Linear Regression

Michael R. Roberts

Department of Finance
The Wharton School
University of Pennsylvania

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Motivation

- Linear regression is arguably the most popular modeling approach across every field in the social sciences.
  1. Very robust technique
  2. Linear regression also provides a basis for more advanced empirical methods.
  3. Transparent and relatively easy to understand technique
  4. Useful for both descriptive and structural analysis
- We're going to learn linear regression inside and out from an applied perspective
  - focusing on the appropriateness of different assumptions, model building, and interpretation
- This lecture draws heavily from Wooldridge's undergraduate and graduate texts, as well as Greene's graduate text.

Terminology

- The **simple linear regression model** (a.k.a. - **bivariate linear regression model**, **2-variable linear regression model**)

$$y = \alpha + \beta x + u \tag{1}$$

- $y =$ **dependent variable**, **outcome variable**, **response variable**, **explained variable**, **predicted variable**, **regressand**
- $x =$ **independent variable**, **explanatory variable**, **control variable**, **predictor variable**, **regressor**, **covariate**
- $u =$ **error term**, **disturbance**
- $\alpha =$ **intercept parameter**
- $\beta =$ **slope parameter**

Details

- Recall model is

$$y = \alpha + \beta x + u$$

- (y, x, u) are random variables
- (y, x) are observable (we can sample observations on them)
- u is unobservable $\implies$ no stat tests involving u
- (α, β) are unobserved but estimable under certain cond's
- Model implies that u captures everything that determines y except for x
- In natural sciences, this often includes frictions, air resistance, etc.
- In social sciences, this often includes a lot of stuff!!!

Assumptions

1. $E(u) = 0$
   - As long as we have an intercept, this assumption is innocuous
   - Imagine $E(u) = k \neq 0$. We can rewrite $u = k + w \implies$

   $$y_i = (\alpha + k) + \beta E(x_i) + w$$

   where $E(\omega) = 0$. Any non-zero mean is absorbed by the intercept.
2. $E(u|x) = E(u)$
   - Assuming $q \perp u$ ($\perp =$ orthogonal) is *not enough!* Correlation only measures *linear* dependence
   - **Conditional mean independence**
   - Implied by full independence $q \perp\!\!\!\perp u$ ($\perp\!\!\!\perp =$ independent)
   - Implies uncorrelated
   - Intuition: avg of u does *not* depend on the value of q
   - Can combine with zero mean assumption to get **zero conditional mean assumption** $E(u|q) = E(u) = 0$

Conditional Mean Independence (CMI)

- This is the key assumption in most applications
- Can we test it?
  - Run regression.
  - Take **residuals** $\hat{u} = y - \hat{y}$ & see if avg $\hat{u}$ at each value of x = 0?
  - Or, see if residuals are uncorrelated with x
  - Does these exercise make sense?
- Can we think about it?
  - The assumption says that no matter whether x is low, medium, or high, the unexplained portion of y is, on average, the same (0).
  - But, what if agents (firms, etc.) with different values of x are different along other dimensions that matter for y?

CMI Example 1: Capital Structure

- Consider the regression

$$Leverage_i = \alpha + \beta Profitability_i + u_i$$

- CMI $\implies$ that average u for each level of *Profitability* is the same
- But, unprofitable firms tend to have higher bankruptcy risk and should have lower leverage than more profitable firms according to tradeoff theory
- Or, unprofitable firms have accumulated fewer profits and may be forced to debt financing, implying higher leverage according to the pecking order
- These e.g.'s show that the average u is likely to vary with the level of profitability
  - 1st e.g., low profitable firms will be less levered implies lower avg u for less profitable firms
  - 2nd e.g., low profitable firms will be more levered implies higher avg u for less profitable firms

CMI Example 2: Investment

- Consider the regression

$$Investment_i = \alpha + \beta q_i + u_i$$

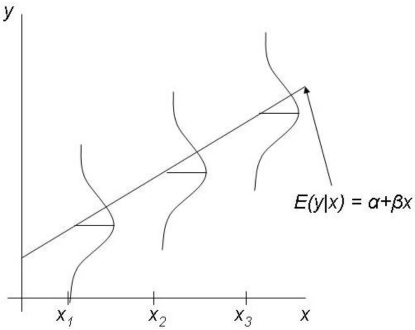
- CMI $\implies$ that average u for each level of q is the same
- But, firms with low q may be in distress and invest less
- Or, firms with high q may have difficultly raising sufficient capital to finance their investment
- These e.g.'s show that the average u is likely to vary with the level of q
  - 1st e.g., low q firms will invest less implies higher avg u for low q firms
  - 2nd e.g., high q firms will invest less implies higher avg u for low q firms

Population Regression Function (PRF)

- PRF is $E(y|x)$. It is fixed but unknown. For simple linear regression:

$$PRF = E(y|x) = \alpha + \beta x \tag{2}$$

- Intuition: for any value of x, distribution of y is centered about $E(y|x)$



OLS Regression Line

- We don't observe PRF, but we can estimate via OLS

$$y_i = \alpha + \beta x_i + u_i \tag{3}$$

  for each sample point i
- What is u_i? It contains all of the factors affecting y_i *other* than x_i.
  $\implies$ u_i contains a lot of stuff! Consider complexity of
  - y is individual food expenditures
  - y is corporate leverage ratios
  - y is interest rate spread on a bond
- **Estimated Regression Line** (a.k.a. **Sample Regression Function (SRF)**)

$$\hat{y} = \hat{\alpha} + \hat{\beta} x \tag{4}$$

  Plug in an x and out comes an estimate of y, $\hat{y}$
- Note: Different sample $\implies$ different $(\hat{\alpha}, \hat{\beta})$

OLS Estimates

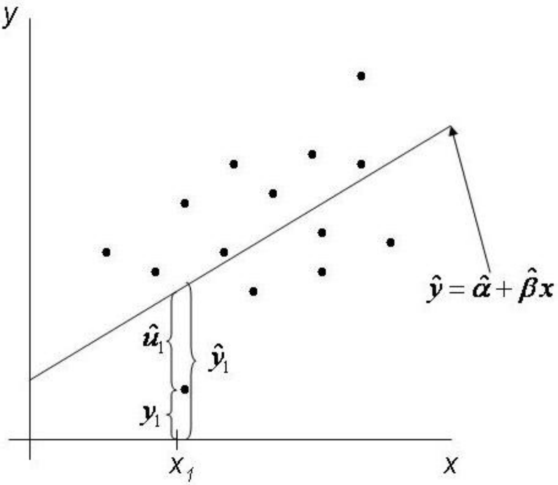
- Estimators:

$$\begin{aligned} \text{Slope} &= \hat{\beta} = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2} \\ \text{Intercept} &= \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \end{aligned}$$

- Population analogues

$$\begin{aligned} \text{Slope} &= \frac{Cov(x,y)}{Var(x)} = Corr(x,y)\frac{SD(y)}{SD(x)} \\ \text{Intercept} &= E(y) - \hat{\beta}E(x) \end{aligned}$$

The Picture



Example: CEO Compensation

- Model

$$salary = \alpha + \beta ROE + y$$

- Sample 209 CEOs in 1990. Salaries in \$000s and ROE in % points.
- SRF

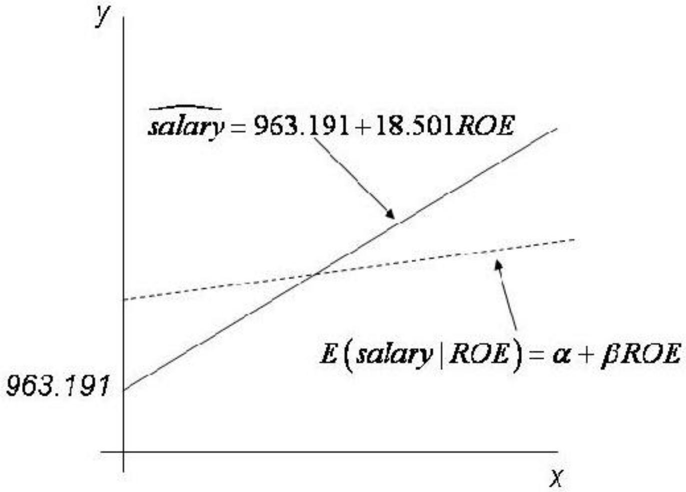
$$salary = 963.191 + 18.501 ROE$$

- What do the coefficients tell us?
- Is the key CMI assumption likely to be satisfied?
  - Is ROE the only thing that determines salary?
  - Is the relationship linear? $\implies$ estimated change is constant across salary and ROE

$$dy/dx = \beta \text{ indep of salary \& ROE}$$

  - Is the relationship constant across CEOs?

PRF vs. SRF


y
$\widehat{salary} = 963.191 + 18.501ROE$
$E(salary \mid ROE) = \alpha + \beta ROE$
963.191
X

Goodness-of-Fit (R^2)

- R-squared defined as

$$R^2 = SSE/SST = 1 - SSR/SST$$

where

$$\begin{aligned} SSE &= \text{Sum of Squares Explained} = \sum_{i=1}^{N} (\hat{y}_i - \bar{\hat{y}})^2 \\ SST &= \text{Sum of Squares Total} = \sum_{i=1}^{N} (y_i - \bar{y})^2 \\ SSR &= \text{Sum of Squares Residual} = \sum_{i=1}^{N} (\hat{u}_i - \bar{\hat{u}})^2 = \sum_{i=1}^{N} \hat{u}_i{}^2 \end{aligned}$$

- $R^2 = [Corr(y, \hat{y})]^2$

Example: CEO Compensation

- Model

$$salary = \alpha + \beta ROE + y$$

- $R^2 = 0.0132$
- What does this mean?

Scaling the Dependent Variable

- Consider CEO SRF

$$salary = 963.191 + 18.501 ROE$$

- Change measurement of salary from $000s to $s. What happens?

$$salary = 963,191 + 18,501 ROE$$

- More generally, multiplying **dependent variable** by constant $c \implies$ OLS intercept and slope are also multiplied by c

$$\begin{aligned} & y = \alpha + \beta x + u \\ \iff \quad & cy = (c\alpha) + (c\beta)x + cu \end{aligned}$$

(Note: variance of error affected as well.)

- Scaling $\implies$ multiplying *every* observation by same #
- No effect on R^2 - invariant to changes in units

Scaling the Independent Variable

- Consider CEO SRF

$$salary = 963.191 + 18.501 ROE$$

- Change measurement of ROE from percentage to decimal (i.e., multiply every observation's ROE by 1/100)

$$salary = 963.191 + 1,850.1 ROE$$

- More generally, multiplying **independent variable** by constant $c \implies$ OLS intercept is unchanged but slope is divided by c

$$\begin{aligned} & y = \alpha + \beta x + u \\ \iff \quad & y = \alpha + (\beta / c) cx + cu \end{aligned}$$

- Scaling $\implies$ multiplying *every* observation by same #
- No effect on R^2 - invariant to changes in units

Changing Units of Both y and x

- Model:

$$y = \alpha + \beta x + u$$

- What happens to intercept and slope when we scale y by c and x by k?

$$\begin{aligned} cy &= c\alpha + c\beta x + cu \\ cy &= (c\alpha) + (c\beta/k)kx + cu \end{aligned}$$

- intercept scaled by c, slope scaled by c/k

Shifting Both y and x

- Model:

$$y = \alpha + \beta x + u$$

- What happens to intercept and slope when we add c and k to y and x?

$$\begin{aligned} c + y &= c + \alpha + \beta x + u \\ c + y &= c + \alpha + \beta(x + k) - \beta k + u \\ c + y &= (c + \alpha - \beta k) + \beta(x + k) + u \end{aligned}$$

- Intercept shifted by $\alpha - \beta k$, slope unaffected

Incorporating Nonlinearities

- Consider a traditional wage-education regression

$$wage = \alpha + \beta education + u$$

- This formulation assumes change in wages is constant for all educational levels
- E.g., increasing education from 5 to 6 years leads to the same $ increase in wages as increasing education from 11 to 12, or 15 to 16, etc.
- Better assumption is that each year of education leads to a constant *proportionate* (i.e., percentage) increase in wages
- *Approximation* of this intuition captured by

$$log(wage) = \alpha + \beta education + u$$

Log Dependent Variables

- Percentage change in wage given one unit increase in education is

$$\%\Delta wage \approx (100\beta)\Delta educ$$

- Percent change in wage is constant for each additional year of education

$\Longrightarrow$ Change in wage for an extra year of education *increases* as education increases.

  - I.e., increasing return to education (assuming $\beta > 0$)

- Log wage is linear in education. Wage is nonlinear

$$\begin{aligned} log(wage) &= \alpha + \beta education + u \\ \Longrightarrow \quad wage &= \exp\left(\alpha + \beta education + u\right) \end{aligned}$$

Log Wage Example

- Sample of 526 individuals in 1976. Wages measured in \$/hour. Education = years of education.
- SRF:

$$log(wage) = 0.584 + 0.083 education, \qquad R^2 = 0.186$$

- Interpretation:
    - Each additional year of education leads to an 8.3% increase in wages (NOT log(wages)!!!).
    - For someone with no education, their wage is exp(0.584)...this is meaningless because no one in sample has education=0.
- Ignores other nonlinearities. E.g., diploma effects at 12 and 16.

Constant Elasticity Model

- Alter CEO salary model

$$log(salary) = \alpha + \beta log(sales) + u$$

- β is the **elasticity** of salary w.r.t. sales
- SRF

$$log(salary) = 4.822 + 0.257 log(sales), \qquad R^2 0.211$$

- Interpretation: For each 1% increase in sales, salary increase by 0.257%
- Intercept meaningless...no firm has 0 sales.

Changing Units in Log-Level Model

- What happens to intercept and slope if we Δ units of dependent variable when it's in log form?

$$
\begin{aligned}
& log(y) = \alpha + \beta x + u \\
\iff \quad & log(c) + log(y) = log(c) + \alpha + \beta x + u \\
\iff \quad & log(cy) = (log(c) + \alpha) + \beta x + u
\end{aligned}
$$

- Intercept shifted by $log(c)$, slope unaffected because slope measures *proportionate* change in log-log model

Changing Units in Level-Log Model

- What happens to intercept and slope if we Δ units of independent variable when it's in log form?

$$
\begin{aligned}
& y = \alpha + \beta log(x) + u \\
\iff \quad & \beta log(c) + y = \alpha + \beta log(x) + \beta log(c) + u \\
\iff \quad & y = (\alpha - \beta log(c)) + \beta log(cx) + u
\end{aligned}
$$

- Slope measures *proportionate* change

Changing Units in Log-Log Model

- What happens to intercept and slope if we Δ units of dependent variable?

$$\begin{aligned} & log(y) = \alpha + \beta log(x) + u \\ \iff \quad & log(c) + log(y) = log(c) + \alpha + \beta log(x) + u \\ \iff \quad & log(cy) = (\alpha + log(c)) + \beta log(x) + u \end{aligned}$$

- What happens to intercept and slope if we Δ units of independent variable?

$$\begin{aligned} & log(y) = \alpha + \beta log(x) + u \\ \iff \quad & \beta log(c) + log(y) = \alpha + \beta log(x) + \beta log(c) + u \\ \iff \quad & log(y) = (\alpha - \beta log(c)) + \beta log(cx) + u \end{aligned}$$

Log Functional Forms

Model	Dependent Variable	Independent Variable	Interpretation of β
Level-level	y	x	$dy = \beta dx$
Level-log	y	log(x)	$dy = (\beta/100)\%dx$
Log-level	log(y)	x	$\%dy = (100\beta)dx$
Log-log	log(y)	log(x)	$\%dy = \beta\%dx$

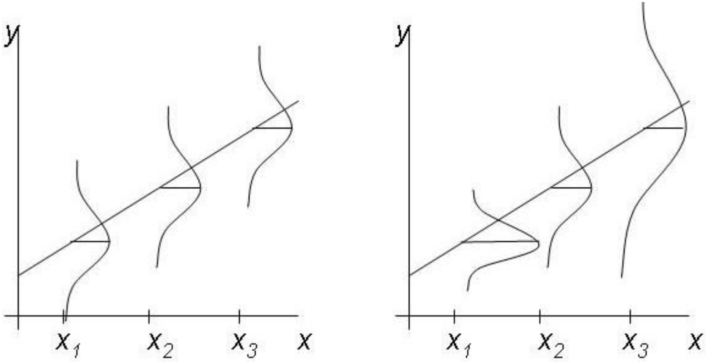
- E.g., In Log-level model, $100 \times \beta =$ % change in y for a 1 unit increase in x ($100\beta =$ **semi-elasticity**)
- E.g., In Log-log model, $\beta =$ % change in y for a 1% change in x (β = **elasticity**)

Unbiasedness

- When is OLS unbiased (i.e., $E(\hat{\beta}) = \beta$)?
    1. Model is linear in parameters
    2. We have a random sample (e.g., self selection)
    3. Sample outcomes on x vary (i.e., no collinearity with intercept)
    4. Zero conditional mean of errors (i.e., $E(u|x) = 0$)
- Unbiasedness is a feature of sampling distributions of $\hat{\alpha}$ and $\hat{\beta}$.
- For a given sample, we hope $\hat{\alpha}$ and $\hat{\beta}$ are close to true values.

Variance of OLS Estimators

- **Homoskedasticity** $\implies Var(u|x) = \sigma^2$
- **Heterokedasticity** $\implies Var(u|x) = f(x) \in \mathbb{R}^+$



Standard Errors

- Remember, larger error variance $\Longrightarrow$ larger $Var(\beta)$ $\Longrightarrow$ bigger SEs
- Intuition: More variation in unobservables affecting y makes it hard to precisely estimate β
- Relatively more variation in x is our friend!!!
- More variation in x means lower SEs for β
- Likewise, larger samples tend to increase variation in x which also means lower SEs for β
- I.e., we like big samples for identifying β!

Basics

- **Multiple Linear Regression Model**

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + u$$

- Same notation and terminology as before.
- Similar key identifying assumptions
  1. No perfect collinearity among covariates
  2. $E(u|x_1, ...x_k) = 0 \implies$ at a minimum no correlation and we have correctly accounted for the functional relationships between y and $(x_1, ..., x_k)$
- SRF

$$y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + ... + \hat{\beta}_k x_k$$

Interpretation

- Estimated intercept $\hat{beta}_0$ is predicted value of y when *all* $x = 0$. Sometimes this makes sense, sometimes it doesn't.
- Estimated slopes $(\hat{\beta}_1, ...\hat{\beta}_k)$ have partial effect interpretations

$$\Delta\hat{y} = \hat{\beta}_1 \Delta x_1 + ... + \hat{\beta}_k \Delta x_k$$

  I.e., given changes in x_1 through x_k, $(\Delta x_1, ..., \Delta x_k)$, we obtain the *predicted* change in y.
- When all but one covariate, e.g., x_1, is held fixed so $(\Delta x_2, ..., \Delta x_k) = (0, ..., 0)$ then

$$\Delta\hat{y} = \hat{\beta}_1 \Delta x_1$$

  I.e., $\hat{\beta}_1$ is the coefficient holding *all else fixed* (ceteris paribus)

Example: College GPA

- SRF of college GPA and high school GPA (4-point scales) and ACT score for $N = 141$ university students

$$\widehat{colGPA} = 1.29 + 0.453 hsGPA + 0.0094 ACT$$

- What do intercept and slopes tell us?
  - Consider two students, Fred and Bob, with identical ACT score but *hsGPA* of Fred is 1 point higher than that of Bob. Best prediction of Fred's *colGPA* is 0.453 points higher than that of Bob.
- SRF without *hsGPA*

$$\widehat{colGPA} = 1.29 + 0.0271 ACT$$

- What's different and why? Can we use it to compare 2 people with same *hsGPA*?

All Else Equal

- Consider prev example. Holding ACT fixed, another point on high school GPA is predicted to inc college GPA by 0.452 points.
- If we could collect a sample of individuals with the same high school ACT, we could run a simple regression of college GPA on high school GPA. This holds all else, ACT, fixed.
- Multiple regression mimics this scenario without restricting the values of any independent variables.

Changing Multiple Independent Variables Simultaneously

- Each β corresponds to the partial effect of its covariate
- What if we want to change more than one variable at the same time?
- E.g., What is the effect of increasing the high school GPA by 1 point and the ACT score by 1 points?

$$\Delta \widehat{colGPA} = 0.453 \Delta hsGPA + 0.0094 \Delta ACT = 0.4624$$

- E.g., What is the effect of increasing the high school GPA by 2 point and the ACT score by 10 points?

$$\begin{aligned} \Delta \widehat{colGPA} &= 0.453 \Delta hsGPA + 0.0094 \Delta ACT \\ &= 0.453 \times 2 + 0.0094 \times 10 = 1 \end{aligned}$$

Fitted Values and Residuals

- **Residual** $= \hat{u}_i = y_i - \hat{y}_i$
- Properties of residuals and fitted values:
  1. sample avg of residuals $= 0 \implies \bar{\hat{y}} = \bar{y}$
  2. sample cov between each indep variable and residuals $= 0$
  3. Point of means $(\bar{y}, \bar{x}_1, ..., \bar{x}_k)$ lies on regression line.

Partial Regression

- Consider 2 independent variable model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

- What's the formula for just $\hat{\beta}_1$?

$$\hat{\beta}_1 = (\hat{r}_1' \hat{r}_1)^{-1} \hat{r}_1' y$$

where $\hat{r}_1$ are the residuals from a regression of x_1 on x_2.

- In other words,
  1. regress x_1 on x_2 and save residuals
  2. regress y on residuals
  3. coefficient on residuals will be identical to $\hat{\beta}_1$ in multivariate regression

Frisch-Waugh-Lovell I

- More generally, consider general linear setup

$$y = XB + u = X_1 B_1 + X_2 B_2 + u$$

- One can show that

$$\hat{B}_2 = (X_2' M_1 X_2)^{-1} (X_2' M_1 y) \tag{5}$$

where

$$M_1 = (I - P_1) = I - X_1 (X_1' X_1)^{-1} X_1'$$

- P_1 is the projection matrix that takes a vector (y) and projects it onto the space spanned by columns of X_1
- M_1 is the orthogonal compliment, projecting a vector onto the space orthogonal to that spanned by X_1

Frisch-Waugh-Lovell II

- What does equation (5) mean?
- Since M_1 is idempotent

$$\begin{aligned}\hat{B}_2 &= (X_2' M_1 M_1 X_2)^{-1}(X_2' M_1 M_1 y) \\ &= (\tilde{X}_2' \tilde{X}_2)^{-1}(\tilde{X}_2' \tilde{y})\end{aligned}$$

- So $\hat{B}_2$ can be obtained by a simple multivariate regression of $\tilde{y}$ on $\tilde{X}_2$
- But $\tilde{y}$ and $\tilde{X}_2$ are just the residuals obtained from regressing y and each component of X_2 on the X_1 matrix

Omitted Variables Bias

- Assume correct model is:

$$y = XB + u = X_1 B_1 + X_2 B_2 + u$$

- Assume we *in*correctly regress y on just X_1. Then

$$\begin{aligned} \hat{B}_1 &= (X_1' X_1)^{-1} X_1' y \\ &= (X_1' X_1)^{-1} X_1' (X_1 B_1 + X_2 B_2 + u) \\ &= B_1 + (X_1' X_1)^{-1} X_1' X_2 B_2 + (X_1' X_1)^{-1} X_1' u \end{aligned}$$

- Take expectations and we get

$$\hat{B}_1 = B_1 + (X_1' X_1)^{-1} X_1' X_2 B_2$$

  Note $(X_1' X_1)^{-1} X_1' X_2$ is the column of slopes in the OLS regression of each column of X_2 on the columns of X_1
- OLS is biased because of omitted variables and direction is unclear — depending on multiple partial effects

Bivariate Model

- With two variable setup, inference is easier

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

- Assume we *in*correctly regress y on just x_1. Then

$$\begin{aligned}\hat{\beta}_1 &= \beta_1 + (x_1' x_1)^{-1} x_1' x_2 \beta_2 \\ &= \beta_1 + \delta \beta_2\end{aligned}$$

- Bias term consists of 2 terms:
  1. $\delta =$ slope from regression of x_2 on x_1
  2. $\beta_2 =$ slope on x_2 from multiple regression of y on (x_1, x_2)
- Direction of bias determined by signs of δ and β_2.
- Magnitude of bias determined by magnitudes of δ and β_2.

Omitted Variable Bias General Thoughts

- Deriving sign of omitted variable bias with multiple regressors in estimated model is hard. Recall general formula

$$\hat{B}_1 = B_1 + (X_1'X_1)^{-1}X_1'X_2B_2$$

  $(X_1'X_1)^{-1}X_1'X_2$ is vector of coefficients.
- Consider a simpler model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

  where we omit x_3
- Note that *both* $\hat{\beta}_1$ and $\hat{\beta}_2$ will be biased because of omission unless *both* x_1 and x_2 are uncorrelated with x_3.
- The omission will infect every coefficient through correlations

Example: Labor

- Consider

$$log(wage) = \beta_0 + \beta_1 education + \beta_2 ability + u$$

- If we can't measure ability, it's in the error term and we estimate

$$log(wage) = \beta_0 + \beta_1 education + w$$

- What is the likely bias in $\hat{\beta}$? recall

$$\hat{\beta}_1 = \beta_1 + \delta\beta_2$$

where δ is the slope from regression of ability on education.
- Ability and education are likely positively correlated $\implies \delta > 0$
- Ability and wages are likely positively correlated $\implies \beta_2 > 0$
- So, bias is likely positive $\implies \hat{\beta}_1$ is too big!

Goodness of Fit

- R^2 still equal to squared correlation between y and $\hat{y}$
- Low R^2 doesn't mean model is wrong
- Can have a low R^2 yet OLS estimate may be reliable estimates of ceteris paribus effects of each independent variable
- Adjust R^2

$$R_a^2 = 1 - (1 - R^2)\frac{n-1}{n-k-1}$$

  where $k = \#$ of regressors excluding intercept
- Adjust R^2 corrects for df and it can be < 0

Unbiasedness

- When is OLS unbiased (i.e., $E(\hat{\beta}) = \beta$)?
  1. Model is linear in parameters
  2. We have a random sample (e.g., self selection)
  3. No perfect collinearity
  4. Zero conditional mean of errors (i.e., $E(u|x) = 0$)
- Unbiasedness is a feature of sampling distributions of $\hat{\alpha}$ and $\hat{\beta}$.
- For a given sample, we hope $\hat{\alpha}$ and $\hat{\beta}$ are close to true values.

Irrelevant Regressors

- What happens when we include a regressor that shouldn't be in the model? (**overspecified**)
- No affect on unbiasedness
- Can affect the variances of the OLS estimator

Variance of OLS Estimators

- Sampling variance of OLS slope

$$Var(\hat{\beta}_j) = \frac{\sigma^2}{\sum_{i=1}^{N}(x_{ij} - \bar{x}_j)^2(1 - R_j^2)}$$

  for $j = 1, ..., k$, where R_j^2 is the R^2 from regressing x_j on all other independent variables including the intercept and σ^2 is the variance of the regression error term.
- Note
  - Bigger error variance (σ^2) $\implies$ bigger SEs (Add more variables to model, change functional form, improve fit!)
  - More sampling variation in x_j $\implies$ smaller SEs (Get a larger sample)
  - Higher collinearity (R_j^2) $\implies$ bigger SEs (Get a larger sample)

Multicollinearity

- Problem of small sample size.
- No implication for bias or consistency, but can inflate SEs
- Consider

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

  where x_2 and x_3 are highly correlated.
- $Var(\hat{\beta}_2)$ and $Var(\hat{\beta}_3)$ may be large.
- But correlation between x_2 and x_3 has no direct effect on $Var(\hat{\beta}_1)$
- If x_1 is uncorrelated with x_2 and x_3, then $R_1^2 = 0$ and $Var(\hat{\beta}_1)$ is unaffected by correlation between x_2 and x_3
- Make sure included variables are not too highly correlated with the variable of interest
- **Variance Inflation Factor (VIF)** $= 1/(1 - R_j^2)$ above 10 is sometimes cause for concern but this is arbitrary and of limited use

Data Scaling

- No one wants to see a coefficient reported as 0.000000456, or 1,234,534,903,875.
- Scale the variables for cosmetic purposes:
  1. Will effect coefficients & SEs
  2. Won't affect t-stats or inference
- Sometimes useful to convert coefficients into comparable units, e.g., SDs.
  1. Can standardize y and x's (i.e., subtract sample avg. & divide by sample SD) before running regression.
  2. Estimated coefficients $\implies$ 1 SD Δ in y given 1 SD Δ in x.
- Can estimate model on original data, then multiply each coef by corresponding SD. This marginal effect $\implies$ Δ in y units for a 1 SD Δ in x

Log Functional Forms

- Consider

$$log(price) = \beta_0 + \beta_1 log(pollution) + \beta_2 rooms + u$$

- Interpretation
  1. β_1 is the elasticity of price w.r.t. pollution. I.e., a 1% change in pollution generates an $100\beta_1$% change in price.
  2. β_2 is the semi-elasticity of price w.r.t. rooms. I.e., a 1 unit change in rooms generates an $100\beta_2$% change in price.
- E.g.,

$$log(price) = 9.23 - 0.718 log(pollution) + 0.306 rooms + u$$

$\Longrightarrow$ 1% inc. in pollution $\Longrightarrow$ -0.72% dec. in price

$\Longrightarrow$ 1 unit inc. in rooms $\Longrightarrow$ -30.6% inc. in price

Log Approximation

- Note: percentage change interpretation is only *approximate*!
- Approximation error occurs because as $\Delta log(y)$ becomes larger, approximation $\%\Delta y \approx 100\Delta log(y)$ becomes more inaccurate. E.g.,

$$log(y) = \hat{\beta}_0 + \hat{\beta}_1 log(x_1) + \hat{\beta}_2 x_2$$

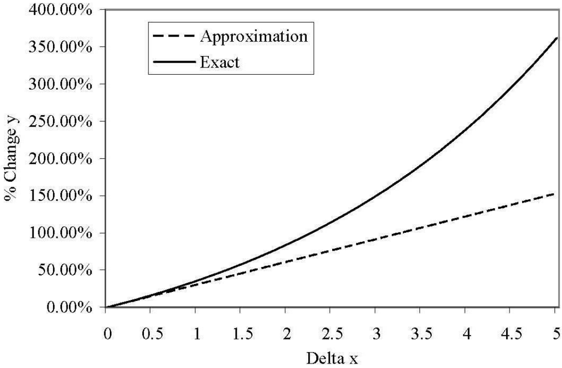
- Fixing x_1 (i.e., $\Delta x_1 = 0$) $\implies \Delta log(y) = \Delta\hat{\beta}_2 x_2$
- *Exact* % change is

$$\begin{aligned}
\Delta log(y) &= log(y') - logy(y) = \hat{\beta}_2 \Delta x_2 = \hat{\beta}_2(x_2' - x_2) \\
log(y'/y) &= \hat{\beta}_2(x_2' - x_2) \\
y'/y &= exp(\hat{\beta}_2(x_2' - x_2)) \\
\left[(y' - y)/y\right]\% &= 100 \cdot \left[exp(\hat{\beta}_2(x_2' - x_2)) - 1\right]
\end{aligned}$$

Figure of Log Approximation

$$\text{Approximate \% change } y \quad : \quad \Delta log(y) = \hat{\beta}_2 \Delta x_2$$

$$\text{Exact \% change } y \quad : \quad (\Delta y/y)\% = 100 \cdot \left[exp(\hat{\beta}_2 \Delta x_2) \right]$$



Usefulness of Logs

- Logs lead to coefficients with appealing interpretations
- Logs allow us to be ignorant about the units of measurement of variables appearing in logs since they're proportionate changes
- If $y > 0$, log can mitigate (eliminate) skew and heteroskedasticity
- Logs of y or x can mitigate the influence of outliers by narrowing range.
- "Rules of thumb" of when to take logs:
  - positive currency amounts,
  - variable with large integral values (e.g., population, enrollment, etc.)

  and when not to take logs
  - variables measured in years (months),
  - proportions
- If $y \in [0, \infty)$, can take log(1+y)

Percentage vs. Percentage Point Change

- Proportionate (or Relative) Change

$$(x_1 - x_0)/x_0 = \Delta x/x_0$$

- Percentage Change

$$\%\Delta x = 100(\Delta x/x_0)$$

- Percentage Point Change is raw change in percentages.
- E.g., let $x =$ unemployment rate in %
- If unemployment goes from 10% to 9%, then
  - 1% percentage point change,
  - (9-10)/10 = 0.1 proportionate change,
  - 100(9-10)/10 = 10% percentage change,
- If you use log of a % on LHS, take care to distinguish between percentage change and percentage point change.

Models with Quadratics

- Consider

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$$

- Partial effect of x

$$\Delta y = (\beta_1 + 2\beta_2 x)\Delta x \implies dy/dx = \beta_1 + 2\beta_2 x$$

$\implies$ must pick value of x to evaluate (e.g., $\bar{x}$)

- $\hat{\beta}_1 > 0, \hat{\beta}_2 < 0 \implies$ parabolic relation
  - Turning point = Maximum = $\left|\hat{\beta}_1/(2\hat{\beta}_2)\right|$
  - *Know where the turning point is!*. It may lie outside the range of x!
  - Odd values may imply misspecification or be irrelevant (above)
- Extension to higher order straightforward

Models with Interactions

- Consider

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + u$$

- Partial effect of x_1

$$\Delta y = (\beta_1 + \beta_3 x_2)\Delta x_1 \implies dy/dx_1 = \beta_1 + \beta_3 x_2$$

- Partial effect of $x_1 = \beta_1 \iff x_2 = 0$. Have to ask if this makes sense.
- If not, plug in sensible value for x_2 (e.g., $\bar{x}_2$)
- Or, reparameterize the model:

$$y = \alpha_0 + \delta_1 x_1 + \delta_2 x_2 + \beta_3 (x_1 - \mu_1)(x_2 - \mu_2) + u$$

where (μ_1, μ_2) is the population mean of (x_1, x_2)
- $\delta_2(\delta_1)$ is partial effect of $x_2(x_1)$ on y at mean value of $x_1(x_2)$.

Models with Interactions

- Reparameterized model

$$\begin{aligned} y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 (x_1 x_2 + \mu_1 \mu_2 - x_1 \mu_2 - x_2 \mu_1) + u \\ &= \underbrace{(\beta_0 + \beta_3 \mu_1 \mu_2)}_{\alpha_0} + \underbrace{(\beta_1 + \beta_3 \mu_2)}_{\delta_1} x_1 \\ &+ \underbrace{(\beta_2 + \beta_3 \mu_1)}_{\delta_2} x_2 + \beta_3 x_1 x_2 + u \end{aligned}$$

- For estimation purposes, can use sample mean in place of unknown population mean
- Estimating reparameterized model has two benefits:
  - Provides estimates at average value $(\hat{\delta}_1, \hat{\delta}_2)$
  - Provides corresponding standard errors

Predicted Values and SEs I

- Predicted value:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + ... + \hat{\beta}_k x_k$$

- But this is just an estimate with a standard error. I.e.,

$$\hat{\theta} = \hat{\beta}_0 + \hat{\beta}_1 c_1 + ... + \hat{\beta}_k c_k$$

  where $(c_1, ..., c_k)$ is a point of evaluation
- But $\hat{\theta}$ is just a linear combination of OLS parameters
- We know how to get the SE of this. E.g., $k = 1$

$$\begin{aligned} Var(\hat{\theta}) &= Var(\hat{\beta}_0 + \hat{\beta}_1 c_1) \\ &= Var(\hat{\beta}_0) + c_1^2 Var(\hat{\beta}_1) + 2c_1 Cov(\hat{\beta}_0, \hat{\beta}_1) \end{aligned}$$

- Take square root and voila'! (Software will do this for you)

Predicted Values and SEs II

- Alternatively, reparameterize the regression. Note

$$\hat{\theta} \;=\; \hat{\beta}_0 + \hat{\beta}_1 c_1 + \dots + \hat{\beta}_k c_k \implies \hat{\beta}_0 = \hat{\theta} - \hat{\beta}_1 c_1 - \dots - \hat{\beta}_k c_k$$

- Plug this into the regression

$$y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + u$$

  to get

$$y = \theta_0 + \beta_1 (x_1 - c_1) + \dots + \beta_k (x_k - c_k) + u$$

- I.e., subtract the value c_j from each observation on x_j and then run regression on transformed data.
- Look at SE on intercept and that's the SE of the predicated value of y at the point $(c_1, \dots, c_k)$
- You can form confidence intervals with this too.

Predicting y with $\log(y)$ I

- SRF:

$$\widehat{log(y)} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + ... + \hat{\beta}_k x_k$$

- Predicted value of y is *not* $exp(\widehat{log(y)})$
- Recall Jensen's inequality for convex function, g:

$$g\left(\int f d\mu\right) \leq \int g \circ f d\mu \iff g(E(f)) \leq E(g(f))$$

- In our setting, $f = log(y)$, g=$exp()$. Jensen $\implies$

$$exp\{E[log(y)]\} \leq E[exp\{log(y)\}]$$

We will underestimate y.

Predicting y with $\log(y)$ II

- How can we get a consistent (no unbiased) estimate of y?
- If $u \perp\!\!\!\perp X$

$$E(y|X) = \alpha_0 exp(\beta_0 + \beta_1 x_1 + ... + \beta_k x_k)$$

  where $\alpha_0 = E(exp(u))$
- With an estimate of α, we can predict y as

$$\hat{y} = \hat{\alpha}_0 exp(\widehat{log(y)})$$

  which requires exponentiating the predicted value from the log model and multiplying by $\hat{\alpha}_0$
- Can estimate α_0 with MOM estimator (consistent but biased because of Jensen)

$$\hat{\alpha_0} = n^-1 \sum_{i=1}^{n} exp(\hat{u}_i)$$

Basics

- Qualitative information. Examples,
  1. Sex of individual (Male, Female)
  2. Ownership of an item (Own, don't own)
  3. Employment status (Employed, Unemployed
- Code this information using **binary** or **dummy** variables. E.g.,

$$Male_i = \begin{cases} 1 & \text{if person i is Male} \\ 0 & \text{otherwise} \end{cases}$$

$$Own_i = \begin{cases} 1 & \text{if person i owns item} \\ 0 & \text{otherwise} \end{cases}$$

$$Emp_i = \begin{cases} 1 & \text{if person i is employed} \\ 0 & \text{otherwise} \end{cases}$$

- Choice of 0 or 1 is relevant only for interpretation.

Single Dummy Variable

- Consider

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u$$

- δ_0 measures difference in wage between male and female given same level of education (and error term u)

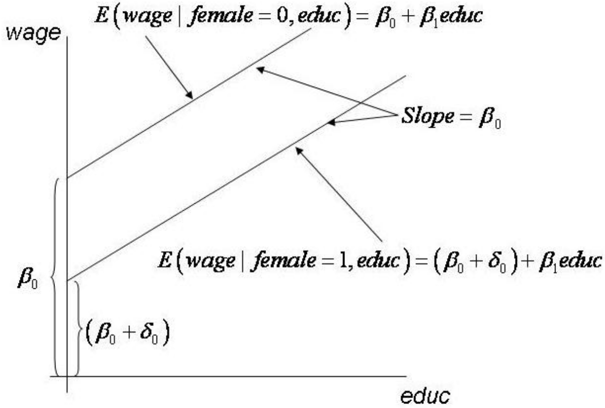
$$\begin{aligned} E(wage|female = 0, educ) &= \beta_0 + \beta_1 educ \\ E(wage|female = 1, educ) &= \beta_0 + \delta + \beta_1 educ \end{aligned}$$

$\Longrightarrow$ $\delta = E(wage|female = 1, educ) - E(wage|female = 0, educ)$

- Intercept for males $= \beta_0$, females $= \delta_0 + \beta_0$

Intercept Shift

- Intercept shifts, slope is same.



Wage Example

- SRF with $n = 526$, $R^2 = 0.364$

$$\widehat{wage} = -1.57 - 1.81 female + 0.571 educ + 0.025 exper + 0.141 tenure$$

  - Negative intercept is intercept for men...meaningless because other variables are never all $= 0$
  - Females earn \$1.81/hour less than men with the same education, experience, and tenure.
- All else equal is important! Consider SRF with $n = 526$, $R^2 = 0.116$

$$\widehat{wage} = 7.10 - 2.51 female$$

- Female coefficient is picking up differences due to omitted variables.

Log Dependent Variables

- Nothing really new, coefficient has % interpretation.
- E.g., house price model with $N = 88$, $R^2 = 0.649$

$$\begin{aligned}\widehat{price} &= -1.35 + 0.168 log(lotsize) + 0.707 log(sqrft) \\ &+ 0.027 bdrms + 0.054 colonial\end{aligned}$$

  - Negative intercept is intercept for non-colonial homes...meaningless because other variables are never all $= 0$
  - A colonial style home costs approximately 5.4% more than "otherwise similar" homes
- Remember this is just an approximation. If the percentage change is large, may want to compare with exact formulation

Multiple Binary Independent Variables

- Consider

$$\begin{aligned} \widehat{log(wage)} &= 0.321 + 0.213 marriedMale - 0.198 marriedFemale \\ &+ -0.110 singleFemale + 0.079 education \end{aligned}$$

- The omitted category is single male $\implies$ intercept is intercept for base group (all other vars $= 0$)
- Each binary coefficient represent the estimated *difference* in intercepts between that group and the base group
- E.g., *marriedMale* $\implies$ that married males earn approximately 21.3% more than single males, all else equal
- E.g., *marriedFemale* $\implies$ that married females earn approximately 19.8% less than single males, all else equal

Ordinal Variables

- Consider credit ratings: $CR \in (AAA, AA, ..., C, D)$
- If we want to explain bond interest rates with ratings, we could convert CR to a numeric scale, e.g., $AAA = 1, AA = 2, ...$ and run

$$IR_i = \beta_0 + \beta_1 CR_i + u_i$$

- This assumes a constant linear relation between interest rates and every rating category.
- Moving from AAA to AA produces the same change in interest rates as moving from BBB to BB.
- Could take log interest rate, but is same proportionate change much better?

Converting Ordinal Variables to Binary

- Or we could create an indicator for each rating category, e.g., $CR_{AAA} = 1$ if CR = AAA, 0 otherwise; $CR_{AA} = 1$ if CR = AA, 0 otherwise, etc.
- Run this regression:

$$IR_i = \beta_0 + \beta_1 CR_{AAA} + \beta_2 CR_{AA} + ... + \beta_{m-1} CR_C + u_i$$

  remembering to exclude one ratings category (e.g., "D")
- This allows the IR change from each rating category to have a different magnitude
- Each coefficient is the different in IRs between a bond with a certain credit rating (e.g., "AAA", "BBB", etc.) and a bond with a rating of "D" (the omitted category).

Interactions Involving Binary Variables I

- Recall the regression with four categories based on (1) marriage status and (2) sex.

$$\begin{aligned}\widehat{log(wage)} &= 0.321 + 0.213 marriedMale - 0.198 marriedFemale \\ &+ -0.110 singleFemale + 0.079 education\end{aligned}$$

- We can capture the same logic using interactions

$$\begin{aligned}\widehat{log(wage)} &= 0.321 - 0.110 female + 0.213 married \\ &+ -0.301 femaile \times married + \ldots\end{aligned}$$

- Note excluded category can be found by setting all dummies $= 0$

$\Longrightarrow$ excluded category $=$ single (married $= 0$), male (female $= 0$)

Interactions Involving Binary Variables II

- Note that the intercepts are all identical to the original regression.
- Intercept for married male

$$\begin{aligned} \widehat{log(wage)} &= 0.321 - 0.110(0) + 0.213(1) \\ &- 0.301(0) \times (1) = 0.534 \end{aligned}$$

- Intercept for single female

$$\begin{aligned} \widehat{log(wage)} &= 0.321 - 0.110(1) + 0.213(0) \\ &- 0.301(1) \times (0) = 0.211 \end{aligned}$$

- And so on.
- Note that the slopes will be identical as well.

Example: Wages and Computers

- Krueger (1993), $N = 13,379$ from 1989 CPS

$$\begin{aligned}\widehat{log(wage)} &= \hat{beta}_0 + 0.177 compwork + 0.070 comphome \\ &+ 0.017 compwork \times comphome + \ldots\end{aligned}$$

  (Intercept not reported)
- Base category = people with no computer at work or home
- Using a computer at work is associated with a 17.7% higher wage. (Exact value is 100(exp(0.177) - 1) = 19.4%)
- Using a computer at home but not at work is associated with a 7.0% higher wage.
- Using a computer at home and work is associated with a 100(0.177+0.070+0.017) = 26.4% (Exact value is 100(exp(0.177+0.070+0.017) - 1) = 30.2%)

Different Slopes

- Dummies only shift intercepts for different groups.
- What about slopes? We can interact continuous variables with dummies to get different slopes for different groups. E.g,

$$\begin{aligned} log(wage) &= \beta_0 + \delta_0 female + \beta_1 educ + \delta_1 educ \times female + u \\ log(wage) &= (\beta_0 + \delta_0 female) + (\beta_1 + \delta_1 female) educ + u \end{aligned}$$

- Males: Intercept $= \beta_0$, slope $= \beta_1$
- Females: Intercept $= \beta_0 + \delta_0$, slope $= \beta_1 + \delta_1$

$\Longrightarrow$ δ_0 measures difference in intercepts between males and females

$\Longrightarrow$ δ_1 measures difference in slopes (return to education) between males and females

Figure: Different Slopes I

$$log(wage) = (\beta_0 + \delta_0 female) + (\beta_1 + \delta_1 female)educ + u$$

$$(\delta_0 < 0, \delta_1 < 0)$$

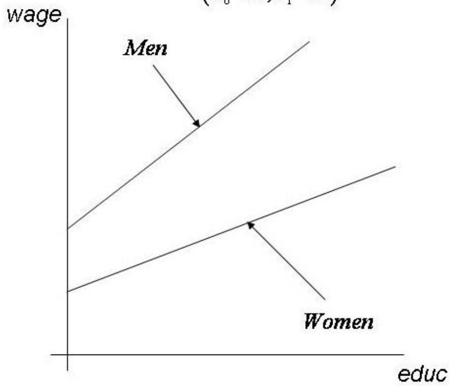
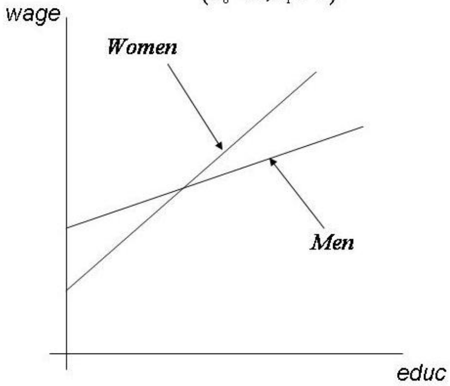


Figure: Different Slopes I

$$log(wage) = (\beta_0 + \delta_0 female) + (\beta_1 + \delta_1 female)educ + u$$

$$(\delta_0 < 0, \delta_1 > 0)$$



Interpretation of Figures

- 1st figure: intercept and slope for women are less than those for men

$\Longrightarrow$ women earn less than men at *all* educational levels

- 2nd figure: intercept for women is less than that for men, but slope is larger

$\Longrightarrow$ women earn less than men at low educational levels but the gap narrows as education increases.

$\Longrightarrow$ at some point, woman earn more than men. But, does this point occur within the range of data?

- Point of equality: Set Women eqn = Men eqn

$$\begin{aligned} \text{Women: } log(wage) &= (\beta_0 + \delta_0) + (\beta_1 + \delta_1)educ + u \\ \text{Men: } log(wage) &= (\beta_0) + \beta_1 educ + u \end{aligned}$$

$\Longrightarrow$ $e* = -\delta_0 / \delta_1$

Example 1

- Consider $N = 526$, $R^2 = 0.441$

$$
\begin{aligned}
\widehat{log(wage)} &= 0.389 - 0.227 female + 0.082 educ \\
&- 0.006 female \times educ + 0.29 exper - 0.0006 exper^2 + \ldots
\end{aligned}
$$

- Return to education for men $= 8.2\%$, women $= 7.6\%$.
- Women earn 22.7% less than men. But statistically insignif...why?
- Problem is multicollinearity with interaction term.
  - Intuition: coefficient on *female* measure wage differential between men and women when $educ = 0$.
  - Few people have very low levels of *educ* so unsurprising that we can't estimate this coefficient precisely.
  - More interesting to estimate gender differential at $\bar{educ}$, for example.
  - Just replace $female \times educ$ with $female \times (educ - \bar{educ})$ and rerun regression. This will only change coefficient on *female* and its standard error.

Example 2

- Consider baseball players salaries $N = 330$, $R^2 = 0.638$

$$\begin{aligned}\widehat{log(salary)} &= 10.34 + 0.0673 years + 0.009 gamesyr + \dots \\ &- -0.198 black - 0.190 hispan \\ &+ 0.0125 black \times percBlack + 0.0201 hispan \times percHisp\end{aligned}$$

- Black players in cities with no blacks ($percBlack = 0$) earn 19.8% less than otherwise identical whites.
- As *percBlack* inc ($\Longrightarrow$ *percWhite* dec since *perchisp* is fixed), black salaries increase relative to that for whites. E.g., if $percBalck = 10\% \implies$ blacks earn -0.198 + 0.0125(10) = -0.073, 7.3% less than whites in such a city.
- When $percBlack = 20\% \implies$ blacks earn 5.2% more than whites.
- Does this $\implies$ discrimination against whites in cities with large black pop? Maybe best black players choose to live in such cities.

Single Parameter Tests

- Any misspecification in the functional form relating dependent variable to the independent variables will lead to bias.
- E.g., assume true model is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_2^2 + u$$

  but we omit squared term, x_2^2.
- Amount of bias in $(\beta_0, \beta_1, \beta_2)$ depends on size of β_3 and correlation among (x_1, x_2, x_2^2)
- Incorrect functional form on the LHS will bias results as well (e.g., log(y) vs. y)
- This is a minor problem in one sense: we have all the sufficient data, so we can try/test as many different functional forms as we like.
- This is different from a situation where we don't have data for a relevant variable.

RESET

- **Regression Error Sepecification Test (RESET)**
- Estimate

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k + u$$

- Compute predicted values $\hat{y}$
- Estimate

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k + \delta_1 \hat{y}^2 + \delta_2 \hat{y}^3 + u$$

  (choice of polynomial is arbitrary.)
- $H_0 : \delta_1 = \delta_2 = 0$
- Use F-test with $F \sim F_{2,n-k-3}$

Tests Against Nonnested Alternatives

- What if we wanted to test 2 nonnested models? I.e., we can't simply restrict parameters in one model to obtain the other.
- E.g.,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

vs.

$$y = \beta_0 + \beta_1 log(x_1) + \beta_2 log(x_2) + u$$

- E.g.,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u$$

vs.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 z + u$$

Davidson-MacKinnon Test

- Test

$$\begin{aligned} \text{Model 1: } y &= \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u \\ \text{Model 2: } y &= \beta_0 + \beta_1 log(x_1) + \beta_2 log(x_2) + u \end{aligned}$$

- If 1st model is correct, then fitted values from 2nd model, $(\hat{\hat{y}})$, should be insignificant in 1st model
- Look at t-stat on θ_1 in

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \theta_1 \hat{\hat{y}} + u$$

- Significant $\theta_1 \implies$ rejection of 1st model.
- Then do reverse, look at t-stat on θ_1 in

$$y = \beta_0 + \beta_1 log(x_1) + \beta_2 log(x_2) + \theta_1 \hat{y} + u$$

  where $\hat{y}$ are predicted values from 1st model.
- Significant $\theta_1 \implies$ rejection of 2nd model.

Davidson-MacKinnon Test: Comments

- Clear winner need not emerge. Both models could be rejected or neither could be rejected.
- In latter case, could use R^2 to choose.
- Practically speaking, if the effects of key independent variables on y are not very different, the it doesn't really matter which model is used.
- Rejecting one model does *not* imply that the other model is correct.

Omitted Variables

- Consider

$$log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 ability + u$$

- We don't observe or can't measure ability.

$\Longrightarrow$ coefficients are unbiased.

- What can we do?
- Find a **proxy variable**, which is correlated with the unobserved variable. E.g., IQ.

Proxy Variables

- Consider

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3^* + u$$

- x_3^* is unobserved but we have proxy, x_3
- x_3 should be related to x_3^*:

$$x_3^* = \delta_0 + \delta_1 x_3 + v_3$$

  where v_3 is error associated with the proxy's imperfect representation of x_3^*
- Intercept is just there to account for different scales (e.g., ability may have a different average value than IQ)

Plug-In Solution to Omitted Variables I

- Can we just substitute x_3 for x_3^*? (and run

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

- Depends on the assumptions on u and v_3.
  1. $E(u|x_1, x_2, x_3^*) = 0$ (Common assumption). In addition, $E(u|x_3) = 0 \implies x_3$ is irrelevant once we control for (x_1, x_2, x_3^*) (Need this but not controversial given 1st assumption and status of x_3 as a proxy
  2. $E(v_3|x_1, x_2, x_3) = 0$. This requires x_3 to be a good proxy for x_3^*

$$E(x_3^*|x_1, x_2, x_3) = E(x_3^*|x_3) = \delta_0 + \delta_1 x_3$$

  Once we control for x_3, x_3^* doesn't depend on x_1 or x_2

Plug-In Solution to Omitted Variables II

- Recall true model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3^* + u$$

- Substitute for x_3^* in terms of proxy

$$y = \underbrace{(\beta_0 + \beta_3 \delta_0)}_{\alpha_0} + \beta_1 x_1 + \beta_2 x_2 + \beta_3 \delta_3 x_3 + \underbrace{u + \beta_3 v_3}_{e}$$

- Assumptions 1 & 2 on prev slide $\implies E(e|x_1, x_2, x_3) = 0 \implies$ we can est.

$$y = \alpha_0 + \beta_1 x_1 + \beta_2 x_2 + \alpha_3 x_3 + e$$

- Note: we get unbiased (or at least consistent) estimators of $(\alpha_0, \beta_1, \beta_2, \alpha_3)$.
- (β_0, β_3) not identified.

Example 1: Plug-In Solution

- In wage example where IQ is a proxy for ability, the 2nd assumption is

$$E(ability|educ, exper, IQ) = E(ability|IQ) = \delta_0 + \delta_3 IQ$$

- This means that the average level of ability only changes with IQ, *not* with education or experience.
- Is this true? Can't test but must think about it.

Example 1: Cont.

- If proxy variable doesn't satisfy the assumptions 1 & 2, we'll get biased estimates
- Suppose
$$x_3^* = \delta_0 + \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + v_3$$
where $E(v_3|x_1, x_2, x_3) = 0$.
- Substitute into structural eqn
$$y = (\beta_0 + \beta_3\delta_0) + (\beta_1 + \beta_3\delta_1)x_1 + (\beta_2 + \beta_3\delta_2)x_2 + \beta_3\delta_3 x_3 + u + \beta_3 v_3$$
- So when we estimate the regression:
$$y = \alpha_0 + \beta_1 x_1 + \beta_2 x_2 + \alpha_3 x_3 + e$$
we get consistent estimates of $(\beta_0 + \beta_3\delta_0)$, $(\beta_1 + \beta_3\delta_1)$, $(\beta_2 + \beta_3\delta_2)$, and $\beta_3\delta_3$ assuming $E(u + \beta_3 v_3|x_1, x_2, x_3) = 0$.
- Original parameters are not identified.

Example 2: Plug-In Solution

- Consider q-theory of investment

$$Inv = \beta_0 + \beta_1 q + u$$

- Can't measure q so use proxy, market-to-book (MB),

$$q = \delta_0 + \delta_1 MB + v$$

- Think about identifying assumptions
  1. $E(u|q) = 0$ theory say q is sufficient statistic for inv
  2. $E(q|MB) = \delta_0 + \delta_1 MB \implies$ avg level of q changes *only* with MB
- Even if assumption 2 true, we're not estimating β_1 in

$$Inv = \alpha_0 + \alpha_1 MB + e$$

We're estimating (α_0, α_1) where

$$Inv = \underbrace{(\beta_0 + \beta_1 \delta_0)}_{\alpha_0} + \underbrace{\beta_1 \delta_1}_{\alpha_1} MB + e$$

Using Lagged Dependent Variables as Proxies

- Let's say we have no idea how to proxy for an omitted variable.
- One way to address is to use the lagged dependent variable, which captures inertial effects of *all* factors that affect y.
- This is unlikely to solve the problem, especially if we only have one cross-section.
- But, we can conduct the experiment of comparing to observations with the same value for the outcome variable last period.
- This is imperfect, but it can help when we don't have panel data.

Model I

- Consider an extension to the basic model

$$y_i = \alpha_i + \beta_i x_i$$

  where α_i is an unobserved intercept and the return to education differs for each person.
- This model is unidentified: more parameters ($2n$) than observations (n)
- But we can hope to identify avg intercept, $E(\alpha_i) = \alpha$, and avg slope, $E(\beta_i) = \beta$ (a.k.a., **Average Partial Effect (APE)**.

$$\alpha_i = \alpha + c_i, \beta_i = \beta + d_i$$

  where c_i and d_i are the individual specific deviation from average effects.

$\implies$ $E(c_i) = E(d_i) = 0$

Model II

- Substitute coefficient specification into model

$$y_i = \alpha + \beta x_i + c_i + d_i x_i \equiv \alpha + \beta x_i + u_i$$

- What we need for unbiasedness is $E(u_i|x_i) = 0$

$$E(u_i|x_i) = E(c_i + d_i x_i|x_i)$$

- This amounts to requiring
  1. $E(c_i|x_i) = E(c_i) = 0 \implies E(\alpha_i|x_i) = E(\alpha_i)$
  2. $E(d_i|x_i) = E(d_i) = 0 \implies E(\beta_i|x_i) = E(\beta_i)$
- Understand these assumptions!!!! In order for OLS to consistently estimate the mean slope and intercept, the slopes and intercepts must be mean independent (at least uncorrelated) of the explanatory variable.

What is Measurement Error (ME)?

- When we use an imprecise measure of an economic variable in a regression, our model contains **measurement error (ME)**
  - The market-to-book ratio is a noisy measure of "q"
  - Altman's Z-score is a noisy measure of the probability of default
  - Average tax rate is a noisy measure of marginal tax rate
  - Reported income is noisy measure of actual income
- Similar statistical structure to omitted variable-proxy variable solution but conceptually different
  - Proxy variable case we need variable that is associated with unobserved variable (e.g., IQ proxy for ability)
  - Measurement error case the variable we don't observe has a well-defined, quantitative meaning but our recorded measure contains error

Measurement Error in Dependent Variable

- Let y be observed measure of y^*

$$y^* = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + u$$

- Measurement error defined as $e_0 = y - y^*$
- Estimable model is:

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + u + e_0$$

- If mean of ME $\neq 0$, intercept is biased so assume mean $= 0$
- If ME independent of X, then OLS is unbiased and consistent and usual inference valid.
- If e_0 and u uncorrelated than $Var(u + e_0) > Var(u) \implies$ measurement error in dependent variable results in larger error variance and larger coef SEs

Measurement Error in Log Dependent Variable

- When $log(y^*)$ is dependent variable, we assume

$$log(y) = log(y^*) + e_0$$

- This follows from multiplicative ME

$$y = y^* a_0$$

where

$$\begin{aligned} a_0 &> 0 \\ e_0 &= log(a_0) \end{aligned}$$

Measurement Error in Independent Variable

- Model

$$y = \beta_0 + \beta_1 x_1^* + u$$

- ME defined as $e_1 = x_1 - x_1^*$
- Assume
  - Mean ME $= 0$
  - $u \perp x_1^*, x_1$, or $E(y|x_1^*, x_1) = E(y|x_1^*)$ (i.e., x_1 doesn't affect y after controlling for x_1^*)
- What are implications of ME for OLS properties?
- Depends crucially on assumptions on e_1
- Econometrics has focused on 2 assumptions

Assumption 1: $e_1 \perp x_1$

- 1^st assumption is ME uncorrelated with *observed* measure
- Since $e_1 = x_1 - x_1^*$, this implies $e_1 \perp x_1^*$
- Substitute into regression

$$y = \beta_0 + \beta_1 x_1 + (u - \beta_1 e_1)$$

- We assumed u and e_1 have mean 0 and are $\perp$ with x_1

$\Longrightarrow$ $(u - \beta_1 e_1)$ is uncorrelatd with x_1.

$\Longrightarrow$ OLS with x_1 produces consistent estimator of coef's

$\Longrightarrow$ OLS error variance is $\sigma_u^2 + \beta_1^2 \sigma_{e_1}^2$

- ME increases error variance but doesn't affect any OLS properties (except coef SEs are bigger)

Assumption 2: $e_1 \perp x_1^*$

- This is the **Classical Errors-in-Variables (CEV)** assumption and comes from representation:

$$x_1 = x_1^* + e_1$$

- (Still maintain 0 correlation between u and e_1)
- Note $e_1 \perp x_1^* \implies$

$$Cov(x_1, e_1) = E(x_1 e_1) = E(x_1^* e_1) + E(e_1^2) = \sigma_{e_1}^2$$

- This covariance causes problems when we use x_1 in place of x_1^* since

$$\begin{aligned} y &= \beta_0 + \beta_1 x_1 + (u - \beta_1 e_1) \text{ and} \\ Cov(x_1, u - \beta_1 e_1) &= -\beta_1 \sigma_{e_1}^2 \end{aligned}$$

- I.e., indep var is correlatd with error $\implies$ bias and inconsistent OLS estimates

Assumption 2: $e_1 \perp x_1^*$ (Cont.)

- Amount of inconsistency in OLS

$$
\begin{aligned}
\text{plim}(\hat{\beta}_1) &= \beta_1 + \frac{Cov(x_1, u - \beta_1 e_1)}{Var(x_1)} \\
&= \beta_1 + \frac{\beta_1 \sigma_{e_1}^2}{\sigma_{x_1^*}^2 + \sigma_{e_1}^2} \\
&= \beta_1 \left(1 - \frac{\sigma_{e_1}^2}{\sigma_{x_1^*}^2 + \sigma_{e_1}^2} \right) \\
&= \beta_1 \left(\frac{\sigma_{x_1^*}^2}{\sigma_{x_1^*}^2 + \sigma_{e_1}^2} \right)
\end{aligned}
$$

CEV asymptotic bias

- From previous slide:

$$\text{plim}(\hat{\beta}_1) = \beta_1 \left(\frac{\sigma^2_{x_1^*}}{\sigma^2_{x_1^*} + \sigma^2_{e_1}} \right)$$

- Scale factor is always $< 1 \implies$ asymptotic bias attenuates estimated effect (**attenuation bias**)
- If variance of error ($\sigma^2_{e_1}$) is small relative to variance of unobserved factor, then bias is small.
- More than 1 explanatory variable and bias is less clear
- Correlation between e_1 and x_1 creates problem. If x_1 correlated with other variables, bias infects everything.
- Generally, measurement error in a single variable casues inconsistency in all estimators. Sizes and even directions of the biases are not obvious or easily derived.

Counterexample to CEV Assumption

- Consider

$$\begin{aligned} colGPA &= \beta_0 + \beta_1 smoked^* + \beta_2 hsGPA + u \\ smoked &= smoked^* + e_1 \end{aligned}$$

  where $smoked^*$ is actual # of times student smoked marijuana and *smoked* is reported
- For $smoked^* = 0$ report is likely to be 0 $\implies$ $e_1 = 0$
- For $smoked^* > 0$ report is likely to be off $\implies$ $e_1 \neq 0$

$\implies$ e_1 and $smoked^*$ are correlated estimated effect (**attenuation bias**)

- I.e., CEV Assumption does not hold
- Tough to figure out implications in this scenario

Statistical Properties

- At a basic level, regression is just math (linear algebra and projection methods)
- We don't need statistics to run a regression (i.e., compute coefficients, standard errors, sums-of-squares, R^2, etc.)
- What we need statistics for is the interpretation of these quantities (i.e., for statistical inference).
- From the regression equation, the statistical properties of y come from those of X and u

What is heteroskedasticity (HSK)?

- Non-constant variance, that's it.
- HSK has no effect on bias or consistency properties of OLS estimators
- HSK means OLS estimates are no longer BLUE
- HSK means OLS estimates of standard errors are incorrect
- We need an HSK-robust estimator of the variance of the coefficients.

HSK-Robust SEs

- Eicker (1967), Huber (1967), and White (1980) suggest:

$$\widehat{Var}(\hat{\beta}_j) = \frac{\sum_{i=1}^{N} \hat{r}_{ij}^2 \hat{u}_i^2}{SSR_j^2}$$

where $\hat{r}_{ij}^2$ is the ith residual from regressing x_j on all other independent variables, and SSR_j is the sum of square residuals from this regression.

- Use this in computation of t-stas to get an HSK-robust t-statistic
- Why use non-HSK-robust SEs at all?
- With small sample sizes robust t-stats can have very different distributions (non "t")

HSK-Robust LM-Statistics

- The recipe:
  1. Get residuals from restricted model $\tilde{u}$
  2. Regress each independent variable excluded under null on all of the included independent variables; q excluded variables $\implies (\tilde{r}_1, ..., \tilde{r}_q)$
  3. Compute the products between each vector $\tilde{r}_j$ and $\tilde{u}$
  4. Regression of 1 (a constant "1" for each observation) on all of the products $\tilde{r}_j \tilde{u}$ without an intercept
  5. HSK-robust LM statistic, LM, is $N - SSR_1$, where SSR_1 is the sum of squared residuals from this last regression.
  6. LM is asymptotically distributed χ^2_q

Testing for HSK

- The model

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k + u$$

- Test $H_0 : Var(y|x_1, ..., x_k) = \sigma^2$
- $E(u|x_1, ..., x_k) = 0 \implies$ this hypothesis is equivalent to $H_0 : E(u^2|x_1, ..., x_k) = \sigma^2$ (I.e., is u^2 related to any explanatory variables?)

$$\hat{u}^2 = \delta_0 + \delta_1 x_1 + ... + \delta_k x_k + u$$

- Test null $H_0 : \delta_1 = ... = \delta_k = 0$

$$\begin{aligned} \text{F-test} \quad &: \quad F = \frac{R^2_{\hat{u}^2}}{(1 - R^2_{\hat{u}^2}/(n-k-1)} \\ \text{LM-test} \quad &: \quad LM = N \times R^2_{\hat{u}^2} \text{ (BP-test sort of)} \end{aligned}$$

Weighted Least Squares (WLS)

- Pre HSK-robust statistics, we did WLS - more efficient than OLS if correctly specified variance form

$$Var(u|X) = \sigma^2 h(X), h(X) > 0 \forall X$$

- E.g., $h(X) = x_1^2$ or $h(x) = exp(x)$
- WLS just normalizes all of the variables by the square root of the variance fxn ($\sqrt{h(X)}$) and runs OLS on transformed data.

$$\begin{aligned} y_i/\sqrt{h(X_i)} &= \beta_0/\sqrt{h(X_i)} + \beta_1/(x_{i1}/\sqrt{h(X_i)}) + \ldots \\ &+ \beta_k/(x_{ik}/\sqrt{h(X_i)}) + u_i/\sqrt{h(X_i)} \\ y_i^* &= \beta_0 x_0^* + \beta_1 x_1^* + \ldots + \beta_k x_k^* + u^* \end{aligned}$$

where $x_0^* = 1/\sqrt{h(X_i)}$

Feasible Generalized Least Squares (FGLS)

- WLS is an example of a **Generalized Least Squares** Estimator
- Consider

$$Var(u|X) = \sigma^2 exp\delta_0 + \delta x_1$$

- We need to estimate variance parameters. Using estimates gives us FGLS

Feasible Generalized Least Squares (FGLS) Recipe

- Consider variance form:

$$Var(u|X) = \sigma^2 exp(\delta_0 + \delta_1 x_1 + ... + \delta_k x_k)$$

- FGLS to correct for HSK:
  1. Regress y on X and get residuals $\hat{u}$
  2. Regress $log(\hat{u}^2)$ on X and get fitted values $\hat{g}$
  3. Estimate by WLS

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_k x_k + u$$

  with weights $1/exp(\hat{g})$, or transform each variable (including intercept) by multiplying by $1/exp(\hat{g})$ and estimate via OLS

- FGLS estimate is biased but consistent and more efficient than OLS.

OLS + Robust SEs vs. WLS

- If coefficient estimates are very different across OLS and WLS, it's likely $E(y|x)$ is misspecified.
- If we get variance form wrong in WLS then
  1. WLS estimates are still unbiased and consistent
  2. WLS standard errors and test statistics are invalid even in large samples
  3. WLS may not be more efficient than OLS

Single Parameter Tests

- Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + u$$

- Under certain assumptions

$$t(\hat{\beta}_j) = \frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1}$$

- Under other assumptions, asymptotically $t \overset{a}{\sim} N(0,1)$
- Intuition: $t(\hat{\beta}_j)$ tells us how far – in standard deviations – our estimate $\hat{\beta}_j$ is from the hypothesized value (β_j)
- E.g., $H_0 : \beta_j = 0 \implies t = \hat{\beta}_j / se(\hat{\beta}_j)$
- E.g., $H_0 : \beta_j = 4 \implies t = (\hat{\beta}_j - 4)/se(\hat{\beta}_j)$

Statistical vs. Economic Significance

- These are not the same thing
- We can have a statistically insignificant coefficient but it may be economically large.
  - Maybe we just have a power problem due to a small sample size, or little variation in the covariate
- We can have a statistically significant coefficient but it may be economically irrelevant.
  - Maybe we have a very large sample size, or we have a lot of variation in the covariate (outliers)
- You need to think about *both* statistical and economic significance when discussing your results.

Testing Linear Combinations of Parameters I

- Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$$

- Are two parameters the same? I.e., $H_0 : \beta_1 = \beta_2 \iff (\beta_1 - \beta_2) = 0$
- The usual statistic can be slightly modified

$$t = \frac{\hat{\beta}_1 - \hat{\beta}_2}{se(\hat{\beta}_1 - \hat{\beta}_2)} \sim t_{n-k-1}$$

- Careful: when computing the SE of difference not to forget covariance term

$$se(\hat{\beta}_1 - \hat{\beta}_2) = \left(se(\hat{\beta}_1)^2 + se(\hat{\beta}_2)^2 - 2Cov(\hat{\beta}_1, \hat{\beta}_2) \right)^{1/2}$$

Testing Linear Combinations of Parameters II

- Instead of dealing with computing the SE of difference, can reparameterize the regression and just check a t-stat
- E.g., define $\theta = \beta_1 - \beta_2 \implies \beta_1 = \theta + \beta_2$ and

$$
\begin{aligned}
y &= \beta_0 + (\theta + \beta_2)x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u \\
&= \beta_0 + \theta x_1 + \beta_2(x_1 + x_2) + \ldots + \beta_k x_k + u
\end{aligned}
$$

- Just run a t-test of new null, $H_0 : \theta = 0$ same as previous slide
- This strategy always works.

Testing Multiple Linear Restrictions

- Consider $H_0 : \beta_1 = 0, \beta_2 = 0, \beta_3 = 0$ (a.k.a., **exclusion restrictions**), $H_1 : H_0 not true$
- To test this, we need a **joint hypothesis test**
- One such test is as follows:
  1. Estimate the **Unrestricted Model**
  $$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + u$$
  2. Estimate the **Restricted Model**
  $$y = \beta_0 + \beta_4 x_4 + \beta_5 x_5 + \ldots + \beta_k x_k + u$$
  3. Compute F-statistic
  $$F = \frac{SSR_R - SSR_U)/q}{SSR_U/(n-k-1)} \sim F_{q,n-k-1}$$
  where q = degrees of freedom (df) in numerator $= df_R - df_U$, $n-k-1$ = df in denominator $= df_U$,

Relationship Between F and t Statistics

- t^2_{n-k-1} has an $F_{1,n-k-1}$ distribution.
- All coefficients being individually statistically significant (significant t-stats) does not imply that they are jointly significant
- All coefficients being individually statistically insignificant (insignificant t-stats) does not imply that they are jointly insignificant
- R^2 form of the F-stat:

$$F = \frac{R_U^2 - R_R^2)/q}{(1 - R_U^2)/(n-k-1)}$$

  (Equivalent to previous formula.)
- "Regression F-Stat" tests $H_0 : \beta_1 = \beta_2 = ... = \beta_k = 0$

Testing General Linear Restrictions I

- Can write any set of linear restrictions as follows

$$H_0 : R\beta - q = 0$$
$$H_1 : R\beta - q \neq 0$$

$dim(R) = \#$ of restrictions $\times$ $\#$ of parameters. E.g.,

$$\begin{aligned}
H_0 &: \beta_j = 0 \implies R = [0, 0, ..., 1, 0, ..., 0], q = 0 \\
H_0 &: \beta_j = \beta_k \implies R = [0, 0, 1, ..., -1, 0, ..., 0], q = 0 \\
H_0 &: \beta_1 + \beta_2 + \beta_3 = 1 \implies R = [1, 1, 1, 0, ..., 0], q = 1 \\
H_0 &: \beta_1 = 0, \beta_2 = 0, \beta_3 = 0 \implies
\end{aligned}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Testing General Linear Restrictions II

- Note that *under the null hypothesis*

$$E(R\hat{\beta} - q|X) = R\beta 0 - q = 0$$
$$Var(R\hat{\beta} - q|X) = RVar(\hat{\beta}|X)R' = \sigma^2 R(X'X)^{-1}R'$$

- Wald criterion:

$$W = (R\hat{\beta} - q)'[\sigma^2 R(X'X)^{-1}R']^{-1}(R\hat{\beta} - q) \sim \chi^2_J$$

where J is the degrees of freedom under the null (i.e., the # of restrictions, the # of rows in R)

- Must estimate σ^2, this changes distribution

$$F = (R\hat{\beta} - q)'[\hat{\sigma}^2 R(X'X)^{-1}R']^{-1}(R\hat{\beta} - q) \sim F_{J,n-k-1}$$

where the $n - k - 1$ are df of the denominator (σ^2)

Differences in Regression Function Across Groups I

- Consider

$$cumgpa = \beta_0 + \beta_1 sat + \beta_2 hsperc + \beta_3 tothrs + u$$

where sat = SAT score, $hsperc$ = high school rank percentile, $tothrs$ = total hours of college courses.

- Does this model describe the college GPA for male *and* females?
- Can allow intercept and slopes to vary by sex as follows:

$$\begin{aligned} cumgpa &= \beta_0 + \delta_0 female + \beta_1 sat + \delta_1 sat \times female \\ &+ \beta_2 hsperc + \delta_2 hsperc \times female \\ &+ \beta_3 tothrs + \delta_3 tothrs \times female + u \end{aligned}$$

- $H_0 : \delta_0 = \delta_1 = \delta_2 = \delta_3 = 0$, H_1 : At least one δ is non-zero.

Differences in Regression Function Across Groups II

- We can estimate the interaction model and compute the corresponding F-test using the statistic from above

$$F = (R\hat{\beta} - q)'[\hat{\sigma}^2 R(X'X)^{-1}R']^{-1}(R\hat{\beta} - q) \sim F_{J,n-k-1}$$

- We can estimate the restricted (assume female $= 0$) and unrestricted versions of the model. Compute F-statistic as (will be identical)

$$F = \frac{SSR_R - SSR_U}{SSR_U} \frac{n - 2(J)}{J}$$

  where $SSR_R =$ sum of squares of restricted model, $SSR_U =$ sum of squares of unrestricted model, $n =$ total # of obs, $k =$ total # of explanatory variables *excluding* intercept, $J = k + 1$ total # of restrictions (we restrict all k slopes and intercept).

- $H_0 : \delta_0 = \delta_1 = \delta_2 = \delta_3 = 0$, H_1 : At least one δ is non-zero.

Chow Test

- What if we have a lot of explanatory variables? Unrestricted model will have a lot of terms.
- Imagine we have two groups, $g = 1, 2$
- Test whether intercept and slopes are same across two groups. Model is:

$$y = \beta_{g,0} + \beta_{g,1}x_1 + ... + \beta_{g,k}x_k + u$$

- $H_0 : \beta_{1,0} = \beta_{2,0}, \beta_{1,1} = \beta_{2,1}, ..., \beta_{1,k} = \beta_{2,k}$
- Null $\implies$ $k + 1$ restrictions (slopes + intercept). E.g., in GPA example, $k = 3$

Chow Test Recipe

- Chow test form of F-stat from above:

$$F = \frac{SSR_P - (SSR_1 + SSR_2)}{SSR_1 + SSR_2} \frac{n - 2(k+1)}{k+1}$$

1. Estimate pooled (i.e., restricted) model with no interactions and save SSR_P
2. Estimate model on group 1 and save SSR_1
3. Estimate model on group 2 and save SSR_2
4. Plug into F-stat formula.

- Often used to detect a structural break across time periods.
- Requires homoskedasticity.

Asymptotic Distribution of OLS Estimates

- If
  1. u are i.i.d. with mean 0 an dvariance σ^2, and
  2. x meet Grenander conditions (look it up), then

$$\hat{\beta} \xrightarrow{a} N\left[\beta, \frac{\sigma^2}{n} Q^{-1}\right]$$

  where $Q = plim(X'X/n)$
- Basically, under fairly weak conditions, OLS estimates are asymptotically normal and centered around the true parameter values.

The Delta Method

- How do we compute variance of nonlinear function of random variables? Use a Taylor expansion around the expectation
- If $\sqrt{n}(z_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ and $g(z_n)$ is continuous function not involving n, then

$$\sqrt{n}(g(z_n) - g(\mu)) \xrightarrow{d} N(0, g'(\mu)^2\sigma^2)$$

- If Z_n is $K \times 1$ sequence of vectgor-valued random variables: $\sqrt{n}(Z_n - M) \xrightarrow{d} N(0, \Sigma)$ and $C(Z_n)$ is a set of J continuous functions not involving n, then

$$\sqrt{n}(C(Z_n) - C(M)) \xrightarrow{d} N(0, G(M)\Sigma G(M)')$$

where $G(M)$ is the $J \times K$ matrix $\partial C(M)/\partial M'$. The jth row of $G(M)$ is the vector of partial derivatives of the jth fxn with respect to M'

The Delta Method in Action

- Consdier two estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ of β_1 and β_2:

$$\begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} \overset{a}{\sim} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma\right] \text{ where } \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

- What is asymptotic distribution of $f(\hat{\beta}_1, \hat{\beta}_2) = \hat{\beta}_1/(1-\hat{\beta}_2)$

$$\begin{aligned} \frac{\partial f}{\partial \beta_1} &= \frac{1}{1-\beta_2} \\ \frac{\partial f}{\partial \beta_2} &= \frac{\beta_1}{(1-\beta_2)^2} \\ \text{AVar } f(\hat{\beta}_1, \hat{\beta}_2) &= \begin{pmatrix} \frac{1}{1-\beta_2} & \frac{\beta_1}{(1-\beta_2)^2} \end{pmatrix} \Sigma \begin{pmatrix} \frac{1}{1-\beta_2} \\ \frac{\beta_1}{(1-\beta_2)^2} \end{pmatrix} \end{aligned}$$

Reporting Regression Results

- A table of OLS regression output should show the following:
  1. the dependent variable,
  2. the independent variables (or a subsample and description of the other variables),
  3. the corresponding estimated coefficients,
  4. the corresponding standard errors (or t-stats),
  5. stars by the coefficient to indicate the level of statistical significance, if any (1 star for 5%, 2 stars for 1%),
  6. the *adjusted* R^2, and
  7. the number of observations used in the regression.
- In the body of paper, focus discussion on variable(s) of interest: sign, magnitude, statistical & economic significance, economic interpretation.
- Discuss "other" coefficients if they are "strange" (e.g., wrong sign, huge magnitude, etc.)

Example: Reporting Regression Results

	Book Leverage			
	(1)	(2)	(3)	(4)
Industry Avg. Leverage	0.067**		0.053**	0.018**
	(35.179)		(25.531)	(7.111)
Log(Sales)		0.022**	0.017**	0.018**
		(11.861)	(8.996)	(9.036)
Market-to-Book		-0.024**	-0.017**	-0.018**
		(-17.156)	(-12.175)	(-12.479)
EBITDA / Assets		-0.035**	-0.035**	-0.036**
		(-20.664)	(-20.672)	(-20.955)
Net PPE / Assets		0.049**	0.031**	0.045**
		(24.729)	(15.607)	(16.484)
Firm Fixed Effects	No	No	No	No
Industry Fixed Effects	No	No	No	Yes
Year Fixed Effects	Yes	Yes	Yes	Yes
Obs	77,328	78,189	77,328	77,328
Adj. R^2	0.118	0.113	0.166	0.187