

Lecture 5: Seasonal Time Series, Trend Analysis & Component Model

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“Business cycle” plays an important role in economics. In time series analysis, business cycle is typically represented by a seasonal (or periodic) model. For deterministic function $f(\cdot)$, we say that $f(\cdot)$ is periodic with a periodicity s if

$$f(t) = f(t + k \times s) \quad k = 0, \pm 1, \pm 2, \dots$$

A typical example of a deterministic periodic function is a trigonometric series, e.g. $\sin(\theta) = \sin(\theta + 2k\pi)$ or $\cos(\theta) = \cos(\theta + 2k\pi)$. The trigonometric series are often used in econometrics to model time series with strong seasonality. [In some cases, seasonal dummy variables are used.]

For stochastic process Z_t , we say that it is a seasonal (or periodic) time series with periodicity s if Z_t and Z_{t+ks} have the same distribution. Such processes are common in business and economics. For instance, the series of monthly sales of a department store in the U.S. tends to peak at December and to be periodic with a period 12.

In what follows, we shall use s to denote periodicity of a seasonal time series. Often $s = 4$ and 12 are used for quarterly and monthly time series, respectively.

A. Pure seasonal time series

General Model: $\Phi(B^s)Z_t = C + \Theta(B^s)a_t$ where C is a constant,

$$\Phi(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \dots - \Phi_P B^{Ps}, \quad \Theta(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \dots - \Theta_Q B^{Qs}$$

A simple example: $Z_t = C + (1 - \Theta B^{12})a_t$. This is a simple seasonal MA model. It is easy to see that

- Invertibility: $|\Theta| < 1$.
- $E(Z_t) = C$.
- $\text{Var}(Z_t) = (1 + \Theta^2)\sigma_a^2$.
- ACF:

$$\rho_\ell = \begin{cases} \frac{-\Theta}{1+\Theta^2} & \text{if } \ell = 12 \\ 0 & \text{if } \ell \neq 0 \text{ or } 12. \end{cases}$$

Another simple example: $Z_t - \Phi Z_{t-12} = C + a_t$. This is a simple seasonal AR model. It is easy to see that

- Stationarity: $|\Phi| < 1$.
- Mean: $E(Z_t) = \frac{c}{1-\Phi}$.

- $\text{Var}(Z_t) = \frac{1}{1-\Phi^2} \sigma_a^2$.

- ACF:

$$\rho_\ell = \begin{cases} \Phi^k & \text{for } \ell = 12k \text{ for } k = 0, \pm 1, \dots \\ 0 & \text{otherwise.} \end{cases}$$

When $\Phi = 1$, the series is non-stationary.

Exercise: Study properties of the seasonal model $(1 - \Phi B^{12})Z_t = (1 - \Theta B^{12})a_t$.

B. Multiplicative seasonal time series

A special, parsimonious class of seasonal time series models that is commonly used in practice is the multiplicative seasonal model $\text{ARIMA}(p, d, q)(P, D, Q)_s$

$$\phi(B)\Phi(B^s)(1 - B)^d(1 - B^s)^D Z_t = c + \theta(B)\Theta(B^s)a_t$$

where all zeros of $\phi(B)$, $\Phi(B^s)$, $\theta(B)$ and $\Theta(B^s)$ lie outside the unit circle. Of course, there are no common factors between $\phi(B)\Phi(B^s)$ and $\theta(B)\Theta(B^s)$.

The basic idea of this model is close to the “two-way table” in analysis of variance in which the seasonal and regular components are approximately orthogonal. For $s = 12$, we have

Year	Jan.	Feb	...	Nov.	Dec.
1	Z_1	Z_2	...	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	...	Z_{23}	Z_{24}
\vdots	\vdots				\vdots

Here the column-effects are the “regular” serial correlations and the row-effects denote the annual correlations.

A special model: The airline model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta B)(1 - \Theta B^{12})a_t$$

where $|\theta| < 1$ and $|\Theta| < 1$. This model is the most used seasonal model in practice. It was proposed by Box and Jenkins (1976) for modeling the well-known monthly series of airline passengers. It has been shown, Cleveland and Tiao (1976), that the X-11 technique of seasonal adjustment used by the US government is in fact close to this model.

Let $W_t = (1 - B)(1 - B^{12})Z_t$, where $(1 - B)$ and $(1 - B^{12})$ are usually referred to as the “regular” and “seasonal” difference, respectively. Obviously, $W_t = c + (1 - \theta B)(1 - \Theta B^{12})a_t$ is a multiplicative MA model. It pays to study carefully this seasonal MA model. For simplicity, assume $c = 0$.

- Mean: $E(W_t) = 0$.

- Variance: $\text{Var}(W_t) = (1 + \theta^2)(1 + \Theta^2)\sigma_a^2$

- ACF:

$$\rho_\ell = \begin{cases} 1 & \text{for } \ell = 0 \\ \frac{-\theta}{1+\theta^2} & \text{for } \ell = 1 \\ \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} & \text{for } \ell = 11 \\ \frac{-\Theta}{1+\Theta^2} & \text{for } \ell = 12 \\ \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} & \text{for } \ell = 13 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\rho_{11} = \rho_{13} \neq 0$, which can be regarded as an “interaction” between the regular and seasonal correlations. Also, the seasonal factor does not affect the regular correlation, neither the regular factor affects the seasonal correlation.

Exercise: Study the ACF of the series $W_t = (1 - \theta_1 B - \theta_2 B^2)(1 - \Theta B^{12})a_t$.

Exercise: Study the ACF of the series $W_t = (1 - \theta B)(1 - \Theta B^4)a_t$ and $R_t = (1 - \Theta B^4)a_t$.

C. Non-multiplicative seasonal model

In some applications, a non-multiplicative model might be suitable. A simple example of the model is

$$Z_t = (1 - \theta B - \Theta B^{12})a_t$$

The ACF of this series is (for $\ell > 0$)

$$\rho_\ell = \begin{cases} \frac{-\theta}{1+\theta^2+\Theta^2} & \text{for } \ell = 1 \\ \frac{\theta\Theta}{1+\theta^2+\Theta^2} & \text{for } \ell = 11 \\ \frac{-\Theta}{1+\theta^2+\Theta^2} & \text{for } \ell = 12 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the difference between this and that of multiplicative model. The ACF structure also highlights the parsimony of the multiplicative model as both models use two parameters, yet the multiplicative model covers serial correlation at lag 13.

Exercise: Study the properties of the model

$$Z_t - \Phi Z_{t-12} = (1 - \theta B)a_t$$

where $|\Phi| < 1$. This model is also useful in practice.

D. The simplifying operator

The seasonal difference $(1 - B^{12})$ can be factorized as

$$(1 - B^{12}) = (1 - B)(1 - \sqrt{3}B + B^2)(1 - B + B^2)(1 + B + B^2)(1 + \sqrt{3}B + B^2)(1 + B)(1 + B^2)$$

All 12 zeros of this polynomial are on the unit circle. Each factor produces different response function. The overall pattern, however, has a period of 12. The factor

$$(1 + B + B^2 + \cdots + B^{11})$$

represents an average of 12 consecutive observations. It can be used as a “filter” to remove the seasonality in a monthly time series. The factor $(1 - B)$ is not included, because it corresponds to a “trend”.

Similar comments apply to $(1 - B^4)$ and $(1 - B^s)$ in general.

E. Trend Analysis

By and large, two types of “trend” are commonly used in business and economic time series analysis, namely deterministic and stochastic trends.

Deterministic trend:

- Linear trend: $Z_t = \beta_0 + \beta t + X_t$, where X_t is a stationary time series, e.g. a white noise series.
- Exponential trend: $\ln(Z_t) = \beta_0 + \beta t + X_t$.
- Cyclical trend: $Z_t = r \cos(\omega t + \theta) + X_t$, where r is amplitude, ω is the frequency with period $\frac{2\pi}{\omega}$, and θ denotes the phase shift. More generally,

$$Z_t = \sum_{i=1}^k r_i \cos(\omega_i t + \theta_i) + X_t = \sum_{i=1}^k [A_i \cos(\omega_i t) + B_i \sin(\omega_i t)] + X_t$$

where $A_i = r_i \cos(\theta_i)$ and $B_i = r_i \sin(\theta_i)$.

Stochastic trend:

- Linear trend: $Z_t = \mu_t + a_t$, where $\mu_t = \mu_{t-1} + \epsilon_t$ with $\{\epsilon_t\}$ a white noise series independent of a_t .
- Quadratic trend: $Z_t = \mu_t + a_t$, where $(1 - B)^2 \mu_t = \epsilon_t$.
- Seasonal trend: $Z_t = \mu_t + a_t$, where $(1 - B^s) \mu_t = \epsilon_t$.

The deterministic trend can be regarded as a special case of stochastic trend. For instance, if $\omega_i = \frac{2\pi i}{12}$ for $i = 1, 2, \dots, 6$, then we have $\cos(\omega_i) = 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -1$. Therefore, by applying $(1 - B^{12})$ to the general cyclical trend model we have

$$(1 - B^{12}) \left[\sum_{i=1}^6 r_i \cos(\omega_i t + \theta_i) \right] = 0$$

and

$$(1 - B^{12}) Z_t = (1 - B^{12}) X_t.$$

This latter equation points out an important fact that is commonly overlooked by data analysts. The model “seems” to indicate that there is a common factor $(1 - B^{12})$ on both sides of the equation, implying that one might say that $Z_t = X_t$. However, this is only part

of the picture, as we know that the original time series Z_t is X_t plus some cyclical trend. Thus, the correct cancellation formula is

$$Z_t = f(t, 12) + X_t$$

where $f(t)$ is a deterministic function of period 12.

F. Component Models

There is a growing literature in considering component models in time series literature. The component model has a long history, it is basically assume that

$$Z_t = T_t + S_t + R_t$$

where T_t, S_t, R_t are respectively the “trend”, “seasonal” and “irregular” components of Z_t . The three components are assumed to be independent. The common approach to component model is the “structural model”, e.g. Harvey (1990), which assumes a particular model for each of the three components, then estimate the parameters involved by maximum likelihood method.

The idea of such a component model is appealing. However, one must use the model with care. Why? Basically, the model is not “identifiable”. In other words, there are infinite many ways to decompose a time series into the three components.

A simple example is in order. Consider the ARIMA(0,1,1) model

$$(1 - B)Z_t = (1 - \theta B)a_t.$$

This is a model we can build from data. However, this model may arise from many sources. Case 1: Write $Z_t = T_t + b_t$ where $T_t = T_{t-1} + e_t$ and $\{e_t\}$ and $\{b_t\}$ are independent white noise series. Then, we have

$$(1 + \theta^2)\sigma_a^2 = \sigma_e^2 + 2\sigma_b^2 \quad \text{and} \quad \theta\sigma_a^2 = \sigma_b^2.$$

Thus, θ and σ_a^2 are determined by σ_b^2 and σ_e^2 .

Case 2: Write $Z_t = T_t + b_t$ where $T_t = T_{t-1} + \epsilon_t - \eta\epsilon_{t-1}$ with $\{\epsilon_t\}$ a white noise independent of $\{b_t\}$. In this case, it is easily seen that θ and σ_a^2 are determined by

$$(1 + \theta^2)\sigma_a^2 = (1 + \eta^2)\sigma_e^2 + 2\sigma_b^2 \quad \text{and} \quad \theta\sigma_a^2 = \eta\sigma_e^2 + \sigma_b^2.$$

Thus, given models for the component T_t and b_t , we can determine θ and σ_a^2 . On the other hand, given θ and σ_a^2 , there is no way we can determine which case is the “true” underlying model. In practice, only Z_t is available (observable), implying that we can only CHECK the model for Z_t . Therefore, the identifiability problem arises.

One can resolve the identifiability problem if he/she is willing to add certain conditions. For example, in the above instance, one may require that T_t is a random walk. Then, case 1 is the solution. Do not overlook this identifiability problem if you make inference about the components.

In summary, the component model is suitable for forecasting. To use it to make inference on components, one must understand the assumption used to obtain the decomposition and the fact that the decomposition obtained is only one of the many possible decompositions.