Handout 3

Introduction to ARMA Time Series Models
Moving Average Models

Class notes for Statistics 451: Applied Time Series
Iowa State University

January 7, 2007
17h 8min

Deviations from the Sample Mean

- The sample mean is computed as

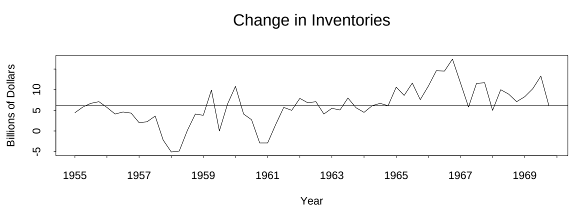
$$\bar{Z} = \frac{\sum_{t=1}^{n} Z_t}{n}$$

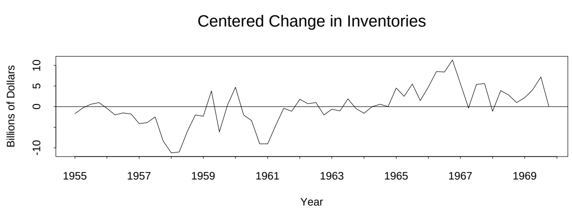
- We use the centered realization $Z_t - \bar{Z}$ in the computation of many statistics. For example

$$S_Z = \sqrt{\frac{\sum_{t=1}^{n}(Z_t - \bar{Z})^2}{n-1}}$$

- Also used in computing sample autocovariances and autocorrelations. Subtracting out the mean ($\bar{Z}$) does not change the sample variance ($\widehat{\gamma}_0$), sample autocovariances ($\widehat{\gamma}_k$), or sample autocorrelations ($\widehat{\rho}_k$) of a realization.

Change in Business Inventories 1955-1969 and Centered Change in Business Inventories 1955-1969





Deviations from the Process Mean

- The process mean is $\mu = \mathrm{E}(Z_t)$.
- Some derivations are simplified by using $\dot{Z}_t = Z_t - \mu$ to compute model properties (only the mean has changed).
- In particular,

$$\mathrm{E}(\dot{Z}_t) = \mathrm{E}(Z_t - \mu) = \mathrm{E}(Z_t) - \mathrm{E}(\mu) = \mu - \mu = 0$$

Similarly,

$$\gamma_0 = \mathrm{Var}(\dot{Z}_t) = \mathrm{Var}(Z_t - \mu) = \mathrm{Var}(Z_t) + \mathrm{Var}(\mu) = \mathrm{Var}(Z_t) = \sigma_Z^2$$

and

$$\begin{aligned}\gamma_k &= \mathrm{Cov}(\dot{Z}_t, \dot{Z}_{t+k}) = \mathrm{Cov}(Z_t - \mu, Z_{t+k} - \mu) = \mathrm{Cov}(Z_t, Z_{t+k}) \\ \rho_k &= \gamma_k/\gamma_0\end{aligned}$$

Backshift Operator Notation

Backshift operators:
$\mathrm{B}Z_t = Z_{t-1}$
$\mathrm{B}^2 Z_t = \mathrm{B}\mathrm{B}Z_t = \mathrm{B}Z_{t-1} = Z_{t-2}$
$\mathrm{B}^3 Z_t = Z_{t-3}$, etc.

If C is a constant, then $\mathrm{B}C = C$

Differencing operators:
$(1-\mathrm{B})Z_t = Z_t - \mathrm{B}Z_t = Z_t - Z_{t-1}$
$(1-\mathrm{B})^2 Z_t = (1 - 2\mathrm{B} + \mathrm{B}^2)Z_t = Z_t - 2Z_{t-1} + Z_{t-2}$

Seasonal differencing operators:
$(1-\mathrm{B}^4)Z_t = Z_t - \mathrm{B}^4 Z_t = Z_t - Z_{t-4}$
$(1-\mathrm{B}^{12})Z_t = Z_t - \mathrm{B}^{12} Z_t = Z_t - Z_{t-12}$

Polynomial Operator Notation

$\phi(\mathrm{B}) = \phi_p(\mathrm{B}) \equiv (1 - \phi_1\mathrm{B} - \phi_2\mathrm{B}^2 - \cdots - \phi_p\mathrm{B}^p)$
is useful for expressing the AR(p) model.

For example, with $p = 2$, an AR(2) model.

$$\begin{aligned}\phi_2(\mathrm{B})\dot{Z}_t &= a_t \\ (1 - \phi_1\mathrm{B} - \phi_2\mathrm{B}^2)\dot{Z}_t &= a_t \\ \dot{Z}_t - \phi_1\dot{Z}_{t-1} - \phi_2\dot{Z}_{t-2} &= a_t\end{aligned}$$

or

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} + a_t$$

Other polynomial operators:
$\theta(\mathrm{B}) = \theta_q(\mathrm{B}) \equiv (1 - \theta_1\mathrm{B} - \theta_2\mathrm{B}^2 - \cdots - \theta_q\mathrm{B}^q)$ [as in $\dot{Z}_t = \theta_q(\mathrm{B})a_t$]
$\psi(\mathrm{B}) = \psi_\infty(\mathrm{B}) \equiv (1 + \psi_1\mathrm{B} + \psi_2\mathrm{B}^2 + \cdots)$ [as in $\dot{Z}_t = \psi(\mathrm{B})a_t$]
$\pi(\mathrm{B}) = \pi_\infty(\mathrm{B}) \equiv (1 - \pi_1\mathrm{B} - \pi_2\mathrm{B}^2 - \cdots)$ [as in $\pi(\mathrm{B})\dot{Z}_t = a_t$]

General ARMA Model in Terms of $\dot{Z}_t$

$$
\begin{aligned}
\dot{Z}_t &\equiv Z_t - \mu \\
\phi_p(\mathrm{B})\dot{Z}_t &= \theta_q(\mathrm{B})a_t \\
(1 - \phi_1\mathrm{B} - \phi_2\mathrm{B}^2 - \cdots - \phi_p\mathrm{B}^p)\dot{Z}_t &= \\
(1 - \theta_1\mathrm{B} &- \theta_2\mathrm{B}^2 - \cdots - \theta_q\mathrm{B}^q)a_t \\
\dot{Z}_t - \phi_1\dot{Z}_{t-1} - \phi_2\dot{Z}_{t-2} - \cdots - \phi_p\dot{Z}_{t-p} &= \\
a_t - \theta_1 a_{t-1} &- \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}
\end{aligned}
$$

or

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} + \cdots + \phi_p\dot{Z}_{t-p} - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} + a_t$$

Mean (μ) and Constant Term (θ_0)

Replace $\dot{Z}_t$ with $\dot{Z}_t = Z_t - \mu$ and solve for $Z_t = \cdots$.

For example, using $p = 2$ and $q = 2$, giving an ARMA(2,2) model:

$$\dot{Z}_t = \phi_1\dot{Z}_{t-1} + \phi_2\dot{Z}_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t$$

and

$$
\begin{aligned}
Z_t - \mu &= \phi_1(Z_{t-1} - \mu) + \phi_2(Z_{t-2} - \mu) - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t \\
Z_t &= \mu - \phi_1\mu - \phi_2\mu + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t \\
Z_t &= \theta_0 + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} - \theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t
\end{aligned}
$$

where $\theta_0 = \mu - \phi_1\mu - \phi_2\mu = \mu(1 - \phi_1 - \phi_2)$ is the ARMA model "constant term." Also, $\mathrm{E}(Z_t) = \mu = \theta_0/(1 - \phi_1 - \phi_2)$.
Note that $\mathrm{E}(Z_t) = \mu = \theta_0$ for an MA(q) model.

Mean and Variance of the MA(1) Model

Model: $Z_t = \theta_0 - \theta_1 a_{t-1} + a_t, \quad a_t \sim \mathrm{nid}(0, \sigma_a^2)$

Mean: $\mu_Z \equiv \mathrm{E}(Z_t) = \mathrm{E}(\theta_0 - \theta_1 a_{t-1} + a_t) = \theta_0$.

Variance: $\gamma_0 \equiv \mathrm{Var}(Z_t) \equiv \mathrm{E}[(Z_t - \mu_Z)^2] = \mathrm{E}(\dot{Z}_t^2)$

$$
\begin{aligned}
\gamma_0 &= \mathrm{E}(\dot{Z}_t^2) \\
&= \mathrm{E}[(-\theta_1 a_{t-1} + a_t)^2] \\
&= \mathrm{E}[(\theta_1^2 a_{t-1}^2 - 2\theta_1 a_{t-1}a_t + a_t^2)] \\
&= \theta_1^2\mathrm{E}(a_{t-1}^2) - 2\theta_1\mathrm{E}(a_{t-1}a_t) + \mathrm{E}(a_t^2) \\
&= \theta_1^2\sigma_a^2 - 0 + \sigma_a^2 \\
&= (\theta_1^2 + 1)\sigma_a^2
\end{aligned}
$$

Autocovariance and Autocorrelation Functions for the MA(1) Model

Autocovariance: $\gamma_k \equiv \mathrm{Cov}(Z_t, Z_{t+k}) \equiv \mathrm{E}(\dot{Z}_t\dot{Z}_{t+k})$

$$
\begin{aligned}
\gamma_1 &\equiv \mathrm{E}(\dot{Z}_t\dot{Z}_{t+1}) \\
&= \mathrm{E}[(-\theta_1 a_{t-1} + a_t)(-\theta_1 a_t + a_{t+1})] \\
&= \mathrm{E}(\theta_1^2 a_{t-1}a_t - \theta_1 a_{t-1}a_{t+1} - \theta_1 a_t^2 + a_t a_{t+1}) \\
&= 0 - 0 - \theta_1\sigma_a^2 + 0 \\
&= -\theta_1\sigma_a^2
\end{aligned}
$$

Thus $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1}{(1+\theta_1^2)}$.

Using similar operations, it is easy to show that $\gamma_2 \equiv \mathrm{E}(\dot{Z}_t\dot{Z}_{t+2}) = 0$ and thus $\rho_2 = \gamma_2/\gamma_0 = 0$. In general, for MA(1), $\rho_k = \gamma_k/\gamma_0 = 0$ for $k > 1$.

Partial Autocorrelation Function

For <u>any</u> model, the <u>process</u> PACF ϕ_{kk} can be computed from the <u>process</u> ACF from $\rho_1, \rho_2, \ldots$ using

$$
\begin{aligned}
\phi_{1,1} &= \rho_1 \\
\phi_{kk} &= \frac{\rho_k - \sum_{j=1}^{k-1}\phi_{k-1,j}\,\rho_{k-j}}{1 - \sum_{j=1}^{k-1}\phi_{k-1,j}\,\rho_j}, \quad k = 2, 3, \ldots \qquad \text{(cf. 2.5.25)}
\end{aligned}
$$

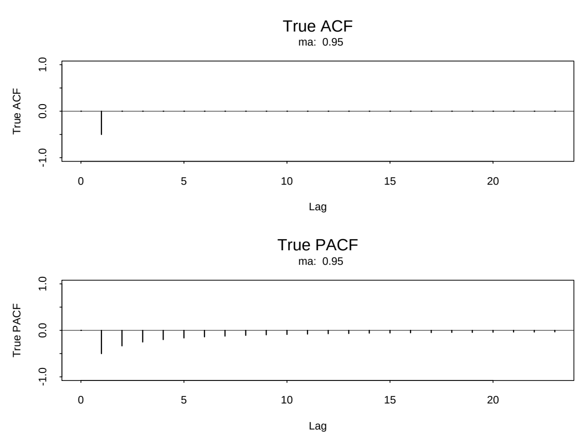
where

$$\phi_{kj} = \phi_{k-1,j} - \phi_{kk}\phi_{k-1,k-j} \quad (k = 3, 4, \ldots;\ j = 1, 2, \ldots, k-1)$$

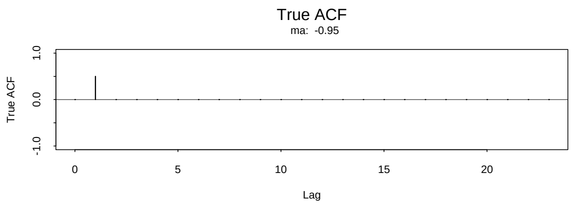
Similarly, the <u>sample</u> PACF can be computed from the <u>sample</u> ACF. That is, $\widehat{\phi}_{kk}$ is a function of $\widehat{\rho}_1, \widehat{\rho}_2, \ldots$.

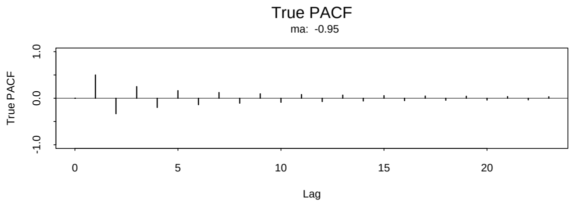
This formula for computing the PACF is based on the solution of the Yule-Walker equations.

True ACF and PACF for MA(1) Model with $\theta_1 = .95$

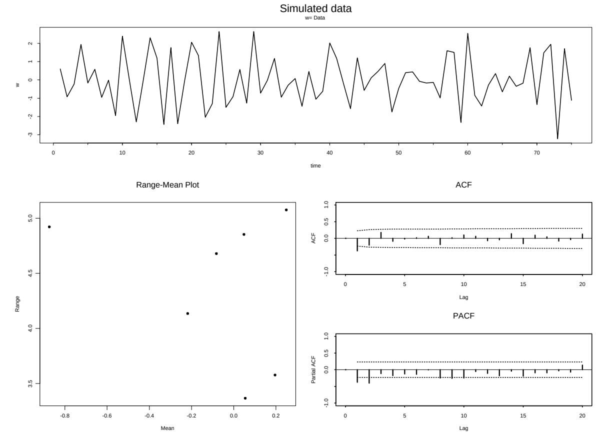


True ACF and PACF for MA(1) Model with $\theta_1 = -.95$

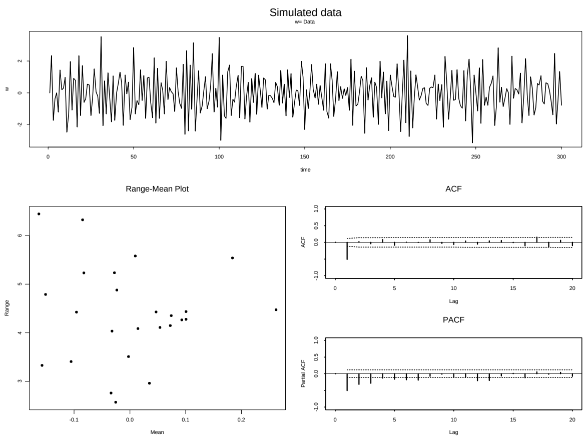




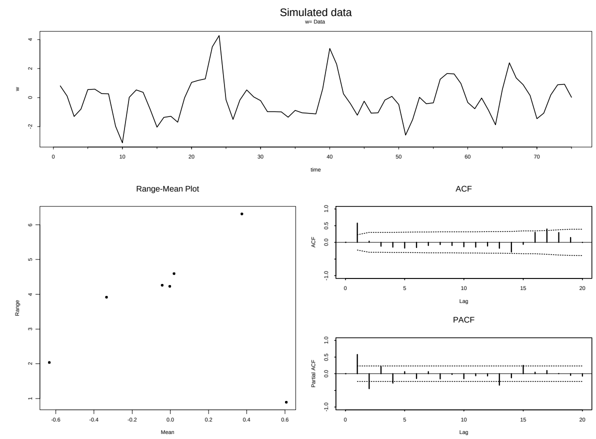
Simulated Realization ($\mathsf{MA}(1), \theta_1 = .95, n = 75$)
Graphical Output from Function `iden`



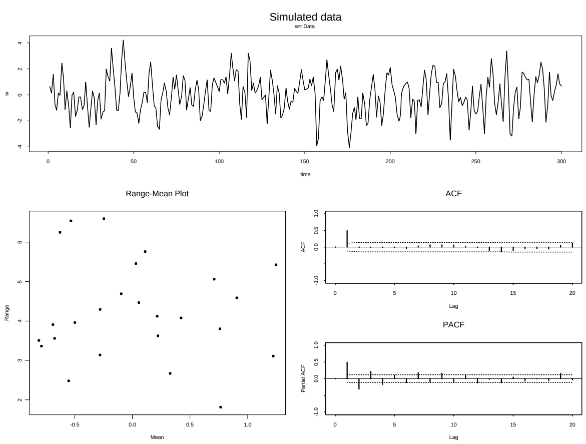
Simulated Realization ($\mathsf{MA}(1), \theta_1 = .95, n = 300$)
Graphical Output from Function `iden`



Simulated Realization ($\mathsf{MA}(1), \theta_1 = -.95, n = 75$)
Graphical Output from Function `iden`



Simulated Realization ($\mathsf{MA}(1), \theta_1 = -.95, n = 300$)
Graphical Output from Function `iden`



Geometric Series

$$\frac{1}{(1-\theta_1 \mathsf{B})} = (1-\theta_1\mathsf{B})^{-1} = (1+\theta_1\mathsf{B}+\theta_1^2\mathsf{B}^2+\cdots)$$

Provides a convenient method for re-expressing parts of some ARMA models.

Also

$$\frac{1}{(1-\phi_1 \mathsf{B})} = (1-\phi_1\mathsf{B})^{-1} = (1+\phi_1\mathsf{B}+\phi_1^2\mathsf{B}^2+\cdots)$$

Using the Geometric Series to Re-express the MA(1) Model as an Infinite AR

$$\begin{aligned}\dot{Z}_t &= (1-\theta_1\mathrm{B})a_t \\ a_t &= (1-\theta_1\mathrm{B})^{-1}\dot{Z}_t \\ a_t &= (1+\theta_1\mathrm{B}+\theta_1^2\mathrm{B}^2+\cdots)\dot{Z}_t \\ \dot{Z}_t &= -\theta_1\dot{Z}_{t-1}-\theta_1^2\dot{Z}_{t-2}-\cdots+a_t\end{aligned}$$

Thus the MA(1) can be expressed as an infinite AR model.

If $-1 < \theta_1 < 1$, then the weight on the old observations is decreasing with age (having more practical meaning). This is the condition of "invertibility" for an MA(1) model.

Using the Back-substitution to Re-express the MA(1) Model as an Infinite AR

$$\dot{Z}_t = -\theta_1 a_{t-1} + a_t$$

$$\begin{aligned}a_{t-1} &= \dot{Z}_{t-1}+\theta_1 a_{t-2} \\ a_{t-2} &= \dot{Z}_{t-2}+\theta_1 a_{t-3} \\ a_{t-3} &= \dot{Z}_{t-3}+\theta_1 a_{t-4}\end{aligned}$$

Substituting, successively, $a_{t-1}, a_{t-2}, a_{t-3}\ldots$, shows that

$$\dot{Z}_t = -\theta_1\dot{Z}_{t-1}-\theta_1^2\dot{Z}_{t-2}-\cdots+a_t$$

This method works, more generally, for higher-order MA(q) models, but the algebra is tedious.

Notes on the MA(1) Model

Given $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{-\theta_1}{(1+\theta_1^2)}$ we can see

- $-.5 \le \rho_1 \le .5$
- Solving the $\rho_1 = \cdots$ quadratic equation for θ_1 gives

  $$\theta_1 = \frac{-1}{2\rho_1} \pm \sqrt{\frac{1}{(2\rho_1)^2}-1}$$

  The two solutions are related by

  $$\theta_1 = \frac{1}{\theta_1'}$$

  The solution with $-1 < \theta_1 < 1$ is the "invertible" parameter value.
- Substituting $\widehat{\rho}_1$ for ρ_1 provides an estimator for θ_1.

Mean, Variance, and Covariance of the MA(q) Model

Model: $Z_t = \theta_0 + \theta_q(\mathrm{B})a_t = \theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t$

Mean: $\mu_Z \equiv \mathrm{E}(Z_t) = \mathrm{E}(\theta_0 - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t) = \theta_0$.

Variance: $\gamma_0 \equiv \mathrm{Var}(Z_t) \equiv \mathrm{E}[(Z_t-\mu_Z)^2] = \mathrm{E}(\dot{Z}^2)$

Centered MA(q)**:** $\dot{Z}_t = \theta_q(\mathrm{B})a_t = -\theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} + a_t$

$$\begin{aligned}\mathrm{Var}(Z_t) \equiv \gamma_0 &= \mathrm{E}(\dot{Z}^2) = (1+\theta_1^2+\cdots+\theta_q^2)\sigma_a^2 \\ \mathrm{Cov}(Z_t, Z_{t+k}) = \gamma_k &= \cdots & (3.2.11) \\ \rho_k &= \frac{\gamma_k}{\gamma_0} = \cdots & (3.2.12)\end{aligned}$$

Re-expressing the MA(q) Model as an Infinite AR

More generally, any MA(q) model can be expressed as

$$\begin{aligned}\dot{Z}_t &= \theta_q(\mathrm{B})a_t \\ \frac{1}{\theta_q(\mathrm{B})}\dot{Z}_t &= \pi(\mathrm{B})\dot{Z}_t = (1-\pi_1\mathrm{B}-\pi_2\mathrm{B}^2-\ldots)\dot{Z}_t = a_t \\ \dot{Z}_t &= \pi_1\dot{Z}_{t-1}+\pi_2\dot{Z}_{t-2}+\cdots+a_t = \sum_{k=1}^{\infty}\pi_k\dot{Z}_{t-k}+a_t\end{aligned}$$

Values of $\pi_1, \pi_2, \ldots$ depend on $\theta_1, \ldots, \theta_q$. For the model to have practical meaning, the π_j values should not remain large as j gets large. Formally, the invertibility condition is met if

$$\sum_{i=1}^{\infty}|\pi_j| < \infty$$

Invertibility of an MA(q) model can be checked by finding the roots of the polynomial $\theta_q(\mathrm{B}) \equiv (1-\theta_1\mathrm{B}-\theta_2\mathrm{B}^2-\cdots-\theta_q\mathrm{B}^q) = 0$. All q roots must lie <u>outside</u> of the "unit-circle."

Checking the Invertibility of an MA(2) Model

Invertibility of an MA(2) model can be checked by finding the roots of the polynomial $\theta_2(\mathrm{B}) = (1-\theta_1\mathrm{B}-\theta_2\mathrm{B}^2) = 0$. Both roots must lie <u>outside</u> of the "unit-circle." From page 39,

$$B = \frac{-\theta_1 \pm \sqrt{\theta_1^2+4\theta_2}}{2\theta_2}$$

Roots have the form

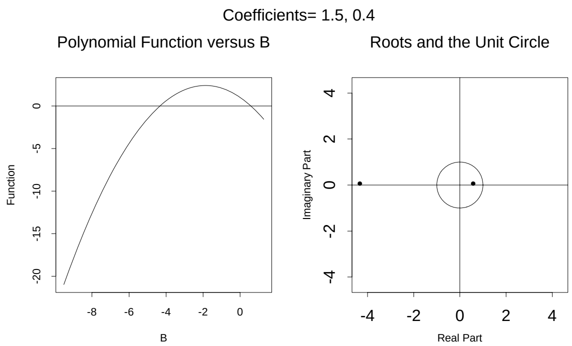
$$z = x + iy$$

where $i = \sqrt{-1}$. A root is "outside of the unit circle" if

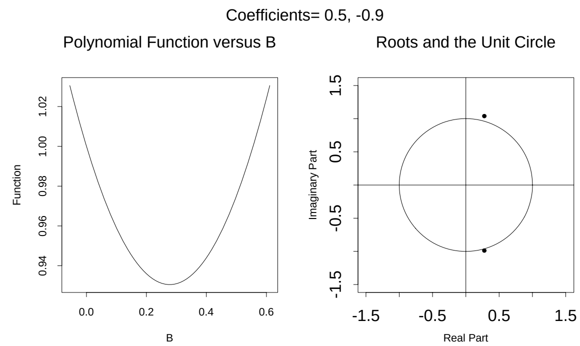
$$|z| = \sqrt{x^2+y^2} > 1$$

This method works for any q, but for $q > 2$ it is best to use numerical methods to find the q roots.

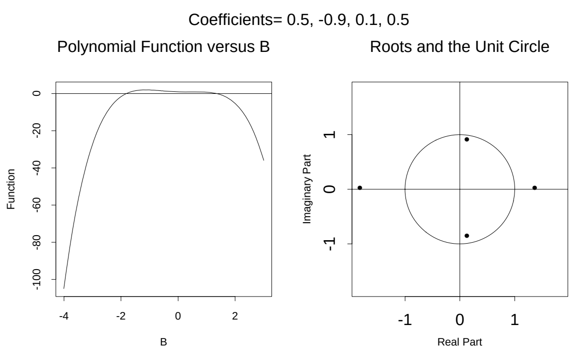
Roots of $(1 - 1.5\mathrm{B} - .4\mathrm{B}^2) = 0$



Roots of $(1 - .5\mathrm{B} + .9\mathrm{B}^2) = 0$



Roots of $(1 - .5\mathrm{B} + .9\mathrm{B}^2 - .1\mathrm{B}^3 - .5\mathrm{B}^4) = 0$



Checking the Invertibility of an MA(2) Model
Simple Rule

Both roots lying <u>outside</u> of the "unit-circle" implies

$$\begin{aligned} \theta_2 + \theta_1 &< 1 \\ \theta_2 - \theta_1 &< 1 \\ -1 < \theta_2 &< 1 \end{aligned}$$

Defines the MA(2) triangle.

See pages 39/40 of Wei for algebraic argument.

Autocovariance and Autocorrelation Functions for the MA(2) Model

$$\begin{aligned} \gamma_1 &\equiv \mathrm{E}(\dot{Z}_t \dot{Z}_{t+1}) \\ &= \mathrm{E}[(-\theta_1 a_{t-1} - \theta_2 a_{t-2} + a_t)(-\theta_1 a_t - \theta_2 a_{t-1} + a_{t+1})] \\ &= \mathrm{E}[\theta_1\theta_2 a_{t-1}^2 - \theta_1 a_t^2] + 0 + 0 + 0 + 0 + 0 + 0 + 0 \\ &= \theta_1\theta_2 \mathrm{E}(a_{t-1}^2) - \theta_1 \mathrm{E}(a_t^2) = (\theta_1\theta_2 - \theta_1)\sigma_a^2 \end{aligned}$$

Thus

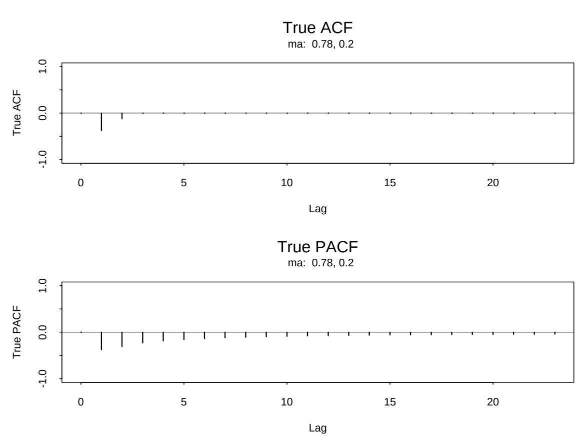
$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta_1\theta_2 - \theta_1}{1 + \theta_1^2 + \theta_2^2}.$$

Using similar operations, it is easy to show that

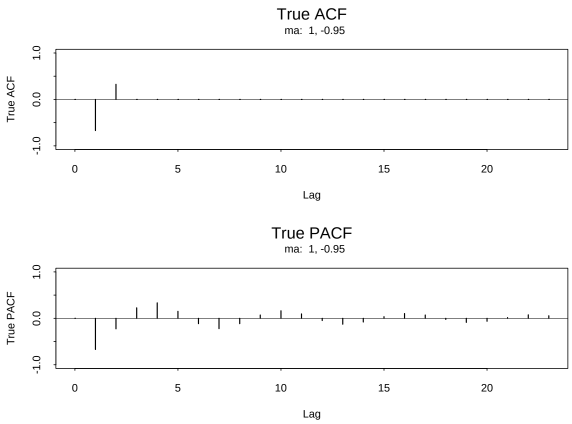
$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

and that, in general, for MA(2), $\rho_k = \gamma_k/\gamma_0 = 0$ for $k > 2$.

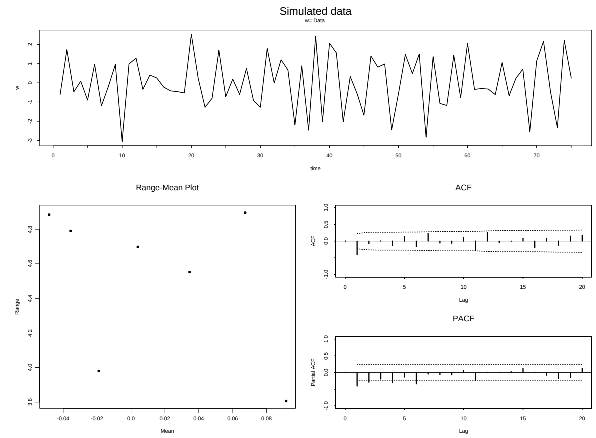
True ACF and PACF for MA(2) Model with $\theta_1 = .78, \theta_2 = .2$



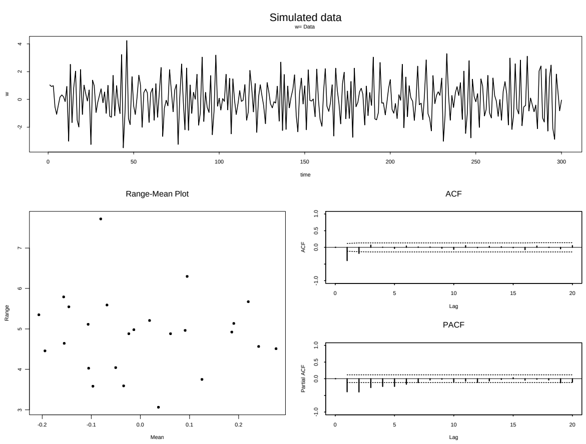
True ACF and PACF for MA(2) Model with $\theta_1 = 1, \theta_2 = -.95$



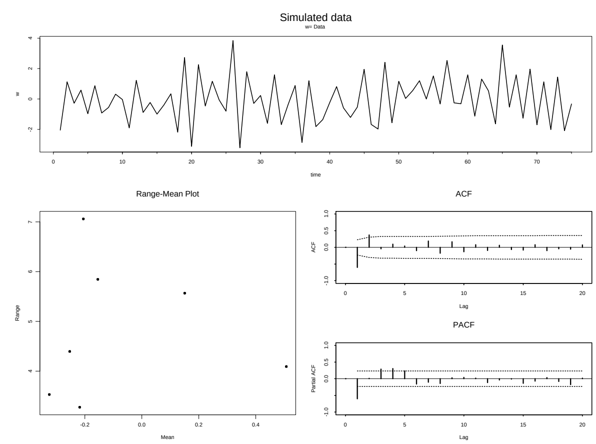
Simulated Realization (MA(2), $\theta_1 = .78, \theta_2 = .2, n = 75$) Graphical Output from Function `iden`



Simulated Realization (MA(2), $\theta_1 = .78, \theta_2 = .2, n = 300$) Graphical Output from Function `iden`



Simulated Realization (MA(2), $\theta_1 = 1, \theta_2 = -.95, n = 75$) Graphical Output from Function `iden`



Simulated Realization (MA(2), $\theta_1 = 1, \theta_2 = -.95, n = 300$) Graphical Output from Function `iden`