Problem Set:
7.4 [2, 4, 18, 20]
7.7 [12, 14]
7.8 [4, 6, 8]

7.4

7.4 [2]

Find the rank. Find a basis for the row space. Find a basis for the column space. *Hint.* Row-reduce the matrix and its transpose. Row-reduce the matrix and its transpose. (You may omit obvious factors from the vectors of these bases.)

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \tag{1}$$

$$\mathbf{R_2} = \mathbf{R_2} + (-\frac{b}{a})\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} a & b \\ b - \frac{b}{a}(a) & a - \frac{b}{a}(b) \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix} \tag{3}$$

$$(1) \quad \mathbf{A^T} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \tag{4}$$

$$(1) \quad = \mathbf{A} \tag{5}$$

$$(3) \quad = \begin{bmatrix} a & b \\ 0 & \frac{a^2-b^2}{a} \end{bmatrix} \tag{6}$$

$$(3),(6) \quad \text{rank } \mathbf{A} = 2 \tag{7}$$

$$(3),(6) \quad \text{Both the row and column bases are} \{ \begin{bmatrix} a & b \end{bmatrix}, \begin{bmatrix} 0 & \frac{b^2-a^2}{b} \end{bmatrix} \} \tag{8}$$

7.4 [4]

Find the rank. Find a basis for the row space. Find a basis for the column space. *Hint.* Row-reduce the matrix and its transpose. (You may omit obvious factors from the vectors of these bases.)

Let

$$\mathbf{B} = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{1}$$

$$\mathbf{R_2} = \mathbf{R_2} + \frac{2}{3}\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 6 & -4 & 0 \\ -4 + \frac{2}{3}6 & 0 - \frac{2}{3}(4) & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 6 & -4 & 0 \\ 0 & -\frac{8}{3} & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{3}$$

$$\mathbf{R_3} = \mathbf{R_3} + \frac{3}{4}\mathbf{R_2} \longrightarrow \quad = \begin{bmatrix} 6 & -4 & 0 \\ 0 & -\frac{8}{3} & 2 \\ 0 & 2 + (\frac{3}{4})(-\frac{8}{3}) & 6 + \frac{3}{4}(2) \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 6 & -4 & 0 \\ 0 & -\frac{8}{3} & 2 \\ 0 & 0 & \frac{15}{2} \end{bmatrix} \tag{5}$$

$$(1) \quad \mathbf{B^T} = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 2 \\ 0 & 2 & 6 \end{bmatrix} \tag{6}$$

$$(1) \quad = \mathbf{B} \tag{7}$$

$$(5),(7) \quad \text{rank } \mathbf{A} = 3 \tag{8}$$

$$(5) \quad \text{A row-space basis: } \{\begin{bmatrix} 6 & -4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{8}{3} & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{15}{2} \end{bmatrix}\} \tag{9}$$

$$(5),(7) \quad \text{A column-space basis: } \{\begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{8}{3} \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{15}{2} \end{bmatrix}\} \tag{10}$$

7.4 [18]

Are the following sets of vectors linearly independent? Show the details of your work.

$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$

Make a matrix:

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \tag{1}$$

$$\mathbf{R_2} = \mathbf{R_2} - \frac{1}{2}\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix} \tag{2}$$

$$\mathbf{R_3} = \mathbf{R_3} - \frac{1}{3}\mathbf{R_1}, \mathbf{R_4} = \mathbf{R_4} - \frac{1}{4}\mathbf{R_1} \longrightarrow \quad = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & \frac{1}{12} & \frac{4}{45} & \frac{1}{12} \\ 0 & \frac{1}{30} & \frac{1}{12} & \frac{9}{112} \end{bmatrix} \tag{3}$$

$$\mathbf{R_3} = \mathbf{R_3} - \mathbf{R_2}, \mathbf{R_4} = \mathbf{R_4} - \frac{2}{5}\mathbf{R_2} \longrightarrow \quad = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & \frac{1}{20} & \frac{141}{2800} \end{bmatrix} \tag{4}$$

$$\mathbf{R_4} = \mathbf{R_4} - 9\mathbf{R_3} \longrightarrow \quad = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & \frac{1}{180} & \frac{1}{120} \\ 0 & 0 & 0 & -\frac{69}{2800} \end{bmatrix} \tag{5}$$

$$\text{rank } = 4 \tag{6}$$

The number of vectors and the rank of the matrix they form are both 4. Hence, these vectors are **linearly independent**.

7.4 [20]

Are the following sets of vectors linearly independent? Show the details of your work.

$\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}, \begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}, \begin{bmatrix}3 & 4 & 5 & 6\end{bmatrix}, \begin{bmatrix}4 & 5 & 6 & 7\end{bmatrix}$
Make a matrix:

$$\begin{bmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 5\\3 & 4 & 5 & 6\\4 & 5 & 6 & 7\end{bmatrix} \tag{1}$$

$$\begin{pmatrix}\mathbf{R_2} = \mathbf{R_2} - 2\mathbf{R_1}\\\mathbf{R_3} = \mathbf{R_3} - 3\mathbf{R_1}\\\mathbf{R_4} = \mathbf{R_4} - 4\mathbf{R_1}\end{pmatrix} \longrightarrow \quad = \begin{bmatrix}1 & 2 & 3 & 4\\0 & -1 & -2 & -3\\0 & -2 & -4 & -6\\0 & -3 & -6 & -9\end{bmatrix} \tag{2}$$

$$\begin{pmatrix}\mathbf{R_3} = \mathbf{R_3} - 2\mathbf{R_2}\\\mathbf{R_4} = \mathbf{R_4} - 3\mathbf{R_2}\end{pmatrix} \longrightarrow \quad = \begin{bmatrix}1 & 2 & 3 & 4\\0 & -1 & -2 & -3\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} \tag{3}$$

$$(3) \qquad \text{rank} = 2 \tag{4}$$

The number of vectors is 4, and the rank of the matrix they form is 2. Hence, these vectors are **linearly dependent**.

7.7

7.7 [12]

Showing the details, evaluate:

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix} \tag{1}$$

$$= a(a*a - b*c) - b(c*a - b*b) + c(c*c - a*b) \tag{2}$$

$$= a^3 - abc - abc + b^3 + c^3 - abc \tag{3}$$

$$= a^3 + b^3 + c^3 - 3abc \tag{4}$$

7.7 [14]

Showing the details, evaluate:

$$\begin{vmatrix} 4 & 7 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -2 & 2 \end{vmatrix} = 4\begin{vmatrix} 8 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & -2 & 2 \end{vmatrix} - 7\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & -2 & 2 \end{vmatrix} + 0 + 0 \tag{1}$$

$$= 4\left(8\begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} + 0 + 0\right) - 7\left(2\begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} + 0 + 0\right) \tag{2}$$

$$= 32(1*2 + 2*5) - 14(1*2 + 2*5) \tag{3}$$

$$= 216 \tag{4}$$

7.8

7.8 [4]

Find the inverse by Gauss-Jordan. Check.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & -0.4 & 0 \\ 2.5 & 0 & 0 \end{bmatrix} \tag{1}$$

Make aug. mx:
$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0.1 & 1 & 0 & 0 \\ 0 & -0.4 & 0 & 0 & 1 & 0 \\ 2.5 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

switch $\mathbf{R_1}$, $\mathbf{R_3} \longrightarrow$
$$\left[\begin{array}{ccc|ccc} 2.5 & 0 & 0 & 0 & 0 & 1 \\ 0 & -0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.1 & 1 & 0 & 0 \end{array}\right] \tag{3}$$

$\mathbf{R_1} = \frac{1}{2.5}\mathbf{R_1} \longrightarrow$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & -0.4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.1 & 1 & 0 & 0 \end{array}\right] \tag{4}$$

$\mathbf{R_2} = -\frac{1}{0.4}\mathbf{R_2} \longrightarrow$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 0 & -2.5 & 0 \\ 0 & 0 & 0.1 & 1 & 0 & 0 \end{array}\right] \tag{5}$$

$\mathbf{R_3} = \frac{1}{0.1}\mathbf{R_3} \longrightarrow$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 0 & -2.5 & 0 \\ 0 & 0 & 1 & 10 & 0 & 0 \end{array}\right] \tag{6}$$

(6)
$$\mathbf{A^{-1}} = \begin{bmatrix} 0 & 0 & 0.4 \\ 0 & -2.5 & 0 \\ 10 & 0 & 0 \end{bmatrix} \tag{7}$$

Check:
$$\mathbf{AA^{-1}} = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & -0.4 & 0 \\ 2.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0.4 \\ 0 & -2.5 & 0 \\ 10 & 0 & 0 \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$= \mathbf{I} \tag{10}$$

7.8 [6]

Find the inverse by Gauss-Jordan. Check.

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix} \tag{1}$$

Make aug. mx.:
$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8 & 13 & 0 & 1 & 0 \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

$\mathbf{R_2} = \mathbf{R_2} - \frac{13}{5}\mathbf{R_3} \longrightarrow$
$$\left[\begin{array}{ccc|ccc} -4 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & 0 & 1 & -\frac{13}{5} \\ 0 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

$$\mathbf{R_3} = \mathbf{R_3} - 15\mathbf{R_2} \longrightarrow \quad \begin{bmatrix} -4 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 & | & 0 & 1 & -\frac{13}{5} \\ 0 & 0 & 5 & | & 0 & -15 & 40 \end{bmatrix} \tag{4}$$

$$\mathbf{R_1} = -\frac{1}{4}\mathbf{R_1} \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 & | & 0 & 1 & -\frac{13}{5} \\ 0 & 0 & 5 & | & 0 & -15 & 40 \end{bmatrix} \tag{5}$$

$$\mathbf{R_2} = 5\mathbf{R_2} \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 5 & -13 \\ 0 & 0 & 5 & | & 0 & -15 & 40 \end{bmatrix} \tag{6}$$

$$\mathbf{R_3} = \frac{1}{5}\mathbf{R_3} \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 5 & -13 \\ 0 & 0 & 1 & | & 0 & -3 & 8 \end{bmatrix} \tag{7}$$

$$(7) \quad \mathbf{A^{-1}} = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix} \tag{8}$$

$$\text{Check:} \quad \mathbf{AA^{-1}} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 8 & 13 \\ 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & 5 & -13 \\ 0 & -3 & 8 \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$= \mathbf{I} \tag{11}$$

7.8 [8]

Find the inverse by Gauss-Jordan. Check.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \tag{1}$$

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \tag{2}$$

$$= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \tag{3}$$

$$= (5*9 - 6*8) - 2(4*9 - 7*6) + 3(4*8 - 5*7) \tag{4}$$

$$= -3 + 12 - 9 \tag{5}$$

$$= 0 \tag{6}$$

Hence, the inverse of $\mathbf{A}$ does not exist.