Distributed Algorithms with Fixed-Time Convergence for Nash Equilibrium Seeking of Non-Cooperative Games

Xingxing Ju, Chuandong Li, *Senior Member, IEEE,* Xiuxian Li, *Senior Member, IEEE,* Gang Feng, *Fellow, IEEE*

***Abstract*—This paper investigates the problem of distributed Nash equilibrium (NE) seeking with fixed time convergence for non-cooperative games over undirected communication graphs. Two distributed NE seeking algorithms are proposed based on distributed observers estimating other players' actions. It is proved that the proposed distributed NE seeking algorithms converge to the NE in fixed-time which is independent of the initial conditions of the players. Moreover, the upper bound of the convergence time for the proposed NE seeking algorithms is explicitly given. The effectiveness of the proposed distributed algorithms are validated via a simulation example on energy consumption games.**

***Index Terms*—Nash equilibrium, non-cooperative games, distributed algorithms, fixed-time convergence.**

I. Introduction

Non-cooperative games are one kind of games with competition between individual players, as opposed to cooperative games. NE seeking in non-cooperative games is one of the most critical research topics attracting considerable interest from both theoretical and practical perspectives [1]–[5]. The NE in these games refers to an action profile on which no player can gain more payoff by changing unilaterally his own actions.

Many centralized approaches have been developed to solve the problem of NE seeking in non-cooperative games in recent years [4]–[7]. However, such centralized approaches would run into difficulties in those network based games, especially large scale ones, where the centralized information is to be shared by or distributed to all players under topological limitations of communication networks. In light of this, a lot of distributed approaches have been proposed for the NE seeking problem in network based games [8]–[18]. For instance, an NE seeking algorithm was proposed with primal-dual dynamics for a class of quadratic games in smart grids [8]. A passivity-based approach to solving the NE seeking problem was proposed for quadratic and non-quadratic games in [9]. The authors in [10] proposed a distributed NE seeking algorithm for a class of quadratic games. Distributed NE seeking algorithms were proposed for quadratic and non-quadratic games based on distributed observers in [11], and the authors in [12] extended the results in [11] to adaptive versions for achieving fully distributed NE seeking. The authors in [13] introduced a distributed observer similar to that in [11], and presented a distributed NE seeking algorithm for non-cooperative constrained games. The authors in [14] proposed distributed NE seeking approaches with or without considering the boundedness of control inputs, and also introduced two centralized algorithms for non-cooperative games. In addition, distributed NE seeking algorithms were developed in [15]–[18] for smooth or nonsmooth non-cooperative games under various constraints. It is, however, noted that the centralized algorithms [4]–[7] and distributed algorithms [8]–[18] focused on only *asymptotical* convergence or *exponential* convergence.

On the other hand, inspired by the newly developed concepts of finite-time ($\mathbf{FIN_t}$) convergence proposed in [19] and fixed-time ($\mathbf{FIX_t}$) convergence proposed in [20]–[22], some $\mathbf{FIN_t}$ or $\mathbf{FIX_t}$ converging algorithms have been developed in various optimization problems such as centralized algorithms for constrained or unconstrained optimization problems [23]–[27], and distributed algorithms for multi-agent optimization problems [28]–[33]. In fact, the centralized algorithms with $\mathbf{FIX_t}$ convergence proposed in [26], [27] can be directly applied to two kinds of game problems, that is, zero-sum games and quadratic non-cooperative games. The authors in [34] studied the solution of time-varying NE seeking and tracking problems in non-cooperative games via nonsmooth, model-based and model-free algorithms with fixed-time convergence. At the same time, similar ideas have also been adopted for NE seeking problems of network based games [35]–[39]. For example, the authors in [35] proposed distributed NE seeking algorithms with prescribed convergence time over either fixed or switching communication topologies. The authors in [36] proposed a fixed-time distributed adaptive NE seeking algorithm, where the distributed observer converges within $\mathbf{FIX_t}$ while the NE seeking was achieved asymptotically. The authors in [37] developed a NE seeking algorithm with $\mathbf{FIX_t}$ convergence for non-cooperative games. The authors in [38] introduced first-order and zeroth-order Nash equilibrium

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X. Ju, and C. Li are with the Chongqing Key Laboratory of Nonlinear Circuits and Intelligent Information Processing, College of Electronic and Information Engineering, Southwest University, Chongqing 400715, China (e-mail: bob211@email.swu.edu.cn, cdli@swu.edu.cn).

X. Li is with the Department of Control Science and Engineering, College of Electronics and Information Engineering, the Shanghai Institute of Intelligent Science and Technology, and the Shanghai Research Institute for Intelligent Autonomous Systems, Tongji University, Shanghai, China (e-mail: xli@tongji.edu.cn).

G. Feng is with the Department of Biomedical Engineering, City University of Hong Kong, Hong Kong (e-mail: megfeng@cityu.edu.hk).

seeking dynamics with $\mathbf{FIX_t}$ and practical $\mathbf{FIX_t}$ convergence for non-cooperative games having finitely many players while authors in [39] extended the results in [11], [14] to distributed NE seeking for quadratic and non-quadratic non-cooperative games with $\mathbf{FIN_t}$ convergence. It should be noted that the convergence time for $\mathbf{FIN_t}$ is dependent on initial states and can grow unbounded with the increasing distance of the initial state from an equilibrium point while the convergence time for $\mathbf{FIX_t}$ is independent of initial conditions and can be prescribed. The above literature review reveals that most distributed algorithms achieve NE seeking for quadratic or non-quadratic non-cooperative games either with *asymptotic* or *exponential* convergence and that there are few results on distributed algorithms of NE seeking for the same games with $\mathbf{FIN_t}$ or $\mathbf{FIX_t}$, which motivates this study.

In this work, we propose two distributed algorithms for NE seeking with $\mathbf{FIX_t}$ convergence for non-cooperative games. The main contributions of this paper in comparison with those relevant existing works lie in the following three aspects.

(i) Two distributed NE seeking algorithms with $\mathbf{FIX_t}$ convergence, where the players have no direct access to the actions of the players who are not their neighbors, are proposed for solving quadratic and non-quadratic non-cooperative games in this work. Unlike those works [4]–[7] with either asymptotic or exponential convergence, and also unlike the distributed approaches [8]–[14], [36], [39], where the settling time for the convergence to the solution cannot be explicitly given, the upper bound on the settling time to reach the solution is explicitly given in this work. Compared with $\mathbf{FIX_t}$ centralized algorithms [26], [27], [34], [38], distributed NE seeking approaches with $\mathbf{FIX_t}$ convergence have better scalability, robustness, adaptability, and privacy protection capabilities.

(ii) Different from the NE seeking approaches proposed in [36], where the NE seeking was achieved asymptotically, our NE seeking is achieved within $\mathbf{FIX_t}$. The upper bound on the settling time can be explicitly given in our work while it is not given in [35]. Different from the NE seeking approaches developed in [37], where the semi-global *practical* NE seeking to its neighborhood was achieved within $\mathbf{FIX_t}$ while our algorithms converge to the NE exactly in $\mathbf{FIX_t}$. In contrast to model-based fully centralized $\mathbf{FIX_t}$ Nash equilibrium seeking dynamic approaches [38], two more accurate upper bounds of settle time are given in this work. Different from the NE seeking approaches proposed in [39], where the distributed observer and NE seeking are achieved in $\mathbf{FIN_t}$, our distributed algorithms achieve the NE seeking within $\mathbf{FIX_t}$. Different from the NE seeking approaches proposed in [13], where the convergence to an arbitrarily small neighborhood of the NE in $\mathbf{FIN_t}$ is ensured while our algorithms converge to the NE exactly in $\mathbf{FIX_t}$.

(iii) Detailed comparisons of the proposed NE seeking algorithms with those existing algorithms [11], [13], [14], [39] via an example of energy consumption games are presented. It is demonstrated that our algorithms perform better.

The rest of the paper is organized as follows. In Section II, some basic definitions and concepts are recalled. In Section III, main results are given, including two novel distributed algorithms for non-cooperative games and their analysis. A numerical example on energy consumption games is provided to illustrate effectiveness as well as advantages of the proposed NE seeking algorithms in Section IV, which is followed by some concluding remarks in Section V.

Notation: $\mathbb{R}^n$ denotes the n-dimensional column vector space, $\mathbb{R}^{m\times n}$ the $m\times n$-dimensional matrix space, $x^\top$ the transpose of vector $x\in\mathbb{R}^n$, $P^\top$ the transpose of matrix $P\in\mathbb{R}^{m\times n}$, $\|\cdot\|_1$ the 1-norm of vectors, $\|\cdot\|$ the 2-norm of vectors, $\mathbf{0}_N$ the N-dimensional column vector with all elements being 0, I_N the N-dimensional identity matrix, and $\otimes$ the Kronecker product. For an arbitrary $P\in\mathbb{R}^{n\times n}$, $\tau_2(P)$ denotes its second smallest eigenvalue. For $x = [x_1,x_2,\ldots,x_n]^\top\in\mathbb{R}^n$, $\lceil x\rfloor^\alpha = [\lceil x_1\rfloor^\alpha,\lceil x_2\rfloor^\alpha,\ldots,\lceil x_n\rfloor^\alpha]^\top$ with $\lceil x_i\rfloor^\alpha = |x_i|^\alpha \mathrm{sgn}(x_i)$, where $|\cdot|$ is the absolute value and $\mathrm{sgn}(\cdot)$ is the sign function.

II. Preliminaries and Problem Formulation

A. Graph theory

An undirected graph can be modelled as $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with the node set $\mathcal{V}=\{1,2,\ldots,N\}$ and the edge set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. The adjacency matrix of $\mathcal{G}$ is defined as $A=[a_{ij}]_{N\times N}$ with $a_{ij}=a_{ji}>0$ if there is an edge between nodes i and j for information exchange and $a_{ij}=0$ otherwise. The Laplacian matrix is defined as $\mathcal{L}=[l_{ij}]_{N\times N}$ with $l_{ii}=\sum_{i=1}^N = a_{ij}$ and $l_{ij}=-a_{ij}$ for $i\neq j$. $\mathcal{G}$ is said to be connected if there is a path between any two distinct nodes $i,j\in\mathcal{V}$.

B. Technical Lemmas

The following results provide sufficient conditions for $\mathbf{FIX_t}$ convergence in terms of a Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}$.

Lemma 1. [20], [21] Consider the following system,

$$\dot{x}=\psi(x(t)),\quad x(0)=x_0\in\mathbb{R}^n, \tag{1}$$

where $\psi:\mathbb{R}^n\to\mathbb{R}^n$ is a mapping and $\psi(\mathbf{0})=\mathbf{0}$. $x^*=\mathbf{0}$ is called an equilibrium point of system (1). Suppose that there exists a continuously differentiable positive definite function $V:\mathbb{R}^n\to\mathbb{R}$ for system (1) such that

$$\dot{V}(x)\le -C_1V(x)^{p_1}-C_2V(x)^{p_2},\ \forall x\in\mathbb{R}^n\backslash\{\mathbf{0}\}, \tag{2}$$

where $C_1,C_2>0$, $0<p_1<1$ and $p_2>1$ are four constants. Then, the equilibrium point of system (1) is $\mathbf{FIX_t}$ convergent with the settling time given by

$$T\le\frac{1}{C_1(1-p_1)}+\frac{1}{C_2(p_2-1)}. \tag{3}$$

In addition, let $p_1=1-\frac{1}{2\theta}$ and $p_2=1+\frac{1}{2\theta}$ in (2), where $\theta>1$. Then, the equilibrium point of system (1) is $\mathbf{FIX_t}$ convergent with the settling time given by

$$T\le\pi\theta(C_1C_2)^{-\frac{1}{2}}, \tag{4}$$

where π is the well-known mathematical constant.

It follows from [21] that eq. (3) provides a quite conservative settling time estimate while eq. (4) provides a more accurate settling time estimate. The following lemma will be utilized in the analysis of the proposed NE seeking algorithms in Section III.

Lemma 2. [28] Given $\zeta_1, \zeta_2, ..., \zeta_n \geq 0$ and $1 \leq \alpha < \beta$, it holds that

$$\left(\sum_{i=1}^{n} \zeta_i\right)^{\frac{1}{\alpha}} \leq \sum_{i=1}^{n} \zeta_i^{\frac{1}{\alpha}}, \quad n^{1-\beta}\left(\sum_{i=1}^{n} \zeta_i\right)^{\beta} \leq \sum_{i=1}^{n} \zeta_i^{\beta}.$$

C. Problem Formulation

Consider N players in a non-cooperative game with $x_i \in \mathbb{R}$ being player i's action. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ denote the set of the players and $\mathbf{x} = [x_1, x_2, \ldots, x_N]^\top \in \mathbb{R}^N$ be a algorithmprofile of all the players' actions. The ith player has a payoff function $\psi_i(\mathbf{x})$, depending on its own action and other players' actions. Let $\mathbf{x}_{-i} = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N]^\top \in \mathbb{R}^{N-1}$ be a vector consisting of all other players' actions except x_i. For notation convenience, denote $\mathbf{x} = (x_i, \mathbf{x}_{-i})$, then $\psi_i(\mathbf{x})$ can also be written as $\psi_i(x_i, \mathbf{x}_{-i})$. All the players have the intention to minimize their own cost functions selfishly.

In this paper, we mainly focus on NE seeking for non-cooperative games, which have wide applications in energy consumption problems in optical communication networks [1], smart grids [8], [10], and vehicle deployment problems in transportation [40]. The NE refers to an action profile on which no player can gain more payoff by changing unilaterally his own actions. The NE seeking problem for a non-cooperative game is defined to seek the NE $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ such that

$$\psi_i(x_i^*, \mathbf{x}_{-i}^*) \leq \psi_i(x_i, \mathbf{x}_{-i}^*), \tag{5}$$

for $x_i \in \mathbb{R}$, $\forall i \in \mathcal{N}$, $\mathbf{x}_{-i}^* = [x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_N^*]^\top$.

Before proceeding, we make the following three assumptions.

A1: The players' objective functions $\psi_i(x_i, \mathbf{x}_{-i})$ for $i \in \mathcal{N}$ are twice continuously differentiable with respect to x_i for any fixed $\mathbf{x}_{-i} \in \mathbb{R}^{N-1}$.

A2: The communication topology $\mathcal{G}$ is undirected and connected.

Define

$$\Theta(\mathbf{x}) = \left[\nabla\psi_1(x_1, \mathbf{x}_{-1}), ..., \nabla\psi_N(x_N, \mathbf{x}_{-N})\right]^\top, \tag{6}$$

$$\frac{\partial\Theta(\mathbf{x})}{\partial\mathbf{x}} = \Phi(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2\psi_1(\mathbf{x})}{\partial^2 x_1} & \frac{\partial^2\psi_1(\mathbf{x})}{\partial x_1\partial x_2} & \cdots & \frac{\partial^2\psi_1(\mathbf{x})}{\partial x_1\partial x_N} \\ \frac{\partial^2\psi_2(\mathbf{x})}{\partial x_2 x_1} & \frac{\partial^2\psi_2(\mathbf{x})}{\partial^2 x_2} & \cdots & \frac{\partial^2\psi_2(\mathbf{x})}{\partial x_2\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\psi_N(\mathbf{x})}{\partial x_N x_1} & \frac{\partial^2\psi_N(\mathbf{x})}{\partial x_N\partial x_2} & \cdots & \frac{\partial^2\psi_N(\mathbf{x})}{\partial^2 x_N} \end{pmatrix}, \tag{7}$$

where $\nabla\psi_i(x_i, \mathbf{x}_{-i}) = \frac{\partial\psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i} = \frac{\partial\psi_i(\mathbf{x})}{\partial x_i}$.

A3: The players' objective functions $\psi_i(x_i, \mathbf{x}_{-i})$ are globally Lipschitz continuous.

A4: There is a constant $\zeta > 0$ such that the matrix $\Phi(\mathbf{x})$ satisfies that $\Phi^\top(\mathbf{x}) + \Phi(\mathbf{x}) \geq 2\zeta I_N$ for any $\mathbf{x} \in \mathbb{R}^N$.

Remark 1. *Assumptions* **A1**-**A4** *are common in the study of non-cooperative games [10]–[14], [18], [37]–[39].* **A3** *guarantees the boundedness of $\psi_i(x_i, \mathbf{x}_{-i})$. Since $\psi_i(x_i, \mathbf{x}_{-i})$ is twice continuously differentiable, via* **A4** *and Lagrange mean value theorem we can obtain that $\langle\Theta(u) - \Theta(v), u - v\rangle \geq \zeta\|u - v\|^2$, holds for all $v, u \in \mathbb{R}^N$, which means that the NE $\mathbf{x}^*$ of game (5) exists uniquely, and the NE is reached at $\mathbf{x}^*$ if and only if $\Theta(\mathbf{x}^*) = \mathbf{0}_N$ [14].*

III. Main Results

In this section, we present two novel $\mathbf{FIX_t}$ converging distributed NE seeking algorithms for game (5) based on the gradients of neighboring players.

A. Distributed $\mathbf{FIX_t}$ Converging NE Seeking Algorithm with Gradients

Inspired by $\mathbf{FIX_t}$ converging distributed approaches [29]–[32], the following distributed NE seeking algorithm is proposed, for $i, j \in \mathcal{N} = \{1, 2, \ldots, N\}$,

$$\dot{z}_{ij} = -\alpha_i\lceil\Lambda_{ij}\rfloor^{\mu} - \beta_i\lceil\Lambda_{ij}\rfloor^{\nu} - \gamma_i\mathrm{sgn}(\Lambda_{ij}), \tag{8a}$$

$$\dot{x}_i = \begin{cases} -\delta_i\frac{\nabla\psi_i(\mathbf{z}_i)}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\omega}} - \eta_i\frac{\nabla\psi_i(\mathbf{z}_i)}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\rho}}, \mathrm{if}\|\psi_i(\mathbf{z}_i)\| \neq 0, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise}, \end{cases} \tag{8b}$$

where z_{ij} is player i's estimate on player j's action,

$$\Lambda_{ij} = \sum_{k=1}^{N} a_{ik}(z_{ij} - z_{kj}) + a_{ij}(z_{ij} - x_j),$$

and a_{ik} is the (i, k)-th entry of the adjacency matrix A corresponding to the communication topology $\mathcal{G}$, $\mathbf{z}_i = [z_{i1}, \ldots, z_{iN}]^\top$ denote player i's estimates on all the players' actions, $\nabla\psi_i(\mathbf{z}_i) = \frac{\partial\psi_i(\mathbf{x})}{\partial x_i}|_{\mathbf{x}=\mathbf{z}_i}$, $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta_i > 0$, $\mu, \omega \in (0, 1)$, and $\nu, \rho \in (1, +\infty)$ are some tunable parameters.

It should be noted that the right hand side of (8) is discontinuous implying that the solution to equation (8) is defined in the sense of Filippov [41]. It can be observed from eq. (8) that the NE seeking algorithm of player i has two components: one for player i to estimate the actions of other players in (8a) and the other for player i to change its own action in (8b).

The concatenated-vector form of (8) can be expressed as follows,

$$\dot{\mathbf{z}} = -\tilde{\alpha}\lceil\tilde{\Lambda}\rfloor^{\mu} - \tilde{\beta}\lceil\tilde{\Lambda}\rfloor^{\nu} - \tilde{\gamma}\mathrm{sgn}(\tilde{\Lambda}), \tag{9a}$$

$$\dot{\mathbf{x}} = \begin{cases} -\delta\mathcal{Q}_1(\mathbf{z}) - \eta\mathcal{Q}_2(\mathbf{z}), & \|\psi_i(\mathbf{z}_i)\| \neq 0, \\ 0, & \text{otherwise}, \end{cases} \tag{9b}$$

where $\mathbf{z} = \left[\mathbf{z}_1^\top, \mathbf{z}_2^\top, \ldots, \mathbf{z}_N^\top\right]^\top$, $\tilde{\alpha} = \mathrm{diag}\{\alpha_i\} \otimes I_N$, $\tilde{\beta} = \mathrm{diag}\{\beta_i\} \otimes I_N$, $\tilde{\gamma} = \mathrm{diag}\{\gamma_i\} \otimes I_N$, $\delta = \mathrm{diag}\{\delta_i\}$, and $\eta = \mathrm{diag}\{\eta_i\}$ are five positive definite matrices,

$$\tilde{\Lambda} = (\mathcal{L} \otimes I_N + \mathcal{M})(\mathbf{z} - 1_N \otimes \mathbf{x}),$$

with $\mathcal{L}$ being the Laplacian matrix of the communication topology $\mathcal{G}$, $\mathcal{M} = \operatorname{diag}\{a_{ij}\}$,

$$\mathcal{Q}_1(\mathbf{z}) := \left[\frac{\nabla\psi_1(\mathbf{z}_1)}{\|\nabla\psi_1(\mathbf{z}_1)\|^{1-\omega}}, \ldots, \frac{\nabla\psi_N(\mathbf{z}_N)}{\|\nabla\psi_N(\mathbf{z}_N)\|^{1-\omega}}\right]^\top,$$

$$\mathcal{Q}_2(\mathbf{z}) := \left[\frac{\nabla\psi_1(\mathbf{z}_1)}{\|\nabla\psi_1(\mathbf{z}_1)\|^{1-\rho}}, \ldots, \frac{\nabla\psi_N(\mathbf{z}_N)}{\|\nabla\psi_N(\mathbf{z}_N)\|^{1-\rho}}\right]^\top.$$

Lemma 3. Under Assumption **A3**, there exists a constant $\vartheta > 0$ such that $\|1_N \otimes \dot{\mathbf{x}}\| \leq \sqrt{N}\vartheta$.

Proof: It follows from (8b) that

$$\begin{aligned}
\|\dot{\mathbf{x}}\| &\leq \sum_{i=1}^N \|\dot{x}_i\| \\
&= \sum_{i=1}^N \left(\left\|\delta_i \frac{\nabla\psi_i(\mathbf{z}_i)}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\omega}} + \eta_i \frac{\nabla\psi_i(\mathbf{z}_i)}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\rho}}\right\|\right) \\
&\leq \sum_{i=1}^N \left(\delta_i \frac{\|\nabla\psi_i(\mathbf{z}_i)\|}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\omega}} + \eta_i \frac{\|\nabla\psi_i(\mathbf{z}_i)\|}{\|\nabla\psi_i(\mathbf{z}_i)\|^{1-\rho}}\right) \\
&= \sum_{i=1}^N (\delta_i \|\nabla\psi_i(\mathbf{z}_i)\|^\omega + \eta_i \|\nabla\psi_i(\mathbf{z}_i)\|^\rho) \\
&\leq N \max\{\delta_i\} M^\omega + N \max\{\eta_i\} M^\rho, \qquad (10)
\end{aligned}$$

where the last inequality holds due to Assumption **A3**, and $M > \|\nabla\psi_i(\mathbf{z}_i)\|$ is a positive constant. Let $\vartheta = N \max\{\delta_i\} M^\omega + N \max\{\eta_i\} M^\rho$, it then follows from eq. (10) that $\|1_N \otimes \dot{\mathbf{x}}\| \leq \sqrt{N}\vartheta$. The proof is thus completed. ■

Remark 2. *It should be noted that if $\psi_i(\mathbf{z}_i)$ is globally Lipschitz continuous, the boundedness of $\nabla\psi_i(\mathbf{z}_i)$ follows directly. Furthermore, the globally Lipschitz continuity of $\psi_i(\mathbf{z}_i)$ can be easily verified for many practical applications such as signal-to-noise ratio Nash games in optical communication networks [1], energy consumption games in smart grids [8], [9], mobile sensor network games [14], [18], and power control games in femtocell networks [36].*

The following theorem establishes the $\mathbf{FIX_t}$ convergence of the proposed estimator (8a).

Theorem 1. Consider the non-cooperative game (5) with the NE seeking algorithm (8) under Assumptions **A1** – **A3**. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then for any $i, j \in N$, the estimate z_{ij} in (8a) converges to x_j in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_1 \leq \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)},$$

where $A_1 > 0$, $A_2 > 0$, $p_1 \in (0.5, 1)$, and $p_2 > 1$ are four constants.

Proof: Since $\mathcal{G}$ is undirected and connected, then 0 is a simple eigenvalue of its corresponding Laplacian matrix L. According to Gershgorin's disc theorem, $(L \otimes I_N + \mathcal{M})$ is symmetric and positive definite [39]. Consider the candidate Lyapunov function as

$$V_1(\mathbf{z}) = \frac{1}{2}(\mathbf{z} - 1_N \otimes \mathbf{x})^\top (L \otimes I_N + \mathcal{M})(\mathbf{z} - 1_N \otimes \mathbf{x}). \quad (11)$$

It follows from positive definiteness of $(L \otimes I_N + \mathcal{M})$ that

$$\xi_{\min}(\Xi)\|\mathbf{z} - 1_N \otimes \mathbf{x}\|^2 \leq 2V_1 \leq \xi_{\max}(\Xi)\|\mathbf{z} - 1_N \otimes \mathbf{x}\|^2, \quad (12)$$

where $\Xi := L \otimes I_N + \mathcal{M}$, $\xi_{\min}(\Xi)$ and $\xi_{\max}(\Xi)$ denotes the minimum and maximum eigenvalue of matrix Ξ, respectively. Eq. (12) implies

$$-\|\mathbf{z} - 1_N \otimes \mathbf{x}\|^{\mu+1} \leq -\frac{2^{\frac{\mu+1}{2}} V_1^{\frac{\mu+1}{2}}}{(\xi_{\max}(\Xi))^{\frac{\mu+1}{2}}}, \quad (13)$$

and

$$-\|\mathbf{z} - 1_N \otimes \mathbf{x}\|^{\nu+1} \leq -\frac{2^{\frac{\nu+1}{2}} V_1^{\frac{\nu+1}{2}}}{(\xi_{\max}(\Xi))^{\frac{\nu+1}{2}}}. \quad (14)$$

Taking the time derivative of V_1 in (11), one has

$$\begin{aligned}
\dot{V}_1(\mathbf{z}) =& (\mathbf{z} - 1_N \otimes \mathbf{x})^\top (L \otimes I_N + \mathcal{M})(\dot{\mathbf{z}} - 1_N \otimes \dot{\mathbf{x}}) \\
=& (\mathbf{z} - 1_N \otimes \mathbf{x})^\top (L \otimes I_N + \mathcal{M})(-\tilde{\alpha}\lceil\tilde{\Lambda}\rfloor^\mu - \tilde{\beta}\lceil\tilde{\Lambda}\rfloor^\nu \\
& - \tilde{\gamma}\operatorname{sgn}(\tilde{\Lambda}) - 1_N \otimes \dot{\mathbf{x}}) \\
=& \tilde{\Lambda}(-\tilde{\alpha}\lceil\tilde{\Lambda}\rfloor^\mu - \tilde{\beta}\lceil\tilde{\Lambda}\rfloor^\nu - \tilde{\gamma}\operatorname{sgn}(\tilde{\Lambda}) - 1_N \otimes \dot{\mathbf{x}}) \\
=& \tilde{\Lambda}(-\tilde{\alpha}|\tilde{\Lambda}|^\mu \operatorname{sgn}(\tilde{\Lambda}) - \tilde{\beta}|\tilde{\Lambda}|^\nu \operatorname{sgn}(\tilde{\Lambda}) - \tilde{\gamma}\operatorname{sgn}(\tilde{\Lambda}) - 1_N \otimes \dot{\mathbf{x}}) \\
\leq& -\min_{i\in\mathcal{N}}\{\alpha_i\}\|\tilde{\Lambda}\|_{\mu+1}^{\mu+1} - \min_{i\in\mathcal{N}}\{\beta_i\}\|\tilde{\Lambda}\|_{\nu+1}^{\nu+1} - \min_{i\in\mathcal{N}}\{\gamma_i\}\|\tilde{\Lambda}\| \\
& + \|\tilde{\Lambda}\|\|1_N \otimes \dot{\mathbf{x}}\| \\
\leq& -\min_{i\in\mathcal{N}}\{\alpha_i\}\|\tilde{\Lambda}\|^{\mu+1} - \min_{i\in\mathcal{N}}\{\beta_i\}N^{-\nu}\|\tilde{\Lambda}\|^{\nu+1} \\
& - (\min_{i\in\mathcal{N}}\{\gamma_i\} - \sqrt{N}\vartheta)\|\tilde{\Lambda}\| \\
\leq& -\min_{i\in\mathcal{N}}\{\alpha_i\}\|\tilde{\Lambda}\|^{\mu+1} - \min_{i\in\mathcal{N}}\{\beta_i\}N^{-\nu}\|\tilde{\Lambda}\|^{\nu+1}, \qquad (15)
\end{aligned}$$

where the first inequality holds due to Lemma 2, the second inequality holds due to Lemma 3, and the last inequality holds due to $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$. Rewrite (15) as

$$\dot{V}_1(\mathbf{z}) \leq -\digamma_1\|z - I_N \otimes x\|^{\mu+1} - \digamma_2\|z - I_N \otimes x\|^{\nu+1}, \quad (16)$$

where $\digamma_1 = \min_{i\in\mathcal{N}}\{\alpha_i\}(\xi_{\min}(\Xi))^{\mu+1}$ and $\digamma_2 = \min_{i\in\mathcal{N}}\{\beta_i\}(\xi_{\min}(\Xi))^{\nu+1}N^{-\nu}$ are two constants. It follows from eqs. (13)-(16) that

$$\dot{V}_1(\mathbf{z}) \leq -\frac{2^{\frac{\mu+1}{2}}\digamma_1}{(\xi_{\max}(\Xi))^{\frac{\mu+1}{2}}} V_1^{\frac{\mu+1}{2}} - \frac{2^{\frac{\nu+1}{2}}\digamma_2}{(\xi_{\max}(\Xi))^{\frac{\nu+1}{2}}} V_1^{\frac{\nu+1}{2}}.$$

It then follows from Lemma 1 that the estimate z_{ij} converges to x_j in $\mathbf{FIX_t}$ with the settling time satisfying

$$T_1 \leq \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)}, \quad (17)$$

where $p_1 = \frac{\mu+1}{2}$, $p_2 = \frac{\nu+1}{2}$, $A_1 = \frac{2^{\frac{\mu+1}{2}}\digamma_1}{(\xi_{\max}(\Xi))^{\frac{\mu+1}{2}}}$, and $A_2 = \frac{2^{\frac{\nu+1}{2}}\digamma_2}{(\xi_{\max}(\Xi))^{\frac{\nu+1}{2}}}$ are four constants. The proof is thus completed. ■

Theorem 1 provides a settling time estimate for the proposed algorithm (8a) by eq. (3). In the investigation of $\mathbf{FIX_t}$ convergence, a significant index is the estimated $\mathbf{FIX_t}$ of the settling time and it is desirable to derive a high-precision estimation. The following corollary with a more accurate settling time estimates can be directly obtained from Theorem 1 and eq. (4), and Corollary 1 in [22] respectively.

Corollary 1. Consider the non-cooperative game (5) with the NE seeking algorithm (8) under Assumptions $\mathbf{A1}-\mathbf{A3}$.

(i) Let $\mu = 1-\frac{1}{\varpi}$, and $\nu = 1+\frac{1}{\varpi}$ in Theorem 1, where $\varpi > 1$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then for any $i,j \in N$, the estimate z_{ij} in (8a) converges to x_j in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_1 \leq \pi\varpi(A_1A_2)^{-\frac{1}{2}}, \tag{18}$$

where $A_1 > 0$ and $A_2 > 0$ are two constants.

(ii) If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then for any $i,j \in N$, the estimate z_{ij} in (8a) converges to x_j in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_1^\sharp \leq \frac{1}{A_1}\left(\frac{A_1}{A_2}\right)^{\frac{1-p_1}{p_2-p_1}}\left(\frac{1}{1-p_1}+\frac{1}{p_2-1}\right), \tag{19}$$

where A_1, A_2, p_1, and p_2 are the same as those in (17).

Moreover, we can prove $T^\sharp_{1_{\max}} - T_{1_{\max}} \leq 0$. It follows from inequalities (17) and (19) that

$$\begin{aligned}
&T^\sharp_{1_{\max}} - T_{1_{\max}}\\
=&\frac{1}{A_1}\left(\frac{A_1}{A_2}\right)^{\frac{1-p_1}{p_2-p_1}}\left(\frac{1}{1-p_1}+\frac{1}{p_2-1}\right)\\
&-\frac{1}{A_1(1-p_1)}+\frac{1}{A_2(p_2-1)}\\
=&\frac{1}{A_1(p_2-1)}\left(\left(\frac{A_1}{A_2}\right)^{\frac{1-p_1}{p_2-p_1}}\left(\frac{p_2-1}{1-p_1}+1\right)-\frac{p_2-1}{1-p_1}-\frac{A_1}{A_2}\right)\\
=&\frac{1}{A_1(p_2-1)}\left(\left(\frac{A_1}{A_2}\right)^{\frac{1}{1+\vartheta}}(1+\vartheta)-\vartheta-\frac{A_1}{A_2}\right),
\end{aligned} \tag{20}$$

where $\vartheta = \frac{p_2-1}{1-p_1}$. If $\frac{2^{\frac{\mu+1}{2}}F_1}{(\xi_{\max}(\Xi))^{\frac{\mu+1}{2}}} = A_1 = A_2 = \frac{2^{\frac{\nu+1}{2}}F_2}{(\xi_{\max}(\Xi))^{\frac{\nu+1}{2}}}$, then equality (20) yields $T^\sharp_{1_{\max}} = T_{1_{\max}}$. If $\frac{2^{\frac{\mu+1}{2}}F_1}{(\xi_{\max}(\Xi))^{\frac{\mu+1}{2}}} \neq \frac{2^{\frac{\nu+1}{2}}F_2}{(\xi_{\max}(\Xi))^{\frac{\nu+1}{2}}}$, we can get $T^\sharp_{1_{\max}} < T_{1_{\max}}$. This can be verified by the following step. Let $g(y) = a^{\frac{1}{1+y}}(1+y)-y-a$, where $y \geq 0$, $a > 0$ and $a \neq 1$, then one obtains that $g(y)$ is decreasing, and $g(y) < g(0) = 0$ for $y > 0$. Therefore, $T^\sharp_{1_{\max}} < T_{1_{\max}}$ if $A_1 \neq A_2$.

Remark 3. *It is noticed that the condition $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$ is a sufficient condition for the $\mathbf{FIX_t}$ convergence of algorithm (8a). It is also noticed from Theorem 1 and Corollary 1 that the upper bounds for the convergence time of algorithm (8a) are dependent on parameters α_i, β_i, μ, and ν, $i = 1,2,\ldots,N$.*

It follows from Theorem 1 or Corollary 1 that the estimate z_{ij} converges to x_j after the settling time $T_{\max}$, that is, $\mathbf{z}_i = \mathbf{x}$ for any $i \in \mathcal{N}$ when $t \geq T_{\max}$. Therefore, $\forall t \geq T_{\max}$, the action of player i can be described as follows,

$$\dot{x}_i = \begin{cases}-\delta_i\frac{\nabla\psi_i(\mathbf{x})}{\|\nabla\psi_i(\mathbf{x})\|^{1-\omega}}-\eta_i\frac{\nabla\psi_i(\mathbf{x})}{\|\nabla\psi_i(\mathbf{x})\|^{1-\rho}}, \text{if } \|\nabla\psi_i(\mathbf{x})\| \neq 0,\\ 0, \qquad\qquad\qquad\qquad \text{otherwise,}\end{cases} \tag{21}$$

where $\nabla\psi_i(\mathbf{x}) = \frac{\partial\psi_i(\mathbf{x})}{\partial x_i}$, $\delta_i > 0$, $\eta_i > 0$, $\omega \in (0,1)$ and $\rho \in (1,+\infty)$ are some tunable parameters. The concatenated-vector form of (21) can be given as follows,

$$\dot{\mathbf{x}} = \begin{cases}-\delta\mathcal{Q}_1(\mathbf{x})-\eta\mathcal{Q}_2(\mathbf{x}), & \text{if } \|\nabla\psi_i(\mathbf{x})\| \neq 0,\\ 0, & \text{otherwise,}\end{cases} \tag{22}$$

where $\delta = \mathrm{diag}\{\delta_i\}$ and $\eta = \mathrm{diag}\{\eta_i\}$ are two positive definite matrices,

$$\mathcal{Q}_1(\mathbf{x}) := \left[\frac{\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}}{\|\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}\|^{1-\omega}}, \frac{\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}}{\|\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}\|^{1-\omega}}, \ldots, \frac{\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}}{\|\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}\|^{1-\omega}}\right]^\top,$$

$$\mathcal{Q}_2(\mathbf{x}) := \left[\frac{\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}}{\|\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}\|^{1-\rho}}, \frac{\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}}{\|\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}\|^{1-\rho}}, \ldots, \frac{\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}}{\|\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}\|^{1-\rho}}\right]^\top.$$

It is noted from algorithm (21) that if $\eta_i = 0$, the algorithm degenerates to a $\mathbf{FIN_t}$ converging NE seeking algorithm; if $\eta_i = 0, \omega = 0$ or $\delta_i = 0, \rho = 0$, then the algorithm degenerates to the normal NE seeking algorithm; and if $\eta_i = 0$ and $\omega = 1$, then the algorithm becomes

$$\dot{x}_i = -\delta_i\nabla\psi_i(\mathbf{x}),$$

which is a conventional gradient descent NE seeking algorithm [5].

Lemma 4. Let Assumption $\mathbf{A4}$ hold. $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ is an equilibrium point of algorithm (21) if and only if it is an NE of the non-cooperative game (5).

Proof: If $\mathbf{x}^*$ is an equilibrium point of algorithm (21), then one has that

$$-\delta\mathcal{Q}_1(\mathbf{x}^*)-\eta\mathcal{Q}_2(\mathbf{x}^*) = \mathbf{0}_N \text{ or } \|\nabla\psi_i(\mathbf{x}^*)\| = 0.$$

It follows that $\delta_i\frac{\|\nabla\psi_i(\mathbf{x}^*)\|}{\|\nabla\psi_i(\mathbf{x}^*)\|^{1-\omega}} + \eta_i\frac{\|\nabla\psi_i(\mathbf{x}^*)\|}{\|\nabla\psi_i(\mathbf{x}^*)\|^{1-\rho}} = 0$ or $\|\nabla\psi_i(\mathbf{x}^*)\| = 0$. Therefore, $\delta_i\|\nabla\psi_i(\mathbf{x}^*)\|^\omega + \eta_i\|\nabla\psi_i(\mathbf{x}^*)\|^\rho = 0$ or $\|\nabla\psi_i(\mathbf{x}^*)\| = 0 \Rightarrow \|\nabla\psi_i(\mathbf{x}^*)\| = 0$, which implies $\Theta(\mathbf{x}^*) = \mathbf{0}_N$. It follows from $\mathbf{A4}$ that $\mathbf{x}^*$ is a NE of the non-cooperative game (5). By following the similar arguments, one can also show that the converse is also true, and the proof is thus completed. ∎

The following theorem establishes the $\mathbf{FIX_t}$ convergence of the proposed algorithm (21).

Theorem 2. Consider the non-cooperative game (5) with the NE seeking algorithm (8) under Assumptions $\mathbf{A1}-\mathbf{A4}$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then

$\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2 \leq \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)}, \tag{23}$$

where $B_1 > 0$, $B_2 > 0$, $p_4 \in (0, 0.5)$, and $p_4 > 1$ are four constants.

Proof: It follows from **A4** that the NE of the non-cooperative game (5) exists uniquely. It then follows from Lemma 4 that $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ is the equilibrium point of algorithm (22). Consider the following candidate Lyapunov function,

$$V(\mathbf{x}) = \frac{1}{2}\|\Theta(\mathbf{x})\|^2 = \frac{1}{2}\sum_{i=1}^{N}\left\|\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right\|^2, \tag{24}$$

where $\Theta(\mathbf{x})$ is defined in eq. (6). It follows from Assumption **A4** that V is radially unbounded. It follows from $\langle \Theta(\mathbf{x}) - \Theta(v), \mathbf{x} - v\rangle \geq \zeta\|\mathbf{x} - v\|^2$ and $\Theta(\mathbf{x}^*) = \mathbf{0}_N$ that

$$\|\Theta(\mathbf{x})\| \geq \zeta\|\mathbf{x} - \mathbf{x}^*\|,$$

which indicates that $V(\mathbf{x}) = 0$ if and only if $\Theta(\mathbf{x}) = \mathbf{0}_N$, i.e., $x = \mathbf{x}^*$. Taking the time derivative of $V(\mathbf{x})$ in (24) yields

$$\begin{aligned}
&\dot{V}(\mathbf{x}) \\
&= -\,\Theta(\mathbf{x})^\top \Phi(\mathbf{x})\left[\delta \mathcal{Q}_1(\mathbf{x}) + \eta \mathcal{Q}_2(\mathbf{x})\right] \\
&= -\left[\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right]_{\text{vec}}^\top \Phi(\mathbf{x}) \left[\frac{\delta_i \frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}}{\|\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\|^{1-\omega}}\right]_{\text{vec}} \\
&\quad - \left[\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right]_{\text{vec}}^\top \Phi(\mathbf{x}) \left[\frac{\eta_i \frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}}{\|\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\|^{1-\rho}}\right]_{\text{vec}} \\
&\leq -\,\zeta \min\{\delta_i\}\left[\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right]_{\text{vec}}^\top I_N \left[\frac{\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}}{\|\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\|^{1-\omega}}\right]_{\text{vec}} \\
&\quad -\,\zeta \min\{\eta_i\}\left[\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right]_{\text{vec}}^\top I_N \left[\frac{\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}}{\|\frac{\partial \psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\|^{1-\rho}}\right]_{\text{vec}} \\
&= -\,\zeta \min\{\delta_i\}\sum_{i=1}^{N}\left\|\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right\|^{1+\omega} \\
&\quad -\,\zeta \min\{\eta_i\}\sum_{i=1}^{N}\left\|\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right\|^{1+\rho},
\end{aligned} \tag{25}$$

where the first inequality holds due to Assumption **A4**. It further follows from (25) and Lemma 2 that

$$\begin{aligned}
\dot{V}(\mathbf{x}) \leq &-\,\zeta \min\{\delta_i\}\left(\sum_{i=1}^{N}\left\|\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\omega}{2}} \\
&-\,\zeta \min\{\eta_i\}N^{1-\frac{1+\rho}{2}}\left(\sum_{i=1}^{N}\left\|\frac{\partial \psi_i(x_i, \mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\rho}{2}} \\
= &-\,\zeta \min\{\delta_i\}2^{\frac{1+\omega}{2}}V(\mathbf{x})^{\frac{1+\omega}{2}} \\
&-\,\zeta \min\{\eta_i\}N^{1-\frac{1+\rho}{2}}2^{\frac{1+\rho}{2}}V(\mathbf{x})^{\frac{1+\rho}{2}}
\end{aligned} \tag{26}$$

It then follows from Lemma 1 that all the players' actions converge to the unique NE in $\mathbf{FIX_t}$ with the settling time satisfying

$$T_2 \leq \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)}, \tag{27}$$

where $p_3 = \frac{1+\omega}{2}$, $p_4 = \frac{1+\rho}{2}$, $B_1 = \zeta\min\{\delta_i\}2^{\frac{1+\omega}{2}}$, and $B_2 = \zeta\min\{\eta_i\}N^{1-\frac{1+\rho}{2}}2^{\frac{1+\rho}{2}}$. The proof is thus completed. ■

Similar to Corollary 1, the following result with more accurate convergence time estimates for algorithm (8) can be directly obtained from Theorem 2 and eq. (4), and Corollary 1 in [22] respectively.

Corollary 2. Consider the non-cooperative game (5) with the NE seeking algorithm (8) under Assumptions $\mathbf{A1} - \mathbf{A4}$.

(i) Let $\omega = 1 - \frac{1}{\varpi}$, and $\rho = 1 + \frac{1}{\varpi}$ in Theorem 2, where $\varpi > 1$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then $\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2 \leq \pi\omega(B_1B_2)^{-\frac{1}{2}}, \tag{28}$$

where $B_1 > 0$ and $B_2 > 0$ are two constants.

(ii) If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then $\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2^\sharp \leq \frac{1}{B_1}\left(\frac{B_1}{B_2}\right)^{\frac{1-p_3}{p_4-p_3}}\left(\frac{1}{1-p_3} + \frac{1}{p_4-1}\right), \tag{29}$$

where B_1, B_2, p_3, and p_4 are the same as those in (27).

On the basis of Theorems 1 and 2, the proposed distributed NE seeking algorithm (8) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T \leq \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)} + \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)},$$

where A_i, B_i and p_i, $i = 1, 2, 3, 4$, are defined in (17) and (27).

On the basis of (i) in Corollaries 1 and 2, the proposed distributed NE seeking algorithm (8) converges to the unique NE in $\mathbf{FIX_t}$ with a more accurate convergence time estimate given as follows,

$$T \leq \pi\varpi\left[(A_1A_2)^{-\frac{1}{2}} + (B_1B_2)^{-\frac{1}{2}}\right],$$

where A_i and B_i, $i = 1, 2$, are defined in (18) and (28), respectively.

Similarly, on the basis of (ii) in Corollaries 1 and 2, the proposed distributed NE seeking algorithm (8) converges to the unique NE in $\mathbf{FIX_t}$ with a more accurate convergence time estimate given as follows,

$$T^\sharp \leq T_1^\sharp + T_2^\sharp,$$

where $T_1^\sharp$ and $T_2^\sharp$ satisfy inequalities (19) and (39), respectively.

Remark 4. *It can be seen from the proofs of Theorems 1 and 2 that the larger the design parameters α_i, β_i, and γ_i, the larger A_1 and A_2, and thus the smaller the settling time, and* *that the larger the design parameter* $\min\{\delta_i\}$ *or* $\min\{\eta_i\}$, *the larger* B_1 *and* B_2, *and thus the smaller the settling time.*

Now we consider a special case of seeking the NE in quadratic non-cooperative games with the payoff function $\psi_i(\mathbf{x})$ defined as follows,

$$\psi_i(\mathbf{x}) = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} x_j \mathcal{A}_{jk}^i x_k + \sum_{j=1}^{N}\phi_j^i x_j + K_i, \tag{30}$$

where $\mathcal{A}_{jk}^i$, ϕ_j^i and K_i are some constants, $i \in \mathcal{N}$, and $\mathcal{A}_{jk}^i = \mathcal{A}_{kj}^i$. The following theorem establishes a upper bound for the $\mathbf{FIX_t}$ convergence of the proposed algorithm (21) in terms of the following constant matrix $\mathcal{A}$,

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{11}^1 & \mathcal{A}_{12}^1 & \cdots & \mathcal{A}_{1N}^1 \\ \mathcal{A}_{21}^2 & \mathcal{A}_{22}^2 & \cdots & \mathcal{A}_{2N}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{N1}^N & \mathcal{A}_{N2}^N & \cdots & \mathcal{A}_{NN}^N \end{pmatrix}.$$

Theorem 3. Consider the quadratic non-cooperative games with the NE seeking algorithm (8) under Assumptions $\mathbf{A1}-\mathbf{A4}$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then $\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2 \leq \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)}, \tag{31}$$

where $B_1 > 0$, $B_2 > 0$, $p_4 \in (0, 0.5)$, and $p_4 > 1$ are four constants.

Proof: It follows from $\mathbf{A4}$ that the NE of the non-cooperative game (5) exists uniquely. It then follows from Lemma 4 that $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ is the equilibrium point of algorithm (22). Consider the following candidate Lyapunov function,

$$V(\mathbf{x}) = \frac{1}{2}\Theta(\mathbf{x})^\top(\delta^{-1}+\eta^{-1})\mathcal{A}^{-1}\Theta(\mathbf{x}). \tag{32}$$

where $\Theta(\mathbf{x})$ is defined in eq. (6), and Φ is defined in eq. (7). It follows from assumption $\mathbf{A4}$ that V is radially unbounded. Noting that $(\delta^{-1}+\eta^{-1})\mathcal{A}^{-1}$ is positive definite, one has

$$\zeta_{\min}(\Omega)\|\Theta(\mathbf{x})\|^2 \leq 2V \leq \zeta_{\max}(\Omega)\|\Theta(\mathbf{x})\|^2,$$

where $\Omega = (\delta^{-1}+\eta^{-1})\mathcal{A}^{-1}$, $\zeta_{\min}(\Omega)$ and $\zeta_{\max}(\Omega)$ denotes the minimum and maximum eigenvalue of matrix Ω respectively. This inequality can be written as follows, $\zeta_{\min}(\Omega)\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2 \leq 2V \leq \zeta_{\max}(\Omega)\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2$, which implies

$$-\left(\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\omega}{2}} \leq -\frac{2^{\frac{\omega+1}{2}}V^{\frac{\omega+1}{2}}}{(\xi_{\max}(\Omega))^{\frac{\omega+1}{2}}}, \tag{33}$$

and

$$-\left(\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\rho}{2}} \leq -\frac{2^{\frac{\rho+1}{2}}V^{\frac{\rho+1}{2}}}{(\xi_{\max}(\Omega))^{\frac{\rho+1}{2}}}. \tag{34}$$

Taking the time derivative of V in (32) yields

$$\begin{aligned}\dot{V}(\mathbf{x}) &= -\Theta(\mathbf{x})^\top[\delta^{-1}\mathcal{A}^{-1}\mathcal{A}\delta\mathcal{Q}_1(\mathbf{x}) + \eta^{-1}\mathcal{A}^{-1}\mathcal{A}\eta\mathcal{Q}_2(\mathbf{x})] \\ &= -\Theta(\mathbf{x})^\top\mathcal{Q}_1(\mathbf{x}) - \Theta(\mathbf{x})^\top\mathcal{Q}_2(\mathbf{x}) \\ &= -\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^{1+\omega} - \sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^{1+\rho},\end{aligned} \tag{35}$$

It further follows from eqs. (33)-(35) and Lemma 2 that

$$\begin{aligned}\dot{V}(\mathbf{x}) \leq &-\left(\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\omega}{2}} \\ &- N^{1-\frac{1+\rho}{2}}\left(\sum_{i=1}^N\left\|\frac{\partial\psi_i(x_i,\mathbf{x}_{-i})}{\partial x_i}\right\|^2\right)^{\frac{1+\rho}{2}} \\ \leq &-\frac{2^{\frac{\omega+1}{2}}}{(\xi_{\max}(\Omega))^{\frac{\omega+1}{2}}}V^{\frac{\omega+1}{2}} - \frac{2^{\frac{\rho+1}{2}}N^{1-\frac{1+\rho}{2}}}{(\xi_{\max}(\Omega))^{\frac{\rho+1}{2}}}V^{\frac{\rho+1}{2}}.\end{aligned} \tag{36}$$

It then follows from Lemma 1 that all the players' actions converge to the unique NE in $\mathbf{FIX_t}$ with the settling time satisfying

$$T_2 \leq \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)}, \tag{37}$$

where $p_3 = \frac{\omega+1}{2}$, $p_4 = \frac{\rho+1}{2}$, $B_1 = \frac{2^{\frac{\omega+1}{2}}}{(\xi_{\max}(\Omega))^{\frac{\omega+1}{2}}}$, and $B_2 = \frac{2^{\frac{\rho+1}{2}}N^{1-\frac{1+\rho}{2}}}{(\xi_{\max}(\Omega))^{\frac{\rho+1}{2}}}$. The proof is thus completed. ■

Similar to Corollary 2, the following result with more accurate convergence time estimates for algorithm (8) can be directly obtained from Theorem 3 and eq. (4), and Corollary 1 in [22] respectively, in the case of the quadratic non-cooperative games.

Corollary 3. Consider the quadratic non-cooperative games with the NE seeking algorithm (8) under Assumptions $\mathbf{A1}-\mathbf{A4}$.

(i) Let $\omega = 1-\frac{1}{\varpi}$, and $\rho = 1+\frac{1}{\varpi}$ in Theorem 2, where $\varpi > 1$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then $\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2 \leq \pi\omega(B_1B_2)^{-\frac{1}{2}}, \tag{38}$$

where B_1 and B_2 are some constants.

(ii) If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then $\forall t \geq T_{\max}$, the proposed algorithm (21) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T_2^\sharp \leq \frac{1}{B_1}\left(\frac{B_1}{B_2}\right)^{\frac{1-p_3}{p_4-p_3}}\left(\frac{1}{1-p_3} + \frac{1}{p_4-1}\right), \tag{39}$$

where B_1, B_2, p_3, and p_4 are the same as those in (37).

B. Distributed $\mathbf{FIX_t}$ Converging NE Seeking Algorithm Avoiding Singularity

The distributed NE seeking algorithm (8b) would become singular when $\|\nabla\psi_i(\mathbf{z}_i)\| = 0$. In order to avoid this shortcoming, the following distributed NE seeking algorithm for the non-cooperative game (5) is proposed, for $i, j \in \mathcal{N} = \{1, 2, \ldots, N\}$,

$$\dot{z}_{ij} = -\alpha_i\lceil\Lambda_{ij}\rfloor^{\mu} - \beta_i\lceil\Lambda_{ij}\rfloor^{\nu} - \gamma_i\mathrm{sgn}(\Lambda_{ij}), \tag{40a}$$
$$\dot{x}_i = -\delta_i\lceil\nabla\psi_i(\mathbf{z}_i)\rfloor^{\omega} - \eta_i\lceil\nabla\psi_i(\mathbf{z}_i)\rfloor^{\rho} - \xi_i\lceil\nabla\psi_i(\mathbf{z}_i)\rfloor, \tag{40b}$$

where all the variables and constants in (40a) are the same as those in algorithm (8a), $\nabla\psi_i(\mathbf{z}_i)$, $\omega \in (0, 1)$, $\rho \in (1, +\infty)$, $\delta_i > 0$, $\eta_i > 0$ in (40b) are the same as those in algorithm (8b), and $\xi_i > 0$ are some design parameters.

Similar to algorithm (8), it can be observed from algorithm (40) that the NE seeking algorithm of player i has two components: one for player i to estimate the actions of other players in (40a) and the other for player i to change its own action in (40b).

The concatenated-vector form of algorithm (40) can be given as follows,

$$\dot{\mathbf{z}} = -\tilde{\alpha}\lceil\tilde{\Lambda}\rfloor^{\mu} - \tilde{\beta}\lceil\tilde{\Lambda}\rfloor^{\nu} - \tilde{\gamma}\mathrm{sgn}(\tilde{\Lambda}), \tag{41a}$$
$$\dot{\mathbf{x}} = -\delta\mathcal{Q}_1(\mathbf{z}) - \eta\mathcal{Q}_2(\mathbf{z}) - \xi\mathcal{Q}_3(\mathbf{z}), \tag{41b}$$

where $\mathbf{z}$, $\tilde{\Lambda}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, δ, and η are the same as those in algorithm (9), $\xi = \mathrm{diag}\{\xi_i\}$ is a positive definite matrix,

$$\mathcal{Q}_1(\mathbf{z}) := \left[\lceil\nabla\psi_1(\mathbf{z}_1)\rfloor^{\omega}, \lceil\nabla_2\psi_2(\mathbf{z}_2)\rfloor^{\omega}, \ldots, \lceil\nabla\psi_N(\mathbf{z}_N)\rfloor^{\omega}\right]^{\top},$$
$$\mathcal{Q}_2(\mathbf{z}) := \left[\lceil\nabla\psi_1(\mathbf{z}_1)\rfloor^{\rho}, \lceil\nabla_2\psi_2(\mathbf{z}_2)\rfloor^{\rho}, \ldots, \lceil\nabla\psi_N(\mathbf{z}_N)\rfloor^{\rho}\right]^{\top},$$
$$\mathcal{Q}_3(\mathbf{z}) := \left[\lceil\nabla\psi_1(\mathbf{z}_1)\rfloor, \lceil\nabla_2\psi_2(\mathbf{z}_2)\rfloor, \ldots, \lceil\nabla\psi_N(\mathbf{z}_N)\rfloor\right]^{\top}.$$

Theorem 4. Consider the non-cooperative game (5) with the NE seeking algorithm (40) under Assumptions **A1** – **A4**. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then the algorithm converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T \le \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)} + \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)},$$

where $A_i, B_i > 0$, $i = 1, 2$, $p_1, p_3 \in (0.5, 1)$, $p_2, p_4 > 1$ are some constants.

Proof: It follows from Theorem 1 that z_{ij} converges to x_j in $\mathbf{FIX_t}$, that is, $\mathbf{z}_i = x$ for any $i \in \mathcal{N}$ when

$$t \ge T_{\max} := \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)}$$

as shown in eq. (17). Therefore, $\forall t \ge T_{\max}$, the action of player i can be described as follows,

$$\dot{x}_i = -\delta_i\lceil\nabla\psi_i(\mathbf{x})\rfloor^{\omega} - \eta_i\lceil\nabla\psi_i(\mathbf{x})\rfloor^{\rho} - \xi_i\lceil\nabla\psi_i(\mathbf{x})\rfloor, \tag{42}$$

The concatenated-vector form of algorithm (42) can be given as follows,

$$\dot{\mathbf{x}} = -\delta\mathcal{Q}_1(\mathbf{x}) - \eta\mathcal{Q}_2(\mathbf{x}) - \xi\mathcal{Q}_3(\mathbf{x}), \tag{43}$$

where

$$\mathcal{Q}_1(\mathbf{x}) := \left[\left\lceil\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}\right\rfloor^{\omega}, \left\lceil\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}\right\rfloor^{\omega}, \ldots, \left\lceil\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}\right\rfloor^{\omega}\right]^{\top},$$
$$\mathcal{Q}_2(\mathbf{x}) := \left[\left\lceil\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}\right\rfloor^{\rho}, \left\lceil\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}\right\rfloor^{\rho}, \ldots, \left\lceil\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}\right\rfloor^{\rho}\right]^{\top},$$
$$\mathcal{Q}_3(\mathbf{x}) := \left[\left\lceil\frac{\partial\psi_1(\mathbf{x})}{\partial x_1}\right\rfloor, \left\lceil\frac{\partial\psi_2(\mathbf{x})}{\partial x_2}\right\rfloor, \ldots, \left\lceil\frac{\partial\psi_N(\mathbf{x})}{\partial x_N}\right\rfloor\right]^{\top}.$$

Consider the candidate Lyapunov function as

$$V_2(\mathbf{x}) = \frac{1}{2}\|\Theta(\mathbf{x})\|^2 = \frac{1}{2}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|^2, \tag{44}$$

where $\Theta(\mathbf{x})$ is defined in eq. (6). It follows from Assumption **A4** that $V_2(\mathbf{x})$ is radially unbounded. It follows from $\langle\Theta(\mathbf{x}) - \Theta(v), \mathbf{x} - v\rangle \ge \zeta\|\mathbf{x} - v\|^2$ and $\Theta(\mathbf{x}^*) = \mathbf{0}_N$ that

$$\|\Theta(\mathbf{x})\| \ge \zeta\|\mathbf{x} - \mathbf{x}^*\|, \tag{45}$$

which indicates that $V_2(\mathbf{x}) = 0$ if and only if $\Theta(\mathbf{x}) = \mathbf{0}_N$. Taking the time derivative of V_2 in (44), one has

$$\begin{aligned}
&\dot{V}(\mathbf{x})\\
=& -\Theta(\mathbf{x})^{\top}\Phi(\mathbf{x})\left[\delta\mathcal{Q}_1(\mathbf{x}) + \eta\mathcal{Q}_2(\mathbf{x}) + \xi\mathcal{Q}_3(\mathbf{x})\right]\\
=& -\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}\Phi(\mathbf{x})\left[\delta_i\left\lceil\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\rfloor^{\omega}\right]_{\mathrm{vec}}\\
&-\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}\Phi(\mathbf{x})\left[\eta_i\left\lceil\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\rfloor^{\rho}\right]_{\mathrm{vec}}\\
&-\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}\Phi(\mathbf{x})\left[\xi_i\left\lceil\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\rfloor\right]_{\mathrm{vec}}\\
\le& -\zeta\min\{\delta_i\}\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}I_N\left[\left|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right|^{\omega}\mathrm{sgn}\left(\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right)\right]_{\mathrm{vec}}\\
&-\zeta\min\{\eta_i\}\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}I_N\left[\left|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right|^{\rho}\mathrm{sgn}\left(\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right)\right]_{\mathrm{vec}}\\
&-\zeta\min\{\xi_i\}\left[\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right]_{\mathrm{vec}}^{\top}I_N\left[\left|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right|\mathrm{sgn}\left(\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right)\right]_{\mathrm{vec}}\\
=& -\zeta\min\{\delta_i\}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|^{1+\omega} - \zeta\min\{\eta_i\}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|^{1+\rho}\\
&-\zeta\min\{\xi_i\}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|\\
\le& -\zeta\min\{\delta_i\}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|^{1+\omega} - \zeta\min\{\eta_i\}\sum_{i=1}^{N}\left\|\frac{\partial\psi_i(\mathbf{x})}{\partial x_i}\right\|^{1+\rho},
\end{aligned}$$

where the first inequality holds due to Assumption **A4**. Then, following the similar arguments of eqs. (25)-(26) in the proof of Theorem 2, one obtains that algorithm (40) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as

$$T_2 \le \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)},$$

where $p_3 = \frac{1+\omega}{2}$, $p_4 = \frac{1+\rho}{2}$, $B_1 = \zeta\min\{\delta_i\}2^{\frac{1+\omega}{2}}$, and $B_2 = \zeta\min\{\eta_i\}N^{1-\frac{1+\rho}{2}}2^{\frac{1+\rho}{2}}$. It then follows that distributed NE

seeking algorithm (40) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as

$$T \leq \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)} + \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)}.$$

The proof is thus completed. ■

Similar to Corollaries 1 and 2, the following result with more accurate convergence time estimates for algorithm (40) can be directly obtained from Theorem 4 and eq. (4), eq. (17), eq. (39), and Corollary 1 in [22] respectively.

Corollary 4. Consider the non-cooperative game (5) with the NE seeking algorithm (40) under Assumptions $\mathbf{A1} - \mathbf{A4}$.

(i) Let $\mu = \omega = 1 - \frac{1}{\varpi}$, and $\rho = \nu = 1 + \frac{1}{\varpi}$ in Theorem 4, where $\varpi > 1$. If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then the distributed NE seeking algorithm (40) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$T \leq \pi\varpi\left[(A_1A_2)^{-\frac{1}{2}} + (B_1B_2)^{-\frac{1}{2}}\right],$$

where $A_i, B_i > 0$, $i = 1, 2$, are some constants.

(ii) If $\min_{i\in\mathcal{N}}\{\gamma_i\} > \sqrt{N}\vartheta$, where ϑ is defined in Lemma 3, then the algorithm (40) converges to the unique NE in $\mathbf{FIX_t}$ with the settling time given as follows,

$$\begin{aligned} T^\sharp \leq \; & \frac{1}{A_1}\left(\frac{A_1}{A_2}\right)^{\frac{1-p_1}{p_2-p_1}}\left(\frac{1}{1-p_1}+\frac{1}{p_2-1}\right) \\ & + \frac{1}{B_1}\left(\frac{B_1}{B_2}\right)^{\frac{1-p_3}{p_4-p_3}}\left(\frac{1}{1-p_3}+\frac{1}{p_4-1}\right), \end{aligned}$$

where A_i, B_i and p_i are the same as those in Theorem 4.

Remark 5. *The algorithm parameters μ, ν, ω and ρ in the distributed NE seeking algorithms (8) and (40) are global information for all the players and are assumed to be known a priori. Thus in this sense, the proposed fixed-time convergent distributed NE seeking algorithms (8) and (40) are distributed but not fully distributed.*

Remark 6. *It is noticed from Theorem 4 and Corollary 4 that the upper bound for the settling time of algorithm (40) depends on tunable design parameters α_i, β_i, δ_i, η_i, ξ_i, μ, ν, ω, ρ, $i = 1, 2, \ldots, N$, and the number of the players. The convergence time can be shortened by proper choices of those design parameters.*

Remark 7. *It should be noted that the action of an individual player is available to itself in general. In this case, the proposed algorithms (8) and (40) can be enhanced by setting $z_{ii} = x_i$ when $i = j$ in (8a) and (40a). A better convergence speed can be expected for the enhanced algorithms.*

Remark 8. *Assumption $\mathbf{A4}$ guarantees the existence of the unique NE $\mathbf{x}^*$ of game (5), which is often called the strong monotone game. It also guarantees that the equilibrium point $\mathbf{x}^*$ of dynamical system (21) or (43) is equivalent to the NE of game (5). On the other hand, if Assumption $\mathbf{A4}$ is replaced by an alternative assumption, that is, there exists a constant $\kappa > 0$ such that $\langle\Theta(u)-\Theta(v), u-v\rangle \leq -\kappa\|u-v\|^2$, for all $v, u \in \mathbb{R}^N$, the proposed distributed NE seeking algorithms (8) and (40) can still be applied. In fact, the alternative assumption also guarantees that the existence of the unique NE $\mathbf{x}^*$ of game (5) exists with "$\leq$" being replaced by "$\geq$", and that the equilibrium point $\mathbf{x}^*$ of dynamical system (21) or (43) is equivalent to the NE of the non-cooperative game (5).*

Remark 9. *There are two ways to implement continuous-time algorithms in this paper. One way is by hardware. For example, it can be realized by analog circuits. The other way is by software, for example, Matlab or Python, though, in this case, some numerical errors would arise. We will realize the continuous-time algorithms (8) and (40) by Matlab in the simulation examples of this paper.*

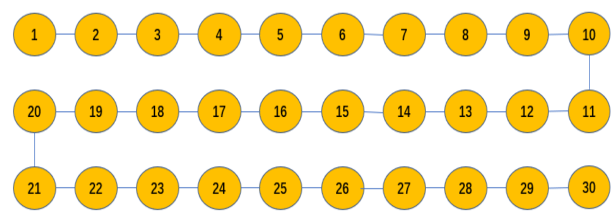


Fig. 1: The communication network for a group of thirty players in game (46).

TABLE I: The constants p_i and q_i, NE $\mathbf{x}^*$ and the values of payoff function $\psi_i(x_i^*, \mathbf{x}_{-i}^*)$ in game (46).

i	p_i	q_i	x_i^*	$\psi_i(x_i^*, \mathbf{x}_{-i}^*)$	i	p_i	q_i	x_i^*	$\psi_i(x_i^*, \mathbf{x}_{-i}^*)$
1	1	10.5	0.3402	110.0184	16	1	10	0.0069	99.9999
2	1	10	0.0069	99.9999	17	1	10.5	0.3402	110.0184
3	1	10.5	0.3402	110.0184	18	1	11	0.6736	120.0923
4	1	10	0.0069	99.9999	19	1	11.5	1.0069	130.2219
5	1	12	0.6736	141.7451	20	1	10	0.0069	99.9999
6	2	5	0.0041	49.9999	21	1	12	1.3402	140.4071
7	2	5.25	0.2041	54.9999	22	1	11	0.6736	120.0923
8	2	5	0.0041	49.9999	23	1	10	0.0069	99.9999
9	2	5.5	0.4041	60.0099	24	1	10	0.0069	99.9999
10	2	5	0.0041	49.9999	25	1	10	0.0069	99.9999
11	1	12	1.3402	140.4071	26	2	5	0.0041	49.9999
12	1	10	0.0069	99.9999	27	2	5	0.0041	49.9999
13	1	11.5	1.0069	130.2219	28	2	6	0.8041	70.0598
14	1	10	0.0069	99.9999	29	2	5	0.0041	49.9999
15	1	10.5	0.3402	110.0184	30	2	5.5	0.4041	60.0099

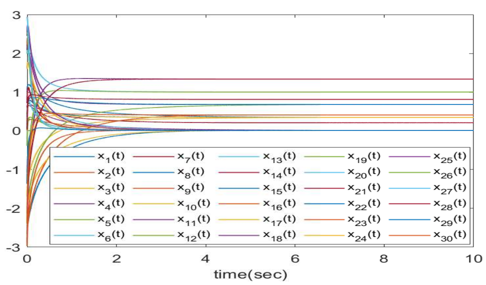
IV. A Numerical Example

In this section, we consider an energy consumption game [8] to demonstrate the effectiveness and advantages of the proposed distributed NE seeking algorithms. All the programs are written in Matlab 2018 and executed on a PC Desktop Intel(R) Core(TM) i7-10710U CPU @ 1.61GHz, RAM 16.0 GB. In this example, the energy consumption game of N players for heating ventilation and air conditioning is to seek the NE $\mathbf{x}^* = (x_i^*, \mathbf{x}_{-i}^*)$ such that

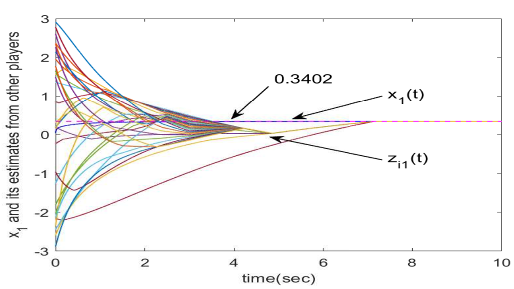
$$\psi_i(x_i^*, \mathbf{x}_{-i}^*) \leq \psi_i(x_i, \mathbf{x}_{-i}^*), \tag{46}$$

TABLE II: The parameters α_i, β_i, γ_i, δ_i and η_i of the proposed distributed NE seeking algorithm (8).

Player i	Design parameters					Player i	Design parameters				
	α_i	β_i	γ_i	δ_i	η_i		α_i	β_i	γ_i	δ_i	η_i
1	2	2	1	1	2	16	5	3	2	5	6
2	2	2	1	1	2	17	5	3	2	5	6
3	2	2	1	1	2	18	5	3	2	5	6
4	2	2	1	1	2	19	5	3	2	5	6
5	2	2	1	1	2	20	5	3	2	5	6
6	3	3	2	2	1	21	6	3	3	4	3
7	3	3	2	2	1	22	6	3	3	4	3
8	3	3	2	2	1	23	6	3	3	4	3
9	3	3	2	2	1	24	6	3	3	4	3
10	3	3	2	2	1	25	6	3	3	4	3
11	4	2	3	3	5	26	8	2	1	2	5
12	4	2	3	3	5	27	8	2	1	2	5
13	4	2	3	3	5	28	8	2	1	2	5
14	4	2	3	3	5	29	8	2	1	2	5
15	4	2	3	3	5	30	8	2	1	2	5



(a)



(b)

Fig. 2: (a) Transient responses of Players1-30 under $\mathbf{FIX_t}$ converging NE seeking algorithm (8); (b) The estimates on x_1 of players under algorithm (8).

where the payoff function of the ith player is defined as follows,

$$\psi_i(x_i, \mathbf{x}_{-i}) = \mathrm{p}_i(x_i - \mathrm{q}_i)^2 + \left(\mathrm{s}\sum_{j=1}^{N} x_j + f\right) x_i, \quad \forall i \in \mathcal{N},$$

with $\mathrm{s}\sum_{j=1}^{N} x_j + f$ being the pricing function.

We consider 30 players $(N = 30)$ in this energy consumption game with the undirected graph as plotted in Fig. 1. Without loss of generality, suppose that $s = 1$ and $f = 10$ for all $i \in \mathcal{N}$ in the simulation. The constants p_i and q_i, NE $\mathbf{x}^*$ and the value of payoff function $\psi_i(x_i^*, \mathbf{x}_{-i}^*)$ are listed in Table I. We show the effectiveness of the proposed two NE seeking strategies (8) and (40), some comparisons, and impacts of the design parameters subsequently.

TABLE III: The comparisons among the real convergence time and the upper bounds of settling time.

	T	$T_{\max}$	$T^{\sharp}_{\max}$
NE seeking algorithm (8)	9.15	69.67	16.37

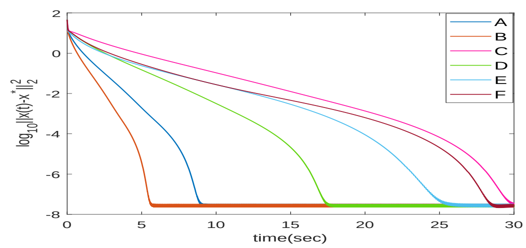


Fig. 3: Error responses for Players 1-30 under distributed NE seeking Algorithms A-F.

A. Distributed NE Seeking Algorithm (8) with $\mathbf{FIX_t}$ Convergence

Choose $\mu = \omega = 0.5$ and $\nu = \rho = 1.5$ in algorithm (8). The parameters α_i, β_i, γ_i, δ_i and η_i are listed in Table II. Simulation results are shown in Fig. 2 (a) with one random initial state. Fig. 2 (b) shows that the evolution of the players' estimates on x_1 under the Nash equilibrium seeking algorithm (8). It can be seen from Fig. 2 that the players' actions converge to the Nash equilibrium $\mathbf{x}^*$. It can also be observed from Fig. 2 that the evolution of players' estimates converge to x_1 within seven seconds and x converges to x^* in ten seconds under the distributed algorithm (8).

We now study the tightness of the estimation on the upper bound of fixed time convergence property. $\mu = \omega = 0.5$, $\nu = \rho = 1.5$, $\min_{i\in\mathcal{N}}\{\alpha_i\}$ =2, $\min_{i\in\mathcal{N}}\{\beta_i\}$ =2, $N = 30$, $\min_{i\in\mathcal{N}}\{\delta_i\}$ =1, $\min_{i\in\mathcal{N}}\{\eta_i\}$ =1, $\zeta = 3$. Thus, $T^1_{\max} = \frac{1}{A_1(1-p_1)} + \frac{1}{A_2(p_2-1)} \approx 67.56$, $T^2_{\max} = \frac{1}{B_1(1-p_3)} + \frac{1}{B_2(p_4-1)} \approx 2.11$, $T^{1\sharp}_{\max} = \frac{1}{A_1}\left(\frac{A_1}{A_2}\right)^{\frac{1-p_1}{p_2-p_1}}\left(\frac{1}{1-p_1} + \frac{1}{p_2-1}\right) \approx 14.33$, $T^{2\sharp}_{\max} = \frac{1}{B_1}\left(\frac{B_1}{B_2}\right)^{\frac{1-p_3}{p_4-p_3}}\left(\frac{1}{1-p_3} + \frac{1}{p_4-1}\right) \approx 2.04$. The comparisons among the real convergence time and the upper bounds of settling time are listed in Table III.

B. Comparison with Related Existing Distributed NE Seeking Algorithms

Let $\xi_i = \delta_i$ in the distributed NE seeking algorithm (40) and choose the same design parameters α_i, β_i, γ_i, δ_i and η_i as those in Table II.

Comparisons among our algorithms and several existing algorithms are shown in Figs. 3, where algorithms A-F refer to algorithm (8), algorithm (40), the $\mathbf{FIN_t}$ converging algorithm in [39], the asymptotical converging distributed algorithm in [14], the exponential converging distributed algorithm in [11], and the distributed algorithm in [13], respectively. It can be observed from Fig. 3 that the convergence rate of distributed algorithm (40) is faster than other five distributed algorithms, and its settling time is strictly lower than those of other five distributed algorithms.

C. Influence of Tunable Parameters

The average convergence time with error tolerance of 10^{-7} under five different sets of tunable parameters α_i, β_i, γ_i, δ_i, η_i and ξ_i in algorithms (8) and (40) for solving game (46) are shown in Table IV where each test was run with 10 random initial conditions. We choose $\mu = \omega = 0.5$ and $\nu = \rho = 1.5$ in algorithms (8) and (40) for all the tests. Test 1 adopts the same parameters as those in Table II with $\xi_i = \delta_i$ in algorithm (40). Tests 2-5 adopt 0.2, 0.5, 1.5, and 2 times of parameters δ_i and η_i as those in Test 1, respectively, while with other parameters being the same as those in Test 1.

It can be observed from Table IV that the average convergence time of algorithm (40) is less than that of algorithm (8), and the average convergence time of (40) in Test 5 is the smallest. It can be also observed from Table IV that the larger the design parameters α_i, β_i, γ_i, δ_i, η_i, and ξ_i, the smaller the convergence time. These results are consistent with the discussions in Remark 4 and Remark 6.

TABLE IV: Average convergence time of distributed algorithms (8) and (40) for problem (46).

	Test 1	Test 2	Test 3	Test 4	Test 5
(8)	10.26	25.36	16.36	8.86	3.93
(40)	7.15	21.21	13.58	6.35	2.12

V. Conclusions

This paper proposes two novel distributed NE seeking algorithms for solving non-cooperative games. It is shown that the settling time of the proposed distributed algorithms are independent of the players' initial states and the fixed-time interval can be prescribed according to the task demands, unlike existing results with asymptotical or exponential convergence. The effectiveness and advantages of the proposed algorithms are illustrated via a simulation of the energy consumption game. Future work will be directed to study non-cooperative games over directed or time-varying networks.

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