let $(\alpha_i)_{i\in\mathbb{N}}$ be a uniformly bounded sequence of independent random variables (i.e., $|\alpha_i|\leq M < \infty$ for all i) with zero mean $\mathbf{E}\,\alpha_i = 0$ and unit variance $\mathbf{E}\,\alpha_i^2 = 1$. Then the sequence of random variables*

$$\left(\frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{n}}\right)_{n\in\mathbb{N}}$$

converges in distribution to a standard normal distribution.

This is a statement about universality since it is independent of the actual distributions of the random variables α_i.

Remark 1.2. *Actually, the condition on the uniform boundedness is unnecessary. It can be shown that the distributional convergence of the normalized sum $n^{-1/2}(\alpha_1 + \cdots + \alpha_n)$ of independent random variables with zero mean and unit variance to a standard normal condition is equivalent to the much weaker Lindeberg condition, namely*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{E}[\alpha_k^2 1(|\alpha_k|> n\epsilon)] = 0$$

for all $\epsilon > 0$ (see, for example [3]).

In the theory of random matrices a different distribution of similar universality emerges.

Theorem 1.3 (Wigner Semicircle Law). *Let*

$$W_n = \frac{1}{\sqrt{n}}\begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{pmatrix}$$

be the Hermitian matrix consisting of possibly complex valued random variables $\alpha_{i,j}$ *of zero mean* $\mathbf{E}\,\alpha_{i,j} = 0$ *and unit variance* $\mathbf{E}|\alpha_{i,j}|^2 = 1$ *which are uniformly bounded (i.e.* $|\alpha_{i,j}| \le M < \infty$ *for all* i, j*), independent for* $i \ge j$ *and satisfy* $\alpha_{i,j} = \overline{\alpha_{j,i}}$ *for all* i, j*. Then the empirical spectral distribution*

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}$$

(where the λ_i *are the eigenvalues of* W_n*, counted with multiplicity) converges almost surely weakly to the semicircular distribution* μ_{sc} *with density function*

$$\rho_{sc}(x) := \frac{\sqrt{(4-x^2)_+}}{2\pi}.$$

More precisely, almost surely the sequence $\left(\int_{\mathbb{R}} f \,\mathrm{d}\mu_n\right)_{n\in\mathbb{N}}$ *of scalar random variables converges to the constant* $\int_{\mathbb{R}} f \,\mathrm{d}\mu_{sc}$ *for any bounded continuous function* $f \in C_b(\mathbb{R})$*.*

The matrices W_n are in some sense the most random matrices under the condition of being Hermitian.

Remark 1.4. *Similarly to the Central Limit Theorem, the version of the Wigner Semicircle Law given above assuming the uniform boundedness of the matrix entries is not the most general one. In fact, the conclusion of the Theorem holds if and only if*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbf{E}[|\alpha_{i,j}|^2 1(|\alpha_{i,j}| \ge \epsilon n)] = 0$$

for all $\epsilon > 0$ *(see, for example [8]).*

Important special cases are the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE). In the latter all entries are $N(0,1)$-normally distributed, in the former the off-diagonal terms are $N(0,1)_{\mathbb{C}}$-complex-normally distributed while the diagonal terms are still $N(0,1)$-normally distributed. It is an important question in the theory of random matrices how much structure and dependence can be added to the random matrix W_n while still having the universal semicircular limiting density of states. In this thesis we give a partial answer to this question for a particular class of physically motivated random matrices.

1.2 Spin Models

A standard model in the study of classical spin glasses is the Edwards–Anderson model (EA; see, for example, [4])

$$H_{\text{EA}}(s) = N^{-d/2} \sum_{\substack{i,j\in\{-N,\dots,N\}^d\subset\mathbb{Z}^d \\ |i-j|=1}} \alpha_{i,j} s_i s_j, \quad s_i \in \{-1, 1\}$$

defined on a finite d-dimensional discrete box of length $2N$. It models next-neighbour interactions with independent random couplings $\alpha_{i,j}$, say with mean zero and unit variance. In contrast, the mean field Sherrington-Kirkpatrick model (SK; see, for example, [11])

$$H_{\text{SK}}(s) = N^{-d} \sum_{i\neq j\in\{-N,\dots,N\}^d} \alpha_{i,j} s_i s_j, \quad s_i \in \{-1, 1\},$$

also with independent random couplings $\alpha_{i,j}$ of zero mean and unit variance, models infinite range interactions. The classical Central Limit Theorem can be used to show that the distribution of energy levels follows a normal distribution in the thermodynamic limit $N \to \infty$ in both cases.

In contrast, the Hamiltonian operators of the corresponding quantum models of n, say spin-$1/2$, particles, i.e.

$$H_n = \frac{1}{\sqrt{9n}} \sum_{i=1}^{n} \sum_{a,b=1}^{3} \alpha_{i,a,b} \sigma_i^{(a)} \sigma_{i+1}^{(b)}, \quad \sigma_i^{(a)} := 1_2^{\otimes(i-1)} \otimes \sigma^{(a)} \otimes 1_2^{\otimes(n-i)}, \quad \sigma_{n+1}^{(a)} := \sigma_1^{(a)}$$

in terms of Pauli matrices

$$1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are essentially $2^n \times 2^n$ dimensional hermitian random matrices and therefore a semicircular spectral density might be expected. It turns out that, despite the inherent noncommutativity, the density of states (meaning the eigenvalue distribution) for a large class of quantum spin glasses still follows the normal law. For the one dimensional spin chain and slightly more general interaction geometries with symmetrically distributed coupling constants this has recently been shown by Keating, Linden and Wells in [10]. More precisely, they showed that the spectral density of Hamiltonians

$$H_n^{(\Gamma_n)} := \frac{1}{\sqrt{9e(\Gamma_n)}} \sum_{(ij)\in\Gamma_n} \sum_{a,b=1}^{3} \alpha_{(ij),a,b} \sigma_i^{(a)} \sigma_j^{(b)}$$

whose interaction geometry is given by a sequence of graphs Γ_n on the vertex sets $[n]$ approaches a normal distribution if the chromatic number of the graphs in bounded uniformly in n.

The first part of this thesis is devoted to a generalisation of their result in several directions by considering more general graphs and also hypergraphs allowing not only two-spin interactions. Moreover, we remove the symmetry condition on the distribution of the coupling constants. We find that the central limit theorem holds for a quantum spin glass on an arbitrary hypergraph, provided that the maximal number of edges connected to any fixed edge is much smaller than the total number of edges. This condition guarantees that the noncommutative effects, related to edges sharing a common vertex, are subleading: most degrees of freedom are still commutative. Thus the system is essential classical as far as the density of states is concerned. We also present an example (the star graph where all edges connect to some fixed vertex) where the degree of a distinguished vertex is comparable with the total number of edges. The density of states is explicitly computable and it is neither Gaussian nor the semicircle law.

In the second part of this thesis we observe a transition from the classical regime dominated by commuting variables to the quantum regime where noncommutativity determines the leading behaviour. This transition is particularly transparent for the quantum p_n-spin model, i.e. a quantum spin glass on a p_n-uniform hypergraph (in this mean field model any collection of p_n spins interact). The case $p_n = 2$ corresponds to the quantum version of the standard Sherrington-Kirkpatrick model and its density of states follows the normal law. The other extreme case, $p_n = n$, is closely related to the GUE model with the semicircle law. We prove a sharp phase transition at $p_n \sim \sqrt{n}$; for $p_n \ll \sqrt{n}$ we have the normal law, while for $p_n \gg \sqrt{n}$ we get the semicircle law. For $p = \lambda\sqrt{n}$ with a fixed $\lambda \in (0, \infty)$, we establish a new family of densities of states, parametrised by λ, that naturally interpolates between the normal distribution and the semicircle law. We emphasise that the regime $p \sim n^\alpha$, $\alpha \in (1/2, 1)$, is still far from the mean field regime in the sense of random matrices: we have only $3^p\binom{n}{p} \ll 2^n$ independent random variables parametrising an operator acting an $N = 2^n$ dimensional Hilbert space. In contrast, Wigner random matrices of dimension $N \times N$ have $N(N+1)/2$ independent degrees of freedom. The p-spin model thus corresponds to a sparse random matrix with a lot of structure, but it still follows the Wigner semicircle law if $p_n \gg \sqrt{n}$. Our result gives a rigorous proof of the transition between the Gaussian and the semicircle density of states that has been numerically observed in [7] for k-body interactions as k approaches the total number of particles.

We mention that this phase transition is apparently present only for the density of states; as far as the local eigenvalue statistics is concerned all these models seem to belong to the random matrix universality class. The numerical tests presented in [10] deal with the one-dimensional quantum chain, one of the sparsest model, and still demonstrate a very strong agreement with the GOE (for odd n) and GSE (Gaussian Symplectic Ensemble, for even n) gap distribution. Certainly the same is expected for spin glasses on denser graphs. Quantum spin glasses are one of the simplest interacting many-body disordered quantum models. Therefore, this remarkable feature is yet another manifestation of Wigner's vision on the ubiquity of the random matrix gap statistics for essentially any disordered quantum system. For more details on the physical motivation and related works we refer to [10].

Chapter 2

Preliminaries

2.1 Moment Method

The following section shall be devoted to a justification of the so called moment method. We will also demonstrate the usefulness of the method by utilizing it to give proofs of the Central Limit Theorem and the Wigner Semicircle Law. While there are multiple different ways to prove these theorems, we still present the moment based proofs here to illustrate the general method before following a similar approach in the study of spin models later.

A sequence of random variables X_n on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is said to converge *in distribution* to a random variable X if the corresponding probability measures $\mu_{X_n}(A) := \mathbf{P}(X_n^{-1}(A)), A \in \mathcal{B}(\mathbb{R})$ converge *weakly* to the measure μ_X, i.e. if for all continuous and bounded $f\colon \mathbb{R} \to \mathbb{R}$ it holds that $\lim_{n\to\infty} \int_{\mathbb{R}} f \, \mathrm{d}\mu_{X_n} = \int_{\mathbb{R}} f \, \mathrm{d}\mu_X$. It is well known that the pointwise convergence of the corresponding distribution functions $F_{X_n}(t) := \mathbf{P}(X_n \leq t) = \mu_{X_n}((-\infty, t]), t \in \mathbb{R}$ to the distribution function F_X in all points of continuity of F_X is equivalent to the weak convergence of the measures μ_{X_n}.

It often suffices to check the condition that $\lim_{n\to\infty} \int f \, \mathrm{d}\mu_{X_n} = \int f \, \mathrm{d}\mu_X$ for certain families of test functions f. Since in our case of measures corresponding to random variables, we have that $\lim_{n\to\infty} \mu_{X_n}(\mathbb{R}) = 1 = \mu_X(\mathbb{R})$, it is sufficient to check this condition for all continuous functions with compact support. Similarly Levy's continuity theorem states that the family of test functions $\{ x \mapsto e^{itx} \}_{t\in\mathbb{R}}$ is sufficient and the pointwise convergence of the characteristic functions $\mathbf{E}\, e^{itX_n}$ to the characteristic function $\mathbf{E}\, e^{itX}$ already implies convergence in distribution. Under some additional assumptions it is even sufficient to consider polynomial test functions:

Theorem 2.1. *Let μ and $\mu_n, n \in \mathbb{N}$ be probability distributions on $\mathbb{R}$ with all finite moments $\int x^k \, \mathrm{d}\mu_n < \infty$ such that for all $k \in \mathbb{N}$ it holds that $\lim_{n\to\infty} \int x^k \, \mathrm{d}\mu_n = \int x^k \, \mathrm{d}\mu$. Then the sequence of distributions $(\mu_n)_{n\in\mathbb{N}}$ converges in weakly to μ if μ is uniquely determined by its moments.*

Proof (as in [3]). First note that since $\left(\int x^2 \, \mathrm{d}\mu_n\right)_{n\in\mathbb{N}}$ is a convergent sequence of finite numbers, we have that $C := \sup_{n\in\mathbb{N}} \int x^2 \, \mathrm{d}\mu_n < \infty$ is finite. Thus the family $\{\mu_n\}_{n\in\mathbb{N}}$ is tight since we can choose $N > 0$ such that $1 - F_n(N) + F_n(-N) \leq C/N^2$ is arbitrarily small.

Now let $\nu = \text{w-lim}_{k\to\infty} \mu_{n_k}$ be any subsequential weak limit (which is guaranteed to exist by Banach-Alaoglu) of mass 1, i.e. an probability measure. It is clear that for all $m \in \mathbb{N}$, the function $x \mapsto x^m$ is uniformly μ_{n_k}-integrable (for odd m just take any even integer exceeding it) and therefore it is also ν-integrable and $\int x^m \,\mathrm{d}\nu = \lim_{k\to\infty} \int x^m \,\mathrm{d}\mu_{n_k} = \int x^m \,\mathrm{d}\mu$. By the uniqueness assumption we find that $\mu = \nu$ and therefore $\mu = \text{w-lim}_{k\to\infty} \mu_{n_k}$ proving that any weakly convergent subsequence converges weakly to the same limit μ.

Now suppose that μ_n does not converge weakly to μ. Then there exists a point of continuity of continuity of F_X, say $x \in \mathbb{R}$, such that $F_n(x)$ does not converge to $F(x)$. This implies that there exists $\epsilon > 0$ and some infinite sequence $(n_k)_{k\in\mathbb{N}}$ s.t. $F_{n_k}(x)$ is not even ϵ-close to $F(x)$. By Banach-Alaoglu, there has to exist a further subsequence $\mu_{n_{k_l}}$ of $(\mu_{n_k})_{k\in\mathbb{N}}$ weakly convergent to some measure ν. Given $\epsilon > 0$, by tightness we can choose N s.t. $\mu_n(-N, N) \geq 1-\epsilon$ for all n and $-N, N$ are continuity points of F. Thus also $\nu(-N, N) \geq 1-\epsilon$ proving that ν is again a probability measure distinct from μ. This clearly contradicts the second paragraph of the proof. $\square$

It remains to find a sufficient condition on probability distributions to be uniquely determined by their moments.

Theorem 2.2 (Perron's continuity condition). *A probability distribution μ is uniquely determined by its finite moments $m_k := \int x^k \,\mathrm{d}\mu$ if* $\limsup_{n\to\infty} |\frac{m_{2n}}{(2n)!}|^{1/2n} < \infty$.

Remark 2.3. *A far more general sufficient condition is due to Carleman (see, for example [1]): A probability distribution μ is uniquely determined by its finite moments $m_k := \int x^k \,\mathrm{d}\mu < \infty$ if*

$$\sum_{n\in\mathbb{N}} m_{2n}^{-1/2n} = \infty.$$

Proof of Theorem 2.2. By the Cauchy-Schwarz inequality we have

$$\int |x|^{2k+1} \mathrm{d}\mu \leq \sqrt{m_{2k} m_{2k+2}}$$

and therefore the growth condition

$$\limsup_{n\to\infty} \left| \frac{1}{n!} \int |x|^n \mathrm{d}\mu \right|^{1/n} =: \frac{1}{\epsilon} < \infty$$

for all $n \in \mathbb{N}$ follows automatically from the one for even n. Therefore the power series

$$\int e^{i(t+s)x} d\mu = \sum_{n=0}^{\infty} \frac{(is)^n}{n!} \int e^{itx} x^n \,\mathrm{d}\mu$$

of the characteristic function in t has a positive radius of convergence of $\epsilon > 0$ uniform in t and therefore is analytic. The characteristic function is therefore uniquely determined by its expansion in $t \in (-\epsilon, \epsilon)$, i.e. $\sum_{n=0}^{\infty} \frac{m_n (it)^n}{n!}$ which in turn is uniquely determined by the moments. Therefore the result follows from Levy's continuity theorem. $\square$

Example 2.4. *Integration by parts shows that the standard normal distribution has moments $m_n = 0$ for odd n and $m_n = (n-1)!! := (n-1)(n-3)\cdots 3 \cdot 1$ for even n. Therefore it fulfils the condition from Theorem 2.2 and is uniquely determined by its moments.*

Theorems 2.1 and 2.2 justify the so called moment method: The convergence of the moments $\int x^k \, \mathrm{d}\mu_n$ to moments $\int x^k \, \mathrm{d}\mu$ not growing too fast implies weak convergence of the measures μ_n to the measure μ. It can be utilized to give a quick proof of the Central Limit Theorem:

Proof of Theorem 1.1. The k-th expected moment is given by

$$m_{k,n} := \left(\frac{\alpha_1 + \cdots + \alpha_n}{\sqrt{n}} \right)^k = n^{-k/2} \sum_{i_1,\ldots,i_k=1}^{n} \mathbf{E}\, \alpha_{i_1} \ldots \alpha_{i_k}.$$

By independence and zero mean hypothesis $\mathbf{E}\, \alpha_{i_1} \ldots \alpha_{i_k} = 0$ if some i_l only appears once. Therefore only terms with at most $\lfloor k/2 \rfloor$ distinct i_l contribute to the sum. On the other hand there are only $\mathcal{O}(n^m)$ (as $n \to \infty$) terms with m distinct i_l, each being bounded by M^k, which don't contribute because of the normalizing factor. Thus the only relevant terms are those where the $(i_1, \ldots, i_l)$ appear in pairs of twos. After defining

$$P_2(I^k) := \left\{ (i_1, \ldots, i_k) \in I^k \;\middle|\; \text{for all } l \in [k] \text{ exists exactly one } m \in [k] \setminus \{l\} \text{ with } i_l = i_m \right\}$$

the k-th moment then reads

$$\overline{m_{k,n}} = n^{-k/2} \sum_{(i_1,\ldots,i_k) \in P_2([n]^k)} \mathbf{E}\, \alpha_{i_1} \ldots \alpha_{i_k} + \mathcal{O}\left(n^{-1/2}\right) = n^{-k/2} |P_2([n]^k)| + \mathcal{O}\left(n^{-1/2}\right)$$

by the independence and unit variance hypotheses. For counting the number of elements of $P_2([n]^k)$ (which is obviously empty for odd k) note that there are $\binom{n}{k/2}$ ways of choosing $k/2$ distinct $i_l \in [n]$ which can be assigned in $\binom{k}{2}\binom{k-2}{2} \ldots \binom{2}{2}$ different ways to the tuples. Therefore we finally arrive at

$$\begin{aligned} \overline{m_{k,n}} &= \frac{n(n-1)\ldots(n-k/2+1)}{n^{k/2}} \frac{k!}{(k/2)! 2^{k/2}} + \mathcal{O}\left(n^{-1/2}\right) = \frac{k!}{(k/2)! 2^{k/2}} + \mathcal{O}\left(n^{-1/2}\right) \\ &= (k-1)!! + \mathcal{O}\left(n^{-1/2}\right) \end{aligned}$$

for even k and $\overline{m_{k,n}} = \mathcal{O}\left(n^{-1/2}\right)$ for odd k agreeing with the moments of the normal distribution. $\square$

In the case of the Wigner semicircle law, the empirical spectral distributions and therefore also their moments are in themselves random. The following proposition establishes a connection between the types of convergence of moments and the corresponding types of weak convergence of distributions.

Proposition 2.5. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of random measures with moments $m_{k,n} := \int x^k \, \mathrm{d}\mu_n < \infty$ and let μ be a deterministic measure with moments $m_k := \int x^k \, \mathrm{d}\mu < \infty$ which uniquely determine μ.*

(i) If the expected moments $\mathbf{E}\, m_{k,n}$ converge to m_k for all $k \in \mathbb{N}$, then $\mathbf{E}\, \mu_n$ (defined by duality in the sense that $\int f \, \mathrm{d}\mathbf{E}\, \mu_n := \mathbf{E} \int f \, \mathrm{d}\mu_n$) converges weakly to μ.

(ii) If the moments $m_{k,n}$ converge almost surely to m_k for all $k \in \mathbb{N}$, then also μ_n converges almost surely weakly to μ. More explicitly, the event that $\int f \,\mathrm{d}\mu_n$ converges to $\int f \,\mathrm{d}\mu$ for all bounded continuous f has probability one. In particular, if $\mathbf{E}\, m_{k,n}$ converges to m_k for all k and the series $\sum_{n\in\mathbb{N}} \mathbf{Var}\, m_{k,n}$ is finite for all k, then μ_n converges almost surely to μ.

(iii) If $\mathbf{E}\, m_{k,n}$ converges to m_k and $\mathbf{Var}\, m_{k,n}$ converges to zero for all $k \in \mathbb{N}$, then μ_n converges weakly in probability to μ. More explicitly, for all bounded continuous f the sequence $\int f \,\mathrm{d}\mu_n$ converges to $\int f \,\mathrm{d}\mu$ in probability.

Proof.

(i) This was already discussed in Theorem 2.1.

(ii) By assumption for all $k \in \mathbb{N}$ there exists a set Ω_k of measure 1 such that $m_{k,n}(\omega)$ converges to m_k as $n \to \infty$ for all $\omega \in \Omega_k$. Since the intersection of countably many sets of measure 1 still has measure 1, we find a set $\Omega = \bigcap_{n\in\mathbb{N}} \Omega_n$ of full measure such that $\lim_{n\to\infty} m_{k,n}(\omega) = m_k$ for all $k \in \mathbb{N}$ and $\omega \in \Omega$. An application of Theorem 2.1 then shows that $\mu_n(\omega)$ converges weakly to μ for all $\omega \in \Omega$ as $n \to \infty$.

If we assume the summability of the variances, Chebyshev's inequality implies that

$$\mathbf{P}\left(|m_{k,n} - \mathbf{E}\, m_{k,n}| > \epsilon\right) \leq \frac{\mathbf{Var}\, m_{k,n}}{\epsilon^2}.$$

and therefore by the Borel–Cantelli Lemma

$$\mathbf{P}\left(|m_{k,n} - \mathbf{E}\, m_{k,n}| > \epsilon \text{ infinitely often}\right) = 0$$

for all $\epsilon > 0$. Taking the union over all $\epsilon = \frac{1}{N}$, $N \in \mathbb{N}$ we then find

$$\begin{aligned} &\mathbf{P}(m_{k,n} \text{ does not converge to } m_k) \\ &\qquad = \mathbf{P}\left(|m_{k,n} - \mathbf{E}\, m_{k,n}| > \frac{1}{N} \text{ infinitely often for some } N \in \mathbb{N}\right) = 0 \end{aligned}$$

and the second claim follows.

(iii) Fix a bounded and continuous f. It clearly suffices to show (in fact, this is equivalent to the claim) that for any subsequence $(n_m)_{m\in\mathbb{N}}$ there exists a further subsequence $(n_{m_l})_{l\in\mathbb{N}}$ such that $\int f \,\mathrm{d}\mu_{n_{m_l}}$ converges to $\int f \,\mathrm{d}\mu$ almost surely as $l \to \infty$.

Now fix any subsequence $(n_m)_{m\in\mathbb{N}}$. By assumption $\mathbf{Var}\, m_{1,n_m}$ converges to zero as $m \to \infty$. Thus it is possible to choose a subsequence $(n_{m_{1,l}})_{l\in\mathbb{N}}$ such that $\mathbf{Var}\, m_{1,n_{m_l}} \leq \frac{1}{l^2}$ which implies (as in (ii)) that $m_{1,n_{m_{1,l}}}$ converges almost surely to m_1. In the same manner we now can find a further subsequence $(n_{m_{2,l}})_{l\in\mathbb{N}}$ for which also $m_{2,n_{m_{2,l}}}$ converges almost surely to m_2. Continuing inductively and by a diagonal argument we end up with a subsequence $(n_{m_l})_{l\in\mathbb{N}}$ such that $m_{k,n_{m_l}}$ converges to m_k for all $k \in \mathbb{N}$ almost surely. Thus (as in (ii)) Theorem 2.1 now gives the almost sure weak convergence of $\mu_{n_{m_l}}$ to μ, in particular the almost sure convergence of $\int f \,\mathrm{d}\mu_{n_{m_l}}$ to $\int f \,\mathrm{d}\mu$. □

2.2 Catalan Numbers

Similarly to the proof of the central limit theorem, the moment- method proof of the Wigner Semicircle Law requires knowledge about the moments of the semicircular distribution.

Definition 2.6 (Catalan Numbers). *The n-th Catalan number is defined to be*

$$C_n := \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

The Catalan numbers occur in various counting problems. The following propositions summarize the two interpretations relevant for the scope of this thesis.

Proposition 2.7 (Dyck Paths). *A Dyck path of length $2k$, $k \in \mathbb{N}$ is a path in $\mathbb{Z}^2$ from $(0,0)$ to $(2k,0)$ such that the vertex following (i,j) is either $(i+1,j+1)$ or $(i+1,j-1)$ and the path is never below the horizontal axis, i.e. for all (i,j) in the path we have $j \geq 0$. The number of such Dyck paths of length $2k$ is given by the Catalan number C_k.*

Proof. In total, there are $\binom{2k}{k}$ paths from $(0,0)$ to $(2k,0)$. If a path is a bad path, it contains some vertex $(i,-1)$ where it first went below the horizontal axis. If you now reverse every stroke to the right of i, you end up having a path ending in $(2k,-2)$. Conversely every path ending in $(2k,-2)$ corresponds to exactly one of those bad paths. The total number of paths from $(0,0)$ to $(2k,-2)$ is given by $\binom{2k}{k+1}$ and therefore the number of Dyck paths is given by $C_k = \binom{2k}{k} - \binom{2k}{k+1}$. □

The Catalan numbers can also be defined recursively:

Lemma 2.8 (Recursive Definition of Catalan Numbers). *The Catalan numbers satisfy the recurrence relation $C_k = C_0C_{k-1} + C_1C_{k-2} + \cdots + C_{k-1}C_0$.*

Proof. Let's use the interpretation of Catalan numbers as the number of Dyck paths. For $k = 0$ everything is trivial. For $k \geq 1$ the first stroke goes to $(1,1)$ and there has to be a first matching downwards stroke from $(2i-1,1)$ to $(2i,0)$ with $1 \leq i \leq k$ minimal. The paths between $(1,1)$ and $(2i-1,1)$ and $(2i,0)$ to $(2k,0)$ have to be Dyck paths of length $2(i-1)$ and $2(k-i)$ again. Therefore the number of Dyck paths satisfies the above recurrence relation. □

A pair partition of the set $[2n]$ into disjoint subsets $\pi_1,\dots,\pi_n$ of size 2 is called crossing if there exist $1 \leq a < b < c < d \leq 2n$ such that $\pi_i = \{a,c\}$ and $\pi_j = \{b,c\}$.

Proposition 2.9 (Non-crossing Pair Partitions). *The number of non-crossing pair partitions of the set $[2n]$ is given by the Catalan number C_n.*

Proof. Let $\pi_1,\dots,\pi_n$ be a non-crossing partition of $[2n]$ into n sets of size 2. There has to exist a block of the form $\pi_i = \{1,2j\}$. Since the partition is assumed to be non-crossing any block that contains an element of $\{2,\dots,2j-1\}$ has to be completely contained in this range. The same statement is true for the range $\{2j+1,\dots,2n\}$. Therefore these ranges are non-crossing partitions of sets of sets of sizes $2(j-1)$ and $2(n-j)$ in their own. Together with the trivially true statement for $n = 1$ this establishes the recurrence satisfied by the Catalan numbers. □

We are now in the position to show that the moments of the semicircular distribution are given by Catalan numbers:

Lemma 2.10. *The moments of the semicircular distribution satisfy*

$$\int x^k \,\mathrm{d}\mu_{sc} = \begin{cases} C_{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{else.} \end{cases}$$

Proof. The integral vanishes for odd k by symmetry. For even $k = 2n$ a trigonometric substitution and integration by parts yield

$$\begin{aligned} m_{2n} &= \frac{1}{2\pi}\int_{-2}^{2} x^{2n}\sqrt{4-x^2}\,\mathrm{d}x = \frac{2^{2n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n} y \cos^2 y \,\mathrm{d}y \\ &= -\frac{2^{2n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^{2n+1} y}{2n+1}(-\sin y)\,\mathrm{d}y = \frac{2^{2n+1}}{\pi(2n+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n+2} y \,\mathrm{d}y. \end{aligned}$$

On the other hand

$$m_{2n} = \frac{2^{2n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n} y - \sin^{2n+2} y \,\mathrm{d}y = \frac{2^{2n+1}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n} y \,\mathrm{d}y - (2n+1)m_{2n}$$

and hence

$$m_{2n} = \frac{2^{2n+1}}{\pi(2n+2)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{2n} y \,\mathrm{d}y = \frac{4(2n-1)}{2n+2} m_{2n-2}$$

for all $n \geq 1$. Since trivially $m_0 = m_1 = 1$ and $C_n = \frac{(2n)!}{n!(n+1)!} = \frac{2n(2n-1)}{n(n+1)} C_{n-1}$ the Catalan numbers fulfill the same recursion relation and the result follows. ◻

The below moment computation is originally due to Wigner (see [12]). It is the key part in the proof of Theorem 1.3 based on the moment method. Note, however, that there are many possible proofs of the Theorem, for example based on the Stieltjes transform method.

Proposition 2.11. *Under the assumptions of Theorem 1.3 it holds that*

$$\lim_{n\to\infty} \mathbf{E} \int x^k \,\mathrm{d}\mu_n = \begin{cases} C_{k/2} & \text{for even } k \\ 0 & \text{else} \end{cases}$$

for all $k \in \mathbb{N}$.

Proof. By linearity of integration we find that

$$\begin{aligned} m_{k,n} := \int x^k \,\mathrm{d}\mu_n &= \frac{1}{n}\sum_{i=1}^{n} \int x^k \,\mathrm{d}\delta_{\lambda_i} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k = \frac{1}{n}\operatorname{Tr} W_n^k \\ &= n^{-1-k/2} \sum_{i_1,\dots,i_k=1}^{n} \alpha_{i_1,i_2}\dots\alpha_{i_{k-1},i_k}\alpha_{i_k,i_1} \end{aligned}$$

and therefore

$$\overline{m_{k,n}} := \mathbf{E}\, m_{k,n} = n^{-1-k/2} \sum_{i_1,\ldots,i_k=1}^{n} \mathbf{E}\, \alpha_{i_1,i_2} \ldots \alpha_{i_{k-1},i_k} \alpha_{i_k,i_1}.$$

This corresponds to the summation over all cycles of length k in the complete directed graph on n vertices. By zero mean and independence hypotheses only those cycles where each undirected edge is visited at least twice contribute to the sum. These cycles therefore visit at most $\lfloor k/2 \rfloor$ undirected edges and thus at most $\lfloor k/2 \rfloor + 1$ vertices. On the other hand the prefactor of $n^{-1-k/2}$ makes the contribution of graphs with strictly less than $k/2+1$ distinct vertices negligible and allows us to concentrate on even k and cycles visiting $k/2+1$ distinct vertices. This is only possible if the undirected edge set of these cycles forms a tree. Furthermore, note that by the unit variance hypothesis all expectations are identically 1 for the contributing cycles and we therefore have reduced the problem to counting the number of these cycles.

Given an equivalence class of such cycles there are $n(n-1)\cdots(n-k/2) = n^{k/2+1} + \mathcal{O}\left(n^{-1}\right)$ ways of choosing the included vertices – exactly cancelling the prefactor. We claim that the set of equivalence classes of cycles of length k on $k/2+1$ vertices is in bijection with the Dyck paths of length k. Indeed, given a Dyck path we choose the cycle in such a way that we go to a so far unexplored vertex (since we are talking about equivalence classes, we don't need to specify the vertex) in case the Dyck path goes up and move back along the current edge in case the Dyck path goes down. This produces a cycle since the numbers of up's and down's are equal. Conversely given an edge in a cycle, we draw the next stroke of a Dyck path upwards if the edge hasn't been traversed so far and down otherwise. This completes the proof. □

2.3 Concentration of Measure

We already saw that the moments of empirical spectral distribution of Wigner random matrices converge in mean to the moment of the semicircular distribution. The so called concentration of measure phenomenon helps, however, to upgrade convergence in mean to almost sure convergence.

Lemma 2.12. *Under the assumptions of the Wigner Semicircle law (Theorem 1.3) we have that*

$$\mathbf{P}(|m_{k,n} - \overline{m_{k,n}}| > \epsilon) = \mathbf{P}\left(\left|\int x^k \,\mathrm{d}\mu_n - \mathbf{E}\int x^k \,\mathrm{d}\mu_n\right| > \epsilon\right) = \mathcal{O}\left(n^{-2}\right)$$

for all $\epsilon > 0$.

Proof. We compute the variance of the moments $m_{k,n} := \int x^k \,\mathrm{d}\mu_n$ to be

$$\begin{aligned}\mathbf{Var}\, m_{k,n} &= \mathbf{E}\left(\frac{1}{n}\operatorname{Tr} W_n^k\right)^2 - \left(\mathbf{E}\,\frac{1}{n}\operatorname{Tr} W_n^k\right)^2 \\ &= n^{-2-k} \sum_{i_1,j_1,\ldots,i_k,j_k=1}^{n} [\mathbf{E}\, \alpha_{i_1,i_2} \ldots \alpha_{i_k,i_1} \alpha_{j_1,j_2} \ldots \alpha_{j_k,j_1} \\ &\qquad\qquad - (\mathbf{E}\, \alpha_{i_1,i_2} \ldots \alpha_{i_k,i_1})(\mathbf{E}\, \alpha_{j_1,j_2} \ldots \alpha_{j_k,j_1})].\end{aligned}$$

The summation goes over all pairs of two cycles of length k each in the complete directed graph with vertex set $[n]$. If some undirected edge is only visited once by the $2k$ directed edges, the corresponding term vanishes identically. If the undirected graph is disconnected, the cycles are disjoint, and by independence also those terms vanish identically. Therefore all contributing graphs are connected and consist of at most k undirected edges and thus at most $k+1$ vertices. But if some graph visits exactly $k+1$ vertices, its undirected edge set forms a tree and each edge is visited exactly twice. This is only possible if the two cycles don't share any edges and in this case the contribution again vanishes. The leading contribution is therefore of order $\mathcal{O}\left(n^k\right)$ which vanishes as $\mathcal{O}\left(n^{-2}\right)$ after normalization. □

Proof of Theorem 1.3. The moments m_k of the semicircular distribution (as computed in Lemma 2.10) satisfy

$$\left(\frac{m_{2k}}{(2k)!}\right)^{1/2k} = \frac{1}{[k!(k+1)!]^{1/2k}} \to 0$$

as $k \to \infty$. The result thus follows from Theorem 2.2, Proposition 2.5 and Lemma 2.12. □

Chapter 3

Model and Main Results

Given a sequence of undirected graphs Γ_n on the vertex sets $\{1, \ldots, n\}$, we are considering Hermitian random matrices $H_n^{(\Gamma_n)}$ defined by

$$H_n^{(\Gamma_n)} := \frac{1}{\sqrt{9e(\Gamma_n)}} \sum_{(ij)\in\Gamma_n} \sum_{a,b=1}^{3} \alpha_{a,b,(ij)} \sigma_i^{(a)} \sigma_j^{(b)}, \tag{3.1}$$

where $e(\Gamma_n)$ denotes the number of edges in Γ_n. The normalisation factor of $(9e(\Gamma_n))^{-1/2}$ corresponds to the $9e(\Gamma_n)$ terms under the sum and is chosen to keep the spectrum of order 1. As a convention, the edge connecting $i < j$ is called (ij) and since the vertex set of the graphs is canonical, we shall, with a slight abuse of notation, identify the graph with its collection of edges. The coefficients $\alpha_{a,b,(ij)}$ are assumed to be independent random variables with zero mean and unit variance. For definiteness we will work with spin-$1/2$ systems, but all our results hold for spin-s models with any fixed s, see Remark 4.9.

We are interested in the eigenvalue density of the operators $H_n^{(\Gamma_n)}$ in the limit $n \to \infty$. The empirical spectral density (ESD) of $H_n^{(\Gamma_n)}$ is given by

$$\mu_n := \frac{1}{2^n} \sum_{j=1}^{2^n} \delta_{\lambda_j},$$

where λ_j are the eigenvalues of $H_n^{(\Gamma_n)}$. The result in [10] shows that under the assumption that the random variables are bounded, symmetric about 0 and that the graphs have a uniformly bounded chromatic number, μ_n converges weakly to a standard normal distribution as $n \to \infty$. Theorem 3.1 generalises this result by removing the symmetry condition and also allowing sequences of graphs for which the maximal vertex degree $d_{\max}(n)$ grows slower than the number of edges $e(\Gamma_n)$. Since graph sequences with uniformly bounded chromatic numbers have a uniformly bounded maximal degree, our degree condition is implied by the condition from [10] on the chromatic number, but it is much more general, and in some sense optimal.

Theorem 3.1. *Let Γ_n be a sequence of graphs on the vertex sets $\{1, \ldots, n\}$ such that*

$$\lim_{n\to\infty} \frac{d_{max}(n)}{e(\Gamma_n)} = 0$$

and let

$$\left\{ \alpha_{a,b,(ij)} \;\middle|\; 1 \le a,b \le 3, (ij) \in \Gamma_n \right\}$$

be a collection of independent (not necessarily identically distributed) random variables with zero mean, unit variance and uniformly bounded k-th moments for each $k \in \mathbb{N}$. Then empirical spectral density of the Hamiltonian $H_n^{(\Gamma_n)}$ defined in (3.1) *converges weakly in probability to a standard normal distribution.*

Theorem 3.1 addresses both the model of nearest neighbour interactions in a 1-dimensional closed chain (where the labelling is cyclic in the sense $\sigma_{n+1}^{(a)} = \sigma_1^{(a)}$)

$$H_n := \frac{1}{\sqrt{9n}} \sum_{j=1}^{n} \sum_{a,b=1}^{3} \alpha_{a,b,j} \sigma_j^{(a)} \sigma_{j+1}^{(b)},$$

as well as the mean field model realised by the complete graph

$$H_n^{(\text{comp})} := \frac{1}{\sqrt{9n(n-1)/2}} \sum_{1 \le i < j \le n} \sum_{a,b=1}^{3} \alpha_{a,b,(ij)} \sigma_i^{(a)} \sigma_j^{(b)}.$$

It also applies to all d_n-regular graphs in between, i.e. those where every vertex has the same degree $d_n \ge 1$ (here $(d_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of parameters). Indeed, these graphs satisfy $nd_n = 2e(\Gamma_n)$ and therefore

$$\frac{d_{\max}(n)}{e(\Gamma_n)} = \frac{d_n}{e(\Gamma_n)} = \frac{2}{n} \to 0$$

as $n \to \infty$.

We can generalise Theorem 3.1 to hypergraphs allowing not only quadratic but also higher order spin interactions. The main condition is that the maximal *hyperedge degree*, i.e. the maximal number of hyperedges intersecting any fixed hyperedge, should be negligible compared with the total number of hyperedges. The precise formulation will be given in Theorem 4.6, here we present only a prominent example of this generalisation, the quantum p-spin glasses. For any $p \ge 1$, the Hamiltonian of a quantum p-spin glass is given by

$$H_n^{(p-\text{glass})} := 3^{-p/2} \binom{n}{p}^{-1/2} \sum_{1 \le i_1 < \cdots < i_p \le n} \sum_{a_1,\ldots,a_p=1}^{3} \alpha_{a_1,\ldots,a_p,(i_1\ldots i_p)} \sigma_{i_1}^{(a_1)} \ldots \sigma_{i_p}^{(a_p)}.$$

The following theorem shows that the limiting density of states is Gaussian if p is fixed or it is n-dependent, $p = p_n$, but grows slower than $\sqrt{n}$ i.e. $\lim_{n\to\infty} \frac{p_n}{\sqrt{n}} = 0$. On the other hand, if p_n grows faster than $\sqrt{n}$ i.e. $\lim_{n\to\infty} \frac{\sqrt{n}}{p_n} = 0$, then the density of states is given by the semicircle law. We shall use the notations $a_n \ll b_n$ and $a_n \gg b_n$ meaning that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, respectively.

Theorem 3.2. *Let $1 \le p_n \le n$ be any sequence in n and assume that the independent random variables $\alpha_{a_1,\ldots,a_{p_n},(i_1\ldots i_{p_n})}$ have zero mean, unit variance and uniformly bounded k-th moments for each $k \in \mathbb{N}$. Then the empirical spectral distribution of the Hamiltonians $H_n^{(p_n-\mathit{glass})}$ converges weakly in probability to* *(i)* *a standard normal distribution if* $p_n \ll \sqrt{n}$,

(ii) *a semicircle distribution with density function* $\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}\chi_{[-2,2]}(x)$ *if* $p_n \gg \sqrt{n}$,

(iii) *a distribution with the compactly supported density function*

$$\rho_\lambda(x) = \begin{cases} v(x|e^{-4\lambda^2/3}) & \textit{if } x \in \left[-\frac{2}{\sqrt{1-e^{-4\lambda^2/3}}}, \frac{2}{\sqrt{1-e^{-4\lambda^2/3}}}\right], \\ 0 & \textit{else} \end{cases} \tag{3.2}$$

where

$$v(x|q) := \frac{\sqrt{1-q}}{\pi\sqrt{1-(1-q)x^2/4}} \prod_{k=0}^{\infty} \left[\frac{1-q^{2k+2}}{1-q^{2k+1}} \left(1 - \frac{x^2(1-q)q^k}{(1+q^k)^2}\right)\right]$$

if $\lim_{n\to\infty} \frac{p_n}{\sqrt{n}} = \lambda$.

Chapter 4

Proofs of Main Results

4.1 Proof of Theorem 3.1 via the Moment Method

To keep the terms simple we introduce the notations

$$\sigma_J := \sigma_i^{(a_1)}\sigma_j^{(a_2)}, \qquad \alpha_J := \alpha_{a_1,a_2,(ij)}$$

for tuples $J = (\mathbf{a},(ij)) = (a_1,a_2,(ij))$ and denote the index sets by

$$\begin{aligned} I_n &:= \{1,2,3\}^2 \times \Gamma_n \\ &= \left\{ (\mathbf{a},(ij)) = (a_1,a_2,(ij)) \;\middle|\; \mathbf{a} = (a_1,a_2) \in \{1,2,3\}^2, (ij) \in \Gamma_n \right\}. \end{aligned}$$

In order to compute the k-th moment m_k we have to evaluate the sum

$$\begin{aligned} m_{k,n} &= 2^{-n}\operatorname{Tr}(H_n^{(\Gamma_n)})^k = 2^{-n}\operatorname{Tr}\Big(\frac{1}{\sqrt{9e(\Gamma_n)}}\sum_{J\in I_n}\sigma_J\alpha_J\Big)^k \\ &= (9e(\Gamma_n))^{-k/2}\sum_{J_1,\ldots,J_k\in I_n} 2^{-n}\operatorname{Tr}\sigma_{J_1}\ldots\sigma_{J_k}\alpha_{J_1}\ldots\alpha_{J_k} \end{aligned} \tag{4.1}$$

in the limit $n \to \infty$. Since the empirical spectral densities are sharply concentrated around their mean, the main part in the computation will lie in computing the limit of the averaged moments $\overline{m_{k,n}} := \mathbf{E}\, m_{k,n}$ which will then imply converge in probability or even almost sure convergence of the empirical moments $m_{k,n}$ to the same limit. We split the sum in (4.1) into three disjoint parts

$$\sum_{J_1,\ldots,J_k\in I_n} = \sum_{D_{n,k}} + \sum_{A_{n,k}} + \sum_{B_{n,k}} \tag{4.2}$$

for a partition $I_n^k = A_{n,k} \cup B_{n,k} \cup D_{n,k}$ into three subsets defined below.

Firstly, we split I_n^k into the disjoint sets $P_2(I_n^k)$ and its complement $D_{n,k} := I_n^k \setminus P_2(I_n^k)$ and then further split $P_2(I_n^k)$ into

$$A_{n,k} := \left\{ ((\mathbf{a}_1,e_1),\ldots,(\mathbf{a}_k,e_k)) \in P_2(I_n^k) \;\middle|\; e_i \cap e_j = \emptyset \text{ if } (\mathbf{a}_i,e_i) \neq (\mathbf{a}_j,e_j) \right\},$$

the family of k-tuples with all edges non-intersecting, and its complement $B_{n,k} := P_2(I_n^k) \setminus A_{n,k}$. The condition $e_i \cap e_j = \emptyset$ (meaning that the edges have no vertex in common) assures that the matrices $\sigma_{(\mathtt{a}_i, e_i)}$ and $\sigma_{(\mathtt{a}_j, e_j)}$ commute. The reasons for these two splits are of entirely different nature. The sum over $D_{n,k}$ is negligible under fairly general circumstances due for the exact same reason it was negligible in the proof of the Central Limit Theorem. The second split of the remaining $P_2(I_n^k)$ into $A_{n,k} \cup B_{n,k}$ is important since for $(J_1, \ldots, J_k) \in A_{n,k}$ the σ_{J_i} corresponding to different J_i commute and the can be reordered in such a way that only squares of Pauli matrices remain and the normalised trace is 1. That means that all relevant quantum effects due to (potential) non-commutativities are isolated in the index set $B_{n,k}$. The system is essentially classical if the contribution of the index set $B_{n,k}$ to the rhs. of eq. (4.1) can be neglected. If this is the case, the asymptotic eigenvalue distribution equals the asymptotic energy histogram of the corresponding classical model where the spin matrices are replaced by commuting spins $s_i \in \{-1, 1\}$ or $s_i \in S^2$.

Lemma 4.1. *Let $(X_n)_{n \in \mathbb{N}}$ be a growing sequence of index sets and let*

$$\{ \alpha_x \mid n \in \mathbb{N}, x \in X_n \}$$

be a family of independent random variables with zero mean and unit variance and uniformly bounded k-th moments for all $k \in \mathbb{N}$, i.e., $|\mathbf{E}\, \alpha_x^k| \leq C_k < \infty$ for all $n, k \in \mathbb{N}$ and $x \in X_n$. Then we have that

$$\frac{1}{|X_n|^{k/2}} \sum_{(x_1, \ldots, x_k) \in X_n^k \setminus P_2(X_n^k)} |\mathbf{E}\, \alpha_{x_1} \ldots \alpha_{x_k}| \leq \begin{cases} \frac{D_k}{|X_n|^{1/2}} + \mathcal{O}\left(\frac{1}{|X_n|^{3/2}}\right) & \textit{if } k \textit{ is odd,} \\ \frac{\tilde{D}_k}{|X_n|} + \mathcal{O}\left(\frac{1}{|X_n|^2}\right) & \textit{if } k \textit{ is even} \end{cases}$$

as $n \to \infty$ while k is fixed with some constants $D_k, \tilde{D}_k$ that grow with k as $(k+1)!!$ and $(k+5)!!$, respectively.

Proof. First note that in the case that some x_i only appears once, by independence and zero mean hypothesis these terms vanish identically. Since the case that all x_i's appear exactly twice is excluded from the index set in the sum above, we only have to consider those terms for which there are strictly less than $\frac{k}{2}$ distinct x_i's. There are only $\mathcal{O}(|X_n|^m)$ (as $n \to \infty$) terms with $m < \frac{k}{2}$ distinct x_i's, so we find, after summation, that the total contribution of these terms vanish as $\mathcal{O}\left(|X_n|^{m-k/2}\right)$ as $n \to \infty$. Let us try to find the coefficient of the highest order term in n. In the case that k is odd the term with the highest order comes from $m = \frac{k-1}{2}$ i.e. vanishes for $k < 3$. The terms with $\frac{k-1}{2}$ distinct x_i's such that all x_i appear at least twice are those for which some x_i appears three times and the rest only two times. There are $\binom{|X_n|}{k/2-1/2}$ ways of choosing the x_i's to appear, then there are $\frac{k-1}{2}$ ways of choosing the x_i that appears three times and there are $\frac{k!}{3 \cdot 2^{k/2-3/2}}$ ways of assigning those pre-described values to the tuples $(x_1, \ldots, x_k)$. In total we find for the number of terms contributing to the leading order

$$\binom{|X_n|}{k/2 - 1/2} \frac{k-1}{2} \frac{k!}{3 \cdot 2^{k/2-3/2}} = \frac{|X_n|^{k/2-1/2} k!}{3(k-3)!!} + \mathcal{O}\left(|X_n|^{k/2-3/2}\right),$$

each having a modulus bounded by C_3. The proof for even k is analogous. □

The estimate in Lemma 4.1 used only the scaling properties of the expectations and applies to the computation of $\overline{m_{k,n}}$ since the normalised traces have uniformly bounded modulus. By using more specifics of the tracial part we can improve the error estimate from Lemma 4.1 significantly:

Lemma 4.2. *Assume that the random variables α_J are independent, have zero mean, unit variance and uniformly bounded moments $|\mathbf{E}\,\alpha_J^k| \leq C_k < \infty$ for all $n \in \mathbb{N}$, $J \in I_n$. Then we have the bound*

$$\left|(9e(\Gamma_n))^{-k/2} \sum_{(J_1,\ldots,J_k)\in D_{n,k}} 2^{-n}\operatorname{Tr}\sigma_{J_1}\ldots\sigma_{J_k}\,\mathbf{E}\,\alpha_{J_1}\ldots\alpha_{J_k}\right| \leq \begin{cases} \frac{D_k}{e(\Gamma_n)^{3/2}} + \mathcal{O}\left(e(\Gamma_n)^{-5/2}\right) & \textit{if } k \textit{ is odd} \\ \frac{\tilde{D}_k}{e(\Gamma_n)} + \mathcal{O}\left(e(\Gamma_n)^{-2}\right) & \textit{if } k \textit{ is even} \end{cases}$$

as $n \to \infty$ while k is fixed with some constants D_k and $\tilde{D}_k$ that grow with k as $(k+8)!!$ and $(k+3)!!$, respectively.

Before going into the proof of Lemma 4.2 we state some properties of the traces of Pauli matrices we shall need. A proof of this technical Lemma is given in the appendix.

Lemma 4.3 (Traces of products of Pauli matrices). *Given $a_1, \ldots, a_k \in \{1,2,3\}$ the normalised traces of products of Pauli matrices*

$$\sigma(a_1,\ldots,a_k) := \frac{1}{2}\operatorname{Tr}\sigma^{(a_1)}\ldots\sigma^{(a_k)}$$

satisfy:

(i) $\sigma(a_1,\ldots,a_k) \in \{0,1,-1,i,-i\}$*;*

(ii) More generally for all $1 \leq j_1,\ldots,j_k \leq n$

$$\frac{1}{2^n}\operatorname{Tr}\sigma_{j_1}^{(a_1)}\ldots\sigma_{j_k}^{(a_k)} \in \{0,1,-1,i,-i\};$$

(iii) $\sigma(a_1,\ldots,a_k)$ *is non-zero if and only if the parities of the numbers of 1's, 2's and 3's among the $a_1,\ldots,a_k$ coincide.*

(iv) If k is even we have the recursion relation

$$\sigma(a_1,\ldots,a_k) = \sum_{j=2}^{k} \delta_{a_1 a_j}(-1)^j \sigma(a_2,\ldots,\widehat{a_j},\ldots,a_k)$$

where $\widehat{a_j}$ means that the j-th entry is omitted;

Proof of Lemma 4.2. For odd k first note that up to a factor of ± 1 we can reorder the σ_{J_i}'s in the expression $2^{-n}\operatorname{Tr}\sigma_{J_1}\ldots\sigma_{J_k}$ since Pauli matrices either commute or anti-commute. Since for any J_i the we have $\sigma_{J_i}^2 = 1_{2^n}$ the normalised trace reduces to $\pm 2^{-n}\operatorname{Tr}\sigma_{J_{i_1}}\ldots\sigma_{J_{i_l}}$ with the $J_{i_1},\ldots,J_{i_l}$ being all exactly those distinct $J_1,\ldots,J_k$ that appear an odd number of times. By Lemma 4.3(iii) in each component of the tensor product we get a zero trace if there are either one or two different Pauli matrices acting on it. Hence the normalised trace is zero if there are one or two distinct J_i appearing an odd number of times. Since k is odd we therefore see that the highest order contribution comes from the term where three distinct J_i appear three times and the rest appear two times. Thus for $k < 9$ we see that $\overline{m_{k,n}}$ is identically zero. For $k \geq 9$ we first choose the $\left(\frac{k-9}{2}+3\right) = \frac{k-3}{2}$ distinct J_i to appear and then those three to appear three times. By counting the number of ways of assigning those J_i to our tuples we therefore find the factor

$$\begin{aligned}&\binom{9e(\Gamma_n)}{k/2-3/2}\binom{k/2-3/2}{3}\frac{k!}{3^3\cdot 2^{k/2-3/2}}\\&=\frac{(9e(\Gamma_n))^{k/2-3/2}}{(k/2-3/2)!}\binom{k/2-3/2}{3}\frac{k!}{3^3\cdot 2^{k/2-3/2}}+\mathcal{O}\left((e(\Gamma_n))^{k/2-5/2}\right)\end{aligned}$$

as $n\to\infty$ from which after dividing by $(9e(\Gamma_n))^{k/2}$ the claimed asymptotics follow. The claims for even k immediately follow from Lemma 4.1 using that the term with C_3^2 vanishes by the above argument (since there are two distinct J_i appearing an odd number of times). □

In particular this already shows that in the limit $n\to\infty$ all odd moments vanish. The situation with the sums over $A_{n,k}$ and $B_{n,k}$ in (4.2) is a little bit more delicate. For a large class of graph sequences the sum over $B_{n,k}$ is also negligible and the only contribution comes from $A_{n,k}$ where all normalised traces are equal to 1 and the system is essentially classical. In this case the non-commutativity is actually only a small perturbation and consequently we see the same result as in the classical central limit theorem rather than a random matrix semicircle law. We are now ready to give a proof of Theorem 3.1, including explicit estimates regarding the rate of convergence of the moments.

Theorem 4.4 (Detailed version of Theorem 3.1). *Denote the maximal vertex degree in the graph Γ_n by $d_{max}(n)$. Let Γ_n be a sequence of graphs on the vertex sets $\{1,\dots,n\}$ such that $\lim_{n\to\infty}\frac{d_{max}(n)}{e(\Gamma_n)}=0$ and let*

$$\left\{\ \alpha_{a,b,(ij)}\ \middle|\ 1\le a,b\le 3,(ij)\in\Gamma_n\ \right\}$$

be a collection of independent (not necessarily identically distributed) random variables with zero mean, unit variance and uniformly bounded k-th moment for each $k\in\mathbb{N}$. Then the Hamiltonian defined by

$$H_n^{(\Gamma_n)}:=\frac{1}{\sqrt{9e(\Gamma_n)}}\sum_{(ij)\in\Gamma_n}\sum_{a,b=1}^{3}\alpha_{a,b,(ij)}\sigma_i^{(a)}\sigma_j^{(b)}$$

(where as a convention the edge between $i<j$ is denoted by (ij)) has an empirical spectral distribution which converges weakly in probability to a standard normal distribution. The convergence rate of the moments is of order $e(\Gamma_n)^{-3/2}$ for odd moments and $\frac{d_{max}(n)}{e(\Gamma_n)}$ for even moments. Under the stronger assumption that $\frac{d_{max}(n)}{e(\Gamma_n)}=\mathcal{O}\left(n^{-1-\epsilon}\right)$ for some $\epsilon>0$, we also have almost sure weak convergence of the ESDs.

Proof. The treatment of the sum from eq. (4.1) is performed in three steps according to the split from eq. (4.2). Lemma 4.2 dealt with the $D_{n,k}$-part of the sum. We now consider the part of the sum over the index set $B_{n,k}$. From the condition $\frac{d_{\max}(n)}{e(\Gamma_n)}\to 0$ as $n\to\infty$ it follows that the number $d_{j,n}$ of choosing j non intersecting edges from the graph Γ_n asymptotically behaves as $e(\Gamma_n)^j/j!$ i.e. $\lim_{n\to\infty}\frac{d_{j,n}}{e(\Gamma_n)^j/j!}=1$ for all fixed j. Indeed, there are $e(\Gamma_n)$ choices for the first edge (il). For the next edge we can pick all edges except those including i and l

i.e. there are at least $e(\Gamma_n) - 2d_{\max}(n)$ choices for the second edge. Continuing we find the bound

$$\begin{aligned}\frac{e(\Gamma_n)^j}{j!} \geq d_{j,n} &\geq \frac{1}{j!} e(\Gamma_n)(e(\Gamma_n) - 2d_{\max}(n)) \ldots (e(\Gamma_n) - 2(j-1)d_{\max}(n)) \\ &= \frac{e(\Gamma_n)^j}{j!} \left(1 - \frac{j(j-1)d_{\max}(n)}{e(\Gamma_n)} + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)^2\right)\right) \end{aligned} \tag{4.3}$$

as $n \to \infty$. Dividing by $e(\Gamma_n)^j/j!$ then proves that $d_{j,n}$ asymptotically behaves as $e(\Gamma_n)^j/j!$.

The estimate from eq. (4.3) also shows that the number of choosing j edges that have at least one intersection is, to leading order, at most given by

$$\frac{e(\Gamma_n)^{j-1} d_{\max}(n)}{(j-2)!} + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)^2 e(\Gamma_n)^j\right).$$

Since there are $\binom{k}{2}\binom{k-2}{2} \ldots \binom{2}{2} = \frac{k!}{2^{k/2}}$ ways of assigning $\frac{k}{2}$ chosen edges to $e_1, \ldots, e_k$ such that each appears twice, the index set $B_{n,k}$ therefore contains at most

$$9^{k/2} \frac{k!}{2^{k/2}} \frac{d_{\max}(n)}{(k/2-2)!} e(\Gamma_n)^{k/2-1} + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)^2 e(\Gamma_n)^{k/2}\right)$$

elements as $n \to \infty$. Using that the modulus of the normalised traces is at most 1 (see Lemma 4.3) and that the expectations are all equal to 1 due to unit variance and independence, we therefore found the bound

$$\begin{aligned}&\left| \frac{1}{(9e(\Gamma_n))^{k/2}} \sum_{(J_1,\ldots,J_k)\in B_{n,k}} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} \right| \\ &\qquad\qquad \leq D_k \frac{d_{\max}(n)}{e(\Gamma_n)} + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)^2\right) \end{aligned} \tag{4.4}$$

as $n \to \infty$ while k is fixed for some k-dependent constants D_k that we, for convenience, allow to change over time.

For the summation over $A_{n,k}$ in (4.2), we first note that all terms under the sum are equal to 1. Indeed, the expectations are again 1 by independence and unit variance. For the traces we find that since all distinct J_i act on distinct qubits, in all components there is either an identity matrix or a product of two identical Pauli matrices i.e. again identity matrices. Again similarly to eq. (4.3) (with $\frac{k}{2}$ playing the role of j) we can estimate $|A_{n,k}|$ to get

$$\begin{aligned}(k-1)!! &\geq (9e(\Gamma_n))^{-k/2}|A_{n,k}| \\ &\geq (k-1)!! \left(1 - D_k \frac{d_{\max}(n)}{e(\Gamma_n)}\right) + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)^2\right) \end{aligned} \tag{4.5}$$

as $n \to \infty$ while k is fixed for some k dependent constants D_k (which we allow to change over time).

The moments m_k of a standard normal distribution are given by $(k-1)!!$ for even k and 0 for odd k and satisfy Carleman's continuity condition. Using the bound in Lemma 4.2 together with eqs. (4.4) and (4.5) we arrive at

$$\begin{aligned}
|\mathbf{E}\, m_{k,n} - (k-1)!!| \leq & \left| (9e(\Gamma_n))^{-k/2} \sum_{(J_1,\ldots,J_k)\in D_{n,k}} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \, \mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} \right| \\
& + \left| (9e(\Gamma_n))^{-k/2} \sum_{(J_1,\ldots,J_k)\in B_{n,k}} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \, \mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} \right| \\
& + \left| (9e(\Gamma_n))^{-k/2} \sum_{(J_1,\ldots,J_k)\in A_{n,k}} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \, \mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} - (k-1)!! \right| \\
\leq & \frac{D_k}{e(\Gamma_n)} + \tilde{D}_k \frac{d_{\max}(n)}{e(\Gamma_n)} + \mathcal{O}\left(\left(\frac{d_{\max}(n)}{e(\Gamma_n)} \right)^2 \right)
\end{aligned}$$

for (fixed) even $k \geq 9$ as $n \to \infty$, whereas

$$\begin{aligned}
|\mathbf{E}\, m_{k,n} - 0| = & \left| (9e(\Gamma_n))^{-k/2} \sum_{(J_1,\ldots,J_k)\in D_{n,k}} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \, \mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} \right| \\
\leq & \frac{D_k}{e(\Gamma_n)^{3/2}} + \mathcal{O}\left(e(\Gamma_n)^{-5/2} \right)
\end{aligned}$$

for (fixed) odd k as $n \to \infty$. This shows the convergence of each expected moment.

It remains to show that the moments are sharply concentrated around their mean. The variance of the k-th moment is given by

$$\begin{aligned}
\mathbf{Var}\, m_{k,n} &= \mathbf{E}\, m_{k,n}^2 - (\mathbf{E}\, m_{k,n})^2 \\
&= (9e(\Gamma_n))^{-k} \sum_{J_1,L_1,\ldots,J_k,L_k \in I_n} 2^{-2n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \operatorname{Tr} \sigma_{L_1} \ldots \sigma_{L_k} \\
&\qquad \cdot [\mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k} \alpha_{L_1} \ldots \alpha_{L_k} - (\mathbf{E}\, \alpha_{J_1} \ldots \alpha_{J_k})(\mathbf{E}\, \alpha_{L_1} \ldots \alpha_{L_k})].
\end{aligned}$$

As seen earlier the leading order terms are those where all $J_1, L_1, \ldots, J_k, L_k$ appear exactly twice and the edges corresponding to non-equal indices share no vertices. Therefore if some pair of equal indices of the form J_a, L_b appears, both traces are identically zero since the trace of any Pauli matrix is zero and therefore also those terms vanish. The remaining pairings where all J_a are paired with some J_b also vanish by independence. The next to leading order stems from those terms with $k-1$ distinct J_i, L_i and therefore vanishes as $\mathcal{O}\left(e(\Gamma_n)^{-1}\right)$. Additionally, the error made by assuming distinct edges is of order $\mathcal{O}\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)$ according to eq. (4.3). Thus $\mathbf{Var}\, m_{k,n} = \mathcal{O}\left(\frac{d_{\max}(n)}{e(\Gamma_n)}\right)$ (as $n \to \infty$) and the convergence in probability or almost sure convergence follow from Proposition 2.5. $\square$

4.2 Counterexample for Localized Graphs

As the following example shows, the assumption on the growth of the maximal degree is necessary. Let

$$H_n^{(\mathrm{star})} := \frac{1}{\sqrt{9(n-1)}} \sum_{j=2}^{n} \sum_{a,b=1}^{3} \alpha_{a,b,j} \sigma_1^{(a)} \sigma_j^{(b)}$$

be the Hamiltonian corresponding to the star graph in which, say, the vertex 1 is connected to all other vertices while there are no edges between the rest. This model shows a significantly different limiting behaviour (a proof is given in the Appendix, see also Figure 4.1):

Proposition 4.5. *Suppose that the random variables*

$$\{ \alpha_{a,b,j} \mid n \in \mathbb{N}, 1 \leq a, b \leq 3, 2 \leq j \leq n \}$$

are independent, have zero mean, unit variance and uniformly bounded k-th moments for each $k \in \mathbb{N}$. The expected spectral density of $H_n^{(star)}$ then converges weakly to a distribution with density

$$\rho(x) = 3\sqrt{\frac{3}{2\pi}} x^2 e^{-3x^2/2}$$

as $n \to \infty$.

Proof. We again start to compute the moments using the short hand notation $\sigma_{J_i} := \sigma_1^{(a_i)} \sigma_{j_i}^{(b_i)}$ for $J = (a_i, b_i, j_i)$ and find by Lemma 4.1 that

$$\overline{m_{k,n}} \approx (9(n-1))^{-k/2} \sum_{\{J_1,\dots,J_{k/2}\} \subset I_n}^{\star} \sum_{\pi \in S_k} 2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \dots \sigma_{J_{\pi(k)}}.$$

In the limit $n \to \infty$ the part of the sum where two different J_i have the same j_i*-coordinate* can be neglected and we find that $\overline{m_{k,n}}$ is approximately given by

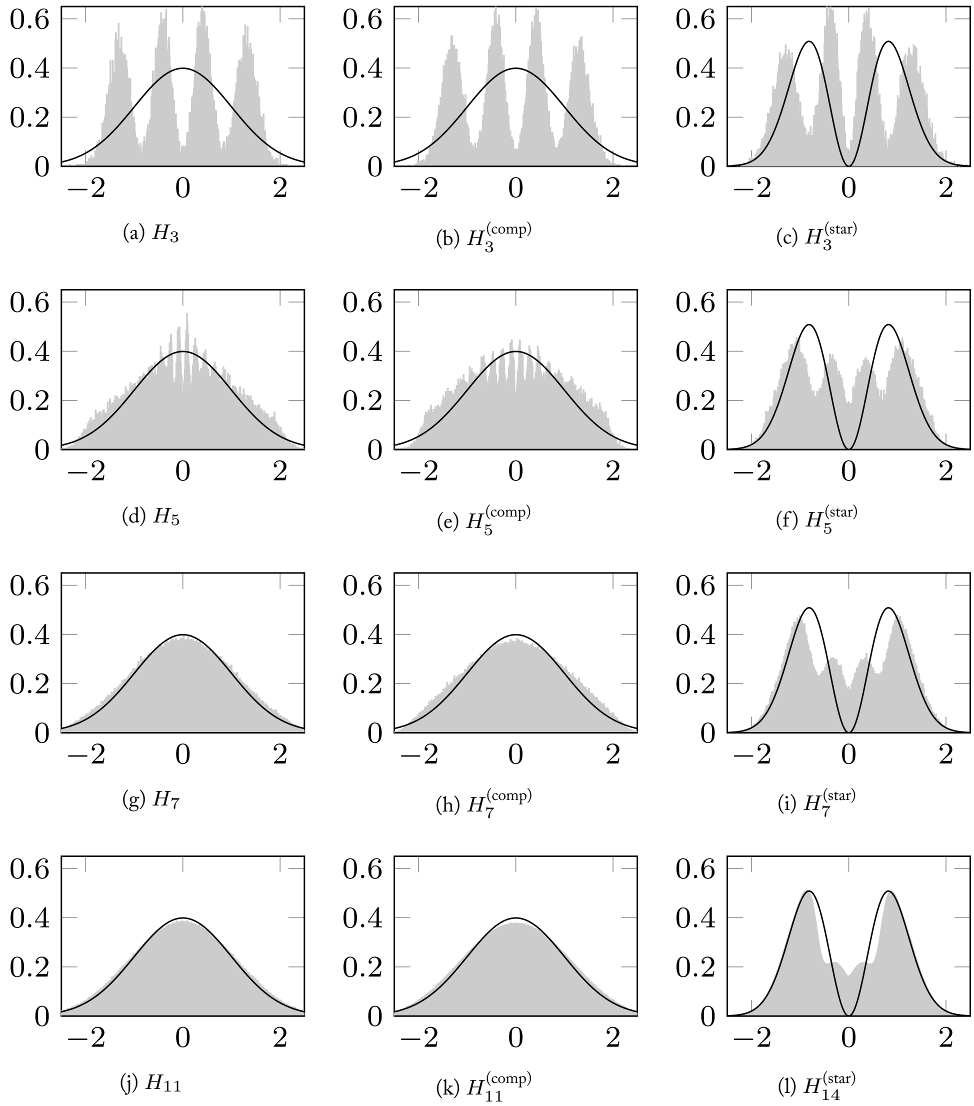
$$(9(n-1))^{-k/2} \sum_{\{j_1,\dots,j_{k/2}\} \subset \{2,\dots,n\}} \sum_{a_1,\dots,a_{k/2}=1}^{3} \sum_{b_1,\dots,b_{k/2}=1}^{3} \sum_{\pi \in S_k} 2^{-1} \operatorname{Tr} \sigma^{(a_{\pi(1)})} \dots \sigma^{(a_{\pi(k)})}$$

where it was used that in all but the first component the matrices commute (since the j_i are mutually distinct), the square of any Pauli matrix is the identity and that the trace of the tensor product is the product of the traces. After performing the sums over the j_i's and b_i's (and using that $\binom{n-1}{k/2}(n-1)^{-k/2} \approx \frac{1}{(k/2)!}$) we arrive at

$$\overline{m_{k,n}} \approx m_k := \frac{3^{-k/2}}{(k/2)!} \sum_{a_1,\dots,a_{k/2}=1}^{3} \sum_{\pi \in S_k} \frac{1}{2} \operatorname{Tr} \sigma^{(a_{\pi(1)})} \dots \sigma^{(a_{\pi(k)})}.$$

We claim that

$$f(k) := \sum_{a_1,\dots,a_{k/2}=1}^{3} \sum_{\pi \in S_k} \frac{1}{2} \operatorname{Tr} \sigma^{(a_{\pi(1)})} \dots \sigma^{(a_{\pi(k)})} = \frac{(k+1)!}{2^{k/2}}$$



(a) H_3 (b) $H_3^{(\text{comp})}$ (c) $H_3^{(\text{star})}$

(d) H_5 (e) $H_5^{(\text{comp})}$ (f) $H_5^{(\text{star})}$

(g) H_7 (h) $H_7^{(\text{comp})}$ (i) $H_7^{(\text{star})}$

(j) H_{11} (k) $H_{11}^{(\text{comp})}$ (l) $H_{14}^{(\text{star})}$

Figure 4.1: Histograms of the empirical eigenvalue distribution averaged over 1000 samples (grey) and the limiting density profile (black)

holds for all even k. While $k=2$ is trivial, for the induction step we compute using Lemma 4.3(iv) and the notation therein

$$\begin{aligned} f(k) &= \sum_{a_1,\ldots,a_{k/2}=1}^{3} \sum_{\pi\in S_k} \frac{1}{2} \operatorname{Tr} \sigma^{(a_{\pi(1)})} \ldots \sigma^{(a_{\pi(k)})} \\ &= \sum_{j=2}^{k} (-1)^j \sum_{a_1,\ldots,a_{k/2}=1}^{3} \sum_{\pi\in S_k} \delta_{a_{\pi(1)} a_{\pi(j)}} \sigma(a_{\pi(2)},\ldots,\widehat{a_{\pi(j)}},\ldots,a_{\pi(k)}) \end{aligned}$$

and then split the sum into to parts $f_1(k)$ and $f_2(k)$ where in $f_1(k)$ we only consider those $\pi\in S_k$ for which $\pi(1)=\pi(j)$ and in $f_2(k)$ those π for which $\pi(1)\neq\pi(j)$. For $f_1(k)$ we can then compute

$$\begin{aligned} f_1(k) &= 3\sum_{j=2}^{k}(-1)^j \sum_{a_1,\ldots,\widehat{a_{\pi(1)}},\ldots,a_{k/2}=1}^{3} \sum_{\substack{\pi\in S_k \\ \pi(1)=\pi(j)}} \sigma(a_{\pi(2)},\ldots,\widehat{a_{\pi(j)}},\ldots,a_{\pi(k)}) \\ &= 3\frac{k}{2}\sum_{j=2}^{n}(-1)^j \sum_{a_1,\ldots,a_{(k-2)/2}=1}^{3} \sum_{\pi\in S_{k-2}} \sigma(a_{\pi(1)},\ldots,a_{\pi(k-2)}) \\ &= 3\frac{k}{2}\sum_{j=2}^{k}(-1)^j f(k-2) = 3\frac{k}{2} f(k-2) \end{aligned}$$

where in the first step we performed the sum over $a_{\pi(1)}=a_{\pi(j)}$ and in the second step took out a factor of $\frac{k}{2}$ corresponding to the $\frac{k}{2}$ possible values of $\pi(1)=\pi(j)$. For $f_2(k)$ we get

$$\begin{aligned} f_2(k) &= \sum_{j=2}^{k}(-1)^j \sum_{\substack{\pi\in S_k \\ \pi(1)\neq\pi(j)}} \sum_{\substack{a_1,\ldots,a_{k/2}=1 \\ a_{\pi(1)}=a_{\pi(j)}}}^{3} \sigma(a_{\pi(2)},\ldots,\widehat{a_{\pi(j)}},\ldots,a_{\pi(k)}) \\ &= \frac{k(k-2)}{2}\sum_{j=2}^{k}(-1)^j \sum_{a_1,\ldots,a_{(k-2)/2}=1}^{3} \sum_{\pi\in S_{k-2}} \sigma(a_{\pi(1)},\ldots,a_{\pi(k-2)}) \\ &= \frac{k(k-2)}{2}\sum_{j=2}^{k}(-1)^j f(k-2) = \frac{k(k-2)}{2} f(k-2) \end{aligned}$$

where in the second step we used that $a_{\pi(1)}=a_{\pi(j)}$ while $\pi(1)\neq\pi(j)$ also implies that $a_{\pi(r)}=a_{\pi(s)}$ for those uniquely determined $r\neq 1, s\neq j$ which satisfy $\pi(r)=\pi(1)$ and $\pi(s)=\pi(j)$. The factor comes from the fact that there are $|S_k|-\frac{k}{2}|S_{k-2}|=\frac{k(k-2)}{2}|S_{k-2}|$ choices for $\pi\in S_k$ such that $\pi(1)\neq\pi(j)$. By adding the two recursion relations we finally arrive at

$$f(k)=f_1(k)+f_2(k)=\frac{k(k+1)}{2}f(k-2)=\frac{k(k+1)}{2}\frac{(k-1)!}{2^{(k-2)/2}}=\frac{(k+1)!}{2^{k/2}}$$

proving the claim. Inserting this into the expression we had for m_k then gives $m_k=\frac{(k+1)!}{6^{k/2}(k/2)!}$ for even k and $m_k=0$ for odd k.

These moments again satisfy the condition of Theorem 2.2 and therefore uniquely correspond to a limiting distribution whose characteristic function ϕ is given by

$$\phi(t) = \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} \frac{(2k+1)!}{6^k k!} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)t^{2k}}{6^k k!} = \left(1 - \frac{t^2}{3}\right) e^{-t^2/6}$$

from which by a Fourier transform we find the density

$$\rho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) e^{-itx} \, \mathrm{d}t = 3\sqrt{\frac{3}{2\pi}} x^2 e^{-3x^2/2}. \qquad \square$$

4.3 Extension to Hypergraphs

A hypergraph is a generalised graph in which any hyperedge can contain a variable number of vertices. Formally a hypergraph on a vertex set V is any subset of $\mathcal{P}(V) \setminus \emptyset$. A hyperedge e containing the (distinct) vertices $i_1 < \cdots < i_l$ will be denoted by $(i_1 \ldots i_l)$. We shall use the notation $|e| := l$ for the number of vertices in a given hyperedge $e = (i_1 \ldots i_l)$. Just as in the traditional graph, the degree of a vertex is defined to be the number of hyperedges containing the given vertex. The total number of hyperedges is again denoted by $e(\Gamma_n)$. For a given hypergraph Γ_n on the vertex set $\{1, \ldots, n\}$ we introduce the notations

$$\alpha_J := \alpha_{(\mathbf{a},e)} := \alpha_{a_1,\ldots,a_l,(i_1 \ldots i_l)}, \qquad \sigma_J := \sigma_{(\mathbf{a},e)} := \sigma_{i_1}^{(a_1)} \ldots \sigma_{i_l}^{(a_l)}$$

and $|J| = |e| = l$ for

$$J = (\mathbf{a}, e) = (a_1, \ldots, a_l, (i_1 \ldots i_l)) \in I_n := \left\{ (\mathbf{a}, e) \;\middle|\; e \in \Gamma_n, \mathbf{a} \in \{1,2,3\}^{|e|} \right\}.$$

The generalised Hamiltonian corresponding to the hypergraph Γ_n is defined to be

$$H_n^{(\Gamma_n)} := \frac{1}{\sqrt{e(\Gamma_n)}} \sum_{e \in \Gamma_n} \frac{1}{3^{|e|/2}} \sum_{\mathbf{a} \in \{1,2,3\}^{|e|}} \alpha_{(\mathbf{a},e)} \sigma_{(\mathbf{a},e)}. \tag{4.6}$$

We again want to study the moments

$$\begin{aligned} m_{k,n} &= 2^{-n} \operatorname{Tr} \mathbf{E} (H_n^{(\Gamma_n)})^k \\ &= (e(\Gamma_n))^{-k/2} \sum_{J_1,\ldots,J_k \in I_n} 3^{-(|J_1|+\cdots+|J_k|)/2} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k} \mathbf{E} \, \alpha_{J_1} \ldots \alpha_{J_k} \end{aligned} \tag{4.7}$$

in the limit $n \to \infty$. Lemma 4.1 again applies and immediately shows that we can restrict our attention to those summands where the $J_1, \ldots, J_k$ appear in pairs of two. If the hyperedges of the $\frac{k}{2}$ distinct J_i's are disjoint we can reorder the σ_{J_i}'s freely and therefore get a normalised trace of 1. As in the proof of Theorem 3.1 we establish a sufficient criterion on the sequence of graphs such that among all families of $\frac{k}{2}$ edges the proportion of those that have mutually disjoint edges approaches 1.

As for conventional graphs, the *line graph* $L(\Gamma_n)$ of a hypergraph Γ_n is graph whose vertices are the hyperedges $\{e_1, \ldots, e_M\}$ of Γ_n. Two vertices of $L(\Gamma_n)$ (i.e. hyperedges of Γ_n) e_1, e_2

are adjacent (connected by an edge in the line graph) if and only if e_1 and e_2 are non-disjoint and so the edges of $L(\Gamma_n)$ are given by

$$\{\, (e_i e_j) \mid 1 \le i, j \le M, e_i \cap e_j \neq \emptyset \,\}\,.$$

Given some fixed hyperedge $e_1 \in \Gamma_n$ there are at least $e(\Gamma_n) - d_{\max}(L(\Gamma_n))$ hyperedges e_2 disjoint from e_1, where the maximal hyperedge degree $d^{(e)}_{\max}(n) := d_{\max}(L(\Gamma_n))$ is the maximal vertex degree of the line graph. Continuing we find for the number $d_{j,n}$ of choices of j disjoint hyperedges from Γ_n the bound

$$\begin{aligned}\frac{e(\Gamma_n)^j}{j!} \ge d_{j,n} &\ge \frac{1}{j!} e(\Gamma_n)(e(\Gamma_n) - d^{(e)}_{\max}(n)) \dots (e(\Gamma_n) - (j-1) d^{(e)}_{\max}(n)) \\ &= \frac{e(\Gamma_n)^j}{j!}\left(1 - \frac{j(j-1)}{2}\frac{d^{(e)}_{\max}(n)}{e(\Gamma_n)} + \mathcal{O}\left(\left(\frac{d^{(e)}_{\max}(n)}{e(\Gamma_n)}\right)^2\right)\right) \end{aligned} \tag{4.8}$$

as $n \to \infty$ while j is fixed if $\lim_{n\to\infty} \frac{d^{(e)}_{\max}(n)}{e(\Gamma_n)} = 0$. Following the proof of Theorem 3.1 we therefore proved its generalisation for hypergraphs:

Theorem 4.6. *Let Γ_n be a sequence of graphs on the vertex sets $\{1, \dots, n\}$ such that $\lim_{n\to\infty} \frac{d^{(e)}_{max}(n)}{e(\Gamma_n)} = 0$ and let*

$$\left\{ \alpha_{(\mathbf{a},e)} \;\middle|\; e \in \Gamma_n, \mathbf{a} \in \{1,2,3\}^{|e|} \right\}$$

be a collection of independent (not necessarily identically distributed) random variables with zero mean, unit variance and uniformly bounded k-th moments for all k. Then the Hamiltonian defined in (4.6) *has a density of states which converges weakly in probability to a standard normal distribution.*

4.4 Phase Transition for p-uniform Hypergraphs

For 2-uniform hypergraphs (meaning that all edges connect 2 vertices) the statement of this Theorem is equivalent to Theorem 3.1. More generally the theorem also covers a sequence of p_n-uniform graphs Γ_n corresponding to the p_n-spin glasses. An interesting special case is the sequence of complete p_n-uniform hypergraphs in which the hyperedges connect any p_n distinct vertices. The corresponding Hamiltonians $H_n^{(p_n-\text{glass})}$ are given by

$$3^{-p_n/2}\binom{n}{p_n}^{-1/2} \sum_{1\le i_1<\dots<i_{p_n}\le n}\;\sum_{a_1,\dots,a_{p_n}=1}^{3} \alpha_{a_1,\dots,a_{p_n},(i_1\dots i_{p_n})}\sigma^{(a_1)}_{i_1}\dots\sigma^{(a_{p_n})}_{i_{p_n}}.$$

In this case the degree of any hyperedge is

$$\deg(i_1 \dots i_{p_n}) = \binom{n}{p_n} - \binom{n-p_n}{p_n},$$

while the total number of hyperedges is given by $e(\Gamma_n) = \binom{n}{p_n}$. Since

$$\lim_{n\to\infty}\frac{\binom{n-p_n}{p_n}}{\binom{n}{p_n}} = \begin{cases} 1 & \text{if } p_n \ll \sqrt{n}, \\ 0 & \text{if } p_n \gg \sqrt{n}, \\ e^{-\alpha^2} & \text{if } \lim_{n\to\infty}\frac{p_n}{\sqrt{n}} = \alpha \in (0,\infty) \end{cases}$$

(see Lemma 4.7, a proof is given in the appendix) this p_n-spin glass model fulfils the condition of Theorem 4.6 if and only if p_n grows slower than $\sqrt{n}$.

We now turn to the question whether for $p_n \gg \sqrt{n}$, the expected density of states of $H_n^{(p_n-\text{glass})}$ indeed exhibits a different limiting behaviour. As Theorem 3.2 shows, this is indeed the case and for p_n growing faster than $\sqrt{n}$ the density of states approaches a semicircle distribution. This also shows that the condition about the maximal edge degree in Theorem 4.6 is in a certain sense optimal. We start with a combinatorial lemma, whose proof is given in the appendix.

Lemma 4.7 (Asymptotics of intersections of growing sets). *Let a_n, b_n and c_n be three sequences taking values in $\{1,\ldots,n\}$.*

(i) Given any subsets $A_n \subset \{1,\ldots,n\}$ with a_n elements, the proportion of $B_n \subset \{1,\ldots,n\}$ with b_n elements that have a non-empty intersection with A_n goes to one if and only if $a_n b_n$ grows faster than n. More precisely it holds that

$$\lim_{n\to\infty} \frac{\binom{n}{b_n} - \binom{n-a_n}{b_n}}{\binom{n}{b_n}} = \begin{cases} 1 & \text{if } a_n b_n \gg n, \\ 0 & \text{if } a_n b_n \ll n, \\ 1-e^{-\lambda} & \text{if } \lim_{n\to\infty} \frac{a_n b_n}{n} = \lambda \in (0,\infty). \end{cases}$$

(ii) Given any subsets $A_n \subset \{1,\ldots,n\}$ with a_n elements, the proportion of $B_n \subset \{1,\ldots,n\}$ with b_n elements that share at least c_n elements with A_n goes to 1, i.e.

$$\lim_{n\to\infty} \frac{|\{\, B_n \subset \{1,\ldots,n\} \mid |B_n| = b_n, |A_n \cap B_n| \geq c_n \,\}|}{\binom{n}{b_n}} = 1,$$

provided $a_n b_n \gg n$ and $c_n \ll \frac{a_n b_n}{n}$.

Proof of Theorem 3.2. As already mentioned the first claim is a immediate consequence of Theorem 4.6 and the estimate from Lemma 4.7.

Now assume that p_n grows faster than $\sqrt{n}$. As before, we compute the moments and due to Lemma 4.1 again know that the odd moments vanish and for even moments we only have to consider those tuples of

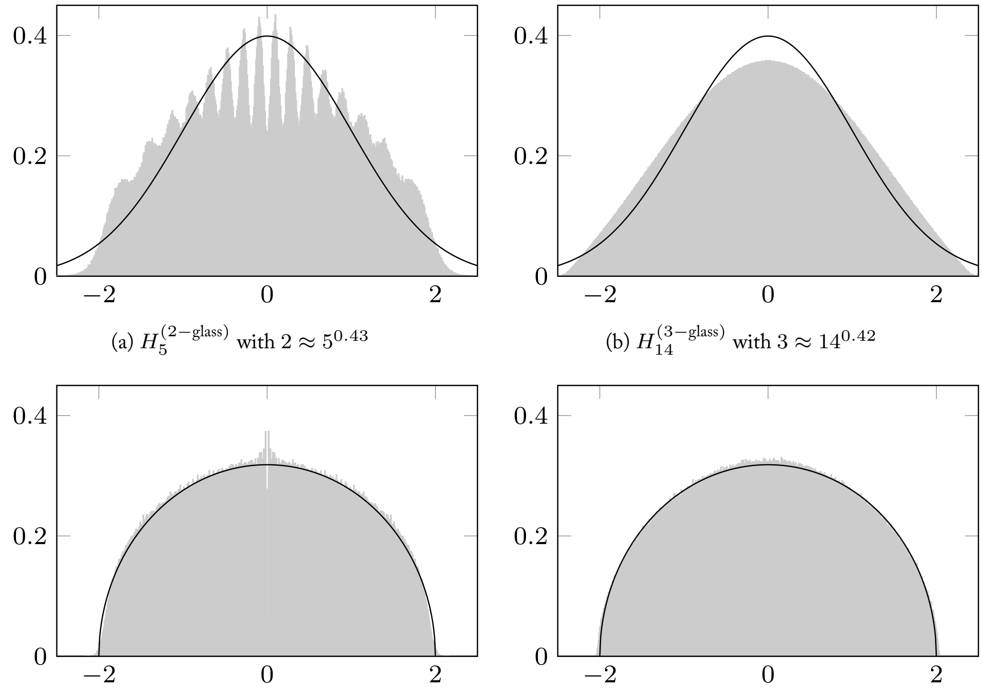
$$J_i \in I_n := \{\, (\mathbf{a}, (i_1 \ldots i_{p_n})) \mid \mathbf{a} \in \{1,2,3\}^{p_n}, 1 \leq i_1 < \cdots < i_{p_n} \leq n \,\}$$

which come in pairs of two. Using the already established short hand notation for the σ_{J_i} we have, for even k,

$$\overline{m_{k,n}} \approx 3^{-kp_n/2} \binom{n}{p_n}^{-k/2} \sum_{(J_1,\ldots,J_k)\in P_2(I_n^k)} 2^{-n} \operatorname{Tr} \sigma_{J_1} \ldots \sigma_{J_k}.$$

(here $\approx$ means *is equal in the limit* $n \to \infty$). We now rephrase condition $(J_1 \ldots J_k) \in P_2(I_n^k)$. The tuples can be thought of being constructed by first drawing $\frac{k}{2}$ distinct J_i from I_n and then assigning those $\frac{k}{2}$ J_i's to the tuples in a way that each J_i appears twice. By defining the family of (labelled) pair-partitions of the set $\{1,\ldots,k\}$ into $\frac{k}{2}$ labelled subsets with 2 elements each;

$$S_k := \left\{\, \pi : \{1,\ldots,k\} \to \{1,\ldots,k/2\} \;\middle|\; |\pi^{-1}(\{j\})| = 2 \text{ for all } 1 \leq j \leq k/2 \,\right\},$$



(a) $H_5^{(2-\text{glass})}$ with $2 \approx 5^{0.43}$

(b) $H_{14}^{(3-\text{glass})}$ with $3 \approx 14^{0.42}$

(c) $H_7^{(3-\text{glass})}$ with $3 \approx 7^{0.56}$

(d) $H_{12}^{(4-\text{glass})}$ with $4 \approx 12^{0.56}$

FIGURE 4.2: Histograms of the empirical eigenvalue distribution averaged over 1000 samples of p-spin glass Hamiltonians with $p \approx n^{0.42}$ and $p \approx n^{0.56}$ (grey) and the limiting density profile (black)

the sum then reads

$$\overline{m_{k,n}} \approx 3^{-kp_n/2} \binom{n}{p_n}^{-k/2} \sum_{\pi \in S_k} \sum_{\{J_1,\ldots,J_{k/2}\} \subset I_n}^{\star} 2^{-n} \operatorname{Tr} \sigma_{J_{\pi(1)}} \ldots \sigma_{J_{\pi(k)}}, \tag{4.9}$$

where $\sum^{\star}$ indicates that the elements $J_1, \ldots, J_{k/2}$ are distinct.

At this point it is useful to introduce the notion of non-crossing pair-partitions which often appear in random matrix theory. An element $\pi \in S_k$ shall be called *crossing* if there exists $1 \leq a < b < c < d \leq k$ such that $\pi(a) = \pi(c)$ and $\pi(b) = \pi(d)$, otherwise it is called *non-crossing*; the corresponding subsets of S_k are denoted by $S_k^{(c)}$ and $S_k^{(nc)}$. These notions emerge in this context since by Lemma 4.8 (a proof of which is given in the appendix) for a non-crossing $\pi \in S_k^{(nc)}$ the matrices $\sigma_{J_{\pi(j)}}$ in the trace in (4.9) can be reordered such that all appear as squares and therefore the normalised traces are all 1 independent of the J_l's.

Lemma 4.8 (Product of Pauli matrices ordered in pair-partitions). *Let k be even, $\pi \in S_k$ and define $I_n := \{ (\mathbf{a}, e) \mid e \in \Gamma_n, \mathbf{a} \in \{1,2,3\}^{|e|} \}$ for some hypergraph Γ_n.*

(i) If π is non-crossing, then
$$\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\dots\sigma^{(a_{\pi(k)})}=1$$
for all $1\le a_1,\dots,a_{k/2}\le 3$.

(ii) If π is crossing, there exist $1\le a_1,\dots a_{k/2}\le 3$ *such that*
$$\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\dots\sigma^{(a_{\pi(k)})}\neq 1.$$

(iii) If π is non-crossing, then
$$2^{-n}\operatorname{Tr}\sigma_{J_{\pi(1)}}\dots\sigma_{J_{\pi(k)}}=1$$
for all $J_1,\dots,J_{k/2}\in I_n$.

For the sum over the non-crossing pair-partitions $S_k^{(nc)}$ we thus find a contribution of
$$\begin{aligned}3^{-kp_n/2}\binom{n}{p_n}^{-k/2}\sum_{\pi\in S_k^{(nc)}}\sum_{\{J_1,\dots,J_{k/2}\}\subset I_n}^{\star}1&=\frac{|S_k^{(nc)}|\binom{|I_n|}{k/2}}{\binom{n}{p_n}^{k/2}3^{kp_n/2}}\approx\frac{|S_k^{(nc)}|}{(k/2)!}\\&=\frac{k!}{(k/2)!(k/2+1)!},\end{aligned}$$
where it was used that the number of non-crossing pair-partitions into unlabelled subsets are given by the Catalan numbers (see e.g. [2, Proposition 2.1.11]). It remains to show that the sum over the crossing pair-partitions gives no contribution. Since the total number of partitions is finite it suffices to show that
$$\lim_{n\to\infty}\frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{J_1,\dots,J_{k/2}\in I_n}2^{-n}\operatorname{Tr}\sigma_{J_{\pi(1)}}\dots\sigma_{J_{\pi(k)}}=0 \tag{4.10}$$
for each crossing $\pi\in S_k$. Notice that this summation is normalised, i.e. the combinatorial prefactor is exactly the number of terms in the sum.

Since π is assumed to be crossing there are $1\le a<b<c<d\le k$ such that $r:=\pi(a)=\pi(c)$ and $s:=\pi(b)=\pi(d)$. From Lemma 4.7 it follows that there exists a sequence $q_n\gg 1$ such that the proportion of pairs of subsets of $\{1,\dots,n\}$, with p_n elements each, that share at least q_n elements approaches 1 as $n\to\infty$. Applied to our normalised sum in (4.10), this means that we can restrict our attention to those terms for which e_r and e_s have at least q_n vertices in common. In this way we arrive at
$$\begin{aligned}&\frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{(\mathbf{a}_1,e_1),\dots,(\mathbf{a}_{k/2},e_{k/2})\in I_n}2^{-n}\operatorname{Tr}\sigma_{(\mathbf{a}_{\pi(1)},e_{\pi(1)})}\dots\sigma_{(\mathbf{a}_{\pi(k)},e_{\pi(k)})}\approx\\&\frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{\substack{e_1,\dots,e_{k/2}\in\Gamma_n\\|e_r\cap e_s|\ge q_n}}\sum_{\mathbf{a}_1,\dots,\mathbf{a}_{k/2}\in\{1,2,3\}^{p_n}}2^{-n}\operatorname{Tr}\sigma_{(\mathbf{a}_{\pi(1)},e_{\pi(1)})}\dots\sigma_{(\mathbf{a}_{\pi(k)},e_{\pi(k)})}\end{aligned}\tag{4.11}$$

for some $q_n \gg 1$ only depending on p_n.

For a hyperedge e, the hyperedge consisting of the first l vertices of e (with respect to the natural ordering) will be denoted by $e_{1:l}$. We introduce the shorthand notation

$$g := (e_r \cap e_s)_{1:q_n},$$

(we recall that hyperedges are subsets of the vertex set, thus set theoretical operations, such as $\in, \cup, \cap, \setminus$, are meaningful for them). With this notation we can factorize the inner sum from eq. (4.11) to get

$$\sum_{\mathbf{a}_1,\dots,\mathbf{a}_{k/2}\in\{1,2,3\}^{pn}} 2^{-n}\operatorname{Tr}\sigma_{(\mathbf{a}_{\pi(1)},e_{\pi(1)})}\cdots\sigma_{(\mathbf{a}_{\pi(k)},e_{\pi(k)})} =$$

$$\left(\sum_{\mathbf{a}_1\in\{1,2,3\}^{|e_1\cap g|}}\cdots\sum_{\mathbf{a}_{k/2}\in\{1,2,3\}^{|e_{k/2}\cap g|}} 2^{-n}\operatorname{Tr}\sigma_{(\mathbf{a}_{\pi(1)},e_{\pi(1)}\cap g)}\cdots\sigma_{(\mathbf{a}_{\pi(k)},e_{\pi(k)}\cap g)}\right)$$

$$\times\left(\sum_{\mathbf{a}_1\in\{1,2,3\}^{|e_1\setminus g|}}\cdots\sum_{\mathbf{a}_{k/2}\in\{1,2,3\}^{|e_{k/2}\setminus g|}} 2^{-n}\operatorname{Tr}\sigma_{(\mathbf{a}_{\pi(1)},e_{\pi(1)}\setminus g)}\cdots\sigma_{(\mathbf{a}_{\pi(k)},e_{\pi(k)}\setminus g)}\right)$$

where the second factor is bounded by $3^{\sum_{l=1}^{k/2}|e_l\setminus g|}$. We then further factorize the first factor to obtain

$$\prod_{j\in g}\sum_{\substack{a_l\in\{1,2,3\}\text{ if } j\in e_l\cap g\\ a_l=0 \text{ else}}}\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\cdots\sigma^{(a_{\pi(k)})}.$$

For any fixed $j\in g$, the number

$$m_j := |\{\, 1\le l\le k/2 \mid j\in e_l\cap g \,\}|$$

of l's such that $e_l \cap g$ contains the vertex j, is always between $2\le m_j\le k/2$ since at least r and s satisfy this condition. By ignoring the $a_l = 0$ factors and writing m for m_j, we see that we can rewrite

$$\sum_{\substack{a_l\in\{1,2,3\}\text{ if } j\in e_l\cap g\\ a_l=0 \text{ else}}}\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\cdots\sigma^{(a_{\pi(k)})} = \sum_{a_1,\dots,a_m=1}^{3}\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\tilde\pi(1)})}\cdots\sigma^{(a_{\tilde\pi(2m)})}$$

for some crossing $\tilde\pi\in S_{2m}$. According to part (ii) of Lemma 4.8, some (but not all) terms in this sum are equal to -1 and therefore there exists a (possibly) $\tilde\pi$ and m-dependent constant $C(\tilde\pi,m)<1$ such that

$$\left|\sum_{a_1,\dots,a_m=1}^{3}\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\tilde\pi(1)})}\cdots\sigma^{(a_{\tilde\pi(2m)})}\right| \le C(\tilde\pi,m)\cdot 3^m.$$

By setting the

$$C := \max_{2\le m\le k/2}\max_{\tilde\pi\in S_{2m}^{(c)}} C(\tilde\pi,m) < 1$$

to be the maximum of those constants, and recalling that $|g| = q_n$, we arrive at

$$\Big|\prod_{j\in g} \sum_{\substack{a_l\in\{1,2,3\} \text{ if } j\in e_l\cap g \\ a_l=0 \text{ else}}} \frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\ldots\sigma^{(a_{\pi(k)})}\Big|$$

$$\leq \prod_{j\in g}\left[C\cdot 3^{|\{\,1\leq l\leq k/2\,\mid\, j\in e_l\cap g\,\}|}\right] = C^{q_n}\cdot 3^{\sum_{l=1}^{k/2}|e_l\cap g|}.$$

After plugging in our estimates into eq. (4.11) we finally find

$$\begin{aligned}
&\Big|\frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{(\mathtt{a}_1,e_1),\ldots,(\mathtt{a}_{k/2},e_{k/2})\in I_n} 2^{-n}\operatorname{Tr}\sigma_{(\mathtt{a}_{\pi(1)},e_{\pi(1)})}\cdots\sigma_{(\mathtt{a}_{\pi(k)},e_{\pi(k)})}\Big| \\
&\leq \frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{\substack{e_1,\ldots,e_{k/2}\in\Gamma_n \\ |e_r\cap e_s|\geq q_n}} C^{q_n}3^{\sum_{l=1}^{k/2}(|e_l\cap g|+|e_l\setminus g|)} + \mathcal{O}(1) \qquad (4.12)\\
&= \frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{e_1,\ldots,e_{k/2}\in\Gamma_n} C^{q_n}3^{kp_n/2} + \mathcal{O}(1) = \frac{C^{q_n}}{(k/2)!} + \mathcal{O}(1) = \mathcal{O}(1)
\end{aligned}$$

as $n\to\infty$, proving that the contribution of any crossing partition vanishes.

We have now proved that the k-th moment of the limiting distribution is given by $\frac{k!}{(k/2)!(k/2+1)!}$ for even k and 0 for odd k. We know from Lemma 2.10 that these are the moments of the semicircular distribution. An application of Proposition 2.5 then gives the claim after a variance bound analogous to the proof of Theorem 3.1.

We now turn to part (iii) of Theorem 3.2, i.e. the case where $\lim_{n\to\infty}\frac{p_n}{\sqrt{n}} = \lambda\in(0,\infty)$. By Lemma 4.1 the odd moments vanish also in this case. For even k an explicit formula for the k-th moment can be derived as follows. For a given partition $\pi\in S_k$ we define the *number of crossings* $\kappa(\pi)$ to be the number of subsets $\{r,s\}\subset\{1,\ldots,k/2\}$ such that for some $1\leq a<b<c<d\leq k$ we have that $\pi(a)=\pi(c)=r$ and $\pi(b)=\pi(d)=s$. We claim that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{3^{-kp_n/2}}{(k/2)!}\binom{n}{p_n}^{-k/2}\sum_{(\mathtt{a}_1,e_1),\ldots,(\mathtt{a}_{k/2},e_{k/2})\in I_n} 2^{-n}\operatorname{Tr}\sigma_{(\mathtt{a}_{\pi(1)},e_{\pi(1)})}\cdots\sigma_{(\mathtt{a}_{\pi(k)},e_{\pi(k)})} \\
&\quad = \frac{(e^{-4\lambda^2/3})^{\kappa(\pi)}}{(k/2)!} \qquad (4.13)
\end{aligned}$$

holds for all partitions π. If $\{r_1,s_2\},\ldots,\{r_{\kappa(\pi)},s_{\kappa(\pi)}\}$ are the crossings of π, by Lemma 4.7 the numbers of vertices in the intersections $e_{r_1}\cap e_{s_1},\ldots,e_{r_{\kappa(\pi)}}\cap e_{s_{\kappa(\pi)}}$ are approximately independently Poisson-λ^2 distributed. It furthermore follows from Lemma 4.7 that in the limit we can restrict our attention to those edges where the sets $e_{r_1}\cap e_{s_1},\ldots,e_{r_{\kappa(\pi)}}\cap e_{s_{\kappa(\pi)}}$ are mutually disjoint. Since the normalised trace of the Hamiltonian acting on a qubit within such a twofold crossing is given by

$$3^{-2}\sum_{a,b=1}^{3}\frac{1}{2}\operatorname{Tr}\sigma^{(a)}\sigma^{(b)}\sigma^{(a)}\sigma^{(b)} = -\frac{1}{3}$$

whereas the normalised trace is 1 for those qubits not involved in any crossings we find that the lhs. of eq. (4.13) can be asymptotically rewritten as

$$\frac{1}{(k/2)!}\sum_{m_1=0}^{\infty}\cdots\sum_{m_{\kappa(\pi)}=0}^{\infty}\frac{\lambda^{2m_1+\cdots+2m_{\kappa(\pi)}}}{m_1!\ldots m_{\kappa(\pi)}!}e^{-\kappa(\pi)\lambda^2}(-1/3)^{m_1+\cdots+m_{\kappa(\pi)}}=\frac{(e^{-4\lambda^2/3})^{\kappa(\pi)}}{(k/2)!},$$

just as claimed. The k-th limiting moment, i.e. the normalised trace of $H_n^{(p_n-\text{glass})}$ in the limit $n\to\infty$, is thus given by

$$m_k(\lambda) := \frac{1}{(k/2)!}\sum_{\pi\in S_k}(e^{-4\lambda^2/3})^{\kappa(\pi)} = \sum_{\pi\in\tilde{S}_k}(e^{-4\lambda^2/3})^{\kappa(\pi)},$$

where $\tilde{S}_k$ denotes the set of unlabelled partitions.

These moments uniquely correspond to the distribution given in eq. (3.2), as known from the theory of the q-Hermite polynomials, see [9, eqs. (3.2) and (3.8)]. For the convenience of the reader we collect some further properties of this distribution in Proposition 4.10. □

Remark 4.9. *The proof of the Theorem 3.2 also works for general spin-s systems (instead of spin-$1/2$) with small changes. Mainly, part (ii) from Lemma 4.8 has to be replaced by a corresponding Lemma for spin-s which can be proved along the lines of the original proof. This replacement (possibly) changes the value of the C-constant from eq.* (4.12) *which is irrelevant for the result since $C<1$ is sufficient for the convergence against zero. In part (iii) the proof also applies to general spin-s systems, except that $\mathrm{e}^{-4\lambda^2/3}$ has to be replaced by $\mathrm{e}^{-4s\lambda^2/(2s+1)}$. Theorems 3.1 and 4.6 also carry over to spin-s since for the important bounds only degree properties of the graph and no specifics of the spin-$1/2$ system were used.*

Proposition 4.10. *Suppose that $\lim_{n\to\infty}\frac{p_n}{\sqrt{n}}=\lambda\in(0,\infty)$, then $\overline{m_{k,n}}(\lambda)$, the normalised trace of the k-th power of $H_n^{(p_n)-glass}$, in the limit $n\to\infty$ takes the form*

$$m_k(\lambda) := \lim_{n\to\infty}\overline{m_{k,n}}(\lambda) = 0$$

if k is odd and

$$\begin{aligned} m_k(\lambda) &:= \lim_{n\to\infty}\overline{m_{k,n}}(\lambda) = \frac{1}{(1-e^{-4\lambda^2/3})^{k/2}}\sum_{j=-k/2}^{k/2}(-1)^j e^{-2\lambda^2\cdot j(j-1)/3}\binom{k}{k/2+j} \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}\left(\frac{2\sinh^2(ix\sqrt{\lambda^2/3}+\lambda^2/3)}{e^{-2\lambda^2/3}\sinh(-2\lambda^2/3)}\right)^{k/2}\mathrm{d}x \end{aligned} \tag{4.14}$$

if k is even. For any fixed even k it furthermore holds that $m_k(\lambda)$ is monotonically decreasing in λ and satisfies

$$\lim_{\lambda\to 0}m_k(\lambda)=(k-1)!! \quad \text{and} \quad \lim_{\lambda\to\infty}m_k(\lambda)=\frac{k!}{(k/2)!(k/2+1)!} \tag{4.15}$$

in agreement with the statements of Theorem 3.2. The corresponding limiting probability distribution μ_λ has the compactly supported density function given in eq. (3.2) *which converges pointwise to the* *semicircular density function when* $\lambda \to \infty$ *and to the density function of the normal distribution when* $\lambda \to 0$. *Furthermore for any fixed* λ *the density function* ρ_λ *has square root singularities in* $\pm 2/\sqrt{1 - e^{-4\lambda^2/3}}$.

Proof. Recall from the proof of Theorem 3.2 that the moments are given by

$$m_k(\lambda) = \sum_{\pi \in \tilde{S}_k} (e^{-4\lambda^2/3})^{\kappa(\pi)}$$

for even k and $m_k(\lambda) = 0$ for odd k. According to an exact formula by Touchard and Riordan and its integral representation (see [6, eqs. (5) and (7) on page 197]) the even moments are given by the formulas in eq. (4.14). The monotone decrease follows by computing the derivative in λ and the claimed limits in (4.15) are also direct computations. The claimed limiting behaviour of the density function as $\lambda \to 0$ and $\lambda \to \infty$ also follows from the discussion in Section 2 of [9]. The $k = 0$ term from eq. (3.2) is responsible for the square root singularity near the edges $\pm 2/\sqrt{1 - e^{-4\lambda^2/3}}$. □

Appendix A

Appendix: Proofs of some Technical Lemmas

Proof of Lemma 4.3.

(i),(ii) Trivial calculations.

(iii) Two Pauli matrices $\sigma^{(a)}, \sigma^{(b)}$ anti-commute if $a \neq b$. We can therefore up to a factor of ± 1 reorder the arguments of $\sigma(a_1, \ldots, a_k)$ in such a way that $a_1 = \cdots = a_{n_1} = 0$, $a_{n_1+1} = \cdots = a_{n_1+n_2} = 1$, $a_{n_1+n_2+1} = \cdots = a_{n_1+n_2+n_3} = 2$, $a_{n_1+n_2+n_3+1} = \cdots = a_k = 3$ where $n_0 + n_1 + n_2 + n_3 = k$ denote the numbers of 0's, 1's, 2's and 3's. Using that the square of any Pauli matrix is the identity we then find

$$\sigma(a_1, \ldots, a_k) = \pm\sigma(\pi_{n_1}, 2\pi_{n_2}, 3\pi_{n_3})$$

where π_n is the parity function i.e. $\pi_n = 0$ if n is even and $\pi_n = 1$ if n is odd.

(iv) Using the anti-commutation relation $\sigma^{(a)}\sigma^{(b)} = 2\delta_{ab}1_2 - \sigma^{(a)}\sigma^{(b)}$ we compute inductively

$$\begin{aligned}\sigma(a_1,a_2, \ldots, a_k) &= 2\delta_{a_1,a_2}\sigma(a_3, \ldots, a_k) - \sigma(a_2, a_1, a_3, \ldots, a_k) \\ &= 2\delta_{a_1a_2}\sigma(a_3, \ldots, a_k) - 2\delta_{a_1a_3}\sigma(a_2, a_4, \ldots, a_k) + \sigma(a_2, a_3, a_1, a_4, \ldots, a_k) \\ &= \cdots = 2\sum_{j=2}^{k}(-1)^j\delta_{a_1a_j}\sigma(a_2, \ldots, \widehat{a_j}, \ldots, a_k) - \sigma(a_2, \ldots, a_k, a_1)\end{aligned}$$

from which the claim follows by the cyclicity of the trace. □

Proof of Lemma 4.7.

(i) We can safely assume that eventually $a_n + b_n \leq n$ since the assertion is trivial otherwise and then compute

$$\binom{n-a_n}{b_n}\Big/\binom{n}{b_n} = \frac{(n-a_n)(n-a_n-1)\ldots(n-a_n-b_n+1)}{n(n-1)\ldots(n-b_n+1)} = \prod_{k=0}^{b_n-1}\left(1 - \frac{a_n}{n-k}\right)$$

where all factors are non-negative. We continue with the obvious bounds

$$\left(1-\frac{a_n}{n}\right)^{b_n} \leq \prod_{k=0}^{b_n-1}\left(1-\frac{a_n}{n-k}\right) \leq \left(1-\frac{a_n}{n-b_n+1}\right)^{b_n}$$

and after applying a logarithm arrive at

$$\begin{aligned}\exp\left(-\frac{a_n b_n}{n}\right) \approx \exp\left(b_n \log\left(1-\frac{a_n}{n}\right)\right) \leq \binom{n-a_n}{b_n}\Big/\binom{n}{b_n}\\ \leq \exp\left(b_n \log\left(1-\frac{a_n}{n-b_n+1}\right)\right) \approx \exp\left(-\frac{a_n b_n}{n-b_n+1}\right)\end{aligned}$$

from which the claim follows immediately.

(ii) For any fixed $k \geq 0$ the proportion of sets B_n of size b_n that share exactly k elements with A_n is given by

$$\binom{a_n}{k}\binom{n-a_n}{b_n-k}\Big/\binom{n}{b_n}$$

i.e. the number of elements in the intersection is hypergeometrically distributed with parameters (n, a_n, b_n) and therefore has a mean of $\frac{a_n b_n}{n}$ and a variance of

$$\frac{a_n b_n}{n}\frac{n-b_n}{n}\frac{n-a_n}{n-1}$$

which shows that for c_n growing slower than $\frac{a_n b_n}{n}$ the proportion of B_n's that share at least c_n elements with A_n converges to 1. □

Proof of Lemma 4.8.

(i) Suppose that π is non-crossing. Let $i < j$ be those indices for which $\pi(i) = \pi(j) = 1$. Since the partition is non-crossing in the tuple $(\pi(1), \ldots, \pi(k))$ there are either zero or two l indices between $\pi(i)$ and $\pi(j)$ for all $1 < l \leq k/2$. Recall that $\sigma^{(a_1)}$ anti-commutes with $\sigma^{(a_l)}$ if $a_l \neq a_1$ and commutes otherwise. Hence we can freely permute $\sigma^{(a_{\pi(j)})}$ to the left next to $\sigma^{(a_{\pi(j)})}$ and then the claim follows inductively since $(\sigma^{(a_1)})^2 = 1_2$ and we therefore proved the claim assuming the result for $k-2$. For $k = 2$ the assertion is trivially true.

(ii) Suppose now that π is crossing, i.e. there exist $a < b < c < d$ such that $r := \pi(a) = \pi(c)$ and $s := \pi(b) = \pi(d)$. Then by setting $a_l = 1$ for $l \notin \{r, s\}$ the expression simplifies to

$$\frac{1}{2}\operatorname{Tr}\sigma^{(a_{\pi(1)})}\ldots\sigma^{(a_{\pi(k)})} = \sigma(a_{\pi(1)}, \ldots, a_{\pi(k)}) = \sigma(\alpha, a_r, \beta, a_s, \gamma, a_r, \delta, a_s, \epsilon)$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in \{0, 1\}$. Using the anti-commutation relations we then find that for $a_r, a_s \in \{2, 3\}$ it holds that

$$\begin{aligned}\sigma(a_{\pi(1)}, \ldots, a_{\pi(k)}) &= (-1)^{\gamma}\sigma(\alpha, a_r, \beta, a_s, a_r, \gamma, \delta, a_s, \epsilon)\\ &= (-1)^{\gamma+1-\delta_{a_r,a_s}}\sigma(\alpha, a_r, \beta, a_r, a_s, \gamma, \delta, a_s, \epsilon)\\ &= \cdots = (-1)^{2\gamma+\beta+\delta+1-\delta_{a_r,a_s}}\sigma(\alpha, a_r, a_r, \beta, \gamma, a_s, a_s, \delta, \epsilon)\\ &= (-1)^{\beta+\delta+1-\delta_{a_r,a_s}}\sigma(\alpha, \beta, \gamma, \delta, \epsilon).\end{aligned}$$

i.e. the result changes sign depending on whether $a_r = 2$ and $a_s = 3$ or $a_r = a_s = 2$ and in particular cannot be equal to 1 for all choices of $a_1, \ldots, a_{k/2}$.

(iii) This is an immediate consequence of applying part (i) to all components of the tensor product separately and using that the trace of a tensor product factorizes. $\square$

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