Handout 4

Properties of AR and ARMA Time Series Models

Class notes for Statistics 451: Applied Time Series
Iowa State University

January 7, 2007
17h 8min

Mean the AR(p) Model

Model: $Z_t = \theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t$

Mean: $\mu_Z \equiv \mathrm{E}(Z_t)$

$$\begin{aligned} \mathrm{E}(Z_t) &= \mathrm{E}(\theta_0 + \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t) \\ &= \mathrm{E}(\theta_0) + \phi_1 \mathrm{E}(Z_{t-1}) + \cdots + \phi_p \mathrm{E}(Z_{t-p}) + \mathrm{E}(a_t) \\ &= \theta_0 + (\phi_1 + \cdots + \phi_p)\mathrm{E}(Z_t) \\ &= \frac{\theta_0}{1-\phi_1-\cdots-\phi_p} \end{aligned}$$

Variance of the AR(1) Model

Model: $Z_t = \theta_0 + \phi_1 Z_{t-1} + a_t$

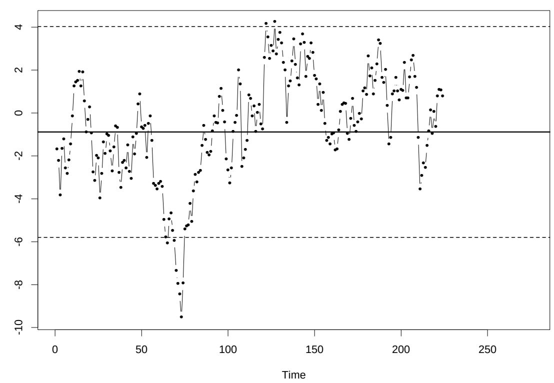
Variance: $\gamma_0 \equiv \mathrm{Var}(Z_t) \equiv \mathrm{E}[(Z_t - \mu_Z)^2] = \mathrm{E}(\dot{Z}^2)$

$$\begin{aligned} \gamma_0 &= \mathrm{E}(\dot{Z}_t^2) \\ &= \mathrm{E}[(\phi_1 \dot{Z}_{t-1} + a_t)^2] \\ &= \mathrm{E}[(\phi_1^2 \dot{Z}_{t-1}^2 + 2\phi_1 \dot{Z}_{t-1} a_t + a_t^2)] \\ &= \phi_1^2 \mathrm{E}(\dot{Z}_{t-1}^2) + 2\phi_1 \mathrm{E}(\dot{Z}_{t-1} a_t) + \mathrm{E}(a_t^2) \\ &= \phi_1^2 \gamma_0 + 0 + \sigma_a^2 \\ &= \frac{\sigma_a^2}{1-\phi_1^2} \end{aligned}$$

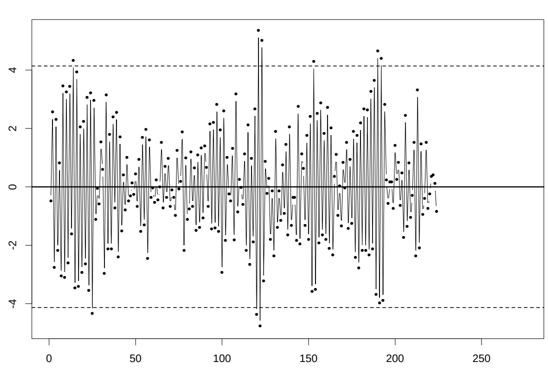
or

$$\mathrm{Var}(Z_t) = \frac{\mathrm{Var}(a_t)}{1-\phi_1^2}$$

Simulated AR(1) Data with $\phi_1 = .9, \sigma_a = 1$ Showing $\pm 2 \times \hat{\sigma}_Z$ limits



Simulated AR(1) Data with $\phi_1 = -.9, \sigma_a = 1$ Showing $\pm 2 \times \hat{\sigma}_Z$ limits



Autocovariance and Autocorrelation Functions for the AR(1) Model

Autocovariance: $\gamma_k \equiv \mathrm{Cov}(Z_t, Z_{t+k}) \equiv \mathrm{E}(\dot{Z}_t \dot{Z}_{t+k})$

$$\begin{aligned} \gamma_1 \equiv \mathrm{E}(\dot{Z}_t \dot{Z}_{t+1}) &= \mathrm{E}[\dot{Z}_t(\phi_1 \dot{Z}_t + a_{t+1})] \\ &= \phi_1 \mathrm{E}(\dot{Z}_t^2) + \mathrm{E}(\dot{Z}_t a_{t+1}) \\ &= \phi_1 \gamma_0 + 0 = \phi_1 \gamma_0 \end{aligned}$$

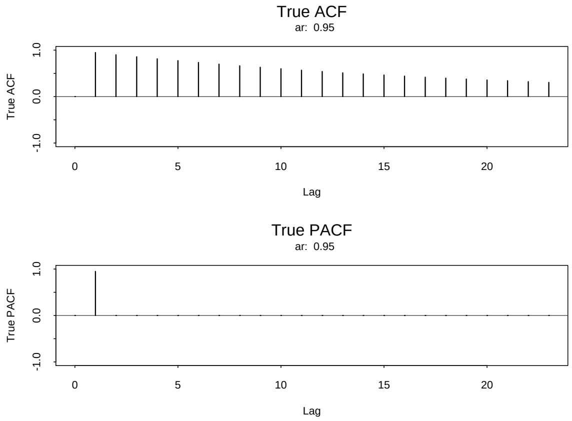
Thus $\rho_1 = \frac{\gamma_1}{\gamma_0} = \phi_1$.

$$\begin{aligned} \gamma_2 \equiv \mathrm{E}(\dot{Z}_t \dot{Z}_{t+2}) &= \mathrm{E}[\dot{Z}_t(\phi_1 \dot{Z}_{t+1} + a_{t+2})] \\ &= \phi_1 \mathrm{E}(\dot{Z}_t \dot{Z}_{t+1}) + \mathrm{E}(\dot{Z}_t a_{t+2}) \\ &= \phi_1 \gamma_1 = \phi_1(\phi_1 \gamma_0) = \phi_1^2 \gamma_0 \end{aligned}$$

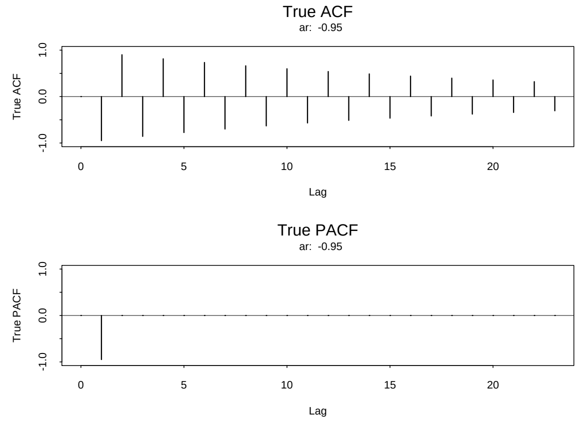
Thus $\rho_2 = \frac{\gamma_2}{\gamma_0} = \phi_1^2$.

In general, for AR(1), $\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1^k$.

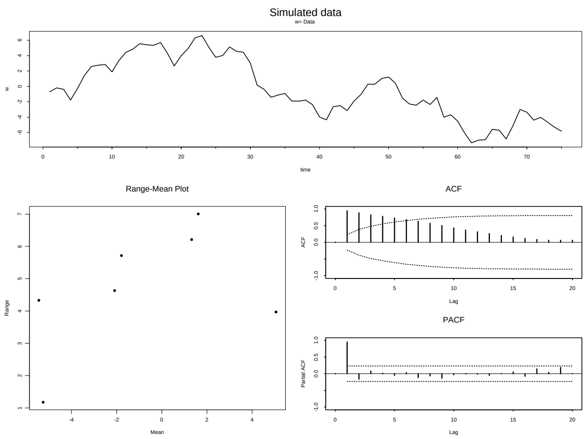
True ACF and PACF for AR(1) Model with $\phi_1 = .95$



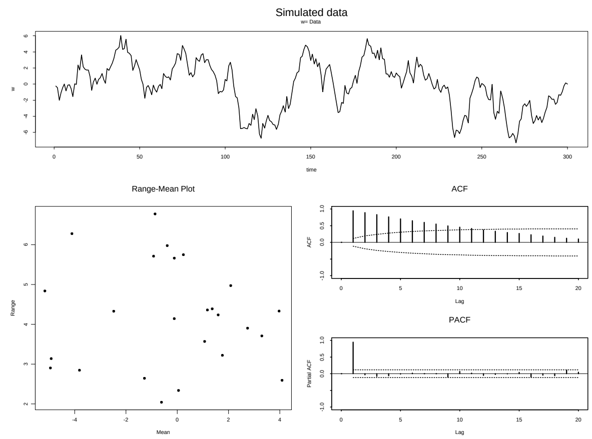
True ACF and PACF for AR(1) Model with $\phi_1 = -.95$



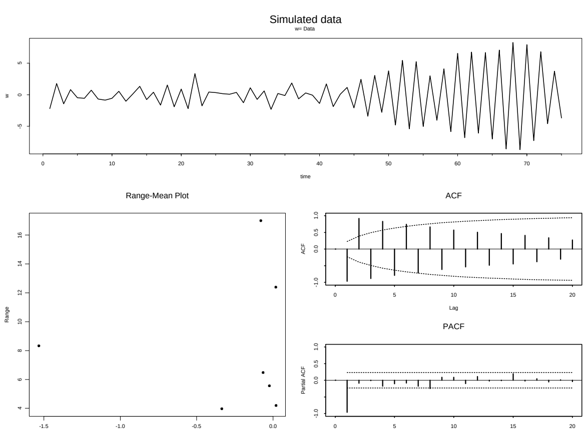
Simulated Realization $(\text{AR}(1), \phi_1 = .95, n = 75)$ Graphical Output from Function `iden`



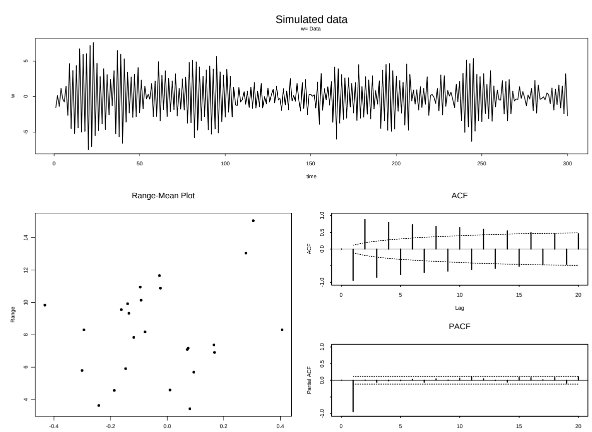
Simulated Realization $(\text{AR}(1), \phi_1 = .95, n = 300)$ Graphical Output from Function `iden`



Simulated Realization $(\text{AR}(1), \phi_1 = -.95, n = 75)$ Graphical Output from Function `iden`



Simulated Realization $(\text{AR}(1), \phi_1 = -.95, n = 300)$ Graphical Output from Function `iden`



Using the Geometric Series to Re-express the AR(1) Model as an Infinite MA

$$
\begin{aligned}
(1-\phi_1 \mathrm{B})\dot{Z}_t &= a_t \\
\dot{Z}_t &= (1-\phi_1 \mathrm{B})^{-1} a_t \\
&= (1+\phi_1 \mathrm{B}+\phi_1^2 \mathrm{B}^2+\cdots)a_t \\
&= \phi_1 a_{t-1}+\phi_1^2 a_{t-2}+\cdots+a_t
\end{aligned}
$$

Thus the AR(1) can be expressed as an infinite MA model.

If $-1 < \phi_1 < 1$, then the weight on the old residuals is decreasing with age. This is the condition of "stationarity" for an AR(1) model.

Using the Back-substitution to Re-express the AR(1) Model as an Infinite MA

$$
\begin{aligned}
\dot{Z}_t &= \phi_1 \dot{Z}_{t-1}+a_t \\
\dot{Z}_{t-1} &= \phi_1 \dot{Z}_{t-2}+a_{t-1} \\
\dot{Z}_{t-2} &= \phi_1 \dot{Z}_{t-3}+a_{t-2} \\
\dot{Z}_{t-3} &= \phi_1 \dot{Z}_{t-4}+a_{t-3}
\end{aligned}
$$

Substituting, successively, $\dot{Z}_{t-1}, \dot{Z}_{t-2}, \dot{Z}_{t-3} \ldots$, shows that

$$\dot{Z}_t = \phi_1 a_{t-1}+\phi_1^2 a_{t-2}+\phi_1^3 a_{t-3}+\cdots+a_t$$

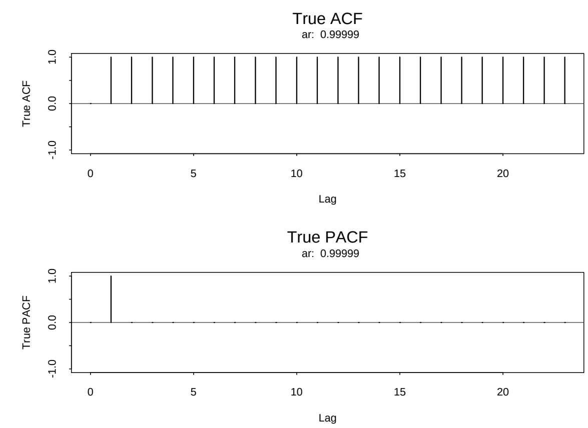
This methods works, more generally, for higher-order AR(p) models, but the algebra is tedious.

Notes on the AR(1) model

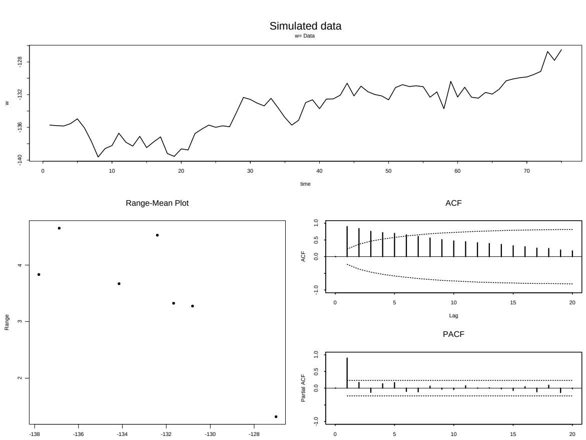
$$Z_t = \phi_1 Z_{t-1} + a_t$$

- Because $\rho_1 = \phi_1$, we can estimate ϕ_1 by $\widehat{\phi}_1 = \widehat{\rho}_1$.
- Root of $(1-\phi_1 \mathrm{B}) = 0$ is $\mathrm{B} = 1/\phi_1$ must be <u>outside</u> $[-1, 1]$ or $-1 < \phi_1 < 1$ so that AR(1) will be stationary.
- $\phi_1 = 1$ implies $Z_t = Z_{t-1} + a_t$, the "random walk" model.
- $\phi_1 > 1$ explosive
- $\phi_1 < -1$ oscillating explosive

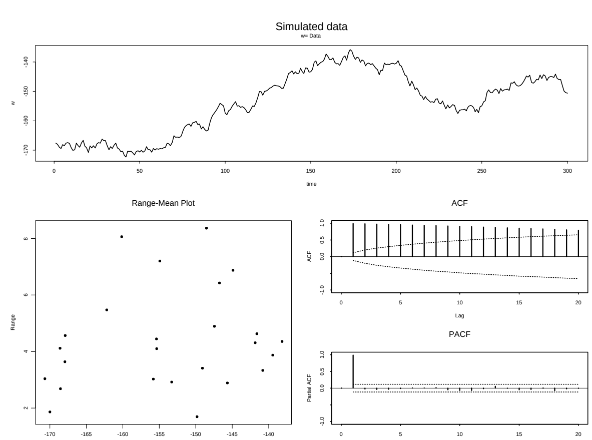
True ACF and PACF for AR(1) Model with $\phi_1 = .99999$



Simulated Realization (AR(1), $\phi_1 = .99999, n = 75$) Graphical Output from Function `iden`



Simulated Realization (AR(1), $\phi_1 = .99999, n = 300$) Graphical Output from Function `iden`



Re-expressing the AR(p) Model as an Infinite MA

More generally, any AR(p) model can be expressed as

$$\begin{aligned}\phi_p(\mathrm{B})\dot{Z}_t &= a_t \\ \dot{Z}_t &= \frac{1}{\phi_p(\mathrm{B})}a_t = \psi(\mathrm{B})a_t \\ &= \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots + a_t = \sum_{k=1}^{\infty}\psi_k a_{t-k} + a_t\end{aligned}$$

Values of $\psi_1, \psi_2, \ldots$ depend on $\phi_1, \ldots, \phi_p$. For the AR model to be stationary, the ψ_j values should not remain large as j gets large. Formally, the stationarity condition is met if

$$\sum_{j=1}^{\infty}|\psi_j| < \infty$$

Stationarity of an AR(p) model can be checked by finding the roots of the polynomial $\phi_p(\mathrm{B}) \equiv (1-\phi_1\mathrm{B}-\phi_2\mathrm{B}^2-\cdots-\phi_p\mathrm{B}^p)$. All p roots must lie <u>outside</u> of the "unit-circle."

Checking the Stationarity of an AR(2) Model

Stationarity of an AR(2) model can be checked by finding the roots of the polynomial $(1-\phi_1\mathrm{B}-\phi_2\mathrm{B}^2) = 0$. Both roots must lie <u>outside</u> of the "unit-circle." From page 39,

$$B = \frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

Roots have the form

$$z = x + iy$$

where $i = \sqrt{-1}$. A root is "outside of the unit circle" if

$$|z| = \sqrt{x^2 + y^2} > 1$$

This method works for any p, but for $p > 2$ it is best to use numerical methods to find the p roots.

Checking the Stationarity of an AR(2) Model
Simple Rule

Both roots lying <u>outside</u> of the "unit-circle" implies

$$\begin{aligned}\phi_2 + \phi_1 &< 1 \\ \phi_2 - \phi_1 &< 1 \\ -1 < \phi_2 &< 1\end{aligned}$$

Defines the AR(2) triangle.

See pages 39/40 of Wei for algebraic argument.

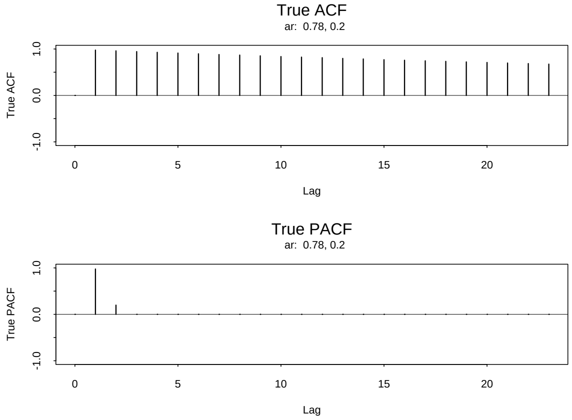
Autocovariance and Autocorrelation Functions for the AR(2) Model

$$\begin{aligned}\gamma_1 \equiv \mathrm{E}(\dot{Z}_t\dot{Z}_{t+1}) &= \mathrm{E}[\dot{Z}_t(\phi_1\dot{Z}_t + \phi_2\dot{Z}_{t-1} + a_{t+1})] \\ &= \phi_1\mathrm{E}(\dot{Z}_t^2) + \phi_2\mathrm{E}(\dot{Z}_t\dot{Z}_{t-1}) + \mathrm{E}(\dot{Z}_t a_{t+1}) \\ &= \phi_1\gamma_0 + \phi_2\gamma_1 + 0 \\ \gamma_2 \equiv \mathrm{E}(\dot{Z}_t\dot{Z}_{t+2}) &= \mathrm{E}[\dot{Z}_t(\phi_1\dot{Z}_{t+1} + \phi_2\dot{Z}_t + a_{t+2})] \\ &= \phi_1\mathrm{E}(\dot{Z}_t\dot{Z}_{t+1}) + \phi_2\mathrm{E}(\dot{Z}_t^2) + \mathrm{E}(\dot{Z}_t a_{t+2}) \\ &= \phi_1\gamma_1 + \phi_2\gamma_0 + 0\end{aligned}$$

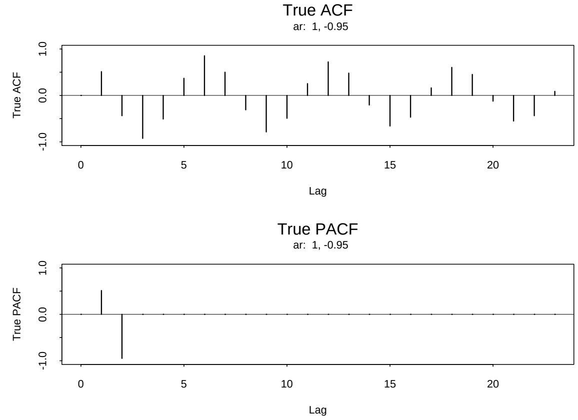
Then using $\rho_k = \gamma_k/\gamma_0$ and $\rho_0 = 1$, gives the AR(2) ACF

$$\begin{aligned}\rho_1 &= \phi_1 + \phi_2\rho_1 \\ \rho_2 &= \phi_1\rho_1 + \phi_2 \\ \rho_3 &= \phi_1\rho_2 + \phi_2\rho_1 \\ &\vdots \\ \rho_k &= \phi_1\rho_{k-1} + \phi_2\rho_{k-2}\end{aligned}$$

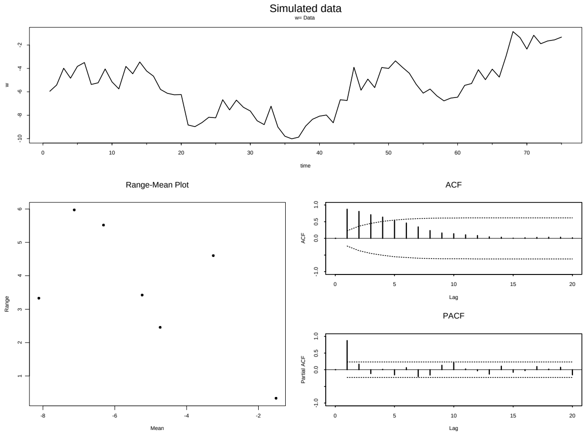
True ACF and PACF for AR(2) Model with $\phi_1 = .78, \phi_2 = .2$



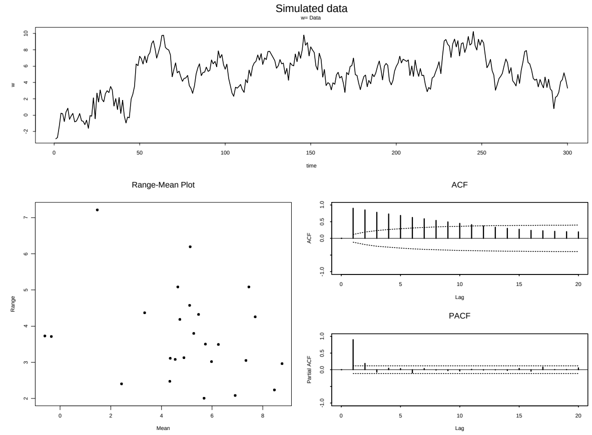
True ACF and PACF for AR(2) Model with $\phi_1 = 1, \phi_2 = -.95$



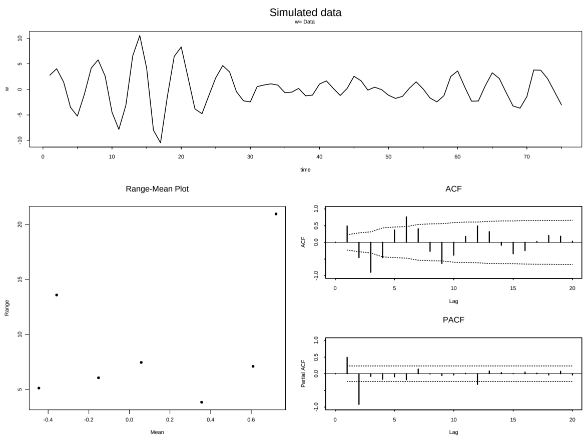
Simulated Realization (AR(2), $\phi_1 = .78, \phi_2 = .2, n = 75$) Graphical Output from Function `iden`



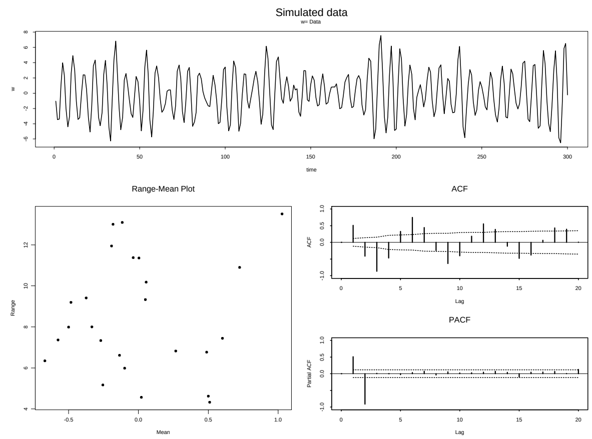
Simulated Realization (AR(2), $\phi_1 = .78, \phi_2 = .2, n = 300$) Graphical Output from Function `iden`



Simulated Realization (AR(2), $\phi_1 = 1, \phi_2 = -.95, n = 75$) Graphical Output from Function `iden`



Simulated Realization (AR(2), $\phi_1 = 1, \phi_2 = -.95, n = 300$) Graphical Output from Function `iden`



Yule-Walker Equations (Correlation Form)

AR(1) Yule-Walker Equation

$$\rho_1 = \phi_1$$

AR(2) Yule-Walker Equations

$$\begin{aligned} \rho_1 &= \phi_1 + \rho_1\phi_2 \\ \rho_2 &= \rho_1\phi_1 + \phi_2 \end{aligned}$$

AR(3) Yule-Walker Equations

$$\begin{aligned} \rho_1 &= \phi_1 + \rho_1\phi_2 + \rho_2\phi_3 \\ \rho_2 &= \rho_1\phi_1 + \phi_2 + \rho_1\phi_3 \\ \rho_3 &= \rho_2\phi_1 + \rho_1\phi_2 + \phi_3 \end{aligned}$$

Applications of the Yule-Walker Equations

- Provides the true ACF of an AR(p) model (given $\phi_1, \phi_2, \ldots, \phi_p$, compute $\rho_1, \rho_2, \ldots$ recursively)
- Provides estimate of $\phi_1, \phi_2, \ldots, \phi_p$ of an AR(p) model (given sample ACF values $\widehat{\rho}_1, \widehat{\rho}_2, \ldots, \widehat{\rho}_p$, substitute in Y-W equations and solve the set of simultaneous equations for $\widehat{\phi}_1, \widehat{\phi}_2, \ldots, \widehat{\phi}_p$)
- Provides the sample PACF $\widehat{\phi}_{1,1}, \widehat{\phi}_{2,2}, \ldots, \widehat{\phi}_{kk}$ for any realization. (substitute $\widehat{\rho}_1, \widehat{\rho}_2, \ldots, \widehat{\rho}_k$ into the AR(k) Y-W equations and solve for $\widehat{\phi}_k$). Same as Durbin's formula.
- Provides the true PACF $\phi_{1,1}, \phi_{2,2}, \ldots, \phi_{kk}$ for any ARMA model. (substitute $\rho_1, \rho_2, \ldots, \rho_k$ into the AR(k) Y-W equations and solve for ϕ_k). Same as Durbin's formula.

Variance of the AR(p) model

$$\begin{aligned}\dot{Z}_t &= \phi_1\dot{Z}_{t-1} + \cdots + \phi_p\dot{Z}_{t-p} + a_t\\ \dot{Z}_t^2 &= \phi_1\dot{Z}_t\dot{Z}_{t-1} + \cdots + \phi_p\dot{Z}_t\dot{Z}_{t-p} + \dot{Z}_t a_t\end{aligned}$$

Taking expectations and noting that $\mathrm{E}(\dot{Z}_t a_t) = \sigma_a^2$ gives

$$\begin{aligned}\gamma_0 &= \mathrm{Var}(Z_t) = \mathrm{E}(\dot{Z}_t^2) = \phi_1\gamma_1 + \cdots + \phi_p\gamma_p + \sigma_a^2\\ &= \tfrac{\sigma_a^2}{1-\phi_1\rho_1-\cdots-\phi_p\rho_p}\end{aligned}$$

Note:

$$\mathrm{E}(\dot{Z}_t a_t) = \mathrm{E}(\phi_1\dot{Z}_{t-1}a_t + \cdots + \phi_p\dot{Z}_{t-p}a_t + a_t^2) = \sigma_a^2$$

Properties of the ARMA(1,1) Model

$$\begin{aligned}\phi_1(\mathrm{B})Z_t &= \theta_0 + \theta_1(\mathrm{B})a_t\\ Z_t &= \theta_0 + \phi_1 Z_{t-1} - \theta_1 a_{t-1} + a_t\end{aligned}$$

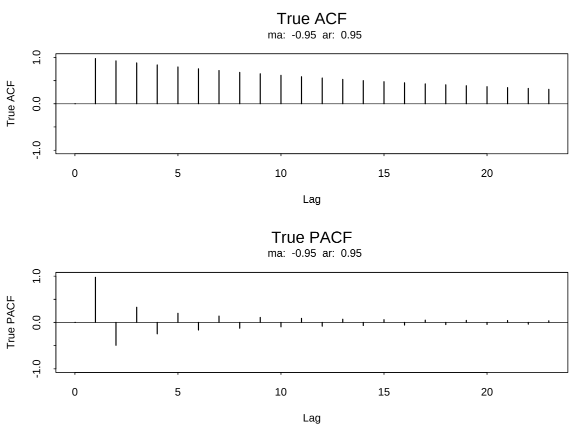
Noting that $\mu_Z = \mathrm{E}(Z_t) = \theta_0/(1-\phi_1)$

$$\begin{aligned}\phi_1(\mathrm{B})\dot{Z}_t &= \theta_1(\mathrm{B})a_t\\ \dot{Z}_t &= \phi_1\dot{Z}_{t-1} - \theta_1 a_{t-1} + a_t\end{aligned}$$

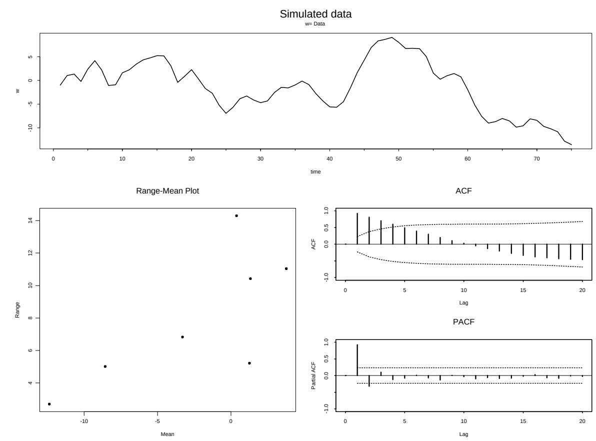
Omitting derivations (page 59),

$$\begin{aligned}\gamma_0 &= \mathrm{Var}(Z_t) = [(1-2\phi_1\theta_1+\theta_1^2)/(1-\phi_1^2)]\sigma_a^2\\ \rho_1 &= (1-\phi_1\theta_1)(\phi_1-\theta_1)/(1+\theta_1^2-2\phi_1\theta_1)\\ \rho_2 &= \phi_1\rho_1\\ \rho_k &= \phi_1\rho_{k-1}\end{aligned}$$

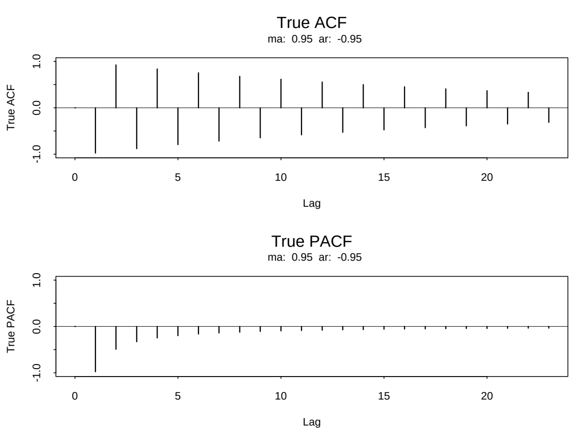
True ACF and PACF for ARMA(1,1) Model with $\phi_1 = .95, \theta_1 = -.95$



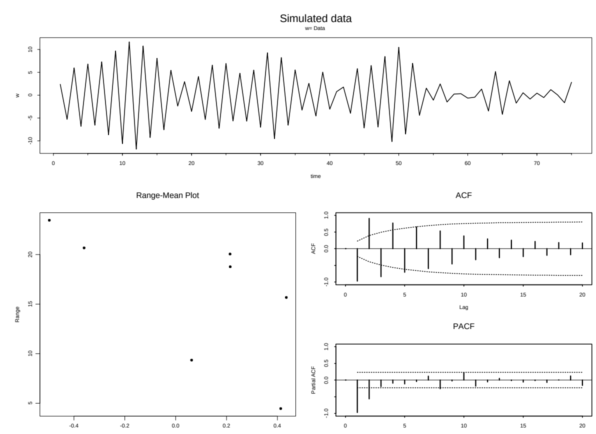
Simulated Realization (ARMA(1,1), $\phi_1 = .95,, \theta_1 = -.95, n = 75$) Graphical Output from Function `iden`



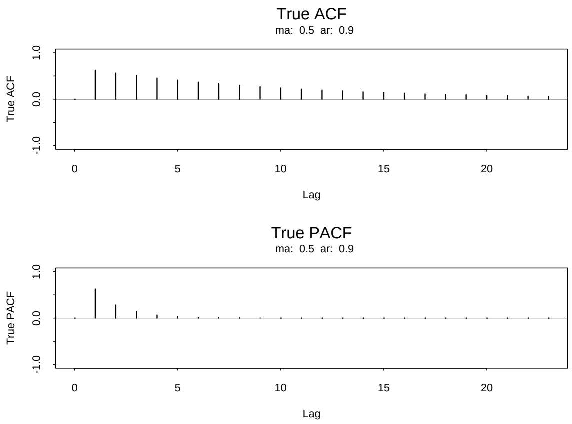
True ACF and PACF for ARMA(1,1) Model with $\phi_1 = -.95, \theta_1 = .95$



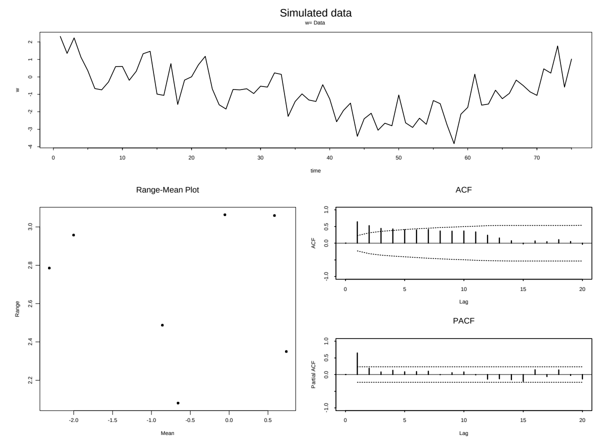
Simulated Realization (ARMA(1,1), $\phi_1 = -.95,, \theta_1 = .95, n = 75$) Graphical Output from Function `iden`



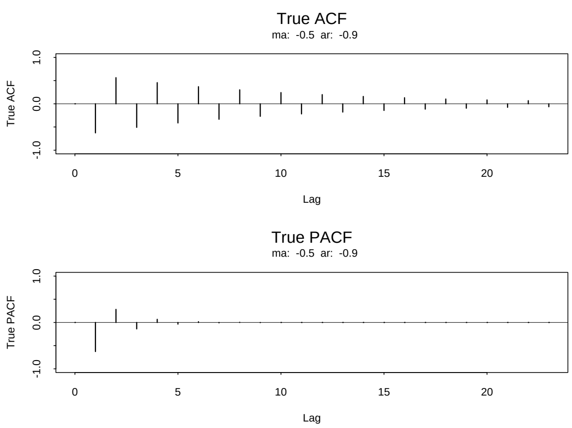
True ACF and PACF for ARMA$(1,1)$ Model with $\phi_1 = .9, \theta_1 = .5$



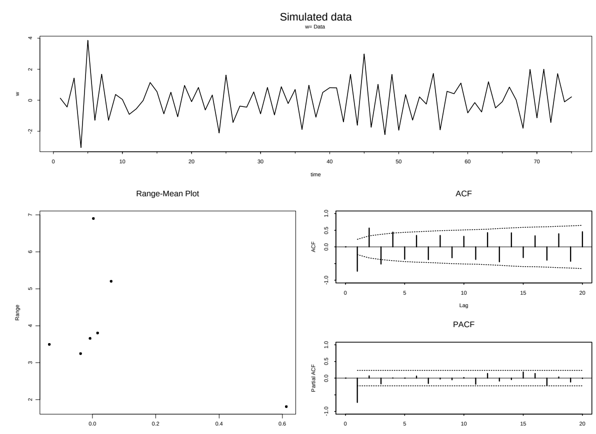
Simulated Realization (ARMA$(1,1), \phi_1 = .9, , \theta_1 = .5, n = 75$) Graphical Output from Function `iden`



True ACF and PACF for ARMA$(1,1)$ Model with $\phi_1 = -.9, \theta_1 = -.5$



Simulated Realization (ARMA$(1,1), \phi_1 = -.9, , \theta_1 = -.5, n = 75$) Graphical Output from Function `iden`



Stationarity of an ARMA$(1,1)$ Model

$$(1-\phi_1\mathrm{B})\dot{Z}_t = (1-\theta_1\mathrm{B})a_t$$

$$\begin{aligned}
\dot{Z}_t &= \frac{(1-\theta_1\mathrm{B})}{(1-\phi_1\mathrm{B})}a_t \\
&= (1-\theta_1\mathrm{B})(1-\phi_1\mathrm{B})^{-1}a_t \\
&= (1-\theta_1\mathrm{B})(1+\phi_1\mathrm{B}+\phi_1^2\mathrm{B}^2+\cdots)a_t \\
&= (\phi_1-\theta_1)a_{t-1}+\phi_1(\phi_1-\theta_1)a_{t-2}+\phi_1^2(\phi_1-\theta_1)a_{t-3}+\cdots+a_t \\
&= \sum_{j=1}^{\infty}\phi_1^{j-1}(\phi_1-\theta_1)a_{t-j}+a_t = \sum_{j=0}^{\infty}\psi_j a_{t-j}
\end{aligned}$$

where $\psi_j = \phi_1^{j-1}(\phi_1-\theta_1)$ and $\phi_1^0 \equiv 1$.

ARMA(1,1) model is stationary if $-1 < \phi_1 < 1$.

ARMA(1,1) model is white noise (trivial model) if $\phi_1 = \theta_1$.

Invertibility of an ARMA$(1,1)$ Model

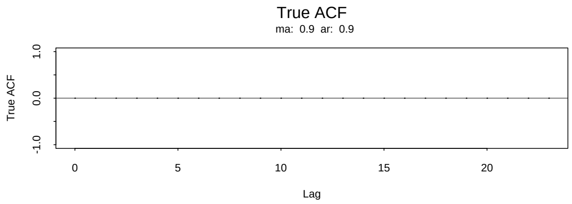
$$(1-\phi_1\mathrm{B})\dot{Z}_t = (1-\theta_1\mathrm{B})a_t$$

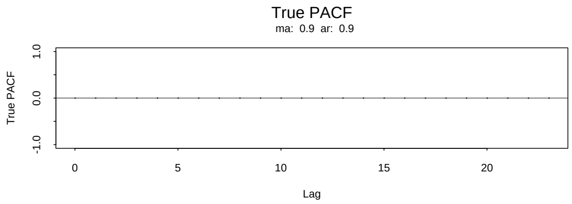
$$\begin{aligned}
a_t &= \frac{(1-\phi_1\mathrm{B})}{(1-\theta_1\mathrm{B})}\dot{Z}_t \\
&= (1-\phi_1\mathrm{B})(1-\theta_1\mathrm{B})^{-1}\dot{Z}_t \\
&= (1-\phi_1\mathrm{B})(1+\theta_1\mathrm{B}+\theta_1^2\mathrm{B}^2+\cdots)\dot{Z}_t \\
&= \dot{Z}_t+(\theta_1-\phi_1)\dot{Z}_{t-1}+\theta_1(\theta_1-\phi_1)\dot{Z}_{t-2}+\theta_1^2(\theta_1-\phi_1)\dot{Z}_{t-3}+\cdots \\
\dot{Z}_t &= (\phi_1-\theta_1)\dot{Z}_{t-1}+\theta_1(\phi_1-\theta_1)\dot{Z}_{t-2}+\theta_1^2(\phi_1-\theta_1)\dot{Z}_{t-3}+\cdots+a_t \\
&= \sum_{j=1}^{\infty}\theta_1^{j-1}(\phi_1-\theta_1)\dot{Z}_{t-j}+a_t = \sum_{j=1}^{\infty}\pi_j\dot{Z}_{t-j}+a_t
\end{aligned}$$

where $\pi_j = \theta_1^{j-1}(\phi_1-\theta_1)$.

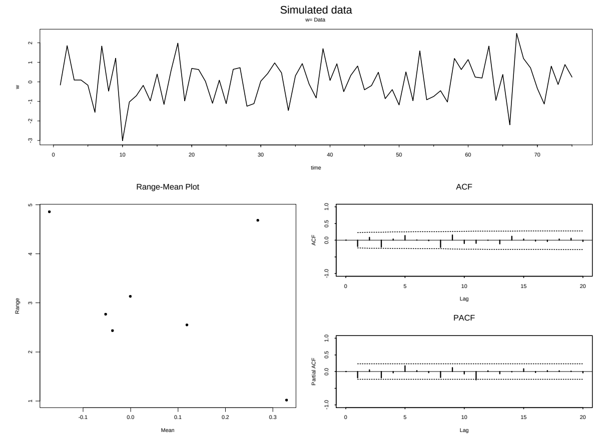
ARMA(1,1) model is invertible if $-1 < \theta_1 < 1$.

True ACF and PACF for ARMA$(1,1)$ Model with $\phi_1 = .9, \theta_1 = .9$





Simulated Realization (ARMA$(1,1), \phi_1 = .9, , \theta_1 = .9, n = 75$) Graphical Output from Function `iden`



Properties of ARMA(p,q) model

$$\phi_p(\mathrm{B})Z_t = \theta_0 + \theta_q(\mathrm{B})a_t$$
$$\phi_p(\mathrm{B})\dot{Z}_t = \theta_q(\mathrm{B})a_t$$
$$(1 - \phi_1\mathrm{B} - \phi_2\mathrm{B}^2 - \cdots - \phi_p\mathrm{B}^p)\dot{Z}_t = (1 - \theta_1\mathrm{B} - \theta_2\mathrm{B}^2 - \cdots - \theta_q\mathrm{B}^q)a_t$$

- $\mu_Z \equiv E(Z_t) = \frac{\theta_0}{1-\phi_1-\cdots-\phi_p}$
- An ARMA(p,q) model is <u>stationary</u> if all of the roots of $\phi_p(\mathrm{B})$ lie <u>outside</u> of the unit circle.
- An ARMA(p,q) model is <u>invertible</u> if all of the roots of $\theta_q(\mathrm{B})$ lie <u>outside</u> of the unit circle.

Expressing an ARMA(p,q) Model as an Infinite MA

$$\begin{aligned}\phi_p(\mathrm{B})\dot{Z}_t &= \theta_q(\mathrm{B})a_t\\ \dot{Z}_t &= [\phi_p(\mathrm{B})]^{-1}\theta_q(\mathrm{B})a_t = \psi(\mathrm{B})a_t\\ \phi_p(\mathrm{B})\psi(\mathrm{B}) &= \theta_q(\mathrm{B})\end{aligned}$$

- Given $\phi_p(\mathrm{B})$ and $\theta_q(\mathrm{B})$, solve for $\psi(\mathrm{B})$.
- See page 54 for an example using AR(2)
- Similar method for expressing an ARMA(p,q) model as an infinite AR (see page 55).

$$\phi_p(\mathrm{B}) = \pi(\mathrm{B})\theta_q(\mathrm{B})$$

Expressing and ARMA(p,q) Model as an Infinite MA

Using ARMA(2,2) as a particular example

$$\begin{aligned}\phi_2(\mathrm{B})\psi(\mathrm{B}) &= \theta_2(\mathrm{B})\\ (1-\phi_1\mathrm{B}-\phi_2\mathrm{B}^2)(1+\psi_1\mathrm{B}+\psi_2\mathrm{B}^2+\psi_3\mathrm{B}^3+\cdots) &= (1-\theta_1\mathrm{B}-\theta_2\mathrm{B}^2)\end{aligned}$$

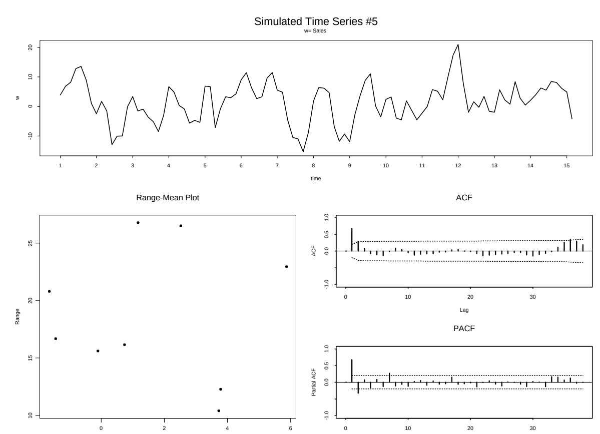
Multiplying out gives

$$\begin{array}{llll} 1+\psi_1\mathrm{B} & +\psi_2\mathrm{B}^2 & +\psi_3\mathrm{B}^3 & +\cdots \\ \quad -\phi_1\mathrm{B} & -\psi_1\phi_1\mathrm{B}^2 & -\psi_2\phi_1\mathrm{B}^3 & -\cdots \\ & -\phi_2\mathrm{B}^2 & -\psi_1\phi_2\mathrm{B}^3 & -\cdots \quad = 1-\theta_1\mathrm{B}-\theta_2\mathrm{B}^2 \end{array}$$

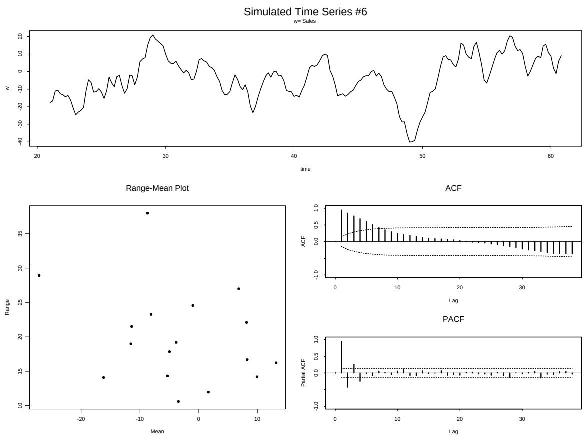
Equating terms with the same power of B gives

$$\begin{array}{lrcll} \mathrm{B}: & \psi_1-\phi_1 & = & -\theta_1 & \rightarrow \psi_1 = -\theta_1+\phi_1 \\ \mathrm{B}^2: & \psi_2-\psi_1\phi_1-\phi_2 & = & -\theta_2 & \rightarrow \psi_2 = -\theta_2+\psi_1\phi_1+\phi_2 \\ \mathrm{B}^3: & \psi_3-\psi_2\phi_1-\psi_1\phi_2 & = & 0 & \rightarrow \psi_3 = \psi_2\phi_1+\psi_1\phi_2 \\ \vdots & \vdots & & & \vdots \\ \mathrm{B}^k: & \psi_k-\psi_{k-1}\phi_1-\psi_{k-2}\phi_2 & = & 0 & \rightarrow \psi_k = \psi_{k-1}\phi_1+\psi_{k-2}\phi_2 \end{array}$$

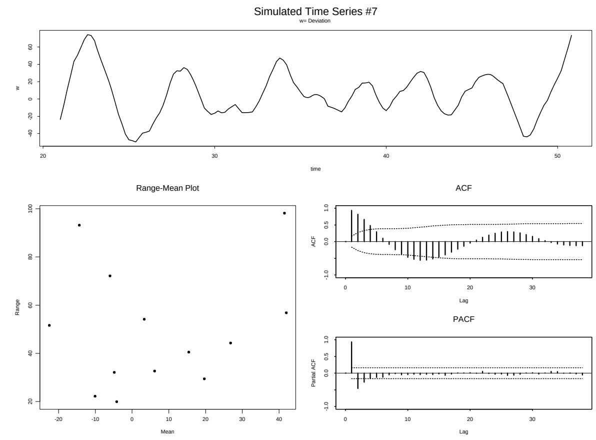
Simulated Time Series #5 Graphical Output from Function `iden(simsta5.d)`



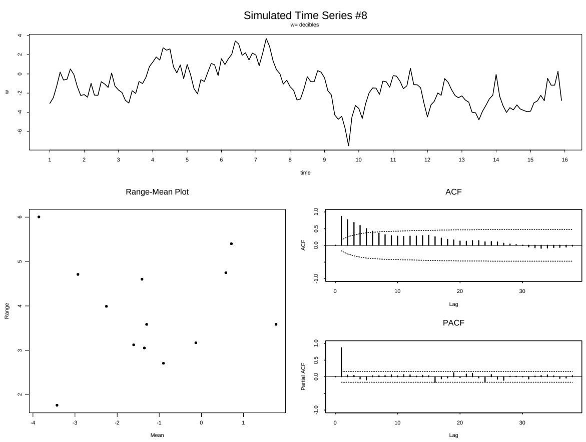
Simulated Time Series #6
Graphical Output from Function `iden`



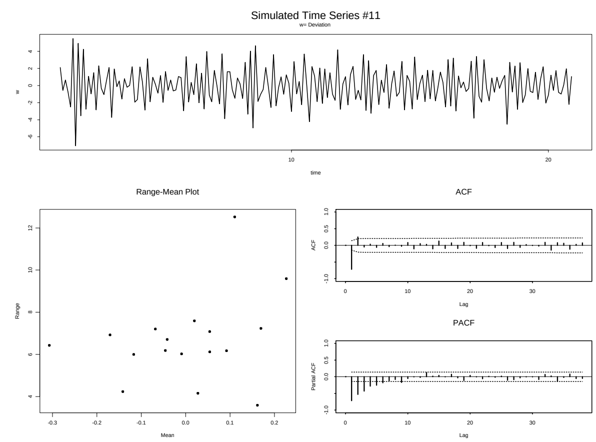
Simulated Time Series #7
Graphical Output from Function `iden`



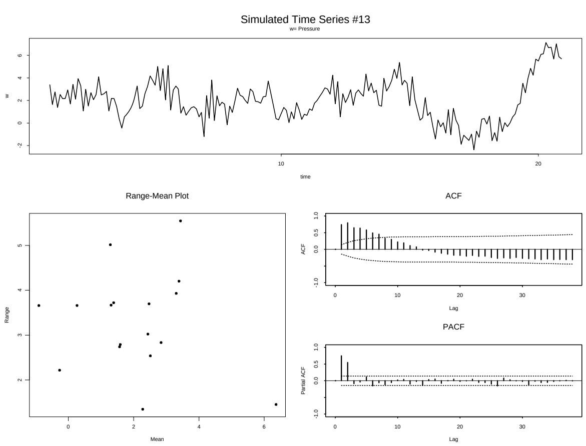
Simulated Time Series #8
Graphical Output from Function `iden`



Simulated Time Series #11
Graphical Output from Function `iden`



Simulated Time Series #13
Graphical Output from Function `iden`



Simulated Time Series #15
Graphical Output from Function `iden`