8.2.2 The BSM Pricing Formula: Standard and General Forms

The final and core piece of the BSM model is the the BSM pricing formula for European call options. For convenience, in this section we shall simply state the standard and general forms of the pricing formulas without any derivations or analysis. The remainder of the chapter is designed to build intuition into the formula using binomial trees and applications with specific numerical values. Chapter 9 will be dedicated to several derivations of the BSM formula in both its standard and general forms.

The standard BSM formula for the price of a European call at the current time t_0 is given by:

$$\boxed{C(t_0) = \mathrm{e}^{-q\tau} S(t_0)\,\mathsf{N}(\mathrm{d}_1) - \mathrm{e}^{-r_f\tau}\,\mathsf{K}\,\mathsf{N}(\mathrm{d}_2),} \tag{8.5}$$

where $S(t_0)$ is the current price of the security, $\tau = T - t_0$ is the time remaining on the contract, $\mathsf{N}(\cdot)$ is the standard normal cumulative distribution function, and

$$\mathrm{d}_1 = \mathrm{d}_2 + \sigma_0, \qquad \mathrm{d}_2 = \frac{\log(S(t_0)/\mathsf{K}) + \mu_0}{\sigma_0},$$

with

$$\mu_0 = \left(r_f - q - \frac{\sigma^2}{2}\right)\tau, \qquad \sigma_0 = \sigma\sqrt{(T-t)}.$$

When emphasizing the explicit dependence of the European call price on the current underlying security price, strike price, security volatility, risk-free rate, term of the option, and dividend yield rate, we shall denote the call price by:

$$C = C(S(t_0), \mathsf{K}, \sigma, r_f, \tau, q).$$

The general BSM pricing formula for a European call is as follows:

$$\boxed{C(t_0) = \mathrm{e}^{-\int_{t_0}^{T} q(s)\,ds} S(t_0)\,\mathsf{N}(\mathrm{d}_1) - \mathrm{e}^{-\int_{t_0}^{T} r(s)\,ds}\,\mathsf{K}\,\mathsf{N}(\mathrm{d}_2)} \tag{8.6}$$

where

$$\mathrm{d}_1 = \mathrm{d}_2 + \sigma_0, \qquad \mathrm{d}_2 = \frac{\log(S_0/\mathsf{K}) + \mu_{(r_f)_0}}{\sigma_0},$$

with

$$\mu_{(r_f)_0} = \int_{t_0}^{T}\left(r_f(s) - q(s) - \frac{\sigma^2(s)}{2}\right) ds, \qquad \sigma_0 = \left(\int_{t_0}^{T}\sigma^2(s)\,ds\right)^{1/2}.$$

The next section uses the first two steps in a binomial tree to give an approximate intuitive approach to the standard BSM formula. The more steps used in the binomial tree, the better the approximation. The general binomial-tree option pricing formula is treated in Chapter 9.

8.3 Binomial-Tree Method: The Standard BSM Model

The binomial tree approach is pedagogically an excellent way to gain intuition into the standard BSM option pricing formula (with dividend). Basically, this approach approximates the standard BSM formula in discrete time using a binomial tree and recovers the exact formula after taking the continuous-time limit.

We shall first summarize the method and then determine explicitly the option pricing formula for general 1- and 2-period trees (see Section 9.1.1 for n-period case). Note that in applications a specific binomial tree is chosen (e.g. a CRR tree) to determine the values of u_n, d_n, and p_n in the tree.

> *Unless stated to the contrary, for simplicity we shall in applications of binomial trees to pricing options suppress notationally the dependence of* u_n, d_n, *and* p_n *on the number* n *of time steps and simply write* u, d, *and* p.

The binomial-tree strategy for pricing options

- ➢ Use a binomial tree to model the prices of both the option and the underlying security.
- ➢ Create at the current time t_0 a portfolio consisting of certain amount—say, $\Delta_C(t_0)$ units—of the underlying security and a loan for amount $\mathsf{B}(t_0)$ of a riskless bond (e.g., T-bond), where the amounts are such that at the termination time T of the option contract, the portfolio payoff replicates the option payoff. This portfolio is naturally called a *replicating portfolio*. Recall that the payoff of the portfolio at some future time T is its future net cash flow (cash in-flow minus cash out-flow), which is the revenues obtained at T from selling the held units of the security (cash in-flow) minus the amount used at T to pay off with interest the riskless bond (cash out-flow). Note that the portfolio's payoff will also be binomial in nature.
- ➢ Determine the current price of the option from the current value of the portfolio. Indeed, by the Law of One Price since the option and the portfolio have the same payoff and because there is no arbitrage opportunity in the market (M-2), the portfolio and the option will have the same current value.

For later reference, it is useful to recall from Chapter 8 that the terminal value of the option is its payoff (otherwise, there is an arbitrage opportunity):

$$C(T) = \max\{S(T) - \mathsf{K}, 0\} = \text{ option payoff}, \tag{8.7}$$

where K is the strike price.

In the next two subsections, we shall employ 1- and 2-step binomial trees to price options. Specifically, for the 1-step case over an interval $[t_0, t_1]$, where $t_1 = T$, we determine the initial values $\Delta_C(t_0)$ and $\mathsf{B}(t_0)$ for which the portfolio replicates the option's future cash flow at the next time increment t_1. A similar analysis is carried out for the 2-step model over a time span $[t_0, t_2]$ with $t_0 < t_1 < t_2 = T$.

8.3.1 1-Period Binomial Pricing: Replicating Portfolio

Let us price a European call using a 1-step binomial tree with $t_1 = T$ the termination date and $t_1 - t_0 = h$ the time remaining on the call. The two possible values of the security at future time t_1 are:

$$S(t_1) = \begin{cases} S_\mathrm{u}(t_1) \equiv S(t_0)\,\mathrm{u} & \text{with probability } p \\ \\ S_\mathrm{d}(t_1) \equiv S(t_0)\,\mathrm{d} & \text{with probability } 1-p, \end{cases} \tag{8.8}$$

where we assume that

$$\mathrm{u} > 1, \qquad 0 < \mathrm{d} < 1, \qquad 0 < p < 1. \tag{8.9}$$

Using (8.7) and (8.8), the possible payoffs of the call option at t_1 are given by:

$$\begin{aligned} C(t_1) &= \begin{cases} C_\mathrm{u}(t_1) \equiv \max\{S_\mathrm{u} - K, 0\} & \text{with probability } p \\ \\ C_\mathrm{d}(t_1) \equiv \max\{S_\mathrm{d} - K, 0\} & \text{with probability } 1-p \end{cases} \\ &= \text{Option Payoff}. \end{aligned} \tag{8.10}$$

These results are summarized in Figure 8.1.

Replicating portfolio (with "real-world" probabilities).

Construct a portfolio at the current time t_0 as follows:
- Long $\Delta_C(t_0)$ units of the underlying security at time t_0.
- Short a risk-free bond (e.g., T-bond) of amount $\mathsf{B}(t_0)$ at time t_0.

The value $\mathcal{V}(t_0)$ of the portfolio today t_0 is:

$$\mathcal{V}(t_0) = S(t_0)\,\Delta_C(t_0) - \mathsf{B}(t_0).$$

The portfolio's value at the future time $t_1 = t + h$ is uncertain and given by the random quantity:

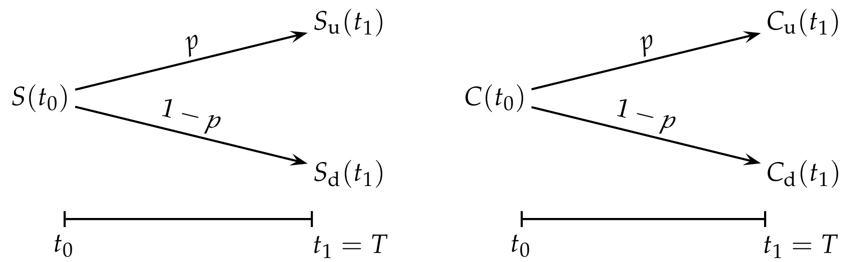

Fig. 8.1 1-period binomial tree models of the prices of the security (left tree) and call option (right tree). Here p is the probability of an upward movement in the tree.

$$\mathcal{V}(t_1) = \underbrace{\underbrace{S(t_1)}_{\text{random}} \cdot \underbrace{\Delta_C(t_1)}_{\text{non-random}}}_{\text{random}} - \underbrace{e^{r_f h}\,\mathsf{B}(t_0)}_{\text{non-random}}. \tag{8.11}$$

Indeed the future value $S(t_1)$ of the underlying risky security cannot be predicted with certainty and so is random. The factor $\Delta_C(t_1)$ is the number of units of the underlying security at time t_1, which has grown from the initial amount $\Delta_C(t_0)$ due to continuous reinvesting of the security's dividend at rate q. In Chapter 1 we showed that

$$\Delta_C(t_1) = e^{q\,h}\,\Delta_C(t_0), \tag{8.12}$$

which is non-random. The first term on the right of (8.11) is then random. The second term on the right of (8.11) is non-random and arises from the initial risk-free bond amount $\mathsf{B}(t_0)$ growing by continuous compounding at rate r_f over the time period $h = t_1 - t_0$.

Portfolio payoff.

The portfolio's payoff (i.e., net cash flow) at time t_1 is the amount received from selling the $\Delta_C(t_1)$ units of the security at price $S(t_1)$ per unit (cash in-flow) minus the amount $e^{r_f h}\,\mathsf{B}(t_0)$ used to pay back the shorted bond with interest (cash out-flow). Consequently, the portfolio payoff coincides with the value (8.11) of the portfolio at t_1 :

$$\begin{matrix}\text{Portfolio}\\ \text{Payoff}\end{matrix} = \underbrace{S(t_1) \cdot \Delta_C(t_1)}_{\text{cash in-flow}} - \underbrace{e^{r_f h}\,\mathsf{B}(t_0)}_{\text{cash out-flow}} = \mathcal{V}(t_1). \tag{8.13}$$

Using equations (8.8) and (8.12), we see that (8.13) becomes:

$$\begin{matrix}\text{Portfolio}\\\text{Payoff}\end{matrix} = \begin{cases} S_{\mathrm{u}}(t_1)\cdot \mathrm{e}^{q\,h}\,\Delta_C(t_0) - \mathrm{e}^{r_f\,h}\,\mathsf{B}(t_0) & \text{with prob. } p \\[2ex] S_{\mathrm{d}}(t_1)\cdot \mathrm{e}^{q\,h}\,\Delta_C(t_0) - \mathrm{e}^{r_f\,h}\,\mathsf{B}(t_0) & \text{with prob. } 1-p, \end{cases} \tag{8.14}$$

Portfolio payoff replicates option payoff.

Next we find the initial values $\Delta_C(t_0)$ and $\mathsf{B}(t_0)$ for which the portfolio payoff (8.14) replicates the option payoff (8.10) at time t_1. Setting (8.14) and (8.10) equal—so $\mathcal{V}(t_1) = C(t_1)$—yields two equations which we can solve for the unknowns $\Delta_C(t_0)$ and $\mathsf{B}(t_0)$:

$$\begin{aligned} S_{\mathrm{u}}(t_1)\cdot \mathrm{e}^{q\,h}\,\Delta_C(t_0) - \mathrm{e}^{r_f\,h}\,\mathsf{B}(t_0) &= \mathsf{C}_{\mathrm{u}}(t_1) \\[2ex] S_{\mathrm{d}}(t_1)\cdot \mathrm{e}^{q\,h}\,\Delta_C(t_0) - \mathrm{e}^{r_f\,h}\,\mathsf{B}(t_0) &= \mathsf{C}_{\mathrm{d}}(t_1). \end{aligned} \tag{8.15}$$

We obtain:

$$\Delta_C(t_0) = \frac{\mathrm{e}^{-q\,h}\,\left(\mathsf{C}_{\mathrm{u}}(t_1) - \mathsf{C}_{\mathrm{d}}(t_1)\right)}{(\mathrm{u}-\mathrm{d})\,S(t_0)}, \qquad \mathsf{B}(t_0) = \frac{\mathrm{e}^{-r_f\,h}\,\left(\mathsf{C}_{\mathrm{u}}(t_1)\,\mathrm{d} - \mathsf{C}_{\mathrm{d}}(t_1)\,\mathrm{u}\right)}{\mathrm{u}-\mathrm{d}}. \tag{8.16}$$

For these values of $\Delta_C(t_0)$ and $B(t_0)$, the replicating portfolio and call option have the same net cash flow at time t_1 and so at time t_1,

$$\text{Portfolio Payoff} \;=\; \text{Option Payoff.} \tag{8.17}$$

Current option price equals portfolio present value.

By the Law of One Price, equation (8.17)—or equivalently (8.15) with (8.16)—yields that the replicating portfolio and the call have the same price today:

$$C(t_0) = \mathcal{V}(t_0) = S(t_0)\,\Delta_C(t_0) - \mathsf{B}(t_0),$$

where $\Delta_C(t_0)$ and $B(t_0)$ are given (8.16). Explicitly, the current price is

$$\mathsf{C}(t_0) = \mathrm{e}^{-r_f\,h}\left[\left(\frac{\mathrm{e}^{(r_f-q)\,h} - \mathrm{d}}{\mathrm{u}-\mathrm{d}}\right)\mathsf{C}_{\mathrm{u}}(t_1) + \left(\frac{\mathrm{u} - \mathrm{e}^{(r_f-q)\,h}}{\mathrm{u}-\mathrm{d}}\right)\mathsf{C}_{\mathrm{d}}(t_1)\right] \tag{8.18}$$

where from (8.10) we have

$$\mathsf{C}_{\mathrm{u}}(t_1) = \max\{S(t_0)\,\mathrm{u} - \mathsf{K}, 0\}, \qquad \mathsf{C}_{\mathrm{d}}(t_1) = \max\{S(t_0)\,\mathrm{d} - \mathsf{K}, 0\}.$$

Equation (8.18) yields *the option's current price (8.18) is independent of m! Hence, the one-period option price is independent of the investor risk preference*. This is consistent with Assumption I1 on page 370.

8.3.2 1-Period Binomial Pricing: Risk-Neutral Approach

The observation above shows that we could have chosen the investor to be risk-neutral (8.3) from the beginning, and the option's current price would still be the same. In fact, we shall see that risk-neutrality provides a simpler way to price options in our discrete-time binomial tree approach.

Now suppose that we are in a risk-neutral world. By (8.3), the expected future values of the option and underlying security are given, respectively, as follows:

$$E_*[C(t_1)] = C(t_0)\,\mathrm{e}^{h\,r_f}, \qquad E_*[S(t_1)] = S(t_0)\,\mathrm{e}^{h\,(r_f-q)}, \tag{8.19}$$

where no dividend is included in the call, rather in the underlying security. The $*$ indicates that we are using a risk-neutral probability. Using (8.8) and (8.10), we see that (8.19) becomes:

$$E_*[S(t_1)] = p_*\,S(t_0)\,\mathrm{u} \;+\; (1-p_*)\,S(t_0)\,\mathrm{d} = S(t_0)\,\mathrm{e}^{h\,(r_f-q)} \tag{8.20}$$

$$E_*[C(t_1)] = p_*\,C_{\mathrm{u}}(t_1) \;+\; (1-p_*)\,C_{\mathrm{d}}(t_1) = C(t_0)\,\mathrm{e}^{h\,r_f}. \tag{8.21}$$

Moreover, equations (8.20) and (8.21) hold if and only if the risk-neutral probability is given by

$$p_* = \frac{\mathrm{e}^{(r_f-q)\,h} - \mathrm{d}}{\mathrm{u}-\mathrm{d}}. \tag{8.22}$$

We saw on page 268 that p_* we saw that the no-arbitrage condition yields $0 < p_* < 1$, i.e. p_* is a probability. Statistical quantities computed using p_* will be given a subscript $*$ (e.g. the risk-neutral expectation of X is $E_*[X]$).

In a risk-neutral world, equations (8.19) and (8.22) immediately show that the current price of the call option is:

$$C(t_0) = \mathrm{e}^{-r_f\,h}\,E_*[C(t_1)] = \mathrm{e}^{-r_f\,h}\,\left(p_*\,C_{\mathrm{u}}(t_1) \;+\; (1-p_*)\,C_{\mathrm{d}}(t_1)\right), \tag{8.23}$$

where

$$C_{\mathrm{u}}(t_1) = \max\{S(t_0)\,\mathrm{u} - \mathsf{K}, 0\}, \qquad C_{\mathrm{d}}(t_1) = \max\{S(t_0)\,\mathrm{d} - \mathsf{K}, 0\}. \tag{8.24}$$

A comparison of the 1-period option pricing formulas (8.23) and (8.18) shows that they are identical. In retrospect, we see that the risk-neutral approach of equation (8.23) is a simpler way to obtain the option's current price than the replicating portfolio method.

Remark 8.1. Indeed most would agree that risk-neutral investors are rare in the marketplace and would not be a realistic representation of typical investors. As we saw in the chapters on portfolio theory, risk-averse investors would be realistic. So, we are clearly *not* assuming that investors in the real-world marketplace are risk-neutral. Why then can we price options using the risk-neutral probability p_* as in (8.23)? Perhaps a simple analogy from calculus may help. Suppose that you want to find the volume of a sphere or cylinder. Since the volume of an object is independent of coordinates, we would naturally choose spherical or cylindrical coordinates to find the desired volume since such coordinates take advantage of the symmetry of the objects, making the volume calculation simpler. Of course, one can also obtain the volume of a sphere or cylinder using rectangular coodinates, but the computations would be more complicated. Similarly, we can take advantage of a call option's current price being independent of investors' risk-preferences by choosing the simplest risk-preference, namely, the risk-neutrality of (8.3). What is remarkable is that by calculating the option price in the artificial risk-neutral world, we obtain mathematically the correct price of real-world call options in a simpler way. □

Example 8.1. Suppose that the risk-free rate is 2% per annum and consider a nondividend paying stock with current price of \$50 and an estimated volatility of 0.0225. Using a CRR tree, compute the 1-period current price of a 3-month European call option on this stock given a strike price of \$50?

Solution. The formula for the 1-period European call price at the current time t_0 is:

$$C(t_0) = \mathrm{e}^{-r_f h}\Big[p_* \max\{S(t_0)\,\mathrm{u} - \mathsf{K}, 0\} \;+\; (1-p_*) \max\{S(t_0)\,\mathrm{d} - \mathsf{K}, 0\}\Big],$$

where

$$p_* = \frac{\mathrm{e}^{(r_f - q)\,h} - \mathrm{d}}{\mathrm{u} - \mathrm{d}}.$$

The CRR tree allows us to determine u and d in terms of σ and period h:

$$\mathrm{u} = \mathrm{e}^{\sigma\sqrt{h}}, \qquad \mathrm{d} = \mathrm{e}^{-\sigma\sqrt{h}}.$$

The inputs of the problem yield:

$$h = 0.25, \quad \sqrt{h} = 0.5, \quad r_f = 0.02, \quad q = 0, \quad \sigma = 0.15$$

$$\mathrm{u} = 1.0778, \quad \mathrm{d} = 0.9277, \quad p_* = 0.5147, \quad S(t_0) = \$50, \quad \mathsf{K} = \$50.$$

Direct calculation then yields the current call price:

$$C(t_0) = \$1.99.$$

□

8.3.3 2-Period Binomial Pricing: Risk-Neutral Approach

Consider the 2-period binomial tree in Figure 8.2, where the term of the European call option is from t_0 to t_2.

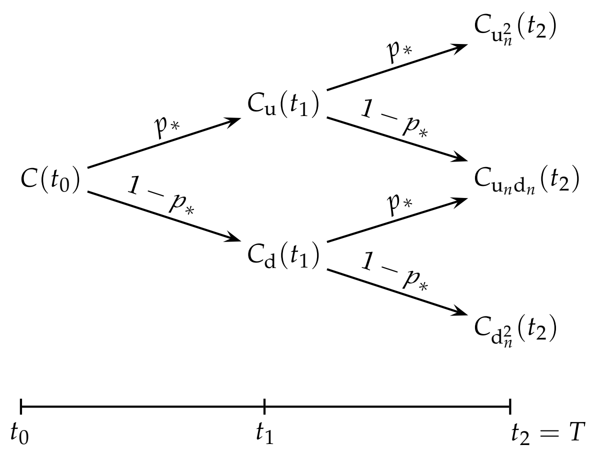

Fig. 8.2 Option pricing using a 2-period binomial tree with risk-neutral probabilities.

To price the call option over the 2-period span $[t_0, t_2]$, the strategy is to break up $[t_0, t_2]$ into two 1-period intervals, namely, $[t_1, t_2]$ and $[t_0, t_1]$, and then apply the 1-period risk-neutral result (8.23) to $[t_1, t_2]$ and then to $[t_0, t_1]$.

Working backwards through the tree, we begin with the 1-period interval $[t_1, t_2]$. To apply the previous 1-period risk-neutral result (8.23) to $[t_1, t_2]$, think of t_1 and t_2 as now playing the respective roles of the "present" time t_0 and the "termination" time t_1 in (8.23). Since the call price $C(t_1)$ at time t_1 has two possible values $C_u(t_1)$ and $C_d(t_1)$, apply (8.23) to each of them with t_0 and t_1 in (8.23) replaced by t_1 and t_2, respectively:

$$C_u(t_1) = e^{-r_f h} E_*[C_u(t_2)], \qquad C_d(t_1) = e^{-r_f h} E_*[C_d(t_2)],$$

where

$$C_u(t_2) = \begin{cases} C_{u^2}(t_2) & \text{with probability } p_* \\ \\ C_{ud}(t_2) & \text{with probability } 1 - p_* \end{cases}$$

and

$$C_d(t_2) = \begin{cases} C_{ud}(t_2) & \text{with probability } p_* \\ \\ C_{d^2}(t_2) & \text{with probability } 1 - p_*. \end{cases}$$

We then obtain the possible call prices at time t_1:

$$\begin{cases} C_{\mathrm{u}}(t_1) = \mathrm{e}^{-r_f h}\,[p_* C_{\mathrm{u}^2}(t_2) \;+\; (1-p_*)\,C_{\mathrm{ud}}(t_2)] \\[2ex] C_{\mathrm{d}}(t_1) = \mathrm{e}^{-r_f h}\,[p_* C_{\mathrm{ud}}(t_2) \;+\; (1-p_*)\,C_{\mathrm{d}^2}(t_2)]\,. \end{cases} \tag{8.25}$$

Continuing to work backwards through the tree, we next consider the 1-period interval $[t_0, t_1]$. Application of (8.23) yields the current call option price as:

$$C(t_0) = \mathrm{e}^{-r_f h} E_*[C(t_1)] = \mathrm{e}^{-r_f h}\,[p_* C_{\mathrm{u}}(t_1) \;+\; (1-p_*)\,C_{\mathrm{d}}(t_1)]\,,$$

where $C_{\mathrm{u}}(t_1)$ and $C_{\mathrm{d}}(t_1)$ are now given by (8.25). The European call option price over the 2-period interval $[t_0, t_2]$ is then given explicity as follows:

$$\boxed{C(t_0) = \mathrm{e}^{-r_f (2h)}\left[p_*^2 C_{\mathrm{u}^2}(t_2) \;+\; 2p_*(1-p_*)\,C_{\mathrm{ud}}(t_2) \;+\; (1-p_*)^2 C_{\mathrm{d}^2}(t_2)\right]} \tag{8.26}$$

Here

$$C_{\mathrm{u}^2}(t_2) = \max\{S(t_0)\,\mathrm{u}^2 - \mathrm{K}, 0\}, \qquad C_{\mathrm{ud}}(t_2) = \max\{S(t_0)\,\mathrm{ud} - \mathrm{K}, 0\},$$
$$C_{\mathrm{d}^2}(t_2) = \max\{S(t_0)\,\mathrm{d}^2 - \mathrm{K}, 0\}.$$

Note that we deliberately separate out $2h$ in (8.26) to emphasize that it is the term of the call option: $2h = t_2 - t_0$.

Now, since

$$\begin{aligned} \mathbb{P}\{C(t_2) = C_{\mathrm{u}^2}(t_2)\} &= p_*^2 \\ \mathbb{P}\{C(t_2) = C_{\mathrm{ud}}(t_2)\} &= 2p_*(1-p_*) \\ \mathbb{P}\{C(t_2) = C_{\mathrm{d}^2}(t_2)\} &= (1-p_*)^2, \end{aligned} \tag{8.27}$$

we get:

$$E_*[C(t_2)] = \left[p_*^2 C_{\mathrm{u}^2}(t_2) \;+\; 2p_*(1-p_*)\,C_{\mathrm{ud}}(t_2) \;+\; (1-p_*)^2 C_{\mathrm{d}^2}(t_2)\right]. \tag{8.28}$$

Therefore, as in the 1-step case we see that the call's price arises from a risk-neutral expectation:

$$\boxed{C(t_0) = \mathrm{e}^{-r_f (2h)} E_*[C(t_2)]} \tag{8.29}$$

Example 8.2. Suppose that the risk-free rate is 2% per annum and consider a nondividend paying stock with current price of \$50 and an estimated annual volatility of 15%. Using a CRR tree, compute the 2-period current price of a 3-month European call option on this stock given a strike price of \$50?

Solution. The 2-period pricing formula is (8.26):

$$C(t_0) = e^{-r_f(2h)} \left[p_*^2 C_{u^2}(t_2) \;+\; 2p_*(1-p_*) C_{ud}(t_2) \;+\; (1-p_*)^2 C_{d^2}(t_2) \right],$$

where

$$C_{u^2}(t_2) = \max\{S(t_0)\,u^2 - K, 0\}, \qquad C_{ud}(t_2) = \max\{S(t_0)\,ud - K, 0\},$$
$$C_{d^2}(t_2) = \max\{S(t_0)\,d^2 - K, 0\}.$$

Since

$$h = \frac{0.25}{2} = 0.125, \quad \sqrt{h} = 0.3536 \quad r_f = 0.02, \quad q = 0, \quad \sigma = 0.15$$

$$u = 1.0545, \quad d = 0.9483, \quad p_* = 0.5104, \quad S(t_0) = \$50, \quad K = \$50,$$

we get

$$C(t_0) = \$1.45.$$

□

8.3.4 *n*-Period Binomial Pricing Formula

In Section 9.1.1 (page 421), we shall show that for the n-period risk-neutral case, the price of the option is given by

$$C(t_0) = e^{-(nh)r_f} \left[\sum_{i=0}^{n} \binom{n}{i} p_*^i (1-p)^{n-i} C_{u^i d^{n-i}}(t_n) \right] \tag{8.30}$$

where $t_0 < t_1 < \cdots < t_n$ and

$$C_{u^i d^{n-i}}(t_n) = \max\{S(t_0) u^i d^{n-i} - K, 0\}$$

for $i = 0, 1, \ldots, n$. It will also be shown that the n-period pricing formula can be expressed as:

$$C(t_0) = e^{-q\tau} N(n, k_*, \hat{p}_*)\, S(t_0) - e^{-r_f\tau} K\, N(n, k_*, p_*), \tag{8.31}$$

where

$$T - t = t_n - t_0, \quad N(n, k_*, p_*) = \sum_{i=k_*}^{n} \binom{n}{i} p_*^i (1-p_*)^{n-i}, \quad \hat{p}_* = (p_* u)/e^{(r_f - q)h}.$$

The reader may notice that the n-period binomial pricing formula (8.31) looks similar to the exact standard BSM formula. In the limit $n \to \infty$, we show in Chapter 10 that (8.31) actually converges to the exact standard BSM formula in (8.5):

$$C(t) = \mathrm{e}^{-q\tau}\mathsf{N}(\mathrm{d}_1)S(t) - \mathrm{e}^{-r_f\tau}\mathsf{K}\mathsf{N}(\mathrm{d}_2)$$

where $\mathsf{N}(x)$ is the standard normal cumulative distribution function and

$$\mathrm{d}_1 = \frac{\log[S(t)/\mathsf{K}] + (T-t)(r_f - q + \sigma^2/2)}{\sigma\sqrt{T-t}}, \qquad \mathrm{d}_2 = d_1 - \sigma\sqrt{T-t}.$$

Example 8.3. Suppose that the risk-free rate is 2% per annum and consider a nondividend paying stock with current price of \$50 and an estimated annual volatility of 15%. Using a CRR tree, compute the 100-period current price of a 3-month European call option on this stock given a strike price of \$50?

Solution. Applying formula (8.30) with the values,

$$\begin{aligned}
&n = 100, \quad h = \frac{0.25}{100} = 0.0025, \quad \sqrt{h} = 0.05, \quad r_f = 0.02, \qquad q = 0, \quad \sigma = 0.15 \\
&\mathrm{u} = 1.0075, \quad \mathrm{d} = 0.9925, \quad p_* = 0.5033, \quad S(t_0) = \$50, \quad \mathsf{K} = \$50,
\end{aligned}$$

we obtain a current call price of

$$C(t_0) = \$1.62.$$

□

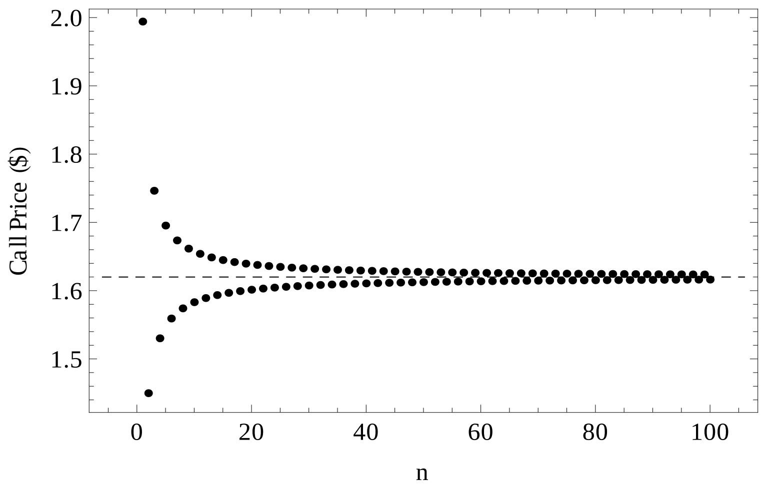

Fig. 8.3 The per share prices of a 3-month European call option, where the strike price is $\mathsf{K} = \$50$ per share, risk-free rate is 0.02 per annum, and underlying stock has current price of \$50 per share and variance of 0.0225 per annum. The bullets show the call prices given by the n-period binomial option-pricing formula using a CRR tree for $n = 1, \ldots, 100$. The horizonal dashed line is the per share call price of \$1.62 obtained using the BSM formula. Even values of n give call prices below the BSM line, while odd values are above. The binomial per share call price is \$1.99 for $n = 1$ and \$1.45 for $n = 2$. The price to two decimal places is already \$1.62 for $n = 80$.