Univariate Time Series Models-1

1. One objective of analyzing economic data is to predict or forecast the future values of economic variables.

2. In this week we utilize a pure Time Series approach where current values of an economic variable are related to past values. The emphasis is purely on making use of the information in past values of a variable for forecasting its future.

3. We consider a time series of observations on some variable, e.g. unemployment Rate, denoted as $Y_1, \ldots, Y_T$. These observations will be considered realizations of random variables that can be described by some stochastic process. We want to describe the properties of this process via simple models. It will be of particular importance how observations corresponding to different time periods are related, so that we can exploit the dynamic properties of the series to generate predictions for future periods.

4 A simple way to model dependence between consecutive observations assumes that Y_t depends linearly upon its previous value Y_{t-1}. That is,

$$Y_t = \delta + \theta Y_{t-1} + \epsilon_t$$

where ϵ_t is a white noise process with mean 0 and constant variance and also exhibits no autocorrelation. {assume $|\theta| < 1$}

This model is called an autoregressive process (AR(1)).

$$E(Y_t) = \delta + \theta E(Y_{t-1})$$

Assuming a stationary process {mean not dependent on time}

$$E(Y_t) = \delta + \theta E(Y_t)$$

$$E(Y_t) - \theta E(Y_t) = \delta$$

$$E(Y_t) = \frac{\delta}{1-\theta}\ ; \quad |\theta| < 1.$$

That is $\mu \equiv E(Y_t) = \dfrac{\delta}{1-\theta}$; $|\theta| < 1$

Define $y_t \equiv Y_t - \mu$, we have

$$y_t = \theta y_{t-1} + \epsilon_t. \quad \{How??\}$$

Using y_t inplace of Y_t is notationally convenient.

Note, variance (y_t) = Variance (Y_t). This is because variance $(\mu) = 0$ as μ is a constant.

The joint distribution of all the values of Y_t is characterized by the so called autocovariances. This is simply the covariances between Y_t and one of its lag Y_{t-k}. For the AR(1) model, the dynamic properties of the Y_t series can be determined if we assume that the variances and autocovariances do not depend on time, t. This is the so called stationary assumption.

Let's take a look at this:

$V(Y_t) = V\{\delta + \theta Y_{t-1} + \epsilon_t\}$

$v(\delta) = 0$ as δ is a constant.

Also, note that $V(ax) = a^2 V(x)$ "a" a constant. Therefore,

$$V(Y_t) = V(\theta Y_{t-1} + \epsilon_t)$$
$$= \theta^2 V(Y_{t-1}) + V(\epsilon_t)$$

Since $V(Y_t)$ is independent of time; then, $V(Y_t) = V(Y_{t-1})$ and

$$V(Y_t) = \theta^2 V(Y_t) + \sigma^2$$
$$\Rightarrow V(Y_t) = \frac{\sigma^2}{1-\theta^2}; \quad |\theta| < 1.$$

Now, what about $Cov(Y_t, Y_{t-1})$?

$$Cov(Y_t, Y_{t-1}) = E(Y_t Y_{t-1}) - E(Y_t)E(Y_{t-1})$$
$$= E(Y_t Y_{t-1})$$
$$= E\{(\theta Y_{t-1} + \epsilon_t) Y_{t-1}\}$$
$$= \theta V(Y_{t-1}) \quad \because E(\epsilon_t) = \emptyset$$
$$= \theta \frac{\sigma^2}{1-\theta^2}$$

It can be shown that, in general for $K = 1, 2, \ldots$

$$\mathrm{Cov}(Y_t, Y_{t-K}) = \theta^K \frac{\sigma^2}{1-\theta^2}.$$

As long as θ is non-zero, any 2 obsvs on Y_t have non-zero correlation, while their dependence is smaller if the observations are further apart. Also note that the covariance between Y_t and Y_{t-K} is independent of t — this reflects the stationarity of the process.

Another Example

A first order moving average (MA(1)) process.

$$Y_t = \mu + \epsilon_t + \alpha\, \epsilon_{t-1}$$

Apart from the mean μ, this says that Y_1 (as an example) is a weighted average of ϵ_1 and ϵ_0.

Y_2 is a weighted average of ϵ_2 and ϵ_1 ... the values of Y_t are defined in terms of drawings from the white noise process ϵ_t.

What is $E(Y_t)$?

$$E(Y_t) = E\{\mu + \epsilon_t + \alpha\epsilon_{t-1}\}$$
$$= E(\mu) + E(\epsilon_t) + \alpha E(\epsilon_{t-1})$$
$$= \mu \quad \because \quad E(\epsilon_t) = \emptyset$$
$$E(\epsilon_{t-1}) = \emptyset$$

What is $V(Y_t)$?

$$V(Y_t) = V\{\mu + \epsilon_t + \alpha\,\epsilon_{t-1}\}$$
$$= V(\epsilon_t) + \alpha^2 E(\epsilon_{t-1})$$
$$= \sigma^2 + \alpha^2\sigma^2$$
$$= (\alpha^2 + 1)\,\sigma^2$$

$$Cov\,(Y_t, Y_{t-1}) = E\{(\epsilon_t + \alpha\,\epsilon_{t-1})\,(\epsilon_{t-1} + \alpha\epsilon_{t-2})\}$$
$$= \alpha E(\epsilon_{t-1}^2)$$
$$= \alpha\sigma^2 \qquad \text{(why? How?)}$$

$Cov(Y_t, Y_{t-2}) = 0$ How? Why?

In general,

$$Cov(Y_t, Y_{t-k}) = 0 \quad \forall k = 2, 3, 4, \ldots$$

How? Why?

So the simple moving average implies that observations that are 2 or more periods apart are uncorrelated – different from what we saw with the simple AR process.

As $n \to \infty$ the AR process $\to$ MA process

Can you prove this?

Stationarity and the autocorrelation function

- A stochastic process is said to be strictly stationary if its properties are unaffected by a change of time origin. {Distribution does not change, mean does not, variance does not, covariance does not}.

Autocovariance

The k-th order autocovariance is given by

$$\gamma_k = Cov(Y_t, Y_{t-k}) = Cov(Y_{t-k}, Y_t)$$

so, when $k=0$

$$\gamma_0 = Cov(Y_t, Y_t) = Var(Y_t).$$

Autocorrelation

$$\rho_k = \frac{Cov(Y_t, Y_{t-k})}{V(Y_t)} = \frac{\gamma_k}{\gamma_0}$$

Note, $\rho_0 = 1$ (why?)

Also, $-1 \leq \rho_K \leq 1$

The autocorrelations considered as a function of K are referred to as the autocorrelation function (ACF).

From the ACF, we can infer the extent to which one value of the process is correlated with previous values and $\therefore$ the length and strength of the memory of the process. It indicates how long (and how strongly) a shock in the process (ϵ_t) affects the values of Y_t.

For the AR(1) process

$$Y_t = \delta + \theta Y_{t-1} + \epsilon_t$$

$$\rho_K = \theta^K \quad \text{(How? Why?)}$$

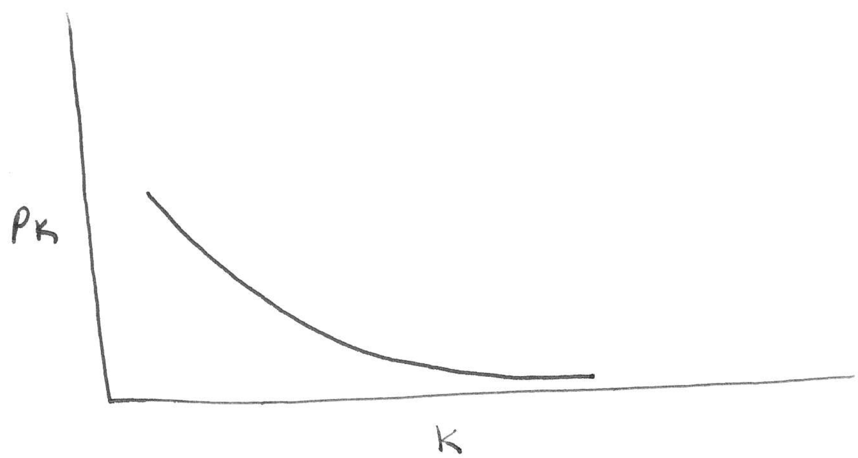
For the MA process

$$Y_t = \mu + \epsilon_t + \alpha \epsilon_{t-1}$$

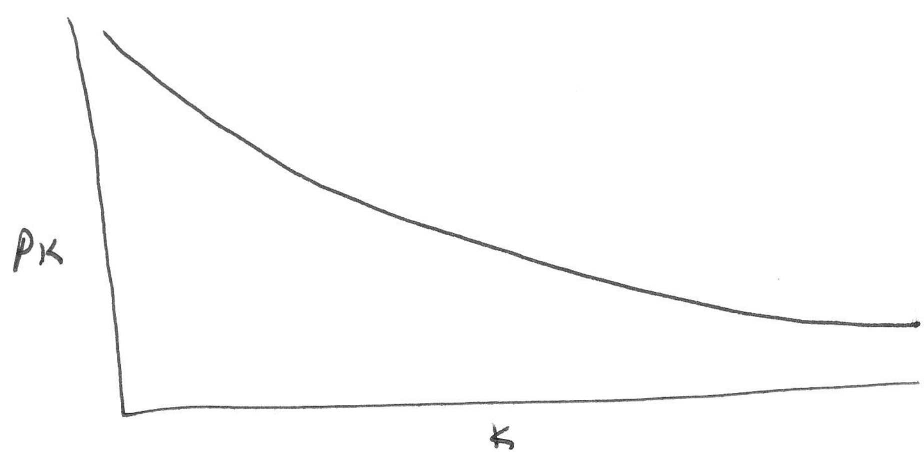
$$\rho_1 = \frac{\alpha}{1+\alpha^2} \quad \text{and} \quad \rho_K = 0 \quad K = 2, 3, 4 \ldots$$

Consequently, a shock in a MA(1) process affects Y_t in 2 periods-only. While a shock in the AR(1) process affects all future observations.

Theoretical autocorrelation function for AR(1) $\theta = .5$

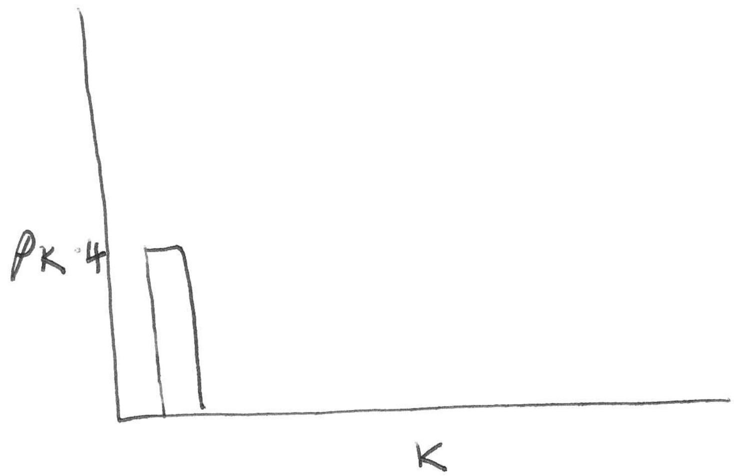


$\theta = .9$



Theoretical autocorrelation function

MA(1) $\alpha = .5$



MA(1) $\alpha = .9$