Solution Sheet I

Introduction to Random Matrices, IST Austria
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László Erdős, Dominik Schröder

1 Derivation of the Sine Kernel for GUE matrix ensemble

The goal of this exercise is to prove that the k-point correlation function for the GUE matrix ensemble indeed follows the sine kernel. The symmetrized probability density of the eigenvalues $\lambda_1, \dots, \lambda_N$ can be computed explicitly and is given by

$$p_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \prod_{i<j} (\lambda_j - \lambda_i)^2 \exp\Big(-\frac{1}{2} \sum_i \lambda_i^2 \Big) \tag{1}$$

where Z_N are appropriate normalization constants.

Problem 1 (Properties of Hermite polynomials)**.** *For $k \geq 0$ define the k-th Hermite polynomial by*

$$H_k(x) := (-1)^k \exp(x^2/2) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp(-x^2/2).$$

(i) Check that H_k is a k-th order polynomial and that the leading coefficient of H_k is 1.

(ii) [Optional][1] Check that the Hermite polynomials are orthogonal with respect to the weight $\exp(-x^2/2)$, i.e., that

$$\int_{\mathbb{R}} H_k(x) H_l(x) \exp(-x^2/2) \, \mathrm{d}x = \delta_{lk} \sqrt{2\pi} k!$$

and conclude that the functions $\psi_k(x) := (\sqrt{2\pi} k!)^{-1/2} H_k(x) \exp(-x^2/4)$ for $k \geq 0$ form an orthonormal set in $L^2(\mathbb{R})$.

(iii) [Optional] Show that the Hermite polynomials H_k satisfy the recurrence relation

$$H_{k+1}(x) = x H_k(x) - k H_{k-1}(x)$$

and conclude that

$$x \psi_k(x) = \sqrt{k+1} \psi_{k+1}(x) + \sqrt{k} \psi_{k-1}(x) \tag{2}$$

for each $k \geq 1$.

Solution.

(i) This is an easy induction because $(\mathrm{d}/\mathrm{d}x)(-x^2/2) = -x$ and the leading order term x^k can only come from k times differentiating $\exp(-x^2/2)$ as all other terms are lower order.

(ii) Integration by parts gives

$$\int H_k(x) H_l(x) \exp(-x^2/2) \, \mathrm{d}x = (-1)^l \int H_k(x) \frac{\mathrm{d}^l}{\mathrm{d}x^l} \exp(-x^2/2) \, \mathrm{d}x = \int \Big[\frac{\mathrm{d}^l}{\mathrm{d}x^l} H_k(x) \Big] \exp(-x^2/2) \, \mathrm{d}x. \tag{3}$$

If $l \neq k$, assume wlog. that $l > k$ and since H_k is a polynomial of degree k the rhs. of (3) vanishes. On the other hand if $k = l$, then we know that $(\mathrm{d}^k/\mathrm{d}x^k) H_k(x) = k!$ from part (i) and the claim follows from the standard Gaussian integral.

[1] Problems marked optional require calculations with no direct relevance for the course and therefore do not have to be handed in.

(iii) Denote the L^2-inner product weighted by $w(x) = \exp(-x^2/2)$ by $\langle\cdot,\cdot\rangle_w$. By linear combination of part (ii) it follows that $\langle p, H_l\rangle_w = 0$ for any polynomial p of degree at most $l-1$. We proceed via induction on k. Since $xH_k(x)$ is a polynomial of degree $k+1$ and $\langle xH_k, H_l\rangle_w = \langle H_k, xH_l\rangle_w$ it follows that

$$xH_k(x) = \sum_{l=k-1}^{k+1} \frac{\langle xH_k, H_l\rangle_w}{\langle H_l, H_l\rangle_w} H_l(x). \tag{4}$$

For $l = k$ the rhs. of (4) vanishes since $\langle xH_k, H_k\rangle_w = \langle x, H_k^2\rangle_w$ and both w, H_k are even functions while x is odd. For $l = k+1$ it follows that $\langle xH_k, H_{k+1}\rangle_w = \langle H_{k+1}, H_{k+1}\rangle_w$ from the fact that the leading coefficient of H_k is 1. Finally, for $l = k-1$ we can use the induction hypothesis to compute

$$\frac{\langle xH_k, H_{k-1}\rangle_w}{\langle H_{k-1}, H_{k-1}\rangle_w} = \frac{\langle H_k, xH_{k-1}\rangle_w}{\langle H_{k-1}, H_{k-1}\rangle_w} = \frac{\langle H_k, H_k + (k-1)H_{k-2}\rangle_w}{\langle H_{k-1}, H_{k-1}\rangle_w} = k$$

and the first claim follows from (4). The second claim follows directly from the first claim and the definition of ψ_k.

□

Problem 2 (Computation of k-point correlation functions via Hermite polynomials)**.**

(i) Use part (i) of Problem 1 to prove that

$$\prod_{i<j}(\lambda_j - \lambda_i) = \det(H_{j-1}(\lambda_i))_{i,j=1}^N$$

and conclude that

$$p_N(\lambda_1,\dots,\lambda_N) = C_N \det(K_N(\lambda_i,\lambda_j))_{i,j=1}^N, \qquad K_N(x,y) := \sum_{k=0}^{N-1}\psi_k(x)\psi_k(y)$$

*for some constants C_N. **Hint.** For the first part it might be helpful the realize that $\prod(\lambda_j - \lambda_i)$ is a* Vandermonde determinant*.*

(ii) Use part (ii) of Problem 1 to show that

$$\int_{\mathbb{R}} K_N(x,y)K_N(y,z)\,\mathrm{d}y = K_N(x,z), \qquad \int_{\mathbb{R}} K_N(x,x)\,\mathrm{d}x = N \tag{5}$$

and use (5) *to prove that the k-point correlation functions are given by*

$$p_N^{(k)}(\lambda_1,\dots,\lambda_k) = C_{N,k}\det(K_N(\lambda_i,\lambda_j))_{i,j=1}^k$$

*for some constants $C_{N,k}$. **Hint.** For the second assertion try integrating the variables one by one and use Laplace expansion of the determinant.*

(iii) Use (2) *and a telescoping sum argument to prove that*

$$K_N(x,y) = \sqrt{N}\left(\frac{\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)}{x-y}\right). \tag{6}$$

(iv) Finally use the Plancherel-Rotach asymptotics *[1, Theorem 8.22.9] for $|2m-N| \le C$*

$$\psi_{2k}(x) \approx \frac{(-1)^k}{N^{1/4}\sqrt{\pi}}\cos(\sqrt{N}x), \qquad \psi_{2k+1}(x) \approx \frac{(-1)^k}{N^{1/4}\sqrt{\pi}}\sin(\sqrt{N}x)$$

to conclude the sine kernel asymptotics

$$K_N(x,y) \approx \frac{\sin\left(\sqrt{N}(x-y)\right)}{\pi(x-y)}$$

from (6)*.*

Solution.

(i) We realize that $\prod_{i<j}(\lambda_j - \lambda_i) = \det V$ is the determinant of the Vandermonde matrix

$$V \coloneqq \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^{N-1} \end{pmatrix}.$$

The first column of V is given by $(1, \dots, 1) = (H_0(\lambda_1), \dots, H_0(\lambda_N))$. Now we can subtract a multiple of the first column from the second column (without changing the determinant) to obtain $(H_1(\lambda_1), \dots, H_1(\lambda_N))$ in the second column and we continue in the same manner with all columns only using that the leading term of $H_k(\lambda)$ is given by λ^k. Thus we find that

$$\prod_{i<j}(\lambda_j - \lambda_i) = \det V = \det \begin{pmatrix} H_0(\lambda_1) & H_1(\lambda_1) & \dots & H_{N-1}(\lambda_1) \\ H_0(\lambda_2) & H_1(\lambda_2) & \dots & H_{N-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(\lambda_N) & H_1(\lambda_1) & \dots & H_{N-1}(\lambda_N) \end{pmatrix},$$

just as claimed. Thus we find that

$$p_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N}\Big[\det(H_{j-1}(\lambda_i))_{i,j=1}^N\Big]^2 \prod_i \exp(-\lambda_i^2/2) = \frac{1}{Z_N'}\Big[\det(\psi_{j-1}(\lambda_i))_{i,j=1}^N\Big]^2 = \frac{1}{Z_N'}\det(K_N(\lambda_i, \lambda_j))_{i,j=1}^N,$$

where in the last step we used that $(\det A)^2 = \det(AA^t)$.

(ii) It follows from the orthonormality of ψ_k that

$$\int K_N(x,y)K_N(y,z)\,\mathrm{d}y = \sum_{k,j=0}^{N-1} \langle \psi_k, \psi_j \rangle\, \psi_k(x)\psi_j(z) = K_N(x,z), \qquad \int K_N(x,x)\,\mathrm{d}x = \sum_{k=0}^{N-1} \langle \psi_k, \psi_k \rangle = N.$$

We now claim that

$$\int \det(K_N(\lambda_i, \lambda_j))_{i,j=1}^{k+1}\,\mathrm{d}\lambda_{k+1} = (N-k)\det(K_N(\lambda_i, \lambda_j))_{i,j=1}^{k} \tag{7}$$

holds for all $k \geq 0$ with $k = 0$ being trivial. For $k \geq 1$ we use a Laplace expansion of the determinant after the last row and it follows that

$$\det(K_N(\lambda_i, \lambda_j))_{i,j=1}^{k+1} = K_N(\lambda_{k+1}, \lambda_{k+1})\det(K_N(\lambda_i, \lambda_j))_{i,j=1}^{k} + \sum_{l=1}^{k}(-1)^{k+1+l}K_N(\lambda_l, \lambda_{k+1})\det(K_N(\lambda_i, \lambda_j))_{i\in[k]}^{j\in[k+1]\setminus l}.$$

Integrating the first term gives $N\det(K_N(\lambda_i, \lambda_j))_{i,j=1}^{k}$ while integrating the $j = k+1$ column of the matrix in the second term together with the $(-1)^{k+1-l}K_N(\lambda_l, \lambda_{k+1})$ prefactor produces the column $(-1)^{k+1-l}(K_N(\lambda_i, \lambda_l))_{i\in[k]}$. Thus after column exchange each of the second terms in the Laplace expansion equals $-\det(K_N(\lambda_i, \lambda_j))_{i,j\in[k]}$, proving (7). By applying (7) repeatedly the claim follows.

(iii) We have to prove that

$$\sum_{k=1}^{N-1} x\psi_k(x)\psi_k(y) - \sum_{k=1}^{N-1} \psi_k(x)y\psi_k(y) = \sqrt{N}\Big[\psi_N(x)\psi_{N-1}(y) - \psi_N(y)\psi_{N-1}(x)\Big]$$

which follows from applying (2) to $x\psi_k(x)$ and $y\psi_k(y)$ and noting that all but two terms cancel each other.

(iv) This follows directly from the trigonometric identity $\sin(a-b) = \sin a \cos b - \cos a \sin b$.

$\square$

2 Differences in level repulsion for real symmetric and complex Hermitian matrix ensembles

The goal of this exercise is to demonstrate that the level repulsion asymptotics can depend on the symmetry class of the matrix ensemble, i.e., whether the matrices are real symmetric or complex Hermitian.

Problem 3. *Let*

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

be a 2×2 real symmetric or complex Hermitian random matrix with independent continuously distributed entries. Denote the real eigenvalues of H by λ_1, λ_2.

(i) If H is real symmetric, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^2$$

in the sense that the probability scales like ϵ^2 for small ϵ.

(ii) If H is complex Hermitian and the real and imaginary part of h_{12} are independent and continuously distributed, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^3.$$

Solution. For both real symmetric and complex Hermitian matrices an elementary computation shows that

$$|\lambda_1 - \lambda_2| = \sqrt{|h_{12}|^2 + (h_{11} - h_{22})^2}.$$

(i) If h_{12} is real then $|\lambda_1 - \lambda_2|$ is of order ϵ if and only if both h_{12} and $h_{11} - h_{22}$ are of order ϵ. For continuous independent distributions this is the case with a probability of ϵ^2.

(ii) On the other hand if h_{12} is complex, then $|\lambda_1 - \lambda_2|$ is of order ϵ if and only if $\Re h_{12}$, $\Im h_{12}$ and $h_{11} - h_{22}$ are all of order ϵ.

□

References

[1] G. Szegő, *Orthogonal polynomials*, Fourth, American Mathematical Society, Colloquium Publications, Vol. XXIII (American Mathematical Society, Providence, R.I., 1975), pp. xiii+432.