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Enhancing the settling time estimation of a class of fixed-time stable systems

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Summary

In this paper, we provide a new nonconservative upper bound for the settling time of a class of fixed-time stable systems. To expose the value and the applicability of this result, we present four main contributions. First, we revisit the well-known class of fixed-time stable systems, to show the conservatism of the classical upper estimate of its settling time. Second, we provide the smallest constant that the uniformly upper bounds the settling time of any trajectory of the system under consideration. Third, introducing a slight modification of the previous class of fixed-time systems, we propose a new predefined-time convergent algorithm where the least upper bound of the settling time is set *a priori* as a parameter of the system. At last, we design a class of predefined-time controllers for first- and second-order systems based on the exposed stability analysis. Simulation results highlight the performance of the proposed scheme regarding settling time estimation compared to existing methods.

KEYWORDS

finite-time stability, fixed-time stability, Lyapunov analysis, predefined-time stability

1 | INTRODUCTION

Convergence time is an important performance specification for a controlled system from a practical point of view.^{1,2} Indeed, the design of controllers which guarantee finite-time stability, instead of asymptotic stability, is one of the desired objectives, which appear in many applications such as missile guidance,³ hybrid formation flying,⁴ group consensus,^{5,6} online differentiators,⁷⁻⁹ state observers,¹ synchronization control,¹⁰ etc. Furthermore, in the case of switching systems, it is frequently required that the observer (or controller) achieves the stability of the observation error (or tracking error) before the next switching.^{11,12}

Lack of uniform boundedness of the settling-time function regardless of the initial conditions causes several restrictions to the practical application of finite time observer/controller.¹³⁻¹⁵ These restrictions can be relaxed using the fixed-time stability concept. It is an extension of global finite-time stability and guarantees the convergence (settling) time to be globally uniformly bounded, ie, the bound does not depend on the initial state of the system.^{3,16,17} To this end, the class of

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systems

$$\dot{x} = -(\alpha|x|^p + \beta|x|^q)^k \text{sign}(x), \quad x(0) = x_0, \quad (1)$$

where x is a scalar state variable and the real numbers $\alpha, \beta, p, q, k > 0$ are system parameters, which satisfy the constraints $kp < 1$, and $kq > 1$, was proposed in the works of Polyakov¹⁶ and Andrieu et al¹⁷ and has been extensively used. Indeed, it represents a wide class of systems, which present the fixed-time stability property through homogeneity^{17,18} and Lyapunov analysis frameworks.^{16,19} However, it is still difficult to derive a relatively simple relationship between the system parameters and the upper bound of the settling time.^{20,21} This drawback yields some difficulties in the tuning of the system parameters to achieve a prescribed-time stabilization (see the work of Cruz-Zavala et al,⁷ for instance).

The computation of the least upper bound of the settling time is usually not an easy task. Therefore, it is quite common to find an upper bound of the settling time as an attempt to estimate the least upper bound. For instance, in the work of Polyakov,¹⁶ it is shown that for system (1), the settling-time function $T(x_0)$ is bounded as

$$T(x_0) \leq \frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)} = T_{\max}, \quad \forall x_0 \in \mathbb{R}. \quad (2)$$

However, as we will show later, this bound significantly overestimates the least upper one. This overestimation can lead to restrictions for the practical implementation of prescribed-time observer/controller. In this case, the gains will be overcalculated to achieve stabilization before a prescribed time. It may lead to poor performances in terms of control magnitude or robustness against measurement noise for instance. To address this problem, there have been some attempts to improve the settling time estimation (2). In the work of Parsegov et al,¹⁹ a nonconservative upper bound was provided for the particular case where $k = 1$, $p = 1 - s$, $q = 1 + s$ with $s \in (0, 1)$. In the work of Zuo and Tie,²² an improved estimation was given for the case where $k = 1$, $q = \frac{\hat{m}}{\hat{n}} > 1$, $p = \frac{\hat{p}}{\hat{q}} < 1$ satisfying

$$0 < v = \frac{\hat{q}(\hat{m} - \hat{n})}{\hat{n}(\hat{q} - \hat{p})} \leq 1. \quad (3)$$

For this case, the upper bound for the settling time given by Zuo and Tie²² is

$$T(x_0) \leq \frac{\hat{q}}{\hat{q} - \hat{p}} \left(\frac{1}{\sqrt{\alpha\beta}} \arctan \left(\sqrt{\frac{\beta}{\alpha}} \right) + \frac{1}{\beta v} \right) = \tau_{\max}, \quad \forall x_0 \in \mathbb{R}. \quad (4)$$

Unfortunately, although (4) is less conservative than (2), this bound still significantly overestimates the least upper one, and it presents the same drawbacks like (2), as we will illustrate later.

Moreover, the settling time estimation problems get worse when analyzing or designing controllers/observers for second-order and higher-order systems, since most of the algorithms are based on the well-known block-control technique^{16,23-25} or on the homogeneity in the bi-limit property.²⁶ In particular, the homogeneity-based algorithms^{17,26} suffer from the drawback that no settling time estimate is provided. On the other hand, several block-control-based algorithms for second-order systems²³⁻²⁵ neglect some time, which the system trajectories stay on a band around a manifold in the plane. These issues add to the drawbacks mentioned before.

Considering the extensive use of the class of systems represented by (1) and the overestimation exhibited by T_{\max} in (2) and τ_{\max} in (4), this paper addresses the computation of the least upper bound of the settling-time function for this class of systems, enhancing the settling-time estimation presented in the mentioned references. With the previous result, we also present the derivation of a new predefined-time convergent algorithm where the least upper bound of the settling time is set a priori as a parameter of the system. Finally, based on this new predefined-time stable system, new predefined-time controllers for first-order and second-order systems are introduced. In contrast to the conventional approaches of homogeneity and Lyapunov analysis, the results in this paper are derived using the well-known geometric conditions proposed in the work of Haimo.¹³ The overall contribution of this paper is divided into the following four main results.

1. The well-known class of fixed-time stable systems (1) is revisited. For this class of systems, the least upper bound of the settling-time function,

$$\gamma = \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k\Gamma(k)(q-p)} \left(\frac{\alpha}{\beta} \right)^{m_p},$$

with $m_p = \frac{1-kp}{q-p}$ and $m_q = \frac{kq-1}{q-p}$, is found.

2. The following new predefined-time convergent system, where the least upper bound of the settling time T_c is set a priori as a parameter of the system, is proposed:

$$\dot{x} = -\frac{\gamma}{T_c}(\alpha|x|^p + \beta|x|^q)^k \text{sign}(x), \quad x(0) = x_0, \quad (5)$$

where x is a scalar state variable, real numbers $\alpha, \beta, p, q, k > 0$ are system parameters, which satisfy the constraints $kp < 1$ and $kq > 1$ and $T_c > 0$. Notice that the only difference between (1) and the modified system (5) is the constant gain γ/T_c . This slight change in the original system represents a considerable improvement in its properties, since the tunable parameter T_c is directly the least upper bound of the convergence time. As a consequence of this desirable feature, we say that the origin of system (5) is predefined-time stable with (strong) predefined-time T_c , a notion formally defined in Section 2.

3. The bound given in the formula (2) for system (5) is shown to be a conservative estimation of the settling time $T_f = \sup_{x_0 \in \mathbb{R}} T(x_0) = T_c$. Moreover, letting $\alpha = \rho$ and $\beta = \frac{1}{\rho}$, it is shown that, even if the least upper bound of the convergence time is T_c , the upper estimate (2), given in the work of Polyakov,¹⁶ goes to infinity as $\rho \rightarrow +\infty$ and as $\rho \rightarrow 0$. A similar argument is made for the upper bound (4), given in the work of Zuo and Tie.²²
4. New predefined-time controllers, with enhanced the settling-time estimation, are introduced for first-order and second-order scalar systems with matched and bounded perturbations.

The rest of this paper is organized as follows. In Section 2, we introduce the preliminaries on finite-time, fixed-time, and predefined-time stability. In Section 3, we present the main result on the least upper bound for the settling time and propose a new strongly predefined-time convergent algorithm where the least upper bound of the settling time is set a priori as a parameter of the system. We also present the analysis of how conservative the bound provided in the work of Polyakov¹⁶ may result and show some numerical results. In Section 4, we apply the previous result to derive new predefined-time controllers for first- and second-order systems. Finally, in Section 5, we present some concluding remarks.

2 | PRELIMINARIES AND DEFINITIONS

Consider the nonlinear system

$$\dot{x} = f(x; \rho), \quad x(0) = x_0, \quad (6)$$

where $x \in \mathbb{R}^n$ is the system state; the vector $\rho \in \mathbb{R}^b$ stands for the system (6) parameters, which are assumed to be constant, ie, $\dot{\rho} = 0$. The function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be nonlinear and continuous, and the origin is assumed to be an equilibrium point of system (6), so $f(0; \rho) = 0$.

Let us first recall some useful definitions and lemma on finite-time, fixed-time, and predefined-time stability.

Definition 1 (Lyapunov stability, definition 4.1 in the work of Khalil and Grizzle²⁷).

The origin of system (6) is said to be *Lyapunov stable* if, for all $\epsilon > 0$, there is $\delta := \delta(\epsilon) > 0$ such that for all $\|x_0\| < \delta$, any solution $x(t, x_0)$ of (6) exists for all $t \geq 0$, and $\|x(t, x_0)\| < \epsilon$ for all $t \geq 0$.

Definition 2 (Finite-time stability¹⁴).

The origin of (6) is said to be *globally finite-time stable* if it is Lyapunov stable, and for any $x_0 \in \mathbb{R}^n$, there exists $0 \leq T < +\infty$ such that the solution $x(t, x_0) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T : x(t, x_0) = 0, \forall t \geq T\}$ is called the *settling-time function*.

Lemma 1 (Finite-time stability characterization for scalar systems, fact 1 in the work of Haimo¹³).

Let $n = 1$ in system (6) (scalar system). The origin is globally finite-time stable if and only if for all $x \in \mathbb{R} \setminus \{0\}$

- (i) $xf(x; \rho) < 0$, and
- (ii) $\int_x^0 \frac{dz}{f(z; \rho)} < +\infty$.

Remark 1. A proof of Lemma 1 shall not be given here, but can be found in lemma 3.1 in the work of Moulay and Perruquetti.²⁸ Nevertheless, intuitively, condition (i) implies Lyapunov stability. Moreover, under the conditions of

Lemma 1, one can note that the settling time function is $T(x_0) = \int_0^{T(x_0)} dt$. Since first-order systems do not oscillate, the solution $x(\cdot, x_0) : [0, T(x_0)) \rightarrow [x_0, 0)$ of system (6) as a function of t defines a bijection. Using it as a variable change, the aforementioned integral is equal to (note that $\frac{1}{f(x; \rho)}$ is defined for all $x \in \mathbb{R} \setminus \{0\}$ from condition (i))

$$T(x_0) = \int_0^{T(x_0)} dt = \int_{x_0}^0 \frac{dx}{f(x; \rho)}. \quad (7)$$

Thus, condition (ii) of Lemma 1 refers to the settling-time function being finite.

Definition 3 (Fixed-time stability¹⁶).

The origin is said to be a *fixed-time stable equilibrium* of (6) if it is globally finite-time-stable and the settling time function $T(x_0)$ is bounded on \mathbb{R}^n , ie, $\exists T_{\max} > 0 : \forall x_0 \in \mathbb{R}^n : T(x_0) \leq T_{\max}$.

Remark 2. Let the origin $x = 0$ of system (6) be fixed-time stable. Notice that there are multiple upper bounds of the settling-time function T_{\max} ; for instance, if $\forall x_0, T(x_0) \leq T_{\max}$; also, $\forall x_0, T(x_0) \leq \lambda T_{\max}$ with $\lambda \geq 1$. However, from this boundedness condition, the least upper bound of the settling-time function $\sup_{x_0 \in \mathbb{R}^n} T(x_0)$ exists.

Remark 3. Notice that fixed-time stability can be derived based on the homogeneity approach introduced in the works of Andrieu et al¹⁷ and Polyakov et al.¹⁸ However, in these cases, the upper bound for the settling time is usually not obtained. To discriminate those cases to one where a settling-time bound T_c is set in advance as a function of system parameters ρ , ie, $T_c = T_c(\rho)$, we recall the concept of predefined-time stability. A strong notion of this class of stability is given when $\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c$, ie, T_c is the least upper bound for the settling-time.

Definition 4 (Predefined-time stability²⁹).

For the parameter vector ρ of system (6) and an arbitrarily selected constant $T_c := T_c(\rho) > 0$, the origin of (6) is said to be *predefined-time stable* if it is fixed-time stable and the settling-time function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ is such that

$$T(x_0) \leq T_c, \quad \forall x_0 \in \mathbb{R}^n.$$

If this is the case, T_c is called a *predefined time*. Moreover, if the settling-time function is such that $\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c$, and then T_c is called the *strong predefined time*.

Remark 4. The stability property, of any kind, refers to equilibrium points of a system. However, since this study only focuses on the global stability of the origin of the system under consideration, it may be referred hereafter, without ambiguity, to the stability of the system in the respective sense (asymptotic, fixed-time, or predefined-time).

3 | ON THE LEAST UPPER BOUND FOR THE SETTLING TIME OF A CLASS OF FIXED-TIME STABLE SYSTEMS

In this section, we present the least upper bound for the settling time of (1). Based on this result, we introduce a modification to (1) to derive a system that is strongly predefined-time stable. Afterward, we present analysis and comparison between our upper bound for the settling-time and those given in the works Polyakov¹⁶ and Zuo and Tie,²² which expose our contribution.

3.1 | Least upper bound of the settling-time function of system (1)

Before presenting the result on the least upper bound for the settling time of (1), let us introduce some auxiliary results that are fundamental to obtain such bound. In our derivation, the beta and the gamma functions³⁰ will play a central role, and the following definition provides a formal setting for those functions.

Definition 5 (See the work of Erdélyi et al^{30p87}).

Let $a, b > 0$. The beta function, denoted by $B(a, b)$, is defined as

$$B(a, b) = \int_0^1 z^{a-1} (1-z)^{b-1} dz = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

where $\Gamma(\cdot)$ is the gamma function defined as $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$ (see chapter 1 in the work of Erdélyi et al³⁰).

Our derivation of the least upper bound of the settling time is based on (7). To solve such integral, the next result is exploited.

Proposition 1. Let $\alpha, \beta, p, q, k > 0$, with $pk < 1, qk > 1$. Hence,

$$\int_0^{+\infty} \frac{dx}{(\alpha x^p + \beta x^q)^k} = \frac{\Gamma(m_p) \Gamma(m_q) \left(\frac{\alpha}{\beta}\right)^{m_p}}{\alpha^k \Gamma(k)(q-p)}. \quad (8)$$

Proof. The left-hand side of (8) can be rewritten as

$$\int_0^{+\infty} \frac{dx}{(\alpha x^p + \beta x^q)^k} = \int_0^{+\infty} \frac{\beta^{-k} x^{-qk} dx}{\left(\frac{\alpha}{\beta} x^{p-q} + 1\right)^k}. \quad (9)$$

Note that the term $z = \left(\frac{\alpha}{\beta} x^{p-q} + 1\right)^{-1}$ goes to 0 when $x \rightarrow 0$ and to 1 when $x \rightarrow +\infty$ if $p - q < 0$. Furthermore, using z as a variable change, with $x = \left(\frac{\beta}{\alpha} \left(\frac{1}{z} - 1\right)\right)^{\frac{1}{p-q}}$ and $dx = \frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-q}}}{z^2(q-p)} \left(\frac{1}{z} - 1\right)^{\frac{1}{p-q}-1} dz$, the integral (9) can be solved by direct substitution and the application of Definition 5 using some algebraic manipulation

$$\begin{aligned} \int_0^{+\infty} \frac{\beta^{-k} x^{-qk} dx}{\left(\frac{\alpha}{\beta} x^{p-q} + 1\right)^k} &= \int_0^1 z^k \beta^{-k} \left(\frac{\beta}{\alpha} \left(\frac{1}{z} - 1\right)\right)^{\frac{qk}{q-p}} \left(\frac{\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-q}}}{z^2(q-p)} \left(\frac{1}{z} - 1\right)^{\frac{1}{p-q}-1}\right) dz \\ &= \frac{\left(\frac{\alpha}{\beta}\right)^{m_p}}{\alpha^k(q-p)} \int_0^1 z^{m_p-1} (1-z)^{m_q-1} dz \\ &= \frac{B(m_p, m_q) \left(\frac{\alpha}{\beta}\right)^{m_p}}{\alpha^k(q-p)} \\ &= \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k \Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p}, \end{aligned}$$

concluding the proof. \square

In the following theorem, based on an appropriate use of the gamma function, the least upper bound of the settling-time function of system (1) is provided.

Theorem 1. Let

$$\gamma = \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k \Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p}, \quad (10)$$

where $m_p = \frac{1-kp}{q-p}$ and $m_q = \frac{kq-1}{q-p}$ are positive parameters. The origin $x = 0$ of system (1) is fixed-time stable and the settling time function satisfies $\sup_{x_0 \in \mathbb{R}^n} T(x_0) = \gamma$.

Proof. Note that, for system (1), the field is $f(x; \rho) = -(\alpha|x|^p + \beta|x|^q)^k \text{sign}(x)$, where the parameter vector is $\rho = [\alpha \beta p q k]^T \in \mathbb{R}^5$. Furthermore, the product $x f(x; \rho) = -(\alpha|x|^p + \beta|x|^q)^k |x| < 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Thus, $V(x) = \frac{1}{2}x^2$ is a radially unbounded Lyapunov function for system (1), so its origin $x = 0$ is Lyapunov stable (see Theorem 4.2 in the work of Khalil and Grizzle²⁷).

Now, let $x_0 \in \mathbb{R} \setminus \{0\}$ (if $x_0 = 0$, then $x(t, 0) = 0$ is the unique solution of (1) and $T(0) = 0$). From (7), the settling time function is

$$\begin{aligned} T(x_0) &= \int_{x_0}^0 \frac{dx}{f(x; \rho)} \\ &= \int_0^{x_0} \frac{\text{sign}(x) dx}{(\alpha|x|^p + \beta|x|^q)^k}, \quad z = |x| \\ &= \int_0^{|x_0|} \frac{dz}{(\alpha z^p + \beta z^q)^k}. \end{aligned}$$

Since the integrand $\frac{1}{(\alpha z^p + \beta z^q)^k}$ is positive for $z \in (0, |x_0|)$, the settling time function is increasing with respect to $|x_0|$. Hence, the least upper bound of $T(x_0)$ (in the extended real numbers set, since we do not know yet if it is finite or not) is obtained using Proposition 1, as

$$\sup_{x_0 \in \mathbb{R}} T(x_0) = \lim_{|x_0| \rightarrow +\infty} T(x_0) = \int_0^{+\infty} \frac{dz}{(\alpha z^p + \beta z^q)^k} = \gamma,$$

with $\gamma < +\infty$ as in (10). Using Lemma 1 and by the definitions of finite-time and fixed-time stability, the origin $x = 0$ of system (1) is fixed-time stable and $T_f = \gamma$, which completes the proof. \square

3.2 | A class of predefined-time stable systems

Now, the result presented in Theorem 1 is used to derive a Lyapunov-like condition for characterizing predefined-time stability of a system.

Theorem 2 (A Lyapunov characterization for predefined-time stable systems).

If there exists a continuous positive definite radially unbounded function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that any solution $x(t, x_0)$ of (6) satisfies

$$\dot{V}(x) \leq -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (11)$$

where $\alpha, \beta, p, q, k > 0$, $kp < 1$, $kq > 1$ and γ is given in (10).

Then, the origin of (6) is predefined-time stable and T_c is a predefined time. If, in addition, the equality holds in (11), then T_c is the strong predefined time.

Proof. The negative definiteness of the time derivative of the function V implies Lyapunov stability of the origin of (6). Now, suppose that there exists a function $w(t) \geq 0$ that satisfies

$$\dot{w} = -(\hat{\alpha} w^p + \hat{\beta} w^q)^k,$$

where $\hat{\alpha} = \alpha \left(\frac{\gamma(\rho)}{T_c}\right)^{\frac{1}{k}}$ and $\hat{\beta} = \beta \left(\frac{\gamma(\rho)}{T_c}\right)^{\frac{1}{k}}$, and $V(x_0) \leq w(0)$. Hence, by Theorem 1, $w(t)$ will converge to the origin in a strong predefined time

$$\frac{\Gamma(m_p)\Gamma(m_q)}{\hat{\alpha}^k \Gamma(k)(q-p)} \left(\frac{\hat{\alpha}}{\hat{\beta}}\right)^{m_p} = \frac{T_c}{\gamma(\rho)} \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k \Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_p} = T_c,$$

which is directly a tunable parameter of the system. Furthermore, by the comparison lemma (lemma 3.4 in the work of Khalil and Grizzle²⁷), it follows that $V(x(t)) \leq w(t)$, with equality only if (11) is an equality. Consequently, the origin of system (6) is predefined-time stable with predefined time T_c . Moreover, if (11) is an equality, then $\sup_{x_0 \in \mathbb{R}^n} T(x_0) = T_c$, ie, T_c is the strong predefined time. \square

Example 1. Consider system (5) and the continuous positive definite radially unbounded Lyapunov candidate function $V(x) = |x|$ for this system. For $x \neq 0$, the derivative of $V(x)$ along the trajectories of system (5) is

$$\dot{V}(x) = -\text{sign}(x) \frac{\gamma}{T_c} (\alpha|x|^p + \beta|x|^q)^k \text{sign}(x) = -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k.$$

Hence, by Theorem 2, the origin of system (5) is predefined-time stable with strong predefined time T_c .

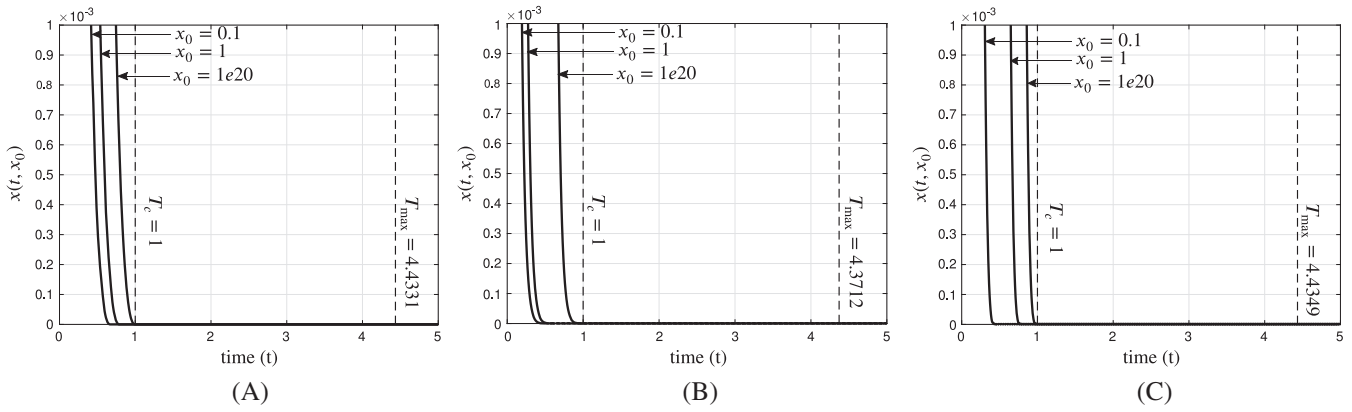


FIGURE 1 Trajectories of (5) for different initial conditions with least upper estimate is $T_c = 1$ s and (A) $\alpha = 4$, $\beta = 0.25$, $p = 0.5$, $q = 3$, $k = 1.5$ and upper estimate (2) in $T_{\max} = 4.4331$ s. (B) $\alpha = 0.2$, $\beta = 10$, $p = 0.8$, $q = 1.2$, $k = 0.9$ and upper estimate in (2) $T_{\max} = 4.7312$ s. (C) $\alpha = 0.06$, $\beta = 100$, $p = 1.2$, $q = 2.4$, $k = 0.5$ and upper estimate in (2) $T_{\max} = 4.4349$ s

To illustrate the aforementioned equation, some numerical simulations of system (5) are conducted for different values of the parameters p, q, k, α and β . The simulations are conducted for several initial conditions x_0 , as presented in Figure 1A to 1C. It can be seen that $\sup_{x_0 \in \mathbb{R}} T(x_0) = T_c = 1$ as stated above.

Remark 5. In other works,^{19,31,32} the least upper estimation of the settling time of (1) was addressed for the case where $k = 1$, $p = 1 - s$, and $q = 1 + s$, with $0 < s < 1$, where $\gamma(\rho)$ reduces to $\gamma(\rho) = \frac{\Gamma(\frac{1}{2})^2}{2s\sqrt{\alpha\beta}} = \frac{\pi}{2s\sqrt{\alpha\beta}}$. In the work of Jiménez-Rodríguez et al.,³³ it was shown that, in the case $\alpha = \beta = \frac{\pi}{2sT_c}$, the least upper bound of the settling time is T_c . Thus, Theorem 2 is a generalization of the results presented in the works of Parsegov et al.¹⁹ and Jiménez-Rodríguez et al.³³ Since only for the case where $k = 1$, $p = 1 - s$, and $q = 1 + s$, with $0 < s < 1$, a nonconservative upper bound estimate is provided; many applications, for instance, fixed-time consensus protocols,³⁴⁻³⁷ have been restricted to this case.

Given the relevance of fixed-time stability argued in the introduction, inequality (11) is a result of paramount importance, since as we show in the following, the upper estimate (2) is often too conservative. Thus, applications based on the upper estimate of the settling time (2) presented in Lemma 1 in the work of Polyakov,¹⁶ are often overengineered.

3.3 | Settling time bound analysis and comparison

Consider the predefined-time stable system (5), with strong predefined-time T_c . From (2), calculated in the work of Polyakov,¹⁶ an upper bound for the settling time $T(x_0)$ is

$$T(x_0) \leq \frac{T_c}{\gamma(\rho)} \left(\frac{1}{\alpha^k(1-pk)} + \frac{1}{\beta^k(qk-1)} \right), \quad \forall x_0 \in \mathbb{R}. \quad (12)$$

Let $\rho > 0$, $\alpha = \rho$ and $\beta = \frac{1}{\rho}$. Assuming that p, q , and k remain constant, and noticing that γ is a function of ρ , it can be seen that varying ρ the least upper bound of the settling time remains constant and equal to T_c . However, the bound (12) becomes

$$T(x_0) \leq T_{\max}(\rho) := \frac{T_c}{K} \left(\frac{1}{\rho^{2m_p}(1-pk)} + \frac{\rho^{2(k-2m_p)}}{(qk-1)} \right), \quad (13)$$

where $K = \frac{\Gamma(m_p)\Gamma(m_q)}{\Gamma(k)(q-p)}$. It is easy to see that

$$\lim_{\rho \rightarrow 0} T_{\max}(\rho) = \lim_{\rho \rightarrow \infty} T_{\max}(\rho) = +\infty,$$

ie, $T_{\max}(\rho)$ in (13) has no upper bound as ρ increases or tends to zero.

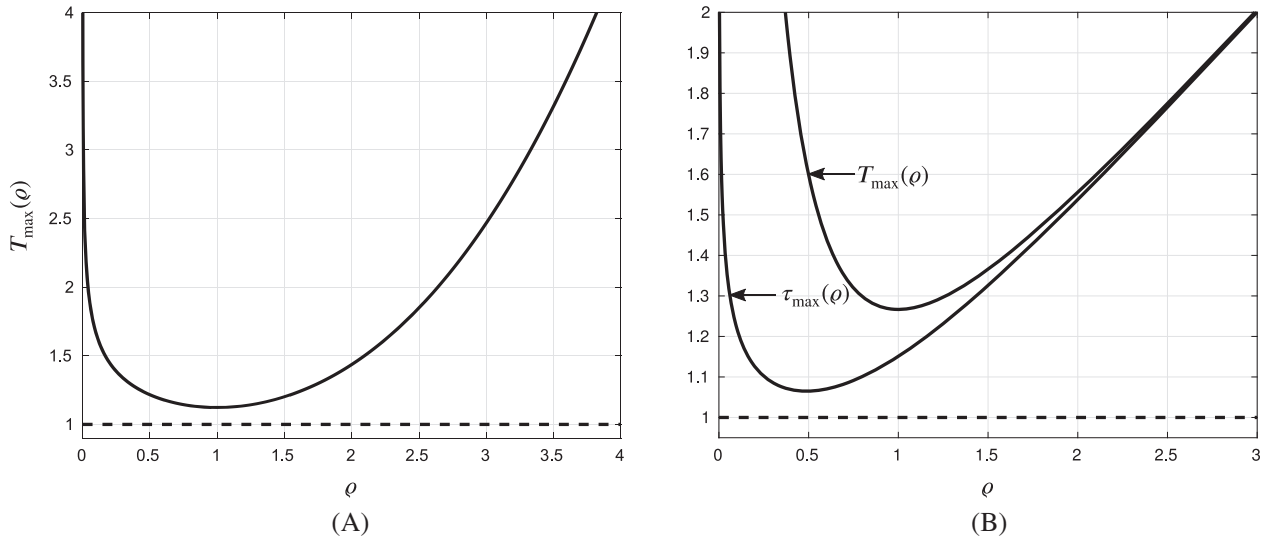


FIGURE 2 A, Function $T_{\max}(\rho)$ for $p = 0.5$, $q = 3$, and $k = 1.5$. The least upper bound of the settling time is $T_c = 1$ s. The smallest value of $T_{\max}(\rho)$ is $\min_{\rho>0} T_{\max}(\rho) = T_{\max}(1) = 1.1249$ s. B, Functions $T_{\max}(\rho)$ and $\tau_{\max}(\rho)$ for $p = 1/10$, $q = 5/3$, and $k = 1$. The least upper bound of the settling time is $T_c = 1$ s

Moreover, the best upper estimate of the bound (13) is achieved at $\arg \min_{\rho>0} T_{\max}(\rho) = 1$, and its value is

$$\min_{\rho>0} T_{\max}(\rho) = T_{\max}(1) = \frac{T_c}{K} \left(\frac{1}{(1-pk)} + \frac{1}{(qk-1)} \right) > T_c.$$

An illustration of this argument, showing $T_{\max}(\rho)$ as a function of ρ , with $T_c = 1$ s, is presented in Figure 2A. Although, by Theorem 2 the least upper bound of the settling time is $T_c = 1$ s, it can be seen that, in the best case, the bound (12) provides an overestimation of εT_c with $\varepsilon = \frac{1}{K} \left(\frac{1}{(1-pk)} + \frac{1}{(qk-1)} \right) |_{p=0.5, q=3, k=1.5} = 1.1249$. Moreover, in Figure 1A, it can be seen that, although the least upper bound of the settling-time function is $T_c = 1$ s, the upper bound estimation provided by (13) is $T_{\max}(\rho)|_{\rho=4} = 4.4331$ s.

A similar argument is done for (4), proposed in the work of Zuo and Tie²² for the case of $k = 1$ and $0 < v < 1$, where v is given in (3). Let $\rho > 0$, $\alpha = \rho$ and $\beta = \frac{1}{\rho}$; then, the upper bound for the settling time $T(x_0)$ in (4) becomes

$$T(x_0) \leq \tau_{\max}(\rho) := \frac{T_c \hat{q}}{K(\hat{q} - \hat{p})} \left(\frac{1}{\rho^{2m_p-1}} \arctan \left(\frac{1}{\rho} \right) + \frac{\rho^{2-2m_p}}{v} \right) \leq T_{\max}(\rho), \quad \forall x_0 \in \mathbb{R}. \quad (14)$$

Likewise, this estimation satisfies

$$\lim_{\rho \rightarrow 0} \tau_{\max}(\rho) = \lim_{\rho \rightarrow \infty} \tau_{\max}(\rho) = +\infty,$$

and therefore does not have an upper bound as ρ increases or near zero. Figure 2B shows $T_{\max}(\rho)$ and $\tau_{\max}(\rho)$ as functions of ρ , with $T_c = 1$ s. Two different parameter selections are illustrated, ie, $p = 0.5$, $q = 3$, and $k = 1.5$; and $p = 1/10$, $q = 5/3$, and $k = 1$; notice that only for the second case the condition (3) is satisfied with $v = 0.8$. It can be noted that, even though (4) is a better upper estimate of the settling time than (2), both are too conservative and its overestimation is not upper bounded.

4 | APPLICATION TO ROBUST PREDEFINED-TIME STABILIZATION FOR FIRST- AND SECOND-ORDER SYSTEMS

In this section, to illustrate the application of our main result provided in Section 3, we exploit the result in Theorem (2) to derive new robust controllers for first- and second-order perturbed systems. The advantage of such controllers with respect to the state of the art, such as the work of Tian et al,²⁶ is two-fold. First, these controllers are designed to satisfy time constraints and the desired convergence bound is selected as a parameter of the controller. Second, our controllers are less overengineered, since the slack between the desired upper bound and the real convergence time is significantly lower in our approach, for instance, when compared to the work of Polyakov.¹⁶

4.1 | First-order predefined-time controllers

Consider the following perturbed first-order system:

$$\dot{x}(t) = u(t) + \Delta(t), \quad (15)$$

where $x \in \mathbb{R}$ is the state variable, $u \in \mathbb{R}$ is the control input, and $\Delta(t) \in \mathbb{R}$ is an unknown but bounded perturbation term of the form $|\Delta(t)| \leq \delta$, with $0 \leq \delta < \infty$ a known constant.

The objective is to design the control input u such that the origin $x = 0$ of system (15) is predefined-time stable, in spite of the unknown perturbation term $\Delta(t)$.

Theorem 3. Let $\alpha, \beta, p, q, k > 0$, $kp < 1$, $kq > 1$, $T_c > 0$, $\zeta \geq \delta$, and γ be as in (10). If the control input u is selected as

$$u = - \left[\frac{\gamma}{T_c} (\alpha |x|^p + \beta |x|^q)^k + \zeta \right] \text{sign}(x), \quad (16)$$

then the origin $x = 0$ of system (15) is predefined-time stable with predefined time T_c .

Proof. Consider the continuous radially unbounded Lyapunov function candidate $V(x) = |x|$. Its derivative along the trajectories of the closed system (15) and (16) yields

$$\begin{aligned} \dot{V}(x) &= -\text{sign}(x) \left[\frac{\gamma}{T_c} (\alpha |x|^p + \beta |x|^q)^k \text{sign}(x) + \zeta \text{sign}(x) - \Delta \right] \\ &= -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k - \zeta + \Delta \text{sign}(x) \\ &\leq -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k - \zeta + |\Delta \text{sign}(x)| \\ &\leq -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k - (\zeta - \delta) \\ &\leq -\frac{\gamma}{T_c} (\alpha V(x)^p + \beta V(x)^q)^k. \end{aligned}$$

Hence, using Theorem 2, the origin $x = 0$ of system (15) is predefined-time stable with predefined time T_c . \square

Remark 6. Notice that, due to the disturbance affecting the system, the convergence to the origin may occurs before T_c even as $|x_0|$ tends to ∞ . In the worse case, when the disturbance is such that $\Delta(t) = \zeta \text{sign}(x)$, the settling time tends to T_c as $|x_0|$ tends to ∞ .

Example 2. An example of this approach is shown in Figure 3, where the control input u given in (16), with $\zeta = 1$, $p = 0.5$, $q = 3$, $k = 1.5$, and $\alpha = 1/\beta = \rho = 4$, is applied to the perturbed system (15) with disturbance $\Delta(t) = \sin(2\pi t/5)$ in solid line and with disturbance $\Delta(t) = 1$ in dashed line. Note that, although the upper bound of the settling-time function is $T_c = 1s$ using the proposed scheme, the upper bound estimation provided by the work of Polyakov¹⁶ is $T_{\max}(\rho)|_{\rho=4} = 4.4331s$.

4.2 | Second-order predefined-time controllers

Consider the following perturbed second-order system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \Delta(t), \end{aligned} \quad (17)$$

where $x_1, x_2 \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the control input, and $\Delta(t) \in \mathbb{R}$ is an unknown but bounded perturbation term of the form $|\Delta(t)| \leq \delta$, with $0 \leq \delta < \infty$ a known constant.

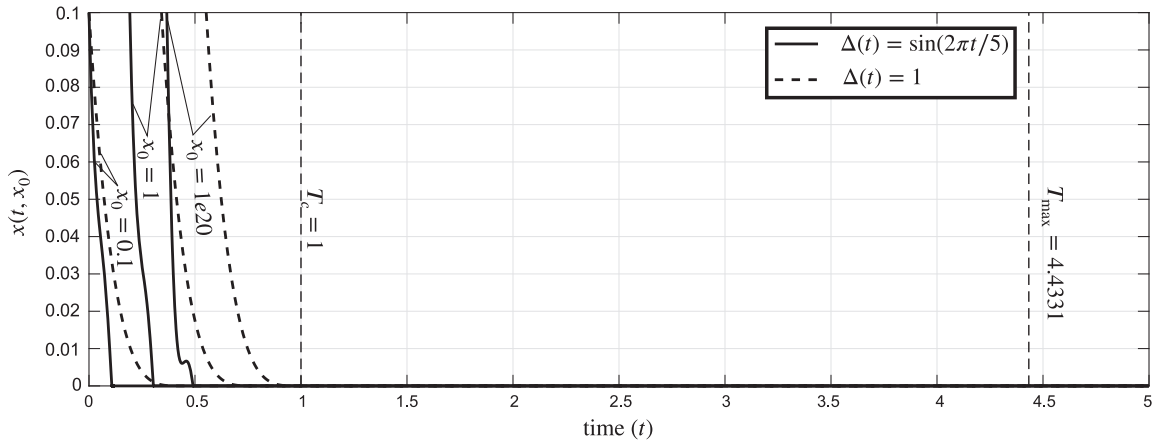


FIGURE 3 State trajectory of the perturbed closed-loop system (15) and (16) with $T_c = 1$ s. In solid line, the state trajectory when $\Delta(t) = \sin(2\pi t/5)$. In dashed line, the state trajectory when $\Delta(t) = 1$. The computation of the upper bound for the settling time using the work of Polyakov¹⁶ results in $T_{\max} = 4.4331$

The objective is to design the control input u such that the origin $(x_1, x_2) = (0, 0)$ of system (17) is predefined-time stable, in spite of the unknown perturbation term $\Delta(t)$. To this end, define the sliding variable σ as in the work of Polyakov,¹⁶

$$\sigma = x_2 + \left[|x_2|^2 + \frac{\gamma_1^2}{T_{c_1}^2} (\alpha_1 |x_1|^1 + \beta_1 |x_1|^3) \right]^{1/2}, \quad (18)$$

with $\alpha_1, \beta_1, T_{c_1} > 0$ and $\gamma_1 = \frac{\Gamma(\frac{1}{4})^2}{2\alpha_1^{1/2}\Gamma(\frac{1}{2})} (\frac{\alpha_1}{\beta_1})^{1/4}$ (note that γ_1 is selected as in (10) with $\alpha = \alpha_1, \beta = \beta_1, p = 1, q = 3$, and $k = 1/2$). Note that, if the manifold $\sigma = 0$ is enforced, the system (17) and (18) reduces to

$$\dot{x}_1 = -\frac{\gamma_1}{T_{c_1}} (\alpha_1 |x_1| + \beta_1 |x_1|^3)^{1/2} \text{sign}(x_1),$$

whose origin is predefined-time stable, by Theorem 2.

With this prior analysis, it only remains to design a controller which enforces the sliding mode on the manifold $\sigma = 0$ in predefined time. A controller with such a feature is introduced in Theorem 4.

The following definition will be useful for stating Theorem 4.

Definition 6. For any real number r , the function $x \mapsto [x]^r$ is defined as $[x] = |x|^r \text{sign}(x)$ for any $x \in \mathbb{R}$ if $r > 0$, and for any $x \in \mathbb{R} \setminus \{0\}$ if $r \leq 0$.

Theorem 4. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, p, q, k > 0, kp < 1, kq > 1, T_{c_1}, T_{c_2} > 0, \zeta \geq \delta$, and

$$\gamma_1 = \frac{\Gamma(\frac{1}{4})^2}{2\alpha_1^{1/2}\Gamma(\frac{1}{2})} \left(\frac{\alpha_1}{\beta_1}\right)^{1/4}, \text{ and } \gamma_2 = \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha_2^k\Gamma(k)(q-p)} \left(\frac{\alpha_2}{\beta_2}\right)^{m_p},$$

with $m_p = \frac{1-kp}{q-p}$ and $m_q = \frac{kq-1}{q-p}$ (note that γ_2 is selected as in (10) with $\alpha = \alpha_2$ and $\beta = \beta_2$). If the control input is selected as

$$u = - \left[\frac{\gamma_2}{T_{c_2}} (\alpha_2 |\sigma|^p + \beta_2 |\sigma|^q)^k + \frac{\gamma_1^2}{2T_{c_1}^2} (\alpha_1 + 3\beta_1 x_1^2) + \zeta \right] \text{sign}(\sigma), \quad (19)$$

where the sliding variable σ is defined as in (18), then the origin $(x_1, x_2) = (0, 0)$ of system (17) is predefined-time stable with predefined time $T_c = T_{c_1} + T_{c_2}$.

Proof. The time-derivative of the sliding variable σ in (18) is

$$\begin{aligned}\dot{\sigma} &= u + \Delta + \frac{|x_2|(u + \Delta) + \frac{\gamma_1^2}{2T_{c1}^2}(\alpha_1 + 3\beta_1 x_1^2)x_2}{\left| |x_2|^2 + \frac{\gamma_1^2}{T_{c1}^2}(\alpha_1 |x_1|^1 + \beta_1 |x_1|^3) \right|^{1/2}} \\ &= -\frac{\gamma_2}{T_{c2}}(\alpha_2 |\sigma|^p + \beta_2 |\sigma|^q)^k \text{sign}(\sigma) - \zeta \text{sign}(\sigma) + \Delta - \frac{\gamma_1^2}{2T_{c1}^2}(\alpha_1 + 3\beta_1 x_1^2) \text{sign}(\sigma) \\ &\quad - \frac{\frac{\gamma_2}{T_{c2}}(\alpha_2 |\sigma|^p + \beta_2 |\sigma|^q)^k \text{sign}(\sigma) + \zeta \text{sign}(\sigma) - \Delta}{\left| |x_2|^2 + \frac{\gamma_1^2}{T_{c1}^2}(\alpha_1 |x_1|^1 + \beta_1 |x_1|^3) \right|^{1/2}} - \frac{\frac{\gamma_1^2}{2T_{c1}^2}(\alpha_1 + 3\beta_1 x_1^2)}{\left| |x_2|^2 + \frac{\gamma_1^2}{T_{c1}^2}(\alpha_1 |x_1|^1 + \beta_1 |x_1|^3) \right|^{1/2}} (|x_2| \text{sign}(\sigma) - x_2).\end{aligned}$$

Now, considering $V_2(\sigma) = |\sigma|$ as a continuous radially unbounded positive definite Lyapunov candidate function, it can be easily checked using the aforementioned equation that

$$\dot{V}_2(\sigma) < -\frac{\gamma_2}{T_{c2}}(\alpha_2 V_2(\sigma)^p + \beta_2 V_2(\sigma)^q)^k, \quad (20)$$

and using Theorem 2, the origin $\sigma = 0$ of the sliding variable dynamics is predefined-time stable with predefined time T_{c2} .

Once the system trajectories are constrained to the manifold $\sigma = 0$, ie, for $t \geq T_{c2}$, the solutions of system (17) satisfy the following reduced-order dynamics (see (18)):

$$\dot{x}_1 = x_2 = -\frac{\gamma_1}{T_{c1}}(\alpha_1 |x_1| + \beta_1 |x_1|^3)^{1/2} \text{sign}(x_1).$$

Thus, considering $V_1(x_1) = |x_1|$ as a continuous radially unbounded positive definite Lyapunov candidate function and using Theorem 2, it is concluded that the origin $x_1 = 0$ of the reduced-order system is predefined-time stable with predefined time T_{c1} . Moreover, from (18), if $\sigma = 0$ and $x_1 = 0$, then $x_2 = 0$. Hence, it is concluded that the origin $(x_1, x_2) = (0, 0)$ of the closed-loop system (17) to (19) is predefined-time stable with predefined time $T_{c1} + T_{c2}$. \square

Remark 7. Notice that, in the second-order controller (19), T_c is not the least upper bound for the settling time of the error dynamics, ie, even in the worse case disturbance, there is a nonzero slack between T_c and the real settling time, even as $|\sigma(0)| \rightarrow \infty$ and $|x_1(0)| \rightarrow \infty$. This is because, in the proof of Theorem 4, the mathematical development

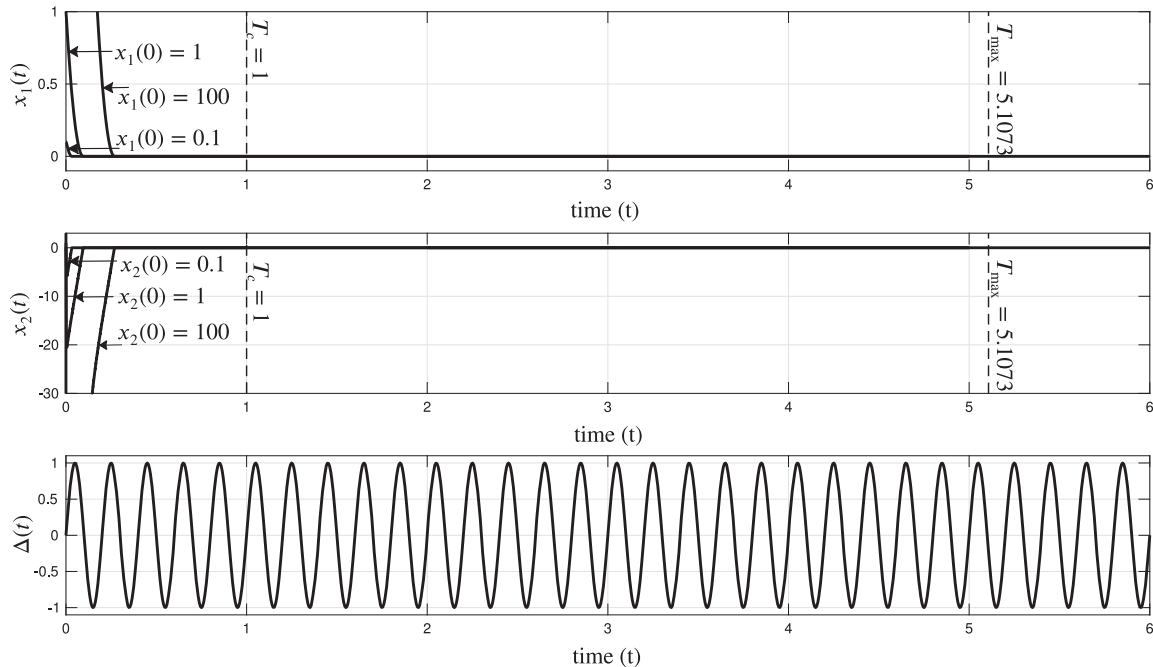


FIGURE 4 State trajectory of the closed-loop system (17) to (19) with $T_c = 1$ s and the considered perturbation

leading to the application of Theorem 2 results in the proper inequality (20). However, the estimation of the settling time using the proposed approach leads to a significantly lower slack between T_c and the real settling time than with existing methods, such as the work of Polyakov,¹⁶ as illustrated in Example 3.

Example 3. An example of this approach is shown in Figure 4, where the control input u given in (19), with $\zeta = 1$, $p = 0.5$, $q = 3$, $k = 1.5$, and $\alpha_1 = \alpha_2 = 1/\beta_1 = 1/\beta_2 = 4$ is applied to the perturbed system (17) with disturbance $\Delta(t) = \sin(2\pi t/5)$. It follows from Theorem 4 that, with $T_{c_1} = T_{c_2} = 0.5s$, the origin $(x_1, x_2) = (0, 0)$ of system (17) is predefined-time stable with predefined time $T_c = 1s$. Note that, although the upper bound of the settling-time function is $T_c = 1s$ using the proposed scheme, the upper bound estimation provided by the work of Polyakov¹⁶ is

$$T_{\max} = \frac{T_{c_2}}{\gamma_2} \left(\frac{1}{\alpha_2^k(1-pk)} + \frac{1}{\beta_2^k(qk-1)} \right) + \frac{2T_{c_1}}{\gamma_1} \left(\frac{1}{\sqrt{\alpha_1}} + \frac{1}{\sqrt{\beta_1}} \right) = 5.1073s.$$

Notice, that γ in (16) and γ_1 and γ_2 in (19) are constants that can be computed offline.

5 | CONCLUSION

In this paper, we have studied the convergence time of a class of fixed-time stable systems to provide a new nonconservative upper bound for its settling time. We showed that the well-known upper bound condition for the settling time of this class of systems is often too conservative. To illustrate our claim, we showed how by changing one parameter the upper estimate of the settling time tends to infinity even though the actual settling time is always bounded by a constant T_c . To address this problem, we proposed a modification to the classical fixed-time algorithm to transform it into a strongly predefined time (in which the upper bound for the settling time is set in advance as a parameter of the system and is the lowest upper estimate of the settling time) with strong predefined-time T_c . With this result, the Lyapunov inequality, which is a sufficient condition for fixed-time stability, was modified in a way that it becomes predefined-time parametrized by T_c . When such inequality becomes equality, T_c becomes the lowest upper estimate of the settling time of the system. At last, predefined-time controllers for first-order and second-order systems were introduced. Some simulation results have shown the performance of the proposed scheme in terms of settling time estimation compared to existing methods.

As future work, we consider the case of consensus algorithms,⁶ online differentiators, observers, and controllers satisfying prescribed-time and prescribed-performance objectives. The extension to fixed-time stability analysis of high-order systems will also be considered in the future. As the first proposal, recent works on controller^{10,38-43} and observer design,^{44,45} including the cases of on-line differentiation,^{46,47} and parameter estimation⁴⁸ can be revisited to improve the calculation of their settling-time.

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