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Time-varying feedback for stabilization in prescribed finite time

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Summary

This paper provides a time-varying feedback alternative to control of finite-time systems, which is referred to as "prescribed-time control," exhibiting several superior features: (i) such time-varying gain–based prescribed-time control is built upon regular state feedback rather than fractional-power state feedback, thus resulting in smooth (C^m) control action everywhere during the entire operation of the system; (ii) the prescribed-time control is characterized with uniformly prespecifiable convergence time that can be preassigned as needed within the physically allowable range, making it literally different from not only the traditional finite-time control (where the finite settling time is determined by a system initial condition and a number of design parameters) but also the fixed-time control (where the settling time is subject to certain constraints and thus can only be specified within the corresponding range); and (iii) the prescribed-time control relies only on regular Lyapunov differential inequality instead of fractional Lyapunov differential inequality for stability analysis and thus avoids the difficulty in controller design and stability analysis encountered in the traditional finite-time control for high-order systems.

KEYWORDS

finite-time control, fixed-time control, prescribed-time control, time-varying feedback

1 | INTRODUCTION

Professor Vadim I. Utkin has built one of the unmatched legacies in the field of control engineering.[1,2] Sliding mode controllers, ie, fathered and championed by Professor Utkin for over 4 decades, are probably more ubiquitous, both in the control practice and in research, than just about any other class of controllers except for PID. Methods such as model predictive control, backstepping, and H_∞ control can only aspire for an impact in the future, which is comparable to the range of cumulative impact of sliding mode control (SMC).

Sliding mode control owes its popularity to its effectiveness. Its use of "variable structure" feedback results, simultaneously, in 2 desirable properties: complete rejection (rather than partial attenuation) of disturbances of a known upper bound and the regulation of the state to zero in finite time.

The ability of the classical SMC[3] to achieve finite-time regulation has inspired many "spin-off" finite-time methodologies that use fractional-order feedback,[4-8] homogeneity,[9-12] etc. Indeed, finite-time control has been an active research

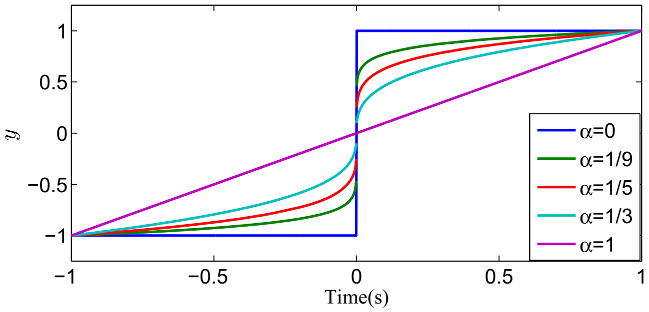

FIGURE 1 $y = \text{sgn}(x)|x|^{\alpha}$ ($|x| \leq 1$) when $\alpha = 0$, $0 < \alpha < 1$ and $\alpha = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

topic during the past decades[13-29] due to its appealing features of faster convergence rate, better disturbance rejection, and stronger robustness against uncertainties. Thus far, most existing finite-time control methods are based on fractional or SMC in that the control schemes are of the form $u = -c\,\text{sgn}(x)|x|^{\alpha}$ (see the work of Bhat and Bernstein[4]) with $c > 0$ and $0 < \alpha < 1$, essentially signum function–based when the fraction $\alpha = 0$, ie, $u = -c\,\text{sgn}(x)$(Figure 1 illustrates the evolution of the function of $y = \text{sgn}(x)|x|^{\alpha}$ for different α, $\alpha \in [0, 1]$).[1,25]

In this work, we present a time-varying feedback alternative to achieving finite-time control, which is referred to as "prescribed-time control"[30] or predefined-time control.[31-33] We show that such prescribed-time control exhibits several superior features. (i) The time-varying gain–based prescribed-time control is built upon regular state feedback, not on fractional-power state feedback, thus resulting in smooth (C^n) control action everywhere during the entire operation of the system. (ii) The prescribed-time control is characterized with uniformly prespecifiable convergence time that is independent of initial condition and any other design parameters and can be preassigned as needed within the physically allowable range. It is noted that in the traditional finite-time control, the settling time is determined by the system initial condition and a number of design parameters and thus cannot be arbitrarily prespecified. In addition, the settling time $T(x_0)$ in fixed-time control still cannot be preassigned arbitrarily within any physically possible range because the upper bound of $T(x_0)$ is subject to certain restrictions. From this aspect, the prescribed-time control differs from the finite-time control and the fixed-time control. (iii) The prescribed-time control relies only on a regular Lyapunov differential inequality rather than a fractional Lyapunov differential inequality for stability analysis and thus avoids the difficulty in controller design and stability analysis encountered in the traditional finite-time control for high-order systems.

2 | FINITE-TIME STABILITY THEORY

Before moving on, we first give the definitions for finite-time stability, fixed-time stability, and prescribed-time stability and the related lemmas.

Definition 1. (See the work of Polyakov et al[29])
Consider a dynamical system

$$\dot{x}(t) = f(x(t), t), \quad f(0, t) = 0, \quad x(0) = x_0, \quad t \in \mathfrak{R}_+, \tag{1}$$

where $x \in \mathfrak{R}^n$ is the state vector, $f : U_0 \times \mathfrak{R}_+ \to \mathfrak{R}^n$ is a nonlinear vector field locally bounded on an open neighborhood U_0 of the origin and uniformly in time. The origin of system (1) is said to be (locally) uniformly finite-time stable if it is uniformly asymptotically stable, and, for any initial condition $x_0 \in U\,(U \subseteq U_0)$, there exists a locally bounded function $T : U \to \mathfrak{R}_+ \bigcup \{0\}$ such that $x(x_0, t) = 0$ for all $t \geq T(x_0)$, where $x(x_0, t)$ is an arbitrary solution of the Cauchy problem (1). The function $T(x_0)$ is called the settling-time function. If $U = U_0 = \mathfrak{R}^n$, the origin of system (1) is said to be globally uniformly finite-time stable.

From Definition 1, we see that the settling time in traditional finite-time stability is not uniform in the initial condition. For example, the origin of system $\dot{x}(t) = -\text{sign}(x(t))|x(t)|^{1/2}$, $x \in \mathfrak{R}$ is globally uniformly finite-time stable because its settling-time function T is locally bounded: $T(x_0) = 2\sqrt{|x_0|}$, but it is obvious that the settling time $T(x_0)$ depends on the initial condition.

Definition 2. (See the work of Polyakov[28])
The origin of system (1) is said to be globally fixed-time stable if it is globally uniformly finite-time stable and the settling-time function T is globally bounded, ie, $\exists T_{\max} \in \mathfrak{R}_+$ such that $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathfrak{R}^n$.

According to Definition 2, the finite settling time in a fixed-time system, although uniform with respect to the initial condition, is still subject to certain restrictions and thus can only be prespecified within a certain range. For example, the origin of the system $\dot{x}(t) = -\text{sign}(x(t))(|x(t)|^{1/2} + |x(t)|^{3/2})$, $x \in \mathfrak{R}$, is globally fixed-time stable since its settling time $T(x_0) = 2\arctan(\sqrt{|x_0|})$ is uniformly bounded by π, from which we see that the upper bound of the finite convergence time is subject to certain constraints, ie, it cannot be preselected (assigned) to be smaller than π.

Lemma 1. (See the work of Bhat and Bernstein[4])
Consider system (1). If there is a continuously differentiable positive definite Lyapunov function $V(x, t)$ defined on $U \times \mathfrak{R}^+$, where $U \subseteq U_0$ is a neighborhood of the origin, real constants $c > 0$ and $0 < r < 1$, such that $\dot{V}(x, t) \leq -cV(x, t)^r$ on U, then the origin of system (1) is finite-time stable. Moreover, the finite settling time T^* satisfies $T^* \leq \frac{V(x_0,t)^{1-r}}{c(1-r)}$ for any given initial condition $x(t_0) \in U$. If $U = \mathfrak{R}^n$, then the origin of system (1) is globally finite-time stable.

Lemma 2. (See the works of Wang et al[22-23])
Consider system (1). If there exist a continuously differentiable positive definite Lyapunov function $V(x, t)$ defined on $U \times \mathfrak{R}^+$ where $U \subseteq U_0$, and there are real constants $c > 0$, $0 < r < 1$, and $\|d\|_{[t_0,t]}$ where $\|d\|_{[t_0,t]} := \sup_{\tau\in[t_0,t]}|d(\tau)|$, such that $\dot{V}(x, t) \leq -cV(x, t)^r + \|d\|_{[t_0,t]}$, then system (1) is locally finite-time convergent within a finite settling time T^*, satisfying $T^* \leq \frac{V(x_0,t)^{1-r}}{c\theta_0(1-r)}$, where $0 < \theta_0 < 1$ is a real constant, for any given initial condition $x(t_0) \in U$. Moreover, when $t \geq T^*$, the state trajectory converges to a compact set

$$\Theta = \left\{ x \,\middle|\, V(x, t) \leq \left(\frac{\|d\|_{[t_0,t]}}{c(1-\theta_0)} \right)^{\frac{1}{r}} \right\}. \tag{2}$$

Now, we introduce a different approach to finite-time control, namely, prescribed-time control. The key in the prescribed-time control design is the utilization of the monotonically increasing function[30]

$$\mu_1(t - t_0) = \frac{T}{T + t_0 - t}, \quad t \in [t_0, t_0 + T), \tag{3}$$

where $T > 0$ with the properties that $\mu_1(0) = 1$ and $\mu_1(T) = +\infty$. We review 2 definitions related to the prescribed-time stability.

Definition 3. (PT-ISS[30])
The system $\dot{x} = f(x, t, d)$ (of arbitrary dimensions of x and d) is said to be *prescribed-time input-to-state stable (PT-ISS) in time T* if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0 + T)$,

$$|x(t)| \leq \beta\left(|x_0|, \mu_1(t - t_0) - 1\right) + \gamma\left(\|d\|_{[t_0,t]}\right). \tag{4}$$

The function $\mu_1(t-t_0)-1 = (t-t_0)/(T+t_0-t)$ starts from zero at $t = t_0$ and grows monotonically to infinity as $t \to t_0+T$. Therefore, a system that is PT-ISS is, in particular, ISS with the additional property that in the absence of the disturbance d, it is prescribed-time globally asymptotically stable in T.

Definition 4. (PT-ISS+C[30])
The system $\dot{x} = f(x, t, d)$ (of arbitrary dimensions of x and d) is said to be *prescribed-time input-to-state stable in time T and convergent to zero (PT-ISS+C)* if there exist class $\mathcal{KL}$ functions β and β_f, and a class $\mathcal{K}$ function γ, such that, for all $t \in [t_0, t_0 + T)$,

$$|x(t)| \leq \beta_f\left(\beta(|x_0|, t - t_0) + \gamma\left(\|d\|_{[t_0,t]}\right), \mu_1(t - t_0) - 1\right). \tag{5}$$

Clearly, a system that is PT-ISS+C is also PT-ISS, with the additional property that its state converges to zero in time T despite the presence of a disturbance.

Lemma 3. (See the works of Song et al[30])
Consider the function

$$\mu(t-t_0)=\frac{T^{n+m}}{(T+t_0-t)^{n+m}}=(\mu_1(t-t_0))^{n+m} \tag{6}$$

on $[t_0, t_0+T)$*, with positive integers* m, n *(*n *is corresponding to the order of the system). If a continuously differentiable function* $V:[t_0, t_0+T)\to[0,+\infty)$ *satisfies*

$$\dot V(t)\le -2k\mu(t-t_0)V(t)+\frac{\mu(t-t_0)}{4\lambda}d(t)^2 \tag{7}$$

for positive constants k *and* λ*, then*

$$V(t)\le \exp^{-\frac{2kT}{m+n-1}\left(\mu_1(t-t_0)^{m+n-1}-1\right)}V(t_0)+\frac{\|d\|^2_{[t_0,t]}}{8k\lambda},\quad \forall t\in[t_0,t_0+T). \tag{8}$$

Corollary 1. *Under the conditions of Lemma 3, if* $d(t)\equiv 0$*, then* $\lim_{t\to t_0+T}V(t)=0$.

Corollary 1 is a time-varying fixed-time counterpart of the basic nonsmooth finite-time result of theorem 4.2 by Bhat and Bernstein,[5] whereas Lemma 3 is a time-varying finite-time counterpart of the robustness result in theorem 5.2 by Bhat and Bernstein.[5]

3 | SOME FEATURES OF THE FINITE-TIME CONTROL AND PRESCRIBED-TIME CONTROL

In this section, we present some comment features between the traditional finite-time control and the prescribed-time control and highlight the major differences between them, with particular attention to the control structure, convergence time, disturbance rejection, and robustness against the uncertainties.

To help with the understanding of the fundamental idea and technical development, we consider the following single-integrator system:

$$\dot x = u, \tag{9}$$

where x and u are the system state and control input, respectively. It is worth mentioning that, for the prescribed-time control proposed in this paper, neither the plant or its solutions nor the control law is expected to exist beyond the terminal time (prescribed time), and all the stability analysis made for the closed loop is valid only in that time interval.

3.1 | Control structure

3.1.1 | Traditional finite-time control

The traditional finite-time controller bears the following form[1,4-8]:

$$u=-\varrho(t)\,\mathrm{sgn}(x)|x|^{\alpha}, \tag{10}$$

where $\varrho(t)$ is the control gain to be designed and $0\le\alpha=p/q<1$ (p and q are positive odd integers). Note that the control law (10) covers several different cases: when $\alpha=1$, it reduces to the typical asymptotical control $u=-\varrho(t)x$; when $\alpha=0$, it reduces to the signum function–based (sliding mode) finite-time control,[8] which is discontinuous due to the using of the signum function; when $0<\alpha<1$, it corresponds to the fractional state–based finite-time control established by Bhat and Bernstein,[4,5] which is continuous but nonsmooth with respect to state variables. It is worth mentioning that with $0\le\alpha<1$, the finite settling time T^* is determined by $T^*=\frac{V(t_0)^{1-r}}{c(1-r)}$ according to Lemma 1, where $c>0$ is some constant related to the control gain $\varrho(t)$ and the fraction index α, and r is related to α.

3.1.2 | Prescribed-time control

The time-varying scaling function utilized is of the form

$$\mu(t-t_0)=\frac{T^{1+m}}{(T+t_0-t)^{1+m}},\quad t\in[t_0,t_0+T), \tag{11}$$

which corresponds to the case of $n = 1$ in (6). With such μ, we conduct the following simple transformation, $w = \mu x$, on $t \in [t_0, t_0 + T)$. Then, the prescribed-time controller based on time-varying gain is of the form

$$u = -k(t)\mu x, \tag{12}$$

with $k(t) > 0$.

3.2 | Convergence time

3.2.1 | Traditional finite-time control

For the traditional finite-time method, applying to system (9), we choose

$$\varrho(t) = \varrho_0 > 0, \tag{13}$$

ie, a constant gain, then we have the following result.

Observation 1. For system (9) with controller (10) and (13), the following result is obtained:

(i) When $0 < \alpha < 1$ (fractional state–based control), the finite settling time satisfies

$$T^* \leq \frac{V(t_0)^{1-\frac{1+\alpha}{2}}}{2\varrho_0\left(1 - \frac{1+\alpha}{2}\right)}, \tag{14}$$

with $V(t_0)$ being a Lyapunov function related to the initial state.

(ii) When $\alpha = 0$ (sliding mode–based control), the finite settling time satisfies $T^* \leq \frac{|x(t_0)|}{\varrho_0}$.

(iii) When $\alpha = 1$ (regular state–based control), the settling time satisfies $T^* = +\infty$.

Proof. System (9) with (10) and (13) is rewritten as

$$\dot{x} = -\varrho_0 \mathrm{sgn}(x)|x|^{\alpha}. \tag{15}$$

The details of the finite-time stability of (15) can be seen in the works of Bhat and Bernstein[4] and Poznyak et al[26] for (i), in the works of Utkin[1] and Polyakov and Poznyak[25] for (ii), and in the work of Khalil[34] for (iii) and thus are omitted here. □

From (i) and (ii) in Observation 1, we see that the finite settling time T^* under the fractional state–based finite-time control is dependent on the initial condition and a number of design parameters, and T^* by the sign function–based control depends on the initial condition.

3.2.2 | Time-varying gain–based prescribed-time control

The prescribed-time controller for system (9) has the following control gain:

$$k = k_0 + \frac{1+m}{T} > 0, \tag{16}$$

which leads to the following result.

Observation 2. System (9) with controller (12) and (16) is PT-ISS+C with the prescribed finite settling time T, and

$$|x(t)| \leq \mu^{-1}\exp^{-k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}|x(t_0)| \qquad \text{for } t \in [t_0, t_0+T), \tag{17}$$

$$|x(t)| \to 0 \qquad \text{as } t \to (t_0+T)^-, \tag{18}$$

where

$$\mu^{-1}(t - t_0) = 1 - \frac{t - t_0}{T} \tag{19}$$

is a monotonically decreasing linear function with the properties that $\mu^{-1}(0) = 1$ and $\mu^{-1}(T) = 0$, which means that, in particular, the state $x(t)$ converges to zero in prescribed time T.

Proof. With the transformed variable $w = \mu x$, system (9) is rewritten as

$$\dot{w} = \mu u + \dot{\mu} x, \tag{20}$$

where

$$\dot{\mu} = \frac{(1+m)T^{1+m}}{(T+t_0-t)^{2+m}} = \frac{1+m}{T}\mu^{\frac{2+m}{1+m}}. \tag{21}$$

Choose the Lyapunov function candidate as $V = w^2/2$ whose derivative along (20) is

$$\dot{V} = w\dot{\mu}x + w\mu u. \tag{22}$$

By noting that

$$w\dot{\mu}x = w\frac{1+m}{T}\mu^{\frac{2+m}{1+m}}x = w\mu\frac{1+m}{T}\mu^{-\frac{m}{1+m}}\mu x \le w\mu\frac{1+m}{T}w, \tag{23}$$

then we have

$$\dot{V} \le w\mu\left(u + \frac{1+m}{T}w\right) = w\mu(-k_0 w) = -k_0\mu w^2 = -2k_0\mu V(t). \tag{24}$$

By using Lemma 3, we arrive at

$$V(t) \le \exp^{-2k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}V(t_0), \tag{25}$$

which implies that

$$x(t)^2 = \frac{1}{\mu^2}w^2 = 2\mu^{-2}V(t) \le 2\mu^{-2}V(t_0) \le \mu^{-2}\exp^{-2k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}x(t_0)^2 \to 0, \tag{26}$$

as $t \to t_0 + T$ by using the fact that $\mu^{-l} \to 0\ (l > 0)$ as $t \to t_0 + T$, and thus, (17) holds. This means that the regulation is achieved in the prescribed time T, which is independent of the initial condition and any other design parameters and thus can be explicitly prespecified. □

3.3 | Disturbance rejection and robustness against uncertainties

Of particular interest in this section is the capability for disturbance rejection and robustness against uncertainties.

Consider

$$\dot{x} = u + f_d(x,t), \tag{27}$$

where x and u are the system state and control input, respectively, and $f_d(x,t)$ denotes the unknown lumped uncertainty that may be nonvanishing.

Assumption 1. (Bound on matched but possibly nonvanishing uncertainty)
The nonlinearity f_d in (27) obeys

$$|f_d(x,t)| \le d(t)\psi(x), \tag{28}$$

where $d(t)$ is a disturbance with an unknown bound $\|d\|_{[t_0,t]} := \sup_{\tau\in[t_0,t]}|d(\tau)|$, and $\psi(x) \ge 0$ is a known scalar-valued continuous function.

3.3.1 | Traditional finite-time method

For the traditional finite-time method to system (27), we choose

$$\varrho(t) = \varrho_0 + \gamma\psi^2, \tag{29}$$

where $\gamma > 0$ is a free design constant parameter, and we have the following result.

Observation 3. Consider system (27) under Assumption 1. If controller (10) with (29) is applied, the following result is obtained:

(i) When $0 < \alpha < 1$ (fractional state–based control), the state trajectory is convergent to a compact set

$$\Theta_1 = \left\{ x \,\middle|\, |x| \leq \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma \varrho_0 (1-\theta_0)} \right)^{\frac{1}{2\alpha}} \right\} \tag{30}$$

within the finite time T^*, which satisfies

$$T^* \leq \frac{V(t_0)^{1-\frac{2\alpha}{1+\alpha}}}{\theta_0 \varrho_0 (1+\alpha)^{\frac{2\alpha}{1+\alpha}} \left(1 - \frac{2\alpha}{1+\alpha}\right)}, \tag{31}$$

where $V(t_0)$ is a Lyapunov function related to the initial state and $0 < \theta_0 < 1$ is defined the same as in Lemma 2.

(ii) When $\alpha = 0$ (sliding mode–based control), the state trajectory is convergent to a compact set

$$\Theta_2 = \left\{ x \,\middle|\, |x| \leq \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma \varrho_0 (1-\theta_0)} \right)^{\frac{1}{2\times 0}} \right\}, \tag{32}$$

which converges to zero if we choose the design parameters such that $\frac{\|d\|^2_{[t_0,t]}}{4\gamma \varrho_0 (1-\theta_0)} < 1$, within the finite time T^*, which satisfies

$$T^* \leq \frac{V(t_0)}{\theta_0 \varrho_0}. \tag{33}$$

(iii) When $\alpha = 0$ (regular state-feedback control), the state trajectory is convergent to a compact set

$$\Theta_3 = \left\{ x \,\middle|\, |x| \leq \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma \varrho_0 (1-\theta_0)} \right)^{\frac{1}{2}} \right\} \tag{34}$$

within the infinite time.

Proof. System (27) with (10) and (29) is rewritten as

$$\dot{x} = -(\varrho_0 + \gamma \psi^2) \mathrm{sgn}(x) |x|^\alpha + f_d(x,t). \tag{35}$$

In the following, we analyze the finite-time stability of (35), especially its disturbance rejection and robust ability against uncertainties.

(i) Choose the Lyapunov function candidate as $V = x^{1+\alpha}/(1+\alpha)$, whose time derivation along (35) is

$$\dot{V} = x^\alpha \dot{x} = -\varrho_0 x^{2\alpha} - \gamma \psi^2 x^{2\alpha} + x^\alpha f_d. \tag{36}$$

Upon using the Young's inequality, we have

$$x^\alpha f_d(\cdot) \leq \gamma \psi^2 x^{2\alpha} + \frac{\|d\|^2_{[t_0,t]}}{4\gamma}. \tag{37}$$

Then, we arrive at

$$\dot{V} \leq -\varrho_0 x^{2\alpha} - \gamma \psi^2 x^{2\alpha} + \gamma \psi^2 x^{2\alpha} + \frac{\|d\|^2_{[t_0,t]}}{4\gamma} = -\varrho_0 x^{2\alpha} + \frac{\|d\|^2_{[t_0,t]}}{4\gamma}. \tag{38}$$

By noting that $V^{\frac{2\alpha}{1+\alpha}} = (\frac{1}{1+\alpha})^{\frac{2\alpha}{1+\alpha}} x^{2\alpha}$, we then have

$$x^{2\alpha} = (1+\alpha)^{\frac{2\alpha}{1+\alpha}} V^{\frac{2\alpha}{1+\alpha}}. \tag{39}$$

According to (38) and (39), we see that

$$\dot{V} \le -\varrho_0(1+\alpha)^{\frac{2\alpha}{1+\alpha}} V^{\frac{2\alpha}{1+\alpha}} + \frac{\|d\|^2_{[t_0,t]}}{4\gamma}. \tag{40}$$

Let $c = \varrho_0(1+\alpha)^{\frac{2\alpha}{1+\alpha}}$, $\bar{d} = \frac{\|d\|^2_{[t_0,t]}}{4\gamma}$, and $r = \frac{2\alpha}{1+\alpha}$. According to Lemma 2, we then have, for $t \ge T^*$, that

$$V(t) \le \left(\frac{\bar{d}}{(1-\theta_0)c}\right)^{\frac{1}{r}} = \frac{1}{1+\alpha}\left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)}\right)^{\frac{1+\alpha}{2\alpha}} \tag{41}$$

with $T^* \le \frac{V_3(t_0)^{1-r}}{c\theta_0(1-r)} = \frac{V_3(t_0)^{1-\frac{2\alpha}{1+\alpha}}}{\theta_0\varrho_0(1+\alpha)^{\frac{2\alpha}{1+\alpha}}(1-\frac{2\alpha}{1+\alpha})}$. From (41), we then arrive at

$$|x| = (x^{1+\alpha})^{\frac{1}{1+\alpha}} = [(1+\alpha)V]^{\frac{1}{1+\alpha}} \le \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)}\right)^{\frac{1}{2\alpha}}. \tag{42}$$

(ii) When $\alpha = 0$, it corresponds to the signum function–based control (SMC)

$$u = -\varrho(t)\mathrm{sgn}(x)|x|^0 = -(\varrho_0 + \gamma\psi^2)\mathrm{sgn}(x). \tag{43}$$

The result in (ii) is straightforward to be obtained by following the similar line as in the proof of (i).

(iii) When $\alpha = 1$, the control in (10) with (29) reduces to the nonfinite-time control

$$u = -\varrho(t)\mathrm{sgn}(x)|x| = -(\varrho_0 + \gamma\psi^2)x, \tag{44}$$

and the bound for the state x is $|x| \le \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)}\right)^{\frac{1}{2}}$. If choosing ϱ_0 and γ such that $\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)} < 1$, and if $0 < \alpha << 1$, then $\left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)}\right)^{\frac{1}{2\alpha}} << \left(\frac{\|d\|^2_{[t_0,t]}}{4\gamma\varrho_0(1-\theta_0)}\right)^{\frac{1}{2}}$ (see the work of Wang et al[22]), which implies that the fractional state-feedback finite-time control proposed in (10) with (29) has better disturbance rejection and robustness against uncertainties than the regular nonfinite-time control given in (44). Especially, the signum function–based finite-time control (43) (SMC) has better disturbance rejection and robustness against uncertainties than the fractional state–based finite-time control (10) with (29) and the regular nonfinite-time control (44). □

3.3.2 | Time-varying gain–based prescribed-time control

In this case, the same controller is used as in (12) with the control gain designed as

$$k(t) = k_0 + \frac{1+m}{T} + \lambda\psi^2. \tag{45}$$

Observation 4. System (27) with the control scheme (12) and (45) is PT-ISS+C with the prespecified finite settling time T, and

$$|x(t)| \le \mu^{-1}\left(\exp^{-k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}|x(t_0)| + \frac{\|d\|_{[t_0,t]}}{2\sqrt{k_0\lambda}}\right) \tag{46}$$

for all $t \in [t_0, t_0+T)$, which means that, in particular, the state converges to zero in prescribed time T.

Proof. By using the time-varying scaling function $\mu(t)$ given in (11) and the transformed variable $w = \mu x$, we have

$$\dot{w} = \mu u + \mu f_d + \dot{\mu}x. \tag{47}$$

Choose the Lyapunov function as $V = w^2/2$ whose derivative along (47) is

$$\dot{V} = w\dot{\mu}x + w\mu f_d + w\mu u. \tag{48}$$

Upon using the Young's inequality, we get

$$w\mu f_d \leq \mu\lambda w^2\psi^2 + \frac{\mu^2}{4\lambda} = w\mu\lambda\psi^2 w + \frac{\mu\|d\|^2_{[t_0,t]}}{4\lambda}. \tag{49}$$

From (23), (48), and (49), we have

$$\begin{aligned}\dot{V} &\leq w\mu\left(u + \frac{1+m}{T}w + \lambda\psi^2 w\right) + \frac{\mu\|d\|^2_{[t_0,t]}}{4\lambda} = w\mu(-k_0 w) + \frac{\mu\|d\|^2_{[t_0,t]}}{4\lambda}\\ &= -k_0\mu w^2 + \frac{\mu\|d\|^2_{[t_0,t]}}{4\lambda} = -2k_0\mu V(t) + \frac{\mu\|d\|^2_{[t_0,t]}}{4\lambda}.\end{aligned} \tag{50}$$

By using Lemma 3, we then arrive at

$$V(t) \leq \exp^{-2k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}V(t_0) + \frac{\|d\|^2_{[t_0,t]}}{8k_0\lambda}, \tag{51}$$

which implies that

$$x(t)^2 = \mu^{-2}w^2 = \mu^{-2}\left(\exp^{-2k_0\frac{T}{m}\left(\mu^{\frac{m}{1+m}}-1\right)}x(t_0)^2 + \frac{\|d\|^2_{[t_0,t]}}{4k_0\lambda}\right) \to 0 \tag{52}$$

as $t \to t_0 + T$, meaning that the state x converges to zero when $t \to t_0 + T$, and therefore, (46) holds naturally. □

4 | HIGH-ORDER NONLINEAR SYSTEMS

In this section, we extend the comparison results in Sections 1 to 3 to nonlinear systems diffeomorphically equivalent to the "normal form"

$$\dot{x}_i = x_{i+1}, \quad \dot{x}_n = b(x,t)u + f_d(x,t), \quad y = x_1, \tag{53}$$

where $x = [x_1, \dots, x_n]^T \in \mathfrak{R}^n$, $u \in \mathfrak{R}$, and $y \in \mathfrak{R}$ are system state, control input, and control output, respectively, $b(x,t)$ represents the control gain of the system, which would pollute any design input and thus imposes technical challenge for control if such gain is unknown and time varying.[22-24,35-38] Here, $b(x,t)$ is assumed to be unknown and bounded, $f_d(x,t)$ is unknown and possibly nonvanishing[30] satisfying Assumption 1.

Assumption 2. (Global controllability)
For system (53), there exists a known $\underline{b} \neq 0$ (and without loss of generality $\underline{b} > 0$) such that $\underline{b} \leq |b(x,t)| < \infty$ for all $x \in \mathfrak{R}^n$, $t \in \mathfrak{R}_+$.

4.1 | Traditional finite-time control method

To develop the finite-time control for system (53), the following virtual errors ξ_q are defined as

$$\xi_i = x_i^{1/q_i} - x_i^{*1/q_i}, \quad i = 1, 2, \dots, n, \tag{54}$$

where $q_i = (2n + 3 - 2i)/(2n + 1) > 0$ $(i = 1, \dots, n)$, and the virtual controllers x_i^* $(i = 1, \dots, n)$ is defined as

$$x_1^* = 0, \quad x_i^* = -\beta_{i-1}\xi_{i-1}^{q_i}, \quad i = 2, \dots, n, \tag{55}$$

with β_{i-1} $(i = 2, \dots, n)$ being a positive constant to be designed.

The actual controller is of the following form:

$$u = -\frac{1}{\underline{b}}\left(\beta_n + \sigma\psi^2\right)\xi_n^{q_{n+1}}. \tag{56}$$

Observation 5. (See the work of Wang et al[23])
Consider the nonlinear system (53) under Assumptions 1 and 2. If the control scheme (55)-(56) is applied, then for any initial conditions satisfying $V(t_0) \le \chi$ ($\chi > 0$ is a bounded constant), the following result is achieved in that the output state y converges to a small residual set Θ, namely,

$$\Theta = \left\{ |y| \le \left(\frac{[(1+s)k_b]^{\frac{2s}{1+s}} \varsigma}{\mu_1 \mu_2 k_a} \right)^{\frac{1}{2s}} \right\}, \tag{57}$$

in a finite time T^* satisfying $T^* \le \frac{V(t_0)^{1-\frac{2s}{1+s}}}{(1-\mu_1)\tilde{c}(1-\frac{2s}{1+s})}$, where $0 < s < 1$, $0 < \mu_1 < 1$, $0 < \mu_2 \le 1$, k_a, k_b, and ς are bounded constants related to a set of design parameters β_i $(i = 1, \ldots, n)$, σ, and s.

Remark 1. In the high-order case, the signum function–based control method is not available because in such case $1/q_i$ $(i = 1, \ldots, n)$ would correspond to $1/0$, which is equal to ∞. It is worth mentioning that the Lyapunov function for the stability analysis of the fractional state–based finite-time control method for high-order systems normally includes an adding power integrator of the form $W_i(x_1, \ldots, x_i) = \int_{x_i^*}^{x_i} \left(s^{\frac{1}{q_i}} - x_k^{*\frac{1}{q_i}} \right)^{2-q_i} ds$ $(i = 2, \ldots, n)$, making stability analysis rather involved.

4.2 | Time-varying gain–based prescribed-time method

For the time-varying gain–based prescribed-time control, the following transformation is conducted:

$$\begin{aligned} w_1(t) &= \mu(t-t_0)x_1(t), \\ w_i(t) &= \mathrm{d}w_{i-1}(t)/\mathrm{d}t, \quad i = 2, \ldots, n+1. \end{aligned} \tag{58}$$

Denote

$$r_1 = [w_1, \ldots, w_{n-1}]^T = J_1 w \in \mathfrak{R}^{n-1} \tag{59}$$

$$r_2 = \dot{r}_1 = [w_2, \ldots, w_n]^T = J_2 w \in \mathfrak{R}^{n-1}, \tag{60}$$

where

$$J_1 = \begin{bmatrix} I_{n-1}, & 0_{(n-1)\times 1} \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0_{(n-1)\times 1}, & I_{n-1} \end{bmatrix}, \tag{61}$$

and $K_{n-1} = [k_1, \ldots, k_{n-1}]^T \in \mathfrak{R}^{n-1}$, where K_{n-1} is an appropriately chosen coefficient vector so that the polynomial $s^{n-1} + k_{n-1}s^{n-2} + \cdots + k_1$ and the matrix

$$\Lambda = \begin{bmatrix} 0 & I_{n-2} \\ -k_1 & -k_2 \cdots - k_{n-1} \end{bmatrix} \tag{62}$$

are both Hurwitz. Now, we replace the state w_n by the new variable z as

$$z = w_n + K_{n-1}^T r_1. \tag{63}$$

This then results in

$$\dot{r}_1 = \Lambda r_1 + e_{n-1} z, \tag{64}$$

where $e_{n-1} = [0, \ldots, 0, 1]^T \in \mathfrak{R}^{n-1}$. Before proceeding, we note that the linear system (64) is ISS with respect to z, which means that there exist positive constants M_1, δ_1, γ_1 such that

$$|r_1(t)| \le M_1 \mathrm{e}^{-\delta_1(t-t_0)}|r_1(t_0)| + \gamma_1 \|z\|_{[t_0,t]}, \qquad \forall t \in [t_0, t_0 + T). \tag{65}$$

The derivative of the new state (63) is

$$\dot{z} = \dot{w}_n + K_{n-1}^T J_2 w, \tag{66}$$

which, by substitution of $\dot{w}_n = w_{n+1}$ and $\dot{x}_n = x_{n+1}$, yields

$$\dot{z} = \mu(\dot{x}_n + L_0 + L_1) = \mu(bu + f + L_0 + L_1) \tag{67}$$

with

$$L_0 := \sum_{i=1}^{n} \binom{n}{i} \frac{\mu^{(i)}}{\mu} x_{n+1-i}, \quad L_1 := \nu^{n+m} K_{n-1}^T J_2 w, \tag{68}$$

in which the quantity L_0 is expressed in terms of w as $L_0 = \nu^m l_0(\nu) w$, where $l_0(\nu) = [l_{0,1}, l_{0,2}, \dots, l_{0,n}]$, and for $j = 1, 2, \dots, n$, $l_{0,j}(\nu) = \bar{l}_{0,j} \nu^{j-1}$ with

$$\bar{l}_{0,j} = \frac{n+m}{T^{n+1-j}} \sum_{i=0}^{n-j} \binom{n}{n-i-j+1} \binom{i+j-1}{i} \times \frac{(-1)^i (2n+m-i-j)!}{(n+m-i)!}. \tag{69}$$

Furthermore, $l_0(\nu)$ is bounded.

The time-varying gain–based prescribed-time controller for system (53) is designed as

$$u = -\frac{1}{\underline{b}} \left(k_0 + \theta + \lambda \psi(x)^2 \right) z, \tag{70}$$

where z is defined via (63), in which the control gains are chosen such that $\rho, k_0, \lambda > 0$,

$$\rho k_0 > \gamma_1 / 4, \tag{71}$$

where γ_1, defined in (65), depends on the choice of the gain vector K_{n-1} in (62) and (64), and where $\theta \geq \theta_*$ and

$$\theta_* = k_{n-1} + \bar{l}_{0,n} + \rho \max_{\nu \in [0,1]} \left| \left(\nu^n K_{n-1}^T J_2 + l_0(\nu) \right) \left(J_1^T - e_n K_{n-1}^T \right) \right|^2. \tag{72}$$

Observation 6. (See the work of Song et al[30])

Consider system (53) under Assumptions 1 and 2. If controller (70) with (71)-(72) is applied, system (53) is PT-ISS+C and there exist $\check{M}, \check{\delta}, \check{\gamma}$ such that

$$|x(t)| \leq \nu(t-t_0)^{m+1} \left(\check{M} e^{-\check{\delta}(t-t_0)} |x(t_0)| + \check{\gamma} \|d\|_{[t_0,t]} \right) \tag{73}$$

for all $\forall t \in [t_0, t_0 + T)$. Furthermore, control u remains uniformly bounded over $[t_0, t_0 + T)$.

5 | NUMERICAL SIMULATIONS

Simulations on 2 numerical examples are conducted in this section to demonstrate and validate the fast convergence time, good disturbance rejection, and robustness against uncertainties of the time-varying gain–based prescribed-time control scheme.

Example 1. The simulation model in this example is the single integrator system described by (9).

The purpose of the numerical simulation is to compare the convergence performance between the traditional fractional state-feedback–based finite-time control given in (10) with (13) and the time-varying gain–based prescribed-time control given in (12) with (16). To make a fair comparison, the design parameters for the traditional control (10) with (13) are taken as $\varrho_0 = 3.5$, $\alpha = 1/3$; the design parameters for the time-varying gain–based control (12) with (16) are taken as $k_0 = 0.5$, $T = 1$ second, and $m = 2$, and thus, the initial control gain is the same in the 2 control methods.

The simulation results are represented in Figures 2 to 4. From Figures 2 to 4, we see that the convergence time of the state under the traditional finite-time control method depends on the initial state, whereas the convergence time under the time-varying gain–based prescribed-time control method can be prespecified uniformly, independent of any initial condition and other design parameters. In addition, the control signal under the signum function–based control ($\alpha = 0$, SMC) is discontinuous when the state approaches to zero.

Example 2. The simulation model in this example is the first-order system subject to unknown nonvanishing uncertainties descried by (27), in which the lumped uncertainty is taken as $f_d(x,t) = a_0 + a_1 x \cos(t)$ with $a_0 = 1$ and $a_1 = 1$.

The purpose of the numerical simulation is to compare the disturbance rejection and the robustness against the uncertainties between the traditional fractional state–based finite-time control given in (10) with (29) (including the

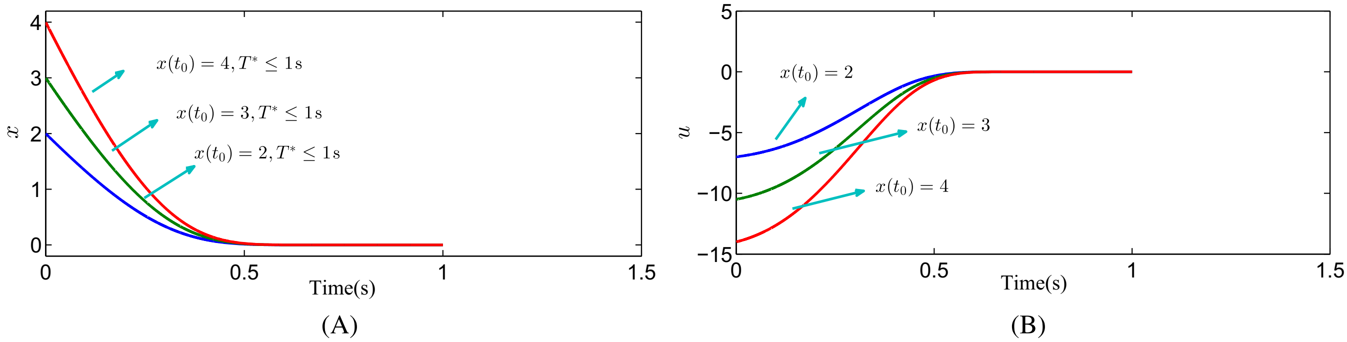

FIGURE 2 Response of the single integrator system with time-varying gain–based prescribed-time control law (12) and (16) under $k_0 = 0.5$, $m = 2$, and $T = 1$ second. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

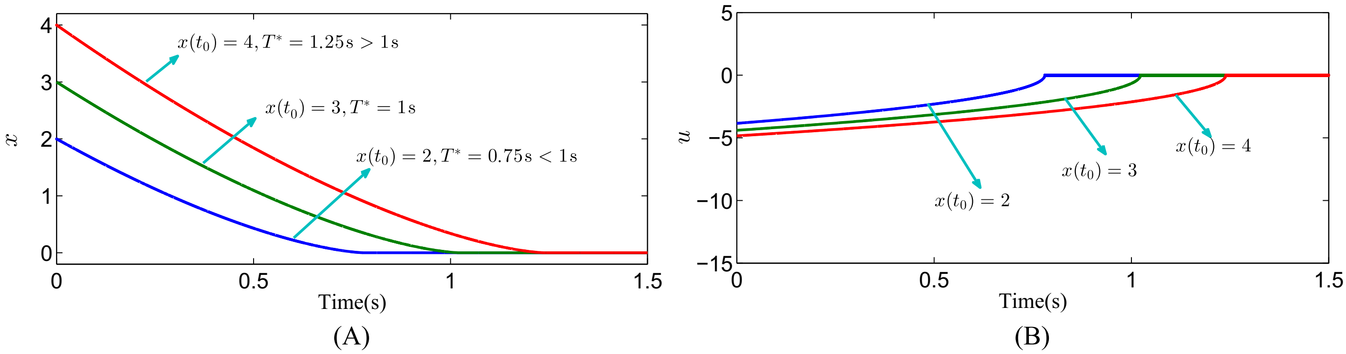

FIGURE 3 Response of the single integrator system with traditional finite-time control law (10) and (13) under $\alpha = 1/3$ and $k_0 = 3.5$. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

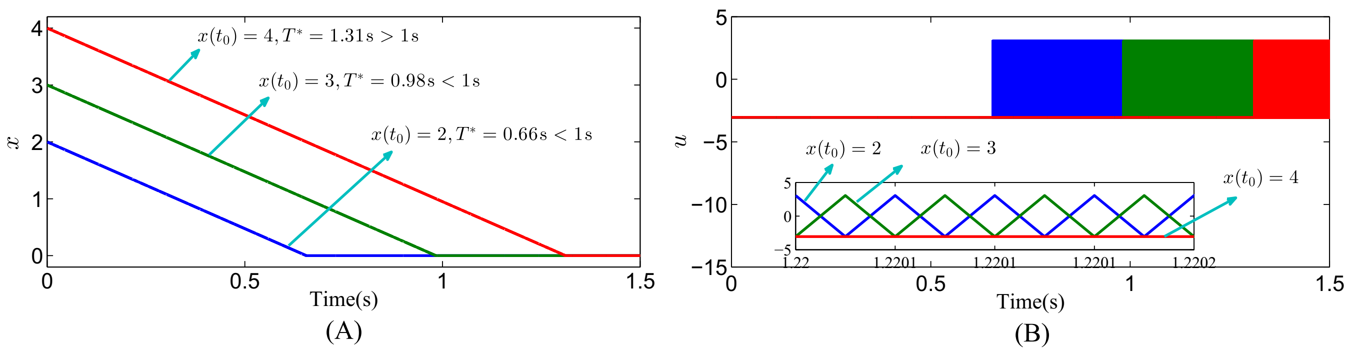

FIGURE 4 Response of the single integrator system with traditional finite-time control law (10) and (13) under $\alpha = 0$ and $k_0 = 3.5$. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

sign function–based control as a specifical case when $\alpha = 0$) and the time-varying gain–based prescribed-time control given in (12) with (45). To make a fair comparison, the design parameters for the traditional control law (10) with (29) are taken as $\varrho_0 = 3.5$, $\alpha = 1/3$, $\gamma = 0.1$, and $\psi = 1 + |x|$ (the sign function–based control is corresponding to $\alpha = 0$); the design parameters for the time-varying gain–based control law (12) with (45) are taken as $k_0 = 0.5$, $\lambda = 0.1$, $\psi = 1 + |x|$, $T = 1$ second, and $m = 2$, and thus, the initial control gain is the same in the 2 control methods.

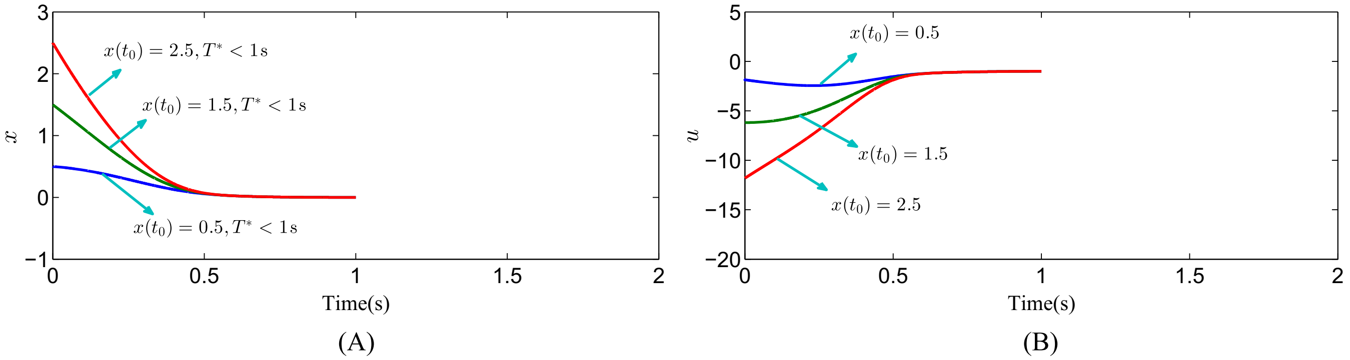

FIGURE 5 Response of first-order system (27) subject to unknown nonvanishing uncertainties with time-varying gain–based prescribed-time control law (12) and (45) under $k_0 = 0.5$, $\lambda = 0.1$, $m = 2$, and $T = 1$ second. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

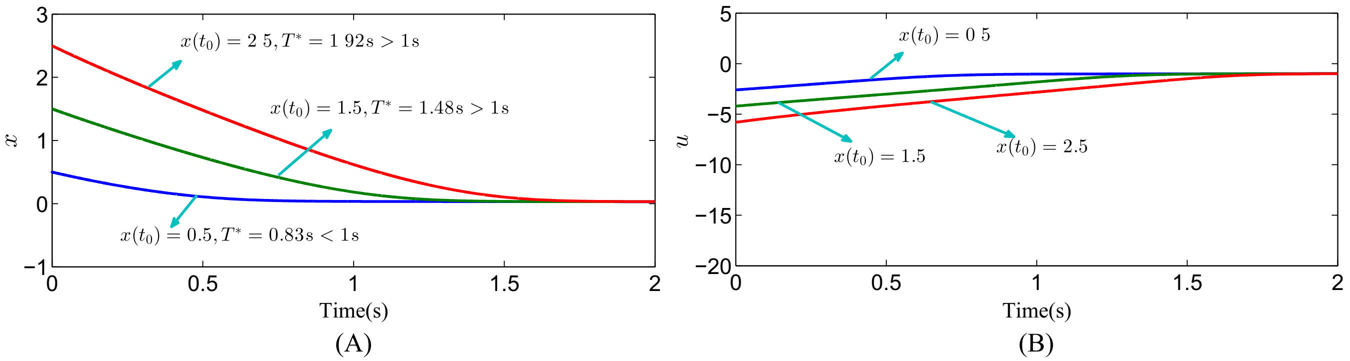

FIGURE 6 Response of first-order system (27) subject to unknown nonvanishing uncertainties with the traditional finite-time control law (10) and (29) under $\alpha = 1/3$, $\varrho_0 = 3.5$, and $\gamma = 0.1$. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

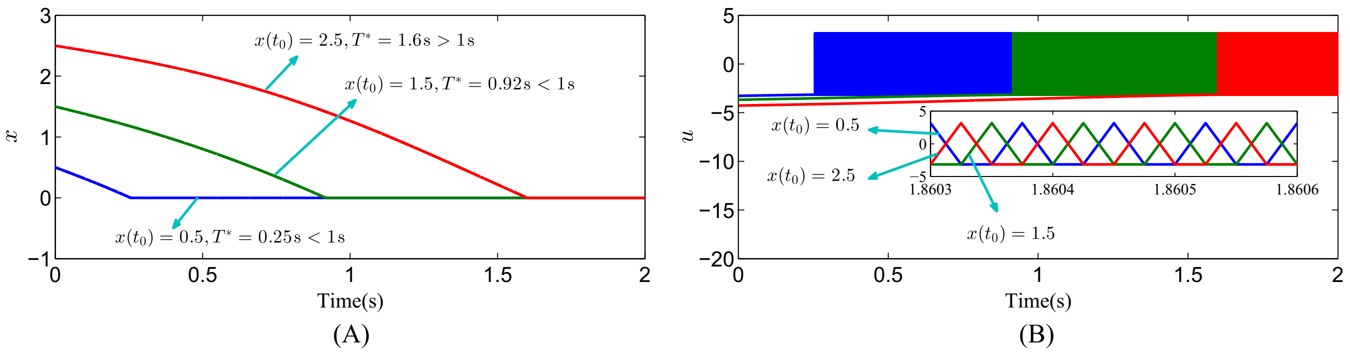

FIGURE 7 Response of first-order system (27) subject to unknown nonvanishing uncertainties with the traditional finite-time control law (10) and (29) under $\alpha = 0$ (SMC), $\varrho_0 = 3.5$, and $\gamma = 0.1$. A, x; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

The simulation results are represented in Figures 5 to 7. The robustness against the uncertainties and the disturbance rejection are reflected by the convergence time in the simulation. From Figures 5 to 7, we see that the convergence time of the state under the traditional finite-time control method does depend on the initial state, whereas the convergence time under the prescribed-time control method can be prespecified uniformly, which does not depend on the initial condition and any other design parameters. In particular, the control signal under the signum function–based control ($\alpha = 0$, SMC) is discontinuous when the state approaches to zero.

Example 3. To further compare the time-varying gain–based prescribed-time control method with the commonly used finite-time control,[19] we conduct simulation on the double integrator system (because the existing method

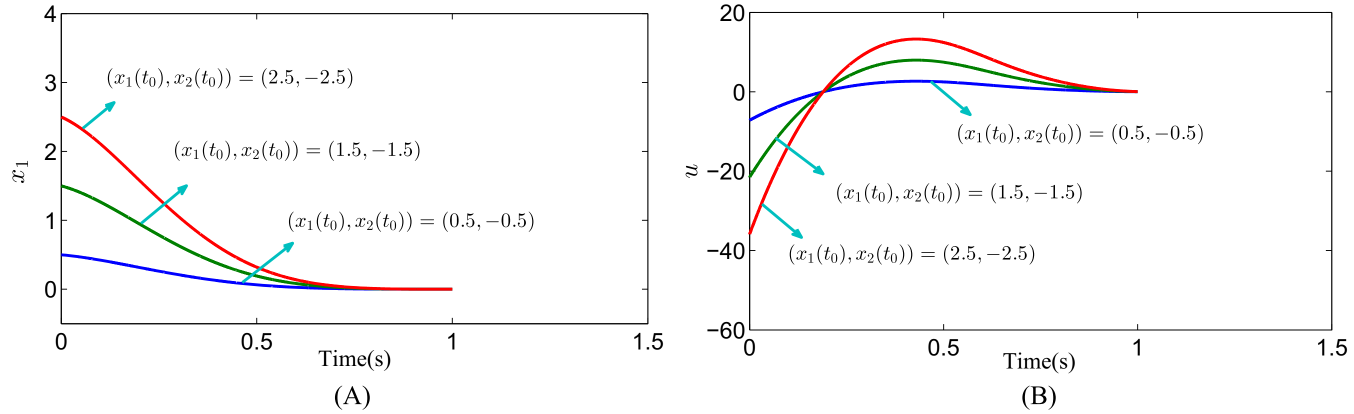

FIGURE 8 Response of the double integrator system with time-varying gain–based prescribed-time control law (74) under $k = 1$, $k_1 = 0.1$, and $T = 1$ second. A, x_1; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

by Shen and Huang[19] is not applicable to systems with time varying and unknown control gain and nonvanishing uncertainty). The prescribed-time controller corresponding to the linear system is of the form

$$u = -\frac{1}{b}(f + L_0 + L_1 + k_0 z), \tag{74}$$

where L_0 and L_1 are defined as in (68), which is used with the design parameters: $k_0 = 1$ and $k_1 = 0.1$, with $T = 1$ second and $m = 2$. According to the algorithm given in the work of Shen and Huang,[19] the corresponding finite-time control law is

$$u = -\xi_2^{q_3}\left[c_2 + l/3\left(1 + \xi_2^2\right)\right], \tag{75}$$

with $\xi_2 = x_2^{1/q_2} - x_2^{*1/q_2}$, $x_2^* = -(c_1 x_1)^{q_2}(1 + l(1 + x_1^2))$, in which the parameters are set as $q_2 = 3/5$, $q_3 = 1/5$, $c_1 = 2.5$, $c_2 = 4.5$, and $l = 0.18$ as in the aforementioned work.[19]

The evolution (versus time) of x_1 of the double integrator under the control of the proposed scheme (74) and the scheme (75) by Shen and Huang[19] is presented in Figure 8A and Figure 9A, respectively. It is seen that our control (74) achieves prescribed-time regulation in $T = 1$ second, whereas the convergence time of the controller by Shen and Huang[19] depends on initial conditions. Besides, from Figure 8B and Figure 9B, it is observed that our scheme demands a lower overall (initial) control effort and exhibits smoother control action, avoiding the sharp transitions (nearly jumps) at sign changes as reflected in Figure 9B.

Remark 2. In all the simulations for time-varying gain–based prescribed-time control above, the terminal time is taken slightly larger than its prescribed value so that, while the state has already been regulated to a very small value (roughly at 10^{-7} in all the 3 examples) at the end of the simulation, the gain is still finite at the prescribed time (here,

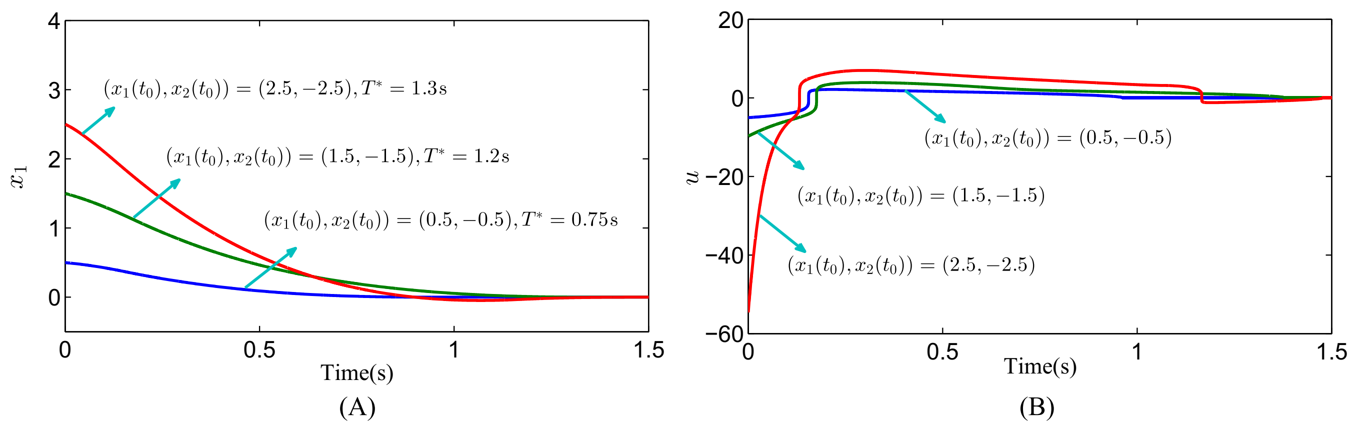

FIGURE 9 Response of the double integrator system with traditional finite-time control law (75) in the work of Shen and Huang[19] under $q_2 = 3/5$, $c_1 = 2.5$, and $c_2 = 4.5$. A, x_1; B, u [Colour figure can be viewed at wileyonlinelibrary.com]

the terminal time of 1.03 seconds is used in the simulation so that the desired result is achieved at the prescribed time of 1 second, and at the same time, the control gain is finite before the terminal time).

Remark 3. It should be mentioned that SMCs also have the property of finite-time,[3,9] fixed-time,[27,28] and the so-called equiuniformly finite-time (see the work of Orlov[10] for the definition) convergence. By combining SMC with fractional power state feedback, predefined-time control can be achieved for a certain class of systems with perturbation.[31-33] However, the control smoothness is always an issue when SMC and fractional state power feedback are utilized. In addition, it is worth noting that for the case when the system is still alive after convergence to the origin the SMC can maintain the solution in the origin. Therefore, the proposed controller can be used to regulate reaching phase of sliding mode controller.

Remark 4. It is interesting to note that the proposed input (11), (12) approaches, in a certain sense, the Dirac control law $x_0\delta(t-t_0)$ as T goes to zero. The latter was established in Orlov's work[39] to impose the perfect regulation on the system in question while minimizing the closed-loop quadratic performance. Moreover, by applying the sequential approach to the Dirac distribution definition, an appropriate approximation of the above Dirac signal is capable of ensuring the prescribed finite-time stability of the closed-loop system. Such an approximation was illustrated by Y. Orlov by means of high gain kx as k escapes to infinity, whereas, in this work, we simply apply the time rescaling to arrive at a time-varying feedback, approaching the delta signal, thereby ensuring the prescribed finite-time stability of the closed-loop system. It is worth mentioning that using the constant high gain all the way from the beginning in control scheme could result in an excessively large initial control demand when the initial state or initial tracking error is large, whereas the proposed control involving time-varying gain does not have such issue.

6 | CONCLUSIONS

This paper focuses on an alternative to finite-time control design for a class of uncertain nonlinear systems, demonstrating the difference between the traditional finite-time control and the newly developed prescribed-time control. It shows that the traditional finite-time control is based on fractional-power state feedback, in which the resultant finite settling time depends on the system initial condition and some other design parameters, whereas the prescribed-time control is based on a regular state feedback with time-varying gain, leading to finite settling time uniform with respect to the system initial condition and any other design parameters.

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