① $X(t) = \log[S(t)/S(0)], \quad t > 0$

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for $S(t) = S(0)\, e^{\mu_{(m)} t + \sigma \mathcal{B}(t)}$

$$X(t) = \log\left[e^{\mu_{(m)} t + \sigma \mathcal{B}(t)}\right] = \mu_{(m)} t + \sigma \mathcal{B}(t)$$

Properties

(i) $\mathbb{P}(X(0) = 0) = \underline{1}$ This is trivial as $\mathcal{B}(0) \overset{def}{=} 0$

(ii) Stationary increments:

Let $s, t \in [0, \infty)$.

Then $X(t) - X(s) = \mu_{(m)} t + \sigma \mathcal{B}(t) - \mu_{(m)} s - \sigma \mathcal{B}(s)$

$= \mu_{(m)}(t-s) + \sigma[\mathcal{B}(t) - \mathcal{B}(s)]$

Since $\mathcal{B}(t)$ has stationary increments, $\mathcal{B}(t) - \mathcal{B}(s) = \mathcal{B}(t-s)$

and so $X(t) - X(s) = \mu_{(m)}(t-s) + \sigma \mathcal{B}(t-s) = X(t-s)$

(iii) Independent Increments:

Let $\{t_1 < t_2 < \dots < t_n\} \in [0, \infty)$

$$\left.\begin{array}{l} \text{Then } X(t_{i+1}) - X(t_i) = X(t_{i+1} - t_i) \\ \text{and } X(t_i) - X(t_{i-1}) = X(t_i - t_{i-1}) \end{array}\right\} \text{ by (ii)}$$

Suppose $X(t_{i+1} - t_i)$ was dependent on $X(t_i - t_{i-1})$.

Since $\mu_{(m)}(t_{i+1} - t_i)$ and $\mu_{(m)}(t_i - t_{i-1})$ are deterministic, the dependence must come between $\sigma \mathcal{B}(t_{i+1} - t_i)$ and $\sigma \mathcal{B}(t_i - t_{i-1})$. This contradicts the independent increments property of $\mathcal{B}(t)$.

see reverse for remainder

Properties (continued)

(iv) Normally distributed.

By construction, $\mathcal{B}(t) \sim \mathcal{N}(0, t)$.

By the properties normally distributed random variables,

$\mu_{(m)} t + \sigma \mathcal{B}(t) \sim \mathcal{N}(\mu_m . t, \sigma^2 t)$ and

so $X(t)$ is normally distributed

(v) $t \to X(t)$ is continuous almost surely (a.s.)

By construction, $\mathcal{B}(t)$ is continuous a.s.

$\mu_{(m)} t$ is linear so it is continuous a.s,

and $\sigma \mathcal{B}(t)$ is continuous a.s. since σ is a constant.

If f and g are continuous a.s, then $f + g$ is also continuous a.s. so

$X(t) = \mu_{(m)} t + \sigma \mathcal{B}(t)$ is continuous a.s.

$X(t) = \mu_{(m)} t + \sigma \mathcal{B}(t)$ satisfies:

$dX(t) = (m - q)\, dt + \sigma\, d\mathcal{B}(t)$. This is the same answer we would get using Itô's lemma, since $X(t)$ is an Itô diffusion-drift process. To check:

$d[\mu_{(m)} t + \sigma \mathcal{B}(t)] = \mu_{(m)}\, dt + \sigma\, d\mathcal{B}(t) + \frac{\sigma^2}{2}\, dt$ by Itô.

$\mu_{(m)} + \sigma^2/2 = m - q$ so both SDEs agree.

$\boxed{2}$ ⓐ $S(t) = S(0)\, e^{\mu(m)t + \sigma \mathcal{B}(t)}$

$$\Rightarrow \quad dS(t) = d\left[S(0)\, e^{\mu(m)t + \sigma \mathcal{B}(t)}\right]$$

$$= S(0)\, e^{\mu(m)t + \sigma \mathcal{B}(t)} \quad d\left[\mu(m)t + \sigma \mathcal{B}(t)\right]$$

$$= S(0)\, e^{\mu(m)t + \sigma \mathcal{B}(t)} \left[\mu(m)dt + \sigma d\mathcal{B}(t) + \frac{\sigma^2}{2} dt\right]$$

$$= S(0)\, e^{\mu(m)t + \sigma \mathcal{B}(t)} \left[(m-q)\, dt + \sigma d\mathcal{B}(t)\right]$$

$$\Rightarrow \quad dS(t) = (m-q)\, S(t)\, dt + \sigma S(t)\, d\mathcal{B}(t)$$

Let δt satisfy: $0 < \delta t << 1$.

$$\Delta S(t) \cong (m-q)\, S(t)\, \delta t + \sigma S(t) \sqrt{\delta t} \cdot Z$$

$$\boxed{E[\Delta S(t)] \cong (m-q)\, S(t) \cdot \delta t}$$

$$Var[\Delta S(t)] \cong \sigma^2 S^2(t) \cdot \delta t \cdot var(Z) = \sigma^2 S^2(t)\, \delta t$$

$$\Rightarrow \boxed{Sdev[\Delta S(t)] \cong \sigma S(t) \sqrt{\delta t}}$$

For δt sufficiently small, the $Sdev[\Delta S(t)]$ will control the security's price fluctuations (in the short term). This is because for $0 < \delta t << 1$, $\sqrt{\delta t} > \delta t$ and so the second term in the SDE dominates.

$\boxed{2}$ (b)

$$\mathbb{E}[S(t)] = S_0 e^{\mu(m)t}\, \mathbb{E}[e^{\sigma B(t)}]$$

$$\mathbb{E}[e^{\sigma B(t)}] = \int_0^\infty \frac{1}{s\sigma\sqrt{2\pi t}} \cdot s \cdot e^{-\frac{\ln^2(s)}{2\sigma^2 t}}\, ds$$

$$= \int_0^\infty \frac{ds}{\sigma\sqrt{2\pi t}}\, e^{-\frac{\ln^2(s)}{2\sigma^2 t}}$$

Let $s = e^x$, $\ln(s) = x$, and $ds = e^x\, dx$

Then $\mathbb{E}[e^{\sigma B(t)}] = \int_0^\infty ds\, \frac{1}{\sigma\sqrt{2\pi t}}\, e^{\frac{-x^2}{2\sigma^2 t}}\, e^x\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{\frac{-1}{2\sigma^2}(x^2 - 2\sigma^2 x t)}\, dx$$

$x^2 - 2\sigma^2 t x = (x-\sigma^2 t)^2 - \sigma^4 t^2$ so substituting

$$= \frac{1}{\sigma\sqrt{2\pi t}} \int_0^\infty e^{\frac{-1}{2\sigma^2 t}(x-\sigma^2 t)^2}\, e^{-\frac{1}{2\sigma^2 t}(-\sigma^4 t^2)}\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi t}} \cdot e^{\sigma^2/2 \cdot t} \cdot \sigma\sqrt{2\pi t}$$

$$= e^{\sigma^2 t/2}$$

Hence $\mathbb{E}[S(t)] = S_0 e^{\mu(m)t} \cdot e^{\sigma^2 t/2}$

$$\boxed{\mathbb{E}[S(t)] = S_0 e^{[\mu(m) + \sigma^2/2]t} = S_0 e^{(m-q)t}}$$

$\boxed{2}$ (b)

$$\mathbb{E}[S^2(t)] = S^2(0)\, e^{2\mu(m)t}\, \mathbb{E}[e^{2\sigma B(t)}]$$

$\mathbb{E}[e^{2\sigma B(t)}] = e^{(2\sigma)^2 t/2}$ by the calculation for $\mathbb{E}[e^{\sigma B(t)}]$ taking $\sigma \to 2\sigma$,

so
$$\mathbb{E}[S^2(t)] = S^2(0)\, e^{2\mu(m)t}\, e^{2\sigma^2 t}$$
$$= S^2(0)\, e^{(2\mu(m)+\sigma^2)t}\, e^{\sigma^2 t}$$

$$\left(\mathbb{E}[S(t)]\right)^2 = S^2(0)\left[e^{(\mu(m)+\sigma^2/2)t}\right]^2$$
$$= S^2(0)\, e^{(2\mu(m)+\sigma^2)t}$$

$$\mathrm{Var}[S(t)] = \mathbb{E}[S^2(t)] - [\mathbb{E}\, S(t)]^2$$
$$= S^2(0)\, e^{(2\mu(m)+\sigma^2)t}\, e^{\sigma^2 t}$$
$$- S^2(0)\, e^{(2\mu(m)+\sigma^2)t}$$

$$\boxed{\mathrm{Var}[S(t)] = S^2(0)\, e^{(2\mu(m)+\sigma^2)t}\left[e^{\sigma^2 t} - 1\right]}$$

$\boxed{2}$ (b)

$\text{Med}[S(t)] = K$ s.t.

$$\mathbb{P}\left(\frac{S(t)}{S(0)} > \frac{K}{S(0)}\right) = 1/2 = \int_{\frac{K}{S(0)}}^{\infty} \frac{1}{s\sigma\sqrt{2\pi t}} e^{-\frac{1}{2\sigma^2 t}[\ln(s) - \mu_{(m)}t]^2} ds$$

Letting $ds = e^x dx$, $s = e^x$, $\ln(s) = x$.

$$1/2 = \int_{\ln\left(\frac{K}{S(0)}\right)}^{\infty} \frac{1}{\sigma\sqrt{2\pi t}} e^{-x} e^{-\frac{1}{2\sigma^2 t}(x - \mu_{(m)})^2} e^{x} dx$$

$$\Rightarrow \frac{1}{2} = \frac{1}{\sigma\sqrt{2\pi t}} \int_{\ln\left(\frac{K}{S(0)}\right)}^{\infty} e^{-\frac{1}{2\sigma^2 t}(x - \mu_{(m)}t)^2} dx$$

By symmetry, this integral is 1/2 when $\ln\left(\frac{K}{S(0)}\right) = \mu_{(m)}t$ hence $K = S(0)e^{\mu_{(m)}t}$

So the median, $\text{Med}[S(t)] = S(0)e^{\mu_{(m)}t}$

$$S(0)e^{\mu_{(m)}t} = S(0)e^{(m-q)t - \sigma^2 t/2} = e^{-\sigma^2 t/2}\,\mathbb{E}[S(t)]$$

(3) $S(t) = S(0)\, e^{\mu_{c,m,}t + \sigma B(t)}$

Let $K > 0$.

$$\mathbb{P}(S(t) > K) = \mathbb{P}\left(\frac{S(t)}{S(0)} > \frac{K}{S(0)}\right)$$

$$= \int_{\frac{K}{S_0}}^{\infty} \frac{ds}{s\sqrt{2\pi}\,\sigma\sqrt{t}} \cdot e^{\frac{-(\ln(s) - \mu_{c,m,}t)^2}{2\sigma^2 t}}$$

Let $x = \ln s$, $e^x = s \Rightarrow ds = e^x dx$

$$\Rightarrow \int_{\ln(\frac{K}{S_0})}^{\infty} \frac{e^x\, dx}{e^x \sigma\sqrt{2\pi t}} \cdot e^{-\frac{(x - \mu_{c,m,}t)^2}{2\sigma^2 t}} = \int_{\ln(\frac{K}{S_0})}^{\infty} \frac{1}{\sqrt{2\pi(\sigma^2 t)}}\, e^{-\frac{(x - \mu_{c,m,}t)}{2(\sigma^2 t)}}$$

This is a normal distribution integral with

$\sigma^2 \to \sigma^2 t$, $\mu \to \mu_{c,m,}t$, going from

$\ln(\frac{K}{S_0})$ to ∞, i.e. it represents $\mathbb{P}(x > \ln(K/S_0))$

when $x \sim N(\mu_{c,m,}t, \sigma^2 t)$.

i.e. $\mathbb{P}(S(t) > K) = \mathbb{P}(x > \ln(K/S_0))$

$$= \mathbb{P}\left(\frac{x - \mu_{c,m,}t}{\sigma\sqrt{t}} > \frac{\ln(K/S_0) - \mu_{c,m,}t}{\sigma\sqrt{t}}\right)$$

$$= \mathbb{P}\left(\frac{x - \mu_{c,m,}t}{\sigma\sqrt{t}} > \frac{-\ln(S_0/K) - \mu_{c,m,}t}{\sigma\sqrt{t}}\right) = \mathbb{P}(Z > -d_{2,m})$$

$N(z) = \mathbb{P}(Z < z)$ so $\mathbb{P}(Z > -d_{2,m}) = 1 - \mathbb{P}(Z < -d_{2,m})$

Hence $\mathbb{P}(Z > -d_{2,m}) = 1 - N(-d_{2,m}) = N(d_{2,m})$ by the properties of $N(\cdot)$.

Likewise: $\mathbb{P}(S(t) < K) = 1 - \mathbb{P}(S(t) > K) = 1 - N(d_{2,m}) = N(-d_{2,m})$

$\therefore$ $\mathbb{P}(S(t) > K) = N(d_{2,m})$ and $\mathbb{P}(S(t) < K) = N(-d_{2,m})$ $\square$

④ I agree. Let $\sigma^2 > 2(m-q)$.

Then $$\mathrm{Med}[S(t)] = e^{-\frac{\sigma^2}{2}t} \cdot \mathbb{E}[S(t)]$$
$$= S(0)\, e^{-\frac{\sigma^2}{2}t}\, e^{(m-q)t}$$
$$= S(0)\, e^{-\frac{t}{2}(m-q-\sigma^2)} < S(0).$$

So $\sigma^2 > 2(m-q) \Rightarrow \mathrm{Med}[S(t)] < S(0)$

Hence, $$\mathbb{P}(S(t) < S(0)) = \mathbb{P}(S(t) < \mathrm{Med}[S(t)]) + \mathbb{P}(\mathrm{Med}[S(t)] < S(t) < S(0))$$
$$= \frac{1}{2} + \mathbb{P}(\mathrm{Med}[S(t)] < S(t) < S(0))$$

Since $\mathrm{Med}[S(t)] < S(0)$, $\mathbb{P}(\mathrm{Med}[S(t)] < S(t) < S(0)) > 0$.

And so $\mathbb{P}(S(t) < S(0)) > \frac{1}{2}$. $\square$

[5] Show that $\mathbb{E}(S(t) \mid S(t) > k) = \frac{N(d_{1,m})}{N(d_{2,m})} \mathbb{E}(S(t))$

where $d_{2,m} = \frac{\ln(S/k) - \mu_{m,t}}{\sigma\sqrt{t}}$ and

$$d_{1,m} = d_{2,m} + \sigma\sqrt{t}$$

$$\mathbb{E}[S(t) \mid S(t) > k] = \int_0^\infty s\,\mathbb{P}(s \mid s>k)\, ds$$

$$\mathbb{P}(s \mid s>k) = \frac{\mathbb{P}(s) \cdot \mathbb{1}(s>k)}{\int_0^\infty \mathbb{P}(s) \cdot \mathbb{1}(s>k)\, ds}$$

$$\Rightarrow \mathbb{E}[S(t) \mid S(t) > k] = \frac{\int_0^\infty s\,\mathbb{P}(s) \cdot \mathbb{1}(s>k)\, ds}{\int_0^\infty \mathbb{P}(s)\, \mathbb{1}(s>k)\, ds}$$

$$= \frac{\int_k^\infty s\,\mathbb{P}(s)\, ds}{\int_k^\infty \mathbb{P}(s)\, ds}$$

The denominator is simply $\mathbb{P}(S(t) > k) = N(d_{2,m})$ from problem (3)

The numerator is $\int_k^\infty S \cdot \frac{ds}{S \cdot \sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(\ln(S/S_0) - \mu_{m,t})^2}{2\sigma^2 t}}$.

Let $x = \ln(S/S_0)$, $S = S_0 e^x$, $ds = S_0 e^x dx$, yielding:

$$\int_{\ln(k/S_0)}^\infty \frac{S_0 e^x}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(x-\mu_{m,t})^2}{2\sigma^2 t}}\, dx = \frac{S_0}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(k/S_0)}^\infty e^{\frac{-1}{2\sigma^2 t}\left[(x-\mu_{m,t})^2 - 2\sigma^2 t x\right]}\, dx$$

$$(x-\mu_{m,t})^2 - 2\sigma^2 t x = x^2 - 2\mu_{m,t} x + \mu_{m,}^2 t^2 - 2\sigma^2 t x$$

$$= x^2 - 2x(\mu_{m,} t + \sigma^2 t) + \mu_{m,}^2 t^2$$

$$= [x - (\mu_{m,} t + \sigma^2 t)]^2 + \mu_{m,}^2 t^2 - (\mu_{m,} t + \sigma^2 t)^2$$

see reverse →

$\boxed{5}$ | continued |

$$\mathbb{E}\left[S(t) \mid S(t) > k\right] = \frac{1}{N(d_{2,m})} \cdot \frac{S_0}{\sqrt{2\pi\sigma^2 t}} \int_{\ln(k/S_0)}^{\infty} e^{\frac{-1}{2\sigma^2 t}\left[(x-\mu_{(m)}t)^2 - 2\sigma^2 t x\right]} dx$$

$$\int_{\ln(k/S_0)}^{\infty} \exp\left\{\frac{-1}{2\sigma^2 t}\left[(x-\mu_{(m)})^2 - 2\sigma^2 t x\right]\right\} dx$$

$$= \int_{\ln(k/S_0)}^{\infty} \exp\left\{\frac{-1}{2\sigma^2 t}\left[x-(\mu_{(m)}t+\sigma^2 t)\right]^2 + \mu_{(m)}^2 t^2 - (\mu_{(m)}t+\sigma^2 t)^2\right\} dx$$

$$\mu_{(m)}^2 t^2 - (\mu_{(m)}t + \sigma^2 t)^2 = \cancel{\mu_{(m)}^2 t^2} - \cancel{\mu_{(m)}^2 t^2} - \sigma^4 t^2 - 2\mu_{(m)} t \sigma^2 t$$

$$\Rightarrow \int_{\ln(k/S_0)}^{\infty} \exp\left\{\frac{-1}{2\sigma^2 t}\left[(x-(\mu_{(m)}t+\sigma^2 t))\right]^2 + \mu_{(m)}^2 t^2 - (\mu_{(m)}t+\sigma^2 t)^2\right\} dx$$

$$= e^{\frac{1}{2\sigma^2 t}\left[2\mu_{(m)} t \sigma^2 t + \sigma^4 t^2\right]} \cdot \int_{\ln(k/S_0)}^{\infty} e^{\frac{-1}{2\sigma^2 t}\left(x-(\mu_{(m)}t+\sigma^2 t)\right)^2} dx$$

$$= e^{\mu_{(m)}t + \frac{\sigma^2}{2}t} \cdot \int_{\ln(k/S_0)}^{\infty} e^{\frac{-1}{2\sigma^2 t}\left(x-(\mu_{(m)}t+\sigma^2 t)\right)^2} dx$$

Let $\dfrac{x-[\mu_{(m)}t+\sigma^2 t]}{\sigma\sqrt{t}} = z \Rightarrow \displaystyle\int_{k^*}^{\infty} e^{-\frac{1}{2}z^2} \cdot \sigma\sqrt{t}\, dz$

where $K^* = \dfrac{\ln(K/S_0) - (\mu_{(m)}t + \sigma^2 t)}{\sigma\sqrt{t}} = -d_{1,m}$

$$\sqrt{2\pi} \cdot \sigma\sqrt{t} \cdot \int_{k^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \sqrt{2\pi\sigma^2 t}\, N(-k^*)$$

$$\Rightarrow \mathbb{E}\left[S(t) \mid S(t) > k\right] = \frac{1}{N(d_{2,m})} \cdot S_0 e^{\mu_{(m)}t + \frac{\sigma^2}{2}t} \cdot N(d_{1,m})$$

$$S_0 e^{\mu_{(m)}t + \frac{\sigma^2}{2}t} = S_0 e^{(m-q)t} = \mathbb{E}[S(t)]$$

$$\Rightarrow \mathbb{E}\left[S(t) \mid S(t) > k\right] = \frac{N(d_{1,m})}{N(d_{2,m})} \mathbb{E}[S(t)] \quad \blacksquare$$