Groupwise Heteroscedasticity

$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{\epsilon}$ with $E(\underset{\sim}{\epsilon}|\underset{\sim}{X}) = \underset{\sim}{0}$ $\quad Var(\underset{\sim}{\epsilon}|\underset{\sim}{X}) = \sigma^2\underset{\sim}{\Omega} = \underset{\sim}{\Sigma}$

$$\sigma^2\underset{\sim}{\Omega} = \sigma^2 \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & \omega_n \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix}$$

$$\sigma^2\Omega = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \cdots & \rho_{n-2} \\ & & \vdots & \\ \rho_{n-1} & \rho_{n-2} & \cdots & 1 \end{bmatrix}$$

Least Squares Inefficient. {We saw this.}

We saw detection methods.

- White's test
- Breusch-Pagan Test
- Godfrey's Test

Greene (272-275) discusses robust variance-covariance matrices. {White's estimator - Pg 274}

Estimation of $\underset{\sim}{\beta}$

We found that $\hat{\underset{\sim}{\beta}}$ no longer $(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{y}$

BUT, $\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T\underset{\sim}{\Omega}^{-1}\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{\Omega}^{-1}\underset{\sim}{y}$.

Consider the most general case where $Var(\epsilon_i | \underset{\sim}{x}) = \sigma^2 w_i$. Then $\underset{\sim}{\Omega}^{-1}$ is a diagonal matrix with diagonal elements $\frac{1}{w_i}$ $i=1,\dots,n$.

As discussed last week; premultiply (transform) $\underset{\sim}{y}$ and $\underset{\sim}{x}$ as

$$\underset{\sim}{\Omega}^{-1/2} \underset{\sim}{y} \quad \text{and} \quad \underset{\sim}{\Omega}^{-1/2} \underset{\sim}{x}$$

i.e.
$$\underset{\sim}{\Omega}^{-1/2} \underset{\sim}{y} = \begin{bmatrix} y_1/\sqrt{w_1} \\ y_2/\sqrt{w_2} \\ \vdots \\ y_n/\sqrt{w_n} \end{bmatrix}, \quad \underset{\sim}{\Omega}^{-1/2} \underset{\sim}{x} = \begin{bmatrix} \underset{\sim}{x}_1^T/\sqrt{w_1} \\ \underset{\sim}{x}_2^T/\sqrt{w_2} \\ \vdots \\ \underset{\sim}{x}_n^T/\sqrt{w_n} \end{bmatrix}$$

and apply OLS to the transformed response and data matrix.

Read 278--279--280.

Groupwise Heteroscedasticity

$$y_i = \underset{\sim}{x}_i^T \underset{\sim}{\beta} + \epsilon_i \qquad i=1,\dots,n$$

$$E(\epsilon_i | \underset{\sim}{x}_i) = 0 \qquad \forall i=1,\dots,n$$

There are $\underline{\underline{G}}$ groups each with n_g observations. That is, $\sum_{g=1}^{G} n_g = n$.

The slope vector is assumed to be the same in all groups, but within group g

$$Var(\epsilon_{ig} | \underset{\sim}{x}_{ig}) = \sigma_g^2 \qquad i=1,\dots,n_g$$

If the variances are known then the GLS estimator is

$$\hat{\underset{\sim}{\beta}} = \left\{ \sum_{g=1}^{G} \left(\frac{1}{\sigma_g^2}\right) \underset{\sim}{X}_g^T \underset{\sim}{X}_g \right\}^{-1} \left\{ \sum_{g=1}^{G} \left(\frac{1}{\sigma_g^2}\right) \underset{\sim}{X}_g^T \underset{\sim}{y}_g \right\}$$

Note that $\underset{\sim}{X}_g^T \underset{\sim}{y}_g = (X_g^T X_g) \underset{\sim}{b}_g$ where $\underset{\sim}{b}_g$ is the OLS estimator in the g^{th} group.

$$\therefore \quad \hat{\underset{\sim}{\beta}} = \left\{ \sum_{g=1}^{G} \left(\frac{1}{\sigma_g^2}\right) \underset{\sim}{X}_g^T \underset{\sim}{X}_g \right\}^{-1} \left\{ \sum_{g=1}^{G} \left(\frac{1}{\sigma_g^2}\right) \underset{\sim}{X}_g^T \underset{\sim}{X}_g \underset{\sim}{b}_g \right\}$$

$$= \left[\sum_{g=1}^{G} \underset{\sim}{V}_g \right]^{-1} \left[\sum_{g=1}^{G} \underset{\sim}{V}_g \underset{\sim}{b}_g \right] = \sum_{g=1}^{G} \underset{\sim}{W}_g \underset{\sim}{b}_g$$

This results in a matrix weighted average of the G least squares estimators. The weighting matrices are $\underset{\sim}{W}g = [\sum_{g=1}^{G} (var(\underset{\sim}{b}g))^{-1}]^{-1} \times (var(\underset{\sim}{b}g))^{-1}$.

Now what

- If n_g $(g=1,\ldots,G) < k$ (# of regressors) then we have a problem. Why?
- Do we need to worry about this?
- All we need are the σ_g^2's.
- How do we get these.
- Run a pooled OLS first.
- Get the OLS residuals.
- Separate these by the groups.
- Calculate $\hat{\sigma}_g^2 = \dfrac{\underset{\sim}{e}_g^T \underset{\sim}{e}_g}{n_g}$.