Continuous-Time Penalty Methods for Nash Equilibrium Seeking of a Nonsmooth Generalized Noncooperative Game

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***Abstract*—In this article, we propose centralized and distributed continuous-time penalty methods to find a Nash equilibrium for a generalized noncooperative game with shared inequality and equality constraints and private inequality constraints that depend on the player itself. By using the ℓ_1 penalty function, we prove that the equilibrium of a differential inclusion is a normalized Nash equilibrium of the original generalized noncooperative game, and the centralized differential inclusion exponentially converges to the unique normalized Nash equilibrium of a strongly monotone game. Suppose that the players can communicate with their neighboring players only and the communication topology can be represented by a connected undirected graph. Based on a leader-following consensus scheme and singular perturbation techniques, we propose distributed algorithms by using the exact ℓ_1 penalty function and the continuously differentiable squared ℓ_2 penalty function, respectively. The squared ℓ_2 penalty function method works for games with smooth constraints and the exact ℓ_1 penalty function works for certain scenarios. The proposed two distributed algorithms converge to an η-neighborhood of the unique normalized Nash equilibrium and an η-neighborhood of an approximated Nash equilibrium, respectively, with η being a positive constant. For each $\eta > 0$ and each initial condition, there exists an ε^* such that for each $0 < \varepsilon < \varepsilon^*$, the convergence can be guaranteed where ε is a parameter in the algorithm.**

***Index Terms*—Multiagent system, Nash equilibrium seeking, nonsmooth analysis, penalty methods.**

I. Introduction

A. Literature Review

1) Generalized Nash Equilibrium Problem (GNEP): Noncooperative games, as one of the most important branches of game theory, are widely applied in modeling and decision making of demand response and road pricing, etc. Nash equilibrium is a critical concept in noncooperative game theory, which refers to a state where no player can improve its utility by changing its own strategy, provided that the other players maintain their current strategies.

Compared with conventional unconstrained Nash games, a generalized noncooperative game assumes that each player's feasible set can be constrained by the rival players' strategies [1], i.e., a player's objective function and constraints might be coupled with other players' actions. There are many applications that can be modeled as a generalized noncooperative game, e.g., market liberalization of electricity, femto-cell power allocation, and environmental pollution control [2].

In multiagent networks, the agent might have limited communication with other agents [3]. Recently, distributed computation algorithms for optimization (e.g., [4], [5]) and game problems (e.g., [6]–[10], [12]–[27]) have attracted much attention. Among these relevant works, in [8], [9], [16], [19]–[21], and [23]–[25], discrete-time algorithms for distributed GNEPs were investigated and in [13] and [22], continuous ones were studied. Most of these works use the Lagrange multiplier methods. For example, in [19] and [20], Lagrange multiplier methods were proposed for solving distributed GNEPs using interference graphs. The works in [16] and [27] solved a class of GNEPs with partial decision information.

2) Methods for Solving GNEPs and Penalty Methods: In centralized GNEP studies, various methods have been developed to study how to seek a Nash equilibrium, including VI methods [28], NI-function methods [29], penalty methods [30], [31], Newton methods [32], ODE methods [33], etc. Most of the algorithms work only for shared constraint sets [30]. Penalty methods are among the few methods that could be used to solve GNEPs with nonshared constraints [2], [30], [31]. Hence, they hold the potential for interesting developments [2].

B. Motivation and Contribution

This article studies continuous-time penalty methods for solving GNEPs. Motivated by the works in [6], [30], [31], and [34]–[36], we propose continuous-time algorithms for noncooperative games with nonsmooth objective functions and nonsmooth constraints using ℓ_1 penalty based methods. The main motivation of using penalty based methods is to remove the requirement of estimating the multiplier in a distributed Nash equilibrium problem. The nondifferentiable ℓ_1 penalty function is selected since the error between the Nash equilibrium of the ℓ_1 penalized game and that of the original game can be zero, and thus the algorithm using ℓ_1 penalty function may result in exact convergence.

The contributions of this article are summarized as follows.

1) According to the authors' knowledge, this is the first article to study continuous-time penalty methods. Among the aforementioned works, [25], [30], and [31] are directly related to this work while all the three works proposed discrete-time penalty methods for solving GNEPs. Several more main differences from these works are summarized as follows.
   a) The algorithms in [30] and [31] are centralized and the algorithm in [25] is based on interference graphs, which is a different problem setting with this article.
   b) Smoothness of the games is an important assumption in [25], [30], and [31], especially for the constraints in the feasibility proof in [30] and [31]. While in this article, nonsmooth games are studied.
2) Benefiting from the penalty methods, the proposed algorithms do not need the evolution of multipliers, which reduces the number of

Manuscript received August 5, 2020; accepted November 21, 2020. Date of publication November 24, 2020; date of current version September 27, 2021. This work was supported in part by the Singapore Ministry of Education Academic Research Fund Tier 1 RG180/17(2017-T1-002-158) and in part by the Wallenberg-NTU Presidential Postdoctoral Fellow Set-Up Grant. This paper was presented in part at the 15th International Conference on Information and Automation, Wuyi Mountain, Fujian, China, August 2018. Recommended by Associate Editor S. Grammatico. *(Corresponding author: Guoqiang Hu.)*

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Color versions of one or more of the figures in this article are available online at https://doi.org/10.1109/TAC.2020.3040377.
Digital Object Identifier 10.1109/TAC.2020.3040377

the communicated variables for distributed implementation and the complexity of the algorithm. From this perspective, this method has its advantage when there exists a large number of constraints in the network, compared with Lagrange multiplier methods in [8], [9], [13], [16], [19], and [20]–[24]. Another advantage of the penalty methods is that they have the potential to be applied to solve GNEPs with general nonshared constraints [2], [30], [31].

3) Most of the existing distributed GNEP studies are based on discrete-time algorithms. This article provides a continuous-time solution based on control theory and nonsmooth analysis. The proposed method is different from the existing multiplier-based continuous-time algorithms in [13] and [22].

II. Notations and Preliminaries

Notations: Throughout this article, $\mathbb{R}, \mathbb{R}^{\geq 0}, \mathbb{R}^N$, and $\mathbb{R}^{N\times N}$ represent the set of reals, nonnegative reals, N-dimensional real column vectors, and $N\times N$ real matrices, respectively. $\mathbf{1}$ and $\mathbf{0}$ are column vectors with all elements being 1 and 0, respectively. For scalars $a_i, i=1,\ldots,N$, $[a_i]_N \in \mathbb{R}^N$ represents a column vector defined as $[a_i]_N = [a_1,\cdots,a_N]^T$. For sets $a_i, i=1,\cdots,N$, $[a_i]_N$ represents a set defined as $[a_i]_N = \{[e_1,\cdots,e_N]^T | e_i \in a_i, i=1,\cdots,N\}$. $|\cdot|$ and $\|\cdot\|$ denote the absolute value and the 2-norm, respectively. For any $a\in\mathbb{R}$, $[a]_+ \triangleq \max\{0,a\}$.

For a convex function $f(x): \mathbb{R}^N \to \mathbb{R}$, the (total) subdifferential of f at $x_0 \in \mathbb{R}^N$ is the set of vectors s satisfying $f(y) \geq f(x_0) + \langle s, y-x_0\rangle$ for all $y\in\mathbb{R}^N$, denoted by $\partial_x f(x_0)$. Any element $s\in\partial_x f(x_0)$ is called a subgradient of f at x_0. v_i is called a partial subgradient of f at $z=[z_1,\cdots,z_n]^T$ with respect to x_i, if $f(z_1,\cdots,z_{i-1},x_i,z_{i+1},\cdots,z_n) \geq f(z) + \langle v_i, x_i - z_i\rangle$ for all $x_i \in \mathbb{R}$. The set of the partial subgradients is called the partial subdifferential, denoted by $\partial_{x_i} f(z)$.

A set-valued map $M(x): \mathbb{R}^N \to \mathbb{R}^N$ is said to be monotone on $\mathbb{R}^N$ if for all $x, \breve{x} \in \mathbb{R}^N$, $x \neq \breve{x}$, $g(x)\in M(x)$, $g(\breve{x}) \in M(\breve{x})$, and $(g(x)-g(\breve{x}))^T(x-\breve{x}) \geq 0$. A set-valued map $M(x): \mathbb{R}^N \to \mathbb{R}^N$ is said to be strongly monotone on $\mathbb{R}^N$ with modulus $\rho > 0$ if for all $x,\breve{x}\in\mathbb{R}^N$, $g(x)\in M(x)$, $g(\breve{x})\in M(\breve{x})$, and $(g(x)-g(\breve{x}))^T(x-\breve{x}) \geq \rho\|x-\breve{x}\|^2$.

III. Problem Formulation

Consider a set of players $\mathcal{V} \triangleq \{1,\cdots,N\}$ where the action of player i is denoted by $x_i\in\mathbb{R}$ and the action vector of i's rival players is denoted by $x_{-i} \in \mathbb{R}^{N-1}$. Each player i minimizes its objective function $f_i(x): \mathbb{R}^N \to \mathbb{R}$ where $x = [x_1,\cdots,x_N]^T \in \mathbb{R}^{\mathbb{N}}$ is the action vector of N players.

The game considered in this article can be described as

$$\min_{x_i} f_i(x), \text{ such that } x_i \in \mathcal{I}_i(x_{-i}) \cap \mathcal{H}_i(x_{-i}) \tag{1}$$

where $\mathcal{I}_i(x_{-i}) \triangleq \{x_i\in\mathbb{R} | g_{ik}(x_i)\leq 0, k\in\{1,\cdots,m_i\}, g_l(x)\leq 0, l\in\{1,\cdots,n_g\}\}$ and $\mathcal{H}_i(x_{-i}) \triangleq \{x_i\in\mathbb{R} | h_s(x) = 0, s\in\{1,\cdots,n_h\}\}$ denote the inequality set and the equality set for player i, where m_i, n_g, and n_h are the numbers of the private constraints of player i, the shared inequality constraints, and the shared equality constraints, respectively.

The following assumptions are made to facilitate the subsequent analysis.

Assumption 1: The objective function $f_i(x)$ is convex in x_i for all $x_{-i}\in\mathbb{R}^{N-1}$.

Assumption 2: First, $g_{ik}(x_i), k\in\{1,\cdots,m_i\}$ is convex for x_i in $\mathbb{R}$, $i\in\mathcal{V}$. $g_l(x), l\in\{1,\cdots,n_g\}$ is convex for x in $\mathbb{R}^N$. $h_s(x)=0, s\in\{1,\cdots,n_h\}$ is a smooth and affine function. Second, the set $\Omega \triangleq \{x\in\mathbb{R}^N | g_{ik}(x_i)\leq 0, k\in\{1,\cdots,m_i\}, i\in\mathcal{V}, g_l(x)\leq 0, l\in\{1,\cdots,n_g\}, h_s(x)=0, s\in\{1,\cdots,n_h\}\}$ is nonvoid, convex, and compact.

Assumption 3: The map $F(x) \triangleq [\partial_{x_i} f_i(x)]_N$ is strongly monotone in $\mathbb{R}^N$ with modulus ρ.

Remark 1: The functions $f_i(x)$, $g_{ik}(x_i)$, and $g_l(x)$ can be nonsmooth.

IV. Centralized Approaches and Convergence

A. Dynamical Systems by Using ℓ_1 Penalty Functions

The ℓ_1 penalty functions are a class of important exact penalty functions in optimization. However, for GNEPs, the ℓ_1 penalty functions are not necessarily exact [37]. A Nash equilibrium of the unconstrained game after penalization is not necessarily a feasible solution to the original GNEP.

Using ℓ_1 penalty functions, the unconstrained game can be described as

$$\min_{x_i} f_i(x) + \alpha\left(\sum_{l=1}^{n_g}[g_l(x)]_+ + \sum_{s=1}^{n_h}|h_s(x)| + \sum_{k=1}^{m_i}[g_{ik}(x_i)]_+\right) \tag{2}$$

where $\alpha > 0$ is a fixed penalty parameter.

Motivated by the gradient descent dynamics in optimization, we propose the following centralized algorithm for player i:

$$\dot{x}_i = -r_i(x) - \alpha\varphi_i(x) \tag{3}$$

where $r_i(x)\in\partial_{x_i} f_i(x)$ is a partial subgradient of f_i at x, $\varphi_i(x) \in \sum_{l=1}^{n_g}\partial_{x_i}[g_l(x)]_+ + \sum_{s=1}^{n_h}\partial_{x_i}|h_s(x)| + \sum_{k=1}^{m_i}\partial_{x_i}[g_{ik}(x_i)]_+$, which is the set sum of the partial subdifferential of the penalty functions.

Moreover, $\varphi_i(x)$ is selected to satisfy that for all x

$$\phi(x) \triangleq [\varphi_i(x)]_N \in \Phi(x) \tag{4}$$

where $\Phi(x) \triangleq \Phi_a(x) + \Phi_b(x)$, $\Phi_a(x) = \sum_{l=1}^{n_g}\partial_x[g_l(x)]_+ + \sum_{s=1}^{n_h}\partial_x|h_s(x)|$, and $\Phi_b(x) = [\sum_{k=1}^{m_i}\partial_{x_i}[g_{ik}(x_i)]_+]_N$.

Example 1: For simplicity, consider a game with a shared constraint $g_1(x) = x_1+2x_2-1$ only. In this case

$$\partial_{x_1}[g_1(x)]_+ = \begin{cases}\{1\}, & x_1+2x_2-1>0\\ \{0\}, & x_1+2x_2-1<0\\ \{z\in\mathbb{R}|0\leq z\leq 1\}, & x_1+2x_2-1=0\end{cases}$$

$$\partial_{x_2}[g_1(x)]_+ = \begin{cases}\{2\}, & x_1+2x_2-1>0\\ \{0\}, & x_1+2x_2-1<0\\ \{z\in\mathbb{R}|0\leq z\leq 2\}, & x_1+2x_2-1=0\end{cases}$$

$$\partial_{x}[g_1(x)]_+ = \begin{cases}\{[1,2]^T\}, & x_1+2x_2-1>0\\ \{[0,0]^T\}, & x_1+2x_2-1<0\\ \{z*[1,2]^T|0\leq z\leq 1\}, & x_1+2x_2-1=0.\end{cases}$$

Then, $\phi_1(x)$ can be chosen as $\phi_1(x)=1$ if $x_1+2x_2-1>0$, $\phi_1(x)=0$ if $x_1+2x_2-1<0$, and $\phi_1(x)=1$ if $x_1+2x_2-1=0$. $\phi_2(x)$ can be chosen as $\phi_2(x)=2$ if $x_1+2x_2-1>0$, $\phi_2(x)=0$ if $x_1+2x_2-1<0$, and $\phi_2(x)=2$ if $x_1+2x_2-1=0$. Note that when $x_1+2x_2-1=0$, if $\phi_1(x)=1$, $\phi_2(x)$ can only be 2. The functions $\phi_1(x)$ and $\phi_2(x)$ can be set when initializing the algorithm.

The concatenated form of (3) and (4) is

$$\dot{x} = -r(x) - \alpha\phi(x) \tag{5}$$

where $r(x) = [r_i(x)]_N$.

According to [38], the Filippov differential inclusion to the discontinuous dynamical system (5) can be written as

$$\dot{x} \in -F(x) - \alpha\Phi(x). \tag{6}$$

Based on Assumptions 1 and 2, the right-hand side of (6) is upper semicontinuous [39]. Furthermore, for each x, it is a nonempty, locally bounded, compact, and convex set. Then, a solution to differential inclusion (6) exists.

Next, we analyze the relationship among the equilibrium of differential inclusion (6), Nash equilibrium of GNEP (1), and Nash equilibrium of unconstrained game (2).

B. Exactness of ℓ_1 Penalty Functions for the Normalized Nash Equilibrium

Let $\mathcal{K}$ be a nonempty convex subset of $\mathbb{R}^N$ and $\mathcal{F}(x)$ is a set-valued map with nonempty, compact, and convex values. The generalized variational inequality problem (GVIP) ([40, p. 228]) is to find $\tilde{x} \in \mathcal{K}$ and $\tilde{u} \in \mathcal{F}(x)$ such that for each $z \in \mathcal{K}$, $\langle \tilde{u}, z - \tilde{x} \rangle \geq 0$.

If $\mathcal{K} = \mathbb{R}^N$, then the aforementioned GVIP is equivalent to the following set-valued differential inclusion problem (SDIP) [40]: to find $\tilde{x} \in \mathbb{R}^N$ such that $\mathbf{0} \in \mathcal{F}(\tilde{x})$.

The equilibrium of (6) is a solution to the following SDIP: to find $\tilde{x} \in \mathbb{R}^N$ such that

$$\mathbf{0} \in F(\tilde{x}) + \alpha\Phi(\tilde{x}). \tag{7}$$

Remark 2: For convex optimization problems, if we replace $F(\tilde{x})$ with the subdifferential of the objective function, then (7) is a necessary and sufficient condition for $\tilde{x}$ being an optimum of the unconstrained optimization problem ([41, Th. 2.2.1]). However, for GNEPs, (7) is only sufficient. The condition is sufficient because $\mathbf{0} \in F(\tilde{x}) + \alpha\Phi(\tilde{x}) \subseteq F(\tilde{x}) + \alpha\Phi_{\text{vec}}(\tilde{x})$, where $\Phi_{\text{vec}}(x) \triangleq [\sum_{l=1}^{n_g} \partial_{x_i}[g_l(x)]_+]_N + [\sum_{s=1}^{n_h} \partial_{x_i}|h_s(x)|]_N + \Phi_b(x)$ while $\mathbf{0} \in F(\tilde{x}) + \alpha\Phi_{\text{vec}}(\tilde{x})$ is a necessary and sufficient condition according to the definition of Nash equilibrium. Since $\Phi_{\text{vec}}(x)$ is not necessarily monotone, it is difficult to design an algorithm according to $\Phi_{\text{vec}}(x)$. This is the reason why (4) is required. In addition, if $\tilde{x}$ is a Nash equilibrium of the unconstrained game (2), $\tilde{x}$ is not necessarily a Nash equilibrium of GNEP (1) since the feasibility need to be guaranteed.

In the following analysis, we prove that under some conditions, a solution to SDIP (7) is a normalized Nash equilibrium of GNEP (1).

Consider the following two GVIPs.

1) Find $\tilde{x}_a \in \mathbb{R}^N$ and $\tilde{u}_a \in F(\tilde{x}_a) + \alpha\Phi(\tilde{x}_a)$ such that for each $z \in \mathbb{R}^n$

$$\langle \tilde{u}_a, z - \tilde{x}_a \rangle \geq 0. \tag{8}$$

2) Find $\tilde{x}_b \in \Omega$ and $\tilde{u}_b \in F(\tilde{x}_b)$ such that for each $z \in \Omega$

$$\langle \tilde{u}_b, z - \tilde{x}_b \rangle \geq 0. \tag{9}$$

As mentioned previously, SDIP (7) is equivalent to GVIP (8). [13, Lemma 2.4] shows that any solution $\tilde{x}_b$ to GVIP (9) is a normalized Nash equilibrium of GNEP (1), under the strong Slater condition [41], which links GVIP (9) with GNEP (1). Thus, we need to find the relationship between the solution to GVIP (8) and the solution to GVIP (9).

Since the subdifferential of any convex function defined in $\mathbb{R}^N$ is monotone, we have the following conclusion.

Lemma 1: Under Assumption 2, the set-valued map $\Phi(x)$ is monotone in $\mathbb{R}^{\mathbb{N}}$.

The following lemma shows the existence and uniqueness of the solution to GVIP (8) and the solution to GVIP (9).

Lemma 2: Under Assumptions 1–3, a solution to GVIP (9) exists and is unique. For a given α, a solution to GVIP (8) exists and is unique.

Proof: According to Assumption 1, $\partial_{x_i} f_i(x)$ is an upper semicontinuous map for x in $\mathbb{R}^N$. Furthermore, for each x, $\partial_{x_i} f_i(x)$ is a nonempty compact convex subset of $\mathbb{R}$. According to the convexity of the constraints and Assumption 3, the existence and uniqueness can be obtained by directly using [42, Th. 4.3]. ■

Then, we can get the following conclusion.

Lemma 3: Under Assumptions 1–3 and supposing that the strong Slater condition holds for Ω, there exists $\alpha^* > 0$ such that if $\alpha > \alpha^*$, the unique solution to SDIP (7) is a normalized Nash equilibrium of GNEP (1).

Proof: See Appendix A. ■

Remark 3: Under the conditions in Lemma 3, it can be proven that the normalized Nash equilibrium of GNEP (1) is unique for the nonsmooth games by using the subdifferential and following [1, Th. 2], where smooth games are considered. We omit it here due to the space limit.

Remark 4: The exactness of ℓ_1 penalty function for the normalized Nash equilibrium under the strong Slater condition was recently studied by [43]. Here, we provide a different idea of proof. In addition, the smoothness of constraints is an important assumption in [43] (for example, for formula (7) in [43]), whereas in our article, this is not needed.

C. Main Result

Theorem 1: Suppose that Assumptions 1–3 hold, the strong Slater condition holds for Ω. Then, there exists a positive constant α^*, such that for all $\alpha > \alpha^*$, the Filippov solutions to (3) and (4) exponentially converge to the unique normalized Nash equilibrium of GNEP (1).

Proof: See Appendix B. ■

D. Approximation by Squared ℓ_2 Penalty Functions

In this section, we use the continuously differentiable squared ℓ_2 penalty function instead of the nondifferentiable ℓ_1 penalty function to obtain an approximated solution.

Suppose that the constraints in GNEP (1) is smooth. Consider the following dynamical system:

$$\dot{x}_i = -r_i(x) - \alpha_p\tilde{\varphi}_i(x) \tag{10}$$

where $\tilde{\varphi}_i(x) = \sum_{l=1}^{n_g} \nabla_{x_i}([g_l(x)]_+)^2 + \sum_{s=1}^{n_h} \nabla_{x_i}(h_s(x))^2 + \sum_{k=1}^{m_i} \nabla_{x_i}([g_{ik}(x_i)]_+)^2$, and α_p is a fixed penalty parameter.

Note that different from (3), condition (4) is automatically satisfied for (10) since for a continuously differentiable convex function, the Cartesian product of the partial gradients coincides with the total gradient.

The Filippov differential inclusion to the discontinuous dynamical system (10) is

$$\dot{x} \in -F(x) - \alpha_p[\tilde{\varphi}_i(x)]_N. \tag{11}$$

Since $([g_l(x)]_+)^2$, $(h_s(x))^2$, and $([g_{ik}(x_i)]_+)^2$ are once continuously differentiable and convex, $[\tilde{\varphi}_i(x)]_N$ is a monotone single-valued map. The equilibrium $\tilde{x}_{\alpha_p}$ of (11) satisfies

$$\mathbf{0} \in F(\tilde{x}_{\alpha_p}) + \alpha_p[\tilde{\varphi}_i(\tilde{x}_{\alpha_p})]_N. \tag{12}$$

The existence and uniqueness of $\tilde{x}_{\alpha_p}$ for each fixed α_p can be guaranteed by [42, Th. 4.3].

Lemma 4: Let $\tilde{x}_{\alpha_p}, \alpha_p \to \infty, p = 1, \cdots, \infty$ be a sequence generated by the solution to (12). Then, any limit point of the sequence is a Nash equilibrium of GNEP (1).

Algorithm 1:

Initialization: Each player $i \in \mathcal{V}$ selects the initial values $x_i(0) \in \mathbb{R}$ and $y_{ij}(0) \in \mathbb{R}$, $j \in \mathcal{V}\backslash\{i\}$; Initialize the function $\varphi_i(x)$.

Information Exchange: Each player $i \in \mathcal{V}$ broadcasts $x_i(t)$ and $y_{ij}(t)$, $j \in \mathcal{V}\backslash\{i\}$ to its neighbors and receives the information from neighbors.

Run: Run the dynamical system in (13) until the expected performance is achieved.

Proof: See Appendix C. ■

Theorem 2: Suppose that Assumptions 1–3 hold, and the constraints are smooth. Then, for each fixed $\alpha_p > 0$, the Filippov solutions to (10) exponentially converge to an approximated Nash equilibrium of GNEP (1) in the sense of Lemma 4.

Proof: The proof is similar to Theorem 1 and is thus omitted. ■

V. Distributed Approaches and Convergence

In this section, we investigate the distributed implementation of the penalty methods proposed in Section IV. The idea is motivated by our previous work [6], where leader-following consensus is used to estimate unknown information and singular perturbation is used to analyze the convergence of the whole system. Due to the nonsmoothness of the system, the techniques used in [6] cannot be directly applied to this article. We discuss the algorithm design for two cases in terms of the availability of $\varphi_i(x)$.

A. $\varphi_i(x)$ Is Available

Let $y_i = [y_{i1}, \cdots, y_{iN}]^T$ be player i's estimation on all the players and $y_{ii} = x_i$. Consider the following continuous-time dynamical system:

$$\begin{aligned}\dot{x}_i &= -\bar{k}_i(r_i(y_i) + \alpha\varphi_i(x))\\ \dot{y}_{ij} &= -w_{ij}\sum_{k=1}^{N} a_{ik}(y_{ij} - y_{kj}), j \neq i\end{aligned} \tag{13}$$

where $\bar{k}_i = \varepsilon k_i > 0$, $\varepsilon > 0$ is a small parameter that will be determined later, k and w_{ij} are positive constants, and $\varphi_i(x)$ is required to satisfy (4). Algorithm 1 describes the exchanged information among the players.

Remark 5: A difficulty when applying the exact penalty method in (3) is that if $\varphi_i(y_i)$ is used instead of $\varphi_i(x)$, (4) is difficult to be satisfied. In this section, we suppose that $\varphi_i(x)$ is available. Specifically, it may happen in the following scenarios.

1) The shared constraints are equal to zero.
2) The shared constraints are related to neighboring players only.
3) The shared constraints are smooth affine functions, and the players know the signs of the shared constraints at any time playing the game.

Specifically, we discuss the scenarios 2) and 3). For 2), a typical example is a constraint defined on an interference graph [19] and [25]. For example, in a six-player game, let $x_1 + x_2 + x_3 \leq 2$ being a shared constraint. Then, players 1, 2, and 3 have to be neighboring to each other. Different from the work in [19] and [25], we do not require the objective functions to be related to neighboring players only. For 3), from Example 1, it can be seen that if the signs of the constraints are known, then $\varphi_1(x)$ and $\varphi_2(x)$ are known and do not depend on the real-time value of x.

For agent i, define a subgraph $\mathcal{G}_i = \{\mathcal{V}_i, \mathcal{E}_i\}$, where $\mathcal{V}_i = \mathcal{V}/\{i\}$ and $\mathcal{E}_i \subset \mathcal{V}_i \times \mathcal{V}_i$ denote the set of vertices and edges, respectively. Let L_i be the Laplacian matrix of $\mathcal{G}_i$ and $B_i = \mathrm{diag}\{a_{1i}, \cdots, a_{(i-1)i}, a_{(i+1)i}, \cdots, a_{Ni}\} \in \mathbb{R}^{(N-1)\times(N-1)}$.

Let $\bar{y}_i = [y_{1i}, \cdots, y_{(i-1)i}, y_{(i+1)i}, \cdots, y_{Ni}]^T \in \mathbb{R}^{N-1}$ and $W_i = \mathrm{diag}\{w_{1i}, \cdots, w_{(i-1)i}, w_{(i+1)i}, \cdots, w_{Ni}\} \in \mathbb{R}^{(N-1)\times(N-1)}$. Define $y = [y_i]_N \in \mathbb{R}^{N^2}$, $r(y) = [r_i(y_i)]_N$, and $\bar{k} = \mathrm{diag}\{\bar{k}_1, \cdots, \bar{k}_N\}$. Then

$$\begin{aligned}\dot{x} &= -\bar{k}(r(y) + \alpha\phi(x))\\ \dot{\bar{y}}_i &= -W_i(L_i + B_i)(\bar{y}_i - x_i\mathbf{1}), i \in \mathcal{V}.\end{aligned} \tag{14}$$

The Filippov differential inclusion to the discontinuous dynamical system (4) and (14) can be written as

$$\begin{aligned}\dot{x} &\in -\bar{k}(\tilde{F}(y) + \alpha\Phi(x))\\ \dot{\bar{y}}_i &= -W_i(L_i + B_i)(\bar{y}_i - x_i\mathbf{1}), i \in \mathcal{V}\end{aligned} \tag{15}$$

where $\tilde{F}(y) = [\partial_{x_i} f_i(y_i)]_N$.

Denote $\tilde{y}_i = [y_{1i} - x_i, \cdots, y_{(i-1)i} - x_i, y_{(i+1)i} - x_i, \cdots, y_{Ni} - x_i]^T \in \mathbb{R}^{N-1}$ and $\tilde{Y} = [\tilde{y}_i]_N \in \mathbb{R}^{N(N-1)}$. Let $k_{\min} = \min_{i\in\mathcal{V}}\{k_i\}$ and $k_{\max} = \max_{i\in\mathcal{V}}\{k_i\}$. The main result of this section can be summarized as follows.

Theorem 3: Suppose that Assumptions 1–3 hold, the strong Slater condition holds for Ω. If $k_{\min} \geq \frac{1}{2\rho}$, then for any positive constant η, there exists an ε^* dependent on η such that for all $0 < \varepsilon < \varepsilon^*$, the Filippov solutions to the distributed seeking law in (13) and (4) guarantee that $[x^T, \tilde{Y}^T]^T$ converges to an η-neighborhood of $[x^{*T}, \mathbf{0}^T]^T$ with x^* being the unique normalized Nash equilibrium of GNEP (1).

Proof: See Appendix D.

Remark 6: Theorem 3 proves an existence result, in terms of α^* and ε^*. To compute α^*, *a priori* knowledge of an upper bound of the multipliers is needed. In practice, one may select a sufficiently large α and a sufficiently small ε.

Remark 7: For the computation of a Filippov solution, usually a numerical scheme is selected instead of finding the explicit expression. The Filippov system in (15) can be numerically solved in a distributed way by using event-based methods (e.g., [44]). The difference with numerically solving differential equations is that an event function relying on the discontinuous point is required, which is known by each related player according to the availability of $\varphi_i(x)$.

B. $\varphi_i(x)$ Is Unavailable

We further study the general case where $\varphi_i(x)$ is unavailable. Suppose that the constraints in GNEP (1) is smooth.

The distributed dynamical system is designed as

$$\begin{aligned}\dot{x}_i &= -\bar{k}_i(r_i(y_i) + \alpha_p\tilde{\varphi}_i(y_i))\\ \dot{y}_{ij} &= -w_{ij}\sum_{k=1}^{N} a_{ik}(y_{ij} - y_{kj}), j \neq i\end{aligned} \tag{16}$$

where $\tilde{\varphi}_i$ was defined in (10).

Let $\tilde{\Phi}(y) \triangleq [\tilde{\varphi}_i(y_i)]_N$. Then, the following Filippov differential inclusion can be obtained:

$$\begin{aligned}\dot{x} &\in -\bar{k}(\tilde{F}(y) + \alpha_p\tilde{\Phi}(y))\\ \dot{\bar{y}}_i &= -W_i(L_i + B_i)(\bar{y}_i - x_i\mathbf{1}), i \in \mathcal{V}.\end{aligned} \tag{17}$$

Remark 8: In (17), when $y_i \to x$, $\tilde{\Phi}(y) \to [\tilde{\varphi}_i(x)]_N$ according to the upper semicontinuity. For (14), consider to replace $\phi(x)$ by $[\varphi_i(y_i)]_N$. Then, in (15), $\Phi(x)$ will become $\tilde{\Phi}_{\mathrm{vec}}(y) \triangleq [\sum_{l=1}^{n_g} \partial_{x_i}[g_l(y_i)]_+]_N + [\sum_{s=1}^{n_h} \partial_{x_i}|h_s(y_i)|]_N + \Phi_b(x)$. When $y_i \to$

TABLE I
CONSTANTS FOR THE RIVER BASIN POLLUTION GAME

Player i	1	2	3	4	5	6
e_{1i}	0.56	1.37	1.75	1	1.5	2
e_{2i}	0.075	0.15	0.2	0.1	0.2	0.3
v_i	0.091	0.161	0.221	0.1	0.242	0.385

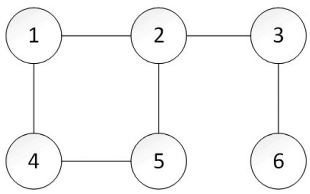

Fig. 1. Communication graph for the six players.

x, $\tilde{\Phi}_{\text{vec}}(y) \rightarrow \Phi_{\text{vec}}(x)$ with $\Phi_{\text{vec}}(x)$ defined in Remark 2, whereas generally, $\Phi_{\text{vec}}(x) \neq \Phi(x)$ for nondifferentiable penalty functions. This is the reason why for (13), we use $\varphi_i(x)$ rather than $\varphi_i(y_i)$.

Theorem 4: Suppose that Assumptions 1–3 hold and the constraints are smooth. If $k_{\min} \geq \frac{1}{2\rho}$, then for any positive constant η, there exists an ε^* dependent on η such that for all $0 < \varepsilon < \varepsilon^*$, the Filippov solutions to the distributed seeking law in (16) guarantees that $[x^T, \tilde{Y}^T]^T$ converges to an η-neighborhood of $[\tilde{x}_{\alpha_p}^T, \mathbf{0}^T]^T$ with $\tilde{x}_{\alpha_p}$ being an approximated Nash equilibrium of GNEP (1) in the sense of Lemma 4.

Proof: The proof is similar to Theorem 3 and is omitted. ■

VI. SIMULATION

We consider a six-player river basin pollution game [44] described as follows.

1) Six players are located along a river and each player is engaged in an economic activity with action x_i. The pollutant of player i is related to its action x_i with coefficient v_i.
2) The profit function for player i is $p_i(x) = -f_i(x) = \mathcal{R}_i(x) - \mathcal{T}_i(x)$ where $\mathcal{R}_i(x) = (d_1 - d_2(\sum_{j=1}^{6} x_j))x_i$ represents the revenue and $\mathcal{T}_i(x) = \begin{cases} (e_{1i} + e_{2i}x_i)x_i, & \text{if } x_i \leq 1 \\ (2e_{1i} + e_{2i}x_i)x_i - 5e_{1i}, & \text{if } x_i > 1 \end{cases}$ represents the expenditure, and d_1, d_2, e_{1i}, and e_{2i} are constants.
3) The constraints include: $x_i \geq 0, i = 1, \cdots, 6$ and the shared emission constraints $\sum_{i=1}^{6} v_i x_i - 2 \leq 0$ and $x_4 + x_5 \leq 10$. The game satisfies Assumptions 1–3 and the strong Slater condition.

The economic constants d_1 and d_2 are set to be $d_1 = 3$ and $d_2 = 0.02$. The other constants of the game are shown in Table I. The communication graph is shown in Fig. 1 .

By calculation, the normalized Nash equilibrium of the game is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (10, 2.0951, 0.1699, 5.7415, 0.5826, 0)$. The initial states of the agents are selected to be $x(0) = \mathbf{0}$ and $y(0) = \mathbf{0}$. Let the centralized algorithms described in (3) and (4), and (10), and the distributed algorithms described in (4) and (13), and (16) be the updating laws, respectively. The parameters are selected as: $\varepsilon = 0.001, \bar{k}_i = \varepsilon k_i = 1$, and $w_{ij} = 20\,000$. The parameter k_i satisfies the conditions in Theorems 3 and 4.

In Figs. 2 and 3, we set the parameters α and α_p as $\alpha = \alpha_p = 10$ and $\alpha = \alpha_p = 100$, respectively. It can be seen that the absolute errors $|x_i - x_i^*|$, $i = 1, \cdots, 6$ are smaller for the ℓ_1 penalty function based method. Furthermore, for a larger α_p, the errors are smaller. In Fig. 4, the simulation results for distributed algorithms are given and the effectiveness of the algorithms is shown.

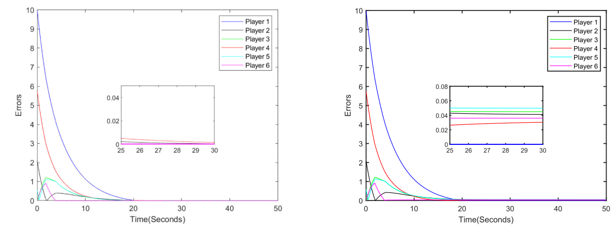

Fig. 2. Convergence errors for $\alpha = 10$ and $\alpha_p = 10$ where the left figure is generated by (3) and (4) and the right figure is generated by (10).

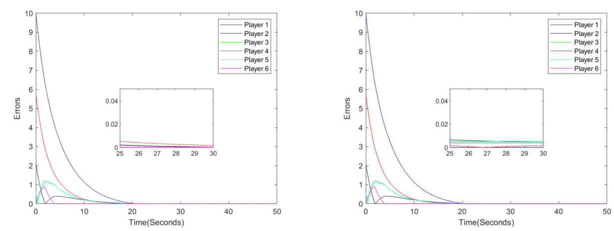

Fig. 3. Convergence errors for $\alpha = 100$ and $\alpha_p = 100$ where the left figure is generated by (3) and (4) and the right figure is generated by (10).

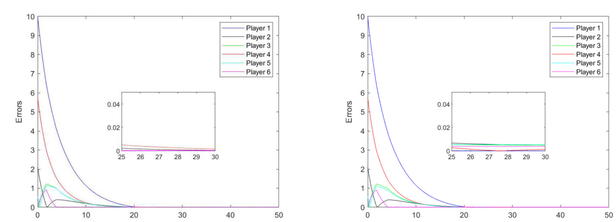

Fig. 4. Convergence errors for $\alpha = 100$ and $\alpha_p = 100$ where the left figure is generated by (4) and (13) and the right figure is generated by (16).

VII. CONCLUSION

In this article, we propose continuous-time penalty-based methods to solve the Nash equilibrium seeking problem of a generalized noncooperative game. Continuous-time dynamical systems are developed for both centralized and distributed implementation, by using exact and inexact penalty functions, respectively. For the continuous-time algorithms, exact exponential convergence can be attained. For distributed implementation, the consensus scheme is used to estimate all players' strategies and singular perturbation analysis is developed to prove the convergence to a neighborhood of a Nash equilibrium.

APPENDIX

A. Proof of Lemma 3

Under the given conditions, we need to prove that the solution $\tilde{x}_a$ to GVIP (8) is also a solution to GVIP (9). The proof is motivated from [45], where we add the equality constraint.

According to (9), $\langle \tilde{u}_b, \tilde{x}_b \rangle = \min_{z \in \Omega} \langle \tilde{u}_b, z \rangle$. For this convex optimization problem, the Lagrange function can be written as $L(z, \lambda) = \langle \tilde{u}_b, z \rangle + \sum_{l=1}^{n_g} \lambda_{1l} g_l(x) + \sum_{s=1}^{n_h} \lambda_{2s} h_s(x) + \sum_{i=1}^{N} \sum_{k=1}^{m_i} \lambda_{3ik} g_{ik}(x_i)$, where $\lambda_{1l} \geq 0$, $\lambda_{2s} \in \mathbb{R}$, $\lambda_{3ik} \geq 0$ and $\lambda = [\lambda_{11}, \cdots, \lambda_{1n_g}, \lambda_{21}, \cdots, \lambda_{2n_h}, \lambda_{3i1}, \cdots, \lambda_{3Nm_N}]^T$.

According to [41, Th. 2.3.2 and 4.4.3], under the strong Slater condition, there exists a KKT multiplier $\lambda_b^* = [\lambda_{b11}^*, \cdots, \lambda_{b1n_g}^*, \lambda_{b21}^*, \cdots, \lambda_{b2n_h}^*, \lambda_{b3i1}^*, \cdots, \lambda_{b3Nm_N}^*]^T$ with $\lambda_{b1l}^* \geq 0$, $\lambda_{b2s}^* \in \mathbb{R}$, $\lambda_{b3ik}^* \geq 0$ such that $(\tilde{x}_b, \lambda_b^*)$ is a saddle point of $L(z,\lambda)$, i.e., $L(\tilde{x}_b, \lambda) \leq L(\tilde{x}_b, \lambda_b^*) \leq L(z, \lambda_b^*)$ $\forall z \in \mathbb{R}^N$, $\lambda_{1l} \geq 0$, $\lambda_{2s} \in \mathbb{R}$, and $\lambda_{3ik} \geq 0$.

According to (8), $\tilde{u}_a \in F(\tilde{x}_a) + \alpha\Phi(\tilde{x}_a)$. Then, there exists $\tilde{u}_{a1} \in F(\tilde{x}_a)$ and $\tilde{u}_{a2} \in \Phi(\tilde{x}_a)$ such that $\tilde{u}_a = \tilde{u}_{a1} + \alpha\tilde{u}_{a2}$. It follows that for all $z \in \mathbb{R}^N$, we have (a): $\langle \tilde{u}_{a1} + \alpha\tilde{u}_{a2}, \tilde{x}_a - z\rangle \leq 0$ and (b): $L(\tilde{x}_b, \lambda_b^*) - L(z, \lambda_b^*) = \langle \tilde{u}_b, \tilde{x}_b - z\rangle + \sum_{l=1}^{n_g} \lambda_{b1l}^*(g_l(\tilde{x}_b) - g_l(z)) + \sum_{s=1}^{n_h} \lambda_{b2s}^*(h_s(\tilde{x}_b) - h_s(z)) + \sum_{i=1}^N \sum_{k=1}^{m_i} \lambda_{b3ik}^*(g_{ik}(\tilde{x}_{bi}) - g_{ik}(z)) \leq 0$.

Letting $z = \tilde{x}_b$ in (a) and $z = \tilde{x}_a$ in (b), and taking the sum of (a) and (b) gives $\langle \tilde{u}_{a1} - \tilde{u}_b, \tilde{x}_a - \tilde{x}_b\rangle + \langle \alpha\tilde{u}_{a2}, \tilde{x}_a - \tilde{x}_b\rangle + \sum_{l=1}^{n_g} \lambda_{b1l}^*(g_l(\tilde{x}_b) - g_l(\tilde{x}_a)) + \sum_{s=1}^{n_h} \lambda_{b2s}^*(h_s(\tilde{x}_b) - h_s(\tilde{x}_a)) + \sum_{i=1}^N \sum_{k=1}^{m_i} \lambda_{b3ik}^*(g_{ik}(\tilde{x}_{bi}) - g_{ik}(\tilde{x}_{ai})) \leq 0$.

According to Assumption 3, convexity, and properties of KKT multipliers, we have

$$\alpha\left(\sum_{l=1}^{n_g}[g_l(\tilde{x}_a)]_+ + \sum_{s=1}^{n_h}|h_s(\tilde{x}_a)| + \sum_{i=1}^{N}\sum_{k=1}^{m_i}[g_{ik}(\tilde{x}_{ai})]_+\right) + \sum_{l=1}^{n_g}(0 - \lambda_{b1l}^* g_l(\tilde{x}_a)) + \sum_{s=1}^{n_h}(0 - \lambda_{b2s}^* h_s(\tilde{x}_a)) + \sum_{i=1}^{N}\sum_{k=1}^{m_i}(0 - \lambda_{b3ik}^* g_{ik}(\tilde{x}_{ai}))) \leq 0. \tag{18}$$

It follows that

$$\alpha\left(\sum_{l=1}^{n_g}[g_l(\tilde{x}_a)]_+ + \sum_{s=1}^{n_h}|h_s(\tilde{x}_a)| + \sum_{i=1}^{N}\sum_{k=1}^{m_i}[g_{ik}(\tilde{x}_{ai})]_+\right) \leq \sum_{l=1}^{n_g}\lambda_{b1l}^*[g_l(\tilde{x}_a)]_+ + \sum_{s=1}^{n_h}|\lambda_{b2s}^*||h_s(\tilde{x}_a)| + \sum_{i=1}^{N}\sum_{k=1}^{m_i}\lambda_{b3ik}^*[g_{ik}(\tilde{x}_{ai})]_+. \tag{19}$$

Thus, $[g_l(\tilde{x}_a)]_+ = 0, |h_s(\tilde{x}_a)| = 0$, and $[g_{ik}(\tilde{x}_{ai})]_+ = 0$ for all $\alpha > \alpha^* \triangleq \max\{\lambda_{b11}^*, \cdots, \lambda_{b1n_g}^*, |\lambda_{b21}^*|, \cdots, |\lambda_{b2n_h}^*|, \lambda_{b3i1}^*, \cdots, \lambda_{b3Nm_N}^*\}$.

Furthermore, if $\alpha > \alpha^*$, then for all $z \in \Omega$, $\langle \tilde{u}_{a1}, \tilde{x}_a - z\rangle \leq \langle \tilde{u}_{a1}, \tilde{x}_a - z\rangle + \alpha(\sum_{l=1}^{n_g}[g_l(\tilde{x}_a)]_+ + \sum_{s=1}^{n_h}|h_s(\tilde{x}_a)| + \sum_{i=1}^N\sum_{k=1}^{m_i}[g_{ik}(\tilde{x}_{ai})]_+) - \alpha(\sum_{l=1}^{n_g}[g_l(z)]_+ + \sum_{s=1}^{n_h}|h_s(z)| + \sum_{i=1}^N\sum_{k=1}^{m_i}[g_{ik}(z_i)]_+) \leq \langle \tilde{u}_{a1} + \alpha\tilde{u}_{a2}, \tilde{x}_a - z\rangle \leq 0$, which implies that $\tilde{x}_a$ is a solution to GVIP (9).

B. Proof of Theorem 1

Define a Lyapunov candidate $V = \frac{1}{2}(x - x^*)^T(x - x^*)$, where x^* is the normalized Nash equilibrium of GNEP (1). Taking the Lie derivative of V along (6) gives $L_f V = -(x - x^*)^T(F(x) + \alpha\Phi(x))$. According to Lemma 3, if $\alpha > \alpha^*$, $\mathbf{0} \in F(x^*) + \alpha\Phi(x^*)$. Taking $\zeta \in L_f V$, we have $\zeta - \mathbf{0} \in -(x - x^*)^T(F(x) + \alpha\Phi(x)) + (x - x^*)^T(F(x^*) + \alpha\Phi(x^*))$. By Lemma 1 and Assumption 3, $\zeta \leq -\rho\|x - x^*\|^2$. Thus, differential inclusion (6) exponentially converges to x^*.

C. Proof of Lemma 4

According to (12) and the strong monotonicity, $\tilde{x}_{\alpha_p}$ is the unique Nash equilibrium of the unconstrained game: $\min_{x_i} f_i(x) + q_i(\alpha_p, x)$, where $q_i(\alpha_p, x) = \alpha_p\tilde{q}_i(x) = \alpha_p(\sum_{l=1}^{n_g}([g_l(x)]_+)^2 + \sum_{s=1}^{n_h}(h_s(x))^2 + \sum_{k=1}^{m_i}([g_{ik}(x_i)]_+)^2)$. Let $\bar{x} = \lim_{p\to\infty}\tilde{x}_{\alpha_p}$ be the limit of the sequence, $\bar{x}_i$ be the ith component and $\bar{x}_{-i}$ consist of the other components. Let $\tilde{x}_{\alpha_{p,i}}$ and $\tilde{x}_{\alpha_{p,-i}}$ be defined in the same way. By viewing $\bar{x}_{-i}$ as a fixed parameter and following [46, p. 404], one can prove that $\lim_{p\to\infty}\tilde{q}_i(\tilde{x}_{\alpha_{p,i}}, \bar{x}_{-i}) = 0$, which implies that $\tilde{q}_i(\bar{x}) = 0$ and $\bar{x}$ is feasible. Furthermore, for each p, $f_i(\tilde{x}_{\alpha_{p,i}}, \bar{x}_{-i}) \leq \min_{x_i} f_i(x_i, \bar{x}_{-i})$ ([46, Lemma 2, p. 404]), which implies that $f_i(\bar{x}) = \lim_{p\to\infty} f_i(\tilde{x}_{\alpha_{p,i}}, \bar{x}_{-i}) \leq \min_{x_i} f_i(x_i, \bar{x}_{-i})$. Thus, $\bar{x}$ is a Nash equilibrium of GNEP (1).

D. Proof of Theorem 3

From Theorem 1, the normalized Nash equilibrium x^* is globally exponentially stable for the following differential inclusion:

$$\dot{x} \in -\bar{k}\,(F(x) + \alpha\Phi(x))$$

and for $V_0 = \frac{1}{2}(x - x^*)^T(\frac{1}{\varepsilon}\bar{k})^{-1}(x - x^*)$, $\max_{w_0 \in L_0}\langle\nabla V_0(x), w_0\rangle \leq -2\rho k_{\min}V_0(x)$ where $L_0 = -\bar{k}(F(x) + \alpha\Phi(x))$. Note that V_0 is independent of small parameter ε.

Define the following auxiliary system:

$$\dot{\tilde{y}}_i = -W_i(L_i + B_i)(\tilde{y}_i), i \in \mathcal{V} \tag{20}$$

which is the boundary layer system [47].

Let $V_1 = \frac{1}{2}\sum_{i=1}^N \tilde{y}_i^T\tilde{y}_i$ be a Lyapunov candidate function of system (20). Under an undirected and connected graph, it can be proven that $\left\langle\nabla V_1(\tilde{Y}), w_1\right\rangle \leq -2\varpi_{\min}V_1(\tilde{Y})$, where $\varpi_{\min} = \min_{i\in\mathcal{V}}\{\lambda_{\min}(W_i(L_i + B_i))\}$ and $w_1 = [-W_i(L_i + B_i)(\tilde{y}_i)]_N$. Let $\tilde{\xi} = x - x^*$ and $\Delta = [\tilde{\xi}^T, \tilde{Y}^T]^T$. Define a set-valued map $L_\varepsilon = -\frac{1}{\varepsilon}\bar{k}(\tilde{F}(y) + \alpha\Phi(x))$.

Define a Lyapunov candidate $V(\Delta, t) = V_0 + cV_1$, where $0 < c < 1$ is a constant. Then, $c_1||\Delta||^2 \leq V \leq c_2||\Delta||^2$ where $c_1 = \min\{\frac{k_{\min}}{2}, \frac{c}{2}\}$ and $c_2 = \max\{\frac{k_{\max}}{2}, \frac{c}{2}\}$.

Suppose that $\Delta(0) \in \Gamma_1 \triangleq \{\Delta|||\Delta|| \leq \sqrt{\frac{c_1\delta_0^2}{c_2}}\}$, where δ_0 is a positive constant. Then, the following conclusion can be obtained.

Lemma 5: Let δ be a positive constant such that for any $\Delta \in \Gamma_2 \triangleq \{\Delta|||\Delta|| \leq \delta_0\}$, we have $V_0(\tilde{\xi}) \leq \delta$, $V_1(\tilde{Y}) \leq \delta$ and $\max_{w_\varepsilon \in L_\varepsilon}\left\langle\nabla V_0(\tilde{\xi}), w_\varepsilon\right\rangle + V_0(\tilde{\xi}) + \left\langle\nabla V_1(\tilde{Y}), w_2\right\rangle \leq \delta$, where $w_2 = -\frac{1}{\varepsilon}\frac{dx}{dt}\otimes\mathbf{1}$. Let $w_3 = [\frac{d\tilde{y}_i}{dt}]_N = [-W_i(L_i + B_i)(\tilde{y}_i)]_N - \frac{dx}{dt}\otimes\mathbf{1} = w_1 + \varepsilon w_2$. According to the outer semicontinuity[1] of the map L_ε, for any positive constant η, there exists a positive constant σ such that if $w_\varepsilon \in L_\varepsilon, \|\tilde{Y}\| \leq \sigma$, then $\left\langle\nabla V_0(\tilde{\xi}), w_\varepsilon\right\rangle + \left\langle\nabla V_1(\tilde{Y}), w_2\right\rangle \leq -2\rho k_{\min}V_0(\tilde{\xi}) + \frac{\eta}{2}$.

Proof: The proof is similar to [49, Claim 1]. Suppose that the claim is false. Then, there exists η such that for each positive integer s, there exists $\Delta_s \in \Gamma_1$, $w_{\varepsilon s} \in L_{\varepsilon s}$, and w_{2s} such that $\|\tilde{Y}\| \leq \frac{1}{s}$ and $\left\langle\nabla V_0(\tilde{\xi}_s), w_{\varepsilon s}\right\rangle + \left\langle\nabla V_1(\tilde{Y}_s), w_{2s}\right\rangle > -2\rho k_{\min}V_0(\tilde{\xi}_s) + \frac{\eta}{2}$. According to the compactness of Γ_2, and local boundedness of L_ε, there is a subsequence converging to $(\Delta_0,\ w_{\varepsilon 0})$, where $\Delta_0 = [\tilde{\xi}_0^T, \mathbf{0}^T]^T$, and $\tilde{\xi}_0$ is a certain vector. Due to the outer semicontinuity of L_ε, $w_{\varepsilon 0} \in L_0$. Since $\nabla V_1(\mathbf{0}) = \mathbf{0}$, it can be obtained that $\left\langle\nabla V_0(\tilde{\xi}_0), w_{\varepsilon 0}\right\rangle \geq -2\rho k_{\min}V_0(\tilde{\xi}_0) + \frac{\eta}{2}$, which contradicts to the fact that $\max_{w_0 \in L_0}\left\langle\nabla V_0(\tilde{Y}), w_0\right\rangle \leq -2\rho k_{\min}V_0(\tilde{Y})$. ∎

Let ε^* be a positive constant satisfying $\varepsilon^* \leq 2c\varpi_{\min}$ and $(2\delta - \eta)\varepsilon^* \leq c\varpi_{\min}\sigma^2$.

[1]According to [48], for locally bounded set-valued mappings with closed values, outer semicontinuity agrees with the often used upper semicontinuity.

Then, for all $||\Delta|| \leq \delta_0$, if $\|\tilde{Y}\| \leq \sigma$

$$\begin{aligned}\frac{1}{\varepsilon}\langle \nabla V, w\rangle &= \left\langle \nabla V_0(\tilde{\xi}), w_\varepsilon \right\rangle + \frac{c}{\varepsilon}\left\langle \nabla V_1(\tilde{Y}), w_3 \right\rangle \\ &= \left\langle \nabla V_0(\tilde{\xi}), w_\varepsilon \right\rangle + c\left\langle \nabla V_1(\tilde{Y}), w_2 \right\rangle \\ &+ \frac{c}{\varepsilon}\left\langle \nabla V_1(\tilde{Y}), w_1 \right\rangle \\ &\leq -2\rho k_{\min} V_0(\tilde{\xi}) + \frac{\eta}{2} - \frac{2c}{\varepsilon}\varpi_{\min} V_1(\tilde{Y}) \\ &\leq -V + c(1 - \frac{2c}{\varepsilon}\varpi_{\min})V_1(\tilde{Y}) + \frac{\eta}{2} \\ &\leq -V + \eta \end{aligned} \tag{21}$$

based on the condition $2\rho k_{\min} \geq 1$ and the fact $1 - \frac{2c}{\varepsilon}\varpi_{\min} \leq 0$ for all $0 < \varepsilon < \varepsilon^*$. If $\|\tilde{Y}\| \geq \sigma$, then

$$\begin{aligned}\frac{1}{\varepsilon}\langle \nabla V, w\rangle &= \left\langle \nabla V_0(\tilde{\xi}), w_\varepsilon \right\rangle + c\left\langle \nabla V_1(\tilde{Y}), w_2 \right\rangle \\ &+ \frac{c}{\varepsilon}\left\langle \nabla V_1(\tilde{Y}), w_1 \right\rangle \\ &\leq -V_0(\tilde{\xi}) + \delta - \frac{2c}{\varepsilon}\varpi_{\min} V_1(\tilde{Y}) \\ &\leq -V + cV_1(\tilde{Y}) + \delta - \frac{c}{\varepsilon}\varpi_{\min}\sigma^2 \\ &\leq -V + \eta \end{aligned} \tag{22}$$

based on the fact $c\delta + \delta - \frac{c}{\varepsilon}\varpi_{\min}\sigma^2 \leq \eta$ for all $0 < \varepsilon < \varepsilon^*$.

Thus, $\dot{V} \leq -\varepsilon V + \varepsilon\eta$, a.e., for all $||\Delta|| \leq \delta_0$ and $\dot{V} \leq -\frac{\varepsilon}{2}V$, a.e., for all $\sqrt{\frac{2\eta}{c_1}} \leq ||\Delta|| \leq \delta_0$, provided that η satisfies $\sqrt{\frac{2\eta}{c_1}} < \delta_0$. Define $\delta_1 = c_1\delta_0^2$ and $\delta_2 = \frac{2\eta c_2}{c_1}$. Let $\eta < \frac{c_1^2\delta_0^2}{2c_2}$. Then, $\dot{V} \leq -\frac{\varepsilon}{2}V$, a.e., for any $\delta_2 \leq V \leq \delta_1$. Based on a similar semiglobal stability analysis as [47, Sec. 4.8, Th. 4.18], if $\Delta(0) \in \Gamma_1$, which implies $V(\Delta(0), 0) \leq \delta_1$, V exponentially converges to $\Gamma_3 \triangleq \{V | V \leq \delta_2\}$ and stays inside (if $V(\Delta(0), 0) \in \Gamma_3$ initially, then it will stay inside at all the time), and there exists a time T such that $\|\Delta\| \leq \frac{\sqrt{2\eta c_2}}{c_1}$, for all $t \geq T$.

Given an initial condition $\Delta(0) \in \Gamma_1$ and the parameter k_i in the updating law, δ_0 can be obtained. Then, δ can be obtained according to Lemma 5. Furthermore, for any $\eta > 0$ satisfying $\sqrt{\frac{2\eta}{c_1}} < \delta_0$ and $\eta < \frac{c_1^2\delta_0^2}{2c_2}$, $\sigma > 0$ exists. Thus, for any ε satisfying $0 < \varepsilon < \varepsilon^*$, $\varepsilon^* \leq 2c\varpi_{\min}$ and $(2\delta - \eta)\varepsilon^* \leq c\varpi_{\min}\sigma^2$, there exists a time T such that $\|\Delta\| \leq \frac{\sqrt{2\eta c_2}}{c_1}$, for all $t \geq T$. Note that σ is independent of ε and $\varpi_{\min}$. A larger $\varpi_{\min}$ will increase the upper bound of ε^* for the same η. In addition, if given ε, since σ and η are correlated, we cannot determine η.

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