Jacobs University | November 5, 2020
Fall 2020

Stochastic Methods + Lab

Assignment Sheet 9

Due on November 12, 2020

Note: Please use one jupyter notebook for the homework submission.

Problem 1 [20 points]

Suppose S_i for $i = 0, \dots, N$ denotes time series data which we believe behaves like geometric Brownian motion with parameters μ and σ. Then estimates $\hat{\mu}$ and $\hat{\sigma}$ for μ and σ can be obtained by considering the log-returns

$$r_i = \ln S_{i+1} - \ln S_i \quad \text{for} \quad i = 0, \dots, N-1,$$

as discussed in class. *For the following exercises no loops are allowed, please implement all operations vectorized.*

(a) [6 points] Generate a sample geometric Brownian path on the time interval $[0, 1]$ with fixed $\mu = 0.3$ and $\sigma = 0.5$, where the number of sub-intervals is $N = 2^k$.

Estimate $\hat{\sigma}$ and $\hat{\mu}$ on a coarsened data set where you use only every 2^i-th data point, where $i = 0, \dots, k$.

Plot $\hat{\sigma}$ and $\hat{\mu}$ vs. the log of the number of sample points (`semilogx`). Do the estimated values converge to the true values of the model?

(b) [4 points] Now repeat the computation on a large ensemble of geometric Brownian paths and estimate the mean and the standard deviation of the estimate for σ and μ as a function of the number of sub-intervals N using a doubly logarithmic graph.

Note that it is known that the variance of the estimate for σ is approximately

$$\mathrm{Var}[\hat{\sigma}] = \frac{\mathbb{E}(\hat{\sigma})^2}{2N}.$$

Does your statistics reproduce this result?

How does the variance of the estimate for μ behave as N increases?

(c) [4 points] Use the single geometric Brownian path from Question (a), estimate its parameters, and use the estimated values $\hat{\mu}$ and $\hat{\sigma}$ to generate an ensemble of geometric Brownian paths as in Question (b). Plot a histogram (`hist`) of the estimates for μ and σ from this ensemble, compute the standard deviation, and visualize the standard deviation and the true value of the original model in this histogram.

(This procedure is a simple example of so-called *parametric bootstrapping.*)

(d) [2 points] How does the result of Question (a) change if

   (1) you add Gaussian noise to the geometric Brownian motion;

   (2) you add a high frequency periodic perturbation?

(e) [2 points] Perform a QQ-plot vs. the normal distribution for the distribution of the log-returns, and the distribution of the two noisy log-returns from Question (d), all into one graph. Briefly discuss the result.

(f) [2 points] Plot the autocorrelation function for the time series of log-returns, and the two noisy versions, all into one graph. Briefly discuss the result.