$\boxed{5.1}$ For an n-step binomial tree, give a financial interpretation of the following:

a) $u_n d_n = 1$. The tree will have a reflectional symmetry, as d steps of increase and d steps of decrease returns the stock to its price of 2d periods ago. Financially, this means that the stock increases and decreases by the same amount. This might be the case if instances of good news drive the price up just as much as bad news drives the price down.

b) $u_n - 1$: The percentage increase of the stock when it goes up. The investor's net gain is then $S_{t-1}(u_n - 1)$ in period t.

c) $d_n - 1$: The percentage decrease of the stock when it goes down. This number will be negative.

CHRISTIAN
DRAPPI 1/22/2014

netid: cld22

5.2 Explain why the condition

$$d_n < e^{(m-q)h_n} < u_n$$

holds for every n-step binomial tree.

$e^{(m-q)h_n}$ is the "risk-free" growth rate for a single period. If $d_n > e^{(m-q)h_n}$, then even in the worst months for the stock, it outperforms the risk-free rate. Hence, no one would invest in the risk-free instrument as the risky one is strictly better. On the other hand, $u_n < e^{(m-q)h_n}$ would mean just the opposite: the risk-free security would strictly outperform the risky one, in both the latter's good times and bad. Hence, there would be no reason to buy the risky security.

(this is equivalent to the no-arbitrage condition)

$\boxed{5.3}$ How many 1-step subtrees are there in an n-step binomial tree?

- After 1 step, there is obviously only 1 one-step subtree
- After 2 steps, there are 2 additional one-step subtrees
- After 3 steps, there are 3 additional one-step subtrees

Following this pattern, in an n step tree there are a total of:

$$1 + 2 + 3 + \dots + n = \sum_{i=1}^{n} i$$

$$= \boxed{\frac{n(n-1)}{2} \text{ one step subtrees}}$$

$\boxed{5.4}$ For an n-step binomial tree, let N_u be the number of security upticks from time t_0 to t_n. Explain why N_u is a binomial random variable. What are its expected value and variance if the tree has 40 steps $(n=40)$ and an uptick probability of 60% $(p_n = 0.6)$?

At each step, there is a fixed probability of going up, p_n. That is, each step is a Bernoulli trial with $p = p_n$. There are n such steps, and summing the number of successes over n Bernoulli trials each with a success probability p gives a Binomial(n, p) random variable.

If X is a Binomial$(n=40, p=0.6)$ random variable, then:

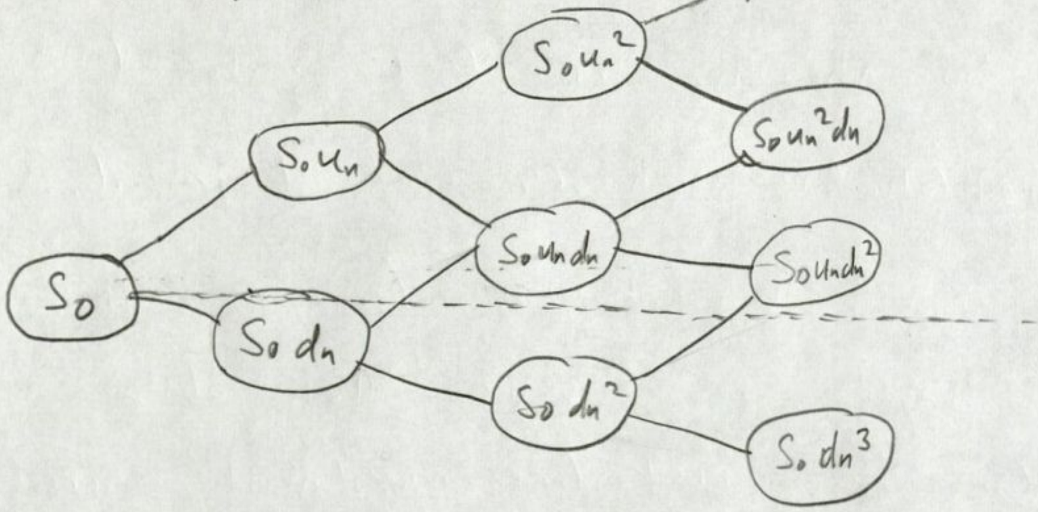
$$\mathbb{E}X = np = 40 \cdot \tfrac{3}{5} = \boxed{24 = \mathbb{E}X}$$

$$\mathrm{Var}(X) = np(1-p) = 40 \cdot \tfrac{3}{5} \cdot \tfrac{2}{5} = \tfrac{48}{5} = \boxed{9.6 = \mathrm{Var}(X)}$$

$\boxed{5.5}$ An n-step CRR tree has reflection symmetry about the horizontal line $S(t_0) = S_0$ since the tree recombines. Agree or Disagree?

Disagree. This is only the case if $u_n d_n = 1$, which is not always true.

If $u_n d_n > 1$, for example, we might see:



In this case, the tree drifts upward, as an uptick and downtick result in a net increase.

$\boxed{5.6}$ For an n-step CRR tree, show that if $u_n < e^{(r_f - q)h_n}$, then there is an arbitrage.

Suppose $u_n < e^{(r_f-q)h_n}$. Construct the following portfolio, and use the no-arbitrage condition:

→ Short 1 share of the risky security $(-S_0)$

→ Invest S_0 at the risk free rate. $(+S_0)$

Now, our portfolio is worth 0. Wait 1 period...

Suppose the <u>worst thing</u> happens for you: the risky security goes up in price. Then after one period, the risk free investment increases in value by a factor of $e^{r_f h_n}$, and you pay out dividends of $(e^{q h_n} - 1) S_0 u_n$

Your portfolio is now:

$$S_0 e^{r_f h_n} - (e^{q h_n} - 1) S_0 u_n - u_n S_0$$

$$= S_0 (e^{r_f h_n} - u_n e^{q h_n}) = S_0 e^{q h_n} (e^{(r_f - q) h_n} - u_n)$$

$$e^{(r_f - q)h_n} > u_n \Rightarrow S_0 e^{q h_n} (e^{(r_f-q)h_n} - u_n) > 0.$$

And we have made money. Since $u_n > d_n$ we make money regardless of the security's directional move.

$\boxed{15.7}$ The risk-neutral uptick probability is related to the real-world probability by $p_n^* = p_n + \eta\sqrt{p_n(1-p_n)}$, where $\eta = \dfrac{E(R_1) - r_f h_n}{\sigma_n \sqrt{h_n}}$

Interpret η.

η has the same form as the Sharpe ratio from CAPM. It captures the ratio of excess returns to risk: the idea is that a risk-averse investor must be compensated with higher returns to take on a riskier investment. This is why $E(R_1) > r_f$, as in the "reel world" investors are assumed to be risk-averse, demanding a rate of return higher than the risk free rate if they are to take on variance.

$\boxed{5.8}$ If in an n-step real world CRR tree, the real world probability p_n is replaced by the risk neutral probability p_n^*, then the expected annualized return rate m is unchanged, but the annualized variance σ^2 changes. Agree or disagree. Justify your answer.

Disagree. The annualized return rate must change if $p_n \neq p_n^*$ and (by assumption) u_n and d_n remain fixed. This is because m depends on only p_n, u_n and d_n, and is strictly increasing in p_n.

On the other hand, the annualized variance remains the same. The prices will be the same in both, the real-world and the risk-neutral world. So there is no more variance in the real-world; rather, investors are compensated for the variance they take by holding a risky security.

$\boxed{5.9}$ As $n \to \infty$, the number of time steps in an n-step binomial tree becomes countably infinite, $\aleph_0 = \text{card } \mathbb{N}$. Since an n-step binomial tree has 2^n paths, what is the cardinality of the set of possible security paths in the continuous-time limit as $n \to \infty$?

$$2^{\text{card } \mathbb{N}} = \text{card } \mathbb{R} = \aleph_1,$$

the cardinality of the continuum.

by Cantor's theorem.

This is because $n \to \text{card } \mathbb{N} \equiv \aleph_0$,

and there are 2^n paths, so

$2^n \to \text{card } \mathbb{R}$ as $n \to \text{card } \mathbb{N}$.

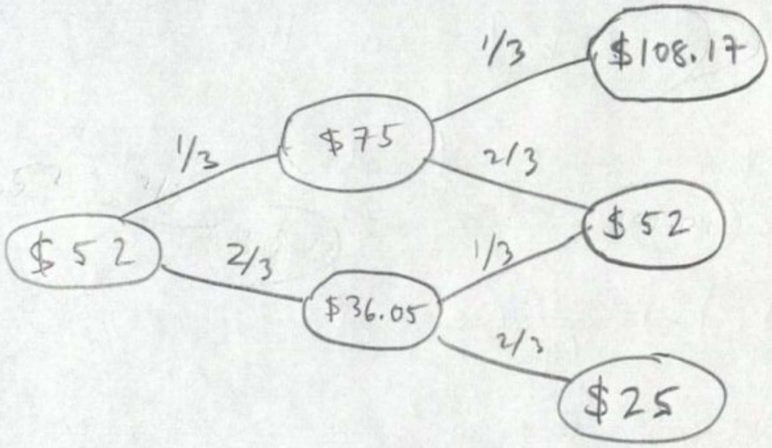
$\boxed{5.10}$ A trader believes that a stock currently at \$51.25/share has a 7/10 chance of increasing by \$0.50 and a 3/10 chance of decreasing by \$0.25. What is the expected price at the end of the day?

$$\mathbb{E}S = \frac{7}{10}(51.25 + 0.50) + \frac{3}{10}(51.25 - 0.25)$$

$$= \frac{7}{10}(51.75) + \frac{3}{10}(51)$$

$$= 51 + \frac{7}{10}\frac{3}{4} = 51 + \frac{21}{40} = \boxed{\$51.53}$$

$\boxed{5.11}$ Complete the following binomial tree and determine whether it is approximately a CRR tree. Compute $\mathbb{E}(S(t_2) \mid S(t_1) = 36.05)$.



$$52\,u_n = 75 \Rightarrow u_n = 75/52 \overset{n}{=} 1.44$$

$$52\,d_n = 36.05 \Rightarrow d_n = 36.05/52 \cong 0.69 \cong 1/1.44$$

Since $u_n d_n \cong 1$, $\boxed{\text{yes,}}$ it is approximately a CRR tree.

$$\mathbb{E}(S(t_2) \mid S(t_1) = 36.05) = \tfrac{2}{3}\$25 + \tfrac{1}{3}\$52 \overset{n}{=} \boxed{\$34}$$

$\boxed{5.12}$ Assume that the current share price of a stock is \$100 and the variance is 0.001/annum. Using a CRR tree model over a trading year with each time step having a length of one day, forecast the spread in the stock's price one trading day from now.

Annualized variance: $0.001/\text{year} = \sigma^2_{\text{annual}}$

$252 \text{ days} = 1 \text{ trading year.} \Rightarrow t_{\text{day}} = \frac{1}{252} \text{ years}$

$\text{Spread} = 100(u-d).$

$u = 1/d = e^{\sqrt{\sigma^2 t}}$

$\sigma^2_{\text{annual}}\left(\frac{1}{252}\text{ years}\right) = \frac{1}{1000}\cdot\frac{1}{252}$

Hence $e^{\sqrt{\sigma^2 t}} = e^{\sqrt{\frac{1}{1000}\cdot\frac{1}{252}}} \doteq e^{0.002} \doteq 1.002 = u.$

So $\text{spread} = 100(1.002 - 0.998) = \boxed{\$0.40}$

$\boxed{5.13}$ For an n-step binomial tree, show that:

$$\mathbb{P}\{S(t_n) = S_0 u_n^i d_n^{n-i} \mid S(t_0) = S_0\} = \binom{n}{i} p_n^i (1-p_n)^{n-i}$$

At each time-step, t_j, there is a probability of increase, p_n, or decrease, $1-p_n$. Let Y_j be 1 iff the security jumps up at time t_j and 0 otherwise. Then let $i = \sum_{j=1}^{n} Y_j$, which is also a random variable. In fact, as discussed in problem 5.4, it is a binomial random variable since it is the sum of Bernoulli Y_j's.

The binomial pmf for i successes in n trials with success probability p_n is:

$\binom{n}{i} p_n^i (1-p_n)^{n-i}$, since there are $\binom{n}{i}$ possible ways to succeed i times and fail $n-i$ times, and the probability of each such sequence is $p_n^i (1-p_n)^{n-i}$.

If there are i upticks and $n-i$ downticks, then the price at t_n will be:

$S(t_n) = S_0 u_n^i d_n^{n-i}$. Putting it all together,

$$\mathbb{P}\{S(t_n) = S_0 u_n^i d_n^{n-i} \mid S(t_0) = S_0\} = \binom{n}{i} p_n^i (1-p_n)^{n-i}$$

$\boxed{5.14}$ Determine the number of elements in the sample space Ω_n of price paths of an n-step binomial tree.

That is, determine $card(\Omega_n)$ or $|\Omega_n|$.

Each time step t_j has a Bernoulli trial, Y_j.

There are 2 possible outcomes in each trial, 0 or 1. There are n such trials, so the entire space has:

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{n\text{ - times}} = 2^n$$

possibilities. So $\boxed{card\ \Omega_n = 2^n}$,

and $\Omega_n = \{0,1\}^n$.