MA1 given by $Y_t = \mu + e_t - \beta e_{t-1}$

Using backshift notation

$$Y_t = \mu + (1 - \beta B^1) e_t$$

MA2 given by $Y_t = \mu + e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2}$

Using backshift notation

$\boxed{e_t \sim N(0, \sigma_e^2)}$

$$Y_t = \mu + (1 - \beta_1 B^1 - \beta_2 B^2) e_t$$

AR(0) given by $Y_t = \mu + e_t$

{WN model}

AR1 given by $Y_t = \mu + \alpha_1 y_{t-1} + e_t$

or $y_t - \alpha_1 y_{t-1} = \mu + e_t$

or $(1 - \alpha_1 B^1) y_t = \mu + e_t$ $\boxed{e_t \sim N(0, \sigma_e^2)}$

AR2 given by $Y_t = \mu + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + e_t$

or $y_t - \alpha_1 y_{t-1} - \alpha_2 y_{t-2} = \mu + e_t$

or $(1 - \alpha_1 B^1 - \alpha_2 B^2) y_t = \mu + e_t$

Various authors will work with centered observations to simplify derivations.

$E(y_t) = \mu$. Define $\dot{y}_t = y_t - \mu$.

Then, $E(\dot{y}_t) = \phi$

$$V(\dot{y}_t) = V(y_t - \mu) = \sigma_y^2.$$

Backshift Operator Notation.

$B^1 y_t = y_{t-1}$

$B^2 y_t = B \cdot B\, y_t = B\, y_{t-1} = y_{t-2}$

$B^3 y_t = y_{t-3}$; etc.

If c is a constant → then, $B^1 c = c$

<u>Differencing Operators</u>

$(1-B)\, y_t = y_t - y_{t-1}$

$(1-B)^2 y_t = (1 - 2B + B^2) = y_t - 2y_{t-1} + y_{t-2}$

$$= y_t - y_{t-1} - y_{t-1} + y_{t-2}$$

$$= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

Seasonal differencing operators

$$(1-B^4)y_t = y_t - B^4 y_t$$
$$= y_t - y_{t-4}$$

$$(1-B^{12})y_t = y_t - B^{12} y_t$$
$$= y_t - y_{t-12}$$

General ARMA models

Again, let $\dot{y}_t = y_t - \mu$.

Then,

$$(1-\alpha_1 B^1 - \alpha_2 B^2 - \dots - \alpha_p B^p)\,\dot{y}_t =$$
$$(1-\beta_1 B^1 - \beta_2 B^2 - \dots - \beta_q B^q)\,e_t$$

$$\Rightarrow \dot{y}_t - \alpha_1 \dot{y}_{t-1} - \alpha_2 \dot{y}_{t-2} - \dots - \alpha_p \dot{y}_{t-p}$$
$$= e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2} - \dots - \beta_q e_{t-q}$$

$$\Rightarrow \dot{y}_t = \alpha_1 \dot{y}_{t-1} + \alpha_2 \dot{y}_{t-2} + \dots + \alpha_p \dot{y}_{t-p}$$
$$+ e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2} - \dots - \beta_q e_{t-q}$$

The relationship between the mean and the constant term.

We know that $\dot{y}_t = y_t - \mu$

Now, for $p = q = 2$ we have

$$\dot{y}_t = \alpha_1 \dot{y}_{t-1} + \alpha_2 \dot{y}_{t-2} + e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2}$$

$$y_t - \mu = \alpha_1 (y_{t-1} - \mu) + \alpha_2 (y_{t-2} - \mu) + e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2}$$

$$y_t = [\underbrace{\mu - \alpha_1 \mu - \alpha_2 \mu}_{\text{constant term}}] + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + e_t - \beta_1 e_{t-1} - \beta_2 e_{t-2}$$

ARIMA Models

ARIMA (p, d, q)

$$\phi_p(B)(1-B)^d y_t = \theta_q(B) e_t$$

Let $W_t = (1-B)^d y_t$ The new series.

Now use ARMA on W_t

$$\phi_p(B) W_t = \theta_q(B) e_t$$

Can unscramble the y_t for forecasting and interpretations.

Examples

1. ARIMA $(0, 1, 1)$

$$(1-B)^1 y_t = (1-\beta_1 B^1) e_t$$

$$y_t - y_{t-1} = e_t - \beta_1 e_{t-1}$$

$$y_t = y_{t-1} + e_t - \beta_1 e_{t-1}$$

2. ARIMA $(1, 1, 0)$

$$(1-\alpha_1 B^1)(1-B)^1 y_t = e_t$$

$$(1-\alpha_1 B)(y_t - y_{t-1}) = e_t$$

$$y_t - y_{t-1} - \alpha_1 B y_t + \alpha_1 B y_{t-1} = e_t$$

$$y_t - y_{t-1} - \alpha_1 y_{t-1} + \alpha_1 y_{t-2} = e_t$$

$$y_t = (1+\alpha_1) y_{t-1} - \alpha_1 y_{t-2} + e_t$$

Relationship between ARIMA (0, 1, 1) and Exponential Smoothing

We saw that $y_t = y_{t-1} - \beta_1 e_{t-1} + e_t$

$$\hat{y}_t = y_{t-1} - \hat{\beta}_1 \hat{e}_{t-1}$$

$$= y_{t-1} - \hat{\beta}_1 [y_{t-1} - \hat{y}_{t-1}]$$

$$= (1-\hat{\beta}_1) y_{t-1} + \hat{\beta}_1 \hat{y}_{t-1}$$ Let $\alpha = 1-\hat{\beta}_1$

$$= \alpha y_{t-1} + (1-\alpha) \hat{y}_{t-1}$$

$$= \alpha y_{t-1} + \alpha(1-\alpha) y_{t-2} + \alpha(1-\alpha)^2 y_{t-3} + \cdots$$

Usually $0.01 < \alpha < 0.1$

exponentially weighted moving average.

Single exponential smoothing

ARIMA (0, 1, 1)

Double

ARIMA (0, 2, 2)

Triple

ARIMA (0, 3, 3)

EWMA charts $\Rightarrow$

$$EWMA_i = \lambda y_i + (1-\lambda) EWMA_{i-1}$$

Choose λ and $EWMA_0$ and plot the data using a control chart.

$$\mu_{EWMA} = \mu_y$$

$$\sigma_{EWMA} = \sigma_y \sqrt{\frac{\lambda}{2-\lambda}}$$

$$UCL = \mu_y + K \sigma_y \sqrt{\frac{\lambda}{2-\lambda}}$$

$$LCL = \mu_y - K \sigma_y \sqrt{\frac{\lambda}{2-\lambda}}$$

Single Exponential Smoothing

$$S_t = \alpha\, y_{t-1} + (1-\alpha) S_{t-1} \qquad 0 < \alpha \leq 1 \qquad t \geq 3$$

Double Exponential Smoothing {Trend}

$$S_t = \alpha y_t + (1-\alpha)(S_{t-1} + b_{t-1}) \qquad 0 \leq \alpha \leq 1$$

$$b_t = \gamma (S_t - S_{t-1}) + (1-\gamma) b_{t-1} \qquad 0 \leq \gamma \leq 1$$

Trend and seasonality → {Triple Exp smoothing}

$$S_t = \alpha \left\{ \frac{y_t}{I_{t-L}} \right\} + (1-\alpha)(S_{t-1} + b_{t-1})$$

overall smoothing

$$b_t = \gamma (S_t - S_{t-1}) + (1-\gamma) b_{t-1}$$

Trend smoothing

$$I_t = \beta \frac{y_t}{S_t} + (1-\beta) I_{t-L}$$

$$F_{t+m} = (S_t + m b_t) I_{t-L+m} \qquad \text{Forecast}$$

y = obsv.

S = smoothed obsv.

b = trend Factor

I = seasonal Index

F = forecast → m periods ahead

t = Time index

α, β, γ need to be estimated.

{NIST}