# Lecture 6: Aggregation

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Aggregation is an interesting subject in time series analysis. Two good references for this topic are the textbook by Wei (1990), which contains various results of aggregation, and the paper by Engel (1984, J. Time Series Analysis, pp. 159-171), which considers models of the sums and aggregations of linear ARMA models.

There are two types of aggregation: sum over series and sum over time (or temporal aggregation). Some relevant results are as follows:

a. Sum of two independent time series: Suppose that $X_{1t}$ and $X_{2t}$ are two independent ARMA series of orders $(p_1, q_1)$ and $(p_2, q_2)$, respectively. Let $Z_t = X_{1t} + X_{2t}$. Then, $Z_t$ is an ARMA$(p, q)$ process with $p \le p_1 + p_2$ and $q \le \max\{p_1 + q_2, p_2 + q_1\}$.

The reason that "$\le$" is used is because of the possibility of common factors in the polynomials involved.

**Proof**: Write the model for $X_{it}$ as

$$\phi_i(B)X_{it} = \theta_i(B)a_{it}.$$

Applying $\phi_1(B)\phi_2(B)$ to $Z_t$, we have

$$\phi_1(B)\phi_2(B)Z_t = \phi_1(B)\phi_2(B)(X_{1t} + X_{2t}) = \phi_2(B)\theta_1(B)a_{1t} + \phi_1(B)\theta_2(B)a_{2t}.$$

The result follows.

Notice that the "independent" assumption is important, otherwise, $X_{1t}$ may be a lagged version of $X_{2t}$, e.g. $X_{1t} = X_{2,t-100}$, which can easily violate the stated result.

b. Temporal aggregation: For a time series $\{Y_t\}$, let $\{Z_\ell\}$ be the series consisting of sums of $m$ non-overlapping points of $Y_t$. For example, for $m = 3$, we can aggregate monthly series to obtain a quarterly series. Mathematically, we can define $Z_\ell$ as

$$Z_\ell = \sum_{t=m(\ell-1)+1}^{m\ell} Y_t = (1 + B + \cdots + B^{m-1})Y_{m\ell}.$$

If $Y_t$ is an ARMA$(p, q)$ process, then $Z_\ell$ is also an ARMA process. To determine the model for $Z_t$, one needs to consider first the implication of the "backshift operator $B$" on $Z_\ell$. By definition,

$$BZ_\ell = Z_{\ell-1} = \sum_{t=m(\ell-2)+1}^{m(\ell-1)} Y_t = (1 + B + \cdots + B^{m-1})Y_{m(\ell-1)} = B^m(1 + B + \cdots + B^{m-1})Y_{m\ell}.$$

Thus, the backshift operator "B" applying to $Z_\ell$ is amount to applying $B^m$ to $Y_t$. In other words, we need to understand that "time scales"are different between the original series $Y_t$

and the aggregated series $Z_t$. For instance, a backshift operator in a quarterly time series is corresponding to $B^3$ in a monthly series.

**Example**: The best way to derive a model for the aggregated series $Z_t$ is to consider a simple example. Suppose that $Y_t$ is an ARMA(1,1) process, say

$$(1-\phi B)Y_t = (1-\theta B)a_t,$$

and $m = 2$. Applying the polynomial operator $(1-\phi^2 B)$ to $Z_\ell$, we have

$$\begin{aligned}
(1-\phi^2 B)Z_\ell &= Z_\ell - \phi^2 Z_{\ell-1} \\
&= (Y_{2\ell} + Y_{2\ell-1}) - \phi^2(Y_{2(\ell-1)} + Y_{2(\ell-1)-1}) \\
&= Y_{2\ell} - \phi^2 Y_{2\ell-2} + Y_{2\ell-1} - \phi^2 Y_{2\ell-3} \\
&= a_{2\ell} + (\phi-\theta)a_{2\ell-1} - \phi\theta a_{2\ell-2} + a_{2\ell-1} + (\phi-\theta)a_{2\ell-2} - \phi\theta a_{2\ell-3} \\
&\stackrel{def}{=} a_{2\ell} + \omega_1 a_{2\ell-1} + \omega_2 a_{2\ell-2} + \omega_3 a_{2\ell-3}.
\end{aligned}$$

In the above, we have used the result

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + a_t - \theta a_{t-1} \\
&= \phi^2 Y_{t-2} + a_t + (\phi-\theta)a_{t-1} - \phi\theta a_{t-2}.
\end{aligned}$$

Since $2\ell - 2$ and $2\ell - 3$ correspond to $Z_{\ell-1}$, thus the model for $Z_\ell$ can be written as

$$(1-\phi^2 B)Z_\ell = (1-\Theta B)b_\ell,$$

which is again an ARMA(1,1) model in the time scale of $\ell$.

**Example**: Consider now that $m = 2$ and $Y_t$ is an AR(2) model, say

$$(1-.5B)(1+.5B)Y_t = (1-.25B^2)Y_t = a_t.$$

Here the coefficients are chosen purposely so that $(.5)^2 = (-.5)^2$. Applying $(1-.25B)$ to $Z_\ell$, we have

$$\begin{aligned}
(1-.25B)Z_\ell &= Z_\ell - .25Z_{\ell-1} \\
&= (Y_{2\ell} + Y_{2\ell-1}) - .25(Y_{2\ell-2} + Y_{2\ell-3}) \\
&= (Y_{2\ell} - .25Y_{2\ell-2}) + (Y_{2\ell-1} - .25Y_{2\ell-3}) \\
&= a_{2\ell} + a_{2\ell-1} \\
&\stackrel{def}{=} b_\ell,
\end{aligned}$$

which is an AR(1) model in the $\ell$ time scale. Thus, the fact that the coefficients satisfy $(.5)^2 = (-.5)^2$ reduces the AR order by 1. This phenomenon holds in general aggregation.

General result: For an ARMA$(p,q)$ process $Y_t$, say $\phi(B)Y_t = \theta(B)a_t$, its $m$-point non-overlapping aggregate $Z_t$ is an ARMA$(p^*, q^*)$ where $p^* \leq p$ and $q^* \leq [p+1+\frac{q-p-1}{m}]$ with $[x]$ the integer part of $x$. The inequality "$<$" occurs when some zeros of $\phi(B)$ satisfy $\lambda_i^m = \lambda_j^m$ for $i \neq j$. If all the zeros of $\phi(B)$ are distinct and $\lambda_i^m \neq \lambda_j^m$ for $i \neq j$, then the equality holds. For further details, see Wei (1990, Chap. 16).

Systematic sampling: Suppose that a time series $Y_t$ is an ARMA$(p,q)$ process. However, one only observes the $m$-th subseries $Z_t$, which is defined as

$$Z_t = Y_{mt} \quad \text{with} \quad m \quad \text{a fixed positive integer.}$$

This situation occurs frequently in business and in process control. For instance, one may inspect every 5-th product of a production line. Here again, a backshift operator of $Z_t$ is corresponding to $B^m$ of the original series $Y_t$. In general, the model for $Z_t$ is ARMA$(p,r)$ where $r \leq [p + \frac{q-p}{m}]$.