Probability and Measure Solutions

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This document will contain notes and solutions corresponding to Probability and Measure, Third Edition, by Patrick Billingsley [amazon].

A note on how I will do the questions. I want to answer every question, but there really are a lot of them. So, I think I will tackle them this way. To get through a chapter, I'll go through every question that is in the back of the book (as they are the most important, and might be needed at a later time). Then, I'll go maybe over 3-4 more. Then, I'll answer one question I skipped from the previous chapters.

Section 1 - Borel's Normal Number Theorem

Notes

For a complete understanding of probability, you need to understand an infinite number of events as well as a finite number of events. We try and present why that must be so here.

The Unit Interval

We take the length of an interval $I = (a, b] = b - a$. Note, for A a disjoint set of intervals in $(0, 1]$, we have that $P(A)$ is well defined. If B is a similar disjoint set, and is disjoint from A, $P(A + B) = P(A) + P(B)$ is well defined as well. Note - we haven't defined anything for intersections yet. These definitions can also directly stem from the Riemann integral of step functions.

The unit interval can give the probability that a single particle is emitted in a unit interval of time. Or a single phone call comes in. However, it can also model an infinite coin toss. This is done as follows - for $\omega \in (0,1]$, define:

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n}$$

Where $d_n(\omega)$ is 0 or 1, and comes from the binary expansion of ω. We take ω as the non terminating representation. Note, we were particular when we defined intervals as half inclusive. Examine the set of ω for which $d_i(\omega) = u_i$ for $i = 1, \cdots, n$, $u_i \in \{0,1\}$. We have that:

$$\sum_{i=1}^{n} \frac{u_i}{2^i} < \omega \leq \sum_{i=1}^{n} \frac{u_i}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}$$

We cannot have the lower extreme value, as this would imply ω takes on its terminating binomial representation, which is what we said we would not do. This is our first taste, I guess, of measure 0 sets, we we still have:

$$\mathbb{P}\left[\omega : d_i(\omega) = u_i, i = 1, \cdots, n\right] = \frac{1}{2^n}$$

Note, probabilities of various familiar events can be written down immediately. Ultimately, note, however, each probability is the sum of disjoint dyadic intervals of various ranks k. Ie, all the events are still well defined by our probability definition above. We have:

$$\mathbb{P}\left[\omega : \sum_{i=1}^{n} d_i(\omega) = k\right] = \binom{n}{k} \frac{1}{2^n}$$

All these results have been for finitely many components of $d_i(\omega)$. What we are interested in, however, is properties of the entire sequence of $\omega = (d_1(\omega), d_2(\omega), \cdots)$.

The Weak Law of Large Numbers

What I like about this chapter, is to me - it *emphasizes* the connection between the *structure of real numbers*, and probability. At the end of the day - probability can be seen as just extracting properties of *frequency* over the real numbers, to be understood as probabilistic statements. However, with just our basic real numbers - we can't really prove a lot of properties about infinite things. That is when measure theory comes in later. However, for now, we look at what we can prove - and that starts with the weak law of large numbers. We have:

Theorem 1.1 - The Weak Law of Large Numbers For each ϵ:

$$\lim_{n\to\infty} \mathbb{P}\left[\omega : \left|\frac{1}{n}\sum_{i=1}^{n} d_i(\omega) - \frac{1}{2}\right| \geq \epsilon\right] = 0$$

Probabilistically - this is saying that if n is large, then there is a small probability that the fraction/relative *frequency* of heads in n tosses will deviate much from 1/2. Think about it as a statement over the real numbers as well - it is also interesting. Ultimately, the intervals containing ω that do not satisfy the above are getting smaller and smaller and smaller. We formalize this with the following concept:

As $d_i(\omega)$ are constant over each dyadic interval of rank n if $i \leq n$, the sums $\sum_{i=1}^{n} d_i(\omega)$ are also constant over rank n. Thus, the set in the theorem is just a disjoint union of dyadic intervals of rank n. Note - the theorem is saying, that the total weight given to those intervals gets smaller and smaller as n goes to infinity.

Now, we go over how to prove the theorem. It relies on rademacher variables:

$$r_n(\omega) = 2d_n(\omega) - 1$$

These are ± 1 when $d_n = 1/0$. Note, these have the same "being constant on dyadic intervals" properties as $d_n(\omega)$. We define:

$$s_n(\omega) = \sum_{i=1}^{n} r_i(\omega)$$

And so, our theorem is equivalent to proving:

$$\lim_{n\to\infty} \mathbb{P}\left[\omega : \left|\frac{1}{n}s_n(\omega)\right| \geq \epsilon\right] = 0$$

Note, rademacher functions also have interpretations, probabilistically, of random walks and such. With these variables, we can ultimately find properties, going all the way to:

$$\int_0^1 s_n^2(\omega) = n$$

However, what interests me is the following: Chebyshev's Lemma, but as a property of the real numbers. We have:

Lemma - Chebyshev's Inequality If f is a nonnegative step function, then $[\omega : f(\omega) \geq \alpha]$ is for $\alpha > 0$ a finite union of intervals, and:

$$\mathbb{P}\left[\omega : f(\omega) \geq \alpha\right] \leq \frac{1}{\alpha} \int_0^1 f(\omega) d\omega$$

Proof: Note, it is all just properties of step functions. Let c_j correspond to the step intervals $(x_{j-1}, x_j]$, and let $\sum'$ be the sum over $c_j \geq \alpha$. Then, we have quite easily:

$$\int_0^1 f(\omega) d\omega = \sum c_j (x_j - x_{j-1}) \geq \sum' c_j (x_j - x_{j-1}) \geq \sum' \alpha (x_j - x_{j-1}) = \alpha \mathbb{P}\left[\omega : f(\omega) \geq \alpha\right]$$

Thus, we have Chebyshev's inequality, and with it, we can easily prove the Weak Law of Large Numbers. However - it is important to note - these are *properties over the real numbers*, as much as they are probabilistic properties.

The Strong Law of Large Numbers

Just to first formalize some terms - the frequency of 1 in ω is $\sum_{i=1}^n d_i(\omega)$, the relative frequency is that number normalized, ie $\frac{1}{n} \sum_{i=1}^n d_i(\omega)$, and the asymptotic relative frequency is the limit. We can derive, with some technical tools outside of discrete probability theory, results on the set:

$$N = \left[\omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n d_i(\omega) = 1/2\right]$$

We call this the set of normal numbers N. The tools themselves are the concepts of negligibility. A set A is negligible if for every $\epsilon > 0$, there is a countable number of intervals (not necessarily disjoint) such that:

$$A \subset \bigcup_k I_k \qquad \sum_k I_k = \sum_k b_k - a_k < \epsilon$$

For one - I like to note here interpretations. Essentially - if A is negligible, it is a practical impossibility that ω randomly drawn will lie within A. And if A^c is negligible, it is a practical certainty that ω randomly drawn will lie within A. These are just how they should be understood - and these understandings are reasonable, as the total "length" that A takes up can be understood to be incredibly incredibly small.

Some properties of negligibility - note, these are the standard properties, stemming from infinite sums ($1/2^k$) summing to values less than ϵ. Individual points are negligible, and so to thus are countable sets. So to are countable unions of countable sets.

With these properties - we understand that the property of our model not including ω with a terminating sequence (all 0 ending) is not a short coming. These ω form a countable set - and so, they can be considered negligible.

Theorem 1.2 The set of normal numbers N has negligible complement.

Proof As an aside - we note that this proof is stronger than just the negligibility properties we noted above. This is because N^c is not countable. The set of $d_i(\omega) = 1$ unless i is a multiple of 3 clearly belongs to N - as for each n, $n^{-1}\sum_{i=1}^{n} d_i(\omega) \geq 2/3$. However, note this set is uncountable (diagonalization argument).

Note, the proof relies on equivalently defining N as:

$$N = \left[\omega : \lim_{n\to\infty} \frac{1}{n} s_n(\omega) = 0\right]$$

Then, we can again make use of Chebyshev's Inequality (step function version) to find that:

$$\mathbb{P}\left[\omega : |s_n(\omega)| \geq n\epsilon\right] \leq \frac{1}{n^4\epsilon^4}\int_0^1 s_n^4(\omega)d\omega = \frac{n + 3n(n-1)}{n^4\epsilon^4} \leq \frac{3}{n^2\epsilon^4}$$

Where the last step is just via an in depth (but simple) investigation of the integrals of multiplications of rademacher variables. With this property, we can find that if $A_n = [\omega : |n^{-1}s_n(\omega)| \geq \epsilon_n]$, then we have a sequence of ϵ_n such that $P(A_n) \leq 3\epsilon_n^{-4}n^{-2}$, and we can find such a sequence such that:

$$\sum_n \mathbb{P}\left[A_n\right] < \infty$$

The final step to proving the theorem is noting that:

$$\bigcap_{n=m}^{\infty} A_n^c \subset N \implies N^c \subset \bigcup_{n=m}^{\infty} A_n$$

Which will ultimately prove the theorem. Note - a lot of details are left out, but I do not consider them important. You should be able to fill in. These are just the major strokes, outlining the proof. It essentially hinges on our integral value, and the relationship between A_n and the set of normal number N. qed.

So, we have N^c is negligible. But, can we have that N itself is negligible? Well, we could say no - using our "practically impossible" notions, and noting that for $\omega \in [0,1]$ randomly drawn, it must be in $[0,1]$, and $N^c \cup N = [0,1]$. But, that is not rigorous. And so, the following theorem will give us our initial basis of *measure*, and also help us note that N is not negligible.

Theorem 1.3 - Lebesgue Measure Starting Point

1. If $\bigcup_k I_k \subset I$, and the I_k are disjoint, then $\sum_k |I_k| \leq |I|$
2. If $I \subset \bigcup I_k$ (the I_k need not be disjoint), then $|I| \leq \sum_k |I_k|$
3. If $\bigcup I_k = I$, and the I_k are disjoint, then $|I| = \sum_k |I_k|$

Note, this Theorem is true for countably infinite intervals as well. **Proof:** Note that the third part follows directly from (1) and (2). We start with the finite cases. For (1), we can prove by induction on the number of intervals n. It is clearly true for $n = 1$, and it is a fairly simple induction hypothesis to prove in general. We similarly have the same for (2).

The difficult part comes when going to infinite intervals. For (1), it is a simple limit, ie:

$$\sum_k |I_k| = \lim_{n\to\infty} \sum_{k=1}^{n} |I_k|$$

Note, each sum is less than $|I|$, as the finite case to 1 applies for each finite sum. And so, the inequality can be expanded to the limit. However - we can't do that for (2). Ultimately, the difference between the two cases is the inclusion of unions. We note:

$$\bigcup_k I_k \subset I \implies \bigcup_{k=1}^{n} I_k \subset I$$

Ie, the inclusion is true for every subset. However, we *do not* necessarily have:

$$I \subset \bigcup_k I_k \implies I \subset \bigcup_{k=1}^{n} I_k$$

Note that in the following way: $I = (a, b]$. We have that $I_i = (a + 1/i, b]$. We do indeed have that:

$$I \subset \bigcup_k I_k$$

As if you take $x \in I$, $a < x \leq b$, and so we must have for i large enough, $a + 1/i < x \leq b$, and so $x \in I_i$. However, note that the inclusion is not actually true for a specific finite subunion. So, we need to take a different strategy to prove the infinite case. This comes from dealing with *open covers of compact spaces*, and relying on the Heine-Borel theorem, which says that intervals $[a, b]$ are indeed compact. In this case, we are able to bridge between infinite unions and finite unions - as we can take a finite sub cover of an open cover on compact spaces. We prove the theorem essentially for $[a+\epsilon, b]$, that:

$$|I| - \epsilon = b - (a + \epsilon) \leq \sum_k |I_k| + \epsilon$$

However, as the ϵ is arbitrary, we can conclude the fact for the infinite case as well. qed.

Note - this implies that N is not negligible. As it it was, $[0, 1]$ would be negligible, but that is incorrect by the above, as any open covering must have total sum at least 1, ie, the total sum is not smaller than arbitrary ϵ.

The Measure Theory of Diophantine Approximation

This section is just additional, so my notes here are sparse. However, I do read through it, and record the theorems, plus some notes I have on them.

Theorem 1.4 If x is irrational, there are infinitely many irreducible fractions p/q such that:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

Honestly, this proof is so good. I like it a lot - it is pretty clever. However, I don't just want to copy it down here - it is in the book. I'm not sure if there is any broad message I can glean from it - just that, it is a property of the real numbers. It just hinges on the following fact (which itself is pretty difficult to prove), that for every Q positive integer, there is an integer $q < Q$ and corresponding p such that:

$$\left| x - \frac{p}{q} \right| < \frac{1}{qQ} \leq \frac{1}{q^2}$$

Note, this is true for x rational or irrational. However, we have an infinite number of such irreducible fractions for the irrational case, and the contradiction derived in the book is nice as well. Anyway - read the book for this. qed.

Anyway, the above essentially means that, apart from a negligible set of x, each real number has an infinite set of irreducible rationals such that the bounds in Theorem 1.4 are true. We now consider a generalization - when can we tighten the inequality in Theorem 1.4 - Consider:

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2\varphi(q)}$$

Let A_φ consist of the real x for which the above has infinitely many irreducible solutions. Under what conditions on φ will A_φ have negligible complement? Note that if $\varphi(q) < 1$, then the condition is weaker than Theorem 1.4, and so A_φ has negligible complement immediately. It becomes interesting if $\varphi(q) > 1$. We will later prove the theorem:

Theorem 1.5 Suppose that φ is positive and nondecreasing. If:

$$\sum_q \frac{1}{q\varphi(q)} = \infty$$

Then A_φ has negligible complement. We will prove this later, but we can now prove:

Theorem 1.6 Suppose that φ is positive. If

$$\sum_q \frac{1}{q\varphi(q)} < \infty$$

Then A_φ is negligible.

We will go over the proof soon for this theorem. However - just note what the theorems are saying. Note that in the second - $\varphi(q)$ must be growing quite quickly. We need the denominator to be quite large, so that the infinite sum is ultimately finite. However, in theorem 1.5, we don't want the $\varphi(q)$ to be too large, lest the sum actually does become finite. Ultimately - both theorems are conditions on how $\varphi(q)$ grows. Which, ultimately does make sense. If $\varphi(q)$ grows to large - it becomes unreasonable to expect our condition to hold infinitely many times. If I ever encounter such situations, where I might want to examine the growth of a function φ - I think examining whether the infinite sum of $1/\varphi$ equals infinity or not is often a good property that is related to the growth of a function.

Proof of Theorem 1.6 I'll give the full proof here, as it is interesting to me, and rather short. We want to show that A_φ is negligible. Well, given that the sum is finite, there is a q_0 large enough such that the tail sum $\sum_{q\geq q_0}\frac{1}{q\varphi(q)} < \epsilon/4$. If $x \in A_\varphi$, then our definition holds for some $q \geq q_0$, and as $0 < x < 1$, we have that the corresponding p lies in $0 \leq p \leq q$. Thus, we have that:

$$A_\varphi \subset \bigcup_{q\geq q_0}\bigcup_{p=0}^{q}\left(\frac{p}{q} - \frac{1}{q^2\varphi(q)}, \frac{p}{q} + \frac{1}{q^2\varphi(q)}\right]$$

Which stems from every $x \in A_\varphi$ being in the right expression, given that x is within one of the intervals on the right by the property we just described. Now, we have a covering interval - we just need to find the length of it. Note, by assumption, we have all of our q must satisfy $q \geq 1$ (or, we can add that in). And so, we have the sum of the intervals is:

$$\sum_{q\geq q_0}\sum_{p=0}^{q}\frac{2}{q^2\varphi(q)} = \sum_{q\geq q_0}\frac{2(q+1)}{q^2\varphi(q)} \leq \sum_{q\geq q_0}\frac{2(q+q)}{q^2\varphi(q)} \leq \sum_{q\geq q_0}\frac{4}{q\varphi(q)} < \epsilon$$

And thus, A_φ is negligible. qed.

Problems

1.1 Infinite Independent Events on a Discrete Space (are impossible)

1. As for why the existence of an infinite sequence of independent events each with probability 1/2 in a discrete probability space would make the section superfluous - I think this is because, in the section, we *rely* on the uncountability of the real numbers. This allows us to make notions like negligible, which helps us make Borel's Number Theorem (the Strong Law of Large numbers). We could then just handle infinite cases with a countable, discrete space - which would make the section unnecessarily in depth ("superfluous").

   As for why a discrete space cannot have an infinite sequence of independent events. Note, we can partition the space into sets $A_1 \cap A_2$, $A_1 \cap A_2^c$, $A_1^c \cap A_2$, and $A_1^c \cap A_2^c$. Note that each has probability 2^{-2}, by independence. And so, each countable point ω belongs to one of these sets, and $P(\omega) \leq 2^{-2}$. We can continue on for arbitrary 2^k partitions, each of probability at most 2^{-k}. Thus, we find that $P(\omega) = 0$ for each point. This is a contradiction, as:

$$\sum P(\omega) = 1 \neq 0 = \sum_{\omega} 0 = \sum P(\omega)$$

2. This portion draws the same contradiction as above, namely, $P(\omega) = 0$ for all ω. Each ω belongs to a sequence of A_1, A_2, A_3^c, something like that. Let $t_i = p_i/1 - p_i$ that corresponds to $\omega \in A_i$ or $\omega \in A_i^c$. We find:

$$P(\omega) \leq \prod_{i=1}^{n} t_i \leq \exp\left(-\sum_{i=1}^{n}(1 - t_i)\right)$$

Where the second step notes a property for $t_i \in [0, 1]$. Note, it is clear that the above is bounded by:

$$\leq \exp\left(-\sum_{i=1}^{n} \alpha_i\right)$$

If $\sum_n \alpha_n$ diverges, the above goes to 0, and we can conclude:

$$P(\omega) = 0$$

Which again, draws out our contradiction. qed.

1.2 Normal Numbers and Complements are Dense in $(0, 1]$

Show that N and N^c are dense in $(0, 1]$. Recall, the definition of *dense* is that N is dense in $(0, 1]$ if for each $x \in (0, 1]$, and each interval J containing x, there is a $y \in N$ such that $y \in J$.

Take $\omega \in (0, 1]$. We note ω has some form:

$$(d_1(\omega), d_2(\omega), \cdots)$$

Note, the problem is equivalent to saying that if $\omega \in N$, can we find $x \in N^c$ arbitrarily close to ω, and vice versa, for $\omega \in N^c$ and $x \in N$. We first assume $\omega \in N$. We can easily find an $x \in N^c$, such that x is arbitrarily close to ω. Just take the first k elements matching, so that we are within $\frac{1}{2^k}$ of ω, and continue with ones. Clearly, such an x can be arbitrarily close to ω, and within N^c. So, N^c is clearly dense.

For the other direction, take $\omega \in N^c$, and now, again, match x on the first k elements of the binary expansion. For the remainder, oscillate between 1 and 0. Clearly, $x \in N$, and arbitrarily close to ω. qed.

1.3 Trifling Set Properties

Definition: Define a set A to be *trifling* if for each ϵ there exists a *finite* sequence of intervals I_k satisfying that they cover A and interval sum less than ϵ. Recall, from Calculus on Manifolds - this is essentially content 0.

1. A trifling set is also negligible. This must is clear - take the remaining infinite intervals as ones that sum up to less than a small enough ϵ'.

2. Show that the *closure* of a trifling set is also trifling. Recall, the closure is all points that are not exterior to A - exterior meaning that they have open neighborhoods not intersecting A. Well, take a finite covering less than $\epsilon/2$ of A. We define:

$$A \subset \bigcup_{k=1}^{n} I_k \qquad \sum_{k=1}^{n} I_k < \frac{\epsilon}{2} \qquad I_k' = \left(a_k - \frac{\epsilon}{2^{k+2}}, b_k + \frac{\epsilon}{2^{k+2}}\right]$$

$$\implies \sum_{k=1}^{n} I_k' < \epsilon$$

   Note that I_k' covers $\overline{A}$. Take $x \in \overline{A}$. By definition, we have $\left(x - \frac{\epsilon}{2^{n+2}}, x + \frac{\epsilon}{2^{n+2}}\right)$ intersects A, and so coincides with some I_k, and so must be contained within I_k'. Thus, it is clear that I_k' covers $\overline{A}$, and so $\overline{A}$ is trifling as well.

3. The rationals in $(0, 1]$ are bounded and negligible (being countable), but not trifling. Assume we have a covering of the rationals that is finite and sums to less than ϵ. Note, we can take the covering to be restricted to $(0, 1]$, as all the rationals are in $(0, 1]$. For $\epsilon < 1$ - Theorem 1.3.2 implies that these intervals do not cover all of $(0, 1]$. Note, if they don't cover a rational, we are done. We now note that these sets must not cover some interval of non negligible length - if they covered every such interval, there sum would be 1. This interval contains a rational, which contradicts the set being covering.

4. Show that the closure of a negligible set may not be negligible. Again, the closure of the rationals in $(0, 1]$ is $(0, 1]$, which is not negligible.

5. Show that finite unions of trifling sets are trifling, but that this can fail for countable unions. Fail for countable unions - take the union of each rational, which is a countable union of trifling singleton sets - by the above, this is negligible, and not trifling. Note, for finite unions of trifling sets - say k such sets - take the covering of size $\frac{\epsilon}{2^k}$, and note that the union of these intervals is a finite covering of total length less than ϵ.

1.4 Trifling Sets In Base r

1. First thing to note. We can look at $A_r(i)$ as iteratively removing intervals from $(0,1]$, where step k corresponds to removing the numbers whose expansions do not contain i for the first $k-1$ digits, but contain i at digit k. At step k, we remove $(r-1)^{k-1}$ intervals (corresponding to the $r-1$ possible digits in the first $k-1$ spaces) of length $\frac{1}{r^k}$ (corresponding to the length of the interval starting with $i \in [r-1]$ in the kth entry going to $i+1$ in the kth entry). We find, that the total length of the disjoint intervals removes is:

$$\sum_{k=1}^{\infty} \frac{(r-1)^{k-1}}{r^k} = \frac{1}{r} \sum_{k=1}^{\infty} \frac{(r-1)^{k-1}}{r^{k-1}} = \frac{1}{r} \sum_{k=0}^{\infty} \frac{(r-1)^k}{r^k} = \frac{1}{r} * \frac{1}{1/r} = 1$$

   Where the second to last step is the sum of a geometric series. So, at the very least, we have that it could be possible that $A_r(i)$ is trifling.

   Here is how it is trifling. We *know* that a finite amount of points is trifling. As the above sum equals 1 - if we go far enough, the amount removed will be arbitrarily close to 1. Say, within $\epsilon/2$ of 1. And so, the remaining *intervals* that are uncovered must have at most a total length of $\epsilon/2$. We can cover those intervals with the intervals themselves. Frankly, I think that is enough. Note, of course, at each step we remove an interval that looks like $(a,b]$. And so, the remaining intervals should be of the form $(c,d]$ as well. Everything should be nice, as the intervals in our iteration are disjoint. I literally think that is it. qed.

2. We want to find a trifling set A such that every point in the unit interval can be represented in the form $x+y$ with x and y in A.

3. Let $A_r(i_1, \cdots, i_k)$ consist of the numbers in the unit interval in whose base r expansion the digits $i_1, \cdots, i_k$ nowhere appear consecutively in that order. Show that it is *trifling.*

   The first observation I have made: if we have that $i_1, \cdots, i_k$ are all equal, whereas $j_1, \cdots, j_k$ are an arbitrary sequence of digits, we have that:

$$|A_r(j_1, \cdots, j_k)| \leq |A_r(i_1, \cdots, i_k)|$$

   The reason is the following: for the first n digits of the base r expansion, there are more numbers without $i_1, \cdots, i_k$ appearing consecutively then there are numbers without $j_1, \cdots, j_k$ appearing consecutively. Consider the following example: Base 3, with $n = 3$. We have the following possible sequences:

000	001	002	010	011	012	020	021	022
100	101	102	110	111	112	120	121	122
200	201	202	210	211	212	220	221	222

Consider the count of sequences above without 11. There are 22 such sequences. Now, count the sequences without 12. There are 21 such sequences. Note, above, we have that the sequence 111 has 11 at the start, and 11 at the end, but only takes up one entry. However, we have that $12X$ and $Y12$ can never be the same, and so there are two such sequences taken up. We can expand this concept in general - when $i_1, \cdots, i_k$ are all equal, we get the most *collisions* between sequences with digits $i_1, \cdots, i_k$ starting at the possible $n - k + 1$ starting points. And so, if we find that $A_r(i_1, \cdots, i_k)$ is trifling for $i_1 = \cdots = i_k$, we can conclude that $A_r(j_1, \cdots, j_k)$ is trifling in general.

To be honest, I am going to skip this one, because I am getting nowhere. However, here is the work I've done so far, for what it is worth. I have made the following definitions:

$S_{k,n} = \{$The length n sequences that contain the digit d repeated k times$\}$

$A_{t,n} = \{$The length n sequences where d repeated k times first appears at pos $t\}$

And so, with these definitions, we have:

$$S_{k,n} = \bigcup_{t=1}^{n-k+1} A_{t,n} \implies |S_{k,n}| = \sum_{t=1}^{n-k+1} |A_{t,n}|$$

Where the first step is just definitional, as if d is repeated k times, that subsequence first starts at position $1, 2$, or up to position $n-k+1$. The second step comes from noting that the sets $A_{t,n}$ are disjoint - if the sequence first starts at position t, it *does not* start at position $t' \neq t$. And so, if we can find that $\lim_{n\to\infty} \frac{|S_{k,n}|}{r^n} = 1$, then for n large enough, it will equal $1 - \epsilon$, and we can take a *finite* number of intervals to cover the remaining intervals of total length ϵ that are not represented in the *finite* union of intervals $\bigcup_{t=1}^{n-k+1} A_{t,n}$. The difficult part is actually finding what the above is in terms of numbers. However, I now note that:

$$|A_{t,n}| = r^{n-(t+k-1)} \cdot 1^k \cdot (r-1) \cdot (r^{t-2} - |S_{k,t-2}|)$$

This comes from examining what each of the possible digits in our expansion of $x \in A_{t,n}$ can be. The remaining $n - (t + k - 1)$ after the digits d repeated k times starting at position t can be anything we want. This gives us our first element in the product. 1^k refers to the k digits starting at t must all be d. $(r - 1)$ refers to position $t - 1$ - that *must* be any number other than d. If it is d, then we get $x \in A_{t-1,n}$, which is incorrect. Finally, the remaining first $t - 2$ entries can be any sequence at all, except for a sequence of d repeated k times. The count of those sequences is removed from the total r^{t-2} sequences possible. And so, we find that essentially, both sides are equal. If we make any sequence described on the right side, we have that it is within $A_{t,n}$. And, any sequence in $A_{t,n}$ can be described on the right side. And so now, the work remains to just simplify the calculation of $|S_{k,n}|$. Note, we could actually calculate this value by a recursive algorithm. That might help us.

Big Note: The calculation of $|A_{t,n}|$ assumes somethings, like the existence of position $t - 1$ (which is not there if $t = 1$) or $t > 2$ for r^{t-2}. Just make sure to keep these exceptions in mind. Anyway. We make a hand wavy assumption - that $r^n - |S_{k,n}| > r^{k-1}$. Note, r is some base n digit, and so r^{k-1} is just some constant. And while we want to eventually prove that the ratio is equal to 1 in limit - ultimately, there will always be some constant distance between r^n and $|S_{k,n}|$. And this constant will continue to grow to infinity. Anyway, for n large enough, I think it is clear. And so, being hand wavy, and including all terms, although they might not be present, we have:

$$\lim_{n\to\infty} \frac{|S_{k,n}|}{r^n} \approx \lim_{n\to\infty} \frac{\sum_{t=1}^{n-k+1} r^{n-(t+k-1)} \cdot (r-1) \cdot (r^{t-2} - |S_{k,t-2}|)}{r^n}$$

$$= \lim_{n\to\infty} \frac{r^{n-k+1}(r-1) \sum_{t=1}^{n-k+1} r^{-t} \cdot (r^{t-2} - |S_{k,t-2}|)}{r^n}$$

$$= r^{-k+1}(r-1) \lim_{n\to\infty} \sum_{t=1}^{n-k+1} r^{-t} \cdot (r^{t-2} - |S_{k,t-2}|)$$

Now, making use of our hand waviness, we have that:

$$\geq r^{-k+1}(r-1) \lim_{n\to\infty} \sum_{t=1}^{n-k+1} r^{-t} r^{k-1} = (r-1) \lim_{n\to\infty} \sum_{t=1}^{n-k+1} r^{-t}$$

Now, we can make use of our geometric series, and have that the above equals:

$$= (r-1) * \frac{1}{r-1} = 1$$

And so yes, the limit does indeed equal 1. Unraveling everything we said above, this allows us to conclude that $A_r(i_1, \cdots, i_k)$ is indeed trifling. Now, there is a saying, that a monkey typing at random for infinity will ultimately write Shakespeare. Well, we can let every word be a digit in some base 10 million language. Ultimately, the probability that a monkey *does not* type our specific Shakespeare sequence is indeed 0, as the amount of "worlds" where the monkey types at random, but does not hit our $A_r(i_1, \cdots, i_k)$ digit sequence is 0.

1.5 Cantor Set Is Trifling, Uncountable, and Perfect

1. The Cantor set C can be defined as the closure of $A_3(1)$. Show that C is uncountable but trifling. First, note uncountable. This can be done by the diagonalization argument. Take any list of numbers in $A_3(1)$, which are also in C. We can make a new number in $A_3(1)$, but not in the list, by taking the *ith* entry, and switching the 0 for 2 or 2 for 0. Thus, $A_3(1)$ is clearly uncountable, and so to must be C.

   Now, we note that $A_3(1)$ is trifling. Take any finite covering of $A_3(1)$ by intervals with total length less than ϵ, than covers $A_3(1)$. Note that we can extend each interval by some length of $\epsilon/2^{i+3}$ on each edge, and this should cover all of C as well. This is because any element within the closure of $A_3(1)$, can also be viewed as within the closure of all the intervals, and so we can extend the interval lengths a bit. Note, this would apply for every trifling set (in $\mathbb{R}$, not sure about $\mathbb{R}^n$ with covering half open rectangles, but I think the idea could be extended).

2. From $[0, 1]$, remove the open middle third $(1/3, 2/3)$. From the remainder, a union of two closed intervals, removed the open middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$. Show that C is what remains when this process continues ad infimum.

   Note, this is the standard definition of C. We have to show this equals our closure definition above. I make the note - the *nth* step is the closure of points in $[0, 1]$ such that the base 3 representation does not contain 1 in the first n digits. This, I think, is clear. And so, taking the process to infinity, we can conclude that C is the closure of the set where none of the digits are 1.

3. Show that C is perfect. A set is *perfect* if it is closed and for each x in A and positive ϵ, there is a y in A such that $0 < |x - y| < \epsilon$.

   Note, C is closed, as it is the closure of a set. Now, take $x \in C$ and $\epsilon > 0$. We can find the corresponding y just by matching say the first n digits of x, and then flipping the $n + 1$ digit from 0 to 2 or vice versa, and then taking any random 0 or 2 for the remaining digit. Note, $0 < |x - y| < \epsilon$ if n is large enough. Note - I guess this doesn't apply for the points in the closure. For a point $x \in C$ in the closure, we can find a y in $A_3(1)$ within $\epsilon/2$ of x, by the limit definition of the closure. Now, take z within $\epsilon/2$ of y by changing a digit. We now know that $z \neq x$, and the property applies. qed.

1.6 Alternate S_n Integral Value Proof

We first show the derivative property. We have that:

$$M(t) = \int_0^1 e^{ts_n(\omega)} d\omega$$

We have that $f(\omega, t) = e^{ts_n(\omega)}$. Note, we can make use of *Leibnitz's rule* to take the derivative of $M(t)$ under the integral. See problem 3-32 in Calculus on Manifolds by Spivak. However, this presupposes that f is continuous. I get around this by the following: note that $s_n(\omega)$ has only finite points of discontinuity, when we switch from one dyadic interval of rank n to the next. We can split $\int_0^1$ so that we ignore those points of discontinuity, and only integrate where $s_n(\omega)$ is continuous (and constant). Note, the points of discontinuity have content 0, and so the integral on $[0, 1]$ is equal to the integral on the set not including those points of discontinuity. Thus, we have:

$$M'(t) = \int_0^1 D_2(e^{ts_n(\omega)}(0) d\omega = \int_0^1 s_n(\omega) e^{ts_n(\omega)} d\omega \implies M'(0) = \int_0^1 s_n(\omega) d\omega$$

Now, we can repeat this operation a finite amount of times, successive differentiation under the integral, to clearly find that:

$$M^{(k)}(0) = \int_0^1 s_n(\omega)^k d\omega$$

Now, noting again that $s_n(\omega)$ is constant on the 2^n dyadic intervals, we have that $M(t)$ is actually easy to evaluate. For each of those 2^n intervals, s_n is some sum of the form $\pm 1 \pm 1 \cdots \pm 1$, and so we have that:

$$M(t) = \frac{1}{2^n} \sum_{i=1}^{2^n} \exp\left(t\left(\pm 1 \pm 1 \cdots \pm 1\right)\right)$$

We note that this can be broken down with a binomial coefficient. If we have that k of the ± 1 are -1, then the value of the sum is $n - 2k$. And so, based off of how many possible sequences of the ± 1 that contain k -1, we have that the above can be expressed as:

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp(t(n-2k)) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp(t(n-k-k))$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp(t(n-k) - t(k)) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \exp(t)^{n-k} \exp(-t)^k$$

By the Binomial Theorem, the above equals:

$$= \frac{1}{2^n} \left(\exp(t) + \exp(-t)\right)^n = \left(\frac{e^t + e^{-t}}{2}\right)^n = (\cosh t)^n$$

Where the last step is just an identity. Now, for a new proof of 1.16:

$$\int_0^1 s_n(\omega) d\omega = M'(0) = n \cosh^{n-1}(0) \sinh(0) = n * 1 * 0 = 0$$

Now, for a new proof of 1.18:

$$\int_0^1 s_n^2(\omega) d\omega = M''(0) = n \left[(n-1) \cosh^{n-2}(0) \sinh^2(0) + \cosh^n(0)\right] = n*1 = n$$

Finally, for a new proof of 1.28, we just have to take the fourth derivative, and plug in 0. I will not be going over the steps, but using derivative calculator, we get the fourth derivative at 0 is:

$$\int_0^1 s_n^4(\omega) d\omega = M''''(0) = n\left(3n - 2\right) = 3n^2 - 2n$$

Which is the final property. qed.

1.7 Vieta's Formula

We first find a similar property to the above. We examine:

$$\int_0^1 \exp\left[i\sum_{k=1}^n a_k r_k(\omega)\right] d\omega$$

We note that the summation is constant on the 2^n dyadic intervals of rank n. In which case, the integral becomes a summation, that looks like:

$$= \frac{1}{2^n}\sum \exp\left[i\left(\pm a_1 \pm a_2 \pm \cdots \pm a_n\right)\right]$$

Now, we extract the first $+a_1$ term and $-a_1$ term from the exponential:

$$= \frac{1}{2^n}\exp(ia_1)\sum \exp\left[i\left(\pm a_2 \pm \cdots \pm a_n\right)\right] + \frac{1}{2^n}\exp(-ia_1)\sum \exp\left[i\left(\pm a_2 \pm \cdots \pm a_n\right)\right]$$

We note that by symmetry, both the summations are equal, and so the above actually equals:

$$= \frac{1}{2^n}\left(\exp(ia_1) + \exp(-ia_1)\right)\sum \exp\left[i\left(\pm a_2 \pm \cdots \pm a_n\right)\right]$$

We can continue this process n times, and then split the 2^{-n}, to find:

$$\int_0^1 \exp\left[i\sum_{k=1}^n a_k r_k(\omega)\right] d\omega = \prod_{k=1}^n \frac{e^{ia_k} + e^{-ia_k}}{2}$$

Using the cos identity, we find:

$$\int_0^1 \exp\left[i\sum_{k=1}^n a_k r_k(\omega)\right] d\omega = \prod_{k=1}^n \cos(a_k)$$

We let $a_k = t2^{-k}$. We note that $\sum_{k=1}^{\infty} r_k(\omega)2^{-k} = 2\omega - 1$ - this is because in each entry, we have $r_k(\omega) = 2d_k(\omega) - 1$, and $\omega = \sum_{k=1}^{\infty} d_k(\omega)2^{-k}$. We can apply the $2\omega - 1$ operations (as it is continuous and passes the summation limit), to derive the above. We thus have:

$$\lim_{n\to\infty}\int_0^1 \exp\left[i\sum_{k=1}^n a_k r_k(\omega)\right] d\omega = \int_0^1 \exp\left[i\lim_{n\to\infty}\sum_{k=1}^n a_k r_k(\omega)\right] d\omega = \int_0^1 \exp\left[ti(2\omega - 1)\right] d\omega$$

Note, we don't have theorems to pass the limit through the integral yet. However, I believe this will be by the monotone convergence theorem - the partial sums should be non decreasing, given the exponential being nonnegative. Anyway, we don't have to prove that here. We now note this is an integral from 0 to 1 - whose value is:

$$-\frac{ie^{it(2x-1)}}{2t} = \frac{-i^2}{t}\left(\frac{e^{it}-e^{-it}}{2i}\right) = \frac{\sin(t)}{t}$$

Taking the limit on the other side, we can conclude:

$$\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\frac{t}{2^k}$$

We can now derive Vieta's Formula:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

We recall the half angle formula for cos:

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos(\theta)}{2}}$$

We can use this to make an induction argument that:

$$\cos\left(\frac{\pi}{2}\frac{1}{2^k}\right) = \frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}$$

For the base case of $k = 1$, we have:

$$\cos\left(\frac{\pi}{2}\frac{1}{2}\right) = \sqrt{\frac{1+\cos(\pi/2)}{2}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

Now, for arbitrary k, we have:

$$\cos\left(\frac{\pi}{2}\frac{1}{2^k}\right) = \cos\left(\frac{1}{2}\frac{\pi}{2}\frac{1}{2^{-1}}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{2}\frac{1}{2^{k-1}}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{4}} = \frac{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}{2}$$

And so, by our identity, we find:

$$\frac{2}{\pi} = \frac{\sin(\pi/2)}{\pi/2} = \prod_{k=1}^{\infty}\cos\left(\frac{\pi}{2}\frac{1}{2^k}\right) = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

And we have found Vieta's formula. qed.

1.8 Differences between the Weak and Strong Law Is Uniform Convergence

A number ω is normal in base 2 if and only if for each positive ϵ there exists an $n_0(\epsilon, \omega)$ such that $|n^{-1} \sum_{i=1}^{n} d_i(\omega) - 1/2| < \epsilon$ for all n exceeding $n_0(\epsilon, \omega)$. That is just the definition. Theorem 1.2 concerns the entire dyadic expansion (ie, the complement of the set of normal numbers whose infinite sum equals 1/2 has probability 0), whereas Theorem 1.1 concerns only the beginning sequence (ie, the limit of probability, of the dyadic expansion numbers whose first n partial sum is not close to 1/2, is 0). Identify the difference by showing that for $\epsilon < 1/2$ the $n_0(\epsilon, \omega)$ above cannot be the same for all ω in N - in other words, $n^{-1} \sum_{i=1}^{n} d_i(\omega)$ converges to 1/2 for all ω in N, but not uniformly. But see problem 13.9.

Well - noting that for $\epsilon < 1/2$, the $n_0(\epsilon, \omega)$ cannot be the same for all ω in N, means finding ω in N for which the n is different. We can take just ω where the first k are all 0, and then the remaining digits alternate between 0 and 1. Note, that the first n for which that sum is actually close to 1/2 (when normalized by n^{-1}), must increase to ∞. And so yes, we have convergence to 1/2 for ω in N, but not uniform convergence. And so, while Theorem 1.1 can rely on this non uniform convergence (ie, the sets are getting smaller, because ultimately, we reach that n value for all ω), we cannot rely on that for Theorem 1.2.

Looking at Problem 13.9 - it looks like, however, we might be able to get uniform convergence on subsets of N. Perhaps this is how we can actually prove Theorem 1.2. Anyway, I like to just note that the underlying difference in the two Theorems comes from an underlying difference in just real number theory - uniform vs non uniform convergence.

1.9 Nowhere Dense Set Existence and Properties

1. Show that a trifling set is nowhere dense. A set E is nowhere dense in B if each open interval I contains some interval J that does not meet E. Use the previous problems (1.3 b) and theorems (1.3ii) to prove this.

   I am going to assume we have a trifling set E in the real line. We take an open interval of the real line I. We want to find some interval J which our trifling set E does not meet. Assume $I = (a, b]$, and so take $\epsilon = (b - a)/2$. By trifling, there is a finite interval cover of E with total length less than ϵ. Assume there is no interval J in I that E does not meet. Ie, all subintervals J meet E. This means that all subintervals J are contained within the finite cover. This is because every point inside of J has an open neighborhood around it that meets E, and so that open neighborhood meets the interval - and if the neighborhood is small enough, the original point is contained within the interval as well (I guess, this can be, via the closure is trifling as well).

And so, every subinterval of I is contained any finite open cover of E - this is a contradiction, as this implies every subinterval of I is vanishing in length. Some subintervals of I are of length $(b-a)/2$, which is not vanishing. qed.

2. Let $B = \bigcup_n (r_n - 2^{-n-2}, r_n + 2^{-n-2}]$, where $r_1, r_2, \cdots$ is an enumeration of the rationals in $(0,1]$. Show that $(0,1] - B$ is nowhere dense but not trifling or even negligible.

   We first show nowhere dense. Take a subinterval I of $(0,1]$. Note that I must contain some rational r_n, as the rationals are dense. We let J be one of the intervals from the intersection of $(r_n - 2^{-n-2}, r_n + 2^{-n-2}] \cap I$. Note that $J \subset I$, and $J \subset B$. And so, $J \cap (0,1] - B = \emptyset$. And so B must be nowhere dense.

   We now note that $(0,1] - B$, however, is not negligible (and thus not trifling). We have that:

$$|B| \leq \sum_{n=1}^{\infty} |(r_n - 2^{-n-2}, r_n + 2^{-n-2}]| = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = 1/2$$

   And so, we must have that:

$$|(0,1] - B| \geq 1 - 1/2 = 1/2$$

   If we had any cover, whose total length is less than $\epsilon < 1/2$, we would get a contradiction. qed.

3. Take a compact negligible set. Take an infinite open interval cover whose total length is less than ϵ. By compactness, a finite subset of those sets covers the compact negligible set as well. And so, the set is trifling.

1.10 Normal Numbers are of the First Category

Definition: Set of the First Category A set of the first category is a set that is a countable union of nowhere dense sets. This is a topological notion of smallness - similar to first countable, I guess. This is similar to the metric notion of smallness, negligibility. Neither condition implies the other.

1. Show that the non negligible set N of normal numbers is of the first category by proving that:

$$A_m = \bigcap_{n=m}^{\infty} \left[\omega : |n^{-1}s_n(\omega)| < 1/2\right]$$

   Is nowhere dense, and $N \subset \bigcup_m A_m$. Note, we can prove A_m is nowhere dense in the following way - if the complement is dense, and each point in that dense set has an interval around it contained in the complement, then A_m is nowhere dense. This is similar to the above problem. Note, this implies nowhere dense, as every interval I has a point in the complement (by the complement being dense), and this point has an interval around it that is not contained in A_m.

   We have that:

$$A_m^c = \bigcup_{n=m}^{\infty} \left[\omega : |n^{-1}s_n(\omega)| \geq 1/2\right]$$

   We note - each point $\omega \in A_m^c$ has an interval around it in A_m^c. This is just the dyadic interval - $\omega \in A_m^c \implies \omega \in [\omega : |n^{-1}s_n(\omega)| \geq 1/2]$. Every point in the same rank n dyadic interval as ω must also belong to A_m^c. Now, we just have to find such an $\omega \in I$. Note, we can just find some dyadic interval of rank n that is contained in I. Then, we can examine:

$$\left[\omega : |4n + 1^{-1}s_{4n+1}(\omega)| \geq 1/2\right]$$

   We can take ω such that the first n bits place ω in the dyadic interval of rank n - then, the remaining bits can be 1, and so $\omega \in I$, and $\omega \in [\omega : |(4n+1)^{-1}s_{4n+1}(\omega)| \geq 1/2] \implies \omega \in A_m^c$. Note, this is because $(4n+1)^{-1}s_{4n+1}(\omega) \geq \frac{3n+1-n}{4n+1} = \frac{2n+1}{4n} \geq 1/2$ Thus, A_m^c is nowhere dense.

   The final part is to prove $N \subset \bigcup_m A_m$. Well, $\omega \in N$ essentially implies for m large enough, $\omega \in A_m$. By the limit definition, we have that at some point, $\frac{1}{n}s_n(\omega)$ remains within ϵ of 0. Just set $\epsilon = 1/2$, and we have it.

2. According to a famous theorem of Baire, a nonempty interval is *not* of the first category. Use this fact to prove that the negligible set

$N^c = (0, 1] - N$ is not of the first category.

Proof by contradiction. Assume N^c is of the first category. Then, N and N^c are both of the first category. Note, the countable union of sets of the first category are of the first category (stemming from the countable union of countable sets is still countable), and so this would imply $[0, 1] = N \cup N^c$ is of the first category as well. This is a contradiction of the Theorem by Baire. Thus, N^c is not of the first category. qed.

1.11 Diophantine Approximation Properties

Prove:

1. If x is rational, (1.33) has only finitely many irreducible solutions. Recall, (1.33) is the formula:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

Well, we have x is some irreducible $\frac{p'}{q'}$. Thus, we have:

$$\left| \frac{p'}{q'} - \frac{p}{q} \right| = \frac{|p'q - q'p|}{q'q}$$

We note that $|p'q - q'p| \geq 1$. This is integer multiplication and subtraction, which results in an integer. The only case when this is less than 1 is when it equals 0 - or $p'q = pq'$. By different rationals, we can only have one of $q = q'$ or $p = p'$. If we are in that case, divide and not equal. If $q \neq q'$ and $p \neq p'$, then the above is equal only if $p' = q'$ and $q = p$ - in which case we are not irreducible. Anyway, this implies that:

$$\left| \frac{p'}{q'} - \frac{p}{q} \right| \geq \frac{1}{q'q}$$

And so, the number of rationals $\frac{p}{q}$ that satisfy $\frac{1}{q^2} \geq \frac{1}{q'q}$ is finite.

2. Suppose that $\varphi(q) \geq 1$ and (1.35) holds for infinitely many pairs p, q but only for finitely many relatively prime ones. Then x is rational.

Just as a reminded, we have that (1.35) is:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2 \varphi(q)}$$

We have that by Theorem 1.4, if x is irrational, there are infinitely many irreducible fractions p/q such that:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

Well, frankly, I don't understand what the "relatively prime" wording here is. Like, if x is rational - we just set p/q as x, and we have infinitely many *reducible* rational fractions that satisfy 1.35. I am just going to take the statement as "finitely many irreducible fractions satisfy 1.35." If $\varphi(q) = 1$, it is clear that this implies x is rational - this is by Theorem 1.4, which says there are infinitely many irreducible solutions for irrational x.

Now, we consider $\varphi(q) > 1$. And, well, what I think we want to note is the following. By part 1 - a rational will always have finitely many solutions, for the p/q that satisfy $\frac{1}{q^2\varphi(q)} > \frac{1}{q'q}$. So, the only distinction to make is that if x is *irrational* - well, then:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2\varphi(q)}$$

Either has infinitely many, or 0 solutions. Suppose irrational x has finitely many solutions to the equation:

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2\varphi(q)}$$

I think we can make a contradiction like in Theorem 1.4. Suppose we had finitely many irreducible solutions - $p_1/q_1, \cdots, p_m/q_m$. Then, $|x - p_k/q_k|$ is positive, and we can take Q where Q^{-1} is smaller than each of the differences. Anyway, I don't actually want to do this.

3. If φ goes to infinity too rapidly, then A_σ is negligible (Theorem 1.6). But however rapidly φ goes to infinity, show that A_φ is nonempty, even uncountable. Note, this one has a solution in the book. I don't want to really go over it - I read the section as it was interesting, but these problems aren't something I'm interested in.

Section 2 - Probability Measures

Notes

Spaces We first go over Ω - a space of points, where each $\omega \in \Omega$ is a possible result or outcome. What is interesting to me here is - the fact that Ω will be interesting from the point of view of *geometry and analysis*, as well as the point of view of probability. In fact, we will probably be able to understand probability properties at times, because of the geometry/analysis properties of the underlying space. Read the section for some examples.

A subset of Ω is an *event*, and an element $\omega \in \Omega$ is a sample point.

Assigning Probabilities This section discusses assigning probabilities to events (subsets) in our space Ω, in a way such that we will be able to get useful properties out of it. It goes over the unit interval $(0, 1]$, and how we have assigned probabilities to disjoint (countable) unions of intervals. We could extend this to say negligible (metric notion) sets are probability 0. However - how can we be sure we have assigned probabilities to all useful sets? The successful procedure, generally, is to assign probabilities to such a large amount of sets, that any set we could possibly think of would be covered (or, not think of, but usefully use).

Another interesting point - can we not just assign a probability to every subset? Well, we will find later, no, we cannot, as this will remove an important property that we need from the spaces - countable additivity. We will go over this later. Given this - we will need to restrict ourselves to a subclass of the class of all subsets of a space.

Classes of Sets This section essentially outlines what our class of subsets should and shouldn't have, for it to satisfy some of the properties we noted above, and to be able to express some of the stuff we were doing previously.

We can express the set of normal numbers as a countable union and intersection of disjoint intervals. This is done by expressing the limit definition (ϵ/δ) in terms of a countable intersection and union. It is pretty simple. However, this tells us - if we want a systematic treatment of section 1 - we need a class of sets that contains the intervals, and is closed under the formation of countable unions and intersections.

A second interesting thing to note is - the singleton $\{x\}$ is a countable intersection $\bigcap_n (x - n^{-1}, x]$. And so, if a class contains singletons, and is closed under *arbitrary unions* - our class is essentially all the subsets of Ω. As the theory does not apply to such an extensive class - in the uncountable Ω case - our attention needs to be restricted to countable set theory operations.

Definition - Fields and σ Fields are sets of subsets of Ω with the following properties:

1. $\Sigma \in F$
2. $A \in F \implies A^c \in F$
3. $A, B \in F$ implies $A \cup B \in F$ for a field, or countable unions for a σ field

Note, via DeMorgan's law, you have that the above definitions are equivalent for finite/countable intersections.

Examples

1. The finite disjoint unions of subintervals on $\Omega = (0, 1]$ is a field, denoted B_0. It is not, however, a σ field, as it doesn't contain singletons (not a subinterval of the form (]), which is a countable intersection of intervals of the form (].
2. Finite and cofinite (ie, A^c is finite) sets of Ω are a field. If Ω is finite, then they are a σ field as well (as F would be all subsets). If Ω is not finite - F is not a σ field.
3. Countable and cocountable sets of Ω are a σ field. If Ω is uncountable, there are sets that F misses, however (via the axiom of choice).

Definition - Generated Sigma Field For a class of sets A, the smallest σ field containing A is called $\sigma(A)$. It is the (non empty) intersection of σ fields containing A. Note, any arbitrary intersection (even uncountable) of σ fields is a σ field - just a definitional property. Some other properties of generated σ fields are:

1. $A \subset \sigma(A)$
2. $\sigma(A)$ is a σ field
3. If $A \subset G$ and G is a σ field, $\sigma(A) \subset G$
4. If F is a σ field, $\sigma(F) = F$
5. If $A \subset A'$ then $\sigma(A) \subset \sigma(A')$
6. If $A \subset A' \subset \sigma(A)$, then $\sigma(A) = \sigma(A')$.

Example - Borel Sets Let I be the subintervals of $\Omega = (0, 1]$, and let $B = \sigma(I)$. Note that $I \subset B_0 \subset B$, and so $\sigma(B_0) = B$. Note that elements of B are called the Borel sets. Note, it contains open sets on the unit interval (intersection of contained rational intervals).

Probability Measures A set function is a real-valued function defined on some class of subsets of Ω. A set function P on a field F is a *probability measure* if it satisfies the conditions:

1. $0 \leq P(A) \leq 1$ for $A \in F$
2. $P(\emptyset) = 0, P(\Omega) = 1$
3. Countable additivity.

Note, countable additivity implies finite additivity. Note, the conditions can be seen as slightly redundant (ie, they can be relaxed, and still imply the same things).

Probability Space If F is a σ field on Ω, and P is a probability measure on F, the triple (Ω, F, P) is called a *probability measure space*, or simply *probability space*. A support for P is any F set A for which $P(A) = 1$.

Discrete Probability Space If Ω is countable, and $p(\omega)$ is a nonnegative function on Ω, which sums to 1 for all ω. Then, $P(A) = \sum_{\omega \in A} p(\omega)$ is a probability measure, and so (Ω, F, P) is a probability space. This forms the basis for discrete probability theory.

An interesting note: why do we call it discrete probability theory, but not *countable probability theory*? Well, I think the reason for this is the following. As noted earlier, and within my measure theory notes - we cannot have a *discrete* probability space on an uncountable space. This is because, any measure which assigns a value to each subset of an uncountable space, cannot have countable additivity. And so - a *discrete probability space* implies that the space Ω must be *countable*.

Massed Discrete Probability Space This just refers to the notion that Ω need not be countable - however, there are finitely/countably many ω_k with corresponding masses m_k, such that $P(A) = \sum_{\omega_k \in A} m_k$ for A in F. Here, P is discrete, but the space need not be. We can write $P(A) = \sum_k m_k I_A(\omega_k)$.

For P probability measure on a field F, we can easily prove the following properties:

1. $P(A) \leq P(B)$ if $A \subset B$
2. $P(A^c) = 1 - P(A)$
3. $P(A) + P(B) = P(A \cup B) + P(A \cap B)$
4. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. Inclusion-exclusion formula, which is just induction on the above formula.
6. Finite subadditivity
7. Continuity from below: If A_n and A lie in F and $A_n \uparrow A$, then $P(A_n) \uparrow P(A)$. Note, the $\uparrow$ notation implies inclusion and total union is A, whereas implies monotone increasing up to $P(A)$ in the limit.
8. Continuity from above: If A_n and A line in F and $A_n \downarrow A$, then $P(A_n) \downarrow P(A)$.
9. Countable subadditivity, which stems from the two easily proved above properties.
10. Finite additivity and $A_n \downarrow \emptyset$ implies $P(A_n) \downarrow 0$ implies countable additivity.

Lebesgue Measure on the Unit Interval In this section - we essentially say that our measure $|I| = |(a, b]| = b - a = \lambda(I)$ is a *finitely additive* probability measure on B_0, the set of finite unions of disjoint intervals I. However, we are also able to extend this - and say that λ is a *countably additive probability measure* on our *field* of B_0. This involves using Theorem 1.3 multiple times, where we concluded that if $A = \bigcup_k I_k$ for disjoint I_k, then:

$$\lambda(A) = \sum_k \lambda(I_k)$$

However, the tricky bit comes in extending this to $A = \bigcup_k A_k$ for $A_k \in B_0$. However, at the end of the day, each of the A and A_k can be expressed as a *finite union of disjoint intervals*, and so we can just break down the summations to where we can use Theorem 1.3 iii on a countable union of disjoint intervals. None of these theorems are particularly ground breaking.

However, I like the note at the end - consider what we might have to do to *extend* λ to the Borel sets $B = \sigma(B_0)$, and hopefully prove that λ is still countably additive on such a set. Note, to prove finite additivity on a field - we just had to rely on basic elementary properties of the real number system. However, to prove countable additivity on our field - we had to rely on something deeper about the real numbers - namely, *compactness of* $[a, b]$. To extend to B, however, and still prove countable additivity - we will need some *new property*, ultimately.

Sequence Space Note, everything here has to do with S, a *finite* set of points, regarded as *outcomes* of a simple observation, or experiment. Note - we can extend this to *multiple repeated experiments* in the following way. However, for me - I often had trouble with understanding how a *sequence of events* could be independent. Well - if we have S is a finite possible outcomes of events, we could just allow our σ algebra to be the *sequence space* - and then, different coordinates *should be independent.* Note, I haven't really proved that statement, but it seems to me to be intuitive enough.

Let $\Omega = S^\infty$ be the space of all infinite sequences:

$$\omega = (z_1(\omega), z_2(\omega), \cdots)$$

of elements of S: $z_k(\omega) \in S$ for all $\omega \in S^\infty$ and $k \geq 1$. The sequence above can be viewed as the result of repeating infinitely often the simple experiment represented by S. For $S = \{0, 1\}$, the space S^∞ is closely related to the unit interval. Note, I think this is the case for $S = \{0, 1, \cdots, n-1\}$ as well - just take S to be some base n representation of the numbers, and we can again identify S^∞ with the unit interval. See problems 1.4 and 1.5, for example.

Definition - Coordinate Functions Note that each $z_k(\cdot)$ is a *coordinate function*, or *natural projection*, or just *projection function.* Such a function takes an element in S^∞, and returns one in S. Let $S^n = S \times \cdots \times S$ be the Cartesian product of n copies of S - it consists of the n long sequences $(u_1, \cdots, u_n)$ of elements of S. For such a sequence, the set:

$$\{\omega : (z_1(\omega), \cdots, z_n(\omega)) = (u_1, \cdots, u_n)\}$$

Represents the event that the first n repetitions of the experiment give the outcomes $u_1, \cdots, u_n$ in sequence.

Definition: Cylinder of Rank n is a set of the form:

$$A = \{\omega : (z_1(\omega), \cdots, z_n(\omega)) \in H\}$$

Note - colloquially, I like to say - cylinders of rank n are just *restrictions* on the first n experiments. Note that $H \subset S^n$, and that if H is nonempty, A is also nonempty. The previous example, where $|H| = 1$, is called a *thin cylinder*.

Define C_0 as the class of cylinders of all ranks. We have that C_0 is a *field*. S^∞ and $\emptyset$ are within C_0, for $H = S^n, \emptyset$. If H is replaced by $S^n - H$, we have C_0 is closed under complements. We also have closed under finite unions - note just that the union of two cylinders of rank $n \leq m$ is just a restriction on the first m experiments, and is also a cylinder.

Definition: Probability Measure on the Field of Cylinders Let p_u, $u \in S$ be probabilities on S - nonnegative and summing to 1. Define a set function P on C_0 in this way:

$$P(A) = \sum_{(u_1, \cdots, u_n) \in H} p_{u_1} \cdots p_{u_n}$$

We have that P is a *probability measure*, with *finite additivity*. $P(S^\infty) = P(A) = 1$, where A is the cylinder with $H = S^n$. Clearly, $0 \leq P(A) \leq 1$, and $P(\emptyset) = 0$. We just need finite additivity - which can also be proved easily, but I won't give it here, just read the book. This is called a *product measure*, given that the definition just is a product of the individual probabilities on S.

Theorem 2.3 Every finitely additive probability measure on the field C_0 of cylinders in S^∞ is in fact countably additive. Note - this theorem is easy to prove, which we will, after stating the following lemma.

Lemma: If $A_n \downarrow A$, where the A_n are nonempty cylinders, then A is nonempty Using this lemma, we quickly prove Theorem 2.3. Recall, we had that if $A_n \downarrow \emptyset$ implies that $P(A_n) \downarrow 0$, for a finitely additive probability measure, then the probability measure is countably additive as well. Take $A_n \downarrow \emptyset$. Note, each A_n is a cylinder in C_0. Assume the lemma is true. Now, assume that A_n does *not* converge to 0. Then, $P(A_n) \geq P(\emptyset) \geq \epsilon > 0$ for some ϵ. But, this implies that A_n is nonempty. By the lemma, this makes the assumption $A_n \downarrow \emptyset$ a contradiction, and so we must have $P(A_n) \downarrow 0$.

Thus, we indeed have that if $A_n \downarrow \emptyset$, we must have $P(A_n) \downarrow 0$, and by our previous fact, the finitely additive probability measure on our *field* C_0 is indeed countably additive. qed.

Proof of Lemma Note, the proof of the lemma in the book is interesting. It is essentially a diagonalization argument, and taking subsequences of sequences that contain an element repeating an infinite number of times. It becomes easy to note that the ω containing these elements appearing infinite number of times is within the intersection of each A_n, and so A is nonempty. However, I want to try and prove the more general argument.

By Tychonoff's theorem, we have that the Cartesian product of countable compact topological spaces is compact as well, with the product topology. Note, we can take the discrete topology on S, and as S is finite, we have that it must be compact as well. For any open cover of S, we can take a finite subcover, by identifying each finite number of elements with one of the sets in the open cover that contains it. And so, we have that S^∞ must be compact as well, by Tychonoff's theorem.

Note, for a proof of Tychonoff's theorem, see my Topology notes. We now note that each $A_n \in C_0$ is a set within the product topology as well. Each A_n can be viewed as the intersection of open cylinders as defined in my topology book (the only difference being that there, open cylinders are restrictions on one coordinate, rather than restrictions on the first n coordinates). And so, the A_n in question in the theorem are also elements of the topological space, and so any cover of S^∞ by sets like A_n must have a finite subcover. And so, the proof of the lemma comes down to the proof of the following fact: For $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ closed nonempty sets on a compact topological space, we must have that:

$$A = \bigcap_{n=1}^{\infty} A_n$$

is nonempty. Proof by contradiction. Assume that $A = \emptyset$. Then, we must have that $A_1^c, A_2^c, \cdots$ is an open cover of S^∞. By compactness, there is a finite subcover, which we will just identify (wlog) the sets $A_1^c, A_2^c, \cdots, A_n^c$. Thus, we have:

$$S^\infty = A_1^c \cup \cdots \cup A_n^c$$

However, as $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$, we must have that $A_n^c \supseteq \cdots \supseteq A_2^c \supseteq A_1^c$, which tells us:

$$S^\infty = A_n^c \implies A_n = \emptyset$$

This contradicts A_n being nonempty, and so, we must have A is nonempty. qed.

Extended Sequence Measure In Chapter 3, we will learn how to extend a countably additive probability measure on a field F to a countably additive probability measure on the sigma field $\sigma(F)$. The term *probability measure* more accurately refers to the extended P. We have that C_0 is the field above, and we let $C = \sigma(C_0)$. Thus, (S^∞, C, P) will be a probability space that we will look at later, an important one.

Problems

2.1 Prove Set Theory Results Using Indicators

Define $x \vee y = \max(x, y)$, and for a collection $\{x_\alpha\}$ define:

$$\bigvee_\alpha x_\alpha = \sup_\alpha x_\alpha$$

Similarly, define $x \wedge y = \min(x, y)$, and for a collection $\{x_\alpha\}$ define:

$$\bigwedge_\alpha x_\alpha = \inf_\alpha x_\alpha$$

Prove that $I_{A\cup B} = I_A \vee I_B$, $I_{A\cap B} = I_A \wedge I_B$, $I_{A^c} = 1 - I_A$, and $I_{A\Delta B} = |I_A - I_B|$, in the sense that there is equality at each point of Ω. Show that $A \subset B$ if and only if $I_A \leq I_B$ pointwise. Check the equation:

$$x \wedge (y \wedge z) = (x \wedge y) \vee (x \wedge z)$$

And deduce the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. By similar arguments prove that:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \qquad A\Delta C \subset (A\Delta B) \cup (B\Delta C)$$

$$\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c \qquad \left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c$$

I think, the first thing to note is - what is the I_A notation? It is just the indicator notation.

1. $I_{A\cup B} = I_A \vee I_B$ is clear, as if a point is in either A or B, the maximum between 0 and 1 is 1.

2. $I_{A\cap B} = I_A \wedge I_B$ also clear

3. $I_{A^c} = 1 - I_A$ Also clear

4. $I_{A\Delta B} = |I_A - I_B|$ If ω is in $A\delta B$, it is in one, and not the other. Thus, $|I_A - I_B| = |1| = |-1| = 1$.

5. $A \subset B$ if and only if $I_A \leq I_B$ pointwise. If $x \in A$ and $x \in B$, then $1 \leq 1$. If $x \in A$, but $x \notin B$, we have $1 \not\leq 0$.

6. The remaining points are clear. Identify the $\wedge$ or $\vee$ with the corresponding set operation.

2.2 Union and Intersection Equality Property

We have:

$$U_k = \bigcup (A_{i_1} \cap \cdots \cap A_{i_k}) \qquad I_k = \bigcap (A_{i_1} \cup \cdots \cup A_{i_k})$$

Where union and intersection are over all k tuples satisfying $1 \leq i_1 \leq \cdots \leq i_k \leq n$. We note that $U_k = I_{n-k+1}$. This is via the pigeon hold principle - take $x \in U_k$, note that it is in some $A_{i_1} \cap \cdots \cap A_{i_k}$ intersction. Now, note that $x \in I_{n-k+1}$, as each of unions being intersecteed has $n-k+1$ unique sets out of the n total sets - so, it is only missing $k-1$ sets. So, each of the unions must include one of the $A_{i_1}, \cdots, A_{i_k}$, and $x \in I_{n-k+1}$. Similar in the other direction.

2.3 Equivalent and Non Equivalent Field Definitions

1. We have that $A^c = \Omega - A \in F$. We note that $A \cup B = \Omega - (A^c - B) \in F$.

2. Let Ω be four points, F the empty set, Ω, and all six of the two-point sets. Note, complements and disjoint unions will not get us a set of size 3, which means that F is not a field.

2.4 Unions of increasing Fields are Fields, but not true for σ fields

1. Note, any two sets ultimately belong to the same F_i, and so their union is within the union. Clearly complements are maintained. Let $\Omega = \bigcup F_n$, and so Ω and $\emptyset$ are also within the big union.

2. Let F_n be the discrete σ field on $\{1, \cdots, n\}$. Note, $\bigcup_n F_n$ does not contain $\{1, 2, \cdots\}$, which can be expressed as the countable union of singletons.

2.5 Generated Field and Explicit Definition

1. To show that $f(A)$ is a field - note, the discrete field is a field, so the intersection is non empty. Note that clearly unions and complements of items in the intersection remain within the intersection.

2. Show that the given representation is indeed a field that contains A, and so $f(A)$ is equivalent to the representation. Note, this is just set theory stuff. See solutions for the complement, represented as a disjoint intersection of unions.

2.6 Comparing Fields and σ fields

1. Note that $f(A)$ must contain finite and cofinite fields, as any finite field is the finite union of the points, and we can take the complement of those to get any cofinite set. Note, the field of cofinite and finite sets contains A, and so we must have it is equal to A.

2. Clearly $f(A) \subset \sigma(A)$, as $\sigma(A)$ is within the intersection definition of $f(A)$. If A is finite, we have that $f(A) = \sigma(A)$, as $\sigma(A) \subset f(A)$, as all unions can be represented as finite unions. Finally, we have $A \subset f(A) \subseteq \sigma(A)$, which implies $\sigma(f(A)) = \sigma(A)$.

3. If A is countable, then $f(A)$ is countable, as each set can be represented as noted in 2.5. There are a countable number of such representations (as the countable product of countable sets is surjective onto the representation).

2.7 Extending a field/σ field by $\{H\}$

Let H be a set outside of F, where F is a field (or σ field). Show that the (σ) field generated by $F \cup \{H\}$ consists of sets of the form:

$$(H \cap A) \cup (H^c \cap B) \qquad A, B \in F$$

Well, first note that the above is a (σ) field containing $F \cup \{H\}$. Clearly contains F and H, and the emptyset and Ω. Now note closed under complements (see exercise 4.10 for in depth argument):

$$[(H \cap A) \cup (H^c \cap B)]^c = (H \cap A^c) \cup (H^c \cap B^c)$$

Now, note closed under unions:

$$\bigcup_k (H \cap A_k) \cup (H^c \cap B_k) = (H \cap \bigcup_k A_k) \cup (H^c \cap \bigcup_k B_k)$$

Finally, note that any σ algebra containing $F \cup \{H\}$ must contain the one above. And so yes, the set above is the (σ) field generated by $F \cup \{H\}$.

2.8 σ algebra for class where A^c is countable union of class elements

Note that the smallest class over A that contains countable unions must be a σ algebra containing A, so this class K satisfies:

$$\sigma(A) \subseteq K$$

Finally, we note that $K \subseteq \sigma(A)$, as every element in K must be within $\sigma(A)$, by definition.

2.9 Equivalent σ algebra definition

Define:

$$G = \{B \in \sigma(A) | \exists \text{ a countable subclass } A_B \text{ of } A \text{ such that } B \in \sigma(A_B)\}$$

Note that $G = \sigma(A)$. This can be done by showing

1. We have that $A \subset G$. We have that for $B \in A$, $B \in \sigma(B)$, where $A_B = B$ is a countable subclass of A.
2. G is a σ algebra. First, note that $\emptyset, \Omega \in G$, as $\emptyset, \Omega \in \sigma(B)$ for $B \in A$. We have closed under countable unions, as for $B_1, B_2, \cdots \in G$, we have that the union is within $\sigma(A_{B_1} \cup A_{B_2} \cup \cdots)$, where $A_{B_1} \cup A_{B_2} \cup \cdots$ is a countable subclass of A. Finally, we have for $B \in G$, we must have that $B^c \in \sigma(A_B) \implies B \in G$.
3. We have $\sigma(A) \subseteq G$ by the above two points.
4. We have that $G \subseteq \sigma(A)$, by definition, as G only consists of elements within $\sigma(A)$

Thus, every $B \in \sigma(A) \implies B \in G$. Thus, every $B \in \sigma(A)$ has a countable subclass A_B of A such that $B \in \sigma(A_B)$. qed.

2.10 Classes Generating The Discrete Field are Large

1. Show that if $\sigma(A)$ contains every subset of Ω, then for each pair ω and ω' of distinct points in Ω there is in A a subset A such that $I_A(\omega) \neq I_A(\omega')$.

   We first note the following: subsets of Ω containing both ω and ω', or neither, is a σ field F. F contains both the empty set and Ω. F is closed under complements (as the complement of a set containing both is neither, and vice versa). F is closed under unions (union of both and neither contains both, union of both and both contains both, union of neither and neither contains neither).

We now examine $\sigma(B)$, where B is subsets of A that contain both ω and ω', or neither. Note that $\sigma(B)$ must be contained within the σ field noted above, and so $\sigma(B) \neq 2^\Omega$. Thus, we must have that $\sigma(B)$ is strictly smaller than $\sigma(A)$, which implies B is strictly smaller than A, and so A must contain sets that have either ω and no ω', or vice versa.

2. Show that the reverse implication holds if Ω is countable.

   The reverse implication is that if Ω is countable, and A is such that for every pair ω and ω', there is a subset A such that $I_A(\omega) \neq I_A(\omega')$, then $\sigma(A) = 2^\Omega$. This can be noted via forming $\{\omega\}$ as a countable intersection of sets within $\sigma(A)$. Note that for every pair ω, ω', we have a set that contains one and not the other, and via complements, we can always find a set $A_{\omega'} \in A$ such that $\omega \in A_{\omega'}$, but $\omega' \notin A_{\omega'}$. By a countable union, we must have:

$$\{\omega\} = \bigcap_{\omega' \in \Omega} A_{\omega'}$$

   Thus, $\sigma(A) = 2^\Omega$.

3. Show by example that the reverse implication need not hold for uncountable Ω.

   Well, we can have that $\Omega = [0, 1]$, and A is the class of all single intervals. This is essentially the example gone over in Chapters 1 and 2. Note, $\sigma(A) = B$, the Borel σ algebra. At the end of Chapter 3, we will note that $B \neq 2^\Omega$, or all possible subsets.

2.11 Countably Generated Fields

A σ field is *countable generated*, or *separable*, if it is generated by some countable class of sets.

1. Show that the σ field B of Borel sets is countable generated. Well, we have that it is generated by intervals with rational endpoints. There are countably many such intervals, there being a 1-1 correspondence

with $\mathbb{Q} \times \mathbb{Q}$. Note that $\sigma(\mathbb{Q} \times \mathbb{Q})$ contains all singletons, as we can express every singleton in $[0, 1]$ as a countable intersection of rational intervals - as we can approach every point from below and above by rational numbers in $[0, 1]$. Thus, $\sigma(\mathbb{Q} \times \mathbb{Q}) = B$.

2. Show that the σ field of Example 2.4 is countable generated if and only if Ω is countable.

   In example 2.4, we have the countable and co-countable sets sigma field F. We assume that Ω is uncountable, and F is countably generated. Thus, F must be generated by a countable number of singletons - namely, the countable singletons in all generating sets (or their complements - the generating sets must be countable or co-countable). We let Ω_0 be the union of these countable sets. We note that F must consist of sets B and $B \cup \Omega_0^c$, with $B \subset \Omega_0$. Clearly, each $B \subset \Omega_0$ is within F, and so is each $B \cup \Omega_0^c$, as $(B \cup \Omega_0^c)^c = B^c \cap \Omega_0$, which is also countable.

   Now, as for why all sets in F are of the form B or $B \cup \Omega_0^c$, for $B \subset \Omega_0$. Note first that these sets form a σ field. We have that $B^c = (\Omega_0 - B) \cup \Omega_0^c$, which is in the field. We have that $(B \cup \Omega_0^c) = B^c \cap \Omega_0 \subset \Omega_0$, which is also in the field, so it is closed under complements. We have that $\emptyset$ is in the field, and so by closed under complements, so is Ω. Finally, it is closed under unions - clearly $B_1 \cup B_2$ and $B_1 \cup \Omega_0^c \cup B_2 \cup \Omega_0^c$ are all within the field, and this can be extended countably many times. Finally, note that $B_1 \cup (B_2 \cup \Omega_0^c) = B_1 \cup B_2 \cup \Omega_0^c$ is within the field as well, and can be extended countably many times.

   And so, we have defined a field. Note, this field contains all the sets within the generating set, and so all the countable and co-countable sets must be of the form defined above. Now, we find our contradiction. Note that the singletons within Ω_0^c are countable sets. However, these singletons are not with Ω_0, and they are not of the form $B \cup \Omega_0^c$. Note, Ω_0^c is nonempty, as Ω is uncountable. And so, we have a contradiction - namely, being that the countable generating sets cannot form every countable and co-countable set.

   So, if Ω is uncountable, F is *not* countably generated. If Ω is countable - we must have that F is countable generated. Take all the singletons that are in the countable or co-countable sets. This must be a countable amount of singletons, as Ω is countable. They generate all the countable and co-countable sets. Note, actually, if Ω is countable, the countable and co-countable sets are just the power sets. And so, the singletons generate all sets, and F.

3. Suppose that F_1 and F_2 are σ fields, $F_1 \subset F_2$, and F_2 is countable generated. Show by example that F_1 may not be countably generated.

   Let F_2 consist of the Borel sets in $\Omega = (0, 1]$, and let F_1 consist of the countable and co-countable sets in $(0, 1]$. Note that F_1 cannot be countably generated, by the previous problem. F_2 is countably generated by part 1. Finally, $F_1 \subset F_2$. Each countable set is a countable union of singletons, which is in F_2. The complement of each co-countable set is a countable union of singletons, which is in F_2, and so is the complement. Thus, we indeed have $F_1 \subset F_2$, F_2 countably generated, by F_1 not. qed.

2.12 Cardinality of Fields and σ Fields

Show that a σ field cannot be countably infinite - its cardinality must be finite, or else at least that of the continuum (the real numbers). Show by example that a field can be countably infinite.

Note, we have examples for finite and uncountably infinite σ fields. Assume a σ field is countably infinite. Then, we can list every set in the σ field like so:

$$A_1, A_2, A_3, \cdots$$

We will $\omega \in \{\pm 1\}^{\mathbb{N}}$. We define:

$$B^{\omega} = \bigcap_{n \in \mathbb{N}} A_n^{\omega_n}$$

Where ω_n is the nth bit in ω. Note, -1 is just the complement. If $\gamma \neq \omega$, then we must have:

$$B^{\omega} \cap B^{\gamma} = \emptyset$$

As they differ on including some A_n and A_n^c in their intersections. We also note that:

$$A_n = \bigcup_{\omega : \omega_n = 1} B^{\omega}$$

As because we must have that $A_1^{\omega_1} \cap \cdots \cap A_{n-1}^{\omega_{n-1}} \cap \widehat{A_n} \cap \cdots$ partitions all of Ω (think about it this way - for each of the sets listed in the intersection, $x \in \Omega$ is in either A_i, or A_i^c. x would be in the intersection that makes the right choice for each i).

So now, we consider the set $\{B^\omega : \omega \in \{\pm 1\}^{\mathbb{N}}\}$. If there are only a finite amount of different B^ω - this would imply that there are only a finite amount of distinct A_n. So, there is an infinite family of ω for which the B^ω are different. We have for some index set, ω_i, $i \in I$, $B^{\omega_i} \neq B^{\omega_j}$. Let $C_n := B^{\omega_n}$. Note, we just assume that the index set is countable. So, what is the reason of the above mess - we had that each of the A_i where *distinct.* However, what we get from the C_n, is that these are countably infinite sets that are *disjoint.* Now, we define:

$$\Gamma : \{0,1\}^{\mathbb{N}} \to F, (\gamma) \to \bigcup_{\gamma_n = 1} C_n$$

Note: the map of Γ is *injective.* As the C_n were disjoint, we see that each set in the disjoint union contains *different elements.* Finally, we note that all possible subsets of $\{0,1\}^{\mathbb{N}}$ is uncountable (as we can identify one point in $[0,1]$ with each element in $\{0,1\}^{\mathbb{N}}$, this is all of chapter 1). And so, we have that if the σ algebra is not finite, it has cardinality of *at least the real line.*

The steps are actually simple. 1. Find countably many disjoint sets. 2. Create uncountable sets in the σ algebra via disjoint unions. qed.

2.13 Probability Measure on Finite and CoFinite Sets

1. The probability measure we define is that $P(A) = 0$ if A is finite and $P(A) = 1$ if A is cofinite. Note, the measure is not well defined if Ω is finite, but we assume that Ω is infinite. We note that it is countably additive - for finite $A_1, \cdots, A_n$, we have the union is finite, and so:

   $$P\left(\bigcup A_n\right) = 0 = \sum P(A_n)$$

   I now note - we cannot have disjoint cofinite sets. Assume that A and B are cofinite. Thus, A^c and B^c are finite - this is because $\Omega - A^c - B^c$ are both non empty, and $\Omega - A^c - B^c \subset A, \Omega - A^c - B^c \subset B$. Finally, we note for a disjoint union of $n-1$ finite sets and a cofinite set, we must have that their complement is cofininte, and so:

   $$P\left(\bigcup A_n\right) = 1 = \sum P(A_n)$$

2. Show that P is not countably additive if Ω is countably infinite.

So, in the case that Ω is countably infinite - the countable union of finite sets can result in a cofinite set ($A_n = \{\omega_1\}$). And so:

$$P\left(\bigcup A_n\right) = 1 \neq 0 = \sum P(A_n)$$

3. Show that P is countably additive if Ω is uncountable.

   We just need countably additive in the case that the *countable union remains within our field*. Note - the countable union of finite sets cannot be cofinite. This union would mean A is countable, and A^c is finite, which would imply that $\Omega = A \cup A^c$ is countable. And so, we can only have a countable union that results in a finite set A. And so:

$$P\left(\bigcup A_n\right) = P(A) = 0 = \sum P(A_n)$$

4. Now let F be the σ field consisting of the countable and the cocountable sets in an uncountable Ω, and define P analogously. Show that P is countably additive.

   Union of countable countable sets is still countable. No such union of disjoint cocountable sets. Union of countable countable sets with one cocountable set is cocountable. qed.

2.14 First Category σ Field

In $(0, 1]$ let F be the class of sets that either (i) are of the first category (countable union of sets that are nowhere dense, which are sets E for which each interval I contains an interval J that does not meet E) or (ii) have complement of the first category. Show that F is a σ field.

The empty set is of the first category, and the complement of $(0, 1]$ (the empty set) is of the first category. Clearly complements of sets in F are still in F, as this is definitional. Finally, we consider countable unions of sets in F. If each set is first countable, the countable union of a countable amount of nowhere dense sets is still a countable union of nowhere dense sets, and so the countable union of first countable sets is first countable.

Now, we assume at least one of the sets in our countable union has a complement in the first category. Note, the complement of the countable union must be contained within the complement of our first set, and so the complement of the countable union is in the first category as well. Thus, the countable union in this case is also within F. In all cases, F contains countable unions of elements of F. Thus, we can conclude F is a σ field.

For A in F, take $P(A)$ to be 0 in case (i) and 1 in case (ii). Show that P is countably additive.

Countably additive only needs to apply for disjoint sets. For disjoint sets in case (i) - the countable union of such disjoint sets is still in case (i), and so $P(\bigcup A_i) = 0 = \sum 0 = \sum P(A_i)$.

Now, we note that we cannot have 2 disjoint sets in case (ii). Assume A and B are disjoint, and the complements of A and B are both first countable. We note that for any two disjoint sets - we must have $\Omega = A^c \cup B^c$. Take $x \in \Omega$. Assume $x \notin A^c$. Thus, $x \in A \implies x \notin B \implies x \in B^c$. So, $x \in A^c$ or $x \in A$, in which case, $x \in B^c$. However, note that $A^c \cup B^c$ is still first countable, which implies that $(0,1]$ is first countable. However, in an earlier problem, we noted that by Baire, a nonempty interval is *not* of the first category. And so, this is a contradiction, and we cannot have disjoint A and B whose complements are first countable.

So, the only remaining case for countably additive is one set of case (ii) and a countable amount of sets of case (i) which are all disjoint. As noted above - the union of such sets is case (ii), and we have:

$$P\left(\bigcup A_i\right) = 1 = 1 + 0 = P(A_1) + \sum_{i=2}^{\infty} P\left(A_i\right) = \sum P(A_i)$$

And thus P is countably additive. qed.

2.15 Differences between B_0 and C_0

On the field B_0 in $(0,1]$ defined $P(A)$ to be 1 or 0 according as there does or does not exist some positive ϵ_A (depending on A) such that A contains the interval $\left(\frac{1}{2}, \frac{1}{2} + \epsilon_A\right]$. Show that P is finitely but not countably additive. No such example is possible for the field C_0 in S^∞ (as by Theorem 2.3, every finitely additive probability measure on the field C_0 must be countably additive. This shows the opposite for B_0).

We let sets of case (i) be those for which $P(A) = 0$, and those of case (ii) be those for which $P(A) = 1$. Note, for finitely additive - we are essentially just doing the same shit we did above. Two sets of case (i) - there finite union must also be of case (i). As for why this is the case:

Assume that A and B are of case (i), but $A \cup B$ is of case (ii), ie, there is some positive $\epsilon_{A\cup B}$ such that:

$$\left(\frac{1}{2}, \frac{1}{2} + \epsilon_{A\cup B}\right] \subseteq A \cup B$$

We will show that this creates a contradiction. For both A and B, as they are of case (i), for every $\epsilon > 0$, there is a point $x_{\epsilon A}$ and $x_{\epsilon B}$ such that:

$$x_{\epsilon A}, x_{\epsilon B} \in \left(\frac{1}{2}, \frac{1}{2} + \epsilon\right] \qquad x_{\epsilon A} \notin A \qquad x_{\epsilon B} \notin B$$

For the entire interval to be contained, we must have for every $\epsilon < \epsilon_{A\cup B}$, we also have:

$$x_{\epsilon A} \in B \qquad x_{\epsilon B} \in A$$

This creates a contradiction. As A and B are within B_0, they must be finite unions of disjoint intervals. Note, a point itself is not actually within B_0, and so these points $x_{\epsilon A}$ must be contained within some interval of B. However, this entire interval is not within $\left(\frac{1}{2}, \frac{1}{2} + \epsilon\right]$, and so it has some starting point $1/2 < \epsilon'_B$, and A similarly has a starting point $1/2 < \epsilon'_A$. We can take a new $\epsilon = \min(\epsilon'_A, \epsilon'_B)$, and find new points that must not be contained in A and B, but create corresponding intervals in the other set, and so on. This can be done a countably infinite number of times. This process thus implies that A and B must both be disjoint unions of countable sets - which contradicts $A, B \in B_0$. Thus, we must have that $A \cup B$ must be of case (i). Thus, we find:

$$P(A \cup B) = 0 = P(A) + P(B)$$

We can use induction to prove for any finite disjoint union of case (i) sets, the measure is finitely countable. Now, we consider case (ii). Like above, it is clear that there is no disjoint union of case (ii) sets. That is because, we just take the minimum of ϵ_A, ϵ_B, and clearly, both intervals are in both sets. So, the final case we consider is a case (ii) set with multiple case (i) sets. Again, this finite disjoint union is of case (ii), and so:

$$P(A \cup B_1 \cup \cdots \cup B_n) = 1 = 1 + 0 + \cdots + 0 = P(A) + P(B_1) + \cdots + P(B_n)$$

The final point to make is that P is not countably additive. We can define:

$$A_n = \left(\frac{1}{2} + \frac{1}{n+2}, 1\right]$$

Each A_n is clearly of case (i). However, the countable union of A_n is $(1/2, 1]$, and so is of case (ii). Thus, we do not have countable additivity, as this would give us a contradiction that $1 = 0$. Thus, P is not countably additive.

I think the most important part is why this is interesting. We noted that if $S = \{0, 1\}$, then C_0 is essentially a field on $(0, 1]$, as we can identify each of the infinite sequences with a point in $(0, 1]$. And, initially, the sets in the field C_0 look like they contain intervals and their unions. However, there must be something substantially different about the field and/or measure on C_0, when compared with B_0.

2.18 Stochastic Arithmetic

Define a set function P_n on the class of all subsets of $\Omega = \{1, 2, \cdots\}$ by:

$$P_n(A) = \frac{1}{n}\# [m : 1 \leq m \leq n, m \in A]$$

among the first n integers, the proportion that lie in A is just $P_n(A)$. Then P_n is a discrete probability measure (a probability measure defined on a discrete space). The set A has *density*:

$$D(A) = \lim_n P_n(A)$$

provided this limit exists. Note, density makes sense, as we are essentially getting a ratio of how many numbers are in A, compared with Ω (as $P_n(\Omega) = 1$ for all cases). Let $\mathcal{D}$ be the class of sets having density.

(a) Show that D is finitely but not countably additive on $\mathcal{D}$. Take $A, B \in \mathcal{D}$ disjoint. We have that:

$$D(A \cup B) = \lim_n P_n(A \cup B) = \lim_n P_n(A) + P_n(B) = \lim_n P_n(A) + \lim_n P_n(B) = D(A) + D(B)$$

Note implicit above is that $\mathcal{D}$ contains finite disjoint unions. Note, we have additivity of P_n on *disjoint* sets, but it is not true for non-disjoint sets. We show not countably additive by a counter example. Define $A_i = \{i\}$. We have that:

$$D(A_i) = \lim_n P_n(A_i) = \lim_n \frac{1}{n} = 0$$

However, $\bigcup A_i = \Omega$. Thus:

$$D\left(\bigcup A_i\right) = D(\Omega) = 1 \neq 0 = \sum D(A_i)$$

Note in the above example, we do have that the countable disjoint union is within $\mathcal{D}$.

(b) Show that $\mathcal{D}$ contains the empty set and Ω and is closed under the formation of complements, proper differences, and finite disjoint unions, but is not closed under the formation of countable disjoint unions or of finite unions that are not disjoint.

In part (a) we concluded closed under finite disjoint unions. We now consider complements. Take $A \in \mathcal{D}$. By definition, we have that the limit exists and

$$\lim_n P_n(A) = \lim_n x_n = x$$

We note that $P_n(A^c) = 1 - x_n$. This is because, A contains some i elements between 1 and n. We have that A^c must contain $n - i$ elements between 1 and n. Thus, $P_n(A^c) = \frac{n-i}{n} = 1 - \frac{i}{n} = 1 - P_n(A) = 1 - x_n$. Thus, we find:

$$\lim_n P_n(A^c) = \lim_n 1 - x_n = 1 - \lim_n x_n = 1 - x$$

Thus, $A^c \in \mathcal{D}$. We now consider proper differences. Take $A, B \in \mathcal{D}$. We note:

$$A \setminus B = A \cap B^c = (A^c \cup B)^c$$

Recall that the *proper* difference implies that $B \subset A$. Thus, $A^c \in \mathcal{D}$, $A^c \cap B \in \mathcal{D}$ being a disjoint union, and finally $(A^c \cup B)^c \in \mathcal{D}$.

Now, we consider the not closed examples. We will try and find counter examples. I first outline a set that is not within $\mathcal{D}$. We need the x_n values to oscillate, so that the limit does not exist. I define such an A like so. If we have:

$$P_n(A) = \frac{1}{4} = \frac{n/4}{n}$$

We can make $P_{4n}(A) = \frac{3}{4}$, by setting $11n/4$ of the final $3n$ entries as 1:

$$P_{4n}(A) = \frac{n/4 + 11n/4}{4n} = \frac{3}{4}$$

Similarly, if we have:

$$P_n(A) = \frac{3}{4} = \frac{3n/4}{n}$$

We can make $P_{4n}(A) = \frac{1}{4}$, by setting $n/4$ of the final $3n$ entries as 1:

$$P_{4n}(A) = \frac{3n/4 + n/4}{4n} = \frac{1}{4}$$

In such a way, we build up A. We start with $A = \{4\}$. Thus, we have:

$$P_4(A) = \frac{1}{4}$$

Then, we add $A = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$, such that:

$$P_{16}(A) = \frac{3}{4}$$

We continue with the process defined above, adding the numbers that correspond to the final entries in the added $3n$ spots. Thus, we have:

$$\lim P_n(A)$$

Does not exist, as it oscillates between $1/4$ and $3/4$. We have that $\mathcal{D}$ is not closed under the formation of countable disjoint unions. We have that:

$$A = \{4\} \cup \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \cup \cdots$$

Note that each set in out countable union has a finite amount of elements - and so for each subset B, $P_n(B) = 0$. However, $\bigcup B = A \notin \mathcal{D}$.

We now consider finite unions that are not disjoint. In this case, we define a new subset, as noted in the solutions. We let A be the set of even integers. We define $C_k = \{m : v_k < m \leq v_{k+1}\}$. Note, for $v_1 = 0$, $\bigcup C_k = \Omega$. Finally, we define B as:

$$B = (A \cap (C_1 \cup C_3 \cup \cdots)) \cup (A^c \cap (C_2 \cup C_4 \cup \cdots))$$

We thus have that:

$$A \cap B = A \cap (C_1 \cup C_3 \cup \cdots)$$

We note that if v_k increases rapidly enough, then the set $A \cap B$ has no density. Note, consider just the sets $A \cap C_{2i-1}$, which in a disjoint union, would form $A \cap B$. We have that:

$$P_{v_{2k-1}}(A \cap B) = \frac{1}{v_{2k-1}} \left| \cup_{i=1}^{k} A \cap C_{2i-1} \right|$$

$$P_{v_{2k}}(A \cap B) = \frac{1}{v_{2k}} \left| \cup_{i=1}^{k} A \cap C_{2i-1} \right|$$

Note, this is because going from v_{2k-1} to v_{2k}, we are not including any of the ones included in C_{2k}. Thus, we have:

$$\frac{P_{v_{2k-1}}}{P_{v_{2k}}} = \frac{v_{2k-1}}{v_{2k}}$$

Note, if $A \cap B$ has a density $D(A \cap B)$, then we must have that:

$$\lim_{k \to \infty} \frac{P_{v_{2k-1}}}{P_{v_{2k}}} = \frac{D(A \cap B)}{D(A \cap B)} = 1$$

Thus, if we have the limit of the ratio $\frac{v_{2k-1}}{v_{2k}}$ does not equal 1, then that implies $A \cap B$ must have no density. Such a sequence would be $v_k = 2^k$, in which case the ratio is 2. Now, as noted above, this set is a union of countable singletons, each with density 0, so like our above example - this shows that $\mathcal{D}$ is not closed under countable unions. However, this example also helps us with finite non disjoint unions. However, we also note that:

$$D(B) = \frac{1}{2}$$

Because, by in large, B contains one of every two points in the pair (x_i, x_i+1). We also have that $A \in \mathcal{D}$, as clearly $D(A) = 1/2$ as well. By the above points, we have that $A^c, B^c \in \mathcal{D}$. However, if we assume that $\mathcal{D}$ is closed under non disjoint unions, we have:

$$A^c \cup B^c = (A \cap B)^c \in \mathcal{D} \implies A \cap B \in \mathcal{D}$$

Which is a contradiction.

(c) Let $\mathcal{M}$ consist of the periodic sets $M_a = \{ka : k = 1, 2, \cdots\}$. Observe that:

$$P_n(M_a) = \frac{1}{n}\left\lfloor \frac{n}{a} \right\rfloor \to \frac{1}{a} = D(M_a)$$

Show that the field $f(\mathcal{M})$ generated by $\mathcal{M}$ is contained in $\mathcal{D}$. Show that D is completely determined on $f(\mathcal{M})$ by the value it gives for each a to the event that m is divisible by a.

First, we note that by problem 2.5, $f(\mathcal{M})$ is the class of sets of the form:

$$\bigcup_{i=1}^{m}\bigcap_{j=1}^{n_i} A_{ij} \quad \text{such that} \quad A_{ij} \in \mathcal{M} \quad \text{or} \quad A_{ij}^c \in \mathcal{M} \quad \text{and the } m \text{ sets} \quad \bigcap_{j=1}^{n_i} A_{ij} \quad \text{are disjoint}$$

Note that by closed under finite disjoint unions, if each $\bigcap_{j=1}^{n_i} A_{ij} \in \mathcal{D}$, then $f(\mathcal{M})$ is completely contained in $\mathcal{D}$. We note that:

$$M_a \cap M_b = M_{lcm(a,b)}$$

Take $x \in M_a \cap M_b$. Note that $lcm(a, b)$ is the smallest integer that is divisible by both a and b. Recall the fundamental theorem of arithmetic - *wiki*. Go over the proofs - they are not to difficult. Every number has a unique prime factorization. Thus, we have:

$$a = p_1^{e_1} \cdots p_n^{e_n} \qquad\qquad b = p_1^{f_1} \cdots p_n^{f_n}$$

Where $p_1, \cdots, p_n$ is a list from 1 to a prime larger than a and b (infinite primes, one exists). We note that:

$$lcm(a, b) = p_1^{\max(e_1, f_1)} \cdots p_n^{\max(e_n, f_n)}$$

Note, the rhs is clearly divisible by a and b - and if we decrease one of the powers by 1, it is not divisible by the two. And so, it is clearly equal, by definition (ie, you cannot have a smaller prime factorization that is divisible by a and b). Note that $x = ka$ and $x = k'b$. Thus, the prime factorization of x must include *at least* $p_i^{\max(e_i, f_i)}$ for each i, and thus $x = k'' lcm(a, b)$, and $x \in M_{lcm(a,b)}$. The other direction is trivial.

What is the point of this? It shows that $\mathcal{M}$ is closed under intersections, and as $\mathcal{M} \subset \mathcal{D}$, we have that:

$$\bigcap_{j=1}^{n_i} A_{ij} \in \mathcal{D} \implies f(\mathcal{M}) \subset \mathcal{D}$$

Now, this implies that D on $f(\mathcal{M})$ is determined by D on $\mathcal{M}$. For each intersection, take the lcm to find the value of D on the intersection. Then, just add up the fractions.

(d) Assume that $\sum p^{-1}$ diverges - this is the sum over all primes. Prove that D, although finitely additive, is not countably additive on the field $f(\mathcal{M})$.

Note, finite additivity is given, by finite additivity on all of $\mathcal{D}$, and $f(\mathcal{M}) \subset \mathcal{D}$. We now make use of the solutions. We define $B_l = M_a - \bigcup_{1<p\leq l} M_{ap}$ for some $a \in \Omega$. We note that $B_l \in f(M)$ - by the definition of a field, it is closed under finite unions, complements, and finite intersections, and these operations are used to form B_l out of sets originally in $f(M)$. We now note:

$$D(B_l) = D(M_a) - D\left(\bigcup_{1<p\leq l} M_{ap}\right)$$

Note that $\mathcal{D}$ is closed under proper differences, and for $p > 1$, we have $M_{ap} \subset M_a$. We also note why the above equation is true - for $B \subset A$, both in D, we have that:

$$D(A-B) = D(A \cap B^c) = D((A^c \cup B)^c) = 1 - D(A^c \cup B) = 1 - D(A^c) - D(B) = D(A) - D(B)$$

Where, each equality above we have proved in the preceding questions, as we have that A^c and B are disjoint. Thus, we have:

$$D(B_l) = \frac{1}{a} - D\left(\bigcup_{1<p\leq l} M_{ap}\right)$$

Now, we note that D is a finite probability measure on $f(M)$. Thus, we can apply the inclusion-exclusion formula, to find:

$$D(B_l) = \frac{1}{a} - \sum_{1<p\leq l} \frac{1}{ap} + \sum_{1<p<q\leq l} \frac{1}{apq} - \cdots$$

$$\implies D(B_l) \leq \frac{1}{a} - \sum_{1<p\leq l} \frac{1}{ap}$$

Note - the implication is given in the notes. However, I do note that:

$$\frac{1}{apq} ? \sum_{q<k\leq l} \frac{1}{apqk} = \frac{1}{apq} \sum_{q<k\leq l} \frac{1}{k}$$

It is hard to compare with the next term in the inclusion-exclusion principle. We are given in the question that $\sum \frac{1}{k}$ diverges - and so it must be greater than one, correct? I guess, you can split it up, for every apq term, like:

$$-\frac{1}{apq} \sum_{m=1}^{l-q} \sum_{q<k_1<k_2<\cdots<k_m\leq l} \left[\frac{1}{k_1} - \frac{1}{k_1 k_2} + \cdots + \frac{1}{k_1 k_2 \cdots k_m}\right]$$

I have spent to much time on this question. I will skip it.

2.19 Nonatomic Probability Spaces

A probability measure space $(\Omega, \mathcal{F}, P)$ is *nonatomic* if $P(A) > 0$ implies that there exists a B such that $B \subset A$ and $0 < P(B) < P(A)$ (A and B in $\mathcal{F}$, of course).

Note: An initial observation I had is that $\mathcal{F}$ cannot contain finite sets - as then we always have the existence of a finite subset with nonzero probability, and we can continue recursively until a single point (which is where I guess the name *nonatomic* comes from), which doesn't satisfy the *nonatomic condition*.

1. Assuming the existence of Lebesgue measure λ on $\mathcal{B}$ (ie, we have a probability space $((0,1], \mathcal{B}, \lambda)$), prove that it is nonatomic.

   We first considered proving that the sets satisfying the nonatomic condition in $\mathcal{B}$ contained the intervals and where a field, and thus equal to $\mathcal{B}$. However, a simpler way is by examining the following function for A with $\lambda(A) > 0$:
   $$f(x) = \lambda(A \cap (0, x])$$
   We first note that $f(0) = 0$ and $f(1) = P(A)$. We now note that f, being a measure, is continuous from below. Ie, take a sequence x_n, with $x_0 = 0$, and $x_n \to 1$, non decreasing. We note that:
   $$B_n = A \cap (0, x_n] \implies B_n \subseteq B_{n+1} \implies \bigcup_n B_n = A$$
   We also note that as $A \in \mathcal{B}$ and $(0, x_n] \in \mathcal{B}$, each $B_n \in \mathcal{B}$, and so is the countable union clearly. By continuity from below we have:
   $$\lim_n f(x_n) = \lim_n \lambda(B_n) = \lambda(A) = P(A)$$
   Or, even better. We let a be the infimum of points in A, and b the supremum - note, not equal and well defined, as A has more than 2 points (a finite set is Lebesgue measure 0). We let x_n start at a, and approach b. We still have $\lambda(B_0) = 0$. Now, we assume that for every arbitrary sequence of such x_n, there is no intermediate value - ie, we either have $\lambda(B_n) = 0$, or $\lambda(B_n) = P(A)$. This must be a contradiction. If it is true for every sequence - there must be a single point $c \in [a, b)$ where the switch happens - ie, if $x_n \geq c$, then $P(B_n) = P(A)$, and 0 otherwise. If there were two such c, we could form a sequence with x_n between the c, and get a contradiction from the limit being nondecreasing. However, the existence of such a c would be a contradiction for the following reasons - it would imply $P(\{c\}) = P(A) > 0$. This is because the above would imply for all $x_n \in (c, b)$, we have:
   $$P(A \cap (c, x_n]) = P(A)$$
   And we could define $C_n = A \cap (c - \frac{1}{n}, x_n]$ for x_n decreasing to c, $\{c\} = \bigcap C_n$, and then probability from above would give $P(\{c\}) = P(A)$.

   Thus, there must be some x_n satisfying:
   $$0 < P(B_n) < P(A)$$
   And thus $\mathcal{B}$ is *nonatomic.*

2. Show that in the nonatomic case that $P(A) > 0$ and $\epsilon > 0$ imply that there exists a B such that $B \subset A$ and $0 < P(B) < \epsilon$.

   I think, this follows if there is some $P(A) = \epsilon$. Or, we have that as a probability measure space is a σ algebra, we assume some nonempty $P(A)$, and then we have the existence of $0 < P(B) < P(A)$. By the probability of a complement, we have either $P(B) \leq P(A)/2$, or $P(A - B) \leq P(A)/2$ (both B and $A - B$ being in the sigma field). We can continue iteratively to ϵ.

3. Show in the nonatomic case that $0 \leq x \leq P(A)$ implies that there exists a B such that $B \subset A$ and $P(B) = x$.

   By the proof in part A - as we take the x from 0 to 1 - $f(x)$ must be *continuous*. It is clear that $f(x)$ is increasing. Assume that it is not continuous. Then, we have some x where a *jump discontinuity* is made. In such a case - we can just make use of the above argument, to assign a nonzero probability to a singleton x, namely $P(\{x\}) > 0$. In which case, we have a contradiction in the nonatomic case, as noted at the start. No finite set can have nonzero probability.

   Note, just more details. The above argument relies on Ω going from 0 to some number t. It works for the Borel sets, however the hint gives us something more general. Assume the existence of some x. Inductively define class $\mathcal{H}_n$ in the following way:

$$\mathcal{H}_0 = \{\emptyset\} \qquad \mathcal{H}_n = \left\{ H : H \subset A - \bigcup_{k<n} H_k, P\left(\bigcup_{k<n} H_k\right) + P(H) \leq x \right\}$$

   We define H_k above in the following way. We let $h_k = \sup\{P(H) : H \in \mathcal{H}_k\}$, and we have that H_k is some set in $\mathcal{H}_k$ with $P(H_k) > h_k - \frac{1}{k}$.

   We first note that $P\left(\bigcup_{k<n} H_k\right) \leq x$. By definition, we have that:

$$P(H_{n-1}) + P\left(\bigcup_{k<n-1} H_k\right) \leq x \implies P\left(\bigcup_{k<n} H_k\right) \leq x$$

   As it is also a disjoint union, by definition, we have the implication above. We also have that each $\mathcal{H}_n$ is nonempty. If we are in the case where $P\left(\bigcup_{k<n} H_k\right) = x$, we are in the clear, as we have found a subset of A satisfying what we need. However, if $P\left(\bigcup_{k<n} H_k\right) < x$, we have that $\epsilon = x - P\left(\bigcup_{k<n} H_k\right) < P(A)$, and we make use of part (b) to find at least one set that satisfies being within $\mathcal{H}_n$. Finally, we have that finding $P(H_n) > h_n - \frac{1}{n}$ does exist - as, by the supremum, it is either obtained, or we reach it in the limit.

And so, the iteration is well defined. We have that $H_1, H_2, \cdots$ is a *strictly increasing* number of disjoint sets. We have that there countable union (if we must continue that long) is within $\mathcal{F}$. We that the unions:

$$\bigcup_{k=1}^{n} H_k$$

Are strictly increasing. We have by disjoint union:

$$P\left(\bigcup_{k=1}^{n} H_k\right) = \sum_{k=1}^{n} P(H_k) > \sum_{k=1}^{n} h_k - \frac{1}{k}$$

Note, I don't like the hanging $1/k$ term, as it goes to infinity. However, we can note that:

$$h_k \in \left(x - \sum_{t=1}^{k-1} h_t - \frac{1}{t}, x\right]$$

We just need a condition to prove h_k grows quick enough. I think this might come from noting that if we exclude members of previous classes of $\mathcal{H}_n$, then we can ensure that the remaining H to define $\mathcal{H}_n$ are big enough, so that h_t grows quick enough.

4. Show in the nonatomic case that if $p_1, p_2, \cdots$ are nonnegative and add to 1, then A can be decomposed into sets B_1, B_2 such that $P(B_n) = p_n P(A)$.

   Well, first we define the subset $B_1 = p_1 P(A)$, then we look at $A - B_1$, which still must contain a set of size $p_2 P(A)$, and so on. Easy enough.

2.21 Generating Sigma Algebras "From the Inside"

1. Suppose that $\mathcal{A} = \{A_1, A_2, \cdots\}$ is a countable partition of Ω. Recall that $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_n = \mathcal{A}_{n-1}^*$, where H^* consists of the sets in H, the complements of sets in H, and the finite and countable unions of sets in H.

   So, $\mathcal{A}_1 = \mathcal{A}_0^*$ is the sets in $\mathcal{A}_0$, the complements of sets in $\mathcal{A}_0$, and the finite and countable unions of sets in $\mathcal{A}_0$. Show that $\mathcal{A}_1$ coincides with $\sigma(\mathcal{A})$. This is a case where $\sigma(\mathcal{A})$ can be constructed "from the inside."

Well - coincides is essentially $\mathcal{A}_1 = \sigma(\mathcal{A})$. This is true if $\mathcal{A}_1$ is a σ field, as $\mathcal{A}_1$ contains $\mathcal{A}$. For $B \in \mathcal{A}_1$, we note $B^c \in \mathcal{A}_1$ if B is in the element of $\mathcal{A}_0$ case. If B is a complement or countable union of elements in $\mathcal{A}_0$ - note that B^c is just another countable union of elements in $\mathcal{A}_0$, in which case $B^c \in \mathcal{A}_1$.

Now, we quickly note Ω is in $\mathcal{A}_1$, and the complement of Ω (being an empty union). Finally, we note $\mathcal{A}_1$ is closed under countable unions - every element of $\mathcal{A}_1$ can be expressed as a finite or countable union of elements of $\mathcal{A}_0$ (as seen with the complement case), and a countable amount of countable/finite unions is still a countable union of elements in $\mathcal{A}_0$. Thus, we have that $\mathcal{A}_1$ is a σ algebra, and $\mathcal{A}_1 \supseteq \sigma(\mathcal{A})$. We also have that $\mathcal{A}_1 \subseteq \sigma(\mathcal{A})$, again noting that every element of $\mathcal{A}_1$ is a countable union of elements of $\mathcal{A}$. Thus, we can conclude:

$$\mathcal{A}_1 = \sigma(\mathcal{A})$$

2. Show that the set or normal numbers lies in $\mathcal{I}_6$. Recall, $\mathcal{I}$ is the class of subintervals of Ω. We recall that the set of normal numbers is of the form:

$$N = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} [\omega : |n^{-1} s_n(\omega)| < k^{-1}]$$

Where we also recall that each $[\omega : |n^{-1} s_n(\omega)| < k^{-1}]$ is a finite disjoint union of intervals. Note, $\mathcal{I}_{n-1}^*$ is also closed under countable intersections. So, we have that $\mathcal{I}_1$ would contain sets of the form $[\omega : |n^{-1} s_n(\omega)| < k^{-1}]$, $\mathcal{I}_2$ would have intersections of those, $\mathcal{I}_3$ would have unions of those, and $\mathcal{I}_4$ would have intersections of those. Thus, the normals are indeed within $\mathcal{I}_6$, give or take.

3. Show that $\mathcal{H}^* = \mathcal{H}$ if and only if $\mathcal{H}$ is a σ field. Show that $\mathcal{I}_{n-1}$ is strictly smaller than $\mathcal{I}_n$ for all n.

Well - clearly if $\mathcal{H}$ is a σ field, then $\mathcal{H}^* = \mathcal{H}$, given that countable unions and complements are already contained. Now, assume that $\mathcal{H}^* = \mathcal{H}$. Then, for any element in $A \in \mathcal{H}$, we know its complement is already in $\mathcal{H}$, and any list of elements in $\mathcal{H}$, the union is in $\mathcal{H}$. Thus, we do have the if and only if.

Assume that $\mathcal{I}_{n-1} = \mathcal{I}_n$ for all n. Then, $\mathcal{I}_{n-1} = \mathcal{B}$, as it is a σ algebra containing the intervals (note how both $\supseteq$ and $\subseteq$ are obtained). Note, this is a contradiction, as we had shown in the chapter that $\bigcup_n \mathcal{I}_n \neq \mathcal{B}$. Thus, as $\mathcal{I}_n \supseteq \mathcal{I}_{n-1}$, and $\mathcal{I}_n \neq \mathcal{I}_{n-1}$, we can conclude that $\mathcal{I}_{n-1} \subset \mathcal{I}_n$.

2.22

Extend (2.27) to the infinite ordinals α by defining $\mathcal{A}_\alpha = \left(\bigcup_{\beta<\alpha} \mathcal{A}_\beta\right)^*$. Show that, if Ω is the first uncountable ordinal, then:

$$\bigcup_{\alpha<\Omega} \mathcal{A}_\alpha = \sigma(\mathcal{A})$$

Show that, if the cardinality of $\mathcal{A}$ does not exceed that of the continuum ($\mathbb{R}$), then the same is true of $\sigma(\mathcal{A})$. Thus $\mathcal{B}$ has the power of the continuum. I will skip this for now, as I don't really know ordinals too well.

Section 3 - Existence and Extension

Notes

The main theorem of the chapter is the following:

Theorem 3.1 - Probability Measure Extension Theorem A probability measure on a field has a unique extension to the generated σ field.

Construction of the Extension

Outer Measure I like the discussion on the intuition for defining the *outer measure* and *inner measure*. Say we have P a probability measure on field $\mathcal{F}_0$. We want to define P on $\sigma(\mathcal{F}_0)$ - but it doesn't need to be restricted to $\sigma(\mathcal{F}_0)$. The outer measure is defined for each subset A of Ω as:

$$P^*(A) = \inf \sum_n P(A_n) \qquad \text{such that} \qquad A \subset \bigcup_n A_n$$

Note - as $\mathcal{F}_0$ is a field on Ω, it contains Ω, so we could take our finite cover as Ω itself, at least. So, the infimum is always non-empty. As for intuition - we have P on $\mathcal{F}_0$ - to extend P to sets outside of $\mathcal{F}_0$ - using what we already have - we just have to approximate A by sets in $\mathcal{F}_0$. The covering is an approximation, and the infimum takes the best such approximation.

Inner Measure Note how "from the inside" applies to the inner measure:

$$P_*(A) = 1 - P^*(A^c)$$

The covering of A^c will perhaps cover more than A^c, and so the complement of that will be slightly less than A. I like considering whether we can take $P_*(A) = \sup \sum_n P(A_n)$ of A_n unions inside of A. However, just practically - this would imply normal numbers N have inner measure of 0 (as there are no nonempty intervals in $\mathcal{B}_0$ inside of N).

P^* **Measurable** So, $P^*(A)$ holds for every set. Which sets do we want our measure to actually apply to? We would want it to apply to sets where our probability measure properties hold - so perhaps sets where:

$$P^*(A) + P^*(A^c) = 1$$

Note, this should always be the case by countable additivity and $P(\Omega) = 1$. And so, we take th is as our condition. Note, this is *equivalent* to:

$$P^*(A) = P_*(A)$$

We also note that it will be helpful to apply the more stringent condition:

$$P^*(A \cap E) + P^*(A^c \cap E) = P^*(E)$$

For every set E - the previous condition held with $E = \Omega$, as it would turn our $P^*(\Omega) = 1$. In a later problem, we will see that all three conditions are equivalent anyway - see problem 3.2. And so, we just for now, define and say a set A is called P^*-*Measurable* if the last condition, with E, holds. Note, it is the most stringent, and so if we have that P^*-Measurable sets contain $\sigma(\mathcal{F}_0)$, it should be enough anyway. We use the variable $\mathcal{M}$ to define the class of P^* measurable sets.

Properties of P^* **and** $\mathcal{M}$ We will need these properties to prove that $\mathcal{M}$ contains $\sigma(\mathcal{F}_0)$ and that the restriction of P^* to $\sigma(\mathcal{F}_0)$ is our required (unique) extension of P.

1. $P^*(\emptyset) = 0$
2. P^* is nonnegative: $P^*(A) \geq 0$ for every $A \subset \Omega$
3. P^* is monotone: $A \subset B$ implies $P^*(A) \leq P^*(B)$
4. P^* is countably subadditive: $P^*(\cup_n A_n) \leq \sum_n P^*(A_n)$

1 is obvious, as the union of empty sets contains $\emptyset$. 2 is obvious, as we already said the infimum is nonempty, and each element is nonnegative (being a sum of P nonnegative). 3 is also obvious, as any cover of B covers A, and so all sums in B's infimum are in A's. The final part is countable subadditivity. This is the normal $\epsilon/2^n$ proof:

For each A_n, take a cover of $\mathcal{F}_0$ sets B_{nk}, where $\sum_k P(B_{nk}) < P^*(A_n)+\epsilon/2^{-n}$. Note, it is possible to find by infimum definition. Note that all B_{nk} cover the total union - so, by the definition of infimum:

$$P^*(\cup_n A_n) \leq \sum_{n,k} P(B_{nk}) < \sum_n P^*(A_n) + \epsilon/2^{-n} = \epsilon + \sum_n P^*(A_n)$$

This is true for all $\epsilon > 0$, and so 4 follows.

Lemma 1 - The class $\mathcal{M}$ is a field First, we note by finite subadditivity above:

$$P^*(E) \leq P^*(A \cap E) + P^*(A^c \cap E)$$

And so, to prove that A is in $\mathcal{M}$, we just need to prove that:

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E)$$

It is clear that $\Omega \in \mathcal{M}$, and $\mathcal{M}$ is closed under complements. The final step is to prove $\mathcal{M}$ is closed under finite unions. Note, it is simple to prove that:

$$P^*(E) \geq P^*((A \cup B) \cap E) + P^*((A \cup B)^c \cap E)$$

Using the E condition being true on A and B (and taking $(B \cap E)$ as the set E, and so on) and subadditivity. qed.

Lemma 2 - P^* Condition on Countable Disjoint Sets in $\mathcal{M}$ If $A_1, A_2, \cdots$ is a finite or infinite sequence of disjoint $\mathcal{M}$-sets, then for each $E \subset \Omega$:

$$P^*\left(E \cap \left(\bigcup_k A_k\right)\right) = \sum_k P^*(E \cap A_k)$$

Proof: For finite disjoint sets, we use induction. Clearly true for $n = 1$. For $n = 2$, it is also easy to prove. If $A_1 \cup A_2 = \Omega$, then $A_2 = A_1^c$, and it is just our condition. Otherwise:

$$P^*(E \cap (A_1 \cup A_2)) = P^*(A_1 \cap E \cap (A_1 \cup A_2)) + P^*(A_1^c \cap E \cap (A_1 \cup A_2))$$

Where the last step again makes use of our condition for $E = E \cap (A_1 \cup A_2)$. The above equals:

$$= P^*(E \cap A_1) + P^*(E \cap A_2)$$

Where we have that $A_1^c \cap E \cap (A_1 \cup A_2) = E \cap A_2$ via disjointness. Ie, $A_2 \subseteq A_1^c$, so $x \in A_2 \implies x \in A_1^c$. With the $n = 2$ case, we can imply our inductive hypothesis and find the formula is true for any finite disjoint set of unions. For the infinite case - note:

$$P^*\left(E \cap \left(\bigcup_k A_k\right)\right) \geq \sum_{k=1}^{n} P^*(E \cap A_k)$$

This is by monotonicity of P^*. We can take $n \to \infty$ on the RHS, and so we can indeed conclude:

$$P^*\left(E \cap \left(\bigcup_k A_k\right)\right) \geq \sum_k P^*(E \cap A_k)$$

And the other inequality $\leq$ follows from countable subadditivity. qed.

Lemma 3 - The class $\mathcal{M}$ is a σ field, and P^* restricted to $\mathcal{M}$ is countably additive Suppose that $A_1, A_2, \cdots$ are disjoint $\mathcal{M}$ sets with union A. Since $F_n = \bigcup_{k=1}^{n} A_k$ lies in $\mathcal{M}$, we have:

$$P^*(E) = P^*(E \cap F_n) + P^*(E \cap F_n^c)$$

By applying lemma 2 and monotonicity, we thus find:

$$P^*(E) \geq \sum_{k=1}^{n} P^*(E \cap A_k) + P^*(E \cap A^c)$$

We can take n to infinity, and using Lemma 2 again, we find:

$$P^*(E) \geq P^*(E \cap A) + P^*(E \cap A^c)$$

Thus, the countable disjoint union is also within $\mathcal{M}$. Note, any countable union can be expressed as a countable disjoint union - thus $\mathcal{M}$ is a σ field. Countable additivity of P^* on $\mathcal{M}$ follows again from Lemma 3.2, taking $E = \Omega$. qed.

Note - Lemmas 1, 2, and 3 only used properties (1) through (4) we defined above, along with an Ω, P^* being defined on all subsets of Ω, and the condition that defined $\mathcal{M}$. Now, we can bring it back to comparing with P on $\mathcal{F}_0$ and extending to P^* on $\sigma(\mathcal{F}_0)$.

Lemma 4 - If P^* is defined by the outer measure, then $\mathcal{F}_0 \subset \mathcal{M}$ Note, this just requires $\mathcal{M}$ containing every set in $\mathcal{F}$. Take $A \in \mathcal{F}_0$. Given E and ϵ, choose $\mathcal{F}_0$ sets A_n such that:

$$E \subset \bigcup_n A_n \qquad \text{and } \sum_n P(A_n) \leq P^*(E) + \epsilon$$

Recall - the goal is to show $F_0 \subset \mathcal{M}$, and so we need to show for our $E \subset \Omega$ defined above:

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E)$$

Define $B_n = A_n \cap A$ and $C_n = A_n \cap A^c$. B_n and C_n are in $\mathcal{F}_0$ by field properties. Also, $E \cap A \subset \cup_n B_n$, and $E \cap A^c \subset \cup_n C_n$. By first P^* outer measure definition, and then by finite additivity of P on $\mathcal{F}_0$, we have:

$$P^*(A \cap E) + P^*(A^c \cap E) \leq \sum_n P(B_n) + \sum_n P(C_n) = \sum_n P(A_n) \leq P^*(E) + \epsilon$$

As this is true for all $\epsilon > 0$, we have that our necessary and sufficient condition is true, and thus $A \in \mathcal{F}_0 \implies A \in \mathcal{M}$. qed.

Lemma 5 - If P^* is defined by the outer measure, then $P^*(A) = P(A)$ for $A \in \mathcal{F}_0$ Clearly, we have that $P^*(A) \leq P(A)$, as we can just take a countable union of A and emptysets. Now, take any union where $A \subset \cup_n A_n$. By countable subadditivity and monotonicity of P on $\mathcal{F}_0$, we have:

$$P(A) = P(A \cap \cup_n A_n) = P(\cup_n A \cap A_n) \leq \sum_n P(A \cap A_n) \leq \sum_n P(A_n)$$

Thus, the infimum clearly takes the value of $P(A)$. qed.

Proof of Extension in Theorem 3.1 Suppose P^* is defined as our outer measure from a (countably additive) probability measure P on the field $\mathcal{F}_0$. Let $\mathcal{F} = \sigma(\mathcal{F}_0)$. By Lemma 3 and 4:

$$\mathcal{F}_0 \subseteq \mathcal{F} \subseteq \mathcal{M} \subseteq 2^\omega$$

By Lemma 5, we have $P^*(\Omega) = P(\Omega) = 1$. By Lemma 3, P^* when restricted to $\mathcal{M}$ is a probability measure there, and thus P^* restricted to $\mathcal{F}$ is also clearly a probability measure on that class as well. By Lemma 5, P^* agrees with P on $\mathcal{F}$, and thus it is our required extension.

Uniqueness of the Extension and the $\pi - \lambda$ Theorem

We first give the following definitions:

π **System** A class $\mathcal{P}$ of subsets of Ω is a π system if it is closed under the formation of finite intersections: $A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$

λ **System** A class $\mathcal{L}$ of subsets of Ω is a λ system if it contains Ω, and is closed under complements and finite and countable disjoint intersections:

1. $\Omega \in \mathcal{L}$
2. $A \in \mathcal{L} \implies A^c \in \mathcal{L}$
3. $A_1, A_2, \dots \in \mathcal{L}$ disjoint $\implies \cup_n A_n \in \mathcal{L}$

Note: As to why these are useful. We ultimately want to prove uniqueness of an extension. We have that an extension is equal on a field - which is also a π system. And ultimately, the *essential* property of the probability measure is its countable additivity, which is defined for a countable disjoint union. These countable disjoint unions are contained within λ.

And so, how we can prove uniqueness can go something like this: if we have a probability measure on a π system, and we can extend it to a λ system (that is somehow related to a σ field), and the measure agrees on the λ system as well - then, the probability measure is unique!

That is a "high flying" statement I would say. Also, it is much more general than what we need it for - like, why should we start with a π system, and not just a field? Well, the generality will be useful later. And so, we shall go and prove portions of that statement:

Lemma 6 π **and** λ **implies** σ A class that is both a π system and a λ system is a σ field. **Proof**: Easy enough, as any countable union can be expressed as a countable disjoint union, using finite intersections. qed. This gives us the connection to a σ field.

Theorem 3-2: $\pi - \lambda$ **Theorem** If $\mathcal{P}$ is a π system and $\mathcal{L}$ is a λ system, then $\mathcal{P} \subset \mathcal{L}$ implies $\sigma(\mathcal{P}) \subset \mathcal{L}$.

Proof: Let $\mathcal{L}_0$ be the λ system generated by $\mathcal{P}$, the intersection of all containing λ systems. It is easy to prove intersections of λ systems are λ systems (essentially the same as the proof for $\sigma(\cdot)$ and nonempty by 2^{Ω}). Thus, we have $\mathcal{P} \subset \mathcal{L}_0 \subset \mathcal{L}$ by definition. If $\mathcal{L}$ can be shown to be a π system, then it will follow by Lemma 6 that $\mathcal{L}_0$ is a σ field. Thus, we *must* have:

$$\mathcal{P} \subseteq \sigma(\mathcal{P}) \subseteq \mathcal{L}_0 \subseteq \mathcal{L}$$

Thus, the theorem follows if $\mathcal{L}_0$ is a π system as well.

For each $A \subset \Omega$, let $\mathcal{L}_A$ be the class of sets B such that $A \cap B \in \mathcal{L}_0$. If $A \in \mathcal{P}$, or $A \in \mathcal{L}_0$, then $\mathcal{L}_A$ is a λ system. Note, $A \in \mathcal{P} \implies A \in \mathcal{L}_0$, and so we just prove it for the second case. First, $\mathcal{L}_A$ contains Ω, as $A \cap \Omega = A \in \mathcal{L}_0$. We now recall that the conditions:

1. $\mathcal{L}_A$ contains Ω
2. $\mathcal{L}_A$ contains proper differences
3. $\mathcal{L}_A$ contains disjoint countable unions

Is equivalent to a λ system - as the complement of an element of the set can be expressed as a proper difference. We have the first point. For the second, take $B_1, B_2 \in \mathcal{L}_A$ with $B_1 \subset B_2$. We have:

$$(A \cap B_2) - (A \cap B_1) = A \cap (B_2 - B_1) \in \mathcal{L}_0$$

By the proper difference property of $\mathcal{L}_0$, and thus $B_2 - B_1 \in \mathcal{L}_1$, fulfilling point 2. The final point is the third. Let B_n be disjoint $\mathcal{L}_A$ sets. $\mathcal{L}_0$ contains $A \cap B_n$, which are disjoint, and so $\mathcal{L}_0$ contains $\bigcup_n A \cap B_n = A \cap (\bigcup_n B_n)$, and thus $\mathcal{L}_A$ contains disjoint unions.

Thus, we have that $\mathcal{L}_A$ is a λ system. If $A \in \mathcal{P}$ and $B \in \mathcal{P}$, then $A \cap B \in \mathcal{P} \subset \mathcal{L}_0$, which is equivalent to $B \in \mathcal{L}_A$. Thus, $A \in \mathcal{P} \implies P \subset \mathcal{L}_A$, and by minimality, we have:

$$\mathcal{L}_0 \subseteq \mathcal{L}_A$$

This is where it gets tricky. By the above - $A \in \mathcal{P}$ and $B \in \mathcal{L}_0$ implies $B \in \mathcal{L}_0 \in \mathcal{L}_A$. As this is symmetric - it implies $A \in \mathcal{L}_B$. Thus, $B \in \mathcal{L}_0$ implies $\mathcal{P} \subset \mathcal{L}_B$ (as for every $A \in \mathcal{P}$, $A \in \mathcal{L}_B$). Thus, again by minimality, we have if $B \in \mathcal{L}_0$:

$$\mathcal{L}_0 \subseteq \mathcal{L}_B$$

We essentially used “symmetry" of the $\mathcal{L}_C$ definition to go from $A \in \mathcal{P}$ implies $\mathcal{L}_0 \subseteq \mathcal{L}_A$, directly to $B \in \mathcal{L}_0$ implies $\mathcal{L}_0 \subseteq \mathcal{L}_B$. Finally - $B \in \mathcal{L}_0$, and $C \in \mathcal{L}_0$, together imply $C \in \mathcal{L}_B$, or $B \cap C \in \mathcal{L}_0$ by definition. Therefore, $\mathcal{L}_0$ is a π system, and the theorem follows. qed.

Uniqueness of our extended probability measure P^* is thus a result of the $\pi - \lambda$ theorem:

Theorem 3-3 Uniqueness of Extensions Suppose that P_1 and P_2 are probability measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a π system. If P_1, P_2 agree on $\mathcal{P}$, then they agree on $\sigma(\mathcal{P})$.

Let $\mathcal{L}$ be the class of sets A in $\sigma(\mathcal{P})$ such that $P_1(A) = P_2(A)$. Clearly, $\Omega \in \mathcal{L}$. If $A \in \mathcal{L}$, then $P_1(A^c) = 1 - P_1(A) = 1 - P_2(A) = P_2(A^c)$, and so $\mathcal{L}$ is closed under complements (note, we have the formula by being a probability measure). If A_n are disjoint sets in $\mathcal{L}$, then:

$$P_1(\bigcup_n A_n) = \sum_n P_1(A_n) = \sum_n P_2(A_n) = P_2(\bigcup_n A_n)$$

Thus, $\mathcal{L}$ is also closed under disjoint countable unions, and so $\mathcal{L}$ is a λ system. Thus, as $\mathcal{P} \subset \mathcal{L}$, the $\pi - \lambda$ theorem tells us that $\sigma(\mathcal{P}) \subseteq \mathcal{L}$, which implies that P_1 and P_2 agree on $\sigma(\mathcal{P})$ as well. qed.

And so - we have proved the uniqueness of the extension, as for any two probability measures on $\sigma(\mathcal{F}_0)$ that agree on $\mathcal{F}_0$, agree on the π system $\mathcal{F}_0$, and thus agree on the extended sigma algebra.

Note - as stated above, our Theorems are pretty general. We could have replaced π system every where with field, as a field has all the properties of a π system. All in total, we have proved:

Theorem 3.1 - Probability Measure Extension Theorem A probability measure on a field has a unique extension to the generated σ field.

Monotone Classes Are classes that are closed under the formation of monotone unions and intersections (like in continuity from above and below). That's it. There is another close relative of the $\pi - \lambda$ theorem for these classes:

Theorem 3.4 - Halmos's Monotone Class Theorem If $\mathcal{F}_0$ is a field and $\mathcal{M}$ is a monotone class, then $\mathcal{F}_0 \subset \mathcal{M} \implies \sigma(\mathcal{F}_0) \subset \mathcal{M}$. **Proof**: Very similar to the $\pi - \lambda$ theorem, in that it relies on first defining a minimal monotone class $m(\mathcal{F}_0)$ containing $\mathcal{F}_0$ - and then finding a simpler condition to proving that it is a σ field and thus contains $\sigma(\mathcal{F}_0)$. That simpler condition is proving that $m(\mathcal{F}_0)$ is a field - which clearly implies it is a σ field.

Proving that $m(\mathcal{F}_0)$ is closed under complements is easy. However - proving it is closed under finite unions is more difficult. The argument, however - is the same in structure as the $\pi - \lambda$ theorem. It involves a "switching" statement, to prove that the class of sets in $m(\mathcal{F}_0)$ that are closed under unions are also a monotone class containing $\mathcal{F}_0$, and thus $m(\mathcal{F}_0)$ is contained in that class. Thus, we have $m(\mathcal{F}_0)$ is a field, and unraveling all the way back proves the theorem. qed.

Lebesgue Measure on the Unit Interval

Recall the unit interval $(0, 1]$ with the field $\mathcal{B}_0$ of finite disjoint unions of subintervals. In Theorem 2.2, we have a probability measure λ on $\mathcal{B}_0$, where λ assigns intervals a measure equal to their length. Thus, by Theorem 3.1, λ extends to $\mathcal{B}$, the extended λ being the Lebesgue measure. The probability space $((0, 1], \mathcal{B}, \lambda)$ will thus be the basis of a lot of the probability theory we will go over in the remaining sections of the first chapter. An interesting note: as the intervals in $(0, 1]$ form a π system generating $\mathcal{B}$, λ is the *only* probability measure on $\mathcal{B}$ that assigns to each interval its length as its measure.

Example 1 - Subintervals around rationals We can assign to each rational $r_1, r_2, \cdots \in (0, 1)$ an interval I_n of length $\epsilon/2^n$. Consider $A = \cup_n I_n$. By subadditivity, $\lambda(A) < \epsilon$. Note, A is dense, and note each interval I must intersect an I_n, so $\lambda(A \cap I) > 0$. Note that $B = (0, 1) - A$ satisfies $\lambda(B) > 1 - \epsilon$ - however, no matter how close to 1 the measure of B is, B will always be nowhere dense (all intervals contain a rational and thus intersect B^c).

Example 2 - k Repeated Sequences This example gives a set with real-world probabilistic interpretations that has similar properties to the set in the previous example. Let $d_n(\omega)$ be the *nth* digit in the dyadic expansion of ω - like in the first section. Define:

$$A_n = \{\omega \in (0, 1] : d_i(\omega) = d_{n+i}(\omega) = d_{2n+i}(\omega), i = 1, \cdots, n\}$$

Then, we define:

$$A = \bigcup_{n=1}^{\infty} A_n$$

Probabilistically, A corresponds to the event that in an infinite sequence of tosses of a coin, some finite initial segmant is immediately duplicated twice over. From $\lambda(A_n) = 2^n \cdot 2^{-3n}$ (consider $3n$ spaces, we have any choice for the first n spaces, and then fixed choices for the final $2n$ spaces) (also, note that the dyadic cylinders can be expressed as disjoint interval unions, and so they do have a λ value), we have $0 < \lambda(A) \leq 1/3$. A is dense in the unit interval - we can find an element in A by taking a midpoint of some interval, and then repeating the sequence again, which remains in the interval. It's measure can be made less than 1/3, by requiring the initial segment is duplicated k times with k being large.

And so, we have a real world example of a dense set, but with very very small probability.

Completeness A probability measure space $(\Omega, \mathcal{F}, P)$ is *complete* if $A \subset B, B \in \mathcal{F}$, and $P(B) = 0$ together imply $A \in \mathcal{F}$ (and hence, $P(A) = 0$ by monotonicity).

Suppose that $(\Omega, \mathcal{F}, P)$ is an arbitrary probability space. Define P^* for $\mathcal{F}_0 = \mathcal{F} = \sigma(\mathcal{F}_0)$ - note, even though $\mathcal{F}$ is a σ field, P^* is still defined for *all subsets of* Ω via the outer measure. Consider the σ field $\mathcal{M}$ of P^* measurable sets (ones for which $P^*(A)+P^*(A^c) = 1$). The arguments outlines above tell us that $(\Omega, \mathcal{M}, P^*)$ is a probability measure space. If $P^*(B) = 0$ and $A \subset B$, then:

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(B) + P^*(E) = P^*(E) \implies A \in \mathcal{M}$$

Thus, $(\Omega, \mathcal{M}, P^*)$ is a *complete probability measure space*. We note: In any probability space, it is therefore possible to enlarge the σ field and extend the measure in such a way as to get a complete space.

Suppose that $((0, 1], \mathcal{B}, \lambda)$ is completed in this way. The sets in the completed σ field $\mathcal{M}$ are called *Lebesgue Sets*, and λ extended to $\mathcal{M}$ is still called the Lebesgue measure.

Nonmeasurable Sets Above, we have focused on what sets are in $\mathcal{B}$, and finding more measurable sets. Now, we find a set that it *outside of* $\mathcal{B}$. We need the following definitions:

Definition: Set Shifts/Translations For $x, y \in (0, 1]$, take $x \oplus y$ to be $x + y$ or $x + y - 1$ according as $x + y$ lies in $(0, 1]$ or not - this is the circle group, essentially. Define $A \oplus x = \{a \oplus x : a \in A\}$. Let $\mathcal{L}$ be the class of Borel sets A such that $A \oplus x$ is a Borel set and $\lambda(A \oplus x) = \lambda(A)$. Then $\mathcal{L}$ is a λ system containing the intervals, and so $\mathcal{B} \subset \mathcal{L}$ by the $\pi - \lambda$ theorem. Note, it is clear that:

1. The intervals are contained in $\mathcal{L}$.

2. The intervals are a π system.

3. Complements are in $\mathcal{L}$, as we can take $\lambda(A^c \oplus x) = 1 - \lambda(A \oplus x) = 1 - \lambda(A) = \lambda(A^c)$.

4. Disjoint unions are in $\mathcal{L}$:

$$\lambda\left(\bigcup_n A_n \oplus x\right) = \lambda\left(\bigcup_n (A_n \oplus x)\right) = \sum_n \lambda(A_n \oplus x) = \sum_n \lambda(A_n) = \lambda\left(\bigcup_n A_n\right)$$

And so, we do indeed have that $\mathcal{B} \subset \mathcal{L}$ by the $\pi - \lambda$ theorem, and $A \in \mathcal{B}$ implies that $A \oplus x \in \mathcal{B}$, and $\lambda(A \oplus x) = \lambda(A)$. In this sense - λ is translation invariant on $\mathcal{B}$.

Vitali Sets Define x and y to be equivalent ($x \sim y$) if $x \oplus r = y$ for some rational r in $(0, 1]$. Let H be a subset of $(0, 1]$ consisting of exactly one representative point from each equivalence class - such a set exists via the *axiom of choice*. Note - I believe that H is uncountable, as I think there are uncountably such equivalence classes - but, I'm not sure, and I'm not gonna try and prove it here.

Now, consider the countably many sets $H \oplus r$ for rational r. These sets are disjoint - as no two distinct points of H are equivalent, and so if $H \oplus r_1$ and $H \oplus r_2$ share the point $h_1 \oplus r_1$ and $h_2 \oplus r_2$, then $h_1 \sim h_2$, then $h_1, h_2 \in H$ satisfy $h_1 \sim h_2$, which contradicts the definition of H.

Note that each point of $(0, 1]$ lies in one of these sets, because H has a representative from each equivalence class, and each point can be expressed as $h \oplus r$ for some $h \in H$ and $r \in \mathbb{Q}$. Thus:

$$(0, 1] = \bigcup_r (H \oplus r)$$

Now, we prove that H is outside of $\mathcal{B}$ via a contradiction. Assume that $H \in \mathcal{B}$ - then, translation invariance implies each $H \oplus r$ is in $\mathcal{B}$. By countable additivity, we have:

$$1 = \lambda((0, 1]) = \lambda\left(\bigcup_r (H \oplus r)\right) = \sum_r \lambda(H \oplus r) = \sum_r \lambda(H)$$

If $\lambda(H) = 0$, we have a contradiction. If $\lambda(H) > 0$, we have a contradiction. Thus, we have a contradiction in all cases, and H is outside of $\mathcal{B}$. H is called a *Vitali Set*.

Problems

3.1 Finite vs. Countable Additivity

1. In the proof of Theorem 3.1 the assumed finite additivity of P is used twice and the assumed countable additivity of P is used once. Where?
   Finite additivity is used once in Lemma 4 to prove the condition for $A \in \mathcal{F}_0 \implies A \in \mathcal{M}$. In Lemma 5, we use countable additivity (via countable subadditivity) to prove $P^*(A) = P(A)$, and finite subadditivity via monotonicity.

2. Show by example that a finitely additive probability measure on a field may not be countably subadditive. Show in fact that if a finitely additive probability measure is countably subadditive, then it is necessarily countably additive as well.

   We do the second part first. By countable subadditivity, we have:

   $$P\left(\bigcup_n A_n\right) \leq \sum_n P(A_n)$$

   Assume A_n are disjoint. Recall - monotonicity *only* requires finite additivity. And so, we have:

   $$\sum_{n=1}^{k} P(A_n) \leq P(A) \implies \lim_{k\to\infty} P(A_n) \leq P(A) \implies P\left(\bigcup_n A_n\right) \geq \sum_n P(A_n)$$

   $$\implies P\left(\bigcup_n A_n\right) = \sum_n P(A_n)$$

   For an example, consider problem 2-15 (which we did above). This is a P that is finitely but not countably additive.

3. Suppose Theorem 2.1 (Continuity From Above, Below, and Countable Subadditivity) were weakened by strengthening its hypothesis to the assumption that $\mathcal{F}$ is not a σ-field. Why would this weakened result not suffice for the proof of Theorem 3.1?

   In Lemma 2 - we make use of countable subadditivity to prove the Lemma 2 property - we cannot prove Theorem 3.1 without first concluding that $\mathcal{M}$ is a σ field, which relies on Lemma 2. Also, in Lemma 5, we rely on countable subadditivity to prove $P^*(A) = P(A)$.

3.2 Redefining the Inner and Outer Measure

Let P be a probability measure on a field $\mathcal{F}_0$ and for every subset A of Ω define $P^*(A)$ by the outer measure. Denote also by P the extension of P to $\mathcal{F} = \sigma(\mathcal{F}_0)$.

1. Show that:
$$P^*(A) = \inf\left[P(B) : A \subset B, B \in \mathcal{F}\right]$$
And:
$$P_*(A) = \sup\left[P(C) : C \subset A, C \in \mathcal{F}\right]$$
And show that the infimum and supremum are always achieved.

For the first case - note that:
$$P^*(A) = \inf\left[\sum_n P(A_n) : A \subseteq \cup A_n, A_n \in \mathcal{F}_0\right] \geq \inf\left[P(B) : A \subset B, B \in \mathcal{F}\right]$$
As for why this is - take any covering of $A \subseteq \cup A_n, A_n \in \mathcal{F}_0$, and note that $\cup A_n \in \mathcal{F}$, and $\sum P(A_n) \geq P(\cup A_n)$. Now, take $B \in \mathcal{F}$, and note that $B \in \mathcal{M}$ by the extension theorem, and so:
$$P(B) = P^*(B) = P^*(A \cap B) + P^*(A^c \cap B) \implies P(B) = P^*(A) + P^*(A^c \cap B)$$
$$\implies P(B) \geq P^*(A)$$
Thus, we have:
$$P^*(A) \leq \inf\left[P(B) : A \subset B, B \in \mathcal{F}\right]$$
Giving us our first equality. Now, we want to show:
$$P_*(A) = 1 - P^*(A^c) = \sup\left[P(C) : C \subset A, C \in \mathcal{F}\right]$$
By the above part, and definitions, we have:
$$P_*(A) = \inf\left[1 - P(C) : A^c \subset C, C \in \mathcal{F}\right]$$
Note that $A^c \subset C$ if and only if $C^c \subset A$. If we consider $C^c \subset A$ - we note that $1 - P(C)$ is minimized if and only if $1 - P(C^c) = P(C)$ is maximized. Thus, the above equals:
$$= \sup\left[1 - P(C^c) : C^c \subset A, C \in \mathcal{F}\right] = \sup\left[1 - P(C^c) : C^c \subset A, C^c \in \mathcal{F}\right]$$

Note, the second step is because $C \in \mathcal{F}$ if and only if $C^c \in \mathcal{F}$. Now, note that the above expression can be simplified to:

$$= \sup\left[P(C) : C \subset A, C \in \mathcal{F}\right]$$

The final step is to show that these are achievable - ie, there are sets B and C such that:

$$P^*(A) = P(B) \qquad\qquad P_*(A) = P(C)$$

Assume not achievable in the first case. Then, we have $P(B_n)$ such that:

$$P^*(A) = \lim_n P(B_n)$$

As $\mathcal{F}$ is a σ field, we have that $\cap B_n \in \mathcal{F}$. Let C_n be the incremental intersections - note $C_n \downarrow \cap B_n$, and:

$$P^*(A) \le \lim_n P(C_n) \le \lim_n P(B_n) \implies P(\cap B_n) = P^*(A)$$

And so, we have a contradiction. Continuity from below will give us a similar result for the supremum, and so in both cases, we have the infimum and supremum are always achieved.

2. Show that A is P^* measurable if and only if $P_*(A) = P^*(A)$. This is equivalent to:

$$1 - P^*(A^c) = P^*(A) \implies P^*(A) + P^*(A^c) = 1$$

Note that if A is P^* measurable, the above is clearly true via definition, taking $E = \Omega$. Now, assume $P_*(A) = P^*(A)$. We can take sets in $\mathcal{F}$ such that $A_1 \subset A \subset A_2$, with:

$$P(A_1) = P_*(A) = P^*(A) = P(A_2)$$

Note, this is because the infimum and supremum are achievable in the previous part. Now, I will show that $A \in \mathcal{M}$, which implies A is P^* measurable. We have that:

$$A = A_1 \cup (A \setminus A_1)$$

Now, we note that, by the properties of a probability measure:

$$P(A_2 \setminus A_1) = P(A_2) - P(A_1) = 0$$

As $\mathcal{M}$ is complete, and $A \setminus A_1 \subseteq A_2 \setminus A_1$, we have that $A \setminus A_1 \in \mathcal{M}$. Thus, as $\mathcal{M}$ is a field, and closed under unions, we have that $A \in \mathcal{M}$ as well.

3. The outer and inner measures associated with a probability measure P on a σ field $\mathcal{F}$ are usually *defined* by the above infimum and supremum. Show that the infimum and supremum are the same as our original outer and inner measure definitions with $\mathcal{F}$ in the role of $\mathcal{F}_0$.

   This much is clear - because $\mathcal{F} = \sigma(\mathcal{F})$, and we showed that the definitions are the same in the first part.

3.3 Countable Additivity and the Outer Measure

This problem relies on 2.13, 2.15, and 3.2 - all of which we have done. For the following examples, described P^* as the original outer measure and $\mathcal{M} = \mathcal{M}(P^*)$ by our condition with subsets E. Sort out the cases in which P^* fails to agree with P on $\mathcal{F}_0$, and explain why:

1. Let $\mathcal{F}_0$ consist of sets $\emptyset, \{1\}, \{2,3\}$ and $\Omega = \{1,2,3\}$ and define probability measures P_1 and P_2 on $\mathcal{F}_0$ by $P_1(1) = 0$ and $P_2(2,3) = 0$. Note that $\mathcal{M}(P_1^*)$ and $\mathcal{M}(P_2^*)$ differ.

   Well, the σ algebras must differ on the sets they include - which would be interesting. I guess, going from the back - this would be because of completeness - maybe they differ on including $\{2\}$. In $\mathcal{M}(P_1^*)$, we find:

   $$1 = P_1^*(2,3) \qquad P_1^*(2 \cap 2,3) + P_1^*(2^c \cap 2,3) = P_1^*(2) + P_1^*(3) = 2$$

   $$0 = P_2^*(2,3) \qquad P_2^*(2 \cap 2,3) + P_2^*(2^c \cap 2,3) = P_2^*(2) + P_2^*(3) = 0$$

   So, $\mathcal{M}(P_2^*)$ contains $\{2\}$. The completeness comment comes from only $\mathcal{M}(P_2^*)$ containing a set with measure 0, and nonempty subsets, which led to the example, I guess.

2. Suppose that Ω is countably infinite, let $\mathcal{F}_0$ be the field of finite and cofinite sets, and take $P(A)$ to be 0 or 1 as A is finite or cofinite.

   By 2-13, as Ω is countably infinite, we have that P is *not* countably additive. And so, our theorem 3.1 does not apply - we need P to be countably infinite to prove that $P^*(A) = P(A)$ for $A \in \mathcal{F}_0$.

   And so, we should have such a set. Take a cofinite set in Ω - which is just a countable set A. We have that:

   $$P(A) = 1$$

However, A can be covered by a union of finite sets - in which case, the infimum would be 0, and $P^*(A) = 0$, which means that $P^*(A) \neq P(A)$. Countable additivity would have implies that the sum of disjoint singletons should be equal to $P(A)$.

3. The same, but suppose that Ω is uncountable. Well, in 2-13, we have that P is a countably additive probability measure on our field, and so it can be properly extended.

4. Suppose that Ω is uncountable, let $\mathcal{F}_0$ consist of the countable and the cocountable sets, and take $P(A)$ to be 0 or 1 as A is countable or cocountable. We have that $\mathcal{F}_0$ is a σ field, and P is a countably additive probability measure - so, the theorems should apply.

5. The probability in Problem 2.15. Well, in that problem, we have that the probability measure is not countably additive. Take the set $A = (1/2, 1]$. We have that $P(A) = 1$. However, we also find:

$$A = \bigcup A_n \qquad A_n = (1/2 + 1/(n+2), 1]$$

   And so, our infimum would give us $P^*(A) = 0$, and we get a disagreement again.

6. Let $P(A) = I_A(\omega_0)$ for $A \in \mathcal{F}_0$, and assume $\{\omega_0\} \in \sigma(\mathcal{F}_0)$. Well, it is countably additive - there are no disjoint sets containing ω_0. We have additivity in all other cases (disjoint sets not containing ω_0, and disjoint sets with one containing ω_0).

   So, P^* and $\mathcal{M}$ are well defined. If we want to describe $\mathcal{M}$ - we need sets where:

$$\inf\left[P(B) : A \subset B, B \in \mathcal{F}\right] = \sup\left[P(C) : C \subset A, C \in \mathcal{F}\right]$$

   I think $\mathcal{M}$ consists of all sets containing ω_0 and their complements. First, note $P(\omega_0) = 1$. Any cover of ω_0 by sets in $\mathcal{F}_0$ would have a sum greater than or equal to 1. Now, take A containing ω_0. The supremum contains 1, and cannot be larger than 1. Similarly, the infimum cannot be smaller than 1 - as any set in $\mathcal{F}$ that contains A is a set in $\mathcal{F}$ that contains ω_0, and so must be $P = 1$ by monotonicity.

   So, if A contains ω_0, it is in $\mathcal{M}$, with $P^*(A) = 1$. Similarly, if it doesn't, it has $P^*(A) = 0$, as the complement is in it. Note, $\mathcal{M}$ thus contains all subsets, and their probabilities are either 1 or 0 based on this condition. qed.

3.6 Extension for a Finitely Additive Probability Measure on a Field

Let P be a *finitely additive* probability measure on a field $\mathcal{F}_0$. For $A \subset \Omega$, in analogy with (3.1), define:

$$P^\circ(A) = \inf \sum_n P(A_n)$$

Over *finite* sequences of $\mathcal{F}_0$ sets A_n. Let $\mathcal{M}^\circ$ be the class of sets A such that:

$$P^\circ(E) = P^\circ(A \cap E) + P^\circ(A^c \cap E)$$

For all $E \subset \Omega$.

1. Show that $P^\circ(\emptyset) = 0$ and that P° is nonnegative, monotone, and *finitely* subadditive.

   $P^\circ(\emptyset) \leq P(\emptyset) = 0$. Note, as all the sums are greater than 0, $P^\circ(\emptyset) = 0$. Clearly, this implies non-negative as well. Monotonicity, as if $A \subset B$, any cover of B covers A, and so that sum is included in A's infimum. Take any finite union $\cup_n A_n$. Examine:

   $$P^\circ(\cup A_n)$$

   Note, finite subadditivity can just be found via the $\epsilon/2^k$ trick. We have a finite cover of each A_n, with total sum within $\epsilon/2^n$ of $P^\circ(A_n)$ - note, a finite amount of finite covers still covers $\cup A_n$, thus, we have:

   $$P^\circ(\cup A_n) \leq \sum_n P^\circ(A_n) + \epsilon/2^n = \epsilon + \sum P^\circ(A_n)$$

   Using these four properties, prove:

   Lemma 1 $\mathcal{M}^\circ$ is a field.

   First, we have:

   $$P^\circ(E) = P^\circ(E) + P^\circ(\emptyset) = P^\circ(\Omega \cap E) + P^\circ(\emptyset \cap E)$$

   And so $\mathcal{M}^\circ$ contains Ω and $\emptyset$. Now, we go over complements - as the E definition is symmetric between complements, we have that $A \in \mathcal{M}^\circ \implies A^c \in \mathcal{M}^\circ$. Now, we prove that $\mathcal{M}^\circ$ is closed under finite unions (or equivalently, finite intersections). Take $A, B \in \mathcal{M}^\circ$, and $E \subset \Omega$. We have:

$$P^\circ(E) = P^\circ(B \cap E) + P^\circ(B^c \cap E)$$

$$= P^\circ(A \cap B \cap E) + P^\circ(A^c \cap B \cap E) + P^\circ(A \cap B^c \cap E) + P^\circ(A^c \cap B^c \cap E)$$

Just by applying $A, B \in \mathcal{M}^\circ$ three times. By *finite subadditivity*, we find:

$$\geq P^\circ(A \cap B \cap E) + P^\circ((A^c \cap B \cap E) \cup (A \cap B^c \cap E) \cup (A^c \cap B^c \cap E))$$

$$= P^\circ(A \cap B \cap E) + P^\circ((A \cap B)^c \cap E)$$

As we already have the other direction via finite subadditivity, we thus have $A \cap B \in \mathcal{M}^\circ$, and $\mathcal{M}^\circ$ is a field.

Lemma 2 If $A_1, A_2, \cdots$ is a *finite* sequence of disjoint $\mathcal{M}^\circ$ sets, then for each $E \subset \Omega$:

$$P^\circ\left(E \cap \left(\bigcup_k A_k\right)\right) = \sum_k P^\circ(E \cap A_k)$$

If $n = 1$, there is nothing to prove. Now, assume $n = 2$ - if $A_1 \cup A_2 = \Omega$, then $A_2 = A_1^c$, and it is just our condition. Otherwise:

$$P^\circ(E \cap (A_1 \cup A_2)) = P^\circ(A_1 \cap E \cap (A_1 \cup A_2)) + P^\circ(A_1^c \cap E \cap (A_1 \cup A_2))$$

Where the second step again makes use of our condition for $E \cap (A_1 \cup A_2)$ and $A_1 \in \mathcal{M}^\circ$. The above equals:

$$= P^\circ(E \cap A_1) + P^\circ(E \cap A_2)$$

Where we have that $A_1^c \cap E \cap (A_1 \cup A_2) = E \cap A_2$ via disjointness. Ie, $A_2 \subseteq A_1^c$, so $x \in A_2 \implies x \in A_1^c$. It is easy to prove via induction for an arbitrary length finite sequence of disjoint $\mathcal{M}^\circ$ sets.

Lemma 3 P° restricted to the field $\mathcal{M}^\circ$ is *finitely additive*. Take a finite disjoint sequence $A_1, A_2, \cdots$ of $\mathcal{M}^\circ$ sets. By Lemma 1, $\cup A_n \in \mathcal{M}$. We want to show:

$$P^\circ(\cup A_n) = \sum P^\circ(A_n)$$

Just apply Lemma 2 with $E = \Omega$. Thus, we have:

$$P^\circ(\cup A_n) = P^\circ(\Omega \cap \cup A_n) = \sum P^\circ(\Omega \cap A_n) = \sum P^\circ(A_n)$$

2. Show that if P° is defined by the finite coverings definition - then, **Lemma 4**: $\mathcal{F}_0 \subset \mathcal{M}^\circ$.

   Take $A \in \mathcal{F}_0$. We want to show, for $E \subset \Omega$:
   $$P^\circ(E) \geq P^\circ(A \cap E) + P^\circ(A^c \cap E)$$
   Well, we have for some finite union $E \subset \sup_n A_n$, $\sum P(A_n) \leq P^\circ(E) + \epsilon$. We have:
   $$P^\circ(A \cap E) + P^\circ(A^c \cap E) \leq \sum_n P(A \cap A_n) + \sum_n P(A^c \cap A_n)$$
   Which is by monotonicity of P° and finite subadditivity. Now, with finite additivity of disjoint sets:
   $$= \sum_n P(A_n) \leq P^\circ(E) + \epsilon$$
   Thus, $A \in \mathcal{M}^\circ$.

   Lemma 5 Show that $P^\circ(A) = P(A)$ for $A \in \mathcal{F}_0$. Clearly, $P^\circ(A) \leq P(A)$. Also, if $A \subset \cup A_n$ for finite A_n, then $P(A) \leq \sum_n P(A \cap A_n) \leq \sum_n P(A_n)$, so $P(A)$ is less than all elements in the infimum, and $P^\circ(A) = P(A)$.

3. Define $P_\circ(A) = 1 - P^\circ(A^c)$. Prove that if $E \subset A \in \mathcal{F}_0$, then:
   $$P_\circ(E) = P(A) - P^\circ(A - E)$$
   We have, by finite additivity:
   $$P_\circ(E) = 1 - P^\circ(E^c) = 1 - P^\circ(E^c \cap A) - P^\circ(E^c \cap A^c) = 1 - P^\circ(A - E) - P^\circ(A^c)$$
   As $A^c \subset E^c$. Note, $P^\circ(A^c) = P(A^c)$, and $1 - P(A^c) = P(A)$, and so:
   $$P_\circ(E) = P(A) - P^\circ(A^c)$$

3.7 Finitely Additive Field Extension

Relies on 2.7 and 3.6, both of which we have solutions for. Suppose that H lies outside the field $\mathcal{F}_0$, and let $\mathcal{F}_1$ be the field generated by $\mathcal{F}_0 \cup \{H\}$. By 2.7, this field consists of sets of the form:
$$(H \cap A) \cup (H^c \cap B) \qquad A, B \in \mathcal{F}_0$$
In this problem, we show that a finitely additive probability measure P on $\mathcal{F}_0$ has a finitely additive extension to $\mathcal{F}_1$. Define Q on $\mathcal{F}_1$ by:
$$Q((H \cap A) \cup (H^c \cap B)) = P^\circ(H \cap A) + P_\circ(H^c \cap B)$$

1. Show that the definition is consistent. Ie, that if:

$$(H \cap A_1) \cup (H^c \cap B_1) = (H \cap A_2) \cup (H^c \cap B_2)$$

Then:

$$Q\left[(H \cap A_1) \cup (H^c \cap B_1)\right] = Q\left[(H \cap A_2) \cup (H^c \cap B_2)\right]$$

Well, if they are equal - then we must have $H \cap A_1 = H \cap A_2$, as it is a disjoint union. Similarly, $H^c \cap B_1 = H^c \cap B_2$. Thus, the definition is consistent, as clearly:

$$P^\circ(H \cap A_1) = P^\circ(H \cap A_2) \qquad P_\circ(H^c \cap B_1) = P_\circ(H^c \cap B_2)$$

2. Show that Q agrees with P on $\mathcal{F}_0$. Take $A \in \mathcal{F}_0$. We have $A = (H \cap A) \cup (H^c \cap A)$. Thus:

$$Q(A) = P^\circ(H \cap A) + P_\circ(H^c \cap A)$$

By Part (c) of the previous question, as $H^c \cap A \subset A \in \mathcal{F}_0$ the above equals:

$$= P^\circ(H \cap A) + P(A) - P^\circ(A - (H^c \cap A) = P(A)$$

3. Show that Q is finitely additive on $\mathcal{F}_1$. Show that $Q(H) = p^\circ(H)$.

Take disjoint sets $C_1, \cdots, C_n$ in $\mathcal{F}_1$, which must be of the form $(H \cap A_i) \cup (H^c \cap B_i)$ for $A_i, B_i \in \mathcal{F}_0$. We find:

$$Q(\cup C_i) = Q\left(\bigcup (H \cap A_i) \cup (H^c \cap B_i)\right) = Q\left(\left(H \cap \bigcup A_i\right) \cup \left(H^c \cap \bigcup B_i\right)\right)$$

$$= P^\circ\left(H \cap \bigcup A_i\right) + P_\circ\left(H^c \cap \bigcup B_i\right)$$

Note, each of the A_i and B_i were disjoint, and so they are disjoint unions above. We note that each A_i is an $\mathcal{M}^\circ$ set, and so by Lemma 2 above, the above equals:

$$= \sum_i P^\circ(H \cap A_i) + P_\circ\left(H^c \cap \bigcup B_i\right)$$

For the second term, note that $H^c \cap \bigcup B_i \subset \bigcup B_i \in \mathcal{F}_0$, and so by part (c) in the previous question, the above equals:

$$= \sum_i P^\circ(H \cap A_i) + P\left(\bigcup B_i\right) - P^\circ\left(H \cap \bigcup B_i\right)$$

Breaking down the middle term by finite additivity on $\mathcal{F}_0$, and the final term by Lemma 2 in the previous question, the above equals:

$$= \sum_i P^\circ(H \cap A_i) + P(B_i) - P^\circ(H \cap B_i)$$

Where the second and middle terms now vary with the i in the sum. Finally, we make use of part (c) above again, noting that $H^c \cap B_i \subset B_i$, and $B_i - H^c \cap B_i = H \cap B_i$:

$$= \sum_i P^\circ(H \cap A_i) + P_\circ(H^c \cap B_i) = \sum_i Q(C_i)$$

Now, we show that $Q(H) = P^\circ(H)$. We have that:

$$Q(H) = Q((H \cap \Omega) \cup (H^c \cap \emptyset)) = P^\circ(H \cap \Omega) + P_\circ(H^c \cap \emptyset) = P^\circ(H)$$

So - what we have now is that Q is finitely additive on $\mathcal{F}_1$. It is also nonnegative, clearly, and $Q(\Omega) = P(\Omega) = 1$. It also clearly respects monotonicity, by P° respecting it. Thus, Q is a finitely additive probability measure on $\mathcal{F}_1$. Further, Q agrees with P on $\mathcal{F}_0$, and so it is indeed an extension. Thus, we have proved the main points of the problem.

4. Define Q' by interchanging the roles of P° and $P_\circ$ on the right hand side of the definition of Q, namely:

$$Q'((H \cap A) \cup (H^c \cap B)) = P_\circ(H \cap A) + P^\circ(H^c \cap B)$$

Show that Q' is another finitely additive extension of P to $\mathcal{F}_1$. The same is true of any convex combination of Q'' of Q and Q'. If Q' is a finitely additive extension - so to would be the additions of Q' and Q'', and we maintain being an extension as the fractions of a convex combination add to 1. Show that $Q''(H)$ can take any value between $P_\circ(H)$ and $P^\circ(H)$. Well, that last part is clear, if the previous parts are true.

So, the only thing to prove is that Q' is another finitely additive extension of P to $\mathcal{F}_1$. We will not show that here - it is just an extension of the above problems, and I don't want to do it again.

3.8 Finitely Additive Field Extension to Power Set Field

Use Zorn's lemma to prove a theorem of Tarski: A finitely additive probability measure on a field has a finitely additive extension to the field of all subsets of the space.

Note - this makes use of the previous problem. Also note - the last part of 3.7d also now has more of a use, I think. For the original finitely additive probability measure P on field $\mathcal{F} \subset 2^{\Omega}$, with each subsequent expansion to $H \in 2^{\Omega} - \mathcal{F}$, we can choose any measure value for it between $P_{\circ}(H)$ and $P^{\circ}(H)$. As for why - note that when we expand the field - we expand the number of sets that can finitely cover H. However, H can still be covered by sets in the original $\mathcal{F}$, and so the values between $P_{\circ}(H)$ and $P^{\circ}(H)$ would actually be a subset of the possible values that H can take on (actually, not sure this is correct... we just proved the existence of an extension, not that all extensions agreed on sets outside of $\mathcal{F}$).

Zorn's Lemma, as stated in my Topology book: Let $(M, \leq)$ be a partially ordered set. Suppose every chain $K \subset M$ is bounded. Then M has a maximal element, ie there is an $a \in M$ such that no $x \in M$ satisfies $x > a$.

I was originally thinking about chains of sets - but maybe chains of fields that extend $\mathcal{F}$. First, take any chain of such fields $\mathcal{I}$:

$$\mathcal{F}_i \qquad i \in \mathcal{I} \qquad \Longrightarrow \qquad i, j \in \mathcal{I} \implies \mathcal{F}_i \leq \mathcal{F}_j \text{ or } \mathcal{F}_j \leq \mathcal{F}_i$$

The notation above is because $\mathcal{I}$ doesn't have to be countable. We note that:

$$\bigcup_{i \in \mathcal{I}} \mathcal{F}_i$$

Is a field as well - over any arbitrary unions. Note, this is not the case in general - but the $\mathcal{F}_i$ are *ordered*. First off, it is clearly closed under complements, and contains Ω and $\emptyset$. Now, take $A_1, \cdots, A_n \in \bigcup_{i \in \mathcal{I}} \mathcal{F}_i$. We note that there is an order between each of the $\mathcal{F}_i$ containing the A_i - so, take the largest one, which contains all A_i, and thus their disjoint union. Thus, we have found a maximal element for our chain.

So, our partially ordered set $(M, \leq)$ where M contains fields containing $\mathcal{F}$ that agree with P, must have a maximal element. This is because every chain is bounded. Now, this maximal element must contain each of the extensions of $\mathcal{F}$ to one arbitrary set in 2^{Ω}. So, the maximal element must contain every set in 2^{Ω}, and so it is just the field on 2^{Ω}.

So, the existence of a maximal element implies our theorem: a finitely additive probability measure P on a field $\mathcal{F}$ has a finitely additive extension to the field of all subsets of the space. qed.

3.14 Lebesgue Measure 0 Sets that are not Borel Sets

This problem relies on problems 1.5 and 2.22. Prove the existence of a Lebesgue set of Lebesgue measure 0 that is not a Borel set.

First, recall that the Lebesgue sets are the λ^* measurable sets, where λ starts as the countably additive probability measure on the field $\mathcal{B}_0$, finite disjoint unions of subintervals.

In problem 1.5, we discussed the Cantor set C. And in 2.22, we showed that $\mathcal{B}$ has the cardinality of the continuum. I first note that C is a Lebesgue set with measure 0. In 1.5, we proved that C was trifling - which means for any $\epsilon > 0$, there is a finite sequence of intervals such that:

$$\sum |I_k| < \epsilon \text{ and } C \subset \cup I_k$$

Thus, it should be clear that $\lambda^*(C) = 0$, straight from the infimum definition. Now, we note:

$$\lambda^*(C^c \cap E) \leq \lambda^*(E) \implies \lambda^*(C \cap E) + \lambda^*(C^c \cap E) \leq \lambda^*(E) \implies C \in \mathcal{L}$$

The final part is showing that C is not a Borel set. I realized on the solutions for this one, as again, I'm not too familiar with all the cardinality stuff. We have, as $\mathcal{L}$ is complete:

$$2^C \subseteq \mathcal{L}$$

Now, we note Cantor's theorem - which I just have found out about, which states that the cardinality of a set A is strictly less than the cardinality of its power set Cantor's Theorem. We note that the cardinality of $(0, 1]$ equals the cardinality of the power set of natural numbers:

$$|(0, 1]| = |2^{\mathbb{N}}|$$

As for why this is - we have a bijection. Let each $x \in \mathbb{N}$ be tied to a decimal spot in the binary expansion of a point in $(0, 1]$. Note - we associate a subset of $2^{\mathbb{N}}$ with a point in $(0, 1]$, by if $x \in \mathbb{N}$, whether the corresponding decimal spot is 0 or 1. Thus, $\{1, 2, 4\}$ corresponds to 0.110100... and so on. Thus, bijection should be clear. By 1.22, we have:

$$|\mathcal{B}| = |(0, 1]| = |2^{\mathbb{N}}| < |2^C|$$

And so, $\mathcal{B}$ cannot contain every element of 2^C. However, every element of 2^C has Lebesgue measure 0, and is a Lebesgue measurable set, and so there must exist a Lebesgue measurable set with measure 0, that is not within $\mathcal{B}$. qed.

3.18 All non-zero outer measure sets contain a non (borel) measurable subset

Let H be the nonmeasurable set constructed at the end of the section (the Vitali Set).

1. Show that, if A is a Borel set and $A \subset H$, then $\lambda(A) = 0$ - that is, $\lambda_*(H) = 0$.

   As for why the "that is" follows. In Problem 3.2, we noted that the inner measure of a subset H was equal to the supremum of the inner measure (or outer measure, or defined as just probability, as they are equal) of sets in $\sigma(\mathcal{B}_0) = \mathcal{B}$ that are contained within H. So, by problem 3.2, if each $A \in \mathcal{B}, A \subset H$ satisfies $\lambda(A) = 0$, then that supremum must also be 0.

   Take $A \subset H$. Assume $\lambda(A) > 0$. Note, we still have that each $A \oplus r \in \mathcal{B}$. We thus have by countable additivity:

   $$1 = \lambda((0,1]) \geq \lambda\left(\bigcup A \oplus r\right) = \sum_r \lambda(A \oplus r) = \infty$$

   Which is a contradiction. So, we must have that $\lambda(A) = 0$.

2. Show that, if $\lambda^*(E) > 0$, then E contains a nonmeasurable set. We note that one of the $E \cap (H \circ r)$ must be nonmeasurable. If each is measurable - ie, $E \cap (H \circ r) \in \mathcal{B}$, by the previous part, we have:

   $$\lambda^*(E \cap (H \circ r)) = 0$$

   By countable additivity, this would imply:

   $$\lambda^*(E) = \sum_r \lambda^*(E \cap (H \circ r)) = 0$$

   Which is a contradiction. So, we must have that one of the $E \cap (H \circ r)$ is nonmeasurable. qed.

3.19 Existence of an Intermediate Borel Set $0 < \lambda(A \cap G) < \lambda(G)$ for all nonempty open G

The aim of this problem is the construction of a Borel set A in $(0,1)$ such that:

$$0 < \lambda(A \cap G) < \lambda(G)$$

For every nonempty open set G in $(0,1)$. Now, note both inequalities are needed, as they are both *strict*. So, we can't do something like let $A = (0,1)$.

1. It is shown in Example 3.1 how to construct a Borel set of positive Lebesgue measure that is nowhere dense. Show that every interval contains such a set.

   Take an interval $I \subset (0, 1]$. Note, it must contain a countable amount of rationals. Not more than countable, as the rationals themselves are countable - and countable, because for $a < b$, we can find a rational $a < q_1 < b$, and then we can find rationals $a < q_2 < q_1$ and $q_1 < q_3 < b$, and so on, an infinite number of times.

   Let $q_1, q_2, \cdots$ be the sequence of countable rationals given above. Take some $\epsilon > 0$, but smaller than $|I|/2$. Around each q_i, center an interval I_i of length $\epsilon/2^i$. Define:
$$A = \bigcup_i I_i$$
   By subadditivity, we have $0 < \lambda(A) < \epsilon$. Now, we define our interval $B = I - A$. Note, as $A \subseteq I$, we have that:
$$\lambda(B) = |I| - \lambda(A)$$
   Note, as $\lambda(A) < \epsilon < |I|$, we have that $\lambda(B) > 0$. Note, $\lambda(B)$ can be made arbitrarily close to $|I|$, actually. Now, we note that B is nowhere dense. Take any open interval J - if $J \subseteq I$, note that J must contain some rational, with an interval around it, that is contained in A, and thus is not in B. Thus, every interval $J \subseteq (0, 1]$ has an interval that does not meet B. Thus, I contains a Borel set of positive Lebesgue measure that is nowhere dense.

2. Let $\{I_n\}$ be an enumeration of the open intervals in $(0, 1)$ with rational endpoints. Construct disjoint, nowhere dense Borel sets $A_1, B_1, A_2, B_2, \cdots$ of positive Lebesgue Measure such that $A_n \cup B_n \subset I_n$.

   First, we note for any interval J, we can construct *two* disjoint, nowhere dense sets such that $A \cup B \subseteq J$. First let A be defined as above. Next, note that there is an interval $J' \subseteq J$ that A does not meet. Let B be the construction above within that interval.

   Next, note that the finite union of nowhere dense sets is nowhere dense. Take A, B nowhere dense. Take an interval J - note that it contains a subinterval J' that A does not meet. Now, note that J' contains a subinterval J'' that B does not meet. Thus, J contains a subinterval

J'' that $A \cup B$ does not meet.

Now, we define $A_1, B_1, A_2, B_2, \cdots$ as follows, inductively. For I_1 - follow the steps above. Now, say we have $A_1, B_1, \cdots, A_n, B_n$ disjoint nowhere dense sets satisfying $A_i \cup B_i \subset I_i$. Let $C_n = \cup_i A_i \cup B_i$. Note that C_n is nowhere dense - so I_{n+1} contains a subinterval J_{n+1} that does not meet C_n. Define A_{n+1}, B_{n+1} as above. Thus, we have found A_{n+1}, B_{n+1} nowhere dense sets disjoint from the previous $2n$ sets, that also have positive Lebesgue measure, and they satisfy:

$$A_{n+1} \cup B_{n+1} \subset I_{n+1}$$

3. Let $A = \cup_k A_k$. A nonempty open G in $(0,1)$ contains some I_n. Show that:

$$0 < \lambda(A_n) \leq \lambda(A \cap G) < \lambda(A \cap G) + \lambda(B_n) \leq \lambda(G)$$

First, we note that yes, a nonempty open G in $(0,1)$ contains one of the I_n - a point in G contains an open interval, which contains an I_n. We have:

$$0 < \lambda(A_n)$$

As A_n has positive Lebesgue measure. As $A_n \subseteq A$, and $A_n \subseteq I_n \subset G$, we have $A_n \subseteq A \cap G$, and so by subadditivity, we have:

$$\lambda(A_n) \leq \lambda(A \cap G)$$

As $\lambda(B_n) > 0$, and B_n disjoint from A gives countable subadditivity, and $B_n \subseteq I_n \subset G$, we have:

$$< \lambda(A \cap G) + \lambda(B_n) = \lambda((A \cap G) \cup B_n) \leq \lambda(G)$$

Note, the final point follows, because if $x \in B_n$, then $x \in G$ and if $x \in A \cap G$, then $x \in G$. Thus, all together, we have:

$$0 < \lambda(A \cap G) < \lambda(G)$$

And so, for every nonempty open set G in $(0,1)$, we have a borel set A such that $0 < \lambda(A \cap G) < \lambda(G)$. qed.

3.20 No Intermediate Borel Set on All Intervals $a\lambda(I) \leq \lambda(A \cap I) \leq b\lambda(I)$

There is no Borel set A in $(0,1)$ such that $a\lambda(I) \leq \lambda(A \cap I) \leq b\lambda(I)$ for every interval I in $(0,1)$, where $0 < a \leq b < 1$. We will prove this via the following stronger results:

1. If $\lambda(A \cap I) \leq b\lambda(I)$ for all I and if $b < 1$, then $\lambda(A) = 0$.

   Following the hint, the first step is too choose an open G such that $A \subset G \subset (0, 1)$ *and*:
   $$\lambda(G) < b^{-1}\lambda(A)$$
   Note, as $b < 1$, $b^{-1} > 1$. By the definition of the borel outer measure:
   $$\lambda(A) = \lambda^*(A) = \inf_n \sum_n P(A_n) \quad \text{where} \quad A \subseteq \bigcup_n A_n \quad \text{and} \quad A_n \in \mathcal{B}_0$$
   As $A_n \in \mathcal{B}_0$, A_n is in the field of disjoint unions of intervals, which is open sets. As $\lambda(A) < b^{-1}\lambda(A)$, we can find a countable union of open sets covering A (which is still open), which also satisfies:
   $$\lambda(\bigcup A_n) \leq \sum_n P(A_n) < b^{-1}\lambda(A)$$
   We let $G = \bigcup_n A_n$ be our open set satisfying the necessary properties. Now, we recall that every open set in $\mathbb{R}$ can be described as a disjoint countable union of intervals I (for a quick proof - each connected portion of G must be an interval, and each of these disjoint intervals contains a rational number, so they must be countable).

   Thus, we have $G = \bigcup_n I_n$ for countable disjoint intervals in $(0, 1)$. We have by countable additivity, our assumption on A, and as $A \cap I_n$ are disjoint sets whose union equals A:
   $$\lambda(G) < b^{-1}\lambda(A) \implies \sum_n \lambda(I_n) < b^{-1}\lambda(A) \implies b^{-1}\sum_n \lambda(A \cap I_n) < b^{-1}\lambda(A)$$
   $$\implies b^{-1}\lambda(A) < b^{-1}\lambda(A)$$
   The above is a contradiction if $\lambda(A) > 0$. Thus, we must have $\lambda(A) = 0$.

2. If $a\lambda(I) \leq \lambda(A \cap I)$ for all I, and if $a > 0$, then $\lambda(A) = 1$.

   This follows just by taking complements of the above result. We note that $\lambda(A) = 1$ if and only if $\lambda(A^c) = 0$. We note that if the above condition applies to $\lambda(A^c)$, then $\lambda(A^c) = 0$. We examine:
   $$\lambda(A^c \cap I) = \lambda(I) - \lambda(A \cap I)$$
   The above follows from set theory, and $A^c \cap I \subseteq I$. By the assumption, this implies:
   $$\implies \lambda(A^c \cap I) \leq \lambda(I) - a\lambda(I) = (1 - a)\lambda(I)$$

And so, give $a > 0$, we have $b = 1 - a < 1$. The above tells us for all I and $b < 1$, we have:

$$\lambda(A^c \cap I) \leq b\lambda(I)$$

Thus, by part a, we have $\lambda(A^c) = 0 \implies \lambda(A) = 1$.

3. Now, as for why the statement is impossible. If we had an A in $(0, 1)$ such that $a\lambda(I) \leq \lambda(A \cap I) \leq b\lambda(I)$ for every open interval I with $0 < a \leq b < 1$, then the above would tell us $\lambda(A) = 0$ and $\lambda(A) = 1$. This is a contradiction. qed.

Section 4 - Denumerable Probabilities

Notes

Just like the starting note - there is a two way street between measure theory and extramathematical probabilistic ideas (ie, probability ideas stemming from outside math). Probability ideas can be made clear and systematic with measure theory, and ideas like independence (which really stem from outside of measure theory) can help illuminate problems of purely mathematical interest. This reciprocal exchange is why measure-theoretic probability is so interesting.

In this section - we are concerned with infinite sequences of events in a probability space - Borel's first paper on the subject was called "Denumerable Probabilities", hence the chapter name. Our examples will be centered in the *unit interval* - ie, the probability space $((0, 1], \mathcal{B}, \lambda)$. However, the Theorems will apply to *all* probability spaces.

General Formulas

Conditional Probability If $P(A) > 0$, the *conditional probability* of B given A is defined in the usual way as:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

We have chain rule formulas:

$$P(A \cap B) = P(A)P(B|A)$$

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

And so on, all of which are clear from the definition. We also have if $A_1, A_2, \cdots$ partition Ω, then:

$$P(B) = \sum_n P(A_n \cap B) = \sum_n P(A_n) P(B|A_n)$$

Note: for a fixed A, the function $P(B|A)$ defines a probability measure as B varies over $\mathcal{F}$. Note that $P(\Omega|A) = 1$, and the rest of the probability measure properties are easy to prove.

If $P(A_n) = 0$, then by subadditivity $P(\cup_n A_n) = 0$. If $P(A_n) = 1$, then $\cap A_n$ has complement $\cup A_n^c$ of probability 0. This gives two facts that are used over and over again:

If $A_1, A_2, \cdots$ are sets of probability 0, so is $\cup_n A_n$. If $A_1, A_2, \cdots$ are sets of probability 1, so is $\cap_n A_n$.

Limit Sets

Lim Inf and Lim Sup For a sequence $A_1, A_2, \cdots$ of sets, define a set:

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \qquad \liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

Note - $\omega \in \limsup_n A_n$ implies that ω is in the A_n infinitely often. $\omega \in \liminf A_n$ implies that ω is in all but finitely many A_n. Essentially - these sets *express* how an ω can appear within the sets A_n as we go to infinity. They capture two levels of infinity - infinitely often but not all, and eventually in all. It should be clear that the second implies the first - and so it is strictly smaller. Thus, we give the second the name of lim inf (infimum being the smaller), and lim sup for the former. Essentially - what are the biggest and smallest sets containing an ω that appears in the limits of the A_n. If the sets are equal, we write:

$$\lim_n A_n = \liminf_n A_n = \limsup_n A_n$$

As one direction always holds, to prove that $A_n \to A$ involves checking:

$$\limsup_n A_n \subset A \subset \liminf_n A_n$$

Example 1 Let $l_n(\omega)$ be the length of the run of 0's starting at $d_n(\omega)$. If $l_n(\omega) = k$, then $d_n(\omega) = \cdots = d_{n+k-1}(\omega) = 0$, and $d_{n+k}(\omega) = 1$. If $l_n(\omega) = 0$, $d_n(\omega) = 1$. We can compute probabilities for this - it is clear that:

$$\mathbb{P}[\omega : l_n(\omega) = k] = \frac{1}{2}^{k+1}$$

As we have $k+1$ spots to choose a specific binary choice. Also, as it is a disjoint union of intervals - the set lies in $\mathcal{B}$. Therefore, we find:

$$\mathbb{P}[\omega : l_n(\omega) \geq r] = \sum_{k \geq r} \frac{1}{2}^{k+1} = 2^{-r}$$

Note, the sum comes from disjoint sets. If A_n is the event above, then $\limsup_n A_n$ is the set of ω such that $l_n(\omega) \geq r$ for *infinitely many* n. Thus, we can regard n as a time index, and we have $l_n(\omega) \geq r$ infinitely often.

When n has the role of time, $\limsup_n A_n$ is frequently written as:

$$\limsup_n A_n = [A_n \text{i.o.}]$$

Theorem 4.1 - Ordering of Limit Probabilities For each sequence $\{A_n\}$:

$$\mathbb{P}\left(\liminf_n A_n\right) \leq \liminf_n \mathbb{P}(A_n) \leq \limsup_n \mathbb{P}(A_n) \leq \mathbb{P}\left(\limsup_n A_n\right)$$

And if $A_n \to A$, then $P(A_n) \to P(A)$.

Proof: Recall the definitions of $\liminf_n$ and $\limsup_n$ on sequences - they are the limits of the infimum and supremum of the sets $\{x_n : n \geq k\}$. Note, if we have the ordering is true, $A_n \to A$ implies $\liminf_n A_n = \limsup_n A_n$, which tells us:

$$\mathbb{P}\left(\liminf_n A_n\right) = \liminf_n \mathbb{P}(A_n) = \limsup_n \mathbb{P}(A_n) = \mathbb{P}\left(\limsup_n A_n\right)$$

Recall, that if $\liminf_n \mathbb{P}(A_n) = \limsup_n \mathbb{P}(A_n)$, then they both equal $\lim P(A_n)$, which tells us:

$$\lim P(A_n) = \mathbb{P}\left(\liminf_n A_n\right) \implies P(A_n) \to P(A)$$

So, all that is left to prove is the ordering. Define $B_n = \bigcap_{k=n}^{\infty} A_k$ and $C_n = \bigcup_{k=n}^{\infty} A_k$. Note $B_n \uparrow \liminf_n A_n$ and $C_n \downarrow \limsup_n A_n$, so by Continuity from below and above, we have:

$$P(A_n) \geq P(B_n) \to \mathbb{P}\left(\liminf_n A_n\right)$$

$$P(A_n) \leq P(C_n) \to \mathbb{P}\left(\limsup_n A_n\right)$$

Given that $\inf_{k \geq n} P(A_k) \geq P(B_n)$ (by monotonicity), and $\sup_{k \geq n} P(A_k) \leq P(C_n)$ (also by monotonicity), and $\liminf \leq \limsup$, we clearly have:

$$\mathbb{P}\left(\liminf_n A_n\right) \leq \liminf_n \mathbb{P}(A_n) \leq \limsup_n \mathbb{P}(A_n) \leq \mathbb{P}\left(\limsup_n A_n\right)$$

Thus, we have an ordering of the limit probabilities. qed.

Example 4.2 Define $l_n(\omega)$ as above, and let $A_n = [\omega : l_n(\omega) \geq r]$ for fixed r. By Example 4.1 and Theorem 4-1, we have $\mathbb{P}[\omega : l_n(\omega) \geq r \text{i.o.}] \geq 2^{-r}$.

Independent Events

Events A and B are *independent* if $P(A \cap B) = P(A)P(B)$. Note - this is a *notation we have put on a general idea.* In life, we see events that we call *independent* from each other, meaning that one happening doesn't have any effect on the other happening. In life - we have four cases - event A happens and B doesn't, vice versa, A and B both happen, or neither happens. We have probabilities for A and B - $P(A)$ and $P(B)$. The world is split into cases for event A - there are n (equally probable) possible outcomes, in which A represents m of those outcomes. Similarly for event B - the world is split into q (equally probable) possible outcomes, in which B represents p of those outcomes. In which case:

$$P(A) = m/n \qquad P(B) = p/q$$

We can split the world into nq cases, by taking the Cartesian product of the n cases that split the world when looking at it in terms of A, and the q cases that split the world when looking at it in terms of B. If A and B have nothing to do with each other - ie, they are independent - then logically, the splits of the world were on *different axes*, and the m events that indicate A happened are Cartesian multiplied with the p events that indicate B happened, and so the probability that both A and B happened is:

$$\frac{m \times p}{n \times q} = P(A)P(B)$$

We take this intuition, and derive our probability theory property of *Independence.*

Independence is equivalent to requiring $P(B|A) = P(B)$, or $P(A|B) = P(A)$. Note, $B|A$ and $A|B$ have a similar intuition as above. More generally, a finite collection $A_1, \cdots, A_n$ of events is independent if:

$$P(A_{k_1} \cap \cdots \cap A_{k_j}) = P(A_{k_1}) \cdots P(A_{k_j})$$

For $2 \leq j \leq n$ and $1 \leq k_1 < \cdots < k_j \leq n$. An *infinite* (perhaps uncountable) collection of events is defined to be independent if each of its finite sub-collections is.

Note the number of constraints required for independence on a set of n events is:

$$\sum_{j=2}^{n} \binom{n}{j} = 2^n - 1 - n$$

Example 4.5 We have that the events $H_n = [\omega : d_n(\omega) = 0]$, $n = 1, 2, \cdots$ are clearly independent, any finite intersection of which having both sides of the probability equation equaling $2^{-j} = 2^{-1} \times \cdots \times 2^{-1}$. It should be intuitive, then, that any events that can be described in terms of disjoint sets of $H_n, n \in \{i_1, i_2, \cdots\}$ should also be independent (note, not that the proof is intuitive, just the assumption). For example, A and B depending on even and odd times, respectively, should be independent. The set-theoretic form of the statement is that for:

$$A \in \sigma(H_2, H_4, \cdots) \qquad\qquad B \in \sigma(H_1, H_2, \cdots)$$

It ought to be possible to deduce the independence of A and B.

Independence of Classes Defined *classes* $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_n$ in the basic σ field $\mathcal{F}$ to be independent if for each choice of A_i from $\mathcal{A}_i$, the events $A_1, \cdots, A_n$ are independent. This is equivalent to the following holding:

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)P(B_2) \cdots P(B_n)$$

For $B_i \in \mathcal{A}_i$ or $B_i = \Omega$ (in which case, B_i is removed from the intersection, and $P(B_i) = 1$ is removed from the multiplication).

Theorem 4.2 - Independence of Generated σ field If $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent and each $\mathcal{A}_i$ is a π system, then $\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$ are independent as well.
Proof: Let $\mathcal{B}_i$ be the class $\mathcal{A}_i$ augmented by Ω - note, it is still a π system, and we must have $\sigma(\mathcal{A}_i) = \sigma(\mathcal{B}_i)$. By the independence hypothesis - we have that for $B_i \in \mathcal{B}_i$:

$$P(B_1 \cap B_2 \cap \cdots \cap B_n) = P(B_1)P(B_2)\cdots P(B_n)$$

Ie, the $\mathcal{B}_i$ are independent as well. Also, note that the above is a criteria for independence. For fixed $B_2, \cdots, B_n$ in $\mathcal{B}_2, \cdots, \mathcal{B}_n$, let $\mathcal{L}$ be the class of $\mathcal{F}$ sets B_1 for which the above holds. Note - $\mathcal{L}$ is a λ system, (contains Ω, clearly contains complements, and disjoint unions by countable additivity of P) that contains $\mathcal{B}_1$, and by the π-λ theorem, it also contains $\sigma(\mathcal{A}_1)$. Thus, we have:

$$\sigma(\mathcal{A}_1), \mathcal{B}_2, \cdots, \mathcal{B}_n$$

Are independent classes. Now, for fixed $B_1, B_3, \cdots, B_n$ in $\sigma(\mathcal{A}_1), \mathcal{B}_3, \cdots, \mathcal{B}_n$, let $\mathcal{L}$ be the class of $\mathcal{F}$ sets for which the independence condition holds. Again, it is a λ system containing $\mathcal{B}_2$, and so we have independence of:

$$\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \cdots, \mathcal{B}_n$$

Continue to find independence of $\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$. qed.

Independence of Infinite Classes $[\mathcal{A}_\theta : \theta \in \Theta]$ are independent if each collection $[A_\theta : \theta \in \Theta]$ for $A_\theta \in \mathcal{A}_\theta$ is. This is equivalent to the independence of each finite sub-collection $\mathcal{A}_{\theta_1}, \cdots, \mathcal{A}_{\theta_n}$. If each finite subcollection is independent - any finite set of A_θ is independent, which means that any finite set in the collection $[A_\theta : \theta \in \Theta]$ is independent, which means that $[\mathcal{A}_\theta : \theta \in \Theta]$ is independent. Theorem 4-2 has an immediate consequence:

Corollary 1 - Independence of Generated σ fields (infinite version) If $\mathcal{A}_\theta, \theta \in \Theta$ are independent and each $\mathcal{A}_\theta$ is a π system, then $\sigma(\mathcal{A}_\theta), \theta \in \Theta$, are independent.

Proof: As $[\mathcal{A}_\theta, \theta \in \Theta]$ are independent, by the above iff, each finite subcollection $[\mathcal{A}_\theta, \theta \in I]$ where $I \subseteq \Theta$, $|I| = n$ is independent. As each $\mathcal{A}_\theta$ is a π system, Theorem 4.2 tells us that $[\sigma(\mathcal{A}_\theta), \theta \in I]$ where $I \subseteq \Theta$, $|I| = n$ is independent. Note, by the iff, this is equivalent to $[\sigma(\mathcal{A}_\theta), \theta \in \Theta]$ is independent. qed.

Corollary 2 Suppose that the array:

$$\begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

of events is independent; here, each row is a finite or infinite sequence, and there are finitely or infinitely many rows. If $\mathcal{F}_i$ is the σ field generated by the *ith* row, then $\mathcal{F}_1, \mathcal{F}_2, \cdots$ are independent.

Proof: If $\mathcal{A}_i$ is the class of all finite intersections of elements of the *ith* row in the matrix above, then $\mathcal{A}_i$ is a π system and $\sigma(\mathcal{A}_i) = \mathcal{F}_i$. Let I be a finite collection of indices (integers), and for each i in I let J_i be a finite collection of indices. Consider for $i \in I$, the element:

$$C_i = \bigcap_{j \in J_i} A_{ij} \in \mathcal{A}_i$$

Since every finite subcollection of the array is independent:

$$\mathbb{P}\left[\bigcap_i C_i\right] = \mathbb{P}\left[\bigcap_i \bigcap_j A_{ij}\right] = \prod_i \prod_j \mathbb{P}(A_{ij}) = \prod_i \mathbb{P}\left[\bigcap_j A_{ij}\right] = \prod_i \mathbb{P}\left[C_i\right]$$

It follows that the classes $\mathcal{A}_1, \mathcal{A}_2, \cdots$ are independent, as every finite subcollection of classes is independent. Thus, Corollary 1 applies, and so the $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2), \cdots$ are independent. qed.

Note, by the above corollary, the events:

$$\begin{pmatrix} H_2 & H_4 & H_6 & \cdots \\ H_1 & H_3 & h_5 & \cdots \end{pmatrix}$$

Are independent, and the σ algebras generated by the rows are independent as well. Note - this is why Corollary 2 is *needed.* Because the previous two theorems assumed π systems - with corollary 2, we can generate π systems on the rows, and then apply the theorems. Note, Corollary 2 can be used to prove that if $A_1, \cdots, A_n$ are independent, so are there complements, as $A_1^c \in \sigma(A_1), \cdots, A_n^c \in \sigma(A_n)$.

Example 4.7 If $\mathcal{A} = \{A_1, A_2, \cdots\}$ is a finite or countable partition of Ω, and $P(B|A_i) = p$ for each A_i of positive probability, then $P(B) = p$ and B is independent of $\mathcal{A}$. Note, B is independent of a class if for each $A \in \mathcal{A}$ with

$P(A) > 0$, then $P(B|A) = P(B)$ (ie, B is pairwise independent with each element of positive probability).

Note, this implies that B is independent of $\sigma(\mathcal{A})$. First, note what $\sigma(\mathcal{A})$ is - it is just all possible unions of elements of $\mathcal{A}$. Note, this is a σ algebra - it contains complements (which is just unions of all elements not in the union), and countable unions. Note that, also:

$$P(B| \cup_i A_i) = \frac{P(B \cap (\cup_i A_i))}{P(\cup_i A_i)} = \frac{\sum P(B \cap A_i)}{\sum P(A_i)} = \frac{b \sum P(A_i)}{\sum P(A_i)} = b = P(B)$$

And so yes, B is independent of $\sigma(\mathcal{A})$.

Subfields

Probability theory differentiates itself from measure theory, in that we are often playing around with lots of σ algebras. Note, in probability, σ fields in $\mathcal{F}$ - ie, sub σ fields, play an important role.

A subclass $\mathcal{A}$ of $\mathcal{F}$ corresponds heuristically to *partial information.* Note - these are not math terms, but just in the real world, partial information. Imagine for a point $\omega \in \Omega$ that we draw, according to probabilities P, where $\omega \in A$ with probability $P(A)$. Imagine an observer, who doesn't know what ω we drew, but does know whether $\omega \in A$ for $A \in \mathcal{A}$ - ie, the value of $I_A(\omega)$ for $A \in \mathcal{A}$. We can *identify* this partial information with the class $\mathcal{A}$ itself. This will help draw connections between the measure theory concepts we are going over, and the real world probabilities we are trying to examine.

Interpreting Theorem 4.2 We have that Theorem 4.2 can be understood in this informal "information" notion. B is independent from the class $\mathcal{A}$ is $P(B|A) = P(B)$ for all sets $A \in \mathcal{A}$ for which $P(A) > 0$. Note, this implies that the classes $\{B\}$ and $\mathcal{A}$ are independent. Thus, if B is independent of $\mathcal{A}$ - even if the observer knows $\mathcal{A}$, then he still has no information about B, as B still occurs with probability $P(B)$. Even if we know whether A happened or not, for every $A \in \mathcal{A}$ - as we don't know the underlying ω, we still can't make conclusions about B.

The point of Theorem 4.2 is - if $\mathcal{A}$ is a π system (so knowing $\mathcal{A}$ means knowing the finite intersection of every $A \in \mathcal{A}$) - then even if the observer is given information about $\sigma(\mathcal{A})$ (generally a strictly larger set) - the observer still knows nothing about B!

Partial Information as Partitions Say that ω and ω' are $\mathcal{A}$ equivalent if, for every $A \in \mathcal{A}$, ω and ω' either both lie in A, or both lie in A^c - ie:

$$I_A(\omega) = I_A(\omega') \qquad A \in \mathcal{A}$$

This is an equivalence relation for every $\omega \in \Omega$ - and thus, the relation partitions Ω into equivalence classes. Call this the $\mathcal{A}$ partition.

Example 4.8 If ω and ω' are $\sigma(\mathcal{A})$ equivalent, then they are clearly $\mathcal{A}$ equivalent. For fixed ω and ω', the class of A such that $I_A(\omega) = I_A(\omega')$ is a σ field (clearly closed under complements and countable unions, and contains the empty set and Ω). If ω and ω' are $\mathcal{A}$ equivalent, then this σ field contains $\mathcal{A}$ and hence $\sigma(\mathcal{A})$, so we know that ω and ω' are also $\sigma(\mathcal{A})$ equivalent.

Thus, $\mathcal{A}$ equivalence and $\sigma(\mathcal{A})$ equivalence are the same thing, and the $\mathcal{A}$ partition coincides with the $\sigma(\mathcal{A})$ partition. And so - an observer with the information in $\sigma(\mathcal{A})$ knows, not the point ω is drawn, but the equivalence class containing it. I think - if B was independent of $\mathcal{A}$ π system - then maybe we could conclude that B contains no elements from any of the equivalence classes? In which case, we could say that knowing $\sigma(\mathcal{A})$ does nothing for knowing B? Perhaps. Note - this "information" understanding can break down.

Example 4.10 - Breakdown of Information Interpretation In the unit interval $(\Omega, \mathcal{F}, P)$, let $\mathcal{G}$ be the σ field consisting of the countable and cocountable sets. Since $P(G)$ is 0 or 1 for each G in $\mathcal{G}$ - each set H in $\mathcal{F}$ is independent of $\mathcal{G}$. But in this case, the $\mathcal{G}$ partition consists of the singletons, so the information in $\mathcal{G}$ tells the observer exactly which ω in Ω has been drawn. So, we have two conclusions:

1. The σ field $\mathcal{G}$ contains no information about H - in the sense that H and $\mathcal{G}$ are independent.

2. The σ field $\mathcal{G}$ contains *all* the information about H - in the sense that it tells the observer exactly which ω was drawn.

Note - this example emphasizes that the information interpretation is just a *heuristic*. But what has broken down here? The book says that it is the unnatural structure of $\mathcal{G}$ - rather than a deficiency in the notion of independence. However, I like that it emphasizes that the heuristic *is not* perfect - as it is always confusing to me when I try and look at it as well.

The Borel-Cantelli Lemmas

Theorem 4.3 First Borel-Cantelli Lemma If $\sum_n P(A_n)$ converges, then $P(\limsup_n A_n) = 0$. **Proof:** Recall:

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \subseteq \bigcup_{k=m}^{\infty} A_k$$

Thus, we have:

$$P\left(\limsup_n A_n\right) \leq P\left(\bigcup_{k=m}^{\infty} A_n\right) \leq \sum_{k=m}^{\infty} P(A_k)$$

Note that as $m \to \infty$, the right hand side goes to 0, which implies:

$$P\left(\limsup_n A_n\right) = 0$$

Thus, we have the theorem. qed.

There are some interesting examples in the text involving $l_n(\omega)$, the run of zero length starting at $d_n(\omega)$, but I won't go over them here. In a way, the theorem makes sense intuitively - as if we have the final sequence of events has probability essentially zero - even their total union will have probability zero, much less the ω that appear infinitely often.

Theorem 4.4 Second Borel-Cantelli Lemma If $\{A_n\}$ is an independent sequence of events and $\sum_n P(A_n)$ diverges, then $P(\limsup_n A_n) = 1$.

Note, this is equivalent to proving:

$$P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^c\right) = 0 \iff P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c\right) = 0$$

Note, the RHS above is implied if we have for each n:

$$P\left(\bigcap_{k=n}^{\infty} A_k^c\right) = 0$$

Note, for a finite number of the intersection j, we have by independence:

$$P\left(\bigcap_{k=n}^{n+j} A_k^c\right) = \prod_{k=n}^{n+j} 1 - P\left(A_k\right) \leq \exp\left[-\sum_{k=n}^{n+j} P(A_k)\right]$$

As the sum diverges, the exponential expression goes to 0, and hence:

$$P\left(\bigcap_{k=n}^{\infty} A_k^c\right) \leq \lim_{j\to\infty} \exp\left[-\sum_{k=n}^{n+j} P(A_k)\right] = 0$$

Thus, we can conclude that $P(\limsup_n A_n) = 1$. qed.

Again, this makes intuitive sense. We have each of the events is independent - so, with our information notion, the result of one event won't impact the result of the other. Given that the sum diverges - an infinite number of the events has nonzero probability, that is fairly large, and so independently, an infinite subset of them should happen.

Again, there are some good examples for the second Borel-Cantelli Lemma. One of them has to do with Diophantine approximation again, ie, approximating numbers by a rational. However, I think you have to dive deep into Diophantine numbers to be able to compare again with previous examples (discussing for an ω, how many rational numbers are actually close to it that satisfy certain properties).

The Zero-One Law

Definition: Tail σ Field For a sequence $A_1, A_2, \cdots$ of events in a probability space $(\Omega, \mathcal{F}, P)$ consider the σ fields for $\sigma(A_n, A_{n+1}, \cdots)$ and their intersection:

$$\mathcal{I} = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \cdots)$$

Recall, the intersection of a σ fields is always a σ field. $\mathcal{I}$ is called the tail sigma field associated with the sequence $\{A_n\}$. Its elements are called *tail events.*

Example 4.18 Lipsup and Liminf are Tail Events Take a sequence $\{A_n\}$. Recall, we have:

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n$$

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n$$

Note, however, that:

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_n = \bigcap_{n=m}^{\infty} \bigcup_{k=n}^{\infty} A_n$$

Take $x \in LHS$. Note, it is in each of the unions. And so, it clearly is in each of the unions on the RHS. Now, take $x \in RHS$. Note, it is in each of the unions. This implies it is in each of the unions on the LHS - as the unions on the LHS contain the unions on the RHS. Thus, they are equal. Similarly, we have:

$$\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n = \bigcup_{n=m}^{\infty} \bigcap_{k=n}^{\infty} A_n$$

Now, note that:

$$\bigcap_{n=m}^{\infty} \bigcup_{k=n}^{\infty} A_n \in \sigma(A_m, A_{m+1}, \cdots) \implies \limsup_n A_n \in \sigma(A_m, A_{m+1}, \cdots)$$

As the above is true for each m, we have that $\limsup_n A_n$ is a tail event. Similarly, we have $\liminf_n A_n$ is a tail event. qed.

Example 4.19 Run Lengths Being a specific length are tail events Let $l_n(\omega)$ be the run length, as before (ie, $l_n(\omega)$ is the number of zeros that appear after $d_n(\omega)$. If $d_n(\omega) = 1$, $l_n(\omega) = 0$). Let $H_n = [\omega : d_n(\omega) = 0]$. For each n_0, we have:

$$[\omega : l_n(\omega) \geq r_n \text{i.o.}] = \bigcap_{n \geq n_0} \bigcup_{k \geq n} [w : l_n(\omega) \geq r_k] = \bigcap_{n \geq n_0} \bigcup_{k \geq n} H_k \cap H_{k+1} \cap \cdots \cap H_{k+r_k-1}$$

Thus, $[\omega : l_n(\omega) \geq r_n \text{i.o.}]$ is a tail event for the sequence $\{H_n\}$.

Theorem 4.5 - Kolmogorov's Zero-One Law If $A_1, A_2, \cdots$ is an independent sequence of events, then for each event A in the tail σ field, $P(A)$ is 0 or 1.

Proof: By corollary 2 to Theorem 4.2 (the independent array implies independent row sigma algebras), we have $\sigma(A_1), \cdots, \sigma(A_{n-1}), \sigma(A_n, A_{n+1}, \cdots)$ are independent. If $A \in \mathcal{I}$ (the tail σ algebra), then $A \in \sigma(A_n, A_{n+1}, \cdots)$. Therefore, $A_1, \cdots, A_{n-1}, A$ are independent (all pick one subsets are independent, and subset is independent). As independence of a collection of events is defined by independence of each finite subcollection, the sequence:

$$A, A_1, A_2, \cdots$$

Is independent (note, any subset can be reordered, for which we have independence). By a second application of Corollary 2, we have $\sigma(A)$ and $\sigma(A_1, A_2, \cdots)$ are all independent. But, $A \in \mathcal{I} \subset \sigma(A_1, A_2, \cdots)$. Thus, as $A \in \sigma(A)$ and $A \in \sigma(A_1, A_2, \cdots)$, we have that A is independent of itself. Thus:

$$P(A \cap A) = P(A)P(A) \implies P(A) = P(A)^2 \implies P(A) = 0 \text{ or } 1$$

Thus, the Zero-One law follows. qed.

I'm just trying to make sure the above is valid. I think it must be. Any probability 0 or 1 set is independent of itself - so, they are always allowed to be within two independent classes, as they would still satisfy the requirements.

Example 4.20 - Borel Cantelli and Zero One Law By Example 4.18. we have $P(\limsup_n A_n)$ is 0 or 1 if the A_n are independent. The Borel-Cantelli Lemmas in this case go further, and give a specific criterion in terms of the convergence or divergence of $\sum P(A_n)$. Namely, if A_n are independent, we have that $\sum_n P(A_n)$ diverges or converges, based on whether $P(\limsup_n A_n) = 1$ or 0 respectively.

Example 4.21 - Run Length At time n longer than r_n infinitely often By Komogorov's Theorem, and example 4.19, we have that:

$$[\omega : l_n(\omega) \geq r_n \text{i.o.}]$$

Is either a 0 or 1. We call the sequence $\{r_n\}$ an *outer boundary* or an *inner boundary* according to whether the probability is 0 or 1, respectively.

In Example 4.11, we used the first Borel-Cantelli lemma to show that $\{r_n\}$ is an *outer boundary* if $\sum 2^{-r_n} < \infty$.

In Example 4.15, we used the second Borel-Cantelli lemma to show that $\{r_n\}$ is an *inner boundary* if $\sum 2^{-r_n} r_n^{-1} = \infty$.

By these criteria, $r_n = \theta \log_2 n$ gives an outer boundary if $\theta > 1$ and an inner boundary if $\theta \leq 1$. This can be noted that $2^{-\theta \log_2 n} = 1/n^\theta$, which converges for $\theta > 1$. Further, $1/n^\theta (\theta \log_2 n)^{-1}$ clearly diverges for $\theta \leq 1$.

Now, we consider the sequence $r_n = \log_2 n + \theta \log_2 \log_2 n$. Here, we have:

$$\sum 2^{-r_n} = \sum \frac{1}{n(\log_2 n)^\theta}$$

This converges for $\theta > 1$, which gives an outer boundary. Similarly:

$$\sum 2^{-r_n} r_n^{-1} \sim \sum \frac{1}{n(\log_2 n)^{1+\theta}}$$

Which diverges if $\theta \leq 0$, which gives an inner boundary. But, we have the range of $0 < \theta \leq 1$ unresolved, although every sequence is an inner or an outer boundary. We will revisit this in Chapter 6.

Problems

4.1 Equating Set $\liminf$ and $\limsup$ definitions with the sequence definitions

The limsup and liminf of a numerical sequence $\{x_n\}$ can be defined as the supremum and infimum of the set of limit points - the set of limits of convergent subsequences. This is the same as defining:

$$\limsup_n x_n = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k \qquad \liminf_n x_n = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k$$

Where $\bigvee$ is the maximum between points, and $\bigwedge$ is the minimum. First note that our symbolic definition above is the normal definition for $\limsup$ and $\liminf$. $\limsup$ is the limit of the supremum s of tail sequences. Over an infinite set, I take $\bigvee$ to be the supremum, and $\bigwedge$ as the infimum. And so, our above definition is the infimum of the supremums of tail sequences. Note, the infimum is equal to the limit - as the sequence of supremums is decreasing. We can note the same for our $\liminf$ symbolic definition - it matches the standard one.

As for why the $\limsup$ definition is the same thing as the supremum of limits of converging sets. Note - the limit of a converging set must be smaller than the supremum of every tail sequence. This is because, every converging subsequence is contained within every tail subsequence. Thus, we have:

$$\limsup_n x_n \geq \text{ supremum of convergent subsequence limits}$$

We now note, the infimum of the supremum of every tail sequence is smaller than or equal to the limit of some converging subsequence. Say the supremum isn't achieved - thus, by definition, it is approached, and the supremum of the tail sequence equals the limit of some converging subsequence. Now, if the limit is achieved for every tail sequence - then, the achieved elements form a convergent subsequence on their own, that is decreasing. In both cases, we have:

$$\limsup_n x_n \leq \text{ supremum of convergent subsequence limits}$$

A similar argument can be made for the infimum.

Now, we get to the actual question. We prove:

$$I_{\limsup_n A_n} = \limsup_n I_{A_n} \qquad I_{\liminf_n A_n} = \liminf_n I_{A_n}$$

We prove it for the $\liminf$, and $\limsup$ will be similar. Say $I_{\liminf_n A_n}(\omega) = 1$. Then, we have that ω appears in every set with index $k \geq n$ for some finite n. Note, $\bigwedge_{k=n}^{\infty} I_{A_k}(\omega) = 1$, as it is the minimum over the set of 1. Thus, the maximum is 1 as well, and we have:

$$I_{\liminf_n A_n}(\omega) = 1 = \liminf_n I_{A_n}(\omega)$$

Now, assume that $I_{\liminf_n A_n}(\omega) = 0$. Then, that means ω is not present in some A_k for $k > n$, for every n. This implies that $\bigwedge_{k=n}^{\infty} I_{A_k}(\omega) = 0$, and we have:

$$I_{\liminf_n A_n}(\omega) = 0 = \liminf_n I_{A_n}(\omega)$$

Thus, both sides are equal. Similar arguments can be made for the $\limsup$.

Finally, prove that $\lim_n A_n$ exists in the sense of $\liminf_n A_n = \limsup_n A_n$ if and only if $\lim_n I_{A_n}(\omega)$ exists for each ω.

Assume the limit exists for each ω. That means, for each ω, $I_{A_n}(\omega)$ is eventually all zeros, or all ones. That is because, if the sequence jumps from 0 to 1 infinitely for at least one ω, the limit does not exist for that one ω. We have a partition of ω, ω_0 and ω_1, for whether the sequence settles on all zeros, or all ones.

We now note that $\omega_1 \in \lim_n A_n$. This is because ω_1 is eventually in every set of some tail sequence. This means that each $\bigcup_{k=n} A_k$ contains ω_1, in which case $\omega_1 \in \limsup_n A_n$. Similarly, as ω_1 is eventually in every set in some tail sequence, $\omega_1 \in \bigcap_{k=n} A_k$ for some k, which means $\omega_1 \in \liminf_n A_n$.

In the other direction, we note that $\omega_0 \notin \lim_n A_n$. Eventually, ω_0 is not in any of the sets of some tail sequence. Thus, eventually each $\bigcup_{k=n} A_k$ does not ω_0, and $\omega_0 \notin \limsup_n A_n$. Similarly, ω_0 is not in any $\bigcap_{k=n} A_k$, which means $\omega_0 \notin \liminf_n A_n$.

Thus, we have that $\limsup_n A_n = \liminf_n A_n$, and the limit is well defined if $\lim_n I_{A_n}(\omega)$ exists. The other direction can be proved with similar arguments. qed.

4.3 Rotated Square $\limsup$ and $\liminf$

Let A_n be the square $[(x, y) : |x| \leq 1, |y| \leq 1]$ rotated through the angle $2\pi n\theta$. Give geometric descriptions of $\limsup_n A_n$ and $\liminf_n A_n$ in case:

1. $\theta = 1/8$. Note, in this case, we have:

$$\text{angle}_n = 2\pi n/8 = \frac{n\pi}{4}$$

   In this case, we have that A_{2k} are equal and A_{2k+1} are all equal for $k = 0, 1, 2, \cdots$, ie:

$$A_1 = A_3 = \cdots \qquad A_2 = A_4 = \cdots$$

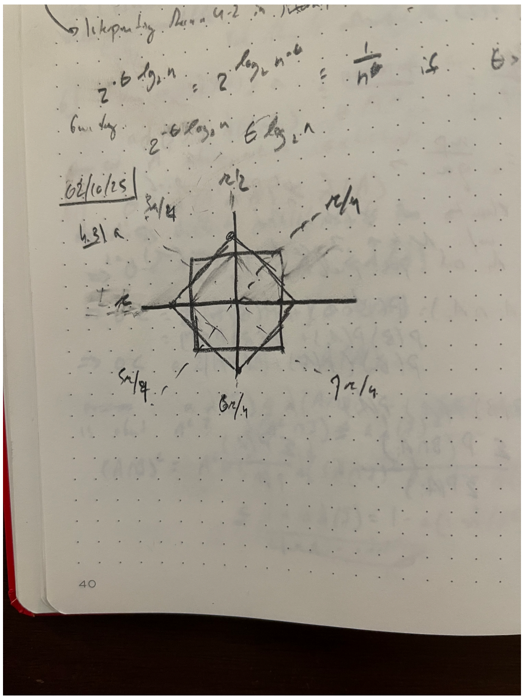


Figure 1: Diagram for Question 4.3.a

Note that the $\limsup$ is the union of the two squares in the above picture, as any point that appears within one of the two squares will appear infinitely often. $\liminf$ is the intersection of the two squares - as only the points in the intersection will appear in every tail set.

2. θ is rational. In this case:

$$\text{angle}_n = 2\pi n * p/q = \frac{2pn\pi}{q}$$

Note - we will have a picture like above, with cycle of q. Note that:

$$A_k = A_{q+k} = A_{2q+k}$$

For $k \in [q]$, as:

$$\frac{2p(tq+k)\pi}{q} = 2\pi pt + 2\pi \frac{pk}{q}$$

Note that as t and p are integers, $2\pi pt$ is an integer multiple of 2π, which just circles back to the angle of zero. And then, the added fraction of $2\pi\frac{pk}{q}$ is always the same. So, the union of the sets A_k is the $\limsup$, like above, and the intersection is the $\liminf$, like above.

3. θ is irrational. Note, in this case, we have that:

$$\text{remainder } 2\pi n\theta/2\pi$$

Is dense in $[0, 2\pi]$. Let's just assume that is the case for now. And so, take $x \notin D$, where D is the unit disk, but $x \in A_n$ for some A_n. Note, I think it might appear infinitely often in some A_n.

Just making a leap here: but I think $\limsup A_n$ is the disk of radius $\sqrt{2}$, whereas $\liminf A_n$ is the disk of radius 1. If you want to appear in every tail sequence - you have to be inside the unit disc.

4.7 Independence Complement Criteria

For events $A_1, \cdots, A_n$ consider the 2^n equations $P(B_1 \cap \cdots \cap B_n) = P(B_1) \cdots P(B_n)$ with $B_i = A_i$ or $B_i = A_i^c$ for each i. Show that $A_1, \cdots, A_n$ are independent if all these equations hold.

We note a similar criteria that we have already approved - consider the 2^n equations $P(C_1 \cap \cdots \cap C_n) = P(C_1) \cdots P(C_n)$ with $C_i = A_i$ or $C_i = \Omega$ for each i. If each of these holds, then the A_i are independent. We prove that if the B equations hold, the C equations holds. Thus, the B equations also imply independence.

First, assume one of the C_i is Ω. Thus, we have by countable additivity:

$$P(C_1 \cap \cdots \cap C_n) = P(A_1 \cap \cdots \cap A_i \cap \cdots \cap A_n) + P(A_1 \cap \cdots \cap A_i^c \cap \cdots \cap A_n)$$

By the B equations, the above equals:

$$= (P(A_i) + P(A_i)^c)(P(A_1) \cdots \widehat{P(A_i)} \cdots P(A_n)) = P(C_1) \cdots P(C_n)$$

Now, we can take this result, and prove the equations hold for two Ω in the same way, and so on. Thus, we have that the C_i equations hold if the B_i equations hold, and the B_i equations imply independence. Note, independence clearly implies the B_i equations hold, so its an if and only if. qed.

4.8 Class $\mathcal{A}$ Partitions of Ω

Recall that ω and ω' are called $\mathcal{A}$ equivalent if, for every $A \in \mathcal{A}$, ω and ω' lie either both in A or both in A^c. That is:

$$I_A(\omega) = I_A(\omega') \qquad A \in \mathcal{A}$$

This is clearly an equivalence relation, and this relation partitions Ω. Note, there are $2^{|\mathcal{A}|}$ possible equivalence classes - as for each $A \in \mathcal{A}$, being in A or A^c defines two difference classes (note, this might not be entirely accurate if $\mathcal{A}$ contains the complement or not. But, the idea is there). Also note - we proved that if ω and ω' are $\mathcal{A}$ equivalent, then they are also $\sigma(\mathcal{A})$ equivalent.

In this problem, we want to describe the $\mathcal{A}$ partition for each of the following classes:

1. The class of finite and cofinite sets. For this class - each ω would belong to a finite set of itself. And so, if $\omega' \neq \omega$, we would have $I_{\{\omega\}}(\omega) \neq I_{\{\omega\}}(\omega')$. The partition would be all singletons.

2. The class of countable and cocountable sets. For this class - I again believe the partition would be the singletons. Say ω, ω' both belong to a countable set A. Then, we have that $A - \{\omega'\}$ is also a countable set. In which case, ω and ω' are not in the same equivalence class.

3. A partition (or arbitrary cardinality) of Ω. So, each $A \in \mathcal{A}$ is disjoint from the rest. We have that the $\omega \in A$ form an equivalence class - as $\omega, \omega' \in A$ also implies $\omega, \omega' \notin B \in \mathcal{A}$ for $B \neq A$.

4. The level sets of $\sin(x)$, where $\Omega = \mathbb{R}$. Again, I believe this would form a partition of Ω, as $\sin(x) = c$ has multiple x that satisfy that value. If we want to describe the partitions - we have for $c \in [-1, 1]$, the level set L is:

$$L = \{x \in \mathbb{R} : x = \arcsin(c) + 2n\pi \text{ or } x = \pi - \arcsin(c) + 2n\pi, n \in \mathbb{Z}\}$$

Think of this in terms of the unit circle - $\arcsin(c)$ gives the x value (angle value) on the positive x axis. The x value on the negative x axis is symmetrical, or mirrored, which is given by $\pi - x$. This gives the set above.

5. The σ field in problem 3.5. In problem 3.5, we have $\Omega = \{(x, y) : 0 < x, y \leq 1\}$ is the unit square, and $\mathcal{F}$ is the class of sets of the form:

$$\{(x, y) : x \in A, 0 < y \leq 1\}$$

And let P have value $\lambda(A)$ on this set. In the problem, we show that $(\Omega, \mathcal{F}, P)$ is a probability space. First, we note that $\mathcal{F}$ is a σ algebra - as countable unions and intersections on the x axis still remain within $\mathcal{B}$, and we still have all the “height" from zero to one. We also clearly have Ω and complements. Probability measure still holds, as we still have countable additivity (if the A are disjoint in $\mathcal{B}$, so are the corresponding sets above). It is also clear that for $A = \{(x, y) : 0 < x \leq 1, y = 1/2\}$, we have the outer measure $P^*(A) = 1$ while the inner measure $P^*(A) = 0$. That is essentially answering question 3.5.

Now, we note that the $\mathcal{A}$ partition of Ω is just straight lines $\{(x, y) : x \in (0, 1], 0 < y \leq 1\}$. Every ω along such a line belongs to the same A or A^c for $A \in \mathcal{F}$ - this is because, each $A \in \mathcal{F}$ either contains the entire line, or it doesn't. Also, note that if ω and ω' are not on the same line - the singleton $\{x\}$ and $\{x'\}$ sets are within $\mathcal{B}$, so $\omega \in \{(x, y) : 0 < y \leq 1\}$, but $\omega' \notin \{(x, y) : 0 < y \leq 1\}$. qed.

4.10 Independence does not imply Independent Information

Where by information, we mean which *partition* of ω was selected. Note, some classes can be independent, but the information one class gives (ie, the partition that ω belongs to) might tell us which set in the other class the ω belongs to. This question essentially just emphasizes that the information notion is not altogether perfect.

There is in the unit interval a set H that is nonmeasurable in the extreme sense that its inner and outer Lebesgue measures are 0 and 1: $\lambda_*(H) = 0$ and $\lambda^*(H) = 1$. See problem 12.4 for its construction.

1. Let $\Omega = (0, 1]$, let $\mathcal{G}$ consist of the Borel sets in Ω, and let H be the set just described. Show that the class $\mathcal{F}$ of sets of the form $(H \cap G_1) \cup (H^c \cap G_2)$ for $G_1, G_2 \in \mathcal{G}$ is a σ field, and that:

$$P\left[(H \cap G_1) \cup (H^c \cap G_2)\right] = \frac{1}{2}\lambda(G_1) + \frac{1}{2}\lambda(G_2)$$

consistently defines a probability measure on $\mathcal{F}$. First, we show it is a σ field. Clearly, it contains complements, as:

$$[(H \cap G_1) \cup (H^c \cap G_2)]^c = (H \cap G_1^c) \cup (H^c \cap G_2^c)$$

Note $x \in (H \cap G_1^c) \implies x \notin (H \cap G_1) \wedge x \notin (H^c \cap G_2)$. Similar can be shown for $x \in (H^c \cap G_2^c)$, and so:

$$(H \cap G_1^c) \cup (H^c \cap G_2^c) \subseteq (H \cap G_1) \cup (H^c \cap G_2)$$

Now, take $x \in [(H \cap G_1) \cup (H^c \cap G_2)]^c$. This implies $x \in H^c \vee x \in G_1^c$ *and* $x \in H \vee x \in G_2^c$. If $x \in H^c$, then $x \notin H$, in which case $x \in G_2^c$, and we have $x \in (H \cap G_2^c$. Similar for $x \in H$. Final case is $x \in G_1^c$ and $x \in G_2^c$, in which case $x \in H$ or $x \in H^c$. In all cases:

$$[(H \cap G_1) \cup (H^c \cap G_2)]^c \subseteq (H \cap G_1^c) \cup (H^c \cap G_2^c)$$

$$\implies [(H \cap G_1) \cup (H^c \cap G_2)]^c = (H \cap G_1^c) \cup (H^c \cap G_2^c)$$

Note, see problem 2.7 for closed under countable unions (it is actually easier than the above. So yes, $\mathcal{F}$ is a σ field. Now, we want to show that:

$$P\left[(H \cap G_1) \cup (H^c \cap G_2)\right] = \frac{1}{2}\lambda(G_1) + \frac{1}{2}\lambda(G_2)$$

consistently defines a probability measure on $\mathcal{F}$. I will show that:

$$\lambda(G_1) = \lambda^*(G_1) = \lambda^*(H \cap G_1)$$

First, by monotonicity of the outer measure, we have:

$$\lambda^*(H \cap G_1) \leq \lambda^*(G_1) \qquad \text{and} \qquad \lambda^*(H \cap G_1^c) \leq \lambda^*(G_1^c) = 1 - \lambda^*(G_1)$$

Now, we note that as $G_1 \in \mathcal{G} = \mathcal{B}$, G_1 is measurable, and so:

$$\lambda^*(H \cap G_1) + \lambda^*(H \cap G_1^c) = \lambda^*(H) = 1 \implies \lambda^*(H \cap G_1) + 1 - \lambda^*(G_1) \geq 1$$

$$\implies \lambda^*(H \cap G_1) \geq \lambda^*(G_1)$$

Thus, we indeed have:

$$\lambda^*(H \cap G_1) = \lambda^*(G_1)$$

We can similarly show:

$$\lambda^*(H \cap G_3) = \lambda^*(G_3)$$

Given that $H \cap G_1 = H \cap G_3$, we thus must have:

$$\lambda(G_1) = \lambda(G_3)$$

And similarly, we have $\lambda(G_2) = \lambda(G_4)$. Thus, we can conclude that:

$$(H \cap G_1) \cup (H^c \cap G_2) = (H \cap G_3) \cup (H^c \cap G_4) \implies \lambda(G_1) + \lambda(G_2) = \lambda(G_3) + \lambda(G_4)$$

And the probability measure is consistent. Now, we note it is a probability measure, as it is clearly between 0 and 1, $P(\Omega) = 1$, and we have countable additivity (disjoint unions, transfer the disjoint unions into the $H\cap$ and $H^c\cap$ terms, and then rely on countable additivity of λ).

2. Show that $P(H) = 1/2$ and that $P(G) = \lambda(G)$ for $G \in \mathcal{G}$. First, note that:

$$P(H) = P\left[(H \cap \Omega) \cup (H^c \cap \emptyset)\right] = 1/2\lambda(\Omega) + 1/2\lambda(\emptyset) = 1/2$$

Now, note that:

$$P(G) = P\left[(H \cap G) \cup (H^c \cap G)\right] = 1/2\lambda(G) + 1/2\lambda(G) = \lambda(G)$$

3. Show that $\mathcal{G}$ is generated by a countable subclass. Note, in problem 2.11, we showed that with the rational intervals, and that it contained the singletons.

4. Show that $\mathcal{G}$ contains all the singletons and that H and $\mathcal{G}$ are independent (in the sense that $\mathcal{G}$ is a class contained within $\mathcal{F}$). Above, we noted that $\mathcal{G}$ contains the singletons. Independence follows if for $G \in \mathcal{G}$, we have:

$$P(H \cap G) = P(H)P(G) = 1/2\lambda(G)$$

Where the last equality is by part 3. Note that:

$$P(H \cap G) = P\left[(H \cap G) \cup (H^c \cap \emptyset)\right] = 1/2\lambda(G) + 1/2 * 0 = 1/2\lambda(G)$$

Thus, H and $\mathcal{G}$ are independent.

This construction proves the following: There exists a probability space $(\Omega, \mathcal{F}, P)$, a σ field $\mathcal{G}$ in $\mathcal{F}$, and a set $H \in \mathcal{F}$, such that $P(H) = 1/2$, H and $\mathcal{G}$ are independent, and $\mathcal{G}$ is generated by a countable subclass and contains all the singletons.

This relates to example 4.10, in which, even though $\mathcal{G}$ and H are independent, the "information" contained in G would tell us whether or not the drawn ω is in H or not. However, note that with this example - the $\mathcal{G}$ is more *natural* - it is countably generated, and contains singletons. In fact - it is the borel sets! However, the example involves the pathological set H, which throws everything for a loop.

4.11 Different Criteria for $P(\limsup_n A_n) = 1$

1. If $A_1, A_2, \cdots$ are independent events, then prove:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} P(A_n)$$

And:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \prod_{n=1}^{\infty} (1 - P(A_n))$$

From these facts, derive the second Borel-Cantelli lemma by the well-known relation between infinite series and products (I would guess, it is the exponential fact).

We note that for a finite sequence, independence implies:

$$P\left(\bigcap_{n=1}^{k} A_n\right) = \prod_{n=1}^{k} P(A_n)$$

We note that by continuity from above, we have:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} P(A_n)$$

We have by De'Morgan's laws, and independence implying independence of complements:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcap_{n=1}^{\infty} A_n^c\right) = 1 - \prod_{n=1}^{\infty}(1 - P(A_n))$$

Recall, the Borel Cantelli Lemma states that if $\{A_n\}$ is an independent sequence of events, and $\sum_n P(A_n)$ diverges, then $P(\limsup_n A_n) = 1$. Note, this follows if the complement of the lim sup equals 0, which follows if $\bigcap_{k=n}^{\infty} A_k^c = 0$ for all n. Note, we have A_k^c are independent by above, and so:

$$P\left(\bigcap_{k=n}^{\infty} A_k^c\right) = \prod_{k=n}^{\infty}(1 - P(A_k)) \leq \exp\left[-\sum_{k=n}^{\infty} P(A_k)\right] = 0$$

As we know the sum diverges, and for $x \in [0, 1]$, $(1 - x) \leq \exp(-x)$. Note - the above is essentially just the proof for the second lemma, but made easier, as we don't have to apply a limit argument in the middle of the above steps, by the facts above. If we wanted to just *derive* the lemma, then we might use the Weierstrass product inequality. We have:

$$1 - \sum_{n=1}^{\infty} P(A_n) \leq \prod_{n=1}^{\infty}(1 - P(A_n)) \leq \exp\left[-\sum_{n=1}^{\infty} P(A_n)\right]$$

From this, we have:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \prod_{n=1}^{\infty}(1 - P(A_n)) \geq 1 - \exp\left[-\sum_{n=1}^{\infty} P(A_n)\right] = 1$$

So, the probability of each union of A_n, and each tail sequence union of A_n, is 1. Note, from this, we can derive $P(\limsup_n A_n) = 1$, as it is an intersection of unions of probability one. Note, that by itself isn't enough, but we can use continuity from above to prove it equals 1 as well (as the unions each contain each other). qed.

2. Show that $P(\limsup_n A_n) = 1$ if for each k the series $\sum_{n>k} P(A_n | A_k^c \cap \cdots \cap A_{n-1}^c)$ diverges. From this deduce the second Borel-Cantelli lemma once again.

Just from an informal "information" level - the independent parts of each A_n, the probability of each A_n when you are given the rest of the previous A_k - diverges (note, I treat knowing A_k^c equivalent to knowing A_k).

Recall, we have:

$$\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

As noted above, we can prove this probability equals one if each $\bigcup_{k=n}^{\infty} A_k$ equals one. Note - I think we can use the ratio test. If the limit of the ratios exists - then it must be greater than or equal to 1, as a ratio less than 1 would imply convergence. For the ratio test, we would have ratios of:

$$\frac{P(A_k^c \cap \cdots \cap A_n^c \cap A_{n+1})}{P(A_k^c \cap \cdots \cap A_n^c)} \frac{P(A_k^c \cap \cdots \cap A_{n-1}^c)}{P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n)}$$

$$= \frac{P(A_k^c \cap \cdots \cap A_n^c \cap A_{n+1})}{P(A_k^c \cap \cdots \cap A_n^c)} \frac{P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n^c) + P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n)}{P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n)}$$

$$= \frac{P(A_k^c \cap \cdots \cap A_n^c \cap A_{n+1})}{P(A_k^c \cap \cdots \cap A_n^c)} \left(\frac{P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n^c)}{P(A_k^c \cap \cdots \cap A_{n-1}^c \cap A_n)} + 1 \right)$$

I'm not sure if these ratios will actually help us.

We will first derive the second Borel-Cantelli lemma, just to get something on the board. Assume that the A_n are independent, and their sum diverges. Then, we have:

$$\sum_{n>k} P(A_n | A_k^c \cap \cdots \cap A_{n-1}^c) = \sum_{n>k} \frac{P(A_n \cap A_k^c \cap \cdots \cap A_{n-1}^c)}{P(A_k^c \cap \cdots \cap A_{n-1}^c)}$$

$$= \sum_{n>k} \frac{P(A_n)P(A_k^c) \cdots P(A_{n-1}^c)}{P(A_k^c) \cdots P(A_{n-1}^c)} = \sum_{n>k} P(A_n) = \infty$$

Where the $= \infty$ comes from every tail sequence diverging as well. Thus, we have that $P(\limsup_n A_n) = 1$, and this tail series fact is actually stronger. Now, we note, that $P(\limsup_n A_n) = 1$ if we have for every k:

$$P\left[\bigcap_{n \geq k} A_n^c\right] = 0$$

We use this fact in the original proof. Assume by contradiction that we have at least one k such that the above is greater than 0. Then, we note by assumption:

$$\sum_{n>k} P(A_n | A_k^c \cap \cdots \cap A_{n-1}^c) = \infty$$

$$\implies \sum_{n>k} P(A_n \cap A_k^c \cap \cdots \cap A_{n-1}^c) \frac{1}{P(A_k^c \cap \cdots \cap A_{n-1}^c)} = \infty$$

Now, by contradiction, we have assumed:

$$P\left[\bigcap_{n \geq k} A_n^c\right] = \epsilon > 0 \implies P(A_k^c \cap \cdots \cap A_{n-1}^c) \downarrow \epsilon$$

Note, this implies the fraction approaches some constant $1/\epsilon$, and we can pull it out to conclude the first series diverges. More rigorously, for some N large enough, for $n \geq N$, we have that:

$$\epsilon \leq P(A_k^c \cap \cdots \cap A_{n-1}^c) \leq 2\epsilon$$

Thus, on the tail, we find:

$$\infty = \sum_{n>N} P(A_n \cap A_k^c \cap \cdots \cap A_{n-1}^c) \frac{1}{P(A_k^c \cap \cdots \cap A_{n-1}^c)} \geq 2\epsilon \sum_{n>N} P(A_n \cap A_k^c \cap \cdots \cap A_{n-1}^c)$$

$$\implies \sum_{n>k} P(A_n \cap A_k^c \cap \cdots \cap A_{n-1}^c) = \infty$$

This is a contradiction, as we have each of the sets in the above sum are disjoint, and so their sum cannot be more than one. Thus, we must conclude for all k:

$$P\left[\bigcap_{n \geq k} A_n^c\right] = 0 \implies P(\limsup_n A_n) = 1$$

Note: I used help online to solve this. My problem was - I was too focused on directly proving it. I had gotten to the intersections equal 0/unions equal one statement - but direct proof was the problem. I think if I had pivoted quicker, I could have solved this one.

3. Show by example that $P(\limsup_n A_n) = 1$ does not follow from the divergence of $\sum_n P(A_n | A_1^c \cap \cdots \cap A_{n-1}^c)$ alone.

Thinking in terms of "information" - I think this might follow if A_1 is a really *small* event. Then, knowing A_1^c might mean that you know *a lot*, in which case the probabilities in the sum are large.

Can't really figure this one out. First, we need A_n to not be disjoint from the A_k^c for $k < n$, as otherwise our conditional probabilities will be 0. Then, we need A_1^c to be restrictive enough so that the fraction becomes something like x/x, which is close to 1.

4. Show that $P(\limsup_n A_n) = 1$ if and only if $\sum_n P(A \cup A_n)$ diverges for each A of positive probability.

Assume that $P(\limsup_n A_n) = 1$ and take A such that $P(A) > 0$. Note, we have that:

$$B_k = \bigcup_{i=k}^{\infty} A_i \implies 1 = P(\limsup_n A_n) = P\left(\bigcap_{n=1}^{\infty} B_n\right) \leq P(B_k)$$

$$\implies P(B_k) = 1$$

Assume by contradiction that $\sum_n P(A \cup A_n)$ converges. Then, for some k large enough, we have for $\epsilon < P(A)$:

$$P(A \cap B_k) = P\left(A \cap \bigcup_{i=k}^{\infty} A_i\right) = P\left(\bigcup_{i=k}^{\infty} A \cap A_i\right) \leq \sum_{i=k}^{\infty} P(A \cap A_i) < \epsilon$$

However, we note:

$$P(A) = P((A \cap B_k) \cup (A \cap B_k^c)) = P(A \cap B_k) + P(A \cap B_k^c) = P(A \cap B_k)$$

As $P(A \cap B_k^c) \leq P(B_k^c) = 0$. Thus, we have a contradiction, as we have found:

$$P(A) = P(A \cap B_k) < \epsilon < P(A)$$

And so, by contradiction, we have that $\sum_n P(A \cup A_n)$ diverges. Now, we go the other direction, and assume that the sum diverges for all A with positive probability. We have $P(\limsup_n A_n) = 1$ if and only if $P(B_k) = 1$. By contradiction, assume that $P(B_k) < 1$, in which case $P(B_k) > 1$. Then, we have:

$$\sum_{i=k}^{\infty} P(B_k^c \cap A_i) \text{ diverges} \implies \sum_{i=k}^{\infty} P(A_i - B_k) \text{ diverges}$$

$$\implies \sum_{i=k}^{\infty} P(A_i - (A_k \cup \cdots \cup A_{i-1})) \text{ diverges}$$

Note, the last step is because $A_k \cup \cdots \cup A_{i-1} \subseteq B_k$. Now, note that the above is a summation of disjoint sets, and so, we have:

$$\implies P\left(\bigcup_{i=k}^{\infty} A_i - (A_k \cup \cdots \cup A_{i-1})\right) = \infty \implies P(B_k) = \infty$$

This is a contradiction, as an event must have less than or equal to 1 probability. And so, by contradiction, we have that $P(B_k) = 1$ and thus $P(\limsup_n A_n) = 1$. qed.

5. If sets A_n are independent and $P(A_n) < 1$ for all n, then $P(A_n \text{i.o.}) = 1$ if and only if $P(\bigcup_n A_n) = 1$.

   We first go in the easy direction. Note, $P(A_n \text{i.o.}) = 1$ if and only if $P(B_k) = 1$ for all k. Note, for $k = 1$, we have:

$$1 = P(B_1) = P\left(\bigcup_n A_n\right)$$

   And so now, we go in the other direction. Recall, one sufficient condition for $P(A_n \text{i.o.}) = 1$ is that:

$$P\left(\bigcap_{i=k} A_i^c\right) = 0$$

   For all i. Note, we have by assumption:

$$P\left(\bigcap_{i=1} A_i^c\right) = 0$$

   As the A_i^c are independent events as well, by part a, we have:

$$P\left(\bigcap_{i=k} A_i^c\right) = \prod_{i=k}^{\infty} P(A_i^c)$$

   For $j = 1, \cdots, k-1$, we note that $P(A_j) < 1$, and so we have that $P(A_j^c) > 0$. And so, we have that:

$$0 = P\left(\bigcap_{i=1} A_i^c\right) = \lim_{n\to\infty} \prod_{i=1}^{n} P(A_i^c) = P(A_1^c) \cdots P(A_{k-1}^c) \lim_{n\to\infty} \prod_{i=k}^{n} P(A_i^c)$$

$$\implies 0 = P(A_1^c)\cdots P(A_{k-1}^c)P\left(\bigcap_{i=k} A_i^c\right)$$

As $P(A_1^c)\cdots P(A_{k-1}^c) > 0$, we must have for all k:

$$0 = P\left(\bigcap_{i=k} A_i^c\right) \implies P(A_n\text{i.o.}) = 1$$

Thus, we can indeed conclude that if sets A_n are independent and $P(A_n) < 1$ for all n, then $P(A_n\text{i.o.}) = 1$ if and only if $P(\cup_n A_n) = 1$. qed.

4.14 Infinite Independent Events Criteria for a Nonatomic Space

Suppose that there are in $(\Omega, \mathcal{F}, P)$ independent events $A_1, A_2, \cdots$ such that, if:

$$\alpha_n = \min\{P(A_n), 1 - P(A_n)\}$$

Then $\sum \alpha_n = \infty$. Show that P is *nonatomic*.

Recall - nonatomic means that if $P(A) > 0$, there exists a B such that $B \subset A$ and $0 < P(B) < P(A)$, and $A, B \in \mathcal{F}$. Note, in problem 2-19, we prove that $\mathcal{B}$ is nonatomic with the lebesgue measure (deriving a contradiction from a single point needing to have 0 lebesgue measure). We also showed that in the nonatomic case, that for $0 \leq x \leq P(A)$, there exists a B such that $B \subset A$ and $P(B) = x$. Finally, we proved that if $p_1, p_2, \cdots$ are nonnegative and add to 1, then A can be decomposed into sets $B_1, B_2, \cdots$ such that $P(B_n) = p_n P(A)$.

Take an $A \in \mathcal{F}$. We just have to show that there is a B such that $B \subset A$, and $0 < P(B) < P(A)$ and $B \in \mathcal{F}$. Take B_i as either A_i or A_i^c, whichever satisfies $P(B_i) = \alpha_i$. Note, by Borel-Cantelli, we have $P(B_i\text{i.o.}) = 1$, and by 4.11e, we have:

$$P(\cup_i B_i) = 1$$

Examine each $A \cap B_i$. We note for some B_i, we have that $0 < P(A \cap B_i) < P(A)$, and thus $(\Omega, \mathcal{F}, P)$ is nonatomic. First, note they all can't be zero, as that would imply:

$$0 = P(A \cap B_i) \implies 0 = P(\cup A \cap B_i) = P(A \cap \cup B_i) = P(A)$$

Which is a contradiction. Note, the only thing to do now is prove that if $P(A \cap B_i) > 0$, we don't have $P(A \cap B_i) = P(A)$. However, this could be the case - what if the B_i are disjoint, and $A = B_j$. This path will not work.

Attempt 2 Using the solution at the back of the book. First, note that:

$$\lim_{n\to\infty} \max P(B_1 \cap \cdots \cap B_n) = 0$$

Here, we take B_i as either A_i or A_i^c - note, we have:

$$P(B_1 \cap \cdots \cap B_n) = \prod_{i=1}^{n} P(B_i) \leq \exp\left(-\sum_{i=1}^{n} P(B_i)\right)$$

Note, no matter the sum, it always diverges, and so yes, the limit is indeed 0, even if you take the maximizing B_i. Now, define:

$$C_x = \left[\omega : \sum_n I_{A_n}(\omega)2^{-n} \leq x\right]$$

First note, C_x contains all ω not in the A_n. Then, it would pick up the ω that appear sporadically, or in tail sequences far away enough such that the sum of the corresponding 2^{-n} does not exceed x. We want to show that:

$$P(A \cap C_x)$$

Is continuous in x. Note that the above is maximized for $x = 1$, in which case $P(A \cap C_1) = P(A)$, as $C_1 = \Omega$, and minimized for $x = 0$, in which case $P(A \cap C_0) = P\left(A \cap (\cup A_i)^c\right) = 0$, as we have $P(\cup A_i) = 1$, and so its complement has probability zero. We want to show it is continuous in x. So, take some $c \in (0, 1)$. We want to show that as $x_n \to c$, we have:

$$\lim_n P(A \cap C_{x_n}) = P(A \cap C_c)$$

I believe this can be done by looking at the symmetric difference:

$$(A \cap C_{x_n}) \triangle (A \cap C_c)$$

And noting that for x_n close enough to c, the ω in one of the sets but not the other must be contained within small enough dyadic intervals. Or, we can make use of the fact that the maximum goes to 0. We first assume that $x_n \uparrow c$. Later, we will generalize the argument for all $x_n \to c$. Note, in this case, we have:

$$A \cap C_{x_n} \subseteq A \cap C_c \implies P(A \cap C_c) - P(A \cap C_{x_n}) = P(A \cap C_c - A \cap C_{x_n})$$

$$= P(A \cap (C_c - C_{x_n})) \leq P(C_c - C_{x_n})$$

So, we examine:

$$\omega \in C_c - C_{x_n}$$

Note, x_n can be arbitrarily close to c, and so $\omega \in C_c - C_{x_n}$ implies:

$$c - \epsilon \leq \sum_n I_{A_n}(\omega)2^{-n} \leq c$$

Note, c has at most two dyadic expansions (terminating and non-terminating):

$$c = \sum_n d_n^1(c)2^{-n} = \sum_n d_n^2(c)2^{-n}$$

If $\epsilon < 2^{-k}$, we have that ω must satisfy for $n = 1, \cdots, k-1$:

$$\forall n \;\; I_{A_n}(\omega) = d_n^1(c) \text{ or } \forall n \;\; I_{A_n}(\omega) = d_n^2(c)$$

If ω doesn't satisfy either of the two above equations, then the difference:

$$\left| \sum_n d_n^i(c)2^{-n} - \sum_n I_{A_n}(\omega)2^{-n} \right| > 2^{-k} > \epsilon$$

Note, by the fact that for any *specific* choice of B_i, we have:

$$\lim_{n\to\infty} \max P(B_1 \cap \cdots \cap B_n) = 0$$

We thus have that:

$$\lim_{n\to\infty} P(C_c - C_{x_n}) \leq \lim_{n\to\infty} P(B_1^1 \cap \cdots \cap B_n^1) + P(B_1^2 \cap \cdots \cap B_n^2) = 0$$

And so, for $x_n \uparrow c$, we indeed have:

$$\lim_n |P(A \cap C_c) - P(A \cap C_{x_n})| = \lim_n P(A \cap C_c) - P(A \cap C_{x_n}) \leq \lim_{n\to\infty} P(C_c - C_{x_n}) \leq 0$$

$$\implies \lim_n P(A \cap C_{x_n}) = P(A \cap C_c)$$

Now, for $x_n \to c$. Note, we can still prove that:

$$\lim_n |P(A \cap C_c) - P(A \cap C_{x_n})| = 0$$

As either $A \cap C_c \subseteq A \cap C_{x_n}$, or vice versa, and we can still apply the argument for the bigger interval $(c - \epsilon, c + \epsilon)$ to conclude that the $\omega \in (A \cap C_{x_n}) \triangle (A \cap C_c)$

have to be in one of two specific sequences of B_i, which both have probability 0. Thus, we indeed have that:

$$P(A \cap C_x)$$

Is continuous in x. Thus, we can conclude there is some x such that:

$$0 < P(A \cap C_x) < P(A)$$

We clearly have $A \cap C_x \subseteq P(A)$. So, the final thing to note is that $A \cap C_x \in \mathcal{F}$, which comes from $C_x \in \mathcal{F}$. Note, the function describing C_x can be shown to be a measurable mapping, in which case the preimage of an interval $[0, x]$ would be a measurable set as well. Note, clearly each partial sum is a measurable mapping (as the preimage is clearly measurable), and thus the infinite sum is measurable as well. This actually gives us a way to directly prove C_x is measurable. First, note that for:

$$C_x^k = \left[\omega : \sum_{n=1}^{k} I_{A_n}(\omega) 2^{-n} \leq x\right]$$

We have that C_x^k is clearly measurable. There are certain sequences of $x \in B_1 \cap \cdots \cap B_k$, where $B_i = A_i$ or A_i^c, that satisfy the above equation. One such example is $x \in A_1^c \cap \cdots \cap A_k^c$. So, $C_x^k \in \mathcal{F}$, as C_x^k is a finite union of finite intersections of measurable B_i. Now, we have that:

$$C_x = \bigcap_{k=1}^{\infty} C_x^k$$

Clearly, $\omega \in C_x^k \implies \omega \in C_x$, as we can just take $\omega \in A_i^c$ for $i > k$. Now, take $\omega \in C_x$. We note that ω must be in $B_1, \cdots, B_k$ such that the sum satisfies:

$$\sum_{n=1}^{k} I_{A_n}(\omega) 2^{-n} \leq x$$

Given that the infinite sum also satisfies being less than x. And so, $\omega \in C_x \implies \omega \in C_x^k$ for all k. Thus, we have that C_x is a countable intersection of $\mathcal{F}$ sets, and so C_x is within $\mathcal{F}$.

Thus, we have proved that if there are in $(\Omega, \mathcal{F}, P)$ independent events $A_1, A_2, \cdots$ such that for:

$$\alpha_n = \min\left\{P(A_n), 1 - P(A_n)\right\}$$

Then $\sum \alpha_n = \infty$, we have that P is *nonatomic*. qed.

4.15 Density of the Square-Free Integers

Let F be the set of square-free integers - those integers not divisible by any perfect square. Let F_l be the set of m such that $p^2|m$ (p^2 divides m) for no $p \leq l$, and show that $D(F_l) = \prod_{p \leq l}(1 - p^{-2})$. Show that $P_n(F_l - F) \leq \sum_{p>l} p^{-2}$, and conclude that the square-free integers have density:

$$\prod_p (1 - p^{-2}) = 6/\pi^2$$

We need to recall a lot of the above notation. Its all from problem 2.18. We have that:

$$P_n(A) = \frac{1}{n} \# \{m : 1 \leq m \leq n, m \in A\}$$

We also have that P_n is a *discrete* probability measure on $\Omega = \{1, 2, \cdots\}$. We define the *density* as:

$$D(A) = \lim_n P_n(A)$$

We first want to show:

$$D(F_l) = \prod_{p \leq l}(1 - p^{-2})$$

Note - I will first go with the assumption that p is *prime*. Note, assuming prime makes things easier, as we have that $lcm(p_1, p_2) = p_1 p_2$, and we similarly have $lcm(p_1^2, p_2^2) = p_1^2 p_2^2$. Further, in problem 2.18 (which this problem seems to be a continuation of), p referred to prime numbers as well.

We recall the notation that $M_a = \{ak : k = 1, 2, \cdots\}$. These are periodic sets, and in problem 2.18, we proved the following (note, it should also be intuitively clear):

$$P_n(M_a) = \frac{1}{n} \left\lfloor \frac{n}{a} \right\rfloor \to \frac{1}{a} = D(M_a) \qquad M_a \cap M_b = M_{lcm(a,b)}$$

We clearly have:

$$F_l = \left[\bigcup_{p \leq l} M_{p^2} \right]^c \implies F_l = \bigcap_{p \leq l} M_{p^2}^c$$

We now note that for prime $p_1, \cdots, p_k$, we have that:

$$P_n \left[M_{p_1^2} \cap \cdots \cap M_{p_k^2} \right] = P_n \left[M_{lcm(p_1^2, \cdots, p_k^2)} \right] = P_n \left[M_{p_1^2 \cdots p_k^2} \right] = \frac{1}{n} \left\lfloor \frac{n}{p_1^2 \cdots p_k^2} \right\rfloor \to \frac{1}{p_1^2 \cdots p_k^2}$$

So, we now find:

$$D(F_l) = \lim_{n\to\infty} P_n(F_l) = 1 - \lim_{n\to\infty} P_n\left(\bigcup_{p\le l} M_{p^2}\right)$$

We now make use of the inclusion-exclusion principle to find the above equals:

$$1 - \lim_{n\to\infty} \sum_{1<p_1\le l} P_n\left(M_{lcm(p_1^2)}\right) + \lim_{n\to\infty} \sum_{1<p_1<p_2\le l} P_n\left(M_{lcm(p_1^2,p_2^2)}\right) + \cdots$$

$$+(-1)^k \lim_{n\to\infty} \sum_{1<p_1<p_2<\cdots<p_k\le l} P_n\left(M_{lcm(p_1^2,p_2^2,\cdots,p_k^2)}\right)$$

$$= 1 - \sum_{1<p_1\le l} \frac{1}{p_1^2} + \sum_{1<p_1<p_2\le l} \frac{1}{p_1^2 p_2^2} + \cdots + (-1)^k \sum_{1<p_1<p_2<\cdots<p_k\le l} \frac{1}{p_1^2\cdots p_k^2}$$

$$= (1-p_1^2)(1-p_2^2)\cdots(1-p_k^2)$$

Where the final equality is a well known formula for the multiplication of $(1-p_i)$ terms. (also called the inclusion-exclusion principle). The second inequality is making use of our limit fact proved above. Thus, we indeed have:

$$D(F_l) = \prod_{p\le l}(1-p^{-2})$$

Now, the task is to show that $P_n(F_l - F) \le \sum_{p>l} p^{-2}$. We note that:

$$F_l - F \subseteq \bigcup_{p>l} M_{p^2}$$

Take $x \in F_l - F$. We have that x is *not* divisible by p^2 for $p \le l$, but x is divisible by some y^2, as $x \notin F$. Note, y^2 has a prime factorization:

$$y = \prod_{t=1}^{k} p_t \implies y^2 = \prod_{t=1}^{k} p_t^2$$

Note, these p_t^2 must divide x. However, we cannot have $p_t \le l$ by $x \in F_l$, and so we must have $p_t > l \implies x \in M_{p^2}$. Thus, we find by the above results and subadditivity:

$$P_n(F_l - F) = P_n\left(\bigcup_{p>l} M_{p^2}\right) \le \sum_{p>l} M_{p^2} = \sum_{p>l} \frac{1}{n}\left\lfloor \frac{n}{p^2}\right\rfloor \le \sum_{p>l} p^{-2}$$

Thus, we can indeed conclude that:

$$P_n(F_l - F) \leq \sum_{p>l} p^{-2}$$

Finally, we want to conclude that:

$$D(F) = \prod_p (1 - p^{-2})$$

Note, we have:

$$0 \leq P_n(F_l) - P_n(F) = P_n(F_l - F)$$

Note, the final term has an inequality that does not depend on n, and so we have:

$$0 \leq P_n(F_l) - P_n(F) \leq \sum_{p>l} p^{-2}$$

As limits maintain boundedness, we find, taking a limit to n on the middle term:

$$0 \leq D(F_l) - D(F) \leq \sum_{p>l} p^{-2}$$

Finally, we note the well known result that $\sum_n \frac{1}{n^2}$ converges, and so for $l \to \infty$, we have the right hand sum goes to 0. Thus, we have:

$$0 \leq \lim_{l\to\infty} D(F_l) - D(F) \leq 0 \implies D(F) = \lim_{l\to\infty} D(F_l)$$

$$\implies D(F) = \prod_p (1 - p^{-2})$$

The final point is an equality that $\prod_p (1 - p^{-2}) = 6/\pi^2$. I will note prove this here. qed.

Section 5 - Simple Random Variables

Notes

Simple Random Variables Definitions

Definition - Simple Random Variable Let X be a real-valued function on Ω for the arbitrary probability space $(\Omega, \mathcal{F}, P)$. X is a *simple random variable* if it has finite range (ie, takes on finitely many values) and if:

$$[\omega : X(\omega) = x] \in \mathcal{F}$$

For each real x. Note, the above set is $\emptyset$ for each $x \in \mathbb{R}$ that is not in the range. It is also customary to omit the ω function input, ie, just call it the simple random variable X. We also have the customary shorthand of:

$$[X = x] = [\omega : X(\omega) = x]$$

Some examples include the n dyadic digit, $d_n(\omega)$ (which takes on finite values of 0 and 1). The run lengths $l_n(\omega)$ of zeros starting at n are *not* simple random variables, as they take on a countable number of values. A finite sum:

$$X = \sum_i x_i I_{A_i}$$

Is a random variable if the A_i form a partition of Ω into $\mathcal{F}$ sets. Moreover, we can take $A_i = [X = x_i]$, and express a simple random variable X as a finite indicator sum. Note, however - this is not a *unique* representation, as we can have A_{ij} forming a partition of A_i.

Definition - Measurable with Respect to a Sub-σ-field If $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$, a simple random variable X is *measurable* $\mathcal{G}$, or *measurable with respect to* $\mathcal{G}$, if $[X = x] \in \mathcal{G}$ for each x. By definition, a simple random variable is always measurable $\mathcal{F}$. Since:

$$[X \in H] = \bigcup [X = x]$$

Where the union extends over the finitely many $x \in H$ and $x \in range(X)$, we have that $[X \in H] \in \mathcal{G}$ for every $H \subset \mathbb{R}$ if X is a simple random variable measurable $\mathcal{G}$ (as finite unions stay within $\mathcal{G}$).

Definition - σ field generated by a simple random variable X The σ field $\sigma(X)$ *generated* by X is the smallest σ field with respect to which X is measurable; that is $\sigma(X)$ is the intersection of all σ fields with respect to which X is measurable. For a finite or infinite sequence $X_1, X_2, \cdots$ of simple random variables, $\sigma(X_1, X_2, \cdots)$ is the smallest σ field with respect to which each X_i is measurable.

Theorem 5.1 - Simple Random Variable Generated σ Fields and Function of Simple Random Variables Let $X_1, \cdots, X_n$ be simple random variables.

1. The σ field $\sigma(X_1, \cdots, X_n)$ consists of the sets:

$$[(X_1, \cdots, X_n) \in H] = [\omega : (X_1(\omega), \cdots, X_n(\omega)) \in H]$$

For $H \subset \mathbb{R}^n$. H in this representation may be taken finite (by taking the intersection of H and the finite amount of possible coordinates by the SRVs X_i).

2. A SRV Y is measurable $\sigma(X_1, \cdots, X_n)$ if and only if:

$$Y = f(X_1, \cdots, X_n)$$

for some $f : \mathbb{R}^n \to \mathbb{R}$.

Proof of Theorem 5.1

1. We start with part (1). Let $\mathcal{M}$ be the class of sets of the form $[(X_1, \cdots, X_n) \in H]$. Note that sets of the form:

$$[(X_1, \cdots, X_n) = (x_1, \cdots, x_n)] = \bigcap_{i=1}^{n} [X_i = x_i] \in \sigma(X_1, \cdots, X_n)$$

As X_i is measurable with respect to $\sigma(X_1, \cdots, X_n)$, we have that $[X_i = x_i] \in \sigma(X_1, \cdots, X_n)$, and a σ algebra would contain a finite intersection. Note that each set in $\mathcal{M}$ is a finite union of sets of the above form - for all coordinates in the range of the tuple that are within H. And so, we have:

$$\mathcal{M} \subseteq \sigma(X_1, \cdots, X_n)$$

Now, we go the other direction. Note that $\mathcal{M}$ is a σ field. We have $\Omega = [(X_1, \cdots, X_n) \in \mathbb{R}^n] \in \mathcal{M}$, $[(X_1, \cdots, X_n) \in H]^c = [(X_1, \cdots, X_n) \in H^c] \in \mathcal{M}$, and:

$$\bigcup_j [(X_1, \cdots, X_n) \in H_j] = [(X_1, \cdots, X_n) \in \bigcup_j H_j] \in \mathcal{M}$$

Note that each X_i is measurable with respect to $\mathcal{M}$. This is because:

$$[X_i = x] = [(X_1, \cdots, X_i, \cdots, X_n) \in \mathbb{R} \times \cdots \times x \times \cdots \times \mathbb{R}] \in H$$

Thus, $\mathcal{M}$ is a σ field with respect to which each X_i is measurable, and so we have:

$$\sigma(X_1, \cdots, X_n) \subseteq \mathcal{M} \implies \mathcal{M} = \sigma(X_1, \cdots, X_n)$$

2. Assume that Y is a function of the $X_1, \cdots, X_n$:

$$Y = f(X_1, \cdots, X_n)$$

Note that:

$$[Y = y] = [(X_1, \cdots, X_n) = (x_1, \cdots, x_n) : f(x_1, \cdots, x_n) = y]$$

Note - $[Y = y] \in \sigma(X_1, \cdots, X_n)$, if we let $H = [(x_1, \cdots, x_n) : f(x_1, \cdots, x_n) = y]$. One confusion I had is - what if a coordinate $(x_1, \cdots, x_n)$ that maps to y cannot be achieved by $(X_1, \cdots, X_n)$ (ie, it is not in the range?). Note, this means for no ω, $[Y = y]$ via that coordinate mapping, as for all ω, $Y = f(X_1, \cdots, X_n)$. So, it is consistent.

Now, we assume that Y is measurable $\sigma(X_1, \cdots, X_n)$. Let $y_1, \cdots, y_r$ be the distinct values Y assumes. By part (1), there exist sets $H_1, \cdots, H_r$ in $\mathbb{R}^n$ such that:

$$[\omega : Y(\omega) = y_i] = [\omega : (X_1(\omega), \cdots, X_n(\omega)) \in H_i]$$

Define $f = \sum_{i=1}^r y_i I_{H_i}$. Note, H_i and H_j are not disjoint only if $y_i = y_j$. Therefore, the H_i partition $\mathbb{R}^n$, and each $(X_1(\omega), \cdots, X_n(\omega))$ lies in exactly one of the H_i, and it follows that:

$$Y(\omega) = f(X_1(\omega), \cdots, X_n(\omega))$$

Thus, a simple random variable Y is measurable $\sigma(X_1, \cdots, X_n)$ if and only if Y can be written as a function of the $X_1, \cdots, X_n$. qed.

Note: By the above theorem, it follows that functions of simple random variables are again simple random variables (as the function would be measurable $\mathcal{F}$, and the function could only take on finite values). Thus, X^2, e^{tX}, and so on are simple random variables along with X. Taking f to be:

$$\sum_{i=1}^n x_i \qquad \prod_{i=1}^n x_i \qquad \max_{i \leq n} x_i$$

Shows that sums, products, and maxima of simple random variables are again simple random variables. What is key in all of this is the finiteness - note that each function can still only take on a finite number of values.

Example 5.2 Let $s_n(\omega) = \sum_{k=1}^{n} r_k(\omega)$ be the partial sums of the Rademacher functions. By Theorem 5.1ii, s_k is measurable $\sigma(r_1, \cdots, r_n)$ for $k \leq n$. And $r_k = s_k - s_{k-1}$ is measurable $\sigma(s_1, \cdots, s_n)$. Thus:

$$\sigma(r_1, \cdots, r_n) = \sigma(s_1, \cdots, s_n)$$

In information terms, this means that the first n positions of a random walk contain the same information as the first n distances moved. Or, knowing the first n fortunes of the gambler is the same as knowing his gains and losses on each of the first n plays.

Convergence of Simple Random Variables

A common problem for probability is: for given random variables X and $X_1, X_2, \cdots$ on a probability space $(\Omega, \mathcal{F}, P)$, look for the probability of the event that:

$$\lim_n X_n(\omega) = X(\omega)$$

The normal number theorem is essentially concerned with $X_n(\omega) = n^{-1}\sum_{i=1}^{n} d_i(\omega)$ and $X(\omega) = 1/2$, as an example. It is easy and convenient to characterize the complementary event: $X_n(\omega)$ *fails* to converge to $X(\omega)$ if and only if there is some ϵ such that for no m, does $|X_n(\omega) - X(\omega)|$ remain below ϵ for all n exceeding m. That is to say, if and only if, for some ϵ, $|X_n(\omega) - X(\omega)| \geq \epsilon$ holds for infinitely many values of n. Therefore:

$$\left[\lim_n X_n = X\right]^c = \bigcup_{\epsilon} [|X_n - X| \geq \epsilon \text{ i.o.}]$$

The union can be restricted to rational (positive) ϵ because the set in the union *increases* as ϵ *decreases* (ie, for any positive ϵ, we can find $1/n$ rational that is smaller, and all ω will be included).

Note, this implies that the event $[\lim_n X_n = X]$ always lies within the basic σ field $\mathcal{F}$. First note, $[|X_n - X| \geq \epsilon] \in \mathcal{F}$. We have that $Y = |X_n - X|$ is a simple random variable on $\mathcal{F}$, and restricting the set greater than ϵ is in the field (note, this section for some reason seems to be agnostic to whether the rv is *simple* or not - in this case, we still have functions of rvs are rvs (measurable), and so to is the inequality). As i.o. is a countable intersection of countable unions, the i.o. terms in the countable union are within $\mathcal{F}$, and as complements stay in $\mathcal{F}$, we indeed have that $[\lim_n X_n = X] \in \mathcal{F}$. This event has probability 1 if and only if:

$$P\left[|X_n - X| \geq \epsilon \text{ i.o.}\right] = 0$$

For each ϵ. Note, by Theorem 4.1, we have the above implies:

$$0 \leq \liminf_n P\left[|X_n - X| \geq \epsilon\right] \leq \limsup_n P\left[|X_n - X| \geq \epsilon\right] \leq P\left[|X_n - X| \geq \epsilon \text{ i.o.}\right] = 0$$

$$\implies \lim_n P\left[|X_n - X| \geq \epsilon\right] = 0$$

Definition: Convergence in Probability The above argument leads to a definition: If $\lim_n P\left[|X_n - X| \geq \epsilon\right] = 0$ holds for each positive ϵ, then X_n is said to *converge to X in probability*, written $X_n \to_p X$. These arguments prove the following theorem:

Theorem 5.2 - Convergence with Probability 1, and Convergence in Probability

1. There is convergence $\lim_n X_n = X$ with probability 1 if and only if $P\left[|X_n - X| \geq \epsilon \text{ i.o.}\right] = 0$ holds for each ϵ.
2. Convergence with probability 1 implies convergence in probability.

Go back to the normal number theorem - it dealt with *convergence with probability 1*, as we ultimately said something along the lines of, the complement had probability 0. However, Theorem 1.1 had to do with convergence in probability of the same sequence. By Theorem 5.2 - we should have that Theorem 1.1 (the weak law of large numbers) is a consequence of the normal number theorem. Note, however, the converse is not generally true.

Example 5.4 Take $X = 0$ and $X_n = I_{A_n}$. Note that $X_n \to_p X$ is equivalent to $P(A_n) \to 0$. This is because:

$$0 = \lim_n P\left[|I_{A_n} - 0X| \geq \epsilon\right] = \lim_n P\left[I_{A_n} = 1\right] = \lim_n P\left[A_n\right]$$

We also have:

$$\left[\lim_n X_n = X\right]^c = \bigcup_\epsilon \left[|I_{A_n} - 0| \geq \epsilon \text{ i.o.}\right] = \bigcup_\epsilon \left[I_{A_n} \text{ i.o.}\right] = \left[A_n \text{ i.o.}\right]$$

And so, any sequence $\{A_n\}$ that satisfies $P(A_n) \to 0$ but $P[A_n\text{i.o.}] > 0$ will therefor give a counterexample to the converse of Theorem 5.2ii - ie, a sequence of random variables that converge in probability to a random variable, but don't converge with probability 1.

We give two such examples. Let $A_n = [\omega : l_n(\omega) \geq \log_2 n]$. Here, it is clear that $P(A_n) \leq 1/n \to 0$. However, by the second Borel Cantelli lemma, we proved that $P(A_n\text{i.o.}) = 1$ in example 4.15 in the previous chapter. A better example is the following. Define the sequence in the following way. The first two sets are:

$$A_1 = (0, 1/2] \qquad A_2 = (1/2, 1]$$

Define the next four by:

$$A_3 = (0, 1/4] \qquad A_4 = (1/4, 1/2] \qquad A_5 = (1/2, 3/4] \qquad A_6 = (3/4, 1]$$

Define the next eight as the dyadic intervals of rank 3, and so on. Clearly, $P(A_n) \to 0$. However, since each point ω is covered by one set in each successive block of length 2^k, the set $[A_n\text{i.o.}]$ is all of $(0,1]$, and so $P[A_n\text{i.o.}] = 1$.

Independence

Definition - Independent Simple Random Variables A sequence $X_1, X_2, \cdots$ (finite or infinite) of simple random variables is by definition *independent* if the classes $\sigma(X_1), \sigma(X_2), \cdots$ are independent in the sense of the preceding section. By Theorem 5.1(i), $\sigma(X_i)$ consists of the sets $[X_i \in H$ for $H \subset \mathbb{R}$. The condition of independence for $X_1, \cdots, X_n$ is therefore that:

$$P[X_1 \in H_1, \cdots, X_n \in H_n] = P[X_1 \in H_1] \cdots P[X_n \in H_n]$$

For linear sets $H_1, \cdots, H_n$. Note, the definition also requires the above equation to hold if any X_i is not included - but this is included above, if we set $H_i = \mathbb{R}$. For a countably infinite sequence - the above finite definition must hold for each n. A special case of the above is:

$$P[X_1 = x_1, \cdots, X_n = x_n] = P[X_1 = x_1] \cdots P[X_n = x_n]$$

Note, however, that summing over disjoint coordinates implies that the above definition gives us the set definition. Thus, simple random variables X_i are independent if and only if the above holds for all coordinates $x_1, \cdots, x_n$.

Independent Functions of Independent SRVs (2d Array Argument) Suppose that:

$$\begin{pmatrix} X_{11} & X_{12} & \cdots \\ X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Is an independent array of simple random variables. Finitely or infinitely many rows, and each row finite or infinite. Say $\mathcal{A}_i$ consists of the finite intersections:

$$\bigcap_j [X_{ij} \in H_j]$$

With $H_j \subset \mathbb{R}^1$. We can apply Theorem 4.2 to show that the σ fields $\sigma(X_{i1}, X_{i2}, \cdots)$ are independent. Note that $\mathcal{A}_i$ is a π system. The intersection of two elements of $\mathcal{A}_i$ will just result in another finite intersection - and note that $[X_{ij} \in H_j]$ and $[X_{ij} \in H'_j]$ can be combined into $[X_{ij} \in H_j \cap H'_j]$. Then, note that:

$$\sigma(\mathcal{A}_i) = \sigma(X_{i1}, X_{i2}, \cdots)$$

As every set in $\mathcal{A}_i$ is within $\sigma(X_{i1}, X_{i2}, \cdots)$, as finite intersections of sets $[X_{ij} \in H_j] \in \sigma(X_{i1}, X_{i2}, \cdots)$ are within the smallest σ field for which each X_{ij} is measurable. This gives us:

$$\sigma(\mathcal{A}_i) \subseteq \sigma(X_{i1}, X_{i2}, \cdots)$$

As $\sigma(\mathcal{A}_i)$ is the smallest σ algebra that contains the finite intersections of $[X_{ij} \in H_j]$. We have:

$$\sigma(X_{i1}, X_{i2}, \cdots) \subseteq \sigma(\mathcal{A}_i)$$

As $\sigma(X_{i1}, X_{i2}, \cdots)$ is the smallest σ algebra such that each X_{ij} is measurable with respect to. Note that each X_{ij} is measurable with respect to $\sigma(\mathcal{A}_i)$, as it contains each $[X_{ij} = x_j]$. Thus, we do have equality.

We now note that each $\mathcal{A}_i$ is independent. Take $A_i \in \mathcal{A}_i$, and note that for any finite subset of the A_i:

$$P(A_{i_1} \cap \cdots \cap A_{i_n}) = P(A_{i_1}) \cdots P(A_{i_n})$$

As we can express:

$$P(A_{i_1} \cap \cdots \cap A_{i_n}) = P\left(\bigcap_{k=1}^{n} \bigcap_{j=1}^{j_n} [X_{i_k,j} \in H_{i_k,j}]\right) = \prod_{k=1}^{n} \prod_{j=1}^{j_n} P\left([X_{i_k,j} \in H_{i_k,j}]\right)$$

$$= \prod_{k=1}^{n} P(A_{i_k})$$

And so, the $\mathcal{A}_i$ are independent, and by Corollary 1 of Theorem 4.2, we have that $\sigma(\mathcal{A}_i)$ are independent. Thus, the σ fields $\sigma(X_{i1}, X_{i2}, \cdots)$ are independent. As a consequence, $Y_1, Y_2, \cdots$ are independent if each Y_i is measurable $\sigma(X_{i1}, X_{i2}, \cdots)$ - this is because, the Y_i will satisfy:

$$\sigma(Y_i) \subseteq \sigma(X_{i1}, X_{i2}, \cdots)$$

And the above criteria easily proves the independence of the $\sigma(Y_i)$. Note, this implies that if Y_i is a function of $X_{i1}, X_{i2}, \cdots$, then the Y_i are independent, as Theorem 5.1 then implies that the Y_i are measurable $\sigma(X_{i1}, X_{i2}, \cdots)$. NOTE: we have only proved this for finite rows of $X_{i1}, \cdots, X_{in}$ however.

Example 5.6 Permutation Cycles We can write every permutation as a *product of cycles*:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 4 & 6 & 2 & 3 \end{pmatrix} = (1562)(37)(4)$$

Note - the cycle form implies that 1 goes to 5, 5 to 6, 6 to 2, and 2 to 1. For a finite permutation, this is intuitive to prove. Continue to create a cycle, until you have reached the first point in the cycle again. If you have not exhausted all the points in the permutation - find the smallest integer not yet encountered, and make a second cycle, and so on until all integers have been exhausted (note, we can't end a cycle at a point in the middle of a cycle, as a permutation is a bijection, and once a position can't be visited twice. As there is a finite number of positions, the process must end).

Note, the above process also gives us a standard cyclic representation.

Let Ω consist of the $n!$ permutations of $1, 2, \cdots, n$, all equally probable (note, $n!$, as a permutation can be identified with a bijection, of which there are $n!$). Let $\mathcal{F}$ consist of all subsets of Ω, and $P(A)$ to be the fraction of points in A. Let $X_k(\omega)$ be a 1 or 0 according to whether the kth position in our cyclic representation of the permutation ω completes the cycle or not. Note - $k \in [n]$, as each cycle representation will have n positions.

Let $S(\omega) = \sum_{k=1}^{n} X_k(\omega)$ be the number of cycles in ω. In the above example, we have $n = 7$, $X_1 = X_2 = X_3 = X_5 = 0$, and $X_4 = X_6 = X_7 = 1$, with $S = 3$. We will now show that the $X_1, \cdots, X_n$ are independent, and:

$$P[X_k = 1] = \frac{1}{n - k + 1}$$

First note - $P[X_1 = 1] = 1/n$, as $X_1(\omega) = 1$ if and only if the random permutation ω sends 1 to itself, and there are n equally likely possibilities of where 1 is sent.

Given that $X_1 = 1$, we look at X_2 - note that as $1 \to 1$, we have that $2 \to \{2, \cdots, n\}$, which implies that the *conditional probability* that $X_2(\omega) = 1$ is $1/(n-1)$. If, on the other hand, $X_1(\omega) = 0$, then $X_2(\omega) = 1$ if $2 \to 1$, but there are still $(n-1)$ outcomes for 1 to compete with, so again, the *conditional probability* that $X_2(\omega) = 1$ is $1/(n-1)$. This implies $P[X_2 = 1] = 1/(n-1)$. Note: this argument generalizes.

I give the generalization here, as I think it is a good argument. We let $Y_1(\omega), \cdots, Y_n(\omega)$ be the integers in successive positions of the cyclic representation of permutation ω. Fix k, and let A_v be the set where:

$$(X_1, \cdots, X_{k-1}, Y_1, \cdots, Y_k) = v = (x_1, \cdots, x_{k-1}, y_1, \cdots, y_k)$$

Note that the A_v *partition* Ω - each $\omega \in \Omega$ belongs to one of the A_v above. Note, the first $k-1$ entries are one or 0, and the remaining k are integers. *If* we have:

$$P[X_k = 1|A_v] = \frac{1}{n-k+1}$$

Then we must have $P[X_k = 1] = \frac{1}{n-k+1}$ - as the A_v form a partition. Further, example 4.7 tells us that this implies X_k is independent of $\sigma(\mathcal{A})$ (where $\mathcal{A}$ is the class of A_v, and $\sigma(\mathcal{A})$ is just the union of elements in $\mathcal{A}$, and the conditional probability is the same for unions of elements, which implies X_k is independent of the entire σ algebra). Hence, X_k would be independent of the smaller σ field $\sigma(X_1, \cdots, X_{k-1})$ - note, any set $[X_j \in H_j]$ can be formed as unions of sets of the form A_v, where the j entry is 0 or 1, and we union over the finite possibilities for the remaining entries. This implies independence of $X_1, \cdots, X_n$ by induction.

And so, we just need to prove that $P[X_k = 1|A_v] = \frac{1}{n-k+1}$. Let j be the rightmost 1 among $x_1, \cdots, x_{k-1}$ ($j = 0$ if there are none). Then ω lies in A_v if and only if it permutes $y_1, \cdots, y_j$ among themselves (as specified by the values $x_1, \cdots, x_{j-1}, x_j = 1, y_1, \cdots, y_j$), and sends each of $y_{j+1}, \cdots, y_{k-1}$ to the y just to the right (as the remaining y must not complete a cycle - they must be a partial cycle). And so, A_v fixes the first $k-1$ positions, and there are $(n-(k-1))! = (n-k+1)!$ possible remaining positions, which corresponds with A_v containing $(n-k+1)!$ sample points. Note, $X_k(\omega) = 1$ if any only if ω also sends y_k to y_{j+1} - and so, $\omega \in A_v \cap [X_k = 1]$ contains $(n-k)$ non fixed positions, meaning the number of sample points in $A_v \cap [X_k = 1]$

is $(n-k)!$. Given that each point is equally likely, we have:

$$P[X_k = 1 | A_v] = \frac{P\left(A_v \cap [X_k = 1]\right)}{P(A_v)} = \frac{(n-k)!/n!}{(n-k+1)!/n!} = \frac{1}{n-k+1}$$

And we have thus concluded our probability fact, and that $X_1, \cdots, X_n$ are independent. qed.

Existence of Independent Sequences

Definition - Distribution The *distribution* of a simple random variable X is the *probability measure* μ defined for all subsets of A of the line by:

$$\mu(A) = P[X \in A]$$

This is a probability measure. We have $\mu(\mathbb{R}) = 1$, $\mu(\emptyset) = 0$, $0 \leq \mu(A) \leq 1$, and for disjoint sequences we do have countable additivity (by countable additivity of $\mathcal{F}$). Note - I think this is different from the case in which we can't have a probability on every subset, *and* have countable additivity, as in this case, countable additivity is equivalent to finite additivity (given only a finite number of disjoint sets can have non zero probabilities).

Note that the distribution for a SRV is *discrete* in the sense that there are finite or countably many points $\omega \in \mathbb{R}$ that have a probability mass. Namely, these are the points in the finite range of X, $x_1, \cdots, x_l$, for which:

$$\mu\{x_i\} = P[X = x_i] = p_i$$

Further, if $A = \{x_1, \cdots, x_l\}$, we have that $\mu(A) = 1$, and so μ is discrete, and has *finite support* (recall, a support is a subset of Ω with probability one - in this case, we have a support that is finite).

Theorem 5.3 - Existence of SRV distributions for arbitrary finite support measures Let $\{\mu_n\}$ be a sequence of probability measures on the class of all subsets of the line, each having ***finite support***. There exists on some probability space $(\Omega, \mathcal{F}, P)$ an independent sequence $\{X_n\}$ of simple random variables such that X_n has distribution μ_n.

Proof of Theorem 5.3 Note, I won't give the full proof here, as it is long. However, the main idea is very simple and clever, and so that is all that I will describe here. As μ_n has finite support, we can take $x_{n,1}, \cdots, x_{n,l_n}$ as the finite distinct points on which μ_n concentrates its mass. Note, by finite support, we have a finite A with probability 1, and A^c has probability 0, and so μ_n must give mass concentrations to only the finite points in A.

The idea of the proof is to take $(\Omega, \mathcal{F}, P)$ as $((0,1], \mathcal{B}, \lambda)$, ie the borel sets on $(0,1]$ and the lebesgue measure. Then, we just split $(0,1]$ into intervals of lengths that correspond to $\mu_n(x_{n,1}), \cdots, \mu_n(x_{n,l_n})$. These intervals will split the intervals that we created for μ_{n-1}, and X_n will assign output values of $x_{n,i}$ for ω in these intervals. Independence comes from, in an intuitive sense, that if we split *each* of the previous random variables intervals - knowing which interval the previous random variable landed in won't give any information about the current one.

As a simple example, we go over the case where μ_n will assign p_n to 0, and $q_n = 1 - p_n$ to 1 - ie, each distribution concentrates its mass on two points, 0 and 1. Split $(0,1]$ into two intervals I_0 and I_1 of lengths p_1 and q_1. Define $X_1(\omega) = 0$ for $\omega \in I_0$ and $X_1(\omega) = 1$ for $\omega \in I_1$. If P is the Lebesgue measure, it is clear:

$$P[X_1 = 0] = p_1 = \mu_1(0) \qquad P[X_1 = 1] = q_1 = \mu_1(1)$$

And so X_1 has distribution μ_1 (ie, the distribution probability measure matches the μ_1 probability measure on all subsets of $\mathbb{R}$). Now, split I_0 into two intervals I_{00} and I_{01} of lengths p_1p_2 and p_1q_2, and I_1 into two intervals I_{10} and I_{11} of lengths q_1p_2 and q_1q_2. Define $X_2(\omega) = 0$ for $\omega \in I_{00} \cup I_{10}$, and $X_2(\omega) = 1$ for $\omega \in I_{01} \cup I_{11}$. It is clear that:

$$P[X_1 = 0, X_2 = 0] = p_1p_2$$

And similarly for all other three possibilities. It is also clear that $P[X_2 = 0] = p_1p_2 + q_1p_2 = p_2$, and so X_2 has distribution μ_2, and it is independent of X_1. We continue iteratively - X_3 splits the four intervals $I_{00}, I_{01}, I_{10}, I_{11}$ and so on. Each X_i is independent of the previous X_j, and by induction all of the X_i are independent.

The argument for general finite support μ_i is pretty much exactly the same. Except, with each next iteration, we split each of the previous intervals into l_n intervals, not just 2 intervals. qed.

Usefulness of the Existence of SRV Distributions for Arbitrary Finite Support Measures Many of the theorems in probability concern themselves with specific distributions, or distributions that have specific properties. However, we want to know that the theorems are not just true *in a vacuous sense* through their conditions never being fulfilled. Existence of actual random variables with the specified distributions helps us understand that actually yes, these theorems can apply to real things. Note, this theorem has fairly restrictive conditions - requiring a finite support measure and simple random variables. In later chapters, more complicated existence theorems will be given.

Expected Value

Definition: Expected Value A simple random variable in one of it's indicator sum representations is assigned an *expected value* or *mean value* of:

$$E[X] = E\left[\sum_i x_i I_{A_i}\right] = \sum_i x_i P(A_i)$$

Another alternative form is:

$$E[X] = \sum_x x P[X = x]$$

Note: If X is a simple random variable on the unit interval, and if the A_i happen to be subintervals, then the expected value definition coincides with the Riemann integral definition. More general notions of integral and expected value will be given later on.

From the indicator definition, it follows that for SRV X, $f(X) = \sum_i f(x_i) I_{A_i}$, and hence:

$$E[f(X)] = \sum_x f(x) P[X = x]$$

This helps us calculate values like the *kth moment* of X, defined as $E[X^k]$.

Properties of the Expected Value Note, it is really easy to prove these properties for the expected value of simple random variables:

1. Linear: $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$
2. Preserves Order: If $X \leq Y$ then $E[X] \leq E[Y]$
3. $|E[X]| \leq E[|X|]$ By the two above.
4. $|E[X - Y]| \leq E[|X - Y|]$ as $X - Y$ is a SRV as well.

These properties will be used repeatedly, as well as the following theorem on expected values and limits.

Definition - Uniformly Bounded If there is a finite K such that $|X_n(\omega)| \leq K$ for all ω and n, the X_n are *uniformly bounded.*

Theorem 5.4 - Uniformly Bounded Sequence implies Uniformly Bounded Limit If $\{X_n\}$ is uniformly bounded, and if $X = \lim_n X_n$ with probability 1, then $E[X] = \lim_n E[X_n]$.

Again, I don't go over the complete proof here. I just will outline some notes for reference to come back to:

1. Recall, convergence with probability 1 implies convergence in probability: $X_n \to_p X$, which means the probability of the set of ω that have X_n and X differ by more than ϵ goes to 0.

2. Note, X is assumed to be a simple random variable, which equals the limit with probability 1. However, also note, it must be a simple random variable with finite range. Assume it didn't have finite range - then there would be sets with positive probability outside of the finite range of each X_n, and then we would have convergence in probability.

3. Expecting that X has finite range (or that, at least, it cannot be infinite, as the bounds of X_n apply to X), then we can find K that bounds both $|X|$ and $|X_n|$, so that $|X - X_n| \leq 2K$. Then, the theorem from there is just applying the expected value properties to $|X - X_n|$ and a cleverly found random variable that bounds $|X - X_n|$ (let $A = [|X - X_n| \geq \epsilon]$, consider $2KI_A(\omega) + \epsilon I_{A^c}(\omega)$).

Thus, we can easily conclude the theorem with the expected value properties above and convergence in probability. qed. Theorems like 5.4 are constantly used. The general version is Lebesgue's dominated convergence theorem.

Example 5.7 - Counter Example to Theorem 5.4 Without Uniform Boundedness Take $X(\omega) = 0$ on the unit interval, and $X_n(\omega) = n^2$ if $0 < \omega \leq n^{-1}$ and 0 if $n^{-1} < \omega \leq 1$. We have $X_n(\omega) \to X(\omega)$ for every ω - and so we have convergence in probability. However, $E[X_n] = n$, which does not converge to $E[X] = 0$. We are missing a additional hypothesis - uniform boundedness gives us one, but there are others in the future as well.

Corollary to 5.4 If $X = \sum_n X_n$ on an $\mathcal{F}$ set of probability 1 (convergence in probability), and if the partial sums of $\sum_n X_n$ are uniformly bounded, then $E[X] = \sum_n E[X_n]$.

Note that the partial sums S_n are uniformly bounded, and $X = \lim_n S_n$ with probability 1, and so $E[X] = \lim_n E[S_n] = \lim_n \sum_n E[X_n]$ by linearity. qed.

Expected Value for Independent Random Variables We have for X and Y independent SRV:

$$XY = \sum_{ij} x_i y_j I_{A_i \cap B_j}$$

If the x_i are distinct from each other, and the y_j are distinct from each other (which can always be made to be the case), we have $A_i = [X = x_i]$ and $B_j = [Y = y_j]$, and by independence, we have $P[A_i \cap B_j] = P[A_i]P[B_j]$. So, we can decompose to find:

$$E[XY] = E[X]E[Y]$$

For *independent simple random variables.* If X, Y, Z are independent, so are XY and Z by our 2d array argument, and so we can conclude:

$$E[XYZ] = E[X]E[Y]E[Z]$$

We can continue iteratively.

Variance If $E[X] = m$, the *variance* of X is:

$$Var[X] = E\left[(X-m)^2\right] = E[X^2] - m^2$$

As $aX + b$ has mean $am + b$, it is easy to simplify to find:

$$Var[aX + b] = a^2 Var[X]$$

With a similar simplification, we can find the variance of the sum of *independent* $X_1, \cdots, X_n$ is:

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i]$$

Alternate Expected Value Definition Suppose X is nonnegative, and order its range $0 \leq x_1 < x_2 < \cdots < x_k$. Then, we have:

$$E[X] = \sum_{i=1}^{k} x_i P[X = x_i] = x_k P[X \geq x_k] + \sum_{i=1}^{k-1} x_i P[(X \geq x_i) \setminus (X \geq x_{i+1})]$$

Where, the last equality comes from the fact that as X_i only takes on the values of x_i, we know that if it is between x_i and x_{i+1}, it must equal x_i. this simplifies to:

$$= \sum_{i=1}^{k-1} x_i \left(P[X \geq x_i] - P[X \geq x_{i+1}]\right) + x_k P[X \geq x_k]$$

$$= x_1 P[X \geq x_1] + \sum_{i=2}^{k} (x_i - x_{i-1}) P[X \geq x_i]$$

As $P[X \geq x] = P[X \geq x_i]$ for $x_{i-1} < x \leq x_i$, we have that the above can be rewritten as a Riemann integral of a step function:

$$E[X] = \int_0^\infty P[X \geq x] dx$$

Note: the x_i values come in along the domain from 0 to ∞ - for $0 \leq x \leq x_1$, the probabilities are 1, and so the riemann integral introduces a rectangle of area $1 \times x_1$, and so on. See the diagram in the book for a better understanding.

Inequalities

Markov's Inequality Note that for nonnegative SRV X, we have for positive a:

$$E[X] = \sum_x x P[X = x] \geq \sum_{x : x \geq a} x P[X = x] \geq a \sum_{x : x \geq a} P[X = x]$$

$$\implies P[X \geq a] \leq \frac{1}{a} E[X]$$

Now, replace X with $|X|^k$, and we have:

$$P[|X| \geq \alpha] = P[|X|^k \geq \alpha^k] \leq \frac{1}{\alpha^k} E\left[|X|^k\right]$$

Chebyshev's Inequality Taking the above Markov inequality for $k = 2$, and subtracting $m = E[X]$ from X, gives us:

$$P\left[|X - m| \geq \alpha\right] \leq \frac{1}{\alpha^2} E\left[|X - m|^2\right] = \frac{1}{\alpha^2} Var[X]$$

Jensen's Inequality A function φ is *convex* if $\varphi(px+(1-p)y) \le p\varphi(x)+(1-p)\varphi(y)$. A sufficient condition for this is φ has a nonnegative second derivative (for all values in it's domain). By induction, we easily have:

$$\varphi\left(\sum_{i=1}^{l} p_i x_i\right) \le \sum_{i=1}^{l} p_i \varphi(x_i)$$

For p_i nonnegative and sum to 1, and x_i in the domain of φ. If SRV X assumes the value x_i with probability p_i, we have Jensen's inequality:

$$\varphi\left(E[X]\right) = \varphi\left(\sum_{i=1}^{l} p_i x_i\right) \le \sum_{i=1}^{l} p_i \varphi(x_i) = E\left[\varphi(X)\right]$$

Which is valid if φ is convex on an interval containing the range of X.

Holder's Inequality Suppose that:

$$\frac{1}{p}+\frac{1}{q}=1 \qquad p>1 \text{ and } q>1$$

Then, we have:

$$E[|XY|] \le \left(E[|X|^p]\right)^{1/p}\left(E[|Y|^q]\right)^{1/q}$$

The proof of this is the following. Assume that the right side of the above is positive. If a and b are positive, there exist s and t such that:

$$a = e^{p^{-1}s} \qquad b = e^{q^{-1}t}$$

As e^x is conved:

$$e^{p^{-1}s+q^{-1}t} \le p^{-1}e^s + q^{-1}e^t \implies ab \le \frac{a^p}{p}+\frac{b^q}{q}$$

This is an interesting inequality, and it holds for nonnegative, as well as positive a and b. Let u and v be $\left(E[|X|^p]\right)^{1/p}$ and $\left(E[|Y|^q]\right)^{1/q}$. For each ω:

$$\left|\frac{X(\omega)Y(\omega)}{uv}\right| \le \frac{1}{p}\left|\frac{X(\omega)}{u}\right|^p + \frac{1}{q}\left|\frac{Y(\omega)}{v}\right|^q$$

By our above inequality. And so, taking expected values, and pulling out scalars, we have:

$$\frac{1}{uv}E[|XY|] \le \frac{1}{pu^p}E[|X|^p] + \frac{1}{qv^q}E[|Y|^q] = \frac{1}{p}+\frac{1}{q} = 1$$

$$\implies E[|XY|] \le uv = \left(E[|X|^p]\right)^{1/p}\left(E[|Y|^q]\right)^{1/q}$$

And so, we have Holder's inequality. qed.

Schwarz's Inequality If $p = q = 2$, Holder's inequality becomes *Schwarz's* inequality:

$$E[|XY|] \leq \left(E[X^2]\right)^{1/2} \left(E[Y^2]\right)^{1/2}$$

Note, we drop the absolute value on X^2, Y^2 as the value is always positive.

Lyapounov's Inequality Suppose $0 < \alpha < \beta$. Take $p = \frac{\beta}{\alpha}$, and $q = \frac{\beta}{\beta - \alpha}$ - note $1/p + 1/q = 1$. From Holder's inequality, let $Y(\omega) = 1$, and replace X by $|X|^\alpha$. Then, we have *Lyapounov's inequality*:

$$E[|X|^\alpha] = E[||X|^\alpha Y|] \leq E\left[(|X|^\alpha)^p\right]^{1/p} (E[|Y|^q])^{1/q} = E\left[|X|^\beta\right]^{\alpha/\beta}$$

$$\implies E[|X|^\alpha]^{1/\alpha} \leq E\left[|X|^\beta\right]^{1/\beta}$$

Problems

5.1 Measurable on a Tail Field implies $X = c$ with probability 1

1. Show that X is measurable with respect to the σ field $\mathcal{G}$ if and only if $\sigma(X) \subset \mathcal{G}$. Show that X is measurable $\sigma(Y)$ if and only if $\sigma(X) \subset \sigma(Y)$.

   First, assume X is measurable with respect to $\mathcal{G}$. Note that $\sigma(X)$ is the smallest σ field to which X is measurable with respect to. Note that $\sigma(X)$ is generated by sets of the form:

   $$[X \in H] \qquad H \subset \mathbb{R}$$

   Note, $[X \in H] \in \mathcal{G}$, as X is measurable with respect to $\mathcal{G}$. Thus, $\sigma(X) \subseteq \mathcal{G}$, as $\mathcal{G}$ would be in the intersection defining $\sigma(X)$. Now, assume that $\sigma(X) \subset \mathcal{G}$. Again, X is measurable with respect to $\mathcal{G}$, as $[X = x] \in \sigma(X) \implies [X = x] \in \mathcal{G}$.

   X is measurable $\sigma(Y)$ if and only if $\sigma(X) \subset \sigma(Y)$ is the same as for arbitrary $\mathcal{G}$.

2. Show that, if $\mathcal{G} = \{\emptyset, \Omega\}$, then X is measurable $\mathcal{G}$ if and only if X is constant.

   First, assume X is measurable $\mathcal{G}$. Assume X is not constant - then, X takes on at least two different values, x_1 and x_2. We note that $[X = x_1] \cap [X = x_2] = \emptyset$, and both are nonempty. Note, neither can be $\emptyset$, as they are nonempty, and neither can be Ω, as otherwise the intersection would be nonempty. As X is measurable $\mathcal{G}$, this implies $[X = x_1] \in \mathcal{G}$, which is a contradiction. Thus, X must be constant.

Now, assume that X is constant. We have that $[X = x]$ for $x \in \mathbb{R}$ is either $\emptyset$, if x is not the constant value, or Ω, if x is the constant value. Thus, in all cases, $[X = x] \in \mathcal{G}$, and X is measurable $\mathcal{G}$.

3. Suppose that $P(A)$ is 0 or 1 for every A in $\mathcal{G}$. This holds, for example, if $\mathcal{G}$ is the tail field of an independent sequence, or $\mathcal{G}$ consists of the countable and cocountable sets on the unit interval with the Lebesgue measure. Show that if X is measurable $\mathcal{G}$, then $P[X = c] = 1$ for some constant c.

   By assumption, we have that X is measurable $\mathcal{G}$. Thus, we must have that $P[X = x] = 1$ or 0 for all $x \in \mathbb{R}$. As $[X = x]$ are disjoint sets for all $x \in \mathbb{R}$, and X takes on finite values, we have:

$$P[X = x_1, \cdots, X = x_n] = P[\Omega] = 1 \implies \sum_{i=1}^{n} P[X = x_i] = 1$$

   Note, for a single x_i, we must have that $P[X = x_i] = 1$. Let $x_i = c$, and we have the statement. qed.

5.2 - Existence of independent SRV X_i with density μ_i for finite support μ_i on a nonatomic space

Show that the unit interval can be replaced by any nonatomic probability measure space in the proof of Theorem 5.3.

The construction in Theorem 5.3 is an inductive one. The theorem hinges on first being able to split Ω into l_1 sets each of probability $p_{1,1}, \cdots, p_{1,l_1}$, and then defining X_1 as the indicator sum that takes $\omega \in A_i$ to $x_{1,i}$ that has probability $p_{1,i}$ by the finitely supported measure μ_1. Note, on a *nonatomic* probability measure space, we can split up Ω into sets $A_1, \cdots, A_{l_i}$ of probability $P(A_i) = p_{1,i}P(\Omega) = p_{1,i}$, and so we can perform the first step on our inductive construction with a nonatomic space.

The second step, the inductive step, requires taking each of the sets $A_{i_1,\cdots,i_n}$, for $1 \leq i_1 \leq l_1, \cdots, 1 \leq i_n \leq l_n$, and splitting them into l_{n+1} sets each of probability $p_{n+1,1}, \cdots, p_{n+1,l_{n+1}}$, like we did for Ω in the first step. Note, on a nonatomic probability measure space, problem 2.19d (which we did above) implies that we can find subsets of $A_{i_1,\cdots,i_n}$, which we denote $A_{i_1,\cdots,i_n,j}$, where:

$$P(A_{i_1,\cdots,i_n,j}) = p_{n+1,j}P(A_{i_1,\cdots,i_n})$$

We define $X_{n+1} = x_j$ if $\omega \in A_{i_1,\cdots,i_n,j}$ for any choice of the first n indices. Note, this construction has the same properties we need to prove that $X_1, X_2, X_3, \cdots$ are independent, and $P(X_n = x_j) = p_{n,j}$ is satisfied as well. And so, we are able to indeed prove Theorem 5.3 by just relying on a nonatomic probability measure space, rather than just the unit interval. qed.

5.3 - Expected Value minimizes Variance

Show that $\mu = E[X]$ minimizes $E[(X-\mu)^2]$. For this one, I assume that X is a SRV. We note that as X is a SRV, we have:

$$f(\mu) = E[(X-\mu)^2] = \sum_{i=1}^{n}(x_i-\mu)^2 P[X=x_i]$$

This is a smooth function in μ. Note that it is maximized (or minimized) where the first derivative equals 0. We have:

$$f'(\mu) = -\sum_{i=1}^{n} 2(x_i-\mu)P[X=x_i]$$

Setting equal to 0, we have a critical point if:

$$\sum_{i=1}^{n} x_i P[X=x_i] = \mu \sum_{i=1}^{n} P[X=x_i]$$

Note, the sum goes over all values in the range of X, and so it equals 1. Thus, we have a critical point at:

$$\mu = \sum_{i=1}^{n} x_i P[X=x_i] = E[X]$$

We have a second derivative of:

$$f''(\mu) = \sum_{i=1}^{n} P[X=x_i] = 1 > 0$$

And so, by the second derivative test, $\mu = E[X]$ indeed minimizes $E[(X-m)^2]$. qed.

5.4 Chebyshev's Inequality can reach equality

Suppose that X assumes the values $m-\alpha, m, m+\alpha$ with probabilities $p, 1-2p, p$, and show that there is equality in Chebyshev's inequality. Thus, Chebyshev's inequality cannot be improved upon without special assumptions on X.

Well, Chebyshev's inequality needs m to be the expected value - however, note that that is indeed the case (by symmetry and directly). We have that:

$$P[|X-m| \geq \alpha] = 2p$$

As when $X = m-\alpha$ or $x = m+\alpha$, we have $|X-m| = \alpha \geq \alpha$. We now note that:

$$\frac{1}{\alpha^2}Var[X] = \frac{1}{\alpha^2}\left(p\alpha^2 + (1-2p)*0^2 + p\alpha^2\right) = 2p$$

And so, we have:

$$P[|X-m| \geq \alpha] = 2p = \frac{1}{\alpha^2}Var[X]$$

And so there are cases where there is equality in Chebyshev's inequality. qed.

5.5 Cantelli's Inequality

Suppose that X has mean m and variance σ^2

1. Prove Cantelli's inequality:

   $$P[X-m \geq \alpha] \leq \frac{\sigma^2}{\sigma^2+\alpha^2} \qquad \alpha \geq 0$$

   First, we assume $m = 0$. We have for $x > 0$:

   $$P[X-m \geq \alpha] = P[X \geq \alpha] = P[X+x \geq \alpha+x] \leq P[(X+x)^2 \geq (\alpha+x)^2]$$

   Where the last inequality comes from the fact that we are including potential negative values of $X+x$ whose magnitude might be larger than $\alpha+x$. Continuing, we have by Markov's inequality:

   $$\leq \frac{1}{(\alpha+x)^2}E[(X+x)^2] = \frac{1}{(\alpha+x)^2}\left(E[X^2]+2xE[X]+x^2\right) = \frac{\sigma^2+x^2}{(\alpha+x)^2}$$

   We now want to find what x minimizes the above. I won't include the first/second derivative test here - but they conclude that $x = \sigma^2/\alpha$ minimizes the fraction, and gives us our strictest inequality. Simplifying, we have:

$$\frac{\sigma^2 + \sigma^4/\alpha^2}{(\alpha + \sigma^2/\alpha)^2} = \frac{\frac{\sigma^2(\alpha^2+\sigma^2)}{\alpha^2}}{\frac{(\alpha^2+\sigma^2)^2}{\alpha^2}} = \frac{\sigma^2(\alpha^2+\sigma^2)}{(\alpha^2+\sigma^2)^2} = \frac{\sigma^2}{\sigma^2+\alpha^2}$$

Thus, if $m = 0$, we can conclude that:

$$P[X \geq \alpha] \leq \frac{\sigma^2}{\sigma^2 + \alpha^2}$$

Now, the only difference comes when we examine $X - m$. However, note that we can treat $X - m$ as our X above with mean 0. And so, we directly have that:

$$P[X - m \geq \alpha] \leq \frac{\sigma^2}{\sigma^2 + \alpha^2} \qquad \alpha \geq 0$$

2. Show that:

$$P[|X - m| \geq \alpha] \leq \frac{2\sigma^2}{\sigma^2 + \alpha^2}$$

When is this better than Chebyshev's inequality? First, we note:

$$P[|X - m| \geq \alpha] = P[X - m \geq \alpha] + P[-(X - m) \geq \alpha] \leq \frac{2\sigma^2}{\sigma^2 + \alpha^2}$$

Where the first equality comes from union of disjoint sets, and the final inequality comes from applying Cantelli's to each set. We note by Chebychev's:

$$P[|X - m| \geq \alpha] \leq \frac{\sigma^2}{\alpha^2}$$

We would want to use Cantelli's when the bound is tighter, ie, when:

$$\frac{2\sigma^2}{\sigma^2 + \alpha^2} \leq \frac{\sigma^2}{\alpha^2}$$

Simplifying, this happens when:

$$\iff 2 \leq (\sigma^2 + \alpha^2)\frac{1}{\alpha^2} \iff 1 \leq \frac{\sigma^2}{\alpha^2} \iff \alpha \leq \sigma$$

Where in the last $\iff$, we note that α is always positive, and we can take σ as positive, and so the absolute value statement follows. So, when we want to examine whether X will exceed one standard deviation, we use Chebychev's, otherwise, use Cantelli's.

3. By considering a random variable assuming two values, show that Cantelli's inequality is sharp.

   Let $P[X = 1] = p$ and $P[X = 0] = 1 - p$. We have $E[X] = p$ and $Var[X] = p - p^2$. Note that, intuitively:

   $$P[X \geq 1] = p$$

   We have that Cantelli's lemma is sharp, as we have:

   $$P[X - p \geq 1 - p] \leq \frac{p - p^2}{p - p^2 + (1-p)^2} = \frac{p - p^2}{1 - p} = p$$

   And in this case, we have actually achieved equality, as:

   $$P[X - p \geq 1 - p] = P[X \geq 1] = p$$

   And so, in this sense, there are cases where Cantelli's lemma is actually *sharp*.

5.8 Multiple Variable Jensen Inequality

1. Let f be a convex real function on a convex set C in the plane. Suppose that $(X(\omega), Y(\omega)) \in C$ for all ω and prove a two-dimensional Jensen's inequality:

   $$f(E[X], E[Y]) \leq E[f(X, Y)]$$

   I give a proof of this without any assumptions. We can note this *without* an independence assumption, as we note that

   $$P[X = x] = P[X = x, \Omega] = P[X = x, \bigcup Y = y] = P[\bigcup X = x, Y = y]$$

   $$= \sum_y P[X = x, Y = y]$$

   Where the last step is over the sum of disjoint sets. And so, we find:

   $$f(E[X], E[Y]) = f\left(\sum_x xP[X = x], \sum_y yP[Y = y]\right)$$

   $$= f\left(\sum_x x \sum_y P[X = x, Y = y], \sum_y y \sum_x P[X = x, Y = y]\right)$$

$$= f\left(\sum_x \sum_y xP[X = x, Y = y], \sum_x \sum_y yP[X = x, Y = y]\right)$$

$$= f\left(\sum_x \sum_y P[X = x, Y = y](x, y)\right)$$

By convexity of f, we have the above is bounded by:

$$\leq \sum_x \sum_y P[X = x, Y = y]f(x, y) = E[f(X, Y)]$$

And so, in total, we have:

$$f(E[X], E[Y]) \leq E[f(X, Y)]$$

2. Show that f is convex if it has continuous second derivatives that satisfy:

$$f_{11} \geq 0 \qquad f_{22} \geq 0 \qquad f_{11}f_{22} \geq f_{12}^2$$

We can make use of the fact that a twice differentiable function *in one dimension* is convex iff its second derivative is greater than or equal to 0. Proofs for the iff. If f is convex, then it's second derivative is larger than 0: stack exchange - this relies on two uses of the mean value theorem, and is easy. Now assume the second derivative is larger than 0, but not convex - then, you could apply the same mean value proof on the line segment that is below the function stack exchange.

So, we indeed have a twice differentiable function is convex iff its second derivative is nonnegative. Now, we examine:

$$\varphi(t) = f(t(x', y') + (1 - t)(x, y))$$

We note that if φ is convex in t then f is. We have that φ being convex in t implies that:

$$\varphi(pt + (1 - p)t') \leq p\varphi(t) + (1 - p)\varphi(t')$$

As then, we can take $t = 1$ and $t' = 0$ to see:

$$\varphi(p) \leq p\varphi(1) + (1 - p)\varphi(0)$$

$$\implies f(p(x', y') + (1 - p)(x, y)) \leq pf(x', y') + (1 - p)f(x, y)$$

So, we just need convexity of $\varphi(t)$ for $t \in [0,1]$. Note, we make use of the second derivative test. We have that:

$$\varphi'(t) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} x'-x \\ y'-y \end{pmatrix} = f_1(t(x',y')+(1-t)(x,y))a+f_2(t(x',y')+(1-t)(x,y))b$$

Where $a = x' - x$ and $b = y' - y$. The above it by the chain rule. Applying again, we have that:

$$\varphi'' = f_{11}a^2 + f_{12}ab + f_{21}ab + f_{22}b^2 = f_{11}a^2 + 2f_{12}ab + f_{22}b^2$$

Noting that continuous second partials implies $f_{12} = f_{21}$. If $f_{11} > 0$, we have:

$$\varphi'' = \frac{1}{f_{11}} (f_{11}a + f_{12}b)^2 + \frac{1}{f_{11}} \left(f_{11}f_{22} - f_{12}^2\right) b^2 \geq 0$$

If $f_{11}f_{22} \geq f_{12}^2$. Note, so if we have:

$$(f_{11} > 0 \text{ and } f_{22} \geq 0) \text{ or } (f_{11} \geq 0 \text{ and } f_{22} > 0) \text{ and } f_{11}f_{22} \geq f_{12}^2$$

Then we have that f is convex. If both equal 0, then $f_{12} = 0$ as well, and we have that the first derivative is constant. In which case, we have that f is still convex, as f is a plane. qed.

5.9 Prove Holders for nonnegative X, Y via Jensen's

Holder's inequality is equivalent to $E[X^{1/p}Y^{1/q}] \leq E^{1/p}[X]E^{1/q}[Y]$ for $p^{-1} + q^{-1} = 1$, where X and Y are nonnegative random variables. Derive this from the previous problem.

I think, this requires noting something like $f(x,y) = x^{1/p}y^{1/q}$ is convex. Or, perhaps concave, in which case we could use symmetry to note the other direction. As f has continuous second derivatives, we can use the second derivative check applied above. We note:

$$f_{11} = \frac{\left(\frac{1}{p} - 1\right) x^{1/p-2}y^{1/q}}{p} \qquad f_{22} = \frac{\left(\frac{1}{q} - 1\right) x^{1/p}y^{1/q-2}}{p} \qquad f_{12} = \frac{x^{1/p-1}y^{1/p-1}}{pq}$$

Note, $f_{11} \leq 0$ and $f_{22} \leq 0$. If $f_{11}f_{22} \leq f_{12}^2$, we will have our *concave* criteria, which is equivalent to $-f$ being convex (note, we can just carry the negative sign and also just prove $-f$ is convex). We have:

$$f_{11}f_{22} = \frac{\left(\frac{1}{p} - 1\right)\left(\frac{1}{q} - 1\right) x^{2/p-2}y^{2/q-2}}{pq} = \frac{x^{2/p-2}y^{2/q-2}}{(pq)^2}$$

$$f_{12}^2 = \frac{x^{2/p-2}y^{2/p-2}}{(pq)^2}$$

So, we indeed have our concave criteria, and by the two dimensional Jensen's inequality (for concave functions), we have:

$$E[X^{1/p}Y^{1/q}] = E[f(X,Y)] \leq f(E[X],E[Y]) = E^{1/p}[X]E^{1/q}[Y]$$

Thus, we have proven Holder's inequality for nonnegative functions using Jensen's inequality. qed.

5.10 Prove Minkowski's Inequality for nonnegative X, Y via Jensen's

Minkowski's inequality is:

$$E^{1/p}[|X+Y|^p] \leq E^{1/p}[|X|^p] + E^{1/p}[|Y|^p]$$

Valid for $p \geq 1$. It is enough to prove for nonnegative X, Y that:

$$E[(X^{1/p} + Y^{1/p})^p] \leq (E^{1/p}[X] + E^{1/p}[Y])^p$$

Note why it is enough: we can bring the p to the otherside, and replace X and Y with X^p and Y^p, which are still nonnegative. Then, if it is true for nonnegative X and Y, then it will be true for $|X|$ and $|Y|$. Note, this problem is the same as before - we have:

$$f(x,y) = (x^{1/p} + y^{1/p})^p$$

If we just show that f is concave (equivalently, $-f$ is convex), the equality comes through. Note the second derivative test:

$$f_{11} = -\frac{(p-1)\, y^{\frac{1}{p}} x^{\frac{1}{p}-2} \left(x^{\frac{1}{p}} + y^{\frac{1}{p}}\right)^{p-2}}{p}$$

$$f_{22} = -\frac{(p-1)\, x^{\frac{1}{p}} y^{\frac{1}{p}-2} \left(y^{\frac{1}{p}} + x^{\frac{1}{p}}\right)^{p-2}}{p}$$

As $p \geq 1$, we have $f_{11} \leq 0$ and $f_{22} \leq 0$. We have that:

$$f_{11}f_{22} = \frac{(p-1)^2\, x^{\frac{2}{p}-2} y^{\frac{2}{p}-2} \left(y^{\frac{1}{p}} + x^{\frac{1}{p}}\right)^{2p-4}}{p^2}$$

We also note:

$$f_{12} = \frac{(p-1)\, x^{\frac{1}{p}-1} y^{\frac{1}{p}-1} \left(y^{\frac{1}{p}} + x^{\frac{1}{p}}\right)^{p-2}}{p}$$

From which it is clear that $f_{12}^2 = f_{11}f_{22}$. Thus, two dimensional Jensen's inequality for concave functions applies, and we can conclude Minkowski's inequality. qed.

5.11 Average Number of Successful Bernoulli Trials approaches p

For events $A_1, A_2, \cdots$ not necessarily independent, let $N_n = \sum_{k=1}^{n} I_{A_k}$ be the number to occur among the first n. Let:

$$\alpha_n = \frac{1}{n}\sum_{k=1}^{n} P(A_k) \qquad \beta_n = \frac{2}{n(n-1)} \sum_{1\leq j<k\leq n} P(A_j \cap A_k)$$

So the first is just the average of the probabilities, and the second is the average of all unique intersection probabilities. Show that:

$$E[n^{-1}N_n] = \alpha_n \qquad Var[n^{-1}N_n] = \beta_n - \alpha_n^2 + \frac{\alpha_n - \beta_n}{n}$$

These can be proved making extensive use of the linearity properties of the expected value (and noting that N_n is a SRV). We note:

$$E[n^{-1}N_n] = \frac{1}{n}E\left[\sum_{k=1}^{n} I_{A_k}\right] = \frac{1}{n}\sum_{k=1}^{n} E[I_{A_k}] = \frac{1}{n}\sum_{k=1}^{n} P(A_k) = \alpha_n$$

$$\begin{aligned}
Var[n^{-1}N_n] = E[(n^{-1}N_n)^2] - m^2 &= \frac{1}{n^2}E\left[\left(\sum_{k=1}^{n} I_{A_k}\right)^2\right] - \alpha_n^2 \\
&= \frac{1}{n^2}\sum_{k=1}^{n} E\left[I_{A_k}^2\right] + \frac{2}{n^2}\sum_{1\leq j<k\leq n} E\left[I_{A_j}I_{A_k}\right] - \alpha_n^2 \\
&= \frac{1}{n^2}\sum_{k=1}^{n} P(A_k) + \frac{2}{n^2}\sum_{1\leq j<k\leq n} P(A_j \cap A_k) - \alpha_n^2 \\
&= \frac{\alpha_n}{n} + (n-1)\frac{\beta_n}{n} - \alpha_n^2 \\
&= \beta_n - \alpha_n^2 + \frac{\alpha_n - \beta_n}{n}
\end{aligned}$$

So, we have found the variance and expected value of $n^{-1}N_n$. We note that $Var[n^{-1}N_n] \to 0$ if and only if $\beta_n - \alpha_n^2 \to 0$. This is because $\frac{\alpha_n - \beta_n}{n} \to 0$ always, as the numerator has an absolute value bounded by 2. We note that if the A_n are independent, and $P(A_n) = p$, then:

$$\beta_n - \alpha_n^2 = \frac{2}{n(n-1)} * \frac{n(n-1)}{2}p^2 - \left(\frac{1}{n} * np\right)^2 = p^2 - p^2 = 0$$

So, then for Bernoulli trials, we can conclude that the average number of trials that are successful approaches p as the variance goes to 0. qed.

5.12 Expected Value as Infinite Probability Sum

Show that, if X has nonnegative integers as values, then:

$$E[X] = \sum_{n=1}^{\infty} P[X \geq n]$$

We recall from the reading that if X is nonnegative, then for the values ordered in its range $0 \leq x_1 < x_2 < \cdots < x_k$, we have:

$$E[X] = \sum_{i=1}^{k} x_i P[X = x_i] = \sum_{i=1}^{k-1} x_i \left(P[X \geq x_i] - P[X \geq x_{i+1}]\right) + x_k P[X \geq x_k]$$

$$= x_1 P[X \geq x_1] + \sum_{i=2}^{k} (x_i - x_{i-1}) P[X \geq x_i]$$

The first step, we rewrite $P[X = x_i]$ as $P[X \geq x_i] - P[X \geq x_{i+1}]$, and then we reorder the sum. However, now we note, we can say that X takes on *every* nonnegative integer as a value - with the realization that, $P[X = i] = 0$ for all but finitely many X. Then, the above becomes:

$$= 0P[X \geq 1] + \sum_{i=1}^{\infty} (i - (i-1)) P[X \geq i] = \sum_{n=1}^{\infty} P[X \geq n]$$

So, in conclusion, we do find:

$$E[X] = \sum_{n=1}^{\infty} P[X \geq n]$$

5.15 Convergence in Probability does not imply Convergence with Probability One

By Theorem 5.3, for any prescribed sequence of probabilities p_n, there exists (on some space) an independent sequence of events A_n satisfying $P(A_n) = p_n$. This can be found via identifying independent random variables with $P(X_n = 1) = p_n$, if we're making direct use of Theorem 5.3.

Show that if $p_n \to 0$, but $\sum p_n = \infty$, we have a counterexample (like Example 5.4) to the converse of Theorem 5.2(ii).

Recall, Theorem 5.2ii says that convergence with probability 1:

$$P\left[\lim_n X_n(\omega) = X(\omega)\right] = 1$$

Implies convergence in probability:

$$\forall \epsilon > 0 \qquad \lim_n P\left[|X_n - X| \geq \epsilon\right] = 0$$

The converse is that convergence in probability implies convergence with probability 1. We want to give a counter example to this statement. We let X_n be as defined above, and $X = 0$. We have:

$$\left[\lim_n X_n = X\right]^c = \{\omega : X_n(\omega) \neq 0\text{i.o.}\} = [A_n\text{i.o.}]$$

As the A_n are independent, the second borel cantelli lemma tells us that $P[A_n\text{i.o.}] = 1$, and so:

$$P\left[\lim_n X_n(\omega) = X(\omega)\right] = 0$$

However, note we have convergence in probability, as:

$$\lim_n P[|X_n - X| \geq \epsilon] = \lim_n P(A_n) = 0$$

So, we have convergence in probability, but we don't have convergence with probability 1. And so, we have a counter example to the converse, for every sequence of p_n satisfying $p_n \to 0$ but $\sum p_n = \infty$. qed.

5.16 Independent Sets on a Discrete Space

Suppose that $0 \leq p_n \leq 1$ and put $\alpha_n = \min\{p_n, 1 - p_n\}$. Show that, if $\sum \alpha_n$ converges, then on some discrete probability space there exist independent events A_n satisfying $P(A_n) = p_n$. Compare Problem 1.1b.

For one, the statement in problem 1.1b notes that if $\sum \alpha_n$ diverges, then it is not possible to have independent events with $P(A_n) = p_n$ on a discrete space, as that would imply each $\omega \in \Omega$ has $P(\omega) = 0$, given that ω belongs to some dyadic set of the form $A_1^{c?} \cap A_2^{c?} \cap \cdots$, which has probability 0.

My first thought is the following. Take the discrete space $\Omega = \{1, 2, \cdots\}$. Try and do a *splitting* like we do in Theorem 5.3. We first split Ω into two sets by taking every other point:

$$A_1 = \{1, 3, 5, \cdots\} \qquad A_1^c = \{2, 4, 6, \cdots\}$$

Then, we split A_1 and A_1^c by a similar splitting:

$$A_1' = \{1, 5, 9, \cdots\} \qquad A_1'' = \{3, 7, 11, \cdots\} \qquad A_1^{c'} = \{2, 6, 10, \cdots\} \qquad A_1^{c''} = \{4, 8, 12, \cdots\}$$

And we set:

$$A_2 = A_1' \cup A_1^{c'}$$

And we do the splits iteratively, like for Theorem 5.3, but on a discrete set. We then let $P(A_1) = p_1$, and $P(A_2) = p_2$, and so on, and then we deduce $P(\omega)$ as the multiplication of the probabilities of the set ω is in. Note, the intuition here is that *unlike* in problem 1.1b, not all ω should have probability 0. For instance, if we have $p_n \geq 1 - p_n$ for each n, we have that $\sum 1 - p_n$ converges. Now, consider:

$$P(\{1\}) = \prod_n p_n$$

Note, each p_n should be substantially large enough, such that the above product *does not* go to zero. However, this is not a proof that it is a well defined space. I think, we can maybe rely on the previous problem, or the Borel-Cantelli lemmas. We note that we *do* have that each A_n is independent. So, we have our independent sets on a discrete space. It is just intuition. I think, we can maybe use the i.o. argument used above, to note that $P(\{i\}) \neq 0$, for those i that appear infinitely often in some good set of ω. And, that these i maybe even have probability 1. Which would be enough to prove that the set is well defined.

Let A_i be the independent sets defined above, with $P(A_i) = p_i$. Let C_i be either A_i or A_i^c, where $P(C_i) = 1 - \alpha_i$. We note that C_i, like A_i, are independent. We also note that:

$$\sum_{i=1}^{\infty} C_i = \sum_{i=1}^{\infty} 1 - P(A_i) = -S + \lim_{n \to \infty} \sum_{i=1}^{n} 1 = \infty$$

Thus, by the Second Borel Cantelli Lemma, we have:

$$P[C_i \text{ i.o.}] = 1$$

We thus also note that for $i \in C_i$, we must have some $P(\{i\}) > 0$, as we note:

$$1 = P[C_i \text{ i.o.}] = \sum_{i \in C_i \text{ i.o.}} P(\{i\})$$

By countable additivity. This should conclude the proof, as we were able to create independent A_n on our discrete space, satisfying $P(A_n) = p_n$. But, I still want to highlight the difference between problem 1.1b. First, note that we can't apply the argument in 1.1b to this problem. As α_n does not diverge, we couldn't make use of the same argument, that *every* ω (or i) had probability 0. Now, also note that the argument in 5.16 couldn't be applied to 1.1b either. That is because, we cannot conclude that the sets A_n are independent, given the contradiction that stems from them being independent. As we had no contradiction stemming from independence in this problem, we could continue with applying the Borel-Cantelli lemma.

However, I think one of the main points is the following. Having a convergent sequence *separates* the $i \in \Omega$ into two sets - those that have probability 0, and those that do not. We couldn't find any separation in 1.1b. However, in this problem, as $\sum \alpha_n$ converged, it would be clear that the converging α_n ω had probability 0, whereas the ones where it didn't converge might not have probability 0, in total. Leading to the conclusion. qed.

5.19 Average Prime Power of Pos Integer Prime Factorizations

For integers m and primes p, let $\alpha_p(m)$ be the exact power of p in the prime factorization of m:

$$m = \prod_p p^{\alpha_p(m)}$$

Let $\delta_p(m)$ be 1 or 0 as p divides m or not. Recall P_n - the discrete probability measure on $\Omega = \{1, 2, \cdots\}$ where:

$$P_n(A) = \frac{1}{n} \# \left[m : 1 \leq m \leq n, m \in A \right]$$

Under each P_n, the α_p and δ_p are random variables. First, note that α_p and δ_p are functions from Ω into $\mathbb{R}$. Then, note that for $A \subset \mathbb{R}$, $[\alpha_p \in A]$ is some subset of Ω. Recall that P_n is well defined for *all* subsets of Ω (as only the first n elements really matter).

For distinct primes $p_1, \cdots, p_u$, there are a couple of things we want to show. First:

$$P_n[\alpha_{p_i} \geq k_i, i \leq u] = \frac{1}{n} \left\lfloor \frac{n}{p_1^{k_1} \cdots p_u^{k_u}} \right\rfloor \to \frac{1}{p_1^{k_1} \cdots p_u^{k_u}}$$

Note, the fundamental theorem of arithmetic says that every integer can be expressed as this prime factorization with exponents form. We recall problem 2.18, where we found for the periodic sets $M_a = [ka : k = 1, 2, \cdots]$, we had:

$$P_n(M_a) = \frac{1}{n} \left\lfloor \frac{n}{a} \right\rfloor \to \frac{1}{a} = D(M_a) \qquad M_a \cap M_b = M_{lcm(a,b)}$$

We note that:

$$\{\alpha_{p_i} \geq k_i, i \leq u\} = M_{p_1^{k_1}} \cap \cdots \cap M_{p_u^{k_u}}$$

As every i on the LHS is divisible by each $p_i^{k_i}$, by definition, and every i on the RHS satisfies that $\alpha_{p_i} \geq k_i$, again by definition. We finally note, like in problem 4.15, that:

$$lcm(p_1^{k_1}, \cdots, p_u^{k_u}) = p_1^{k_1} \cdots p_u^{k_u}$$

Assume that it wasn't - then, $x = p_1^{k_1} \cdots p_u^{k_u}$ would have a smaller prime factorization, which would be a contradiction. And so, we have:

$$P_n[\alpha_{p_i} \geq k_i, i \leq u] = P_n\left[M_{p_1^{k_1}} \cap \cdots \cap M_{p_u^{k_u}}\right] = P_n\left[M_{p_1^{k_1} \cdots p_u^{k_u}}\right] = \frac{1}{n}\left\lfloor \frac{n}{p_1^{k_1} \cdots p_u^{k_u}} \right\rfloor \to \frac{1}{p_1^{k_1} \cdots p_u^{k_u}}$$

Next, we want to show:

$$P_n\left[\alpha_{p_i} = k_i, i \leq u\right] \to \prod_{i=1}^{u} \left(\frac{1}{p_i^{k_i}} - \frac{1}{p_i^{k_i+1}}\right)$$

First, to make the notation slightly easier, we define:

$$B = \bigcap_{i=1}^{u} \alpha_{p_i} \geq k_i \qquad \text{and for } C \subseteq [u] \quad B_C = \bigcap_{i=1}^{u} \begin{cases} \alpha_{p_i} \geq k_i + 1 & \text{if } i \in C \\ \alpha_{p_i} \geq k_i & \text{otherwise} \end{cases}$$

From above, it is clear that:

$$\lim_n P_n[B] = \frac{1}{p_1^{k_1} \cdots p_u^{k_u}} \qquad \lim_n P_n[B_C] = \left(\prod_{i \notin C} \frac{1}{p_i^{k_i}}\right)\left(\prod_{i \in C} \frac{1}{p_i^{k_i+1}}\right)$$

We have:

$$P_n\left[\alpha_{p_i} = k_i, i \leq u\right] = P_n\left[B - \bigcup_{i=1}^{u} B_i\right] = P_n[B] - P_n\left[\bigcup_{i=1}^{u} B_i\right]$$

$$= P_n[B] - \left[\sum_{i=1}^{U} P[B_i] - \sum_{1 \leq i < j \leq u} P[B_i \cap B_j] + \cdots\right]$$

Where the above is by the inclusion-exclusion principle. Clearly, $B_i \cap B_j = B_{ij}$, and so taking limits, we have the above equals:

$$= \frac{1}{p_1^{k_1} \cdots p_u^{k_u}} - \sum_{1 \leq i \leq n} \frac{1}{\cdots p_i^{k_i+1}} + \sum_{1 \leq i < j \leq u} \frac{1}{\cdots p_i^{k_i+1} p_j^{k_j+1}}$$

Note, the above is just the expanded form of the multiplication:

$$\prod_{i=1}^{u}\left(\frac{1}{p_i^{k_i}}-\frac{1}{p_i^{k_i+1}}\right)$$

And thus, we can indeed conclude that:

$$P_n\left[\alpha_{p_i}=k_i, i\le u\right]\to\prod_{i=1}^{u}\left(\frac{1}{p_i^{k_i}}-\frac{1}{p_i^{k_i+1}}\right)$$

Similarly, we note:

$$P_n\left[\delta_{p_i}=1, i\le u\right]=P_n[\alpha_{p_i}\ge 1, i\le u]=\frac{1}{n}\left\lfloor\frac{n}{p_1\cdots p_u}\right\rfloor\to\frac{1}{p_1\cdots p_u}$$

By 5.44 and 5.45 (the above conclusions) we can make the statement that α_p and δ_p are "approximately independent for large n under P_n." To be independent, we would have to have that $\sigma(\alpha_p)$ are independent, which would follow if $[\alpha_p = k_i]$ are independent, which does follow from the above conclusions for large n under P_n.

For a function f of positive integers, let:

$$E_n[f]=\frac{1}{n}\sum_{m=1}^{n}f(m)$$

be its expected value under the probability measure P_n. Note, this follows from the definition of the expected value of simple random variables, where f is defined on $\Omega=\{1,2,\cdots\}$, but only the first n integers have nonzero probability mass under P_n. Show that:

$$E_n[\alpha_p]=\sum_{k=1}^{\infty}\frac{1}{n}\left\lfloor\frac{n}{p^k}\right\rfloor\to\frac{1}{p-1}$$

From 5.12, we recall that for nonnegative SRV X:

$$E[X]=\sum_{n=1}^{\infty}P[X\ge n]$$

We thus note:

$$E_n[\alpha_p]=\sum_{k=1}^{\infty}P_n[\alpha_p\ge k]=\sum_{k=1}^{\infty}\frac{1}{n}\left\lfloor\frac{n}{p^k}\right\rfloor$$

$$\to\sum_{k=1}^{\infty}\frac{1}{p^k}=\frac{1}{p}*\frac{p-1}{p}=\frac{1}{p-1}$$

By the limit of a geometric series. This says *roughly* that $(p-1)^{-1}$ is the average power of p in the factorization of large integers.

5.20

First, we note Stirling's Approximation:

$$\log n! = n \log n - n + O(\log n)$$

Ie, we have that:

$$|\log n! - (n \log n - n)| \leq M |\log(n)|$$

For some positive real M. This is actually proved in problem 27.18. See the wiki. We also have

1. We note:

$$E_n[\log] = \frac{1}{n} \sum_{m=1}^{n} \log(m) = \frac{1}{n} \log(n!)$$

Where the final step makes use of the multiplicative law of the logarithm. By Stirling's Approximation, the above equals:

$$= \log n - 1 + \frac{1}{n} O(\log(n)) = \log(n) + O(1)$$

As for the final step - note that $\frac{M \log(n)}{n} \to 0$, and so for some n_0 large enough, we must have the difference $\frac{\log n!}{n} - \log(n)$ is bounded by some scalar to infinity. Thus, we can deduce:

$$E_n[\log] = \log n + O(1)$$

We also note:

$$E_n[\alpha_p] \leq 1/(p-1) \leq 2/p$$

Which we can note just by graphing it - note, the inequality only holds for $x \geq 2$, but 2 is our first prime, so everything is fine. Finally, we also note:

$$m = \prod_p p^{\alpha_p(m)} \implies \log(m) = \sum_p \alpha_p(m) \log(p)$$

With all of these facts, we can note the following. We have that:

$$\frac{\log(n!)}{2n} \to \infty$$

As $n \to \infty$. This is clear by the approximation, as the big O term will go to zero, and $\log(n)$ does to infinity. We have by the above:

$$\frac{\log(n!)}{2n} = \frac{1}{2} * \frac{1}{n} * \sum_{m=1}^{n} \log(m) = \frac{1}{2} E_n[\log]$$

We now note that we are treating log as an integer valued function, and so it is equivalent to:

$$= \frac{1}{2} E_n \left[\sum_p \alpha_p \log(p) \right] = \frac{1}{2} \sum_p E_n[\alpha_p] \log(p) \leq \sum_p p^{-1} \log(p)$$

We note that this sum is larger than $\log(n!)/2n$, which goes to infinity, and so the sum diverges:

$$\sum_p p^{-1} \log(p) = \infty$$

This implies there are infinitely many primes. If there were finitely many - then the sum would clearly be finite. qed.

2. Let $\log^* m = \sum_p \delta_p(m) \log(p)$. Show that:

$$E_n[\log^*] = \sum_p \frac{1}{n} \left\lfloor \frac{n}{p} \right\rfloor \log(p) = \log n + O(1)$$

In truth, I'm going to stop doing this question here, as I'm just copying from the solutions at this point. It can be walked through, however, so if you absolutely need to understand how the results of the problem are obtained (the results being ratios between the *rth* prime and $r \log r$, where the ratio is bounded away from 0 and ∞, stuff like that), you can just walk through it again yourself.

Section 6 - The Law of Large Numbers

Notes

The Strong Law

Definition - Identically Distributed Let $X_1, X_2, \cdots$ be a sequence of simple random variables on some probability space $(\Omega, \mathcal{F}, P)$. They are *identically distributed* if their distributions are all the same. Ie:

$$\mu_i(A) = P[X_i \in A] = P[X_j \in A] = \mu_j(A)$$

Theorem 6.1 - The Strong Law of Large Numbers (SRV) If the X_n are independent and identically distributed and $E[X_n] = m$, then:

$$P\left[\lim_n n^{-1} S_n = m\right] = 1$$

Proof First note, we can assume wlog that $E[X_n] = m$. Then, recalling 5.2, we have that convergence with probability 1 is equivalent to showing for all $\epsilon > 0$ that:

$$P\left[|n^{-1}S_n| \geq \epsilon \,\text{i.o.}\right] = 0$$

We will ultimately use the first borel cantelli lemma - we want to show that:

$$\sum_n P\left[|S_n| \geq n\epsilon\right] < \infty$$

Note, this can be done with Markov's inequality, where:

$$P\left[|S_n| \geq n\epsilon\right] = P\left[|S_n|^4 \geq n^4\epsilon^4\right] \leq \frac{1}{n^4\epsilon^4}E[S_n^4]$$

As S_n is a summation, we have that:

$$E[S_n^4] = \sum E[X_\alpha X_\beta X_\gamma X_\delta]$$

Note that $E[X_i] = 0$, and so the only nonzero terms above are the n terms of the form $E[X_i^4] = \xi^4$, and the $3n(n-1)$ terms of the form $E[X_i^2 X_j^2] = E[X_i^2]E[X_j^2] = \sigma^4$ (noting that X_i^2 and X_j^2 are in $\sigma(X_i)$ and $\sigma(X_j)$, and so they are independent). Thus, we have:

$$\sum E[X_\alpha X_\beta X_\gamma X_\delta] \leq n\xi^4 + 3n(n-1)\sigma^4 \leq Kn^2$$

Where K does not depend on n. Thus, we can bound:

$$P\left[|S_n| \geq n\epsilon\right] \leq \frac{K}{n^2\epsilon^4}$$

As $\sum 1/n^2$ converges, we have by the first Borel-Cantelli Lemma:

$$P\left[|n^{-1}S_n| \geq \epsilon \,\text{i.o.}\right] = 0$$

Thus, we have the theorem. qed.

This is actually pretty simple, once all the prerequisite work has been finished up. Not bad!

The Weak Law

Theorem: The Weak Law of Large Numbers Take the same hypotheses as in the strong law - ie, X_n are independent and identically distributed and $E[X_n] = m$. Then,

$$n^{-1}S_n \to_p m$$

This is because convergence with probability 1 implies convergence in probability. Recall, this means for all $\epsilon > 0$, we have:

$$\lim_{n\to\infty} P[|n^{-1}S_n - m| \geq \epsilon] = 0$$

Note, this could also have just been proved by Chebyshev's Inequality:

$$P[|n^{-1}S_n - m| \geq \epsilon] \leq \frac{Var[S_n]}{n^2\epsilon^2} = \frac{nVar[X_1]}{n^2\epsilon^2} \to 0$$

As $Var[X_1]$ is some constant.

Example 6.3 - Length of Cycles in a Permutation using the Weak Law This is an interesting continuation of example 5.6, where we had, for Ω_n the $n!$ permutations, $P(A)$ the fraction of points in A, and $X_{nk}(\omega)$ 1 or 0 corresponding with whether the kth position of the permutation completes a cycle or not. We proved that $X_{n1}, \cdots, X_{nn}$ are independent, and:

$$P[X_{nk} = 1] = \frac{1}{n-k+1} = E[X_{nk}] = m_{nk} \qquad Var[X_{nk}] = \sigma^2_{nk} = m_{nk}(1-m_{nk})$$

We can make use of a similar Chebyshev argument as above. Let $L_n = \sum_{k=1}^n k^{-1}$, and note that $S_n = \sum_{k=1}^n X_{nk}$ has mean $\sum_{k=1}^n m_{nk} = L_n$, and variance $\sum_{k=1}^n m_{nk}(1-m_{nk}) < L_n$ (as each $0 < m_{nk} < 1$, and independent RV sum variance). So, by Chebyshev's inequality, we have:

$$P\left[\left|\frac{S_n - L_n}{L_n}\right|\right] < \frac{L_n}{\epsilon^2 L_n^2} = \frac{1}{\epsilon^2 L_n} \to 0$$

As when n increases, L_n goes to infinity, and the fraction goes to 0. And so, we have that a proportion of the permutations, larger than $1 - \epsilon^{-2}L_n^{-1}$, have their cycle number in the range $(1 \pm \epsilon)L_n$. With low n, that might not say much, but as $L_n = \log n + O(1)$, for large n, this says most permutations on n letters have about $\log n$ cycles. Not bad! This is pretty interesting.

Bernstein's Theorem

Let f be a function on $[0, 1]$. The *Bernstein polynomial* of degree n associated with f is:

$$B_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \times \binom{n}{k} x^k(1-x)^{n-k}$$

Note, the Bernstein polynomials of degree n (not associated with an f) are just:

$$b_{nk}(x) = \binom{n}{k} x^k(1-x)^{n-k} \qquad k = 0, \cdots, n$$

Theorem 6.2 If f is continuous, $B_n(x)$ converges to $f(x)$ uniformly on $[0, 1]$.

Just taking a step back, and looking at what this theorem says. We have this function $B_n(x)$. We are taking a sum over the values of f at each part of a fraction, k/n. Then, we are multiplying by the binomial theorem term, essentially, $(1 + x)^n$.

Can we make any intuitive sense out of this? Intuitively, we should have that the $f(k/n)$ terms have more weight, when k/n is closer to x. If x is close to 0, when k is lower, I think the $x^k(1 - x)^{n-k}$ term is bigger. And then we have the big coefficient $\binom{n}{k}$. When x is close to 1, we have that $x^k(1 - x)^{n-k}$ is maximized when k is bigger. And when x is close to 0.5, $x^k(1-x)^{n-k}$ is the same for all k, but $\binom{n}{k}x^k(1-x)^{n-k}$ is maximized. I guess, this must rely on the fact that exponentials have a higher growth rate than factorials, so when x is 0 or 1, the endpoint exponentials take over, but when x is close to 0.5, and the exponentials are the same, the factorial takes over.

Proof: f continuous on a compact interval implies f is *uniformly continuous* This can be noted by taking the sequence of x_n that break uniform continuity, finding a convergent subsequence, and noting a contradiction at the limit of that convergent subsequence on the continuity of f. qed.

Proof of Theorem 6.2 Let $M = \sup_x |f(x)|$, and let $\delta(\epsilon) = \sup [|f(x) - f(y)| : |x - y| \leq \epsilon]$ be the "modulus of continuity" of f (similar to the oscillation definition). Note: *continuity on a compact interval implies uniform continuity*, and that the modulus of continuity goes to 0 as ϵ does. We will show that:

$$\sup_x |f(x) - B_n(x)| \leq \delta(\epsilon) + \frac{2M}{n\epsilon^2}$$

If we let $\epsilon = n^{-1/3}$, the above becomes:

$$= \delta(n^{-1/3}) + \frac{2M}{n^{1/3}}$$

And note that the above goes to 0 as n goes to infinity, implying uniform continuity. We let $X_1, \cdots, X_n$ be independent random variables (on some probability space) such that $P[X_i = 1] = x$ and $P[X_i = 0] = 1 - x$. Note that Theorem 5.3 proves the existence of such SRVs. Put $S = X_1 + \cdots + X_n$.

It should be clear that:

$$P[S=k] = P\left[k \text{ of the } X_i \text{ are 1 and the remaining are 0}\right] = \binom{n}{k} x^k (1-x)^{n-k}$$

By the expected value of functions of SRVs, we have:

$$E[f(S/n)] = \sum_{k=0}^{n} f(k/n) P[S=k] = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \times \binom{n}{k} x^k (1-x)^{n-k} = B_n(x)$$

By the law of large numbers - there should be a high probability that S/n is near x and hence (as f is continuous) $f(S/n)$ is near $f(x)$, and so $E[f(S/n)]$ should also be near $f(x)$. This is the intuition.

We note that when $|n^{-1}S - x| < \epsilon$, we must have that $|f(n^{-1}S) - f(x)| \leq \delta(\epsilon)$ for arbitrary x. This is by the definition of the modulus of continuity, and true only for ω that satisfy the condition. Also, it is clear that in all cases, $|f(n^{-1}S) - f(x)| \leq 2M$, the bound on f. So, we have:

$$|f(n^{-1}S) - f(x)| \leq \delta(\epsilon) \mathbb{1}\left\{|n^{-1}S - x| < \epsilon\right\} + 2M \mathbb{1}\left\{|n^{-1}S - x| \geq \epsilon\right\}$$

We will use this later. Now, we prove our bounding property:

$$|B_n(x) - f(x)| = \left|E[f(n^{-1}S) - f(x)]\right| \leq E\left[\left|f(n^{-1}S) - f(x)\right|\right]$$

$$\leq \delta(\epsilon) P\left[|n^{-1}S - x| < \epsilon\right] + 2MP\left[|n^{-1}S - x| \geq \epsilon\right]$$

Where the last step uses the monotonicity of the expected value. Now, recall that $Var[S] = \sum Var[X_i] = nx(1-x)$, and so we can make use of Chebyshev's to find:

$$\leq \delta(\epsilon) + 2M \frac{Var[S]}{n^2 \epsilon^2} \leq \delta(\epsilon) + \frac{2M}{n\epsilon^2}$$

And so, we have our bound. Thus, $B_n(x)$ does indeed converge uniformly to $f(x)$. qed.

Refinement of the Second Borel-Cantelli Lemma

To refine the second Borel-Cantelli lemma, we first note a set equivalent to infinitely often. Let $A_1, A_2, \cdots$ be a sequence of events, and consider the number:

$$N_n = I_{A_1} + \cdots + I_{A_n}$$

N_n is the number of A_i that occur. Note that:

$$[A_n \,\text{i.o.}] = \left[\omega : \sup_n N_n(\omega) = \infty\right]$$

Note, the supremum is just the limit. Note that if ω is in A_n infinitely often, it is in an infinite amount, and so the sum is infinity. Note that if the sum is infinity, ω appears in an infinite number of A_n, and this infinitely often. So, we can study $P[A_n \text{i.o.}]$ be looking at N_n.

Note: what is the point of this? Well, the I_{A_i} are essentially Bernoulli random variables, as we have only two events - ω is in A_i or not. So, we can easily find expressions for the expected value and variance of N_n. Then, once we have values for those - we can start to use Chebyshev's inequality, and take limits. Note, we couldn't really make use of our inequalities for just the A_i, as they don't really relate to each other. N_n is an object that pulls all the A_i together, and there is a relation between N_n and N_{n+1} that we can exploit in limits.

Reproving The Second Borel-Cantelli Lemma with N_n Suppose that the A_n are independent. Put $p_k = P(A_k)$ and $m_n = p_1 + \cdots + p_n$. We clearly have $E[I_{A_k}] = p_k$ and $Var[I_{A_k}] = p_k(1-p_k) \leq p_k$. Independence easily gives us that $E[N_n] = m_n$ and $Var[N_n] \leq m_n$. For $m_n > x$, we have:

$$P[N_n \leq x] = P\left[N_n - m_x \leq x - m_n\right] \leq P\left[|N_n - m_n| \geq m_n - x\right] \leq \frac{Var[N_n]}{(m_n - x)^2} \leq \frac{m_n}{(m_n - x)^2}$$

Note, if $\sum p_n = \infty$, for n large enough, $m_n > x$ always, and so the above inequality will apply. Then, taking a limit on n, we have that the RHS goes to 0, as the squared term will take over. Thus, we have for $\sum p_n = \infty$, if A_n are independent:

$$\lim_n P[N_n \leq x] = 0$$

Note that:

$$P\left[\sup_k N_k \leq x\right] \leq P[N_n \leq x] \implies P\left[\sup_k N_k \leq x\right] = 0$$

As the supremum being smaller implies N_n is smaller. Taking a union over the x, we clearly have:

$$P\left[\sup_k N_k < \infty\right] = 0 \implies P\left[\sup_k N_k = \infty\right] = 1 \implies P[A_n \,\text{i.o.}] = 1$$

Thus, we have once again proved the second Borel-Cantelli lemma. qed.

Note, independence was used to estimate $Var[N_n]$. However, we can refine the argument to not require independence, and instead just give us a bound on the variance that implies the fraction still goes to 0 in the limit.

Theorem 6.3: Second Borel Cantelli Lemma Refinement If $\sum P(A_n)$ diverges and:

$$\liminf_n \frac{\sum_{j=1}^n \sum_{k=1}^n P(A_j \cap A_k)}{\left(\sum_{k=1}^n P(A_k)\right)^2} \leq 1$$

Then $P[A_n \text{i.o.}] = 1$. Note, the ratio is at least one, so the inequality holding actually implies equality.

Proof of Theorem 6.3 Let θ_n be the ratio above, ie:

$$\theta_n = \frac{\sum_{j=1}^n \sum_{k=1}^n P(A_j \cap A_k)}{\left(\sum_{k=1}^n P(A_k)\right)^2}$$

Note that:

$$Var[N_n] = E[N_n^2] - m_n^2 = \sum_{j=1}^n \sum_{k=1}^n E[I_{A_j} I_{A_k}] - m_n^2 = \sum_{j=1}^n \sum_{k=1}^n P[A_j \cap A_k] - m_n^2$$

$$= \left(\frac{\sum_{j=1}^n \sum_{k=1}^n P(A_j \cap A_k)}{m_n^2} - 1 \right) m_n^2 = (\theta_n - 1) m_n^2$$

Note, as the variance is always nonnegative, we must have that $\theta_n - 1 \geq 0 \implies \theta_n \geq 1$. Following the arguments in the Reproof of the Second Borel Cantelli Lemma, we have that:

$$P\left[N_n \leq x\right] \leq \frac{(\theta_n - 1) m_n^2}{(m_n - x)^2}$$

For $x < m_n$. Now, we take the assumed condition, and we find that:

$$\liminf_n P[N_n \leq x] \leq \liminf_n \frac{(\theta_n - 1) m_n^2}{(m_n - x)^2}$$

We now note a property of the liminf - that we can break it into a product of liminf and limit - see wiki. And so, the above equals:

$$= \liminf_n (\theta_n - 1) \times \lim_n \frac{m_n^2}{(m_n - x)^2} = \liminf_n \theta_n - 1$$

By our assumption, we have that it equals 0, and so:

$$\liminf_n P[N_n \leq x] \leq 0 \implies \liminf_n P[N_n \leq x] = 0$$

Note: this is *only the case if* $m_n \to \infty$, as then the limit will be one for the m_n term, and going far enough down the sequence will ensure $m_n > x$ and the equality will apply. Note that, we still have:

$$P\left[\sup_k N_k \leq x\right] \leq P[N_n \leq x]$$

And as the lim inf is the limit of the infimum, we know it upper bounds the infimum, and taking infimums over n on both sides still gives us:

$$P\left[\sup_k N_k < \infty\right] = 0 \implies P[A_n \text{ i.o.}] = 1$$

qed.

Note: as for why we take the lim inf in our ratio - it is just a weaker condition that limit. Clearly, the argument would work with limit, but the limit being less than or equal to one implies the lim inf is as well. We want to have the weakest conditions possible on the limit to give a better theorem. Again, the theorem just hinged on Chebyshev's inequality, and noting that redefining the Second Borel Cantelli based on the variance of N_n was able to give us weaker conditions that still implied $P[A_n\text{i.o.}] = 1$, which is always good. Not bad!

Example 6.5 I won't go over the details of this example - they are too technical for learning anything. However, the running example through these chapters have been a study of run length $l_n(\omega)$, where $A_n = [l_n \geq r_n]$ is the set of ω where $d_n(\omega) = \cdots = d_{n+r_n-1}(\omega) = 0$. Using the improved Second Borel-Cantelli lemma, we have that $\{r_n\}$ is an *outer or inner boundary* according as to whether or not $\sum 2^{-r_n}$ converges or diverges. Note, by *outer boundary*, we mean:

$$P[l_n \geq r_n\text{i.o.}] = 0$$

And by inner boundary, we mean:

$$P[l_n \geq r_n\text{i.o.}] = 1$$

Previously, with the basic second Borel-Cantelli lemma, we were only able to conclude that the r_n where an inner boundary if $\sum 2^{-r_n} r_n^{-1} = \infty$. The improved version closed the gap. Note: the probability would always be 0 or 1, based off of Kolmogorov's Zero One law - the set $[l_n \geq r_n \text{i.o.}]$ could be show to be a member of the tail sigma algebra of independent events, namely $H_n = [d_n(\omega) = 0]$.

Problems

6.1 Convergence With Probability "Finite" Terms Condition

Show that $Z_n \to Z$ with probability 1 if and only if for every positive ϵ there exists an n such that:

$$P[|Z_k - Z| < \epsilon, n \leq k \leq m] > 1 - \epsilon$$

For all m exceeding n. This describes convergence with probability 1 in "finite terms."

Note, this is kind of like taking our limsup condition, and turning it into a liminf condition. We note that $P[\lim_k Z_k = z]$ if and only if for all $\epsilon > 0$, we have:

$$P\left[|Z_k - Z| > \epsilon, \text{ i.o.}\right] = 0$$

We take the complement of the inside, to find that we have convergence with probability 1 iff:

$$P\left[\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}|Z_k - Z| < \epsilon\right] = 1$$

We note that for $A_n = \bigcap_{k=n}^{\infty}|Z_k - Z| < \epsilon$, $A_n \uparrow A$, where A is the expression inside the above probability. Thus, by convergence from below, we have:

$$\lim_{n\to\infty} P\left[\bigcap_{k=n}^{\infty}|Z_k - Z| < \epsilon\right] = 1$$

Thus, by the definition of the limit, we have convergence with probability 1 iff for n large enough:

$$P\left[\bigcap_{k=n}^{\infty}|Z_k - Z| < \epsilon\right] > 1 - \epsilon$$

Now, we note that we are almost there, because for all m exceeding n, we have by monotonicity:

$$P[|Z_k - Z| < \epsilon, n \leq k \leq m] \geq P\left[\bigcap_{k=n}^{\infty}|Z_k - Z| < \epsilon\right] > 1 - \epsilon$$

And so we have found the equivalent condition for convergence with probability 1. qed.

6.2 Number of Cycles In a Length n permutation bound

Show in Example 6.3 that $P\left[|S_n - L_n| \geq L_n^{1/2+\epsilon}\right] \to 0$.

I assume this means as $n \to \infty$. Note that L_n is the mean of S_n, and so just by Chebyshev's inequality we have:

$$P\left[|S_n - L_n| \geq L_n^{1/2+\epsilon}\right] \leq \frac{Var[S_n]}{L_n^{1+2\epsilon}} \leq \frac{L_n}{L_n^{1+2\epsilon}} \leq \frac{1}{L_n^{2\epsilon}}$$

Recall that $L_n = \sum_{k=1}^{n} k^{-1}$, which we know goes to infinity as $n \to \infty$. And so, the exponential similarly goes to infinity, which tells us:

$$\frac{1}{L_n^{2\epsilon}} \to 0$$

Which gives us the proof. qed.

6.3 Approximate Number of Inversions in a Random Permutation

As in Example 5.6 and 6.3, let ω be a random permutation of $1, 2, \cdots, n$. Each k, $1 \leq k \leq n$, occupies some position in the bottom row of the permutation ω:

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

Let $X_{nk}(\omega)$ be the number of smaller elements (between 1 and $k-1$) lying to the *right* of k in the bottom row. The sum:

$$S_n = X_{n1} + \cdots + X_{nn}$$

Is the total number of *inversions* - the number of pairs appearing in the bottom row in reverse order of size. For the example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 7 & 4 & 6 & 2 & 3 \end{pmatrix}$$

Note, for X_{nk}, we look at *where k is located on the final row*, and then look to the right. Ie, 1 is in the second place, so we look to the right from there. 3 is in the final position, and we look to the right from there. The values of $X_{71}, \cdots, X_{77}$ are thus $0, 0, 0, 2, 4, 2, 4$. Recall, the *parity* of a permutation is the parity of the number of transposition that ω can be decomposed into, and the wiki outlines proofs that tell us this is equal to the parity of the number of *inversions* in ω. Both are useful in defining the alternating tensor, but that is just a related note.

For the example, we have that $S_n = 12$. Show that $X_{n1}, \cdots, X_{nn}$ are independent and $P[X_{nk} = i] = k^{-1}$ for $0 \leq i < k$. Calculate $E[S_n]$ and $Var[S_n]$. Show that S_n is likely to be near $n^2/4$.

We first note that:

$$P[X_{nk} = i] = k^{-1}$$

For $0 \leq i < k$. Note, the distribution over Ω is just the uniform distribution - ie, each ω is equally likely. And so, when we examine X_{nk}, we want to know how many of the integers $1, \cdots, k-1$ are to the *right* of k in the permutation two row representation. Note, in each ω, we have that the integers $1, \cdots, k$ are listed out in sequence from left to right:

$$i_1, i_2, \cdots, i_k$$

Note each sequence is equally likely. Each event $X_{nk} = i$ corresponds to k being in a different position in the sequence. However, note that each sequence is equally likely, given that each ω is equally likely. And so, we have that for $i = 0, \cdots, k-1$, we have that $P[X_{nk} = 0] = \cdots = P[X_{nk} = k-1]$, and as these sum to one, we can conclude:

$$P[X_{nk} = i] = k^{-1}$$

Now, we want to show independence. I think this can be done with a permutation argument. We examine:

$$P\left[X_{nk} = i, X_{np} = q\right]$$

Wlog, assume $p > k$. So, we can just consider a sequence of indices:

$$t_1, t_2, \cdots, t_p$$

For $X_{nk} = i$, we need k to be in position $1, 2, \cdots, p-i$, and i of the positions to the right have to be one of the indices below k. However, we also *require* for p to be in position $p-q$ (and any chosen value to the right is fine, given that p is larger than the remaining integers). We make the first calculation easy, and assume $q = i$. In which case, k cannot be in position $p-i$, and must be in one of the previous entries, making the combinatorial sum:

$$\frac{1}{p!} \sum_{t=1}^{p-i-1} \binom{p-t-1}{i}$$

Or something like that. However, the calculation becomes difficult to simplify. We look to the solutions. We first note that the map:

$$\omega \to (X_{n1}(\omega), \cdots, X_{nn}(\omega))$$

Is one to one. I think this can be proved by induction. First, note that $X_{n1}(\omega)$ can have only one possible value - 0. Then, note that for $(X_{n1}(\omega), X_{n2}(\omega))$, $X_{n2}(\omega)$ has 2 possible values - 0 or 1. By induction, we note that $X_{nk}(\omega)$ would have k possible values when being added to $(X_{n1}(\omega), \cdots, X_{n,k-1}(\omega))$ - as k can be switched from any position in the sequence, essentially (if $X_{n,k-1}(\omega) = k-2$, then $X_{nk} = k-1$ by putting k to the left of the position of $k-1$).

Thus, by induction, there are $n!$ possible *different vectors* for $(X_{n1}(\omega), \cdots, X_{nn}(\omega))$. Thus, as there are only $n!$ possible permutations, $\omega \to (X_{n1}(\omega), \cdots, X_{nn}(\omega))$ must be one to one.

Now, note that X_{ni} can take on values x_i such that $0 \leq x_i < i$. Consider $1 \leq i \leq k$. Note that the number of permutations ω satisfying:

$$X_{ni}(\omega) = x_i$$

Must be $(k+1)(k+2)\cdots n$. As, we have $(k+1)(k+2)\cdots n$ possible values for each $X_{n,k+1}, \cdots, X_{n,n}$, and each of those maps to a unique ω. Those unique ω are the ones satisfying $X_{ni}(\omega) = x_i$. Thus, as Ω has uniform probability, we find:

$$P[X_{ni} = x_i, 1 \leq i \leq k] = \frac{(k+1)(k+2)\cdots n}{n!} = \frac{1}{k!} = P[X_{n1} = x_1] \cdots P[X_{nk} = x_k]$$

Where the final equality comes from what we noted above. Thus, we have that the X_{ni} are *independent* as well.

Now, we want to calculate $E[S_n]$ and $Var[S_n]$. These are easy if we find $E[X_{nk}]$ and $Var[X_{nk}]$. We have:

$$E[X_{nk}] = \frac{1}{k}\sum_{i=0}^{k-1} i = \frac{1}{k}\frac{(k-1)k}{2} = \frac{k-1}{2}$$

$$Var[X_{nk}] = E[X_{nk}^2] - \frac{(k-1)^2}{4} = \frac{1}{k}\sum_{i=0}^{k-1} i^2 - \frac{(k-1)^2}{4}$$
$$= \frac{(k-1)(2k-1)}{6} - \frac{(k-1)^2}{4} = \frac{k^2-1}{12}$$

So now, we have:

$$E[S_n] = \sum_{k=1}^{n} \frac{k-1}{2} = \frac{1}{2} * \frac{n(n-1)}{2} = \frac{n(n-1)}{2} \sim \frac{n^2}{4}$$

$$Var[S_n] = \frac{1}{12} \sum_{k=1}^{n} (k^2 - 1) \sim \frac{n^3}{56}$$

Finally, we want to show that S_n is likely to be near $n^2/4$. We apply Chebyshev's inequality to find:

$$P[|S_n - E[S_n]| \geq a] \leq \frac{Var[S_n]}{a^2}$$

Replacing with our approximations, we have:

$$P\left[\left|S_n - \frac{n^2}{4}\right| \geq n^{1.5}\right] \leq \frac{n^3}{n^3 36} = \frac{1}{36}$$

6.5 Poisson's Theorem

If $A_1, A_2, \cdots$ are independent events, $\bar{p}_n = n^{-1} \sum_{i=1}^{n} P(A_i)$, and $N_n = \sum_{i=1}^{n} I_{A_i}$, then $n^{-1} N_n - \bar{p}_n \to_p 0$.

Recall, to show that $n^{-1} N_n - \bar{p}_n \to_p 0$, we just need for every $\epsilon > 0$, we have:

$$\lim_{n \to \infty} P\left[\left|n^{-1} N_n - \bar{p}_n\right| > \epsilon\right] = 0$$

Now, we note that:

$$E[N_n] = \sum_{i=1}^{n} E[I_{A_i}] = \sum_{i=1}^{n} P(A_i) = n\bar{p}_n$$

So really, the value inside of the probability above is just a random variable minus its mean. We can use Chebyshev's to bound the probability:

$$P\left[\left|n^{-1} N_n - \bar{p}_n\right| > \epsilon\right] \leq \frac{Var[N_n]}{n^2 \epsilon^2}$$

To be honest, this looks like the refinement of the second borel cantelli lemma. However, I now note that each A_i is independent, and so to must be the I_{A_i} (as $\sigma(A_i)$ is generated by a π system $\{\emptyset, A_i, A_i^c, \Omega\}$, which are independent). So, the variance can be simplified:

$$= \frac{\sum_{i=1}^{n} Var[I_{A_i}]}{n^2 \epsilon^2}$$

And we note $Var[I_{A_i}] = E[I_{A_i}^2] - P(A_i)^2 = P(A_i) - P(A_i)^2 = P(A_i)(1 - P(A_i)) \leq P(A_i)$. So, the above is bounded by:

$$\leq \frac{\bar{p}_n}{n\epsilon^2} \to 0$$

And thus, we have convergence in probability. qed.

6.6 Cantelli's Theorem

If $X_1, X_2, \cdots$ are independent, $E[X_n] = 0$, and $E[X_n^4]$ is bounded (ie, for all n, $E[X_n^4] < B$), then $n^{-1}S_n \to 0$ with probability 1. The X_n need not be identically distributed.

We first note that $E[X_n^4] < B$ implies $E[X_n^2] < C$ for some C. This follows directly from Jensen's inequality for simple random variables. Note that $f(x) = x^{1/2}$ is concave, and so the inverse of Jensen's tells us $E[f(x)] \leq f(E[X])$. Thus, we have:

$$E[X_n^2] = E[f(X_n^4)] \leq f(E[X_n^4]) < f(B) = B^{1/2}$$

Now, we can prove Cantelli's Theorem in a very similar way to the strong law. Note that $n^{-1}S_n \to 0$ with probability 1 if and only if for every ϵ:

$$P[|n^{-1}S_n| > \epsilon \text{ i.o.}] = P[S_n^4 > n^4\epsilon^4 \text{ i.o.}] = 0$$

The above will follow via the first Borel-Cantelli lemma. Note via Markov's, independence, and $E[X_i] = 0$:

$$P[S_n^4 > n^4\epsilon^4] \leq \frac{E[S_n^4]}{n^4\epsilon^4} = \frac{\sum_{i=1}^n E[X_i^4] + \sum_{j\neq i} 6E[X_i^2]E[X_j^2]}{n^4\epsilon^4}$$

Where the second equality is by the Multinomial Theorem, and if $k_i = 1$ we have that the expected value term equals 0. This is bounded by:

$$\leq \frac{nB + 3n(n-1)B}{n^4\epsilon^4} \leq \frac{3n^2B}{n^4\epsilon^4} = \frac{3B}{\epsilon^4} * \frac{1}{n^2}$$

Note, we thus have the sum converges, and so we can conclude that $n^{-1}S_n \to 0$ with probability 1. qed. Also note, the multinomial theorem with coefficient of $6 = \frac{4!}{2!2!}$ bridges us to the $3n(n-1) = 6 * \binom{n}{2}$ term.

6.7 Convergence of $n^{-2}S_{n^2}$ implies convergence of $n^{-1}S_n$

1. Let $x_1, x_2, \cdots$ be a sequence of real numbers, and put $s_n = x_1 + \cdots + x_n$. Suppose that $n^{-2}s_{n^2} \to 0$ and that the x_n are bounded, and show that $n^{-1}s_n \to 0$.

   We prove by contradiction. We assume $n^{-1}s_n \not\to 0$, which means that we have some $\epsilon > 0$ such that $|n^{-1}s_n| > \epsilon$ infinitely often. Let p_i be this sequence. Note, each p_i is in between some m_i^2 and $(m_i+1)^2$. We find:
   $$|p_i^{-1}s_{p_i}| \le \frac{|s_{m_i^2}| + B(p_i - m_i^2)}{p_i}$$
   Where B is our bound on the x_i. This comes from the triangle inequality:
   $$s_{p_i} = s_{m_i^2} + \sum_{t=m_i^2+1}^{p_i} x_i \implies |s_{p_i}| \le |s_{m_i^2}| + \sum_{t=m_i^2+1}^{p_i} |x_i|$$
   Now, as $p_i \ge m_i^2$, and $p_i \le (m_i+1)^2$, we have:
   $$|p_i^{-1}s_{p_i}| \le \frac{|s_{m_i^2}| + B((m_i+1)^2 - m_i^2)}{m_i^2} = m_i^{-2}s_{m_i^2} + \frac{B(2m_i+1)}{m_i^2}$$
   Note, both sides go to 0, we clearly have:
   $$\lim_{i\to\infty} |p_i^{-1}s_{p_i}| = 0$$
   Which contradicts $|p_i^{-1}s_{p_i}| > \epsilon$ infinitely often.

2. Suppose that $n^{-2}S_{n^2} \to 0$ with probability 1 and the X_n are uniformly bounded ($\sup_{n,\omega} |X_n(\omega)| < \infty$). Show that $n^{-1}S_n \to 0$ with probability 1. Here the X_n need not be identically distributed or even independent.

   We have:
   $$P\left[\lim_{n\to\infty} n^{-2}S_{n^2} = 0\right] = 1$$
   Note, given that $\sup_{n,\omega} |X_n(\omega)| < \infty$ for all ω, we have that the above tells us:
   $$\lim_{n\to\infty} n^{-2}S_{n^2}(\omega) = 0 \implies \lim_{n\to\infty} n^{-1}S_n(\omega) = 0$$
   And so, we must have:
   $$\left\{\lim_{n\to\infty} n^{-2}S_{n^2} = 0\right\} = \left\{\lim_{n\to\infty} n^{-1}S_n = 0\right\}$$
   Which thus implies:
   $$P\left[\lim_{n\to\infty} n^{-1}S_n = 0\right] = 1$$

6.11 m dependent, uniformly bounded, and mean 0 X_i convergence with probability 1

Suppose that $X_1, X_2, \cdots$ are m independent in the sense that random variables more than m apart in the sequence are independent. More precisely, let $A_j^k = \sigma(X_j, \cdots, X_k)$, and assume that $A_{j_1}^{k_1}, \cdots, A_{j_l}^{k_l}$ are independent if $k_{i-1} + m < j_i$ for $i = 2, \cdots, l$. Note, independent random variables are 0-dependent. Suppose that the X_n have this property and are uniformly bounded and that $E[X_n] = 0$ for all n. Show that $n^{-1}S_n \to 0$ with probability 1.

Hint: Consider the subsequences $X_i, X_{i+m+1}, X_{i+2(m+1)}, \cdots$ for $1 \leq i \leq m+1$.

Let's first make use of the hint. Let $Y_1, Y_2, \cdots$ correspond to one of the subsequences for $i \in [1, m+1]$. Note that each Y_i is independent, uniformly bounded, and $E[Y_i] = 0$. Recall, uniform boundedness tells us:

$$|Y_i| \leq M \implies Y_i^2 \leq M^2 \implies E[Y_i^2] \leq M^2$$

And similarly:

$$E[Y_i^4] \leq M^4$$

Let S_n correspond to the sum of Y_n. Note that:

$$P\left[\lim_{n\to\infty} n^{-1}S_n = 0\right] = 1 \Leftrightarrow \forall \epsilon > 0 \ , \ P\left[|n^{-1}S_n| > \epsilon \text{ i.o.}\right] = 0$$

We note that infinitely often events equals 0 if the probability of the individuals events converges, by BC-1. So, we examine:

$$P\left[|n^{-1}S_n| > \epsilon\right] = P\left[S_n^4 > n^4\epsilon^4\right] \leq \frac{E[S_n^4]}{n^4\epsilon^4} = \frac{nE[Y_i^4] + 3n(n-1)E[Y_i^2][Y_j^2]}{n^4\epsilon^4}$$

Where we make use of Markov's inequality, and the independence of Y_i. The above is bounded by:

$$\leq \frac{nM^4 + 6n(n-1)M^2}{n^4\epsilon^4}$$

Clearly, this sum converges with the exponential n^4 term in the bottom. And so, in conclusion, we note that each of the subsequences $X_i, X_{i+m+1}, X_{i+2(m+1)}, \cdots$ for $1 \leq i \leq m+1$ satisfies that their sequence average goes to 0 with probability 1. We now want to turn this fact into a conclusion about the entire sequence. Let S_n^i correspond to the sum of the Y_i for $1 \leq i \leq m+1$. We note that:

$$S_n = S_{n/m+1}^1 + \cdots + S_{n/m+1}^{m+1}$$

Note, of course, there will be some ± 1 differences in the subscript, but essentially, the above is the case. I don't want to deal with floor/ceiling for now, the analysis still applies. We note that:

$$P\left[\lim_{n\to\infty} n^{-1}S_n = 0\right] = P\left[\lim_{n\to\infty} n^{-1}\left[S^1_{n/m+1} + \cdots + S^{m+1}_{n/m+1}\right] = 0\right]$$

Note, as $n/m+1 < n$, we have the above probability is lower bounded by:

$$\geq P\left[\lim_{n\to\infty} \frac{m+1}{n}S^1_{n/m+1} + \cdots + \frac{m+1}{n}S^{m+1}_{n/m+1} = 0\right]$$

As if the above converges to 0, then it definitely converges to zero when divided by a larger number. We now note that an ω for which each of the terms converges to zero, is an ω for which the entire sum converges to zero. And so, note that the above is lower bounded by:

$$\geq P\left[\bigcap_{i=1}^{m+1} \lim_{n\to\infty} \frac{m+1}{n}S^i_{n/m+1} = 0\right]$$

Again, note, for the ω in this set, each limit equals 0, and so for any $\epsilon > 0$, there is an n large enough such that $|\frac{m+1}{n}S^i_{n/m+1}| < \epsilon/m+1$, we take the maximum of the ns, and the entire sum is bounded by ϵ. We now note that the probability of the finite intersection of probability 1 sets is probability 1 (examine the complement, which is clearly probability 0). And so, we have:

$$P\left[\lim_{n\to\infty} n^{-1}S_n = 0\right] \geq P\left[\bigcap_{i=1}^{m+1} \lim_{n\to\infty} \frac{m+1}{n}S^i_{n/m+1} = 0\right] = 1$$

And so, yes, we can conclude that for m dependent X_i uniformly bounded and mean 0, we have their average converges to 0 with probability 1.

6.12 For X_i iid, Relative Asymptotic Frequencies Are What They Should Be

Suppose that the X_n are independent and assume the values $x_1, \cdots, x_l$ with probabilities $p(x_1), \cdots, p(x_l)$. For $u_1, \cdots, u_k$ a k tuple of the x_i's, let $N_n(u_1, \cdots, u_k)$ be the frequency of the k tuple in the first $n+k-1$ trials. That is, the number of t such that $1 \leq t \leq n$ and $X_t = u_1, \cdots, X_{t+k-1} = u_k$. Show that with probability 1, all asymptotic relative frequencies are what they should be - that is, with probability 1, $n^{-1}N_n(u_1, \cdots, u_k) \to p(u_1)\cdots p(u_k)$ for every k and every k tuple $u_1, \cdots, u_k$.

Define the random variable Y_n as follows:

$$Y_n = \begin{cases} 1 & \text{if } X_n = u_1, \cdots, X_{n+k-1} = u_k \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the Y_n are k dependent. We also note that:

$$E[Y_n] = P[X_n = u_1, \cdots, X_{n+k-1} = u_k] = p(u_1) \cdots p(u_k)$$

So, we have that $Y_n - p(u_1) \cdots p(u_k)$ are 0 mean, k dependent, uniformly bounded (by 1) simple random variables. By the previous question, this implies:

$$P\left[\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} Y_i - p(u_1) \cdots p(u_k) = 0\right] = 1$$
$$\implies P\left[\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} Y_i = p(u_1) \cdots p(u_k)\right] = 1$$
$$\implies P\left[\lim_{n\to\infty} n^{-1} N_n(u_1, \cdots, u_k) = p(u_1) \cdots p(u_k)\right] = 1$$

Thus, we have with probability 1, all asymptotic relative frequencies are what they should be. qed.

6.14 Shannon's Theorem

Suppose that $X_1, X_2, \cdots$ are independent, identically distributed random variables taking on the values $1, \cdots, r$ with positive probabilities $p_1, \cdots, p_r$. If $p_n(i_1, \cdots, i_n) = p_{i_1} \cdots p_{i_n}$ and $p_n(\omega) = p_n(X_1(\omega), \cdots, X_n(\omega))$, then $p_n(\omega)$ is the probability that a new sequence of n trials would produce the particular sequence $X_1(\omega), \cdots, X_n(\omega)$ of outcomes that happens actually to have been observed. Show that:

$$-\frac{1}{n} \log p_n(\omega) \to h = -\sum_{i=1}^{r} p_i \log(p_i)$$

with probability 1. Note that $p_n(\omega)$ essentially just equals $P[(X_1, \cdots, X_n) = (i_1, \cdots, i_n)]$, as independence implies the values multiply, and identically distributed implies that for each X_j, $P[X_j = i_j] = p_{i_j}$. We define:

$$Y_j = -\log(f(X_j)) \qquad f(i) = p_i$$

Note, Y_j are clearly independent, as $\sigma(Y_j) \subseteq \sigma(X_j)$, and also identically distributed, as the X_j are. We also note that:

$$E[Y_j] = \sum_{i=1}^{r} -\log(p_i) * P[Y_j = -\log(p_i)] = \sum_{i=1}^{r} -\log(p_i) * P[f(X_j) = p_i]$$

$$= -\sum_{i=1}^{r} \log(p_i) * P[X_j = p_i] = h$$

We note the strong law of large numbers states that:

$$-\frac{1}{n}\sum_{j=1}^{r} Y_j \to h$$

With probability 1. Now, we replace Y_j with its definition:

$$-\frac{1}{n}\sum_{j=1}^{r} Y_j = -\frac{1}{n}\sum_{j=1}^{r} \log(f(X_j)) = -\frac{1}{n}\log\left[f(X_1)\cdots f(X_n)\right] = -\frac{1}{n}\log\left[p_n(X_1, \cdots, X_n)\right]$$

So, we clearly have, the set of ω that satisfy:

$$-\frac{1}{n}\log p_n(\omega) \to h = -\sum_{i=1}^{r} p_i \log(p_i)$$

Have probability 1. qed. Some notes: this is kind of hard to parse. *NOTE:* we are taking the *log of the probability*. We are not just taking the log of the value of the random variable X_i. We note, the log of the probability, for all sequences, approaches its entropy, essentially. Or, for almost all *long sequences* produced by the source, the log-probability per sequence converges to the entropy.

In information theory, $1, \cdots, r$ are interpreted as the *letters* of an *alphabet*, $X_1, X_2, \cdots$ are the successive letters produced by an information *source*, and h is the *entropy* of the source. Prove the *asymptotic equipartition property*: For large n, there is probability exceeding $1 - \epsilon$ that the probability $p_n(\omega)$ of the observed n long sequence, or *message*, is in the range $e^{-n(h \pm \epsilon)}$.

Note, the proof of this statement is essentially just going from the strong law of large numbers, to the weak law of large numbers. Our convergence with probability 1 above is equivalent to:

$$P\left[\lim_{n\to\infty} p_n(\omega) = e^{-nh}\right] = 1$$

By strong convergence implying weak convergence, and taking a complement, the above implies for all $\epsilon' > 0$:

$$\implies \lim_{n\to\infty} P\left[|p_n(\omega) - e^{-nh}| \leq \epsilon'\right] = 1$$

By the definition of the limit, there is an N large enough, where it is guaranteed that if $n \geq N$, we have:

$$P\left[|p_n(\omega) - e^{-nh}| \leq \epsilon'\right] \geq 1 - \epsilon$$

For all $\epsilon > 0$. Now, we note that:

$$|p_n(\omega) - e^{-nh}| \leq \epsilon' \iff e^{-nh} - \epsilon' \leq p_n(\omega) \leq e^{-nh} + \epsilon'$$

Now, we just have to choose our ϵ', based off of the ϵ (note, we could fix ϵ first, then choose our ϵ', then use the definition of the limit to find an N - all of it is consistent). Note that as the above is true for all ϵ', we have the above is equivalent to (for a perhaps larger N):

$$e^{-n(h+\epsilon)} - \epsilon' \leq p_n(\omega) \leq e^{-n(h-\epsilon)}$$

We just have to choose an ϵ small enough that it satisfies:

$$e^{-n(h+\epsilon)} \leq e^{-nh} - \epsilon' \qquad e^{-nh} + \epsilon' \leq e^{-n(h-\epsilon)}$$

So, for all $\epsilon > 0$, there exists an N such that if $n \geq N$, we have:

$$P\left[p_n(\omega) \in e^{-n(h\pm\epsilon)}\right] \geq 1 - \epsilon$$

This is interesting, I think. It means that there is a very high probability that for the message we ultimately observe, it had a probability of essentially e^{-nh}.

I think, we can go on a bit of a side note, as to why it is important. We can prove all of the above with $\log_2$, rather than the natural logarithm. The above says that with *high probability*, the sequence $(X_1, X_2, \cdots, X_n)$ is one of our "typical sequences" with probability $2^{-nH(X)}$ (where we let $H(X)$ denote the entropy for random variable X, again using $\log_2$ instead of the natural log). Note, if the sequences we see have probability $2^{-nH(X)}$, and those are all of the sequences we see, essentially - then we have only $2^{nH(X)}$ possible sequences, as we must have:

$$\text{\# Of Sequences} \times \text{probability of each sequence} = 1$$

So, if you want to describe $2^{nH(X)}$ different sequences - you only need to use $nH(X)$ bits of information, as each sequence can correspond to one of the

$2^{nH(X)}$ possible bit sequences. *This is a lower bound on lossless data compression.*

This gives us an interpretation as to what entropy is - *the average number of bits required to encode a single letter from the alphabet, when using an optimal code.* Why? As n becomes larger, we need $nH(X)$ bits to encode the message of n letters, which means each letter gets on average, $H(X)$ bits. NOT BAD!

Section 7 - Gambling Systems

Notes

To be honest - I'm not too interested in this topic at all. However, I do think it has valuable information, if we want to study probability of like a discrete system. However, because I am not interested - I will skip the optional topics, bold and timid play.

However, I might not actually. I have read a bit of the bold play section - the overall idea, is what betting amounts will improve your odds. I think it ultimately comes down to this: in the case of *subfair play* (which is $p < q$, odds of losing are greater) - the best strategy of bold play is, bet your entire fortune every turn. And intuitively - this makes sense to me. The best odds you are going to get, in a repeating subfair game - is the initial probability p of success. If you bet less, and rely on multiple trials - you will just be relying on large numbers to ensure failure.

Ultimately, I think this is probably a life lesson. If you're in a subfair situation - go for broke. I'll put this on a flash card to be honest, it is a good lesson to remember.

Gambler's Ruin

The setup is as follows. A gambler starts with capital a, and his strategy is to keep on betting until his fortune increases to c, or his funds are exhausted. Every time he bets, his fortune increases by 1 with probability p, and decreases by 1 with probability $q = 1 - p$. What are the probabilities of success and ruin, reaching c or exhaustion respectively?

Let X_i be the gambler's gain on the *ith* play. Let:

$$S_n = X_1 + \cdots + X_n$$

We note, by Theorem 5.3, X_i iid with probability p/q exist, which can be seen as drawing a random number between 0 and 1. The event:

$$A_{a,n} = [a + S_n = c] \cap \bigcap_{k=1}^{n} [0 < a + S_k < c]$$

Represents success for the gambler at time n, and:

$$B_{a,n} = [a + S_n = 0] \cap \bigcap_{k=1}^{n} [0 < a + S_k < c]$$

Represents ruin at time n. I like taking this back to the Borel set - ruin and success can be seen as the probability that a random draw on $(0, 1]$ will be in one of $A_{a,n}$ or $B_{a,n}$ defined above. Probability is always, at the end of the day - trying to find the measure of the sets in question. I think it is clear X_i are measurable $\mathcal{B}$, for $X_i = \pm 1$. Then, it is clear each S_n is as well, being the sum of such functions. And so, each of $A_{a,n}$ and $B_{a,n}$ are measurable $\mathcal{B}$.

If $s_c(a)$ denotes the probability of ultimate success, then:

$$s_c(a) = P\left[\bigcup_{n=1}^{\infty} A_{a,n}\right] = \sum_{n=1}^{\infty} P(A_{a,n})$$

For $0 < a < c$. Now, we note some boundary conditions to make our study consistent. We have for $n \geq 1$, and $0 < a < c$, adopt:

$$A_{a,0} = \emptyset \text{ for } 0 \leq a < c, \text{ as no chance for success at time 0}$$

$$A_{c,0} = \Omega$$

$$A_{0,n} = A_{c,n} = \emptyset \text{ for } n \geq 1, \text{ as play never starts if } a \text{ is 0 or } c$$

This gives us $s_c(0) = 0$, and $s_c(c) = 1$, ie, if our starting fortunes are 0 or c, or chances of success are 0 and 1, respectively. Given that $X_2, X_3, \cdots$ is a replica of $X_1, X_2, \cdots$, intuition would tell us that the chance of success for a gambler with initial fortune a would be the chance of success starting with $a+1$ times the probability we win the first play, plus the chance of success starting at $a-1$, times the probability we lose the first play, giving us an equation:

$$s_c(a) = ps_c(a+1) + qs_c(a-1)$$

With our boundary conditions, this would hopefully give us equations to solve for $s_c(a)$. However, we first want to prove the above rigorously.

Proof of $s_c(a) = ps_c(a+1) + qs_c(a-1)$ Define $A'_{a,n}$ just as $A_{a,n}$ but with $S'_n = X_2 + \cdots + X_{n+1}$ in place of S_n in our previous definition. Note that:

$$P[X_i = x_i, i \leq n] = P[X_{i+1} = x_i, i \leq n]$$

For each sequence $x_1, \cdots, x_n$ of plus or minus 1. Therefore, we have for $H \subset \mathbb{R}^n$:

$$P[(X_1, \cdots, X_n) \in H] = P[(X_2, \cdots, X_{n+1}) \in H]$$

For $H \subset \mathbb{R}^n$ (we can take the left side, break it up by disjoint unions, and use the identity derived above, which came from independence). Take H to be the set of $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ satisfying $x_i + \pm 1, a + x_1 + \cdots + x_n = c$, and $0 < a + x_1 + \cdots + x_k < c$ for $k < n$. It follows that:

$$P(A_{a,n}) = P[(X_1, \cdots, X_n) \in H] = P[(X_2, \cdots, X_{n+1}) \in H] = P(A'_{a,n})$$

Note, in the middle equality - we jump to another set of ω. However, we note that the ω on the RHS has equal probability, and so both events have equal probability.

So, we have the above property. Now, we note that:

$$A_{a,n} = \left([X_1 = +1] \cap A'_{a+1,n-1}\right) \cup \left([X_1 = -1] \cap A'_{a-1,n-1}\right)$$

This comes from noting that $A'_{a+1,n-1}$ means that you are starting at fortune $a + 1$, and we have that $X_2, \cdots, X_{n-1+1} = X_n$ reaches a total sum of c without reaching ruin/success before hand. So yeah, with that intuition, we see that the sets of ω on both side are the same, for $n \geq 1$ and $0 < a < c$. By independence (noting the independent random variable row theorem corollary) and disjoint events, we have:

$$\begin{aligned} P(A_{a,n}) &= P\left[[X_1 = +1] \cap A'_{a+1,n-1}\right] + P\left[[X_1 = -1] \cap A'_{a-1,n-1}\right] \\ &= pP\left[A'_{a+1,n-1}\right] + qP\left[A'_{a-1,n-1}\right] \\ &= pP\left[A_{a+1,n-1}\right] + qP\left[A_{a-1,n-1}\right] \end{aligned}$$

So, we have:

$$s_c(a) = \sum_{n=1}^{\infty} P(A_{a,n}) = \sum_{n=1}^{\infty} pP\left[A_{a+1,n-1}\right] + qP\left[A_{a-1,n-1}\right]$$

We can bring out the scalar multiplies, given that monotone increasing sums converge (either to a point or infinity):

$$= p\left[\sum_{n=1}^{\infty} P\left[A_{a+1,n-1}\right]\right] + q\left[\sum_{n=1}^{\infty} P\left[A_{a-1,n-1}\right]\right]$$

$$= ps_c(a+1) + qs_c(a-1)$$

Note, the above is true for $0 < a < c$ (if $a = 1$ or $a = c - 1$, it is still true, as we get the first value of $0/1$, added with the following 0 values for infinity). So, we have rigorously proved the difference equations we need to solve. In summary, it took:

1. Noting that the probability of one sequence of X_i being some specific value is the same as the shifted sequence
2. We took that fact, and proved that the probabilities of success/ruin of the shifted sums are equal
3. We expressed $A_{a,n}$ as X_1 taking a value, times probabilities of success/ruin of shifted sums
4. Brought back the expressions to the original sequence

So, we now turn to solving the difference equation. We have:

$$s_c(0) = 0 \qquad s_c(c) = 1 \qquad s_c(a) = ps_c(a+1) + qs_c(a-1) \text{ for } 0 < a < c$$

We should be able to solve for every in-between $s_c(a)$ value. Let $\rho = q/p$ - these are the *odds against the gambler*. Recall (Appendix) we have for $a < b$ integers, and difference equation:

$$x_n = px_{n+1} + qx_{n-1} \qquad a < n < b$$

Where $p + q = 1$, we have a general solution of the form:

$$x_n = \begin{cases} A + B(q/p)^n & \text{for } a \leq n \leq b \text{ if } p \neq q \\ A + Bn & \text{for } a \leq n \leq b \text{ if } p = q \end{cases}$$

We can just check this manually. Note:

$$p\left[A + B(q/p)^{n+1}\right] + q\left[A + B(q/p)^{n-1}\right] = A + B\frac{q^{n+1}}{p^n} + B\frac{q^n}{p^{n-1}}$$

$$= A + B\frac{q^{n+1} + q^n p}{p^n} = A + B\frac{q^n}{p^n} = x_n$$

If we have two values of x_n, then A and B can be solved for (just two *linear* equations with two unknowns). We note that for integer capitals a, we have for constants A and B:

$$s_c(a) = \begin{cases} A + B\rho^a & \text{if } p \neq q \\ A + Ba & \text{if } p = q \end{cases}$$

Solving for A and B gives us:

$$s_c(a) = \begin{cases} \frac{\rho^a - 1}{\rho^c - 1} & \text{if } \rho \neq 1 \\ \frac{a}{c} & \text{if } \rho = 1 \end{cases}$$

And so, we above, we have found *the probability that the gambler can attain his goal of c from an initial capital of a before ruin.*

Probability of Ruin Note, we can replace the equations above with expressions for the probability of ruin (mainly switching $A_{a,n}$ and $B_{a,n}$, and redoing the analysis) to find:

$$r_c(a) = \begin{cases} \frac{\rho^{-(c-a)} - 1}{\rho^{-c} - 1} & \text{if } \rho \neq 1 \\ \frac{c-a}{c} & \text{if } \rho = 1 \end{cases}$$

And so we have that $s_c(a) + r_c(a) = 1$. This implies - the *probability is* 0 *that play continues forever.*

Selection Systems

The overall idea behind selection systems is - the gambler will only make a bet at time n, based on if he sees some sort of pattern in the $X_1, \cdots, X_{n-1}$ beforehand. Note, the gambler supposedly knows the X_i are independent with probabilities p and q - but he decides to follow one of these selection systems anyway. We start by giving some notation.

Selection Systems Framework

1. X_i are the iid random variables distributed according to $P[X_i = +1] = p$ and $P[X_i = -1] = 1 - p = q$. X_i describes the result of the *ith* trial. X_i are within some $(\Omega, \mathcal{F}, P)$.

2. B_i describes the gambler's strategy. $B_i = 1$ if the gambler will place a bet on the *ith* trial, and $B_i = 0$ if the gambler chooses not to. Note, the strategy can only depend on the values of the X_i seen before hand, and so B_i must be some function $b_n : \mathbb{R}^{n-1} \to \mathbb{R}$ with:

$$B_i = b_i(X_1, \cdots, X_{i-1})$$

3. We define the generated σ fields:

$$\begin{cases} \mathcal{F}_n = \sigma(X_1, \cdots, X_n) & n = 1, 2, \cdots \\ \mathcal{F}_0 = \{\emptyset, \Omega\} \end{cases}$$

By our requirement on B_i, we have that B_i is measurable $\mathcal{F}_{i-1}$. Recall - any simple random variable that is a function of other simple random variables is measurable their generated σ field.

4. For $i = 1, 2, \cdots$, define N_i to be the time at which the gambler places his *ith* bet. This *ith* bet is placed at time k or earlier, if and only if the number $\sum_{j=1}^{i} B_j \leq k$. In fact, N_i is the *smallest* k *for which* $\sum_{j=1}^{k} B_j = i$. We thus have:

$$[N_n(\omega) \leq k] = \left[\sum_{i=1}^{k} B_i(\omega) \geq n \right]$$

Thus, we must have that $[N_n \leq k]$ lies within $\sigma(B_1, \cdots, B_k)$, as it can be expressed as a union of the various $H \subset \mathbb{R}^k$ that contain coordinates with at least n ones. This is field is contained within $\sigma(X_1, \cdots, X_{k-1}) \subset \mathcal{F}_{k-1}$. Therefore, we have:

$$[N_n = k] = [N_n \leq k] - [N_n \leq k-1] \in \mathcal{F}_{k-1}$$

As clearly both sets in the subtraction are part of the set. Note that N_n, as a function on Ω - is generally not a simple random variable, as it has infinite range. Note, however, the only essential property we will use is the above one. Also, note why it is intuitive. We want to know whether the gambler will make his *nth* play at time k. Well, we will know if it is the *nth* play, based off of whether we had $n-1$ plays in the $k-1$ trials (which must be measurable the first $k-1$ X_i), and whether $B_k = +1$, which is also measurable $\mathcal{F}_{k-1}$.

5. To ensure that play continues forever, and that the N_n have finite values with probability 1, make the assumption that:

$$P[B_n = 1 \text{i.o.}] = 1$$

A sequence of $\{B_n\}$ simple random variables assuming values of 0 and 1, being measurable $\mathcal{F}_{n-1}$, and equaling 1 infinitely often with probability 1, is called a *selection system.*

6. Finally, define Y_n as the gambler's gain on the *nth* trial in which he does bet. Ie:

$$Y_n = X_{N_n}$$

Note, only on the set ω such that $[B_n = 1\text{i.o.}]$ is Y_n well defined. So that Y_n is defined on *all* ω, we actually have to take cases:

$$Y_n(\omega) = \begin{cases} X_{N_n(\omega)}(\omega) & \text{if } \omega \in [B_n = 1\text{i.o.}] \\ -1 & \text{if } \omega \in [B_n = 1\text{i.o.}]^c \end{cases}$$

Note that Y_n is a complicated function on Ω. However, we do have that:

$$[\omega : Y_n(\omega) = 1] = \bigcup_{k=1}^{\infty} [[\omega : N_n(\omega) = k] \cap [\omega : X_k(\omega) = 1]]$$

And so $[\omega : Y_n(\omega) = 1] \in \mathcal{F}$. Similarly, we can derive an expression for $Y_n(\omega) = -1 \in \mathcal{F}$, and so we *do have that Y_n is a SIMPLE RANDOM VARIABLE.*

So, as a wrap up. We have that B_n, X_n, Y_n are all simple random variables. We introduced a notation for N_n, which wasn't really a simple random variable - but we didn't need it to be. We just needed it as notation to make expressing the intersection of sets in $\mathcal{F}_{k-1}$, that ultimately make up the $[N_n(\omega) = k] \in \mathcal{F}_{k-1}$ term that appears in the Y_n definition.

Theorem 7.1 - Selection Systems Do Nothing For every selection system, $\{Y_n\}$ is independent and $P[Y_n = +1] = p$, $P[Y_n = -1] = q$. Ie, Y_n has the same structure as X_n, and so using a selection system is identical to just betting on each of the n trials.

Proof of Theorem 7.1 We relabel p and q as $p(+1)$ and $p(-1)$, so we can easily write $P[\omega : X_k(\omega) = x] = p(x)$. If $A \in \mathcal{F}_{k-1} = \sigma(X_1, \cdots, X_{k-1})$, by independence of the X_i (and our array theorem), we note that:

$$P[A \cap [\omega : X_k(\omega) = x]] = P(A)p(x)$$

And so:

$$P[\omega : Y_n(\omega) = x] = P\left[\omega : X_{N_n(\omega)}(\omega) = x\right] = \sum_{k=1}^{\infty} P[\omega : N_n(\omega) = k, X_k(\omega) = x]$$

Where, just to point it out - the summation comes from disjoint sets of ω. As $N_n \in \mathcal{F}_{k-1}$, the above equals:

$$= \sum_{k=1}^{\infty} P[\omega : N_n(\omega) = k]\, p(x) = p(x) \sum_{k=1}^{\infty} P[\omega : N_n(\omega) = k] = p(x) P[\omega : N_n(\omega) < \infty]$$

Now, $B_n = 1$ infinitely often is the same set of ω that implies $N_n(\omega) < \infty$ for all n, and so the first set equaling one is equivalent to $P[\omega : N_n(\omega) < \infty] = 1$ for all n. Thus, the above simplifies to:

$$P[\omega : Y_n(\omega) = x] = p(x)$$

Which gives us our distribution of the Y_n property. Now, we want to focus on independence. We note that the same argument as above can be extended to a sequence of $x_1, \cdots, x_n$ of ± 1 to find:

$$\begin{aligned} P[\omega : Y_i(\omega) = x_i, i \leq n] &= P\left[\omega : X_{N_i(\omega)}(\omega) = x_i, i \leq n\right] \\ &= \sum_{k_1 < \cdots < k_n} P[\omega : N_i(\omega) = k_i, X_{k_i}(\omega) = x_i, i \leq n] \end{aligned}$$

Where the sum extends over n tuples of positive integers satisfying $k_1 < \cdots < k_n$. Note, this is still a countable number of tuples, and countable additivity of disjoint sets gives us the break down. We note that:

$$[\omega : N_i(\omega) = k_i, i \leq n] \cap [\omega : X_{k_i}(\omega) = x_i, i < n] \in \mathcal{F}_{k_n - 1}$$

They must be -> note that $N_n \in k_n \in \mathcal{F}_{k_n-1}$, which clearly implies the same for smaller k_i. Similarly, for $i < n$, $X_{k_i} \in \mathcal{F}_{k_i} \subseteq \mathcal{F}_{k_n-1}$. And so, we can simplify each term in the sum to:

$$= \sum_{k_1 < \cdots < k_n} P[[\omega : N_i(\omega) = k_i, i \leq n] \cap [\omega : X_{k_i}(\omega) = x_i, i < n]]\, p(x_n)$$

Via independence, and $P[X_{k_n}(\omega) = x_n] = p(x_n)$. We can then break up the sum as (noting all positive sum, reordering is allowed)

$$= \sum_{k_1 < \cdots < k_{n-1}} p(x_n) \sum_{k_n = k_{n-1}+1}^{\infty} P[[\omega : N_i(\omega) = k_i, i \leq n] \cap [\omega : X_{k_i}(\omega) = x_i, i < n]]$$

Now, we note that the only set on the LHS that is changing is $N_n(\omega) = k_n$. Now, this makes the sum a countable sum of disjoint probabilities. We can bring this into a single set, with the event $N_n(\omega) < \infty$, which is just some set A with probability 1. We note that for all $B \in \mathcal{F}$, we have:

$$P[A \cap B] + P[A^c \cap B] = P[B] \implies P[A \cap B] = P[B]$$

As $P[A^c \cap B] \leq P[A^c] = 0$. And so, we can just remove the $N_n(\omega) < \infty$ set, giving us:

$$= \sum_{k_1 < \cdots < k_{n-1}} p(x_n) P[[\omega : N_i(\omega) = k_i, i < n] \cap [\omega : X_{k_i}(\omega) = x_i, i < n]]$$

$$\implies P\left[\omega : Y_i(\omega) = x_i, i \leq n\right] = P[\omega : Y_i(\omega) = x_i, i < n]p(x_n)$$

We can use induction to easily conclude that:

$$P\left[\omega : Y_i(\omega) = x_i, i \leq n\right] = \prod_{i \leq n} p(x_i) = \prod_{i \leq n} P\left[\omega : Y_i(\omega) = x_i\right]$$

And so the Y_i are independent (note, an arbitrary finite subset of the Y_i is independent can clearly be concluded from the above property). qed.

Not bad! This is kind of our first real world result. Any selection system, based purely on previous values of iid random variables, is equivalent to just betting on a similar stream of iid random variables.

Gambling Policies

These refer to schemes that go beyond selection systems, and tell the gambler not only whether to bet or not - but how much as well.

Gambling Policy Framework

1. Let F_n denote the fortune the gambler has on the *nth* play. F_0 is taken as his initial, nonrandom fortune.

2. The wager specified for the *nth* trial is W_n. The gambler cannot see the future, and so W_n must depend only on $X_1, \cdots, X_{n-1}$, and his nonrandom initial fortune F_0, and so:

$$W_n = g_n(F_0, X_1, \cdots, X_{n-1}) \geq 0$$

   Note that the wager is nonnegative, and in effect includes selection systems, as $W_n = 0$ corresponds to not betting at all. We note that W_n is measurable $\mathcal{F}_{n-1}$, being a function of $X_1, \cdots, X_{n-1}$ (and F_0 is fixed and can be taken as part of the function). Finally, we note that:

$$F_n = F_{n-1} + W_n X_n$$

   And that F_n must be measurable $\mathcal{F}_n$.

3. Let $\tau(F_0, \omega)$ be a nonnegative integer for each $\omega \in \Omega$ and $F_0 \geq 0$. If $\tau = n$, the gambler plays on the *nth* trial (betting W_n) and then stops; if $\tau = 0$, he does not bet in the first place. The event $[\omega : \tau(F_0, \omega) = n]$ represents the decision to stop just after the *nth* trial, and so, for

whatever value of F_0, it must depend only on $X_1, \cdots, X_n$. Thus, we assume that:

$$[\omega : \tau(F_0, \omega) = n] \in \mathcal{F}_n \qquad n = 0, 1, 2, \cdots$$

A τ satisfying this requirement is called a *stopping time*. Note, it is not a SRV, as it has an infinite range. We do assume however that $P[\tau < \infty] = 1$, to make it certain that play will terminate.

4. A betting system together with a stopping time is a *gambling policy*. Let π denote such a policy.

5. With a stopping time, the gamblers fortune at time n is:

$$F_n^* = \begin{cases} F_n & \text{if } \tau \geq n \\ F_\tau & \text{if } \tau \leq n \end{cases}$$

It should also be clear that if $W_n^* = I_{\tau \geq n} W_n$, then:

$$F_n^* = F_{n-1}^* + I_{\tau \geq n} W_n X_n = F_{n-1}^* + W_n^* X_n$$

These are the *stopping time aware fortunes and wagers* (my name for them). Note that like W_n, W_n^* is also measurable $\mathcal{F}_{n-1}$, and can itself be treated as a betting system. This is because:

$$[\tau \geq n] = [\tau < n]^c = \left[\bigcup_{k=1}^{n-1} \tau = k \right]^c \in \mathcal{F}_{n-1}$$

As each $\tau = k \in \mathcal{F}_{n-1}$ by our definition, and σ algebras are closed unions and complements. And so, $I_{\tau \geq n}$ is measurable $\mathcal{F}_{n-1}$, and can be expressed as a function of $X_1, \cdots, X_{n-1}$, and thus so can W_n^*. Thus, we have that W_n^* is independent X_n, and:

$$E[F_n^*] = E[F_{n-1}^*] + E[W_n^*] E[X_n]$$

Given that X_n are iid distributed like p, q, if $p \leq q$, $E[X_n] \leq 0$. As $E[W_n^*] \geq 0$, the subfair case thus implies:

$$F_0 = F_0^* \geq E[F_1^*] \geq E[F_2^*] \geq \cdots$$

Theorem 7.2: Expected Stopping Time Aware Fortune Value Note that $\lim_n F_n^* = F_\tau$ with probability 1. This is because, when $n \geq \tau$, we have that $F_n^* = F_\tau$ by definition. And so, if ultimately $n \geq \tau$ with probability one, than we have the equality. As $\tau < \infty$ with probability 1, we can conclude:

$$P\left[\lim_n F_n^* = F_\tau\right] = 1$$

However, this *does not immediately imply* $\lim_n E[F_n^*] = E[F_\tau]$. $E[F_\tau]$ takes in the possible values of τ, and so it could take into account the infinite possible stopping times. NOTE: Technically, all we have proved so far has been for simple random variables. So, in reality, this portion is NOT rigorous. However, let's make the assumption that F_τ is a simple random variable. Note, F_n^* is a simple random variable, as it could only be, at a maximum, some value of $F_0 + MaxWager_1 + MaxWager_2$, and I think the maximum wager at each time step is bounded (it certainly is if the gambler cannot take out loans). Actually, note that as we assumed $W_n = g_n(F_0, X_1, \cdots, X_n)$, it can only be a simple random variable (as a function of simple random variables), and that can easily be seen to carry over to F_n^*. So, really, we only need an assumption on F_τ being a simple random variable.

However, assume that $\lim_n E[F_n^*] = E[F_\tau]$. Then, we note that by our decreasing sequence property, in the subfair case, we can conclude:

$$E[F_\tau] \leq F_0$$

If we just say that F_τ is a simple random variable, and then have a *bounded policy* which implies:

$$0 \leq F_n^* \leq M \qquad n = 0, 1, 2, \cdots$$

Then note, we can make use of Theorem 5.4, which implies that we indeed have $\lim_n E[F_n^*] = E[F_\tau]$. This yields the theorem:

For every policy, if $p \leq 1/2$, we have:

$$F_0 = F_0^* \geq E[F_1^*] \geq E[F_2^*] \geq \cdots$$

If the policy is bounded, and F_τ has finite range, then $E[F_\tau] \leq F_0$.

Bold Play

Note, the above tells us that, for whatever bounded policy we take - the expected value of our fortune at the stopping time will be less than our initial fortune. However - this is *only* a result on the expected value. It is actually possible, and *not* contradictory for us to say - there are specific policies that are better than others. Ie, better in the sense that, they will give us a higher probability of success than other policies - even though, in all cases, the policies are expected to yield us a fortune smaller than what we started at.

Bold Play First, we will give an overview of the bold play policy, and then we will describe the framework to compare different policies. Note, the framework will be one that is compatible with the above, in the sense that we will require F_τ to be a simple random variable, which will allow theorem 7.2 to apply.

For bold play, we rescale, so that the initial fortune satisfies $0 \leq F_0 \leq 1$ and the goal is 1. The policy of bold play is: at each stage, the gambler bets his entire fortune, unless a win carries him past 1, in which case he bets enough just to win. This is described as:

$$W_n = \begin{cases} F_{n-1} & \text{if } 0 \leq F_{n-1} \leq \frac{1}{2} \\ 1 - F_{n-1} & \text{if } \frac{1}{2} \leq F_{n-1} \leq 1 \end{cases}$$

Bold play has not terminated by time k if it hasn't by time $k-1$, and we win if $F_k < 1/2$ or lose if $F_k > 1/2$. It follows by induction that the probability that bold play continues beyond time n is at most m^n, for $m = max(p, q)$.

Policy Framework

1. We consider only policies π that are bounded by 1, ie:

$$0 \leq F_n^* \leq 1 \qquad n = 0, 1, 2, \cdots$$

2. Assume play stops as soon as F_n reaches 0 or 1, and that this is certain eventually to happen (ie, $P\left[\bigcup_n F_n = 0 \text{ or } 1\right] = 1$. Since F_τ assumes the values 0 and 1, and since:

$$[F_\tau = x] = \bigcup_{n=0}^{\infty} [\tau = n] \cap [F_n = x]$$

For $x = 0$ and $x = 1$, F_τ is a simple random variable. Recall, F_n is a simple random variable (being a function of W_n and X_n and F_0), and so $[F_n = x]$ is measurable $\mathcal{F}_n$. Also recall our assumption is that $[\tau = n] \in \mathcal{F}_n$ as well, and so F_τ is indeed measurable $\mathcal{F}$.

3. The policy π leads to success if $F_\tau = 1$. Let $Q_\pi(x)$ be the probability of this for an initial fortune $F_0 = x$:

$$Q_\pi(x) = P[F_\tau = 1] \qquad \text{for } F_0 = x$$

Again, since F_n is a function $\psi_n(F_0, X_1(\omega), \cdots, X_n(\omega)) = \psi_n(F_0, \omega)$, in expended notation this is:

$$Q_\pi(x) = P\left[\omega : \psi_{\tau(x,\omega)}(x, \omega) = 1\right]$$

4. As π specifies that play stops at the boundaries 0 and 1:

$$Q_\pi(0) = 0 \qquad Q_\pi(1) = 1 \qquad 0 \leq Q_\pi(x) \leq 1 \text{ for } 0 \leq x \leq 1$$

5. Finally, note that bold play is one such policy π. We have that play is certain to terminate, as $m^n \to 0$. Further, F_τ only assumes the values of 0 or 1 and F_n is bounded between 0 and 1. Let Q be the $Q\pi$ for bold play.

Theorem 7.3 - Bold Play is Optimal in the subfair case In the subfair case, $Q_\pi(x) \leq Q(x)$ for all π and all x.

Proof: To be honest, I don't care about proving this. As this subsection, and the remaining subsections are optional, I'll end the notes for Gambling Systems here.

Problems

7.1 Comparing Gambler's Ruin Success with Different Fortunes

A gambler with initial capital a plays until his fortune increases b units or he is ruined. Suppose that $\rho = q/p > 1$, and p is the probability of success of an independent trial ($X_i = +1$). The chance of ultimate success is multiplied by $1 + \theta$ is his initial capital is infinite instead of a. Show that $0 < \theta < (\rho^a - 1)^{-1} < (a(\rho - 1))^{-1}$. Relate to example 7.3.

Recall that if the initial capital $a < \infty$, the probability of ultimate success before ruin for $\rho \neq 1$ is:

$$\frac{\rho^a - 1}{\rho^{a+b} - 1}$$

If his capital is infinite, however, and $\rho > 1 \implies q > p$, the probability of success (ie, gain b units from where he started, ie, go up b before going down forever) is:

$$(p/q)^b = (1/\rho)^b$$

So, given that the chance is multiplied by $1 + \theta$ in the infinite capital case, we have:

$$(1+\theta)\left(\frac{\rho^a - 1}{\rho^{a+b} - 1}\right) = \rho^{-b} \implies (1+\theta)(\rho^a - 1) = \rho^a - \rho^{-b}$$

$$\implies \theta = \frac{\rho^a - \rho^{-b}}{\rho^a - 1} - 1$$

We first note, as $1 < \rho^b$, we have $\rho^{-b} < 1$. And so, we have that $\rho^a - \rho^{-b} > \rho^a - 1$, and so it is clear that $0 < \theta$. We can simplify some more to find:

$$\theta = \frac{\rho^a - \rho^{-b} - \rho^a + 1}{\rho^a - 1} = \frac{1 - \rho^{-b}}{\rho^a - 1}$$

Note that $0 < 1 - \rho^{-b} < 1$, and so the above is a multiplication by a number between 0 and 1, which implies:

$$\theta < (\rho^a - 1)^{-1}$$

Finally, we want to show that:

$$(\rho^a - 1)^{-1} < (a(\rho - 1))^{-1}$$

Which follows if we show $a(\rho - 1) < \rho^a - 1$. We first note, that if $\rho = 1$, we have:

$$a(\rho - 1) = 0 = \rho^a - 1$$

So, we just need the function:

$$f(\rho) = \rho^a - 1 - a(\rho - 1)$$

To be *increasing* for $\rho > 1$. The derivative of the above, with respect to ρ, is:

$$a\rho^{a-1} - a = a(\rho^{a-1} - 1)$$

Given that $\rho^{a-1} - 1 > 0$, and $a > 1$, we can indeed conclude that:

$$(\rho^a - 1)^{-1} < (a(\rho - 1))^{-1}$$

7.3 Collectives and Asymptotic Frequencies of Sequences

If V_n is the set of n long sequences of ± 1's (ie, contains 2^n sequences), the function b_n (from our selection system) maps V_{n-1} into $\{0, 1\}$. A selection system is a sequence of such maps. Although there are uncountably many selection systems, how many have an *effective* description in the sense of an algorithm or finite set of instructions by means of which a deputy (perhaps a machine) could operate the system for the gambler? An analysis of the question is a matter of mathematical logic, but one can see that there can be only countably many algorithms or finite sets of rules expressed in finite alphabets.

Let $Y_1^\sigma, Y_2^\sigma, \cdots$ be the random variables of Theorem 7.1 for a particular system σ (ie, Y_i^σ is the result when the gambler makes his ith bet under the selection system σ), and let C_σ be the ω set where every k tuple of ± 1's (k arbitrary, just all possible tuples of size k) occurs in $Y_1^\sigma(\omega), Y_2^\sigma(\omega), \cdots$ with the right asymptotic frequency. This is in the sense of problem 6.12, ie: for the k tuple $u_1, \cdots, u_k$, and $N_n(u_1, \cdots, u_k)$ the number of t such that $1 \leq t \leq n$ and $Y_t^\sigma = u_1, \cdots, Y_{t+k-1}^\sigma = u_k$, we have:

$$n^{-1} N_n(u_1, \cdots, u_k) \to p(u_1) \cdots p(u_k)$$

where each $p(+1) = p$ and $p(-1) = q$. Let C be the intersection of C_σ over all effective selection systems σ (ie, I believe this is just a countable intersection). Show that C lies in $\mathcal{F}$ (the σ-field in the probability space $(\Omega, \mathcal{F}, P)$ on which the X_n are defined) and that $P(C) = 1$. A sequence $(X_1(\omega), X_2(\omega), \cdots)$ for ω in C is called a *collective*: a subsequence chosen by any of the effective rules σ contains all k tuples in the correct proportions.

I think, this problem might require us to do problem 6.12 first. Once 6.12 is done, this is trivial. In 6.12, we proved exactly that the set C_σ consisting of ω such that each k tuple occurs with the right relative frequency is both within $\mathcal{F}$, ie $C_\sigma \in \mathcal{F}$, being a limit of $\mathcal{F}$ measurable random variables, and $P[C_\sigma] = 1$. We have defined:

$$C = \bigcap_{\sigma \in \Sigma} C_\sigma$$

Given the assumption of the problem that Σ is countable, as the *effective* description of algorithms in the real world are limited by finite alphabets/length, we have that $C \in \mathcal{F}$, being a countable intersection of $\mathcal{F}$ sets. We also note that a countable intersection of probability 1 sets is probability 1, and so $P(C) = 1$ as well. qed.

Mainly, what this gives us is, that a sequence $(X_1(\omega), X_2(\omega), \cdots)$ for $\omega \in C$ is a *collective* and occurs with probability one, and with probability one a subsequence chosen by any of the effective rules σ contains all k tuples in the correct proportions.

7.7 Progress and Pinch Gambling Policy

In "progress and pinch," the wager, initially some integer, is increased by 1 after a loss and decreased by 1 after a win, the stopping rule being to quit if the next bet is 0. Show that play is certain to terminate if and only if $p \geq \frac{1}{2}$. Show that $F_\tau = F_0 + \frac{1}{2}W_1^2 + \frac{1}{2}(\tau - 1)$. Infinite capital is required.

First note, that each W_n is a SRV. $W_n = g_n(F_0, X_1, \cdots, X_{n-1})$, and only takes on integer values between 0 and $W_1 + n$. Similarly, F_n is a SRV. We have that play is certain to terminate if and only if $P[\tau < \infty] = 1$. We want to show that this is true if and only if $p \geq \frac{1}{2}$. Note, a way we can think about this is similar to the initial section. We have infinite capital - and we keep on playing until the number of wins is larger than the number of losses, plus W_1. The moment this happens, $W_{n+1} = 0$ and the stopping rule tells us to quit at time n.

So, we want to know under what conditions on p we have:

$$\# \text{ wins} = \# \text{ losses} + W_1$$

With probability 1. Intuitively, this will only be the case if $p \geq \frac{1}{2}$ - ie, the drift is either towards more wins or completely random. Note, this follows rigorously from our gambling ruin framework as well. This can be be rephrased in terms of winning bets exceeding losing bets. We let S_n be the sum of X_i:

$$S_n = \sum_{i=1}^{n} X_i$$

We note that $\#$ wins $= \#$ losses $+ W_1$ is equivalent to the condition that at some point, $S_n \geq W_1$. This is because again, $S_n \geq W_1$ implies that at the *nth* trial, we have won W_1 times more than we have lost. In our gambling ruin framework, we proved:

$$P\left[\sup_n S_n \geq W_1\right] = \begin{cases} 1 & \text{if } p \geq q \\ (p/q)^b & \text{if } p < q \end{cases}$$

So, in total, the above analysis is:

$$P[\tau < \infty] = 1 \Leftrightarrow P[\# \text{ wins} = \# \text{ losses} + W_1] = 1 \Leftrightarrow P\left[\sup_n S_n \geq W_1\right] = 1 \Leftrightarrow p \geq 1/2$$

Given that, we now want to show that $F_\tau = F_0 + \frac{1}{2}W_1^2 + \frac{1}{2}(\tau - 1)$ in the case of play terminating. Note that by the above, we can see that our wager at time k is:

$$W_1 - S_{k-1}$$

That much is clear. And so, we have that:

$$F_\tau = F_0 + \sum_{k=1}^{\tau-1}(W_1 - S_{k-1})X_k = F_0 + \sum_{k=1}^{\tau-1} W_1X_k - S_{k-1}X_k$$

$$= F_0 + W_1S_{\tau-1} - \sum_{k=1}^{\tau-1} S_{k-1}X_k$$

We now note an interesting identity. We have that:

$$\sum_{k=1}^{y} S_{k-1}X_k = \frac{1}{2}(S_y^2 - y)$$

Why? Note that:

$$\frac{1}{2}(S_y^2 - y) = \frac{1}{2}\left[\sum_{k=1}^{y} X_k^2 + 2\sum_{i<j} X_iX_j - y\right] = \sum_{1\leq i<j\leq y} X_iX_j$$

Now, we also note that we have:

$$\sum_{k=1}^{y} S_{k-1}X_k = \sum_{1\leq i<j\leq y} X_iX_j$$

This can be noted by laying out the S_k values in an array, and looking at how each of the X_i are added up. Or, more rigorously, we can simplify to find the above:

$$\sum_{k=1}^{y} S_{k-1}X_k = \sum_{j=1}^{y}\left[\sum_{i=1}^{j-1} X_i\right]X_j = \sum_{1\leq i<j\leq y} X_iX_j$$

And so, in conclusion, we find:

$$F_\tau = F_0 + W_1S_{\tau-1} - \frac{1}{2}(S_{\tau-1}^2 - (\tau - 1)) = F_0 + \frac{1}{2}(\tau - 1) + W_1S_{\tau-1} - \frac{1}{2}S_{\tau-1}^2$$

Now, we just note that it is clear $S_\tau = W_1$, and so the above simplifies to:

$$F_\tau = F_0 + \frac{1}{2}W_1^2 + \frac{1}{2}(\tau - 1)$$

And so, we would like τ to be as large as possible, actually. qed.

7.8 Pattern Martingale Betting

Note - this is a bit tricky to understand. The idea is - the gambler in front of him has his list of numbers, and he uses that list of numbers to make bets on each of the n iid trials, with positive probability p and negative probability $q = 1 - p$. Just before the nth trial, the gambler has before him a pattern $x_1, \cdots, x_k$ - note that k varies with n. He bets $x_1 + x_k$, or x_1 in the case $k = 1$. If he loses, at the next stage he uses the pattern $x_1, \cdots, x_k, x_1 + x_k$ (or x_1, x_1 in case $k = 1$). If he wins, at the next stage he uses the pattern $x_2, \cdots, x_{k-1}$, unless the pattern is empty, in which case he quits. Intuitively, we only get the x_1 case above if the initial pattern is just x_1.

Show that play is certain to terminate if $p > 1/3$ and that the ultimate gain is the sum of the numbers in the initial pattern. Infinite capital is again required.

We first note the ultimate gain that comes, if play does terminate. Let the initial pattern be $x_1, \cdots, x_k$. Note that it is clear if the first k trials are success, the ultimate gain is indeed the initial sum, as we will have:

$$(x_1 + x_k) + (x_2 + x_{k-1}) + \cdots + [\text{choose } (x_i + x_{i+1}) \text{ or } (x_i)]$$

Note, I had to change the rules, the explanation wasn't 100% on point. It should be intuitive that if we add $x_1 + x_k$ to the pattern, it means we lost $x_1 + x_k$ in a bet. And so, for whenever we lose, we will add an amount to bet that exactly cancels out the lost amount. And so ultimately, intuitively, the only amounts that don't cancel out, in the case of termination, are the amounts from the initial pattern. Frankly, I think this intuitive explanation is enough for ultimate gain.

Now, all that remains is termination. We let $L_0 = k$. We note that if we win, we remove 2 from the pattern, and if we lose, we add 1 to the pattern. And so, we can define:

$$L_n = L_{n-1} - (3X_n + 1)/2$$

And it is clear that L_n describes the pattern after the nth trial. And so, τ is the smallest n such that $L_n \leq 0$ (edge case, when $L_{n-1} = 1$, we should subtract only 1 in this case. However, subtracting 2 is fine when we consider that we look for $L_n \leq 0$, not $L_n = 0$). We note that $\frac{1}{n} \sum_{n=1}^{\infty} (3X_n + 1)/2 > 0$ with probability 1 if $p > 1/3$. And so, with probability 1, we have that the subtractions go to negative infinity, and there exists $L_n < 0$ with probability 1 if $p > 1/3$. So, we have that play is certain to terminate if $p > 1/3$. qed.

7.9 Stock Option Stopping Time

Suppose that $W_k = 1$, so that $F_k = F_0 + S_k$. Suppose that $p \geq q$ and τ is a stopping time such that $1 \leq \tau \leq n$ with probability 1. Show that $E[F_\tau] \leq E[F_n]$, with equality in case $p = q$. Interpret this result in terms of a stock option that must be exercised by time n, where $F_0 + S_k$ represents the price of the stock at time k.

Note, even though we have that the F_k and F_τ are uniformly bounded (by F_0+n), we can't make use of Theorem 7.2, as that only makes use of stopping time aware fortunes, which F_n is not. However, we note that:

$$E[F_n - F_\tau] = E\left[\sum_{k=1}^{n} X_k \mathbb{1}\{\tau < k\}\right] = \sum_{k=1}^{n} E[X_k] P[\tau < k]$$

Where the first step notes that when $\tau < k$, say $k-1$, F_τ only consists of the bets made up to X_{k-1}, and then the remaining X_k are not part of F_τ. This difference is what is included in F_n. The second step notes that $[\tau(F_0, \omega) = k] \in \mathcal{F}_k$, and so independence allows us to break out the multiplication of the indicator. Note that $E[X_k] = p - q$, and so the above simplifies to:

$$= (p-q)\sum_{k=1}^{n} P[\tau < k] = p - q$$

When $p \geq q$, we have $E[F_\tau] \leq E[F_n]$. When $p = q$, we have equality. qed.

We want to interpret this result in terms of a stock option that must be exercised by time n, where $F_0 + S_k$ is the price of the stock at time k. We have some strike price V. Ultimately, we want to exercise the option when $F_0 + S_k \geq V$, as otherwise, we should just buy the stock at the current price and save the option for later. If $p \geq q$, it is always better to wait until the end, as the expected value of the stock price keeps rising, making the strike price more valuable.

Section 8 - Markov Chains

Notes

8.1 Markov Chain Setup

Let S be a finite or countable set. Suppose that to each pair (i, j) in S there is assigned a nonnegative number p_{ij} and that these numbers satisfy the constraint:

$$\sum_{j \in S} p_{ij} = 1 \qquad i \in S$$

Let $X_0, X_1, X_2, \cdots$ be a sequence of random variables whose ranges are contained in S. The sequence is a *Markov Chain* or *Markov Process* if:

$$P[X_{n+1} = j | X_0 = i_0, \cdots, X_n = i_n] = P[X_{n+1} = j | X_n = i_n] = p_{i_n j}$$

For every sequence $i_0, \cdots, i_n$ in S for which $P[X_0 = i_0, \cdots, X_n = i_n] > 0$ (note, otherwise, conditional probability is not defined). The set S is the *state space* and the p_{ij} are the *transition probabilities*. Part of the defining condition is that:

$$P[X_{n+1} = j | X_n = i] = p_{ij}$$

Does not vary with n. The *initial probabilities* are:

$$\alpha_i = P[X_0 = i]$$

The α_i are nonnegative and add up to 1, but the setup places no further restriction on them. Later, we will prove that there exists an underlying probability space $(\Omega, \mathcal{F}, P)$. The p_{ij} form a *transition matrix* $P = [p_{ij}]$ of the process. A *stochastic matrix* is one whose entries are nonnegative and satisfy the $\sum_{j \in S} p_{ij} = 1$ condition - ie, that the values along one row sum up to 1. The transition matrix has this property.

Markov Chain States Recall, S can be countably infinite. And so, the Markov Chain can consist of functions X_n on $(\Omega, \mathcal{F}, P)$ that have *infinite range*, and hence will not be simple random variables. This will not cause an issue, as we will not examine the expected values of these X_n. All that is required is that for each $i \in S$, the set $[\omega : X_n(\omega) = i] \in \mathcal{F}$ and thus has a probability.

We also have that the X_n do not have to assume real values. As an example, there is the random walk on the integer lattice of a k dimensional space - each X_n takes a value in $\mathbb{R}^k$. Note, this is still countable. But again, like the note above, the values of the X_n will play no role - we just need that $[\omega : X_n(\omega) = i] \in \mathcal{F}$ for $i \in S$. Do note: there *will* be expected values $E[f(X_n)]$ for real functions f on S with finite range, but then $f(X_n(\omega))$ will be a simple random variable as defined previously.

8.2 High Order Transitions

First, recall the chain rule for conditional probabilities:

$$P[A \cap B \cap C] = P[A]P[B|A]P[C|A \cap B]$$

And so on. This is just by fraction cancellation. Thus, we have:

$$P[X_0 = i_0, X_1 = i_1, X_2 = i_2] = P[X_0 = i_0]P[X_1 = i_1|X_0 = i_0]P[X_2 = i_2|X_0 = i_0, X_1 = i_1]$$
$$= \alpha_{i_0} p_{i_0,i_1} p_{i_1,i_2}$$

This can be extended to an arbitrary length finite sequence:

$$P[X_t = i_t, 0 \leq t \leq m] = a_{i_0} p_{i_0,i_1} \cdots p_{i_{m-1},i_m}$$

For any sequences $i_0, i_1, \cdots, i_m$ of states. Further, we can find:

$$P[X_{m+t} = j_t, 1 \leq t \leq n | X_s = i_s, 0 \leq s \leq m] = p_{i_m,j_1} p_{j_1,j_2} \cdots p_{j_{n-1},j_n}$$

Note, this is easily found again by describing as the conditional probability ratio, making use of our previous result, and applying cancellations. Adding out the intermediate states, we find that:

$$p_{ij}^{(n)} = P[X_{m+n} = j | X_m = i] = \sum_{k_1, \cdots, k_{n-1}} p_{ik_1} p_{k_1 k_2} \cdots p_{k_{n-1} j}$$

This is the *nth order transition probability*, and clearly comes from separating out the probabilities across disjoint unions, and making use of our previous results. Note, it is also clear that $p_{ij}^{(n)}$ is the (i, j) entry of P^n, the nth power of the transition matrix P. If S is infinite, P is a matrix with infinitely many rows and columns. However, as the terms in P are nonnegative, there are no convergence problems. It is natural to put:

$$p_{ij}^{(0)} = \delta_{ij}$$

Then, P^0 is the identity I, as it should be. It is also clear that:

$$p_{ij}^{(m+n)} = \sum_v p_{iv}^{(m)} p_{vj}^{(n)}$$

Via $P^m P^n$, and that we also have:

$$\sum_j p_{ij}^{(n)} = 1$$

As we can replace $p_{ij}^{(n)}$ by its expanded term above, and note that if we are at state i, it will be in some state j in n steps with probability 1.

8.3 An Existence Theorem

Theorem 8.1 Suppose that $P = [p_{ij}]$ is a stochastic matrix and that α_i are nonnegative numbers satisfying $\sum_{i \in S} \alpha_i = 1$. There exists on some $(\Omega, \mathcal{F}, P)$ a Markov Chain $X_0, X_1, X_2, \cdots$ with initial probabilities α_i and transition probabilities p_{ij}.

Proof: This is similar to Theorem 5.3, which we proved the existence of independent sequences of random variables for a sequence of finite support measures on the set of all subsets of $\mathbb{R}$. There, the space $(\Omega, \mathcal{F}, P)$ was $((0,1], \mathcal{B}, \lambda)$. The central part of the argument was decomposing intervals into subintervals of a specific length.

Note. If $\delta_1 + \delta_2 + \cdots = b - a$, and $\delta_i \geq 0$, then $I_i = (b - \sum_{j \leq i} \delta_j, b - \sum_{j<i} \delta_j]$ decomposes $(a, b]$ into intervals of length δ_i for $i = 1, 2, \cdots$. This will be used repeatedly in the proofs - we will be decomposing intervals $(a, b]$ into subintervals of length $p_i(a, b]$, where p_i adds to one. This construction directly corresponds with the δ_i construction.

1. Initial Probability Intervals Suppose our state space is $S = \{1, 2, \cdots\}$. Really, this is only important to recognize that our stochastic matrix is countable, and helps us with our subscript indices. First, decompose $(0,1]$ into a countable partition $I_1^{(0)}, I_2^{(0)}, \cdots$ subintervals of length (and thus probability via $P = \lambda$) $P(I_i^{(0)}) = \alpha_i$.

2. Transition Probability Intervals Next, decompose each $I_i^{(0)}$ into a countable partition $I_{ij_1}^{1}, I_{ij_2}^{(1)}, \cdots$ subintervals of length $P(I_{ij}^{(1)}) = \alpha_i p_{ij}$. Continuing inductively gives a sequence of partitions $\{I_{i_0, \cdots, i_n}^{(n)} : i_0, \cdots, i_n \in S\}$ where each refines the preceding, and:

$$P\left(I_{i_0, \cdots, i_n}^{(n)}\right) = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$$

Now, we define our random variables. We have $X_n(\omega) = i$ if $\omega \in \bigcup_{i_0, \cdots, i_{n-1}} I_{i_0, \cdots, i_{n-1}, i}^{(n)}$. This is just a simple function sum. It should be clear that:

$$[X_0 = i_0, \cdots, X_n = i_n] = I_{i_0, \cdots, i_n}^{(n)}$$

The LHS is a finite intersection of unions of intervals, where each previous union limits what intervals we are actually looking at. Thus, it is clear that:

$$P\left[X_0 = i_0, \cdots, X_n = i_n\right] = P\left[I_{i_0, \cdots, i_n}^{(n)}\right] = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}$$

It should thus be clear that we have:

$$P[X_0 = i] = \alpha_i$$

Which was one of our Markov Chain conditions. We also have:

$$P[X_{n+1} = j | X_0 = i_0, \cdots, X_n = i_n] = \frac{P\left[I^{(n+1)}_{i_0,\cdots,i_n,j}\right]}{P\left[I^{(n)}_{i_0,\cdots,i_n}\right]} = \frac{\alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} p_{i_n j}}{\alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}} = p_{i_n j}$$

Which is another one of our Markov Chain conditions. We also need that this equals $P[X_{n+1} = j | X_n = i_n]$, which we find via:

$$= \frac{P[X_{n+1} = j, X_n = i_n]}{P[X_n = i_n]} = \frac{\sum_{i_0,\cdots,i_{n-1}} \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n} p_{i_n j}}{\sum_{i_0,\cdots,i_{n-1}} \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}}$$

$$= p_{i_n j} \frac{\sum_{i_0,\cdots,i_{n-1}} \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}}{\sum_{i_0,\cdots,i_{n-1}} \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}} = p_{i_n j}$$

Where the only tricky part is that the first numerator probability extends over the union of all disjoint intervals of the form $I_{i_0,\cdots,i_{n-1},i_n,j}$ with the fixed i_n and j. This completes the construction - as we have all the probabilities are correct, and $X_n = i$ is measurable. X_i is still not a simple random variable. For other countable spaces S - let g be a one-to-one mapping between $\{1, 2, \cdots\}$ and S. Replace the X_n as already constructed with $g(X_n)$. Easy enough. And, it is clear that the argument works for S finite. qed.

And so, although the Markov Chain *is* the sequence $X_0, X_1, \cdots$ - we often just speak of the chain as if it were matrix P, with initial probabilities α_i. The above theorem just justifies our use of probability theory apparatus - such as the Borel-Cantelli lemmas, other convergence lemmas, or inequalities.

Cylinder Sequences Split Property From now on, we will fix a chain $X_0, X_1, \cdots$ satisfying $\alpha_i > 0$ for all i. Denote by P_i probabilities conditional on $[X_0 = i]$ - ie, we have $P_i(A) = P[A_i | X_0 = i]$. thus, we have:

$$P_{i_0}[X_t = i_t\,,\ 1 \leq t \leq n] = \frac{P[X_t = i_t\,,\ 0 \leq t \leq n]}{P[X_0 = i_0]} = p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

With this formulation, the initial probabilities α_i are largely irrelevant. From this equation, it follows:

$$P_i[X_1 = i_1, \cdots, X_m = i_m, X_{m+1} = j_1, \cdots, X_{m+n} = j_n]$$

$$= P_i\left[X_1 = i_1, \cdots, X_m = i_m\right] P_{i_m}\left[X_1 = j_1, \cdots, X_n = j_n\right]$$

This follows directly via substitution of the above expression. We can now extend the above to sets. Let I be a finite or countable set of m long sequences of states, and J a finite or countable set of n long sequences of states. Further suppose that every sequence in I ends with state j - ie, $i_m = j$. Then, we can make use of additivity and the above to easily conclude:

$$P_i\left[(X_1, \cdots, X_m) \in I, (X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

$$= P_i\left[(X_1, \cdots, X_m) \in I\right] P_j\left[(X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

8.4 Transience and Persistence

Define:

$$f_{ij}^{(n)} = P_i\left[X_1 \neq j, \cdots, X_{n-1} \neq j, X_n = j\right]$$

$f_{ij}^{(n)}$ is the probability of a first visit to j at time n for a system that starts at i, and let:

$$f_{ij} = P_i\left(\bigcup_{n=1}^{\infty}[X_n = j]\right) = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

Be the probability of an eventual visit to state j for a system that starts at i. Note, the second equality is breaking down the union into an equivalent disjoint union and making use of countable additivity. A state i is *persistent* if a system starting at i is certain sometime to return to i: $f_{ii} = 1$. The state is *transient* in the opposite case: $f_{ii} < 1$.

Suppose that $n_1, \cdots, n_k$ are integers satisfying $1 \leq n_1 < \cdots < n_k$ and consider the event that the system visits j at times $n_1, \cdots, n_k$ but not in between; this event is determined by the conditions:

$$X_1 \neq j, \cdots, X_{n_1-1} \neq j, X_{n_1} = j, X_{n_1+1} \neq j, \cdots, X_{n_2-1} \neq j, X_{n_2} = j, X_{n_2+1} \neq j, \cdots, X_{n_k} = j$$

You should be able to see the pattern. Making use of our property:

$$P_i\left[(X_1, \cdots, X_m) \in I, (X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

$$= P_i\left[(X_1, \cdots, X_m) \in I\right] P_j\left[(X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

We see that under P_i, the probability of this event (which we will denote as $A_{n_1,\cdots,n_k}$) is:

$$P_i\left[A_{n_1,\cdots,n_k}\right] = P_i\left[A_{n_1}\right] P_j\left[A_{n_2-n_1,n_3\cdots,n_k}\right] = P_i\left[A_{n_1}\right] P_j\left[A_{n_2-n_1}\right] \cdots P_j\left[A_{n_k-n_{k-1}}\right]$$

Now, note that $P_i[A_{n_1}]$ is just the event that we start at state i, and don't enter j until state X_{n_1} - this is, by definition, $f_{ij}^{(n_1)}$. We find that, we thus have:

$$P_i[A_{n_1,\cdots,n_k}] = f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_k-n_{k-1})}$$

Now, we can sum over each of the possible k tuples $n_1, \cdots, n_k$ (noting that they each correspond to a different disjoint set) to find that:

$$P_i[X_n = j \text{ at least } k \text{ times}] = P_i\left[\bigcup_{1 \le n_1 < \cdots < n_k} A_{n_1,\cdots,n_k}\right] = \sum_{1 \le n_1 < \cdots < n_k} P_i[A_{n_1,\cdots,n_k}]$$

$$= \sum_{1 \le n_1 < \cdots < n_k} f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_k-n_{k-1})}$$

$$= \sum_{1 \le n_1 < \cdots < n_{k-1}} f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_{k-1}-n_{k-2})} \sum_{t=1}^{\infty} f_{jj}^{(t)}$$

$$= f_{jj} \sum_{1 \le n_1 < \cdots < n_{k-1}} f_{ij}^{(n_1)} f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_{k-1}-n_{k-2})}$$

$$= f_{ij} f_{jj}^{k-1}$$

And so now, taking $k \to \infty$, and making use of continuity from below, we have:

$$P_i[X_n = j \text{ i.o.}] = \lim_{k \to \infty} f_{ij} f_{jj}^{k-1} = \begin{cases} 0 & \text{if } f_{jj} < 1 \\ f_{ij} & \text{if } f_{jj} = 1 \end{cases}$$

Note, just to clear up the conditioning. We *directly* have that:

$$P_i[X_n = j \text{ at least } k \text{ times}] = P[X_n = j \text{ at least } k \text{ times} | X_0 = i]$$

$$= \frac{P[X_n = j \text{ at least } k \text{ times}, X_0 = i]}{P[X_0 = i]}$$

Note that:

$$\{X_n = j \text{ at least 1 times}, X_0 = i\} \supseteq \{X_n = j \text{ at least 2 times}, X_0 = i\} \supseteq \cdots$$

And so, we can make use of monotone convergence theorem to find that:

$$\lim_{k \to \infty} P[X_n = j \text{ at least } k \text{ times}, X_0 = i] = P\left[\bigcap_{k=1}^{\infty} X_n = j \text{ at least } k \text{ times}, X_0 = i\right]$$

Note that that ω inside of that intersection are also inside of $X_n = j$i.o., and so we find:

$$\lim_{k\to\infty} f_{ij} f_{jj}^{k-1} = \frac{P\left[X_n = j \text{ i.o.}, X_0 = i\right]}{P[X_0 = i]} = P_i\left[X_n = j \text{ i.o.}\right]$$

So, we do see that all the standard theorems worked under P_i. Note now, we don't actually have to perform these long derivations over the event $X_0 = i$, as we do have that P_i is a probability measure. Ie, we have that $(\Omega, \mathcal{F}, P_i)$ is a probability space given that $(\Omega, \mathcal{F}, P_i)$ is a probability space. And so, we can apply the normal theorems to all sets that appear within our P_i expressions. And so, if we take $i = j$, then we find:

$$P_i\left[X_n = i \text{ i.o.}\right] = \begin{cases} 0 & \text{if } f_{ii} < 1 \\ 1 & \text{if } f_{ii} = 1 \end{cases}$$

Thus, transience is equivalent to not returning to i infinitely often, whereas persistence is equivalent to returning to i infinitely often. These statements might have been intuitive, but the above argument proved it rigorously. We now give an expanded equivalent definition.

Theorem 8.2 Transience and Persistence Criteria

1. Transience of i is equivalent to $P_i\left[X_n = i \text{ i.o.}\right] = 0$ and to $\sum_n p_{ii}^{(n)} < \infty$
2. Persistence of i is equivalent to $P_i\left[X_n = i \text{ i.o.}\right] = 1$ and to $\sum_n p_{ii}^{(n)} = \infty$

Note, the infinitely often statement has already been proven, we just need to add the sum criteria.

Proof By the first Borel-Cantelli lemma, $\sum_n p_{ii}^{(n)} < \infty$ implies that:

$$P_i\left[X_n = i \text{ i.o.}\right] = 0$$

For one, this needs to be broken down a bit. We do note that conditioning on a set of nonzero probability $(X_0 = i)$ does yield another probability distribution. However, this seems a bit confusing. We recall, we have:

$$p_{ii}^{(n)} = P\left[X_{m+n} = i | X_m = i\right]$$

Note, that we can take $m = 0$, and so:

$$p_{ii}^{(n)} = P_i\left[X_n = i\right]$$

And as P_i is a probability measure on $\mathcal{F}$, we indeed have that if the sum of the probability of the above events converges, we have that:

$$P_i\left[X_n = i \text{ i.o.}\right] = 0$$

By our previous statement, that is equivalent to transience of i. And so, if we number the three statements in the first line as 1, 2, 3, and the three in the second line 4, 5, 6, we so far have:

$$3 \implies 2 \qquad 2 \implies 1$$

If we prove $1 \implies 3$, then we have the first three conditions are equivalent via a circle. Note, it will also prove the final three conditions are equivalent, as we have $\not{1} = 4$, $\not{2} = 5$, $\not{3} = 6$. And so, as $1, 2, 3$ are equivalent, we will also have $\not{1}$, $\not{2}$, $\not{3}$ are equivalent, which means the second line of the theorem is true as well (note, we already have 5 and 4 are equivalent like we have 1 and 2 where equivalent, but we don't need them). So now, we just try and prove that transience of i, ie $f_{ii} < 1$, implies $\sum_n p_{ii}^{(n)} < \infty$. Note that:

$$p_{ij}^{(n)} = P_i\left[X_n = j\right] = \sum_{s=0}^{n-1} P_i\left[X_1 \neq j, \cdots, X_{n-s-1} \neq j, X_{n-s} = j, X_n = j\right]$$

This is known as a *first passage* argument. We are analyzing $p_{ij}^{(n)}$ as *equivalent* to entering j for the first time at some time step before n, and then entering j at time n as well. Note that this is just breaking down a disjoint union. However, its use comes from giving us a recurrent term. We note the above equals:

$$= \sum_{s=0}^{n-1} \frac{P\left[X_1 \neq j, \cdots, X_{n-s-1} \neq j, X_{n-s} = j, X_n = j\right]}{P\left[X_0 = i\right]}$$

$$= \sum_{s=0}^{n-1} P_i\left[X_1 \neq j, \cdots, X_{n-s-1} \neq j, X_{n-s} = j\right] P_j\left[X_s = j\right]$$

Note, this follows *directly* from our earlier statement that:

$$P_i\left[(X_1, \cdots, X_m) \in I, (X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

$$= P_i\left[(X_1, \cdots, X_m) \in I\right] P_j\left[(X_{m+1}, \cdots, X_{m+n}) \in J\right]$$

Given all sequences in I end with j. Above, we can let I be the sequences that for the first $n - s - 1$ states does not equal j, and then equals j at the final state. We can let J be the sequences such that $X_{n-s+1}, \cdots, X_{n-1}$ are anything, and $X_n = j$. The above sum simplifies to the following, directly via definitions:

$$p_{ij}^{(n)} = \sum_{s=0}^{n-1} f_{ij}^{(n-s)} p_{jj}^{(s)}$$

Therefore:

$$\sum_{t=1}^{n} p_{ii}^{(t)} = \sum_{t=1}^{n} \sum_{s=0}^{t-1} f_{ii}^{(t-s)} p_{ii}^{(s)} = \sum_{s=0}^{n-1} p_{ii}^{(s)} \sum_{t=s+1}^{n} f_{ii}^{(t-s)}$$

This, this is just a reshuffling. We note that the $p_{ii}^{(s)}$ terms only appear for $s = 0, \cdots, n-1$. We distribute out the multiplications for each $p_{ii}^{(s)}$ term. Only if $t \geq s+1$, is there a non zero $f_{ii}^{(t-s)} p_{ii}^{(s)}$ term. Finally, note that f_{ii} is the sum of all $f_{ii}^{(n)}$ terms, and so we can easily bound the above by:

$$\leq \sum_{s=0}^{n-1} p_{ii}^{(s)} f_{ii} \leq \sum_{s=0}^{n} p_{ii}^{(s)} f_{ii}$$

We can use this inequality to find a relation between f_{ii} and $\sum_n p_{ii}^{(n)}$. Note, as $p_{ii}^{(0)} = 1$, we have that:

$$\sum_{t=1}^{n} p_{ii}^{(t)} - f_{ii} \sum_{t=1}^{n} p_{ii}^{(t)} \leq f_{ii} \implies (1 - f_{ii}) \sum_{t=1}^{n} p_{ii}^{(t)} \leq f_{ii}$$

If $f_{ii} < 1$, the division is well defined, and we have:

$$\sum_{t=1}^{n} p_{ii}^{(t)} \leq \frac{f_{ii}}{1 - f_{ii}}$$

And as the partial sums are bounded, so is the infinite sum. Thus, we have proved that all the statements are equivalent. qed.

Example 8.6 - Polya's Theorem The statement is that for the symmetric k dimensional random walk (over the k dimensional integer lattice), all states are persistent if $k = 1$ or $k = 2$, or are transient for $k \geq 3$. This first notes that, by symmetry, it is clear that $p_{ii}^{(n)}$, the return in n states to the state i, is the same for all i. And so, we denote this probability as a_n^k to denote the dependence on the dimension k. Also, clearly we have that $a_{2n+1}^k = 0$, as we cannot return in an odd number of steps. Now, we just go about proving the sums or infinite or convergent. We note that:

$$a_{2n}^1 = \binom{2n}{n} \frac{1}{2^{2n}}$$

Each choice of left or right is the same probability (of 1/2), and we need n of the $2n$ steps to be left, and the other n right. By Stirling's formula, we have that $a^1_{2n} \sim (pn)^{-1/2}$. This sum converges, and the state is transient.

For $k = 2$, it is the same argument. Find the equivalent combinatorial form for a choice of 4 directions, use Stirling's formula, and show that the sum is infinite. It is the same for $k \geq 3$, but note that the combinatorial shit is harder, and more in depth. And the sum converges. I don't think these details are important for now, and so I skip them.

Definition: Irreducible A Markov chain is called *irreducible* if $p_{ij}^{(n)} > 0$ for some n, for each i and j. Essentially, it is possible to get from any one state to another.

Theorem 8.3 - Irreducible Markov Chains are persistent or transient If the Markov chain is irreducible, then one of the following two alternatives holds:

1. All states are transient, $P_i\left[\bigcup_j [X_n = j \text{ i.o.}]\right] = 0$ for all i, and $\sum_n p_{ij}^{(n)} < \infty$ for all i and j.
2. All states are persistent, $P_i\left[\bigcap_j [X_n = j \text{ i.o.}]\right] = 1$ for all i, and $\sum_n p_{ij}^{(n)} = \infty$ for all i and j.

Proof: For each i and j there exists r and s such that $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$ - this is irreducibility. Note:

$$p_{ii}^{(r+s+n)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)}$$

Intuitively, this is because not all paths from i to i have to go from i to j and then j to i. Rigorously, this comes from making use of our cylinder sequences property two times, restricting the cylinders on indexes $n + s$ and then s. From $p_{ij}^{(r)} p_{ji}^{(s)}$ we note that $\sum_n p_{ii}^{(n)} < \infty$ implies $\sum_n p_{jj}^{(n)} < \infty$: and so, if one state is transient, they all are. Thus, if one state is persistent, they all must be as well. This gives the split between the two options.

If all the states are transient, recall that we proved earlier:

$$P_i [X_n = j \text{ i.o.}] = 0$$

And so, it is clear that for all i:

$$P_i \left[\bigcup_j [X_n = j \text{ i.o.}] \right] = 0$$

Now note:

$$\sum_{n=1}^{\infty} p_{ij}^{(n)} = \sum_{n=1}^{\infty} \sum_{\nu=1}^{n} f_{ij}^{(\nu)} p_{jj}^{(n-\nu)}$$

This comes directly from our previous theorem. Rearranging the sum gives us:

$$= \sum_{\nu=1}^{\infty} f_{ij}^{(\nu)} \sum_{m=0}^{\infty} p_{jj}^{(m)} \leq \sum_{m=0}^{\infty} p_{jj}^{(m)}$$

Noting that $f_{ij}^{(\nu)}$ represents probabilities of disjoint events. Thus, it follows that if j is transient (which we have assumed is true for all states), then $\sum_{n=1}^{\infty} p_{ij}^{(n)}$ converges for all i. Thus, we have proved the properties of the first case, assuming all states are transient.

Now, we assume that all the states in our irreducible chain are persistent. In this case, by Theorem 8.2, we have:

$$P_j \left[X_n = j \text{ i.o.} \right] = 1$$

And we have:

$$p_{ji}^{(m)} = P_j \left[[X_m = i] \cap [X_n = j \text{ i.o.}] \right]$$

As intersecting with a probability 1 set does not reduce the probability. This is bounded by:

$$\leq \sum_{n>m} P_j \left[X_m = i, X_{m+1} \neq j, \cdots, X_{n-1} \neq j, X_n = j \right]$$

As clearly, each ω in the first probability would appear in one of the sets in the disjoint union described above. By our cylinder sequence property, the above equals:

$$= \sum_{n>m} p_{ji}^{(m)} f_{ij}^{n-m} = p_{ji}^{(m)} f_{ij}$$

Note - as there is an m for which $p_{ji}^{(m)} > 0$, we must have that $f_{ij} = 1$, as f_{ij} is between 0 and 1, as it represents a probability, and the above property can only be satisfied by $f_{ij} \geq 1$. Note, this property can only be concluded when we note all states are persistent. Again, by our earlier transience definition, we proved that:

$$P_i\left[X_n = j \text{ i.o.}\right] = f_{ij} = 1$$

Which allows us to conclude:

$$P_i\left[\bigcap_j [X_n = j \text{ i.o.}]\right] = 1$$

As the intersection of probability one sets is probability 1. Further, if we have that $\sum_n p_{ij}^{(n)} < \infty$ for some i and j, it would follow via the first Borel-Cantelli lemma that $P_i\left[X_n = j \text{ i.o.}\right] = 0$, which again, cannot be the case. And so, we have proved all the statements in the second case defined by the Theorem. qed.

Example: Irreducible Finite Markov Chains are Persistent Recall that we have $\sum_j p_{ij}^{(n)} = 1$. We note that if $\sum_n p_{ij}^{(n)} < \infty$ for all i and j, then we must have:

$$\sum_j \sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$$

Note, we can conclude this is finite, because j is just a finite sum of finite values. As the series converges, we can switch the order:

$$= \sum_{n=1}^{\infty} \sum_j p_{ij}^{(n)} = \sum_{n=1}^{\infty} 1 = \infty$$

Which gives us a contradiction. So, we must have that for some pair i and j, $\sum_n p_{ij}^{(n)} = \infty$. This implies we are in case 2, and so a finite irreducible Markov chain must always be persistent. qed.

Example 8.8: Polya's Theorem Note that we proved in a symmetric random walk, for dimensions 1 and 2 every state was persistent, and dimensions greater than or equal to 3, every state is transient. Given that the chain is irreducible, this matches with the above theorem. However, also note, the above theorem implies that no matter *where* we start, in a dimension 1 or 2 symmetric random walk, we visit each state infinitely often. In dimensions 3 or more, we visit no state infinitely often, and so we must go to infinity.

Example 8.9: Random Walk On the Line Properties Note that, for the random walk on the line, we have that every state is persistent, and:

$$\sum_n p_{ij}^{(n)} = \infty$$

However, also note that if n and $j-i$ have the same parity:

$$p_{ij}^{(n)} = \binom{n}{\frac{n+j-i}{2}} \frac{1}{2^n} \qquad |j-i| \leq n$$

This is easy to note just by choosing paths which go to the right $j-i$ times more, and:

$$\frac{n+j-i}{2} - \left[n - \frac{n+j-i}{2}\right] = j - i$$

This is maximal is $j = i$ or $j = i \pm 1$. By Stirling's formula, the maximal value is order $n^{-1/2}$. Therefore:

$$\lim_n p_{ij}^{(n)} \approx \lim_n n^{-1/2} = 0$$

And so, even though the probabilities sum to infinity, they go to zero. So, think of something like $1/n$. This always holds in the transience case (that the limit is zero), but we see it can also hold in the persistent case.

8.5 Another Criterion for Persistence

Let $Q = [q_{ij}]$ be a matrix whose rows and columns are indexed by the elements of a finite or countable U. Suppose that it is *substochastic* in the sense that $q_{ij} \geq 0$ and $\sum_j q_{ij} \leq 1$ - ie, the rows sum to less than 1. Let $Q^n = [q_{ij}^{(n)}]$ be the *nth* power, defined inductively as:

$$q_{ij}^{(n+1)} = \sum_\nu q_{i\nu} q_{\nu j}^{(n)} \qquad q_{ij}^{(0)} = \delta_{ij}$$

Note, this just replicates matrix multiplication, but is also defined for infinite matrices. We also have that it converges, being a nonnegative sum bounded from above by 1 (note inductively, each $q_{i\nu}$ sums to less than 1 across ν, and the multiplicative factor $q_{\nu j}^{(n)} < 1$).

Consider the row sums:

$$\sigma_i^{(n)} = \sum_j q_{ij}^{(n)}$$

It follows that:

$$\sigma_i^{(n+1)} = \sum_j q_{ij}^{(n+1)} = \sum_j \sum_\nu q_{i\nu} q_{\nu j}^{(n)} = \sum_\nu q_{i\nu} \sum_j q_{\nu j}^{(n)} = \sum_\nu q_{i\nu}\sigma_\nu^{(n)} = \sum_j q_{ij}\sigma_j^{(n)}$$

Since Q is substochastic and $\sigma_i^{(1)} < 1$, we can find another similar derivation to above (noting the equivalent nth power definition where the n matrix is on the left):

$$\sigma_i^{(n+1)} = \sum_j \sum_\nu q_{i\nu}^{(n)} q_{\nu j} = \sum_\nu q_{i\nu}^{(n)} \sum_j q_{\nu j} = \sum_\nu q_{i\nu}^{(n)} \sigma_\nu^{(1)} \leq \sigma_i^{(n)}$$

Equality can be seen by expanding out:

$$q_{ij}^{(n)} = \sum_{k_{n-1}} q_{ik_{n-1}}^{(n-1)} q_{k_{n-1}j} = \cdots = \sum_{k_{n-1}} \cdots \sum_{k_1} q_{ik_1} q_{k_1 k_2} \cdots q_{k_{n-2}k_{n-1}} q_{k_{n-1}j}$$

$$= \sum_{k_1} \cdots \sum_{k_{n-1}} q_{ik_1} q_{k_1 k_2} \cdots q_{k_{n-2}k_{n-1}} q_{k_{n-1}j} = \sum_{k_1} q_{ik_1} q_{k_1 j}^{(n-1)}$$

Where we can switch the order of the sums via convergence, and switching the sums signifies we are collecting from the right side first. Therefore, the monotone limits:

$$\sigma_i = \lim_n \sigma_i^{(n)} = \lim_n \sum_j q_{ij}^{(n)}$$

exist (monotonically decreasing, bounded below by zero). We actually note that we can find the limit by the Weierstrass M-test - recall:

$$\sigma_i = \lim_n \sigma_i^{(n)} = \lim_n \sigma_i^{(n+1)} = \lim_n \sum_j q_{ij}\sigma_j^{(n)}$$

For an individual term in the sum, we have:

$$\lim_n q_{ij}\sigma_j^{(n)} = q_{ij}\sigma_j$$

Again, we have noted that all the limits exist, and everything is bounded. And so, by the Weierstrass M-test, we have:

$$\sigma_i = \sum_j q_{ij}\sigma_j$$

Thus, the σ_i are a solution to the system:

$$\begin{cases} x_i = \sum_{j\in U} q_{ij}x_j & i \in U \\ 0 \leq x_i \leq 1 & i \in U \end{cases}$$

Now, we note something about all possible solutions to the system. We have that $x_i = \sum_j q_{ij} x_j \leq \sum_j q_{ij} = \sigma_i^{(1)}$, as the x_j are between zero and 1. If we have that $x_i \leq \sigma_i^{(n)}$ for all i, this implies:

$$x_i \leq \sum_j q_{ij} \sigma_j^{(n)} = \sigma_i^{(n+1)}$$

Given that we have $x_i \leq \sigma_i^{(1)}$, induction implies that $x_i \leq \sigma_i^{(n)}$ for all n. By the monotonicity of limits, we must have that $x_i \leq \sigma_i$. Thus, we have that σ_i is the *maximal* solution to the above system of equations. This gives us the following lemma:

Lemma 1: Substochastic Matrix Row Sum Limits Criteria For a substochastic matrix Q, the limits $\sigma_i = \lim_n \sigma_i^{(n)}$ are the *maximal* solution of:

$$\begin{cases} x_i = \sum_{j \in U} q_{ij} x_j & i \in U \\ 0 \leq x_i \leq 1 & i \in U \end{cases}$$

Now suppose that U is a subset of the state space S. The p_{ij} for i and j in U give a substochastic matrix Q. It is easy enough to visualize how we would pick out the rows and columns of P to form Q. The row sums are:

$$\sigma_i^{(n)} = \sum_{j_1, \cdots, j_n \in U^n} p_{ij_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n}$$

Note: this is just by noting that:

$$q_{ij}^{(n)} = \sum_{j_1 \in U} \sum_{j_2 \in U} \cdots \sum_{j_{n-1} \in U} q_{ij_1} q_{j_1 j_2} \cdots q_{j_{n-1} j}$$

And noting that the row sum would also vary the j across U. Note, by our cylinder sequence property, we can find that this is equal to the probability:

$$\sigma_i^{(n)} = P_i \left[X_t \in U, t \leq n \right]$$

If we let $n \to \infty$, we have:

$$\sigma_i = P_i \left[X_t \in U, t = 1, 2, \cdots \right] \qquad i \in U$$

Note, the RHS comes from continuity from above. And so, σ_i is the probability that the system remains forever in U, given that it starts at i. And so, the following theorem is an immediate consequence of the above lemma, and the discussion above:

Theorem 8.4 Probability to Stay in a Subset Forever For $U \subset S$ the probabilities (indexed by $i \in U$):

$$\sigma_i = P_i\left[X_t \in U, t = 1, 2, \cdots\right] \qquad i \in U$$

are the maximal solution of the system:

$$\begin{cases} x_i = \sum_{j \in U} p_{ij} x_j & i \in U \\ 0 \leq x_i \leq 1 & i \in U \end{cases}$$

Note - $x_i \geq 0$ is slightly redundant, as we have $x_i = 0$ is a solution, and we take a maximal solution.

Example 8.10 If we take the random walk, and let $U = \{0, 1, 2, \cdots\}$, we have the system of equations is:

$$\begin{cases} x_i & = p x_{i+1} + q x_{i-1} \\ x_0 & = p x_1 \end{cases}$$

Note: This is a *difference equation*, which we can find solutions for.

Now, we just consider the same setup, but with $U = S - \{i_0\}$ for an arbitrary state i_0:

$$\begin{cases} x_i = \sum_{j \neq i_0} p_{ij} x_j & \text{for } i \neq i_0 \\ 0 \leq x_i \leq 1 & \text{for } i \neq i_0 \end{cases}$$

We have that this system gives us another definition for transience. Again, note intuitively - the event that we stay in $U = S - \{i_0\}$ forever with nonzero probability must imply that i_0 is transient, as it means that starting from i_0, there is a non zero probability that we never come back to i_0.

Theorem 8.5 Transience is Equivalent to Staying in a Subset Forever An irreducible chain is transient if and only if:

$$\begin{cases} x_i = \sum_{j \neq i_0} p_{ij} x_j & \text{for } i \neq i_0 \\ 0 \leq x_i \leq 1 & \text{for } i \neq i_0 \end{cases}$$

Has a non trivial solution for some arbitrary $i_0 \in S$. Note, non trivial implies that there is a solution outside of $x_i = 0$, which always satisfies the equations.

Proof of Theorem 8.5 The probabilities:

$$1 - f_{ii_0} = P_i \left[\left(\bigcup_{n=1}^{\infty} X_n = i_0 \right)^c \right] = P_i \left[X_n \neq i_0,\ n \geq 1 \right] \text{ for } i \neq i_0$$

Are, by Theorem 8.4, the maximal solution to our equation. Ie, we have that maximal $\sigma_i = 1 - f_{ii_0}$. Therefore, we have a nontrivial solution for an arbitrary i_0 if and only if $f_{ii_0} < 1$ for some $i \neq i_0$. If the chain is persistent - Theorem 8.3 implies that for persistent, irreducible chains, we have that $f_{ii_0} = 1$, and so for persistent chains, it is impossible to have a nontrivial solution.

So, all we need to show is that a transient irreducible chain has a nontrivial solution, and then we get our theorem. As a nontrivial solution implies not persistent, and a transient chain implies nontrivial. Suppose the chain is transient. We have:

$$f_{i_0 i_0} = P_{i_0} \left[X_1 = i_0 \right] + \sum_{n=2}^{\infty} \sum_{i \neq i_0} P_{i_0} \left[X_1 = i, X_2 \neq i_0, \cdots, X_{n-1} \neq i_0, X_n = i_0 \right]$$

Where above, we use the $f_{i_0 i_0}^{(n)}$ definition, and we decompose it based on the first step. By our cylinder sequence property, the above equals:

$$= p_{i_0 i_0} + \sum_{n=2}^{\infty} \sum_{i \neq i_0} p_{i_0 i} f_{ii_0}^{(n-1)} = p_{i_0 i_0} + \sum_{i \neq i_0} p_{i_0 i} \sum_{n=2}^{\infty} f_{ii_0}^{(n-1)} = p_{i_0 i_0} + \sum_{i \neq i_0} p_{i_0 i} f_{ii_0}$$

Where we could switch the sums by convergence (increasing, bounded by 1). Since $f_{i_0 i_0} < 1$, and $p_{i_0 i_0} + \sum_{i \neq i_0} p_{i_0 i} = 1$, it follows that $f_{ii_0} < 1$ for some $i \neq i_0$. Thus, we have a nontrivial solution. Thus, transience implies a nontrivial solution to our system of equations. qed.

Example 8.11 Again, we can make use of difference equation solutions to find the x_k solutions for a queuing model, one in which at evert step, one person leaves the line, and 0, 1, or 2 people join it. There is a non trivial solution if $t_0 < t_2$, ie the probability of zero customers joining the line is less than the probability of 2 customers joining the line. We can see intuitively, $t_0 < t_2$ implies that the line will grow unboundedly, and so every state must be transient.

8.6 Stationary Distributions

Suppose the chain has initial probabilities π_i satisfying:

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j \qquad j \in S$$

By induction, we have that:

$$\sum_{i \in S} \pi_i p_{ij}^{(n)} = \sum_{i \in S} \pi_i \sum_{k \in S} p_{ik}^{(n-1)} p_{kj} = \sum_{k \in S} p_{kj} \sum_{i \in S} \pi_i p_{ik}^{(n-1)} = \sum_{k \in S} \pi_k p_{kj} = \pi_j \qquad j \in S$$

If π_i is the probability that $X_0 = i$, then the left side above is the probability that $X_n = j$. Note, we have that:

$$P[X_n = j] = \sum_{i \in S} P[X_0 = i] P[X_n = j | X_0 = i] = \sum_{i \in S} \pi_i p_{ij}^{(n)}$$

The above is the law of total probability for conditional expectation. Note we just remove the probability 0 starting states i from the summation (as $P[X_n = j]$ is the same on a probability 1 set), and reintroduce them in the sum with $\pi_i = 0$. And so, our starting condition thus implies that the probability of $[X_n = j]$ is the same for all n. A set of probabilities satisfying:

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j \qquad j \in S$$

Is thus called a *stationary distribution.* The existence of such a distribution implies that the chain is very stable. I think, we might later prove that the limit of the chain will go towards this distribution.

To discuss this, we need the notion of periodicity. The state j has *period* t if $p_{jj}^{(n)} > 0$ implies that t divides n, and if t is the largest integer with this property. Ie, if t doesn't divide n, we have that $p_{jj}^{(n)} = 0$. In other words, the period of j is the greatest common divisor of the set of integers:

$$\left[n : \; n \geq 1, \; p_{jj}^{(n)} > 0 \right]$$

If the chain is irreducible, there are r and s for which $p_{ij}^{(r)}$ and $p_{ji}^{(s)}$ are positive, and so:

$$p_{ii}^{(r+s+n)} \geq p_{ij}^{(r)} p_{jj}^{(n)} p_{ji}^{(s)}$$

Via or cylinder sequences property. Let t_i and t_j be the periods of i and j. Taking $n = 0$ above implies that t_i divides $r + s$. By the inequality, it follows that $p_{jj}^{(n)} > 0$ implies that t_i divides $r+s+n$ - and hence, it must divide n as well. Thus, t_i divides each integer in the periodicity set for j, and so $t_i \leq t_j$. We can interchange and find $t_j \leq t_i$. Thus, for an irreducible chain, we have:

$$t_i = t_j$$

For all states i, j, and we can thus talk about the *period of the chain.* An irreducible chain is called *aperiodic* if the period is 1, ie each state has a period of 1.

Lemma 2 In an irreducible, aperiodic chain, for each i and j, $p_{ij}^{n} > 0$ for all n exceeding some $n_0(i, j)$.

Proof As $p_{jj}^{(m+n)} \geq p_{jj}^{(m)} p_{jj}^{(n)}$, if M is the set $\left[n :\ n \geq 1,\ p_{jj}^{(n)} > 0\right]$, then $m, n \in M$ implies $m + n \in M$. It is a fact of number theory that if a set of positive integers is closed under addition, and has a greatest common divisor of 1, then it contains all integers exceeding some n_1. This isn't intuitive - because 1 is not necessarily in the set, so we can't just add 1 continuously.

Given i and j, choose r so that $p_{ij}^{(r)} > 0$. If $n > n_0 = n_1 + r$, then $p_{ij}^{(n)} \geq p_{ij}^{(r)} p_{jj}^{(n-r)} > 0$.

Ie, for i, j, find the n_1 for j that we know exists, and set $n_0 = n_1 + r$. qed.

Bezout's Identity Let a, b be integers with $gcd(a, b) = d$. Then there exist integers x and y such that $ax + by = d$. **Proof:** Let $S = \{ax + by | x, y \in \mathbb{Z} \text{ and } ax + by > 0\}$. Clearly, S is nonempty, and has a minimum element $d = as + bt$. To prove that d is the gcd of a and b, we need d is a divisor of both, and that for any other common divisor, $c \leq d$.

We can write $a = dq + r$ for $0 \leq r < d$. We note that $r = a - qd = a - q(as + bt) = a(1 - qs) - b(qt) \in S$. Thus, we must have $r = 0$. Thus, d is a divisor of a, and similarly is a divisor of b.

Now, let c be any common divisor of a and b. Ie, $a = cu$ and $b = cv$ for integers u and v. One has that:

$$d = as + bt = cus + cvt = c(us + vt)$$

Thus, c is a divisor of d. As $d > 0$ - we cannot have that $c > d$. Thus, $c \leq d$. qed.

A21 Suppose that M is a set of positive integers closed under addition and that M has greatest common divisor 1. Then M contains all integers exceeding some n_0.

Write $1 = m - m'$ for m and m' in M. Note, we do this for the case where 1 is not in M - as otherwise, the proof is clearly true. As for why we can find such an m and m' - first note that we have $x, x' \in M$ where $x \neq x'$, $gcd(x, x') = 1$, and $x, x' > 0$. By Bezout's identity, there exists a and b such that $ax - bx' = 1$. While either $a \leq 0$ or $b \leq 0$, update using $a = a + x'$ and $b = b + x$. Note that:

$$(a + x')x - (b + x)x' = ax + x'x - bx' - xx' = ax - bx' = 1$$

So yes, such an m and m' can be found in M, as we have $ax \in M$ and $bx' \in M$, when a and b are positive, given that M is closed under addition.

Take $n_0 = (m + m')^2$. Given $n > n_0$, write:

$$n = q(m + m') + r \qquad \text{where } 0 \leq r < m + m'$$

From $n > n_0 \geq (r + 1)(m + m')$, it follows that:

$$q = \frac{n - r}{m + m'} > \frac{(r + 1)(m + m') - r}{m + m'} = r + 1 - \frac{r}{m + m'} > r$$

And so now, we will be able to express n as a linear combination of m and m', which is in M, as M is closed under addition. We have:

$$n = q(m + m') + r \cdot 1 = q(m + m') + r(m - m') = (q + r)m + (q - r)m'$$

And since $q + r \geq q - r > 0$, our coefficients are positive, and n lies in M. Thus, for all $n > n_0$, we have that $n \in M$. qed.

Theorem 8.6 Aperiodic Irreducible Chains Approach The Stationary Distribution Suppose for an irreducible, aperiodic chain that there exists a stationary distribution - a solution to:

$$\sum_{i \in S} \pi_i p_{ij} = \pi_j$$

Satisfying $\pi_i \geq 0$ and $\sum_i \pi_i = 1$ (note, we could always rescale the sum if the previous two properties applied). Then, the chain is persistent, and we have:

$$\lim_n p_{ij}^{(n)} = \pi_j$$

For all i and j, with the π_j are all positive, and the stationary distribution is unique.

The main point of the conclusion is that the effect of the initial state wears off. Whatever initial distribution α_i that the chain may have, if $\lim_n p_{ij}^{(n)} = \pi_j$ holds, then it follows by the Weierstrass M test that the probability $\sum_i \alpha_i p_{ij}^{(n)} = P[X_n = j]$ converges to π_j as well.

Proof of Theorem 8.6 This is a complex proof. It involves taking a *coupled chain* or the original chain. Overall, the main point of the proof is to conclude the $\lim_n p_{ij}^{(n)} = \pi_j$ property. We do that by first looking at the coupled chain, and noting that it is irreducible and persistent. Then, we can find no matter what starting position (i, j) we have, the chain will always reach state (i_0, i_0). And after that point - both coordinates of the coupled chain act the same probabilistically. This allows us to conclude that for k large enough, $p_{ik}^{(n)}$ and $p_{jk}^{(n)}$ are the same. With this property, we can conclude our stationary distribution limit.

Definition: Coupled Markov Chain Say we have an initial state space S and stochastic matrix P. Theorem 8.1 says that there does exist a Markov chain for any given initial probabilities. For an existing Markov Chain, we can also just take its state space S and stochastic matrix P. Now, consider a state space $S \times S$ and transition probabilities:

$$p(ij, kl) = p_{ik} p_{jl}$$

Note that as clearly $0 \leq p(ij, kl) \leq 1$ and:

$$\sum_{kl} p(ij, kl) = \sum_{kl} p_{ik} p_{jl} = \sum_{k} p_{ik} \sum_{l} p_{jl} = \sum_{k} p_{ik} = 1$$

$p(ij, kl)$ forms a stochastic matrix P. By Theorem 8.1, as $S \times S$ is still countable, there does exist a Markov chain $(X_n, Y_n), n = 0, 1, \cdots$ having positive initial probabilities and transition probabilities:

$$P\left[(X_{n+1}, Y_{n+1}) = (k, l) | (X_n, Y_n) = (i, j)\right] = p(ij, kl)$$

Note, the (X_n, Y_n) are just notation, Theorem 8.1 above proves for a random variable S_n, and we break S_n down into components. We also note that:

$$p^{(n)}(ij, kl) = \sum_{(t_1, t_1'), \cdots, (t_{n-1}, t_{n-1}') \in (S \times S)^{n-1}} p(ij, t_1 t_1') p(t_1 t_1', t_2 t_2') \cdots p(t_{n-1} t_{n-1}', kl)$$

$$= \sum_{(t_1,t_1'),\cdots,(t_{n-1},t_{n-1}')\in(S\times S)^{n-1}} p_{it_1}p_{jt_1'}\cdot p_{t_1t_2}p_{t_1't_2'}\cdots p_{t_{n-1}k}p_{t_{n-1}'l}$$

$$= \sum_{t_1,\cdots t_{n-1}\in S^{n-1}} \sum_{t_1',\cdots t_{n-1}'\in S^{n-1}} p_{it_1}p_{jt_1'}\cdot p_{t_1t_2}p_{t_1't_2'}\cdots p_{t_{n-1}k}p_{t_{n-1}'l}$$

$$= \sum_{t_1,\cdots t_{n-1}\in S^{n-1}} p_{it_1}p_{t_1t_2}\cdots p_{t_{n-1}k} \sum_{t_1',\cdots t_{n-1}'\in S^{n-1}} p_{jt_1'}p_{t_1't_2'}\cdots p_{t_{n-1}'l}$$

$$= p_{jl}^{(n)} \sum_{t_1,\cdots t_{n-1}\in S^{n-1}} p_{it_1}p_{t_1t_2}\cdots p_{t_{n-1}k}$$

$$= p_{ik}^{(n)}p_{jl}^{(n)}$$

SubLemma 8.6.1: An irreducible aperiodic chain creates an irreducible coupled chain From lemma 2, we recall that in an irreducible aperiodic chain, for each i and j, $p_{ij}^{(n)} > 0$ for all n exceeding some $n_0(i,j)$. Take two arbitrary states ij, kl in our coupled chain. Note for:

$$n_0 = \max(n_0(i,k), n_0(j,l))$$

We have $n \geq n_0$ implies:

$$p^{(n)}(ij,kl) = p_{ik}^{(n)}p_{jl}^{(n)} > 0$$

And thus, the coupled chain is *irreducible*.

SubLemma 8.6.2: Irreducible Chain with Stationary Distribution is Persistence Assume that an irreducible Markov Chain has a stationary distribution, ie for $\pi_i \geq 0$, $\sum_i \pi_i = 1$ we have:

$$\sum_{i\in S} \pi_i p_{ij} = \pi_j \implies \sum_{i\in S} \pi_i p_{ij}^{(n)} = \pi_j$$

We will prove the chain is persistent. Assume otherwise, that it is transient. Theorem 8.3 implies that:

$$\sum_n p_{ij}^{(n)} < \infty \implies \lim_n p_{ij}^{(n)} = 0$$

We note that $\lim_n \pi_i p_{ij}^{(n)} = 0$, $\pi_i p_{ij}^{(n)} < \pi_i$, and $\sum_i \pi_i = 1$. Thus, by the Weierstrass M test, we have for each j:

$$\pi_j = \lim_n \pi_j = \lim_n \sum_{i\in S} \pi_i p_{ij}^{(n)} = \lim_n \sum_{i\in S} 0 = 0$$

This contradicts $\sum_i \pi_i = 1$, and so an Irreducible Chain with Stationary Distribution must be persistent.

SubLemma 8.6.3: Coupled Chain of Stationary Distribution Chain has a Stationary Distribution Note that $\pi(ij) = \pi_i \pi_j$ forms a stationary distribution for the coupled chain. This is because:

$$\sum_{ij \in S \times S} \pi_{ij} p(ij, kl) = \sum_{ij \in S \times S} \pi_i \pi_j p_{ik} p_{jl} = \sum_{i \in S} \pi_i p_{ik} \sum_{j \in S} \pi_j p_{jl} = \pi_l \sum_{i \in S} \pi_i p_{ik} = \pi_l \pi_k = \pi(kl)$$

With the three lemmas above, we note that the coupled chain based off of the original irreducible, aperiodic chain with stationary distribution, is an irreducible chain with a stationary distribution that is also persistent.

SubLemma 8.6.4: Irreducible and Persistent Coupled Chains imply $\lim_n |p_{ik}^{(n)} - p_{jk}^{(n)}| = 0$ We have our coupled Markov Chain (X_n, Y_n) with transition probabilities:

$$P\left[(X_{n+1}, Y_{n+1}) = (k, l) | (X_n, Y_n) = (i, j)\right] = p(ij, kl) = p_{ik} p_{jl}$$

Which we assume is irreducible and persistent. It follows that, for an arbitrary initial state (i, j), for an arbitrary $i_0 \in S$, we have:

$$P_{ij}\left[(X_n, Y_n) = (i_0, i_0) \text{i.o.}\right] = 1$$

This is directly by Theorem 8.3 for an irreducible persistent chain. Let τ be the smallest integer such that $X_\tau = Y_\tau = i_0$. We note that τ is finite with probability 1, as:

$$P_{ij}\left[\omega : \tau < \infty\right] = P_{ij}\left[\bigcup_n X_n = Y_n = i_0\right] \geq P_{ij}\left[(X_n, Y_n) = (i_0, i_0) \text{i.o.}\right] = 1$$

Using the cylinder sequence property on the coupled chain, if $m \leq n$, we have:

$$P_{ij}\left[(X_n, Y_n) = (k, l), \tau = m\right]$$

$$= P_{ij}\left[(X_t, Y_t) \neq (i_0, i_0), t < m, (X_m, Y_m) = (i_0, i_0)\right] \times P_{i_0 i_0}\left[(X_{n-m}, Y_{n-m}) = (k, l)\right]$$

Note, we just replaced $\tau = m$ with its corresponding essentially $f_{ij, i_0 i_0}^m$ term, and split the cylinders. The above equals:

$$= P_{ij}\left[\tau = m\right] p_{i_0 k}^{(n-m)} p_{i_0 l}^{(n-m)}$$

Where the above uses our higher order transition probabilities on the coupled chain we proved above. We can now add out all the states l:

$$P_{ij}\left[X_n = k, \tau = m\right] = \sum_{l \in S} P_{ij}\left[(X_n, Y_n) = (k, l), \tau = m\right] = \sum_{l \in S} P_{ij}\left[\tau = m\right] p_{i_0 k}^{(n-m)} p_{i_0 l}^{(n-m)}$$

$$= P_{ij}\left[\tau = m\right] p_{i_0 k}^{(n-m)}$$

We can similarly find that:

$$P_{ij}\left[Y_n = l, \tau = m\right] = P_{ij}\left[\tau = m\right] p_{i_0 l}^{(n-m)}$$

If we take $k = l$, we find:

$$P_{ij}\left[X_n = k, \tau = m\right] = P_{ij}\left[Y_n = k, \tau = m\right]$$

We can add over $m = 1, \cdots, n$ to find that:

$$P_{ij}\left[X_n = k, \tau \leq n\right] = P_{ij}\left[Y_n = k, \tau \leq n\right]$$

This property makes sense - after τ, ie if we are at time $n \geq \tau$ - a time at which X and Y are equal - X_n and Y_n enter states with the same probability. From this, it follows:

$$P_{ij}\left[X_n = k\right] = P_{ij}\left[X_n = k, \tau \leq n \cup \tau > n\right] \leq P_{ij}\left[X_n = k, \tau \leq n\right] + P_{ij}\left[\tau > n\right]$$

$$= P_{ij}\left[Y_n = k, \tau \leq n\right] + P_{ij}\left[\tau > n\right] \leq P_{ij}\left[Y_n = k\right] + P_{ij}\left[\tau > n\right]$$

Basic probability rules are used above. We can interchange to also find:

$$P_{ij}\left[Y_n = k\right] \leq P_{ij}\left[X_n = k\right] + P_{ij}\left[\tau > n\right]$$

We also note:

$$P_{ij}\left[X_n = k\right] = \sum_l P_{ij}\left[(X_n, Y_n) = (k, l)\right] = \sum_l p^{(n)}(ij, kl) = p_{ik}^{(n)} \sum_l p_{jl}^{(n)} = p_{ik}^{(n)}$$

Putting it all together, we have:

$$\left|p_{ik}^{(n)} - p_{jk}^{(n)}\right| = \left|P_{ij}\left[X_n = k\right] - P_{ij}\left[Y_n = k\right]\right| \leq P_{ij}\left[\tau > n\right]$$

Where in the last step we note that the inequalities we have above proved that the absolute difference between the probabilities was no more than $P_{ij}\left[\tau > n\right]$. As τ is finite with probability 1, we can conclude:

$$\lim_n \left|p_{ik}^{(n)} - p_{jk}^{(n)}\right| \leq \lim_n P_{ij}\left[\tau > n\right] = 0$$

Proof Of Theorem 8.6 We tie all the 4 sublemmas together. As our initial markov chain is irreducible and aperiodic, we note that our coupled markov chain is irreducible. As our initial markov chain has a stationary distribution, our coupled chain has one as well. We note that as our coupled chain is irreducible and has a stationary distribution - it is persistent. Finally, we note that as our coupled chain is persistent and irreducible, we have that for the higher order transition probabilities on our initial chain:

$$\lim_n \left| p_{ik}^{(n)} - p_{jk}^{(n)} \right| = 0$$

We now note:

$$\pi_k - p_{jk}^{(n)} = \sum_{i \in S} \pi_i p_{ik}^{(n)} - p_{jk}^{(n)}$$

By our higher order stationary distribution property. As $\sum_i \pi_i = 1$, the above equals:

$$= \sum_{i \in S} \pi_i p_{ik}^{(n)} - \sum_{i \in S} \pi_i p_{jk}^{(n)} = \sum_{i \in S} \pi_i \left[p_{ik}^{(n)} - p_{jk}^{(n)} \right]$$

And so, we find by the Weierstrass M Test:

$$\lim_n \pi_k - p_{jk}^{(n)} = \lim_n \sum_{i \in S} \pi_i \left[p_{ik}^{(n)} - p_{jk}^{(n)} \right] = \lim_n \sum_{i \in S} 0 = 0$$

And thus, we can conclude that for an irreducible aperiodic markov chain with a stationary distribution, we have:

$$\lim_n p_{ij}^{(n)} = \pi_j$$

Note, that this equality would hold for any stationary distribution - and so the stationary distribution is unique, as we could define it by the above limit. All that remains is to show that the stationary distribution is positive - ie, $\pi_j > 0$. Choose r and s so that $p_{ij}^{(r)}$ and $p_{ji}^{(s)}$ are positive, which exist as our chain is irreducible. Assume that $\pi_j = 0$. Then we note:

$$0 = \pi_j = \lim_n p_{jj}^{(n)} \geq \lim_n p_{ji}^{(s)} p_{ii}^{(n-s-r)} p_{ij}^{(r)} = p_{ji}^{(s)} p_{ij}^{(r)} \lim_n p_{ii}^{(n-s-r)} = p_{ji}^{(s)} p_{ij}^{(r)} \pi_i$$

$$\implies 0 = p_{ji}^{(s)} p_{ij}^{(r)} \pi_i \implies \pi_i = 0$$

And so, if one π_j is zero, we can conclude the same for every state. This contradicts $\sum_i \pi_i = 1$, as a countable sum of zeros is zero. Thus, we must have that for all states i, $\pi_i > 0$. qed.

A14 - The Diagonal Method Suppose that each row of the array:

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Is a bounded sequence of real numbers. Then there exists an increasing sequence $n_1, n_2, \cdots$ of integers such that the limit:

$$\lim_k x_{r,n_k}$$

Exists for each r.

Proof of the Diagonal Method First note - the Heine-Borel theorem implies that every bounded sequence of real numbers has a convergent subsequence. This is just Bolzano Weierstrass - which says something along the lines of every sequence has a monotone subsequence, and boundedness means we can take the monotone convergence theorem. However, I think it can also be proved by Heine-Borel. Assume we have a sequence of real numbers between $[a, b]$. Then, we can take an interval of length $(b-a)/2$ around each point, find a finite cover - and note the sequence is contained within this cover. And decrease the length of the cover, find an infinite amount of points remain, and so on. Anyway - by whatever method you choose, we do have every compact sequence has a convergence subsequence.

From the first row, select a convergence subsequence $x_{1,n_{1,1}}, x_{1,n_{1,2}}, x_{1,n_{1,3}}, \cdots$. Note that $x_{2,n_{1,1}}, x_{2,n_{1,2}}, x_{2,n_{1,3}}, \cdots$ is still a bounded sequence, and so has a convergence subsequence - which we index as:

$$x_{2,n_{2,1}}, x_{2,n_{2,2}}, x_{2,n_{2,3}}, \cdots$$

Note that $\{n_{2,k}\}$ is an increasing sequence of integers, a subsequence of the original $\{n_{1,k}\}$, and so both $\lim_k x_{2,n_{2,k}}$ and $\lim_k x_{1,n_{2,k}}$ exists. We continue inductively, to find an array:

$$\begin{array}{cccc} n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\ n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

This array has three properties:

1. Each row is an increasing sequence of integers

2. The rth row is a subsequence of the $(r-1)$ row
3. For each r, $\lim_k x_{t,n_{r,k}}$ exists for $t \leq r$

Let $n_k = n_{k,k}$. As each row is increasing, and contained in the preceding row - $n_1, n_2, n_3, \cdots$ is an increasing sequence of integers. Furthermore, $n_r, n_{r+1}, n_{r+2}, \cdots$ is a subsequence of the rth row. Thus, $x_{r,n_r}, x_{r,n_{r+1}}, x_{r,n_{r+2}}, \cdots$ is a subsequence of a convergence subsequence, and therefore convergence. Thus, $\lim_k x_{r,n_k}$ exists for each r. qed.

This is actually such a cool theorem. I like it a lot. I think it is because of the $n_k = n_{k,k}$ portion. I was confused of how we would actually take the sequences into infinity, and combine them into one. It seemed to be a finite property to me. But, we took it to infinity by taking a portion of each of the sequences. Not fucking bad!

Theorem 8.7 Aperiodic Irreducible Chains with no Stationary Distribution have a Null Long Term Distribution If an irreducible, aperiodic chain has no stationary distribution, then:

$$\lim_n p_{ij}^{(n)} = 0$$

For all i and j.

Proof of Theorem 8.7 If the chain is transient - note that Theorem 8.3 implies $\sum_n p_{ij}^{(n)} < \infty \implies \lim_n p_{ij}^{(n)} = 0$. So, we consider just the persistent case.

In the persistent case, we consider the coupled chain. Recall that the coupled chain of an aperiodic irreducible chain is irreducible as well. If the coupled chain is transient - well, again, as in Theorem 8.3, we have:

$$\sum_n p^{(n)}(ii,jj) < \infty \implies \sum_n (p_{ij}^{(n)})^2 < \infty \implies \lim_n (p_{ij}^{(n)})^2 = 0 \implies \lim_n p_{ij}^{(n)} = 0$$

So yet again, we examine one more layer down - we have the chain is aperiodic, irreducible, and persistent, and the coupled chain is irreducible and persistent. By SubLemma 8.6.4 - we thus have:

$$\lim_n \left| p_{ik}^{(n)} - p_{jk}^{(n)} \right| = 0$$

So, now we assume that for some state pairs - we do *not* have that $\lim_n p_{ij}^{(n)} = 0$. By the definition of not converging - we must have a sequence of n_u

such that $p_{ij}^{n_u}$ is bounded away from 0. Let M be the set of j such that $\lim_n p_{ij}^{(n)} \neq 0$. Note that we only consider j, as it is independent of i - given that $\lim_n \left|p_{ik}^{(n)} - p_{jk}^{(n)}\right| = 0$, if we have one i such that $\lim_n p_{ij}^{(n)} \neq 0$, then that must be the case for all i.

Anyway. Note that as S is countable, $S \times S$ must be countable as well, and so we can list each sequence $p_{ij}^{(n)}$ into a countable array of rows corresponding to each pair (i, j). If $\lim_n p_{ij}^{(n)} \neq 0$, we replace that row sequence with the infinite sequence of $p_{ij}^{n_u}$ which is bounded away from zero. Now, we have each row in the array is bounded - and so, we can apply the diagonal method to find a sequence of n_u such that each row converges - ie, $\lim_n p_{ij}^{(n_u)}$ exists. Now, we note that $\lim_n p_{ij}^{(n_u)} = \lim_n p_{kj}^{(n_u)}$, by our subtraction property above. And so, for every state pair (i, j), we have a t_j such that:

$$\lim_u p_{ij}^{(n_u)} = t_j$$

By contradiction, our assumption is that at least one t_j does not equal zero. t_j is nonnegative for each j, and positive for some, at least one j.

The next portion of the proof relies on taking arbitrary finite subsets $M \subset S$, and using finiteness to extract out some necessary properties. If M is a finite set of states, then note:

$$\sum_{j \in M} t_j = \sum_{j \in M} \lim_u p_{ij}^{(n_u)} = \lim_u \sum_{j \in M} p_{ij}^{(n_u)} \leq 1$$

As each term in the limit is bounded by 1. Thus, we have:

$$0 < t = \sum_j t_j \leq 1$$

Clearly, we have $0 < t$, as at least one t_j is positive. We also note that it is bounded above by 1, as $\sum_j t_j$ is the limit of finite sub sums, each of which is bounded by 1. Now note:

$$\sum_{k \in M} p_{ik}^{(n_u)} p_{kj} \leq p_{ij}^{(n_u+1)} = \sum_k p_{ik} p_{kj}^{(n_u)}$$

As for why the above is the case - note the inductive definition of $p_{ij}^{(n_u+1)}$, which has two expressions as a sum with the $p^{(n_u)}$ term on the left or the right. The first $\leq$ comes form the fact that M is finite. Note that this implies:

$$\implies \lim_u \sum_{k \in M} p_{ik}^{(n_u)} p_{kj} \leq \lim_u \sum_k p_{ik} p_{kj}^{(n_u)}$$

$$\implies \sum_{k \in M} \lim_u p_{ik}^{(n_u)} p_{kj} \leq \sum_k \lim_u p_{ik} p_{kj}^{(n_u)}$$

We can pass the first limit inside by a finite sum, and the second limit can be passed inside by the Weierstrass M test ($p_{ik} p_{kj}^{(n_u)} \to p_{ik} t_j$, each of the terms $p_{ik} p_{kj}^{(n_u)}$ is bounded by p_{ik}, the sum across k of which is 1). And so, we have:

$$\implies \sum_{k \in M} t_k p_{kj} \leq \sum_k p_{ik} t_j = t_j$$

As this is true for each finite set of states M, we have:

$$\sum_k t_k p_{kj} \leq t_j$$

Assume the inequality is strict for one t_j. Then, we must have:

$$t = \sum_k t_k = \sum_k \sum_j t_k p_{kj} = \sum_j \sum_k t_k p_{kj} < \sum_j t_j = t$$

This is a contradiction. So, we must assume that for all j:

$$\sum_k t_k p_{kj} = t_j$$

Note that if we then set $\pi_j = t_j / t$, we have a stationary distribution - which gives us another contradiction. So, in our final case, where our chain is irreducible, aperiodic, and persistent - and the coupled chain is irreducible and persistent - the assumption that there was some pair (i, j) such that $\lim_n p_{ij}^{(n)} \neq 0$ always leads to a contradiction. And so, we must in fact therefore have that in this final case, for all pairs (i, j), we have:

$$\lim_n p_{ij}^{(n)} = 0$$

And therefore, we can conclude that the theorem holds in all cases. qed.

Mean Return Time Define:

$$\mu_j = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$$

This can be taken as the *mean return time*, ie the average number of steps it takes to come back to state j if you started at state j. If the series diverges, write $\mu_j = \infty$. In the persistent case, this sum is to be thought of as the average number of steps to first return to j, given that $X_0 = j$. Note, this isn't a simple random variable, as there is no upper bound to the number of steps to return to j. However, we won't use expected values here.

Also, note that the value doesn't make sense in the transient case - as there is a chance that we never come back. Ie, the probability of first return is less than 1. And so, μ_j wouldn't be an average return time - as it doesn't take into account the ω where we don't return.

Lemma 3 - Persistent States Have Consistent Mean Return Times And Limiting Distributions Suppose that j is persistent and that $\lim_n p_{jj}^{(n)} = u$. Then $u > 0$ if and only if $\mu_j < \infty$, in which case $u = 1/\mu_j$. Under the convention that $0 = 1/\infty$, the case $u = 0$ and $\mu_j = \infty$ are consistent as well.

Proof of Lemma 3 For $k \geq 0$, let $\rho_k = \sum_{n>k} f_{jj}^{(n)}$. Note, we are only considering a single persistent state j here. Consider the double series:

$$\begin{array}{cccccc} f_{jj}^{(1)} & + f_{jj}^{(2)} & + f_{jj}^{(3)} & + \cdots \\ & f_{jj}^{(2)} & + f_{jj}^{(3)} & + \cdots \\ & & f_{jj}^{(3)} & + \cdots \\ & & & + \cdots \end{array}$$

The sum of the kth row is ρ_k. The nth column sums to $nf_{jj}^{(n)}$. By arbitrary order is the nonnegative series converges (noting that we can take the double sum as equivalent to supremum of finite sums, which doesn't depend on order) - we thus have:

$$\mu_j = \sum_{k=0}^{\infty} \rho_k$$

Since j is persistent, the P_j probability that the system *does not* hit j up to time n is the probability that it hits j after time n (as it will hit j with probability 1), and this is ρ_n. Ie:

$$P_j\left[X_1 \neq j, \cdots, X_n \neq j\right] = P_j\left[X_1 \neq j, \cdots, X_n \neq j \cap \left[\bigcup_k X_k = j\right]\right]$$

$$= P_j\left[X_1 \neq j, \cdots, X_n \neq j \cap \left[\bigcup_{k>n} X_k = j\right]\right]$$

Recall that:

$$\bigcup_{k>n}\left[X_k = j\right] = \bigcup_{k>n}\left[X_{n+1} \neq j, \cdots, X_{k-1} \neq j, X_k = j\right]$$

So, continuing with our equality:

$$= P_j\left[X_1 \neq j, \cdots, X_n \neq j \cap \left[\bigcup_{k>n} X_{n+1} \neq j, \cdots, X_{k-1} \neq j, X_k = j\right]\right]$$

$$= P_j\left[\bigcup_{k>n} X_1 \neq j, \cdots, X_n \neq j, X_{n+1} \neq j, \cdots, X_{k-1} \neq j, X_k = j\right]$$

$$= \sum_{k>n} f_{jj}^{(k)} = \rho_n$$

So, in total, we have that:

$$P_j\left[X_1 \neq j, \cdots, X_n \neq j\right] = \rho_n$$

With this identity, we can conclude:

$$1 - p_{jj}^{(n)} = P_j\left[X_n \neq j\right]$$

$$= P_j\left[X_1 \neq j, \cdots, X_n \neq j\right] + \sum_{k=1}^{n-1} P_j\left[X_k = j, X_{k+1} \neq j, \cdots, X_n \neq j\right]$$

Note, this is via a *last passage* argument, and disjoint unions - ie, split it up by the ω that last hit j at some time between 1 and n. With our identity, and the cylinder sequence property, this equals:

$$= \rho_n + \sum_{k=1}^{n-1} p_{jj}^{(k)} \rho_{n-k}$$

As $\rho_0 = 1$, we can rearrange the equality to give us something nice: for persistent j, we have:

$$1 = \rho_0 p_{jj}^{(n)} + \rho_1 p_{jj}^{(n-1)} + \cdots \rho_{n-1} p_{jj}^{(1)} + \rho_n p_{jj}^{(0)}$$

For the equation on the right - only keep the first k terms. Note that inductively, we have that:

$$1 \geq \rho_0 p_{jj}^{(n)} + \rho_1 p_{jj}^{(n-1)} + \cdots + \rho_k p_{jj}^{(n-k)}$$

This is for *all* n. And so, we can take $n \to \infty$, which gives us:

$$1 \geq (\rho_0 + \cdots + \rho_k) \lim_n p_{jj}^{(n)} = (\rho_0 + \cdots + \rho_k) u$$

Note - again, we have assumed that the limit exists. Therefore, $u > 0$ implies that $\sum_k \rho_k$ converges, and so $u > 0$ implies:

$$\mu_j = \sum_{k=0}^{\infty} \rho_k < \infty$$

So, at the very least, we have that $u > 0$ implies that μ_j converges. Now, we want to know $u = 1/\mu_j$. Write $x_{nk} = \rho_k p_{jj}^{(n-k)}$ for $0 \leq k \leq n$ and $x_{nk} = 0$ for $n < k$ - note these are our sums above, ie:

$$\sum_{k=0}^{\infty} x_{nk} = \sum_{k=0}^{n} \rho_k p_{jj}^{(n-k)} = 1$$

Then, $0 \leq x_{nk} \leq \rho_k$, and $\lim_n x_{nk} = \rho_k u$. If $\mu_j < \infty$, then $\sum \rho_k$ converges, and it follows by the Weierstrass M test that:

$$1 = \sum_{k=0}^{\infty} x_{nk} \to \sum_{k=0}^{\infty} \rho_k u \implies 1 = \mu_j u \implies u = 1/\mu_j$$

Before we conclude, I want to clear up the if and only if. Assume $u > 0$ - then, our first equality derivation implies $\sum_k \rho_k$ converges, in which case the sum equals μ_j which also converges. So that gives us:

$$u > 0 \implies \mu_j < \infty$$

Now, what about the other direction. Assume that $\mu_j < \infty$. Then, we know that our sum $\sum_k \rho_k$ converges. Ok - then, we can apply our m test to find that:

$$1 = \mu_j u \implies u > 0$$

And in the case that $u > 0$ - then we have $\mu_j < \infty$, $\sum_k \rho_k$ converges, and the above equality can still be derived. qed.

We can now note the law of large numbers bears on the relation $u = 1/\mu_j$ in the persistent case. Let V_n be the number of visits to state j up to time n. If the time from one visit to the next is about μ_j, then theoretically, V_n should be about $n/\mu_j \implies V_n/n \approx 1/\mu_j$. But (if $X_0 = j$) V_n/n has expected value $n^{-1}\sum_{k=1}^n p_{jj}^{(k)}$, which goes to u under the hypothesis of Lemma 3. Just something interesting to note - I'm not really gonna try to remember this, don't think it is too important.

Consider an irreducible, aperiodic, persistent chain. There are two possibilities. If there is a stationary distribution, the the limits $\lim_n p_{ij}^{(n)} = \pi_j$ are positive, and the chain is called *positive persistent*. By Lemma 3, it then follows that $\pi_j = u$, which implies that $\mu_j < \infty$ (the mean return time) and $\pi_j = 1/\mu_j$ for all j. In this case, it is not actually necessary to assume persistence, as the existence of a stationary distribution implies persistence.

The second case is if we have no stationary distribution, but we are still persistent. Then the limits $\lim_n p_{ij}^{(n)} = 0$, and the mean return times $\mu_j = \infty$ again by lemma 3 for all j. This, taken with Theorem 8.3 (irreducible chains are either transient or persistent) give us a complete classification:

Theorem 8.8: Classification of Irreducible Aperiodic Markov Chains
For an irreducible, aperiodic chain there are three possibilities:

1. The chain is transient; then for all i and j, $\lim_n p_{ij}^{(n)} = 0$ (by Theorem 8.3) and in fact $\sum_n p_{ij}^{(n)} < \infty$ (again by Theorem 8.3)

2. The chain is persistent but there exists no stationary distribution (the null persistent case). Then for all i and j, $p_{ij}^{(n)}$ goes to 0 (Theorem 8.7) but so slowly that $\sum_n p_{ij}^{(n)} = \infty$ and thus $\mu_j = \infty$ for all j (Lemma 3).

3. There exists a stationary distribution π_j and (hence) the chain is persistent (the positive persistent case). Then for all i and j, $\lim_n p_{ij}^{(n)} = \pi_j > 0$ and $\mu_j = 1/\pi_j < \infty$.

Note the asymptotic properties of the $p_{ij}^{(n)}$ are distinct in all three cases, and so they characterize the three cases. qed.

8.7 Exponential Convergence

This section - I will only take notes, and record theorem statements. I will not have proof flashcards.

Theorem 8.9 Exponential Convergence to Stationary Distribution in Finite Case If the state space is *finite* and the chain is irreducible and aperiodic, then there is a stationary distribution $\{\pi_i\}$ and:

$$|p_{ij}^{(n)} - \pi_j| < A\rho^n$$

Where $A \geq 0$ and $0 \leq \rho < 1$.

Proof: Let $m_j^{(n)} = \min_i p_{ij}^{(n)}$ and $M_j^{(n)} = \max_i p_{ij}^{(n)}$. Note that:

$$m_j^{(n+1)} = \min_i \sum_\nu p_{i\nu} p_{\nu j}^{(n)} \geq \min_i \sum_\nu p_{i\nu} m_j^{(n)} = m_j^{(n)}$$

Similarly, we can find:

$$M_j^{(n+1)} \leq M_j^{(n)}$$

So, we have:

$$0 \leq m_j^{(1)} \leq m_j^{(2)} \leq \cdots \leq M_j^{(2)} \leq M_j^{(1)} \leq 1$$

Let s be the number of states. Let $\delta = \min_{ij} p_{ij}$. It is clear that $\sum_j p_{ij} \geq s\delta$, and so $0 \leq \delta \leq s^{-1}$. Fix states u and v for now. Let $\sum'$ denote the summation over j in S satisfying $p_{uj} \geq p_{vj}$. Similarly, let $\sum''$ denote the summation over j in S satisfying $p_{uj} < p_{vj}$. Note, both sums taken together go over all j in S. Then:

$$\sum{}'(p_{uj} - p_{vj}) + \sum{}''(p_{uj} - p_{vj}) = 1 - 1 = 0$$

As $\sum' p_{vj} + \sum'' p_{uj} \geq \sum_j \delta = s\delta$, we have:

$$\sum{}'(p_{uj} - p_{vj}) = \sum{}' p_{uj} - \sum{}' p_{vj} = 1 - \sum{}'' p_{uj} - \sum{}' p_{vj} = 1 - \left[\sum{}' p_{vj} + \sum{}'' p_{uj}\right] \leq 1 - s\delta$$

And so, with all of our equations above, we find:

$$p_{uk}^{n+1} - p_{vk}^{n+1} = \sum_j (p_{uj} - p_{vj}) p_{jk}^{(n)} = \sum{}'(p_{uj} - p_{vj}) p_{jk}^{(n)} + \sum{}''(p_{uj} - p_{vj}) p_{jk}^{(n)}$$

Note, the LHS sum is positive, and the RHS sum is negative, and so it is bounded by:

$$\leq \sum{}'(p_{uj} - p_{vj})M_k^{(n)} + \sum{}''(p_{uj} - p_{vj})m_k^{(n)}$$

Now, as $\sum'(p_{uj}-p_{vj})+\sum''(p_{uj}-p_{vj}) = 0$, we have $\sum''(p_{uj}-p_{vj}) = -\sum'(p_{uj}-p_{vj})$, and so the above equals:

$$= \sum{}'(p_{uj} - p_{vj})(M_k^{(n)} - m_k^{(n)}) \leq (1 - s\delta)(M_k^{(n)} - m_k^{(n)})$$

Since the chosen u and v were arbitrary, we have (ie, applied to all the choices of u and v):

$$(M_k^{(n+1)} - m_k^{(n+1)}) \leq (1 - s\delta)(M_k^{(n)} - m_k^{(n)})$$

Therefore, we have $M_k^{(n)} - m_k^{(n)} \leq (1-s\delta)^n$. It follows (by the first inequality we had) that $m_j^{(n)}$ and $M_j^{(n)}$ have a common limit π_j, and that:

$$\left|p_{ij}^{(n)} - \pi_j\right| \leq (1 - s\delta)^n$$

For all $i \in S$. Take $A = 1$ and $\rho = 1 - s\delta$. Note that π_i are stationary probabilities, and we get our theorem. Note - this proof doesn't rely on any of the preceding theory. qed.

8.8 Optimal Stopping

S is finite for the remainder of the Markov chain section.

Stopping Time/Markov Time A function τ on Ω for which $\tau(\omega)$ is a nonnegative integer for each ω. This function becomes a stopping time/markov time if:

$$[\omega : \tau(\omega) = n] \in \mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$$

As S is finite, the X_n are simple random variables, and the generated sigma algebra is well defined for us. Note, it will be necessary to allow $\tau(\omega)$ to take a special value of ∞ on a set of probability 0.

Now, we note that if f is a function on S, $f(X_0), f(X_1), \cdots$ are SRV. For one - I don't know why we have to add a function, why not just encode it as the state? But, I don't think that matters. We make a game, where an observer will follow the states of the system, and receive a payout of $f(X_\tau)$. τ encodes the strategy - and so, the problem for the observer is to find a τ

such that he maximizes $E[f(X_\tau)]$.

Some things to note. I think - if S is finite, then couldn't we just always wait an infinite amount of time until we see the $s \in S$ that maximizes $f(s)$? I'm not sure what goes into the strategy. I think the setup needs to be more clear. I think - perhaps we don't know the complete output of f, or all of the states S.

Conditional Expected Value If $P(A) > 0$ and $Y = \sum_j y_j I_{B_j}$ is a simple random variable, the B_j forming a finite decomposition of Ω into $\mathcal{F}$ sets, the conditional expected value of Y given A is defined by:

$$E[Y|A] = \sum_j y_j P(B_j|A)$$

Compare this with the normal equation for simple random variables:

$$E[Y] = \sum_j y_j P(B_j)$$

Denote by E_i conditional expected values for the case $A = [X_0 = i]$:

$$E_i[Y] = E[Y|X_0 = i] = \sum_j y_j P_i(B_j)$$

The stopping-time problem becomes finding a τ so as to maximize simultaneously $E_i[f(X_\tau)]$ for all initial states i. We do want to find that $f(X_\tau)$ is a *simple* random variable. If x lies in the range of f, which is finite (on S), and if τ is everywhere finite, then:

$$\left[\omega : f(X_{\tau(\omega)}(\omega)) = x\right] = \bigcup_{n=0}^{\infty} [\omega : \tau(\omega) = n, f(X_n(\omega)) = x]$$

This lies in $\mathcal{F}$, as each $\sigma(X_0, X_1, \cdots, X_n) \subseteq \mathcal{F}$, and $\sigma(X_i) \subseteq \mathcal{F}$, and $\mathcal{F}$ is closed under intersections and countable unions. So, $f(X_\tau)$ has a finite range, and is measurable $\mathcal{F}$. Note, some edge cases - we want to allow stopping times where $\tau(\omega) = \infty$, with probability zero. We can just set $f(X_{\tau(\omega)}(\omega)) = 0$ when $\tau(\omega) = \infty$, and note that this case is still measurable $\mathcal{F}$, as $P[\omega : \tau(\omega) = \infty] = 0$ (implicitly assuming it is measurable as it has probability zero).

Value at i The game with payoff function f has at i the value:

$$v(i) = \sup E_i \left[f(X_\tau) \right]$$

The supremum extending over all Markov times τ. It will turn out that the supremum is actually *achieved* - there always exists an optimal stopping time. It will also turn out that the achieved τ is a supremum for all i. The problem is to calculate $v(i)$, and find the best τ. If the chain is irreducible, the system must pass through every state, and the best strategy is obviously to wait until the system enters a state for which f is maximal. Note - this is what I was saying above. This describes an optimal τ, and $v_i = \max f$ for all i. For this reason - the interesting cases are those in which some states are transient, and others are absorbing ($p_{ii} = 1$).

Finally, it is starting to become clear. This is where it becomes tricky - there will be some transient states that give you a pretty high score. The question is - do you take that score, or do you continue on the chain, risking entering the absorbing state with a probably suboptimal score.

Excessive/Superharmonic Function A function φ on S is *excessive* or *superharmonic* if:

$$\sum_j p_{ij} \varphi(j) \leq \varphi(i) \qquad \forall i \in S$$

Note, this is similar to our another criterion for persistence equation, but we have $\leq$ instead of $=$. In terms of conditional expectation, the requirement is:

$$E_i \left[\varphi(X_1) \right] \leq \varphi(i)$$

Lemma 8.4: The value function v is excessive Note, to show that the function $v(i)$ is excessive, we need to show that the supremum across τ for a single i $\sup E_i \left[f(X_\tau) \right]$ is greater than or equal to the other supremums, multiplied by the p_{ij} probabilities. We do this by examining specific τ, which we will describe below. First, given $\epsilon > 0$, choose for each j in S a "good" stopping time τ_j, satisfying:

$$E_j \left[f(X_{\tau_j}) \right] > v(j) - \epsilon$$

Note, by the definition of supremum, such a τ_j exists for each $v(j)$. Recall that by definition, as $\tau_j = n \in \mathcal{F}_n$:

$$[\tau_j = n] = [(X_0, \cdots, X_n) \in I_{jn}]$$

For some set I_{jn} of $(n+1)$ long sequences of states (this is by the set being in $\mathcal{F}_n$, which is equal to all such sets on the RHS. We choose the specific I_{jn} corresponding to our event). Set $\tau = n+1$ (for $n \geq 0$) on the set:

$$[X_1 = j] \cap [(X_1, \cdots, X_{n+1}) \in I_{jn}]$$

So, note that we should have $\bigcup_{n=0}^{\infty} [\tau_j = n]$ is probability 1 - and so they are a disjoint union that forms a probability 1 set. I just want to find that τ is finite with probability 1 as well - which we can see is the case, because:

$$P[\tau < \infty] = P\left[\bigcup_{n=0}^{\infty} \tau = n\right] \geq P\left[\bigcup_{n=0}^{\infty} \tau = n+1\right]$$

$$= P\left[\bigcup_{n=0}^{\infty} \bigcup_{j \in S} X_1 = j \cap (X_1, \cdots, X_{n+1}) \in I_{jn}\right]$$

Now, we note that each of these sets is disjoint across the first step j, so the above equals:

$$= \sum_{j \in S} P\left[X_1 = j \cap \bigcup_{n=0}^{\infty} (X_1, \cdots, X_{n+1}) \in I_{jn}\right]$$

If we find $P\left[\bigcup_{n=0}^{\infty} (X_1, \cdots, X_{n+1}) \in I_{jn}\right] = 1$, then we note that $P[X_1 = j]$ equals the intersected probability (as probability 1 does not minimize the value). I believe, we should have:

$$P[(X_1, \cdots, X_{n+1}) \in I_{jn}] = P[(X_0, \cdots, X_n) \in I_{jn}]$$

As, we can find:

$$P[(X_1, \cdots, X_{n+1}) \in I_{jn}] = \sum_{i \in S} P[X_0 = i, (X_1, \cdots, X_{n+1}) \in I_{jn}]$$

We can break this up by specific sequence, use the conditional chain rule to break up into $\alpha_i p_{ij_1} p_{j_2 j_3} \cdots p_{j_{n-1} j_n}$ probability sums, bring out the α_i with a common factor, sum the α_i to 1, and get the $P[(X_0, \cdots, X_n) \in I_{jn}]$ value. So we have equality. Now, we can thus conclude:

$$P\left[\bigcup_{n=0}^{\infty} (X_1, \cdots, X_{n+1}) \in I_{jn}\right] = \sum_{n=0}^{\infty} P[(X_1, \cdots, X_{n+1}) \in I_{jn}] = \sum_{n=0}^{\infty} P[(X_0, \cdots, X_n) \in I_{jn}]$$

$$= P\left[\bigcup_{n=0}^{\infty} (X_0, \cdots, X_n) \in I_{jn}\right] = P\left[\bigcup_{n=0}^{\infty} \tau_j = n\right] = P[\tau < \infty] = 1$$

Where we note that each I_{jn} corresponds to disjoint coordinate value requirements. And so, we can find:

$$P[\tau < \infty] \geq \sum_{j \in S} P\left[X_1 = j \cap \bigcup_{n=0}^{\infty} (X_1, \cdots, X_{n+1}) \in I_{jn}\right] = \sum_{j \in S} P[X_1 = j] = 1$$

And so we do have that τ is a stopping time. It essentially takes one step (always, it doesn't stop at zero or one), and then from the new state X_1, it adds the "good" stopping time conditions we found for τ_j. Anyway. We have everything defined. And so, we find:

$$E_i[f(X_\tau)] = \sum_k \sum_{n=0}^{\infty} P_i[\tau = n+1, X_{n+1} = k] f(k)$$

$$E_i[f(X_\tau)] = \sum_{n=0}^{\infty} \sum_j \sum_k P_i[X_1 = j, (X_1, \cdots, X_{n+1}) \in I_{jn}, X_{n+1} = k] f(k)$$

Note: the first line is just the definition of the expectation (noting that $\tau = 0, \tau = 1$ are probability zero sets), the second line breaks down $\tau = n+1$, and we break across the disjoint $X_1 = j$ cases. Using the cylinder sequence property, the above is:

$$= \sum_{n=0}^{\infty} \sum_j \sum_k p_{ij} P_j[(X_0, \cdots, X_n) \in I_{jn}, X_n = k] f(k)$$

$$= \sum_j p_{ij} \sum_{n=0}^{\infty} \sum_k P_j[\tau_j = n, X_n = k] f(k)$$

$$= \sum_j p_{ij} E_j\left[f(X_{\tau_j})\right]$$

By the definition of τ_j, and noting that $v(i)$ is the supremum across τ, we have:

$$v(i) \geq E_i[f(X_\tau)] = \sum_j p_{ij} E_j\left[f(X_{\tau_j})\right] \geq \sum_j p_{ij}(v(j) - \epsilon) = \sum_j p_{ij} v(j) - \epsilon$$

Note, the ϵ was arbitrary, and so we can indeed conclude:

$$v(i) \geq \sum_j p_{ij} v(j)$$

And by definition, v is thus *excessive*. qed. At a high level, we created a stopping time τ as a conglomeration of stopping times τ_j that came from v_i - and showed that this τ satisfied the superharmonic property, which therefore implied v is excessive/superharmonic.

Lemma 8.5 Excessive Functions Beat Stopping Times Suppose that φ is excessive. Then:

1. For all stopping times τ, $\varphi(i) \geq E_i[\varphi(X_\tau)]$
2. For all pairs of stopping times satisfying $\sigma \leq \tau$, $E_i[\varphi(X_\sigma)] \geq E_i[\varphi(X_\tau)]$

The first I am not sure is intuitive, but the second seems like it might be. The excessive property seems to directly correlate with the idea that the expected value of the following step is less than the value of the current one. And note what the first part is saying. For an excessive payoff function, we have that $\tau = 0$ is an optimal strategy. This is because, E_i essentially assumes that $[X_0 = i]$, in which case, $\tau = 0$ gives us a payoff of $\varphi(i)$, which is greater than the expected value of any other payoff function. Ie, for $\tau = 0$, $E_i[\varphi(X_\tau)] = \varphi(i) \geq v_\varphi(i) = \sup_\tau E_i[\varphi(X_\tau)]$. In the case where the payoff function is excessive - the value is actually achievable. And, the optimal τ works for all initial states i.

Proof We first try and prove 1. Define $\tau_n = \min\{\tau, N\}$. Note that τ_N is a stopping time as well (we could do it rigorously, but it seems intuitive that $P[\tau_n < \infty] = 1$, as it won't be greater than N, and $\tau_N = k \in \mathcal{F}_k$, which we can take from τ). Note that:

$$E_i[\varphi(X_{\tau_N})] = \sum_{n=0}^{N-1} \sum_k P_i[\tau_N = n, X_n = k]\varphi(k) + \sum_{n=N}^{\infty} \sum_k P_i[\tau_N = n, X_n = k]\varphi(k)$$

We can rewrite the above as:

$$= \sum_{n=0}^{N-1} \sum_k P_i[\tau = n, X_n = k]\varphi(k) + \sum_k P_i[\tau \geq N, X_N = k]\varphi(k)$$

The first sum, we just replaced τ_N with τ, as they are equal for $n < N$. For the second sum, we note that if τ_N can only equal N, which happens in the case if $\tau \geq N$. Essentially, we removed probability zero terms, and redefined the $\tau_N = N$ term to an equivalent event. As $\tau \geq N = [\tau < N]^c \in \mathcal{F}_{N-1}$, we can rewrite the second sum as (using a final passage argument + cylinder sequences, we can make use of cylinder sequences as $\tau \geq N$ just imposes conditions on the first $N-1$ Markov random variables):

$$\sum_k P_i[\tau \geq N, X_N = k]\varphi(k) = \sum_k \sum_j P_i[\tau \geq N, X_{N-1} = j, X_N = k]\varphi(k)$$

$$= \sum_k \sum_j P_i[\tau \geq N, X_{N-1} = j]\, p_{jk}\varphi(k)$$

Now, using the fact that φ is excessive, we have:

$$\leq \sum_j P_i\left[\tau \geq N, X_{N-1} = j\right] \varphi(j)$$

We substitute this back into our above expression:

$$E_i\left[\varphi(X_{\tau_N})\right] \leq \sum_{n=0}^{N-1} \sum_k P_i\left[\tau = n, X_n = k\right] \varphi(k) + \sum_k P_i\left[\tau \geq N, X_{N-1} = k\right] \varphi(k)$$

$$= \sum_{n=0}^{N-2} \sum_k P_i\left[\tau = n, X_n = k\right] \varphi(k) + \sum_k P_i\left[\tau = N-1, X_{N-1} = k\right] \varphi(k) +$$

$$\sum_k P_i\left[\tau \geq N, X_{N-1} = k\right] \varphi(k)$$

$$= \sum_{n=0}^{N-2} \sum_k P_i\left[\tau = n, X_n = k\right] \varphi(k) + \sum_k P_i\left[\tau \geq N-1, X_{N-1} = k\right] \varphi(k)$$

Where we are adding together disjoint unions in the final step. We note that:

$$E_i\left[\varphi(X_{\tau_{N-1}})\right] = \sum_{n=0}^{N-2} \sum_k P_i\left[\tau = n, X_n = k\right] \varphi(k) + \sum_k P_i\left[\tau \geq N-1, X_{N-1} = k\right] \varphi(k)$$

And so, we can conclude, for excessive φ:

$$E_i\left[\varphi(X_{\tau_N})\right] \leq E_i\left[\varphi(X_{\tau_{N-1}})\right]$$

As $\tau_0 = 0$, and so $E_i\left[\varphi(X_0)\right] = \varphi(i)$, we have for all N:

$$E_i\left[\varphi(X_{\tau_N})\right] \leq \varphi(i)$$

On the probability 1 set of ω where $\tau(\omega)$ is finite, we have:

$$\varphi(X_{\tau_N(\omega)}(\omega)) \to \varphi(X_{\tau(\omega)}(\omega))$$

Recall Theorem 5.4: If the $\varphi(X_{\tau_N})$ are uniformly bounded (and they are, as they each have the same range that is finite), and if $\varphi(X_\tau) = \lim_N \varphi(X_{\tau_N})$ with probability 1 (which is the case, as noted above), then we have:

$$E_i\left[\varphi(X_\tau)\right] = \lim_N E_i\left[\varphi(X_{\tau_N})\right]$$

Note, as discussed before, P_i satisfies all of the properties of a probability measure. Thus, we can conclude that:

$$E_i\left[\varphi(X_\tau)\right] = \lim_N E_i\left[\varphi(X_{\tau_N})\right] \leq \lim_N \varphi(i) = \varphi(i)$$

2. Now, we want to prove the second statement, for all pairs of stopping times satisfying $\sigma \leq \tau$, $E_i[\varphi(X_\sigma)] \geq E_i[\varphi(X_\tau)]$. It is very similar to the first proof. Define $\tau_N = \min\{\tau, \sigma + N\}$. Note that τ_N is similarly a stopping time. Further:

$$E_i[\varphi(X_{\tau_N})] = \sum_{n=0}^{\infty} \sum_k P_i[\tau_N = n, X_{\tau_n} = k]\,\varphi(k)$$

We break this up into cases: $\tau < \sigma + N$, and $\tau \geq \sigma + N$. If $m + n \geq m + N$, we have that $\tau_n = m + N$. Otherwise, $\tau_n = m + n$. So, we can break the expectation up as (noting that $\sigma \leq \tau$ allows us to express τ as an addition on top of the value of σ):

$$E_i[\varphi(X_{\tau_N})] = \sum_{m=0}^{\infty} \sum_{n=0}^{N-1} \sum_k P_i[\sigma = m, \tau = m + n, X_{m+n} = k]\,\varphi(k)$$

$$+ \sum_{m=0}^{\infty} \sum_k P_i[\sigma = m, \tau \geq m + N, X_{m+N} = k]\,\varphi(k)$$

We can make the same cylinder sequence argument as before on the second sum, and find that:

$$E_i[\varphi(X_{\tau_N})] \leq E_i\left[\varphi(X_{\tau_{N-1}})\right] \leq E_i[\varphi(X_{\tau_0})] = E_i[\varphi(X_\sigma)]$$

Make use of the same limit argument as in part 1, and we can indeed conclude that:

$$E_i[\varphi(X_\tau)] \leq E_i[\varphi(X_\sigma)]$$

And thus, we have proved both statements. qed.

Lemma 8.6 If an excessive function φ dominates the payoff function f, then it dominates the value function v as well Note, g dominates h means for all i, $g(i) \geq h(i)$.

Proof: Note that $\varphi(i) \geq E_i[\varphi(X_\tau)]$ for all stopping times τ, by the previous lemma. By the well orderedness of expectation, this implies that $E_i[\varphi(X_\tau)] \geq E_i[f(X_\tau)]$ for all τ. As this is true for all τ, we must have:

$$\varphi(i) \geq \sup_\tau E_i[f(X_\tau)] = v(i)$$

qed.

Theorem 8.10 The value function v is the minimal excessive function dominating f. **Proof:** Examine φ, the minimal excessive function dominating f. Note, by lemma 8.6, this implies:

$$\varphi(i) \geq v(i)$$

Now note, for all other excessive functions φ' dominating f, we have:

$$\varphi'(i) \geq \varphi(i)$$

By the definition of minimal. Examine the stopping time $\tau = 0$. We have that:

$$v(i) = \sup_{\tau} E_i\left[f(X_\tau)\right] \geq E_i\left[f(X_0)\right] = f(i)$$

So the value function v dominates f. We now recall, lemma 4 tells us that v is excessive. So, v is an excessive function φ' that dominates f, and so we have:

$$v(i) \geq \varphi(i) \qquad \varphi(i) \geq v(i) \quad \Longrightarrow \quad v(i) = \varphi(i)$$

And thus, we can conclude that the value function v is the minimal excessive function dominating f.

So, now we know what the value of v is. Can we find a stopping time τ that achieves it, and it achieves $v(i)$ for all i? Consider the set of states M for which $v(i) = f(i)$. We first note that M is nonempty. At the very least, it contains the $i \in M$ on which $f(i)$ is maximized. Recall, we assume S is finite, so this is a nonempty set of states. Why does $f(i)$ being maximized imply $v(i) = f(i)$? Well, we have that $\tau = 0$ is a stopping time, and for those i, we have:

$$v(i) = E_i\left[f(X_\tau)\right]$$

As considering going to any other state would lower the expected value. So yes, M is nonempty, and we call it the *support set*.

Define:

$$A = \bigcap_{n=0}^{\infty} \left[X_n \notin M\right]$$

This is the event that the chain never enters M. We want to show that $P_i(A) = 0$ for all i. This is trivial if $M = S$. So, assume $M \subset S$. Choose $\delta > 0$ such that $f(i) \leq v(i) - \delta$ for all $i \in S - M$. Again, such a δ is possible, given that the set is finite. Note that:

$$E_i\left[f(X_\tau)\right] = \sum_{n=0}^{\infty} \sum_{k} P_i\left[\tau = n, X_n = k\right] f(k)$$

$$\leq \sum_{n=0}^{\infty} \sum_{k \in M} P_i [\tau = n, X_n = k] \, v(k) + \sum_{n=0}^{\infty} \sum_{k \notin M} P_i [\tau = n, X_n = k] \, (v(k) - \delta)$$

Distribute the $v(k)$, we find:

$$= E_i [v(X_\tau)] - \delta P_i [X_\tau \in S - M]$$

Now, we note that $P_i(A) \leq P_i [X_\tau \in S - M]$, as ω in the set on the left are contained in the ω that make the set on the right. And so, we have:

$$E_i [f(X_\tau)] \leq E_i [v(X_\tau)] - \delta P_i(A)$$

Take the supremum over τ. This implies:

$$v(i) \leq \sup_\tau E_i [v(X_\tau)] - \delta P_i(A)$$

As v is excessive, lemma 5 tells us that $\tau = 0$ is an optimal strategy, ie maximizes the supremum, and so, we have:

$$v(i) \leq v(i) - \delta P_i(A)$$

Thus, we must have $P_i(A) = 0$.

And so, we can conclude, whatever the initial state, the system is certain to enter the support set M. Define:

$$\tau_0(\omega) = \min [n : X_n(\omega) \in M]$$

This is the hitting time for M. τ_0 is a Markov time - τ_0 is finite with probability 1, and $\tau_0(\omega) = n$ is in $\mathcal{F}_n$. This is because:

$$[\tau_0(\omega) = n] = \bigcup_{k=1}^{n-1} X_k(\omega) \notin M \cap X_n(\omega) \in M$$

This is clearly just a restriction on the sequences of the first n Markov states. It may be the case that $X_n(\omega) \notin M$ for all n, in which case $\tau_0(\omega) = \infty$ - but this is probability zero.

Theorem 8.11 - The Hitting Time τ_0 is Optimal We will prove that $E_i [f(X_{\tau_0})] = v(i)$ for all i.

Proof: The goal of the proof is to show that $\varphi(i) = E_i [f(X_{\tau_0})]$ is both greater than or equal to and less than or equal to $v(i)$. This will be done making use of the fact that $v(i)$ is the minimal excessive function dominating f. However, note that $v(i) = \sup_\tau E_i\left[f(X_\tau)\right] \geq \varphi(i)$, so all we need to prove is the other direction, that $\varphi(i) \geq v(i)$. This can be done by showing $\varphi(i)$ is an excessive function, and it dominates f.

By the definition of the hitting time, we have that $f(X_{\tau_0}) = v(X_{\tau_0})$. Note that we must then have:

$$\varphi(i) = E_i\left[f(X_{\tau_0})\right] = E_i\left[v(X_{\tau_0})\right]$$

First, we want to show φ is excessive, as this will allow us to then show it dominates f, and thus greater than v. Define:

$$\tau_1(\omega) = \min\left[n \geq 1 : X_n(\omega) \in M\right]$$

τ_1 is similarly a Markov time. We have:

$$E_i\left[v(X_{\tau_1})\right] = \sum_{n=1}^{\infty}\sum_{k\in M} P_i\left[X_1 \notin M, \cdots, X_{n-1} \notin M, X_n = k\right] v(k)$$

The above is equivalent to $\tau_1 = n, X_n = k$. Note, we can sum over the possible j values for X_0, to get:

$$= \sum_{n=1}^{\infty}\sum_{k\in M}\sum_{j\in S} P_i\left[X_0 = j, X_1 \notin M, \cdots, X_{n-1} \notin M, X_n = k\right] v(k)$$

By cylinder sequences, this gives us:

$$= \sum_{n=1}^{\infty}\sum_{k\in M}\sum_{j\in S} p_{ij} P_j\left[X_0 \notin M, \cdots, X_{n-2} \notin M, X_{n-1} = k\right] v(k)$$

Switching the order of the sums, we get:

$$= \sum_{j\in S} p_{ij} \sum_{n=1}^{\infty}\sum_{k\in M} P_j\left[X_0 \notin M, \cdots, X_{n-2} \notin M, X_{n-1} = k\right] v(k)$$

$$= \sum_{j\in S} p_{ij} E_j\left[v(X_{\tau_0})\right]$$

As $\tau_0 \leq \tau_1$, and v is excessive by lemma 4, we can conclude by lemma 5:

$$\varphi(i) = E_i\left[v(X_{\tau_0})\right] \geq E_i\left[v(X_{\tau_1})\right] = \sum_{j\in S} p_{ij} E_j\left[v(X_{\tau_0})\right] = \sum_{j\in S} p_{ij}\varphi(i)$$

Thus, we have that by definition, φ is excessive. Now, we just need to show that it dominates f, ie $\varphi(i) \geq f(i)$. As $\tau_0 = 0$ for $X_0 \in M$, if $i \in M$, then:

$$\varphi(i) = E_i [f(X_0)] = f(i)$$

So, now we use proof by contradiction. Assume that $\varphi(i) < f(i)$ for some values of i in $S - M$, and choose i_0 to maximize $f(i) - \varphi(i)$. Then:

$$\psi(i) = \varphi(i) + f(i_0) - \varphi(i_0)$$

Dominates f, and is excessive, being the sum of a constant $f(i_0) - \varphi(i_0)$ and excessive φ. By Theorem 8.10, ψ must dominate v, so that:

$$\psi(i_0) \geq v(i_0) \implies \varphi(i_0)+f(i_0)-\varphi(i_0) \geq v(i_0) \implies f(i_0) \geq v(i_0) \implies f(i_0) = v(i_0)$$

Recall, this implies $i_0 \in M$, a contradiction. So, we have that φ dominates f, and is excessive, and thus:

$$\varphi(i) \geq v(i) \implies E_i [f(X_{\tau_0})] = v(i)$$

And with that, we have proven that the stopping time τ_0, the hitting time of M, the support set on which $f(i) = v(i)$, is an optimal strategy that equals $\sup_\tau E_i [f(X_\tau)]$ for all starting states i. qed.

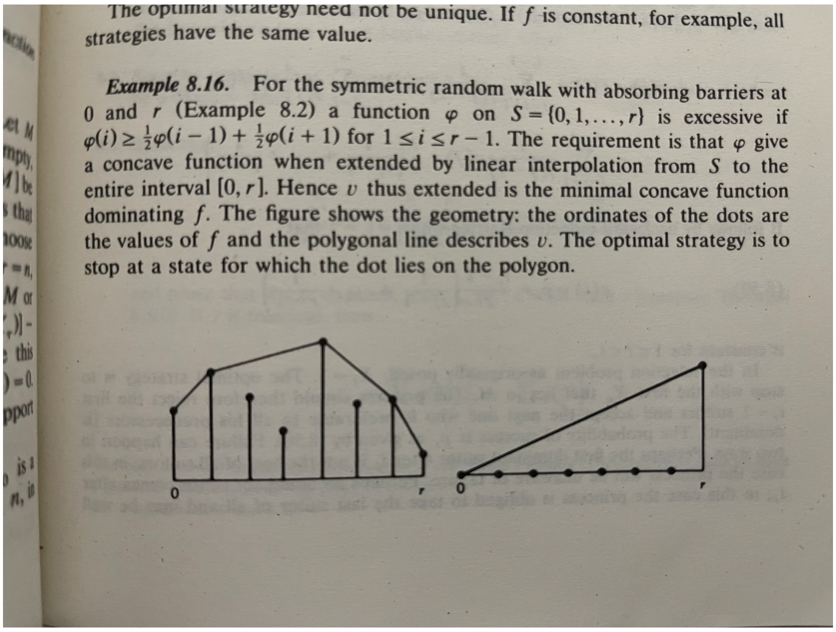
Example 8.16 Consider the symmetric random walk with absorbing barriers at 0 and r. Let f take values on each of the states 0 to r. What is the optimal stopping time? First, we need to find the support set M, where:

$$f(i) = \sup_\tau E_i [f(X_\tau)] = v(i)$$

Well, simpler than trying to calculate that expected value - $v(i)$ is the minimal excessive function that dominates f. Note, to be excessive on the symmetric random walk, requires:

$$\varphi(i) \geq \frac{1}{2}\varphi(i-1) + \frac{1}{2}\varphi(i)$$

This is essentially a concave requirement on S. So, v is the minimal concave function that dominates f. Note, we can do this geometrically - I will give a picture from the book here:



The optimal strategy need not be unique. If f is constant, for example, all strategies have the same value.

Example 8.16. For the symmetric random walk with absorbing barriers at 0 and r (Example 8.2) a function φ on $S = \{0, 1, \ldots, r\}$ is excessive if $\varphi(i) \geq \frac{1}{2}\varphi(i-1) + \frac{1}{2}\varphi(i+1)$ for $1 \leq i \leq r-1$. The requirement is that φ give a concave function when extended by linear interpolation from S to the entire interval $[0, r]$. Hence v thus extended is the minimal concave function dominating f. The figure shows the geometry: the ordinates of the dots are the values of f and the polygonal line describes v. The optimal strategy is to stop at a state for which the dot lies on the polygon.

Figure 2: v geometrically

Note that to dominate f, the function v has to be above the points that meet the dots in the picture. However, we also require v to be concave. The minimal "concavity" is a straight line - as this gives us equality on the concave condition. And so, for the symmetric random walk with absorbing barriers - the set of M are the points that are on the convex hull (smallest polygon enclosing all points, with no inward curves. The bottom line and left and right line are included).

I like this, because it ties a theoretical result to something that I think could be concluded intuitively. Note that in the left example, you would never stop at the inner points, as you will (with probability 1) go to a higher point. However, you do stop at the left and right line, because there is a high probability (1/2) that you will have to stop at a lower value.

Section 9 - Large Deviations and The Iterated Logarithm

Notes

Note, I am skipping, as this is an optional chapter, and I am eager to continue on to general random variables and measure.

Section 10 - General Measures

Notes

Classes of Sets

We want to find analogues to the Borel sets for the entire real line, and k dimensional Euclidean space.

Example 10.1 - k dimensional Borel Sets Let $x = (x_1, \cdots, x_k)$ be the generic point of Euclidean k space $\mathbb{R}^k$. The bounded rectangles:

$$[x = (x_1, \cdots, x_k) : a_i < x_i \leq b_i, i = 1, \cdots, k]$$

Will play in $\mathbb{R}^k$ the role intervals $(a, b]$ played in $(0, 1]$. Let $\mathcal{R}^k$ be the σ field generated by these rectangles. This is the analogue of the class $\mathcal{B}$ of Borel sets in $(0, 1]$. The elements of $\mathcal{R}^k$ are the k-dimensional borel sets. For $k = 1$, they are also called the linear Borel sets.

Note that $\mathcal{R}^k$ contains the open sets. Each point in an open set contains an open rational rectangle - the countable union of these rectangles forms the open set, and so the open set is in the sigma field, being a countable union of sets in the field. As complements, the closed sets are also within $\mathcal{R}^k$. We can also find that $\mathcal{R}^k$ is generated by the open or closed sets (as intersections of open sets form the rectangles above). This also implies the rational rectangles generate $\mathcal{R}^k$.

Now, note the following. We have the sigma field $\mathcal{R}^1$, on the line $\mathbb{R}^1$, is by definition generated by the finite intervals. The field $\mathcal{B}$ in $(0, 1]$ is generated by the subintervals of $(0, 1]$. Note, $\mathbb{R}$ vs. $(0, 1]$. The question is - are the elements of $\mathcal{B}$ the elements of $\mathcal{R}$ that happen to lie inside $(0, 1]$? The short answer is yes - via the following theorem. We first note the definition: if $\mathcal{A}$ is a class of sets in a space Ω and Ω_0 is a subset of Ω, let $\mathcal{A} \cap \Omega_0 = [A \cap \Omega_0 : A \in \mathcal{A}]$. We find:

Theorem 10.1

1. If $\mathcal{F}$ is a σ field in Ω, then $\mathcal{F} \cap \Omega_0$ is a σ field in Ω_0.
2. If $\mathcal{A}$ generates the σ field $\mathcal{F}$ in Ω, then $\mathcal{A} \cap \Omega_0$ generates the σ field $\mathcal{F} \cap \Omega_0$ in $\Omega_0 : \sigma(\mathcal{A} \cap \Omega_0) = \sigma(\mathcal{A}) \cap \Omega_0$.

Proof: The first proof is literally just your standard sigma field proof. Prove $\mathcal{F} \cap \Omega_0$ has Ω_0, sets in it have their complements using $\mathcal{F}$, and countable unions of sets in it are in it as well, again using $\mathcal{F}$.

Now, we get to the second part. Define $\mathcal{F}_0$ as $\sigma(\mathcal{A} \cap \Omega_0)$, the generated σ field *in* Ω_0. We must have that $\mathcal{A} \cap \Omega_0 \subset \mathcal{F} \cap \Omega_0$, as every set in $\mathcal{A}$ is in $\mathcal{F}$. As $\mathcal{F}$ is a σ field, then $\mathcal{F} \cap \Omega_0$ is a σ field by part 1, and so, by definition of the generated σ field, we must have:

$$\mathcal{F}_0 \subseteq \mathcal{F} \cap \Omega_0 \text{ or, written another way } \sigma(\mathcal{A} \cap \Omega_0) \subseteq \sigma(\mathcal{A}) \cap \Omega_0$$

We now want to prove the other direction, $\mathcal{F} \cap \Omega_0 \subseteq \mathcal{F}_0$. This will follow if it is shown $A \in \mathcal{F}$ implies $A \cap \Omega_0 \in \mathcal{F}_0$. Why? We want to show that for all $A \cap \Omega_0 \in \sigma(\mathcal{A}) \cap \Omega_0$, we have that $A \cap \Omega_0 \in \sigma(\mathcal{A} \cap \Omega_0) = \mathcal{F}_0$. Well, the $A \in \mathcal{F}$ by definition, so we are just kind of rewriting the statement. I think, we want to rewrite it in terms of just sigma fields in Ω. We have $A \cap \Omega_0 \in \mathcal{F}_0$ if $A \in \mathcal{G} = [A \subset \Omega : A \cap \Omega_0 \in \mathcal{F}_0]$. So, we have:

$$\mathcal{F} \cap \Omega_0 \subseteq \mathcal{F}_0$$

Is equivalent to us proving:

$$\mathcal{F} \subseteq \mathcal{G}$$

So, we try and prove this. $A \in \mathcal{A}$ implies $A \cap \Omega_0$ is in $\mathcal{A} \cap \Omega_0 \in \mathcal{F}_0$, by definition. Thus, it follows $\mathcal{A} \subset \mathcal{G}$. So, if we prove that $\mathcal{G}$ is a σ field, we would have $\mathcal{F} = \sigma(\mathcal{A}) \subseteq \mathcal{G}$. Note that $\Omega \in \mathcal{G}$, as $\Omega \cap \Omega_0 = \Omega_0 \in \mathcal{F}_0$. If $A \in \mathcal{G}$, then $(\Omega - A) \cap \Omega_0 = \Omega_0 - (A \cap \Omega_0) \in \mathcal{F}$, straight forward enough. Finally, if $A_n \in \mathcal{G}$ for all n, then $(\cup_n A_n) \cap \Omega_0 = \cup_n (A_n \cap \Omega_0) \in \mathcal{F}_0$.

So, $\mathcal{G}$ is a σ field, $\mathcal{F} \subseteq \mathcal{G}$, and we have thus proved:

$$\mathcal{F}_0 \supseteq \mathcal{F} \cap \Omega_0 \text{ or, written another way } \sigma(\mathcal{A} \cap \Omega_0) \supseteq \sigma(\mathcal{A}) \cap \Omega_0$$

And thus, we have equality, and have proven the second statement. qed.

The Borel Sets lie in the Linear Borel Sets The borel sets $\mathcal{B}$ refer to the sets in $(0,1]$. The linear borel sets refer to the sets in $\mathbb{R}^1$. If $\Omega = \mathbb{R}^1$, $\Omega_0 = (0,1]$, and $\mathcal{A}$ is the set of half open intervals on the line, we have that $\mathcal{R}^1 = \sigma(\mathcal{A})$. By the above theorem, we thus have:

$$\sigma(\mathcal{A} \cap (0,1]) = \sigma(\mathcal{A}) \cap (0,1]$$

Ie, $\mathcal{R}^1$, restricted to the unit interval (the RHS) is equal to the generated sigma algebra from the half open intervals in $(0,1]$ (the LHS), which is $\mathcal{B}$. Ie, we have:

$$\mathcal{B} = \mathcal{R}^1 \cap (0,1]$$

Measures

Definition: Measure A set function μ on a field $\mathcal{F}$ in Ω is called a *measure* if it satisfies these conditions:

1. $\mu(A) \in [0, \infty]$ for $A \in \mathcal{F}$
2. $\mu(\emptyset) = 0$
3. If $A_1, A_2, \cdots$ is a disjoint sequence of $\mathcal{F}$ sets and if $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$, then:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

I will note - the third condition doesn't require that all such unions are in $\mathcal{F}$, and $\mathcal{F}$ being a field doesn't guarantee it either. The measure μ is *finite* or *infinite* as $\mu(\Omega) < \infty$ or $\mu(\Omega) = \infty$. It is a probability measure if $\mu(\Omega) = 1$.

Definition: σ finite If $\Omega = A_1 \cup A_2 \cup \cdots$ for some finite or countable sequence of $\mathcal{F}$ sets satisfying $\mu(A_k) < \infty$, then μ is σ-finite. This notion will be significant later on. By our above definitions of finite/infinite measure, a finite measure is clearly σ finite, and a σ finite measure may be finite or infinite. If $\mathcal{A}$ is a subclass of $\mathcal{F}$, and $\Omega = \cup_k A_k$ for some finite or infinite sequence of $\mathcal{A}$ sets satisfying $\mu(A_k) < \infty$, then μ is σ *finite on* $\mathcal{A}$.

It is important to note that σ finiteness is a joint property of the space Ω, the measure μ, and the class $\mathcal{A}$.

Definition: Measure Space If μ is a measure on a σ field $\mathcal{F}$ in Ω, the triple $(\Omega, \mathcal{F}, \mu)$ is a *measure space* - note, not used if $\mathcal{F}$ is only a field. It is σ finite, infinite, or finite based on whether μ has that property. If $\mu(A^c) = 0$ for an $\mathcal{F}$ set A, then A is a *support* of μ, and μ is *concentrated* on A. For a finite measure, A is a support if and only if $\mu(A) = \mu(\Omega)$.

Example 10.2 - Discrete Space A measure μ on $(\Omega, \mathcal{F})$ (called a measurable space) is *discrete* if there are finitely or countable many points ω_i in Ω and masses m_i in $[0, \infty]$ such that $\mu(A) = \sum_{\omega_i \in A} m_i$ for $A \in \mathcal{F}$.

Example 10.3 - The Counting Measure Let $\mathcal{F}$ be the σ field of all subsets of an arbitrary Ω, and let $\mu(A)$ be the number of points in A, where $\mu(A) = \infty$ if A is not finite. Then, μ is the *counting measure*. It is finite if and only if Ω is finite, and is σ finite if and only if Ω is countable.

Carry Over from Probability Measures to General Measures A lot of the properties we derived for probability measures carry over to general measures. Note that:

1. μ is monotone: examine $\mu(B) = \mu(A) + \mu(B - A)$
2. We can only write $\mu(B - A) = \mu(B) - \mu(A)$ if $\mu(B) < \infty$
3. We still have the inclusion-exclusion principle
4. We still have the proof of finite subadditivity, as before:

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) \leq \sum_{k=1}^{n} \mu(A_k)$$

The proof being just replace A_k with B_k, which is disjoint from all the previous A_k. Note $\mu(B_k) \leq \mu(A_k)$.

Theorem 10.2 Let μ be a measure on a field $\mathcal{F}$. We have:

1. **Continuity From Below** If A_n and A lie in $\mathcal{F}$ and $A_n \uparrow A$, then:

$$\mu(A_n) \uparrow \mu(A)$$

2. **Continuity form Above** If A_n and A lie in $\mathcal{F}$ and $A_n \downarrow A$, and if $\mu(A_1) < \infty$ (or any $\mu(A_k) < \infty$) then:

$$\mu(A_n) \downarrow \mu(A)$$

3. **Countable Subadditivity** If $A_1, A_2, \cdots$, and $\bigcup_{k=1}^{\infty} A_k$ lies in $\mathcal{F}$, then:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$

4. If μ is σ finite on $\mathcal{F}$, then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive μ-measure.

Proof: The proof for 1 is the same as it is for probability measure: express A as a disjoint union of B_k (each B_k represents the increment added to A_k from A_{k-1}), and note limit sums are equal to A_k. The proof for 3 is also the same as it is for probability measure: use part 1 expressing the union as $\uparrow$, and then use the same reasoning as for finite additivity.

The proof is essentially the same for 2: if $\mu(A_1) < \infty$, subtraction is possible and $A_1 - A_n \uparrow A_1 - A$ implies that $\mu(A_1) - \mu(A_n) = \mu(A_1 - A_n) \uparrow \mu(A_1 - A) = \mu(A_1) - \mu(A)$, now just rearrange.

So, the real meat comes in part 4. Let $[B_\theta : \theta \in \Theta]$ be a disjoint collection of $\mathcal{F}$ sets satisfying $\mu(B_\theta) > 0$. We will prove that Θ must be countable. Consider an $\mathcal{F}$ set A for which $\mu(A) < \infty$. If $\theta_1, \cdots, \theta_n$ are distinct indices satisfying:

$$\mu\left(A \cap B_{\theta_i}\right) \geq \epsilon > 0$$

Then we must have:

$$n\epsilon \leq \sum_{i=1}^{n} \mu(A \cap B_{\theta_i}) \leq \mu(A)$$

And so:

$$n \leq \mu(A)/\epsilon$$

Thus, we must have that the index set:

$$[\theta : \mu(A \cap B_\theta) > \epsilon]$$

Is finite. If it wasn't - we could make a contradiction proving for some $B \subset A$, we have $\mu(B) > \mu(A)$. If we take a union over positive rational ϵ, we must have:

$$[\theta : \mu(A \cap B_\theta) > 0]$$

Is a countable index set. As μ is σ finite, $\Omega = \cup_k A_k$ for $\mu(A_k) < \infty$, $A_k \in \mathcal{F}$. But then, define the index set:

$$\Theta_k = [\theta : \mu(A_k \cap B_\theta) > 0]$$

Θ_k must be countable. As $\mu(B_\theta) > 0$, there must be a k such that $\mu(A_k \cap B_\theta) > 0$ - if they all equaled zero, we would have a contradiction with countable subadditivity. Thus, each B_θ must appear in one of the Θ_k sets. And so:

$$\Theta = \bigcup_k \Theta_k$$

And thus Θ is countable. qed.

Uniqueness

According to Theorem 3.3, probability measures agreeing on a π system $\mathcal{P}$ agree on $\sigma(\mathcal{P})$ - this is by the π λ theorem. There is an extension in the general case

Theorem 10.3 - Uniqueness of General Measure Extension Suppose that μ_1 and μ_2 are measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a π system, and suppose they are σ finite on $\mathcal{P}$ (ie, countable $\mathcal{P}$ sets with non infinite measure union to Ω). If μ_1 and μ_2 agree on $\mathcal{P}$, then they agree on $\sigma(\mathcal{P})$.

Proof Suppose that $B \in \mathcal{P}$ and $\mu_1(B) = \mu_2(B) < \infty$, and let $\mathcal{L}_B$ be the class of sets A in $\sigma(\mathcal{P})$ for which $\mu_1(B \cap A) = \mu_2(B \cap A)$. Then $\mathcal{L}_B$ is a λ system (clearly contains Ω, complements by subtraction and $\mu(B) < \infty$, and countable disjoint unions via additivity of μ) containing $\mathcal{P}$ (π system contains intersections) and hence containing $\sigma(\mathcal{P})$ via the $\pi - \lambda$ theorem.

By σ-finiteness, there exists $\mathcal{P}$ sets B_k satisfying $\Omega = \bigcup_k B_k$ and $\mu_1(B_k) = \mu_2(B_k) < \infty$. By inclusion-exclusion:

$$\mu_\alpha\left(\bigcup_{i=1}^{n}(B_i \cap A)\right) = \sum_{1\leq i\leq n} \mu_\alpha(B_i \cap A) - \sum_{1\leq i<j\leq n} \mu_\alpha(B_i \cap B_j \cap A) + \cdots$$

For $\alpha = 1, 2$ and all n. Since $\mathcal{P}$ is a π system, it contains all the intersections of B_i, and via the above argument, the terms on the RHS above are equal for μ_1 and μ_2. The LHS is thus equal for both as well. Now, we take $n \to \infty$, and use continuity from below, to find:

$$\mu_1(A) = \lim_{n\to\infty} \mu_1\left(\bigcup_{i=1}^{n}(B_i \cap A)\right) = \mu_2\left(\bigcup_{i=1}^{n}(B_i \cap A)\right) = \mu_2(A)$$

For $A \in \sigma(\mathcal{P})$. Thus, we have equality on $\sigma(\mathcal{P})$. qed.

Theorem 10.4 Suppose μ_1 and μ_2 are finite measures on $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a π system and Ω is a finite or countable union of sets in $\mathcal{P}$. If μ_1 and μ_2 agree on $\mathcal{P}$, then they agree on $\sigma(\mathcal{P})$.

Proof By hypothesis, $\Omega = \bigcup_k B_k$ for $\mathcal{P}$ sets B_k, and we of course have $\mu_\alpha(B_k) < \mu_\alpha(\Omega) < \infty$. So, μ_1 and μ_2 are σ finite on $\mathcal{P}$, and Theorem 10.3 above applies. qed.

Problems

10.1 Equivalent Measure Definition

Show that if conditions (i) and (iii) in the definition of measure hold, and if $\mu(A) < \infty$ for some $A \in \mathcal{F}$, then condition (ii) holds.

We note that by countable additivity:

$$\mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset) \implies \mu(A) - \mu(A) = \mu(\emptyset) \implies \mu(\emptyset) = 0$$

Note, we require $\mu(A) < \infty$, as otherwise subtraction is not defined.

10.2 Finite vs. Countable Additivity on a Countable Ω

On the σ field of all subsets of $\Omega = \{1, 2, \cdots\}$ put $\mu(A) = \sum_{k \in A} 2^{-k}$ if A is finite and $\mu(A) = \infty$ otherwise. Is μ finitely additive? Countably additive?

I would first say that μ is finitely additive. If you have the finite disjoint union of finite sets - then we will just get out the sum. If you add a countable set in there - both sides will be infinite. Note, a finite amount of finite sets won't get to countable additivity.

The issue comes with a countable amount of finite sets. Take $A_k = \{k\}$. We have:

$$\mu(\bigcup A_k) = \infty$$

Clearly. However, we also have:

$$\sum_k \mu(A) = \sum_k 2^{-k} = 1$$

So that is where countable additivity breaks down.

10.3 Continuity From Above Extension

1. In connection with Theorem 10.2(ii), show that if $A_n \downarrow A$ and $\mu(A_k) < \infty$ for some k, then $\mu(A_n) \downarrow \mu(A)$. Note, I actually noted this in the theorem. You can make use of the first $\mu(A_k) < \infty$, instead of A_1.

2. Find an example in which $A_n \downarrow A$, $\mu(A_n) = \infty$ for all n, and $A = \emptyset$.

   I think, we could take $\mathcal{F} = \mathcal{R}^1$, and I think its corresponding measure in the next section, λ. We have that $A_n = \bigcup_{k=n}^{\infty}(n, n+1]$ (note, I could write (n, ∞), but I think this makes it clearer that it is within the first linear borel set). Clearly $A_n \downarrow A = \emptyset$. Take an $x \in \mathbb{R}$, clearly it doesn't belong to some A_n. We also have $\mu(A_n) = \infty$, again via additivity of the measure.

Perhaps, that isn't the best example, as we haven't defined it yet. We could do the counting measure on $\Omega = \{1, 2, \cdots\}$ with $\mathcal{F} = 2^{\Omega}$. Same idea, $A_n = \{n, n+1, \cdots\}$.

10.4 Generalization of $\lim, \liminf, \limsup$ **ordering**

Recall Theorem 4.1 part 1: For each sequence of sets A_n:

$$P\left[\liminf_n A_n\right] \leq \liminf_n P(A_n) \leq \limsup_n P(A_n) \leq P\left[\limsup_n A_n\right]$$

We want to generalize for arbitrary measures:

$$\mu\left[\liminf_n A_n\right] \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu\left[\limsup_n A_n\right]$$

Show that the left inequality always holds, and the right inequality holds if $\mu\left(\bigcup_{k\geq n} A_k\right) < \infty$ for some n.

Note, these restrictions come from when we can conclude continuity from above/below. Let $B_n = \bigcap_{k\geq n} A_k$ and $C_n = \bigcup_{k\geq n} A_k$. Note that $B_n \uparrow \liminf_n A_n$. So, by Theorem 10.2, we have:

$$\mu\left[\liminf_n A_n\right] = \lim_n \mu(B_n) = \liminf_n \mu(B_n) \leq \liminf_n \mu(A_n)$$

Note, the liminf always exists. We similarly note that $C_n \downarrow \limsup_n A_n$ - by the extension to Theorem 10.2 in the previous problem, if there is some $\mu(C_n) < \infty$, then we have:

$$\mu\left[\limsup_n A_n\right] = \lim_n \mu(C_n) = \limsup_n \mu(C_n) \geq \limsup_n \mu(A_n)$$

10.6 Characterizing σ finiteness

The condition in Theorem 10.2(iv) essentially characterizes σ finiteness.

1. Suppose that $(\Omega, \mathcal{F}, \mu)$ has no "infinite atoms" in the sense that for every $A \in \mathcal{F}$, if $\mu(A) = \infty$, then there is in $\mathcal{F}$ a B such that $B \subset A$ and $0 < \mu(B) < \infty$. Show that if $\mathcal{F}$ does not contain an uncountable, disjoint collection of sets of positive measure, then μ is σ finite (Use Zorn's lemma).

   Zorn's Lemma States: If P is a partially ordered set (has an inequality, but not every element is comparable) such that every ordered chain has an upper bound in P (ie, every chain of inequalities has an element greater than every element in the chain), then P has at least one maximal element (an element such that no other element is greater than it).

So now, we know that $\mathcal{F}$ does not contain an uncountable disjoint collection of sets of positive measure, and $\mathcal{F}$ does not have any infinite atoms.

I think it can be done in the following way - iteratively. Assume μ is not σ finite. Define $\mathcal{F}_1$ as the set of $A \in \mathcal{F}$ such that $\mu(A) < \infty$, or $A = \bigcup A_i$ where $\mu(A_i) < \infty$ and $A_i \cap A_j = \emptyset$. Note, $\mathcal{F}_1$ is partially ordered by $\subseteq$. Each chain of elements is bounded above - a countable chain is clear, but consider an uncountable one. Say we have an uncountable chain, with an uncountable amount of distinct sets. Then, this is a contradiction - because:

$$A_1 \subset A_2 \implies A_2 - A_1 \in \Omega$$

But then, we have an uncountable amount of disjoint sets in Ω. So, this is a contradiction. So, our chains are countable, are all bounded - Zorn's lemma tells us there exists a maximal element $C_1 \in \mathcal{F}_1$.

This is actually incorrect, I think. We are allowed to have uncountable, disjoint sets, if they have zero measure. We could have $\mu(A_1) = \mu(A_2)$, with $A_1 \subseteq A_2$.

Real Proof First, we consider the case where $\mathcal{F}$ is countable. Take a union of every $\mu(A) < \infty$. Examine:

$$\Omega - \bigcup A = C$$

We want to show $C = \emptyset$. If that is not the case, note that $\mu(C) < \infty$ is a contradiction, as then C would be included in the union. So, we must have $\mu(C) = \infty$. Note, we have assumed there are no infinite atoms, and so there must be some $B \subset C$ where $\mu(B) < \infty$. However, this leads to another contradiction, B is then within our union, but B is also outside of it. So, if $C \neq \emptyset$, we always have a contradiction, and we must thus have Ω is σ finite (as the union above is uncountable).

Now, we consider the case where $\mathcal{F}$ is uncountable. Let $\mathcal{I}$ index every countable union of non-infinite measure sets. Ie, for $i \in \mathcal{I}$, we have that:

$$C_i = \bigcup_{j=1}^{\infty} C_{ij} \quad \text{such that } \mu(C_{ij}) < \infty$$

By contradiction, assume that Ω is not σ finite. Then, for every $i \in \mathcal{I}$, we must have:

$$A_i = \Omega - C_i \quad \implies \quad A_i \neq \emptyset$$

We note that $\mu(A_i) < \infty$ leads to a contradiction, as then $C_i \cup A_i$ is a countable union of non infinite measure sets that equals Ω. So, we must have $\mu(A_i) = \infty$. As we have no infinite atoms, we must have a $B_i \subset A_i$ such that $0 < \mu(B_i) < \infty$.

Use the axiom of choice, I guess, to select the B_k that are disjoint from B_i. We note that there must be an uncountable amount of them. Assume that there is only a countable amount - note, we can easily create a countable amount of them. List them all out, from $B_1, B_2, \cdots$. As they are countable, we are able to list them in such a way. Now note:

$$C_t = B_i \cup \bigcup_{k=1}^{\infty} B_k$$

Forms a countable union in our index, which is why we set the above equal to C_t. This yields B_t, another positive measure set that is disjoint from the countable list we gave above, of every disjoint positive measure set from B_i. Thus, assuming that the set of B_k that are disjoint from B_i is countable is a contradiction, as any such countable set containing everything can yield another set that should be in there, but isn't.

Thus, the B_k that are disjoint from B_i must be uncountable. Thus, assuming that A_i is not empty for each countable union of non infinite measure sets leads to a contradiction that $\mathcal{F}$ contains an uncountable, disjoint collection of sets of positive measure. So, we must have for some i, $A_i = \emptyset$, and thus Ω is σ finite. qed.

2. Show by example that this is false without the condition that there are no infinite atoms.

   Maybe let $\mathcal{F}$ be the σ algebra containing the two sets, the even integers and the odd integers, in $\mathbb{Z}$. Note, this $\mathcal{F}$ doesn't contain an uncountable disjoint collection of sets of positive measure. Note, however, $\mathcal{F}$ contains infinite atoms. μ is not σ finite either, as we cannot express Ω as a countable union of finite measure sets.

Section 11 - Outer Measure

Notes

Outer Measure

Definition - Outer Measure An *outer measure* is a set function μ^* that is defined for all subsets of a space Ω and has these four properties:

1. $\mu^*(A) \in [0, \infty]$ for every $A \subset \Omega$
2. $\mu^*(\emptyset) = 0$
3. μ^* is monotone; $A \subset B$ implies $\mu^*(A) \leq \mu^*(B)$
4. μ^* is countable subadditive; $\mu^*\left(\bigcup_n A_n\right) \leq \sum_n \mu^*(A_n)$

The set function P^*, defined by (3.1) is an example. This generalizes:

Example 11.1 - General Outer Measure of a Set Function Let ρ be a set function on a class $\mathcal{A}$ in Ω. Assume that $\emptyset \in \mathcal{A}$ and $\rho(\emptyset) = 0$, and that $\rho(A) \in [0, \infty]$ for $A \in \mathcal{A}$; ρ and $\mathcal{A}$ are otherwise arbitrary. Put:

$$\mu^*(A) = \inf \sum_n \rho(A_n)$$

Where the infimum extends over all finite and countable coverings of A by $\mathcal{A}$ sets A_n. If no such covering exists - take $\mu^*(A) = \infty$, following the convention that the infimum over the empty set in ∞.

We have that μ^* is an outer measure. It clearly satisfies $1, 2, 3$. We just go over 4. If $\mu^*(A_n) = \infty$ for some n, then obviously the inequality will trivially hold. Assume that each A_n is finite. Cover each A_n by $\mathcal{A}$ sets B_{nk} satisfying:

$$\sum_k \rho(B_{nk}) < \mu^*(A_n) + \epsilon/2^n$$

Then, we note that all of these covers together cover the union, and so:

$$\mu^*\left(\bigcup_n A_n\right) \leq \sum_{n,k} \rho(B_{nk}) < \sum_n \mu^*(A_n) + \epsilon$$

Take ϵ to zero as normal. Thus, μ^* is an outer measure.

μ^* **Measurable** Define A to be μ^*-measurable if:

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$$

For every E. This is the general version of (3.4) used in section 3. By subadditivity, it is equivalent to saying A satisfies:

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E)$$

Denote $\mathcal{M}(\mu^*)$ as the class of μ^* measurable sets. The book now notes - we can replace P^* with μ^*, and $\mathcal{M}$ with $\mathcal{M}(\mu^*)$, symbol by symbol, and the proof still holds for lemmas 3.1, 3.2, 3.3. I will just list out their statements here:

1. Lemma 3.1 - The class $\mathcal{M}(\mu^*)$ is a field. Yeah, it makes sense - $\mathcal{M}(\mu^*)$ is clearly closed under complements and similarly closed under intersections.

2. Lemma 3.2 - If $A_1, A_2, \cdots$ is a finite or infinite sequence of disjoint $\mathcal{M}(\mu^*)$ sets, then for each $E \subset \Omega$:

$$\mu^* \left(E \cap \left(\bigcup_k A_k \right) \right) = \sum_k \mu^* \left(E \cap A_k \right)$$

   Again, yeah, the lemma still makes sense.

3. Lemma 3.3 - The class $\mathcal{M}(\mu^*)$ is a σ field, and μ^* restricted to $\mathcal{M}$ is countably additive. Yeah, this still makes sense.

If μ^* is countably additive, and also satisfies the properties 1, 2, 3, 4, we can also conclude it is a measure. So, we have the following theorem:

Theorem 11.1 - Induced Measure If μ^* is an outer measure, then $\mathcal{M}(\mu^*)$ is a σ field, and μ^* restricted to $\mathcal{M}(\mu^*)$ is a measure.

Extension

Theorem 11.2 - Extension of a Measure A measure on a field has an extension to the generated σ field. **Proof:** Note, we don't need to prove this, as the result will actually follow from a strong form of the proof in Theorem 11.3. qed. Also note - if the original measure on the field is σ finite, Theorem 10.3 implies that the extension is unique. Recall, Theorem 10.3 says that if there are two measure on a generated sigma field, and the generator is a π system, and the measures agree on the π system and are σ finite (note, one is σ finite implies the other is), then they agree on the entire generated sigma field.

Definition - Semi-Ring A class of sets $\mathcal{A}$ is a *semiring* if:

1. $\emptyset \in \mathcal{A}$
2. $A, B \in \mathcal{A}$ implies $A \cap B \in \mathcal{A}$
3. $A, B \in \mathcal{A}$ and $A \subset B$ implies there exists disjoint $C_1, \cdots, C_n \in \mathcal{A}$ such that:
$$B - A = \bigcup_{k=1}^{n} C_k$$

Some simple examples - the class of finite intervals in $\Omega = \mathbb{R}^1$. Clearly the intersection of two such finite intervals is a finite interval. The set minus is also at most a pair of disjoint intervals. Also, the class of subintervals of $\Omega = (0, 1]$ is a semiring as well, with the same reasoning. Every field is a semiring as well, as:
$$B - A = B \cap A^c \in \mathcal{F}$$

Theorem 11.3 Measure Semi-Ring Extension Suppose that μ is a set function on a semiring of $\mathcal{A}$. Suppose that μ has values in $[0, \infty]$, $\mu(\emptyset) = 0$, and μ is finitely additive and countably subadditive. Then μ extends to a measure on $\sigma(\mathcal{A})$.

First note - this contains Theorem 11.2. A field is clearly a semiring. A measure also satisfies all the given properties, definitionally, on $\mathcal{A}$.

Proof: Let A, B, C_k be as in condition 3 of the semiring. By finite additivity:
$$\mu(B) = \mu(A) + \sum_{k=1}^{n} \mu(C_k) \geq \mu(A)$$
Thus, our set function μ is monotone on $\mathcal{A}$. Take our outer measure μ^* - recall, defined as:
$$\mu^*(A) = \inf \sum_{n} \mu(A_n)$$
Where the infimum extends over coverings of A by $\mathcal{A}$ sets, and an empty infimum yields infinity. We first want to show that $\mathcal{A} \subset \mathcal{M}(\mu^*)$. Then, if we show μ^* and μ agree on $\mathcal{A}$, we can say that μ^* is our extension - Theorem 11.1 tells us that μ^* is a measure on $\mathcal{M}(\mu^*)$, and it would continue to be one when restricted to $\sigma(\mathcal{A})$.

Take $A \in \mathcal{A}$. If $\mu^*(E) = \infty$, then our condition for being in $\mathcal{M}(\mu^*)$:

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \mu^*(E)$$

holds trivially. So, assume $\mu^*(E) < \infty$. For an $\epsilon > 0$, use the definition of infimum to find A_n such that:

$$\sum_n \mu(A_n) < \mu^*(E) + \epsilon$$

As A is a semiring, $B_n = A \cap A_n$ is in $\mathcal{A}$, and $A^c \cap A_n = A_n - B_n$ has the form:

$$\bigcup_{k=1}^{m_n} C_{nk}$$

Note that:

$$A_n = A \cap A_n \bigcup A^c \cap A_n = B_n \cup \bigcup_{k=1}^{m_n} C_{nk}$$

Which is a disjoint union. Note that:

$$A \cap E \subset A \cap \bigcup_n A_n = \bigcup_n B_n$$

Finally, note that:

$$A^c \cap E \subset A^c \cap \bigcup_n A_n = \bigcup_n A_n - B_n = \bigcup_n \bigcup_{k=1}^{m_n} C_{nk}$$

We now conclude:

$$\mu^*(A \cap E) + \mu^*(A^c \cap E) \leq \sum_n \mu(B_n) + \sum_n \sum_{k=1}^{m_n} \mu(C_{nk}) = \sum_n \mu(A_n) < \mu^*(E) + \epsilon$$

The first step is the definition of the outer measure - we take the infimum of covering sums, and as the B_n and C_{nk} are covering, we get less than the corresponding sums. The next equality is via finite additivity on $\mathcal{A}$, and the B_n and C_{nk} being disjoint. Take $\epsilon \to 0$, and yes, we can conclude that $\mathcal{A} \subset \mathcal{M}(\mu^*)$.

Onto the next step. We want to show that μ^* and μ agree on $\mathcal{A}$. If $A \subset \bigcup_n A_n$ for $\mathcal{A}$ sets A and A_n, then by countable subadditivity and monotonicity, we have:

$$\mu(A) \leq \sum_n \mu(A \cap A_n) \leq \sum_n \mu(A_n)$$

Thus, $A \in \mathcal{A}$ implies $\mu(A) \leq \mu^*(A)$. We note that $\mu^*(A) \leq \mu(A)$, just by the definition of the infimum, and so we can easily conclude that:

$$\mu(A) = \mu^*(A)$$

And so now, we wrap up. We have:

$$\mathcal{A} \subseteq \mathcal{M}(\mu^*) \implies \sigma(\mathcal{A}) \subseteq \mathcal{M}(\mu^*)$$

And μ^* restricted to $\sigma(\mathcal{A})$ is a measure. As μ^* agrees with μ on $\mathcal{A}$, μ^* is a measure that is an extension of μ on $\sigma(\mathcal{A})$. qed.

Example 11.2 - Second Example of the Lebesgue Measure on the Unit Interval Note that the subintervals of $\Omega = (0, 1]$ is a *semiring*. This is distinct from the disjoint finite unit of intervals, which is a field. For μ take length $\lambda(a, b] = b - a$. By Theorem 1.3, we already have that λ is finitely additive and countably subadditive. By theorem 11.3, λ extends to a measure on the class $\sigma(\mathcal{A}) = \mathcal{B}$ of Borel sets in $(0, 1]$.

Example 11.3 - The Lebesgue Measure on the Real Line Take $\mathcal{A}$ as the semi ring of finite intervals on the real line $\mathbb{R}$, and $\lambda_1(a, b] = b - a$. We have that λ_1 is finitely additive and countably subadditive on $\mathcal{A}$ (note Theorem 1.3 doesn't need to be restricted to $(0, 1]$). Thus, λ_1 extends to the σ field $\mathcal{R}^1$ of linear Borel sets, which is generated by $\mathcal{A}$ definitionally. This defines the Lebesgue measure λ_1 over the whole real line.

The Lebesgue Measure on the Borel Set and the first Linear Borel Set Coincide Recall, from Theorem 10.1, we have that a subset of $(0, 1]$ lies in $\mathcal{B}$ iff it lies in $\mathcal{R}^1$ as well. We have that $\lambda_1(A) = \lambda(A)$, for A subinterval of $(0, 1]$ - just the length. It follows by uniqueness of extension that $\lambda_1(A) = \lambda(A)$ for all $A \in \mathcal{B}$. So, we don't need the 1 subscript - just call is λ as well.

An Approximation Theorem

By Theorem 10.3, we have that if two measures are sigma finite on a π system $\mathcal{P}$, agree on $\mathcal{P}$, and are measures on $\sigma(\mathcal{P})$, they also agree on $\sigma(\mathcal{P})$. In this way, if $\mathcal{A}$ is a semiring which is σ finite, a measure on $\sigma(\mathcal{A})$ is thus determined by its values on $\mathcal{A}$. We can thus approximate the measure of a $\sigma(\mathcal{A})$ set by measures of $\mathcal{A}$ sets.

Lemma 11.1 - Semi Ring Intersection Equality If $A, A_1, \cdots, A_n$ are sets in a semiring $\mathcal{A}$, then there are disjoint $\mathcal{A}$ sets $C_1, \cdots, C_m$ such that:

$$A \cap A_1^c \cap \cdots \cap A_n^c = C_1 \cup \cdots \cup C_m$$

Proof: The case $n = 1$ follows from the definition of the semiring applied to $A \cap A_1^c = A - (A \cap A_1) = C_1 \cup \cdots \cup C_m$. Use induction. If the result holds for n, then:

$$A \cap A_1^c \cap \cdots \cap A_{n+1}^c = \bigcup_{j=1}^{m} (C_j \cap A_{n+1}^c)$$

And apply the case $n = 1$ to each set in the union.

Theorem 11.4 - Approximating the Measure With Semiring Values Suppose that $\mathcal{A}$ is a semiring, μ is a measure on $\mathcal{F} = \sigma(\mathcal{A})$, and μ is σ finite on $\mathcal{A}$.

1. If $B \in \mathcal{F}$ and $\epsilon > 0$, there exists a finite or infinite disjoint sequence $A_1, A_2, \cdots$ of $\mathcal{A}$ sets such that $B \subset \bigcup_k A_k$ and:

$$\mu\left(\bigcup_k A_k - B\right) < \epsilon$$

2. If $B \in \mathcal{F}$ and $\epsilon > 0$, and if $\mu(B) < \infty$, then there exists a finite disjoint sequence $A_1, \cdots, A_n$ of $\mathcal{A}$ sets such that:

$$\mu\left(B \triangle \bigcup_{k=1}^{n} A_k\right) < \epsilon$$

Proof: Recall the proof of Theorem 11.3. If μ^* is the outer measure, we have $\mathcal{F} \subset \mathcal{M}(\mu^*)$. We also have that μ^* agrees with μ on $\mathcal{A}$ (Theorem 11.3) and by uniqueness of extension agrees with μ on $\mathcal{F}$ (μ^* restricted to $\mathcal{F}$ is a measure).

Take our $B \in \mathcal{F}$, and suppose $\mu(B) = \mu^*(B) < \infty$. There exist $\mathcal{A}$ sets A_k such that:

$$B \subseteq \bigcup_k A_k \qquad \mu\left(\bigcup_k A_k\right) \leq \sum_k \mu(A_k) < \mu(B) + \epsilon$$

By subadditivity, and definition of the outer measure. Thus, we have:

$$\mu\left(\bigcup_k A_k - B\right) < \epsilon$$

To make the A_k disjoint, replace A_k with:

$$A_k \cap A_1^c \cap \cdots \cap A_{k-1}^c = C_{k1} \cup \cdots \cup C_{km_k}$$

The equality follows by Lemma 11.1. So, we have prove 11.4.1 in the finite case. Now assume $\mu(B) = \mu^*(B) = \infty$. By σ-finiteness, there exists $\mathcal{A}$ sets C_m such that $\Omega = \bigcup_m C_m$ and $\mu(C_m) < \infty$. Note that $B \cap C_m \in \mathcal{A}$ by semi ring closed under finite intersection, and $\mu(B \cap C_m) < \infty$. By the above finite case, there exists $A_{mk} \in \mathcal{A}$ such that:

$$\mu\left(\bigcup_k A_{mk} - (B \cap C_m)\right) < \epsilon/2^m$$

The sets A_{mk}, taken together, provide a countable sequence $A_1, A_2, \cdots$ of $\mathcal{A}$ sets satisfying $\mathcal{B} \subset \bigcup_k A_k$ and:

$$\mu\left(\bigcup_k A_k - B\right) < \epsilon$$

We can make the A_k disjoint as before. This thus completely proves 11.4.1.

Now, we go onto 11.4.2. Consider the A_k of part 1 for B with finite measure. For $A = \bigcup_k A_k$, we have $\mu(A) < \mu(B) + \epsilon$, and so A has finite measure, and we can apply continuity from above to see (note, continuity from above on the sets $A - \bigcup_{k \le 1} A_k, A - \bigcup_{k \le 2} A_k, \cdots$) there is a finite n such that:

$$\mu(A - \bigcup_{k \le n} A_k) < \epsilon$$

But then:

$$\mu\left(B \triangle \bigcup_{k \le n} A_k\right) = \mu\left(\bigcup_{k \le n} A_k - B\right) + \mu\left(B - \bigcup_{k \le n} A_k\right)$$

By finite additivity. By monotonicity:

$$\le \mu\left(A - B\right) + \mu\left(A - \bigcup_{k \le n} A_k\right) = 2\epsilon$$

And thus, we have proved part 2 as well. qed.

Just an interesting consequence. For any linear Borel Set B of finite lebesgue measure, there is a disjoint finite collection of finite intervals $A_1, \cdots, A_n$ such that:

$$\lambda\left(B \triangle \bigcup_{k=1}^{n} A_k\right) < \epsilon$$

Corollary 11.4.1 Approximation on Finite Measures If μ is a finite measure on a σ field $\mathcal{F}$ generated by a field $\mathcal{F}_0$, then for each $\mathcal{F}$ set A and each positive ϵ there is an $\mathcal{F}_0$ set B such that $\mu(A \triangle B) < \epsilon$.

Proof: An immediate consequence of the above. For every $A \in \mathcal{F}$, we have $\mu(A) < \infty$, so there are A_n $\mathcal{F}_0$ sets whose finite union symmetric difference with A is less than ϵ. However, $\mathcal{F}_0$ is a field, and so their finite union makes the B in the statement.

However, there is a simpler proof. Let $\mathcal{G}$ be the sets A with the above property, note closed under complements (as $A^c \triangle B^c = A \triangle B$), closed under countable unions, and so $\mathcal{G}$ is a σ field.

Corollary 11.4.2 Monotonicty Extends to σ field Suppose that $\mathcal{A}$ is a semiring, Ω is a countable union of $\mathcal{A}$ sets, and μ_1, μ_2 are measures on $\mathcal{F} = \sigma(\mathcal{A})$. If $\mu_1(A) \leq \mu_2(A) < \infty$ for $A \in \mathcal{A}$, then $\mu_1(B) \leq \mu_2(B)$ for $B \in \mathcal{F}$.

Proof: As μ_2 is σ finite on $\mathcal{A}$ (as $\mathcal{A}$ sets union to Ω, and all $\mathcal{A}$ sets have finite measure), Theorem 11.4 applies. Note the statement is trivial if $\mu_2(B) = \infty$. If $\mu_2(B) < \infty$, choose disjoint $\mathcal{A}$ sets A_k such that $B \subset \bigcup_k A_k$ and:

$$\sum_k \mu_2(A_k) < \mu_2(B) + \epsilon$$

Then, note:

$$\mu_1(B) \leq \sum_k \mu_1(A_k) \leq \sum_k \mu_2(A_k) < \mu_2(B) + \epsilon$$

Take ϵ to zero. qed.

Lemma 11.2 Monotonicity of Set Functions on a Semi-ring Suppose that μ is a nonnegative and finitely additive set function on a semiring $\mathcal{A}$, and let $A, A_1, \cdots, A_n$ be sets in $\mathcal{A}$.

1. If $\bigcup_{i=1}^n A_i \subset A$ and the A_i are disjoint, then $\sum_{i=1}^n \mu(A_i) \leq \mu(A)$
2. If $A \subset \cup_{i=1}^n A_i$, then $\mu(A) \leq \sum_{i=1}^n \mu(A_i)$

Proof For part 1 - make use of lemma 1 and choose disjoint $\mathcal{A}$ sets C_j such that:

$$A - \bigcup_{i=1}^n A_i = \bigcup_{j=1}^m C_j$$

As μ is finitely additive and nonnegative, it follows that:

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i) + \sum_{j=1}^{n} \mu(C_j) \geq \sum_{i=1}^{n} \mu(A_i)$$

For part 2 - take $B_1 = A \cap A_1$, and $B_i = A \cap A_i \cap A_1^c \cap \cdots \cap A_{i-1}^c$ for $i > 1$. By Lemma 1, each B_i is a finite disjoint union of $\mathcal{A}$ sets C_{ij}. Note:

$$A = \bigcup_i B_i = \bigcup_{ij} C_{ij} \qquad\qquad \cup_j C_{ij} \subset A_i$$

As the B_i are disjoint, all the C_{ij} are disjoint, and finite additivity and part 1 tells us that:

$$\mu(A) = \sum_{ij} \mu(C_{ij}) \leq \sum_i \mu(A_i)$$

Problems

11.1 Measure Extension to σ field without Theorem 11.3

The proof of Theorem 3.1 (a probability measure on a field has a unique extension to the generated σ field) applies if the probability measure is replaced by a finite measure - it really is just a matter of rescaling, and above we proved all the steps are essentially the same.

Take as a starting point then the fact that a finite measure on a field extends uniquely to the generated σ field. With the following steps, we can prove Theorem 11.2 - ie, removing the assumption of finiteness - without relying on Theorem 11.3.

1. Let μ be a measure (not necessarily even σ-finite) on a field $\mathcal{F}_0$, and let $\mathcal{F} = \sigma(\mathcal{F}_0)$. If A is a nonempty set in $\mathcal{F}_0$ and $\mu(A) < \infty$, restrict μ to a finite measure μ_A on the field $\mathcal{F}_0 \cap A$, and extend μ_A to a finite measure $\hat{\mu}_A$ on the σ field $\mathcal{F} \cap A$ generated in A by $\mathcal{F}_0 \cap A$.

   To be honest - I don't think there is a question here. We do have that $\mathcal{F}_0 \cap A$ is still a field. We note that μ_A is still a measure on that field. It is finite, as we have A is our Ω, and $\mu_A(A) = \mu(A) < \infty$. We thus have, by the starting point, that μ_A finite measure on the field $\mathcal{F}_0 \cap A$ extends uniquely to:

   $$\sigma(\mathcal{F}_0 \cap A)$$

Then, make use of Theorem 10-1. We have that $\mathcal{F}_0$ generates $\mathcal{F}$ in Ω, and so $\mathcal{F}_0 \cap A$ generates the restricted σ field $\sigma(\mathcal{F}_0) \cap A = \mathcal{F} \cap A$:

$$\sigma(\mathcal{F}_0 \cap A) = \mathcal{F} \cap A$$

2. Suppose that $E \in \mathcal{F}$. If there exist disjoint $\mathcal{F}_0$ sets A_n such that $E \subset \bigcup_n A_n$ and $\mu(A_n) < \infty$, put:

$$\hat{\mu}(E) = \sum_n \hat{\mu}_{A_n}(E \cap A_n)$$

And prove consistency. Otherwise put $\hat{\mu}(E) = \infty$.

So, suppose that we have two satisfying sequences, $E \subset \bigcup_n A_n$ and $E \subset \bigcup_n B_n$. We want to prove that $\hat{\mu}(E)$ is the same for both. We note - how is:

$$\hat{\mu}_{A_n}(E \cap A_n)$$

Defined? Well - note that $E \cap A_n \in \mathcal{F} \cap A_n$, and that $\hat{\mu}_{A_n}$ is well defined for that set - as an extension.

To be honest - I don't think this is well defined. What if we are double counting? Like I feel we should at least take an infimum of some kind. Ohh - we assume disjoint. Ok. This actually makes it easier - because I think we should consider for two such disjoint sequences A_n and B_m:

$$\hat{\mu}(E) = \sum_n \sum_m \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m)$$

We want to show that the above is equal to $\hat{\mu}(E)$ defined on A_n and B_m, in which case we will have equality between the two definitions. Mainly, I want to show:

$$\hat{\mu}_{A_n}(E \cap A_n) = \sum_m \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m)$$

Recall - $\hat{\mu}_{A_n}$ is a measure on $\mathcal{F} \cap A_n$. Note that as $B_m \in \mathcal{F}_0$, we must have that $E \cap B_m \cap A_n \in \mathcal{F} \cap A_n$. Note that we have the B_m are disjoint, and:

$$E \cap A_n = E \cap A_n \cap \bigcup_m B_m$$

And so, via countable additivity of a measure, we have:

$$\hat{\mu}_{A_n}(E \cap A_n) = \sum_m \hat{\mu}_{A_n}(E \cap A_n \cap B_m)$$

So really, we have consistency will follow, if we can show:

$$\hat{\mu}_{A_n}(E \cap A_n \cap B_m) = \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m)$$

First, examine the core definition:

$$\hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m) = \inf \sum_k \mu_{A_n \cap B_m}(D_k) = \inf \sum_k \mu(D_k)$$

Where $D_k \in \mathcal{F}_0 \cap A_n \cap B_m$, and $\bigcup_k D_k \supseteq E \cap A_n \cap B_m$. Note, we must have that:

$$D_k = D_k' \cap A_n \cap B_m \qquad D_k' \in \mathcal{F}_0$$

Thus, we have that via $\mathcal{F}_0$ being a field, $D_k' \cap B_m \in \mathcal{F}_0$, and so $D_k \in \mathcal{F}_0 \cap A_n$. Thus, every covering of $E \cap A_n \cap B_m$ in $\mathcal{F}_0 \cap A_n \cap B_m$ corresponds to a covering of $E \cap A_n \cap B_m$ in $\mathcal{F}_0 \cap A_n$, and so we must have via an infimum over a larger set:

$$\hat{\mu}_{A_n}(E \cap A_n \cap B_m) \leq \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m)$$

Now, we want to go in the opposite direction. Examine a C_k such that:

$$\sum_k \mu_{A_n}(C_k) < \hat{\mu}_{A_n}(E \cap A_n \cap B_m) + \epsilon$$

This via the definition of the infimum. Note that $\bigcup_k C_k \cap B_m$ still covers $E \cap A_n \cap B_m$, and so:

$$\hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m) \leq \sum_k \mu_{A_n \cap B_m}(C_k \cap B_m) = \sum_k \mu(C_k \cap B_m) \leq \sum_k \mu(C_k)$$

$$= \sum_k \mu_{A_n}(C_k) < \hat{\mu}_{A_n}(E \cap A_n \cap B_m) + \epsilon$$

Take ϵ to zero, and so we also have:

$$\hat{\mu}_{A_n}(E \cap A_n \cap B_m) \geq \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m)$$

Note, what we have implicitly used is that for sets in $\mathcal{F}_0 \cap A_n$ and $\mathcal{F}_0 \cap A_n \cap B_m$, $\mu_{A_n}(U) = \mu(U)$ and $\mu_{A_n \cap B_m}(U) = \mu(U)$, as the measures are just restrictions to smaller fields - ie, the values don't change. So, in all, we have:

$$\hat{\mu}_{A_n}(E \cap A_n \cap B_m) = \hat{\mu}_{A_n \cap B_m}(E \cap A_n \cap B_m) = \hat{\mu}_{B_m}(E \cap A_n \cap B_m)$$

Via symmetry. And so, we can conclude that the definition of $\hat{\mu}(E)$ is consistent.

3. Show that $\hat{\mu}$ is a measure on $\mathcal{F}$ and agrees with μ on $\mathcal{F}_0$.

   First, we show that it is a measure on $\mathcal{F}$. We need it to be countably additive, and to have the empty set be measure 0. Well, lets start with something easy. We have:

$$\hat{\mu}(\emptyset) = \hat{\mu}_{A_n}(\emptyset \cap A_n) = 0$$

   Now, countable additivity. We take disjoint $E_1, E_2, \cdots$. If any of the $\hat{\mu}(E_n) = \infty$, we have the statement is true, so assume all are finite. We examine:

$$\hat{\mu}\left(\bigcup_n E_n\right)$$

   I think, for each E_n, we can take a sequence A_{nk} that are disjoint, in $\mathcal{F}_0$, and cover E_n - that is what we get via assuming finiteness. Note that these A_{nk} are still countable, and can be listed like $A_1, A_2, \cdots$ (a countable amount of countable sets is still countable). They can be formed into a disjoint cover of $\bigcup_n E_n$. This is via taking:

$$B_1 = A_1 \qquad B_2 = A_2 \setminus A_1 \qquad B_3 = A_3 \setminus (A_1 \cup A_2)$$

   And so on. So, we have that we can find a cover of disjoint $\mathcal{A}_0$ sets that cover the disjoint union in the case that each $\hat{\mu}(E_n) < \infty$. Thus, by definition, we have:

$$\hat{\mu}\left(\bigcup_n E_n\right) = \sum_t \hat{\mu}_{B_t}\left(\bigcup_n E_n \cap B_t\right) = \sum_t \sum_n \hat{\mu}_{B_t}(E_n \cap B_t)$$

   The first equality is definitional, and the last one makes use of the fact that $\hat{\mu}$ is a measure, and so countable additivity applies for disjoint sets. Now, note that the sequence $B_1, B_2, \cdots$ is a disjoint sequence of $\mathcal{F}_0$ sets that cover each E_n - so, they can also be used to apply the definition of $\hat{\mu}$. We find:

$$= \sum_n \sum_t \hat{\mu}_{B_t}(E_n \cap B_t) = \sum_n \hat{\mu}(E_n)$$

   Where in the first equality we just switched the order of the sum (non-negative sum, fine to switch). And so, we do have countable additivity. So, in total: $\hat{\mu}$ is defined for each $E \in \mathcal{F}$, returns a number in $[0, \infty]$, is zero for the $\emptyset$, and is countably additive. So, it is a measure. Now, we just want to show that $\hat{\mu}$ agrees with μ on $\mathcal{F}_0$. Well, we have that

$\hat{\mu}$ is consistent. Note for $E \in \mathcal{F}_0$, we have that E itself is a disjoint sequence covering E. And so, we have:

$$\hat{\mu}(E) = \hat{\mu}_E(E \cap E) = \mu(E)$$

And so in conclusion: we have constructed an extension for μ on a field $\mathcal{F}_0$, even if the measure is not finite on $\mathcal{F}_0$.

11.2 Countable Subadditivity on Rings

Suppose that μ is a nonnegative and finitely additive set function on a semiring $\mathcal{A}$.

1. Use Lemmas 1 and 2, without reference to Theorem 11.3, to show that μ is countably subadditive if and only if it is countably additive.

2. Find an example where μ is not countably subadditive.

So, for the first part. We have an if and only if. First, assume that μ is countably additive on its semiring. Ie, if $A_1, A_2, \cdots$ are in $\mathcal{A}$ and disjoint, and the union is in $\mathcal{A}$ as well, we have:

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu\left(A_n\right)$$

We want to show this proves countable subadditivity. Take $B_1, B_2, \cdots$ in $\mathcal{A}$. Assume their union is within $\mathcal{A}$ as well. We have by Lemma 1:

$$B_n \cap B_1^c \cap \cdots \cap B_{n-1}^c = C_{n1} \cup \cdots \cup C_{nm_n}$$

These C form a disjoint union, whose union is B, the union of the B_i, which is in $\mathcal{A}$. Countable additivity gives us:

$$\mu\left(\bigcup_n B_n\right) = \mu\left(\bigcup_{n,i} C_{ni}\right) = \sum_{n,i} \mu(C_{ni})$$

Via finite additivity, it is clear that:

$$\sum_{i=1}^{m_n} \mu(C_{ni}) \le \mu(B_n)$$

So, that gives us:

$$\mu\left(\bigcup_n B_n\right) \le \sum_n \mu(B_i)$$

So countable additivity implies countable subadditivity. Now, go the other direction. Assume we have countable subadditivity. Thus, to show equality, we really just need to show for disjoint A_n, we have:

$$\mu\left(\bigcup_n A_n\right) \geq \sum_n \mu\left(A_n\right)$$

This can come pretty directly from Lemma 2. We have that $A = \bigcup_n A_n$. We have that $\bigcup_{n=1}^k A_n \subseteq A$ clearly, and so by lemma 2 part 1 (as the A_n are disjoint), we have:

$$\sum_{n=1}^{k} \mu\left(A_n\right) \leq \mu\left(\bigcup_n A_n\right)$$

Now, just take a limit of $k \to \infty$ (we can take a limit on the sum), which will allow us to conclude countable additivity:

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu\left(A_n\right)$$

qed. We now go to the second part - find an example of a semiring $\mathcal{A}$ where μ is not countably subadditive. One simple semiring is the half open finite intervals on the real line $\mathbb{R}^1$. The intersection of two intervals is clearly a finite interval, and the set minus of finite intervals is a union of disjoint intervals. We are looking for a nonnegative, finitely additive, but not countably subadditive set function on this $\mathcal{A}$.

I will actually try a different ring - inspired by question 10.2. Let $\mathcal{A}$ consist of subsets of $\Omega = \{1, 2, \cdots\}$, being intervals with no jumps - ie, we can not have something like $\{x, x+2\}$ - if $x, y \in I$, then every integer between x and y must also be in the interval. We clearly contain intersections (as if x, y are in both I_1, and I_2, each number between them is in the intersection) and setminuses are disjoint unions of sets in $\mathcal{A}$.

We let $\mu(I) = \sum_{k \in I} 2^{-k}$ if I is finite and $\mu(I) = \infty$ if it contains a non finite amount of elements. It is clear that μ is finitely additive and non-negative, similar to what was proved in 10.2. However, μ is not countably additive. Take sets $I_n = \{n\}$. There disjoint union is within $\mathcal{A}$, but the disjoint sum is 1, whereas the set function on the union is infinity.

Now, we note by the above theorem - this implies that μ is also not countably subadditive. This can be the seen with the same example of $I_n = \{n\}$, as we have:

$$\infty = \mu(\cup_n I_n) \geq \sum_n \mu(I_n) = 1$$

11.4 - Function Lattice Semi-ring + Measure: Riesz Representation Part 1

Let Λ be a real linear functional on a vector lattice $\mathcal{L}$ of (finite) real functions on a space Ω.

Let us unpack what this means. First, what is $\mathcal{L}$. It is a vector space of finite real valued functions. If $f \in \mathcal{L}$, we have $f : \Omega \to \mathbb{R}$, and $f(\omega) < \infty$ for $\omega \in \Omega$. We also have that $\mathcal{L}$ is closed under addition of functions, and scalar multiplication - this comes from being a vector space.

We now go on to what a lattice means. In general, it means for each pair of elements of $\mathcal{L}$, we have a join and meet value also contained within $\mathcal{L}$, where the join is greater than the two, and the meet is less than the two. Less abstractly, we have for $f, g \in \mathcal{L}$:

$$f \vee g : \Omega \to \mathbb{R} \qquad (f \vee g)(\omega) = \max(f(\omega), g(\omega)) \qquad f \vee g \in \mathcal{L}$$

$$f \wedge g : \Omega \to \mathbb{R} \qquad (f \wedge g)(\omega) = \min(f(\omega), g(\omega)) \qquad f \wedge g \in \mathcal{L}$$

Now, we also have that Λ is a real linear functional. A functional is a map from a vector space V to its field. So, $\Lambda : \mathcal{L} \to \mathbb{R}$. We also have that Λ is linear, and so:

$$\Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g)$$

Assume further of $\mathcal{L}$ that $f \in \mathcal{L}$ implies $f \wedge 1 \in \mathcal{L}$ - note, I think that comes from the lattice definition. I guess, $1 \in \mathcal{L}$ might not actually be immediate from the vector space definition (we have a scalar 1, but not an element 1). Assume further of Λ that it is positive in the sense that $f \geq 0$ (pointwise) implies $\Lambda(f) \geq 0$ and continuous from above at 0 in the sense that $f_n \downarrow 0$ (pointwise) implies $\Lambda(f_n) \to 0$. Those are indeed extra conditions on Λ.

1. If $f \leq g$ and $f, g \in \mathcal{L}$, define $\Omega \times \mathbb{R}$ an "interval":

   $$(f, g] = \{(\omega, t) : f(\omega) < t \leq g(\omega)\}$$

   Show that these sets form a semiring $\mathcal{A}_0$.

   To be a semi ring - we need finite intersections, set minus (of nested sets) equal to a disjoint union of elements in $\mathcal{A}_0$, and $\emptyset \in \mathcal{A}_0$.

First, note that $f \leq f$, and so:

$$(f, f] = \{(\omega, t) : f(\omega) < t \leq f(\omega)\} = \emptyset \in \mathcal{A}_0$$

Now, consider intersections. We have $f_1 \leq g_1$ and $f_2 \leq g_2$. We have:

$$(f_1, g_1] \cap (f_2, g_2] = \{(\omega, t) : f_1(\omega) < t \leq g_1(\omega)\} \cap \{(\omega, t) : f_2(\omega) < t \leq g_2(\omega)\}$$

Take (ω, t) in the intersection. We have that $f_1(\omega) < t \leq g_1(\omega)$, and $f_2(\omega) < t \leq g_2(\omega)$. Thus, t is bigger than the maximum of the f, and smaller than (or equal to) the minimum of the g. And so, we have:

$$(\omega, t) \in (f_1 \vee f_2, g_1 \wedge g_2]$$

We can similarly go in the other direction to show equality. And so, as $f_1 \vee f_2, g_1 \wedge g_2 \in \mathcal{L}$, we have:

$$(f_1, g_1] \cap (f_2, g_2] = (f_1 \vee f_2, g_1 \wedge g_2] \in \mathcal{A}_0$$

The last item we need to show is the set minus property. Suppose that $(f_1, g_1] \subseteq (f_2, g_2]$. Examine:

$$(f_2, g_2] - (f_1, g_1] = \{(\omega, t) : f_2(\omega) < t \leq g_2(\omega)\} - \{(\omega, t) : f_1(\omega) < t \leq g_1(\omega)\}$$

Take an (ω, t) in the above set. We know that (ω, t) satisfies $f_2(\omega) < t \leq g_2(\omega)$, but we also must have either $t \leq f_1(\omega)$, or $t > g_1(\omega)$. In the first case, we have that:

$$(\omega, t) \in (f_2, g_2 \wedge f_1]$$

As $f_2(\omega) < t \leq \min(g_2(\omega), f_1(\omega))$, as t is less than or equal to both items in the minimum. In the other case, we have that:

$$(\omega, t) \in (f_2 \vee g_1, g_2]$$

Now, note that both of these sets are disjoint. That is because, if $t > g_1(\omega)$, we have $t > f_1(\omega)$, as $g_1 \geq f_1$. Similarly, if $t \leq f_1$, we must have $t \leq g_1$. And so, we have:

$$(f_2, g_2] - (f_1, g_1] = (f_2, g_2 \wedge f_1] \cup (f_2 \vee g_1, g_2]$$

Note, we only showed $\subseteq$, but the other direction is clear as well.

2. Define a set function ν_0 on $\mathcal{A}_0$ by:

$$\nu_0(f, g] = \Lambda(g - f)$$

Show that ν_0 is finitely additive and countably subadditive on $\mathcal{A}_0$ (note, proving this will allow us to make use of Theorem 11.3). Take finite disjoint $(f_1, g_1], \cdots, (f_n, g_n]$. Examine:

$$\nu_0\left(\bigcup_{i=1}^{n}(f_i, g_i]\right)$$

To be honest, we only have a definition for ν_0 when there is a single function. Finite additivity for a semiring is predicated on the fact that we have:

$$\bigcup_{i=1}^{n}(f_i, g_i] = (f, g]$$

Similar, for countable subadditivity, we need:

$$\bigcup_{i}(f_i, g_i] = (f, g]$$

And, we want to prove:

$$\nu_0(f, g] \leq \sum_i \nu_0(f_i, g_i]$$

Now, consider the case where $(f, g] \subseteq \bigcup_i (f_i, g_i]$ (this includes the countable additivity case). Then, we know for all ω:

$$(f(\omega), g(\omega)] \subseteq \bigcup_i (f_i(\omega), g_i(\omega)]$$

This is clear from the definitions. Theorem 1.3 directly tells us that we thus have:

$$g(\omega) - f(\omega) \leq \sum_i g_i(\omega) - f_i(\omega)$$

Define:

$$h_n = \left[g - f - \sum_{i \leq n} g_i - f_i\right] \vee 0$$

Note that as for each ω, we have $\lim_{n\to\infty} h_n(\omega) = 0$. The pointwise limit of the first functions is ≤ 0, while the $\vee 0$ ensures we don't go below zero. We also have:

$$g - f \leq \left[\sum_{i \leq n} g_i - f_i\right] + h_n$$

Again, it would be equality if we didn't have the $\vee$, but the $\vee$ increases the function giving us the above inequality. Now, we have continuity of Λ gives us:

$$\Lambda(h_n) \to 0$$

With these facts, we will prove countable subadditivity. We have that:

$$\nu_0\left(\bigcup_i (f_i, g_i]\right) = \nu_0\left((f, g]\right) = \Lambda(g - f)$$

Now, as $g - f \leq \left[\sum_{i \leq n} g_i - f_i\right] + h_n$ for all n, we have for all n:

$$\Lambda(g - f) \leq \Lambda\left[\left[\sum_{i \leq n} g_i - f_i\right] + h_n\right]$$

As for why that is. Take functions p, q, where $p \leq q$. Then, $q - p \geq 0$, which implies $\Lambda(q - p) \geq 0$, which implies $\Lambda(q) - \Lambda(p) \geq 0$ (via linearity) which implies $\Lambda(q) \geq \Lambda(p)$. By linearity, we have:

$$\Lambda\left[\left[\sum_{i \leq n} g_i - f_i\right] + h_n\right] = \sum_{i \leq n} \Lambda\left[g_i - f_i\right] + \Lambda\left[h_n\right] = \sum_{i \leq n} \nu_0(f_i, g_i] + \Lambda\left[h_n\right]$$

And so, for all n, we have:

$$\nu_0\left(\bigcup_i (f_i, g_i]\right) \leq \sum_{i \leq n} \nu_0(f_i, g_i] + \Lambda\left[h_n\right]$$

Take a $\lim_{n \to \infty}$ on both sides, note that $\lim_{n \to \infty} \Lambda(h_n) = 0$, and conclude:

$$\nu_0\left(\bigcup_i (f_i, g_i]\right) \leq \sum_i \nu_0(f_i, g_i]$$

Now, we consider finite additivity. We have for disjoint intervals:

$$\bigcup_i (f_i, g_i] = (f, g]$$

By Theorem 1.3, we have as the $(f_i, g_i]$ are disjoint:

$$g(\omega) - f(\omega) = \sum_{i=1}^{n} g_i(\omega) - f_i(\omega)$$

This equality is for all ω, and so the functions are equal. As Λ is linear, we have:

$$\nu_0(f, g] = \Lambda(g - f) = \Lambda\left(\sum_{i=1}^{n} g_i - f_i\right) = \sum_{i=1}^{n} \Lambda(g_i - f_i)$$

Thus, we have proved finite additivity, and countable subadditivity.

11.5: Riesz Representation Part 2

Note, this problem is essentially a continuation of the previous problem. Use the previous definitions.

1. Assume $f \in \mathcal{L}$ and let $f_n = (n(f - f \wedge 1)) \wedge 1$. Show that $f(\omega) \leq 1$ implies $f_n(\omega) = 0$ for all n and $f(\omega) > 1$ implies $f_n(\omega) = 1$ for all sufficiently large n. Conclude that for $x > 0$:

   $$(0, xf_n] \uparrow [\omega : f(\omega) > 1] \times (0, x]$$

   Assume $f(\omega) \leq 1$. Then, we have $f(\omega) \wedge 1 = f(\omega)$. $f(\omega) - f(\omega) = 0$, $n(0) = 0$, and $0 \wedge 1 = 0$. Thus, if $f(\omega) \leq 1$, we have:

   $$f_n(\omega) = 0$$

   For all n. Now, assume $f(\omega) > 1$. Then, $f(\omega) \wedge 1 = 1$. And so:

   $$f_n(\omega) = n(f(\omega) - 1) \wedge 1$$

   I think it is clear, that for $n > 0$, 1 will be the minimum, and so $f_n(\omega) = 1$ for sufficiently large n. Now, take $x > 0$. We examine:

   $$(0, xf_n] = \{(\omega, t) : 0 < t \leq xf_n(\omega)\}$$

   If $f(\omega) \leq 1$, then there are no satisfying (ω, t). If $f(\omega) > 1$, then for n large enough, the right hand side $xf_n(\omega) \to 1$. Note ,as n increases, xf_n can only increase. And so, in terms of sets, we have as $n \to \infty$:

   $$(0, xf_n] \uparrow [\omega : f(\omega) > 1] \times (0, x]$$

2. Let $\mathcal{F}$ be the smallest σ-field with respect to which every f in $\mathcal{L}$ is measurable: $\mathcal{F} = \sigma\left[f^{-1}H : f \in \mathcal{L}, H \in \mathcal{R}^1\right]$. Note, f is measurable on a σ field if $f^{-1}(H)$ is within that sigma field, for all $H \in \mathcal{R}^1$. Note, we didn't worry about $\mathcal{R}^1$ earlier with Simple Random Variables, because we only really had to consider finite element $H \subseteq \mathbb{R}$. So, $\mathcal{F}$ is the σ field generated by all the sets $f^{-1}(H), f \in \mathcal{L}, H \in \mathcal{R}^1$.

   Let $\mathcal{F}_0$ be the class of A in $\mathcal{F}$ for which $A \times (0, 1] \in \sigma(\mathcal{A}_0)$. Show that $\mathcal{F}_0$ is a semiring and that $\mathcal{F} = \sigma(\mathcal{F}_0)$. Recall, $\mathcal{A}_0$ is the semiring defined in the previous problem.

   First note that $\mathcal{F}_0$ is a semiring on Ω. Functions $f : \Omega \to \mathbb{R}$, and so $f^{-1}(H)$ for $H \in \mathcal{R}^1$ are subsets of Ω. Recall, $\mathcal{A}_0$ is the sets of the form $(f, g]$, for $f, g \in \mathcal{L}$ and $f \leq g$. So, $\mathcal{F}_0$ is sets of $A \in \mathcal{F}$ (where $A \subseteq \Omega$) such that:

$$A \times (0, 1] \in \sigma(\mathcal{A}_0)$$

First, we want to show the empty set is contained within $\mathcal{F}_0$. Do we have:

$$\emptyset \times (0, 1] \in \sigma(\mathcal{A}_0)$$

Note, $\emptyset \times (0, 1] = \emptyset$, which we already determined is within $\mathcal{A}_0$, as $\mathcal{A}_0$ is a semiring. Now, take $A, B \in \mathcal{F}_0$. We want to show:

$$(A \cap B) \times (0, 1] \in \sigma(\mathcal{A}_0)$$

Well, both A and B satisfy:

$$A \times (0, 1] \in \sigma(\mathcal{A}_0) \qquad B \times (0, 1] \in \sigma(\mathcal{A}_0)$$

As $\sigma(\mathcal{A}_0)$ is a sigma algebra, we have:

$$[A \times (0, 1]] \cap [B \times (0, 1]] = [A \cap B \times (0, 1]] \in \sigma(\mathcal{A}_0)$$

Also note that as $\mathcal{F}$ is a σ field, $A \cap B \in \mathcal{F}$.

So now, the final ring property would be for $A \subset B \in \mathcal{F}_0$, we have disjoint $C_1, \cdots, C_n \in \mathcal{F}_0$ such that:

$$B - A = C_1 \cup \cdots \cup C_n$$

Note that as $\sigma(\mathcal{A}_0)$ is a σ algebra, we have:

$$(B - A) \times (0, 1] = [B \times (0, 1]] - [A \times (0, 1]] \in \sigma(\mathcal{A}_0)$$

So note, we have that $B - A \in \mathcal{F}$, and the above property implies $B - A \in \mathcal{F}_0$. So, $\mathcal{F}_0$ is actually closed under proper differences - which is enough to prove our semiring fact.

Now, we want to prove that $\mathcal{F} = \sigma(\mathcal{F}_0)$. Note that $\mathcal{F}_0 \subseteq \mathcal{F}$. And so, if we have:

$$\mathcal{F} \subseteq \sigma(\mathcal{F}_0)$$

Then equality must follow - because if we didn't have equality, then $\mathcal{F}$ would be a smaller σ algebra containing $\mathcal{F}_0$ than $\sigma(\mathcal{F}_0)$, which would be a contradiction. First note - for $f \in \mathcal{L}$, we have that:

$$(0, f_n] \uparrow [\omega : f(\omega) > 1] \times (0, 1]$$

That is by the previous part. Note that $[\omega : f(\omega) > 1] \in \mathcal{F}$. Now, we want to show that:

$$[\omega : f(\omega) > 1] \times (0,1] \in \sigma(\mathcal{A}_0)$$

As that will imply $[\omega : f(\omega) > 1] \in \mathcal{F}_0$. Well, $[\omega : f(\omega) > 1] \times (0,1]$ is the union of sets $(0, f_n] \in \mathcal{A}_0$, and so yes, we do have the above fact. Note that $(0, f_n] \in \mathcal{A}_0$, by definition, and so the union is within $\sigma(\mathcal{A}_0)$. And so, $[\omega : f(\omega) > 1] \times (0,1]$ is within $\sigma(\mathcal{A}_0)$, which means $[\omega : f(\omega) > 1] \in \mathcal{F}_0$. And so, we can conclude for all $f \in \mathcal{L}$, we have:

$$[\omega : f(\omega) > 1] \in \mathcal{F}_0$$

Now, we note that $xf \in \mathcal{L}$ and $-xf \in \mathcal{L}$ for $x > 0$. This is because $\mathcal{L}$ is a vector space, and $f + \cdots + f \in \mathcal{L}$. We can make replace f in the above argument with xf to find: Thus, we can also conclude:

$$[f > x] \in \mathcal{F}_0 \qquad [f > -x] \in \mathcal{F}_0 \implies [\omega : f(\omega) > -x] \in \mathcal{F}_0 \implies [\omega \in f^{-1}(-x, \infty)] \in \mathcal{F}_0$$

Note that the sets $(-x, \infty)$ and (x, ∞) generate $\mathcal{R}^1$. Any finite interval of the form $(x, y]$ can be formed by a finite amount of intersections, unions, and complements of the given form - and those intervals generate $\mathcal{R}^1$. We let B be the sets of the form $(-x, \infty), (x, \infty)$ for $x > 0$. We thus have that:

$$f^{-1}(B) \subseteq \mathcal{F}_0 \implies \sigma(f^{-1}(B)) \subseteq \sigma(\mathcal{F}_0)$$

By above, we have $\sigma(B) = \mathcal{R}^1$. And so, by the pullback lemma below (proved as the following problem) - we have that:

$$f^{-1}(\sigma(B)) = \sigma(f^{-1}(B)) \subseteq \sigma(\mathcal{F}_0)$$

Now, recall what we are trying to prove. We want to show that:

$$\mathcal{F} = \sigma\left[\left\{f^{-1}(H) : f \in \mathcal{L}, H \in \mathcal{R}^1\right\}\right] \subseteq \sigma(\mathcal{F}_0)$$

Note, we kind of are approaching this statement. We have that $f^{-1}(\mathcal{R}^1)$, for a single f, is a σ algebra, and for each $f \in \mathcal{L}$, we have $f^{-1}(\mathcal{R}^1) \subseteq \sigma(\mathcal{F}_0)$. We thus have that:

$$\bigcup_{f \in \mathcal{L}} f^{-1}(\mathcal{R}^1) \subseteq \sigma(\mathcal{F}_0)$$

But we don't necessarily have that this union is a σ algebra. What we want to prove is:

$$\mathcal{F} = \sigma\left[\bigcup_{f \in \mathcal{L}} f^{-1}(\mathcal{R}^1)\right] \subseteq \sigma(\mathcal{F}_0)$$

However, note that this is the wrong way of looking at it. We can equivalently prove:

$$\mathcal{F} = \sigma\left[\bigcup_{f \in \mathcal{L}} \sigma(f^{-1}(B))\right] \subseteq \sigma(\mathcal{F}_0)$$

And this will follow from our "Generated Sigma Algebra of Union of Generated Sigma Algebras Lemma", which follows as a problem. By that problem, we have:

$$\sigma\left[\bigcup_{f \in \mathcal{L}} \sigma(f^{-1}(B))\right] = \sigma\left[\bigcup_{f \in \mathcal{L}} f^{-1}(B)\right]$$

As noted above, we have:

$$\bigcup_{f \in \mathcal{L}} f^{-1}(B) \subseteq \mathcal{F}_0$$

And so, by the Union of Sigma Algebras Lemma, we have:

$$\implies \sigma\left[\bigcup_{f \in \mathcal{L}} f^{-1}(B)\right] \subseteq \mathcal{F}_0 \implies \sigma\left[\bigcup_{f \in \mathcal{L}} \sigma(f^{-1}(B))\right] \subseteq \sigma(\mathcal{F}_0)$$

$$\implies \sigma\left[\bigcup_{f \in \mathcal{L}} f^{-1}(\mathcal{R})\right] \subseteq \sigma(\mathcal{F}_0) \implies \mathcal{F} \subseteq \sigma(\mathcal{F}_0)$$

And thus, we have proved that $\mathcal{F}_0$ is a semiring, and that $\mathcal{F} = \sigma(\mathcal{F}_0)$.

3. Let ν be the extension of ν_0 (from the previous problem) to $\sigma(\mathcal{A}_0)$, and for $A \in \mathcal{F}_0$, define:

   $$\mu_0(A) = \nu(A \times (0, 1])$$

   Show that μ_0 is finitely additive and countably subadditive on the semiring $\mathcal{F}_0$.

   First - recall that ν_0 is finitely additive and countably subadditive on

$\mathcal{A}_0$, which is a semiring. Further $\nu_0(\emptyset) = 0$, and is nonnegative, and so by Theorem 11.3, it extends to a measure on $\sigma(\mathcal{A}_0)$. This is what we let ν be. Note - given that $\mathcal{A}_0$ is a semiring, I don't think we have uniqueness of ν.

Now, consider μ_0, on $\mathcal{F}_0$, which we proved is a semiring. First, note that μ_0 is nonnegative, as ν is nonnegative, and:

$$\mu_0(\emptyset) = \nu(\emptyset \times (0,1]) = \nu(\emptyset) = 0$$

So, we would have all the prerequisites for extending μ_0 to $\sigma(\mathcal{F}_0) = \mathcal{F}$ if we can prove that μ_0 is finitely additive and countably subadditive on $\mathcal{F}_0$. Take disjoint $A_1, \cdots, A_n \in \mathcal{F}_0$. Assume that their union is A, and $A \in \mathcal{F}_0$ as well. We have:

$$\mu_0[A] = \mu_0\left[\bigcup_{i=1}^n A_i\right] = \nu\left[\left[\bigcup_{i=1}^n A_i\right] \times (0,1]\right] = \nu\left[\bigcup_{i=1}^n [A_i \times (0,1]]\right]$$

First - note this is well defined. As $A_i \in \mathcal{F}_0$, we have $A_i \times (0,1] \in \sigma(\mathcal{A}_0)$ - by definition. As $\sigma(\mathcal{A}_0)$ is a sigma algebra, their union is contained within the σ algebra as well, and has a measure defined by ν. Now, as ν is countably additive (being a measure), we have:

$$= \sum_{i=1}^n \nu[A_i \times (0,1]] = \sum_{i=1}^n \mu_0(A_i)$$

We can use a similar proof to prove countable subadditivity:

$$\mu_0[A] = \mu_0\left[\bigcup_i A_i\right] = \nu\left[\left[\bigcup_i A_i\right] \times (0,1]\right] = \nu\left[\bigcup_i [A_i \times (0,1]]\right]$$

$$\leq \sum_i \nu[A_i \times (0,1]] = \sum_i \mu_0(A_i)$$

And thus, μ_0 is a finitely additive and countably subadditive set function on the semi ring $\mathcal{F}_0$. qed.

Pullback Lemma For Inverse Functions and Generated Sigma Algebras

The lemma is this. Let f be a function from $\Omega \to X$. Let $C \subseteq 2^X$ be any class of sets on X. We have:

$$\sigma(f^{-1}(C)) = f^{-1}(\sigma(C))$$

Where σ denotes the generated sigma algebras. We first show:

$$\sigma(f^{-1}(C)) \subseteq f^{-1}(\sigma(C))$$

First note. $f^{-1}(\sigma(C))$ is a σ algebra. It contains the emptyset - $f^{-1}(\emptyset)$. It contains complements. Take $A = f^{-1}(H)$ for $H \in \sigma(C)$ - note $H^c \in \sigma(C)$, and $A^c = f^{-1}(H^c) \in f^{-1}(\sigma(C))$. Finally, note that $f^{-1}(\sigma(C))$ is closed under countable unions. If we have:

$$A_1 = f^{-1}(H_1) \quad A_2 = f^{-1}(H_2) \quad \cdots \text{note:} \qquad H_1 \cup H_2 \cup \cdots \in \sigma(C)$$

And we clearly have:

$$A_1 \cup A_2 \cup \cdots = f^{-1}(H_1 \cup H_2 \cup \cdots)$$

So, we have $f^{-1}(\sigma(C))$ is a σ algebra. Now, note that $f^{-1}(C) \subseteq f^{-1}(\sigma(C))$ - this is clear. And so, by the definition of σ yielding the smallest sigma algebra, we must have:

$$\sigma(f^{-1}(C)) \subseteq f^{-1}(\sigma(C))$$

Now, we want to prove:

$$f^{-1}(\sigma(C)) \subseteq \sigma(f^{-1}(C))$$

Consider:

$$D = \left\{H \subseteq X : f^{-1}(H) \in \sigma(f^{-1}(C))\right\}$$

First, note that $C \subseteq D$. For $H \in C$, we have $f^{-1}(H) \in \sigma(f^{-1}(C))$, clearly. Now, we note that D is a σ algebra. We have D contains the emptyset - as $f^{-1}(\emptyset) = \emptyset \in \sigma(f^{-1}(C))$. We have that D is closed under intersections - note that if $H \in D$, we have:

$$A = f^{-1}(H) \in \sigma(f^{-1}(C)) \implies f^{-1}(H^c) = A^c \in \sigma(f^{-1}(C)) \implies H^c \in D$$

And we have D is closed under countable unions. For $H_1, H_2, \cdots \in D$, we have:

$$A_1 = f^{-1}(H_1) \in \sigma(f^{-1}(C)), A_2 = f^{-1}(H_2) \in \sigma(f^{-1}(C)), \cdots$$

$$\implies f^{-1}(H_1 \cup H_2 \cup \cdots) = A_1 \cup A_2 \cup \cdots \in \sigma(f^{-1}(C)) \implies H_1 \cup H_2 \cup \cdots \in D$$

So, D is a σ algebra that contains C. Thus, $\sigma(C) \subseteq D$. And so:

$$f^{-1}(\sigma(C)) \subseteq f^{-1}(D) \subseteq \sigma(f^{-1}(C))$$

Where the last step is definitional - applying f^{-1} on the sets in D yields the sets $f^{-1}(H) \in \sigma(f^{-1}(C))$. And so, we have:

$$\sigma(f^{-1}(C)) \subseteq f^{-1}(\sigma(C)) \text{ and } f^{-1}(\sigma(C)) \subseteq \sigma(f^{-1}(C)) \implies \sigma(f^{-1}(C)) = f^{-1}(\sigma(C))$$

And thus we have the pullback lemma for inverse functions and generated sigma algebras. qed.

Generated Sigma Algebra of Union of Generated Sigma Algebras Lemma

Note that for any class of families $\{S_i\}_{i\in\mathcal{I}}$, we have:

$$\sigma\left[\bigcup_{i\in\mathcal{I}}\sigma(S_i)\right] = \sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right]$$

One direction is immediate: as $\bigcup_{i\in\mathcal{I}}S_i \subseteq \bigcup_{i\in\mathcal{I}}\sigma(S_i)$, we have:

$$\sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right] \subseteq \sigma\left[\bigcup_{i\in\mathcal{I}}\sigma(S_i)\right]$$

Now, we go the other direction. We want to prove:

$$\sigma\left[\bigcup_{i\in\mathcal{I}}\sigma(S_i)\right] \subseteq \sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right]$$

Note that:

$$\sigma(S_i) \subseteq \sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right]$$

As the RHS is included in the intersection definition of $\sigma(S_i)$. This is true for each i, and so:

$$\bigcup_{i\in\mathcal{I}}\sigma(S_i) \subseteq \sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right]$$

Now, this implies:

$$\sigma\left[\bigcup_{i\in\mathcal{I}}\sigma(S_i)\right] \subseteq \sigma\left[\bigcup_{i\in\mathcal{I}}S_i\right]$$

As the LHS is the smallest sigma algebra containing the inner union. And so, we have proved equality. qed.

Section 12 - Measures in Euclidean Space

Notes

Lebesgue Measure

We have that the finite intervals on the real line are a semiring. With $\lambda_1(a, b] = b - a$, we have a finite and countable set function on this semiring (Theorem 1.3). By Theorem 11.3, this extends to a measure λ on $\mathcal{R}^1$.

By Theorem 10.3, we have that this is the only measure on $\mathcal{R}^1$ such that $\lambda(a,b] = b-a$. This is because a semiring is a π system first, and because second, there is a covering of Ω by finite measure sets in our semi ring (and so the semi ring is σ finite as well).

In $\mathbb{R}^k$, there is an analogous k dimensional Lebesgue measure λ_k on the class $\mathcal{R}^k$ of k dimensional borel sets. Example 11.4 shows us that the class of bounded rectangles is indeed a semiring. However - I'm not sure if we have been given a set function yet. I think this chapter is so that we can build up to a measure on $\mathcal{R}^k$, starting off with the semiring of bounded rectangles.

λ_k will be our k dimensional Lebesgue measure λ_k on the class $\mathcal{R}^k$ of k dimensional borel sets. It is specified by the requirement that bounded rectangles have measure:

$$\lambda_k\left[x : a_i < x_i \leq b_i, i = 1, \cdots, k\right] = \prod_{i=1}^{k}(b_i - a_i)$$

This is ordinary volume of a bounded rectangle. Length if $k=1$, area if $k=2$, volume if $k=3$, or hypervolume if $k \geq 4$.

Note that the intersection of rectangles is indeed a rectangle - a semiring is also a π system. So, if we had a measure on $\mathcal{R}^k$ - the uniqueness Theorem in 10.3 would allow us to conclude that the value of λ_k on the rectangles completely determines its value on all of $\mathcal{R}^k$ (as the rectangles generate $\mathcal{R}^k$, and we have that Ω is σ finite over the rectangle values). However - we still need to prove existence of a measure on $\mathcal{R}^k$.

One way to do this would be via Theorem 11.3, I guess - prove that λ_k on the rectangles is finitely additive and countably subadditive. A second construction is given in this chapter. A third, independent construction can be done via the general Theory of product measures (section 18). For the moment, assume existence on $\mathcal{R}^k$ of a measure λ_k that agrees with the set function on the semiring above - ie, the hypervolume definition for bounded rectangles.

Theorem 12.1 - λ_k is Translation Invariant If $A \in \mathcal{R}^k$,then $A + x = \{a + x : a \in A\} \in \mathcal{R}^k$ and $\lambda_k(A) = \lambda_k(A + k)$.

Proof: If $\mathcal{G}$ is the class of A such that $A + x$ is in $\mathcal{R}^k$ for all x, then $\mathcal{G}$ is a σ field containing the bounded rectangles. Clearly contains $\emptyset$. We have it contains complements:

$$A^c + x = (A + x)^c \in \mathcal{R}^k \text{ for all } x \implies A^c \in \mathcal{G}$$

And countably unions

$$(\bigcup_k A_k) + x = \bigcup_k (A_k + x) \in \mathcal{R}^k \text{ for all } x \implies \bigcup_k A_k \in \mathcal{G}$$

And so, by definition of generation, $\mathcal{R}^k \subseteq \mathcal{G}$. As $\mathcal{G} \subseteq \mathcal{R}^k$ (as $\mathcal{G}$ only contains elements of $\mathcal{R}^k$), we have equality. So, we note that $\mathcal{R}^k$ contains all translations of sets within itself.

For fixed x define a measure μ on $\mathcal{R}^k$ by $\mu(A) = \lambda_k(A + x)$. First - note that μ is defined on all of $\mathcal{R}^k$. Second, note that μ must be a measure, as we can take the measure properties from λ_k, and give them to μ. Finally, note that for a bounded rectangle B, we have:

$$\mu(B) = \lambda_k(B + x) = \lambda_k(B)$$

As it is clear, for the definition of λ_k on bounded rectangles, λ_k is translation invariant on the bounded rectangles. Now, as the bounded rectangles both generate $\mathcal{R}^k$, and are σ finite π-systems - Theorem 10.3 implies μ and λ_k agree on all of $\mathcal{R}^k$. This can be proved for all x. Thus, for any $A \in \mathcal{R}^k$, and any x, we have:

$$\lambda_k(A + x) = \mu(A) = \lambda_k(A)$$

And thus, we have that λ_k is translation invariant on all of $\mathcal{R}^k$. qed.

Measure of $k-1$ Dimensional Hyperplanes in $\mathcal{R}^k$ Proof: Now, we examine subspaces. Assume that A is a $(k-1)$ dimensional subspace of $\mathcal{R}^k$. So, a vector space. Assume x lies outside of A. Note that the hyperplanes:

$$A + tx$$

Are disjoint for real t. If $A \in \mathcal{R}^k$ - by Theorem 12.1, we have that for each t, $A+tx$ has the same measure. Note that λ_k is σ finite on $\mathcal{R}^k$ - and so Theorem 10.2 implies that $\mathcal{R}^k$ cannot contain an uncountable, disjoint collection of sets of positive λ_k measure. And so if A had positive measure, or any of the $A + tx$ had positive measure - translation invariance would imply that each of the uncountable disjoint sets would have positive measure, which would contradict Theorem 10.2.4. And so, the common measure of each $A+tx$ must be zero. Thus, we can conclude that *every $k-1$ dimensional hyperplane has k dimensional Lebesgue measure 0.*

Theorem 12.2 Measure λ_k across linear transformations If $T : \mathbb{R}^k \to \mathbb{R}^k$ is linear and nonsingular, then $A \in \mathcal{R}^k$ implies that $TA \in \mathcal{R}^k$ and:

$$\lambda_k(TA) = |\det T| \cdot \lambda_k(A)$$

Note: before we begin the proof. This is giving me vibes similar to change of variables. We just have different notions of volumes - and in all cases, the determinant seems to pop up, when we compare volumes of different spaces under linear transformations.

Just a funny idea. Say we want to calculate the determinant of a linear transformation. We could just describe a volume with n sample points on the surface, examine their location under T, and then try and find an approximate volume of the space spanned by the mapped points.

Also note what this theorem is telling us. If $\det T = \pm 1$ - such as the case for orthogonal (rotation) or unitary (reflection) matrices - then the volume of TA is the same as that for A. Thus, rigid transformations or isometries (which are orthogonal transformations followed by a translation) - preserve Lebesgue measure, when we combine Theorems 12.1 and 12.2. An affine transformation has the form:

$$Fx = Tx + x_0$$

It is nonsingular (invertible) if T is. This is clear, as a translation is easily invertible. It follows by Theorems 12.1 and 12.2 that $\lambda_k(FA) = |\det T| \cdot \lambda_k(A)$ in the non singular case.

Proof of Theorem 12.2 First note, non singularity of T allows us to conclude:

$$T \bigcup_n A_n = \bigcup_n TA_n \qquad TA^c = (TA)^c$$

And so, the class $\mathcal{G} = \left[A : TA \in \mathcal{R}^k\right]$ is a σ field. We now want to show that it is equal to $\mathcal{R}^k$, which comes from showing a generating set is contained in $\mathcal{G}$. Since TA is open for open A (if T is nonsingular, T is continuous, we can find open balls etc), it follows that $\mathcal{G}$ contains the open sets. The open sets generate $\mathcal{R}^k$, and so we can conclude:

$$\mathcal{G} = \mathcal{R}^k \text{ and so } A \in \mathcal{R}^k \implies TA \in \mathcal{R}^k$$

Actually - I don't think this is correct. If A is open, TA is indeed open - but why does A being open imply that $TA \in \mathcal{R}^k$? Oh, because TA is also open,

$TA \in \mathcal{R}^k$ via all open sets being in $\mathcal{R}^k$, and so $A \in \mathcal{G}$ definitionally. So we can continue on.

Now, we need to prove:

$$\lambda_k(TA) = |\det T| \cdot \lambda_k(A)$$

For $A \in \mathcal{R}^k$, set $\mu_1(A) = \lambda_k(TA)$ and $\mu_2(A) = |\det T| \cdot \lambda_k(A)$. Then μ_1 and μ_2 are measures, and by Theorem 10.3 they will agree on $\mathcal{R}^k$ if they agree on the π-system of rational bounded rectangles (which, recall, generate $\mathcal{R}^k$, and $\mathcal{R}^k$ is σ finite on this set). This setup is the same as how we prove Theorem 12.2. Note, μ_1 and μ_2 being measures comes directly from λ_k being a measure.

So, we want to prove for rectangles A with rational sides, $\mu_1(A) = \mu_2(A)$. Now, note that any such rational rectangle is a finite disjoint union of translated rectangles:

$$A = [x : 0 < x_i \leq c, i = 1, \cdots, k]$$

If we prove equality for all such A (rational squares with lower corner at 0), and B is a rational rectangle with $B = \bigcup_{i=1}^n A_i + x_i$, then we have (note, all such B can be described by these squares - take each rational side length $\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n}$, and then take squares of side length $\frac{1}{q_1 \cdots q_n}$):

$$\mu_1(B) = \lambda_k \left[T \left[\bigcup_{i=1}^n A_i + x_i \right] \right] = \lambda_k \left[\bigcup_{i=1}^n T \left[A_i + x_i \right] \right] :$$

Note, the above is still a disjoint union (via invertibility):

$$= \sum_{i=1}^n \lambda_k \left[T \left[A_i + x_i \right] \right] = \sum_{i=1}^n \lambda_k \left[T \left[A_i \right] + T(x_i) \right]$$

Note, linearity of T allows us to break up the summation. Translation invariance, and to be proved equality on A_i as described, and finite countability, give us:

$$= \sum_{i=1}^n \lambda_k \left[T \left[A_i \right] \right] = \sum_{i=1}^n \mu_1(A_i) = \sum_{i=1}^n \mu_2(A_i) = \sum_{i=1}^n |\det T| \lambda_k \left[A_i \right]$$

$$= |\det T| \sum_{i=1}^n \lambda_k \left[A_i + x_i \right] = |\det T| \lambda_k \left[B \right] = \mu_2(B)$$

And so, for Theorem 12.2 to be proved, to show the equality of $\lambda_k(TA) = |\det T| \cdot \lambda_k(A)$, we just need to prove equality on A of the form:

$$A = [x : 0 < x_i \leq c, i = 1, \cdots, k]$$

For rational c. Now recall linear algebra: a general linear transformation T can be described as the finite product of elementary matrices (Gaussian-Jordan Elimination). Ie, T can be represented as a finite product of elementary linear translations of the form:

1. $T(x_1, \cdots, x_k) = (x_{\pi 1}, \cdots, x_{\pi k})$, where π is a permutation of $\{1, \cdots, k\}$.
2. $T(x_1, \cdots, x_k) = (\alpha x_1, x_2, \cdots, x_k)$
3. $T(x_1, \cdots, x_k) = (x_1 + x_2, x_2, \cdots, x_k)$

(Note, the usual appearance of these 3 elementary operations is just a single swap, scalar multiplication in any entry, and addition of any two entries. However, any permutation, combined with just scalar multiplication/entry addition in the first slot is equivalent).

Note, we can make use of this product fact, and just check equality of $\mu_1(A) = \mu_2(A)$ with T of the elementary form, to prove equality for all T. So first, we will prove our equality for A of the given form, across the 3 different elementary T:

1. Note that for a permutation matrix, $\det T = \pm 1$. Think of just the volume of the spanned surface - it is unchanged. Or just think of cofactor expansion. Now, note:

   $$TA = A$$

   This can be seen by noting: $T(ce_i) = ce_j$. Note, how the square of side length c at the origin doesn't change under T. And so:

   $$\mu_1(A) = \lambda_k(TA) = \lambda_k(A) = |\det T| \lambda_k(A) = \mu_2(A)$$

2. Now, consider a scalar matrix T. It is clear that $\det(T) = \alpha$. Also, it is clear that:

   $$TA = [x : 0 < x_1 \leq \alpha c, 0 < x_i \leq c, i = 2, \cdots, k]$$

   And so:

   $$\lambda_k(TA) = c\alpha \cdot c^{k-1} = \det |T| c^k = \det |T| \lambda_k(A)$$

That is for positive α. If $\alpha = 0$ T would be non singular, so we can ignore that case. If $\alpha < 0$, we have:

$$TA = [x : \alpha c \leq x_1 < 0, 0 < x_i \leq c, i = 2, \cdots, k]$$

$$\implies \lambda_k(TA) = c\alpha \cdot c^{k-1} = \det |T| c^k = \det |T| \lambda_k(A)$$

Same thing, equality for all valid α.

3. The final case is row adding T. We have that $\det T = 1$. This is clear via cofactor expansion along the first row (the one will be ignored). Let $B = [x : 0 < x_i \leq c, i = 3, \cdots, k]$ where $B = \mathbb{R}^k$ if $k < 3$, and define:

$$B_1 = [x : 0 < x_1 \leq x_2 \leq c] \cap B \qquad B_2 = [x : 0 < x_2 < x_1 \leq c] \cap B$$

$$B_3 = [x : c < x_1 \leq c + x_2, 0 < x_2 \leq c]$$

Note that $A = B_1 \cup B_2$, a disjoint union accounting for all cases of $0 < x_1 \leq c, 0 < x_2 \leq c$.

Note that $TA = B_2 \cup B_3$. This can be seen by first noting that B_2 are points in the original A that can be expressed as points mapped into TA. Take for example, in the square $[0, 5] \times [0, 5]$. We have the point $(3, 1)$. It is clear that $0 < 1 < 3 \leq 5$. $(3, 1)$ can be expressed as $T(2, 1)$. The point is - if $x_2 < x_1$, then we have:

$$(x_1, x_2) = T(x_1 - x_2, x_2)$$

And we still have that $x_1 - x_2 > 0 \in B$.

Trying another proof. Take $(y_1, \cdots, y_k) \in TA$. Then:

$$(y_1, \cdots, y_k) = T(x_1, \cdots, x_k) \implies y_1 = x_1 + x_2$$

We have y_1 is in two cases. We can have $0 < y_2 < y_1 \leq c$. Or, we have $c < y_1 \leq c + y_2$ and $0 < y_2 \leq c$. Split it up into two cases. Assume $y_1 \leq c$. Then, we must have that $y_2 < y_1$. This is because:

$$y_1 = x_1 + y_2 > y_2$$

Note, we also have $y \in B$, as we have that $y_2 < y_2 \leq c$. Now, assume that $c < y_1$. It is clear that $0 < y_2 \leq c$ is always satisfied. We also have that $y_1 \leq c + y_2$. Again, this is because:

$$y_1 = x_1 + y_2 \leq c + y_2$$

And so, we have $TA \subseteq B_2 \cup B_3$. We can go the other way as well - express each point in B_2 and B_3 as a point in A under T. This is done above.

It is also easy to see that $B_1 + (c, 0, \cdots, 0) = B_3$. And so, we have:

$$\mu_1(A) = \lambda_k(TA) = \lambda_k(B_2)+\lambda_k(B_3) = \lambda_k(B_2)+\lambda_k(B_1) = \lambda_k(A) = |\det T|\lambda_k(A) = \mu_2(A)$$

And so, under elementary transformations T, for rational squares A with corner at 0, we have $\mu_1(A) = \mu_2(A)$. Now, take any T. Express it as a composition of a finite number of elementary transformations:

$$T = T_n \circ \cdots \circ T_1$$

Note - I think Billingsley messes up here. We need equality for all rectangles, with lower left corner at the origin. Note - T will always keep the origin at the origin. Or - maybe note that $T_1 A$ can be expressed as a union of translated squares at the origin.

Note. I make the following argument. Take a rational rectangle. Note that each rational rectangle can be expressed as a disjoint union of rational squares translated from the origin:

$$A = \bigcup_{i=1}^{n} A_i + x_i$$

Take an elementary linear transformation T. We have shown that:

$$\mu_1(A_i) = \lambda_k(TA_i) = |\det T|\lambda_k(A_i) = \mu_2(A_i)$$

Via the argument made above (ie, using linearity of T, countable additivity of μ_j, and equality under translations), we thus have for any rational rectangle A for linear transformations T:

$$\lambda_k(TA) = |\det T|\lambda_k(A)$$

Now, we have that rational rectangles A generate $\mathcal{R}^k$ (and are a π system). We note that by Theorem 10.3, as μ_1 and μ_2 are measures on which the rational rectangles are equal for elementary T (and σ finite on the rational rectangles) we have that for elementary T, μ_1 and μ_2 are equal on all of $\mathcal{R}^k$. NOTE: the difference is, we are applying this argument not for all T, but just elementary T. Now, take a non elementary T. We have:

$$T = T_n \circ \cdots \circ T_1$$

Examine for any $A \in \mathcal{R}^k$:

$$|\det T|\lambda_k(A) = |\det T_n| \cdots |\det T_1|\lambda_k(A) =$$

This is via the multiplicativity of the determinant. Now, we have via the argument above:

$$= |\det T_n| \cdots |\det T_2|\lambda_k(T_1(A)) = |\det T_n| \cdots |\det T_3|\lambda_k(T_2(T_1(A)))$$

This is because, each $T_k \circ \cdots T_1(A)$ is still within $\mathcal{R}^k$, and we proved equality of bringing in the linear transformation for elementary transformations, for all $A \in \mathcal{R}^k$. Continuing, the above becomes:

$$= \lambda_k(T_n(\cdots T_2(T_1(A)))) = \lambda_k(TA)$$

Thus, we are able to conclude, for all nonsingular linear transformations $T : \mathbb{R}^k \to \mathbb{R}^k$, for all $A \in \mathcal{R}^k$, we have $TA \in \mathcal{R}^k$, and:

$$\lambda_k(TA) = |\det T|\lambda_k(A)$$

qed.

Regularity

Definition - Regularity As far as I can tell, regularity just means that the measure of a set can be approximated by closed subsets or open supersets.

Theorem 12.3 Property of Regularity - Finite Measures on Bounded Sets of $\mathcal{R}^k$ implies Measure is Regular Suppose that μ is a measure on $\mathcal{R}^k$ such that $\mu(A) < \infty$ if A is bounded.

1. For $A \in \mathcal{R}^k$ and $\epsilon > 0$, there exist a closed C and an open G such that:

   $$C \subset A \subset G$$

   And $\mu(G - C) < \epsilon$.

2. If $\mu(A) < \infty$, then $\mu(A) = \sup \mu(K)$, the supremum extending over compact subsets K of A.

First note - the second part follows from the first. Without relying on the first part - we have that $\mu(A) < \infty$ implies that:

$$\mu(A - A_0) < \epsilon$$

For a bounded subset A_0 of A. Let C_n be any increasing sequence of bounded Borel Sets that exhausts $\mathbb{R}^k$. Ie, we can let C_n be the rectangles $[-n, n]^k$. Set $A_n := A \cap C_n$. Then, $A_n \uparrow A$. Further, each A_n is bounded, as C_n is bounded. By continuity from below, we have:

$$\lim_n \mu(A_n) = \mu(A)$$

Note, this is all valid so far, because we are considering μ a measure on $\mathcal{R}^k$. For any $\epsilon > 0$, there is thus a bounded A_n such that:

$$|\mu(A) - \mu(A_n)| < \epsilon \implies \mu(A - A_n) < \epsilon$$

Note, we can only compare a subtraction like this, as $\mu(A) < \infty$. So, we indeed have our fact - if $\mu(A) < \infty$, for $A \in \mathcal{R}^k$, then there is a bounded subset A_0 of A such that:

$$\mu\left(A - A_0\right) < \epsilon$$

It follows from the first part that $\mu(A_0 - K) < \epsilon$ for a closed and hence compact subset K of A_0. So, just making both ϵ small enough, we can find a compact subset K of A such that $\mu\left(A - K\right) < \epsilon$, which implies that the supremum across $\mu(K)$ of those compact subsets is equal to $\mu(A)$.

So now, we just need to prove part (1). Consider first a bounded rectangle $A = [x : a_i < x_i \leq b_i, i \leq k]$. The set $G_n = [x : a_i < x_i < b_i + n^{-1}, i \leq k]$ is open, and $G_n \downarrow A$. Since $\mu(G_1)$ is finite by hypothesis, continuity from above gives us:

$$\mu(G_n - A) < \epsilon$$

For large n. A bounded rectangle can thus be approximated from the outside by open sets. Recall the rectangles form a semi ring - This is Example 11.4. For an arbitrary set $A \in \mathcal{R}^k$, as the semiring generates $\mathcal{R}^k$ - theorem 11.4(i) tells us there exist disjoint bounded rectangles A_k such that:

$$A \subset \bigcup_k A_k \quad \text{and} \quad \mu\left[\left(\bigcup_k A_k\right) - A\right] < \epsilon$$

Now, choose open sets G_k such that $A_k \subset G_k$ and $\mu(G_k - A_k) < \epsilon/2^k$. Then, $G = \bigcup_k G_k$ is open, and:

$$\mu(G - A) < 2\epsilon$$

And remember ϵ was arbitrary. Thus, the general k dimensional Borel set can be approximated from the outside by open sets. To approximate from the inside by closed sets - pass to complements.

Specifying Measures on the Line

There are on the line $\mathbb{R}$ many measures other than λ that are important for probability theory. Think about it - we have things like the normal distribution, poisson distribution, exponential distribution - all with different cummulative distribution functions, each describing different probabilities on the real line. This section kind of dives into that.

There is a useful way to describe the collection of all measures on $\mathcal{R}^1$ that assign finite measure to each bounded set. Let μ be such a measure on $\mathcal{R}^1$. Define a real function F by:

$$F(x) = \begin{cases} \mu(0, x] & \text{if } x \geq 0 \\ -\mu(x, 0] & \text{if } x \leq 0 \end{cases}$$

Note how $F(x)$ is non decreasing. μ is always positive - and so as x goes from negative to positive, $F(x)$ goes from negative to positive. Also note, nondecreasing comes from the monotonicity of the measure. As $\mu(A) < \infty$ for bounded A, F is also finite for each x.

Suppose that $x_n \downarrow x$. If $x \geq 0$, we can make use of continuity from above to find:

$$\mu(0, x_n] \downarrow \mu(0, x] \implies F(x_n) \downarrow F(x)$$

If $x < 0$, apply continuity from below (once x_n becomes less than zero, $x_n \downarrow x$)

$$\mu(x_n, 0] \uparrow \mu(x, 0] \implies F(x_n) \downarrow F(x)$$

Thus, in all cases, $x_n \downarrow x$ implies $F(x_n) \downarrow F(x)$. Thus, F is continuous from the right. Not bad! Finally, we have:

$$\mu(a, b] = F(b) - F(a)$$

For every bounded interval $(a, b]$. If $a \leq 0$ and $0 \leq b$, we have:

$$\mu(a, b] = \mu(a, 0] + \mu(0, b] = F(b) - F(a)$$

If $0 \leq a \leq b$, we have:

$$\mu(a, b] = \mu(0, b] - \mu(0, a] = F(b) - F(a)$$

And the $a \leq b \leq 0$ case is similar. So, we have the properties of F:

1. $F(x)$ is finite for each x

2. F is nondecreasing

3. F is continuous from the right

4. $\mu(a,b] = F(b) - F(a)$

Note, if μ is the Lebesgue measure, we have:

$$F(x) = \begin{cases} x = \mu(0,x] & \text{if } x \geq 0 \\ x = -\mu(x,0] & \text{if } x \leq 0 \end{cases}$$

So, the Lebesgue measure is the case $F(x) = x$.

Recall, the finite intervals form a π system generating $\mathcal{R}^1$ - and so, as:

$$\mu(a,b] = F(b) - F(a)$$

If we are given the function F, it completely determines μ via Theorem 10.3 (tied with the finiteness of F implying that μ is σ finite on the π system of finite intervals). However - note that μ does not determine F. As, if F satisfies the above equation, so does $F(x) + c$. However, μ does determine F, within an additive constant.

Cumulative Distribution Function For finite μ, it is customary to standardize F not by our cases, but:

$$F(x) = \mu(-\infty, x]$$

Note, we need μ to be finite, as otherwise, the above value could be infinite. I do want to note that the two functions agree, in the finite case. We have:

$$F(x) = \mu(-\infty, x] \text{ compared with } F(x) = \begin{cases} \mu(0,x] & \text{if } x \geq 0 \\ -\mu(x,0] & \text{if } x \leq 0 \end{cases}$$

Oh note -it is clear that the F value is actually different. However, we just use the first case for finite μ. In such a case, it is clear that:

$$\lim_{x\to\infty} F(x) = 0$$

Why? we have that $\mu(\emptyset)$ is zero, and for $x_n \downarrow -\infty$, we have $\emptyset = \bigcap_n (-\infty, x_n]$. We also have $\lim_{x\to\infty} F(x) = \mu(\mathcal{R}^1)$, which is clear. If μ is a probability measure, F is called a *distribution function*, or a *cumulative distribution function*.

Measures μ are often specified by means of the function F. The following theorem ensures that to each F, there does exist a μ:

Theorem 12.4 - Existence of Measure on $\mathcal{R}^1$ Via a Function F If F is a nondecreasing, right continuous real function on the line, there exists on $\mathcal{R}^1$ a unique measure μ satisfying:

$$\mu(a,b] = F(b) - F(a)$$

For all a and b. **Proof:** Note, uniqueness comes from Theorem 10.3. The proof of existence is almost the same as the construction of the Lebesgue measure, which is the case $F(x) = x$. However, we don't give it here - as we will prove a more general version in Theorem 12.5.

Some things to also note. I think for both of the F we have:

$$F(x) = \mu(-\infty, x] \text{ for finite } \mu \text{ (like a Probability Measure) vs } F(x) = \begin{cases} \mu(0,x] & \text{if } x \geq 0 \\ -\mu(x,0] & \text{if } x \leq 0 \end{cases}$$

F in both cases is nondecreasing, clearly. Right continuous, also clearly (we can apply continuity from above in all cases for a finite measure). Also, in both cases, satisfies the equation for bounded intervals.

Specifying Measures in $\mathcal{R}^k$

The σ field $\mathcal{R}^k$ of k dimensional Borel sets is generated by the class of bounded rectangles:

$$A = [x : a_i < x_i \leq b_i : i = 1, \cdots, k]$$

Consider sets of the form:

$$S_x = \{y : y_i \leq x_i, i = 1, \cdots, k\}$$

Note, sets like S_x are kind of the higher dimensional analogues of the half infinite intervals $(-\infty, x]$. S_x is closed, and we can express bounded rectangles something like:

$$A = S_{b_1,\cdots,b_k} - \left[S_{(a_1,b_2,\cdots,b_k)} \cup S_{(b_1,a_1,\cdots,b_k)} \cup \cdots \cup S_{(b_1,b_2,\cdots,a_k)}\right]$$

Note, $S_{b_1,\cdots,b_k}$ contains all of A, and some infinite portions, which we remove via the unions (which keep an open interval on the correct side). Therefore, we can see that the class of sets S_x actually generate $\mathcal{R}^k$. We have:

$$\{A\} \subseteq \sigma(S_k) \implies \mathcal{R}^k \subseteq \sigma(S_k)$$

However, it is also clear that $\sigma(S_k) \subseteq \mathcal{R}^k$, as $\{S_k\} \subseteq \{A\} \implies \sigma(S_k) \subseteq \mathcal{R}^k$. This class of S_k is also a π system, clearly.

We now want to find an analogue of the equation $\mu(a, b] = F(b) - F(a)$ in higher dimensions. A bounded rectangle has 2^k vertices - for each entry in $(x_1, \cdots, x_k)$, we have a choice of 2 - a_i or b_i. Define the signum of a vertex on the rectangle as:

Signum Of Rectangle Vertex Take a bounded rectangle:

$$A = [x : a_i < x_i \leq b_i : i = 1, \cdots, k]$$

There are 2^k vertices. For x a vertex of the rectangle, define $\text{sgn}_A(x)$, the signum of the vertex, be $+1$ or -1, corresponding with whether the number of $1 \leq i \leq k$ satisfying $x_i = a_i$ is even or odd. For a bounded rectangle in $\mathbb{R}^2$, the top right vertex has number 0, and thus signum $+1$. The bottom left vertex has count 2, and thus signum $+1$. The other two vertices (top left, bottom right) have count 1, and thus signum -1.

Difference Around The Rectangle A A is our bounded rectangle. For a real function F on $\mathbb{R}^k$, the difference of F around the vertices of A is:

$$\Delta_A F = \sum \text{sgn}_A(x) \cdot F(x)$$

In the case that $k = 1$ - then, we have $A = (a, b]$, and $\Delta_A F = F(b) - F(a)$. In the case $k = 2$, we have:

$$\Delta_A F = F(b_1, b_2) - F(b_1, a_2) - F(a_1, b_2) + F(a_1, a_2)$$

So, we can see that $\Delta_A F$ is kind of a generalization of what we need in $\mu(a, b] = F(b) - F(a)$.

Suppose that μ is a finite measure on $\mathcal{R}^k$ and consider the analogue of $F(x) = \mu(-\infty, x]$:

$$F(x) = \mu \left[y : y_i \leq x_i, i = 1, \cdots, k \right]$$

Note, these are half infinite rectangles with top right corner at x. Suppose that S_x is defined as previously. Suppose that A is a bounded rectangle. Then:

$$\mu(A) = \Delta_A F$$

To see this - make use of:

$$A = S_{b_1, \cdots, b_k} - \left[S_{(a_1, b_2, \cdots, b_k)} \cup S_{(b_1, a_1, \cdots, b_k)} \cup \cdots \cup S_{(b_1, b_2, \cdots, a_k)} \right]$$

Apply the inclusion-exclusion principle:

$$\begin{aligned}\mu(A) &= \mu\left[S_{b_1,\cdots,b_k} - \left[S_{(a_1,b_2,\cdots,b_k)} \cup S_{(b_1,a_1,\cdots,b_k)} \cup \cdots \cup S_{(b_1,b_2,\cdots,a_k)}\right]\right] \\ &= \mu\left[S_{b_1,\cdots,b_k}\right] - \mu\left[S_{(a_1,b_2,\cdots,b_k)} \cup S_{(b_1,a_1,\cdots,b_k)} \cup \cdots \cup S_{(b_1,b_2,\cdots,a_k)}\right] \\ &= F(b_1,\cdots,b_k) - \sum_i \mu(S_i) + \sum_{i<j} \mu(S_i \cap S_j) - \cdots + (-1)^k \mu(S_1 \cap \cdots \cap S_k)\end{aligned}$$

Where S_i refers to $S_{(b_1,\cdots,b_{i-1},a_i,b_{i+1},\cdots,b_k)}$. Now - note what each $S_i \cap S_j$ is. This is just:

$$S_i \cap S_j = S_{(b_1,\cdots,b_{i-1},a_i,b_{i+1},\cdots,b_{j-1},a_j,b_{j+1},\cdots,b_k)}$$

So, each of the intersection terms just is another half infinite rectangle. We can again replace the μ term with F:

$$\mu(A) = F(b_1,\cdots,b_k) - \sum_i F(x_i) + \sum_{i<j} F(x_{ij}) - \cdots + (-1)^k F(x_{1,\cdots,k})$$

Where $x_{i_1,\cdots,i_p}$ refers to the vertex of A, with a_i in the index parameters, and b_i in the non indexed parameters. note that the sign added to the term is $+1/-1$ corresponding to whether or not the number of a_i is in the vertex an even/odd amount of times. Thus, we have:

$$\mu(A) = \Delta_A F$$

Overall, we expressed A in terms of half infinite rectangles S, and then matched the terms with $\Delta_A F$. Note - we require μ to be a finite measure, so that the measure of these half infinite rectangles is finite, and subtraction will be well defined. Suppose that $x^{(n)} \downarrow x$ in the sense that $x_i^{(n)} \downarrow x_i$ as $n \to \infty$ for each $i = 1,\cdots,k$. Then $S_{x^{(n)}} \downarrow S_x$, and hence $F(x^{(n)}) \to F(x)$ by continuity from above. In this sense, F is continuous from above. Thus, we have that F is a function of the sort that we talk about in the next theorem.

Note, the book says that we could also have an analogue of:

$$F(x) = \begin{cases} x = \mu(0,x] & \text{if } x \geq 0 \\ x = -\mu(x,0] & \text{if } x \leq 0 \end{cases}$$

In $\mathcal{R}^k$ for a measure μ that is finite on bounded sets - but, it is more complicated. The corresponding example for finite μ that we give above is easier.

Theorem 12.5 - Obtaining Measures on $\mathcal{R}^k$ from Functions on $\mathbb{R}^k$ Suppose that the real function F on $\mathbb{R}^k$ is continuous from above, and satisfies $\Delta_A F \geq 0$ for bounded rectangles A. Then there exists a unique measure μ on $\mathcal{R}^k$ satisfying:

$$\mu(A) = \Delta_A F$$

For Bounded Rectangles on A.

Theorem Notes For one, I just took a quick look at the theorem. It just makes use of Theorem 11.3 - we just need to prove finite additivity and countable subadditivity. This comes from noting the bounded rectangles are a semi-ring that generate $\mathcal{R}^k$. Note that $\mu(\emptyset) = \Delta_\emptyset F = 0$, as we can just take the emptyset as a rectangle where $a_i = b_i$, and the sum will cancel out. Thus, $\mu(A)$ on the bounded rectangles would be a finite valued set function. Note how we will also get uniqueness immediately out of Theorem 10.3 (as a consequence of existence), as the bounded rectangles are a sigma finite π system generating $\mathcal{R}^k$.

Also, note where the Lebesgue Measure fits in. Take $F(x) = x_1 \cdot x_2 \cdots x_k$. Here, we have that:

$$\mu(A) = \Delta_A F = (b_1 - a_1)(b_2 - a_2)\cdots(b_k - a_k)$$

For a bounded rectangle A. Note, we used distribution to conclude the above. And so, $\mu(A)$ is our k-dimensional Lebesgue measure on the bounded rectangles. Note that F is continuous, and so continuous from above, and satisfies $\Delta_A F \geq 0$ for bounded rectangles A, and so this Theorem 12.5 will prove existence of a unique extension on $\mathcal{R}^k$. Not bad!

Proof Of Theorem 12.5 So, we just need to prove the existence of a measure extension. To make use of Theorem 11.3 - we will first show that μ as defined is finitely additive on the class of bounded rectangles.

Suppose that each side $I_i = (a_i, b_i]$ of a bounded rectangle is partitioned into n_i subintervals:

$$J_{ij} = (t_{i,j-1}, t_{ij}) \quad \text{for } j = 1, \cdots, n_i \text{ and } a_i = t_{i0} < t_{i1} < \cdots < t_{in_i} = b_i$$

Then, we would have $n_1 n_2 \cdots n_k$ rectangles of the form:

$$B_{j_1,\cdots,j_k} = J_{1j_1} \times \cdots \times J_{kj_k}$$

That partition A. We call such a partition *regular*. We will first note that μ is additive across regular partitions:

$$\mu(A) = \sum_{j_1,\cdots,j_j} \mu(B_{j_1,\cdots,j_k})$$

And note - this should hopefully be intuitive, in the sense that the values that make up $\Delta_{B_{j_1,\cdots,j_k}} F$ cancel out on shared vertices, leaving only the vertices that make $\Delta_A F$. We have:

$$\sum_{j_1,\cdots,j_j} \mu(B_{j_1,\cdots,j_k}) = \sum_{j_1,\cdots,j_j} \Delta_{B_{j_1,\cdots,j_k}} F = \sum_{j_1,\cdots,j_j} \sum_{x} \mathrm{sgn}_B x \cdot F(x)$$

Where the sum across x is just the vertices of B. Now, we rearrange as:

$$= \sum_{x} F(x) \sum_{B} \mathrm{sgn}_B x$$

Where the left sum is all the vertices noted above, and the right sum is all the B that share the vertex x. Now, assume x is one of the interior vertices - ie, an x that is a vertex of one or more B, but not a vertex of A. Then, there must be an i such that $x_i \neq a_i, x_i \neq b_i$ - this is because x_i is not a vertex of A. There could be several such i - but fix only one. Then:

$$x_i = t_{ij} \quad \text{for } 0 < j < n_i$$

The rectangles B of which x is a vertex therefore come in pairs:

$$B' = B_{j_1,\cdots,j_i,\cdots,j_k} \qquad B'' = B_{j_1,\cdots,j_i+1,\cdots,j_k}$$

What does this mean - Note, we have $J_{ij} = (t_{i,j-1}, t_{ij}]$ and $J_{i,j+1} = (t_{ij}, t_{i,j+1}]$. So, if we increase j_i by 1, we get the $J_{i,j+1}$ interval that includes t_{ij} as the lower bound, rather than the upper bound. We can see this easier with a picture:

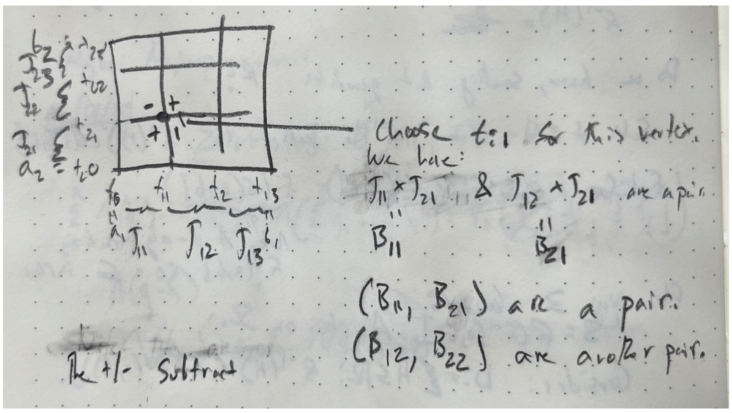


Figure 3: Theorem 12.5 Shared Vertex Rectangles

So, yeah, each B with a vertex on x has a pair along the i axis, corresponding to whether or not $x_i = t_{ij}$ is an upper bound or lower bound in that dimension. Note:

$$\text{sgn}_{B'}x = -\text{sgn}_{B''}x$$

As for why this is - note all the other intervals are the same. Ie, we have something like:

$$J_{1j_1} \times \cdots \times J_{ij} \times \cdots \times J_{kj_k} \quad \text{vs} \quad J_{1j_1} \times \cdots \times J_{i,j+1} \times \cdots \times J_{kj_k}$$

Note, sgn$x$ is the count related to the rectangle, oh how many coordinates in x are in the lower bound. Note, the x should have the same lower bound count, except in the J_{ij} vs. $J_{i,j+1}$ interval, and so the signs must be negative of each other. Thus, we have the inner sum on the right below is zero if x is on the interior:

$$= \sum_x F(x) \sum_B \text{sgn}_B x$$

So, now we consider the vertices x that are not in the interior - ie, shared by A. For those vertices, we have that x is the vertex of only one of the B as well. Further, it is clear that the $\text{sgn}_B x = \text{sgn}_A x$ - you can think of this visually, with the B lining up in the corner of A, and x having the same count of x_i being a lower bound for B and A. Thus, we have:

$$\sum_{j_1,\cdots,j_j} \mu(B_{j_1,\cdots,j_k}) = \sum_{x \in A} F(x)\text{sgn}_B x = \sum_{x \in A} \text{sgn}_A x F(x) = \mu(A)$$

And so, we have countable additivity across regular partitions of a bounded rectangle A. Now, suppose that $A = \bigcup_{u=1}^n A_u$, where A is a bounded rectangle, which has the form of a Cartesian product:

$$A = I_1 \times \cdots \times I_k$$

We also have:

$$A_u = I_{1u} \times \cdots I_{ku}$$

Across the k dimensions. Assume the A_u are disjoint. For each $1 \leq i \leq k$, the intervals $I_{i1}, \cdots, I_{in}$ must equal I_i. We want to show countable additivity. The overall idea will be - to break up each A_u in such a way, that each A_u has a regular partition, and the union of these regular partitions forms a regular partition of A. And then, we can just use additivity of regular partitions to find additivity.

Note, each I_i is split up by the endpoints of I_{iu}. We have I_i is partitioned by disjoint subintervals $J_{i1}, \cdots, J_{in_i}$, such that each I_{iu} is the union of certain of the J_{ij}. Easy enough! Look at the picture in the book for understanding, it is really helpful. The rectangles B of the form:

$$B_{j_1, \cdots, j_k} = J_{1j_1} \times \cdots \times J_{kj_k}$$

Form a regular partition of A - this is clear. Furthermore, the B's contained in a single A_u form a regular partition of A_u. As the A_u are disjoint, we have:

$$\mu(A) = \sum_B \mu(B) = \sum_{u=1}^{n} \sum_{B \subset A_u} \mu(B) = \sum_{u=1}^{n} \mu(A_u)$$

Therefore, our μ is finitely additive on the class $\mathcal{I}^k$ of bounded k-dimensional rectangles.

As noted initially - and as in Example 11.4 - $\mathcal{I}^k$ is a semiring. And so, Theorem 11.3 can apply. So we now work on countable additivity. If $A_, A_1, \cdots, A_n$ are sets in $\mathcal{I}^k$, then by Lemma 11.2 of the preceding section, we have:

$$A \subseteq \bigcup_{u=1}^{n} A_u \implies \mu(A) \leq \sum_{u=1}^{n} \mu(A_u)$$

So, we have finite subadditivity on $\mathcal{I}^k$. Suppose then that $A \subseteq \bigcup_{u=1}^{\infty} A_u$. We want to extend to/show countable subadditivity:

$$\mu(A) \leq \sum_{u=1}^{\infty} \mu(A_u)$$

Suppose that $\epsilon > 0$. Take the definition of bounded rectangle A as:

$$A = [x : a_i < x_i \leq b_i : i = 1, \cdots, k]$$

Now, define:

$$B = [x : a_i + \delta < x_i \leq b_i : i = 1, \cdots, k]$$

Note, we assume that $a_i < b_i$ for this to work with δ small enough - and note, that is fine, because if $a_i = b_i$, then we immediately have $\mu(A) = 0$ via cancellations on $\Delta_A F$, and countable subadditivity is thus immediate as well. Note for δ small enough, we have:

$$\mu(A) - \epsilon < \mu(B)$$

This is because F is continuous from above, so take any δ approaching zero, and we have $\lim \mu(B_\delta) = \mu(A)$. Also, we have that $\mu(B) < \mu(A)$, via finite additivity. Note that A contains B, and the closure of B:

$$\overline{B} = [x : a_i + \delta \leq x_i \leq b_i : i = 1, \cdots, k]$$

Similarly, for each u there is an $\mathcal{I}^k$ a set:

$$B_u = [x : a_{iu} < x_i \leq b_{iu} + \delta_u : i = 1, \cdots, k]$$

Such that:

$$\mu(B_u) < \mu(A_u) + \epsilon/2^u$$

And A_u is in the interior of B_u:

$$\mathring{B}_u = [x : a_{iu} < x_i < b_{iu} + \delta_u : i = 1, \cdots, k]$$

And so, in total, we have:

$$\overline{B} \subset A \subseteq \bigcup_{u=1}^{\infty} A_u \subseteq \bigcup_{u=1}^{\infty} \mathring{B}_u$$

Now, via Heine-Borel, we have closed rectangles are compact, and so every cover by open sets has a finite subcover. And so, there is some n such that:

$$B \subset \overline{B} \subseteq \bigcup_{u=1}^{n} \mathring{B}_u \subset \bigcup_{u=1}^{n} B_u$$

Note now, our Lemma 11.2 note applies, and so:

$$\mu(A) - \epsilon < \mu(B) \leq \sum_{u=1}^{n} \mu(B_u) < \sum_{u=1}^{\infty} \mu(A_u) + \epsilon$$

As ϵ is arbitrary, we can take ϵ to zero, to find:

$$\mu(A) \leq \sum_{u=1}^{\infty} \mu(A_u)$$

Thus, μ, as defined, is finitely additive and countably subadditive on the semiring $\mathcal{I}^k$. By Theorem 11.3, μ extends to a measure on $\mathcal{R}^k = \sigma(\mathcal{I}^k)$. By Theorem 10.3, this measure is unique. Thus, we have proved the theorem. qed.

Strange Euclidean Sets

This is an extra section. It just goes over (note, at a very high level) the Banach Tarski theorem. Recall what congruent means - two shapes are the same size and shape. And they can be moved by rotation/reflection/translation to fit on one another. Ie, one can be obtained from the other via a non-singular linear transformation with determinant 1, or a translation. Take A and B are in $\mathbb{R}^k$, such that they can be decomposed into $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ where A_i and B_i are congruent. A and B are called *congruent by dissection*. If all the pieces of A_i and B_i are borel sets, then of course:

$$\lambda_k(A) = \lambda_k(B)$$

Via additivity, Theorem 12.1, and Theorem 12.2. But if nonmeasurable sets are allowed in the dissections, then something crazy happens. If $k \geq 3$, and if A and B are bounded sets in $\mathbb{R}^k$ and have nonempty interiors, then A and B are congruent by dissection. Ie, we can find nonmeasurable sets that are congruent, and dissect A and B in a disjoint union - even if A and B are completely different.

This is usually illustrated this way: it is possible to break a solid ball the size of a pea into finitely many pieces, and then put them back together in such a way (after a congruent transformation) to get a solid ball the size of the sun.

Note - the issue is, the pieces are not solids, but an infinite scattering of points - recall, this matches what a non measurable set it, like with Vitali's construction.

Problems

12.1 Translation Invariant Measures

Suppose that μ is a measure on $\mathcal{R}^1$ that is finite for bounded sets and is translation-invariant: $\mu(A + x) = \mu(A)$. Show that $\mu(A) = a\lambda(A)$ for some $a \geq 0$. Extend to $\mathbb{R}^k$.

First, note that length 1 bounded intervals generate $\mathcal{R}^1$. We have that:

$$(a, \infty) = \bigcup_{k=1}^{\infty} (a + k - 1, a + k]$$

Something similar for $(-\infty, b)$. We found in problem 11.5 that $(a, \infty), (-\infty, b)$ generate $\mathcal{R}^1$. And so, we have for B the set of length one intervals, and C

the half infinite intervals:

$$B \subseteq C \implies \sigma(B) \subseteq \sigma(C) = \mathcal{R}^1$$

$$C \subseteq \sigma(B) \implies \mathcal{R}^1 \subseteq \sigma(B) \implies \mathcal{R}^1 = \sigma(B)$$

Now, note that via translation invariance, for every length 1 interval $A \in B$, we have:

$$\mu(A) = a$$

Note that $a\lambda(A)$ is a measure. We have that:

$$\mu(A) = a = a * 1 = a\lambda(A)$$

For all $A \in B$.

Note, I tried to make use of Theorem 10.3, but the length 1 intervals are not a π system - an intersection could lead to a non length 1 interval. Anyway. Update B to the bounded intervals - that is indeed a π system. I think we can still conclude:

$$\mu(A) = a\lambda(A)$$

For $A \in B$. I think - we can conclude Theorem 12.2 for our measure μ as well. I am going to go over it - but if it does apply to μ as well, we can note that scaling by s an interval is a linear transformation T with determinant s. Which would be helpful. Or, we can conclude that $\mu(0, 1/n] = 1/n\mu(0, 1]$. And use rational bounded intervals.

Note - the rational intervals generate $\mathcal{R}^1$. The rational intervals are also a π system. So, if $\mu(I) = a\lambda(I)$ for all rational intervals, then we can conclude that μ and $a\lambda$, as measures, are equal on all of $\mathcal{R}^1$ via the uniqueness of extension Theorem 10.3. As noted above, all length 1 intervals $I = (a, a+1]$ for rational a have the same measure - say $\mu(A) = a = a\lambda(A)$ - by translation invariance. Now, I will note that all intervals I_n of length $1/n$ have measure:

$$\mu(I_n) = a/n = a\lambda(I_n)$$

By translation invariance, we know the intervals:

$$(0, 1/n], \cdots, (n - 1/n, 1]$$

All have the same measure. Via countable additivity, we have:

$$n\mu(0, 1/n] = \mu(0, 1] \implies \mu(0, 1/n] = a/n \implies \mu(I_n) = a\lambda(I_n)$$

Now, take any rational interval $A = (p_1/q_1, p_2/q_2]$. Take $n = q_1 * q_2$. We have $p_2q_1 - p_1q_2$ length $1/n$ intervals that partition A:

$$(p_1q_2/q_1q_2, (p_1q_2 + 1)/q_1q_2], \cdots, ((p_2q_1 - 1)/q_1q_2, p_2q_1/q_1q_2]$$

Via countable additivity, we have:

$$\mu(A) = (p_2q_1 - p_1q_2)\mu(0, 1/q_1q_2] = a * \left[\frac{p_2q_1 - p_1q_2}{q_1q_2}\right] = a\left[\frac{p_2}{q_2} - \frac{p_1}{q_1}\right] = a\lambda(A)$$

Thus, the equality holds over the rational intervals. Thus, if μ is a measure on $\mathcal{R}^1$ that is finite for bounded sets and is translation invariant, we have:

$$\mu(A) = a\lambda(A)$$

For some $a \geq 0$. We can extend to $\mathbb{R}^k$ via examining rational rectangles. I guess we could use rational squares - we find a by examining $\mu([0,1]^k)$, then we partition $[0,1]^k$ into n^k squares $[0, 1/n]^k$. Same overall argument, to get the same overall result.

12.2 Measure of a Borel Set is Greater than an Interval

Suppose that $A \in \mathcal{R}^1$, $\lambda(A) > 0$, and $0 < \theta < 1$. Show that there is a bounded open interval I such that:

$$\lambda(A \cap I) \geq \theta\lambda(I)$$

First, note that we can assume $\lambda(A)$ is finite. This is because the finite case proves the infinite case. Note that $(\mathbb{R}, \mathcal{R}^1, \lambda)$ contains no infinite atoms. This is because, we have:

$$A \cap [-n, n] \uparrow A \implies \lim \lambda(A \cap [-n, n]) = \infty$$

And so, there must be some subset $A_n = A \cap [-n, n] \subseteq A$ such that $0 < \lambda(A_n) < \infty$. This is because each $\lambda(A_n) < 2n < \infty$. Also, if each was zero, that would contradict the limit. Anyway, if we have provde the finite case, there for chosen $\lambda(A)$ infinite with θ, we have a A_n where $\lambda(A_n)$ is finite, and the finite case proves there is an I such that:

$$\lambda(A \cap I) \geq \lambda(A_n \cap I) \geq \theta\lambda(I)$$

So now, we assume A is finite, and try and prove the statement. We can find an open G such that:

$$A \subset G \text{ and } \lambda(A) \geq \theta\lambda(G)$$

First note Theorem 12.3 - as λ is a measure on $\mathcal{R}^1$ that is finite if $A \in \mathcal{R}^1$ is bounded, we have for $A \in \mathcal{R}^1$ and any $\epsilon > 0$, there is an open set G such that $A \subseteq G$ and:

$$\lambda(G - A) < \epsilon$$

Note - if $\lambda(A) < \infty$, we have $\lambda(G) = \lambda(A) + \lambda(G - A) < \lambda(A) + \epsilon < \infty$. So, we can break up the subtraction - for any ϵ, we can find an open G such that:

$$\lambda(G) - \lambda(A) < \epsilon \implies \lambda(A) > \lambda(G) - \epsilon$$

Choose ϵ small enough such that $\lambda(G) - \epsilon > \theta\lambda(G)$, which is possible as $0 < \theta < 1$. And so, we can indeed conclude that there is an open G such that:

$$A \subset G \text{ and } \lambda(A) \geq \theta\lambda(G)$$

Now, we note that every open set can be written as a disjoint union of open intervals I_n:

$$G = \bigcup_n I_n$$

We always go over this proof - write two points in G are in the same equivalence class if they are in an interval contained within G. There are only countably many such intervals, as each one contains a rational. Note that via countable additivity and $A \subseteq G \implies A \cap G = A$:

$$\sum_n \lambda\left[A \cap I_n\right] = \lambda\left[A \cap \bigcup_n I_n\right] = \lambda(A) \geq \theta\lambda\left[\bigcup_n I_n\right] = \theta\sum_n \lambda(I_n)$$

Note, if for each I_n, we have $\lambda(A \cap I_n) < \theta\lambda(I_n)$, then the above inequality would be contradicted. Thus, there is an I_n such that:

$$\lambda(A \cap I_n) \geq \theta\lambda(I_n)$$

We have thus shown that for $A \in \mathcal{R}^1$, $\lambda(A) > 0$, and $0 < \theta < 1$, there is a bounded (each I_n must be bounded, or G would have infinite measure) open interval I such that:

$$\lambda(A \cap I) \geq \theta\lambda(I)$$

qed.

12.3 Difference Set of a Positive Measure Borel Set Contains the Origin

If $A \in \mathcal{R}^1$ and $\lambda(A) > 0$, then the origin is interior to the difference set:

$$D(A) = [x - y : x, y \in A]$$

I guess - this is kind of a density thing to me. The origin being in the *interior* of the difference set implies that it has an interval around it - so, we have at least one set of points in A that kind of has the thickness of an interval.

Take $A \in \mathcal{R}^1$ such that $\lambda(A) > 0$. For $\theta = 3/4$, by the above problem 12.2, there is a bounded open interval I such that:

$$\lambda(A \cap I) \geq \theta\lambda(I)$$

Suppose that $|z| < \lambda(I)/2$. Note that both $A \cap I$ and $(A \cap I) + z$ are contained in an interval of length less than $3\lambda(I)/2$. This is because both are contained within $I \cup (I + \lambda(I)/2)$. Thus, the two sets cannot be disjoint, as both sets satisfy:

$$\lambda(A \cap I) \geq \frac{3}{4}\lambda(I) \qquad \lambda(A \cap I + z) \geq \frac{3}{4}\lambda(I)$$

And disjoint would imply $\lambda((A \cap I) \cup (A \cap I) + z) \geq 3\lambda(I)/2$, which would contradict $\lambda((A \cap I) \cup (A \cap I) + z) < 3\lambda(I)/2$. So, there is some $x \in A \cap I$ such that $x \in (A \cap I) + z$, which implies $y = x - z \in A \cap I$. Thus, we have:

$$x - y = x - x + z = z \in D(A)$$

And this is true for all $|z| < \lambda(I)/2$, which does imply zero has an interval around it in $D(A)$, which implies zero is in the interior of the difference set. qed.

12.4 Extreme NonMeasurable Sets in $\mathcal{R}$

The following constructions leads to a subset H of the unit interval that is nonmeasurable in the extreme sense that its inner and outer Lebesgue measures are 0 and 1: $\lambda_*(H) = 0$ while $\lambda^*(H) = 1$. I will be filling in the details. The ideas are similar to the construction of the Vitali set. It will be convenient to work in $G = [0, 1)$ - let $\oplus$ and $\ominus$ denote addition and subtraction modulo 1 in G, which is a group with identity zero.

1. Fix an irrational θ in G and for $n = 0, \pm 1, \pm 2, \cdots$ let θ_n be $n\theta$ reduced modulo 1. Show that $\theta_n \oplus \theta_m = \theta_{n+m}, \theta_n \ominus \theta_m = \theta_{n-m}$, and the θ_n are distinct. Show that $\{\theta_{2n} : n = 0, \pm 1, \cdots\}$ and $\{\theta_{2n+1} : n = 0, \pm 1, \cdots\}$ are dense in G.

   So, we have a lot of stuff to prove. First, I will tackle $\theta_n \oplus \theta_m = \theta_{n+m}$. This is essentially proving something like mod addition is additive. We have:

   $$n\theta = b_n + r_n \qquad m\theta = b_m + r_m$$

For some $b_n, r_m \in \mathbb{Z}$, and $r_n, r_m \in [0, 1]$. And so, we have:

$$(n+m)\theta = n\theta + m\theta = (b_n + b_m) + (r_n + r_m)$$

And so now, we have:

$$\theta_{n+m} = (n+m)\theta \mod 1 = (b_n+b_m)+(r_n+r_m) \mod 1 = (r_n+r_m) \mod 1$$

This is clear, as $(b_n + b_m)$ is an integer. Now, note that:

$$\theta_n \oplus \theta_m = (r_n + r_m) \mod 1$$

And so, both sides are equal. We similarly have $\theta_n \ominus \theta_m = \theta_{n-m}$. Now, we want to find that the θ_n are distinct. Assume for $n \neq m$, we have:

$$\theta_n = \theta_m \implies (n\theta \mod 1) = (m\theta \mod 1)$$

This implies that:

$$n\theta = b_n + r_n \qquad m\theta = b_m + r_m$$

For $b_n, b_m \in \mathbb{Z}$ and $r_n, r_m \in [0, 1)$ and $r_n = r_m$. This implies:

$$n\theta - m\theta = b_n - b_m \implies \theta = \frac{b_n - b_m}{n - m}$$

Which would imply θ is rational, which would be a contradiction. And so, each θ_n must be distinct. Finally, we want to show that both:

$$\{\theta_{2n} : n = 0, \pm 1, \cdots\} \qquad \text{and} \qquad \{\theta_{2n+1} : n = 0, \pm 1, \cdots\}$$

Are dense in G. Recall, this means that for every $x \in G$, and every open interval containing x in G, a member of both sets is in that interval. More simply, every open interval in G has a nonzero intersection with the two sets above. Take $I = (a, b)$. We can restrict a and b to be rational - that is because, there is a rational interval contained within every interval, as the rationals are dense, and so if the sets meet the rational interval, they meet the bigger interval containing the rational interval as well. So:

$$I = \left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)$$

Also, I think we can maybe prove for θ rational first - as we have that the rationals are dense, and we can take a rational close enough to our irrational. Prove something like that. Actually no, I think the rationals might repeat, because for a/b, we have $a(b+a)/b = ab/b+a/b = a+a/b$, for which $\mod 1$ would be a/b.

Maybe consider it like this. Assume that your set hits:

$$(0, 1/n]$$

Then, your set must also hit:

$$(1/n, 2/n], \cdots, (n-1/n, n/n]$$

As we can just take $\theta_m \in (0, 1/n]$, and multiply it by $2, 3, 4, \cdots, n$ (as $\theta_{km} = \theta_m \oplus \cdots \oplus \theta_m$, which becomes normal plus as θ_m is a remainder within $(0, 1/n]$). If I can show something similar - that being within $(k/n, k+1/n]$ implies within each of the other sets (mod 1), then I think we are there - as we can just take n arbitrarily large, partition $(0, 1]$ such that one of the sets is in I.

Let's try this. Split G into finitely many intervals of length less than $\epsilon > 0$ - one of them must contain points θ_{2n} and θ_{2m}, with $\theta_{2n} < \theta_{2m}$. This is because each θ_n is distinct, and so we are putting an infinite amount of points into a finite number of intervals. If the number of intervals is k, consider $\theta_{2*1}, \cdots, \theta_{2*2k}$.

If $k = m - n$, then $0 < \theta_{2m} - \theta_{2n} = \theta_{2m} \ominus \theta_{2n} = \theta_{2k} < \epsilon$. First step is obvious, second step is because the subtraction is entirely within $[0, 1]$, and third step is because they share an interval. The points θ_{2kl} for $1 \leq l \leq \lfloor \theta_{2k}^{-1} \rfloor$ form a chain - in which the distance from each to the next is less than ϵ, the first is to the left of ϵ, and the last is to the right of $1 - \epsilon$.

Ahhh! This is essentially what I wanted to prove above - for each irrational, we have a θ_{2k} between $(0, 1/n]$ - and so, there is a $2kl$ inside of each of the intervals:

$$(0, 1/n], (1/n, 2/n], \cdots, (n-1/n, n/n]$$

Clearly, that implies there is a θ_{2kl} inside of *any* interval I. This proves the first set is dense, and the second set is clearly dense via a similar argument as well.

2. Take x and y to be equivalent if $x \ominus y$ lies in $\{\theta_n : n = 0, \pm 1, \cdots\}$, which is a subgroup of G. Note - G is a group w/ $\oplus$, with associativity clear,

identity 0, and inverse element negative $y = 1 - x \in G$. And by the above, $\oplus$ stays inside of $\{\theta_n : n = 0, \pm 1, \cdots\}$, 0 is in the set, and the inverse would just be θ_{-n}, as $\theta_{-n} + \theta_n = \theta_{-n+n} = \theta_0 = 0$.

Let S contain one representative from each equivalence class (each irrational defines an equivalence class with a countable number of other irrationals). Show that:

$$G = \bigcup_n (S \oplus \theta_n)$$

Where the union is disjoint. Put:

$$H = \bigcup_n (S \oplus \theta_{2n})$$

And show that:

$$G - H = H \oplus \theta$$

Problem Start: Note, it is clear that $\bigcup_n (S \oplus \theta_n)$ is a disjoint union. Take $x \in (S \oplus \theta_n)$ and $x \in (S \oplus \theta_m)$. Note, as adding θ_n stays inside the subgroup - we have that $x = \theta_k \oplus \theta_n = \theta_{k+n}$ and $x = \theta_k + \theta_m = \theta_{k+m}$ for some irrational θ. Thus, we must have $\theta_{k+m} = \theta_{k+n} \implies m = n$, which is a contradiction. So, we have a disjoint union. Now, we want to show this union equals G. Clearly, we have:

$$G \supseteq \bigcup_n (S \oplus \theta_n)$$

So, we go the other direction. Take $x \in G$. Assume x is not in the union. So, $x \neq G$ clearly. We have that every interval containing x intersects with some $(S \oplus \theta_n)$. Note - I think the definition $\{\theta_n : n = 0, \pm 1, \cdots\}$ allows for rational θ as well - as the multiplication $k\theta$ for irrational θ would always be rational. Anyway, in this case, I think it is clear that $x \in [x]$, the corresponding equivalence class, and so we get equality.

Note - still confused by this. I think, perhaps we have that S takes one representative from each coset? These would be sets of the form $g \in G$, $gH = \{g \oplus h : h \in H\}$. The question also includes (each coset) when it mentioned equivalence classes. I think this would allow for rationals. As, if g is rational, and $n = 0$, we have $\theta_0 = 0 \implies g \oplus \theta_0 = g$. And, we fix a single irrational θ. I think this is how it must be done, actually.

So, we will pick up from here.

Problem Start 2 Define:

$$AB = \{\theta_n : n = 0, \pm 1, \cdots\}$$

For some irrational number θ. Recall, B is a subgroup of G - we talked about this above. Define the set:

$$[x] = xB$$

Which is the left coset of the subgroup of B. Recall from group theory - left cosets are equivalence classes. This is equal to the equivalence relation:

$$a \sim b \Leftrightarrow a^{-1}b \in B \Leftrightarrow (1-a) \oplus b \in B \Leftrightarrow b \ominus a \in B$$

Which is the statement in the problem (note, both subtractions differ by 1, modulo 1, so we get the equality). So, if we let S take one element from each of the distinct left cosets, we want to show:

$$G = \bigcup_n (S \oplus \theta_n)$$

For a disjoint union. Equality is clear - as we take $x \in G$, note that for some $y \in [x]$, we have $y \in S$, and definitionally, we note that there is some θ_n such that $x = y \oplus \theta_n$, which implies x is in the union. And so, we just need to show it is a disjoint union. Again, similar argument as when we were doing the problem incorrectly. If not disjoint, there is an $x \in S \oplus \theta_n$ and $x \in S \oplus \theta_m$. As $[x]$ belongs to the same equivalence class, that tells us $x = y \oplus \theta_n = y \oplus \theta_m \implies m = n$. And so, we have that the union is disjoint as well.

We now move onto the final part of the problem. Define:

$$H = \bigcup_n (S \oplus \theta_{2n})$$

And show that:

$$G - H = H \oplus \theta$$

I think it is similar to showing that H and $H \oplus \theta$ are disjoint, and their union equals G. Clearly, their union equals G, as:

$$H \cup H \oplus \theta = \bigcup_n (S \oplus \theta_{2n}) \cup \bigcup_n (S \oplus \theta_{2n} \oplus \theta) = \bigcup_n (S \oplus \theta_{2n}) \cup \bigcup_n (S \oplus \theta_{2n+1}) = \bigcup_n (S \oplus \theta_n) = G$$

And now, we just have to show that they are disjoint. Well, if $x \in H$, and $x \in H \oplus \theta$, then we have:

$$x = y + \theta_{2n} \qquad x = y + \theta_{2m+1}$$

Again, this would imply $2n = 2m+1$. However, this is a contradiction, as for integer m and n, this implies an even number equals an odd number, which is not possible. And so, we cannot have an x in both sets, and so they are disjoint. Thus, we can conclude:

$$G - H = H \oplus \theta$$

3. Suppose that A is a Borel set contained in H. If $\lambda(A) > 0$, then $D(a)$ contains an interval $(0, \epsilon)$; but then some θ_{2k+1} lies in $(0, \epsilon) \subset D(A) \subset D(H)$, and so $\theta_{2k+1} = h_1 - h_2 = h_1 \ominus h_2 = (s_1 \oplus \theta_{2n_1}) \ominus (s_2 \oplus \theta_{2n_2})$ for some h_1, h_2 in H and some s_1, s_2 in S. Deduce that $s_1 = s_2$ and obtain a contradiction. Conclude that $\lambda_*(H) = 0$.

   We have that:

$$s_1 \ominus s_2 = \theta_{2k+1} \ominus \theta_{2n_1} \oplus \theta_{2n_2} = \theta_{2k+1} \ominus \theta_{2(n_1 - n_2)} = \theta_{2(k+n_1-n_2)+1}$$

   This is just via using associativity rules of the group operation. We note that s_1 and s_2 must be in the same subgroup - as the difference between them is in B, ie a multiple of our original irrational. However, as s_1 and s_2 are both a member of S - which only contains one member from each subgroup - we have that $s_1 = s_2$. Thus, we have:

$$0 = \theta_{2(k+n_1-n_2)+1} \implies 0 = 2(k + n_1 - n_2) + 1$$

   This is a contradiction, as we cannot have an odd number equal to zero. And so, our first statement is incorrect. We cannot have a Borel set contained in H, with $\lambda(A) > 0$. So, all Borel sets contained in H must satisfy:

$$\lambda(A) = 0$$

   Note, this implies:

$$\lambda_*(A) = 0$$

   This comes from problem 3.2, where we found for a probability measure P on a field $\mathcal{F}_0$, for every subset of Ω:

$$P^*(A) = \inf\left[P(C) : A \subset C, C \in \mathcal{F}\right]$$

$$P_*(A) = \sup\left[P(C) : C \subset A, C \in \mathcal{F}\right]$$

Note, we can extend the argument to λ, to find:

$$\lambda^*(A) = \inf\left[\lambda(C) : A \subset C, C \in \mathcal{R}^1\right]$$

$$\lambda_*(A) = \sup\left[\lambda(C) : C \subset A, C \in \mathcal{R}^1\right]$$

And so, we have $\lambda_*(H) = 0$, as the supremum of a set containing only zeros is zero.

4. Show that $\lambda_*(H \oplus \theta) = 0$ and $\lambda^*(H) = 1$.

   I think it is clear, via translation invariance, that $\lambda_*(H) = \lambda_*(H \oplus \theta) = 0$. At the very least, we can make a similar argument to above. For $\lambda^*(H)$ - note that:

$$\lambda_*(G - H) = \lambda_*(H \oplus \theta) = 0$$

$$\implies 1 - \lambda^*((G - H)^c)) = 0 \implies \lambda^*((G - H)^c)) = 1 \implies \lambda^*(H) = 1$$

   This comes from noting that $G - H = H^c$, and so $(G - H)^c = (H^c)^c = H$.

12.5 Extreme NonMeasurable Sets in $\mathcal{R}$ - 2

The construction here gives sets H_n such that $H_n \uparrow G$ and $\lambda_*(H_n) = 0$. If $J_n = G - H_n$, then $J_n \downarrow \emptyset$ and $\lambda^*(J_n) = 1$.

1. Let $H_n = \bigcup_{k=-n}^{n}(S \oplus \theta_k)$, so that $H_n \uparrow G$. show that the sets $H_n \oplus \theta_{(2n+1)v}$ are disjoint for different v.

   So, it is clear $H_n \uparrow G$, and $\lambda_*(H_n) \leq \lambda_*(H) = 0$. We just need to show the disjoint part. Well, note that $H_n \oplus \theta_{(2n_1)v}$ is just sets $S \oplus \theta_k$, for $k = \{-n, \cdots, -1, 0, 1, \cdots, n\} + (2n+1)v$. For each v, this is a different disjoint subset of k, (say $k = 1$, then our set looks like $\{n+1, \cdots, 3n+1\}$, etc). And so clearly, each $H_n \oplus \theta_{(2n+1)v}$ is disjoint for different integer v.

2. Suppose that A is a Borel set contained in H_n. Show that A and indeed all the $A \oplus \theta_{(2n+1)v}$ have Lebesgue measure 0. To be honest - I'm just going to use the monotonicity of the outer measure, which just gives all of these sets are subsets of some H_n, which has lower measure zero, which transfers to the lower measure, and thus measure, of A.

12.10 k dimensional analogue of Specifying Measures

Before I state the problem, recall that if μ is a measure on $\mathcal{R}^1$ that assigns finite measures to a bounded set, we could define a real finite function F by:

$$F(x) = \begin{cases} \mu(0, x] & \text{if } x \geq 0 \\ -\mu(x, 0] & \text{if } x \leq 0 \end{cases}$$

This was continuous from above, and satisfied $\mu(a, b] = F(b) - F(a)$. For a finite μ, we could standardize F with:

$$F(x) = \mu(-\infty, x]$$

This was the cumulative distribution function. This gave rise to Theorem 12.4, which was a special case of Theorem 12.5, which we proved. 12.4 said - if F is a nondecreasing, right-continuous real function on the line, there exists on $\mathcal{R}^1$ a unique measure μ satisfying $\mu(a, b] = F(b) - F(a)$ for all a and b. Theorem 12.5 said - if there was a real function F on $\mathbb{R}^k$ that is continuous from above (ie, $x_n \to x$, and each coordinate is nondecreasing) and satisfies $\Delta_A F \geq 0$ for bounded rectangles A, then there exists a unique measure μ on $\mathcal{R}^k$ satisfying $\mu(A) = \Delta_A F$ for all bounded rectangles A.

Now note - $\mu(A) = \Delta_A F$ makes sense, if we take $F(x) = \mu(S_x)$, for the half infinite rectangle S_x that has top vertex at $x \in \mathbb{R}^k$. Then, $\Delta_A F$ subtracts out the other rectangles, giving the correct definition. However, this requires μ to be finite on half infinite intervals - ie, this is the analogue of $F(x) = \mu(-\infty, x]$. We want the analogue of the other $F(x)$, in $\mathbb{R}^k$. I think this problem gives that analogue.

Problem Part 1: Let I_t be $(0, t]$ for $t \geq 0$ and $(t, 0]$ for $t \leq 0$, and let $A_x = I_{x_1} \times \cdots \times I_{x_k}$. Let $\varphi(x)$ be $+1$ or -1 according as the number of i, $1 \leq i \leq k$, for which $x_i < 0$ is even or odd. Show that, if $F(x) = \varphi(x)\mu(A_x)$, then:

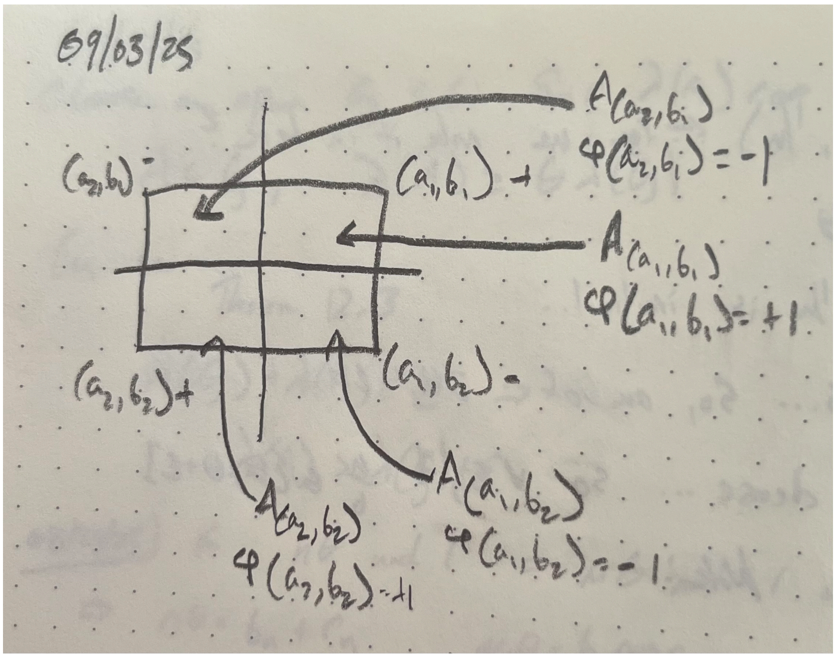
$$\mu(A) = \Delta_A F$$

Holds for bounded rectangles A.

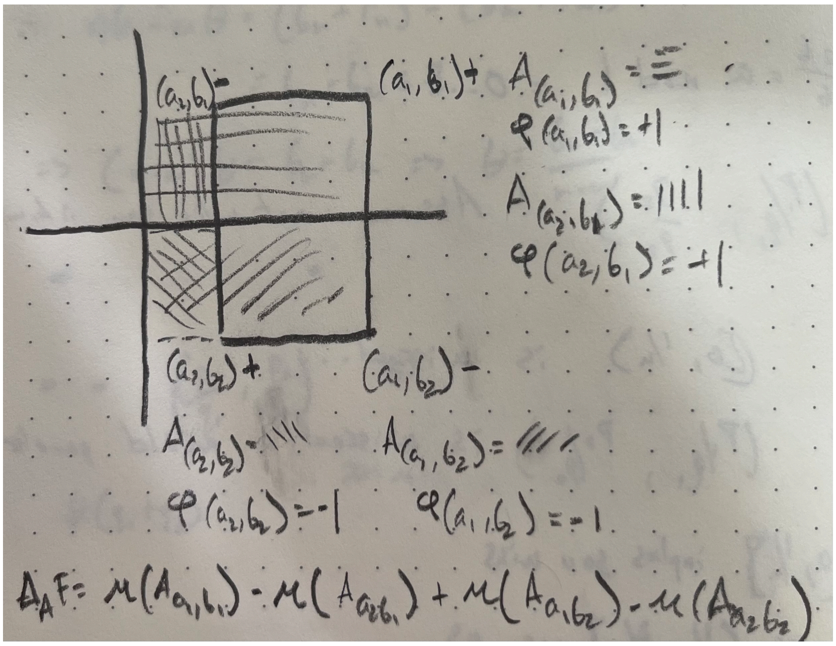
We will go from the right side to the left side. We have:

$$\Delta_A F = \sum_x \operatorname{sgn}_A(x) \cdot F(x) = \sum_x \operatorname{sgn}_A(x)\varphi(x)\mu(A_x)$$

Where x are the vertices of A. Note, if A is a bounded rectangle, where in each dimension, $a_i < 0$ and $b_i > 0$, this is just the sum of measures of a partition of A around the origin - see the picture:



The trickier part is if A is a bounded rectangle, and either $0 \leq a_i < b_i$, or $a_i < b_i \leq 0$. However, the cancellations should still work out:



I guess the difficult part is generalizing this argument. I think, consider $A = [a_1, b_1] \times \cdots [a_n, b_n]$. Each interval $[a_i, b_i]$ can be expressed as:

$$[a_i, b_i] = I_{a_i} \cup I_{b_i} \text{ or } = I_{b_i} - I_{a_i} \text{ or } = I_{a_i} - I_{b_i}$$

Corresponding to whether or not a_i, b_i are positive and negative, both positive, or both negative. We then consider 2^k rectangles

$$A_x = I_{x_1} \times \cdots \times I_{x_k}$$

Where each x_i is either a_i or b_i. At a high level - we know, at least via the pictures - there is a partition of the vertices - P and Q, such that if $x \in P$, we want to add the measure of A_x, and if $x \in Q$, we want to remove the measure of A_x, to get the measure of A - something like:

$$\mu(A) = \sum_{x \in P} \mu(A_x) - \sum_{x \in Q} \mu(A_x)$$

the question is - does $\operatorname{sgn}_A(x)\varphi(x)$ being positive imply $x \in P$, and $\operatorname{sgn}_A(x)\varphi(x)$ being negative imply $x \in Q$? Well - consider A_x. I think $\operatorname{sgn}_A(x)\varphi(x)$ is positive if A_x intersects with A, and negative if it doesn't. This is actually incorrect - I think there is a case where it is positive, and we want to add back a section we double subtracted.

I'm going to stick with intuition on this one. Thought about it for a while, couldn't really think of how to prove it rigorously. Other than, maybe going case by case?

Problem Part 2: Call F degenerate if it is a function of some $k-1$ of the coordinates, the requirement in the case $k = 1$ being that F is constant. Show that $\Delta_A F = 0$ for every bounded rectangle if and only if F is a finite sum of degenerate functions - $\mu(A) = \Delta_A F$ determines F to within addition of a function of this sort.

I'm just going to try and prove that if F is degenerate, then $\Delta_A F = 0$ for every bounded rectangle. This should be clear - as we can just pair upper/lower coordinates on the degenerate point, and the sum should be zero. Define:

$$A = [x : a_i < x_i \leq b_i : i = 1, \cdots, k]$$

We have:

$$\Delta_A(F) = \sum_{x_1 = a_1, b_1} \cdots \sum_{x_k = a_k, b_k} \operatorname{sgn}_A(x_1, \cdots, x_k) F(x_1, \cdots, x_k)$$

Note, the order of a finite sum doesn't matter via commutativity. Let F be degenerate in the i dimension - ie, F doesn't change with changes in the ith coordinate. Rewrite the sum as:

$$= \sum_{x_1 = a_1, b_1} \cdots \sum_{x_k = a_k, b_k} F(x_1, \cdots, x_k) \left[\operatorname{sgn}_A(x_1, \cdots, a_i, \cdots, x_k) + \operatorname{sgn}_A(x_1, \cdots, b_i, \cdots, x_k)\right]$$

Where the sums on the left don't include the x_i sum. Note that the sgn_A is opposite signs, when only one coordinate is changed, and so the above equals:

$$= \sum_{x_1=a_1,b_1} \cdots \sum_{x_k=a_k,b_k} F(x_1,\cdots,x_k)\,[0] = 0$$

12.11 Measures On $\mathcal{R}^2$ Supported By Curves

Let G be a nondecreasing, right-continuous function on the line, and put $F(x,y) = \min\{G(x),y\}$. Show that F satisfies the conditions of Theorem 12.5, that the curve $C = [(x,G(x)) : x \in \mathbb{R}^1]$ supports the corresponding measure, and that $\lambda_2(C) = 0$.

If μ is our resultant measure, to show that C supports it, we want to show $\mu(C) = 1$. First, we will prove that $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies the conditions for Theorem 12.5. We need F is continuous from above, and satisfies $\Delta_A F \geq 0$ for bounded rectangles A. Take $(x_i,y_i) \to (x,y)$, where $x_i \downarrow x$ and $y_i \downarrow y$. To show continuity from above, we need:

$$\lim_i F(x_i,y_i) = F(x,y)$$

Note, min is a continuous function. And so:

$$\lim_i F(x_i,y_i) = \lim_i \min\{G(x_i),y_i\} = \min\{\lim_i G(x_i), \lim_i y_i\} = \min\{G(x),y\} = F(x,y)$$

That is one part. Take a bounded rectangle A. Examine:

$$\Delta_A F = \sum_{x_i=a_1,b_1} \sum_{y_i=a_2,b_2} \text{sgn}_A(x_i,y_i) F(x_i,y_i)$$

$$= F(b_1,b_2) - F(a_1,b_2) - F(b_1,a_2) + F(a_1,a_2)$$

$$= \min\{G(b_1),b_2\} - \min\{G(a_1),b_2\} - \min\{G(b_1),a_2\} + \min\{G(a_1),a_2\}$$

We have some cases. If $G(b_1) \geq b_2$, we have $G(b_1) \geq a_2$. So then, the above becomes:

$$(b_2 - a_2) + \min\{G(a_1),a_2\} - \min\{G(a_1),b_2\}$$

Note, if $G(a_1) \leq a_2$, the above is nonnegative. If $a_2 \leq G(a_1) \leq b_2$, then we have $(b_2 - a_2) - (b_2 - G(a_1)$, which is still nonnegative. Finally, if $b_2 \leq G(a_1)$, the above is zero. So in all cases, $\Delta_A F \geq 0$ if $G(b_1) \geq b_2$. You can similarly analyze if $G(b_1) \leq b_2$. Anyway, we do have for all bounded rectangles A:

$$\Delta_A F \geq 0$$

So, we can make use of Theorem 12.5. There exists a unique measure μ on $\mathcal{R}^2$ satisfying $\mu(A) = \Delta_A F$ for bounded rectangles $A \in \mathcal{R}^2$. We now want to calculate:

$$\mu(C)$$

All I can really think of is describing C as a union of open rectangles, and making use of continuity from above. Say we partition $\mathbb{R}$ into the intervals of length $1/n$, starting from 0. Let $t_0 = 0, t_1 = 1/n$, etc, and $s_0 = 0, s_1 = -1/n$, etc. We have:

$$C \subseteq A_n = \left[\bigcup_{i=0}^{n} [t_i, t_{i+1}] \times [G(t_i), G(t_{i+1})]\right] \cup \left[\bigcup_{i=0}^{n} [s_{i+1}, s_i] \times [G(s_{i+1}), G(s_i)]\right]$$

Where A_n refers to the intervals of being length n. I think it is clear that:

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

My one question is - do we have:

$$C = \bigcap_{n=1}^{\infty} A_n$$

Clearly, we have $\subseteq$. So we need the other direction. I think one issue is - given that G is right continuous, I think our intersection contains any vertical line segments that might appear in jump discontinuities. Let us first assume that G is continuous. In which case, I think it is clear that:

$$C \supseteq \bigcap_{n=1}^{\infty} A_n$$

As take an $x \in \bigcap_{n=1}^{\infty} A_n$. Clearly, it is a part of an infinite number of $[a_n/n, (a_n+1)/n] \times [G(a_n/n), G((a_n+1)/n)]$ rectangles (assume positive wlog). Note, we can find a subsequence of $b_n, G(b_n)$ of these interval edges, where b_n is decreasing, which implies $x = \lim_n (b_n, G(b_n)) \implies x \in G$. Now, for the jump discontinuity case - I think from each A_n, we just remove the intervals where $[t_i, t_{i+1}]$ contains a jump discontinuity. Note, this can be done, because given that G is monotone, it can only have a countable amount of jump discontinuities. Take a look at Calculus on Manifolds, like problem 1-30. Not a proof, but I know I have proved this before.

So now, let us look at the measure of A_n. If we prove:

$$\mu(A_n^c) = 0$$

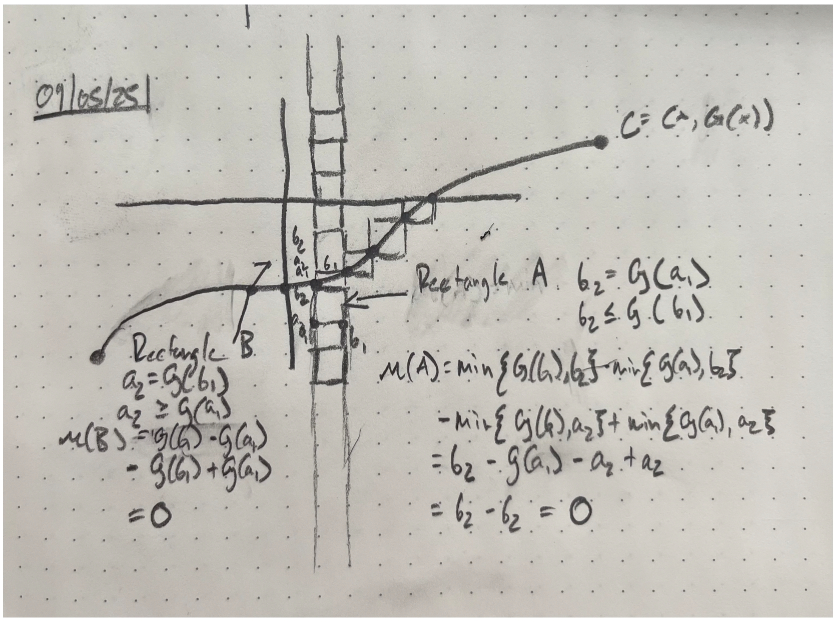
Because then, we would have:

$$\mu(C^c) = \mu\left[\left[\bigcap_{n=1}^{\infty} A_n\right]^c\right] = \mu\left[\bigcup_{n=1}^{\infty} A_n^c\right] = 0$$

And this is actually the correct definition of support - that the measure of the complement of the set is zero. Note - if μ was a probability measure, we could just conclude this is $\mu(C) = 1$, which is where my confusion from earlier came. Now, the issue is - A_n^c, visually, would be like a bunch of unbounded sets, going upward. I think we just need to prove the measure of each of these unbounded rectangles is zero. I think, a starting point is - if A bounded rectangle is completely below the line C, and we examine:

$$\mu(A) = \Delta_A F = \min\{G(b_1), b_2\} - \min\{G(a_1), b_2\} - \min\{G(b_1), a_2\} + \min\{G(a_1), a_2\}$$

I think, it will help more with a picture.



We have that A_n^c is the countable disjoint union of these infinite rectangles above and below each rectangle that makes up A_n. Each of these infinite rectangles is the countable disjoint union of smaller bounded rectangles as shown above, like A and B. Each of these satisfy $\mu(A) = \mu(B) = 0$. And so, each of these infinite rectangles has measure zero, and so A_n^c is the disjoint union of measure 0 rectangles. Thus, we have:

$$\mu(A_n^c) = 0$$

And we can thus conclude that $\mu(C^c) = 0$, and thus the curve C supports the corresponding measure μ.

We finally want to note that $\lambda_2(C) = 0$. I think, we can note that on an interval $[-n, n]$, the curve C must be uniformly continuous. See Bernstein's Theorem in these notes. C will be uniformly continuous outside of the jump discontinuities. If we make the same construction as above, keeping out the jump discontinuities, we will have for some $1/n$ small enough, all rectangles will have height less than ϵ via uniform continuity. The total area covering $C \cap [-n, n]$ would thus be $\leq [-n, n] * \epsilon$. And so, for all $\epsilon > 0$, we have:

$$\lambda_2(C \cap [-n, n]) \leq 2n * \epsilon \implies \lambda_2(C \cap [-n, n]) = 0 \implies \lambda_2(C) = 0$$

12.12

Let F_1 and F_2 be nondecreasing, right-continuous functions on the line and put:

$$F(x_1, x_2) = F(x_1)F(x_2)$$

Show that F satisfies the conditions of Theorem 12.5. Note, nondecreasing will give us $\Delta_A F \geq 0$ for bounded rectangles in $\mathcal{R}^2$, and F is continuous from above via both F_1 and F_2 being right-continuous.

Let μ, μ_1, μ_2 be the measures corresponding to F, F_1, F_2. Note, we can make use of Theorem 12.5 to get these unique measures on $\mathcal{R}^2, \mathcal{R}^1, \mathcal{R}^1$ respectively. Prove that $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ for intervals A_1 and A_2. This μ is the product of μ_1 and μ_2 - product measures are studied in a general setting in Section 18.

So, take intervals $A_1 = (a_1, b_1]$ and $A_2 = (a_2, b_2]$. We have:

$$\mu(A_1 \times A_2) = \Delta_{A_1 \times A_2} F = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$$

$$= F_1(b_1)F_2(b_2) - F_1(a_1)F_2(b_2) - F_1(b_1)F_2(a_2) + F_1(a_1)F_2(a_2)$$

$$= F_1(b_1)\left[F_2(b_2) - F_2(a_2)\right] - F_1(a_a)\left[F_2(b_2) - F_2(a_2)\right]$$

$$= \left[F_1(b_1) - F_1(a_1)\right]\left[F_2(b_2) - F_2(a_2)\right] = \mu_1(A_1)\mu_2(A_2)$$

Thus, we have proved all the statements in the question. qed.

Section 13 - Measurable Functions And Mappings

Notes

Recall our definition for a simple random variable from Section 5. If a real function X on Ω has a finite range, it is a simple random variable if $[\omega : X(\omega) = x]$ lies in the basic σ field $\mathcal{F}$ for each x. The requirement for a general (ie non finite) real function X is stronger - namely $[\omega : X(\omega) \in H]$ must lie in $\mathcal{F}$ for every linear Borel set H. We give an abstract version of this definition to simplify the theory.

Measurable Mappings

Definition - Measurable Mapping Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two *measurable spaces* - ie, they have measures (which I will denote μ and μ', but that is not needed yet). For a mapping $T : \Omega \to \Omega'$, consider the inverse images:

$$T^{-1}A' = [\omega \in \Omega : T\omega \in A'] \qquad \text{for} \qquad A' \subset \Omega'$$

The mapping T is *measurable* $\mathcal{F}/\mathcal{F}'$ if $T^{-1}A' \in \mathcal{F}$ for each $A' \in \mathcal{F}'$.

For a real function f, the image space Ω' is the line $\mathbb{R}^1$, and $\mathcal{R}^1$ is always tacitly understood to play the role of $\mathcal{F}'$. A real function f on Ω is thus measurable $\mathcal{F}$ (or just measurable, we can exclude the $\mathcal{F}$ when it is understood) if it is measurable $\mathcal{F}/\mathcal{R}^1$ - that is, $f^{-1}H \in \mathcal{F}$ for every $H \in \mathcal{R}^1$.

Definition - Random Variable A random variable is a real function on some measurable space $(\Omega, \mathcal{F})$ that is *measurable* (ie, measurable $\mathcal{F}/\mathcal{R}^1$). The point of the definition is to ensure that $[\omega : f(\omega) \in H]$ has a measure or probability for all sufficiently regular sets H or real numbers - that is, for all borel sets H.

Connections I will note some connections, that I am seeing over and over and over again. A lot of math, it seems like, is the practice of taking complicated areas, back into non complicated areas. Or, into areas that we understand. With manifolds, we have coordinate functions/atlases, that take us back to euclidean space. With probability, we have random variables, which take us back into measurable spaces. In both cases - the functions and their inverses have additional structure, that allow us to make better conclusions.

Whether it be the full rank for manifolds, or the measurability or random variables.

Example 13.1 A real function f with finite range is measurable if $f^{-1}\{x\} \in \mathcal{F}$ for *each singleton* $\{x\}$. This, however, is too weak of a condition to impose on the general f. On the other hand, for a measurable f with finite range, $f^{-1}H \in \mathcal{F}$ for *every* $H \subset \mathbb{R}$ (take the finite elements of the subset to a finite union). But, this is too strong of a condition to impose - note that for $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{R}^1)$, even $f(x) = x$ fails to satisfy the condition (take a nonmeasurable H). So, our measurable condition should hopefully be not to weak, not to strong.

Theorem 13.1 - Measurable Generator Sets and Composition of Measurable Functions Take measurable spaces $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}'), (\Omega'', \mathcal{F}'')$. Let $T : \Omega \to \Omega'$ and $T' : \Omega' \to \Omega''$.

1. If $T^{-1}A' \in \mathcal{F}$ for each $A' \in \mathcal{A}'$ and $\mathcal{A}'$ generates $\mathcal{F}'$, then T is measurable $\mathcal{F}/\mathcal{F}'$
2. If T is measurable $\mathcal{F}/\mathcal{F}'$ and T' is measurable $\mathcal{F}'/\mathcal{F}''$, then $T'T$ is measurable $\mathcal{F}/\mathcal{F}''$.

Proof We first prove part 1. Note that $T^{-1}(A'^C) = (T^{-1}A')^C$. Similarly, note $T^{-1}\left[\bigcup_n A'_n\right] = \bigcup_n T^{-1}A'_n$. As $\mathcal{F}$ is a σ field in Ω, the class $[A' : T^{-1}A' \in \mathcal{F}]$ is a σ field in Ω' (as it contains unions, being unions stay inside $\mathcal{F}$, and complements, as complements stay inside $\mathcal{F}$). If this σ field contains $\mathcal{A}'$, it must also contain $\sigma(\mathcal{A}')$. Ie, say that $\mathcal{A}' \subseteq [A' : T^{-1}A' \in \mathcal{F}]$. As $\sigma(\mathcal{A}')$ is the smallest sigma field containing $\mathcal{A}'$, we have:

$$\sigma(\mathcal{A}') \subseteq [A' : T^{-1}A' \in \mathcal{F}]$$

Now, the condition of part 1, is that $\mathcal{F}' = \sigma(\mathcal{A}')$, and that $\mathcal{A}' \subseteq [A' : T^{-1}A' \in \mathcal{F}]$. Thus, we also have that $\mathcal{F}' \subseteq [A' \in \Omega' : T^{-1}A' \in \mathcal{F}]$. Thus, definitionally, we have T must be measurable $\mathcal{F}/\mathcal{F}'$.

Now, we prove part 2. We have that $A'' \in \mathcal{F}''$ implies that $(T')^{-1}A'' \in \mathcal{F}'$, which in turn implies that $(T'T)^{-1}A'' = [\omega : T'T\omega \in A''] = [\omega : T\omega \in (T')^{-1}A''] = T^{-1}\left[(T')^{-1}A''\right] \in \mathcal{F}$. Anyway, long story short, take $A'' \in \mathcal{F}''$ back through two measurable functions, to find $(T'T)^{-1}A'' \in \mathcal{F}$. And so, $T'T$ is measurable $\mathcal{F}/\mathcal{F}''$. qed.

Mappings into $\mathbb{R}^k$

For a mapping $f : \Omega \to \mathbb{R}^k$ carrying Ω into k space, $\mathcal{R}^k$ is always understood to be the σ field of the image space. In probabilistic contexts, a measurable mapping into $\mathbb{R}^k$ is called a *random vector*. Now f must have the form:

$$f(\omega) = (f_1(\omega), \cdots, f_k(\omega))$$

For real functions $f_j(\omega)$. Since the sets $S_x = [y : y_i \leq x_i, i = 1, \cdots, k]$ ("southwest regions") generate $\mathcal{R}^k$, Theorem 13.1 implies that f is measurable $\mathcal{F}$ if and only if the set:

$$[\omega : f_1(\omega) \leq x_1, \cdots, f_k(\omega) \leq x_k] = \bigcap_{j=1}^{k} [\omega : f_j(\omega) \leq x_j]$$

Lies in $\mathcal{F}$ for each $(x_1, \cdots, x_k)$. The condition clearly holds if each f_j is measurable $\mathcal{F}$. On the other hand, if $x_j = x$ is fixed and $x_1 = \cdots = x_{j-1} = x_{j+1} = \cdots = x_k = n$ goes to ∞, the sets defined above increase to $\{\omega : f_j(\omega) \leq x\}$. Therefore, f is measurable $\mathcal{F}$ if and only if each component function f_j is measurable $\mathcal{F}$.

Theorem - $f : \Omega \to \mathbb{R}^k$ measurable $\mathcal{F}/\mathcal{R}^k$ if and only if each component is measurable $\mathcal{F}/\mathcal{R}$ First, assume that each component is measurable $\mathcal{F}/\mathcal{R}$. Then, $[\omega : f_j(\omega) \leq x_j] \in \mathcal{F}$ for any x_j, which implies $[\omega : f_1(\omega) \leq x_1, \cdots, f_k(\omega) \leq x_k] \in \mathcal{F}$, which implies f is measurable $\mathcal{R}^k$ via Theorem 13.1.

Now, assume that f is measurable $\mathcal{F}/\mathcal{R}^k$. The countable union argument above implies that each f_j must be measurable $\mathcal{F}/\mathcal{R}$. qed.

This is a very practical criterion for mappings into $\mathbb{R}^k$.

Definition - Borel Functions A mapping $f : \mathbb{R}^i \to \mathbb{R}^k$ that is measurable $\mathcal{R}^i/\mathcal{R}^k$ is called a *Borel Function*.

Theorem 13.2 - Continuity Implies Measurability If $f : \mathbb{R}^i \to \mathbb{R}^k$ is continuous, then it is measurable. This just requires checking that

$$[\omega : f_1(\omega) \leq x_1, \cdots, f_k(\omega) \leq x_k] \in \mathcal{R}^i$$

However, note, this is checking that the inverse of a closed set S_x is in $\mathcal{R}^i$ - ie, $f^{-1}(S_x) \in \mathcal{R}^i$. Recall the definition of continuity implies that the inverse of a closed set is closed. Also, recall that all closed sets are borel sets. And so, f must be measurable if it is continuous, as $f^{-1}(S_x)$ is a closed set and within $\mathcal{R}^i$. qed.

Theorem 13.3 - Composition of Measurable Function With Measurable Components is Measurable If $f_j : \Omega \to \mathbb{R}^1$ is measurable $\mathcal{F}$, $j = 1, \cdots, k$, then $g(f_1(\omega), \cdots, f_k(\omega))$ is measurable $\mathcal{F}$ if $g : \mathbb{R}^k \to \mathbb{R}^1$ is measurable - in particular, if it is continuous.

This proof is just an amalgamation of the previous statements. $f = (f_1, \cdots, f_k)$ is measurable by our component measurability theorem. $g \circ f$ is measurable by Theorem 13.1.ii. And so we have that $g \circ f = g(f_1(\omega), \cdots, f_k(\omega))$ is measurable if each of the components and g is. qed.

Some consequences of the theorem. Taking $g(x_1, \cdots, x_k)$ to be $\sum_{i=1}^k x_i, \prod_{i=1}^k x_i, \max(x_1, \cdots, x_k)$ in turn shows that sums, products, and maxima of measurable functions are measurable. If $f(\omega)$ is real and measurable, then so are $\sin f(\omega), e^{tf(\omega)}$, and so on, as we are composing continuous (and thus measurable) functions on top of a measurable f. If $f(\omega)$ never vanishes, then $1/f(\omega)$ is measurable as well.

Limits and Measurability

For a real function f - it is convenient to admit the artificial values ∞ and $-\infty$ - to work on the extended real line $[-\infty, \infty]$. We define a new definition of measurability:

Measurable on the Extended Real Line $f : \Omega \to [-\infty, \infty]$ is measurable $\mathcal{F}$ if $[\omega : f(\omega) \in H]$ lies in $\mathcal{F}$ for each Borel Set H of (finite) real numbers, and if $[\omega : f(\omega) = \infty]$ and $[\omega : f(\omega) = -\infty]$ both lie in $\mathcal{F}$.

This notion is needed/convenient for when we are using limits/suprema.

Theorem 13.4 - Measurability of Supremums and Limits Suppose that $f_1, f_2, \cdots$ are real functions measurable $\mathcal{F}$.

1. The functions $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are measurable $\mathcal{F}$
2. If $\lim_n f_n$ exists everywhere, then it is measurable $\mathcal{F}$
3. The ω-set where $\{f_n(\omega)\}$ converges lies in $\mathcal{F}$
4. If f is measurable $\mathcal{F}$, then the ω-set where $f_n(\omega) \to f(\omega)$ lies in $\mathcal{F}$

Proof: Part One. Recall, to prove that $g : \Omega \to \mathbb{R}$ is measurable, we just need $g^{-1}(-\infty, x] \in \mathcal{F}$ - as then each $A \in \mathcal{R}$ satisfies $g^{-1}(A) \in \mathcal{F}$ via Theorem

13.1.i. Extended measurability also needs us to check $g^{-1}(-\infty), g^{-1}(\infty) \in \mathcal{F}$. Note that:

$$\left[\omega : \sup_n f_n(\omega) \leq x\right] = \bigcap_n [\omega : f_n(\omega) \leq x] \in \mathcal{F}$$

$$\left[\omega : \sup_n f_n(\omega) = \infty\right] = \bigcap_{j=0}^{\infty} \bigcup_n [\omega : f_n(\omega) > j] \in \mathcal{F}$$

$$\left[\omega : \sup_n f_n(\omega) = -\infty\right] = \emptyset \in \mathcal{F}$$

For the supremum, we want ω such that there is some $f_n(\omega) \geq j$ for each j - as then the supremum for that ω would be infinite. For $-\infty$, note that each f_n is real, so the supremum across f_n for each ω cannot be $-\infty$. So, $\sup_n f_n$ is measurable. Measurability of $\inf_n f_n$ follows in much the same way. We thus have:

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k \qquad \text{and} \qquad \liminf_n f_n = \sup_n \inf_{k \geq n} f_k$$

Are both measurable, being the inf / sup of measurable functions. Part Two. If $\lim_n f_n$ exists, then it coincides with both $\limsup_n f_n$ and $\liminf_n f_n$, and so f_n is measurable. Part Three. Note that the ω set where $\{f_n(\omega)\}$ converges is:

$$\left[\omega : \limsup_n f_n(\omega) = \liminf_n f_n(\omega)\right]$$

Note, both the functions are measurable. We prove that the set where measurable functions are equal is measurable. Say f and g are measurable. We have:

$$A = [\omega : f(\omega) = g(\omega)] = [\omega : f(\omega) \leq g(\omega)] \cap [\omega : g(\omega) \leq f(\omega)]$$

So now, we just have to prove that the inequality sets are within $\mathcal{F}$. We have:

$$\{f < g\} = \bigcup_{q \in \mathbb{Q}} [\{f \leq q\} \cap \{q < g\}]$$

As if $f(\omega) < g(\omega)$, there must be a rational between them via density of the rationals. Thus, via measurability of g and f, we have $\{f < g\} \in \mathcal{F}$. Now, note $\{g \leq f\} = \{f < g\}^c \in \mathcal{F}$. And so, unwrapping to the top, we have if f and g are measurable:

$$[\omega : f(\omega) = g(\omega)] \in \mathcal{F} \implies \left[\omega : \limsup_n f_n(\omega) = \liminf_n f_n(\omega)\right] \in \mathcal{F}$$

Part Four. We have the ω set where $f_n(\omega) \to f(\omega)$ can be expressed as:

$$\left[\omega : \limsup_n f_n(\omega) = \liminf_n f_n(\omega)\right] \cap \left[\omega : \limsup_n f_n(\omega) = f(\omega)\right] \in \mathcal{F}$$

Where we make use of our equality fact above, and the fact that f is measurable $\mathcal{F}$. qed.

Definition - Simple Function A *simple* real function is one with finite range; it can be put in the form:

$$f = \sum_i x_i \mathbb{1}\{A_i\}$$

Where the A_i form a finite decomposition of Ω. If it clearly measurable $\mathcal{F}$ if each A_i lies in $\mathcal{F}$. Most results for measurable functions are usually first proved for simple functions, and then extended to general measurable functions, by the following theorem:

Theorem 13.5 - Sequence of Simple Functions To Measurable Functions If f is real and measurable $\mathcal{F}$, there exists a sequence $\{f_n\}$ of simple functions, each measurable $\mathcal{F}$, such that:

$$0 \leq f_n(\omega) \uparrow f(\omega) \text{ if } f(\omega) \geq 0$$

And

$$0 \geq f_n(\omega) \downarrow f(\omega) \text{ if } f(\omega) \leq 0$$

Proof: Define:

$$f_n(\omega) = \begin{cases} -n & \text{if } -\infty \leq f(\omega) \leq -n \\ -(k-1)2^{-n} & \text{if } -k2^{-n} < f(\omega) \leq -(k-1)2^{-n} \text{ and } 1 \leq k \leq n2^n \\ (k-1)2^{-n} & \text{if } (k-1)2^{-n} \leq f(\omega) < k2^{-n} \text{ and } 1 \leq k \leq n2^n \\ n & \text{if } n \leq f(\omega) \leq \infty \end{cases}$$

We just need to prove the desired properties for this sequence. First, I want to note that $f_n(\omega)$ is well defined for each pair of (n, ω) - ie, we belong to one of the cases above. Assume that $0 \leq f(\omega)$, but $f(\omega) < n$. So, we can't be in one of the first two cases, and we can't be the last case. Need to prove the third case applies. We have that there must be some k such that:

$$(k-1)2^{-n} \leq f(\omega) < k2^{-n}$$

With $1 \leq k \leq n2^n$. Note the values that this sequence ranges through:

$$0 \leq f(\omega) < 2^{-n} \text{ value: } 0$$
$$2^{-n} \leq f(\omega) < 2 * 2^{-n} \text{ value: } 2^{-n}$$
$$2 * 2^{-n} \leq f(\omega) < 3 * 2^{-n} \text{ value: } 2 * 2^{-n}$$
$$3 * 2^{-n} \leq f(\omega) < 4 * 2^{-n} \text{ value: } 3 * 2^{-n}$$
$$\vdots\vdots$$
$$n - 2 * 2^{-n} \leq f(\omega) < n - 2^{-n} \text{ value: } n - 2 * 2^{-n}$$
$$n - 2^{-n} \leq f(\omega) < n \text{ value: } n - 2^{-n}$$

So, we have these buckets, that clearly partition all the possible places that $f(\omega)$ could be, so $f_n(\omega)$ is well defined. It is also measurable, being a simple function, where each set A is $f^{-1}(a, b)$, which is within $\mathcal{F}$ via measurability of f. Finally, note that the buckets are essentially splitting the interval of f between 0 and n finer and finer and finer. Also note, that when $0 \leq f(\omega)$, we are taking the value of $f(\omega)$ that is smaller than f. Clearly, we have $f_n(\omega) \uparrow f(\omega)$ when $0 \leq f(\omega)$. This is because, we are essentially dividing how "fine" our value for $f(\omega)$ is by two. Here is what we are doing. $f(\omega)$ has a dyadic expansion - see chapter 1. But essentially, $f(\omega) = n + 0.(d_1)(d_2)(d_3)\cdots$. $f_n(\omega)$ increases to that maximum n - and when it reaches it, it then adds the first n decimals (in binary) from below, and keeps on adding them. We clearly must have that $f_1(\omega) \leq f_2(\omega) \leq \cdots$. We also have $f_n(\omega) \to f(\omega)$, as for any $\epsilon > 0$, we have some big enough N such that $2^{-N} < \epsilon$ - and our expansion will be within that distance.

Note, a similar argument can be made for when $f(\omega) \leq 0$. Also, note in the case that $f(\omega) = \pm\infty$, we have $f_n(\omega) \to \pm\infty$ via $\pm n$. So, we have indeed found that if f is real and measurable $\mathcal{F}$, there exists a sequence $\{f_n\}$ of simple functions, each measurable $\mathcal{F}$, such that:

$$0 \leq f_n(\omega) \uparrow f(\omega) \text{ if } f(\omega) \geq 0$$

And

$$0 \geq f_n(\omega) \downarrow f(\omega) \text{ if } f(\omega) \leq 0$$

And we have proved the theorem. qed.

Transformation of Measures

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces, and suppose that the mapping $T : \Omega \to \Omega'$ is measurable $\mathcal{F}/\mathcal{F}'$. Given a measure μ on $\mathcal{F}$, define a set function μT^{-1} on $\mathcal{F}'$ be:

$$\mu T^{-1}(A') = \mu(T^{-1}A') \qquad \text{for } A' \in \mathcal{F}'$$

We have that μT^{-1} is a *measure on* $\mathcal{F}'$. First, note that μT^{-1} is a set function on $\mathcal{F}'$, ie, it is well defined on $\mathcal{F}'$. Take $A' \in \mathcal{F}'$. Note that $T^{-1}A' \in \mathcal{F}$ as T is measurable, and hence $\mu T^{-1}(A')$ has a well defined value. Now, as:

$$T^{-1}\bigcup_n A'_n = \bigcup_n T^{-1}A'_n$$

And the union maintains disjointness (ie, the $T^{-1}A'_n$ are disjoint if the A'_n are), we have countable additivity of μT^{-1} follows from the countable additivity of μ. And as μT^{-1} has the range of μ, and $\mu T^{-1}(\emptyset) = \mu(\emptyset) = 0$, μT^{-1} satisfies all the properties of a *measure on* $\mathcal{F}'$. We will use this transformation many times in our notes.

If μ is finite, so is μT^{-1}; if μ is a probability measure, so is μT^{-1}.

Problems

13.1 Piecewise Functions of Measurable Functions are Measurable

Functions are often defined in pieces (for example, let $f(x)$ be x^3 or x^{-1} as $x \geq 0$ or $x < 0$), and the following result shows that the function is measurable if the pieces are.

Consider the measurable spaces $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ and a map $T : \Omega \to \Omega'$. Let $A_1, A_2, \cdots$ be a countable covering of Ω by $\mathcal{F}$ sets. Consider the σ field in A_n:

$$\mathcal{F}_n = [A : A \subset A_n, A \in \mathcal{F}]$$

And the restriction T_n of T to A_n. Show that T is measurable $\mathcal{F}/\mathcal{F}'$ if and only if T_n is measurable $\mathcal{F}_n/\mathcal{F}'$ for each n.

First, assume that T is measurable $\mathcal{F}/\mathcal{F}'$ for each n. Note, it is possible for these T_n to be measurable. If $B = T^{-1}(B')$ for $B' \in \mathcal{F}'$, we have that $T_n^{-1} = B \cap A_n$. Note, as $A_n \in \mathcal{F}$, and $B \in \mathcal{F}$, we have $B \cap A_n \in \mathcal{F}$, and $B \cap A_n \subset A_n$, so $B \cap A_n \in \mathcal{F}_n$. Also, it is easy to show that $\mathcal{F}_n$ is a σ field on A_n. Both of these facts prove that T_n is measurable on $\mathcal{F}_n/\mathcal{F}'$ for each n, if T is measurable $\mathcal{F}/\mathcal{F}'$.

Anyway. Now, we just assume that T_n is measurable $\mathcal{F}_n/\mathcal{F}'$ for each n. Take $B' \in \mathcal{F}'$. We have that:

$B = T^{-1}(B')$ and we can decompose B as the disjoint union: $B = \bigcup_n (A_n \cap B)$

Note, $(A_n \cap B) = T_n^{-1}(B') \implies A_n \cap B \in \mathcal{F}_n$. Thus, by definition, we have $A_n \cap B \in \mathcal{F}$ for each n. Thus, B is a disjoint countable union of $\mathcal{F}$ sets, and T is measurable $\mathcal{F}/\mathcal{F}'$. qed.

And so, we have concluded that T is measurable $\mathcal{F}/\mathcal{F}'$ if and only if T_n is measurable $\mathcal{F}_n/\mathcal{F}'$ for each n. As for why this proves the piecewise point. Well, it proves the piecewise statement, if the individual pieces are defined on disjoint, measurable sets. Then, T_n represents each of the pieces on measurable set A_n, and if T_n is measurable on the entire space, it will also be measurable with respect to $(A_n, \mathcal{F}_n)$, when restricted to A_n. This proves the piecewise statement.

13.2 Sigma Fields Generated by a map T

1. For a map T and σ fields $\mathcal{F}$ and $\mathcal{F}'$, define $T^{-1}\mathcal{F}' = [T^{-1}A' : A' \in \mathcal{F}']$ and $T\mathcal{F} = [A' : T^{-1}A' \in \mathcal{F}]$. Show that $T^{-1}\mathcal{F}'$ and $T\mathcal{F}$ are σ fields, and that measurability $\mathcal{F}/\mathcal{F}'$ is equivalent to $T^{-1}\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F}' \subset T\mathcal{F}$.

   First, we want to prove that $T^{-1}\mathcal{F}'$ is a σ field. I think it is clear that it is closed under unions, by what we found above:

   $$T^{-1} \bigcup_n A'_n = \bigcup_n T^{-1} A'_n$$

   So, the RHS is an arbitrary union, and the LHS is an inverse of a set within $\mathcal{F}'$, which is in the σ field. We also note that it is closed under complements:

   $$\left[T^{-1}A'_n\right]^c = T^{-1}\left[(A'_n)^c\right] \in T^{-1}\mathcal{F}'$$

   Finally, note that $T^{-1}\mathcal{F}'$ must contain the emptyset. Thus, $T^{-1}\mathcal{F}'$ is a σ field on Ω. Now, as for $T\mathcal{F}$, we make use of the same facts as above (union and complement expressions above apply for $T\mathcal{F}$) to prove it is a σ algebra.

   So, we just need to prove that measurability $\mathcal{F}/\mathcal{F}'$ is equivalent to $T^{-1}\mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F}' \subset T\mathcal{F}$. First, assume T is measurable $\mathcal{F}/\mathcal{F}'$. Take $A \in T^{-1}\mathcal{F}' \implies A = T^{-1}A' : A' \in \mathcal{F}' \implies A \in \mathcal{F}$. Now, take

$A' \in \mathcal{F}' \implies T^{-1}A' \in \mathcal{F} \implies A' \in T\mathcal{F}$. We thus have one direction. Now, assume:

$$T^{-1}\mathcal{F}' \subset \mathcal{F} \quad \text{and} \quad \mathcal{F}' \subset T\mathcal{F}$$

We want to show that this means T is measurable $\mathcal{F}/\mathcal{F}'$. Take $A' \in \mathcal{F}'$. By assumption, we have:

$$A' \in T\mathcal{F} \implies T^{-1}A' \in \mathcal{F}$$

And thus, T is measurable, by definition.

2. For given $\mathcal{F}'$, $T^{-1}\mathcal{F}'$, which is the smallest σ field for which T is measurable $\mathcal{F}/\mathcal{F}'$, is by definition the σ field *generated* by T. Ie - we have a σ field in the range, and we create a σ field $T^{-1}\mathcal{F}'$ in the domain by just saying the inverse of $\mathcal{F}'$ sets are in our sigma field $\mathcal{F} = T^{-1}\mathcal{F}'$. Note, any other σ field $\mathcal{F}$ such that T is measurable $\mathcal{F}/\mathcal{F}'$ must satisfy:

$$T^{-1}\mathcal{F}' \subset \mathcal{F}$$

For simple random variables describe $\sigma(X_1, \cdots, X_n)$ in these terms.

Recall what $\sigma(X_1, \cdots, X_n)$ means. We have that $X_1, \cdots, X_n$ are simple random variables, which in this section, for generalized functions, means that:

$$X_i : \Omega \to \Omega' \qquad X_i(\omega) = \sum_{j=1}^{n_i} x_i \mathbb{1}\{A_j\}$$

Where the A_j form a finite decomposition of Ω, and as we assume X_i is measurable, then the A_j must be measurable. $\sigma(X_i)$ is the smallest σ algebra such that X_i, as a function, is measurable. And so, we must have:

$$\sigma(X_i) = X_i^{-1}\mathcal{F}'$$

Where $\mathcal{F}'$ is the σ algebra corresponding to Ω'. This comes from noting that $X_i^{-1}\mathcal{F}'$ is the smallest σ algebra for which X_i is measurable $\mathcal{F}/\mathcal{F}'$ - which we just noted above, and which corresponds with the definition of the generated σ algebra for X_i. In terms of a *real* simple random variable, this would be:

$$\sigma(X_i) = X_i^{-1}\mathcal{R}$$

Now, the final point to consider is - how can we express $\sigma(X_1, \cdots, X_n)$? This would be the smallest σ algebra such that each X_i is measurable. Note, we must have:

$$\sigma(X_i) \subseteq \sigma(X_1, \cdots, X_n)$$

I think, if we had some notion of product σ algebra, like $(\mathcal{F}')^n$, we could define it as something like:

$$(X_1, \cdots, X_n)^{-1}((\mathcal{F}')^n)$$

Recall that Theorem 5.1 stated that the σ-field $\sigma(X_1, \cdots, X_n)$ consists of the sets:

$$[\omega : (X_1, \cdots, X_n) \in H]$$

For $H \subset \mathbb{R}^n$. Note, in chapter 5, we weren't dealing with the borel sets yet. However, I think perhaps my initial intuition wasn't off, at least for real simple random variables. We have $\mathcal{R}^n$, which could be considered the product σ algebra, I guess. I want to say something like, for *real simple random variables*, we have:

$$\sigma(X_1, \cdots, X_n) = (X_1, \cdots, X_n)^{-1}(\mathcal{R}^n)$$

Note, I think equality can be noted like this. We have:

$$\sigma(X_1, \cdots, X_n) = [\omega : (X_1, \cdots, X_n) \in H \subseteq \mathbb{R}^n]$$

However, note that while $H \in \mathbb{R}^n$, it really only matters which finite points H contains, as their are only finite points that $(X_1, \cdots, X_n)$ can take on. And for any set of finite points - we can create a borel set that contains those points, and none of the other points in the list. Ie, say we have points that the product can take on:

$$x_1, \cdots, x_{n+m} \text{ and } x_1, \cdots, x_n \in H \qquad x_{n+1}, \cdots, x_{n+m} \notin H$$

As the points are disjoint, we can create tiny open balls containing each point $x_1, \cdots, x_n$ that are disjoint from *all* the other points (Hausdorff-ness). Open balls are open, and thus in $\mathcal{R}^n$, and the union of open balls is open, and also in $\mathcal{R}^n$. Call this union $B \in \mathcal{R}^n$. Note:

$$[\omega : (X_1, \cdots, X_n) \in H] = [\omega : (X_1, \cdots, X_n) \in B]$$

For the specific H and B. And so, it should be equivalent, for the case of real simple random variables, that:

$$[\omega : (X_1, \cdots, X_n) \in H \subseteq \mathbb{R}^n] = [\omega : (X_1, \cdots, X_n) \in H \subseteq \mathcal{R}^n] = (X_1, \cdots, X_n)^{-1}(\mathcal{R}^n)$$

3. Let $\sigma'(\mathcal{A}')$ be the σ-field in Ω' generated by $\mathcal{A}'$. Show that $\sigma(T^{-1}\mathcal{A}') = T^{-1}(\sigma'(\mathcal{A}'))$. Prove Theorem 10.1 by taking T to be the identity map from Ω_0 to Ω.

First, we note $T : \Omega \to \Omega'$, and we want to show that for some set of Ω' sets $\mathcal{A}'$, we have:

$$\sigma(T^{-1}\mathcal{A}') = T^{-1}(\sigma'(\mathcal{A}'))$$

First note that:

$$T^{-1}(\sigma'(\mathcal{A}')) \subseteq \sigma(T^{-1}\mathcal{A}')$$

The LHS is the smallest σ algebra for which T is measurable $T^{-1}(\sigma'(\mathcal{A}'))/\sigma'(\mathcal{A}')$. Note that by Theorem 13.1.i, if $T^{-1}(A') \in \mathcal{F}$ for all $A' \in \mathcal{A}'$, then T is measurable $\mathcal{F}/\sigma'(\mathcal{A}')$. Note that $T^{-1}(A') \in T^{-1}\mathcal{A}' \subseteq \sigma(T^{-1}\mathcal{A}')$ for all $A' \in \mathcal{A}'$, and so T is measurable $\sigma(T^{-1}\mathcal{A}')/\sigma'(\mathcal{A}')$. Smallest σ algebra gives the inclusion above. Now, we want to show:

$$\sigma(T^{-1}\mathcal{A}') \subseteq T^{-1}(\sigma'(\mathcal{A}'))$$

By definition, $\sigma(T^{-1}\mathcal{A}')$ is generated by $T^{-1}\mathcal{A}'$. Note that:

$$T^{-1}\mathcal{A}' \subseteq T^{-1}(\sigma'(\mathcal{A}'))$$

As the RHS is a σ field (proved in part a), we have definition of generation implies:

$$\sigma(T^{-1}\mathcal{A}') \subseteq T^{-1}(\sigma'(\mathcal{A}'))$$

As we have proved both inclusions, we can indeed conclude that:

$$\sigma(T^{-1}\mathcal{A}') = T^{-1}(\sigma'(\mathcal{A}'))$$

The last part is to prove Theorem 10.1. Theorem 10.1.i states that if $\Omega_0 \subseteq \Omega$, then if $\mathcal{F}$ is a σ field in Ω, then $\mathcal{F} \cap \Omega_0$ is a σ field in Ω_0. We prove this with the above results. Take T to be the identity map from Ω_0 to Ω - note, it only maps into a subset of Ω.

We have, part a implies that $T^{-1}(\mathcal{F})$ is a σ field - note, we have:

$$T^{-1}(\mathcal{F}) = \mathcal{F} \cap \Omega_0$$

As the inverse only yields the portion of $A \in \mathcal{F}$ that is within Ω_0. Now, for Theorem 10.1.ii, recall that it states if $\mathcal{A}$ generates the σ-field $\mathcal{F}$ in Ω, then $\mathcal{A} \cap \Omega_0$ generates the σ field $\mathcal{F} \cap \Omega_0$. Ie, we need to show:

$$\sigma(\mathcal{A} \cap \Omega_0) = \sigma(\mathcal{A}) \cap \Omega_0$$

Well, let T still be the identity function, and we have by this part of the question:

$$\sigma(T^{-1}\mathcal{A}') = T^{-1}(\sigma'(\mathcal{A}'))$$

As T^{-1} only takes the part in Ω_0, the above really is just saying:

$$\implies \sigma(\mathcal{A} \cap \Omega_0) = \sigma(\mathcal{A}) \cap \Omega_0$$

13.3 Existence of Intermediate Measurable Function

Suppose that $f : \Omega \to \mathbb{R}^1$. Show that f is measurable $T^{-1}\mathcal{F}'$ if and only if there exists a map $\varphi : \Omega' \to \mathcal{R}^1$ such that φ is measurable $\mathcal{F}'$, and $f = \varphi \circ T$.

First, one direction is easy. Suppose that $\varphi : \Omega' \to \mathcal{R}^1$ is measurable $\mathcal{F}'/\mathcal{R}$, and $f = \varphi \circ T$. Note, T is measurable $T^{-1}\mathcal{F}'/\mathcal{F}'$ automatically. By Theorem 13.1.ii, we directly have that f must be measurable $T^{-1}\mathcal{F}'/\mathcal{R}^1$.

Now, we assume that f is measurable $T^{-1}\mathcal{F}'/\mathcal{R}^1$, and we want to show that there exists $\varphi : \Omega' \to \mathcal{R}^1$ measurable $\mathcal{F}'/\mathcal{R}$ such that $f = \varphi \circ T$, where $T : \Omega \to \Omega'$.

We follow the hint. We first consider f as a simple function. And so, we must have:

$$f = \sum_{i=1}^{n} x_i \mathbb{1}\{A_i\}$$

For $A_i \in T^{-1}\mathcal{F}'$, as f is measurable $T^{-1}\mathcal{F}'/\mathcal{R}^1$. We thus have $A_i' \in \mathcal{F}'$ such that:

$$A_i = T^{-1}A_i'$$

Define $\varphi : \Omega' \to \mathcal{R}^1$ as:

$$\varphi = \sum_{i=1}^{n} x_i \mathbb{1}\{A_i'\}$$

Clearly, φ is measurable $\mathcal{F}'$. We also clearly have:

$$f = \varphi \circ T$$

So, we have proved the theorem for f simple function. Now, assume that f is not a simple function. We have by Theorem 13.5, there exists a sequence $\{f_n\}$ of simple functions, each measurable $T^{-1}\mathcal{F}'/\mathcal{R}$, such that:

$$0 \leq f_n(\omega) \uparrow f(\omega) \text{ if } f(\omega) \geq 0 \text{ and } 0 \geq f_n(\omega) \downarrow f(\omega) \text{ if } f(\omega) \leq 0$$

Take our φ_n to be the one defined above for f_n simple function. Let C' be the set of ω' for which $\varphi_n(\omega')$ has finite limit, and define:

$$\begin{cases} \varphi(\omega') = \lim_n \varphi_n(\omega') & \text{for } \omega' \in C' \\ \varphi(\omega') = 0 & \text{otherwise} \end{cases}$$

Note, we have that φ is measurable $\mathcal{F}'$, being the limit of measurable functions on the set where the limit exists, and 0 (simple) on the complement of that set. Now, we have:

$$f(\omega) = \lim_n f_n(\omega) = \lim_n \varphi_n \circ T(\omega) = \varphi(T(\omega))$$

And we have proved the if and only if. qed.

13.4 Equivalence Between Problem 13.3 And Theorem 5.1(ii)

Relate the result in Problem 13.3 to Theorem 5.1(ii).

Recall, Theorem 5.1(ii) said a simple random variable Y is measurable $\sigma(X_1, \cdots, X_n)$ if and only if:

$$Y = f(X_1, \cdots, X_n)$$

For some $f : \mathbb{R}^n \to \mathbb{R}^1$. Problem 13.3, which we proved above, states that $f : \Omega \to \mathbb{R}^1$ is measurable $T^{-1}\mathcal{F}'$ for $T : \Omega \to \Omega'$ if and only if there exists a map $\varphi : \Omega' \to \mathcal{R}^1$ such that φ is measurable $\mathcal{F}'$, and $f = \varphi \circ T$.

By Problem 13.2(b), we can equate SRV Y being measurable $\sigma(X_1, \cdots, X_n)$ as Y being measurable:

$$(X_1, \cdots, X_n)^{-1}(\mathcal{R}^n)$$

For X_i SRV. We let $T : \Omega \to \mathbb{R}^n$ defined by $(X_1, \cdots, X_n)$. We thus have, $Y : \Omega \to \mathbb{R}^1$ is measurable $T^{-1}(\mathcal{R}^n)$, if and only if for some $f : \mathbb{R}^n \to \mathbb{R}^1$, we have:

$$Y = f \circ T$$

The only difference from this statement, and the one from problem 13.3, is that problem 13.3 states that f is also measurable $\mathcal{R}^n/\mathcal{R}$. I think, we can get rid of this extra line in Theorem 5.1(ii), as we are only dealing with SRV.

13.5 Measurability of Addition Directly

Show of real functions f and g that $f(\omega) + g(\omega) < x$ if and only if there exist rationals r and s such that $r + s < x$, $f(\omega) < r$ and $g(\omega) < s$. Prove directly that $f + g$ is measurable $\mathcal{F}$ if f and g are.

Note, there is a rational between both $f(\omega) < x - g(\omega)$, call it r, and a rational between $g(\omega) < x - r$, call it s. Note that:

$$f(\omega) + g(\omega) < r + s < r + x - r = x$$

So, we have proved the first statement. Now, assume both f and g are measurable ($\mathcal{R}$). $f+g$ is measurable $\mathcal{F}$ if we have:

$$(f+g)^{-1}(-\infty, x) \in \mathcal{F}$$

For each x. Note, we have:

$$(f+g)^{-1}(-\infty, x) = \bigcup_{(r,s)\in\mathbb{Q}^2} f^{-1}(-\infty, r) \cap g^{-1}(-\infty, s)$$

Where r and s are defined as above. This is a countable union, and it is non empty. Clearly, we have $\supseteq$ - but what about the other direction, $\subseteq$? Well, for $\omega \in \Omega$ that satisfies $f(\omega) + g(\omega) < x$, it will be in one of those rational pairs. So, we have equality. It is a countable union because $\mathbb{Q}^2$ is countable. Finally, as f and g are measurable, the set on the left is within $\mathcal{F}$, for each x. And so, $f+g$ is measurable. qed.

13.7 Baire and Borel Function Equivalence

Consider on $\mathbb{R}^1$ the smallest class H (that is, the intersection of all classes) of real functions containing all the continuous functions and closed under pointwise passages to the limit. The elements of H are called *Baire* functions. Show that the Baire functions and the Borel functions on $\mathbb{R}^1$ are the same thing.

I am going to start with another definition of the Baire functions. Note that the continuous functions $C(\mathbb{R}, \mathbb{R})$ are not closed under pointwise limits. And so, we can form the following Baire classes:

1. Baire Class 0 - All continuous functions.
2. Baire Class 1 - Pointwise limits of sequences of Baire class 0 functions (some places say either includes/excludes Baire class 0 functions, I will leave ambiguous for now).
3. Baire Class 2 - Pointwise limits of sequences of Baire class 1 functions.
4. And so on.

Note - the Baire functions would take a union of these classes. However, this gets into ordinals - something I am not super familiar with yet. We have class ω is:

$$\text{Baire Class } \omega = \bigcup_{i=0}^{n} \text{Baire Class } n$$

But then, we can take pointwise limits of class ω sets to get Baire Class $\omega+1$. Note, after going through $\omega+1, \omega+2, \cdots$, we get to $\omega \cdot 2$, or $\omega+\omega$. We can continue on, taking unions, getting $\omega \cdot 2+1, \cdots$ to $\omega \cdot 3, \omega \cdot 4$, and so on. We form the set of ordinals:

$$\omega \cdot m + n$$

For natural numbers m, n. This has an ordinal associated it with it as well - the union of these, which would be represented by ω^2. Note, I am describing it as "union," but it really means something like "upper bound" (see the Ordinal Number Wiki). We will ultimately get something like ω^3, as the union of sets:

$$\omega^2 \cdot m + n$$

And os on, and so on. We can even get to ϵ_0 (epsilon naught), which would be:

$$\epsilon_0 = \omega^{\omega^{\omega^{\cdots}}} = \sup\left\{\omega, \omega^\omega, \omega^{\omega^\omega}, \cdots\right\}$$

Note, in each case - the ordinal is the upper bound/union of a *countable class*. Call these *countable ordinals*. The smallest *uncountable ordinal* is the set of all countable ordinals, expressed as ω_1. I believe the Baire functions might be the union of all countable ordinals - see the Baire Function Wiki, which defines the Baire functions for each countable ordinal.

So, for the Borel functions on $\mathbb{R}^1$, are the functions $f : \mathbb{R} \to \mathbb{R}$ that are measurable $\mathcal{R}/\mathcal{R}$. In the previous chapter, we proved that continuous functions are measurable, and pointwise limits of measurable functions are measurable. And so, we must have:

$$H \subseteq B$$

Where we let B here stand for the Borel functions. This is because, the pointwise limits of any measurable function are still contained within the borel set B - ie, we contain all the unions describing the Baire functions above.

Now, for the other direction. Taken from the solutions. First, note that H is closed under $f+g, fg, f-g$, and $f \vee g$ (which is maximum). First, take $f, g \in H$. I think, the book says to note that:

$$[g : f+g \in H]$$

Is closed under the passage to limits. We need to note - if we have some $g_1, g_2, \cdots$, such that each $f+g_1 \in H, f+g_2 \in H, \cdots$, which has a limit g - then $f+g \in H$. Well, as $f+g_i \in H$ - we have:

$$\lim_i f + g_i \in H$$

As H is closed under pointwise limits. As $\lim_i f + g_i = f + g \in H$, we have that g is in the above set. Now, why does this imply that if $f, g \in H$, then $f + g \in H$? Well, I think it is true, if we note:

$$[g : f + g \in H] = H$$

And I think that is indeed the case, if we prove that it contains the continuous functions (as then it is the Baire functions, definitionally). It should - as the addition of two continuous functions are continuous - and $f + g$ can then be expressed all the way back as a pointwise limit of some sort. Note, it is true if f is continuous, and if f is the pointwise limit of continuous functions, and so on.

So yes, we can conclude that $f + g \in H>$ We can similarly conclude $fg \in H$, as the $\lim_i fg_i = fg$, and the same limit property also applies to $f - g$, $f \vee g$. And so, we do have H is closed under addition, multiplication, subtraction, and maximum.

Define:

$$f_n(x) = \begin{cases} 1 & \text{if } x \leq \alpha \\ 1 - n(x - \alpha) & \text{if } \alpha \leq x \leq \alpha + n^{-1} \\ 0 & \text{if } \alpha + n^{-1} \leq x \end{cases}$$

This is just the line to 1 at α, and then a downward slope. Clearly, the pointwise limit is:

$$\lim_n f_n(x) = \mathbb{1}\{x \in (-\infty, a]\}$$

We will now show that:

$$L = [A : \mathbb{1}\{A\} \in H]$$

Is a λ system. Ie - $\mathbb{R} \in L$, $A \in L \implies A^c \in L$, and *disjoint* union $\bigcup_{i=1}^{\infty} A_i \in L$. If we do that, as L clearly contains sets $(-\infty, a]$ by the above (which is a π system, clearly closed under intersection), the $\pi - \lambda$ theorem would tell us that the above set contains $\sigma(-\infty, a] = \mathcal{R}$. In which case, H would contain all indicators of Borel Sets first, then all simple borel functions second via closed under addition. Finally, H would contain all Borel functions via pointwise limits of Borel functions, and Theorem 13.5.

First, $\mathbb{R} \in L$, as $\mathbb{1}\{\mathbb{R}\}$ is the continuous constant function 1, which is in H via continuity. Now, we prove complements. We have $\mathbb{1}\{A^c\} = 1 - \mathbb{1}\{A\} \in H$

as H is closed under subtraction. Finally, we clearly have for disjoint A_i:

$$\mathbb{1}\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \lim_{i\to\infty} \mathbb{1}\left\{A_1\right\} + \cdots + \mathbb{1}\left\{A_i\right\} \in H$$

The limit clearly exists pointwise, and each sum is within H closed under addition. And so, we indeed have that L is a λ system, and using the steps outlined above, we can conclude:

$$B \subset H$$

In total, we thus have:

$$B = H$$

And so the Baire functions and the Borel functions are the same thing. qed.

13.9 Egoroff's Theorem (Uniform Convergence on a Pointwise Convergent Subset)

Suppose that f_n and f are finite-valued, $\mathcal{F}$-measurable functions such that $f_n(\omega) \to f(\omega)$ for $\omega \in A$, where $\mu(A) < \infty$ and μ is a measure on σ field $\mathcal{F}$.

Prove *Egoroff's Theorem:* For each ϵ there exists a subset B of A such that $\mu(B) < \epsilon$ and $f_n(\omega) \to f(\omega)$ uniformly on $A - B$.

We follow the hint. Define:

$$B_n^{(k)} = \left\{\omega \in A : |f(\omega) - f_i(\omega)| > k^{-1} \text{ for some } i \geq n\right\}$$

I think, we should first note that $B_n^{(k)}$ is within $\mathcal{F}$. $f - f_i$ is a measurable function via subtraction, and an absolute value is just a maximum of g and $-g$, both of which are measurable. So, $B_n^{(k)}$ is a countable union (for each f_i) of inverse images of $\mathcal{R}$ sets (k^{-1}, ∞) on measurable functions, and so indeed $B_n^{(k)} \in \mathcal{F}$.

Now, note that $B_n^{(k)} \downarrow \emptyset$ as $n \to \infty$. Clearly, $B_n^{(k)} \supseteq B_{n+1}^{(k)}$, as $B_n^{(k)}$ contains an additional function f_n for which the condition on ω can apply. Further, we have:

$$\bigcap_{n=1}^{\infty} B_n^{(k)} = \emptyset$$

As if the intersection contained an ω, we would have an $\omega \in A$ such that $f_n(\omega) \not\to f(\omega)$ in a pointwise limit, which would be a contradiction. Finally, as $\mu(A) < \infty$, we have that $\mu(B_n^{(k)}) < \infty$, and so we can apply continuity from above to find an n_k for each k such that:

$$\mu(B_{n^k}^{(k)}) < \frac{\epsilon}{2^k}$$

Define:

$$B = \bigcup_{k=1}^{\infty} B_{n^k}^{(k)}$$

First, note $\mu(B) < \epsilon$ via countable subadditivity. Now, note that $f_n \to f$ uniformly on $A - B$. Take an $\epsilon > 0$. There exists a k such that $0 < k^{-1} < \epsilon$. Note for all $i \geq n_k$, and any $\omega \in A - B$, we have:

$$\omega \notin B_{n^k}^{(k)} \implies |f(\omega) - f_i(\omega)| \leq k^{-1} < \epsilon$$

Thus, the definition of uniform convergence is satisfied.

13.12 Circular Lebesgue Measure

Let C be the unit circle in the complex plane, and defined:

$$T : [0, 1) \to C \qquad T(\omega) = e^{2\pi i\omega}$$

Let $\mathcal{B}$ consist of the Borel subsets of $[0, 1)$, and let λ be the Lebesgue measure on $\mathcal{B}$. Show that $\mathcal{C} = [A : T^{-1}A \in \mathcal{B}]$ consists of the sets in $\mathcal{R}^2$ (identify $\mathbb{R}^2$ with the complex plane) that are contained in C. Show that $\mathcal{C}$ is generated by the arcs of C. Circular Lebesgue measure is defined as:

$$\mu = \lambda T^{-1}$$

Show that μ is invariant under rotations: $\mu[\theta z : z \in A] = \mu(A)$ for $A \in \mathcal{C}$ and $\theta \in C$.

1. So, I think a lot of the properties we can just forward from λ - rotation invariance via translation invariance, arc generation via interval generation. However, we need to first show that $\mathcal{C}$ is the sets in $\mathcal{R}^2$ within C.

Realize that we proved $\mathcal{C}$ is a σ algebra in question 13.2. First, note that $C \in \mathcal{R}^2$. We can define $f : \mathbb{R}^2 \to \mathbb{R}^2$ as $f(\theta, r) = (\theta, r - 1)$. This is a continuous function. And so Theorem 13.2 implies that f is measurable, and $C = f^{-1}\left(\{(t, 0) : t \in [0, 2\pi]\}\right) \in \mathcal{R}^2$ (this is the inverse of an intersection of bounded rectangles which is in $\mathcal{R}^2$). Thus, the sets in $\mathcal{R}^2$ that are contained in C can be defined as:

$$C \cap \mathcal{R}^2 = \left\{A \cap C : A \in \mathcal{R}^2\right\}$$

As if $A \in \mathcal{R}^2$, the intersection with an $\mathcal{R}^2$ set will remain within $\mathcal{R}^2$, and if $A \in \mathcal{R}^2$ is already contained within C, we have $A \cap C = A$. We thus want to show:

$$\mathcal{C} = C \cap \mathcal{R}^2$$

Take an $A \in \mathcal{C}$. We have $T^{-1}A \in \mathcal{B}$. I think, we can use f again. Examine $T^{-1}A \times \{0\}$. I believe we can conclude that the product of two borel sets is within $\mathcal{R}^2$, and so $f^{-1}(T^{-1}A \times \{0\}) = A$ must be within $\mathcal{R}^2$. This would imply $\mathcal{C} \subseteq C \cap \mathcal{R}^2$. Note - we have $T^{-1}A \times \{0\} \in \mathcal{R}^2$ via example 18.1 - read ahead, it is easy.

Sub Proof: $\mathcal{R} \times \mathcal{R} = \mathcal{R}^2$ For one, recall that $\mathcal{R}^2$ is generated by the bounded rectangles (a semiring). The bounded rectangles are clearly contained within $\mathcal{R} \times \mathcal{R}$, so $\mathcal{R}^2 \subseteq \mathcal{R} \times \mathcal{R}$. The other direction is more difficult. If A is an interval, $[B : A \times B \in \mathcal{R}^2]$ contains $\mathbb{R}^1$ ($A \times \mathbb{R}^1$ can be represented as a union of bounded rectangles), and is closed under the formation of countable unions, and proper differences: $A \times B_1 - A \times B_2 = A \times (B_1 - B_2) \in \mathcal{R}^2 \implies B_1 - B_2$ is in the set. This is equivalent to closed under complements, and so $[B : A \times B \in \mathcal{R}^2]$ is a σ field containing the intervals, and thus contains the Borel sets. Therefore, if B is a Borel set, $[A : A \times B \in \mathcal{R}^2]$ contains the intervals, and hence, being a σ field, contains the borel sets. Thus, all measurable rectangles (product of measurable $A, B \in \mathcal{R}$) are within $\mathcal{R}^2$. thus, we do have:

$$\mathcal{R} \times \mathcal{R} = \mathcal{R}^2$$

qed.

In which case, we can indeed conclude that:

$$\mathcal{C} \subseteq C \cap \mathcal{R}^2 \text{ as } A \in \mathcal{C} \implies A = f^{-1}(T^{-1}A \times \{0\}) \in C \cap \mathcal{R}^2$$

Now, for the other direction. Note that $T : [0, 1) \to C$ is actually continuous from $[0, 1) \to \mathbb{R}^2$. It just stays inside of C. As it is continuous, it is thus measurable $\mathcal{B}/\mathcal{R}^2$. Take $A \cap C$ for $A \in \mathcal{R}^2$. Note that $T^{-1}(A \cap C) \in \mathcal{B}$. Thus, by definition, we must have $A \cap C \in \mathcal{C}$. Thus, we have:

$$\mathcal{C} \supseteq C \cap \mathcal{R}^2$$

And so, we have equality - $\mathcal{C} = C \cap \mathcal{R}^2$.

2. Now, we want to prove that $\mathcal{C}$ is generated by the arcs of C. We make use of Theorem 10.1. We have that the bounded rectangles in $\mathbb{R}^2$, which we call

$\mathcal{A}$, generate $\mathcal{R}^2$. Theorem 10.1.ii thus implies that $\mathcal{A} \cap C$ generates $C \cap \mathcal{R}^2$. By the above, we thus have:

$$\mathcal{C} = \sigma(\mathcal{A} \cap C)$$

Note that for a bounded rectangle A, $A \cap C$ is just an arc of C. I don't think I need to prove that rigorously, it is just clear visually.

3. Show that $\mu = \lambda T^{-1}$ is invariant under rotation. Note, as T is measurable $\mathcal{B}/\mathcal{C}$, λT^{-1} is indeed a measure on $\mathcal{C}$. Recall that λ is invariant under translation. Take $A \in \mathcal{C}$. For $\theta \in [0,1]$, let θz just be z rotated. Anyway - it should be clear that:

$$T^{-1}(\theta A) = T^{-1}(A) + \theta$$

This is because, we have $z = e^{2\pi i \omega}$ is rotated a $\theta \in [0,1]$ fraction of 2π via multiplying:

$$e^{2\pi i \omega} e^{2\pi i \theta} = e^{2\pi i(\omega+\theta)} = \cos(2\pi(\omega+\theta)) + i\sin(2\pi(\omega+\theta))$$

And, we have:

$$T^{-1}(\theta z) = T^{-1}(e^{2\pi i(\omega+\theta)}) = \omega + \theta$$

And so, in total:

$$\mu(\theta A) = \lambda \circ T^{-1}(\theta A) = \lambda(T^{-1}A + \theta) = \lambda(T^{-1}(A)) = \mu(A)$$

And μ is thus invariant under rotation. qed.

13.13 - Finite Points On The Circle Rotated Into a Small Set

Suppose that the circular Lebesgue measure of A satisfies $\mu(A) > 1 - n^{-1}$ and that B contains at most n points. Show that some rotation carries B into A - ie, $\theta B \subset A$ for some $\theta \in C$.

Looking at the solutions. $B = \{b_1, \cdots, b_k\}$ for $k \leq n$. Note, each element $b_i \in C$ can be considered a rotation in C. Let $E_i = C - b_i^{-1}A$, ie C with A translated b_i^{-1} removed. Let:

$$E = \bigcup_{i=1}^{k} E_i = C - \bigcup_{i=1}^{k} b_i^{-1} A$$

Note, as μ is invariant under rotation, we have:

$$\mu(E_i) = 1 - \mu(A) < n^{-1}$$

This in turn implies that $\mu(E) < 1$, and $C - E$ is nonempty. Thus:

$$C - E = \bigcap_{i=1}^{k} b_i^{-1} A$$

Is nonempty. Take any $\theta \in C - E$. Note, $\theta B \subset A$. This is because, $\theta = b_i^{-1} a$ for some $a \in A$, and $\theta b_i = b_i^{-1} a b_i = a b_i^{-1} b_i = a$. Note - rotations are commutative. qed.

At a high level - the argument here is just - examine the image of A under the rotation b_i^{-1} for each b_i. The intersection of these images is nonempty - and so there is one rotation that will put each b_i into A.

Section 14 - Distribution Functions

Notes

Distribution Functions

Definition - Random Variable A random variable X is a measurable real function X on a probability measure space $(\Omega, \mathcal{F}, P)$.

Definition - Distribution/Law of a Random Variable The *distribution* or *law* of a random variable X is the probability measure on $(\mathbb{R}^1, \mathcal{R}^1)$ *induced* by the random variable, defined by:

$$\mu(A) = P[X \in A] \qquad A \in \mathcal{R}^1$$

Note, the argument ω is usually omitted - it is:

$$P[\omega | X(\omega) \in A] = P[X \in A]$$

Previously, we showed that μ was indeed a measure on $(\mathbb{R}, \mathcal{R})$, and the notation we gave for it was:

$$\mu = PX^{-1}$$

Definition - Distribution Function of X Is defined by:

$$F(x) = \mu(-\infty, x] = P[X \leq x]$$

For real x. Note, via continuity from above for μ (as P is a finite probability measure), we have that F is *right continuous*. F is also clearly *nondecreasing*.

As F is nondecreasing, we have that the left hand limit for x at least *exists.* This is via monotone convergence - the limit is just the supremum of the set. So, while F might not be continuous from the left - at least a limit exists. Define it as $F(x-) = \lim_{y \uparrow x} F(y)$. By continuity from below on μ, we have:

$$F(x-) = \lim_{y \uparrow x} F(y) = \lim_{y \uparrow x} \mu(-\infty, y] = \mu(-\infty, x) = P[X < x]$$

Thus, the jump or *saltus* in F at x is:

$$F(x) - F(x-) = \mu\{x\} = P[X = x]$$

Now, recall Theorem 10.2iv. If P is σ finite on $\mathcal{F}$, then $\mathcal{F}$ cannot contain an uncountable, disjoint collection of sets of positive P measure. Now, P is finite, and thus clearly σ finite. And so, by the Theorem, as $[X = x] \in \mathcal{F}$ (as $\{x\}$ a singleton is within $\mathcal{R}$), we can have at most countably many $x \in \mathbb{R}$ such that:

$$P[X = x] > 0 \implies F(x) - F(x-) > 0$$

And so, F can have at most countably many points of discontinuity. Note, as our last property of F, we have:

$$\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1$$

These are again clear.

Theorem 14.1 - Distribution Functions Have Corresponding Random Variables If F is a nondecreasing, right-continuous function satisfying:

$$\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1$$

Then there exists on some probability space a random variable X for which $F(x) = P\left[X \leq x\right]$.

Side Note: This is also technically a proof that any F, as defined, cannot have an uncountable number of jumps. The existence of an uncountable amount of jumps in such an F would imply an uncountable amount of sets $X = x$ that are disjoint, which our finite probability measure would given non zero probability to. However, you can prove it much more directly:

Assume an uncountable number of jumps, for a nondecreasing function bounded between zero and one. You must have a countable number greater than some ϵ. If not - take $1/n, 1/(n+1), 1/(n+2)$, so on - the limit of which is zero. Note the union of jumps greater than $1/n, 1/(n+1), 1/(n+2)$ and so on must contain each jump - as each jump is positive. If each element of the union was finite (or even countable), the union would be countable - leading to a countable set equaling an uncountable one (a contradiction). So, for some $1/n$, you have a countable number of jumps larger than $1/n$ - the sum of which is infinite, greater than the bound. qed.

First Proof: This relies on Theorem 12.4, which states that if F is a non-decreasing, right-continuous real function on the line, there exists on $\mathcal{R}^1$ a unique measure μ satisfying:

$$\mu(a,b] = F(b) - F(a)$$

For all $a, b \in \mathbb{R}$. So, we take the corresponding μ. Note, we have:

$$\lim_{y\to-\infty} F(y) = 0 \implies \mu(-\infty, x] = \lim_{y\to-\infty} \mu(y,x] = \lim_{y\to-\infty} F(x) - F(y) = F(x)$$

Where the first equality after $\implies$ comes from continuity from below. Further:

$$\lim_{x\to\infty} F(x) = 1 \implies \mu(\mathbb{R}) = \lim_{x\to\infty} \mu(-\infty, x] = \lim_{x\to\infty} F(x) = 1$$

Where again, the first equality after $\implies$ comes from continuity from below. For the probability space, take $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{R}, \mu)$,. Note, this is a probability space by μ being a measure on $\mathcal{R}$, and $\mu(\mathbb{R}) = 1$. For X take the identity function: $X(\omega) = \omega$. Clearly, X is real valued, and measurable with respect to $\mathcal{R}/\mathcal{R}$, and so is a Random Variable. Finally, note:

$$P[X \leq x] = \mu[\omega \in \mathbb{R} : \omega \leq x] = \mu(-\infty, x] = F(x)$$

So, overall, the steps of this proof are:

1. For a function F as defined, prove the existence of a measure μ on $\mathcal{R}$ with Theorem 12.4.

2. Note that $\mu(-\infty, x] = F(x)$ and $\mu(\mathbb{R}) = 1$. So μ itself is a probability measure on $(\mathbb{R}, \mathcal{R})$.

3. Let μ be a probability measure for $(\mathbb{R}, \mathcal{R}, \mu)$, and this space will be the domain for our random variable X.

4. Finally, define the random variable X as the identity on $(\mathbb{R}, \mathcal{R}, \mu)$, and note the required property $P[X \leq x] = F(x)$ comes directly from $\mu(-\infty, x] = F(x)$.

qed.

Note, this is only the first proof of the fact. I like it cause it just easily builds on what we already know. However, the book gives a second proof of the distribution fact.

Second Proof: This proof only relies on the existence of Lebesgue measure *on the unit interval* - so, the stuff we proved in chapter 3-5. Our probability space will be $((0,1), \mathcal{B}, \lambda)$, the borel sets of $(0,1)$ with the Lebesgue measure on $(0,1)$.

First, consider the case where F is continuous and strictly increasing. Then F is a one-to-one mapping of $\mathbb{R}$ onto $(0,1)$. Note, strictly increasing implies that for $x \neq y$, wlog $x < y$, $F(x) < F(y) \implies F(x) \neq F(y)$. This is the definition of injectivity. Note, F thus also has an inverse function $\varphi : (0,1) \to \mathbb{R}$. This is because F is surjective on $(0,1)$, via the limit restrictions on F.

For $0 < \omega < 1$, let $X(\omega) = \varphi(\omega)$. Note that φ is increasing - if $u < v$ for $u, v \in (0,1)$, we have some $F(x) = u < v = F(y)$ - which via F being increasing implies $x < y$ - and so $\varphi(u) = x < y = \varphi(v)$. Finally, note that increasing functions are measurable. As the half infinite intervals generate $\mathbb{R}$, examine:

$$X^{-1}(-\infty, a)$$

It must equal $(0, b)$ for some $b \in (0,1)$. This is because if $c < b$, we must have $X(c) < X(b) = a \implies X(c) \in (-\infty, a)$. And so, Theorem 13.1 implies that X is measurable. By definition, the b satisfies $F(a) = b$.

And so, X as defined is measurable on $((0,1), \mathcal{B}, \lambda)$. The final question is whether or not F is equal to the distribution function of X. Examine:

$$P[X \leq x] = P[\omega \in (0,1) : \varphi(\omega) \leq x] = P[\omega \in (0,1) : \omega \leq F(x)] = F(x)$$

Where the second equality makes use of our inverse definition, and the third equality is via the definition of the Lebesgue measure.

So, we have proved the theorem in the F continuous case. Now, we need to consider general F (ie, right-continuous satisfying the nondecreasing and limit properties). Define:

$$\varphi(u) = \inf\left[x : u \leq F(x)\right]$$

Note, the infimum across this set makes sense, as $u \in (0, 1)$, and $F(x)$ increasing to 1 implies that ∞ would be the supremum. Indeed, $[x : u \leq F(x)]$ is an interval stretching to ∞, and as F is right continuous, the interval must be closed on the left. For $0 < u < 1$ then, we have:

$$[x : u \leq F(x)] = [\varphi(u), \infty)$$

And so $\varphi(u) \leq x$ if and only if $u \leq F(x)$. If $X(\omega) = \varphi(\omega)$ for $0 < \omega < 1$, then by the same reasoning as above, X is a random variable (increasing) and $P[X \leq x] = F(x)$. qed.

Exponential Distributions

There are a number of results which for their interpretation require random variables, independence, and other probabilistic concepts, but which can be discussed technically in terms of distribution functions alone, and do not require measure theory. This is what basic statistics is, essentially. This section examines one of those distributions, the exponential distribution.

Suppose that F is the distribution function of the waiting time to the occurrence of some event - say the arrival of the next customer at a queue, or the next call at a telephone exchange. As the waiting time must be positive, assume $F(0) = 0$ - ie, we want:

$$P\left[X \leq 0\right] = F(0) = 0$$

Suppose that $F(x) < 1$ for all x (which allows it to express a probability), and furthermore suppose that:

$$\frac{1 - F(x + y)}{1 - F(x)} = 1 - F(y) \qquad x, y \geq 0$$

The right side of the equation is the probability that the waiting time exceeds y - and if we think about conditional probability, the left side is the probability that the waiting time exceeds $x + y$, given that it exceeds x. In the real world, we understand this as a "lack of memory" property. Ie, if we have waited till time x - the probability that we have to wait till time $x + y$ is the same as the probability we have to wait until time y in a new world starting from zero.

Two things about this condition. One, we see in practice that waiting times often have this property. At least that is what the book says - I think, to see this in the real world, you would take a bunch of waiting time intervals, and do some sort of test to see if the intervals had that property, which I think would be to match it with the exponential distribution. Kind of chicken or the egg, I guess, in regards to which intuition came first.

The second thing is - these conditions on F completely determine the form of F. If $U(x) = 1 - F(x)$, our property is just:

$$U(x + y) = U(x)U(y)$$

We do see that $U(x) = e^{-\alpha x}$ would satisfy this equation. But is it the only one? This relies on some stuff from the appendix.

Theorem A20 - Cauchy's Equation Let f be a real function on $(0, \infty)$, and suppose that f satisfies Cauchy's equation:

$$f(x + y) = f(x) + f(y)$$

For $x, y > 0$. If there is some interval on which f is bounded above (ie, for $x \in (a, b)$, $f(x) < B$), then:

$$f(x) = xf(1) \text{ for } x > 0$$

Proof: The problem is to prove that $g(x) = f(x) - xf(1)$ vanishes identically (ie equals zero). Clearly, $g(1) = 0$. We also have g satisfies Cauchy's equation:

$$g(x+y) = f(x+y) - (x+y)f(1) = f(x) - xf(1) + f(y) - yf(1) = g(x) + g(y)$$

And, on some interval (a, b), g is bounded above. By induction, we note $g(nx) = ng(x)$ - we have that it is true for $n = 1$, and for $n > 1$, we assume true for $n - 1$, and note:

$$ng(x) = g(x) + (n - 1)g(x) = g(x) + g((n - 1)x) = g(x + (n - 1)x) = g(nx)$$

I guess this is true for all Cauchy equations. Hence:

$$ng(m/n) = g(m) = mg(1) = 0 \implies g(m/n) = 0$$

And so g is zero for rational r - $g(r) = 0$. Suppose that $g(x_0) \neq 0$ for some x_0. If $g(x_0) < 0$, then:

$$g(r_0 - x_0) = -g(x_0) > 0$$

For every rational $r_0 > x_0$ (the case where $r_0 - x_0$ is in the domain). It is thus no restriction to assume that $g(x_0) > 0$ - as if we have some $g(x_0) \neq 0$, we can find such a positive one. Let I be an open interval in which g is bounded above. Given a number M, choose n so that:

$$ng(x_0) > M$$

And then choose a rational r so that $nx_0 + r$ lies within I. Note, this is done by density of the rationals. Also, r might be negative. If $r > 0$, then:

$$g(r + nx_0) = g(r) + g(nx_0) = g(nx_0) = ng(x_0)$$

If $r < 0$, then:

$$ng(x_0) = g(nx_0) = g((-r) + (nx_0 + r)) = g(-r) + g(nx_0 + r) = g(nx_0 + r)$$

In either case, we have:

$$g(nx_0 + r) = ng(x_0)$$

Since $g(nx_0+r) = ng(x_0) > M$, and M was arbitrary, g is not bounded above in I, which is a contradiction. Thus, our first assumption that $g(x_0) \neq 0$ was false. And so indeed, we have that g is identically equal to zero on $(0, \infty)$, and so:

$$f(x) = xf(1) \qquad \text{for } x > 0$$

qed.

Corollary A20 - Multiplicative Cauchy Is Exponential Let U be a real function on $(0, \infty)$ and suppose that:

$$U(x + y) = U(x)U(y)$$

For $x, y > 0$. Suppose further that there is some interval on which U is bounded above. Then either $U(x) = 0$ for $x > 0$, or else there is an A such that:

$$U(x) = e^{Ax} \text{ for } x > 0$$

Proof: Since $U(x) = U^2(x/2)$, U is nonnegative. The equation is true for each $x > 0$, and the square is nonnegative. If $U(x) = 0$, then $U(x/2^n) = 0$ (just apply the definition 2^n times). And so, U vanishes at points arbitrarily near 0. If U vanishes at a point, it must by the above equation vanish everywhere to the right of the point:

$$U(x + z) = U(x)U(z) = 0U(z) = 0$$

Hence, U is identically zero if U equals zero at some point (as we can get arbitrarily close to zero, and say every point to the right is zero. Note, the domain is $(0, \infty)$). And so, U is either identically zero, or else everywhere positive.

So, we consider the everywhere positive case. If U is everywhere positive, $f(x) = \log U(x)$ is well defined. Note that:

$$f(x+y) = \log U(x+y) = \log U(x)U(y) = \log U(x) + \log U(y) = f(x) + f(y)$$

As U is bounded above on some interval, so is $f(x)$. Thus, the previous theorem applies, and so:

$$f(x) = Ax \qquad\qquad A = \log U(1)$$

This implies that:

$$\log U(x) = Ax \implies U(x) = e^{Ax} \quad \text{for } x > 0$$

qed.

Exponential Distribution Derivation So, we have that our conditions for our distribution function F are that $F(x) = 0$ for $x \leq 0$ and:

$$\frac{1 - F(x+y)}{1 - F(x)} = 1 - F(y) \qquad x, y \geq 0$$

So really, we want to consider F restricted to $(0, \infty)$. Define:

$$U(x) = 1 - F(x) \implies U(x+y) = U(x)U(y) \quad \text{for } x, y > 0$$

We have that F is bounded between 0 and 1, so U is bounded by 1. By the Corollary to A20, we must have that:

$$U(x) = e^{-\alpha x}$$

As U is bounded by 1, this implies that α must be positive. We can also conclude this from $\lim_{x\to\infty} U(x) = 0 \implies \lim_{x\to\infty} e^{-\alpha x} = 0$. Thus, our conditions have completely identified F as:

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\alpha x} & \text{if } x \geq 0 \end{cases}$$

Weak Convergence

Random variables $X_1, \cdots, X_n$ are defined to be independent if the events $[X_1 \in A_1], \cdots, [X_n \in A_n]$ are independent for all Borel sets $A_1, \cdots, A_n$, so that:

$$P[X_i \in A_i, i = 1, \cdots, n] = \prod_{i=1}^{n} P[X_i \in A_i]$$

Note, this is just an extension of our independence definition in chapters 4 and 5 from simple random variables to general random variables. Recall, the theorems we have for independence rely on independent π systems, and their σ algebras.

Distribution Function of Maximum Of Independent RVs Define $M_n = \max\{X_1, \cdots, X_n\}$. Note, M_n is measurable, as we proved maximum functions were measurable. Note that:

$$P\left[M_n \leq x\right] = P[X_i \leq x, i = 1, \cdots, n] = \prod_{i=1}^{n} P[X_i \leq x]$$

Where the first step is by equal sets, and the second is by independence. If the X_i have common distribution function G, then we have the distribution function F_n for M_n is given by:

$$F_n(x) = G^n(x)$$

Example 14.1: Examining the Asymptotic Maximum Distribution Function without Measure Theory Say that we are examining the the maximum arrival time out of n waiting times. The n waiting times each have distribution function as described above:

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-\alpha x} & \text{if } x \geq 0 \end{cases}$$

Suppose that they are each independent, and identically distributed, in which case we can conclude that the distribution function for the (measurable) maximum random variable is:

$$F_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (1 - e^{-\alpha x})^n & \text{if } x \geq 0 \end{cases}$$

Note, this makes sense, as the probability the maximum is less than 0 will always be zero, and the probability that it is less than some value $x > 0$ is smaller than for each individual random variable. Now, we note that:

$$\lim_n F_n(x) = 0$$

As n grows to infinity, we are multiplying more and more decimals. This is true for each x. Thus, if n is large, we have that M_n will tend to be large as well. Now note:

$$P\left[M_n - \alpha^{-1}\log n \leq x\right] = F_n(x + \alpha^{-1}\log n) = \left(1 - e^{-(\alpha x + \log n)}\right)^n$$

This is the distribution function for the random variable $M_n - \alpha^{-1}\log n$. Examine as $n \to \infty$. We have:

$$= \left(1 - \frac{e^{-\alpha x}}{n}\right)^n$$

Recall that the limit definition of e is:

$$e^x = \lim_n (1 + x/n)^n$$

And so, we have that:

$$\lim_{n\to\infty} P\left[M_n - \alpha^{-1}\log n \leq x\right] = e^{-e^{-\alpha x}}$$

So, we have the approximate distribution of the normalized random variable $M_n - \alpha^{-1}\log n$ for large n.

Distribution Weak Convergence If F_n and F are distribution functions, then by definition, F_n *converges weakly* to F, written $F \Rightarrow F$, if:

$$\lim_n F_n(x) = F(x)$$

For each x at which F *is continuous*. This distinction seems important for later on.

To study the approximate distribution of a random variable Y_n, it is often necessary to study instead the normalized or rescaled random variable $(Y_n - b_n)/a_n$ for appropriate constants a_n and b_n. If Y_n has distribution function F_n and if $a_n > 0$, then:

$$P\left[(Y_n - b_n)/a_n\right] = P\left[Y_n \leq a_n x + b_n\right]$$

So we have a distribution function $F_n(a_n x + b_n)$. This allows us to do some nice scaling/translation. For this reason, weak convergence is sometimes written in the form:

$$F_n(a_n x + b_n) \Rightarrow F(x)$$

Example 14.2 Limiting Distribution of Scaled Maximum Random Variable 1 Consider again the distribution function of the maximum for n iid random variables:

$$F_n(x) = G^n(x)$$

But now, suppose distribution function G is:

$$G(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - x^{-\alpha} & \text{if } x \geq 1 \end{cases}$$

For $\alpha > 0$. Note, again, this is a continuous function, nondecreasing and has the correct 0/1 limits, so it is a distribution function. Thus, we have:

$$F_n(n^{1/\alpha}x) = \left[1 - (n^{1/\alpha}x)^{-\alpha}\right]^n = \left[1 - \frac{x^{-\alpha}}{n}\right]^n$$

For $x \geq n^{-1/\alpha}$ (as it is zero otherwise). Again, we have a nice limit, where:

$$\lim_n F_n(n^{1/\alpha}x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-x^{-\alpha}} & \text{if } x > 0 \end{cases}$$

Example 14.3 Limiting Distribution of Scaled Maximum Random Variable 2 Same setup, but now we let

$$G(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - (1-x)^{\alpha} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Again, this is a distribution function for $\alpha > 0$. This time, let $a_n = n^{-1/\alpha}$, and $b_n = 1$, in which case for $-n^{1/\alpha} \leq x \leq 0$

$$F_n(n^{-1/\alpha}x + 1) = \left[1 - (n^{-1/\alpha}x)^{\alpha}\right]^n = \left(1 - \frac{(-x)^{\alpha}}{n}\right)^n$$

Therefore, we have:

$$\lim_n F_n(n^{-1/\alpha}x + 1) = \begin{cases} e^{-(-x)^{\alpha}} & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Constant 0 Distribution Function Let Δ be the distribution function with a unit jump at the origin:

$$\Delta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

If $X(\omega) = 0$, then X has a distribution function Δ.

Example 14.4 - The Weak Law of Large Numbers And Δ Distribution Let $X_1, X_2, \cdots$ be independent random variables for which $P[X_k = 1] = P[X_k = -1] = 1/2$. Note, X_i are iid, and they are also simple random variables. And so, we can apply the weak law of large numbers to $S_n = X_1 + \cdots + X_n$, Theorem 6.1 (Corollary) which states:

$$P\left[|n^{-1}S_n| > \epsilon\right] \to 0$$

For $\epsilon > 0$. We let F_n be the distribution function of $n^{-1}S_n$. If $x > 0$, then:

$$F_n(x) = P\left[n^{-1}S_n \leq x\right] = 1 - P\left[n^{-1}S_n > x\right] \to 1$$

If $x < 0$, then:

$$F_n(x) = P\left[n^{-1}S_n \leq x\right] = P\left[|n^{-1}S_n| \geq |x|\right] \to 0$$

Note, this accounts for all continuity points of Δ (the only discontinuity of Δ describe above being at 0. Thus, from the weak law of large numbers, we can deduce that:

$$F_n \Rightarrow \Delta$$

Ie, F_n converges to Δ at continuity points, which we defined as F_n *converges weakly* to Δ. Note, we can similarly turn the argument around. Assume that the distribution function of $n^{-1}S_n$ *converges weakly* to Δ. Then, for any $\epsilon > 0$, we have:

$$P\left[n^{-1}S_n > \epsilon\right] = 1 - P\left[n^{-1}S_n \leq \epsilon\right] \to 0$$

And:

$$P\left[-n^{-1}S_n > \epsilon\right] = P\left[n^{-1}S_n \leq -\epsilon\right] \to 0$$

So, both cases together imply:

$$P\left[|n^{-1}S_n| > \epsilon\right] \to 0$$

So, both facts imply each other. And so, we have - The Weak Law of Large Numbers is *equivalent* to the assertion that the distribution function of $n^{-1}S_n$ *converges weakly* to Δ.

As a final note, examine the discontinuity zero. If n is odd - note that $S_n = 0$ is impossible (as we are adding an odd number of ± 1). And so, by symmetry, we can conclude:

$$P[S_n \leq 0] = P[S_n \geq 0] = 1/2 \implies F_n(0) = P[n^{-1}S_n \leq 0] = 1/2$$

So, if the limit of $F_n(0)$ exists, it must be $1/2$ (or DNE). In either case - $F_n(0)$ *does not* converge to $\Delta(0) = 1$. However, as Δ is discontinuous at zero, the definition of weak convergence does not require they are equal at the discontinuity of zero.

Weak Limits Are Unique So, we have that $F_n \Rightarrow F$. If $F_n \to G$ as well - must F and G be equal? Well, they of course must be at shared continuity points. However, we don't necessarily need equality at discontinuity points (which we specify for situations like above, so it is possible to bring the weak law of large numbers under the theory of weak convergence).

So, consider the discontinuity points. As F and G are distribution functions - we note that they are increasing, right continuous, and bounded between zero and one. As discussed earlier - this implies that F and G can only have countably many jump discontinuities. The set of common continuity points must thus be dense. Take a discontinuity point x. Via density, we have $x_1, x_2, \cdots$ such that $x_n \downarrow x$. We thus have by right continuity:

$$F(x) = \lim_n F(x_n) = \lim_n G(x_n) = G(x_n)$$

Note - I was thinking about the left side of a discontinuity point x. However, recall that F and G take the right side of the jump - ie, the left side is not inclusive - with an open parenthesis.

Problems

14.1 Nondecreasing Implies Countable Discontinuities

The general nondecreasing function F has at most countably many discontinuities. Prove this by considering the open intervals:

$$\left(\sup_{u<x} F(u), \inf_{v>x} F(v)\right)$$

Each nonempty one contains a rational.

So, I proved it above, by examining the sum of an uncountable amount of discontinuities. However, I like these kind of problems - so I will try and prove this one with the method as stated. Note, if F is continuous at x - the interval is empty. That is because the supremum at the left, and infimum at the right, will both equal $F(x)$. So, the only time that these intervals are nonempty would be if x is a discontinuity point of F.

So, for each discontinuity point of F, we have a nonempty interval. To each discontinuity point x, we assign a rational that is contained within the interval, r_x. We now note that this mapping is injective. This is because, if y is a discontinuity point such that $x \neq y$, wlog assume $x < y$. This implies the lower bound of the interval for y, at most equals the upper bound for the interval at x - and given we are using parenthesis, this implies the intervals are disjoint.

And so, we have an injective map from the discontinuities to the rationals - which implies the discontinuities are at most countable.

14.3 Probability Transformation - Passing X into its Distribution Function - $F(X)$

1. Suppose that X has a continuous, strictly increasing distribution function F. Show that the random variable $F(X)$ is uniformly distributed over the unit interval, in the sense that:

$$P\left[F(X) \leq u\right] = u \text{ for } 0 \leq u \leq 1$$

Passing from X to $F(X)$ is called the *probability transformation.*

If F is strictly increasing - then I think we have that F must be invertible on $(0,1)$. Take $x \neq y$ - assume wlog that $x < y$. Then, we have $F(x) < F(y) \implies F(x) \neq F(y)$. And so, F is injective. F is surjective by the limits:

$$\lim_{x \to -\infty} F(x) = 0 \qquad \lim_{x \to \infty} F(x) = 1$$

So, F^{-1} exists. We must have that:

$$F(X) \leq u \Leftrightarrow X \leq F^{-1}(u)$$

Again, this is by strictly increasing. And so, we have:

$$P\left[F(X) \leq u\right] = P\left[X \leq F^{-1}(u)\right] = F(F^{-1}(u)) = u$$

One thing to note - if $F(X)$ a random variable? Well, it is if $F \circ X : \Omega \to \mathbb{R}$ is measurable, which is true if F is measurable. F is strictly increasing, which implies that F is measurable as a function from $\mathbb{R}$ to $\mathbb{R}$ (inverse of half open intervals is half open interval, and within $\mathcal{R}$, which implies the inverse of a generating set is measurable). Also, F continuous implies that F is measurable, and so $F \circ X$ is the composition of measurable functions, which is measurable.

2. Show that the function $\varphi(u)$ defined by:

$$\varphi(u) = \inf\left[x : u \leq F(x)\right]$$

Satisfies:

$$F(\varphi(u)-) \le u \le F(\varphi(u))$$

And that, if F is continuous (but not necessarily strictly increasing), then $F(\varphi(u)) = u$ for $0 < u < 1$.

Note, here, I think the point is to assume that F is a distribution function. First, I want to show that:

$$u \le F(\varphi(u))$$

Which is the same as:

$$u \le F\left[\inf\left[x : u \le F(x)\right]\right]$$

Note, the infimum can be calculated as a right hand limit as F is increasing - namely, we have $x_1, x_2 \downarrow \varphi(u)$. By definition, we have that:

$$u \le F(x_i) \text{ for all } i \implies u \le \lim_i F(x_i) \implies u \le F(\varphi(u))$$

Where the last step uses right continuity. Now, we want to examine the other direction:

$$F(\varphi(u)-) \le u$$

Note, $\varphi(u)-$, by definition, is a left handed limit - ie, we have for $x_i \uparrow \varphi(u)$, we have:

$$\lim_i F(x_i) = F(\varphi(u)-)$$

Note, the left hand limit, definitionally, is taken outside the function. We note that as $x_i < \varphi(u)$, and $\varphi(u)$ is an infimum of points satisfying a property, x_i must not satisfy that property. And so, we must have $F(x_i) < F(u)$. This allows us to conclude that:

$$F(\varphi(u)-) = \lim_i F(x_i) \le u$$

And so, we indeed have the equation:

$$F(\varphi(u)-) \le u \le F(\varphi(u))$$

Note, continuity of F would directly give us:

$$F(\varphi(u)) = F(\varphi(u)-) \le u$$

So $\le$ and $\ge$ gives us:

$$F(\varphi(u)) = u$$

3. Show that $P[F(X) < u] = F(\varphi(u)-)$ and hence that the result in part (a) holds as long as F is continuous.

   We have that:

   $$P[F(X) < u] = 1 - P[F(X) \geq u]$$

   Note that $F(X) \geq u$ if and only if $\varphi(u) \leq X$. So:

   $$= 1 - P[\varphi(u) \leq X] = P[X < \varphi(u)] = F(\varphi(u)-)$$

   Where the last step is from the chapter discussion, using continuity from below. If F is continuous, then we have:

   $$P[F(X) \leq u] = P[F(X) < u] = F(\varphi(u)-) = F(\varphi(u)) = u$$

14.4 Probability Transformation on Borel Sets

Let C be the set of continuity points of F.

1. Show that for every Borel set A, $P[F(X) \in A, X \in C]$ is at most the Lebesgue measure of A.

   I believe, we need to show:

   $$P[F(X) \in A, X \in C] \leq \lambda(A)$$

   Recall, we found in the previous section:

   $\varphi(u) = \inf\,[x : u \leq F(x)] \qquad P\,[F(X) < u] = F(\varphi(u)-)$ and $= u$ if F is continuous

   We can go back to Theorem 11.3 - we have that the bounded intervals are a semiring on $\mathcal{R}$ - and so via Theorem 11.3, which extends λ on the bounded intervals to all borel sets - we have that:

   $$\lambda(A) = \inf \sum_n \lambda(A_n)$$

   Where we cover A by intervals. Take a covering of A by bounded intervals A_n, note we can assume they are disjoint via Lemma 1 in Chapter 11. And so:

   $$P[F(X) \in A, X \in C] \leq P\left[F(X) \in \bigcup_n (a_n, b_n], X \in C\right]$$

$$= \sum_n P\left[F(X) \in (a_n, b_n], X \in C\right]$$

Note, the first step uses the fact that the union is larger than A, and the second step is probability of disjoint intervals. Now, note that:

$$P\left[F(X) \in (a_n, b_n], X \in C\right] = P\left[F(X) \leq b_n, X \in C\right] - P\left[F(X) \leq a_n, X \in C\right]$$

Note, as we are restricting to $X \in C$, we have that:

$$\leq b_n - a_n$$

Note, we can break this down whether or not $\varphi(b_n), \varphi(a_n) \in C$ or not, but in any case, the bound holds. So, for any disjoint covering A_n of A, we have:

$$P[F(X) \in A, X \in C] \leq \sum_n \lambda(A_n)$$

We can take an infimum on both sides, to find:

$$P[F(X) \in A, X \in C] \leq \lambda(A)$$

2. Show that if F is continuous at each point of $F^{-1}A$, then $P[F(X) \in A]$ is at most the lebesgue measure of A. Well, this assumption gives us equivalent information to $X \in C$ in the above argument. So, we can remove $X \in C$, and still get continuity of $\varphi(a_n) = F^{-1}(a_n)$, and so we can conclude:

$$P[F(x) \in A] \leq \lambda(A)$$

14.5 The Levy Distance Definition and Properties

The *Levy Distance* $d(F, G)$ between two distribution functions is the infimum of those ϵ such that $G(x - \epsilon) - \epsilon \leq F(x) \leq G(x + \epsilon) + \epsilon$ for all x. Verify that this is a metric on the set of distribution functions. Ie, on the set of distribution functions, we have:

1. Distance from a point to itself: $d(F, F) = 0$

2. Positivity: $d(F, G) > 0$ if $F \neq G$

3. Symmetry: $d(F, G) = d(G, F)$

4. Triangle Equality: $d(F, H) \leq d(F, G) + d(G, H)$

Show that a necessary and sufficient condition for $F_n \Rightarrow F$ is that $d(F_n, F) \to 0$ - ie, they are equivalent.

First, we try and prove the metric properties: **Distance From a Point To Itself** We have:

$$d(F, F) = \inf\left[\epsilon : \forall x, F(x-\epsilon) - \epsilon \le F(x) \le F(x+\epsilon) + \epsilon\right]$$

Note, that as F is a distribution function, it is non decreasing, so all $\epsilon > 0$ satisfy the above definition, the infimmum of which must be zero. **Positivity** We want to show:

$$d(F, G) = \inf\left[\epsilon : \forall x, G(x-\epsilon) - \epsilon \le F(x) \le G(x+\epsilon) + \epsilon\right] > 0$$

I think, we prove by contradiction. Assume that it equals zero. Then, for all x, we have $\epsilon_n \to 0$ such that:

$$F(x) \le \lim_n G(x+\epsilon_n) + \epsilon_n = G(x)$$

Where we made use of right continuity. If we use symmetry (which we will prove next), we can also conclude for all x:

$$G(x) \le F(x) \implies G = F$$

Which is a contradiction. All we have to make sure is we don't use positivity for the symmetry proof. **Symmetry** We want to show:

$$d(F, G) = d(G, F)$$

Take the $d(F, G) = d$. We have $d_n \downarrow d$ such that for all x:

$$G(x-d_n) - d_n \le F(x) \implies G(x) - d_n \le F(x+d_n) \implies G(x) \le F(x+d_n) + d_n$$

Where the above makes use of the fact that $x + d_n$ is included in the domain of "for all x". Similarly:

$$F(x) \le G(x+d_n) + d_n \implies F(x-d_n) \le G(x) + d_n \implies F(x-d_n) - d_n \le G(x)$$

So, for all x, we have at the very least, a sequence of $d_n \to d$ such that:

$$F(x-d_n) - d_n \le G(x) \le F(x+d_n) + d_n$$

And so, we must have:

$$d(F, G) \le d(G, F)$$

We can go in the other direction, starting from $d(G, F)$, to find:

$$d(G, F) \leq d(F, G) \implies d(F, G) = d(G, F)$$

Triangle Inequality This is the last property we need to prove. Take distribution functions F, H, G. We want to show:

$$d(F, H) \leq d(F, G) + d(G, H)$$

Prove by contradiction. Assume that $d(F, H) = x, d(F, G) = y, d(G, H) = z$, and by contradiction, assume $y + z < x$. I will show that we can set $d(F, H) = y + z$. We have sequences $y_n \downarrow y$, and $z_n \downarrow z$. By the definition of y_n, for all x, we have:

$$F(x) \leq G(x + y_n) + y_n$$

Now, by the definition of z_n, we have:

$$G(x + y_n) \leq H(x + y_n + z_n) + z_n$$

In total, we have for all x:

$$F(x) \leq H(x + y_n + z_n) y_n + z_n$$

Similarly, on the otherside, we have for all x:

$$G(x - y_n) - y_n \leq F(x) \implies H(x - y_n - z_n) - y_n - z_n \leq F(x)$$

In total, we have found, for all x and all n:

$$H(x - y_n - z_n) - y_n - z_n \leq F(x) \leq H(x + y_n + z_n) + y_n + z_n$$

$$\implies d(F, H) = \inf\left[\epsilon : \forall x, H(x - \epsilon) - \epsilon \leq F(x) \leq H(x + \epsilon) + \epsilon\right] \leq y_n + z_n$$

So, we must have $d(F, H) \leq y + z$. Thus, we have proved the triangle inequality.

$F_n \Rightarrow F$ **is equivalent to** $d(F_n, F) \to 0$ Now, we want to show that a necessary and sufficient condition for $F_n \Rightarrow F$ is that $d(F_n, F) \to 0$. Recall, $F_n \Rightarrow F$ just means that for all points x where F is continuous:

$$\lim_n F_n(x) = F(x)$$

So, to prove necessary, I think we need to show $F_n \Rightarrow F$ implies $d(F_n, F) \to 0$ - ie, it is necessary that the Levy distance goes to zero given that $F_n \Rightarrow F$. To prove sufficient, I think we need to show $d(F_n, F) \to 0$ implies $F_n \Rightarrow F$, as the base assumption is sufficient to prove the next one.

Let us first start with necessary - prove $F_n \Rightarrow F$ implies $d(F_n, F) \to 0$. We have:

$$d(F_n, F) = d(F, F_n) = \inf\left[\epsilon : \forall x, F_n(x-\epsilon) - \epsilon \leq F(x) \leq F_n(x+\epsilon) + \epsilon\right] = d_n$$

We want to show $d_n \to 0$. Proof by contradiction - assume that $\inf_n d_n = d > 0$. Take $0 < d' < d$. Thus, for all n, we have an x_n such that:

$$F_n(x_n - d') - d' > F(x_n) \text{ or } F_n(x_n + d') + d' < F(x_n)$$

We have three cases: the x_n are bounded and thus have a convergent subsequence, or the x_n have a subsequence to infinity or negative infinity. Consider case one - let x_n be the increasing n, which converges to x. Note, we have either the $-d'$ or $+d'$ property holds infinitely - wlof assume $+d'$, and let x_n correspond to that subsequence. Further, note that there is some d' such that $0 < d' < d$ such that $x + d'$ is a continuity point - as the discontinuous points are countable, but the $0 < d' < d$ are uncountable. And so, as $F_n \to F$ (limit equality at continuity points), and $x_n + d' \to x + d'$, we have:

$$\lim_n F_n(x + d') = F(x + d') \text{ and } \lim_n F(x_n + d') = F(x + d')$$

As $x + d'$ is a continuity point. Now note:

$$0 = \lim_n F(x_n+d') - F(x+d') = \lim_n F(x_n+d') - F_n(x_n+d') + \lim_n F_n(x_n+d') - F(x+d')$$

Note, I have been trying this, but I can't get the limit argument to work.

Necessity - $(F_n \Rightarrow F) \Rightarrow d(F_n, F) \to 0$ **- From the book** We are proving necessity. Choose continuity points x_i of F in such a way that:

$$x_0 < x_1 < \cdots < x_k$$

Where:

$$F(x_0) < \epsilon \qquad F(x_k) > 1 - \epsilon \qquad x_i - x_{i-1} < \epsilon$$

Clearly, this is jumps by less than ϵ, can always choose such a finite number of k given that discontinuity points are only countable in the uncountable real numbers $\mathbb{R}$. If n exceeds some n_0, we must have:

$$|F(x_i) - F_n(x_i)| < \epsilon/2$$

This is for all i, given continuity points, and just have to choose a maximum out of finite numbers satisfying the $\epsilon - \delta$ definitions. Suppose that $x_{i-1} \leq x \leq x_i$. Then:

$$F_n(x) \leq F_n(x_i) \leq F(x_i) + \epsilon/2 \leq F(x+\epsilon) + \epsilon/2$$

This is by each F_n and F being nondecreasing, and choosing n large enough Similarly, we must have:

$$F(x-\epsilon) - \epsilon/2 \leq F_n(x)$$

And so all together, we can conclude:

$$F(x-\epsilon) - \epsilon \leq F_n(x) \leq F(x+\epsilon) + \epsilon$$

This is for all x between x_0 and x_k. Note, we have proven this for each ϵ, so if our infimum only covered x between x_0 and x_k, we could conclude that $d(F, F_n) = d(F_n, F) \to 0$ (note, for each ϵ, just have to choose n large enough, so for each $\epsilon > 0$, there is an n_0 such that if $n \geq n_0$, $|d(F_n, F)| < \epsilon$).

So, we have almost proved that $F_n \Rightarrow F$ implies $d(F, F_n) \to 0$. However, what about $x \leq x_0$, or $x \geq x_k$? Consider $x \leq x_0$. In that case, we can still conclude by the same argument above for $n \geq n_0$:

$$0 \leq F_n(x) \leq F(x+\epsilon) + \epsilon$$

However, note that we also have:

$$F(x-\epsilon) - \epsilon \leq F(x_0) - \epsilon \leq 0$$

As $F(x_0) < \epsilon$ by our construction. And so, we still have for $x \leq x_0$, for $n \geq n_0$:

$$F(x-\epsilon) - \epsilon \leq F_n(x) \leq F(x+\epsilon) + \epsilon$$

Similarly, the argument applies for $x \geq x_k$. And so, for any $\epsilon > 0$, we can find an n_0 such that for $n \geq n_0$, for all x, we have:

$$F(x-\epsilon) - \epsilon \leq F_n(x) \leq F(x+\epsilon) + \epsilon \implies d(F_n, F) \leq \epsilon$$

Thus, by the definition of the limit, we must have:

$$d(F_n, F) \to 0$$

Sufficiency - $d(F_n, F) \to 0 \Rightarrow (F_n \Rightarrow F)$ Now, we assume that $d(F_n, F) \to 0$. Take x on which F is continuous. We want to show:

$$\lim_n F_n(x) = F(x)$$

Take $\epsilon > 0$. Note for $n \geq n_0$ for n_0 large enough, we have:

$$F(x-\epsilon) - \epsilon \leq F_n(x) \leq F(x+\epsilon) + \epsilon$$

By F nondecreasing, we also have:

$$F(x-\epsilon) - \epsilon \leq F(x) \leq F(x+\epsilon) + \epsilon$$

Note, if $a \leq x \leq b$ and $a \leq y \leq b$, we must have $|x-y| \leq b-a$ - the largest difference occurs when points are at both boundaries. And so:

$$|F_n(x) - F(x)| \leq F(x+\epsilon) - F(x-\epsilon) + 2\epsilon$$

Now, take $\epsilon' > 0$. By continuity of F at x, there is an ϵ small enough such that $F(x+\epsilon) - F(x-\epsilon) < \epsilon'$. So, the steps are:

1. Take x continuity point of F. Note, we only need pointwise continuity.
2. Choose $\epsilon > 0$ - we will bound $|F_n(x) - F(x)|$ by this ϵ.
3. Choose ϵ' such that if $\epsilon'' \leq \epsilon'$, we have $F(x+\epsilon'') - F(x-\epsilon'') < \epsilon/3$.
4. Finally, take $\epsilon''' = \min(\epsilon/3, \epsilon')$. By $d(F_n, F) \to 0$, we can find an n_0 large enough such that if $n \geq n_0$, we have:

$$F(x-\epsilon''') - \epsilon''' \leq F_n(x) \leq F(x+\epsilon''') + \epsilon'''$$

5. Thus, for $n \geq n_0$, we have:

$$|F_n(x) - F(x)| \leq F(x+\epsilon''') - F(x-\epsilon''') + 2\epsilon''' \leq \epsilon/3 + 2\epsilon''' \leq \epsilon/3 + 2\epsilon/3 \leq \epsilon$$

   The first inequality is by n_0 defined for ϵ'''. The second inequality is by $\epsilon''' \leq \epsilon'$. The third is by $\epsilon''' \leq \epsilon/3$. The final is just addition.

Thus, we have proved both directions. qed.

14.6 Borel Functions Are Bounded in Some Interval

Prove that a borel function is automatically bounded in some interval to the right of 0. Recall, this implies that if the function satisfies Cauchy's equation, then we can conclude that $f(x) = xf(1)$.

Just as a note, recall that Cauchy's equation is $f(x+y) = f(x) + f(y)$. A borel function is just $f : \mathbb{R} \to \mathbb{R}$ that is measurable $\mathcal{R}/\mathcal{R}$. I'm just going to prove here that a borel function is bounded in some interval. I don't know how Cauchy's equation can apply, given that it requires $f : (0, \infty) \to \mathbb{R}$.

We make use of problem 12.3. If $A \in \mathcal{R}^1$, and $\lambda(A) > 0$, and $0 < \theta < 1$, there is a bounded open interval I such that:

$$\lambda(A \cap I) \geq \theta\lambda(I)$$

Note, we must have integers K and n such that:

$$\lambda\left[x : x > n, |f(x)| \leq K\right] > 0$$

This is because, we have that as K increases and n decreases:

$$\{x : |f(x)| \leq K\} \uparrow \mathbb{R} \qquad \{x : x > n\} \uparrow \mathbb{R} \qquad \implies \qquad \{x : x > n, |f(x)| \leq K\} \uparrow \mathbb{R}$$

Note, all the sets we are dealing with here a borel sets via measurability of intervals, and measurability of f. We can apply problem 11.3, and for a $\theta \in (0, 1)$, we have for our chosen K and n, for some interval I:

$$\lambda\left[x : x > n, |f(x)| \leq K, x \in I\right] > \theta\lambda(I)$$

Now recall. We must have that $x > n, |f(x)| \leq K, x \in I$ is an open interval. As f is increasing, we have that $|f(x)| \leq K$ is an interval. $x > n$ is an interval, and so is $x \in I$. So, we have found an intersection of intervals - which must contain an interval - with nonzero measure. So, we can indeed conclude that a Borel function is automatically bounded in some interval.

14.8 Modulus of Continuity

1. Show that if a distribution function F is everywhere continuous, then it is uniformly continuous.

   This is similar to what we examined in Bernstein's Theorem, where we found a continuous $f : [0, 1] \to \mathbb{R}$ is uniformly continuous - namely, continuity on a compact set implied uniform continuity.

   Here, we will have to make use of f being increasing, continuous, and having a domain on $[0, 1]$. Intuitively, increasing would remove "oscillations" that make uniform continuity difficult, and we don't have any limits to infinity, or jumps.

   Take $2/n > 0$. Note, by continuity of F, and $\lim_{x \to -\infty} F(x) = 0$, and $\lim_{x \to \infty} F(x) = 1$, we can find $x_1, \cdots, x_{n-1}$ such that $F(x) =$

$1/n, \cdots, F(x_{n-1}) = (n-1)/n$. By continuity, there is a δ_i, such that if $|x_i - y| < \delta_i$, we have $|F(x_i) - F(y)| < 1/n$. Let $\delta = \min(\delta_1, \cdots, \delta_{n-1})$.

Take $x \in \mathbb{R}$. We have $F(x) \in [i/n, i+1/n]$ for some $i = 0, \cdots, n-1$. Take y such that $|x-y| < \delta$. Examine:

$$|F(x) - F(y)|$$

We note that $(-\infty, x_1, \cdots, x_{n-1}, \infty)$ form intervals that partition $\mathbb{R}$. Ie, intervals like $(-\infty, x_1), (x_1, x_2), \cdots, (x_{n-1}, \infty)$. Let $x_0 = -\infty, x_n = \infty$. We must have $x_i - \delta > x_{i-1}$, $x_i + \delta < x_{i+1}$. If this wasn't the case, we would have $F(x_{i+1}) - F(x_i) < 1/n$, which would be a contradiction, as the difference equals $1/n$.

Note that x and y are in the same interval, or two adjacent intervals. Assume they are not in the same interval. If they were not in two adjacent intervals - the difference between them would be larger than δ, a contradiction.

Finally, note that this implies:

$$|F(x) - F(y)| < 2/n$$

As the greatest distance between two points in adjacent intervals is $2/n$. Note, this is true for all x, for arbitrary $2/n$, which implies uniform continuity of F.

2. Let $\delta_F(\epsilon) = \sup\left[F(x) - F(y) : |x-y| < \epsilon\right]$ be the modulus of continuity of F. Note, if F is uniformly continuous, we must have $\delta_F(\epsilon) \to 0$ as $\epsilon \to 0$.

   Show that $d(F,G) < \epsilon$ implies that $\sup_x |F(x) - G(x)| \leq \epsilon + \delta_F(\epsilon)$.

   Recall:

   $d(F,G) = \inf\{\epsilon$ such that for all $x : F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon\}$

   Note, we have $d(F,G) \leq \epsilon$ implies that for all x:

   $$F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon$$

   Note, this implies:

   $$\begin{aligned} |F(x) - G(x)| &= F(x) - G(x) \leq F(x) - F(x-\epsilon) - \epsilon && \text{if } G(x) \leq F(x) \\ |F(x) - G(x)| &= G(x) - F(x) \leq F(x+\epsilon) + \epsilon - F(x) && \text{if } F(x) \leq G(x) \end{aligned}$$

In either case, we must have:

$$\sup_x |F(x)-G(x)| \leq \max\left\{\sup_x F(x) - F(x-\epsilon), \sup_x F(x+\epsilon) - F(x)\right\}+\epsilon$$

As every x term in the left is bounded by an x term on the right. Note for all x, $F(x) - F(x-\epsilon) < \delta_F(\epsilon)$ and $F(x+\epsilon) - F(x) \leq \delta_F(\epsilon)$, by definition. And so:

$$\sup_x |F(x) - G(x)| \leq \delta_F(\epsilon) + \epsilon$$

3. Show that, if $F_n \Rightarrow F$ and F is everywhere continuous, then $F_n(x) \to F(x)$ uniformly in x. What if F is continuous over a closed interval?

   To show that $F_n(x) \to F(x)$ uniformly in x, we need for $\epsilon > 0$, there is an N such that if $n \geq N$, we have for all x:

$$|F_n(x) - F(x)| \leq \epsilon$$

   Take $\epsilon > 0$. As $F_n \Rightarrow F$, problem 14.5 tells us this is equivalent to $d(F_n, F) \to 0$. Note that:

$$d(F_n, F) < \delta \implies \text{for all } x \in \mathbb{R}: \quad F(x-\delta) - \delta \leq G(x) \leq F(x+\delta) + \delta$$

   Note, via increasing, we must have $F(x-\delta) \leq F(x) \leq F(x+\delta)$, and so $F(x)$ is also part of the same interval. Thus, $d(F_n, F) < \delta$ implies for all x:

$$|F_n(x) - F(x)| \leq 2\delta + [F(x+\delta) - F(x-\delta)]$$

   Choose $\delta_1 < \epsilon/3$. By F everywhere continuous, and thus uniformly continuous, we can choose δ_2 small enough such that:

$$\delta_F(2\delta_2) < \epsilon/3$$

   Let $\delta = \min(\delta_1, \delta_2)$. Note, as $d(F_n, F) \to 0$, there exists an N such that if $n \geq N$:

$$\begin{aligned} d(F_n, F) < \delta \implies |F_n(x) - F(x)| &\leq 2\delta + [F(x+\delta) - F(x-\delta)] \\ &\leq 2\delta_1 + [F(x+\delta_2) - F(x-\delta_2)] \end{aligned}$$

   Note, the right F difference term is monotone increasing as δ increases. By the definition of δ_1 and δ_2, we have:

$$\leq 2\epsilon/3 + \epsilon/3 = \epsilon$$

So, we have proven that there exists an N such that if $n \geq N$:

$$|F_n(x) - F(x)| \leq \epsilon$$

For all x. Thus, we have that if $F_n \Rightarrow F$ and F is everywhere continuous, then $F_n(x) \to F(x)$ uniformly in x.

The final point to consider is if F is continuous over a closed interval. Note, on that interval, F will again be uniformly continuous. I think, we can conclude that across the intervals that F is continuous over - F_n goes to F uniformly. Now, think about this - if the rational jumps in distribution F are finite - then there are finite number of intervals, across which F_n goes to F uniformly. Thus, I think we can conclude - F_n goes to F uniformly, almost everywhere. In fact, note that a countable number of jumps wouldn't change the almost everywhere point.

So, in conclusion - I think we have for any distribution function such that $F_n \Rightarrow F$, then $F_n(x) \to F(x)$ uniformly almost everywhere. Ie, we can find a uniform rate on a probability one set. Not bad.

Section 15 - The Integral

Notes

Integral Definition

From now on, f, g and so on will denote real measurable functions, the values $\pm\infty$ allowed, on a measure space $(\Omega, \mathcal{F}, \mu)$. The object of this section is to define and study the definite integral:

$$\int f d\mu = \int_\Omega f(\omega) d\mu(\omega) = \int_\Omega f(\omega)\mu(d\omega)$$

all the above are just different notations. Suppose that f is nonnegative. Then, for each *finite* decomposition $\{A_i\}$ of Ω into $\mathcal{F}$ sets, consider the sum:

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

Note, some conventions in the products. $0 \cdot \infty = 0, a \cdot \infty = \infty$ for $0 < a < \infty$, and $\infty \cdot \infty = \infty$. Note, if A_i is the emptyset, we get the ∞ as the infimum by convention, but when multiplied with $\mu(\emptyset) = 0$, the result is zero.

Integral Definition The integral of f is defined as the supremum of the sums:

$$\int f d\mu = \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i)$$

The supremum extends over all finite decompositions $\{A_i\}$ of Ω into $\mathcal{F}$ sets. For general f, consider its positive part:

$$f^+(\omega) = \begin{cases} f(\omega) & \text{if } 0 \leq f(\omega) \leq \infty \\ 0 & \text{if } -\infty \leq f(\omega) \leq 0 \end{cases}$$

And its negative part:

$$f^-(\omega) = \begin{cases} -f(\omega) & \text{if } -\infty \leq f(\omega) \leq 0 \\ 0 & \text{if } 0 \leq f(\omega) \leq \infty \end{cases}$$

So, we just negate the negative part to make it a positive function. These functions are nonnegative and measurable (maximums and minimums essentially, which are measurable via being continuous). The general integral is thus defined by:

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Unless both $\int f^+ d\mu = \infty$ and $\int f^- d\mu = \infty$, in which case the subtraction is not defined. If both are *finite*, f is called *integrable*, and the value is its *definite integral*. If one of the two is finite, f is not integrable, but it is still assigned the $\pm\infty$ integral value.

Just as an aside. I was thinking of something like:

$$f(x) = \begin{cases} 5x & \text{if } x \geq 0 \\ 0.5x & \text{if } x \leq 0 \end{cases}$$

Note, the integral across $(\mathbb{R}, \mathcal{R}, \lambda)$ would not be defined. I think it is clear that both f^+ and $f-$ have infinite area. However, consider something like:

$$\int_{-n}^{n} f d\lambda = \int_0^n 5x d\lambda - \int_{-n}^{0} 0.5x = \frac{5}{2}n^2 - \frac{1}{4}n^2 = \frac{9}{4}n^2$$

Ie, the positive area is larger than the negative area. As $n \to \infty$, the above value goes to $+\infty$. Is there some kind of theory for this? I think this is called something like the *extended improper integral*.

Nonnegative Functions

Theorem 15.1 - Integral Properties

1. If $f = \sum_i x_i I_{A_i}$ is a nonnegative simple function, $\{A_i\}$ being a finite decomposition of Ω into $\mathcal{F}$-sets, then:

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

2. If $0 \leq f(\omega) \leq g(\omega)$ for all ω, then $\int f d\mu \leq \int g d\mu$.

3. If $0 \leq f_n(\omega) \uparrow f(\omega)$ for all ω, then $0 \leq \int f_n d\mu \uparrow \int f d\mu$

4. For nonnegative functions f and g and nonnegative constants α and β:

$$\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Proof

1. Let $\{B_j\}$ be a finite decomposition of Ω and let β_j be the infimum of f over B_j. If $A_i \cap B_j \neq \emptyset$, then $\beta_j \leq x_i$. Ie, the infimum can only be smaller than a known value on the set. Therefore:

$$\sum_j \beta_j \mu(B_j) = \sum_{ij} \beta_j \mu(A_i \cap B_j) \leq \sum_{ij} x_i \mu(A_i \cap B_j) = \sum_i x_i \mu(A_i)$$

Where all the equalities come from countable additivity of the measure μ. On the other hand, we get equality if $\{B_j\}$ coincides with A_i, as the infimum on A_i is the value x_i, which is the only value of f over A_i. Thus, it is clear that for all finite decompositions of Ω into $\mathcal{F}$ sets B_i, we have:

$$\sum_i \left[\inf_{\omega \in B_i} f(\omega)\right] \mu(B_i) \leq \sum_i x_i \mu(A_i)$$

As the value is reached setting B_i to A_i, we have the supremum is the sum. Thus, definitionally:

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

2. Clearly, the sum:

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

Does not decrease if f is replaced by g, and so the supremum with g is clearly greater. Thus, it is clear that if $0 \leq f(\omega) \leq g(\omega)$ for all ω, then $\int f d\mu \leq \int g d\mu$.

3. By (2), we have that if $0 \leq f_n(\omega) \uparrow f(\omega)$ for all ω, then the sequence $\int f_n d\mu$ is non decreasing and bounded above by $\int f d\mu$. Therefore, $\uparrow$ comes if we just show equality, which follows if we show:

$$\int f d\mu \leq \lim_n \int f_n d\mu$$

Note, as $\int f d\mu$ is a supremum, the above follows if we show the RHS bounds every value the supremum is across, ie if for any decomposition $A_1, \cdots, A_m$ of Ω info $\mathcal{F}$ sets, we have:

$$\sum_{i=1}^{m} \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i) = S \leq \lim_n \int f_n d\mu$$

Let $v_i = \inf_{\omega \in A_i} f(\omega)$ for ease of notation. We do this in cases.

1. S is finite and $\mu(A_i)$ and v_i are positive and finite Fix an ϵ that is positive and less than each v_i. Put:

$$A_{in} = [\omega \in A_i : f_n(\omega) > v_i - \epsilon]$$

First, note A_{in} is measurable, being the intersection of measurable $f_n(\omega) > v_i - \epsilon$ and A_i. Now, note that $A_{in} \uparrow A_i$, as $f_n \uparrow f$, so for each ω in A_i, as v_i is the infimum of f across ω in A_i, there is some f_n for which $f(\omega) - f_n(\omega) < \epsilon$, which implies that for that n, $\omega \in A_{in}$.

Decompose Ω into $A_{1n}, \cdots, A_{mn}$ and the complement of their union - lets just call it A'_n. Observe that, as μ is continuous from below:

$$\int f_n d\mu \geq \sum_{i=1}^{m} (v_i - \epsilon)\mu(A_{in}) + \left[\inf_{\omega \in A'_n} f(\omega) \right] (\mu(A'_n))$$

$$\geq \sum_{i=1}^{m} (v_i - \epsilon)\mu(A_{in})$$

Continuity from below implies we can take the limit on the right side, so we have:

$$\lim_n \int f_n d\mu \geq \sum_{i=1}^{m} (v_i - \epsilon)\mu(A_i) = S - \epsilon \sum_{i=1}^{m} \mu(A_i)$$

As the $\mu(A_i)$ are finite, the sum of $\mu(A_i)$ is finite. As ϵ was arbitrary, we have:

$$S \leq \int f_n d\mu$$

2. S **is finite** Thus, each product $v_i \mu(A_i)$ must be finite. Note, one of the two can be infinite, but the other must then be zero. Suppose it is finite for $i \leq m_0$, and 0 for $i > m_0$. Note, $m_0 \leq m$. Now, v_i and $\mu(A_i)$ must be positive and finite for $i \leq m_0$. Define A_{in} as before, but only for $i \leq m_0$, and decompose Ω into $A_{1n}, \cdots, A_{m_0 n}$, and the complement of their union. The proof still follows.

3. $S = \infty$ Then, $v_{i_0}\mu(A_{i_0}) = \infty$ for some i_0. Note, no limit, as we only consider finite decompositions of Ω. So, both values are positive in the multiplication, and at least one is infinite. We use this one item in the sum to prove that the limit is infinite as well, which will conclude this part.

Suppose $0 < x < v_{i_0} \leq \infty$ and $0 < y < \mu(A_{i_0}) \leq \infty$, and put $A_{i_0 n} = [\omega \in A_{i_0} : f_n(\omega) > x]$. As $0 < x < v_{i_0}$, we do have $A_{i_0 n} \uparrow A_{i_0}$, for the same reason as in part 1, that $f_n \uparrow f$. Each is measurable via being the intersection of measurable sets. And hence, $\mu(A_{i_0 n}) > y$ for n exceeding some n_0. This is just the definition of $\uparrow$.

If we compose Ω into $A_{i_0 n}$ and its complement, we have:

$$\int f_n d\mu \geq x\mu(A_{i_0 n}) \geq xy$$

Where the last step is only true for $n \geq n_0$. If $v_{i_0} = \infty$, let $x \to \infty$. If $\mu(A_{i_0}) = \infty$, let $y \to \infty$. In either case it follows that:

$$\lim_n \int f_n d\mu = \infty$$

Thus, for all cases, we have proven: If $0 \leq f_n(\omega) \uparrow f(\omega)$ for all ω, then $0 \leq \int f_n d\mu \uparrow \int f d\mu$.

4. For nonnegative functions f and g and nonnegative constants α and β:

$$\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Suppose first that f and g are simple:

$$f = \sum_i x_i \mathbb{1}\{A_i\} \qquad g = \sum_j y_j \mathbb{1}\{B_j\}$$

Then:

$$\alpha f + \beta g = \sum_{ij} (\alpha x_i + \beta y_j) \mathbb{1} \{A_i \cap B_j\} =$$

And so, we have by (1):

$$\int \alpha f + \beta g d\mu = \sum_{ij} (\alpha x_i + \beta y_j) \mu(A_i \cap B_j) = \sum_i \alpha x_i \mu(A_i) + \sum_j \beta y_j \mu(B_j)$$

$$= \alpha \int f d\mu + \beta \int g d\mu$$

Where the breaking up into sums and pulling back together comes from disjointness and countable additivity. So, we have integral additivity for nonnegative simple functions. Now, consider general nonnegative f and g. By Theorem 13.5, we have simple functions f_n and g_n such that $0 \leq f_n(\omega) \uparrow f(\omega)$ and $0 \leq g_n(\omega) \uparrow g(\omega)$. But then, $0 \leq \alpha f_n + \beta g_n \uparrow \alpha f + \beta g$. And so, by (3), we have:

$$0 \leq \int \alpha f_n + \beta g_n d\mu \uparrow \int \alpha f + \beta g d\mu$$

Now, we can make use of our additivity for simple functions to find:

$$\lim_n \alpha \int f_n d\mu + \beta \int g_n d\mu = \int \alpha f + \beta g d\mu$$

Finally, use (3) again to find the individual limits:

$$\implies \alpha \int f d\mu + \beta \int g d\mu = \int \alpha f + \beta g d\mu$$

Theorem 15.2 - Almost Everywhere Properties Assume that f and g are nonnegative.

1. If $f = 0$ almost everywhere, then $\int f d\mu = 0$
2. If $\mu [\omega : f(\omega) > 0]$, then $\int f d\mu > 0$
3. If $\int f d\mu < \infty$, then $f < \infty$ almost everywhere
4. If $f \leq g$ almost everywhere, then $\int f d\mu \leq \int g d\mu$
5. If $f = g$ almost everywhere, then $\int d f \mu = \int g d\mu$

Proof:

1. Suppose that $f = 0$ almost everywhere. If A_i meets $[\omega : f(\omega) = 0]$, then the $\inf_{\omega \in A_i} f(\omega) = 0$. If A_i does not meet $[\omega : f(\omega) = 0]$, ie $A_i \subseteq [\omega : f(\omega) = 0]^c$, then $\mu(A_i) = 0$. This is by definition of equals zero almost everywhere - $\mu\left([\omega : f(\omega) = 0]^c\right) = 0$ - and monotonicity.

   Thus, each term in any sum of:

$$\sup \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

   Is zero, and so the sum is zero, and the supremum is 0. Thus, by definition, $\int f d\mu = 0$.

2. If $A_\epsilon = [\omega : f(\omega) > \epsilon]$, then clearly $A_\epsilon \uparrow [\omega : f(\omega) > 0]$ as $\epsilon \downarrow 0$. Thus, if $\mu[\omega : f(\omega) > 0] > 0$, there is some ϵ via limits for which $\mu(A_\epsilon) > 0$. Decomposing Ω into A_ϵ and its complement shows that $\int f d\mu \geq \epsilon \mu(A_\epsilon) > 0$.

3. Prove by contradiction. If $\mu[f = \infty] > 0$, decompose Ω into $[f = \infty]$ and its complement: $\int f d\mu \geq \infty \cdot \mu[f = \infty] = \infty$.

4. Let $G = [f \leq g]$. Note, we have proven in Theorem 13.4 that $G \in \mathcal{F}$ for f and g measurable. Note that for any $A \in \mathcal{F}$, $\mu(A) = \mu(G \cap A)$. This is because $\mu(A^c) = \mu(A^c \cup G^c) = \mu((G \cap A)^c)$, where the first equality comes from subadditivity and $\mu(G^c) = 0$. For any finite decomposition $\{A_1, \cdots, A_m\}$ of Ω:

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) = \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i \cap G) \leq \sum_i \left[\inf_{\omega \in A_i \cap G} f(\omega)\right] \mu(A_i \cap G)$$

$$\leq \sum_i \left[\inf_{\omega \in A_i \cap G} g(\omega)\right] \mu(A_i \cap G) \leq \int g d\mu$$

   Where the last step comes from a consideration of the decomposition $A_1 \cap G, \cdots, A_m \cap G, G^c$, and $\mu(G^C) = 0$. Thus, the supremum of sums of the first type, which is by definition $\int f d\mu$, is less than or equal to $\int g d\mu$.

5. This follows immediately from (4). $f \leq g$ and $g \leq f$ almost everywhere, which implies both $\int f d\mu \leq \int g d\mu$ and $\int g d\mu \leq \int f d\mu$, which implies equality.

Thus we have proved all the almost everywhere properties. qed.

A small note on negatives. Suppose that $f = g$ almost everywhere, where f and g need not be nonnegative. It is clear that $f^+ = g^+$ and $f^- = g^-$ almost everywhere as well. If f has a definite integral, it follows that the subtraction is well defined, and $\int f d\mu = \int g d\mu$ from Theorem 5.2(v).

Uniqueness

This is more of a discussion on different ways to define the integral. The book says that the ultimate value of $\int f d\mu$ is always the same, because it is uniquely determined by certain simple properties that it is natural for the integral to have.

Start by noting, it is natural to want to have properties (i) and (iii) of Theorem 5.1 - namely:

1. If $f = \sum_i x_i I_{A_i}$ is a nonnegative simple function, $\{A_i\}$ being a finite decomposition of Ω into $\mathcal{F}$-sets, then:

$$\int f d\mu = \sum_i x_i \mu(A_i)$$

3. If $0 \leq f_n(\omega) \uparrow f(\omega)$ for all ω, then $0 \leq \int f_n d\mu \uparrow \int f d\mu$

Note, however, these two facts uniquely determine the value of an integral for nonnegative functions. For f nonnegative measurable, there exists simple f_n such that $0 \leq f_n(\omega) \uparrow f(\omega)$ - and so, the value of $\int f d\mu$ would be determined by the limit. In fact - this is how we defined it in my probability class, I believe. We defined the value for simple functions, and then defined for non simple as the limit.

Now, note that the first property [1.] can be derived from linearity, property [4.] in Theorem 15.1:

4. For nonnegative functions f and g and nonnegative constants α and β:

$$\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Together with the assumption that $\int I_A d\mu = \mu(A)$ for indicators. This is clear via just breaking up the interval.

Finally, note that the subtraction definition for general integrals:

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Is also inevitable via linearity - as:

$$\int f d\mu = \int f^+ - f^- d\mu = \int f^+ d\mu - \int f^- d\mu$$

So, in total - if we want the integral to satisfy linearity, non decreasing limits, and $\int I_A d\mu = \mu(A)$, which agrees with our notion of area - then the value of the general integral $\int f d\mu$ is uniquely defined, and our definition:

$$\int f d\mu = \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i)$$

Coincides with that unique value.

Problems

These problems all have to deal with alternative integral definitions, which allow us to understand different properties about the integral.

Call our definition in the chapter the *lower integral*:

$$\int_* f d\mu = \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega) \right] \mu(A_i)$$

This is different from the *upper integral*, which we define as:

$$\int^* f d\mu = \inf \sum_i \left[\sup_{\omega \in A_i} f(\omega) \right] \mu(A_i)$$

15.1 - The Upper Integral is Too Big

Suppose that f is measurable and nonnegative. Show that $\int^* f d\mu = \infty$ if $\mu[\omega : f(\omega) > 0] = \infty$ or if $\mu[\omega : f(\omega) > a] > 0$ for all a.

For the first part, take any partition $A_1, \cdots, A_n$ of Ω into $\mathcal{F}$ sets. Let $B = [\omega : f(\omega) > 0]$. Note that we must have for some i, $\mu(A_i) \geq \mu(A_i \cap B) = \infty$ - if they were all finite, it would imply $\mu(B) < \infty$, which contradicts our assumption. Note that $\sup_{\omega \in A_i} f(\omega) > 0$, as $A_i \cap B$ is nonempty, and $\omega \in B$ implies $f(\omega) > 0$. Thus, we have:

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i) = \infty$$

As this is true for all partitions, the infimum is still infinite. Thus, if $\mu[\omega : f(\omega) > 0] = \infty$, we have:

$$\int^* f d\mu = \infty$$

Now, assume that $\mu[\omega : f(\omega) > a] > 0$ for all a. We need to again show for any partition $A_1, \cdots, A_n$ of Ω into $\mathcal{F}$ sets, the integral sum is infinite. Assume:

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i) < \infty$$

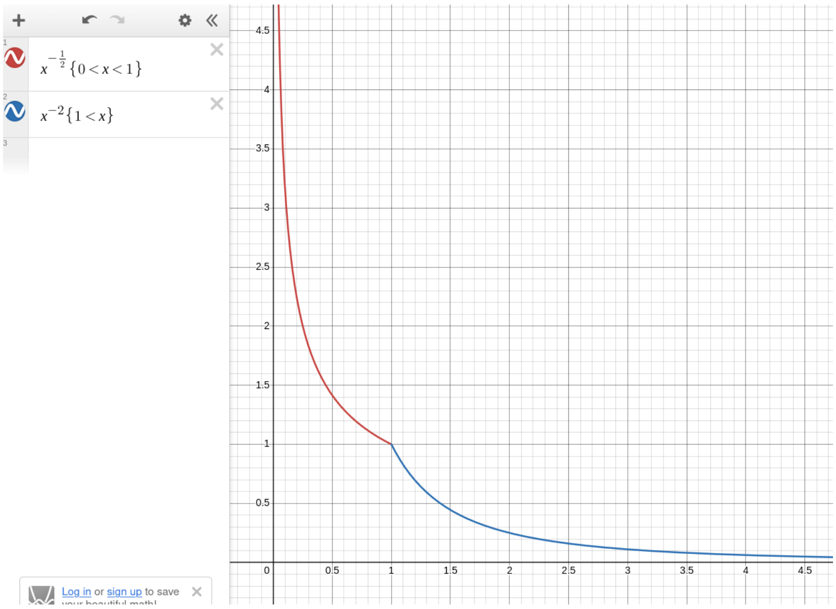
Then $a_i = \sup_{A_i} f < \infty$ for each i in the set of indices for which $\mu(A_i) > 0$. If $a = \max_i a_i$, then $\mu[f > a] = \sum_I \mu(A_i \cap [f > a]) \leq \sum_I \mu(A_i \cap [f > a_i]) = \sum_I 0 = 0$, because we know that a_i is the supremum of f on A_i (and while $f(\omega)$ may equal a_i, we have $f(\omega) > a_i$ must be false).

So, we have $\mu[f > a] = 0$. This contradicts $\mu[\omega : f(\omega) > a] > 0$ for all a, and so we must have for any partition $A_1, \cdots, A_n$:

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i) = \infty \implies \int^* f d\mu = \infty$$

qed.

Note - there are many familiar functions that ought to be integrable, but are of the preceding types and thus have infinite upper integral. Consider $f = x^{-2} \mathbb{1}\{x \geq 1\}$. Here, we have that the area that f is nonnegative in is infinite. Also consider $f = x^{-1/2} \mathbb{1}\{x \in (0, 1)\}$. Here, we have that the measure of $f > a$ for any a is positive. Thus, both functions would have infinite upper integral. However, we generally think they should have finite integrals, as their contained area is finite, even though they go to infinity in the range/domain. These are the classic examples - having an infinite height over a very small width, or a very small height over an infinite width. Also fun to note - these two graphs are essentially symmetrical, so they should both have the same area:



15.2 - Upper and Lower Integral Inequalities

1. Show that:

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) \leq \sum_j \left[\inf_{\omega \in B_j} f(\omega)\right] \mu(B_j)$$

If $\{B_j\}$ is a partition that refines $\{A_i\}$. Note, this should be clear, as for $B_j \subseteq A_i$, we have $\sum_j \mu(B_j) = \mu(A_i)$, and $\inf_{\omega \in A_i} f(\omega) \leq \inf_{\omega \in B_j} f(\omega)$. We similarly can conclude that:

$$\sum_j \left[\sup_{\omega \in B_j} f(\omega)\right] \mu(B_j) \leq \sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

If $\{B_j\}$ refines $\{A_i\}$. These two facts allow us to conclude that:

$$\int f d\mu = \int_* f d\mu \leq \int^* f d\mu$$

Note, the supremum the left is smaller than the infimum on the right, because every element in the left infimum is upper bounded by every element on the right infimum. Indeed, we will compare:

$$\sup \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) \qquad \text{versus} \qquad \inf \sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

Take an arbitrary sum on the left and right, for partitions A_i and C_i. Note, we can find a partition B_{ij} that refines both A_i and C_i. Define $B_{ij} = A_i \cap C_j$. We can take disjoint unions across j for a fixed i to recover A_i, and disjoint unions across i for a fixed j to recover C_j. Note that:

$$\sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) \leq \sum_{ij} \left[\inf_{\omega \in B_{ij}} f(\omega)\right] \mu(B_{ij})$$

$$\leq \sum_{ij} \left[\sup_{\omega \in B_{ij}} f(\omega)\right] \mu(B_{ij}) \leq \sum_j \left[\sup_{\omega \in C_j} f(\omega)\right] \mu(C_j)$$

Where the second inequality comes from comparing infimums and supremums. And so, we can indeed conclude that:

$$\int_* f d\mu \leq \int^* f d\mu$$

2. Assume f is bounded and that $\mu(\Omega) < \infty$. Now assume that f is measurable $\mathcal{F}$ and let M be a bound for $|f|$. Consider the partition $A_i = [\omega : i\epsilon < f(\omega) \leq (i+1)\epsilon]$, where i ranges from $-N$ to N and N is large enough such that $N\epsilon > M$. Show that:

$$\sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i) - \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i) < \epsilon\mu(\Omega)$$

I think this is clear. We have that:

$$= \sum_i \left[\sup_{\omega \in A_i} f(\omega) - \inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

And the difference between the inf and sup on A_i must be less than ϵ:

$$\leq \sum_i \epsilon\mu(A_i) = \epsilon\mu(\Omega)$$

Conclude that:

$$\int_* f d\mu = \int^* f d\mu$$

Note that the difference goes to zero as $\epsilon \to 0$, as $\mu(\Omega) < \infty$. Note that the limits as $\epsilon \to \infty$ equal the values for the lower and upper integral. If they didn't - then we would find that the upper integral is smaller than the lower integral, which would contradict:

$$\int_* f d\mu \leq \int^* f d\mu$$

And so we must have equality.

Section 16 - Properties of the Integral

Notes

Equalities and Inequalities

Integrable Definition 2 Recall, being integrable requires both $\int f^+ d\mu$ and $\int f^- d\mu$ to be finite. Via additivity of nonnegative functions from Theorem 15.1, that implies:

$$\int f^+ d\mu + \int f^- d\mu < \infty \quad \Leftrightarrow \quad \int f^+ + f^- d\mu < \infty \quad \Leftrightarrow \quad \int |f| d\mu < \infty$$

By Theorem 15.2 for almost everywhere monotonicity, this implies that if $|f| \leq |g|$ almost everywhere and g is integrable, then f is as well. If $\mu(\Omega) < \infty$ and f is bounded, f is also integrable (Ω itself is a partition).

Theorem 16.1 - Integral Monotonicity and Linearity

1. Monotonicity: If f and g are integrable and $f \leq g$ almost everywhere, then:
$$\int f d\mu \leq \int g d\mu$$

2. Linearity: If f and g are integrable and α, β are finite real numbers, then $\alpha f + \beta g$ is integrable and:
$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Proof

1. Monotonicity. For nonnegative f and g such that $f \leq g$ almost everywhere, the Theorem follows directly from Theorem 15.2.4. For general integrable f and g, if $f \leq g$ almost everywhere, then $f^+ \leq g^+$ and $g^- \leq f^-$ almost everywhere. So, the theorem follows directly by definition:
$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu \leq \int g^+ d\mu - \int g^- d\mu = \int g d\mu$$

2. Linearity. First, note that $\alpha f + \beta g$ is integrable via Theorem 15.1:
$$\int |\alpha f + \beta g| d\mu \leq \int |\alpha||f| + |\beta||g| d\mu = |\alpha| \int |f| d\mu + |\beta| \int |g| d\mu < \infty$$

The first step is directly Theorem 15.1.2, and the second step is directly Theorem 15.1.4. Now, we consider $g = 0$. It is clear that for $a \geq 0$:

$$\int afd\mu = \int af^+d\mu - \int af^-d\mu = a\int f^+d\mu - a\int f^-d\mu = a\int fd\mu$$

Flipping the subtraction gives us the same conclusion for $a < 0$. Therefore, we just need to prove the general case for $\alpha = \beta = 1$ - if we do, then we can treat $f = af$ and $g = bf$, split the addition, and then pull out the constants. We have $(f+g)^+ - (f+g)^- = f+g = f^+ - f^- + g^+ - g^-$ Rearranging gives us:

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

Being nonnegative sums on both sides, Theorem 15.1.4 gives us:

$$\int (f+g)^+d\mu + \int f^-d\mu + \int g^-d\mu = \int (f+g)^-d\mu + \int f^+d\mu + \int g^+d\mu$$

Rearranging, we can conclude that:

$$\int f + gd\mu = \int fd\mu + \int gd\mu$$

And so the statement follows.

We have thus proven both monotonicity and linearity of f and g integrable. qed.

We note that:

$$\left|\int fd\mu\right| = \left|\int f^+d\mu + \int f^-d\mu\right| \leq \left|\int f^+d\mu\right| + \left|\int f^-d\mu\right| = \int f^+d\mu + \int f^-d\mu = \int |f|d\mu$$

Where the first inequality comes from the triangle theorem, and the second equality comes from f^+ and f^- both being nonnegative. We also can apply to integrable f and g, in which case $f - g$ is integrable and making use of Theorem 16.1 gives us:

$$\left|\int fd\mu - \int gd\mu\right| = \left|\int f - gd\mu\right| \leq \int |f-g|\, d\mu$$

Not sure why this would be useful.

Example 16.1 - Counting Measure and Comparisons With Series Theory Suppose that Ω is countable - $\Omega = \{1, 2, \cdots\}$ to be definite. Let $\mathcal{F}$ consist of all subsets of Ω, and μ is the counting measure - each singleton has measure 1.

A function is thus a sequence, essentially, where $x_i = f(i)$. If x_{nm} is x_m if $m \leq n$ or 0 if $m > n$, the function f_n corresponding to $x_{n1}, x_{n2}, \cdots$ has integral $\sum_{m=1}^{n} x_m$ by Theorem 15.1.1, as it is a simple function essentially on the decomposition $\{1\}, \cdots, \{n\}, \{n+1, n+2, \cdots\}$. It follows by Theorem 15.1.3, that for nonnegative f, the integral of the function $f(m) = x_m$ is the sum:

$$\sum_m x_m$$

Finite or infinite, of the corresponding infinite series. In the general case - we have f^+ and $f-$ are nonnegative, with integrals that can be similarly expressed as $\sum_m x_m$. Thus, general f is *integrable* iff both of these sums are finite, which follows iff $\sum_m |x_m|$ is a convergent infinite series (ie sum is less than infinity). In this case, the integral is $\sum_{m=1}^{\infty} x_m^+ - \sum_{m=1}^{\infty} x_m^-$.

The function $x_m = (-1)^{m+1} m^{-1}$ is not integrable by this definition. That is because the absolute value sum is the non-convergent harmonic series. It fails to even have a definite integral, as $\sum_{m=1}^{\infty} x_m^+ = \sum_{m=1}^{\infty} x_m^- = \infty$ (every other sum on the harmonic series is still infinite).

Compare this with infinite series theory. We have that the alternating harmonic series does converge in the sense that:

$$\lim_M \sum_{m=1}^{M} (-1)^{m+1} m^{-1} = \log(2)$$

However - note that this sum is for the *first M elements.* It requires an order on the space Ω. Consider instead, for example the 3d integer coordinates - the integer lattice points in 3 space. It has no canonical linear ordering. If $\sum_m x_m$ is to have the same finite value no matter the order of summation, the series must be absolutely convergent, which is standard real analysis theory. This helps explain why f is defined to be integrable if and only if both $\int f^+ d\mu$ and $\int f^- d\mu$ are both finite - then, we have absolute convergence (at least in the counting measure example), which implies the sum is the same, no matter the reordering.

Example 16.2 - Principal Value I've included this example because it was something I was wondering about in section 15, defining integrals for which $\int f^+ d\mu = \int f^- d\mu = \infty$ by examining $\lim_n \int_{-n}^n f$.

Consider the function $f(x) = 3\mathbb{1}\{a \leq x\} - 2\mathbb{1}\{x < a\}$. There is no natural value for $\int f d\lambda$, as it is the $\infty - \infty$ case. But, if a function f is bounded on bounded intervals, then $f_n(x) = f(x)\mathbb{1}\{-n \leq x \leq n\}$ is integrable with respect to λ. Since $f = \lim_n f_n$, the limit of $\int f_n d\lambda$, it if exists, is sometimes called the *principal value* of the integral f.

Note, however, this is not the right definition in the context of general measure theory. The functions $g_n(x) = f(x)\mathbb{1}\{-n \leq x \leq n+1\}$ also converges to f - however, we may have $\lim_n \int g_n d\lambda \neq \lim_n \int f_n d\lambda$. Take $f(x) = x$ as one such example. There is no general reason why f_n, which I guess is symmetrical around the origin, should take precedence over g_n.

Integration to the Limit

Theorem 16.2 - Monotone Convergence Theorem If $0 \leq f_n \uparrow f$ almost everywhere, then $\int f_n d\mu \uparrow \int f d\mu$.

Proof: This is essentially Theorem 15.1.3, except we have added an almost everywhere. We have $0 \leq f_n \uparrow f$ on a set A with $\mu(A^c) = 0$. So, we can use an indicator, and make use of Theorem 15.2.5:

$$\int f_n d\mu = \int f_n \mathbb{1}\{A\} d\mu \uparrow \int f \mathbb{1}\{A\} d\mu = \int f d\mu$$

Theorem 16.3 - Fatou's Lemma For nonnegative f_n:

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$$

Proof: If $g_n = \inf_{k \geq n} f_k$, then $0 \leq g_n \uparrow g = \liminf_n f_n$ (this is by definition, the limit exists for each x as it is monotone increasing). Theorem 16.1 gives monotonicity for $g_n \leq f_n$, and Theorem 16.2 monotone convergence theorem gives:

$$\int f_n d\mu \geq \int g_n d\mu \uparrow \int g d\mu$$

As this is true for all n, we have:

$$\int g d\mu = \int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$$

And thus we have concluded the theorem. I think it might be better to note that:

$$\liminf_n \int f_n d\mu \geq \liminf_n \int g_n d\mu = \int g d\mu = \int \liminf_n f_n d\mu$$

We start with liminfs on both sides, as we only have the equality for fixed n, $\int f_n d\mu \geq \int g_n d\mu$. We can take liminfs of both side, which exist, and then note that the liminf equals the limit if it exists, which is our integral of g. That gives us the theorem for nonnegative f_n. qed.

Example 16.6 - Fatou's Lemma Strict Inequality On $(\mathbb{R}^1, \mathcal{R}^1, \lambda)$, the functions $f_n = n^2 \mathbb{1}\left\{(0, n^{-1})\right\}$ have integral value $n^2 * n^{-1} = n$. Note that for $f = 0$ though, $f_n(x) \to f(x)$ for each x.

Thus, $\liminf_n f_n = f$, so:

$$0 = \int f d\mu = \int \liminf_n f d\mu \qquad \infty = \liminf_n n = \liminf_n \int f_n d\mu$$

In this case, the inequality is strict. Note, it also is not always possible ot ingrate to the limit.

Theorem 16.4 - Dominated Convergence Theorem If $|f_n| \leq g$ almost everywhere, where g is integrable, and if $f_n \to f$ almost everywhere, then f and the f_n are integrable and $\int f_n d\mu \to \int f d\mu$.

Proof: Assume at the outset only that the f_n are dominated by an integrable g. Note, $|f_n| \leq g$ implies g is nonnegative, and so $|f_n| \leq |g|$ almost everywhere implies that f_n is integrable given that g is, by the note at the start of the chapter. Note, we also must have:

$$|\liminf_n f_n| \leq |g| \qquad |\limsup_n f_n| \leq |g|$$

This again is clear via liminf and limsup maintaining inequalities. Thus, both $f^* = \limsup_n f_n$ and $f_* = \liminf f_n$ are integrable as well. Recall, we already have f^* and f_* are measurable by Theorem 13.4. As g is measurable and nonnegative, we have $g + f_n$ and $g - f_n$ are nonnegative. And so, we can make use of Fatou's lemma to find:

$$\int g d\mu + \int f_* d\mu = \int \liminf_n (g + f_n) d\mu \leq \liminf_n \int (g + f_n) d\mu = \int g d\mu + \liminf_n \int f_n d\mu$$

Note, we had to pass to $g + f_*$ as Fatou's lemma can't be applied to not necessarily non-negative f_n. Similarly, we find:

$$\int g d\mu - \int f^* d\mu = \int g - \limsup_n f_n d\mu = \int \liminf (g - f_n) d\mu$$

Note, the equality is valid at each x. By Fatou's lemma, we have:

$$\leq \liminf_n \int g - f_n d\mu = \int g d\mu - \limsup_n \int f_n d\mu$$

Subtract $\int g d\mu$ from both sides of the inequalities we have derived above, and reorder to find:

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu \leq \limsup_n \int f_n d\mu \leq \int \limsup_n f_n d\mu$$

Note, we had something similar in Theorem 4.1. If we assume that $f_n \to f$ almost everywhere, then note that $\limsup_n f_n(x) = \liminf_n f_n(x)$ almost everywhere. Both the upper and lower values of the inequality are equal, to the common value of $\int f d\mu$, which exists as f must be similarly dominated by g. Thus, the limit of the integrals exist, as the inner liminf and limsup are equal, and we can conclude:

$$\lim_n \int f_n d\mu = \int f d\mu$$

Thus, we have proven the Dominated Convergence Theorem. qed.

Note - the Dominated Convergence Theorem could be considered just like a corollary to Fatou's lemma. Essentially, if $|f_n| \leq g$ almost everywhere, and g is integrable, then we can conclude:

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu \leq \limsup_n \int f_n d\mu \leq \int \limsup_n f_n d\mu$$

Note, the left hand side is given to us by Fatou's Lemma, but the right hand side is derived from Fatou's lemma, making use of our dominating g to create the integrable function $g - \limsup_n f_n = \liminf_n g - f_n$ to which we can apply Fatou's lemma.

Theorem 16.5 - Bounded Convergence Theorem If $\mu(\Omega) < \infty$ and the f_n are uniformly bounded, then $f_n \to f$ almost everywhere implies $\int f_n d\mu \to \int f d\mu$.

Proof: This is just a special case of the dominated convergence theorem. As the f_n are uniformly bounded, this means there is a finite K such that $|f_n(\omega)| \leq K$ for all ω and n (recall, f_n are all measurable on a measurable space $(\Omega, \mathcal{F}, \mu)$). We let $g = \mathbb{1}\{\Omega\} K$. Theorem 15.1.i tells us that $\int g d\mu = \mu(\Omega)K < \infty$, and so g is integrable. As $|f_n| \leq g = K$ almost everywhere, where g is integrable, and $f_n \to f$ almost everywhere, then Dominated Convergence Theorem tells us that f and f_n are integrable, and:

$$\int f_n d\mu \to \int f d\mu$$

Theorem 16.6 - Monotone Convergence Theorem For Series If $f_n \geq 0$, then $\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$. This is clear via monotone convergence theorem for $g_n = \sum_{k=1}^n f_k$ and $g = \sum_n f_n$. qed.

Theorem 16.7 - Dominated Convergence Theorem For Series If $\sum_n f_n$ converges almost everywhere and $|\sum_{k=1}^n F_k| \leq g$ almost everywhere, where g is integrable, then $\sum_n f_n$ and f_n are integrable, and $\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$. Again, same proof as before. qed.

Corollary 16.7.1 If $\sum_n \int |f_n| d\mu < \infty$, then $\sum_n f_n$ converges absolutely almost everywhere and is integrable, and $\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$.

Proof: The function $g = \sum_n |f_n|$ is integrable be Theorem 16.6, and is finite almost everywhere by Theorem 15.2.iii. Hence, we can apply Theorem 16.7, the Dominated Convergence Theorem For Series. qed.

Integration to Continuous Limits

This is a pretty technical section. In place of a sequence $\{f_n\}$ of real measurable functions on $(\Omega, \mathcal{F}, \mu)$, consider a family $[f_t : t \geq 0]$ indexed by a continuous parameter t. Suppose of a measurable f that:

$$\lim_{t \to \infty} f_t(\omega) = f(\omega)$$

on a set A, where:

$$A \in \mathcal{F} \qquad \mu(\Omega - A) = 0$$

A technical point arises here - since $\mathcal{F}$ doesn't need to necessarily contain A. Consider the following example. Let $\mathcal{F}$ consist of the Borel subsets of

$\Omega = [0, 1)$, and let H be a nonmeasurable set - a subset of Ω that does not lie in $\mathcal{F}$ (such as the Vitali Sets). Define:

$$f_t(\omega) = \begin{cases} 1 & \text{if } \omega = t - \lfloor t \rfloor \text{, the fractional part of } t\text{, AND } \omega = t - \lfloor t \rfloor \in H^c \\ 0 & \text{otherwise} \end{cases}$$

Each f_t is measurable $\mathcal{F}$. That is what the book states at least - I think this would rely on the set where $f_t(\omega) = 1$ being measurable, which is a subset of H^c. I think it is measurable - there should only be one point $\omega = t - \lfloor t \rfloor \in [0, 1)$ - so f_t is either identically equal to 0, or a one hot function, which is measurable (single points are an infinite intersection of $[\omega, \omega + \epsilon)$). However, if $f = 0$, the set where $\lim_{t \to \infty} f_t(\omega) = f(\omega)$ holds is H exactly - indeed, if $\omega \in H^c$, then the limit does not exist, as there will always be points $t > \infty$ such that $\omega = t - \lfloor t \rfloor$, and if $\omega \in H$, we have $f_t(\omega) = 0$ for all t.

Because of the above example, we actually need to assume that A is measurable. Now, assume that f and f_t are integrable. If $I_t = \int f_t d\mu$ converges to $I = \int f d\mu$ as $t \to \infty$, we have convergence for $I_{t_n} \to I$ for $t_n \to \infty$ a sequence. If $I_t \not\to I$, this similarly implies there is a sequence $t_n \to \infty$ such that $|I_{t_n} - I| > \epsilon$. The question of whether or not I_{t_n} converges to I can be answered with our previous theorems, which have to do with convergence on sequences.

Suppose that $|f_t(\omega)| \leq g(\omega)$ holds for $\omega \in A$, where $A \in \mathcal{F}$ and $\mu(A^c) = 0$, and g is integrable. Also, suppose that $\lim_{t \to \infty} f_t(\omega) = f(\omega)$ on A as well. We can make use of the Dominated Convergence Theorem to 1. Find that both f and f_t are integrable and 2. $I_{t_n} \to I$ for every sequence $t_n \to \infty$. Note, as this is true for all sequences t_n, by definition, $I_t \to I$. Note, this result would follow if t went continuously to 0 or some other value. Also note, Dominated Convergence Theorem applies as we have $|f_t| \leq g$ almost everywhere.

Theorem 16.8 - Leibnitz's Rule For Lebesgue Integrals suppose that $f(\omega, t)$ is a measurable and integrable function of ω for each $t \in (a, b)$. Let $\varphi(t) = \int f(\omega, t)\mu(d\omega)$ (note, the notation here helps specify that we are integrating across ω).

1. Suppose that for $\omega \in A$, where A satisfies $A \in \mathcal{F}$ and $\mu(\Omega - A) = 0$, $f(\omega, t)$ is continuous in t at $t_0 \in (a, b)$ - ie, $\lim_{t \to t_0} f(\omega, t) = f(\omega, t_0)$. Suppose further that $|f(\omega, t)| \leq g(\omega)$ for $\omega \in A$ and $|t - t_0| < \delta$, where

δ is independent of ω and g is integrable. Then $\varphi(t)$ is continuous at t_0.

2. Suppose that for $\omega \in A$, where A satisfies $A \in \mathcal{F}$ and $\mu(\Omega - A) = 0$, $f(\omega, t)$ has in (a, b) a derivative $f'(\omega, t)$; suppose further that $|f'(\omega, t)| \leq g(\omega)$ for $\omega \in A$ and $t \in (a, b)$, where g is integrable. Then $\varphi(t)$ has a derivative on (a, b):

$$\varphi'(t) = \int f'(\omega, t)\mu(d\omega)$$

Proof:

1. Note, this follows directly from our previous discussion. We have for each sequence $t_n \to t$, eventually $|f(\omega, t_n)| \leq g(\omega)$ once we have $|t - t_0| < \delta$ for $\omega \in A$, at which point we can switch over to the dominated convergence theorem, as we have $f_{t_n}(\omega) = f(\omega, t_n) \to f(\omega, t_0)$ via continuity (note, the domination of g and convergence all occur on the same almost everywhere set A). Thus, $I_{t_n} \to I_{t_0}$ for all sequences $t_n \to t_0$, and so:

$$\lim_{t_n \to t_0} \varphi(t_n) = \lim_{t_n \to t_0} I_{t_n} = I_{t_0} = \varphi(t_0)$$

2. Note, this is essentially the same proof through the Dominated Convergence Theorem. Consider a fixed t. If $\omega \in A$, then by the mean value theorem:

$$\frac{f(\omega, t + h) - f(\omega, t)}{h} = f'(\omega, s)$$

For some $s \in (t, t + h)$. By definition of the derivative, the ratio on the left goes to $f'(\omega, t)$ as $h \to 0$ (note, this can be done through a sequence). By hypothesis, we have that each entry in the sequence is dominated by g on A - ie, $|f'(\omega, s)| \leq g(\omega)$. Therefore, via the dominated convergence theorem:

$$\frac{\varphi(t + h) - \varphi(t)}{h} = \int \frac{f(\omega, t + h) - f(\omega, t)}{h}\mu(d\omega) \to \int f'(\omega, t)\mu(d\omega)$$

Where the above follows for any sequence of $h \to 0$, and the dominated convergence theorem applies. Thus, we have:

$$\varphi'(t) = \int f'(\omega, t)\mu(d\omega)$$

For $t \in (a, b)$. Note, the condition involving g can be weakened. It suffices to assume that for each t, there is an integrable $g(\omega, t)$ such that $|f'(\omega, s)| \leq g(\omega, t)$ for $\omega \in A$ and s in some neighborhood of t - note, this is essentially what we needed to be able to apply the Dominated Convergence Theorem. qed.

Integration Over Sets

The integral of f over a set A in $\mathcal{F}$ is defined by:

$$\int_A f d\mu = \int \mathbb{1}\{A\} f d\mu$$

Note that the integral equals zero if $\mu(A) = 0$. All the concepts and theorems previously stated carry over in an obvious way to integrals over A.

Theorem 16.9 - Sum Of Integrals Over Sets If $A_1, A_2, \cdots$ are disjoint, and if f is either nonnegative or integrable, then:

$$\int_{\bigcup_n A_n} f d\mu = \sum_n \int_{A_n} f d\mu$$

Proof: This follows directly from Theorem 16.6, the series version of monotone convergence, and Theorem 16.7, the series version of dominated convergence (as each $\|\mathbb{1}\{A_n\} f\| \leq f$, and f is integrable). Just express the union as a summation. qed.

Theorem 16.10 - Integral Values Determine f

1. If f and g are nonnegative and $\int_A f d\mu = \int_A g d\mu$ for all A in $\mathcal{F}$, and if μ is σ-finite, then $f = g$ almost everywhere.

2. If f and g are integrable and $\int_A f d\mu = \int_A g d\mu$ for all A in $\mathcal{F}$, then $f = g$ almost everywhere.

3. If f and g are integrable and $\int_A f d\mu = \int_A g d\mu$ for all A in $\mathcal{P}$, where $\mathcal{P}$ is a π system generating $\mathcal{F}$ and Ω is a finite or countable union of $\mathcal{P}$ sets, then $f = g$ almost everywhere.

Note, we need different cases, as nonnegative does not necessarily imply integrable. The first case will deal with infinite values.

Proof:

1. Suppose that f and g are nonnegative and that $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in \mathcal{F}$. If μ is σ finite, we can find $A_n \in \mathcal{F}$ such that $A_n \uparrow \Omega$ and $\mu(A_n) < \infty$. Note, take the $B_1, B_1 \cup B_2, B_1 \cup B_2 \cup B_3$ and so on, and note that each union measure is bounded by a finite sum of finite values. Define now:

$$B_n = [0 \leq g < f, g \leq n]$$

$B_n \in \mathcal{F}$ via g, f measurable. Note, the hypothesis applies to $A_n \cap B_n$, so:

$$\int_{A_n \cap B_n} f d\mu \leq \int_{A_n \cap B_n} g d\mu < \infty$$

Where the final equality comes from $\mu(A_n \cap B_n) \leq \mu(A_n) < \infty$ and $g \leq n$ on that set, so the integral is less than $\mu(A_n) * n$. Hence, we have that the subtraction is well defined:

$$\int_{A_n \cap B_n} f - g d\mu = 0$$

Where $= 0$ comes from the initial hypothesis. By Theorem 15.2(ii), if the measure where the function $\mathbb{1}\{A_n \cap B_n\} f - g$ was greater than 0, the integral would be greater than 0. So, we must have:

$$\mu(\mathbb{1}\{A_n \cap B_n\}(f - g) > 0) = 0$$

Now, note that:

$$A_n \cap B_n \subseteq \{\mathbb{1}\{A_n \cap B_n\}(f - g) > 0\}$$

As if $\omega \in A_n \cap B_n$, we have $\omega \in B_n \implies g(\omega) < f(\omega) \implies f(\omega) - g(\omega) > 0$, and $\mathbb{1}\{A_n \cap B_n\}(\omega) = 1$. And so, we can conclude that $\mu(A_n \cap B_n) = 0$ via the monotonicity of the measure μ. As $A_n \cap B_m \uparrow \Omega \cap B_m$, this implies:

$$\mu(B_m) = \lim_{n \to \infty} \mu(A_n \cap B_m) = 0$$

Note, everything above would have followed for $m \neq n$. Now, note that $B_m \uparrow \{0 \leq g < f, g < \infty\}$, and so:

$$\mu[0 \leq g < f, g < \infty] = 0$$

This implies that $f \leq g$ almost everywhere. Indeed, we find:

$$\{f \leq g\}^c = \{g < f\} = \{0 \leq g < f, g < \infty\}$$

The first equality is clear. As for the second - note first that $0 \leq g$, so we can add it easily. Note second that if $g = \infty$, $g < f$ is not possible, as there is no number on our extended real line that exceeds infinity. So, we can add that extra condition without losing any information. And so, we have $f \leq g$ almost everywhere, and $g \leq f$ almost everywhere symmetrically. The intersection of almost everywhere events is still almost everywhere, and so:

$$g = f \text{ almost everywhere}$$

2. If f and g are integrable, we have a simpler argument, as we can take the integral subtraction without any added $A_n \cap B_n$ headache. We have $\int_A f d\mu \leq \int_A g d\mu$ for all $A \in F$, and so:

$$\int \mathbb{1}\{g < f\}(f - g)d\mu = 0 \implies \mu(g < f) = 0$$

Where the implication comes from the same argument we made for $\mu(A_n \cap B_n) = 0$ above with Theorem 15.2(ii). And so, $g = f$ almost everywhere, as $\mu((g = f)^c) \leq \mu(g < f) + \mu(f < g) = 0$.

3. First, assume that f and g are nonnegative. We have that the following are measures on $\mathcal{F}$:

$$\nu_f(A) = \int_A f d\mu \qquad \nu_g(A) = \int_A g d\mu$$

Recall, to be a measure, we must have the measure of the emptyset is zero, the value of the measure is nonnegative for all $A \in \mathcal{F}$, and countable additivity. As f and g are nonnegative, the integrals and thus measures are always nonnegative; clearly, $\nu_f(\emptyset) = 0 = \nu_g(\emptyset)$ as well. Also just a note, f and g being nonnegative imply that $\mathbb{1}\{A\} f$ always has an integral value. Finally, note that countable additivity comes from Theorem 16.9. So, we indeed have measures. Our condition implies equality of the measures for all $A \in \mathcal{P}$.

Further, we have for all $A \in \mathcal{F}$, $\nu_f(A) < \infty, \nu_g(A) < \infty$, as f and g are integrable, and so the corresponding integrals must be finite. So, together with the assumption that Ω is a finite or countable union of $\mathcal{P}$ sets, we can make use of Theorem 10.4 to conclude that:

$$\nu_f(A) = \nu_g(A) \text{ for all } A \in \mathcal{F} \implies \int_A f d\mu = \int_A g d\mu \text{ for all } A \in \mathcal{F}$$

So now, we need to extend to the general f and g integrable. Well, note for $A \in \mathcal{P}$, we still have:

$$\int_A f d\mu = \int_A g d\mu \implies \int_A f^+ d\mu - \int_A f^- d\mu = \int_A g^+ d\mu - \int_A g^- d\mu$$

$$\implies \int_A f^+ + g^- d\mu = \int_A g^+ + f^- d\mu$$

Note, as $f^+ + g^-$ and $g^+ + f^-$ are nonnegative, we make use of the same argument to find $f^+ + g^- = g^+ + f^-$ almost everywhere, which implies $f = g$ almost everywhere in the general case. qed.

Densities

Suppose that δ is a nonnegative measurable function and defined a measure ν by:

$$\nu(A) = \int_A \delta d\mu$$

In the previous Theorem 16.10.3, we noted that ν is indeed a measure. You don't even need to assume that ν is integrable - just nonnegative, in which case the integral can be finite or infinite. Note that $\mu(A) = 0 \implies \nu(A) = 0$ via Theorem 15.2ii. Further, ν is finite if and only if δ is integrable μ - this should be clear as well.

Another function δ' gives rise to the same ν if $\delta = \delta'$ almost everywhere. On the other hand, both having the same ν imply the functions are equal via Theorem 16.10.1 (the first assumption of densities is that δ are nonnegative).

The measure ν defined by above is said to have *density* δ with respect to μ. Note - this is a good name - as density times volume (given by A here) should give you a mass (given by $\nu(A)$).

Theorem 16.11 - Integral Across Measures With Density If ν has density δ with respect to μ, then:

$$\int f d\nu = \int f \delta d\mu$$

Holds for nonnegative f. Moreover, f (not necessarily nonnegative) is integrable with respect to ν if and only if $f\delta$ is integrable with respect to μ, in which case:

$$\int f d\nu = \int f \delta d\mu \text{ and } \int_A f d\nu = \int_A f \delta d\mu$$

For nonnegative f, the above always holds as well.

Proof: First, take $f = \mathbb{1}\{A\}$ for $A \in \mathcal{F}$. Note that both ν and μ are measures on $\mathcal{F}$. We have:

$$\int f d\nu = \nu(A) = \int_A \delta d\mu = \int f \delta d\mu$$

Where the first equality makes use of Theorem 15.1.1 and the remaining inequalities are definitions. If f is a simple nonnegative function, then equality follows via linearity of the integral. Finally, for f nonnegative, then:

$$\int f_n d\nu = \int f_n \delta d\mu$$

For $f_n \uparrow f$ simple, which exists by Theorem 13.5. Then, we can pass to the integral via the monotone convergence theorem - Theorem 16.2. So, we have proved the first statement for nonnegative f.

Now, if f is not nonnegative, we can apply the argument to $|f|$, and so if f is integrable with respect to ν, we have:

$$\int |f| d\nu = \int |f| \delta d\mu = \int |f\delta| d\mu$$

The second equality follows by definition of a density, which must be nonnegative, and so it is clear that f is integrable wrt ν iff $f\delta$ is integrable wrt μ. And if f is integrable, we have that:

$$\int f d\nu = \int f^+ d\nu - \int f^- d\nu = \int (f\delta)^+ d\mu - \int (f\delta)^- d\mu = \int f\delta d\mu$$

Replacing f by $f\mathbb{1}\{A\}$ gives us the proofs for integrals across subsets. qed.

Theorem 16.12 - Convergence of Densities Implies Convergence Of Measures Suppose that $\nu_n(A) = \int_A \delta_n d\mu$ and $\nu(A) = \int_A \delta d\mu$ for densities δ_n and δ. If:

$$\nu_n(\Omega) = \nu(\Omega) < \infty \text{ for } n = 1, 2, \cdots$$

And if $\delta_n \to \delta$ except on a set of μ measure 0, then:

$$\sup_{A \in \mathcal{F}} |\nu(A) - \nu_n(A)| \leq \int_\Omega |\delta - \delta_n| d\mu \to 0$$

Proof: Earlier, we proved via monotonicity, that $\left|\int f d\mu\right| \leq \int |f| d\mu$. And so, we can immediately conclude our inequality:

$$\sup_{A\in\mathcal{F}} |\nu(A) - \nu_n(A)| = \sup_{A\in\mathcal{F}} \left|\int_A \delta - \delta_n d\mu\right| \leq \sup_{A\in\mathcal{F}} \int_A |\delta - \delta_n| d\mu \leq \int_\Omega |\delta - \delta_n| d\mu$$

Note, the subtraction is well defined by monotonicity, and $\nu(\Omega) < \infty$. Let $g_n = \delta - \delta_n$. The positive part g_n^+ of g_n converges to 0 except on a set of μ measure 0. As δ and δ_n are nonnegative densities, we also have $0 \leq g_n^+ \leq \delta$ and δ is integrable across μ (again, via $\nu(\Omega) < \infty$ and everything is nonnegative), and so the dominated convergence theorem applies:

$$\int g_n^+ d\mu \to 0$$

Also note that:

$$\int g_n d\mu = \int \delta - \delta_n d\mu = \nu(\Omega) - \nu_n(\Omega) = 0$$

Therefore, we have:

$$\int_\Omega |g_n| d\mu = \int_{[g_n \geq 0]} g_n d\mu - \int_{[g_n < 0]} g_n d\mu = 2\int_{[g_n \geq 0]} g_n d\mu = 2\int_\Omega g_n^+ d\mu \to 0$$

The second equality coming from the fact that as the complete integral of g_n equals zero, the negative and positive integral portions must be equal. And thus, we have found that:

$$\int_\Omega |\delta - \delta_n| d\mu = \int_\Omega |g_n| d\mu \to 0$$

Note, we had to go through the above fuss because we were dealing with the absolute value of the difference, not just the difference. The difference equaling zero came directly from our assumptions. qed.

Corollary 16.12 - Infinite Series If $\sum_m x_{nm} = \sum_m x_m < \infty$, the terms being nonnegative, and if $\lim_n x_{nm} = x_m$ for each m, then:

$$\lim_n \sum_m |x_{nm} - x_m| = 0$$

If y_m is bounded, then $\lim_n \sum_m y_m x_{nm} = \sum_m y_m x_m$.

Note, this is similar to the Weierstrauss M test. Note that it follows from taking μ as the counting measure on $\Omega = \{1, 2, \cdots\}$, and using Theorem 16.12.

Change of Variable

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces, and suppose that the mapping $T : \Omega \to \Omega'$ is measurable by $\mathcal{F}/\mathcal{F}'$. For a measure μ on $\mathcal{F}$, define measure μT^{-1} on $\mathcal{F}'$ by:

$$\mu T^{-1}(A') = \mu(T^{-1}(A')) \text{ where } A' \in \mathcal{F}'$$

Recall, we already discussed this at the end of section 13. This is a pullback measure, as we pull A' back into $\mathcal{F}$ via T^{-1}.

Suppose f is a real function on Ω' that is measurable $\mathcal{F}'$, so that the composition fT is a real function on Ω that is measurable $\mathcal{F}$. Composition of measurable functions is measurable, this is a pushforward I believe.

Theorem 16.13 - Change Of Variables If f is nonnegative real measurable function on Ω', then:

$$\int_{\Omega} f(T\omega)\mu(d\omega) = \int_{\Omega'} f(\omega')\mu T^{-1}(d\omega')$$

A function f (not necessarily nonnegative) is integrable with respect to μT^{-1} if and only if fT is integrable with respect to μ, in which case the above holds, as well as:

$$\int_{T^{-1}A'} f(T\omega)\mu(d\omega) = \int_{A'} f(\omega')\mu T^{-1}(d\omega')$$

For nonnegative f, the above always holds as well.

Note, this is the same problem as in Theorem 16.13 - it is badly stated. I think we just have both expressions hold for nonnegative f, and as well for general f integrable, where we also have the if and only if statement. Note also - the $'$ and $\mu(d\omega)$ help us understand what we are integrating over. On $\mathcal{F}'$, we use the pull back measure, along with f as defined, whereas on $\mathcal{F}$ we use the original measure, and the pushforward fT. Oh, I think the Theorem is stated as is, as we first have to prove the expression for nonnegative f, which gives us the iff, which gives us the expression for integrable f, along with the integral over subsets expression, which we can then conclude for nonnegative f as well.

Proof: If $f = \mathbb{1}\{A'\}$, then $fT = \mathbb{1}\{T^{-1}A'\}$, and so we have that:

$$\int_{\Omega} f(T\omega)\mu(d\omega) = \mu(T^{-1}(A')) = \int_{\Omega'} f(\omega')\mu T^{-1}(d\omega')$$

By linearity, the equation holds for nonnegative simple functions as well. If f_n are simple functions for which $0 \leq f_n \uparrow f$, then $0 \leq f_n T \uparrow fT$ as well. Indeed, for ω, we have:

$$\lim_n f_n T(\omega) = \lim_n f_n(T\Omega)) = f(T(\Omega)) = fT(\Omega)$$

And so, we have for nonnegative f, via the Monotone Convergence Theorem:

$$\int_\Omega f(T\omega)\mu(d\omega) = \lim_n \int_\Omega f_n T(\omega)\mu(d\omega) = \lim_n \int_{\Omega'} f_n(\omega')\mu T^{-1}(d\omega') = \int_{\Omega'} f(\omega')\mu T^{-1}(d\omega')$$

Now, we just have for any f, that:

$$\int_\Omega |f|(T\omega)\mu(d\omega) = \int_{\Omega'} |f|(\omega')\mu T^{-1}(d\omega')$$

So, it is clear that $|f|T = |fT|$ is integrable with respect to μ if and only if $|f|$ is integrable with respect to μT^{-1}. Note, the first expression holds for integrable f if we decompose into $f = f^+ - f^-$, and finally, we get integrals over subsets by replacing f with $f\mathbb{1}\{A\}$. qed.

Example 16.10 Suppose that $(\Omega', \mathcal{F}') = (\mathbb{R}^1, \mathcal{R}^1)$ and $T = \varphi$ is an ordinary real function, measurable $\mathcal{F}$. If $f(x) = x$, the above becomes:

$$\int_\Omega \varphi(x)\mu(d\omega) = \int_{\mathbb{R}^1} x\mu\varphi^{-1}(dx)$$

Actually, I think this example isn't great. However, what we can do is note that - we can get the general Change of Variables Theorem from multivariable calculus out of this Theorem. Let our measurable spaces be $(\mathbb{R}^n, \mathcal{R}^n)$ and $(\mathbb{R}^n, \mathcal{R}^n)$, and let T^{-1} be nonsingular and linear. Let μ be the Lebesgue measure λ. We can just replace T with T^{-1}, and so for integrable/nonnegative f, Theorem 16.13 gives us for $A \in \mathcal{R}^n$:

$$\int_{T(A)} (f \circ T^{-1})d\lambda = \int_A f(\omega')d(\lambda \circ T)$$

Now, recall, Theorem 12.2 tells us that:

$$\lambda \circ T = |\det T|\lambda$$

So, we can conclude (note, this can be done with the similar simple function, nonnegative function, integrable function argument):

$$\int_{T(A)} (f \circ T^{-1})d\lambda = \int_A f(\omega')|\det T|d\lambda$$

And we can just replace f with $f \circ T$ to give us:

$$\int_{T(A)} f d\lambda = \int_A (f \circ T) |\det T| d\lambda$$

Now, the only difference here is that in calculus, we can assume g is 1-1 continuously differentiable, not just linear. I think there is a way to extend the above for such g as well (not just T), but I'll leave it there for now.

Uniform Integrability

This section essentially introduces another criteria which can be used to find the convergence of integrals.

If f is integrable, then $|f| \mathbb{1} \{|f| \geq a\}$ goes to 0 almost everywhere as $a \to \infty$. If that was not the case, then there is a set of positive measure where $|f| = \infty$, which would contradict $\int |f| d\mu < \infty$.

Hence, as $|f| \mathbb{1} \{|f| \geq a\}$ is dominated by $|f|$, the dominated convergence theorem gives us for integrable f:

$$\lim_{a \to \infty} \int_{[|f| \geq a]} |f| d\mu = 0$$

We can use this notion to define a property of a sequence of functions f_n.

Definition - Uniformly Integrable A sequence $\{f_n\}$ is *uniformly integrable* if:

$$\lim_{a \to \infty} \sup_n \int_{[|f_n| \geq a]} |f_n| d\mu = 0$$

Assume the property holds for f_n, and $\mu(\Omega) < \infty$. If a is large enough such that the supremum is less than 1, then:

$$\int |f_n| d\mu = \int_{[|f_n| \geq a]} |f_n| d\mu + \int_{[|f_n| < a]} |f_n| d\mu \leq 1 + \int_{[|f_n| < a]} |f_n| d\mu \leq 1 + a\mu(\Omega)$$

Which implies that the f_n are integrable.

Now, assume that the f_n are uniformly bounded, ie that $f_n(\omega) \leq K$ for each n, ω. In this case, it is clear that uniform integrability always holds, as we just need $a \geq K$. In this case, the f_n don't even need to be integrable, which can happen if $\mu(\Omega) = \infty$.

If h is the maximum of $|f|$ and $|g|$, then:

$$\int_{|f+g|\geq 2a} |f+g| d\mu \leq 2\int_{h\geq a} h d\mu = 2\int_{|f|\geq a, |g|\geq a} h d\mu + 2\int_{|f|\geq a, |g|<a} h d\mu + 2\int_{|f|<a, |g|\geq a} h d\mu$$

$$= 2\int_{|f|\geq a, |g|\geq a} h d\mu + 2\int_{|f|\geq a, |g|<a} f d\mu + 2\int_{|f|<a, |g|\geq a} g d\mu$$

$$= 2\int_{|f|\geq a, |g|\geq a, |f|\geq |g|} |f| d\mu + 2\int_{|f|\geq a, |g|\geq a, |f|<|g|} |g| d\mu + 2\int_{|f|\geq a, |g|<a} f d\mu + 2\int_{|f|<a, |g|\geq a} g d\mu$$

$$= 2\int_{|f|\geq a, |f|\geq |g|} |f| d\mu + 2\int_{|g|\geq a, |f|\leq |g|} |g| d\mu$$

$$\leq 2\int_{|f|\geq a} |f| d\mu + 2\int_{|g|\geq a} |g| d\mu$$

The first inequality holds as $2h \geq f+g$, and if $f+g \geq 2a$ then $2h \geq f+g \geq 2a \implies h \geq a$. So, the domain in the right integral is larger, and the function $2h \geq |f|+|g| \geq |f+g|$ is larger as well. The remaining equalities/inequalities are self-explanatory.

This implies that if $\{f_n\}$ and $\{g_n\}$ are uniformly integrable, so is $\{f_n+g_n\}$. This is because:

$$\lim_{2a\to\infty} \sup_n \int_{[|f_n+g_n|\geq 2a]} |f_n+g_n| d\mu \leq \lim_{2a\to\infty} \sup_n 2\int_{|f_n|\geq a} |f_n| d\mu + 2\int_{|g_n|\geq a} |g_n| d\mu = 0$$

Where the right side goes to zero via uniform integrability of f_n and g_n.

Theorem 16.14 - Uniform Integrability Convergence Suppose that $\mu(\Omega) < \infty$ and $f_n \to f$ almost everywhere.

1. If the f_n are uniformly integrable, then f is integrable and:

$$\int f_n d\mu \to \int f d\mu$$

2. If f and the f_n are nonnegative and integrable, then:

$$\int f_n d\mu \to \int f d\mu$$

   Implies that the f_n are uniformly integrable.

Proof We first prove that if the f_n are uniformly integrable, then f is integrable. We want to show:

$$\int |f| d\mu < \infty$$

We have via our starting notes that give $\mu(\Omega) < \infty$, for some a finite and large enough:

$$\int |f_n| d\mu \leq a\mu(\Omega) + 1 < \infty$$

As $f_n \to f$ almost everywhere, we have $|f_n| \to |f|$ almost everywhere, ie on a set A where $\mu(A^c) = 0$, and so:

$$\int |f| d\mu = \int_A |f| d\mu + \int_{A^c} |f| d\mu = \int_A \liminf |f_n| d\mu + \int_{A^c} |f| d\mu$$

Where the second equality just comes from the limit. Now, via Fatou's Lemma, we have:

$$\leq \liminf_{n\to\infty} \int_A |f_n| d\mu + \int_{A^c} |f| d\mu \leq a\mu(\Omega) + 1 + \int_{A^c} |f| d\mu = a\mu(\Omega) + 1 < \infty$$

Where we note that $f\mathbb{1}\{A^c\} = 0$ almost everywhere. And so, with uniform integrability of f_n, $\mu(\Omega) < \infty$, $f_n \to f$ almost everywhere, and Fatou's Lemma, we can conclude that f is integrable. Define:

$$f_n^{(a)} = \begin{cases} f_n & \text{if } |f_n| < a \\ 0 & \text{if } |f_n| \geq a \end{cases} \qquad f^{(a)} = \begin{cases} f & \text{if } |f| < a \\ 0 & \text{if } |f| \geq a \end{cases}$$

If $\mu[|f| = a] = 0$, then $f_n^{(a)} \to f^{(a)}$ almost everywhere - we need the condition, because at $|f| = a$, for the ω that satisfy the condition, $|f_n^{(a)}|(\omega)$ could be bouncing from 0 to the $|f_n|(\omega) \to |f|(\omega) \neq 0$ value. But, if this is the case, we can make use of the Bounded Convergence Theorem to find:

$$\int f_n^{(a)} d\mu \to \int f^{(a)} d\mu$$

Since:

$$\int f_n d\mu - \int f_n^{(a)} d\mu = \int_{|f_n| \geq a} f_n d\mu$$

Which can be found by using indicator definitions for $f_n^{(a)}$, and similarly:

$$\int f d\mu - \int f^{(a)} d\mu = \int_{|f| \geq a} f d\mu$$

It follows from the above equations that:

$$\limsup_n \left| \int f_n d\mu - \int f d\mu \right| \leq \sup_n \int_{|f_n| \geq a} |f_n| d\mu + \int_{|f| \geq a} |f| d\mu$$

As for why that is:

$$\limsup_n \left| \int f_n d\mu - \int f d\mu \right| = \limsup_n \left| \int f_n d\mu - \int f_n^{(a)} d\mu + \int f^{(a)} d\mu - \int f d\mu \right|$$

Where the middle portion goes to zero, as the limit between the two values exists. This becomes via our above equalities:

$$= \limsup_n \left| \int_{|f_n| \geq a} f_n d\mu - \int_{|f| \geq a} f d\mu \right| \leq \sup_n \left| \int_{|f_n| \geq a} f_n d\mu - \int_{|f| \geq a} f d\mu \right|$$

Where the above comes from the supremum always being greater than the lim sup. Continuing with the triangle inequality and our note at the start of the chapter:

$$\leq \sup_n \left| \int_{|f_n| \geq a} f_n d\mu \right| + \left| \int_{|f| \geq a} f d\mu \right| \leq \sup_n \int_{|f_n| \geq a} |f_n| d\mu + \int_{|f| \geq a} |f| d\mu$$

Now, we just take $a \to \infty$ on the RHS. Note, as f is measurable, there are only countably many points where $\mu[|f| = a] \neq 0$ - if there were more than countably many, we would contradict Theorem 10.2.iv. We take a sequence $a_n \to \infty$ that avoids these points, which allows us to conclude:

$$\limsup_n \left| \int f_n d\mu - \int f d\mu \right| \leq \lim_{a_n} \sup_n \int_{|f_n| \geq a_n} |f_n| d\mu + \int_{|f| \geq a_n} |f| d\mu$$

$$\leq \lim_{a_n} 0 + \int_{|f| \geq a_n} |f| d\mu$$

Where the 0 comes from uniform integrability. As for the remaining portion, a_n being integrable implies that the right integral goes to 0 as well, and so:

$$\limsup_n \left| \int f_n d\mu - \int f d\mu \right| = 0 \implies \lim_n \left| \int f_n d\mu - \int f d\mu \right| = 0$$

$$\implies \int f_n d\mu \to \int f d\mu$$

Now, suppose that the limit holds, where f_n and f are nonnegative and integrable. We want to show that f_n is uniformly integrable. If $\mu[|f| = a] = 0$ holds, like before, we can find:

$$\int f_n^{(a)} d\mu \to \int f^{(a)} d\mu$$

And from our subtractions, and $\int f_n d\mu \to \int f d\mu$ as well, we can find:

$$\int_{f_n \geq a} f_n d\mu \to \int_{f \geq a} f d\mu$$

Absolute values in the domains removed by non-negativity. As f is integrable, there is, for given ϵ, an a such that the right hand side above is less than ϵ and $\mu[f = a] = 0$. But then, the integral on the left is similarly less than ϵ for all n exceeding some n_0. Thus:

$$\sup_{n \geq n_0} \int_{f_n \geq a} f_n d\mu < \epsilon$$

For n_0 corresponding to our a. Now, if we let a be large enough, for the first $n_0 - 1$ functions, they can be found to be less than ϵ as well, given integrability implies there is always some $a_1, \cdots, a_{n_0 - 1}$ for each of the first $n_0 - 1$ functions for which the integrals are less than ϵ. So, for ϵ chosen, and corresponding a, and then corresponding n_0, and then $a = \max(a, a_1, \cdots, a_{n_0-1})$, we have:

$$\sup_n \int_{f_n \geq a} f_n d\mu < \epsilon$$

And this implies:

$$\lim_{a \to \infty} \sup_n \int_{f_n \geq a} f_n d\mu = 0$$

For nonnegative f_n, this is the uniform integrable condition. qed.

Corollary to Uniform Integrability Convergence Suppose that $\mu(\Omega) < \infty$. If f and the f_n are integrable, and if $f_n \to f$ almost everywhere, then these conditions are equivalent:

1. f_n are uniformly integrable
2. $\int |f - f_n| d\mu \to 0$
3. $\int |f_n| d\mu \to \int |f| d\mu$

Proof: Assume that 1. holds. Then, the differences $|f - f_n|$ are uniformly integrable as well. Make use of our previous note - if f_n and g_n are uniformly integrable, so is $\{f_n + g_n\}$. Let $g_n = -f$, and note integrability of $-f$ clearly implies uniform integrability of g_n. So, $f_n - f$ are uniformly integrable. Note that the absolute value $|f_n - f|$ is uniformly integrable as well - this is because we already take the absolute value in the definition of uniformly integrable.

And so, $|f - f_n| = |f_n - f|$ is uniformly integrable.

Now, note that $|f - f_n| \to 0$ almost everywhere via $f_n \to f$ almost everywhere. Thus, Theorem 16.14 implies that $\int |f - f_n| d\mu \to 0$.

Now, note that (2) implies (3), because $||f| - |f_n|| \leq |f - f_n|$. This follows, because the triangle inequality implies $|a| \leq |a-b| + |b| \implies |a| - |b| \leq |a-b|$, and with symmetry, this implies:

$$-|a-b| \leq |a| - |b| \leq |a-b| \implies ||a| - |b|| \leq |a-b|$$

And so, monotonicity of the integral implies:

$$\left| \int |f| - |f_n| d\mu \right| \leq \int ||f| - |f_n|| d\mu \leq \int |f - f_n| \to 0 \implies \lim_n \left| \int |f| - |f_n| d\mu \right| = 0$$

$$\implies \int |f_n| d\mu \to \int |f| d\mu$$

Now, note that (3) implies (1), as Theorem 16.14 part 2 with (3) directly implies the $|f_n|$ are uniformly integrable. Note, this implies f_n are uniformly integrable as well, from what we noted above. qed.

Complex Functions

A complex-valued function on Ω has the form $f(\omega) = g(\omega) + ih(\omega)$, where g and h are ordinary finite valued real functions on Ω. We defined it as measurable f if both h and g are. I guess, essentially, we have the borel sets on $\mathbb{R}^2$ - $\mathcal{R}^2$ - generated from the rectangles. I don't think if $A \in \mathcal{R}^2$, we have the projection $\pi_i(A)$ is a borel set in $\mathcal{R}$.

If g and h are integrable, then f is by definition integrable, and its integral is taken as:

$$\int (g + ih) d\mu = \int g d\mu + i \int h d\mu$$

Now, we note that:

$$\max\{|g|, |h|\} \leq |f| \leq |g| + |h|$$

Both come from $|f| = \sqrt{|g^2| + |h|^2}$. We have $|g|^2 + |h|^2 \leq (|g| + |h|)^2$, which is clear, and so the second inequality comes through. For the first one, note that the square of the max is $\max\{|g|^2, |h|^2\}$, which is less than $|g|^2 + |h|^2 = |g^2| + |h^2| = g^2 + h^2$. So, f is integrable if and only if $\int |f| d\mu < \infty$, just as in the real case.

Note that pretty much all of our theorems on real integral linearity and limits extend to the complex case as well.

Problems

16.1 Alternate Bounded Convergence Theorem Proof

Note, this problem says to make use of problem 13.9, in which we proved Egoroff's Theorem: Suppose that f_n and f are finite valued, $\mathcal{F}$ measurable functions such that $f_n(\omega) \to f(\omega)$ for $\omega \in A$, where $\mu(A) < \infty$ and μ is a measure on σ field $\mathcal{F}$. For each $\epsilon > 0$, there exists a subset B of A such that $\mu(B) < \epsilon$, and $f_n(\omega) \to f(\omega)$ uniformly on $A - B$.

Now, for the problem. Suppose that $\mu(\Omega) < \infty$ and the f_n are uniformly bounded.

1. Assume $f_n \to f$ uniformly and deduce $\int f_n d\mu \to \int f d\mu$ from what we proved earlier, that for integrable f and g:

   $$\left| \int f d\mu - \int g d\mu \right| \leq \int |f - g| d\mu$$

   Recall, f_n uniformly bounded means that $|f_n(\omega)| \leq K$ for all ω, n.

   Can't we just apply the bounded convergence theorem to automatically prove this? I think maybe we want to prove convergence without the theorem. Note, the next part of the problem wants us to use this part to give another proof for the bounded convergence theorem, so yes, we need to go ahead without it.

   It is clear that uniformly bounded and $\mu(\Omega) < \infty$ implies that f_n are integrable. Can we also conclude that f is integrable? Yes. For $\epsilon > 0$, there is an n such that $|f_n(\omega) - f(\omega)| < \epsilon$ for all ω, via uniform convergence. And so, $|f(\omega)| < K + 2\epsilon$, which implies that $\int |f| d\mu < \mu(\Omega)(K + 2\epsilon) < \infty$, so we have integrability. Thus, we can find:

   $$\left| \int f_n d\mu - \int f d\mu \right| \leq \int |f_n - f| d\mu$$

   For each ϵ, there is an N large enough such that the difference $|f_N - f| < \epsilon$. So, for N large enough, for all $n \geq N$, we have via monotonicity of the integral:

$$\int |f_n - f| d\mu < \epsilon\mu(\Omega)$$

As ϵ can be arbitrarily small, we must have that:

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0 \implies \int f_n d\mu \to \int f d\mu$$

2. Use part 1 and Egoroff's theorem to give another proof of Theorem 16.5.

   So, we just assume that $\mu(\Omega) < \infty$, the f_n are uniformly bounded, and $f_n \to f$ almost everywhere. By Egoroff's Theorem, for $\epsilon > 0$, there is a $B \subseteq \Omega$ such that $\mu(B) < \epsilon$, for which $f_n \to f$ uniformly on $\Omega - B$ (union B with the measure 0 set on which $f_n \not\to f$). Thus, we have:

$$\int_{\Omega - B} f_n d\mu \to \int_{\Omega - B} f d\mu$$

   Now, examine:

$$\int_\Omega f_n d\mu = \int_{\Omega - B_n} f_n d\mu + \int_{B_n} f_n d\mu$$

   Where $\mu(B_n) < 1/n$. We have that:

$$\int_{B_n} f_n d\mu \leq 2K\mu(B_n)$$

   Where K is the uniform bound. This clearly goes to zero. So, we have:

$$\lim_n \int_\Omega f_n d\mu = \lim_n \int_{\Omega - B_n} f_n d\mu + \int_{B_n} f_n d\mu \leq \lim_n \int_{\Omega - B_n} f_n d\mu + \lim_n 2k\mu(B_n)$$

$$= \lim_n \int_{\Omega - B_n} f_n d\mu = \int_\Omega f d\mu$$

   And so we have the bounded convergence theorem. qed.

16.2 - Alternate Fatou's Lemma

Prove that if $0 \leq f_n \to f$ almost everywhere and $\int f_n d\mu \leq A < \infty$, then f is integrable and $\int f d\mu \leq A$.

Note, the book says this is essentially the same as Fatou's Lemma, which was just a consequence of monotone convergence theorem. We can use Fatou's

Lemma fo prove it. We have that $\lim_n f_n = f$, which implies $\liminf_n f_n = f$ (almost everywhere). And so:

$$\int f = \int \liminf_n f_n$$

As to functions that are equal almost everywhere have the same integral. By Fatou's lemma, as the f_n are nonnegative:

$$\leq \liminf \int f_n \leq \liminf A = A$$

16.3 Beppo Levi's Theorem

Suppose that f_n are integrable and $\sup_n \int f_n d\mu < \infty$. Show that, if $f_n \uparrow f$, then f is integrable and $\int f_n d\mu \to \int f d\mu$.

Note that we can't make automatic use of Monotone Convergence Theorem because the f_n are not nonnegative. Note that $0 \leq f_n - f_1 \uparrow f - f_1$. Note, non-negativity comes from the f_n getting larger. Thus, we have via Monotone convergence theorem:

$$\int f_n - f_1 d\mu \uparrow \int f - f_1 d\mu$$

If we find that f is integrable, then we should be able to make use of addition/subtraction to find the limit.

As $f_n \uparrow f$, we have $f_n \to f$ everywhere. As the limit exists, we have that:

$$\liminf_n f_n = \lim_n f_n = f$$

And so, we can make use of Fatou's lemma, as well as our supremum bound, to find:

$$\int f d\mu = \int \lim_n f_n d\mu = \int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu \leq \limsup_n \int f_n d\mu \leq \sup_n \int f_n d\mu < \infty$$

However, this doesn't prove that f is integrable. I think, we can instead show that $f - f_1$ is integrable. As $f - f_1 \geq 0$, we just need to show it has a finite integral, which would imply integrable, as $|f - f_1| = f - f_1$. We note:

$$\int f - f_1 d\mu = \lim_n \int f_n - f_1 d\mu = \lim_n \int f_n d\mu - \int f_1 d\mu \leq -\int f_1 d\mu + \sup_n \int f_n d\mu < \infty$$

Where we have the supremum is finite via our assumption, and $-\int f_1 d\mu < \infty$ by integrable, and we can split the subtraction in the limit by Theorem 16.1. As $f - f_1$ is integrable, Theorem 16.1 tells us that $f - f_1 + f_1 = f$ is integrable as well, and we can again make use of Theorem 16.1 to find:

$$\lim_n \int f_n d\mu = \lim_n \int f_n - f_1 + f_1 d\mu = \int f_1 d\mu + \lim_n \int f_n - f_1 d\mu = \int f_1 d\mu + \int f - f_1 d\mu$$

$$= \int f - f_1 + f_1 d\mu = \int f d\mu$$

So, we have proven Beppo Levi's Theorem. qed.

16.8 Integrable Functions Bounded Integral On Small Enough Set

Show that if f is integrable, then for each ϵ there is a δ such that $\mu(A) < \delta$ implies:

$$\int_A |f| d\mu < \epsilon$$

As f is integrable, we have that $\int |f| d\mu < \infty$. Take $\epsilon > 0$. Note, we can't necessarily assume that f is bounded (think of going to infinity slowly on the unit interval). I think, we have that this is true vacuously? If $\delta = 0$, we have:

$$\int_A |f| d\mu = 0 < \epsilon$$

I think, we need to assume that $\delta > 0$ as well. Let $f_n = |f| \mathbb{1}\{|f| \leq n\}$. As $f_n \uparrow |f|$, by monotone convergence, we have:

$$\lim_n \int f_n d\mu \uparrow \int |f| d\mu$$

As both sides are integrable (and finite), we have this implies:

$$\lim_n \int_{|f|>n} |f| d\mu = \lim_n \int |f| - f_n d\mu = 0$$

So, there is some n large enough such that:

$$\int_{|f|>n} |f| d\mu < \epsilon/2$$

Now, note that for any measurable A, we have:

$$\int_A |f| d\mu = \int_{A \cap \{|f| \leq n\}} |f| d\mu + \int_{A \cap \{|f| > n\}} |f| d\mu$$

Note, the RHS is bounded by:

$$\leq \int_A n d\mu + \int_{\{|f|>n\}} |f| d\mu \leq \mu(A)n + \epsilon/2$$

So, if $\delta = \epsilon/2n$ (note, n depends on ϵ, but that is fine) and $\mu(A) < \delta$, we have:

$$\int_A |f| d\mu \leq \epsilon/2 + \epsilon/2 = \epsilon$$

And so we have proved the theorem. qed.

16.9 Uniform Integrability Equivalent Condition

Suppose that $\mu(\Omega) < \infty$. Show that $\{f_n\}$ is uniformly integrable if and only if the following two conditions apply:

1. $\int |f_n| d\mu$ is bounded by some M for each n
2. For each ϵ there is a δ such that $\mu(A) < \delta$ implies $\int_A |f_n| d\mu < \epsilon$ for all n.

$\Rightarrow$ First, we assume that $\{f_n\}$ are uniformly integrable. Note, in the chapter, we found that $\mu(\Omega) < \infty$ together with uniform integrability implies that f_n are integrable. Hence, we do have:

$$\int |f_n| d\mu \leq \infty$$

Note, I think maybe it wants a bound for all n. Well, this also comes from the chapter - if α is large enough such that the supremum:

$$\sup_n \int_{|f_n| \geq \alpha} |f_n| d\mu < 1$$

Note, such an α exists, as uniform integrability implies that the limit is 0 - we have that for all n:

$$\int |f_n| d\mu = \int_{|f_n| \geq \alpha} |f_n| d\mu + \int_{|f_n| < \alpha} |f_n| d\mu \leq 1 + \int \alpha d\mu = 1 + \alpha\mu(\Omega)$$

Now, for the second point. By the previous problem, as each n is integrable, the for ϵ, there is a δ_n such that $\mu(A) < \delta_n$ implies:

$$\int_A |f_n| d\mu < \epsilon$$

However, we need an argument for each n. Well, we do have:

$$\int_A |f_n| d\mu = \int_{A\cap|f_n|\geq\alpha} |f_n| d\mu + \int_{A\cap|f_n|<\alpha} |f_n| d\mu \leq \int_{|f_n|\geq\alpha} |f_n| d\mu + \alpha\mu(A)$$

This is true for all n. So, choose our $\epsilon > 0$. Now, choose α large enough such that for all n:

$$\int_{|f_n|\geq\alpha} |f_n| d\mu \leq \frac{\epsilon}{2}$$

Now, choose $\delta = \frac{\epsilon}{2\alpha}$, and we have for $\mu(A) < \delta$:

$$\int_A |f_n| d\mu \leq \int_{|f_n|\geq\alpha} |f_n| d\mu + \alpha\mu(A) \leq \frac{\epsilon}{2} + \alpha\frac{\epsilon}{2\alpha} = \epsilon$$

And os we have proven the first direction.

$\Leftarrow$ We assume the two conditions hold, and we want to prove that $\{f_n\}$ are uniformly integrable. Note, for $\epsilon > 0$, there is a δ such that $\mu(A) < \delta$ implies:

$$\sup_n \int_A |f_n| d\mu < \epsilon$$

As if it is true for all n, it is true for the supremum by (2). Now, as each $\int |f_n| d\mu < M$, there is an α large enough such that:

$$\alpha^{-1} \int |f_n| d\mu < \delta$$

This is by (1). Then, we have:

$$\mu\left[|f_n| \geq \alpha\right] = \int \mathbb{1}\left\{|f_n| \geq \alpha\right\} d\mu = \alpha^{-1} \int \alpha \mathbb{1}\left\{|f_n| \geq \alpha\right\} d\mu$$

$$\leq \alpha^{-1} \int |f_n| \mathbb{1}\left\{|f_n| \geq \alpha\right\} d\mu \leq \alpha^{-1} \int |f_n| d\mu < \delta$$

Note, the first equality makes use of Theorem 15.1, and then the following are just standard from moving scalars through a nonnegative function. Thus, we have $\mu\left[|f_n| \geq \alpha\right] < \delta$, and we can make use of our first fact that for each n:

$$\int_{|f_n|\geq\alpha} |f_n| d\mu \leq \epsilon$$

This directly implies uniform integrability - namely, there is an α large enough such that:

$$\sup_n \int_{|f_n|\geq\alpha} |f_n| d\mu \leq \epsilon \implies \lim_{\alpha\to\infty} \sup_n \int_{|f_n|\geq\alpha} |f_n| d\mu = 0$$

Thus, we have proven that for $\mu(\Omega) < \infty$, the two conditions taken together are equivalent to uniform integrability. qed.

16.10 Uniform Integrability Equivalent Condition Properties

Assume $\mu(\Omega)$

1. Show by examples that neither of the conditions (1) and (2) in the preceding problem implies the other.

   Define $f_n : (0, 1] \to \mathbb{R}$ as:

   $$f_n = \mathbb{1}\{(0, 1/n)\} n$$

   We have that:

   $$\int f_n d\lambda = \lambda(0, 1/n)n = 1$$

   And so the f_n are all bounded. Now, I will show that f_n as defined do not satisfy property 2. Take $0 < \epsilon < 1$ and assume there is a δ such that $\mu(A) < \delta$ implies:

   $$\int_A |f_n| d\mu < \epsilon \text{ for all } n$$

   We will provide a contradiction to this argument. Indeed, take $1/n < \delta$, we have $\mu(0, 1/n) = 1/n < \delta$. Now, we note:

   $$\int_{(0,1/n)} f_n d\lambda = \int \mathbb{1}\{(0, 1/n)\}^2 n d\lambda = 1 > \epsilon$$

   So, we have our contradiction, and for the given example, (1) does not imply (2).

   Now, we go the other direction. Define $f_n : \mathbb{R} \to \mathbb{R}$:

   $$f_n = \mathbb{1}\{(0, n)\}$$

   Take $\epsilon > 0$. Let $\delta = \epsilon$, and assume that $\mu(A) < \delta = \epsilon$. We have:

   $$\int_A f_n d\lambda = \int_A \mathbb{1}\{(0, n)\} d\lambda \leq \int_A 1 d\lambda = \mu(A) = \epsilon$$

   So, the f_n as defined do satisfy (2). However, we find that:

   $$\int f_n d\lambda = n$$

And so the integrals of f_n are not bounded. Thus, for this example, (2) does not imply (1).

Note, in the above second example, I let $\mu(\Omega) = \infty$, but we need to assume $\mu(\Omega) < \infty$. Considering the second problem, we also need μ to be nonatomic. If we did the counting measure on $\{1, 2, \cdots\}$, we would still have $\mu(\Omega) = \infty$. I guess we could give weights to each entry - say 2^{-n}, and so $\mu(\Omega) = 1$... but I'm not sure we could have a non bounded f_n? maybe, we can define:

$$f_n(\omega) = \mathbb{1}\{\omega = n\} n2^n$$

So then, we should have:

$$\int f_n d\mu = \int \mathbb{1}\{\omega = n\} n2^n d\mu = \mu(\omega = n)n2^n = n$$

Which is not bounded. I think this is approaching what we need, but not quite right.

Instead, define $f_1(\omega) = \mathbb{1}\{\omega = 1\} \cdot \infty$, and $f_n(\omega) = \mathbb{1}\{\omega = n\}$ otherwise. The f_n are not bounded via f_1 - but if you have $\epsilon > 0$, set $\delta = \min(1/2, \epsilon)$, which should exclude the point mass at 1, and otherwise $\int f_n d\mu = 2^{-n} < \epsilon$.

2. Show that (ii) implies (i) for all sequences $\{f_n\}$ if and only if μ is nonatomic.

   First - I need to think - does this not contradict my example above? I defined f_n for the Lebesgue measure on $\mathbb{R}$, which should be nonatomic. Oh - I let $\mu(\Omega) = \infty$, which is incorrect.

   For the rest, just look at the solutions.

16.12 Daniell-Stone Representation Theorem

Note - I am just going to follow with the solutions for this problem. That is because I don't really want to do it, but I do think it is a pretty important theorem to understand, at least.

Review Consider the vector lattice $\mathcal{L}$ and the functional Λ of problems 11.4 and 11.5. $\mathcal{L}$ is thus a vector space of finite real valued functions f :

$\Omega \to \mathbb{R}$, with $f(\omega) < \infty$, and the functions are closed under addition, scalar multiplication. Also, as $\mathcal{L}$ is a lattice, it is closed under join and meet, essentially maximum and minimum:

$$f \vee g : \Omega \to \mathbb{R} \qquad (f \vee g)(\omega) = \max(f(\omega), g(\omega)) \qquad f \vee g \in \mathcal{L}$$

$$f \wedge g : \Omega \to \mathbb{R} \qquad (f \wedge g)(\omega) = \min(f(\omega), g(\omega)) \qquad f \wedge g \in \mathcal{L}$$

And Λ is a real linear functional, which is a map from a vector space V to its field, which is also linear. So, $\Lambda : \mathcal{L} \to \mathbb{R}$, and:

$$\Lambda(af + bg) = a\Lambda(f) + b\Lambda(g)$$

We further assumed of Λ that it is positive in the sense that $f \geq 0$ (pointwise) implies $\Lambda(f) \geq 0$ and continuous from above at 0 in the sense that $f_n \downarrow 0$ (pointwise) implies $\Lambda(f_n) \to 0$.

For $\mathcal{L}$ and Λ as defined, we were able to show a couple of properties in problems 11.4 and 11.5. I will list those properties here:

1. For $f \leq g$ (with $f, g \in \mathcal{L}$), we defined an "interval" in $\Omega \times \mathbb{R}^1$ as:

   $$(f, g] := \{(\omega, t) : f(\omega) < t \leq g(\omega)\}$$

   We found that these sets formed a semi ring $\mathcal{A}_0$, with:

   $$(f_1, g_1] \cap (f_2, g_2] = (f_1 \vee f_2, g_1 \wedge g_2]$$

   $$(f_2, g_2] - (f_1, g_1] = (f_2, g_2 \wedge f_1] \cup (f_2 \vee g_1, g_2]$$

2. For a set function ν_0 on $\mathcal{A}_0$ defined as:

   $$\nu_0(f, g] = \Lambda(g - f)$$

   We found that ν_0 was finitely additive, and countably subadditive. Via Theorem 11.3, this enabled us to extend ν_0 to a measure on $\sigma(\mathcal{A}_0)$.

3. We were able to conclude a property - that for $f \in \mathcal{L}$, with $f_n = (n(f - f \wedge 1)) \wedge 1$, for $x > 0$, we had:

   $$(0, xf_n] \uparrow [\omega : f(\omega) > 1] \times (0, x]$$

   Note, this means the unions of the sets in $\mathcal{A}_0$ go up to the set on the RHS (note, the set on the RHS might not necessarily be in $\mathcal{A}_0$).

4. We defined $\mathcal{F}$ as the smallest σ field with respect to which every f in $\mathcal{L}$ is measurable:

$$\mathcal{F} = \sigma\left[f^{-1}H : f \in \mathcal{L}, H \in \mathcal{R}^1\right]$$

We defined $\mathcal{F}_0$ as the class of A in $\mathcal{F}$ for which $A \times (0,1] \in \sigma(\mathcal{A}_0)$. We were able to show that $\mathcal{F}_0$ is a semiring, and $\mathcal{F} = \sigma(\mathcal{F}_0)$. This was proved using the above property.

5. Finally - we let ν be the extension of ν_0 to $\sigma(\mathcal{A}_0)$, and for $A \in \mathcal{F}_0$, we defined:

$\mu_0(A) = \nu(A \times (0,1])$ well defined, as $A \in \mathcal{F}_0 \implies A \times (0,1] \in \sigma(\mathcal{A}_0)$

We showed the μ_0 is finitely additive and countably subadditive on the semiring $\mathcal{F}_0$ - and thus, via Theorem 11.3, has a measure extension μ on $\mathcal{F}$.

So, these are a lot of properties. But, what it gives us, at the end of the day, is a measure μ on $\mathcal{F}$, the smallest σ field with respect to which every $f \in \mathcal{L}$ is measurable.

Problems

1. Show by (11.7) (our property 3 above) that for positive x, y_1, y_2 one has:
$$\nu([f > 1] \times (0, x]) = x\mu_0[f > 1] = x\mu[f > 1]$$
And:
$$\nu([y_1 < f \le y_2] \times (0, x]) = x\mu[y_1 < f \le y_2]$$
Well - first, I will examine $[f > 1]$. Clearly, as $(1, \infty) \in \mathcal{R}$, we have that $[f > 1] \in \mathcal{F}$. We want to show that $[f > 1] \times (0, x] \in \sigma(\mathcal{A}_0)$, and so ν will be well defined on that set. Recall:

$$f_n = (n(f - f \wedge 1)) \wedge 1$$

We have that if $f(\omega) > 1$ - then, for some n, we have that $(n(f - f \wedge 1)) > 1$, and so $f_n(\omega) = 1$ for all sufficiently large n. If $f(\omega) \le 1$, $f - f \wedge 1 = 0$, and so $f_n(\omega) = 0$. In all cases, $f_n \ge 0$, and so $(0, xf_n]$ is well defined for $x > 0$. Recall, we also proved that:

$$(0, xf_n] \uparrow [f > 1] \times (0, x]$$

And so, at the very least, $\nu([f > 1] \times (0, x])$ is well defined, given that the set is a union of $\mathcal{A}_0$ sets. We have that via continuity from below:

$$\nu([f > 1] \times (0, x]) = \lim_{n\to\infty} \nu(0, xf_n] = \lim_{n\to\infty} \nu_0(0, xf_n]$$

Anyway. We have that $[f > 1] \times (0, x] \in \sigma(\mathcal{A}_0)$ for all $x > 0$. This implies that $[f > 1] \times (0, 1] \in \sigma(\mathcal{A}_0)$, which implies that the following is well defined:

$$\mu_0[f > 1] = \nu((f > 1] \times (0, 1])$$

Anyway. Going back, we have:

$$\nu([f > 1] \times (0, x]) = \lim_{n\to\infty} \nu_0(0, xf_n] = \lim_{n\to\infty} \Lambda(xf_n) = \lim_{n\to\infty} x\Lambda(f_n)$$

As Λ as a linear functional. Thus, we have:

$$\nu([f > 1] \times (0, x]) = \lim_{n\to\infty} x\nu(0, f_n] = x\nu([f > 1] \times (0, x]) = x\mu_0[f > 1]$$

And we have concluded the first expression. The second one we want to prove is:

$$\nu([y_1 < f \leq y_2] \times (0, x]) = x\mu[y_1 < f \leq y_2]$$

This should just be a continuation of the first. We note that for positive y:

$$\nu([f > y] \times (0, x]) = \nu([f/y > 1] \times (0, x]) = x\mu[f/y > 1]$$

Also:

$$[y_1 < f \leq y_2] \times (0, x] = [f > y_1] \times (0, x] - [f > y_2] \times (0, x]$$

As the first set contains the second, we can make use of measure subtraction to find:

$$\nu([y_1 < f \leq y_2] \times (0, x]) = \nu([f/y_1 > 1] \times (0, x]) - \nu([f/y_1 > 1] \times (0, x])$$

$$= x\left[\mu[f/y_1 > 1] - \mu[f/y_2 > 1]\right] = xx\mu[y_1 < f \leq y_2]$$

Note, we have to use μ instead of μ_0, as we don't necessarily have the subtraction property for non measure μ_0. Also, we need positive y_1, y_2 to make the f/y inequalities valid.

2. Show that if $f \in \mathcal{L}$, then f is integrable and:

$$\Lambda(f) = \int f d\mu$$

Here, I just follow the solution in the book, I don't really want to spend time trying to prove this theorem.

First, note we just need to prove that if $f \geq 0$, we have:

$$\Lambda(f) = \int f d\mu$$

Then, we note that $f^+ = f \vee 0$, and $f- = -f \vee 0$. In both cases, we have:

$$\Lambda(f) = \Lambda(f^+ - f^-) = \Lambda(f^+) - \Lambda(f^-) = \int f^+ d\mu - \int f^- d\mu = \int f$$

Where subtraction is well defined, because $\Lambda(f)$ is *real*, and so doesn't return infinity as a value.

Take $f \in \mathcal{L}$ with $f \geq 0$. If $f_n = (1 - n^{-1})f \vee 0$, then $f_n \in \mathcal{L}$ and $f_n \uparrow f$ (note, I don't think we need the $\vee 0$ as f is nonnegative, but it is fine to keep). $f_n \uparrow f$ is clear, as $n \to \infty$, $(1 - n^{-1}) \to 1$. Thus, we note that:

$$\nu(f_n, f] = \Lambda(f - f_n) \downarrow 0$$

Via continuity from above of Λ. Note, as Λ is finite, we have that continuity from above can be applied. We note that:

$$\bigcap (f_n, f] = \{(\omega, t) : f(\omega) = t\}$$

As consider $(\omega, t) \in LHS$. We have that $f_n(\omega) < t \leq f(\omega)$ for all n, and as $f_n(\omega) \uparrow f(\omega)$, we must have that $f(\omega) = t$, and so $(\omega, t) \in RHS$. We can similarly go the other way. And so, via continuity from above, we have that the following is well defined:

$$\nu\left(\{(\omega, t) : f(\omega) = t\}\right) = 0$$

Look at the disjoint union:

$$B_n = \bigcup_{i=1}^{n2^n} \left(\left[\frac{i}{2^n} < f \leq \frac{i+1}{2^n} \right] \times \left(0, \frac{i}{2^n}\right] \right)$$

Some things to note - as n increases, the total range bounding f increases, and gets finer and finer. Further, the range on the RHS gets larger and larger. Note that:

$$B_n \uparrow B \subseteq (0, f]$$

First, note that $B_n \subseteq B_{n+1}$. Indeed, if $(\omega, t) \in B_n$, we have that for some i:

$$\frac{i}{2^n} < f(\omega) \leq \frac{i+1}{2^n} \text{ and } 0 < t \leq \frac{i}{2^n}$$

$(\omega, t) \in B_{n+1}$, as we have that for some $j = 2i, 2i+1$:

$$\frac{j}{2^{n+1}} < f(\omega) \leq \frac{j+1}{2^{n+1}}$$

And for both of these j, we have that $0 < t \leq \frac{j}{2^{n+1}}$. So, B is a well defined union. Note, $B_n \in \sigma(\mathcal{A}_0)$, as we found a ν measure value for each set in the disjoint union. Disjoint, as the ω are disjoint for each set in the union. Now, we need to show that:

$$B \subseteq (0, f]$$

Which follows if we have $B_n \subseteq (0, f]$ for each n. Recall, we have f is nonnegative, and so:

$$(0, f] := \{(\omega, t) : 0 < t \leq f(\omega)\}$$

Note that as $(\omega, t) \in B_n$, we have that $\frac{i}{2^n} < f(\omega)$ and $0 < t \leq \frac{i}{2^n}$, so we indeed have that $B_n \subseteq (0, f]$ for each n, and $B \subseteq (0, f]$.

Now, we note that:

$$(0, f] - B \subseteq [(\omega, t) : f(\omega) = t]$$

Indeed, if we have $(\omega, t) \in (0, f]$, but $(\omega, t) \notin B_n$ for each n, that means that $0 < t \leq f(\omega)$ is satisfied, but $0 < t \leq \frac{i}{2^n} < f(\omega)$ is not satisfied for each possible i and n. Note, if $t < f(\omega)$, there would be an i and n in our range above that would satisfy $t \leq \frac{i}{2^n} < f(\omega)$, so we must conclude that $t < f(\omega)$ is not true if $(t, \omega) \notin B$. Thus, we must indeed have that $t = f(\omega)$, which gives us our above fact. And so, pulling our facts together, we must have:

$$\Lambda(f) = \Lambda(f - 0) = \nu_0(0, f] = \nu(0, f] = \nu(B)$$

As for a measure, given that $B \subseteq (0, f]$, we have:

$$\nu(0, f] - \nu(B) = \nu((0, f] - B) \leq \nu\left(\{(\omega, t) : f(\omega) = t\}\right) = 0$$

Using our continuity from below, we have:

$$\Lambda(f) = \nu(B) = \lim_n \nu(B_n) = \lim_n \sum_{i=1}^{n2^n} \frac{i}{2^n} \mu\left[\frac{i}{2^n} < f \leq \frac{i+1}{2^n}\right]$$

Where we used our property from part 1 for the last equality. Finally, we note that this final limit equals:

$$= \int f d\mu$$

Recall, we only defined:

$$\int f d\mu = \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

Where A_i is a finite decomposition of Ω. Note, our sums in the limit are of the infimum sum form (as for the final A_i not expressed by the intervals decomposing the range of f, the infimum is zero and doesn't contribute to the sum). As for why the limit is equal to the supremum - ultimately, we have that each term is increasing, given that we are taking the lower bound of finite intervals. However, instead, think of it this way. Each term in the limit is the integral for some simple function f_n, as seen in Theorem 13.5. We have that $f_n \uparrow f$, and so we can actually make use of Monotone Convergence Theorem:

$$\Lambda(f) = \lim_n \sum_{i=1}^{n2^n} \frac{i}{2^n} \mu\left[\frac{i}{2^n} < f \leq \frac{i+1}{2^n}\right] = \lim_n \int f_n d\mu = \int f d\mu$$

And so, for nonnegative $f \in \mathcal{L}$, we have found that:

$$\Lambda(f) = \int f d\mu$$

For μ found in problems 11.4 + 11.5. This implies the same for all $f \in \mathcal{L}$ as noted initially. qed.

Section 17 - The Integral With Respect to Lebesgue Measure

Notes

The Lebesgue Integral on the Line

A real measurable function on the line is Lebesgue integrable if it is integrable with respect to the Lebesgue measure λ, and its *Lebesgue integral* $\int f d\lambda$ is denoted by $\int f(x)dx$, or in the case of integrating over an interval, by $\int_a^b f(x)dx$.

The Riemann Integral

A real function f on an interval $(a, b]$ is by definition (and there are many definitions) *Riemann integrable*, with integral r, if this condition holds: for each ϵ there exists a δ with the property that:

$$\left| r - \sum_i f(x_i)\lambda(I_i) \right| < \epsilon$$

If $\{I_i\}$ is any finite partition or $(a, b]$ into subintervals satisfying $\lambda(I_i) < \delta$ and $x_i \in I_i$ for each i.

The Riemann Integral, when it exists, coincides with the Lebesgue Integral Suppose that f is Borel measurable, and suppose that f is bounded, so that it is Lebesgue integrable (on an interval). If f is also Riemann integrable, the r above must coincide with the Lebesgue integral:

$$\int_a^b f(x)dx$$

First note that letting x_i vary over I_i leads to r also satisfying:

$$\left| r - \sum_i \sup_{x \in I_i} f(x) \cdot \lambda(I_i) \right| \leq \epsilon$$

Indeed, we could take limits for each of the finite I_i, to the supremum on I_i, and only lose the strict inequality. Now, consider a simple function g with value $\sup_{x \in I_i} f(x)$ on I_i. We have that $f \leq g$, but also:

$$\int_a^b g(x)dx = \sum_i \sup_{x \in I_i} f(x) \cdot \lambda(I_i)$$

Via Theorem 15.1. By monotonicity of the integral, this must imply that:

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx \leq r + \epsilon$$

The reverse inequality also follows in the same way, if we take the $\inf_{x \in I_i} f(x) \cdot \lambda(I_i)$, and so we find for all $\epsilon > 0$:

$$r + \epsilon \leq \int_a^b f(x)dx \leq r + \epsilon \implies \int_a^b f(x)dx = r$$

A continuous function on a closed interval is Riemann Integrable Suppose that f is continuous on $[a, b]$. A continuous function is uniformly continuous on the closed interval, and so for each ϵ there exists a δ such that $|f(x) - f(y)| < \epsilon/(b - a)$ if $|x - y| < \delta$. If $\lambda(I_i) < \delta$ and $x_i \in I_i$, then:

$$g = \sum_i f(x_i) \mathbb{1}\{I_i\} \implies |f - g| < \epsilon/(b - a)$$

Thus, for the *Lebesgue integral*, we have:

$$\left| \int_a^b f dx - \int_a^b g dx \right| = \left| \int_a^b f - g dx \right| \leq \left| \int_a^b \epsilon/(b - a) dx \right| \leq \epsilon$$

Note, both f and g are integrable, so everything above is well defined. But note, the LHS also simplifies to:

$$\left| \int_a^b f dx - \sum_i \sup_{x \in I_i} f(x) \cdot \lambda(I_i) \right| \leq \epsilon$$

Which is just the definition of Riemann integrable - where $r = \int_a^b f dx$ (and they must be equal, by our previous statement). So, a continuous function f has an r that satisfies the Riemann integrable definition on a closed interval. qed.

Example 17.1 - A Lebesgue But Not Riemann Integrable f Let f be the indicator of the set of rationals in $(0, 1]$. Then, the Lebesgue integral is zero, because $f = 0$ almost everywhere. However, f is not Riemann integrable, as every I_i contains an $x_i = 0$ and $x_i = 1$, so no r can satisfy the definition for $\epsilon < 1/2$.

Example 17.2 - Lebesgue Theory Is Not Reducible To Riemann Theory via Restricting Measure Zero Sets For f in example 17.1, there is a g such that $f = g$ almost everywhere and g is Riemann integrable ($g = 0$). We want an example to show that Lebesgue Theory is not reducible to Riemann Theory by casting out sets of measure 0, and so we want to produce an f (bounded and measurable) such that no g exists.

In Examples 3.1 and 3.2 in the book, we constructed a borel set $A \subseteq (0, 1]$ such that $0 < \lambda(A) < 1$ and $\lambda(A \cap I) > 0$ for each subinterval I of $(0, 1]$. Take $f = \lambda_A$. Suppose that $f = g$ almost everywhere and that $\{I_i\}$ is a decomposition of $(0, 1]$ into subintervals. Since:

$$\lambda(I_i \cap A \cap [f = g]) = \lambda(I_i \cap A) > 0$$

It follows that $g(y_i) = f(y_i) = 1$ for some $y_i \in I \cap A$, and therefore:

$$\sum_i g(y_i)\lambda(I_i) = 1 > \lambda(A)$$

If g were Riemann integrable, its Riemann integral would coincide with the Lebesgue integrals $\int g dx = \int f dx = \lambda(A)$, in contradiction to the other Riemann integral value 1 we found above. So, g cannot be Riemann integrable, and so for f as defined, there is no $f = g$ almost everywhere such that g is Riemann integrable.

Note 17.1 + 17.2 The reason the functions in example 17.1 and 17.2 fail to be Riemann integrable, is because of their extreme oscillations (we proved in our Calculus On Manifolds notes that f is integrable iff the set of its discontinuities has Lebesgue measure 0). This cannot happen in the case of the Lebesgue integral of a measurable function - recall, we have already identified the two reasons why such an f would fail to be Lebesgue integrable - because its positive or negative part is too large, not because one or the other is too irregular.

Theorem 17.1 - Approximating Lebesgue Integrable Functions With Riemann Integrable Functions Suppose that $\int |f| dx < \infty$ and $\epsilon > 0$.

1. There is a step function $g = \sum_{i=1}^{k} x_i \mathbb{1}\{A_i\}$ with bounded intervals at the A_i, such that:
$$\int |f - g| dx < \epsilon$$

2. There is a continuous integrable h with bounded support such that $\int |f - h| dx < \epsilon$.

Proof: 1. By the construction in Theorem 13.5 (sequence of simple functions approaching f real and measurable) and the dominated convergence theorem (as f is integrable, and dominates the simple functions), we have that there is a simple function g such that $\int |f - g| dx < \epsilon$. Note here, we *don't* assume that the A_i are intervals. However, we do have that $\lambda(A_i) < \infty$ for each i for which $x_i \neq 0$ (otherwise this would imply an infinity integral via $x_i \mathbb{1}\{A_i\}$).

By Theorem 11.4, and because the bounded intervals $\mathcal{A}$ are a semiring that generate $\mathcal{B}$ (recall, we are dealing with f defined on the line here), there is a finite disjoint union B_{ij} of intervals such that:

$$\lambda(A_i \triangle (\cup_{j=1}^n B_{ij})) < \frac{\epsilon}{k|x_i|}$$

Note, the union $B_i = \cup_{j=1}^n B_{ij}$ is still a series of bounded intervals. Define:

$$g' = \sum_{i=1}^{k} x_i \mathbb{1}\{B_i\}$$

Note, if any of the B_i intersect, just take the addition of the x_i values. In total, g' is still a finite sum step function on bounded interval steps. We have:

$$\int |f - g'| dx \leq \int |f - g| dx + \int |g - g'| dx \leq \epsilon + \sum_{i=1}^{k} x_i \lambda(A_i \triangle (\cup_{j=1}^n B_{ij})) \leq 2\epsilon$$

Where the second sum comes from noting, that g and g' only differ by x_i on the portions where A_i and B_i don't meet up exactly (added up w/ all the differences for the other x_i). Thus, we have found a g' that satisfies the first part. Just expanding on the sum - say that ω is in A_i and in B_i but also in B_j. Then, we have:

$$g(\omega) = x_i \qquad g'(\omega) = x_i + x_j \implies |g(\omega) - g'(\omega)| = x_j$$

This difference holds for all the $\omega \in A_j \triangle B_j$, and this difference is accounted for in the integral by taking the sum $x_j \lambda(A_j \triangle B_j)$. Going further, I think we just have:

$$|g - g'|(\omega) = \sum_{i=1}^{k} x_i \mathbb{1}\{A_i \triangle B_i\}$$

Going further, in general, the difference between indicators is:

$$|f-g| = \sum_{i=1}^{k}\sum_{j=1}^{k} |a_i - b_j| \mathbb{1}\{A_i \cap B_j\}$$

Where each is disjoint. But, as $a_i = b_i$ in our case, when $i = j$, the difference is zero. If we consider:

$$\sum_{j=1}^{k} |x_i - x_j| \mathbb{1}\{A_i \cap B_j\} \leq \sum_{j=1}^{k} (x_i + x_j) \mathbb{1}\{A_i \cap B_j\}$$

We know that this is included within the $x_i \mathbb{1}\{A_i \triangle B_i\}$ and $x_j \mathbb{1}\{A_j \triangle B_j\}$ terms, given that the A_i are disjoint.

2. Recall, bounded support means the set of points where the function is not zero is bounded. Note, we need this condition, as we only proved continuous functions were Riemann integrable on a closed interval. Take the $\epsilon > 0$, and using part 1, take the g step function on bounded intervals A_i where:

$$\int |f-g| dx < \epsilon$$

We will show there is such a continuous integrable h with bounded support such that:

$$\int |h-g| dx < \epsilon$$

This will imply:

$$\int |f-h| dx \leq \int |f-g| dx + \int |h-g| dx \leq 2\epsilon$$

Which is enough. Suppose that $A_i = (a_i, b_i]$. Let $h_i(x)$ be 1 on $(a_i, b_i]$ and 0 outside of $(a_i - \delta, b_i + \delta]$, and let it increase linearly from 0 to 1 over $(a_i - \delta, a_i]$ and decrease linearly from 1 to 0 over $(b_i, b_i + \delta]$. As clearly:

$$\int |\mathbb{1}\{A_i\} - h| dx \to 0$$

As $\delta \to 0$, $h = \sum_i x_i h_i$ will satisfy for sufficiently small δ. This is because:

$$\int |h-g| dx \leq \sum_i x_i \int |\mathbb{1}\{A_i\} - h| dx \to 0$$

Thus, we have proven the Theorem. As a small note, h clearly has bounded support, just taking the maximum b_i and minimum a_i, and add δ. qed.

Thus, the Lebesgue integral is determined by its values for continuous functions with bounded support. And each of these is Riemann integrable.

The Fundamental Theorem of Calculus

Note, for all of these related theorems, I have already proven them for the Riemann integral. So, I guess I could pass to the Riemann integral whenever it exists, and get the Theorem for the Lebesgue Integral. However, I think these theorems might be more general, ie, we will be able to apply them for the cases in which the Lebesgue integral exists but the Riemann integral doesn't.

Adopt the convention that $\int_\alpha^\beta = -\int_\beta^\alpha$ if $\alpha > \beta$. For positive h:

$$\left| \frac{1}{h} \int_x^{x+h} f(y)dy - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(y) - f(x)| dy \leq \sup \left[|f(y) - f(x)| : x \leq y \leq x + h \right]$$

As the $1/h$ and h factor cancel out in the supremum. The RHS goes to 0 with h if f is continuous at x. The same thing holds for negative h, and therefore $\int_a^x f(y)dy$ has derivative $f(x)$:

$$\frac{d}{dx} \int_a^x f(y)dy = f(x)$$

If f is continuous at x. Note, the function varies with x, ie $g(x) = \int_a^x f(y)dy$. The bottom limit a is arbitrary - and this is because the derivative (in the one dimensional case) is the k that satisfies

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x) - k}{h} = \left| \frac{1}{h} \int_x^{x+h} f(y)dy - k \right| = 0$$

And we found $k = f(x)$ is the satisfying point. Also note, this is all done with the *Lebesgue Integral* with respect to the *Lebesgue measure* λ. The integral calculation made use of the absolute value bound. However, note that all the subtractions require that f be integrable, or at the very least $f \mathbb{1} \{[a, x+h]\}$ is integrable.

Suppose that F is a function with continuous derivative $F' = f$. That is, F is the primitive (or anti-derivative) of f. Then:

$$\int_a^b f(x)dx = \int_a^b F'(x) = F(b) - F(a)$$

As we have $F(x) - F(a) = \int_a^x f(y)dy$ if we set $x = a$, and both functions have identical derivatives, namely $f(x)$ by the above fact. And so, two functions $\mathbb{R} \to \mathbb{R}$ that agree at one point and have the same derivative everywhere are equal.

Change Of Variable

For:

$$[a,b] \xrightarrow{T} [u,v] \xrightarrow{f} \mathbb{R}^1$$

The change of variable formula is:

$$\int_a^b f(Tx)T'(x)dx = \int_{T(a)}^{T(b)} f(y)dy$$

If T' exists and is continuous, and if f is continuous, the two integrals are finite because the integrands are bounded (continuous functions on intervals are bounded). To prove the above, it is enough to let b be a variable and differentiate with respect to it. This is a homework problem, but using the FTC, we have:

$$\frac{d}{db}\int_a^b f(Tx)T'(x)dx = f(Tb)T'(b)$$

$$\frac{d}{db}\int_{T(a)}^{T(b)} f(y)dy = ...$$

Note, I think this will just require the chain rule. Say $F' = f$. Then $(F \circ T)' = (f \circ T)T'$. We have:

$$\int_{T(a)}^{T(b)} f(y)dy = F(T(b)) - F(T(a))$$

Whereas we also have:

$$\int_a^b f(Tx)T'(x)dx = \int_a^b (F \circ T)'dx = (F \circ T)(b) - (F \circ T)(a) = F(T(b)) - F(T(a))$$

So we do have that both sides are equal. Also go to our calculus on manifold book to find the change-of-variable formula in one variable for the Riemann integral. Anyway, if we want to ever spend more time on this, I would say just directly use the derivative definition, with the derivative set at $f \circ T$, and I would guess the limits go to 0 on both sides, if both sides are treated as a function of b. They also have equal values at $b = a$.

Example 17.5 Put $T(x) = \tan(x)$ on $(-\pi/2, \pi/2)$. Note that $T'(x) = 1 + T^2(x)$. Via the change of variable formula, we have:

$$\int_{-\infty}^{\infty} \frac{1}{1+y^2}dy = \int_{\tan(-\pi/2)}^{\tan(\pi/2)} \frac{1}{1+y^2}dy = \int_{-\pi/2}^{\pi/2} \frac{1}{1+T(x)^2}(1+T(x)^2)dx \int_{-\pi/2}^{\pi/2} 1dx = \pi$$

The theorem applies to unbounded intervals, as we can use limits to infinity. Both sides of the limit are the same. For the above, note that we can take $\int_{-t}^{t}$ as $t \to \infty$, but for each individual t, use the change of variable formula, and then note the RHS approaches π.

The Lebesgue Integral in $\mathbb{R}^k$

Actually, just a note, for this section, the probability book references Spivak. That is actually the book where I learned the Change of Variables Theorem as well, so that just made me happy.

The k dimensional Lebesgue integral, the integral in $(\mathbb{R}^k, \mathcal{R}^k, \lambda)$, is denoted $\int f(x)dx$, it being understood that $x = (x_1, \cdots, x_k)$. In can also be denoted $\int \int_A f(x_1, x_2)dx_1 dx_2$ and so on.

Suppose that $T : U \to \mathbb{R}^k$ where U is an open set in $\mathbb{R}^k$. The map has the form $Tx = (t_1(x), \cdots, t_k(x))$, which is continuously differentiable if the partial derivatives exist and are continuous in U. This is the definition of continuously differentiable. Let $D_x = [t_{ij}(x)]$ be the Jacobian matrix, and let $J(x) = \det D_x$ be the Jacobian determinant, and let $V = TU$.

Theorem 17.2 - Change of Variables Let T be a continuously differentiable map of the open set U onto V. Suppose that T is one-to-one and that $J(x) \neq 0$ for all x. If f is nonnegative then:

$$\int_U f(Tx)|J(x)|dx = \int_V f(y)dy$$

This matches the Riemann definition.

Some things to note immediately. V is open - it is equal to $T(U)$, where T is the inverse image of T^{-1}. T^{-1} exists and is continuously differentiable locally for every $x \in U$ via the Inverse Function Theorem - see Calculus on Manifolds for the proof, note it doesn't have anything to do with measurability, just the fact that T is continuously differentiable and has nonzero Jacobian. Note, we also need the assumption that T is one-to-one (as otherwise, we could have T is invertible on disjoint neighborhoods of $x, y \in U$, but $T(x) = T(y)$ or $T(v) = T(u)$ for v and y in the corresponding neighborhoods).

It is assumed that $f : V \to \mathbb{R}^1$ is a Borel Function, as otherwise the integral wouldn't exist definitionally. For the general f, the integral holds with $|f|$ in place of f, and if the two sides are finite, the absolute-value bars can be removed, and of course f can be replaces with $f\mathbb{1}\{B\}$ or $f\mathbb{1}\{A\}$.

We go through some examples before proving.

Example 17.6 Suppose that T is a nonsingular linear transformation on $U = V = \mathbb{R}^k$. Then D_x is for each x the matrix of the transformation. If T is identified with this matrix, then the equation becomes:

$$|\det T| \int_U f(Tx)dx = \int_V f(y)dy$$

If $f = \mathbb{1}\{TA\}$, then this holds because of our earlier Theorem 12.2 for the Lebesgue measure:

$$\lambda_k(TA) = |\det T|\lambda_k(A)$$

I actually went though this in Chapter 12 as well. Then, we can use the standard combine, and prove for simple f and for general nonnegative f. Theorem 17.2 is easy in the linear case.

Example 17.7 In $\mathbb{R}^2$ take $U = [(\rho, \theta) : \rho > 0, 0 < \theta < 2\pi]$ and $T(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$. The Jacobian determinant is thus:

$$J(\rho, \theta) = \rho$$

And so the Theorem gives us the formula for integrating polar coordinates:

$$\int\int_{\rho>0, 0<\theta<2\pi} f(\rho\cos(\theta), \rho\sin(\theta))d\rho d\theta = \int\int_{\mathbb{R}^2} f(x,y)dxdy$$

Note, excluding $(0,0)$ and the line of $\theta = 0$ still gives us equality almost everywhere via Theorem 15.2. We have just $f = f\mathbb{1}\{x \geq 0, y = 0\}$ almost everywhere - they are not equal on the set:

$$A = \{\omega \in \mathbb{R}^2 : \omega_1 \geq 0, \omega_2 = 0\}$$

This is a measurable set in $\mathcal{R}^2$, as it equals a countable intersection of smaller and smaller height rectangles. It is measure 0, as we can find it has measure less than ϵ for all $\epsilon > 0$, by covering it in rectangles of the form $[n, n+1] \times (-\epsilon/2^{n+1}, \epsilon/2^{n+1})$.

Note, this example also shows why we need one-to-one. If we replaced the constraint on θ to $0 < \theta < 4\pi$, for example, the equation is false, as we would need a factor of 2 on the right.

Proof of Theorem 17.2, Change of Variables I'm just going to give the highlights. First, assume that we prove:

$$\int_U f(Tx)|J(x)|dx \geq \int_V f(y)dy$$

Well, note that this symmetric, and we could let $f = f(T(x))|J(x)|$ and $T = T^{-1}$ (and we know the determinant of T^{-1}, and can apply the Theorem to it as the Inverse Function Theorem gives us continuous differentiability of T^{-1}), to show the inequality in the other direction. So, proving the above equation is enough.

Then, consider f of the form $f = \mathbb{1}\{TA\}$. Then, the equation reduces to:

$$\int_A |J(x)|dx \geq \lambda_k(TA)$$

Each side is a measure on $\mathcal{U} = U \cap \mathcal{R}^k$. Note, $\mathcal{U}$ is a σ algebra defined by Theorem 10.1, and $A \subseteq U$. $\lambda_k(TA)$ is clearly a measure, as it is easy enough to show emptyset, countable additivity, and we already proved it in Theorem 12.2. We have that $\int_A |J(x)|dx$ is a measure via our densities discussion in chapter 16, given that $|J(x)|$ is nonnegative, and continuous (via continuous differentiability) and thus measurable.

If $\mathcal{A}$ consists ot the rectangles A satisfying $A^- \subseteq U$ (ie, interior is within U) then $\mathcal{A}$ is a semiring generating $\mathcal{U}$. Semiring property is clear, and generating $\mathcal{U}$ again comes from Theorem 10.1. U is also a countable union of $\mathcal{A}$ sets (common fact), and the integral left side is finite for A in $\mathcal{A}$ (as the supremum of the determinant J on a rectangle is bounded). It follows by Corollary 2 to Theorem 11.4 that if:

$$\int_A |J(x)|dx \geq \lambda_k(TA)$$

Holds for A in $\mathcal{A}$, then it holds for A in $\mathcal{U}$. But then, we can use our standard complex, and find that we can extend to f simple, and f nonnegative, and f general to find:

$$\int_U f(Tx)|J(x)|dx \geq \int_V f(y)dy$$

So, to prove the Change of Variables Theorem, we just have to prove for $A \in \mathcal{A}$ (ie for rectangles in U):

$$\int_A |J(x)|dx \geq \lambda_k(TA)$$

So we prove for such A. Split the rectangle A into finitely many subrectangles Q_i satisfying:

$$\mathrm{diam}(Q_i) < \delta$$

Ie, the farthest distance between two points is less than δ. It is obvious that such a partition is always possible, for any $\delta > 0$. We will determine δ later. Let x_i be some point in Q_i. Given ϵ, choose δ initially so that:

$$|J(x) - J(x')| < \epsilon \text{ if } x, x' \in A^- \text{ and } |x - x'| < \delta$$

Note, $J(x)$ continuously differentiable is uniformly continuous on a closed rectangle rectangle (a small corollary to uniform continuity for continuous functions on intervals). Then, the diameter implies:

$$\sum_i |J(x_i)|\lambda_k(Q_i) \leq \int_A |J(x)|dx + \epsilon\lambda_k(A)$$

Note I am reading through the proof, but it does get into the unhelpful region of like δ ϵ hell. I am already convinced via the theorem for Riemann integrals, and to be honest, the theorem here is very very similar.

Stieltjes Integrals

Suppose that F is a function on $\mathbb{R}^k$ satisfying the hypotheses of Theorem 12.5 (continuous from above real function that satisfies $\Delta_A F \geq 0$ for bounded rectangles A). Then, there is a unique measure μ on $\mathcal{R}^k$ satisfying:

$$\mu(A) = \Delta_A F$$

For such a μ, we often notationally describe the integral across μ as:

$$\int_A f(x)dF(x) = \int_A f(x)\mu(dx)$$

The left side is the *Stieltjes Integral of* f with respect to F; since it is defined by the right side, it really isn't anything new.

Suppose that f is uniformly continuous on a rectangle A, and suppose that A is decomposed into rectangles A_m small enough such that $|f(x) - f(y)| < \epsilon/\mu(A)$ for $x, y \in A_m$. Then:

$$\left|\int_A f(x)dF(x) - \sum_m f(x_m)\Delta_{A_m} f\right| = \left|\int_A f(x) - \sum_m f(x_m)\mathbb{1}\left\{A_m\right\}\mu(dx)\right|$$

Note, $x_m \in A_m$ is fixed, so the ω is the x in $f(x)$, and the x in $\mathbb{1}\{A_m\}$ - however, when $x \in A_m$, the difference between $f(x)$ and $f(x_m)$ is less than $\epsilon/\mu(A)$. So, we find:

$$\leq \epsilon/\mu(A) * \mu(A) = \epsilon$$

And so, in this case, we can define $\int_A f(x)dF(x)$ not as the Lebesgue integral, but the limit of these approximating sums without any reference to the general theory of measure. This is how Stieltjes Integrals are defined, I guess.

Problems

17.1 - Riemann and Lebesuge Upper and Lower Integrals

So, there are a lot of prerequisite properties, just read from the book, I'm not gonna copy them. We want to prove:

1. D_f is a Borel set
2. (i) implies (ii)
3. (ii) implies (iii)
4. (iii) implies (i)
5. The r of (i) must coincide with the $R_*f = R^*f$ of (ii).

Just some of the notation this problem deals with, we have the upper and lower integrals as before:

$$L_*f = \sup \sum_i \left[\inf_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

$$L^*f = \inf \sum_i \left[\sup_{\omega \in A_i} f(\omega)\right] \mu(A_i)$$

Where the A_i extend over all finite partitions $\{A_i\}$ of Ω. Now, R_*f and R^*f are the upper and lower sums, but we only extend over finite partitions of $[0,1]$ into subintervals. Note, we are considering only f as a bounded function on $[0,1]$. Clearly, we have:

$$R_*f \leq L_*f \leq L^*f \leq R^*f$$

As R is going over a subset of the possible partitions of $[0,1]$ into borel sets. We have the following conditions:

1. There is an r with the property that for each ϵ there is a δ such that:

$$\left| r - \sum_i f(x_i)\lambda(I_i) \right|$$

holds if $\{I_i\}$ partitions $[0, 1]$ into subintervals with $\lambda(I_i) < \delta$ and if $x_i \in I_i$.

2. $R_* f = R^* f$

3. If D_f is the set of points of discontinuity of f, then $\lambda(D_f) = 0$.

So, now we begin the properties of the conditions.

(a) D_f is a borel set. Well, D_f is the set of points of discontinuity of f. To be honest - I think we need more than just f is bounded? In the solutions, I think there is an assumption that the first condition holds. Let A_ϵ be the set of x such that for every δ there are points y and z satisfying $|y - x| < \delta$, $|z - x| < \delta$, and $|f(y) - f(z)| \geq \epsilon$.

First, note that D_f is the union of A_ϵ. Indeed, if f is not continuous as x, there is some ϵ such that for all δ, being within δ of x does not imply within ϵ of $f(x)$. So, we do have the union is D_f. We just need to show that each A_ϵ is a borel set, which follows if A_ϵ is closed (open sets are countable unions of rectangles).

So, we just need to show that A_ϵ is closed. I mean, it would make sense, because if you are close enough to an x, you should also satisfy the A_ϵ property above. Well, take $t \notin A_\epsilon$. This implies for ϵ, there is a δ such that there are no points y, z satisfying the above condition. This implies the same for all $\delta' < \delta$. So, take the open set around t, $|k - t| < \delta/2$. I think for all k in this set, we have for ϵ, $\delta/2$ doesn't satisfy the property, given that all y, z in the $\delta/2$ range of t should also be within the δ range of x. So, A_ϵ is indeed closed, as its complement is open, and so D_f is a borel set, being the countable union of borel sets.

(b) Prove that (i) implies (ii). Namely, the existence of such an r implies $R_* f = R^* f$.

As noted in the chapter, as we let x_i vary over the interval, we also have:

$$\left| r - \sum_i \sup_{x \in I_i} f(x_i)\lambda(I_i) \right| < \epsilon$$

For I_i with δ small enough. Similarly:

$$\left| r - \sum_i \inf_{x \in I_i} f(x_i)\lambda(I_i) \right| < \epsilon$$

Both of these inequalities allow us to conclude that $r = R_* f = R^* f$. Indeed, consider from the infimum side. As we refine the intervals, the sum gets larger. So, for any interval consider in the sums of $R_* f$, we can refine it further, into finite interval decompositions of $[0, 1]$, to get a value arbitrarily close to ϵ. It is clear how an argument can be made to show this equality, and so (i) implies (ii).

(c) Prove that (ii) implies (iii). Well, if we show each $\lambda(A_\epsilon)$ is arbitrarily small, we will have that D_f has measure 0 as well (or, the union of measure 0 sets is measure 0). Take arbitrary ϵ and η. Choose a partition of intervals I_i for which the corresponding upper and lower sums ($R_* f$ and $R^* f$) differ by at most $\epsilon\eta$. We can always find such a partition by taking two partitions arbitrarily close to r by above and below and refining them. We have:

$$\sum_i \left[\sup_{\omega \in I_i} f(\omega) - \inf_{\Omega \in I_i} f(\omega) \right] \mu(I_i) < \epsilon\eta$$

How many of these intervals I_i have interiors that meet A_ϵ? Note if I_i has an interior that meets A_ϵ, the difference above would be larger than ϵ (as in the definition of A_ϵ, we could take δ less than half the length of the interval I_i).