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A linear Bayesian stochastic approximation to update project duration estimates

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ABSTRACT

By relaxing the unrealistic assumption of probabilistic independence on activity durations in a project, this paper develops a hierarchical linear Bayesian estimation model. Statistical dependence is established between activity duration and the amount of resource, as well as between the amount of resource and the risk factor. Upon observation or assessment of the amount of resource required for an activity in near completion, the posterior expectation and variance of the risk factor can be directly obtained in the Bayesian scheme. Then, the expected amount of resources required for and the expected duration of upcoming activities can be predicted. We simulate an application project in which the proposed model tracks the varying critical path activities on a real time basis, and updates the expected project duration throughout the entire project. In the analysis, the proposed model improves the prediction accuracy by 38.36% compared to the basic PERT approach.

1. Introduction and overview of former studies

To date, considerable effort has been put forth into the field of project management by many researchers seeking to provide better problem-solving techniques. In examining the project duration estimation problem, there seem to be at least three main strategies. In the first strategy, early contributors tried to suggest expedient solutions with computational simplicity. In the second strategy, they pursued highly detailed and computationally intensive methods in search of prediction accuracy. The major source of complexity in modeling processes and obtaining solutions in the second strategy stems from the allowance of statistical dependence among activities in a project. The third strategy is to lie in the middle position by retaining the relative computational simplicity while providing a mechanism for accounting for dependence among activities.

There are several example studies within the first strategy category. Despite its early contribution to systematical project analysis, the Program Evaluation and Review Technique (PERT) could not deal with uncertainty problems realistically. Even though PERT is called a probabilistic approach, it shares a similar weakness with the Critical Path Method (CPM), a static view of the entire project, due to the assumption of probabilistic independence among activities (MacCrimmon and Ryavec, 1964). Original PERT-type models were interested in expediently assessing the expert belief about activity duration and estimating total project duration (Chatzogou and Macaulay, 1996; Keefer and Bodily, 1983; Magott and Skudlarski, 1993). According to the critical path analysis of these methods, the critical path activities are determined once at the onset of a project and no future revision is allowed. However, activity durations are affected by many risk factors from manpower, technique, resource, and utility. That is, activities lie on the critical path, not in a deterministic way, but in a stochastic way (Dodin and Elmaghraby, 1985).

In the second strategy, studies are categorized into three approaches: the one-time update approach; the Markov Chain Monte Carlo (MCMC) approach; and the full Bayesian approach. Van Dorp and Duffey (1999) modeled a positive probabilistic dependence between activity durations by introducing a risk node in the networks. The risk factor used in their study was engineering change orders and rework whose sources include owner-requested changes, inadequate design specifications, and interface problems. They claimed that the amount of uncertainty associated with estimating the minimal completion time of a project should be increased in the networks that allow dependence, compared to the ones without dependence. Given the predetermined degree of dependence, the durations of all activities are updated once after observing the risk factors, and further updating is not allowed. In this one-time update approach, the risk factor was just another observable variable, not a parameter.

Another branch of the second strategy is using the MCMC methods. Jenzarli (1994) introduced a PERT belief network, by combining the PERT network with a belief network. The network includes two kinds of variables – activity duration nodes as well as other nodes that might affect activity duration. The other variables include budget, the number of workers for a specific task, the effectiveness of workers, the quality of design, and the education of workers. The project duration is estimated by applying the forward Monte Carlo method and the Gibbs sampling technique.

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Some variables are observable while others are not. For example, the effectiveness of workers and the quality of design should be treated as parameters.

The MCMC methods were used to solve project management problems under uncertainty and resource constraints. Virto et al. (2002) identified a set of good project schedules by using a specific version of Metropolis algorithm. This non-sequential decision model focused on utility maximization rather than prediction accuracy. The utility maximization approach was further modified for sequential decision processes evolving stochastically in time (Diaz et al., 2003).

The last category in the second strategy is the full Bayesian approach. Covaliu and Soyer (1996) proposed a sequential project decision problem in which the probability distributions of upcoming activity durations are updated given observing a completed activity in a Bayesian scheme. By extrapolating multivariate distributions such as Lindley and Singpurwalla's (1986) and Marshall and Olkin's (1967) from reliability models, they described three separate models: the resource reallocation model, the common environment/resource model, and the resource sharing model under three different circumstances. Although the study successfully conceptualized a sequential nature of projects, its applicability was limited to a project consisting of two or three activities only.

The extension of the above study involved the common environment/resource model and suggested solutions for a project consisting of many activities (Covaliu and Soyer, 1997). In updating parameter distributions and computing predictive distributions for upcoming activity durations, they used the MCMC methods, a Gibbs sampler and rejection sampling algorithm as in Smith and Gelfand (1992) in particular.

The third strategy is to provide models that allow dependence among activities with relatively light computational burden. In the full Bayesian scheme, the probability distributions of unknown parameters are derived after heavy computations. The posterior probability distribution often cannot be derived in a closed form. In this case, the posterior predictive mean of variables (e.g., activity duration) might be obtained by using the MCMC method. To relieve this computational difficulty, Cho and Covaliu (2003) introduced a sequential project decision model using a linear Bayesian approach. In the linear Bayesian template, the expected value and variance can be directly updated given observation. The expected duration of upcoming activities and their variances are directly updated given observing a completed activity.

The linear Bayesian approximation approach affords considerable computational benefits when compared to the full Bayesian approach. However, it still retains the basic Bayesian form because it combines prior and present data for the posterior inference. With relatively simple linear equations, researchers can easily update the first two moments of distribution, mean and variance, without the expertise in the area of stochastic modeling. The linear Bayesian approximation gives exactly same results as the full Bayesian approach if random variables are jointly normally distributed. However, this also serves as a major limitation of the linear Bayesian approximation. Hartigan (1969) mentioned that the ideal condition for using the linear Bayesian approximation is that the parameter of interest is normally distributed given the observations, with the mean that equals to the linear expectation, and the variance that equals to the estimated mean square error. In the absence of the normality assumption, the usefulness of the linear Bayesian approximation needs to be examined.

The estimation of the correlation coefficient matrix for all pairs of activities would be a practical burden in the linear Bayesian approach. Cho (2006) suggested two ways of eliciting the correlation coefficients between a pair of activities using a concordance probability in the linear Bayesian model. By making the field experts estimate the concordance probabilities or the expected durations of a pair of activities, we can inversely estimate the correlation coefficient of the two activities, since the concordance probability is a monotone increasing function of correlation coefficient.

The limitation of the above works is that they considered activity durations only, not including resources or risk factors in their models. In fact, the risk parameter that might affect activity duration did not appear in the model. Instead, activity duration was directly updated under the assumption that the correlation coefficient between activity durations can be properly assessed. In order to improve the applicability of the model, the current study presents a sequential project estimation model using a linear Bayesian scheme, which includes activities, resources, and risk factors.

The approach starts from the assumption that activities in a project might be correlated with each other and the statistical dependence is measurable in terms of duration. In the traditional PERT approach, the assumption of independence led to the absence of reality and practicality. Early completion or delay of preceding activities affects only the starting and finishing point of time of succeeding activities, but not the durations itself. We relax the unrealistic assumption of independence by taking into account the existence of resource sharing and common risks.

It is common that several activities in a particular project share resources and/or risks. Activities share human beings, raw materials, equipment, facilities, and utilities. Consider the situation where several activities are carried out by a single team. If the preceding activity took longer than expected because of overestimated labor productivity, it signals that the succeeding activity is more likely to take longer than initially expected. The risk associated with human resource sharing might include worker's knowledge, skills, motivation, and experiences. Another major risk might concern common environment such as weather, geographical characteristics, and market stability. For example, if weather is assumed to be a risk factor, it includes various aspects such as temperature, moisture, sunlight, and wind. The market stability risk might reflect various interest rates, foreign exchange rates, real estate prices, and more importantly people's expectation of these rates.

Given the fact that sharing of resource and risk is a usual circumstance in most projects, it is natural to assume that some activity durations are correlated with each other. The dependence structure can be modeled in the hierarchy of resource and risk factors. The observation on an activity might reveal valuable information of the risk factor, which will, in turn, affect other activities. The updating processes are sequentially repeated until the completion of projects. We propose this type of sequential updating model by graphically using Bayesian belief network, and by mathematically using the linear Bayesian framework. The research also demonstrates improvement in prediction accuracy using an application example.

2. Linear Bayesian model

2.1. Bayesian belief network and updating algorithm

This research introduces a graphical model as well as a mathematical model to sequentially update project duration estimates. The graphical model is named the Bayesian belief network, while the mathematical model refers to the linear Bayesian stochastic approximation model in this study.

An influence diagram (or alternatively called, a belief network) is a network modeling tool that consists of a directed graph with no directed cycles (Howard and Matheson, 1981). The nodes store detailed data and the arcs connect with the nodes. Influence diagrams have been widely used for structuring decision problems since information flow is effectively represented and so is the conditional independence (Smith et al., 1993). In general, the variables contained in the influence diagram do not necessarily have the Bayesian-type relationships. During the last decade, researchers in the area of artificial intelligence have worked on the Bayesian network, which is a special case of influence diagram where only discrete variables are included (Jensen, 1996; Heckerman et al., 1995).

The current study applies the Bayesian solution algorithm on the influence diagram. A good reference of performing Bayesian decision analysis on influence diagram would be Smith's study (1989). In this study, we name our graphical model "Bayesian belief network", because the network includes continuous random variables and the relationship between variables can be explained in a Bayesian sense.

Let $\mathscr{R}$ be a set of random variables and $\mathscr{A}$ be a set of directed arcs for paired random variables. A directed graph $(\mathscr{R}, \mathscr{A})$ is a Bayesian belief network that represents probabilistic dependence between the paired random variables. For $(i,j) \in \mathscr{A}$, i refers to an immediate predecessor of j, whereas j for an immediate successor of i. In addition, $\mathscr{A}^{\mathscr{R}}$ denotes a subset of arcs that excludes the arc from the most recently observed random variable from set $\mathscr{A}$.

The basic premise for the hierarchical linear Bayesian model to be effective is that some or all activities in a particular project share a certain type of resource that is affected by a common risk factor. For the sake of explaining the linear Bayesian updating algorithm, consider a set of three activities that belongs to a project consisting of a number of activities.

This study introduces three types of random variables whose distribution is specified with a mean and a standard deviation.

- T_i: duration of activity i.
- K_i: amount of resource required for the completion of activity i.
- α: risk factor.

This research uses the following notations and symbolic notations:

- $\widehat{E}[.]$: the estimation of expectation (e.g., $\widehat{E}[X]$ means the estimation of expected value of X).
- $V^{-1}[.]$: the precision (i.e., the inverse of variance) (e.g., $V^{-1}[X]$ means the precision of X).
- $|$: 'conditioned on' (e.g., $X|Y$ means X conditioned on Y).
- $\coprod$: 'statistical independence' (e.g., $X\coprod Y$ means that X and Y are statistically independent of each other).
- $\sim$: 'as distributed as' (e.g., $X \sim (\mu, \sigma)$ means that X follows a distribution with mean, μ, and standard deviation, σ).

The Bayesian belief network in Fig. 1a depicts a situation in which the amounts of resource required for activities are statistically correlated with each other, and so are the durations of activities, through the risk factor. It must be pointed out that the precedence relationship is not specified in the belief network. In cases where these activities are carried out on different paths, the sequence of work completion cannot be fixed, which is a more natural assumption compared to all serially proceeding case. Accordingly, activities are named in an arbitrary order. Any activity that is about to be finished first is named Activity A, and the second earliest activity about to be finished is called Activity B.

The durations of activities are conditionally independent of each other given the risk factor, and so are the amounts of resource, which can be symbolically denoted as:

$$K_i \coprod K_j | \alpha \quad \text{and} \quad T_i \coprod T_j | \alpha,$$

where $i \neq j$. Upon removing α, the durations of activities might be dependent upon each other, and so might the amounts of resource.

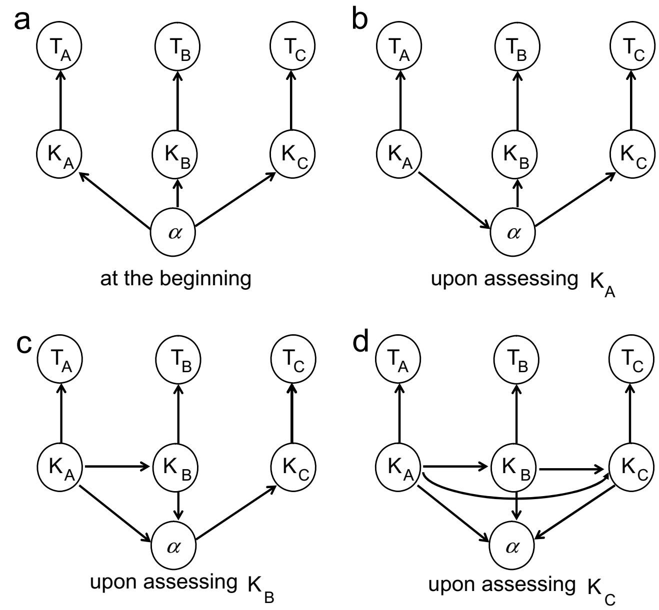

Fig. 1. Graphical representation of the updating processes in the Bayesian belief network.

In this section, we introduce the graphical model to link activity duration, resource, and risk factor, and then present how these random variables can be modeled in the linear Bayesian framework. A detailed description of updating processes will be illustrated in Section 2.2, and then a detailed interpretation of random variables such as K_i and α will be provided in Section 2.3.

To relieve the computational load required in the full Bayesian approach, a few linear Bayesian methods have been introduced outside the area of project estimation problem.

Bunn (1975) and Li et al. (2001) suggested a linear approach in which several individual forecasts are combined into a single forecast. In order to obtain the posterior mean (i.e., weight) of an individual forecasting method's outperforming others in the future, the experiment must be repeated over time and thus the frequency of performance in the past must be available. These two models were basically made for the purpose of time series data analysis. Based on the cumulative data about a random variable, they predict its future value. In other cases such as the regression-type problems, we predict a dependent variable based on multiple pairs of independent and dependent variables. The common feature of these techniques is that the sample size must be greater than one. Note that in the domain of project estimation problem, we need to update other random variables (e.g., the duration of upcoming activities) given a single, one-time event of one random variable (e.g., assessing the amount of resource requirement for a near completion activity). In other words, the present data size equals one. In this sense, neither of models can be applied to this type of project duration estimation problem. Besides, the update of variance was not considered, either.

Farrow et al. (1997) developed a Bayes linear decision support system for the brewery problem. In their linear formula for obtaining the adjusted expectation and variance of a variable given another variable, the covariance matrix must be known. In the problem structure of this study as illustrated in Fig. 1, the statistical dependence between T_i is modeled in a hierarchical way through K_i and α, rather than directly modeled with known covariance structure. For the cases of unknown covariance structures, their method is not applicable.

The linear Bayesian approach that is applicable to project estimation problems, to the best of our knowledge, is Hartigan's linear

Bayes' theorem (1969). Hartigan applied his algorithm to predict the final score of a baseball game from partial scores available as the game progresses. This theorem is applicable to the case where the distribution of one variable can be updated given a single observation of another variable, which is exactly what we need in project estimation. Unlike the full Bayesian scheme where the prior distribution of a parameter and the conditional distribution of observations given parameters must be specified, the linear Bayesian method uses only the first two moments, mean and variance, of a parameter and observation.

The linear Bayes' theorem by Hartigan can be described as follows. Let X, Y, $\mathbf{Z}(=Z_1, Z_2, \ldots, Z_n)$ be random variables with finite variances. "Prior" denotes before observing X and "posterior" denotes after observing X, where X is the most recently observed variable. The linear expectation of X given Y and $\mathbf{Z}$ is set by $\widehat{E}[X \mid Y, \mathbf{Z}] = cY + d$, where $d = a_0 + \sum_{i=1}^{n} a_i Z_i$. Here, c and d are chosen to minimize the variance of the equation. Then, the posterior precision of Y is obtained as the sum of its prior precision and the present data precision:

$$V^{-1}[Y|X, \mathbf{Z}] = V^{-1}[Y|\mathbf{Z}] + c^2 V^{-1}[X|Y, \mathbf{Z}].$$

The posterior mean of Y is estimated as the weighted average of its prior mean and a term due to the present data, where the weights are the ratio of the prior precision over the posterior precision, and the ratio of the present data precision over the posterior precision, respectively:

$$\widehat{E}[Y|X, \mathbf{Z}] = \widehat{E}[Y|\mathbf{Z}] \frac{V^{-1}[Y|\mathbf{Z}]}{V^{-1}[Y|X, \mathbf{Z}]} + c(X - d) \frac{V^{-1}[X|Y, \mathbf{Z}]}{V^{-1}[Y|X, \mathbf{Z}]}.$$

Now, we present how the linear Bayesian method can be applied into the Bayesian belief network illustrated in Fig. 1. Let $\mathbf{K}$ be a set of directed arcs that go from K_i to α for $\forall\, i(i, \alpha) \in K$. Define $K^{\mathbf{R}} \subseteq K$ such that $K^{\mathbf{R}}$ is a subset of arcs that excludes the arc from the most recently observed amount of resource to α. "Prior" denotes before observing K and "posterior" denotes after observing K, where K is the most recently observed variable. The linear equation is set such that $\widehat{E}[K_i|\alpha, K^{\mathbf{R}}] = c\alpha + d$, where $d = a_0 + \sum_{(i,\alpha)\in K^{\mathbf{R}}} a_i K_i$. Here, c and d are chosen to minimize the variance of the equation.

In the updating processes, the following notations are used for a fixed quantity:

- c, d: regression coefficients of the linear equation, which minimize the variance of the equation. These coefficients can be generalized as c_i and d_i.
- c_i, d_i: regression coefficients of the linear equation for the i'th update.
  For example, for the first update of α, $\widehat{E}[K_1|\alpha] = c_1\alpha + d_1$ is used.
  For the second update of α, $\widehat{E}[K_2|\alpha, K_1 = k_1] = c_2\alpha + d_2$ is used.
  For the third update of α, $\widehat{E}[K_3|\alpha, K_2 = k_2, K_1 = k_1] = c_3\alpha + d_3$ is used.
  Similarly, for the i'th update of α, $\widehat{E}[K_i|\alpha, K_{i-1} = k_{i-1}, \ldots, K_1 = k_1] = c_i\alpha + d_i$ is used.
- $a_0, \ldots, a_i$: coefficients as a result of decomposing d_i in the above regression coefficients, as functions of up to the $(i-1)$'th observations.
  For example, for the first update of α, $d_1 = a_0$.
  For the second update of α, $d_2 = a_0 + a_1 k_1$.
  For the third update of α, $d_3 = a_0 + a_1 k_1 + a_2 k_2$.
  Similarly, for the i'th update of α, $d_i = a_0 + a_1 k_1 + a_2 k_2 + \cdots + a_{i-1} k_{i-1}$.

The posterior precision and expectation of α given K_i and $K^{\mathbf{R}}$ can be obtained by Eqs. (1) and (2), respectively

$$V^{-1}[\alpha|K_i, K^{\mathbf{R}}] = V^{-1}[\alpha|K^{\mathbf{R}}] + c^2 V^{-1}[K_i|\alpha, K^{\mathbf{R}}], \quad (1)$$

$$\widehat{E}[\alpha|K_i, K^{\mathbf{R}}] = \widehat{E}[\alpha|K^{\mathbf{R}}] \frac{V^{-1}[\alpha|K^{\mathbf{R}}]}{V^{-1}[\alpha|K_i, K^{\mathbf{R}}]} + c(K_i - d) \frac{V^{-1}[K_i|\alpha, K^{\mathbf{R}}]}{V^{-1}[\alpha|K_i, K^{\mathbf{R}}]}. \quad (2)$$

Once K_i is observed, K_j and T_j of upcoming activities can be predicted using Eqs. (3) and (4) by taking the average with respect to α, and α as well as K_j, respectively

$$E[K_j|K_i] = E_\alpha[E[K_j|\alpha, K_i]], \quad (3)$$

$$E[T_j|K_i] = E_\alpha[E_{K_j}[E[T_j|K_j, \alpha, K_i]]]. \quad (4)$$

As can be identified in the above equations, learning in this study is defined such that by observing or assessing the amount of resource for an activity in near completion status, we can update the parameters of the risk factor, and then predict the expectations of the amount of resource required for and the duration of upcoming activities.

2.2. Updating processes

In this section, we present the updating processes for Fig. 1.

The duration of activity conditioned on the amount of resource requirement, $T_i|K_i$ is assumed to be distributed with the following mean and standard deviation:

$$T_{\mathrm{A}}|K_{\mathrm{A}} \sim (lK_{\mathrm{A}}, \sigma_{T_{\mathrm{A}}}), \quad T_{\mathrm{B}}|K_{\mathrm{B}} \sim (mK_{\mathrm{B}}, \sigma_{T_{\mathrm{B}}}), \quad \text{and} \quad T_{\mathrm{C}}|K_{\mathrm{C}} \sim (nK_{\mathrm{C}}, \sigma_{T_{\mathrm{C}}}),$$

where l, m, and n are the contribution coefficients of activity duration.

The amount of resource required for an activity conditioned on the risk factor, $K_i|\alpha$, is assumed to be distributed with the following mean and standard deviation:

$$K_{\mathrm{A}}|\alpha \sim (p\alpha, \sigma_{k_{\mathrm{A}}}) \quad K_{\mathrm{B}}|\alpha \sim (q\alpha, \sigma_{K_{\mathrm{B}}}), \quad \text{and} \quad K_{\mathrm{C}}|\alpha \sim (r\alpha, \sigma_{K_{\mathrm{C}}}),$$

where p, q, and r are the contribution coefficients of the amount of resource. Note that the standard deviation terms, σ_{T_i} and σ_{K_i}, are fixed quantities.

Lastly, the risk factor is assumed to be distributed with the following mean and standard deviation:

$$\alpha \sim (\mu_{\alpha(t)}, \sigma_{\alpha(t)}),$$

where $\mu_{\alpha(t)}$ denotes the mean of α at time sequence t, and $\sigma_{\alpha(t)}$ for the standard deviation of α at time sequence t. The time sequence $t = 0$ denotes the prior quantities, and $t = 1, 2, \ldots, T$ denotes the posterior quantities.

Here, we need to clarify the distinction between two terms, a conditional expectation and a posterior expectation. In the Bayesian context, "conditional" is a term used for two observable random variables. Although "posterior" takes the form of conditional relationship, it is a term used for an unobservable parameter given an observable variable. Two random variables, T_i and K_i, are observable variables. Thus, $E[T_i]$ is called a marginal expected duration, whereas $E[T_i|K_i]$ is called a conditional expected duration. Although K_i acts as a parameter of T_i, it might not be a genuine parameter in the sense that it is observable. On the other hand, random variable, α, is not observable. One can say that α is a genuine parameter. In this regard, $E[\alpha]$ is called a prior expectation, whereas $E[\alpha|K_i]$ is called a posterior expectation.

Before observing K_i, which corresponds to the graphical representation as in Fig. 1a, the marginal expectations of K_i and T_i can be initially obtained. The marginal expected amount of resource for Activity A is, by taking the average with respect to α, computed

$$E[K_{\mathrm{A}}] = E_\alpha[E[K_{\mathrm{A}}|\alpha]] = p\mu_{\alpha(0)}.$$

The marginal expected duration of Activity A is computed

$$E[T_{\mathrm{A}}] = E_\alpha[E_{K_{\mathrm{A}}}[E[T_{\mathrm{A}}|K_{\mathrm{A}}, \alpha]]] = lp\mu_{\alpha(0)}.$$

Similarly, the estimates for Activity B are

$$E[K_{\mathrm{B}}] = q\mu_{\alpha(0)} \quad \text{and} \quad E[T_{\mathrm{B}}] = mq\mu_{\alpha(0)}.$$

The two estimates for Activity C are

$$E[K_{\mathrm{C}}] = r\mu_{\alpha(0)} \quad \text{and} \quad E[T_{\mathrm{C}}] = nr\mu_{\alpha(0)}.$$

In the proposed diagram, there exist three milestone events, which invoke the update of the risk factor and the re-estimation of the remaining project activities. The detailed updating processes are as follows.

2.2.1. Milestone event K_A

Near the completion of Activity A, suppose the amount of resource required for work completion is assessed. The effect of milestone event is reflected as the arc reversal from K_{A} to α in Fig. 1b. An arc reversal indicates that the variable is either observed or known; it is used to make inferences based on posterior probabilities (Marshall and Oliver, 1995). An important consequence of arc reversal is that even after observing T_{A}, α is not affected by T_{A} given K_{A}. T_{A} is uninformative in estimating α given K_{A}, since α and T_{A} are conditionally independent given K_{A}:

$$\text{since } \alpha \coprod T_{\mathrm{A}}|K_{\mathrm{A}} \quad (\alpha|T_{\mathrm{A}}, K_{\mathrm{A}}) = (\alpha|K_{\mathrm{A}}).$$

This is the reason why observing K_i is defined as a milestone event in this study. Updating the estimates of upcoming activities should be made at the milestone event rather than at the point when the preceding activity is completed.

The updating processes include re-estimating T_i of the corresponding activity, the first two moments of α, and K_i and T_i for upcoming activities.

First, $E[T_{\mathrm{A}}|K_{\mathrm{A}}]$ can be immediately obtained

$$E[T_{\mathrm{A}}|K_{\mathrm{A}} = k_{\mathrm{A}}] = lk_{\mathrm{A}}.$$

Second, the observation of K_{A} invokes the update of the mean and variance of the risk factor. The posterior precision and expectation of α given K_{A} can be obtained using Eqs. (1) and (2), respectively. Let $\widehat{E}[K_{\mathrm{A}}|\alpha] = c_1\alpha + d_1$. Since $E[K_{\mathrm{A}}|\alpha] = p\alpha$, we can instantly notice that $c_1 = p$ and $d_1 = 0$. The posterior variance of α can be obtained from

$$V^{-1}[\alpha|K_{\mathrm{A}}] = V^{-1}[\alpha] + c_1^2 V^{-1}[K_{\mathrm{A}}|\alpha]. \tag{5}$$

For Eqs. (5)–(14), see Appendix for detailed computations. The posterior mean of α can be obtained

$$\widehat{E}[\alpha|K_{\mathrm{A}} = k_{\mathrm{A}}] = \widehat{E}[\alpha]\frac{V^{-1}[\alpha]}{V^{-1}[\alpha|K_{\mathrm{A}}]} + c_1(k_{\mathrm{A}} - d_1)\frac{V^{-1}[K_{\mathrm{A}}|\alpha]}{V^{-1}[\alpha|K_{\mathrm{A}}]}. \tag{6}$$

Table 1 summarizes the milestone event in the network and the update of estimates at each milestone event.

Third, once the posterior mean and variance of the risk factor is obtained, the expected amount of resource required for and the expected duration of upcoming activities can be predicted using Eqs. (3) and (4)

Table 1
Updating estimates at the milestone event

Milestone event	Estimates to be updated	Corresponding diagram
Assessing K_{A}	$E[T_{\mathrm{A}}\|K_{\mathrm{A}}]$, $\widehat{E}[\alpha\|K_{\mathrm{A}}]$, $V[\alpha\|K_{\mathrm{A}}]$, $E[K_{\mathrm{B}}\|K_{\mathrm{A}}]$, $E[T_{\mathrm{B}}\|K_{\mathrm{A}}]$, $E[K_{\mathrm{C}}\|K_{\mathrm{A}}]$, $E[T_{\mathrm{C}}\|K_{\mathrm{A}}]$	Fig. 1b
Assessing K_{B}	$E[T_{\mathrm{B}}\|K_{\mathrm{B}}]$, $\widehat{E}[\alpha\|K_{\mathrm{B}}, K_{\mathrm{A}}]$, $V[\alpha\|K_{\mathrm{B}}, K_{\mathrm{A}}]$, $E[K_{\mathrm{C}}\|K_{\mathrm{B}},K_{\mathrm{A}}]$, $E[T_{\mathrm{C}}\|K_{\mathrm{B}},K_{\mathrm{A}}]$	Fig. 1c
Assessing K_{C}	$E[T_{\mathrm{C}}\|K_{\mathrm{C}}]$	Fig. 1d

$$E[K_{\mathrm{B}}|K_{\mathrm{A}} = k_{\mathrm{A}}], \tag{7}$$
$$E[T_{\mathrm{B}}|K_{\mathrm{A}} = k_{\mathrm{A}}]. \tag{8}$$

Similarly, the two estimates of Activity C can be obtained

$$E[K_{\mathrm{C}}|K_{\mathrm{A}} = k_{\mathrm{A}}], \tag{9}$$
$$E[T_{\mathrm{C}}|K_{\mathrm{A}} = k_{\mathrm{A}}]. \tag{10}$$

2.2.2. Milestone event K_B

The second milestone event refers to the observation of K_{B}.

First, $E[T_{\mathrm{B}}|K_{\mathrm{B}}]$ can be immediately obtained

$$E[T_{\mathrm{B}}|K_{\mathrm{B}} = k_{\mathrm{B}}] = mk_{\mathrm{B}}.$$

Second, the arc reversal occurs from K_{B} to α, which is shown in Fig. 1c. It should be noted that both nodes inherit each other's conditional predecessors upon the arc reversal (Shachter, 1988). In this case, K_{B} inherits an arc from K_{A}. The linear Bayes' theorem states that $\widehat{E}[K_{\mathrm{B}}|\alpha, K_{\mathrm{A}} = k_{\mathrm{A}}] = c_2\alpha + d_2$, where $d_2 = a_0 + a_1K_{\mathrm{A}}$.

Since $E[K_{\mathrm{B}}|\alpha, K_{\mathrm{A}} = k_{\mathrm{A}}] = q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_{\mathrm{A}}}}\right)\alpha + pqk_{\mathrm{A}}\left(\frac{1/\sigma^2_{K_{\mathrm{A}}}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_{\mathrm{A}}}}\right)$, we know that $c_2 = q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_{\mathrm{A}}}}\right)$ and $d_2 = pqk_{\mathrm{A}}\left(\frac{1/\sigma^2_{K_{\mathrm{A}}}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_{\mathrm{A}}}}\right)$.

Thus, the posterior variance of the risk factor given observing K_{B} and K_{A} can be obtained

$$V^{-1}[\alpha|K_{\mathrm{B}}, K_{\mathrm{A}}] = V^{-1}[\alpha|K_{\mathrm{A}}] + c_2^2 V^{-1}[K_{\mathrm{B}}|\alpha, K_{\mathrm{A}}]. \tag{11}$$

Next, the posterior mean of the risk factor given observing K_{B} and K_{A} can be obtained

$$\widehat{E}[\alpha|K_{\mathrm{B}} = k_{\mathrm{B}}, K_{\mathrm{A}} = k_{\mathrm{A}}] = \widehat{E}[\alpha|K_{\mathrm{A}} = k_{\mathrm{A}}]\frac{V^{-1}[\alpha \mid K_{\mathrm{A}}]}{V^{-1}[\alpha|K_{\mathrm{B}}, K_{\mathrm{A}}]} + c_2(k_{\mathrm{B}} - d_2)\frac{V^{-1}[K_{\mathrm{B}}|\alpha, K_{\mathrm{A}}]}{V^{-1}[\alpha|K_{\mathrm{B}}, K_{\mathrm{A}}]}. \tag{12}$$

Third, the expected amount of resource for Activity C can be obtained by substituting the posterior mean of the risk factor that is derived above

$$E[K_{\mathrm{C}}|K_{\mathrm{B}} = k_B, K_{\mathrm{A}} = k_{\mathrm{A}}]. \tag{13}$$

Similarly, the expected duration of Activity C can be obtained

$$E[T_{\mathrm{C}}|K_{\mathrm{B}} = k_B, K_{\mathrm{A}} = k_{\mathrm{A}}]. \tag{14}$$

2.2.3. Milestone event K_C

The last milestone event, the assessment of K_{C}, is demonstrated in Fig. 1d, where the arc is reversed from K_{C} to α, and thus K_{C} inherits the arcs from K_{A} and K_{B}. In the current network, we merely update the duration of Activity C

$$E[T_{\mathrm{C}}|K_{\mathrm{C}} = k_{\mathrm{C}}] = nk_{\mathrm{C}}.$$

Of course, the posterior mean and variance of the risk factor must be updated in a similar fashion if more activities are connected with the risk factor concerning the full network.

2.3. Interpretation of the model

In the proposed linear Bayesian model, the set of K_i and α can be interpretable in three ways given the quantities assumed:

$$T_i \mid K_i \sim (0.5K_i, 2), \quad K_i \mid \alpha \sim (5\alpha, 3), \quad \text{and} \quad \alpha \sim (50, 10).$$

K_i can be interpreted as the amount of material resource, the amount of labor resource, and the amount of overhead resource, all of which are required for work completion. Since the amount of various forms of resources is affected by the risk factor, K_i should be treated as a random variable, rather than a decision variable in this study.

One way of treating K_i is to interpret it as the amount of material resource required for work completion. It may be reasonable, in many cases, to assume that the amount of resources required for the completion of an activity can possibly be observed, or at least assessed near work completion. For example, if we paint the same type of surfaces at a constant speed, the total amount of paint required for work completion can be assessed before work completion. If nine gallons of paint is used up for completing the first 90% of work, ten gallons of paint will be used for the total work.

Suppose α represents the weather unsuitability index (unit: %) measuring the degree of weather unsuitability for work progression. The amount of paint required for work completion is appreciably influenced by the weather risk factor in construction projects. The weather risk is a multidimensional concept reflecting various aspects such as temperature, humidity, shade, and stability of airflow (American Association of State Highway Officials, 1973). For example, in cold weather, paint tends to thicken and painters often use thinner to facilitate application, resulting in a thin coat of paint which must be corrected by adding extra coats of paint.

The contribution coefficient of activity duration might be measured in the unit of days/gallon. In this case, it is 0.5 days/gallon, which is the inverse of 2 gallons/day, meaning the rate of paint use in a normal condition. The coefficient of the amount of paint required is 5 gallons/%, meaning that 5 gallons of paint should be additionally allocated given a 1% increase in the weather unsuitability index. The marginal expected amount of paint required for activity i is 250 gallons:

$$E[K_i] = 250 \text{ gallons } (5 \text{ gallons}/\% \times 50\%).$$

The marginal expected duration is 125 days:

$$E[T_i] = 125 \text{ days } (0.5 \text{ days/gallon} \times 250 \text{ gallons}).$$

Another approach to interpret K_i would be treating it as the amount of labor resource required for work completion, which can reasonably be observed or at least assessed near work completion. For example, for a labor-intensive activity, K_i would be measured in the unit of man-hour or man-day. The coefficient of activity duration would be 0.5/man, which is the inverse of 2 men, meaning that a team of two men works in a normal condition. The coefficient of the amount of work resource is 5 man-days/%, meaning that additional work equivalent to 5 man-days' job is required given a 1% increase in the weather unsuitability index. Here, the marginal expected amount of labor required is 250 man-days:

$$E[K_i] = 250 \text{ man-days } (5 \text{ man-days}/\% \times 50\%).$$

The marginal expected duration is 125 days:

$$E[T_i] = 125 \text{ days } (0.5/\text{man} \times 250 \text{ man-days}).$$

The last form of interpreting K_i is the amount of overhead resource required for work completion. For example, K_i would be measured in the unit of utility. Kilo watt-day (kw-day) can be used in case of electricity. The coefficient of activity duration would be 0.5/kw, which is the inverse of 2 kw, meaning that 2 kw of electricity is required for work completion under a normal condition. The coefficient of the amount of work resource is 5 kw/%, meaning that additional electricity equivalent to 5 kw is required given a 1% increase in the weather unsuitability index. Here, the marginal expected amount of electricity is:

$$E[K_i] = 250 \text{ kw-days } (5 \text{ kw-days}/\% \times 50\%).$$

The marginal expected duration is 125 days:

$$E[T_i] = 125 \text{ days } (0.5/\text{kw} \times 250 \text{ kw-days}).$$

The following simple example helps understand how the updating processes proposed in the previous section apply to the estimation in project management. The duration of activity is given as,

$$T_A|K_A \sim (0.2K_A, 5) \quad \text{and} \quad T_B|K_B \sim (0.5K_B, 6).$$

The amount of resource required for activity is given as,

$$K_A|\alpha \sim (3\alpha, 20) \quad \text{and} \quad K_B|\alpha \sim (2\alpha, 25).$$

The risk factor is specified as,

$$\alpha \sim (50, 10).$$

Recall that the unit of activity duration is the number of days; that of the amount of resource is the number of gallons of paint; and that of the risk factor is the weather unsuitability index in %. Prior to the onset of the project, the marginal expectations are

$$E[K_A] = 150 \text{ gallons}, \quad E[T_A] = 30 \text{ days}, \quad E[K_B] = 100 \text{ gallons}, \quad \text{and} \quad E[T_B] = 50 \text{ days}.$$

Suppose K_A is assessed as 180 gallons near the completion of Activity A. Then, the expected duration of Activity A is immediately updated

$$E[T_A|K_A = 180] = 36 \text{ days}.$$

The posterior mean and variance of the risk factor is updated, using Eqs. (1) and (2)

$$\widehat{E}[\alpha|K_A = 180] = 53.08\% \quad \text{and} \quad V[\alpha|K_A = 180] = 8.32^2\%^2.$$

Thus, the amount of resource for and the duration of upcoming activity can be predicted, using Eqs. (3) and (4)

$$E[K_B|K_A = 180] = 106.15 \text{ gallons} \quad \text{and} \quad E[T_B|K_A = 180] = 53.08 \text{ days}.$$

3. Application: a data warehouse system implementation project

To demonstrate how the proposed model can be applied into real projects, consider a data warehouse system implementation project. Table 2 describes the set of 14 activities, the immediate precedence relationships, and the means and standard deviations of variables. The precedence diagram representation in Fig. 2a indicates that eight paths exist in the project network. According to the field experts in the area of data warehouse systems, one of the most critical success factors in the data warehouse system implementation projects is teamwork. Teamwork has a significant influence on the project duration, the amount of rework, and the system quality. Thereby, teamwork can be regarded as a risk factor, reflecting various aspects such as individual skill set, interpersonal communication, code quality, and software unit procedures. In particular, α is defined as the teamwork unsuitability index in which the neutral point is set to 50 (unit: %). The prior distribution of α is set with the following mean and standard deviation

$$\alpha \sim (50, 15).$$

In the hierarchy, as in Fig. 2b, α might affect the amount of labor resource required for work completion, K_i (unit: man-day), which in turn might affect the duration of activity, T_i (unit: day). For example, the expected amount of labor required for Activity B increases by 2 man-days for one unit increase in the risk factor. For one unit increase in the amount of labor required for Activity B, its expected duration increases by 0.25 days.

We simulated the project progression one-time by generating K_i and T_i, and recorded the milestone event over a length of time, t. In the analysis, a milestone event is referred to the assessment of K_i, which invokes the updating of α as well as K_i and T_i for the remaining activities. Note that the number of possible sequences of activity realization increases exponentially as the size of project network grows. The purpose of simulation is to demonstrate that the proposed model monitors the critical path activities on a real time basis and re-estimates the project duration by updating the

Table 2
Data warehouse system implementation project

Activity	Description	Predecessors	$T_i\|K_i$	$K_i\|\alpha$
A	Preliminary training	–	(21, 3)	—
B	User requirement analysis	A	$(0.25K_B, 5)$	$(2\alpha, 38)$
C	Analytic process definition	B	$(0.2K_C, 6)$	$(3\alpha, 40)$
D	Analytic data definition	B	(32, 5)	—
E	Data modeling	C, D	$(0.1K_E, 4)$	$(4\alpha, 45)$
F	Extract, transform, and load (ETL) flow design	E	$(0.15K_F, 6)$	$(5\alpha, 55)$
G	On-line analytical processing (OLAP) design	E	$(0.2K_G, 5)$	$(3\alpha, 33)$
H	Database design	E	$(0.33K_H, 4)$	$(2\alpha, 24)$
I	Test criteria and method design	E	(34, 5)	—
J	Data migration system implementation	F	$(0.25K_J, 3)$	$(2\alpha, 27)$
K	OLAP implementation	G	$(0.2K_K, 4)$	$(3\alpha, 35)$
L	Database implementation	H, I	$(0.2K_L, 4)$	$(3\alpha, 35)$
M	Integration	J, K, L	(11, 2.5)	—
N	Testing and system open	M	(8, 1.5)	—

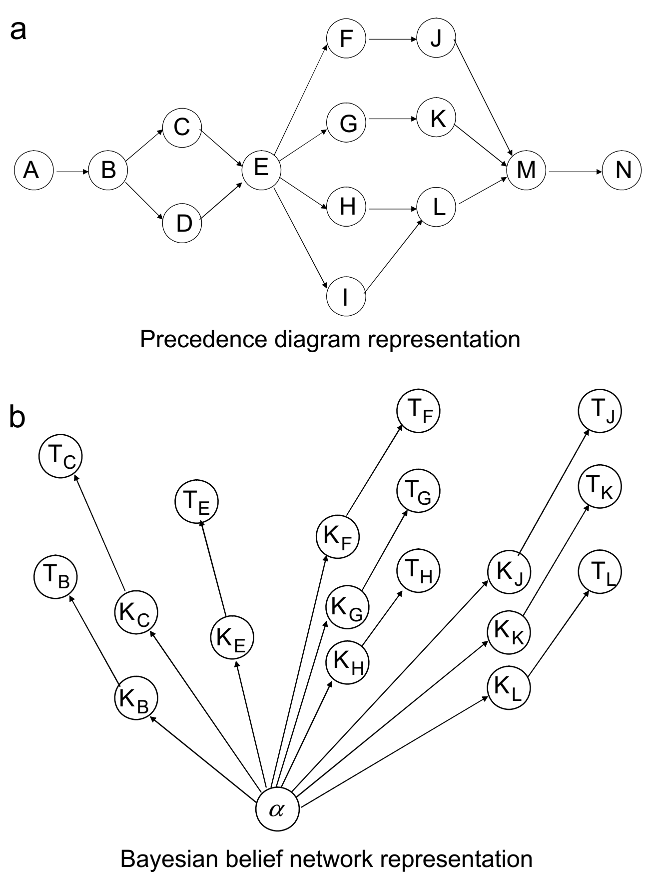

Fig. 2. Graphical representation for the data warehouse system implementation project.

expected amount of resource for and the expected duration of upcoming activities. Updating the estimates is very important in high risk, high technology projects because a great deal of volatility involves estimating activity duration and project duration.

A normal distribution is used to generate the random variables in this project. For the startup of an activity, the earliest starting time approach is adopted; as soon as all the immediate preceding activities are finished, the succeeding activity begins. Activity duration that is independent of another activity duration is omitted in the Bayesian belief network: T_A, T_D, T_I, T_M, and T_N. Other activities might be correlated with each other through the risk factor. Prior to the onset of the project, the critical path activities are determined as: A–B–D–E–I–L–M–N, resulting in the expected project duration of 181 days.

Table 3
Simulation results: data warehouse system implementation project

t	K_i	Posterior α	Critical path activities	$E[T]$	$t + E[T]$
45	$K_B = 115$	(52.88, 11.77)	B–D–E–H–L–M–N	144.53	189.53
83	$K_C = 170$	(54.54, 8.83)	C–E–H–L–M–N	110.53	193.53
104	$K_E = 230$	(55.67, 6.94)	E–H–L–M–N	92.14	196.14
135	$K_H = 90$	(52.99, 6.01)	F–J–M–N	57.24	192.24
141	$K_G = 178$	(54.45, 5.27)	G–K–M–N	53.27	194.27
147	$K_F = 280$	(54.74, 4.76)	F–J–M–N	48.37	195.37
172	$K_K = 155$	(54.30, 4.40)	L–M–N	22.58	194.58
173	$K_L = 160$	(54.18, 4.12)	L–M–N	20	193
174	$K_J = 106$	(54.08, 3.94)	J–M–N	20.5	194.5

Table 3 summarizes the findings of the simulation. The generation of the milestone event on the time line invokes the update of posterior α, which affects the expected duration of the remaining activities. Hence, the critical path activities must be updated, and the re-estimate of the duration of the remaining project follows. Even for a 14-activity project, both the critical path activities and the expected project duration change considerably compared to the initial expectation.

In a way to evaluate the performance of the proposed model, we estimated the durations of activities using the basic PERT approach. The basic PERT approach assumes the probabilistic independence among activity durations, which means that the duration of activity is determined by the marginal distribution. On the other hand, the linear Bayesian model of this study updates the posterior mean and standard deviation of α at each milestone event, and then sequentially updates the conditional expected durations of remaining activities. The prediction accuracy is measured by the mean absolute deviation (MAD), where the deviation represents the difference between the simulated duration and the expected duration. On an individual activity level, the MAD using the proposed model is 2.17 days whereas the MAD using the basic PERT is 3.52 days. The proposed model improves the prediction accuracy by 38.36% in the data warehouse implementation project. Since activities A, D, I, M, and N are independent of each other, both approaches give the same expected durations for these activities. If these activities are excluded in computing the MAD, the improvement in prediction accuracy is 50.50% (MAD = 2.06 days for the proposed model vs. MAD = 4.16 days for the basic PERT).

Needless to say, it is crucial to effectively deal with enormous uncertainty associated with the estimation of project duration for the successful accomplishment of projects. Allowance of the probabilistic dependence among activities and the sequential update of activity duration, resource, and the risk factor in the linear Bayesian framework contribute to more realistic estimate and update in currently prevalent high technology projects.

4. Concluding remarks

In this paper, we considered the duration estimation problem in projects. Usually, the durations of activities need to be updated throughout the entire project period. Unlike the basic PERT-type approaches, the research allowed the probabilistic dependence among activities in a particular project using the hierarchical structure of activity duration, resource, and a risk factor. This dependence structure was graphically modeled by introducing the Bayesian belief network. In so doing, the linear Bayesian model was applied to sequentially update the means and standard deviations of random variables. The application project showed that the proposed method provides 38.36% higher prediction accuracy in estimating the duration of an individual activity, compared to the basic PERT approach. This study contributes to project estimation in the following three perspectives.

First, one can better understand the duration estimation problem in projects through a new graphical paradigm. The traditional PERT-type approaches were concerned with the precedence relationship of activities only, which gave no room for learning in duration estimation. The Bayesian belief network can clearly depict both statistical dependence and conditional independence among activity duration as well as among resource requirements. The dependence of these random variables can be structured through a hierarchy of a risk factor.

The second contribution is the explicit modeling of probabilistic dependence and inferences. At each step of a milestone event, we can obtain the posterior mean and standard deviation of the risk factor, and then update the expected amount of resource required for and the expected durations of remaining activities. Thus, we can keep track of the critical path activities, which vary over time, on a real time basis and update the expectation about the total project duration.

Lastly, the approach seems to be more practical to project managers in the field. The traditional PERT approach was expedient, but gave a static view for the entire project due to the lack of learning in the estimation. Although mending approaches such as the full Bayesian approach and the MCMC technique could model the probabilistic dependence among activities, tremendous computational load restricts the applicability. The proposed method in this study occupied a middle ground by retaining the computational simplicity while providing a probabilistic dependence mechanism among activities. Without the expertise in stochastic modeling, the managers can implement the linear Bayesian model on spreadsheets with minimal algebraic computations.

The limitations of the present study are related to the introduction of additional variables such as resource and the risk factor. In order to build the Bayesian belief network, it is necessary to define the prior distribution of the risk factor and estimate the contribution coefficients of activity duration and the amount of resource requirement.

For eliciting the prior distribution of the risk factor, field experts may involve and provide their subjective judgments. Then, several judgments should be combined into a single distribution. In the current research, the prior mean and standard deviation are to be provided. A good reference for this work would be found from Winkler (1968). Several weighted average methods such as equal weights, weights proportional to a ranking, and weights proportional to a self-rating are introduced. With these methods, one can form the prior distribution of the risk factor for a Bayesian analysis.

In order to combine the field experts' subjective judgments about a single point estimate, that is, the contribution coefficients in this study, one can apply a few linear weighting algorithms. For example, the weight of each individual expert's estimate can be determined by the out-performance probability, which means the fraction of occurrences in which its respective forecasting has performed the best in the past. An alternative way of determining the weight is that the weights are calculated to minimize the error variance of the combination. A detailed method about these types of weighting methods can be referred from de Menezes et al. (2000).

In future studies, the risk factor, which was the only one in the current study, might be extended into multiple risk cases. Presumably, it might be natural to assume that several risk factors simultaneously influence projects. In information technology projects like the one used in the application example, the risk factors might include teamwork, individual expertise, and change in system requirements. In construction projects, several factors such as weather, financial stability, geographical uncertainty, and teamwork might involve with project risks. In the current study, the risk factor serves as a parameter for resource requirement, which in turn serves as a parameter for activity duration. In order to be a comprehensive model that includes multiple risk factors, the distribution specified with multiple parameters should be examined.

Appendix

Eqs. (5)–(14) can be computed in detail as follows:

$$1/\sigma^2_{\alpha(1)} = 1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}, \tag{5}$$

$$\mu_{\alpha(1)} = \mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right), \tag{6}$$

$$\begin{aligned} E[K_B|K_A = k_A] &= E_\alpha[E[K_B|\alpha, K_A = k_A]] = q\widehat{E}[\alpha|K_A = k_A] \\ &= q\left\{\mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}, \end{aligned} \tag{7}$$

$$\begin{aligned} E[T_B|K_A = k_A] &= E_\alpha[E_{K_B}[E[T_B|K_B, \alpha, K_A = k_A]]] = mq\widehat{E}[\alpha|K_A = k_A] \\ &= mq\left\{\mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}, \end{aligned} \tag{8}$$

$$\begin{aligned} &E[K_C|K_A = k_A] \\ &\quad = r\left\{\mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}, \end{aligned} \tag{9}$$

$$\begin{aligned} &E[T_C|K_A = k_A] \\ &\quad = nr\left\{\mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}, \end{aligned} \tag{10}$$

$$1/\sigma^2_{\alpha(2)} = (1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}) + \left\{q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}^2 \Big/ \sigma^2_{K_B}, \tag{11}$$

$$\begin{aligned} \mu_{\alpha(2)} = &\left\{\mu_{\alpha(0)}\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right) + pk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\} \\ &\times\left[\frac{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}{(1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}) + \left\{q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}^2 \Big/ \sigma^2_{K_B}}\right] \\ &+\left[q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\left\{k_B - pqk_A\left(\frac{1/\sigma^2_{K_A}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}\right] \\ &\times\left[\frac{1/\sigma^2_{K_B}}{(1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}) + \left\{q\left(\frac{1/\sigma^2_{\alpha(0)}}{1/\sigma^2_{\alpha(0)} + p^2/\sigma^2_{K_A}}\right)\right\}^2 \Big/ \sigma^2_{K_B}}\right], \end{aligned} \tag{12}$$

$$\begin{aligned} E[K_C|K_B = k_B, K_A = k_A] &= E_\alpha[E[K_C|\alpha, K_B = k_B, K_A = k_A]] \\ &= r\widehat{E}[\alpha|K_B = k_B, K_A = k_A], \end{aligned} \tag{13}$$

$$E[T_C|K_B = k_B, K_A = k_A] = E_\alpha[E[T_C|\alpha, K_B = k_B, K_A = k_A]] = nr\widehat{E}[\alpha|K_B = k_B, K_A = k_A]. \quad (14)$$

References

American Association of State Highway Officials, 1973. Construction Manual for Highway Bridges and Incidental Structures. American Association of State Highway Officials. pp. 48–50.

Bunn, D.W., 1975. A Bayesian approach to the linear combination of forecasts. Operational Research Quarterly 26, 325–329.

Chatzogou, P., Macaulay, L.A., 1996. A review of existing models for project planning and estimation and the need for a new approach. International Journal of Project Management 14, 173–183.

Cho, S., 2006. An exploratory project expert system for elicitation of correlation coefficient and sequential updating of duration estimation. Expert Systems with Applications 30, 553–560.

Cho, S., Covaliu, Z., 2003. Sequential estimation and crashing in PERT networks with statistical dependence. International Journal of Industrial Engineering 10, 391–399.

Covaliu, Z., Soyer, R., 1996. Bayesian project management. In: Proc. Conf. ASA section on Bayesian Statistical Science, pp. 208–213.

Covaliu, Z., Soyer, R., 1997. Bayesian learning in project management networks. In: Proc. Conf. ASA section on Bayesian Statistical Science, pp. 257–260.

de Menezes, L.M., Bunn, D.W., Taylor, J.W., 2000. Review of guidelines for the use of combined forecasts. European Journal of Operational Research 120, 190–204.

Diaz, A.M., Virto, M.A., Martin, J. Insua, D.R., 2003. Approximate solutions to semi Markov decision processes through Markov chain Montecarlo methods. In: Lecturer Notes in Computer Science, vol. 2809, pp. 151–162.

Dodin, B.M., Elmaghraby, S.E., 1985. Approximating the criticality indices of the activities in PERT networks. Management Science 31, 207–223.

Farrow, M., Goldstein, M., Spiropoulos, T., 1997. Developing a Bayes linear decision support system for a brewery. In: French, S., Smith, J.Q. (Eds.), The Practice of Bayesian Analysis. Edward Arnold, pp. 71–106.

Hartigan, J.A., 1969. Linear Bayesian methods. Journal of the Royal Statistical Society Series B 31, 446–454.

Heckerman, D., Geiger, D., Chickering, D., 1995. Learning Bayesian networks: The combination of knowledge and statistical data. Machine Learning 20, 197–243.

Howard, R.A., Matheson, J.E., 1981. Influence diagrams. In: Readings on the Principles And Applications of Decision Analysis, vol. II, Strategic Decisions Group, Menlo park, California, pp. 719–762.

Jensen, F.V., 1996. An Introduction to Bayesian Networks. Taylor & Francis, Group. p. 18.

Jenzarli, A., 1994. PERT belief networks, Report 535, The University of Tampa, FL.

Keefer, D.L., Bodily, S.E., 1983. Three-point approximations for continuous random variables. Management Science 29 (5), 595–609.

Li, K.H., Wong, H., Troutt, M., 2001. An approximate Bayesian algorithm for combining forecasts. Decision Sciences 32, 453–471.

Lindley, D.V., Singpurwalla, N.D., 1986. Multivariate distributions for the life lengths of components of a system sharing a common environment. Journal of Applied Probability 23, 418–431.

MacCrimmon, K.R., Ryavec, C.A., 1964. An analytic study of the PERT assumptions. Operations Research 14, 16–37.

Magott, J., Skudlarski, K., 1993. Estimating the mean completion time of PERT networks with exponentially distributed durations of activities. European Journal of Operational Research 71, 70–79.

Marshall, K.T., Oliver, R.M., 1995. Decision Making and Forecasting. McGraw-Hill.

Marshall, A.W., Olkin, I., 1967. A multivariate exponential distribution. Journal of the American Statistical Association 62, 30–44.

Shachter, R.D., 1988. Probabilistic inference and influence diagrams. Operations Research 36, 589–604.

Smith, J.Q., 1989. Influence diagrams for Bayesian decision analysis. European Journal of Operational Research 40, 363–376.

Smith, A.F.M., Gelfand, A.E., 1992. Bayesian statistics without tears: A sampling-resampling perspective. American Statistician 46, 84–88.

Smith, J.E., Holtzman, S., Matheson, J.E., 1993. Structuring conditional relationships in influence diagrams. Operations Research 41, 280–297.

Van Dorp, J.R., Duffey, M.R., 1999. Statistical dependence in risk analysis for project networks using Monte Carlo methods. International Journal of Production Economics 58, 17–29.

Virto, M.A., Martin, J., Insua, D.R., 2002. An approximate solutions of complex influence diagrams through MCMC methods. In: Gamez, Salmeron (Eds.), Proc. First European Workshop on Probabilistic Graphical Models, pp. 169–175.

Winkler, R.L., 1968. The consensus of subjective probability distribution. Management Science 15 (2), B-61–B-75.