A common proxy (i.e. substitute or model) for the risk-free rate r_f is the coupon rate of a U.S. Treasury. The specific type of U.S. Treasury chosen in applications depends on the time horizon over which an analysis is conducted. In the modeling of derivatives, however, traders typically choose LIBOR as a proxy for r_f (see [8, p. 74]).

When inflation constitutes a major portion of the market risk-free rate r_f, the market risk-free rate is even referred to sometimes as the inflation rate. It is also possible for the inflation rate to be above the market risk-free rate, which for instance can due to the government lowering interest rates to increase liquidity. One then cannot always assume $r_f \geq \mathtt{i}$, but would expect it to hold under normal market conditions.

Readers are referred to Chapter 1 of reference [7] for a detailed and extended discussion of the required return rate.

2.3.2 *Total Return Rate*

The *total return rate* of an invested capital specifies the compensation from the investment (e.g. stock, fund) over a given time span as a percentage of the initially invested capital. In general, it should not be confused with the simple or compound interest rate discussed earlier.

Example 2.4. Suppose that you lend \$1,000 for a year at 12% per year. Under simple interest growth, at the end of the year you will receive back your \$1,000 and a compensation of \$120 (see page 39). The total return rate for the year is also 12% since the percent of \$1,000 that yields \$120 is 12%. See Section 2.4.2 for more on the differences and similarities between the total return rate and simple interest rate.

On the other hand, for an annual interest rate of 12% that is compounded monthly, at the end of the year you will receive back your \$1,000 and a compensation of \$126.83 instead (see page 40). The total return rate over the year is now not 12%, but the percent of \$1,000 that gives the compensation of \$126.83, which is:

$$R(1) = \$126.83/\$1{,}000 = 12.68\%,$$

or, equivalently,

$$R(1) \times \$1{,}000 = \$126.83.$$

The one-year total return rate, $R(1) = 12.68\%$, is higher than the annual interest rate $r = 12\%$ because $R(1)$ involves compounding r monthly (equation (2.46) on page 74 gives an explicit expression). This is also the case over longer time spans (see Section 2.5.5). From a borrower's perspective the percent r appearing as the growth rate in the compounding model is the *quoted rate*, while the total return rate $R(1)$ is the *actual cost* of the loan expressed as a percent of the investment. $\square$

The discussion so far assumed implicitly that the investment yields a positive compensation, which is not guaranteed, and did not allow for income from the investment. We now cast the total return amount of an initial investment for a given time interval as what comes back to you, beyond the initial investment, from the change in the investment's value and from any cash dividend the investment pays. The return amount will then be expressed as a percent of the initial investment. This percent is the total rate return on the investment for the given time interval. In other words, *the total rate return is a performance measure of an investment*.

To model the total return rate, suppose that your investment has a per-unit market value $V(t_0)$ at an initial time $t_0 \geq 0$ and per-unit market value $V(t_f)$ at a final time $t_f > t_0$. *The total return amount of the investment from t_0 to t_f is assumed to arise from its change in market value, $V(t_f) - V(t_0)$, and the total cash-dividend $D(t_0, t_f)$ per unit (share) from the investment during the interval $[t_0, t_f)$.* For simplicity, we exclude complications like share splits and non-cash payouts, and tally any cash dividend at t_f as part of the return amount over a subsequent time interval starting at t_f. The latter bookkeeping for the cash dividend is convenient mathematically for modeling continuous dividend reinvesting (see Section 2.3.4). It is also common practice to assume that $D(t_0, t_f)$ excludes any income such as interest from the cash dividend during $[t_0, t_f)$. This is not a serious concern for sufficiently short investment time intervals.

It is important to recall from Section 1.4.6 that for modeling purposes the ex-dividend date is the important date, rather than the actual payment date (see page 16). In other words, a cash dividend is modeled as the downward adjustment by the amount of the declared dividend of the closing stock price on the trading date just prior to the ex-dividend date. This downward adjustement is not reflected on the actual payment date. *Instead of working with the actual cash dividend payment dates, we shall then use the ex-dividend dates to model the payment dates.* Readers are referred to Hull [8, Sec. 13.12] for more.

For securities like stocks and bonds, the cash dividends flow in discretely—e.g. quarterly, semiannually, and even annually in some cases. The ex-dividend dates of stocks typically do coincide with an exact quarter and are not the same for all companies. The dividend stream for a sufficiently broad stock index funds is then often modeled as continuous due to the many constituent cash dividend paying stocks with different ex-dividend dates (cf. Section 2.3.4)

Expressing the per-unit total return amount on your investment from t_0 to t_f as a percent $R(t_0, t_f)$ of the initial value $V(t_0)$, we obtain:

$$\underbrace{R(t_0, t_f)\, V(t_0)}_{\text{return amount}} = \underbrace{V(t_f) - V(t_0)}_{\text{capital gain/loss}} + \underbrace{D(t_0, t_f)}_{\text{cash dividend}}.$$

When the spread $V(t_f) - V(t_0)$ is positive, it is called a *capital gain*, while a negative spread is a *capital loss*.

The percent,

$$R(t_0,t_f) = \underbrace{\frac{V(t_f) - V(t_0)}{V(t_0)}}_{\text{capital gain/loss return}} + \underbrace{\frac{D(t_0,t_f)}{V(t_0)}}_{\text{dividend yield}}, \qquad (0 \le t_0 \le t_f), \tag{2.5}$$

is called the *total rate of return* or *holding-period return* of the investment from t_0 to t_f. We shall refer to $R(t_0,t_f)$ simply as the *return rate*. Note that if your ownership in the investment consisted of n units (shares), then the return rate is still given by (2.5) since the numerator and denominator of each term would be multiplied by n and so n would drop out.

When the return rate depends on the length τ of $[t_0,t_f]$ rather than on the location of $[t_0,t_f]$ on the positive time axis $[0,\infty)$,[4] we set:

$$R(t_0,t_f) = R(\tau). \tag{2.6}$$

If the annual return rate $R(1)$ is random, then its statistics are typically estimated using historical annual return rates or employing historical daily, weekly, etc. return rate data and then annualizing under certain assumptions.

The ratio $D(t_0,t_f)/V(t_0)$ in (2.5) is called the *dividend yield* and represents the per-unit cash dividend from the investment as a percent of the initially invested capital $V(t_0)$. Additionally, the ratio $V(t_f)/V(t_0)$ is called the *gross return*.[5] It expresses the final value $V(t_f)$ as a percent of the initial value $V(t_0)$.

Example 2.5. Suppose that after one year the return rate on your investment is 50%. Then the gain to you above the initial amount you invested is 50% of your initial investment. If the return rate were -100%, then you have a complete loss, i.e. the value of your investment drops to zero and you would receive no cash dividend. A 200% return rate means that your gain beyond the initial investment is twice the initial investment, i.e. your initial investment tripled in value over the year. □

Finally, in (2.5) we did not specify whether the initial time t_0 is the present time or in the past. Most of the applications in this chapter of the time value of money will have t_0 the current time, t_f a future time, and the current value $V(t_0)$ and future $V(t_f)$ either known or calculable at the current time. In this case, the return rate is known. However, the return rate becomes random if the

[4] See the discussion after equation (2.37) on page 71 for an example where the location of the interval matters.

[5] Some authors call $V(t_f)/V(t_0)$ the *return rate*, but we shall not abide by that usage.

future value $V(t_f)$ and/or the cash dividend $D(t_0, t_f)$ are unknown. Almost all the return rates we encounter in this chapter are non-random, while all the return rates in Chapter 3 are random.

2.3.3 *Stock Return Rate with Historical Data*

Given that a stock's return rate over a future time interval is random, historical stock data is usually used to estimate past stock return rates in an attempt to forecast further stock price behavior. A sample average of the estimated return rates is then computed based on historical closing prices, stock cash dividends, and stock splits going back to a certain time and sampled with a certain frequency (daily, weekly, etc.)[6] Without a doubt, the entire process is part art, part science, and part opinion. Assuming one has confidence in a historical mean return rate estimate, then it is used as a forecast of expected future performance of the stock. Of course, other factors like the company's current and anticipated near-future financial state, the business sector, the macroeconomic outlook, etc., play an important role as one considers expected future performance of a stock.

Let us now consider more closely the return rate (2.5) in the context of a stock and historical data. Assume that t and t' are times in the past with t' occuring before t, i.e. $0 \leq t' < t$. By (2.5), the past return rate from t' to t on the stock is then:

$$\widehat{R}(t',t) = \frac{\widehat{S}(t) - \widehat{S}(t')}{\widehat{S}(t')} \quad + \quad \frac{\widehat{D}(t',t)}{\widehat{S}(t')}, \qquad (0 \leq t' < t). \tag{2.7}$$

Here $\widehat{S}(t)$ and $\widehat{S}(t')$ are the prices of the stock at past times t and t', respectively, and $\widehat{D}(t',t)$ is the stock's total cash dividend per share during $[t',t)$.

Remark 2.4. (Return rate versus yield) The return rate and yield are often confused. In the book, the return rate of a stock over a time interval measures performance arising from capital gain/loss and income (cash dividends), while "yield" deals primarily with income. Though the word "yield" is used in various ways by different authors, we shall focus mainly on its usage in the context of bonds, where the income is from coupon payments. □

There are some complicating issues with estimating return rates using a stock's historical closing prices. For example, a stock's cash dividend payment

---

[6] Strictly speaking, historical closing prices should be adjusted for more than just stock cash dividends and stock splits. In fact, they should be adjusted for corporate actions such as stock dividends, distributions (e.g. selling a lot of shares), rights offerings, etc. However, there is no universal agreement on how to make such adjustments. Most data sources like YahooFinance adjust simply for cash dividends and stock splits.

date is accounted for by its ex-dividend date rather than its actual dividend-payment date (page 15). This means that when a stock qualifies for a declared dividend (cum-dividend stock), the daily closing prices of the stock during that period should be adjusted downward by using the dividend yield determined from the declared dividend and the closing price on the trading date immedidately prior to the ex-dividend date.[7] Sometimes an adjustment may take into account subtracting taxes from the dividend to get a post-tax value and in such cases the stock price reduction is not the full amount of the dividend (see Hull [8, p. 298]). In addition, a stock split can occur, which changes the share price. To adjust for these complications and to allow for comparison of a stock's closing prices at different times, the *adjusted closing price* of a stock was introduced in the historical closing price data. These adjustments are usually done for stock splits and cash dividends. One should surely check how the adjusted closing prices are computed in a given data base.

The University of Chicago's Booth School of Business provides historical adjusted closing prices and the associated historical return rates of stocks through its Center for Research in Security Prices (CRSP).[8] The center provides a highly comprehensive amount of data and documentation on security prices. For this reason, we shall not attempt to compute historical return rates, but will simply present these historical return rates as given (e.g. from a data source like CRSP).

2.3.4 *Continuous Cash Dividend Reinvesting*

For later applications, particularly to derivative pricing, *we shall assume that the cash dividend from a security is reinvested to purchase more units of the security*. The specific issue we shall consider is how the number of units of the security will grow over time under cash dividend reinvesting. We address this issue in some detail since our approach will serve as a precursor to the framework of compound interest in Section 2.5.1, page 59.

To get a grip on the problem, first consider a time interval $[t_0, t_f]$ and partition it into m subintervals,

$$[t_0, t_1], \quad [t_1, t_2], \quad \ldots, [t_{m-1}, t_m],$$

of equal length

[7] For example, at Yahoo! Finance, if a declared dividend is D_0 and the closing price of the stock on the trading day immediately before the ex-dividend date is $S(t_{\text{ex}}^-)$, then the dividend yield used is $y_0 = D_0/S(t_{\text{ex}}^-)$ and a closing price $S(t)$ during the cum-dividend period is adjusted downward to $S(t) - y_0 S(t)$. The quantity $1 - y_0$ is called a *dividend multiplier*.

[8] www.crsp.com

$$h = \frac{\tau}{m},$$

where

$$t_i = t_{i-1} + h, \qquad t_m = t_f, \qquad \tau = t_f - t_0, \qquad\qquad (i = 0,1,2,\ldots,m).$$

Note that as m increases, the final time t_f is always fixed, but the number of subintervals increases as they get smaller and, hence, the label t_m of t_f changes with m. Set

$$\Delta(t_i) = \text{the number of units of the security at } t_i, \qquad (i = 0,1,2,\ldots,m).$$

Second, *assume that the price $S(t_i)$ of the security at t_i remains constant during $[t_{i-1}, t_i)$ for $i = 0,1,2,\ldots,m$*. Third, *assume that the per-unit cash dividend $D(t_{i-1}, t_i)$ for the time interval $[t_{i-1}, t_i)$ is modeled as a percent of the security's initial price $S(t_i)$ using a constant, continuous, proportional, annual dividend yield rate q*. In other words:

$$D(t_{i-1}, t_i) = q\,S(t_i)h, \qquad\qquad (i = 0,1,2,\ldots,m). \tag{2.8}$$

Note that q is a percent and, hence, unitless. Fourth, *assume that each cash dividend $D(t_{i-1}, t_i)$ is immediately reinvested during $[t_{i-1}, t_i)$ to buy more units of the security.*

Given the above assumptions and an initial number $\Delta(t_0)$ of units of the security, what is the number $\Delta(t_f)$ of units it will grow to at time t_f? We answer the question in detail using a "compounding" approach (cf. Section 2.5.1):[9]

➢ *Subinterval $[t_0, t_1]$, where $t_1 = t_0 + h$.*
At time t_0, we have $\Delta(t_0)$ units of the security. During $[t_0, t_0 + h)$, the price of the security is constant at $S(t_0)$ and the total cash dividend from the security is

$$D(t_0, t_0 + h)\,\Delta(t_0) = q\,S(t_0)\,h\,\Delta(t_0).$$

We use the cash dividend during $[t_0, t_0 + h)$ to buy immediately (i.e. before time $t_0 + h$) the following more units:

$$\frac{D(t_0, t_0 + h)\,\Delta(t_0)}{S(t_0)} = q\,h\,\Delta(t_0).$$

At time $t_1 = t_0 + h$, the number of units we then have is:

$$\Delta(1) = \Delta(0) + q\,h\,\Delta(t_0) = (1 + q\,h)\,\Delta(t_0).$$

[9] See page 72 for a differential equations approach.

➢ *Subinterval* $[t_1, t_2]$, *where* $t_2 = t_1 + h$.
At time t_1, we have $\Delta(t_1)$ units. During $[t_1, t_1 + h)$ the price of the security is $S(t_1)$ and the total cash dividend paid is

$$D(t_1, t_1 + h)\,\Delta(t_1) = q\,S(t_1)\,h\,\Delta(t_1).$$

We buy immediately (i.e. before $t_1 + h$) the following more units:

$$\frac{D(t_1, t_1 + h)\,\Delta(t_1)}{S(t_1)} = q\,h\,\Delta(t_1).$$

At time $t_2 = t_1 + h$, the number of units is:

$$\Delta(t_2) = \Delta(t_1) + q\,h\,\Delta(t_1) = (1 + q\,h)^2\,\Delta(t_0).$$

➢ *Subinterval* $[t_{m-1}, t_m]$, *where* $t_m = t_{m-1} + h$.
Continuing the above process, at the end of the mth subinterval, i.e. at time t_m, we will have the following number of units:

$$\Delta(t_m) = (1 + q\,h)^m\,\Delta(t_0) = \left(1 + \frac{q\,\tau}{m}\right)^m \Delta(t_0).$$

➢ *Continuous cash dividend reinvesting*.
As m increases the subintervals get smaller and the number of times cash dividend reinvesting occurs increases. In the limit $m \to \infty$, we have continuous cash dividend reinvesting and obtain the following number of units of the security at time t_f:[10]

$$\Delta(t_f) = \lim_{m\to\infty} \left(1 + \frac{q\,\tau}{m}\right)^m \Delta(t_0) = \mathrm{e}^{q\tau}\,\Delta(t_0), \tag{2.9}$$

where we used $\lim_{x\to\infty}\left(1 + \frac{a}{x}\right)^x = \mathrm{e}^a$. *Equation (2.9) shows that* $\mathrm{e}^{-q\,\tau}$ *units of the security at* t_0 *will grow to 1 unit at* t_f, *while 1 unit at* t_0 *will grow to* $\mathrm{e}^{q\,\tau}$ *units at* t_f.

[10] Strictly speaking, we should write $\Delta_{\mathrm{cts}}(t_f)$ instead of $\Delta(t_f)$ to indicate the continuous-time limit. However, this would make the notation unnecessarily cumbersome. The meaning should be clear from the context.

Equation (2.39) coincides with (2.37) for a principal $V(t_0) = \mathcal{F}_0$. When $r(t)$ is a constant r,[12] we obtain the usual continuous compounding growth:

$$V(t_f) = V(t_0)\,\mathrm{e}^{r\tau}.$$

Applications to continuous dividend reinvesting

The differential equations approach used to obtain (2.39) also yields a quick way to obtain the growth in the number of units of a security from continuous reinvesting, namely, equation (2.9) on page 51.

At time t let $S(t)$ be the unit price of a security and $\Delta(t)$ the number of units of the security. Consider the time interval $[t, t+dt]$. By the assumptions for dividend reinvesting (see Section 2.3.4, page 49), during $[t, t+dt)$ the price of the security stays constant at $S(t)$ and the cash dividend paid[13] for a unit ownership of the security is:

$$D(t, t+dt)\,\Delta(t) = q\,S(t)\,(dt)\,\Delta(t).$$

The cash dividend received during $[t, t+dt)$ is then used to purchase immediately (i.e. before time $t+dt$) the following additional units:

$$\frac{D(t, t+dt)\,\Delta(t)}{S(t)} = \big(q\,\Delta(t)\big)\,dt.$$

At time $t+dt$, the number of units is then:

$$\Delta(t+dt) = \Delta(t) + \big(q\,\Delta(t)\big)\,dt.$$

In other words, the change in the number of units from t to $t+dt$ is:

$$d\Delta(t) = \Delta(t+dt) - \Delta(t) = \Big(q\,\Delta(t)\Big)\,dt, \tag{2.40}$$

which is equivalent to the differential equation,

$$\frac{d\Delta}{dt}(t) = q\,\Delta(t).$$

Solving (2.40) similarly to (2.38) from time t_0 to t_f, we obtain the number of units at time t_f:

$$\Delta(t_f) = \mathrm{e}^{q\tau}\,\Delta(t_0), \qquad\qquad (\tau = t_f - t_0). \tag{2.41}$$

[12] It is common practice to abuse notation by writing a constant interest rate by r and when it is a function of time, write $r(t)$.

[13] Recall that for theoretical purposes it is best to model the cash dividend payment date as the ex-dividend date.