Yo!! I'm back - Univariate Time Series Models - II

Choosing a model

Most of the time, there are no economic reasons to choose a particular specification of the model. To a large extent the data will determine which time series model is more appropriate. Before estimating any model, it is common to estimate ACF and PACF directly from the data. Often this gives some idea about which model might be appropriate. After one or models are estimated, their quality can be judged by checking whether the residuals are more or less white noise and by comparing them with alternative specifications.

ACF

This describes the correlation between y_t and y_{t-k}; as a function of k. It is given by

$$\rho_k = \frac{Cov(y_t, y_{t-k})}{V(y_t)} = \frac{\gamma_k}{\gamma_0}$$

For MA(1)

$$\rho_1 = \frac{\alpha}{1+\alpha^2}, \quad \rho_2 = 0, \quad \rho_3 = 0 \dots$$

For MA(2) where

$$y_t = \epsilon_t + \alpha_1 \epsilon_{t-1} + \alpha_2 \epsilon_{t-2}$$

$$E(y_t^2) = (1+\alpha_1^2+\alpha_2^2)\sigma^2$$

$$E(y_t y_{t-1}) = (\alpha_1 + \alpha_1\alpha_2)\sigma^2$$

$$E(y_t y_{t-2}) = \alpha_2 \sigma^2$$

$$E(y_t y_{t-k}) = 0 \quad \forall k = 3, 4, 5 \dots$$

i.e. we can show that ACF=0 after 2 lags.

In general, for an MA(q) model, the ACF is ϕ after q lags.

The sample autocorrelation function $\hat{\rho}_K$ is given by

$$\hat{\rho}_K = \frac{\frac{1}{T-K} \sum_{t=K+1}^{T} (Y_t - \bar{Y})(Y_{t-K} - \bar{Y})}{\frac{1}{T} \sum_{t=1}^{T} (Y_t - \bar{Y})^2}$$

It will usually not be the case that $\hat{\rho}_K$ is zero after q in a MA(q) process. However, we can test $H_0: \rho_K = 0$ by using $\hat{\rho}_K$.

It can be shown that

$$\sqrt{T}(\hat{\rho}_K - \rho_K) \xrightarrow{asym} N(0, V_K)$$

with

$$V_K = 1 + 2\rho_1^2 + 2\rho_2^2 + \cdots + 2\rho_q^2 \quad \text{if } q < K.$$

For a MA(1) process

$$H_0: \rho_1 = 0 \qquad H_1: \rho > 0 \text{ or } \rho < 0$$

can be tested by comparing $\sqrt{T}\,\hat{\rho}_1$ with critical values of a std. normal dist.

Testing MA(K-1) v/s MA(K) is done by testing $\rho_K = 0$ (H_0) and comparing the test statistic

$$\sqrt{T} \frac{\hat{\rho}_K}{\sqrt{1+2\hat{\rho}_1^2+\cdots+2\hat{\rho}_{K-1}^2}}$$ with critical

values of the standard normal dist.

Typically, 2-σ bounds for $\hat{\rho}_K$ based on the estimated variance $1+\sum_{i=1}^{K-1} 2\hat{\rho}_i^2$ are graphically displayed. The order of a MA model can $\therefore$ be estimated this way.

For AR models, the ACF is less helpful. For AR(1) we saw that the ACF does not cut-off after a finite lag length. Instead, they go to ϕ exponentially corresponding to $\rho_K = \theta^K$. For higher order AR models we get into some complexity.

Consider the AR(2) model

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \epsilon_t.$$

It can be shown that,

$$\mathrm{Cov}(y_t, y_{t-k}) = \theta_1 \mathrm{Cov}(y_{t-1}, y_{t-k}) + \theta_2 \mathrm{Cov}(y_{t-2}, y_{t-k}) + \mathrm{Cov}(\epsilon_t, y_{t-k}).$$

For $k = 0, 1, 2$ we get

$$\gamma_0 = \theta_1 \gamma_1 + \theta_2 \gamma_2 + \sigma^2$$

$$\gamma_1 = \theta_1 \gamma_0 + \theta_2 \gamma_1$$

$$\gamma_2 = \theta_1 \gamma_1 + \theta_2 \gamma_0$$

This set of equations, known as the Yule-Walker equations can be solved for the autocovariances $\gamma_0, \gamma_1, \gamma_2$ as a function of the model parameters $\theta_1, \theta_2, \sigma^2$. The higher order covariances can be determined recursively from

$$\gamma_k = \theta_1 \gamma_{k-1} + \theta_2 \gamma_{k-2} \quad (k = 2, 3, \ldots)$$

PACF

The k-th order sample partial auto corr. coefficient is an estimate of $\hat{\theta}_K$ in an AR(K) model. We denote this by $\hat{\theta}_{KK}$. So, estimating

$$y_t = \delta + \theta_1 y_{t-1} + \epsilon_t$$

gives us $\hat{\theta}_{11}$. Estimating

$$y_t = \delta + \theta_1 y_{t-1} + \theta_2 y_{t-2} + \epsilon_t$$

gives us $\hat{\theta}_{22}$.

The PAC $\hat{\theta}_{KK}$ measures the additional correlation between y_t and y_{t-K} after adjustments are made for the intermediate values $y_{t-1}, \dots, y_{t-(K-1)}$.

If the true model is an AR(p) model then estimating an AR(K) model by OLS gives consistent estimators for the model parameters if $K \geq p$.

It can be shown that

$$\sqrt{T}\,(\hat{\theta}_{KK} - 0) \xrightarrow{asymp} N(0,1) \quad \text{if } K > p.$$

We can use the PACF to determine the order of the AR process.

Testing an AR(K-1) vs. AR(K) model implies testing the null hypothesis $\theta_{KK} = 0$. Under the null hypothesis where AR(K-1) is more appropriate the approx. std. err. of $\hat{\theta}_{KK}$ is $\frac{1}{\sqrt{T}}$ so that $H_0: \hat{\theta}_{KK} = 0$ is rejected if

$|\sqrt{T}\, \hat{\theta}_{KK}| > 1.96$.

For a genuine AR(p) model, the partial autocorrelations will be close to 0 after the p^{th} lag.

For MA models, it can be shown that the PACFs do not have cut-off points but tail off to 0.

<u>In summary</u>

An AR(p) process is described by

1. An ACF that is infinite (it tails off).
2. A PACF that is (close to) 0 for lags > p.

For a MA(q) process

1. An ACF that is (close to) 0 for lags > q.
2. A PACF that is infinite in extent (It tails off).