Department of Mathematics
Federal University Lafia
Course Code: MTH 223
Course Title: Introduction to Numerical Analysis
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Solution of equations $f(x) = 0$ by integration.

To find a solution of the simple equation $f(x) = 0$………..(1), where f is a given function.

A solution of (1) is a number $x = 5$ such that $f(s) = 0$.

To solve (1) when there is no formula for exact solution available, we can use an approximate method, such as an iteration method. An iteration method is a method in which we start from an initial guess x_0 and compute step by step approximations x_1, x_2……….of an unknown solution of (1).

(a). Fixed-point Iteration for solving equations $f(x) = 0$. By some algebraic steps, we transform $f(x) = 0$ into the form $x = g(x)$……………………(2). Then we choose $a_n x_0$ and compute $x_1 = g(x_0), x_2 = g(x_1)$, and in general $x_{n+1} = g(x_n)$…………(3) $n = 0, 1 \dots \dots \dots \dots \dots$

A solution of $x = g(x)$ is called a fixed point of g, motivating the name of the method.

Example:

Set up an iteration process for the equation $f(x) = x^2 - 3x + 1 = 0$

The solution is $x = \frac{3 \pm \sqrt{9-4}}{2}$

$x = 2.618034 \ and \ 0.381966.$

We can watch the behaviour of the error as the iteration proceeds.

Solution: The equation may be written as $x = g(x)$, i.e. $x^2 + 1 = 3x$

= $x = \frac{1}{3}(x^2 + 1) = \frac{1}{3}(x^2{}_n + 1)$

Therefore, $x_{n+1} = \frac{1}{3}(x^2{}_n + 1)$

If we choose $x_0 = 1.000$, we get $x_1 = 0.667, x_2 = 0.481, x_3 = 0.411, x_4 = 0.390$, and so on, which seems to approach the smallest solution.

Choose $x_0 = 2$ and $x_0 = 3$

(b). our equation $f(x) = x^2 - 3x + 1 = 0$, may be written as

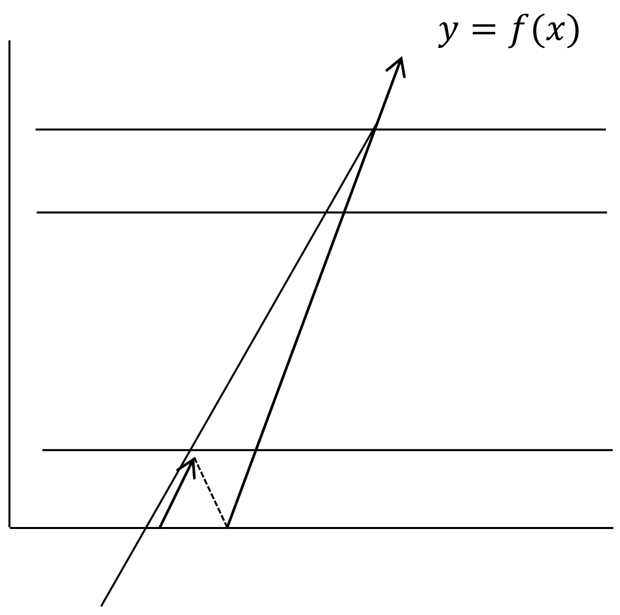
$$g_1(x) = 3 - \frac{1}{x} = x$$

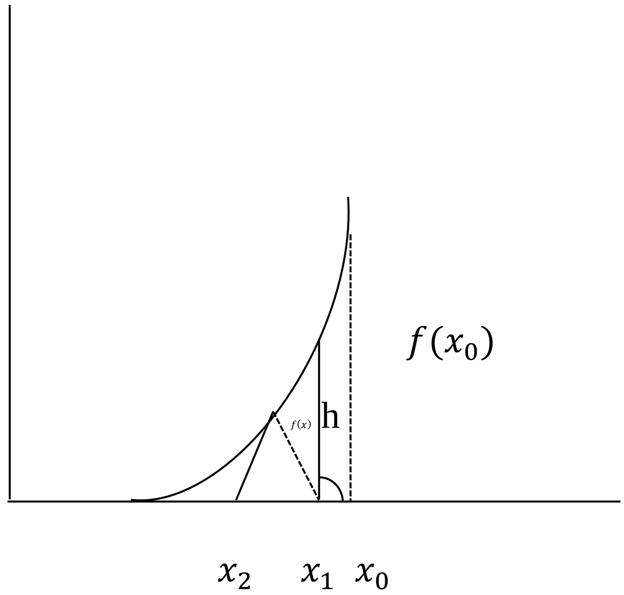
Therefore, $x_{n+1} = 3 - \frac{1}{x_n}$

Choose $x_0 = 1, x_0 = 3$

Definition: An iteration process defined by $x_{n+1} = g(x_n)$ is called convergent for an x_0 if the corresponding sequence x_0, x_1……….is convergent.

(b). Newton's method of solving equations $f(x) = 0$





$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \left(\frac{f(x_0)}{h}\right) / \frac{(x_0 - x_1)}{h} = \frac{f(x_0)}{x_0 - x_1}$$

$$= f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly we have,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_0)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}\text{s}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is called Newton's method or Newton's-Raphson's method.

Example: (Square root)

Set up a Newton iteration for computing the square root of a given positive number and apply it to $c = 2$.

Solution: Let $= \sqrt{c}$, hence, $f(x) = x^2 - \sqrt{c}$ = 0

$f'(x) = 2x$.

Using $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

$=x_n - \frac{x^2{}_n - c}{2x_n}$

$=x_n - \frac{1}{2}\left(x_n - \frac{c}{x_n}\right)$

For $c = 2$, choose $x_0 = 1$, we get $x_1 = 1.50000, x_2 = 1.416667, x_3 = 1.414216$

Example: Iteration for transcendental Equation

Find the positive solution of $2sinx = x$.

Solution: setting$f(x) = x - 2sinx,$

We have $f(x) = 1 - 2cosx$ and using

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, gives

$$x_{n+1} = x_n - \frac{x_n - 2s_n x_n}{1 - 2cosx_n}$$

$= \frac{x_n\,(1-2cosx_n) - x_n - 2sinx_n}{1-2cosx_n}$

$= \frac{-2x_n\,cosx_n - 2sinx_n}{1-2cosx_n}$

$= \frac{2[x_n cosx_n + sinx_n]}{1-2cosx_n}$

Let $x_o = 2.00000$

n	x_n	N_n	D_n	$x'_n = 1$
0	2.00000	3.48318	1.83229	1.90100
1				
2				
3				

Example: Newton's method applied to an Algebraic Equation.

Apply newton's method to the equation

$f(x) = x^3 - x - 1 = 0$

Solution: $f'(x) = 3x^2 - 1$

Using $x_{n+1} = x_n - \frac{f(x_n)}{f\prime(x_n)}$, we have

$$x_{n+1} = x_n = \frac{x_n^3 + x_n - 1}{3x^2{}_n - 1}$$

$= \frac{x_n\ (3x^2{}_n - 1) - x^3{}_n + x_n + 1}{3x^2{}_n - 1}$

$= \frac{3x^3{}_n - x_n - x^3{}_n + x_x + 1}{3x^2{}_n - 1}$

$= \frac{2x^3{}_n + 1}{3x^2{}_n - 1}$

Choosing$x_o = 1$, calculate

$x_1 = 0.750000$, $x_2 = 0.686047$, x_3?,

$x_3 =$?

Secant method for solving $f(x) = 0$.

The secant method is given as

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n - f(x_{n+1})} = \frac{N_n}{D_n}$$

It is not good to write the above equation as

$x_{n+1} = \frac{x_{n+1}f(x_n) - x_n f(x_{n-1})}{f(x_n - f(x_{n-1})}$, because this may lead to loss of significant digits if x_n and x_{x-1} are about equal.

Exercise:

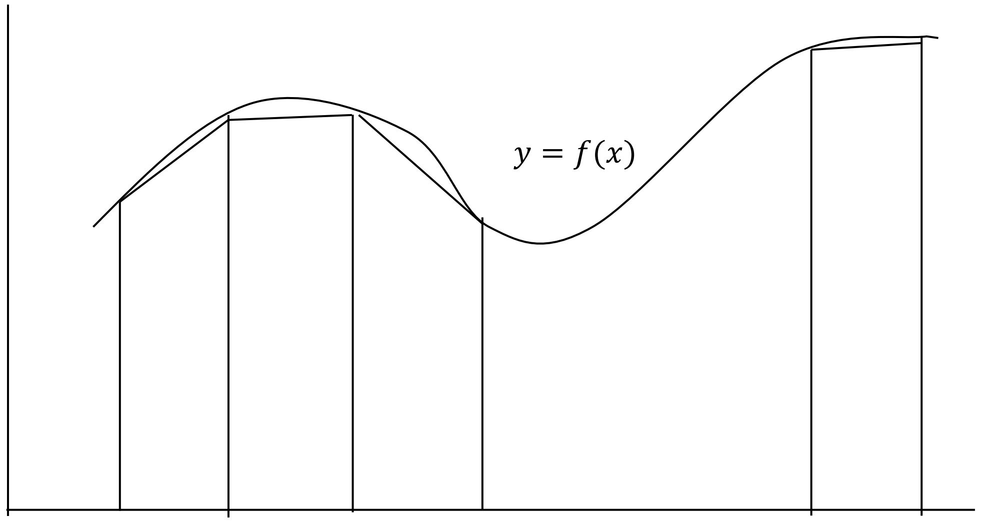
Find the positive solution of $f(x) = x - 2sinx = 0$, by the secant method n

n	x_{n-1}	p_n	$x_{n-1} - x_n$	

Numerical Integration

Trapezoidal Rule:

Consider the diagram below:



The area of the trapezium on the internal $[x_{i-1}, x_i]$ is $\frac{1}{2}[f(x_{i-1}) + f(x_i)]\ \Delta x_i$, $\Delta x_i = x_i - x_{i-1}\ for\ i = 1, 2, 3, \dots \dots \dots n.$

Adding, the n estimated contributions of the n trapezia will give $\frac{1}{2}[f(x_0)+f(x_1)]\Delta x_i+\frac{1}{2}[f(x_1)+f(x_2)]\Delta x_2+\frac{1}{2}[f(x_2)+f(x_3)]\Delta x_3+\cdots+\frac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x_n$

The exact area under the shaded area is

$\int_a^b f(x)dx$…………..(2)

: - $\int_a^b f(x)dx=\frac{1}{2}[f(x_o)+f(x_1)]\Delta x_1+\frac{1}{2}[f(x_1)+f(x_2)]\Delta x_2+\cdots+\frac{1}{2}[f(x_{n-1})+f(x_n)]\Delta x_n$ … … … … (3)

Equation (3) is called the general trapezoidal rule. If the interval $[a,b]$ is divided into n equal parts of length $h=\frac{(b-a)}{n}$ $i.e.\Delta x_i=x_i-x_{i-1}=h$, for $i=1,2,\dots\dots n$, therefore, eqn(3) for equal intervals, is given by

$$\begin{aligned}\int_a^b f(x)dx &= \frac{1}{2}h[f(x_0)+f(x_1)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+f(x_{n-1})\\&\quad+f(x_n)]\\&= h\left[\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_{n-1})\right.\\&\quad\left.+\frac{1}{2}f(x_n)\right] \dots\dots\dots.(4)\end{aligned}$$

Exercises:

a. Calculate the definite integral $I=\int_1^2\frac{dx}{x}$ by the trapezoidal rule, taking
    i. Ten integration steps,
    ii. Five integration steps.

Hint: use the table below and use $I=\int_1^2\frac{dx}{x}$ to get the starting value of x.

x	$f(x)=\frac{1}{x}$	

Simpson's Rule:

Now, we divide $[a, b]$ into an n even number of sub integrals of equal length $h = \frac{b-a}{n}$ and then approximating the function over consecutive pairs of sub-integrals by a quadratic polynomial.

Consider the first consecutive pair of sub-integrals $[a, a + 2h]$ and represent the function $y = f(x)$ in this interval by the quadratic

$y = c_0 + c_1x + c_2x^2 \ldots\ldots..(1)$.

Then the first five approximation in interval is $\int_a^{a+2h} ydx = \int_a^{a+2h}[c_0 + c_1x + c_2x^2]\,dx = \left[c_0x + \frac{c_1x^2}{2} + \frac{c_2x^3}{3}\right] = c_0 2h + \left[\frac{a^2 \pm 4_{a^2}h^2 + 4ah}{2}\right] c_1 + c_2 \frac{[(a+2h)^2 - a^3]c_2}{3}$ ….(2)

In order that the quadratic should pass through the three points at $x = a, x = a + h, and\ x = a + 2h,$ it means

$f(a) = c_0 + c_1a + c_2a^2 \ldots..(3)$

$f(a + h) = c_0 + c_1(a + h) + c_2(a + h)^2 \ldots..(4)$

$f(a + 2h) = c_0 + c_1(a + 2h) + c_2(a + 2h)^2 \ldots\ldots(5)$

To find $c_0, c_1\ and\ c_2$ we must solve (3) to (5) simultaneously.

From (3) $c_0 = f(a) - c_1a - c_2a^2 \ldots\ldots\ldots\ldots.(6)$

Therefore, (4) & (5) become

$f(a + h) = f(a) - c_1a - c_2a^2 + c_1(a + h) + c_2(a + h)^2 \ldots\ldots\ldots..(7)$

$f(a + 2h) = f(a) - c_1a - c_2a^2 + c_1(a + 2h) + c_2(a + 2h)^2 \ldots\ldots.(8)$

We now solve (7) & (8) simultaneously to get the values of $c_1\ and\ c_{2.}$

We now substitute $c_1\ and\ c_2$ into (6) to get the value of c_0.

Now substitute the values of $c_0, c_1\ and\ c_2$ into eqn (2), we get

$$\int_a^{a+2h} f(x)dx = \frac{h}{3}[f(a) + 4f(a + h) + f(a + 2h)]$$

Now for the interval $[a + 2h, \quad a + 4h]$ which is the next interval with n even.

$$\int_{a+2h}^{a+4h} f(x)dx = \frac{h}{3}[f(a + 2h) + 4f(a + 3h + f(a + 4h)]$$

Numerical Solution of Ordinary Differential Equations:

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary function of x. such a solution is known as the closed of finite form of solution.

When such a solution is absent, we have recourse to numerical methods of solution.

We shall consider the first order differential equation $\frac{\mathrm{dy}}{\mathrm{dx}} = f(x, y)$, given $y(x_0)$ $=y_0$,

To study the various numerical methods of solving such equations.

a. **Picard's Method.**

Consider the first order equation $\frac{\mathrm{dy}}{\mathrm{dx}} = f(x, y)$…………………..(1)

Integrating (1) with respect to x, we get

$\int \frac{\mathrm{dy}}{\mathrm{dx}} dx = \int f(x, y) dx$ ……………………………………………………...(2)

Now evaluate the R.H.S between x_0 to x and the L.H.S from y_0 to y, we get

$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y) dx$ ……………………………………………………..(3)

$y - y_0 = \int_{x_0}^{x} f(x, y) dx$

$y = y_0 + \int_{x_0}^{x} f(x, y) dx$ ……………………………………………………...(4)

We shall now use equation (4) to calculate a number of approximations to the value of y in equation (4).

For a first approximation, y_1, we put $y = y_0$ in $f(x, y)$ in equation (4) we get

$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)$

Second approximation on y_2,

$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1) dx$

$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2) dx$

So that

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$

This method, called Picard's Method gives a sequence of approximation y_1, y_2………., y_n. each giving a better result than the preceding one.

Example/ Exercise

a. Using Picard's process of successive approximations, obtain a solution up to the fifth approximation of the equation
$\frac{dy}{dx} = y + x$, such that $y = 1$ when $x = 0$

b. Find the first approximate value of y for $x = 0.1$ by Picard's method given that
$\frac{dy}{dx} - \frac{(y-x)}{y+x} = 0, y(0) = 1.$

Taylor's Method

Consider the first order equation

$$\frac{dy}{dx} = f(x, y) \quad \text{……………………………..(1)}$$

Differentiate (1) we get

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$y'' = f_x + f_y\, f' \quad \text{………………………… (2)}$$

If we repeat the differentiation, we get

$$y''' = f_{xx} + f_{yy}\, f'' \text{, etc.}$$

The Taylor's series for $f(x) = y$, is given by

$$y = y_0 + xy' + \frac{x^2}{2!}y'' + \frac{x^3}{3!}y''' + \text{………………….. (3)}$$

Putting $x = x_0$ and $y = 0$

The values of $(y')_o$, $(y'')_o$, etc can be obtained.

Hence, the Taylor's series,

$y = y_0 + (x - x_0)\,(y')_o + \frac{(x-x)_o{}^2}{2!}\,(y'')_o + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

Gives the values of y for every values of x for which the Taylor's series converges.

On finding the value y_1 for $x = x_1$ from (4), y', y'', etc can be evaluated at

$x = x_1$ by means of their expressions.

Example/Exercise

(a). Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five decimal places, from $\frac{dy}{dx} = x^2\,y - 1,\ y(0) = 1.$

Solution: $y(0) = 1 = y_0 = 1,\ x_0 = 0$

$\frac{dy}{dx} = x^2\,y - 1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Differentiate (1), several times, say,

$y'' = 2xy + x^2y' = (y'')_o = 0$

$y''' = 2y + 4xy' = (y''')_o = 2$

$y^{iv} = 6y' + 6xy'' + x^2\,y''' = \left(y^{iv}\right)_o = -6,$ etc.

Putting these values in T. S, i.e. $y = y_0 + (x - x_0)\,(y')_o + \frac{(x-x)_o{}^2}{2!}\,(y'')_o$ we get

$y = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6)$

$= 1 - x + \frac{x^3}{3!} - \frac{x^4}{4!} + ..$

Hence, at $x = 0.1,$

i.e. $y(0.1) = 1 - 0.1 + \frac{(0.1)^3}{3!} - \frac{(0.1)^4}{4!} + \ldots\ldots$

$= 0.90033$

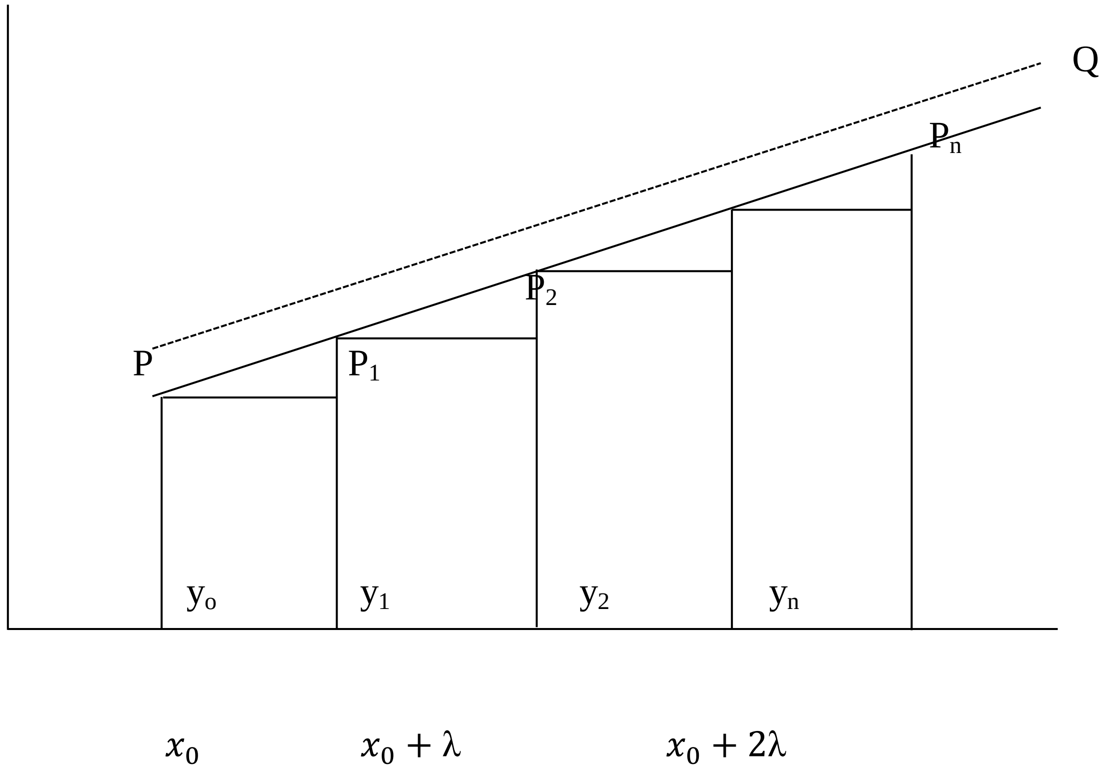
Find that $y(0.2) = 0.80227$

(b). Employ Taylor's method to obtain approximate value of $y \; at \; x = 0.2$ for the differential equation $\frac{dy}{dx} = 2y + 3e^{x}, y(0) = 0.$ Compare the numerical solution obtained with the exact solution.

Euler's Method

Consider the equation $\frac{dy}{dx} = f(x, y),$ given that $y(x_0) = y_0$

Let its curve of solution be the dotted lines PQ in the diagram below.



$y_1 = y_0 + R_1P_1$

$= y_0 + y_1 - y_0$

i.e. $R_1P_1 = y_1 - y_0$

$$\frac{R_1\,P_1}{x_0 + h - x_0} = \frac{y_1 - y_0}{x_0 + h - x_0}$$

$= \frac{R_1P_1}{h} = \frac{y_1 - y_0}{x_0 + h - x_0} = \left(\frac{dy}{dx}\right)$ at $P(x_0y_0)$

Therefore, $y_1 = h\left(\frac{dy}{dx}\right)P$:

$y_1 = y_0 + h\,f(x_0y_0)$.

Similarly,$y_2 = y_1 + h\,f(x_1y_1), where\ x_1 = x_0 + h$

$y_3 = y_2 + h\,f(x_2, y_2), \quad x_2 = x_1 + h$

Up to $y_n = y_{n-1} + h\,f(x_0 + (n-1)h, y_{n-1}) = x_0 + 2h$

Example/Exercise

(a). Using Euler's method, find an approximate value of y corresponding to $x = 1,\ given\ that\ \frac{dy}{dx} = x + y\ and\ y(0) = 1.$

Solution: $y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1})$

= $y_{n-1} + h\left(\frac{dy}{dx}\right)\ at\ (x_{n-1}, y_{n-1})$

= New $y = old\ y + h\left(\frac{dy}{dx}\right), h = \frac{1-0}{n}$ from the interval (0, 1), take

$n = 10,\ \ h = \frac{1}{10} = 0.1$

x	y	$x + y = \dfrac{dy}{dx}$	New $y = old\ y + h\dfrac{dy}{dx}$
0.0	1.00 (given)	1.00	$y_n = y_{n-1} + h + \dfrac{dy}{dx}$
0.1			
0.2			
0.3			
0.4			
0.5			
0.6			
0.7			
0.8			
0.9			
0.1			

(b). Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, with initial condition $y = 1\ at\ x = 0, find\ y\ for\ x = 0.1$ by Euler's method. Take $n = 5$.

Modified Euler's Method

In the Euler's method, the curve of solution for an interval is approximated by a tangent, i.e. the curve of solution between $P(x_0, y_0)\ and\ P_1(x_1, y_1)$ is approximated by the tangent at P.

Now, we shall find a better approximation $y_1^{(1)}$of $y(x_1)$ by taking the slope of the curve as the mean (average) of the slopes of the tangents at P and P_1 since the slope at $P(x_1, y_1)$ is $f(x_1, y_1)$ we then have $y_1 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$…(1)

By Euler's method
$y_1 = y_0 + h\,f(x_0, y_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

And insert y_1 in the R. H. S of (1) to get the better value of $y_1^{(2)}$ i.e.

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$……………………….(3)

We repeat this step, till two consecutive values (i.e. find $y^{(3)}$, $y^{(4)}$….., $y^{(n)}$)till two consecutive values of y agrees. This is then taken as the starting point for the next interval. Once y_1 obtained to desired degree of accuracy, y corresponding to $x_0 + 2h$ $is\ found\ from$ $y_1 = y_0 + h\, f(x_0, y_0)$

Next step

$y_2 = y_1 + h\, f(x_1, y_1)$ and a better approximation $y_2^{(1)}$ is obtained from (3) we have

$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$………………….(4)

We repeat this step until y_2 becomes stationery. Then we proceed to calculate y_3 as above, etc.

Example/Exercise

(a) Using modified Euler's method, find an approximate value of $y\ when\ x = 0.3$, given that $\frac{dy}{dx} = x + y\ and\ y = 1\ when\ x = 0.$ Take $h = 0.1.$

(b)

x	y	$x + y = \frac{dy}{dx}$	Mean slope	$Old\ y + 0.1$ X mean slope $N_e y$
For 1st interval 0.0				
0.1 0.1 0.1				
For 2nd interval 0.1 0.2 0.2 0.2				
For 3rd interval 0.2 0.3 0.3 0.3				