

Probability and Measure Solutions

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Forward

This document will contain notes and solutions corresponding to Probability and Measure, Third Edition, by Patrick Billingsley [amazon].

Chapter 1.1 - Borel's Normal Number Theorem

Notes

For a complete understanding of probability, you need to understand an infinite number of events as well as a finite number of events. We try and present why that must be so here.

The Unit Interval

We take the length of an interval $I = (a, b] = b - a$. Note, for A a disjoint set of intervals in $(0, 1]$, we have that $P(A)$ is well defined. If B is a similar

disjoint set, and is disjoint from A , $P(A + B) = P(A) + P(B)$ is well defined as well. Note - we haven't defined anything for intersections yet. These definitions can also directly stem from the Riemann integral of step functions.

The unit interval can give the probability that a single particle is emitted in a unit interval of time. Or a single phone call comes in. However, it can also model an infinite coin toss. This is done as follows - for $\omega \in (0, 1]$, define:

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n}$$

Where $d_n(\omega)$ is 0 or 1, and comes from the binary expansion of ω . We take ω as the non terminating representation. Note, we were particular when we defined intervals as half inclusive. Examine the set of ω for which $d_i(\omega) = u_i$ for $i = 1, \dots, n$, $u_i \in \{0, 1\}$. We have that:

$$\sum_{i=1}^n \frac{u_i}{2^i} < \omega \leq \sum_{i=1}^n \frac{u_i}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}$$

We cannot have the lower extreme value, as this would imply ω takes on its terminating binomial representation, which is what we said we would not do. This is our first taste, I guess, of measure 0 sets, we we still have:

$$\mathbb{P}[\omega : d_i(\omega) = u_i, i = 1, \dots, n] = \frac{1}{2^n}$$

Note, probabilities of various familiar events can be written down immediately. Ultimately, note, however, each probability is the sum of disjoint dyadic intervals of various ranks k . Ie, all the events are still well defined by our probability definition above. We have:

$$\mathbb{P}\left[\omega : \sum_{i=1}^n d_i(\omega) = k\right] = \binom{n}{k} \frac{1}{2^n}$$

All these results have been for finitely many components of $d_i(\omega)$. What we are interested in, however, is properties of the entire sequence of $\omega = (d_1(\omega), d_2(\omega), \dots)$.

The Weak Law of Large Numbers

What I like about this chapter, is to me - it *emphasizes* the connection between the *structure of real numbers*, and probability. At the end of the

day - probability can be seen as just extracting properties of *frequency* over the real numbers, to be understood as probabilistic statements. However, with just our basic real numbers - we can't really prove a lot of properties about infinite things. That is when measure theory comes in later. However, for now, we look at what we can prove - and that starts with the weak law of large numbers. We have:

Theorem 1.1 - The Weak Law of Large Numbers For each ϵ :

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\omega : \left| \frac{1}{n} \sum_{i=1}^n d_i(\omega) - \frac{1}{2} \right| \geq \epsilon \right] = 0$$

Probabilistically - this is saying that if n is large, then there is a small probability that the fraction/relative *frequency* of heads in n tosses will deviate much from $1/2$. Think about it as a statement over the real numbers as well - it is also interesting. Ultimately, the intervals containing ω that do not satisfy the above are getting smaller and smaller and smaller. We formalize this with the following concept:

As $d_i(\omega)$ are constant over each dyadic interval of rank n if $i \leq n$, the sums $\sum_{i=1}^n d_i(\omega)$ are also constant over rank n . Thus, the set in the theorem is just a disjoint union of dyadic intervals of rank n . Note - the theorem is saying, that the total weight given to those intervals gets smaller and smaller as n goes to infinity.

Now, we go over how to prove the theorem. It relies on rademacher variables:

$$r_n(\omega) = 2d_n(\omega) - 1$$

These are ± 1 when $d_n = 1/2$. Note, these have the same "being constant on dyadic intervals" properties as $d_n(\omega)$. We define:

$$s_n(\omega) = \sum_{i=1}^n r_i(\omega)$$

And so, our theorem is equivalent to proving:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\omega : \left| \frac{1}{n} s_n(\omega) \right| \geq \epsilon \right] = 0$$

Note, rademacher functions also have interpretations, probabilistically, of random walks and such. With these variables, we can ultimately find prop-

erties, going all the way to:

$$\int_0^1 s_n^2(\omega) = n$$

However, what interests me is the following: Chebyshev's Lemma, but as a property of the real numbers. We have:

Lemma - Chebyshev's Inequality If f is a nonnegative step function, then $[\omega : f(\omega) \geq \alpha]$ is for $\alpha > 0$ a finite union of intervals, and:

$$\mathbb{P}[\omega : f(\omega) \geq \alpha] \leq \frac{1}{\alpha} \int_0^1 f(\omega) d\omega$$

Proof: Note, it is all just properties of step functions. Let c_j correspond to the step intervals $(x_{j-1}, x_j]$, and let \sum' be the sum over $c_j \geq \alpha$. Then, we have quite easily:

$$\int_0^1 f(\omega) d\omega = \sum c_j(x_j - x_{j-1}) \geq \sum' c_j(x_j - x_{j-1}) \geq \sum' \alpha(x_j - x_{j-1}) = \alpha \mathbb{P}[\omega : f(\omega) \geq \alpha]$$

Thus, we have Chebyshev's inequality, and with it, we can easily prove the Weak Law of Large Numbers. However - it is important to note - these are *properties over the real numbers*, as much as they are probabilistic properties.

The Strong Law of Large Numbers

Just to first formalize some terms - the frequency of 1 in ω is $\sum_{i=1}^n d_i(\omega)$, the relative frequency is that number normalized, ie $\frac{1}{n} \sum_{i=1}^n d_i(\omega)$, and the asymptotic relative frequency is the limit. We can derive, with some technical tools outside of discrete probability theory, results on the set:

$$N = \left[\omega : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n d_i(\omega) = 1/2 \right]$$

We call this the set of normal numbers N . The tools themselves are the concepts of negligibility. A set A is negligible if for every $\epsilon > 0$, there is a countable number of intervals (not necessarily disjoint) such that:

$$A \subset \bigcup_k I_k \quad \sum_k I_k = \sum_k b_k - a_k < \epsilon$$

For one - I like to note here interpretations. Essentially - if A is negligible, it is a practical impossibility that ω randomly drawn will lie within A . And

if A^c is negligible, it is a practical certainty that ω randomly drawn will lie within A . These are just how they should be understood - and these understandings are reasonable, as the total "length" that A takes up can be understood to be incredibly incredibly small.

Some properties of negligibility - note, these are the standard properties, stemming from infinite sums $(1/2^k)$ summing to values less than ϵ . Individual points are negligible, and so to thus are countable sets. So to are countable unions of countable sets.

With these properties - we understand that the property of our model not including ω with a terminating sequence (all 0 ending) is not a short coming. These ω form a countable set - and so, they can be considered negligible.

Theorem 1.2 The set of normal numbers N has negligible complement.

Proof As an aside - we note that this proof is stronger than just the negligibility properties we noted above. This is because N^c is not countable. The set of $d_i(\omega) = 1$ unless i is a multiple of 3 clearly belongs to N - as for each n , $n^{-1} \sum_{i=1}^n d_i(\omega) \geq 2/3$. However, note this set is uncountable (diagonalization argument).

Note, the proof relies on equivalently defining N as:

$$N = \left[\omega : \lim_{n \rightarrow \infty} \frac{1}{n} s_n(\omega) = 0 \right]$$

Then, we can again make use of Chebyshev's Inequality (step function version) to find that:

$$\mathbb{P}[\omega : |s_n(\omega)| \geq n\epsilon] \leq \frac{1}{n^4 \epsilon^4} \int_0^1 s_n^4(\omega) d\omega = \frac{n + 3n(n-1)}{n^4 \epsilon^4} \leq \frac{3}{n^2 \epsilon^4}$$

Where the last step is just via an in depth (but simple) investigation of the integrals of multiplications of rademacher variables. With this property, we can find that if $A_n = [\omega : |n^{-1} s_n(\omega)| \geq \epsilon_n]$, then we have a sequence of ϵ_n such that $P(A_n) \leq 3\epsilon_n^{-4} n^{-2}$, and we can find such a sequence such that:

$$\sum_n \mathbb{P}[A_n] < \infty$$

The final step to proving the theorem is noting that:

$$\bigcap_{n=m}^{\infty} A_n^c \subset N \implies N^c \subset \bigcup_{n=m}^{\infty} A_n$$

Which will ultimately prove the theorem. Note - a lot of details are left out, but I do not consider them important. You should be able to fill in. These are just the major strokes, outlining the proof. It essentially hinges on our integral value, and the relationship between A_n and the set of normal number N . qed.

So, we have N^c is negligible. But, can we have that N itself is negligible? Well, we could say no - using our "practically impossible" notions, and noting that for $\omega \in [0, 1]$ randomly drawn, it must be in $[0, 1]$, and $N^c \cup N = [0, 1]$. But, that is not rigorous. And so, the following theorem will give us our initial basis of *measure*, and also help us note that N is not negligible.

Solutions