AMM Problem 12372

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2023-06-03

Exercise 1

For $\alpha > 0$ evaluate
$$\int_0^1 \frac{\ln|x^\alpha - (1-x)^\alpha|}{x}\mathrm{d}x$$

1 Numerical work

```
gp("default(realprecision, 100)")
gp("t(x,v)=if(abs(x-1/2)>2^-precision(x),x^v - (1 - x)^v,4*v*(x-1/2))")
gp("f(v) = intnum(x=0,1/2, log(abs(t(x,v)))/x)+intnum(x=1/2,1,
↪  log(abs(t(x,v)))/x)")
gp("f(sqrt(2)-1) + (Pi^2 + 3*sqrt(2)*Pi^2)/12")
gp("n(a)=f(sqrt(a^2+1)-a)/zeta(2)")
gp("algdep(-n(7), 3, flag=10)")
gp("d(a)= f(a) + zeta(2)*(1/a + a/2)")
gp("d(21)")
```

```
0
(x,v)->if(abs(x-1/2)>2^-precision(x),x^v-(1-x)^v,4*v*(x-1/2))
(v)->intnum(x=0,1/2,log(abs(t(x,v)))/x)+intnum(x=1/2,1,log(abs(t(x,v)))/x)
1.019427226765879400631304056300185184427275635974762517829661545074487591672 97 E-40
(a)->f(sqrt(a^2+1)-a)/zeta(2)
4*x^2 - 28*x - 401
(a)->f(a)+zeta(2)*(1/a+a/2)
5.168341577092213532598827118662800493623121123384443600552073477451662260517 4 E-39
```

Let $F(\alpha)$ denote the integral in question. A little numerical experimentation in SageMath suggests that $F(\sqrt{2}-1) = -\zeta(2)\frac{1+3\sqrt{2}}{2}$ We then conjecture that $N(k) = -\frac{F(\sqrt{k^2+1}-k)}{\zeta(2)}$ is always a quadratic integer, and seek its form. Some more experiments suggest that $N(k)$ is a root of $4x^2 - 4kx -$

$(8k^2+9)$ i.e. $N(k) = \frac{k+3\sqrt{k^2+1}}{2}$ In other words, we have established numerically that

$$F(\sqrt{k^2+1}-k) = -\zeta(2)\frac{k+3\sqrt{k^2+1}}{2}$$

Attempting to express the RHS in terms of the argument of the LHS we are led to conjecture:

$$F(\alpha) = -\zeta(2)\left(\frac{1}{\alpha}+\frac{\alpha}{2}\right)$$

We check this holds numerically. We could probably have guessed this form much more quickly by other means.

2 Proof

Denote this integral by $F(\alpha)$ and split about $x = 1/2$:

$$F(\alpha) = \int_0^{\frac{1}{2}} \frac{\ln|x^\alpha-(1-x)^\alpha|\,\mathrm{d}x}{x} + \int_{\frac{1}{2}}^1 \frac{\ln|x^\alpha-(1-x)^\alpha|\,\mathrm{d}x}{x}.$$

We then have that

$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{\ln|x^\alpha-(1-x)^\alpha|\,\mathrm{d}x}{x} &= \int_0^{\frac{1}{2}} \frac{\ln\left|(1-x)^\alpha\left[\left(\frac{x}{1-x}\right)^\alpha-1\right]\right|\mathrm{d}x}{x} \\
&= \int_0^{\frac{1}{2}} \frac{\ln(1-x)^\alpha \mathrm{d}x}{x} + \int_0^{\frac{1}{2}} \frac{\ln\left|\left(\frac{x}{1-x}\right)^\alpha-1\right|\mathrm{d}x}{x} \\
&= \frac{\alpha\ln^2 2}{2} - \frac{\zeta(2)}{2} + \int_0^{\frac{1}{2}} \frac{\ln\left|\left(\frac{x}{1-x}\right)^\alpha-1\right|\mathrm{d}x}{x}
\end{aligned}$$

(the last equality coming from a dilogarithm special value $\mathrm{Li}_2(\frac{1}{2})$ computed by Euler)[1] and

$$\begin{aligned}
\int_{\frac{1}{2}}^1 \frac{\ln|x^\alpha-(1-x)^\alpha|\,\mathrm{d}x}{x} &= \int_{\frac{1}{2}}^1 \frac{\ln\left|x^\alpha\left[1-\left(\frac{1-x}{x}\right)^\alpha\right]\right|\mathrm{d}x}{x} \\
&= \alpha\int_{\frac{1}{2}}^1 \frac{\ln x\mathrm{d}x}{x} + \int_{\frac{1}{2}}^1 \frac{\left|1-\left(\frac{1-x}{x}\right)^\alpha\right|\mathrm{d}x}{x} \\
&= \frac{-\alpha\ln^2 2}{2} + \int_{\frac{1}{2}}^1 \frac{\left|1-\left(\frac{1-x}{x}\right)^\alpha\right|\mathrm{d}x}{x}
\end{aligned}$$

[1]See Lewin's Polylogarithms and Associated Functions pg. 6

So we have

$$F(\alpha) = \frac{-\alpha\zeta(2)}{2} + A(\alpha) + B(\alpha)$$

where

$$A(\alpha) = \int_0^{\frac{1}{2}} \frac{\ln\left|(\frac{x}{1-x})^\alpha - 1\right| \mathrm{d}x}{x}$$

and

$$B(\alpha) = \int_{\frac{1}{2}}^1 \frac{\left|1 - \left(\frac{1-x}{x}\right)^\alpha\right| \mathrm{d}x}{x}.$$

In $A(\alpha)$, substitute $y = \frac{x}{1-x}$. Then $y + 1 = \frac{1}{1-x}$, and

$$\begin{aligned} A(\alpha) &= \int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y(y+1)} \\ &= \int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y} - \int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y+1} \end{aligned}$$

Similarly, in $B(\alpha)$, substitute $y = \frac{1-x}{x}$. Then $y + 1 = \frac{1}{x}$ and

$$B(\alpha) = \int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y+1}.$$

Putting everything together;

$$F(\alpha) = \frac{-\alpha\zeta(2)}{2} + \int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y}$$

But

$$\int_0^1 \frac{\ln|y^\alpha - 1| \,\mathrm{d}y}{y} = \frac{1}{\alpha} \int_0^1 \frac{\ln|w - 1| \,\mathrm{d}w}{w} = \frac{-\zeta(2)}{\alpha}$$

so

$$F(a) = -\zeta(2) \left[\frac{\alpha}{2} + \frac{1}{\alpha}\right].$$