Problem Set:
8.3 [1, 2, 3, 4, 5]
8.4 [1, 2, 9, 10]
9.1 [2, 6, 22]
9.2 [2, 4, 6, 8, 22, 36]

8.3

8.3 [1]

Is the following matrix symmetric, skew-symmetric, or orthogonal? Find the spectrum of the matrix, thereby illustrating Theorems 1 and 5.

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \tag{1}$$

$$\mathbf{A^T} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \tag{2}$$

$$(1), (2) \qquad \mathbf{AA^T} = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

$$= \mathbf{I} \tag{5}$$

Hence, $\mathbf{A}$ is an **orthogonal** matrix.

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \rightarrow \qquad \begin{vmatrix} (0.8-\lambda) & 0.6 \\ -0.6 & (0.8-\lambda) \end{vmatrix} = 0 \tag{6}$$

$$\lambda^2 - 0.6\lambda + 1 = 0 \tag{7}$$

$$\lambda_1 = 0.8 + 0.6j \tag{8}$$

$$\lambda_2 = 0.8 - 0.6j \tag{9}$$

$$\left\|\lambda_1\right\| = 1 \tag{10}$$

$$\left\|\lambda_2\right\| = 1 \tag{11}$$

The absolute magnitudes of each eigenvalue of this orthogonal matrix are 1, illustrating Theorem 5.

8.3 [2]

Is the following matrix symmetric, skew-symmetric, or orthogonal? Find the spectrum of the matrix, thereby illustrating Theorems 1 and 5.

$$\mathbf{A} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \tag{1}$$

$$\mathbf{A^T} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{2}$$

$\mathbf{A}$ is symmetric iff $b = 0$, skew-symmetric iff $a = 0$.

$$\mathbf{A}\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{3}$$
$$= \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} \tag{4}$$
$$= \mathbf{I} \tag{5}$$

if $a^2 + b^2 = 1$, making $\mathbf{A}$ orthogonal.
Find eigenvalues:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \rightarrow \quad \begin{vmatrix} (a-\lambda) & b \\ -b & (a-\lambda) \end{vmatrix} = 0 \tag{6}$$
$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0 \tag{7}$$

If symmetric ($b = 0$):

$$\lambda^2 - 2a\lambda + a^2 = 0 \tag{8}$$
$$(\lambda - a)^2 = 0 \tag{9}$$
$$\lambda = a, \tag{10}$$

which means a is real.
If skew-symmetric ($a = 0$):

$$\lambda^2 + b^2 = 0 \tag{11}$$
$$(\lambda + b)(\lambda - b) = 0 \tag{12}$$
$$\lambda = -b, b, \tag{13}$$

which means b is imaginary or zero.
If orthogonal ($a^2 + b^2 = 1$):

$$\lambda^2 - 2a\lambda + 1 = 0 \tag{14}$$
$$\text{If } \left\|a\right\| = 1: \quad \lambda^2 - 2a\lambda + a^2 = 0 \tag{15}$$
$$(\lambda - a)^2 = 0 \tag{16}$$
$$\lambda = a \tag{17}$$
$$\left\|\lambda\right\| = 1, \tag{18}$$

illustrating Theorem 5.

8.3 [3]

Is the following matrix symmetric, skew-symmetric, or orthogonal? Find the spectrum, illustrating Theorems 1 and 5.

$$\mathbf{A} = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} \tag{1}$$
$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 2 & -8 \\ -8 & 2 \end{bmatrix} \neq \mathbf{A}, -\mathbf{A} \tag{2}$$

$$\mathbf{A}\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 2 & -8 \\ 8 & 2 \end{bmatrix} \tag{3}$$
$$= \begin{bmatrix} 68 & 0 \\ 0 & 68 \end{bmatrix} \neq \mathbf{I} \tag{4}$$

A is not symmetric, skew-symmetric, or orthogonal. Find eigenvalues:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| 0 \qquad \begin{vmatrix} (2-\lambda) & 8 \\ -8 & (2-\lambda) \end{vmatrix} = 0 \tag{5}$$
$$\lambda^2 - 4\lambda + 68 = 0 \tag{6}$$
$$\lambda = 2 \pm 8j \tag{7}$$

The eigenvalues are not real, are not purely imaginary or zero, and do not have absolute values of 1.

8.3 [4]

Is the following matrix symmetric, skew-symmetric, or orthogonal? Find the spectrum.

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{1}$$
$$\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \neq \mathbf{A}, -\mathbf{A} \tag{2}$$
$$\mathbf{A}\mathbf{A}^{\mathbf{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{3}$$
$$= \begin{bmatrix} (\cos^2\theta + \sin^2\theta) & (\cos\theta\sin\theta - \cos\theta\sin\theta) \\ (\sin\theta\cos\theta - \sin\theta\cos\theta) & (\sin^2\theta + \cos^2\theta) \end{bmatrix} \tag{4}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \tag{5}$$

Hence, **A** is orthogonal. Find the eigenvalues:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0: \qquad \begin{vmatrix} (\cos\theta - \lambda) & -\sin\theta \\ \sin\theta & (\cos\theta - \lambda) \end{vmatrix} = 0 \tag{6}$$
$$\lambda^2 - (2\cos\theta)\lambda + 1 = 0 \tag{7}$$
$$\lambda = \cos\theta \pm (\sin\theta)j \tag{8}$$
$$\left\|\lambda\right\| = \sqrt{\cos^2\theta + \sin^2\theta} \tag{9}$$
$$= 1 \tag{10}$$

The eigenvalues have an absolute value of 1, illustrating Theorem 5 (that the eigenvalues of orthogonal matrices have absolute values of 1).

8.3 [5]

Is the following matrix symmetric, skew-symmetric, or orthogonal? Find the spectrum.

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix} \tag{1}$$

$$\mathbf{A^T} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{bmatrix} = \mathbf{A} \tag{2}$$

Hence, **A** is symmetric. Find the eigenvalues:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0: \quad \begin{bmatrix} (6-\lambda) & 0 & 0 \\ 0 & (2-\lambda) & -2 \\ 0 & -2 & (5-\lambda) \end{bmatrix} = 0 \tag{3}$$

$$-\lambda^3 + 13\lambda^2 - 48\lambda + 36 = 0 \tag{4}$$

$$\text{from MATLAB}: \quad \lambda = 6, 6, 1 \tag{5}$$

All eigenvalues are real, illustrating Theorem 1.

8.4

8.4 [1]

Verify that $\mathbf{A}$ and $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$ have equal eigenvalues. If $\mathbf{y}$ is an eigenvector of $\hat{\mathbf{A}}$ and $\mathbf{x}$ an eigenvector of $\mathbf{A}$, show that $\mathbf{x} = \mathbf{Py}$.

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \tag{1}$$

$$\det \mathbf{P} = -2 \neq 0 \tag{2}$$

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \mathbf{C}^{\mathbf{T}} \tag{3}$$

$$= -\frac{1}{2} \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \tag{5}$$

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP} \tag{6}$$

$$= \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} \frac{11}{2} & -1 \\ \frac{25}{2} & 0 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} -25 & 12 \\ -50 & 25 \end{bmatrix} \tag{9}$$

Check if $\mathbf{A}$ and $\hat{\mathbf{A}}$ have the same eigenvalues:

$$\mathbf{A} : \left| \mathbf{A} - \lambda \mathbf{I} \right| = 0 \rightarrow \quad \begin{vmatrix} (3-\lambda) & 4 \\ 4 & (-3-\lambda) \end{vmatrix} \tag{10}$$

$$\lambda^2 - 25 = 0 \tag{11}$$

$$\lambda_{\mathbf{A}} = \pm 5 \tag{12}$$

$$\hat{\mathbf{A}} : \left| \mathbf{A} - \lambda \mathbf{I} \right| = 0 \rightarrow \quad \begin{vmatrix} (-25-\lambda) & 12 \\ -50 & (25-\lambda) \end{vmatrix} = 0 \tag{13}$$

$$\lambda^2 - 25 = 0 \tag{14}$$

$$\lambda_{\hat{\mathbf{A}}} = \pm 5 = \lambda_{\mathbf{A}} \tag{15}$$

$\hat{\mathbf{A}}$ and $\mathbf{A}$ have the same eigenvalues. Find eigenvectors of $\mathbf{A}$:

$$\lambda_{\mathbf{A}} = 5 : \quad \begin{bmatrix} (3-5) & 4 \\ 4 & (-3-5) \end{bmatrix} \mathbf{x}_{\lambda_{\mathbf{A}}=5} = \mathbf{0} \tag{16}$$

$$\begin{bmatrix} -2 & 4 & | & 0 \\ 4 & -8 & | & 0 \end{bmatrix} \tag{17}$$

$$\mathbf{R}_2 = \mathbf{R}_2 + 2\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -2 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{18}$$

$$2x_1 = 4x_2 \tag{19}$$

$$x_1 = 2, x_2 = 1 \tag{20}$$

$$\mathbf{x}_{\lambda_{\mathbf{A}}=5} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{21}$$

$$\lambda_{\mathbf{A}} = -5: \quad \begin{bmatrix} (3+5) & 4 \\ 4 & (-3+5) \end{bmatrix} \mathbf{x}_{\lambda_{\mathbf{A}}=-5} = \mathbf{0} \tag{22}$$

$$\begin{bmatrix} 8 & 4 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix} \tag{23}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - \frac{1}{2}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} 8 & 4 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{24}$$

$$8x_1 = -4x_2 \tag{25}$$

$$x_1 = 1, x_2 \quad = -2 \tag{26}$$

$$\mathbf{x}_{\lambda_{\mathbf{A}}=-5} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{27}$$

Find eigenvectors of $\hat{\mathbf{A}}$:

$$\lambda_{\hat{\mathbf{A}}} = 5: \quad \begin{bmatrix} (-25-5) & 12 \\ -50 & (25-5) \end{bmatrix} \mathbf{x}_{\lambda_{\hat{\mathbf{A}}}=5} = \mathbf{0} \tag{28}$$

$$\begin{bmatrix} -30 & 12 & | & 0 \\ -50 & 20 & | & 0 \end{bmatrix} \tag{29}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - \frac{5}{3}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -30 & 12 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{30}$$

$$30x_1 = 12x_2 \tag{31}$$

$$x_2 = 5, x_1 = 2 \tag{32}$$

$$\mathbf{x}_{\hat{\lambda}=5} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \tag{33}$$

$$\hat{\lambda} = -5: \quad \begin{bmatrix} (-25+5) & 12 \\ -50 & (25+5) \end{bmatrix} \mathbf{x}_{\hat{\lambda}=-5} = \mathbf{0} \tag{34}$$

$$\begin{bmatrix} -20 & 12 & | & 0 \\ -50 & 30 & | & 0 \end{bmatrix} \tag{35}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - \frac{5}{2}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -20 & 12 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{36}$$

$$20x_1 = 12x_2 \tag{37}$$

$$x_1 = 3, x_2 = 5 \tag{38}$$

$$\mathbf{x}_{\hat{\lambda}=-5} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \tag{39}$$

Show that $\mathbf{x} = \mathbf{P}\mathbf{y}$, where $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, and $\mathbf{y}$ is the corresponding eigenvector of $\hat{\mathbf{A}}$:

$$\lambda = 5 \rightarrow \mathbf{x} = \mathbf{P}\mathbf{y}: \quad \mathbf{x}_{\lambda=5} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} \tag{40}$$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow (21) \tag{41}$$

$$\lambda = -5 \rightarrow \mathbf{x} = \mathbf{P}\mathbf{y}: \quad \mathbf{x}_{\lambda=-5} = \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \tag{42}$$

$$= \begin{bmatrix} -2 \\ 4 \end{bmatrix} \rightarrow \text{a scalar multiple of (27)} \tag{43}$$

Hence, $\mathbf{x} = \mathbf{P}\mathbf{y}$, where $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, and $\mathbf{y}$ is the corresponding eigenvector of $\hat{\mathbf{A}}$.

8.4 [2]

Verify that $\mathbf{A}$ and $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ have equal eigenvalues. If $\hat{\mathbf{x}}$ is an eigenvector of $\hat{\mathbf{A}}$ and $\mathbf{x}$ an eigenvector of $\mathbf{A}$, show that $\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \mathbf{P} = \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \tag{1}$$
$$\det \mathbf{P} = 1 \neq 0 \tag{2}$$
$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \mathbf{C}^{\mathbf{T}} \tag{3}$$
$$= \begin{bmatrix} -7 & 5 \\ -10 & 7 \end{bmatrix} \tag{4}$$
$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{5}$$
$$= \begin{bmatrix} -7 & 5 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \tag{6}$$
$$= \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \tag{7}$$
$$= \begin{bmatrix} -29 & 20 \\ -42 & 29 \end{bmatrix} \tag{8}$$

Find eigenvalues λ of $\mathbf{A}$:

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \qquad \begin{vmatrix} (1-\lambda) & 0 \\ 2 & (-1-\lambda) \end{vmatrix} = 0 \tag{9}$$
$$(\lambda+1)(\lambda-1) = 0 \tag{10}$$
$$\lambda = \pm 1 \tag{11}$$

Find eigenvalues $\hat{\lambda}$ of $\hat{\mathbf{A}}$:

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 \qquad \begin{vmatrix} (-29-\hat{\lambda}) & 20 \\ -42 & (29-\hat{\lambda}) \end{vmatrix} = 0 \tag{12}$$
$$\hat{\lambda}^2 - 1 = 0 \tag{13}$$
$$\hat{\lambda} = \pm 1 = \lambda \tag{14}$$

Hence, the eigenvalues λ and $\hat{\lambda}$ of $\mathbf{A}$ and $\hat{\mathbf{A}}$ are equal.
Find eigenvectors $\mathbf{x}_\lambda$ of $\mathbf{A}$:

$$\lambda = 1: \qquad \begin{bmatrix} (1-1) & 0 \\ 2 & (-1-1) \end{bmatrix} \mathbf{x}_1 = \mathbf{0} \tag{15}$$
$$\begin{bmatrix} 0 & 0 & | & 0 \\ 2 & -2 & | & 0 \end{bmatrix} \tag{16}$$
$$2x_1 = 2x_2 \tag{17}$$
$$x_1 = 1, x_2 = 1 \tag{18}$$
$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{19}$$
$$\lambda = -1: \qquad \begin{bmatrix} (1+1) & 0 \\ 2 & (-1+1) \end{bmatrix} \mathbf{x}_{-1} = \mathbf{0} \tag{20}$$

$$\begin{bmatrix} 2 & 0 & | & 0 \\ 2 & 0 & | & 0 \end{bmatrix} \tag{21}$$

$$x_1 = 0, x_2 = 1 \tag{22}$$

$$\mathbf{x}_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{23}$$

Find eigenvectors $\hat{\mathbf{x}}$ of $\hat{\mathbf{A}}$:

$$\hat{\lambda} = 1 : \quad \begin{bmatrix} (-29-1) & 20 \\ -42 & (29-1) \end{bmatrix} \hat{\mathbf{x}}_1 = \mathbf{0} \tag{24}$$

$$\begin{bmatrix} -30 & 20 & | & 0 \\ -42 & 28 & | & 0 \end{bmatrix} \tag{25}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - \frac{42}{30}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -30 & 20 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{26}$$

$$30x_1 = 20x_2 \tag{27}$$

$$x_1 = 2, x_2 = 3 \tag{28}$$

$$\hat{\lambda} = -1 : \quad \begin{bmatrix} (-29+1) & 20 \\ -42 & (29+1) \end{bmatrix} \hat{\mathbf{x}}_{-1} = \mathbf{0} \tag{29}$$

$$\begin{bmatrix} -28 & 20 & | & 0 \\ -42 & 30 & | & 0 \end{bmatrix} \tag{30}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - \frac{42}{28}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -28 & 20 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{31}$$

$$28x_1 = 20x_2 \tag{32}$$

$$x_1 = 5, x_2 = 7 \tag{33}$$

$$\hat{\mathbf{x}}_{-1} = \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{34}$$

Check that $\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$:

$$\text{For } \lambda, \hat{\lambda} = 1 : \quad \mathbf{x}_1 = \mathbf{P}\hat{\mathbf{x}}_1 \tag{35}$$

$$= \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \rightarrow \text{ a scalar multiple of (19)} \tag{37}$$

$$\text{For } \lambda, \hat{\lambda} = -1 : \quad \mathbf{x}_{-1} = \mathbf{P}\hat{\mathbf{x}}_{-1} \tag{38}$$

$$= \begin{bmatrix} 7 & -5 \\ 10 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \tag{39}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow (23) \tag{40}$$

Hence, $\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$.

8.4 [9]

Find an eigenbasis and diagonalize.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{1}$$

Find λ's:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0: \quad \begin{vmatrix} (1-\lambda) & 2 \\ 2 & (4-\lambda) \end{vmatrix} = 0 \tag{2}$$

$$\lambda^2 - 5\lambda = 0 \tag{3}$$

$$\lambda = 0, 5 \tag{4}$$

Find eigenvectors:

$$\lambda = 0: \quad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbf{x}_0 = \mathbf{0} \tag{5}$$

$$\begin{bmatrix} 1 & 2 & | & 0 \\ 2 & 4 & | & 0 \end{bmatrix} \tag{6}$$

$$\mathbf{R}_2 = \mathbf{R}_2 - 2\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} 1 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{7}$$

$$x_1 = -2x_2 \tag{8}$$

$$x_1 = -2, x_2 = 1 \tag{9}$$

$$\mathbf{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{10}$$

$$\lambda = 5: \quad \begin{bmatrix} (1-5) & 2 \\ 2 & (4-5) \end{bmatrix} \mathbf{x}_5 = \mathbf{0} \tag{11}$$

$$\begin{bmatrix} -4 & 2 & | & 0 \\ 2 & -1 & | & 0 \end{bmatrix} \tag{12}$$

$$\mathbf{R}_2 = \mathbf{R}_2 + \frac{1}{2}\mathbf{R}_1 \rightarrow \quad \begin{bmatrix} -4 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \tag{13}$$

$$4x_1 = 2x_2 \tag{14}$$

$$x_1 = 1, x_2 = 2 \tag{15}$$

$$\mathbf{x}_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{16}$$

Hence, an eigenbasis of $\mathbf{A}$ is given by

$$\mathbf{X} = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \tag{17}$$

Find $\mathbf{X}^{-1}$:

$$\mathbf{X}^{-1} = \frac{1}{\det \mathbf{X}} \mathbf{C}^{\mathbf{T}} \tag{18}$$

$$\det \mathbf{X} = -4 - 1 = -5 \tag{19}$$

$$\mathbf{X}^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \tag{21}$$

Diagonalize $\mathbf{A}$:

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \tag{22}$$

$$= \begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{25}$$

8.4 [10]

Find an eigenbasis and diagonalize.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \tag{1}$$

Find eigenvalues:

$$\left|\mathbf{A} - \lambda \mathbf{I}\right| = 0 : \qquad \begin{vmatrix} (1-\lambda) & 0 \\ 2 & (-1-\lambda) \end{vmatrix} = 0 \tag{2}$$

$$(1-\lambda)(-1-\lambda) = 0 \tag{3}$$

$$\lambda = -1, 1 \tag{4}$$

Find eigenvectors:

$$\lambda = -1 : \qquad \begin{bmatrix} (1+1) & 0 \\ 2 & (-1+1) \end{bmatrix} \mathbf{x}_{-1} = \mathbf{0} \tag{5}$$

$$\begin{bmatrix} 2 & 0 & | & 0 \\ 2 & 0 & | & 0 \end{bmatrix} \tag{6}$$

$$x_1 = 0, x_2 = 1 \tag{7}$$

$$\mathbf{x}_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8}$$

$$\lambda = 1 : \qquad \begin{bmatrix} (1-1) & 0 \\ 2 & (-1-1) \end{bmatrix} \mathbf{x}_{1} = \mathbf{0} \tag{9}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ 2 & -2 & | & 0 \end{bmatrix} \tag{10}$$

$$2x_1 = 2x_2 \tag{11}$$

$$\mathbf{x}_{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{12}$$

Hence, an eigenbasis $\mathbf{X}$ of $\mathbf{A}$ is

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{13}$$

Find the inverse of $\mathbf{X}$:

$$\mathbf{X}^{-1} = \frac{1}{\det \mathbf{X}} \mathbf{C}^{\mathbf{T}} \tag{14}$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \tag{16}$$

A diagonalization of A is thus given by

$$\begin{aligned}
\mathbf{D} &= \mathbf{X}^{-1}\mathbf{A}\mathbf{X} && (17)\\
&= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} && (18)\\
&= \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} && (19)\\
&= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} && (20)
\end{aligned}$$

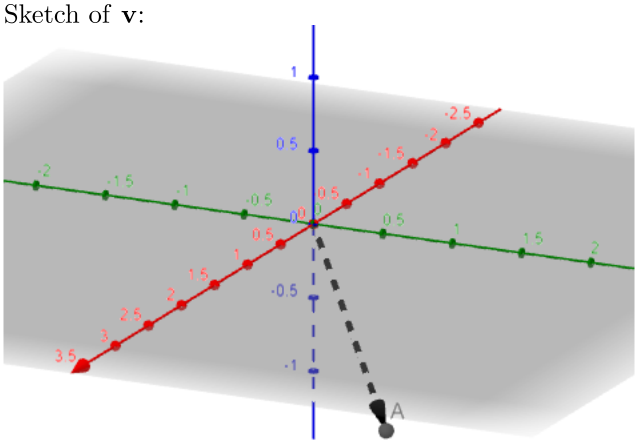
9.1

9.1 [2]

Find the components of the vector $\mathbf{v}$ with initial point P and terminal point Q. Find $|\mathbf{v}|$. Sketch $\mathbf{v}$. Find the unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{v}$.

$$
\begin{aligned}
P : (1, 1, 1), Q : (2, 2, 0) \quad (1)\\
\mathbf{v} &= \mathbf{Q} - \mathbf{P} \quad (2)\\
&= \langle 2, 2, 0\rangle - \langle 1, 1, 1\rangle \quad (3)\\
&= \langle 1, 1, -1\rangle \quad (4)\\
|\mathbf{v}| &= \sqrt{1^2 + 1^2 + (-1)^2} \quad (5)\\
&= \sqrt{3} \quad (6)
\end{aligned}
$$

Sketch of $\mathbf{v}$:



$$
\begin{aligned}
\hat{\mathbf{u}} &= \frac{1}{|\mathbf{v}|}\mathbf{v} \quad (7)\\
&= \frac{1}{\sqrt{3}}\langle 1, 1, -1\rangle \quad (8)\\
&= \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\rangle \quad (9)
\end{aligned}
$$

9.1 [6]

Find the terminal point Q of the vector $\mathbf{v}$ with components as given and initial point P. Find $|\mathbf{v}|$.

$$
\begin{aligned}
\mathbf{v} = \langle 4, 0, 0\rangle \; P : (0, 2, 13) \quad (1)\\
Q &= P + v \quad (2)\\
&= (0, 2, 13) + (4, 0, 0) \quad (3)\\
&= (4, 2, 13) \quad (4)\\
|\mathbf{v}| &= \sqrt{4^2 + 0^2 + 0^2} \quad (5)\\
&= 4 \quad (6)
\end{aligned}
$$

9.1 [22]

Find the resultant in terms of components and its magnitude.

$$
\begin{aligned}
\mathbf{p} &= \langle 1, -2, 3 \rangle && (1)\\
\mathbf{q} &= \langle 3, 21, -16 \rangle && (2)\\
\mathbf{u} &= \langle -4, -19, 13 \rangle && (3)\\
\mathbf{r} &= \mathbf{p} + \mathbf{q} + \mathbf{r} && (4)\\
&= \langle 1, -2, 3 \rangle + \langle 3, 21, -16 \rangle + \langle -4, -19, 13 \rangle && (5)\\
&= \langle 0, 0, 0 \rangle && (6)\\
\left|\mathbf{r}\right| &= 0 && (7)
\end{aligned}
$$

9.2

9.2 [2]

Given

$$\mathbf{a} = \langle 1, -3, 5\rangle \qquad \mathbf{b} = \langle 4, 0, 8\rangle \qquad \mathbf{c} = \langle -2, 9, 1\rangle, \tag{1}$$

find

$$\begin{aligned}
(-3\mathbf{a} + 5\mathbf{c}) \cdot \mathbf{b} : \quad & (-3\langle 1, -3, 5\rangle + 5\langle -2, 9, 1\rangle) \cdot \mathbf{b} && (2)\\
& = (\langle -3, 9, -15\rangle + \langle -10, 45, 5\rangle) \cdot \mathbf{b} && (3)\\
& = \langle -13, 54, -10\rangle \cdot \langle 4, 0, 8\rangle && (4)\\
& = -13 * 4 - 10 * 8 && (5)\\
& = -138 && (6)\\
15(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} : \quad & 15(\langle 1, -3, 5\rangle - \langle -2, 9, 1\rangle) \cdot \mathbf{b} && (7)\\
& = 15\langle 3, -12, 4\rangle \cdot \langle 4, 0, 8\rangle && (8)\\
& = 15(12 + 0 + 32) && (9)\\
& = 660 && (10)
\end{aligned}$$

9.2 [4]

Given

$$\mathbf{a} = \langle 1, -3, 5\rangle \qquad \mathbf{b} = \langle 4, 0, 8\rangle \qquad \mathbf{c} = \langle -2, 9, 1\rangle, \tag{1}$$

find

$$\begin{aligned}
\left|\mathbf{a} + \mathbf{b}\right| : \quad & \left|\langle 1, -3, 5\rangle + \langle 4, 0, 8\rangle\right| && (2)\\
& = \left|\langle 5, -3, 13\rangle\right| && (3)\\
& = \sqrt{5^2 + (-13)^2 + 13^2} && (4)\\
& = \sqrt{203} \approx 14.86 && (5)\\
\left|\mathbf{a}\right| + \left|\mathbf{b}\right| : \quad & \left|\langle 1, -3, 5\rangle\right| + \left|\langle 4, 0, 8\rangle\right| && (6)\\
& = \sqrt{1^2 + (-3)^2 + 5^2} + \sqrt{4^2 + 0^2 + 8^2} && (7)\\
& = \sqrt{35} + \sqrt{80} \approx 14.25 && (8)
\end{aligned}$$

9.2 [6]

Given

$$\mathbf{a} = \langle 1, -3, 5\rangle \qquad \mathbf{b} = \langle 4, 0, 8\rangle \qquad \mathbf{c} = \langle -2, 9, 1\rangle, \tag{1}$$

find

$$\left|\mathbf{a} + \mathbf{c}\right|^2 + \left|\mathbf{a} - \mathbf{c}\right|^2 - 2\left(\left|\mathbf{a}\right|^2 + \left|\mathbf{c}\right|^2\right) \tag{2}$$

$$= \left|\langle 1, -3, 5\rangle + \langle -2, 9, 1\rangle\right|^2 + \left|\langle 1, -3, 5\rangle - \langle -2, 9, 1\rangle\right|^2 - 2\left(1^2 + (-3)^2 + 5^2 + (-2)^2 + 9^2 + 1^2\right) \tag{3}$$

$$= \left|\langle -1, 6, 6\rangle\right|^2 + \left|\langle 3, -12, 4\rangle\right|^2 - 242 \tag{4}$$
$$= (-1)^2 + 6^2 + 6^2 + 3^2 + (-12)^2 + 4^2 - 242 \tag{5}$$
$$= 0 \tag{6}$$

9.2 [8]

Given

$$\mathbf{a} = \langle 1, -3, 5\rangle \qquad \mathbf{b} = \langle 4, 0, 8\rangle \qquad \mathbf{c} = \langle -2, 9, 1\rangle, \tag{1}$$

find

$$5\mathbf{a} \cdot 13\mathbf{b} : \qquad 65\left(\langle 1, -3, 5\rangle \cdot \langle 4, 0, 8\rangle\right) \tag{2}$$
$$= 65(4 + 0 + 40) \tag{3}$$
$$= 2860 \tag{4}$$
$$65\mathbf{a} \cdot \mathbf{b} : \qquad = 2860 \tag{5}$$

9.2 [22]

Find the angle between

$$\mathbf{a} = \langle 1, 1, 0\rangle, \mathbf{b} = \langle 3, 2, 1\rangle \tag{1}$$
$$\cos\gamma = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \tag{2}$$
$$= \frac{\left|\langle 1, 1, 0\rangle \cdot \langle 3, 2, 1\rangle\right|}{\sqrt{2}\sqrt{14}} \tag{3}$$
$$= \frac{5}{2\sqrt{7}} \tag{4}$$
$$\gamma = \cos^{-1}\left(\frac{5}{2\sqrt{7}}\right) \tag{5}$$
$$= 0.33 \text{ rad} \tag{6}$$

9.2 [36]

Find the component of **a** in the direction of **b**:

$$\mathbf{a} = \langle 1, 1, 1\rangle, \mathbf{b} = \langle 2, 1, 3\rangle \tag{1}$$
$$p = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} \tag{2}$$
$$= \frac{\langle 1, 1, 1\rangle \cdot \langle 2, 1, 3\rangle}{\sqrt{14}} \tag{3}$$
$$= \frac{6}{\sqrt{14}} \tag{4}$$
$$\mathbf{a_b} = (p)\frac{\mathbf{b}}{|\mathbf{b}|} \tag{5}$$
$$= \langle \frac{6}{7}, \frac{3}{7}, \frac{9}{7}\rangle \tag{6}$$

Make a sketch: