# BIMCT Individual Round High School Division Solutions

BIMCT Team

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# Individual Round Solutions

1. Alice and Bob share the same birthday. Six years ago, Alice was 5 times as old as Bob. If Alice is only 3 times as old as Bob now, how old will Bob be when Alice is 57 years old? *Problem Proposed by Ary Cheng*

   **Solution:** $\boxed{33}$

   Let Alice's age (in years) today be $a$, and Bob's age (in years) today be $b$. This means that 6 years ago, Alice was $a - 6$ years old, and Bob was $b - 6$ years old. With the information given, we can write the following 2 equations:

   $$a - 6 = 5(b - 6)$$
   $$a = 3b$$

   Substituting $a = 3b$ into the first equation, we get:

   $$3b - 6 = 5(b - 6)$$
   $$\longrightarrow 2b = 24 \longrightarrow b = 12$$
   $$\longrightarrow a = 3b = 36 \longrightarrow a - b = 36 - 12 = 24$$

   Therefore when Alice is 57 years old, Bob will be $57 - 24 = \boxed{33}$ years old. *Solution by Ary Cheng*

2. Workers Charlie and Doyle take 40 minutes and 60 minutes to paint a wall by themselves, respectively. When they work together, they each work 50% slower because they constantly talk. If $N$ is the time in minutes it takes for Charlie and Doyle to paint a wall together, find $N$. *Problem Proposed by Derrick Liu*

   **Solution:** $\boxed{36}$

   When Charlie and Doyle work together, it will take them $(1.5) \cdot 40 = 60$ and $(1.5) \cdot 60 = 90$ minutes to paint a wall by themselves, respectively. Therefore, their rate in walls per hour are 1 and $\frac{2}{3}$, respectively. Adding their rates, we see that it takes $\frac{3}{5}$ of an hour for Charlie and Doyle working together to paint a wall, which is equal to $\boxed{36}$ minutes. *Solution by Derrick Liu*

3. Bob rolls two standard 6-sided dice and adds the sum of the faces. If the probability that the sum of the faces is a perfect square can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime, find the value of $a + b$. *Problem Proposed by Ary Cheng*

**Solution:** $\boxed{43}$

Notice that the only perfect squares that can be made are 4 and 9. There are 3 combinations that can add to 4, namely (1,3), (2,2), and (3,1). There are 4 combinations that can add to 9, namely (3,6), (4,5), (5,4), and (6,3). This means there are a total of $3 + 4 = 7$ combinations that add to a perfect square, and since there are $6 \cdot 6 = 36$ total possible combinations, the desired probability is $7/36$, which gives us an answer of $7 + 36 = \boxed{43}$. *Solution by Ary Cheng*

4. Amanda has a pile of $n$ candies that she wants to distribute to 4 friends. The $i-$th friend will receive $\frac{1}{i+1}$ of the remaining candies that Amanda has and will be the $i-$th person to receive their share of the candy. What is the minimum value of $n$ such that each friend receives an integer amount of candies? *Problem Proposed by Derrick Liu*

   **Solution:** $\boxed{60}$

   Consider how many of the original $n$ candies each friend receives. We can write each amount out as such:

$$
\begin{aligned}
i = 1 &\longrightarrow \frac{1}{2}n \\
i = 2 &\longrightarrow (1 - \frac{1}{2})\frac{1}{3}n = \frac{1}{6}n \\
i = 3 &\longrightarrow (1 - \frac{1}{2} - \frac{1}{6})\frac{1}{4}n = \frac{1}{12}n \\
i = 4 &\longrightarrow (1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{12})\frac{1}{5}n = \frac{1}{20}n
\end{aligned}
$$

   We see that $n$ is both a multiple of 20 and 12, so our answer is $\text{LCM}(20, 12) = \boxed{60}$ *Solution by Derrick Liu.*

5. 7 students are voting for one of two activities. Each student independently votes for one of the activities, with a 1/2 chance of picking either activity. The teacher then selects the activity with more votes. If the expected number of people who voted for the activity selected is $\frac{a}{b}$, what is the value of $a + b$? *Problem Proposed by Adam Tang*

   **Solution:** $\boxed{179}$

   WLOG, assume that activity one is selected. (Note that, at the end, we must multiply our answer by 2 because there are 2 choices for which activity wins). Then, activity one received 4, 5, 6, or 7 votes. Now, the problem is reduced to simple casework.

   4 votes: $\frac{1}{2^7} \cdot \binom{7}{3} \cdot 4 = \frac{140}{128}$

   5 votes: $\frac{1}{2^7} \cdot \binom{7}{2} \cdot 5 = \frac{105}{128}$

   6 votes: $\frac{1}{2^7} \cdot \binom{7}{1} \cdot 6 = \frac{42}{128}$

   7 votes: $\frac{1}{2^7} \cdot 7 = \frac{7}{128}$

   $\frac{140+105+42+7}{128} \cdot 2 = \frac{147}{32} \longrightarrow 147 + 32 = \boxed{179}$. *Solution by Derrick Liu*

6. Zack is walking along a number line from 0 to 10. He starts at position 4. Every minute, he flips a coin. If the coin shows heads, he moves forward by 1, and if the coin shows tails, he moves backward by 1.

If the probability that he arrives at 10 before he arrives at 0 can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime, find $a+b$. *Problem Proposed by Rohan Das*

**Solution:** $\boxed{25}$

**Solution 1:** Let $p_i$ denote the probability of reaching position 10 before position 0 when he starts from position $i$. Clearly, we have $p_0 = 0$ and $p_{10} = 1$. Moreover, for $1 \le i \le 9$, the probability of moving to position $i-1$ is equal to the probability of moving to position $i+1$, which implies $p_i = \frac{1}{2}(p_{i-1} + p_{i+1})$ or

$$p_{i+1} - 2p_i + p_{i-1} = 0$$

This is a linear recurrence, so we can solve the characteristic equation $r^2 - 2r + 1 = 0$ for the general solution. Since the equation has a repeated root of 1, the general solution will be of the form $p_n = (an+b)(1)^n = an + b$ for arbitrary constants $a$ and $b$. Applying our edge conditions of $p_0 = 0$ and $p_{10} = 1$, we obtain the equations $0 = b$ and $1 = 10a$ respectively, so we have $p_i = i/10$. The problem asks for $p_4$, which is $^25$. Therefore, our final answer is $10 \cdot 2 + 5 = \boxed{25}$. *Solution by Felix Liu*

**Solution 2:** At each step, the expected value of Zack's displacement is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$. Thus the expected value of Zack's placement at any time is always 4. Consider the time when Zack reaches position 0 or 10. Let $p$ be the probability that he is at position 10. Then the expected value is $10p + 0(1-p) = 10p = 4$, so $p = \frac{2}{5}$. Then the final answer is $10 \cdot 2 + 5 = \boxed{25}$. *Solution by Rohan Das*

7. In triangle $ABC$, altitudes $AD$, $BE$, and $CF$ intersect at $H$. Given that $AB = 24$, $AH = 20$, and $BH = 8$, find the value of $AE \cdot EC$. *Problem Proposed by Derrick Liu*

   **Solution:** $\boxed{105}$

   First, reflect point $H$ across $AE$ and call the new point $H'$. Notice that $H'$ lies on the circumcircle of $ABC$, and therefore quadrilateral $ABCH'$ is cyclic. We can find that $HF = \sqrt{39}$, and by similar triangles, $BD = \frac{6\sqrt{39}}{5}$, meaning that $HD = \frac{14}{5}$. By power of a point on cyclic quadrilateral $AEDB$, we find that $AH \cdot HD = BH \cdot HE$, meaning that $HE = \frac{\frac{14}{5} \cdot 20}{8} = 7$. Finally, by power of a point on cyclic quadrilateral $ABCH'$, we see that $AE \cdot EC = BE \cdot EH' = (8+7) \cdot 7 = \boxed{105}$. *Solution by Derrick Liu*

8. The newest world cup soccer ball is a polyhedron consisting only of faces which are regular octagons, regular heptagons, and squares. At each vertex, there is one octagon, one heptagon, and one square. If there are $a$ octagons, $b$ heptagons, and $c$ squares, what is $10000a + 100b + c$? *Problem Proposed by Larry Xing*

   **Solution:** $\boxed{141628}$

   We will use Euler's formula $V - E + F = 2$, where $V$, $E$, and $F$ are the number of vertices, edges, and faces respectively for a polyhedron. Since 3 shapes meet at a vertex, and each edge is shared by 2 shapes, we can write the following equations:

$$\begin{aligned} F &= a + b + c \\ V &= \frac{8a + 7b + 4c}{3} \\ E &= \frac{8a + 7b + 4c}{2} \end{aligned}$$

   Combining these equations, we see that

$$\begin{aligned} V - E + F &= \frac{-8a - 7b - 4c}{6} + a + b + c = 2 \\ -2a - b + 2c &= 12 \end{aligned}$$

Furthermore, because each vertex has exactly one octagon, one heptagon, and one square, we know that

$$V = 8a = 7b = 4c$$

Therefore, we know that $c = \frac{7b}{4}$ and $a = \frac{7b}{8}$. Plugging this into the above equation, we see that

$$\frac{-14b}{8} - b + \frac{14b}{4} = 12$$

Therefore, the value of $b$ is 16, the value of $a$ is 14, and the value of $c$ is 28, so our final answer is $\boxed{141628}$.
*Solution by Derrick Liu*

9. How many ordered pairs of (not necessarily distinct) permutations $(\sigma_1,\ \sigma_2)$ of the set $S = \{1, 2, 3, 4\}$ satisfy

$$\sigma_1(\sigma_2(i)) = \sigma_2(\sigma_1(i))$$

for all $i \in S$? (A permutation $\sigma$ is a function that associates each $i \in S$ to some $j \in S$, such that no two $i \in S$ are associated with the same $j$. An example permutation $\sigma$ of the set $S = \{1, 2, 3, 4\}$ is $\{1, 2, 3, 4\} \to \{2, 1, 3, 4\}$. In this case $\sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3, \sigma(4) = 4$. *Problem Proposed by Adam Tang*

**Solution:** $\boxed{120}$

**Solution 1** First, define the cycle length of a permutation to be the number of times the permutation must be applied in order for the starting number to return to itself. For example, the permutation $\sigma = \{1, 2, 3, 4\} \to \{2, 1, 4, 3\}$ has two cycles of cycle length 2. The numbers 1 and 2 form one cycle of cycle length 2 ($1 \to 2 \to 1$ and $2 \to 1 \to 2$), and the numbers 3 and 4 form one cycle of cycle length 2.

Now, we can use casework on $\sigma_1$ by breaking down all possible sums of cycle lengths. Notice that the sum of the cycle lengths must equal 4.

Case 1: If $\sigma_1$ has only one cycle of length 4. There are 6 permutations $\sigma_1$ that have only one cycle of cycle length 4. To verify this, 1 has 3 distinct numbers to map to, 2 had 2 distinct numbers to map to (namely, the two numbers other than itself and the number 1 maps to), and the numbers that 3 and 4 are uniquely determined at this point. For each $\sigma_1$ of only one cycle of cycle length 4, there are exactly 4 permutations $\sigma_2$ that satisfy $\sigma_1(\sigma_2(i)) = \sigma_2(\sigma_1(i))$. Consider the number that 1 maps to after the operation $\sigma_2(\sigma_1(1))$. There are 4 choices for which number $\sigma_2$ maps to. After we determine this number, the rest of the mappings can be uniquely determined. Therefore, for this case, there are $6 \cdot 4 = 24$ pairs of permutations $\sigma_1$ and $\sigma_2$.

Case 2: If $\sigma_1$ has one cycle of cycle length 3 and one cycle of cycle length 1. There are 8 permutations in this case: 4 to choose which number maps to itself, and 2 choices for the cycle of cycle length 3, following the logic above. If we consider the operation $\sigma_2(\sigma_1(i))$ for the number that maps to itself, then $\sigma_2(i) = \sigma_1(\sigma_2(i))$. To satisfy this equation, $\sigma_2(i)$ must equal $i$. Then, by the same logic above, $\sigma_2$ has 3 choices for the three numbers that do not map to themselves, meaning that we have $8 \cdot 3 = 24$ pairs of permutations $\sigma_1$ and $\sigma_2$ for this case.

Case 3: If $\sigma_1$ has two cycles of cycle length 2. There are 3 permutations in this case: 1 has 3 choices to map to, which determines one cycle of cycle length 2, leaving the other two numbers as the other cycle of cycle length 2. WLOG, consider the permutation $\sigma_1 = \{1, 2, 3, 4\} \to \{3, 4, 1, 2\}$. Then $\sigma_2(\sigma_1(1)) = \sigma_2(3) = \sigma_1(\sigma_2(1))$ and $\sigma_2(\sigma_1(3)) = \sigma_2(1) = \sigma_1(\sigma_2(3))$. We have 4 choices for $\sigma_2(3)$, and now $\sigma_2(1)$ is uniquely determined. Now, $\sigma_2(2)$ has 2 choices, and $\sigma_2(4)$ has one choice, meaning that for this case, there are $3 \cdot 4 \cdot 2 = 24$ pairs of permutations $\sigma_1$ and $\sigma_2$.

Case 4: If $\sigma_1$ has two cycles of cycle length 1 and one cycle of cycle length 2. For this case, there are $\binom{4}{2} = 6$ ways to choose which two numbers map to themselves, leaving the other two numbers as a unique cycle of cycle length 2. Again, consider the two numbers that map to themselves. There are two cases: Either the two numbers map to themselves in $\sigma_2$, or they map to each other. In both cases, the other two numbers have 2 possibilities for their permutation in $\sigma_2$: either the two numbers map to each other or to themselves. Therefore, there are $6 \cdot 2 \cdot 2 = 24$ pairs of permutations $\sigma_1$ and $\sigma_2$ for this case.

Case 5: If $\sigma_1$ has 4 cycles of cycle length 1. Because each number in $\sigma_1$ maps to itself (i.e, the identity permutation), it does not matter which permutation $\sigma_2$ is: the equality will always be satisfied, because by definition, $\sigma_2$ multiplied by identity = identity multiplied by $\sigma_2$. Since there are $4! = 24$ total permutations (1 can map to 4 choices, then 2 can map to 3 choices, etc..), we have 24 pairs of permutations $\sigma_1$ and $\sigma_2$ for this case.

Thus, our final answer is $24 + 24 + 24 + 24 + 24 = \boxed{120}$. *Solution by Derrick Liu*

**Solution 2: Group Theory** For those familiar with the group theory, the key idea is to prove that number of ordered pairs of elements that commute in group is the number of conjugacy classes times the size of the group, then use the fact that each conjugacy class of $S_n$ corresponds to a different partition of $n$. This is shown in detail below.

Let $S_4$ be the set of all permutations on $\{1, 2, 3, 4\}$. Restating the question, we wish the find the size of the set $K = \{(\sigma_1, \sigma_2) \in S_4 \times S_4 \mid \sigma_1\sigma_2 = \sigma_2\sigma_1\}$, where the operation is function composition. Let $C(\sigma) = \{\tau \in S_4 \mid \sigma\tau = \tau\sigma\}$; that is, the set of elements that commute with $\sigma$. Since $(\sigma_1, \sigma_2) \in K$ if and only if $\sigma_2 \in C(\sigma_1)$, it follows that

$$|K| = \sum_{\sigma_1 \in S_4} |C(\sigma_1)|.$$

Define the conjugacy class of an element $\sigma$ to be the set $\mathrm{cl}(\sigma) = \{\tau\sigma\tau^{-1} \mid \tau \in S_4\}$. It can be shown that the conjugacy classes partition $S_4$. By mapping each coset $\tau C(\sigma) = \{\tau\pi \mid \pi \in C(\sigma)\}$ to the conjugate $\tau\sigma\tau^{-1}$, it follows that the number of distinct cosets $\tau C(\sigma)$ is equal to number of conjugates of $\sigma$. Because all cosets have the same size, this implies that $|\mathrm{cl}(\sigma)| = |S_4|/|C(\sigma)|$ or $|S_4| = |\mathrm{cl}(\sigma)|\ |C(\sigma)|$, so if $\sigma_1$ and $\sigma_2$ are in the same conjugacy class, it follows that $|C(\sigma_1)| = |C(\sigma_2)|$. For more rigor, look for the properties of cosets. Now consider an arbitrary conjugacy class $\mathrm{cl}(\sigma) = \{\tau_1, \tau_2, \ldots, \tau_n\}$ so $|\mathrm{cl}(\sigma)| = n$. We now have

$$|C(\tau_1)| + |C(\tau_2)| + \cdots + |C(\tau_n)| = n|C(\sigma)| = |S_4|.$$

Let the number of conjugacy classes of $S_4$ be $m$. Taking one representative from each conjugacy class in our above equation for $|K|$, we have

$$|K| = m \cdot |S_4| = 24m.$$

Hence, we will have succeeded if we can find the number of conjugacy classes of $|S_4|$. We claim that the number of conjugacy classes in $|S_4|$ is the number of ways to partition 4. To do this, it suffices to show that two permutations are conjugates (in the same conjugacy class) if and only if they have the same cycle type; that is, if two permutations are decomposed into the product of disjoint cycles, then the number of cycles of a given length for each permutation is the same. There are proofs for this online, but it can intuitively be understood as follows: if $\sigma_1$ is a conjugate of $\sigma_2$, then $\sigma_2 = \tau\sigma_1\tau^{-1}$ for some permutation $\tau$, which implies $\sigma_2\tau = \tau\sigma_1$. In particular, if $\sigma_1$ maps $i$ to $j$, then $\sigma_2$ maps $\tau(i)$ to $\tau(j)$ because $\sigma_2(\tau(i)) = \tau(\sigma_1(i)) = \tau(j)$, so $\sigma_2$ is essentially the same permutation as $\sigma_1$, the only difference being it acts on the set $\{\tau(1), \tau(2), \tau(3), \tau(4)\}$ instead of $\{1, 2, 3, 4\}$.

With this, we know the number of conjugacy classes of $S_4$ is simply the number of ways to partition 4. There are 5 ways to partition 4, represented by $1 + 1 + 1 + 1 = 1 + 1 + 2 = 1 + 3 = 2 + 2 = 4$, so there are 5 conjugacy classes in $S_4$. Hence, the answer is $5 \cdot 24 = \boxed{120}$. *Solution by Felix Liu*

10. Let $w, x, y$, and $z$ be positive real numbers that satisfy the following equations

$$\begin{aligned} 4x^2 + y^2 + xy &= 496 \\ x^2 + z^2 + xz &= 121 \\ 4z^2 + w^2 - zw &= 496 \\ w^2 + y^2 - yw &= 484 \end{aligned}$$

Then, $xw + yz$ can be expressed as $a\sqrt{b}$, where $b$ is squarefree. Find $a + b$. *Problem Proposed by Derrick Liu*

**Solution:** $\boxed{75}$

Recall the formula for the Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$. If we multiply the second equation by 4 so that the system of equations becomes

$$\begin{aligned} 4x^2 + y^2 + xy &= 496 \\ 4x^2 + 4z^2 + 4xz &= 484 \\ 4z^2 + w^2 - zw &= 496 \\ w^2 + y^2 - yw &= 484 \end{aligned}$$

We notice that the first and third equations and the second and fourth equations have the same sum. This leads us to think of a cyclic quadrilateral. In fact, if we set $\sqrt{496}$ and $\sqrt{484}$ to be the lengths of the diagonals and $2x, y, w$ and $2z$ to be the sidelengths in clockwise order, we notice that the 4 equations resemble the Law of Cosines for the 4 pairs of two consecutive sidelengths. To reinforce this idea, let the angles in the quadrilateral be $\phi$, $\theta$, $180 - \theta$, and $180 - \phi$ (the $180 - \theta$ and $180 - \phi$ arise from the fact that opposite angles are supplementary). Using the Law of Cosines formula, we can rewrite the equation as:

$$\begin{aligned} &4x^2 + y^2 - 2(2x)y\cos 180 - \phi = 496 \\ &4x^2 + 4z^2 - 2(2x)(2z)\cos 180 - \theta = 484 \\ &4z^2 + w^2 - 2(2z)w\cos\phi = 496 \\ &w^2 + y^2 - 2yw\cos\theta = 484 \end{aligned}$$

If we let both $\cos\phi$ and $\cos\theta$ be equal to $\frac{1}{2}$, since $\cos 180 - \theta = -\cos\theta$ for any angle $\theta$, after plugging in the values, we see that this system of equations reduces into

$$\begin{aligned} 4x^2 + y^2 + xy &= 496 \\ 4x^2 + 4z^2 + 4xz &= 484 \\ 4z^2 + w^2 - zw &= 496 \\ w^2 + y^2 - yw &= 484 \end{aligned}$$

thus proving that $2x, y, w$, and $2z$ are the sidelengths of a cyclic quadrilateral. By Ptolemy's Theorem, $2xw + 2yz = \sqrt{484} \cdot \sqrt{496}$, so $xw + yz = \frac{22 \cdot 4\sqrt{31}}{2} = 44\sqrt{31}$, so our final answer is $44 + 31 = \boxed{75}$. *Solution by Derrick Liu*