

Problem Sheet I

Introduction to Random Matrices, IST Austria

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1 Derivation of the Sine Kernel for GUE matrix ensemble

The goal of this exercise is to prove that the k -point correlation function for the GUE matrix ensemble indeed follows the sine kernel. The symmetrized probability density of the eigenvalues $\lambda_1, \dots, \lambda_N$ can be computed explicitly and is given by

$$p_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \prod_{i < j} (\lambda_j - \lambda_i)^2 \exp\left(-\frac{1}{2} \sum_i \lambda_i^2\right) \quad (1)$$

where Z_N are appropriate normalization constants.

Problem 1 (Properties of Hermite polynomials). For $k \geq 0$ define the k -th Hermite polynomial by

$$H_k(x) := (-1)^k \exp(x^2/2) \frac{d^k}{dx^k} \exp(-x^2/2).$$

(i) Check that H_k is a k -th order polynomial and that the leading coefficient of H_k is 1.

(ii) [OPTIONAL]¹ Check that the Hermite polynomials are orthogonal with respect to the weight $\exp(-x^2/2)$, i.e., that

$$\int_{\mathbb{R}} H_k(x) H_l(x) \exp(-x^2/2) dx = \delta_{lk} \sqrt{2\pi} k!$$

and conclude that the functions $\psi_k(x) := (\sqrt{2\pi} k!)^{-1/2} H_k(x) \exp(-x^2/4)$ for $k \geq 0$ form an orthonormal set in $L^2(\mathbb{R})$.

(iii) [OPTIONAL] Show that the Hermite polynomials H_k satisfy the recurrence relation

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x)$$

and conclude that

$$x\psi_k(x) = \sqrt{k+1}\psi_{k+1}(x) + \sqrt{k}\psi_{k-1}(x) \quad (2)$$

for each $k \geq 1$.

Problem 2 (Computation of k -point correlation functions via Hermite polynomials).

(i) Use part (i) of Problem 1 to prove that

$$\prod_{i < j} (\lambda_j - \lambda_i) = \det(H_{j-1}(\lambda_i))_{i,j=1}^N$$

and conclude that

$$p_N(\lambda_1, \dots, \lambda_N) = C_N \det(K_N(\lambda_i, \lambda_j))_{i,j=1}^N, \quad K_N(x, y) := \sum_{k=0}^{N-1} \psi_k(x) \psi_k(y)$$

for some constants C_N . **Hint.** For the first part it might be helpful to realize that $\prod(\lambda_j - \lambda_i)$ is a Vandermonde determinant.

¹Problems marked OPTIONAL require calculations with no direct relevance for the course and therefore do not have to be handed in.

(ii) Use part (ii) of Problem 1 to show that

$$\int_{\mathbb{R}} K_N(x, y) K_N(y, z) dy = K_N(x, z), \quad \int_{\mathbb{R}} K_N(x, x) dx = N \quad (3)$$

and use (3) to prove that the k -point correlation functions are given by

$$p_N^{(k)}(\lambda_1, \dots, \lambda_k) = C_{N,k} \det(K_N(\lambda_i, \lambda_j))_{i,j=1}^k$$

for some constants $C_{N,k}$. **Hint.** For the second assertion try integrating the variables one by one and use Laplace expansion of the determinant.

(iii) Use (2) and a telescoping sum argument to prove that

$$K_N(x, y) = \sqrt{N} \left(\frac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{x - y} \right). \quad (4)$$

(iv) Finally use the Plancherel-Rotach asymptotics [1, Theorem 8.22.9]

$$\psi_{2k}(x) \approx \frac{(-1)^k}{N^{1/4}\sqrt{\pi}} \cos(\sqrt{N}x), \quad \psi_{2k+1}(x) \approx \frac{(-1)^k}{N^{1/4}\sqrt{\pi}} \sin(\sqrt{N}x)$$

to conclude the sine kernel asymptotics

$$K_N(x, y) \approx \frac{\sin(\sqrt{N}(x - y))}{\pi(x - y)}$$

from (4).

2 Differences in level repulsion for real symmetric and complex Hermitian matrix ensembles

The goal of this exercise is to demonstrate that the level repulsion asymptotics can depend on the symmetry class of the matrix ensemble, i.e., whether the matrices are real symmetric or complex Hermitian.

Problem 3. Let

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

be a 2×2 real symmetric or complex Hermitian random matrix with independent continuously distributed entries. Denote the real eigenvalues of H by λ_1, λ_2 .

(i) If H is real symmetric, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^2$$

in the sense that the probability scales like ϵ^2 for small ϵ .

(ii) If H is complex Hermitian and the real and imaginary part of h_{12} are independent and continuously distributed, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^3.$$

References

¹G. Szegő, *Orthogonal polynomials*, Fourth, American Mathematical Society, Colloquium Publications, Vol. XXIII (American Mathematical Society, Providence, R.I., 1975), pp. xiii+432.