Lecture 3: ARMA Models

Bus 41910, Univariate Time Series Analysis, Mr. R. Tsay

Autoregressive Moving-Average (ARMA) models form a class of linear time series models which are parsimonious in parameterization. By allowing the order of an ARMA model to increase, one can approximate any linear time series model with desirable accuracy. [This is similar to using rational polynomials to approximate a general polynomial; see the impluse response function.]

A. Autoregressive (AR) Processes

1. AR(p) Model: $Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = c + a_t$ or $\phi(B)Z_t = c + a_t$, where c is a constant. If Z_t is stationary with mean $E(Z_t) = \mu$, then the model can be written as $\phi(B)(Z_t - \mu) = a_t$.

2. Stationarity: All zeros of the polynomial $\phi(B)$ lie outside the unit circle. (Why?)

3. Moments: (Assume stationarity)

   - Mean: Taking expectation of both sides of the model equation, we have
   $$E(Z_t) - \phi_1 E(Z_{t-1}) - \cdots - \phi_p E(Z_{t-p}) = c + E(a_t)$$
   $$\mu - \phi_1\mu - \cdots - \phi_p\mu = c$$
   $$(1 - \phi_1 - \cdots - \phi_p)\mu = c$$
   $$\mu = \frac{c}{1 - \phi_1 - \cdots - \phi_p}$$
   This relation between c and μ is important for stationary time series.

   - Autocovariance function: For simplicity, assume $c = \mu = 0$.
   Multiplying both sides of the AR model by $Z_{t-\ell}$ and taking expectations, we have
   $$E[(Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p})Z_{t-\ell}] = E(Z_{t-\ell}a_t)$$
   For $\ell = 0$,
   $$\gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p = \sigma^2.$$
   For $\ell > 0$,
   $$\gamma_\ell - \phi_1\gamma_{\ell-1} - \cdots - \phi_p\gamma_{\ell-p} = 0.$$

   - Autocorrelation function (ACF): $\rho_k = \frac{\gamma_k}{\gamma_0}$. From autocovariance function, we have
   $$\rho_\ell - \phi_1\rho_{\ell-1} - \cdots - \phi_p\rho_{\ell-p} = \begin{cases} \frac{\sigma^2}{\gamma_0} & \text{for } \ell = 0 \\ 0 & \text{for } \ell > 0. \end{cases}$$
   Since the ACFs satisfy the p-th order difference equation, namely $\phi(B)\rho_\ell = 0$ for $\ell > 0$, they decay exponentially to zero as $\ell \to \infty$.

4. Yule-Walker equation: Consider the above equations jointly for $\ell = 1, \cdots, p$, we have

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_p \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{2-p} & \rho_{1-p} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{3-p} & \rho_{2-p} \\ \vdots & \vdots & & & & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \cdots & \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix}$$

which is the p-order Yule-Walker equation. For given ρ_ℓ's, the equation can be used to solve for ϕ_i. Of course, we can obtain ρ_ℓ's for given ϕ_i's and σ^2 via the moment generating function or the ψ-weight representation to be discussed shortly.

5. MA representation:

$$Z_t - \mu = \frac{1}{\phi(B)} a_t = \sum_{i=0}^{\infty} \psi_i a_{t-i}$$

where the ψ_i's are referred to as the ψ-weights of the model. These ψ-weights are also called the impulse response function and can be obtained from the ϕ_i's by equating the coefficients of B^i

$$\frac{1}{\phi(B)} = 1 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$1 = (1 - \phi_1 B - \cdots - \phi_p B^p)(1 + \psi_1 B + \psi_2 B^2 + \cdots)$$

Thus, we have

$$\begin{aligned} \psi_1 &= \phi_1 \\ \psi_2 &= \phi_1 \psi_1 + \phi_2 \\ \psi_3 &= \phi_1 \psi_2 + \phi_2 \psi_1 + \phi_3 \\ &\vdots \\ \psi_p &= \phi_1 \psi_{p-1} + \phi_2 \psi_{p-2} + \cdots + \phi_{p-1} \psi_1 + \phi_p \\ \psi_\ell &= \sum_{i=1}^{p} \phi_i \psi_{\ell-i} \quad \text{for} \quad \ell > p. \end{aligned}$$

Again, the ψ-weights satisfy the difference equation $\phi(B)\psi_\ell = 0$ for $\ell > p$ so that they also decay exponentially to zero as ℓ goes to infinite.

6. The moment generating function:

$$\Gamma(z) = \frac{\sigma^2}{\phi(z)\phi(z^{-1})}.$$

7. A simple example: the AR(1) case

$$Z_t - \phi_1 Z_{t-1} = c + a_t.$$

- Stationarity condition: $|\phi_1| < 1$.
- Mean: $\mu = \frac{c}{1-\phi_1}$.
- ACF: $\rho_\ell = \phi_1^\ell$.
- Yule-Walker equation: $\rho_1 = \phi_1 \rho_0 = \phi_1$.
- MA representation:

$$Z_t = \frac{c}{1-\phi_1} + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \phi_1^3 a_{t-3} + \cdots$$

  so that the ψ-weights are $\psi_\ell = \phi_1^\ell$.
- The variance of Z_t: $\mathrm{Var}(Z_t) = \frac{\sigma^2}{1-\phi_1^2}$.

8. An AR(2) model:

$$(1 - \phi_1 B - \phi_2 B^2) Z_t = c + a_t.$$

- Stationarity condition: Zeros of $\phi(B)$ are outside the unit circle.
- Mean: $\mu = \frac{c}{1-\phi_1 - phi_2}$.
- ACF: $\rho_0 = 1$, $\rho_1 = \phi_1/(1-\phi_2)$, and $\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$, for $j > 1$. Why?
- If $\phi_1^2 + 4\phi_2 < 0$, then the ACF exhibits a demping sine and cosine pattern. The model is said to exhibit some business cycle behavior. [Recall the solution of 2nd-order difference equation of Lecture 1.]
- Yule-Walker equation:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

- MA representation:

$$Z_t = \frac{c}{1-\phi_1-\phi_2} + a_t + \psi_1 a_{t-1} + \psi_1^2 a_{t-2} + \psi_1^3 a_{t-3} + \cdots$$

  where the ψ-weights satisfy

$$(1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1.$$

B. Moving-Average (MA) Processes

1. MA(q) Model: $Z_t = c + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$ or $Z_t = c + \theta(B) a_t$.
2. Stationarity: Finite order MA models are stationary.

3. Mean: By taking expectation on both sides, we obtain

$$E(Z_t) = c + E(a_t) - \theta_1 E(a_{t-1} - \cdots - \theta_q E(a_{t-q}).$$

$$\mu = c$$

Thus, for MA models, the constant term c is the mean of the process.

4. Invertibility: Can we write an MA model as an AR model? Consider

$$\frac{1}{\theta(B)}(Z_t - \mu) = a_t$$

Let $\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots = \frac{1}{\theta(B)}$. We call the π's the π weights of the process Z_t.

Similar to $\frac{1}{\phi(B)}$, for the above AR representation to be meaningful we require that

$$\sum_{i=1}^{\infty} \pi_i^2 < \infty.$$

The necessary and sufficient condition for such a convergent π-weight sequence is that all the zeros of $\theta(B)$ lie outside the unit circle.

5. Autocovariance function: Again for simplicity, assume $\mu = 0$.

Multiple both sides by $Z_{t-\ell}$ and take expectation. We have

$$\gamma_\ell = \begin{cases} (1 + \theta_1^2 + \cdots + \theta_q^2)\sigma^2 & \text{for } \ell = 0 \\ \sigma^2 \sum_{i=0}^{q-\ell} \theta_i \theta_{\ell+i} & \text{for } \ell = 1, \cdots, q \\ 0 & \text{for } \ell > q \end{cases}$$

where $\theta_0 = -1$. This shows that for an MA(q) process, the autocovariance function γ_ℓ is zero for $\ell > q$. This is a special feature of MA processes and it provides a convenient way to identify an MA model in practice.

6. Autocorrelation function: $\rho_\ell = \frac{\gamma_\ell}{\gamma_0}$. Again, from the autocovariance function we have

$$\rho_\ell = \begin{cases} \neq 0 & \text{for } \ell = q \\ 0 & \text{for } \ell > q. \end{cases}$$

In other words, the ACF has only a finite number of non-zero lags. Thus, for an MA(q) model, Z_t and $Z_{t-\ell}$ are uncorrelated provided that $\ell > q$. For this reason, MA models are referred to as short memory time series.

7. The moment generating function:

$$\Gamma(z) = \sigma^2 \theta(z)\theta(z^{-1}).$$

8. A simple example: the MA(1) case

$$Z_t = c + a_t - \theta a_{t-1}$$

- Invertibility condition: $|\theta| < 1$.
- Mean: $E(Z_t) = \mu = c$.
- ACF:

$$\rho_\ell = \begin{cases} 1 & \text{for } \ell = 0 \\ \frac{-\theta}{1+\theta^2} & \text{for } \ell = 1 \\ 0 & \text{for } \ell > 1. \end{cases}$$

  The above result shows that $\rho_1 \leq 0.5$ for an MA(1) model.
- Variance: $\gamma_0 = (1+\theta^2)\sigma^2$.
- Moment generating fucntion:

$$\Gamma(z) = [(1+\theta^2) - \theta(z + z^{-1})]\sigma^2.$$

From the MGF, we have $\gamma_0 = (1+\theta^2)\sigma^2$, $\gamma_1 = -\theta\sigma^2$, and $\gamma_j = 0$ for $j \geq 2$.

C. Mixed ARMA Processes

1. ARMA(p, q) Model: $Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = c + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$ or $\phi(B)Z_t = c + \theta(B)a_t$. Again, for a stationary process, we can rewrite the model as

$$\phi(B)(Z_t - \mu) = \theta(B)a_t.$$

2. Stationarity: All zeros of $\phi(B)$ lie outside the unit circle.
3. Invertibility: All zeros of $\theta(B)$ lie outside the unit circle.
4. AR representation: $\pi(B)Z_t = \frac{c}{\theta(1)} + a_t$, where

$$\pi(B) = \frac{\phi(B)}{\theta(B)}.$$

   The π-weight π_i can be obtained by equating the coefficients of B^i in

$$\pi(B)\theta(B) = \phi(B).$$

5. MA representation: $Z_t = \frac{c}{\phi(1)} + \psi(B)a_t$, where

$$\psi(B) = \frac{\theta(B)}{\phi(B)}.$$

   Again, the ψ-weights can be obtained by equating coefficients.

   Note that $\psi(B)\pi(B) = 1$ for all B for an ARMA model. This identity has many applications.

6. Moments: (Assume stationarity)

   - Mean: $\mu = \frac{c}{1-\phi_1-\cdots-\phi_p}$.
   - Autocovariance function: (Assume $\mu = 0$) Using the result

$$E(Z_t a_{t-\ell}) = \begin{cases} \sigma^2 & \text{for } \ell = 0 \\ \psi_\ell \sigma^2 & \text{for } \ell > 0 \\ 0 & \text{for } \ell < 0, \end{cases}$$

     and the same technique as before, we have

$$\gamma_\ell - \phi_1\gamma_{\ell-1} - \cdots - \phi_p\gamma_{\ell-p} = \begin{cases} (1-\theta_1\psi_1 - \cdots - \theta_q\psi_q)\sigma^2 & \text{for } \ell = 0 \\ -(\theta_\ell + \theta_{\ell+1}\psi_1 + \cdots + \theta_q\psi_{q-\ell})\sigma^2 & \text{for } \ell = 1, \cdots, q \\ 0 & \text{for } \ell \geq q+1 \end{cases}$$

     where $\psi_0 = 1$ and $\theta_j = 0$ for $j > q$.
   - Autocorrelation function: ρ_ℓ satisfies

$$\rho_\ell - \phi_1\rho_{\ell-1} - \cdots - \phi_p\rho_{\ell-p} = 0 \quad \text{for} \quad \ell > q.$$

     You may think that the ACF satisfy the difference equation $\phi(B)\rho_\ell = 0$ for $\ell \geq q+1$ with $\rho_1, \cdots, \rho_q$ as initial conditions.

7. Generalized Yule-Walker Equation: Consider the above equations of ACF for $\ell = q+1, \cdots, q+p$, we have

$$\begin{bmatrix} \rho_{q+1} \\ \rho_{q+2} \\ \vdots \\ \rho_{q+p} \end{bmatrix} = \begin{bmatrix} \rho_q & \rho_{q-1} & \cdots & \rho_{q+2-p} & \rho_{q+1-p} \\ \rho_{q+1} & \rho_q & \cdots & \rho_{q+3-p} & \rho_{q+2-p} \\ \vdots & & & & \vdots \\ \rho_{q+p-1} & \rho_{q+p-2} & \cdots & \rho_{q+1} & \rho_q \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix}$$

   which is referred to as a p-th order generalized Yule-Walker equation for the ARMA(p, q) process. It can be used to solve for ϕ_i's given the ACF ρ_i's.

8. Moment generating function:

$$\Gamma(z) = \frac{\theta(z)\theta(z^{-1})}{\phi(z)\phi(z^{-1})}\sigma^2$$

9. A simple example: The ARMA(1,1) case

$$Z_t - \phi Z_{t-1} = c + a_t - \theta a_{t-1}$$

   - Stationarity condition: $|\phi| < 1$.
   - Invertibility condition: $|\theta| < 1$.

- Mean: $\mu = \frac{c}{1-\phi}$.
- Variance: $\gamma_0 = \frac{\sigma^2(1+\theta^2-2\phi\theta)}{1-\phi^2}$
- ACF:

$$\rho_1 = \frac{(1-\phi\theta)(\phi-\theta)}{1+\theta^2-2\phi\theta},$$
$$\rho_\ell = \phi\rho_{\ell-1} \quad \text{for} \quad \ell > 1.$$

10. AR representation:

$$Z_t = \frac{c}{1-\theta} + \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots + a_t$$

where $\pi_i = \theta^{i-1}(\phi-\theta)$.

11. MA representation:

$$Z_t = \frac{c}{1-\phi} + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

where $\psi_i = \phi^{i-1}(\phi-\theta)$.