Problem Set:
8.1 [3, 4, 6, 12, 14]

8.1

8.1 [3]

Find the eigenvalues. Find the corresponding eigenvalues.

$$\begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix} \tag{1}$$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \rightarrow \quad \begin{vmatrix} (5-\lambda) & -2 \\ 9 & (-6-\lambda) \end{vmatrix} = 0 \tag{2}$$

$$\lambda^2 - \lambda - 12 = 0 \tag{3}$$

$$\lambda = -4, 3 \tag{4}$$

$$\lambda = -4 \rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0: \quad \begin{bmatrix} (5-(-4)) & -2 \\ 9 & (-6-(-4)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

$$\left[\begin{array}{cc|c} 9 & -2 & 0 \\ 9 & -2 & 0 \end{array}\right] \tag{6}$$

$$\mathbf{R_2} = \mathbf{R_1} - \mathbf{R_2} \quad \left[\begin{array}{cc|c} 9 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{7}$$

$$\mathbf{R_1}\text{: Let} \quad x_1 = 2 \tag{8}$$

$$x_2 = 9 \tag{9}$$

$$(8), (9) \quad \mathbf{x}_{\lambda\,=\,-4} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} \tag{10}$$

$$\lambda = 3 \rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0: \quad \begin{bmatrix} (5-3) & -2 \\ 9 & (-6-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 9 & -9 & 0 \end{array}\right] \tag{12}$$

$$\mathbf{R_2} = \mathbf{R_2} - \frac{2}{9}\mathbf{R_1} \rightarrow \quad \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{13}$$

$$\mathbf{R_1}\text{: Let} \quad x_1 = 1 \tag{14}$$

$$x_2 = 1 \tag{15}$$

$$(14), (15) \quad \mathbf{x}_{\lambda\,=\,3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{16}$$

The pairs of eigenvalues and eigenvectors are $-4, \begin{bmatrix} 2 \\ 9 \end{bmatrix}$ and $3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8.1 [4]

Find the eigenvalues. Find the corresponding eigenvectors.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{1}$$

$$\left|\mathbf{A}-\lambda\mathbf{I}\right| = 0 \rightarrow \quad \begin{vmatrix}(1-\lambda) & 2 \\ 2 & (4-\lambda)\end{vmatrix} = 0 \tag{2}$$

$$\lambda^2 - 5\lambda = 0 \tag{3}$$

$$\lambda = 0, 5 \tag{4}$$

$$\lambda = 0 \rightarrow (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0 : \quad \begin{bmatrix}(1-0) & 2 \\ 2 & (4-0)\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{5}$$

$$\begin{bmatrix}1 & 2 & | & 0 \\ 2 & 4 & | & 0\end{bmatrix} \tag{6}$$

$$\mathbf{R_2} = \mathbf{R_2} - 2\mathbf{R_1} \rightarrow \quad \begin{bmatrix}1 & 2 & | & 0 \\ 0 & 0 & | & 0\end{bmatrix} \tag{7}$$

$$\mathbf{R_1}\text{: Let} \quad x_1 = 2 \tag{8}$$

$$x_2 = -1 \tag{9}$$

$$(8), (9) \quad \mathbf{x}_{\lambda\,=\,0} = \begin{bmatrix}2 \\ -1\end{bmatrix} \tag{10}$$

$$\lambda = 5 \rightarrow (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0 : \quad \begin{bmatrix}(1-5) & 2 \\ 2 & (4-5)\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{11}$$

$$\begin{bmatrix}-4 & 2 & | & 0 \\ 2 & -1 & | & 0\end{bmatrix} \tag{12}$$

$$\mathbf{R_2} = \mathbf{R_2} + \frac{1}{2}\mathbf{R_1} \rightarrow \quad \begin{bmatrix}-4 & 2 & | & 0 \\ 0 & 0 & | & 0\end{bmatrix} \tag{13}$$

$$\mathbf{R_1}\text{: Let} \quad x_1 = 1 \tag{14}$$

$$x_2 = 2 \tag{15}$$

$$(14), (15) \quad \mathbf{x}_{\lambda\,=\,5} = \begin{bmatrix}1 \\ 2\end{bmatrix} \tag{16}$$

The pairs of eigenvalues and eigenvectors are 0, $\begin{bmatrix}2 \\ -1\end{bmatrix}$ and 5, $\begin{bmatrix}1 \\ 2\end{bmatrix}$.

8.1 [6]

Find the eigenvalues. Find the corresponding eigenvectors.

$$\begin{bmatrix}1 & 2 \\ 0 & 3\end{bmatrix} \tag{1}$$

$$\left|\mathbf{A}-\lambda\mathbf{I}\right| = 0 \rightarrow \quad \begin{vmatrix}(1-\lambda) & 2 \\ 0 & (3-\lambda)\end{vmatrix} = 0 \tag{2}$$

$$(\lambda-3)(\lambda-1) = 0 \tag{3}$$

$$\lambda = 1, 3 \tag{4}$$

$$\lambda = 1 \rightarrow (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0 : \quad \begin{bmatrix}(1-1) & 2 \\ 0 & (3-1)\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{5}$$

$$\begin{bmatrix}0 & 2 \\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{6}$$

$$\mathbf{R_1}, \mathbf{R_2} : \quad x_2 = 0 \tag{7}$$

$$\text{Let} \quad x_1 = 1 \tag{8}$$

$$(7), (8) \quad \mathbf{x}_{\lambda\,=\,1} = \begin{bmatrix}1 \\ 0\end{bmatrix} \tag{9}$$

$$\lambda = 3 \rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 : \quad \begin{bmatrix} (1-3) & 2 \\ 0 & (3-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

$$\mathbf{R_1} : \quad x_1 = x_2 \tag{12}$$

$$\text{Let} \quad x_1 = 1 \tag{13}$$

$$x_2 = 1 \tag{14}$$

$$(13),(14) \quad \mathbf{x}_{\lambda\,=\,3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{15}$$

The pairs of eigenvalues and eigenvectors are 1, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and 3, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8.1 [12]

Find the eigenvalues. Find the corresponding eigenvalues.

$$\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \rightarrow \quad \begin{vmatrix} (3-\lambda) & 5 & 3 \\ 0 & (4-\lambda) & 6 \\ 0 & 0 & (1-\lambda) \end{vmatrix} = 0 \tag{2}$$

$$(3-\lambda) \begin{vmatrix} (4-\lambda) & 6 \\ 0 & (1-\lambda) \end{vmatrix} - 0 + 0 = 0 \tag{3}$$

$$(3-\lambda)(4-\lambda)(1-\lambda) = 0 \tag{4}$$

$$\lambda = 1, 3, 4 \tag{5}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0, \lambda = 1 : \quad \begin{bmatrix} (3-1) & 5 & 3 \\ 0 & (4-1) & 6 \\ 0 & 0 & (1-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

$$\text{R}_3,\ \text{R}_2\text{: Let} \quad x_3 = 2 \tag{8}$$

$$\text{R}_2\text{:} \quad 3x_2 + 6(2) = 0 \tag{9}$$

$$x_2 = -4 \tag{10}$$

$$\text{R}_1\text{:} \quad 2x_1 + 5(-4) + 3(2) = 0 \tag{11}$$

$$x_1 = -7 \tag{12}$$

$$(8),(10),(12) \quad \mathbf{x}_{\lambda\,=\,1} = \begin{bmatrix} -7 \\ -4 \\ 2 \end{bmatrix} \tag{13}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0, \lambda = 3 : \quad \begin{bmatrix} (3-3) & 5 & 3 \\ 0 & (4-3) & 6 \\ 0 & 0 & (1-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

$$\text{R}_3\text{:} \quad x_3 = 0 \tag{16}$$

$$\text{R}_2: \quad x_2 = 0 \tag{17}$$
$$\text{R}_1: \quad 0 = 0 \tag{18}$$
$$\text{Let} \quad x_1 = 1 \tag{19}$$
$$(16), (17), (19) \quad \mathbf{x}_{\lambda\,=\,3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$
$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0, \lambda = 4: \quad \begin{bmatrix} (3-4) & 5 & 3 \\ 0 & (4-4) & 6 \\ 0 & 0 & (1-4) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$
$$\begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$
$$\text{R}_2, \text{R}_3: \quad x_3 = 0 \tag{23}$$
$$\text{R}_1: \quad -x_1 + 5x_2 + 3(0) = 0 \tag{24}$$
$$5x_2 = x_1 \tag{25}$$
$$\text{Let} \quad x_2 = 1, \tag{26}$$
$$x_1 = 5 \tag{27}$$
$$(23), (26), (27) \quad \mathbf{x}_{\lambda\,=\,4} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \tag{28}$$

The pairs of eigenvalues and eigenvectors are 1, $\begin{bmatrix} -7 \\ -4 \\ 2 \end{bmatrix}$, 3, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and 4, $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

8.1 [14]

Find the eigenvalues. Find the corresponding eigenvectors.

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix} \tag{1}$$
$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0: \quad \begin{vmatrix} (2-\lambda) & 0 & -1 \\ 0 & (\frac{1}{2}-\lambda) & 0 \\ 1 & 0 & (4-\lambda) \end{vmatrix} = 0 \tag{2}$$
$$(2-\lambda)\begin{vmatrix} (\frac{1}{2}-\lambda) & 0 \\ 0 & (4-\lambda) \end{vmatrix} - 0 + (-1)\begin{vmatrix} 0 & (\frac{1}{2}-\lambda) \\ 1 & 0 \end{vmatrix} = 0 \tag{3}$$
$$-\lambda^3 + \frac{13}{2}\lambda^2 - 12\lambda + \frac{9}{2} = 0 \tag{4}$$
$$\lambda = 3, \frac{1}{2} \tag{5}$$
$$\lambda = 3 \to (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}: \quad \begin{bmatrix} (2-3) & 0 & -1 \\ 0 & (\frac{1}{2}-3) & 0 \\ 1 & 0 & (4-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$
$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \tag{7}$$
$$\mathbf{R_3} = \mathbf{R_3} + \mathbf{R_1} \to \quad \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -\frac{5}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{8}$$

$$\text{R}_2: \quad x_2 = 0 \tag{9}$$

$$\text{R}_1: \quad -x_1 - x_3 = 0 \tag{10}$$

$$\text{Let} \quad x_1 = 1, \tag{11}$$

$$x_3 = -1 \tag{12}$$

$$(9), (11), (12) \quad \mathbf{x}_{\lambda\,=\,3} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{13}$$

$$\lambda = \frac{1}{2} \rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}: \quad \begin{bmatrix} (2-\frac{1}{2}) & 0 & -1 \\ 0 & (\frac{1}{2}-\frac{1}{2}) & 0 \\ 1 & 0 & (4-\frac{1}{2}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$\left[\begin{array}{ccc|c} \frac{3}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{7}{2} & 0 \end{array}\right] \tag{15}$$

$$\mathbf{R_3 = R_3 - \frac{2}{3}R_1} \quad \left[\begin{array}{ccc|c} \frac{3}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{25}{6} & 0 \end{array}\right] \tag{16}$$

$$\text{R}_3: \quad x_3 = 0 \tag{17}$$

$$\text{R}_1: \quad \frac{3}{2}x_1 - x_3 = 0 \tag{18}$$

$$x_1 = 0 \tag{19}$$

$$\text{Let} \quad x_2 = 1 \tag{20}$$

$$(17), (19), (20) \quad \mathbf{x}_{\lambda\,=\,\frac{1}{2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{21}$$

The pairs of eigenvalues and eigenvectors are 3, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\frac{1}{2}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.