BIMCT Team Round

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October 2022

Team Round

1. What is the coefficient of the $(xyz)^3$ term in the expansion of $(x+2y+3z)^9$?

2. Let ABC be a triangle such that $AB = 5,\ AC = 6,$ and $BC = 7$. Let D be the foot of the altitude from A to BC, and let E be the midpoint of AD. Define E' as the reflection of E across C. If the area of $\triangle BEE'$ can be expressed as $a\sqrt{b}$, for an integer a and squarefree b, what is $a+b$?

3. An evil number is a positive integer that has an even sum of digits when expressed in binary. Find the number of evil numbers less than 2021.

4. Let a, b, and c be the 3 solutions to the equation $x^3 + 9x^2 - 98x + 27 = 0$. Find the sum $\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}$.

5. For how many many ordered pairs of (not necessarily distinct) positive integers $1 \le a, b \le 10000$ is $ab \equiv 2022 \pmod{10000}$?

6. Real numbers x and y are chosen uniformly at random from the interval $[-1, 1]$. If the probability that they satisfy:
$$x^2 + 2|x| + y^2 + 2|y| \le 2$$
is p, p can be expressed as $\frac{a\pi}{b} + c\sqrt{d} + e$, where $a, b, c, d,$ and e are integers, $\gcd(a, b) = 1$, and d is not divisible by the square of any prime number. Find $a+b+c+d+e$.

7. Real numbers $w, x, y,$ and z satisfy $wxyz = 36$. Over all combinations of $w, x, y,$ and z, what is the minimum value of $6w^2 + 3x^4 + y^{12} + 2z^6$?

8. Evan has a paper icosahedron. He fully cuts the icosahedron along a series of edges, laying it out on a flat table all in one piece. How many edges did he cut? Recall that an icosahedron is a regular polyhedron with equilateral triangular faces, with 20 faces, 30 edges, and 12 vertices.

9. Let x and y be numbers such that
$$x + y = \sqrt{x^2 - 8x + 19} + \sqrt{y^2 - 8y + 19} = \sqrt{x^2 - 24x + 147} + \sqrt{y^2 - 24y + 147}$$
Then, $\max(x, y) = a + \sqrt{b}$, where a and b are integers. Find $a+b$.

10. Let acute triangle ABC be such that $AB = 13$, $AC = 15$, and $BC = 14$. Let D be the foot of the altitude from A, let M be the midpoint of BC, and let O be the circumcenter of $\triangle ABC$. Then, let X be the unique point within $\triangle ABC$ such that X is on line OM and $\overline{AO} = \overline{DX}$. If E is the foot of the altitude from B, XE can be expressed as $\frac{a}{b}$, where a and b are relatively prime. Find $a+b$.

11. Let a sequence a_n be defined by $a_{n+2} = 9a_n - 3a_{n+1}$. If $a_{2021} = 1$ and $a_{2022} = 2$, $\sum_{n=0}^{2020} a_n$ can be expressed as $a.bc\cdots$. Find $100a + 10b + c$.

12. Two identical regular hexagons are inscribed within a rectangle with side lengths $115\sqrt{3}$ and 36 without overlapping. If the maximum possible area of one of the hexagons is can be expressed as $a\sqrt{b}$, find $a+b$.

Note: This problem was incorrectly copied over. As such, it was dropped. The correct problem statement is as follows:

Two identical regular hexagons are inscribed within a rectangle with side lengths 116 and $36\sqrt{3}$ without overlapping. If the maximum possible area of one of the hexagons is can be expressed as $a\sqrt{b}$, find $a+b$.

13. Let a list s of $2k$ integers be considered $k-pretty$ if it has median k, mode k, mean k, and standard deviation k. If S is the set of all $k-pretty$ lists, what is $E(x^2)$, where $x \in s$ and $s \in S$ is chosen uniformly at random between all lists within S? (Note that the set S is finite, since the standard deviation puts an upper bound on the largest value within s).

Note: This problem was incorrectly copied over. As such, it was unsolvable and dropped. The correct problem statement is as follows:

Let a list s of $2k$ integers be considered $k-pretty$ if it has median k, mode k, mean k, and standard deviation k. If S is the set of all $2023-pretty$ lists, what is $E(x^2)$, where $x \in s$ and $s \in S$ is chosen uniformly at random between all lists within S? (Note that the set S is finite, since the standard deviation puts an upper bound on the largest value within s).

14. Bob has a fair 6-sided die, numbered 1 through 6. Bob rolls the die until he rolls the sequence, $123456123456\cdots123456$, where 123456 is repeated 2022 times. If the expected number of rolls it takes for Bob to roll that sequence is E, find $E \pmod{1000}$.

15. John is in charge of a math class with 99 perfectly logical students, numbered 1 through 99, with all students knowing each other's numbers. One day, he thinks of a number x, and comes up with the following challenge: To each student, John tells them $x \pmod{i+1}$, where i is the number corresponding to that student. Then, John tells the entire class that $x \leq 150$. John then performs the following operation until at least half the class (50 students) raise their hands. He asks the entire class, "Raise your hand if you know what the value of x is!" Let $f(x)$ be the number of times John needs to perform the operation until half the class raises their hands, if the value he thought of was x. What is $\sum_{i=1}^{150} f(i)$?