Problem Set:
- 7.1 [6, 8, 14]
- 7.2 [6, 12, 16, 18]
- 7.3 [2, 8]

7.1

7.1 [6]

If a 12×12 matrix $\mathbf{A}$ shows the distances between 12 cities in kilometers, how can you obtain from $\mathbf{A}$ the matrix $\mathbf{B}$ showing these distances in miles?

$$1 \text{ mi} \approx 1.609 \text{ km}$$

$$\mathbf{B} \approx \frac{1}{1.609}\mathbf{A} \tag{1}$$

$$\mathbf{B} \approx 0.6214\mathbf{A} \tag{2}$$

7.1 [8]

Find the following expressions, indicating which of the rules in (3) or (4) they illustrate, or give reasons why they are not defined.

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix} \tag{1}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} \tag{2}$$

i) 2A + 4B

$$(3),(4) \qquad 2\mathbf{A} + 4\mathbf{B} = 2\begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix} + 4\begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 12 & 10 & 10 \\ 2 & 0 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 20 & 8 \\ 20 & 12 & 16 \\ -8 & 16 & -8 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix} \tag{5}$$

ii) 4B + 2A

$$\text{by 3a} \qquad 4\mathbf{B} + 2\mathbf{A} = 2\mathbf{A} + 4\mathbf{B} \tag{6}$$

$$(7) \qquad = \begin{bmatrix} 0 & 24 & 16 \\ 32 & 22 & 26 \\ -6 & 16 & -14 \end{bmatrix} \tag{7}$$

iii) 0A + B

$$
\begin{aligned}
& & 0\mathbf{A} + \mathbf{B} &= \mathbf{0} + \mathbf{B} & (8)\\
&\text{by 3c} & &= \mathbf{B} & (9)\\
&(4) & &= \begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} & (10)
\end{aligned}
$$

iv) 0.4B - 4.2A

$$
\begin{aligned}
&(3),(4) & 0.4\mathbf{B} - 4.2\mathbf{A} &= 0.4\begin{bmatrix} 0 & 5 & 2 \\ 5 & 3 & 4 \\ -2 & 4 & -2 \end{bmatrix} - 4.2\begin{bmatrix} 0 & 2 & 4 \\ 6 & 5 & 5 \\ 1 & 0 & -3 \end{bmatrix} & (11)\\
& & &= \begin{bmatrix} 0 & 2 & 0.8 \\ 2 & 1.2 & 1.6 \\ -0.8 & 1.6 & -0.8 \end{bmatrix} + \begin{bmatrix} 0 & -8.4 & -16.8 \\ -25.2 & -21 & -21 \\ -4.2 & 0 & 12.6 \end{bmatrix} & (12)\\
& & &= \begin{bmatrix} 0 & -6.4 & -16 \\ -23.2 & -19.8 & -19.4 \\ -5 & 1.6 & 11.8 \end{bmatrix} & (13)
\end{aligned}
$$

7.1 [14]

Find the following expressions, indicating which of the rules in (3) or (4) they illustrate, or give reasons why they are not defined.

$$
\mathbf{u} = \begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 0 & 2 \\ 3 & 4 \\ 3 & -1 \end{bmatrix} \qquad (1)
$$

i) (5u + 5v) - $\frac{1}{2}$w

$$
\begin{aligned}
&\text{by 4a} & (5\mathbf{u} + 5\mathbf{v}) - \frac{1}{2}\mathbf{w} &= 5(\mathbf{u} + \mathbf{v}) - \frac{1}{2}\mathbf{w} & (2)\\
&(1) & &= 5(\begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}) + (-\frac{1}{2})\begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} & (3)\\
& & &= 5\begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix} + (-\frac{1}{2})\begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} & (4)\\
& & &= 5\begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix} & (5)\\
&\text{by 4a} & &= 5(\begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix}) & (6)\\
&\text{by 4b} & &= 5(1+1)\begin{bmatrix} 0.5 \\ 3 \\ -1 \end{bmatrix} & (7)
\end{aligned}
$$

$$= \begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix} \tag{8}$$

ii) -20(u + v) + 2w

$$\begin{aligned}
&\text{by 4a} && -20(\mathbf{u}+\mathbf{v}) + 2\mathbf{w} = -4[5(\mathbf{u}+\mathbf{v}) - \frac{1}{2}\mathbf{w}] && (9)\\
&(8) && = -4\begin{bmatrix} 5 \\ 30 \\ -10 \end{bmatrix} && (10)\\
& && = \begin{bmatrix} -20 \\ -120 \\ 40 \end{bmatrix} && (11)
\end{aligned}$$

iii) E - (u + v)

$$(1) \qquad \mathbf{E} - (\mathbf{u}+\mathbf{v}) = \begin{bmatrix} 0 & 2 \\ 3 & 4 \\ 3 & -1 \end{bmatrix} - (\begin{bmatrix} 1.5 \\ 0 \\ -3.0 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}) \tag{12}$$

Not defined. Matrices of different size (i.e. 3×2 vs. 3×1) cannot be added/subtracted.

iv) 10(u + v) + w

$$\begin{aligned}
&(1) && 10(\mathbf{u}+\mathbf{v}) + \mathbf{w} = 10(\mathbf{u}+\mathbf{v}) + \begin{bmatrix} -5 \\ -30 \\ 10 \end{bmatrix} && (13)\\
& && = 10(\mathbf{u}+\mathbf{v}) + 10\begin{bmatrix} -0.5 \\ -3 \\ 1 \end{bmatrix} && (14)\\
&\text{by 4a} && = 10(\mathbf{u}+\mathbf{v} + \begin{bmatrix} -0.5 \\ -3 \\ 1 \end{bmatrix}) && (15)\\
&(4) && = 10(\begin{bmatrix} 0.5 \\ 0 \\ -3.0 \end{bmatrix} + \begin{bmatrix} -0.5 \\ -3 \\ 1 \end{bmatrix}) && (16)\\
&\text{by 3d} && = (10)\mathbf{0} && (17)\\
& && = \mathbf{0} && (18)
\end{aligned}$$

7.2

7.2 [6]

If $\mathbf{U_1}$, $\mathbf{U_2}$ are upper triangle and $\mathbf{L_1}$, $\mathbf{L_2}$ are lower triangular, which of the following are triangular?
Assuming the matrices are all the same size, let

$$\mathbf{U_1} = \begin{bmatrix} u & u & u \\ 0 & u & u \\ 0 & 0 & u \end{bmatrix} \qquad \mathbf{U_2} = \begin{bmatrix} v & v & v \\ 0 & v & v \\ 0 & 0 & v \end{bmatrix} \qquad \mathbf{L_1} = \begin{bmatrix} l & 0 & 0 \\ l & l & 0 \\ l & l & l \end{bmatrix} \qquad \mathbf{L_2} = \begin{bmatrix} m & 0 & 0 \\ m & m & 0 \\ m & m & m \end{bmatrix} \tag{1}$$

i) $\mathbf{U_1} + \mathbf{U_2}$ is triangular:

$$(1) \qquad \mathbf{U_1} + \mathbf{U_2} = \begin{bmatrix} u+v & u+v & u+v \\ 0 & u+v & u+v \\ 0 & 0 & u+v \end{bmatrix} \tag{2}$$

ii) $\mathbf{U_1 U_2}$ is triangular:

$$(1) \qquad \mathbf{U_1 U_2} = \begin{bmatrix} uv & 2uv & 3uv \\ 0 & uv & 2uv \\ 0 & 0 & uv \end{bmatrix} \tag{3}$$

iii) $\mathbf{U_1}^{\mathbf{2}}$ is triangular:

$$(1) \qquad \mathbf{U_1}^{\mathbf{2}} = \begin{bmatrix} u^2 & 2u^2 & 3u^2 \\ 0 & u^2 & 2u^2 \\ 0 & 0 & u^2 \end{bmatrix} \tag{4}$$

iv) $\mathbf{U_1} + \mathbf{L_1}$ is not triangular:

$$(1) \qquad \mathbf{U_1} + \mathbf{L_1} = \begin{bmatrix} u+l & u & u \\ l & u+l & u \\ l & l & u+l \end{bmatrix} \tag{5}$$

v) $\mathbf{U_1 L_1}$ is not triangular:

$$(1) \qquad \mathbf{U_1 L_1} = \begin{bmatrix} 3ul & 2ul & ul \\ 2ul & 2ul & ul \\ ul & ul & ul \end{bmatrix} \tag{6}$$

vi) $\mathbf{L_1} + \mathbf{L_2}$ is triangular:

$$(1) \qquad \mathbf{L_1} + \mathbf{L_2} = \begin{bmatrix} l+m & 0 & 0 \\ l+m & l+m & 0 \\ l+m & l+m & l+m \end{bmatrix} \tag{7}$$

7.2 [12]

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{1}$$

i) $\mathbf{AA^T}$

$$(1) \qquad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix} \tag{2}$$

$$(1),(2) \qquad \mathbf{AA}^{\mathrm{T}} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 6 & 2 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 16+4+9 & -8-2+18 & 4-4+6 \\ -8-2+18 & 4+1+36 & -2+2+12 \\ 4-4+6 & -2+2+12 & 1+4+4 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 29 & 8 & 6 \\ 8 & 41 & 12 \\ 6 & 12 & 9 \end{bmatrix} \tag{5}$$

ii) $\mathbf{A^2}$

$$(1) \qquad \mathbf{A}^2 = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} 16+4+3 & -8-2+6 & 12-12+6 \\ -8-2+6 & 4+1+12 & -6+6+12 \\ 4-4+2 & -2+2+4 & 3+12+4 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 23 & -4 & 6 \\ -4 & 17 & 12 \\ 2 & 4 & 19 \end{bmatrix} \tag{8}$$

iii) $\mathbf{BB^T}$

$$(1) \qquad \mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{9}$$

$$(1) \qquad = \mathbf{B} \tag{10}$$

$$(10) \qquad \mathbf{BB}^{\mathrm{T}} = \mathbf{B}^2 \tag{11}$$

$$(1) \qquad = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} 1+9+0 & -3-3+0 & 0+0+0 \\ -3-3+0 & 9+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{14}$$

iv) $\mathbf{B^2}$

$$(11),(14) \qquad \mathbf{B}^2 = \begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{15}$$

7.2 [16]

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

$$\mathbf{B} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

i) BC

$$(1) \qquad \mathbf{BC} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -2 & 0 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 0-9+0 & 1-6+0 \\ 0+3+0 & -3+2+0 \\ 0+0+4 & 0+0+0 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} -9 & -5 \\ 3 & -1 \\ 4 & 0 \end{bmatrix} \tag{4}$$

ii) $\mathbf{BC^T}$

$$(1) \qquad \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 2 & 0 \end{bmatrix} \tag{5}$$

$$\mathbf{BC}^{\mathrm{T}} \quad \text{DNE} \tag{6}$$

Multiplication of a 3×3 matrix (**B**) by a 2×3 matrix ($\mathbf{C}^{\mathrm{T}}$) is undefined. The number of columns in **B** (in this case, 3) has to equal the number of rows in $\mathbf{C}^{\mathrm{T}}$ (in this case, 2) for matrix multiplication to be defined.

iii) Bb

$$(1) \qquad \mathbf{Bb} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 3-3+0 \\ -9+1+0 \\ 0+0+2 \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix} \tag{9}$$

iv) $\mathbf{b^TB}$

$$(1) \qquad \mathbf{b}^{\mathrm{T}} = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \tag{10}$$

$$(1),(10) \qquad \mathbf{b}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{11}$$

$$= \begin{bmatrix} 3-3+0 & -9+1+0 & 0+0+2 \end{bmatrix} \tag{12}$$

$$= \begin{bmatrix} 0 & -8 & 2 \end{bmatrix} \tag{13}$$

7.2 [18]

Showing all intermediate results, calculate the following expressions or give reasons why they are undefined:

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{1}$$

i) ab

(1)

$$\mathbf{ab} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \tag{2}$$
$$= \begin{bmatrix} 3 - 2 + 0 \end{bmatrix} \tag{3}$$
$$= \begin{bmatrix} 1 \end{bmatrix} \tag{4}$$

ii) ba

(1)

$$\mathbf{ba} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \tag{5}$$
$$= \begin{bmatrix} 3 & -6 & 0 \\ 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix} \tag{6}$$

iii) aA

(1)

$$\mathbf{aA} = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ -2 & 1 & 6 \\ 1 & 2 & 2 \end{bmatrix} \tag{7}$$
$$= \begin{bmatrix} 4 + 4 + 0 & -2 - 2 + 0 & 3 - 12 + 0 \end{bmatrix} \tag{8}$$
$$= \begin{bmatrix} 8 & -4 & -9 \end{bmatrix} \tag{9}$$

iv) Bb

from 7.2 [16] (iii), eq. (9)

$$\mathbf{Bb} = \begin{bmatrix} 0 \\ -8 \\ 2 \end{bmatrix} \tag{10}$$

7.3

7.3 [2]

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$\begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 1.5 & 4.5 & 6.0 \end{bmatrix} \tag{1}$$

$$\mathbf{R_2} = \mathbf{R_2} + (-\frac{1}{2})\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 1.5 - 1.5 & 4.5 + 0.25 & 6.0 - 0.3 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 0 & 4.75 & 5.7 \end{bmatrix} \tag{3}$$

$$\mathbf{R_2} = \frac{1}{4.75}\mathbf{R_2} \longrightarrow \quad = \begin{bmatrix} 3.0 & -0.5 & 0.6 \\ 0 & 1 & 1.2 \end{bmatrix} \tag{4}$$

This represents the following linear system:

$$(4) \quad 3.0x_1 + 0.5x_2 = 0.6 \tag{5}$$

$$x_2 = 1.2 \tag{6}$$

$$(6) \rightarrow (5) \quad 3.0x_1 + 0.5(1.2) = 0.6 \tag{7}$$

$$x_1 = 0.4 \tag{8}$$

From lines (8), (6):

$$\boxed{\begin{aligned} x_1 &= 0.4 \\ x_2 &= 1.2 \end{aligned}}$$

7.3 [8]

Solve the linear system given explicitly or by its augmented matrix. Show details.

$$4y + 3z = 8 \tag{1}$$

$$2x - z = 2 \tag{2}$$

$$3x + 2y = 5 \tag{3}$$

This can be represented by the matrix

$$\begin{bmatrix} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 & 5 \\ 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \end{bmatrix} \tag{4}$$

$$\mathbf{R_2} = \mathbf{R_2} + (-\frac{2}{3})\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 3 & 2 & 0 & 5 \\ 2-2 & 0-\frac{4}{3} & -1 & 2-\frac{10}{3} \\ 0 & 4 & 3 & 8 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \\ 0 & 4 & 3 & 8 \end{bmatrix} \tag{6}$$

$$\mathbf{R_3} = \mathbf{R_3} + (3)\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \\ 0 & 4-4 & 3-3 & 8-4 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & 0 & 4 \end{bmatrix} \tag{8}$$

$\mathbf{R_3}$ represents the equation $0 = 4$, which is not valid. Hence, **there is no solution** to the linear system.