Probability and Measure Solutions

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Contents

Forward

This document will contain notes and solutions corresponding to Probability and Measure, Third Edition, by Patrick Billingsley [amazon].

Chapter 1.1 - Borel's Normal Number Theorem

Notes

For a complete understanding of probability, you need to understand an infinite number of events as well as a finite number of events. We try and present why that must be so here.

The Unit Interval

We take the length of an interval $I = (a, b] = b - a$. Note, for A a disjoint set of intervals in $(0, 1]$, we have that $P(A)$ is well defined. If B is a similar disjoint set, and is disjoint from A, $P(A + B) = P(A) + P(B)$ is well defined as well. Note - we haven't defined anything for intersections yet. These definitions can also directly stem from the Riemann integral of step functions.

The unit interval can give the probability that a single particle is emitted in a unit interval of time. Or a single phone call comes in. However, it can also model an infinite coin toss. This is done as follows - for $\omega \in (0, 1]$, define:

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n}$$

Where $d_n(\omega)$ is 0 or 1, and comes from the binary expansion of ω. We take ω as the non terminating representation. Note, we were particular when we defined intervals as half inclusive. Examine the set of ω for which $d_i(\omega) = u_i$ for $i = 1, \cdots, n$, $u_i \in \{0, 1\}$. We have that:

$$\sum_{i=1}^{n} \frac{u_i}{2^i} < \omega \leq \sum_{i=1}^{n} \frac{u_i}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}$$

We cannot have the lower extreme value, as this would imply ω takes on its terminating binomial representation, which is what we said we would not do. This is our first taste, I guess, of measure 0 sets, we we still have:

$$\mathbb{P}\left[\omega : d_i(\omega) = u_i, i = 1, \cdots, n\right] = \frac{1}{2^n}$$

Note, probabilities of various familiar events can be written down immediately. Ultimately, note, however, each probability is the sum of disjoint dyadic intervals of various ranks k. Ie, all the events are still well defined by our probability definition above. We have:

$$\mathbb{P}\left[\omega : \sum_{i=1}^{n} d_i(\omega) = k\right] = \binom{n}{k} \frac{1}{2^n}$$

Solutions