Handout 8

Seasonal ARMA (SARIMA) Models

Class notes for Statistics 451: Applied Time Series
Iowa State University

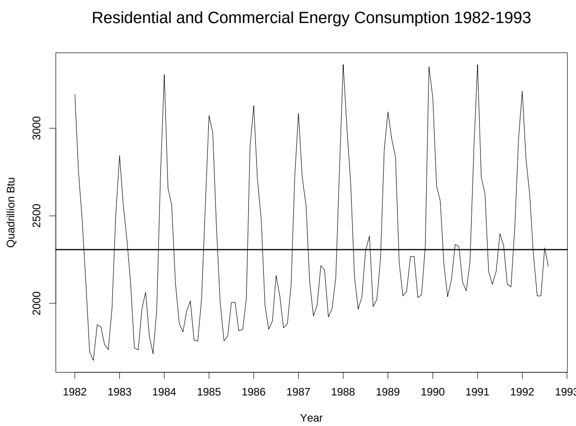
Copyright 2004 W. Q. Meeker.

March 2, 2006
12h 55min

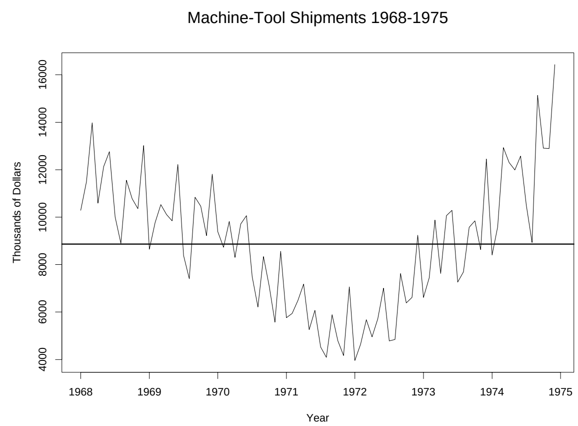
Seasonal Time Series

- "Periodic" is a better term; "seasonal" commonly used
- Seasonal (periodic) model with S observations per period
  - ▶ Monthly data has 12 observations per year
  - ▶ Quarterly data has 4 observations per year
  - ▶ Daily data has 5 or 7 (or some other number) of observations per week.
- Notes:
  - ▶ Sunspot data is cyclical, not seasonal, because distance between peaks is random.
  - ▶ Just because we have 12 observations per year, does not mean that there is seasonal behavior (e.g., stock prices show no regular seasonal patterns)

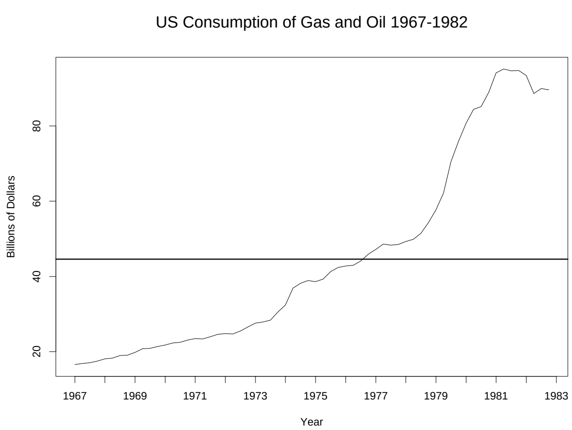
Energy Consumption Data



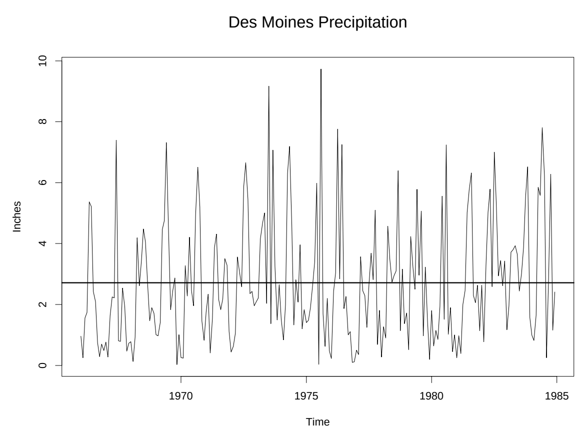
Machine Tool Shipments



US Gas and Oil Consumption



Des Moines Precipitation



Seasonal Differencing

- Seasonal differencing is usually needed. For example,

$$\begin{aligned} W_t &= (1-\mathrm{B}^{12})Z_t = Z_t - \mathrm{B}^{12}Z_t = Z_t - Z_{t-12} \\ Z_t &= Z_{t-12} + W_t \end{aligned}$$

$$\begin{aligned} W_t &= (1-\mathrm{B})(1-\mathrm{B}^{12})Z_t = Z_t - Z_{t-1} - Z_{t-12} + Z_{t-13} \\ Z_t &= Z_{t-1} + Z_{t-12} - Z_{t-13} + W_t \end{aligned}$$

- More generally, the "working series" is

$$W_t = (1-\mathrm{B})^d(1-\mathrm{B}^S)^D Z_t$$

Even more generally

$$W_t = (1-\mathrm{B})^d(1-\mathrm{B}^{S_1})^{D_1}(1-\mathrm{B}^{S_2})^{D_2} Z_t$$

and so on.

Seasonal ARIMA Model (SARIMA)

The SARIMA$(p,d,q)\times(P,D,Q)_S$ model is

$$\begin{aligned} \Phi_P(\mathrm{B}^S)\phi_p(\mathrm{B})(1-\mathrm{B})^d(1-\mathrm{B}^S)^D Z_t &= \Theta_Q(\mathrm{B}^S)\theta_q(\mathrm{B})a_t \\ \Phi_P(\mathrm{B}^S)\phi_p(\mathrm{B})W_t &= \Theta_Q(\mathrm{B}^S)\theta_q(\mathrm{B})a_t \end{aligned}$$

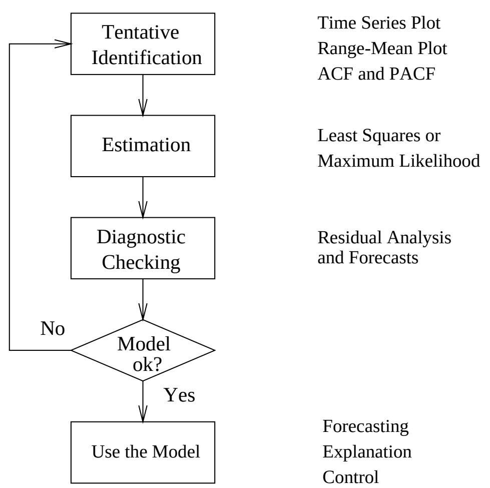
For example using $P=1, p=1, Q=2, q=1, S=12$ gives

$$\begin{aligned} \Phi_1(\mathrm{B}^{12}) &= (1-\Phi_1\mathrm{B}^{12}) \\ \phi_1(\mathrm{B}) &= (1-\phi_1\mathrm{B}) \\ \Theta_2(\mathrm{B}^{12}) &= (1-\Theta_1\mathrm{B}^{12}-\Theta_2\mathrm{B}^{24}) \\ \theta_1(\mathrm{B}) &= (1-\theta_1\mathrm{B}) \end{aligned}$$

The modeling problem is to choose a transformation (γ and m), differencing (d and D), and the SARIMA model (p,q,P,Q) for the working series W_t. As before, use tsplot, range-mean plot, ACF and PACF.

The SARIMA model can be unscrambled to be in the ARMA form: $\phi^*_{p^*}(\mathrm{B})Z_t = \theta^*_{q^*}(\mathrm{B})a_t$.

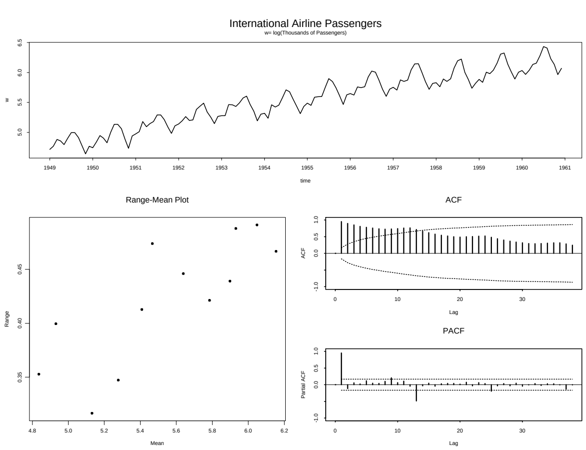
Data Analysis Strategy



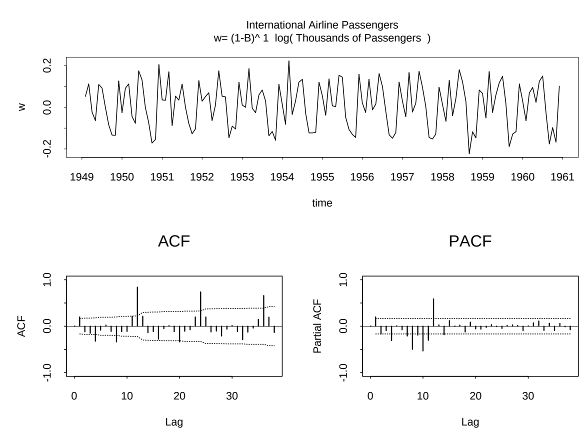
Strategy for Seasonal Time Series Modeling

- Plot data
- Use a range-mean plot to see if a transformation might be needed; choose tentative γ value.
- Choose differencing scheme(s).
  - Look at $3\times S+3$ lags on the ACF and PACF of W_t for all combinations of $d=0,1$ and $D=0,1$.
  - Go higher with d or D as needed.
  - Choose the stationary W_t with the smallest d and D. Avoid over-differencing
- Tentatively identify models(s) from the ACF and PACF of the chosen differencing scheme(s) [i.e., choose $(p,d,q)(P,D,Q)$].
- Fit, check, and compare models.
- Iterate as necessary
- At the end, choose alternative value of γ, if needed

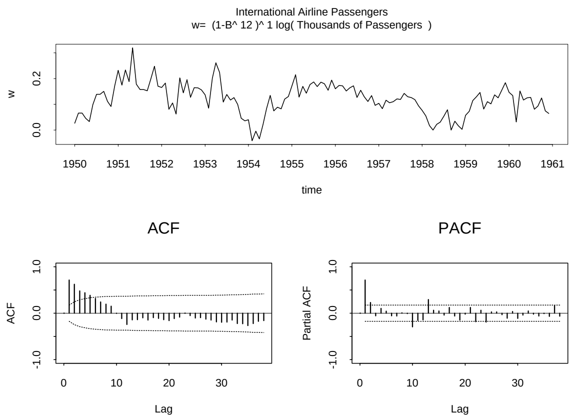
Graphical Output from Function `iden` for the Airline Data Log Transformation and No Differencing



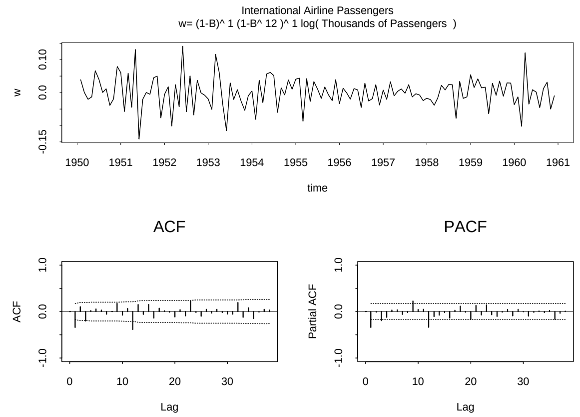
Graphical Output from Function `iden` for the Airline Data Log Transformation and 1 Regular Difference



Graphical Output from Function `iden` for the Airline Data Log Transformation and 1 Seasonal Difference



Graphical Output from Function `iden` for the Airline Data Log Transformation and 1 Regular and 1 Seasonal Difference



Methods of Obtaining the True ACF and PACF for Seasonal Models

- Derivations similar to nonseasonal models. Use the unscrambled form of the model for W_t.
- Formula sheets from Box and Jenkins give the ACF. Note the use of special cases (e.g., setting $\theta = 0$ in model 1 gives the ACF of a particularly simple seasonal model $W_t = -\Theta_1 a_{t-S} + a_t$)
- PACF computed using the same Yule-Walker based "Durbin formula" as in the nonseasonal models
- `plot.true.acfpacf()` S-PlusTS function. For example

  `plot.true.acfpacf(model=list(ma=c(.5,0,0,0,0, 0,0,0,0,0, 0,0.9,-.45)),nacf=38)` or

  `plot.true.acfpacf(model=list(ma=c(.5,rep(0,10),.9,-.45)),nacf=38)`

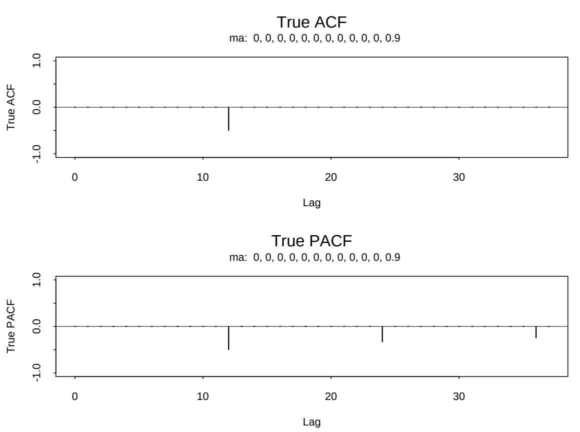
Example: $W_t = (1 - \Theta_1 \mathrm{B}^{12}) a_t = -\Theta_1 a_{t-12} + a_t$

This is a special case of Model 1 from the Box and Jenkins formula sheet with $\theta_1 = 0$. Everything is easy to derive.

$$\begin{aligned}
\gamma_0 &= \mathrm{E}(W_t^2) = (1 + \Theta_1^2)\sigma_a^2 \\
\gamma_1 &= \gamma_2 = \cdots = \gamma_{11} = 0 \\
\gamma_{12} &= \mathrm{E}(W_t W_{t+12}) = \mathrm{E}[(-\Theta_1 a_{t-12} + a_t)(-\Theta_1 a_t + a_{t+12})] \\
&= 0 + \cdots + 0 + \mathrm{E}(-\Theta_1 a_t^2) = -\Theta_1 \sigma_a^2 \\
\gamma_1 &= \gamma_2 = \cdots = \gamma_{11} = 0 \\
\rho_j &= \gamma_j / \gamma_0 \\
\rho_1 &= \rho_2 = \cdots = \rho_{11} = 0 \\
\rho_{12} &= \gamma_{12}/\gamma_0 = -\Theta_1 / (1 + \Theta_1^2) \\
\rho_j &= 0, \quad j > 12.
\end{aligned}$$

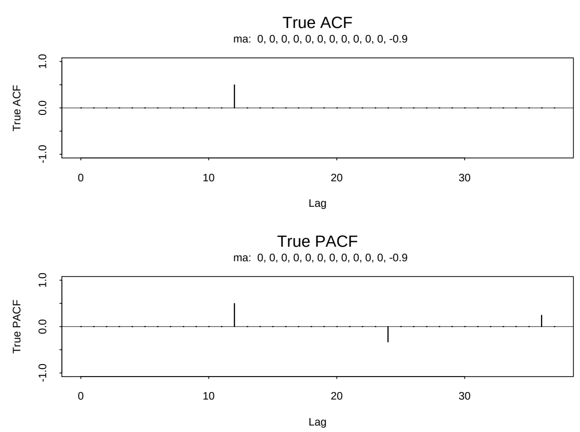
True ACF/PACF for $W_t = -\Theta_1 a_{t-12} + a_t$, $\Theta_1 = .9$

`plot.true.acfpacf(model=list(ma=c(0,0,0,0,0, 0,0,0,0,0, 0,0.9)),nacf=38)`



True ACF/PACF $W_t = -\Theta_1 a_{t-12} + a_t$, $\Theta_1 = -.9$

`plot.true.acfpacf(model=list(ma=c(rep(0,11),-.9)),nacf=38)`



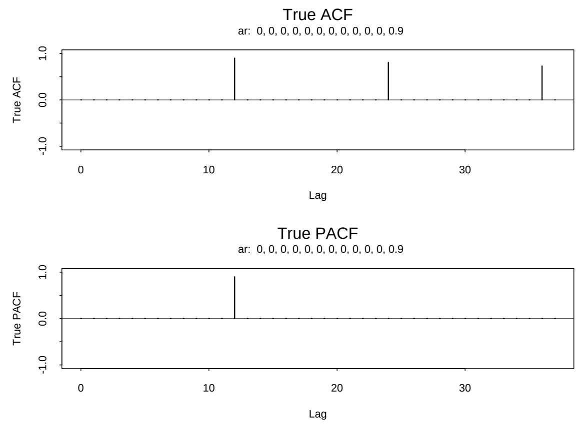
Example: $(1-\Phi_1 \mathrm{B}^{12})W_t = a_t, \quad W_t = \Phi_1 W_{t-12} + a_t$

This is a special case of Model 2 from the Box and Jenkins formula sheet with $\theta_1 = \Theta_1 = 0$.

$$
\begin{aligned}
\gamma_0 &= \mathrm{E}(W_t^2) = \left[1 + \frac{\Phi_1^2}{1-\Phi_1^2}\right]\sigma_a^2 \\
\gamma_{12} &= \mathrm{E}(W_t W_{t+12}) = \Phi_1 \left[1 + \frac{\Phi_1^2}{1-\Phi_1^2}\right]\sigma_a^2 \\
\gamma_1 &= \gamma_2 = \cdots = \gamma_{11} = 0 \\
\gamma_j &= \Phi_1 \gamma_{j-12}, \quad j > 12. \\
\rho_j &= \gamma_j/\gamma_0 \\
\rho_1 &= \rho_2 = \cdots = \rho_{11} = 0 \\
\rho_{12} &= \gamma_{12}/\gamma_0 = \Phi_1 \\
\rho_j &= \Phi_1 \rho_{j-12}, \quad j > 12.
\end{aligned}
$$

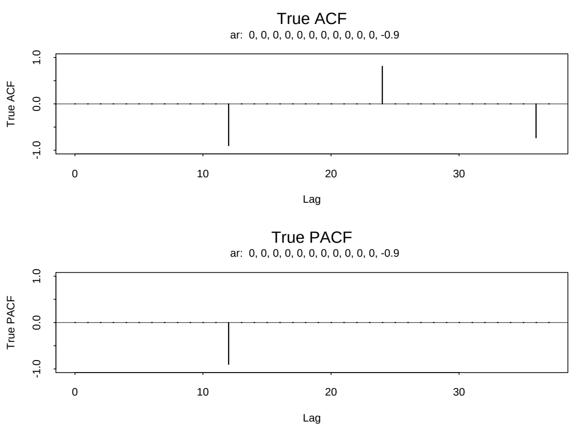
True ACF and PACF for $W_t = +\Phi_1 W_{t-12} + a_t$ with $\Phi_1 = .9$

```
plot.true.acfpacf(model=list(ar=c(rep(0,11),.9)),nacf=38)
```



True ACF and PACF for $W_t = +\Phi_1 W_{t-12} + a_t$ with $\Phi_1 = -.9$

```
plot.true.acfpacf(model=list(ar=c(rep(0,11),-.9)),nacf=38)
```



Example: $W_t = (1-\theta_1 \mathrm{B})(1-\Theta_1 \mathrm{B}^{12})a_t$

This is Model 1 from the Box and Jenkins formula sheet.

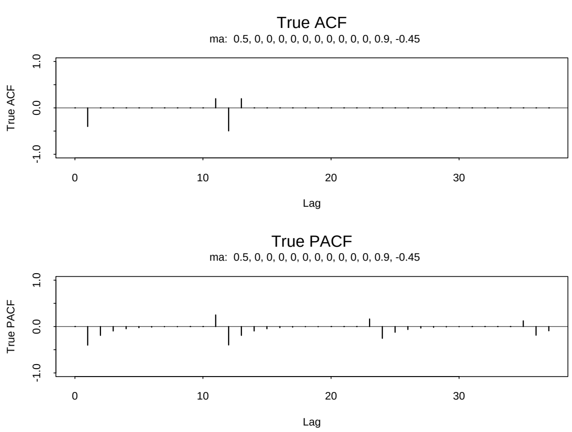
$$
\begin{aligned}
W_t &= (1-\theta_1 \mathrm{B})(1-\Theta_1 \mathrm{B}^{12})a_t \\
&= (1-\theta_1 \mathrm{B} - \Theta_1 \mathrm{B}^{12} + \theta_1\Theta_1 \mathrm{B}^{13})a_t \\
&= -\theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1\Theta_1 a_{t-13} + a_t
\end{aligned}
$$

This is like an MA(13) and, again, everything is easy to derive. For example,

$$
\begin{aligned}
\gamma_{11} &= \mathrm{E}(W_t W_{t-11}) \\
&= 0 + 0 + \cdots + \mathrm{E}[(-\Theta_1 a_{t-12})(-\theta_1 a_{t-12})] + 0 + \cdots \\
&\vdots \\
&= \theta_1\Theta_1\sigma_a^2
\end{aligned}
$$

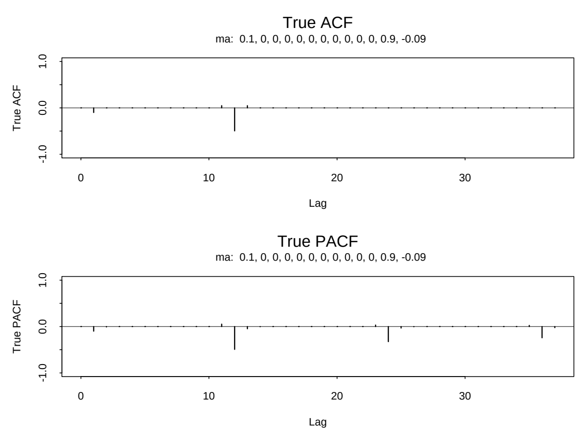
True ACF/PACF for $W_t = (1-\theta_1 \mathrm{B})(1-\Theta_1 \mathrm{B}^{12})a_t$, $\theta_1 = .5$, $\Theta_1 = .9$

```
plot.true.acfpacf(model=list(ma=c(.5,rep(0,10),.9,-.45)),nacf=38)
```



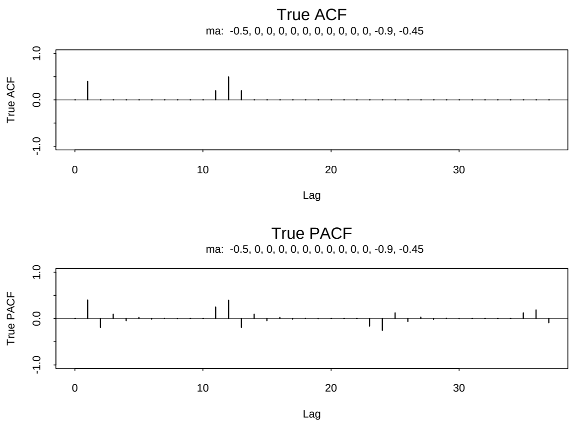
True ACF/PACF for $W_t = (1-\theta_1 \mathrm{B})(1-\Theta_1 \mathrm{B}^{12})a_t$, $\theta_1 = .1$, $\Theta_1 = .9$

```
plot.true.acfpacf(model=list(ma=c(.1,rep(0,10),.9,-.09)),nacf=38)
```



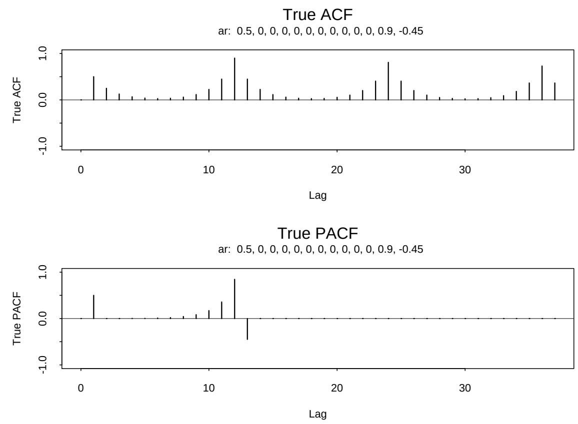
True ACF/PACF for $W_t = (1 - \theta_1 \mathrm{B})(1 - \Theta_1 \mathrm{B}^{12})a_t$, $\theta_1 = -.5$, $\Theta_1 = -.9$

```
plot.true.acfpacf(model=list(ma=c(-.5,rep(0,10),-.9,-.45)),nacf=38)
```



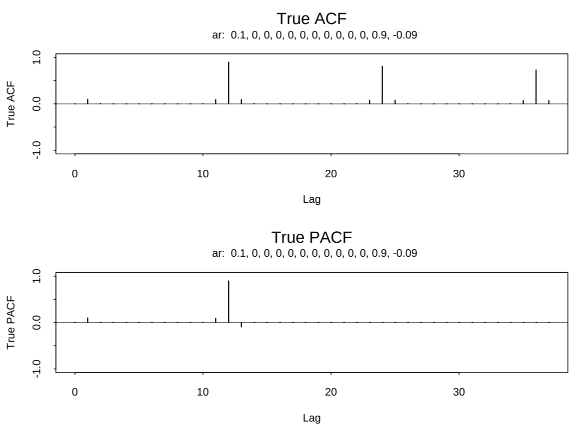
True ACF/PACF for $(1 - \phi_1 \mathrm{B})(1 - \Phi_1 \mathrm{B}^{12})W_t = a_t$, $\phi_1 = .5$, $\Phi_1 = .9$

```
plot.true.acfpacf(model=list(ar=c(.5,rep(0,10),.9,-.45)),nacf=38)
```



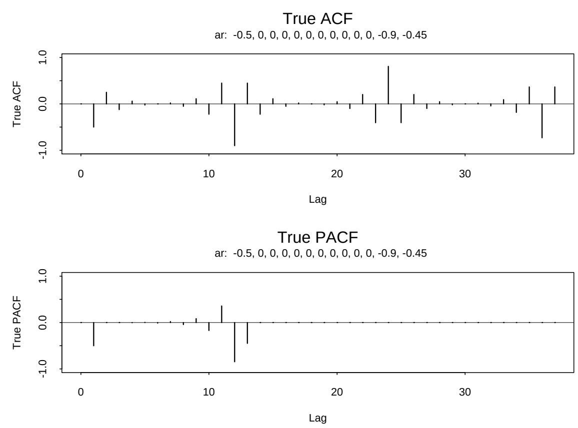
True ACF/PACF for $(1 - \phi_1 \mathrm{B})(1 - \Phi_1 \mathrm{B}^{12})W_t = a_t$, $\phi_1 = .1$, $\Phi_1 = .9$

```
plot.true.acfpacf(model=list(ar=c(.1,rep(0,10),.9,-.09)),nacf=38)
```

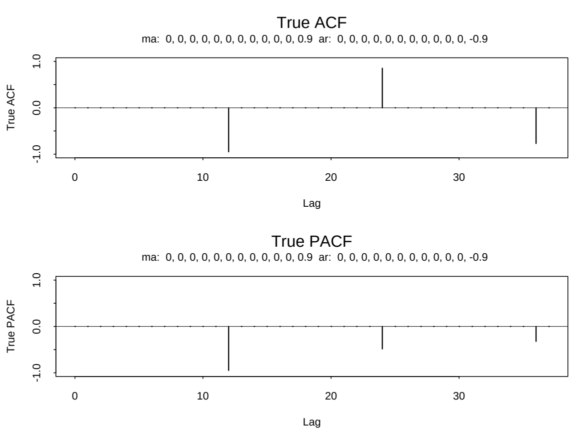


True ACF/PACF for $(1 - \phi_1 \mathrm{B})(1 - \Phi_1 \mathrm{B}^{12})W_t = a_t$, $\phi_1 = -.5$, $\Phi_1 = -.9$

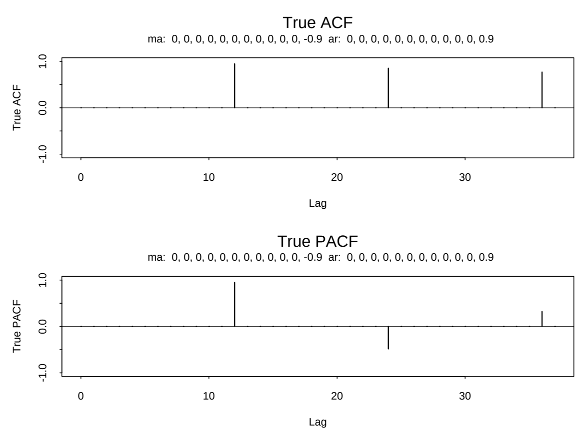
```
plot.true.acfpacf(model=list(ar=c(-.5,rep(0,10),-.9,-.45)),nacf=38)
```



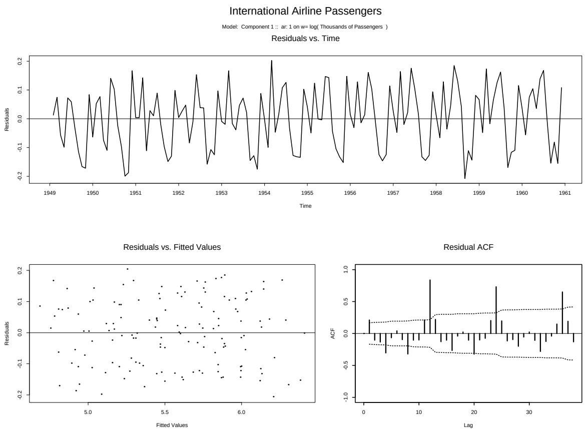
True ACF/PACF for $(1 - \Phi_1 \mathrm{B}^{12})W_t = (1 - \Theta_1 \mathrm{B}^{12})a_t$, $\Phi_1 = -.9, \Theta_1 = .9$



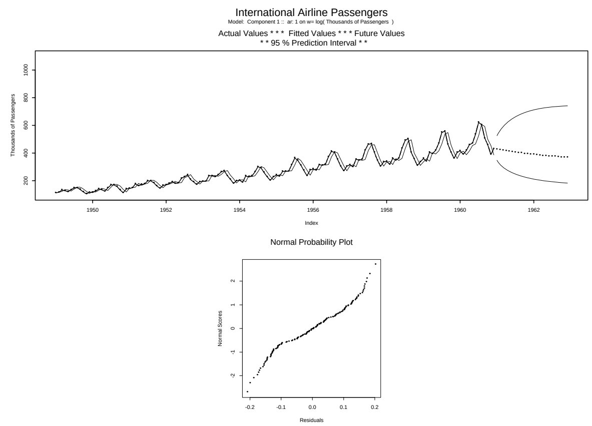
True ACF/PACF for $(1 - \Phi_1 \mathrm{B}^{12})W_t = (1 - \Theta_1 \mathrm{B}^{12})a_t$, $\Phi_1 = .9, \Theta_1 = -.9$



Function `esti` Output for the Airline Data AR(1) Model—Part 1 `esti(airline.d, gamma=0,model = model.pdq(p=1),y.range=c(100,1100))`



Function `esti` Output for the Airline Data AR(1) Model—Part 2 `esti(airline.d, gamma=0,model = model.pdq(p=1),y.range=c(100,1100))`



The "Airline Model"

Differencing $d = 1$ and $D = 1$ to give the working series

$$\begin{aligned} W_t &= (1 - \mathrm{B})(1 - \mathrm{B}^{12})Z_t \\ &= Z_t - Z_{t-1} - Z_{t-12} + Z_{t-13} \end{aligned}$$

Model for the working series W_t

$$\begin{aligned} W_t &= (1 - \theta_1 \mathrm{B})(1 - \Theta_1 \mathrm{B}^{12})a_t \\ &= -\theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13} + a_t \end{aligned}$$

The "unscrambled (multiplicative) airline model" for Z_t is

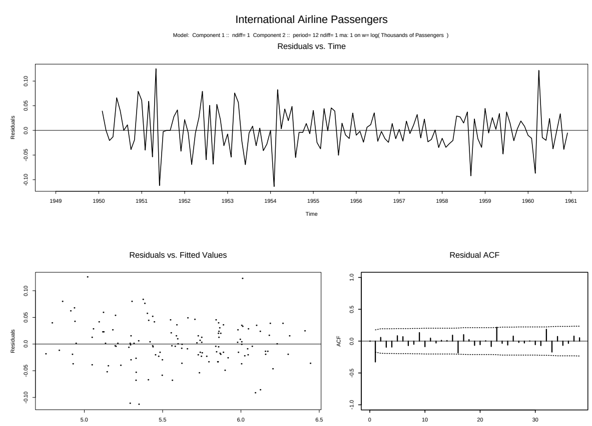
$$Z_t = Z_{t-1} + Z_{t-12} - Z_{t-13} - \theta_1 a_{t-1} - \Theta_1 a_{t-12} + \theta_1 \Theta_1 a_{t-13} + a_t$$

and has only 2 parameters. The "unscrambled (additive) airline model" for Z_t is

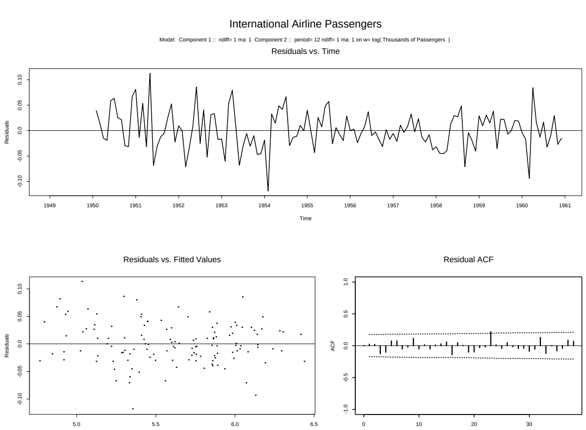
$$Z_t = Z_{t-1} + Z_{t-12} - Z_{t-13} - \theta_1 a_{t-1} - \theta_{12} a_{t-12} - \theta_{13} a_{t-13} + a_t$$

has three parameters.

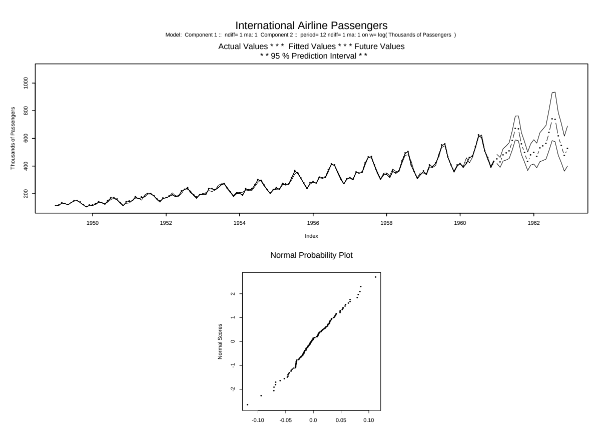
Function `esti` Output for the Airline Data SARMA$(0,1,0)(0,1,1)_{12}$ Model—Part 1



Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Model—Part 1 `esti(airline.d, gamma=0,model=airline.model)`

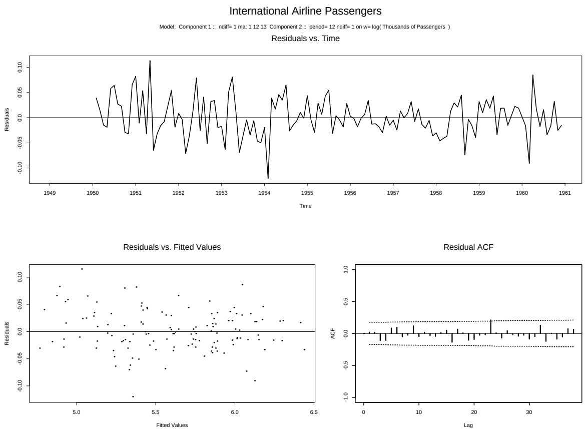


Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Model—Part 2 `esti(airline.d, gamma=0,model=airline.model)`



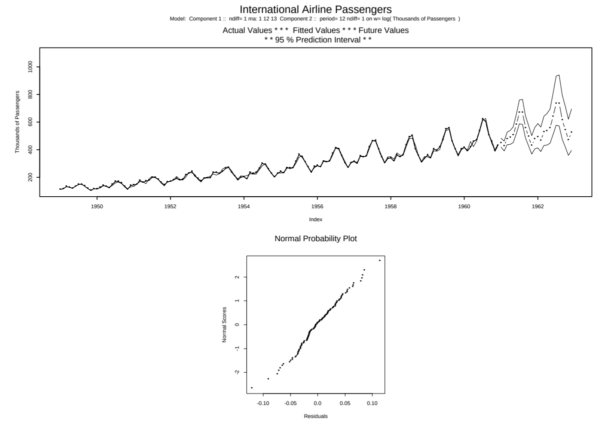
Function `esti` Output for the Airline Data Additive Airline Model—Part 1

`esti(airline.d, gamma=0,model=add.airline.model)`



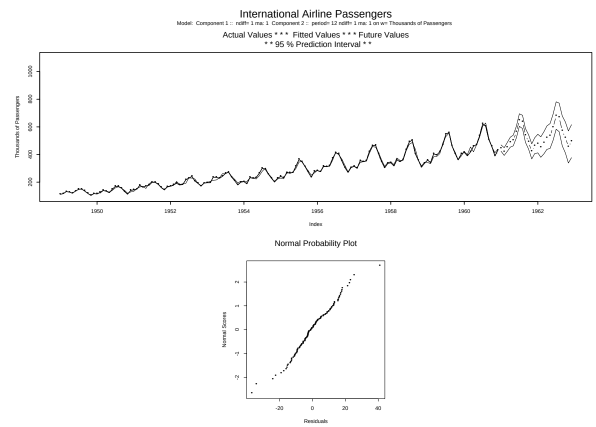
Function `esti` Output for the Airline Data Additive Airline Model—Part 2

`esti(airline.d, gamma=0,model=add.airline.model)`

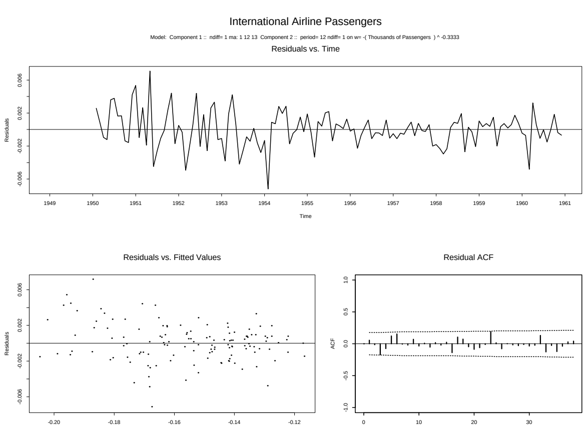


Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Model with No Transformation—Part 1

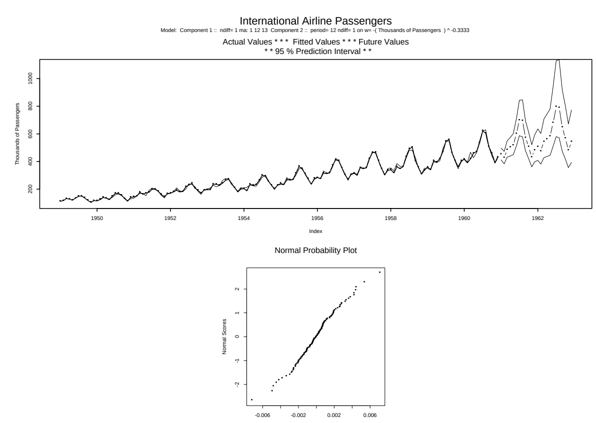
Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Modell with No Transformation—Part 2



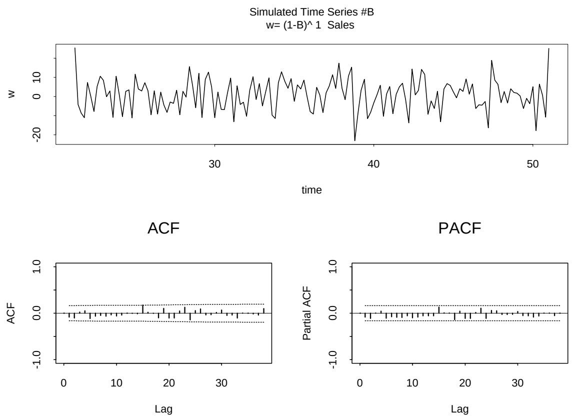
Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Model with $\gamma = -.333$ Transformation—Part 1



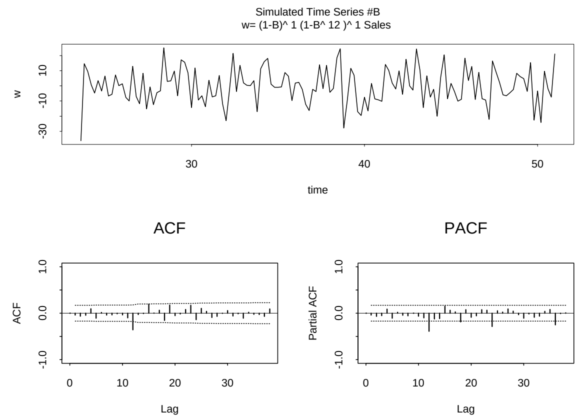
Function `esti` Output for the Airline Data SARMA$(0,1,1)(0,1,1)_{12}$ Modell with $\gamma = -.333$ Transformation—Part 2



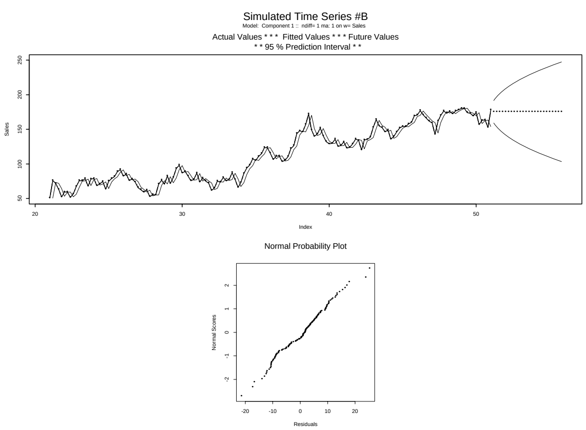
Graphical Output from Function `iden` for Simulated Series #B with 1 Regular Difference



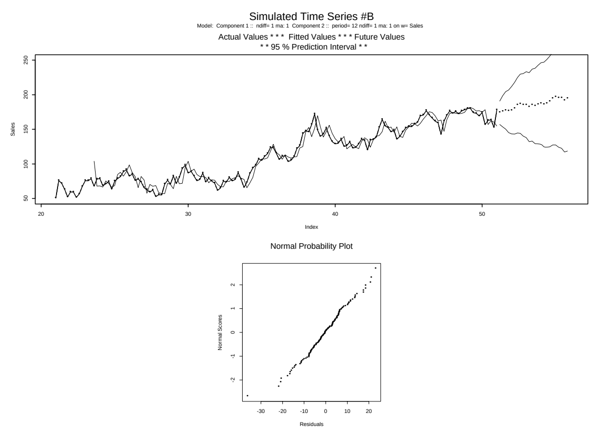
Graphical Output from Function `iden` for Simulated Series #B with 1 Regular and 1 Seasonal Difference



Function `esti` Output for Simulated Series #B SARMA$(0,1,1)(0,0,0)_{12}$ Model—Part 2



Function `esti` Output for Simulated Series #B SARMA$(0,1,1)(0,1,1)_{12}$ Model—Part 2



Seasonal Over-Differencing

Assume that

$$(1-\mathrm{B})Z_t = a_t$$

so that Z_t has a random walk model. Then

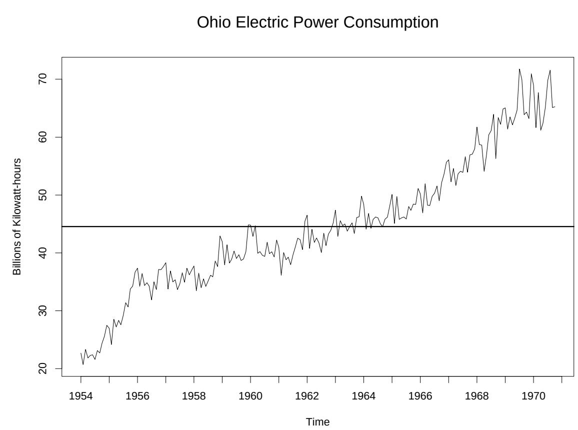
$$W_t = (1-\mathrm{B})Z_t = a_t$$

will follow a "trivial model" and we will not expect to see anything in the ACF/PACF. What if we take seasonal differences?

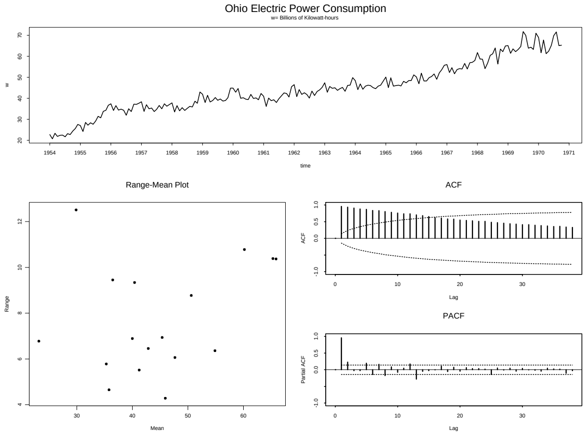
$$\begin{aligned} W_t = (1-\mathrm{B}^{12})(1-\mathrm{B})Z_t &= (1-\mathrm{B}^{12})a_t \\ &= -a_{12} + a_t \end{aligned}$$

which is a noninvertible MA(12)!

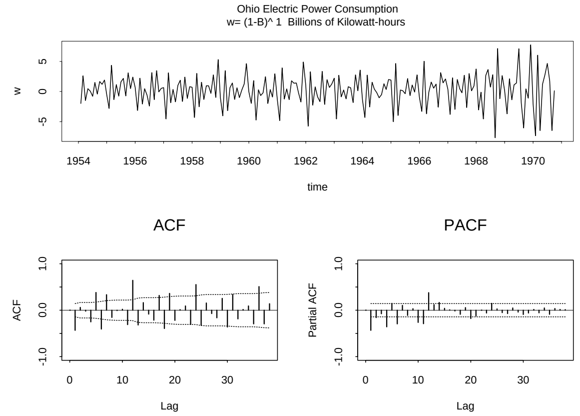
Ohio Power Consumption



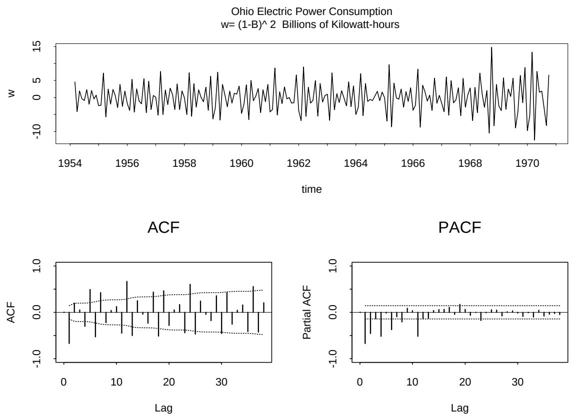
Graphical Output from Function `iden` for the Ohio Power Consumption Data with No Differencing



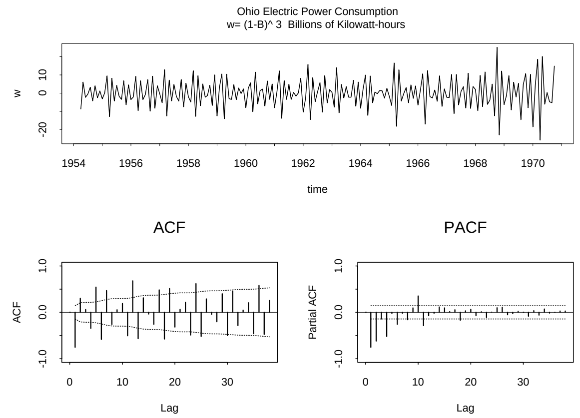
Graphical Output from Function `iden` for the Ohio Power Consumption Data with 1 Regular Difference



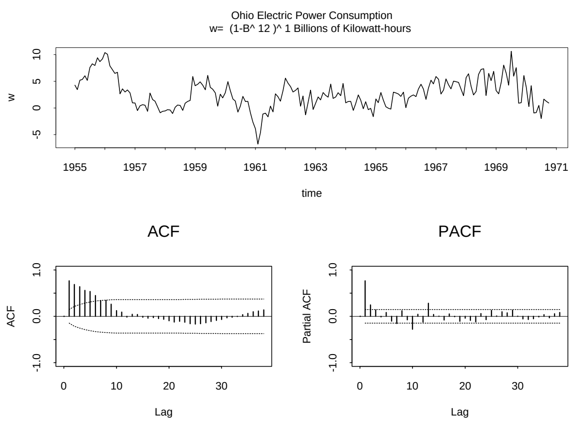
Graphical Output from Function `iden` for the Ohio Power Consumption Data with 2 Regular Differences



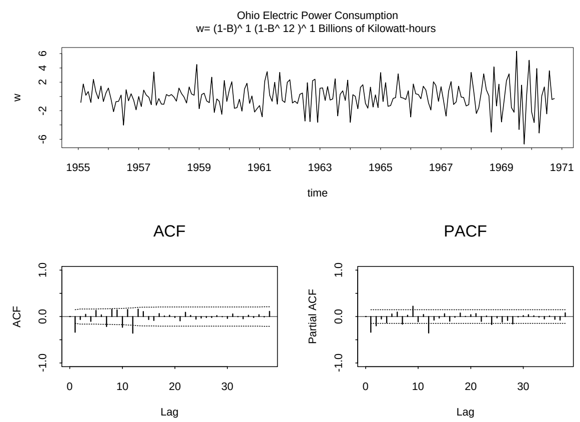
Graphical Output from Function `iden` for the Ohio Power Consumption Data with 3 Regular Differences



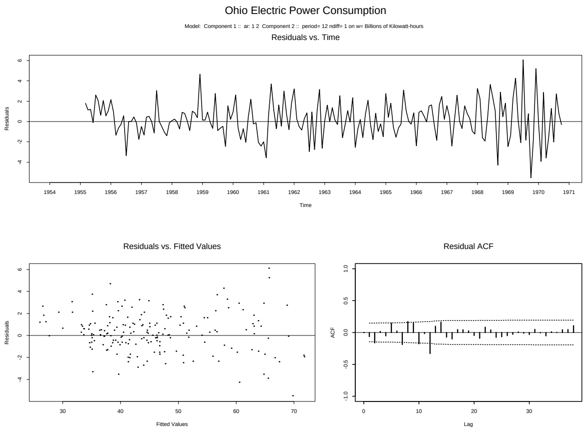
Graphical Output from Function `iden` for the Ohio Power Consumption Data with 1 Seasonal Difference



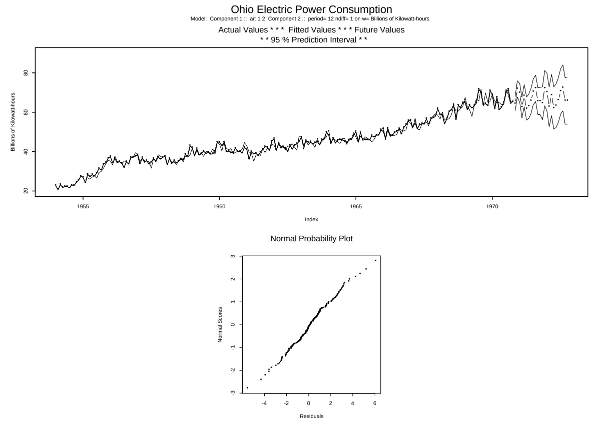
Graphical Output from Function `iden` for the Ohio Power Consumption Data with 1 Seasonal Difference and 1 Regular Difference



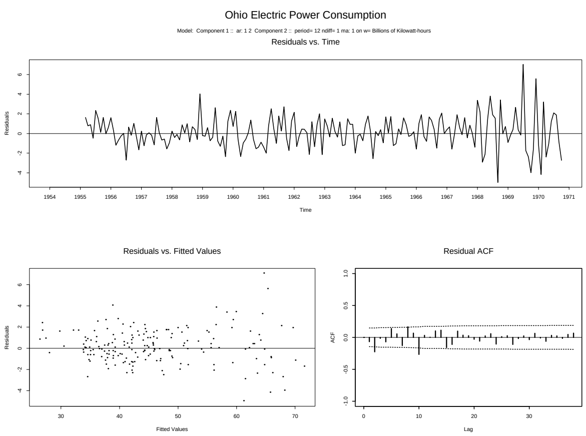
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,0)(0,1,0)_{12}$ Model—Part 1



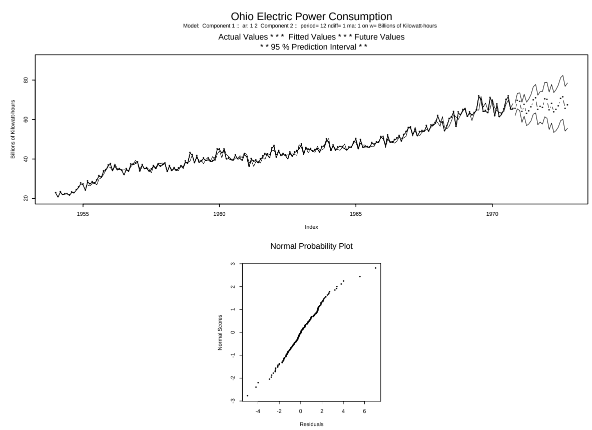
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,0)(0,1,0)_{12}$ Model—Part 2



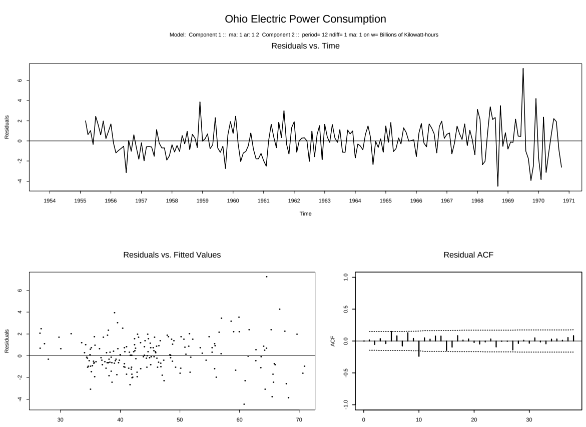
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,0)(0,1,1)_{12}$ Model—Part 1



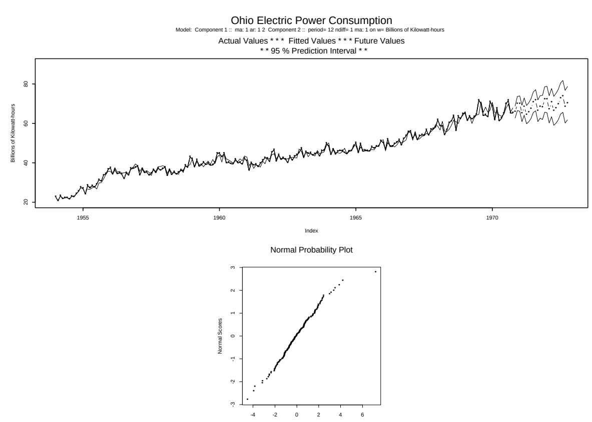
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,0)(0,1,1)_{12}$ Model—Part 2



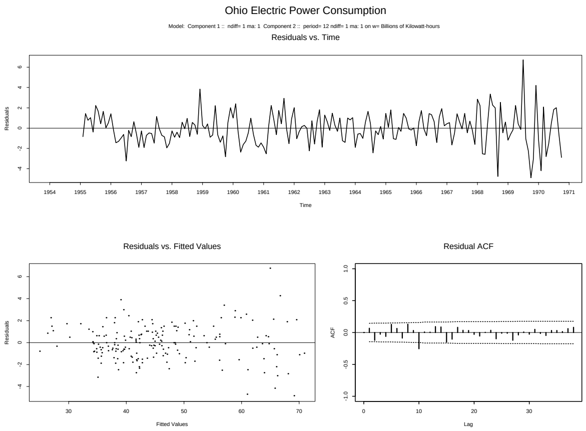
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,1)(0,1,1)_{12}$ Model—Part 1



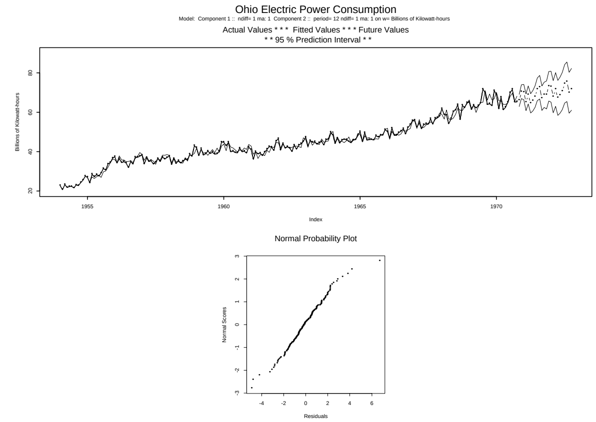
Function `esti` Output for the Ohio Power Consumption Data SARMA$(2,0,1)(0,1,1)_{12}$ Model—Part 2



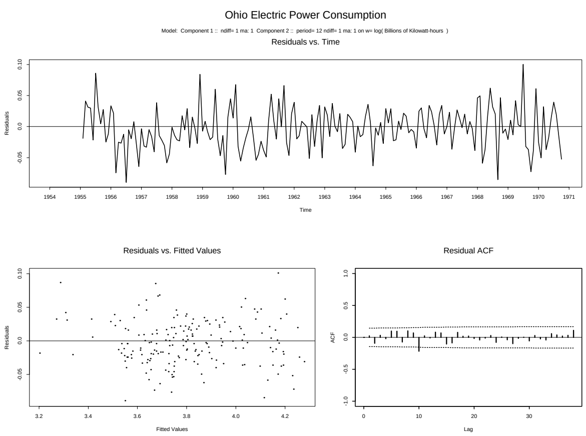
Function esti Output for the Ohio Power Consumption Data $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model—Part 1



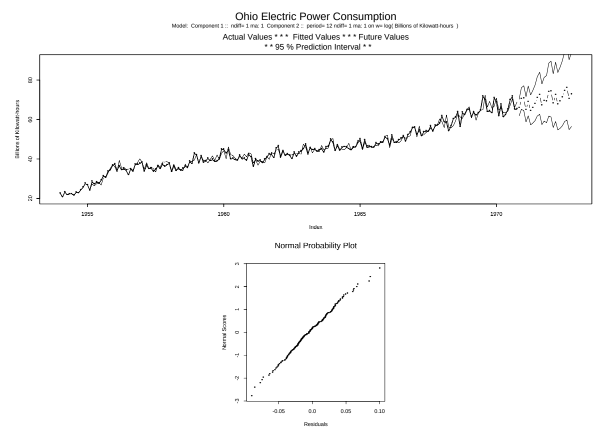
Function esti Output for the Ohio Power Consumption Data $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model—Part 2



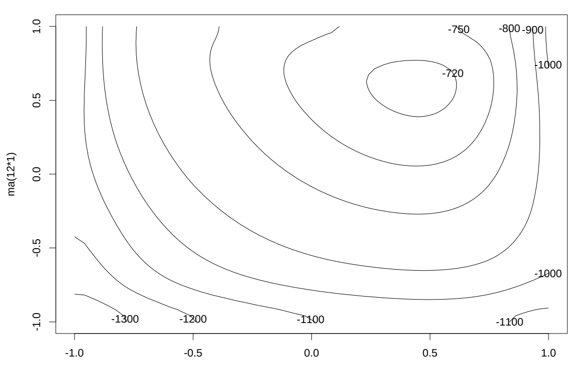
Function esti Output for the Ohio Power Consumption Data with Log Transform and $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model—Part 1



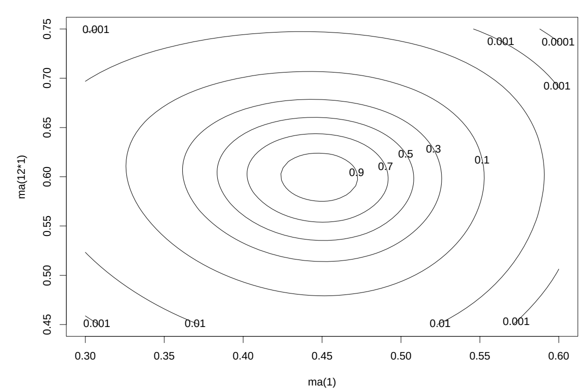
Function esti Output for the Ohio Power Consumption Data with Log Transform and $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model—Part 2



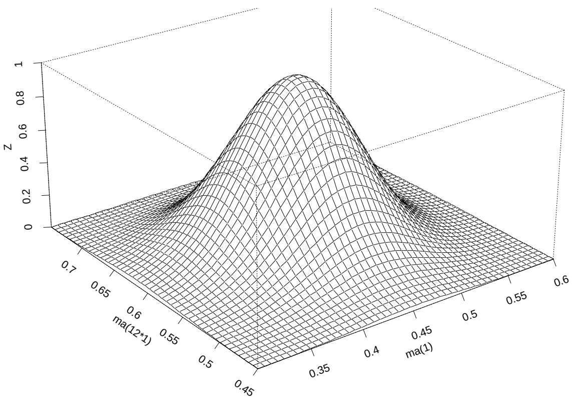
Contour Plot of $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model -2[Log-likelihood] Surface for the Ohio Data



Contour Plot of $\mathrm{SARMA}(0,1,1)(0,1,1)_{12}$ (Airline) Model Relative Likelihood Surface for the Ohio Data



Perspective Plot of SARMA$(0,1,1)(0,1,1)_{12}$ (Airline) Model Relative Likelihood Surface for the Ohio Data



ACF of the Residuals for the Ohio Power using the SARMA$(0,1,1)(0,1,1)_{12}$ (Airline) Model

`show.acf(ohio.model6.out$resid)`