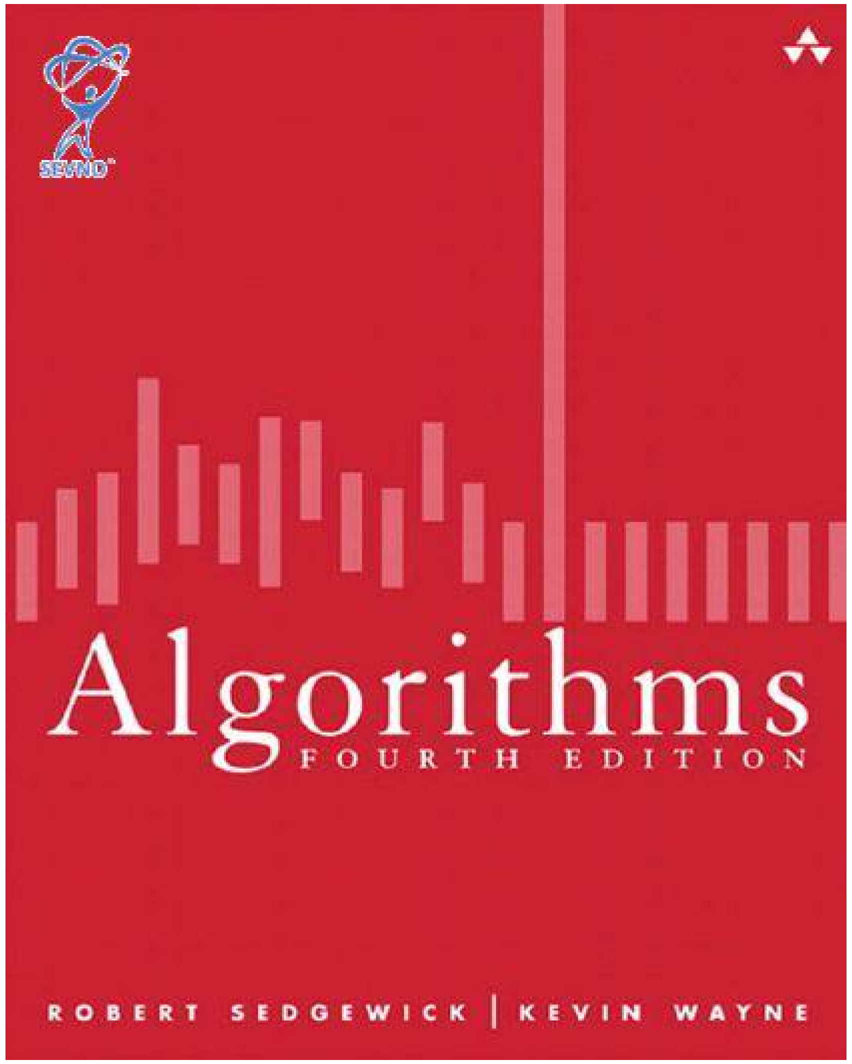
# Introduction to Algorithms

*adapted from Kevin Wayne's slides @ Princeton Univ.*

# Book

- Slide content mostly extracted from :



# Agenda

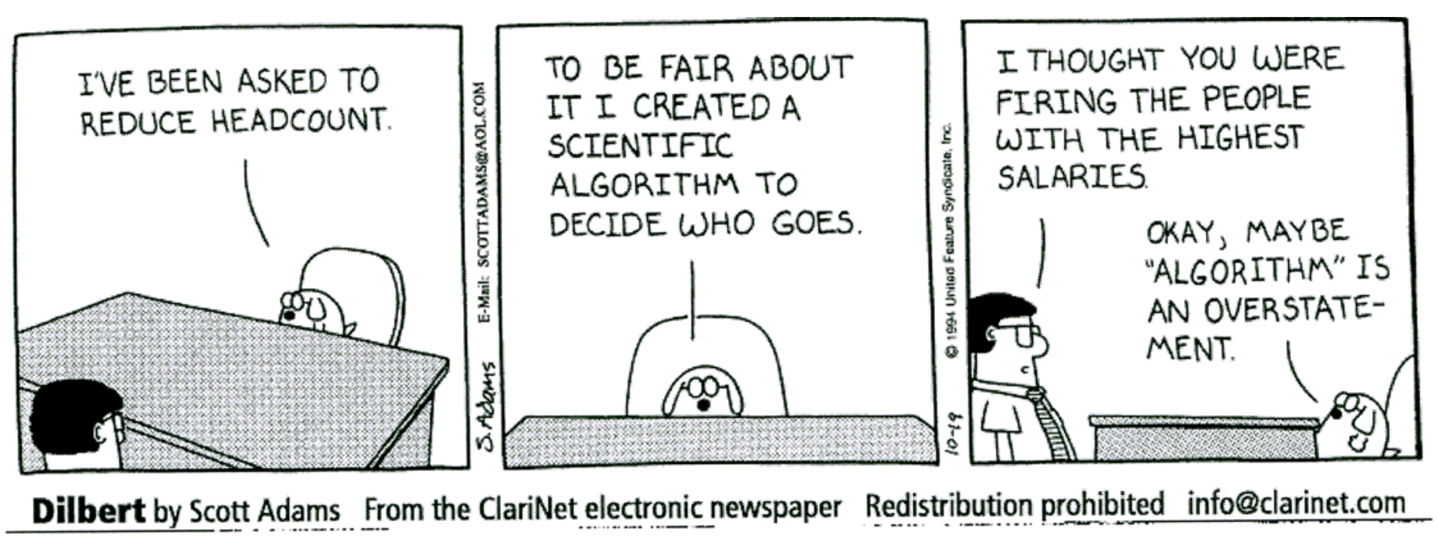
- Introduction to Algorithms
- Algorithm case study.
  - Model the problem.
  - Find an algorithm to solve it.
  - Fast enough? Fits in memory?
  - If not, figure out why.
  - Find a way to address the problem.
  - Iterate until satisfied.

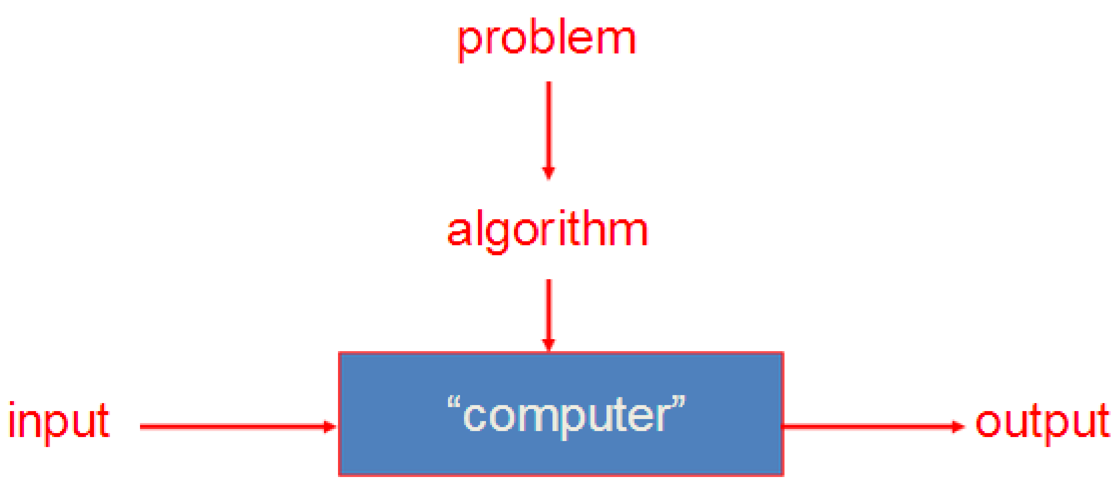
- The scientific method.

- Mathematical analysis.

# What’s an Algorithm

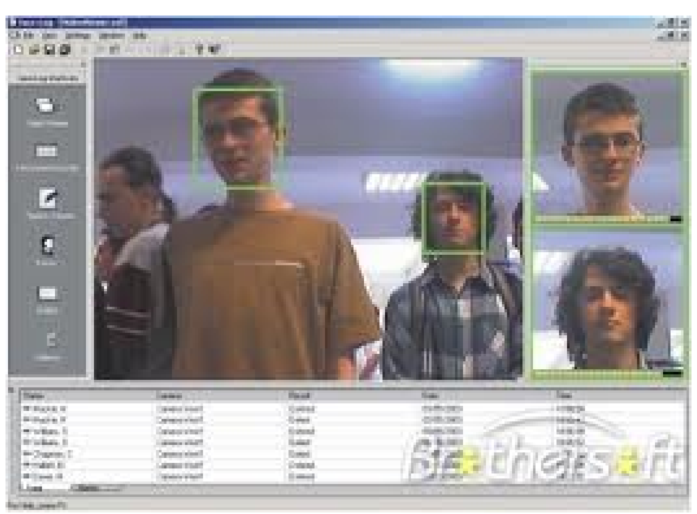
- An ***algorithm*** is a sequence of **unambiguous instructions** for **solving a problem**, i.e., for obtaining a required output for any **legitimate input** in a **finite amount of time**.

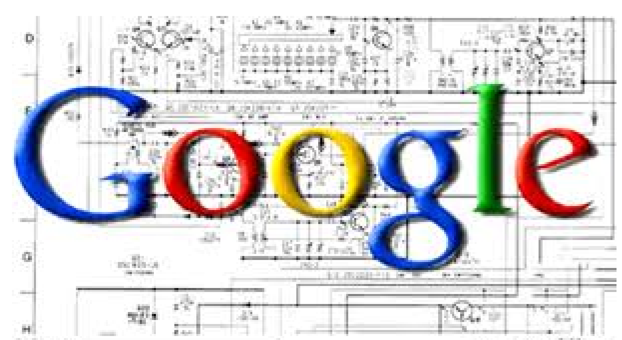


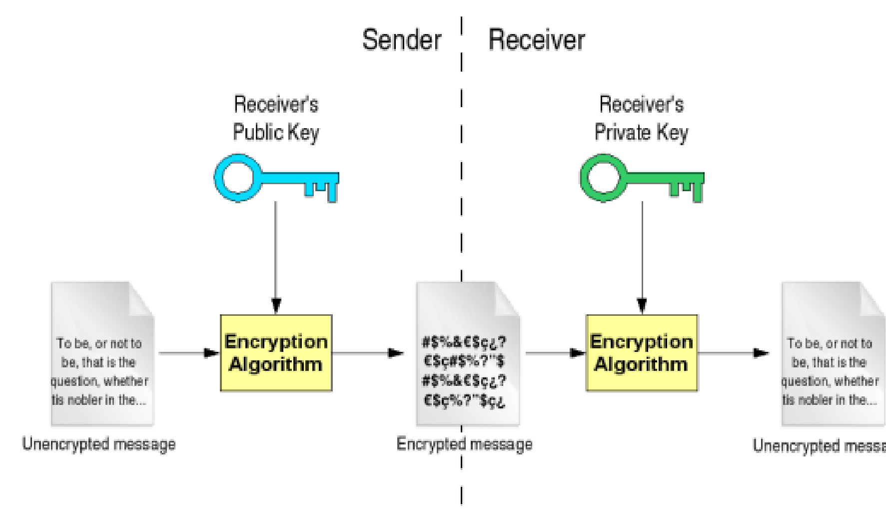



# Why Algorithms

- Algorithms are everywhere…
    - Web Searching, packet routing, peer-to-peer/file sharing
    - Human genome project
    - Circuit layout on silicon
    - Multimedia Image and signal processing(e.g. MP3, divx…
    - Security and Encryption.
    - Biometrics(fingerprint scanning/face recognition)
    - …





# Why Algorithms

- *"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing. " — Francis Sullivan*

- http://en.wikipedia.org/wiki/John_G.F._Francis

- You can have good "poems" and then you have better "poems"…

# Why Algorithms

- *“ Algorithms: a common language for nature, human, and computer. ” — Avi Wigderson*

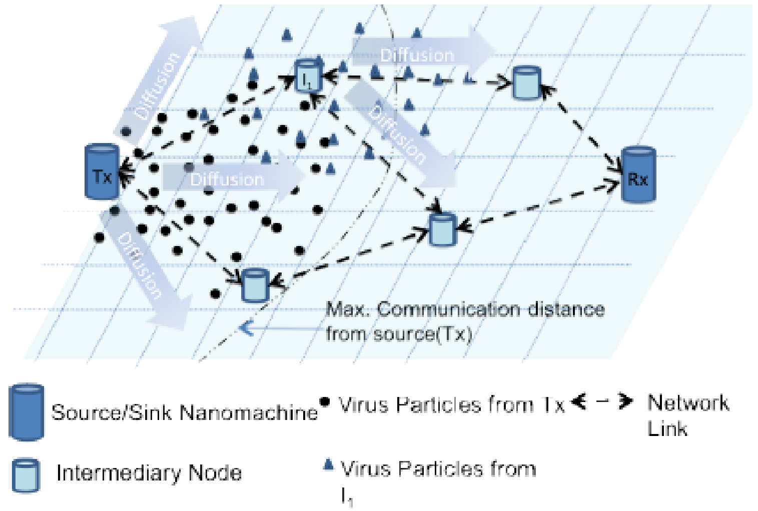
## Mathematical Models

Fick’s Law (Diffusion Equation) and solution

$$\frac{\delta V}{\delta t} = D\frac{\delta^2 V}{\delta r^2} - k_d V$$

$$c_{ij}(\tau_\eta) = \frac{Q}{4\pi D\tau} exp(\frac{-r_{ij}^2}{4D\tau} - k_d\tau)$$

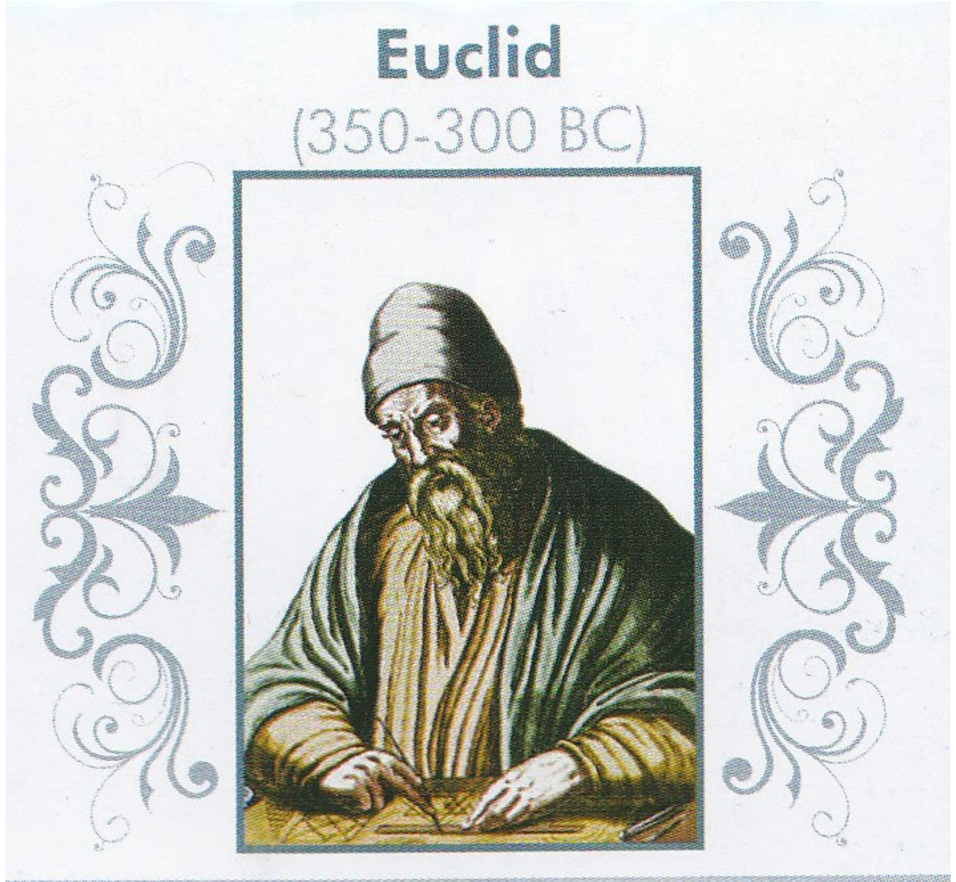
## Computational Models (& Algorithms)




```
function [ out ] = instantPoint2D( r,t,Q,D )
%Calculates Concentration of particles at
%distance r at time t from release of instantaneous conc. Q at t=0
kd=3.34e-5;
a=Q/(4*pi*D*t);
b=exp((-(r^2)/(4*D*t))-(kd*t));
out=a*b;
end
```

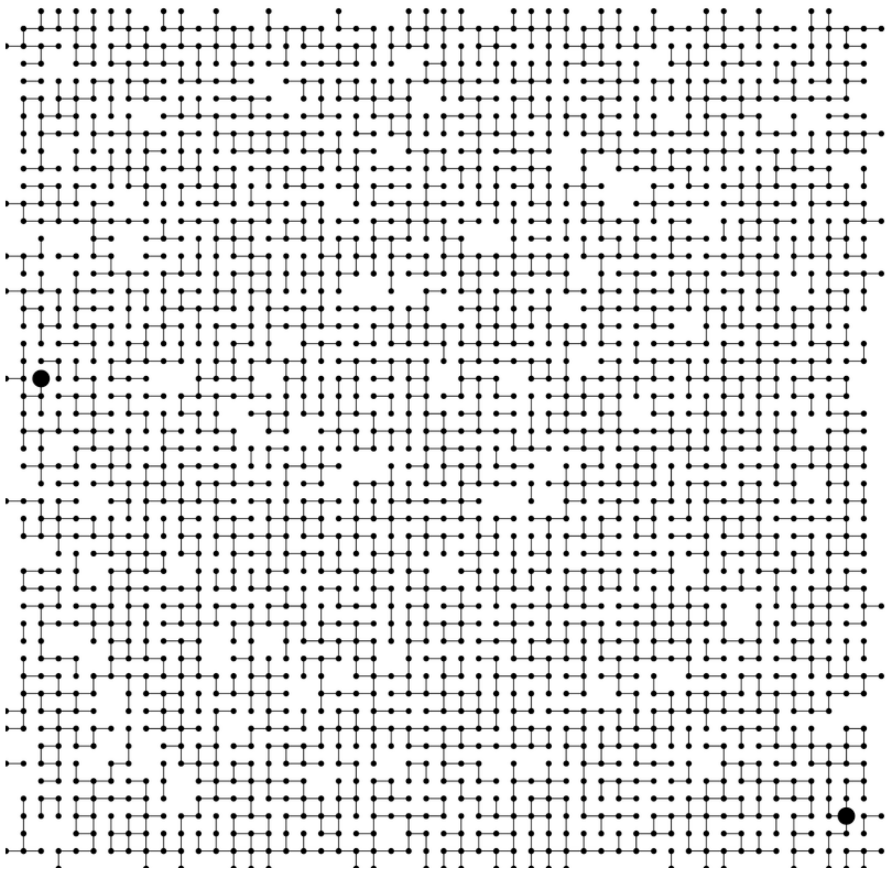
# Historical Perspective

- Euclid's algorithm for finding the greatest common divisor dates back to 300bc (they've been around for a long time…)
- http://en.wikipedia.org/wiki/Greatest_common_divisor
- The name "Algorithm" is derived from Muhammad ibn Musa al-Khwarizmi – 9th century mathematician
- www.lib.virginia.edu/science/parshall/khwariz.html




# Example Problems

- Network connectivity – “Can you get from a to b”



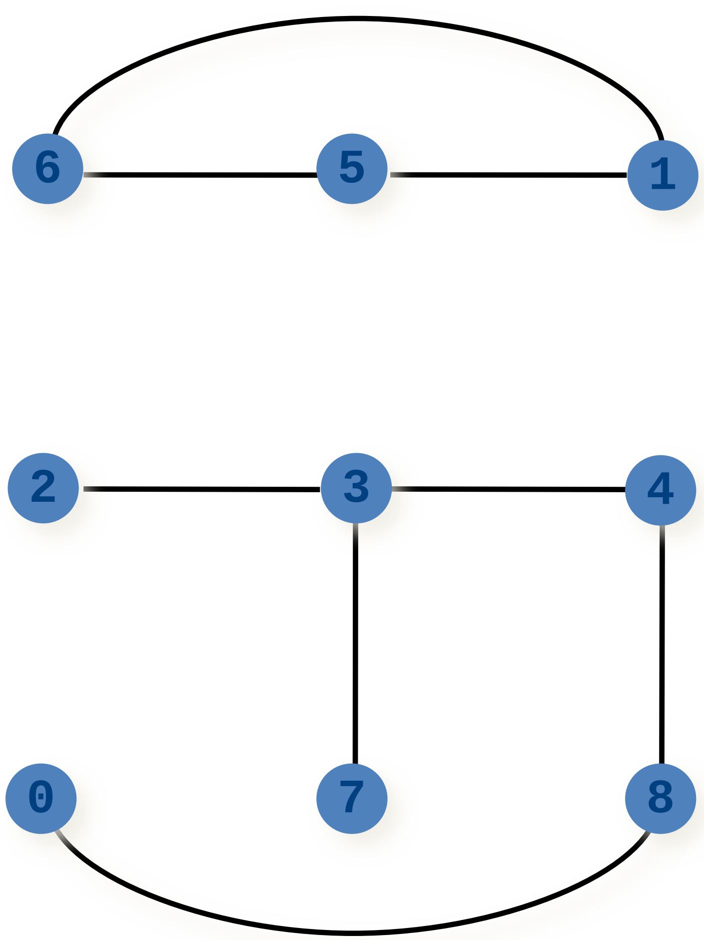
- Sort Class Results for Algorithms in ascending order:
  - *Input:* A sequence of *n* numbers $<a_1, a_2, \ldots, a_n>$
  - *Output:* A reordering of the input sequence $<a'_1, a'_2, \ldots, a'_n>$ so that $a'_i \leq a'_j$ whenever $i < j$
    - *Algorithms: Selection Sort, Insertion Sort, Merge Sort…*

# Dynamic connectivity

- Given a set of objects
  - Union: connect two objects.
  - Connected: is there a path connecting the two objects?

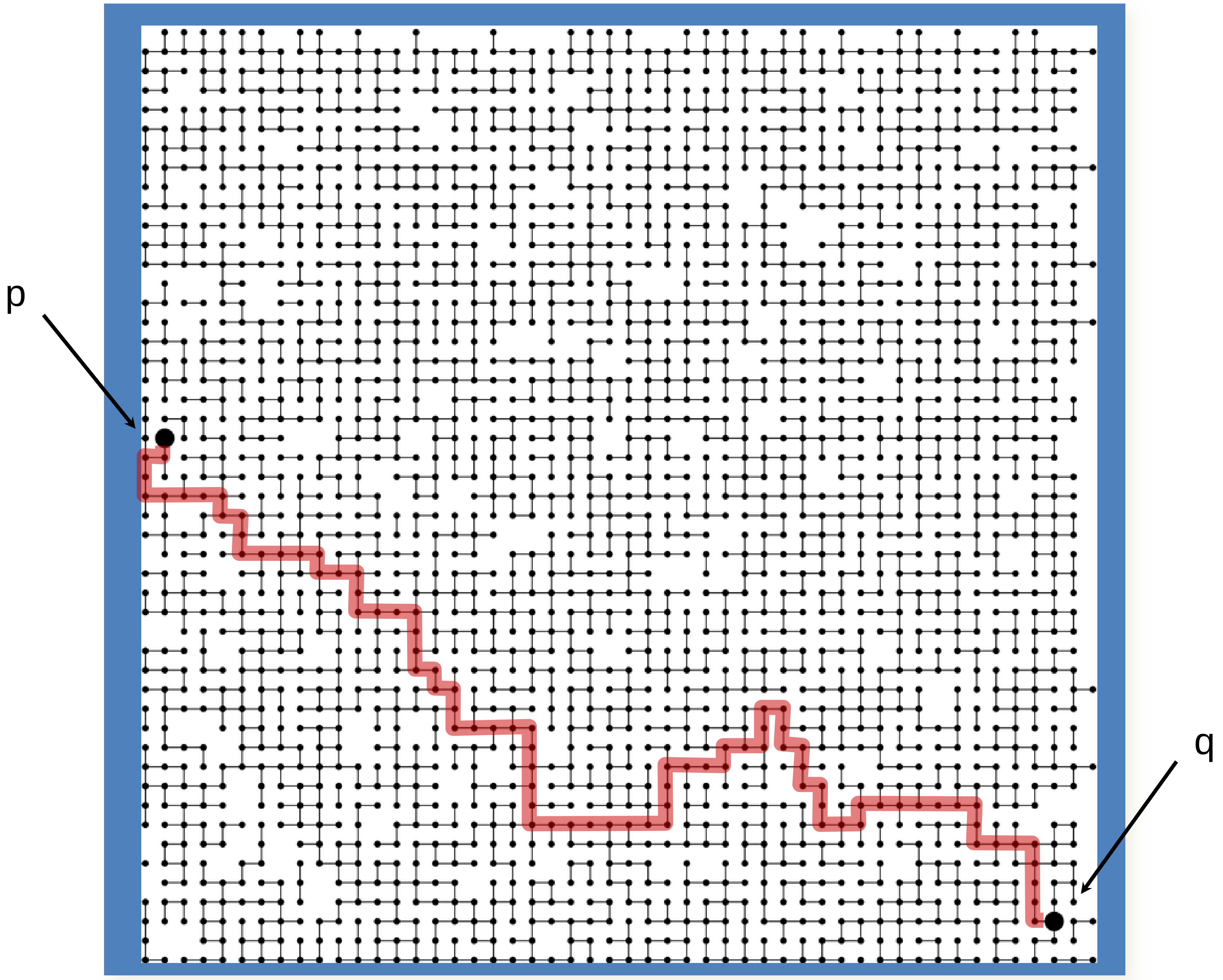
more difficult problem: find the path

```
union(3, 4)
union(8, 0)
union(2, 3)
union(5, 6)
connected(0, 2)        no
connected(2, 4)        yes
union(5, 1)
union(7, 3)
union(1, 6)
union(4, 8)
connected(0, 2)        yes
connected(2, 4)        yes
```



# Connectivity example

Q. Is there a path from p to q?




A. Yes.

# Modeling the objects

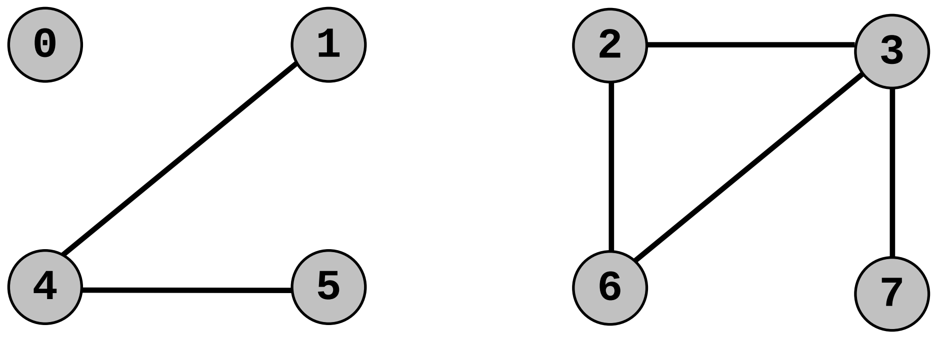
- Dynamic connectivity applications involve manipulating objects of all types.
  - Pixels in a digital photo.
  - Computers in a network.
  - Friends in a social network.
  - Transistors in a computer chip.
  - Elements in a mathematical set.
  - Metallic sites in a composite system.

- When programming, convenient to name sites 0 to N-1.
  - Use integers as array index.
  - Suppress details not relevant to union-find.

# Modeling the connections

- We assume "is connected to" is an equivalence relation:
  - Reflexive: $p$ is connected to $p$.
  - Symmetric: if $p$ is connected to $q$, then $q$ is connected to $p$.
  - Transitive: if $p$ is connected to $q$ and $q$ is connected to $r$, then $p$ is connected to $r$.

- Connected components. Maximal set of objects that are mutually connected.



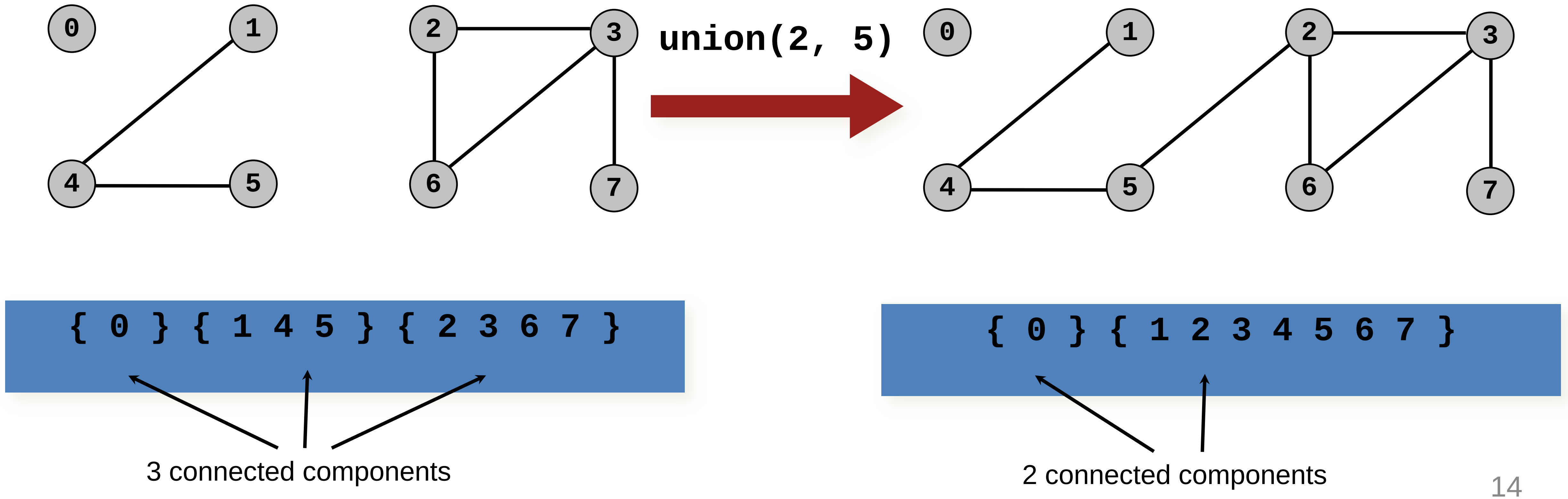

```
{ 0 } { 1 4 5 } { 2 3 6 7 }
```

3 connected components

# Implementing the operations

- Connected query. Check if two objects are in the same component.

- Union command. Replace components containing two objects with their union.



# Union-find data type (API)

- Goal. Design efficient data structure for union-find.
  - Number of objects $N$ can be huge.
  - Number of operations $M$ can be huge.
  - Find queries and union commands may be intermixed.

| public class UF | | |
|---|---|---|
| | UF(int N) | initialize union-find data structure with N objects (0 to N-1) |
| void | union(int p, int q) | add connection between p and q |
| boolean | connected(int p, int q) | are p and q in the same component? |
| int | find(int p) | component identifier for p (0 to N-1) |
| int | count() | number of components |

# Dynamic-connectivity client

- Read in number of objects $N$ from standard input.
- Repeat:
  - read in pair of integers from standard input
  - write out pair if they are not already connected

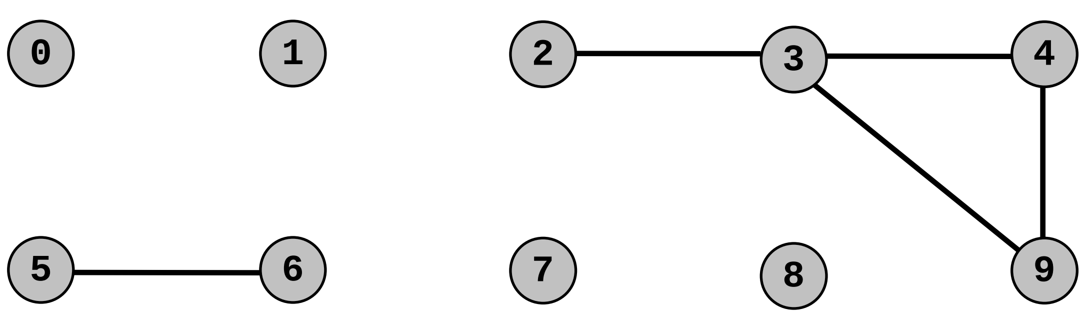
```
public static void main(String[] args)
{
   int N = StdIn.readInt();
   UF uf = new UF(N);
   while (!StdIn.isEmpty())
   {
      int p = StdIn.readInt();
      int q = StdIn.readInt();
      if (uf.connected(p, q)) continue;
      uf.union(p, q);
      StdOut.println(p + " " + q);
   }
}
```

```
% more tiny.txt
10
4 3
3 8
6 5
9 4
2 1
8 9
5 0
7 2
6 1
1 0
6 7
```

# Quick-find [eager approach]

- Data structure.
  - Integer array **`id[]`** of size **`N`**.
  - Interpretation: **`p`** and **`q`** in same component iff they have the same id.

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |




5 and 6 are connected

2, 3, 4, and 9 are connected

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| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |

5 and 6 are connected

2, 3, 4, and 9 are connected

`id[3] = 9`; `id[6] = 6`

3 and 6 in different components

- Find. Check if `p` and `q` have the same id.

# Quick-find [eager approach]

- Data structure.
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|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 9 | 9 | 6 | 6 | 7 | 8 | 9 |

5 and 6 are connected

2, 3, 4, and 9 are connected

- Find. Check if `p` and `q` have the same id.

`id[3] = 9`; `id[6] = 6`

3 and 6 in different components

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 6 | 6 | 6 | 6 | 6 | 7 | 8 | 6 |

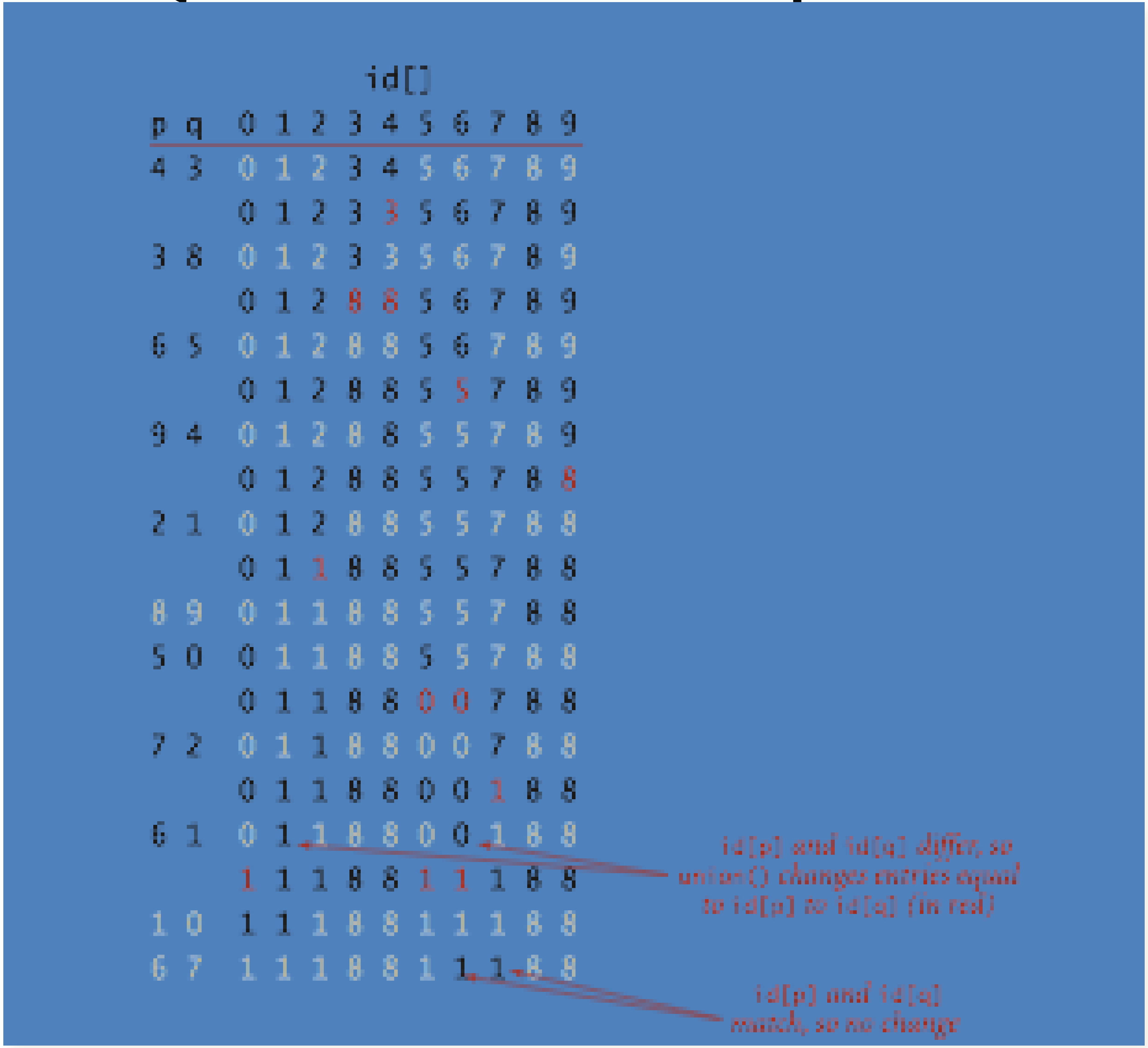
union of 3 and 6

2, 3, 4, 5, 6, and 9 are connected

problem: many values can change

- Union. To merge sets containing `p` and `q`, change all entries with `id[p]` to `id[q]`.

# Quick-find example



| p | q | id[] 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| 3 | 8 | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 6 | 7 | 8 | 9 |
| 6 | 5 | 0 | 1 | 2 | 8 | 8 | 5 | 6 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 9 |
| 9 | 4 | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 9 |
| | | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 2 | 1 | 0 | 1 | 2 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 8 | 9 | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| 5 | 0 | 0 | 1 | 1 | 8 | 8 | 5 | 5 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 7 | 8 | 8 |
| 7 | 2 | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 7 | 8 | 8 |
| | | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |
| 6 | 1 | 0 | 1 | 1 | 8 | 8 | 0 | 0 | 1 | 8 | 8 |
| | | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |
| 1 | 0 | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |
| 6 | 7 | 1 | 1 | 1 | 8 | 8 | 1 | 1 | 1 | 8 | 8 |

id[p] and id[q] differ, so union() changes entries equal to id[p] to id[q] (in red)

id[p] and id[q] match, so no change



http://algs4.cs.princeton.edu/lectures/15DemoQuickFind.mov

# Quick-find: Java implementation

```java
public class QuickFindUF
{
   private int[] id;

   public QuickFindUF(int N)
   {
      id = new int[N];
      for (int i = 0; i < N; i++)
         id[i] = i;
   }

   public boolean connected(int p, int q)
   {  return id[p] == id[q];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each object to itself
(N array accesses)

check whether **`p`** and **`q`**
are in the same component
(2 array accesses)

change all entries with **`id[p]`** to **`id[q]`**
(linear number of array accesses)

# Quick-find is too slow

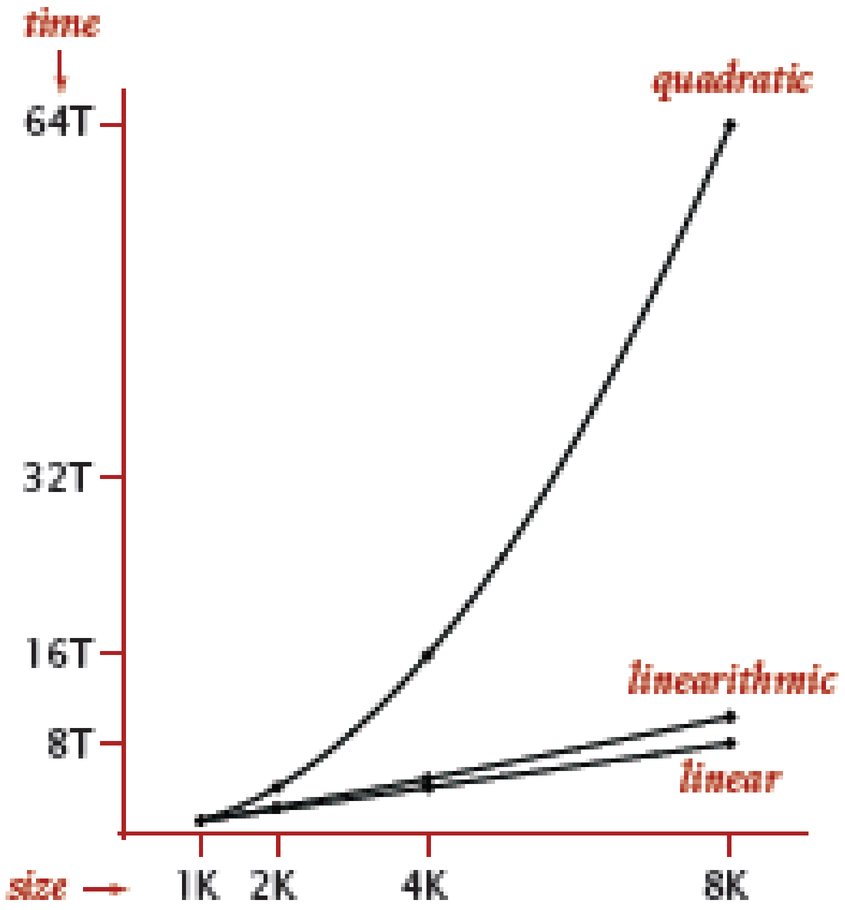
- Cost model. Number of array accesses (for read or write).

| algorithm | init | union | find |
|---|---|---|---|
| quick-find | N | N | 1 |

- Quick-find defect.
  - Union too expensive.
  - Trees are flat, but too expensive to keep them flat.
  - Ex. Takes $N^2$ array accesses to process sequence of $N$ union commands on $N$ objects.

# Quadratic Algorithms

- Currently, a computer can do:
  - $10^9$ operations per second.
  - $10^9$ "words" in main memory.
- Therefore can "touch" every memory location in 1 second!



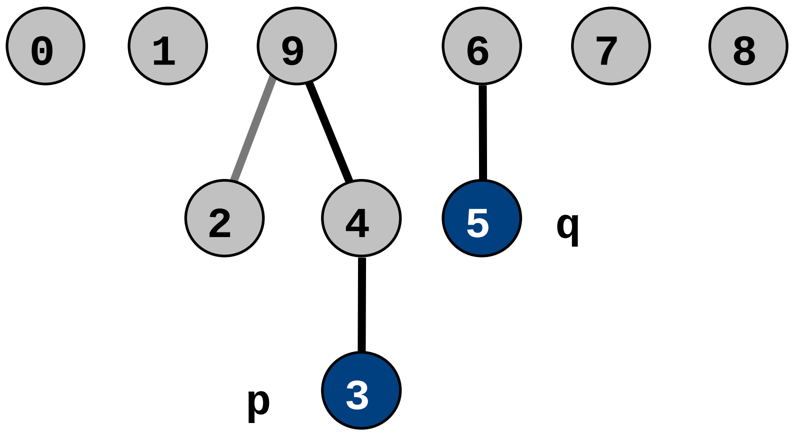

- For Quick-Find
  - $10^9$ union commands on $10^9$ objects = $10^{18}$ operations. Prize for whoever can tell me how long this will take…

# Quick-union [lazy approach]

- Data structure.
  - Integer array `id[]` of size `N`.
  - Interpretation: `id[i]` is parent of `i`.
  - Root of `i` is `id[id[id[...id[i]...]]]`.

keep going until it doesn't change

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 9 |




3's root is 9; 5's root is 6

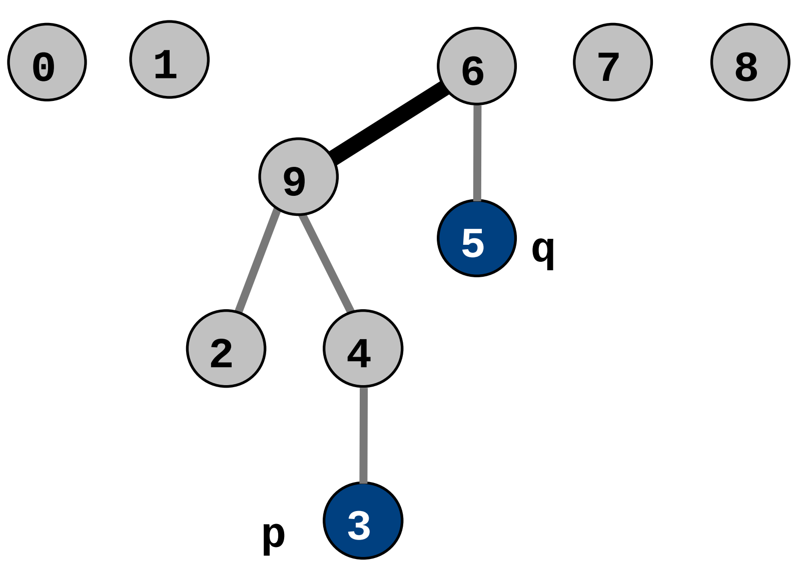
# Quick-union [lazy approach]

- Find. Check if `p` and `q` have the same root.

- Union. To merge sets containing `p` and `q`, set the id of `p`'s root to the id of `q`'s root.

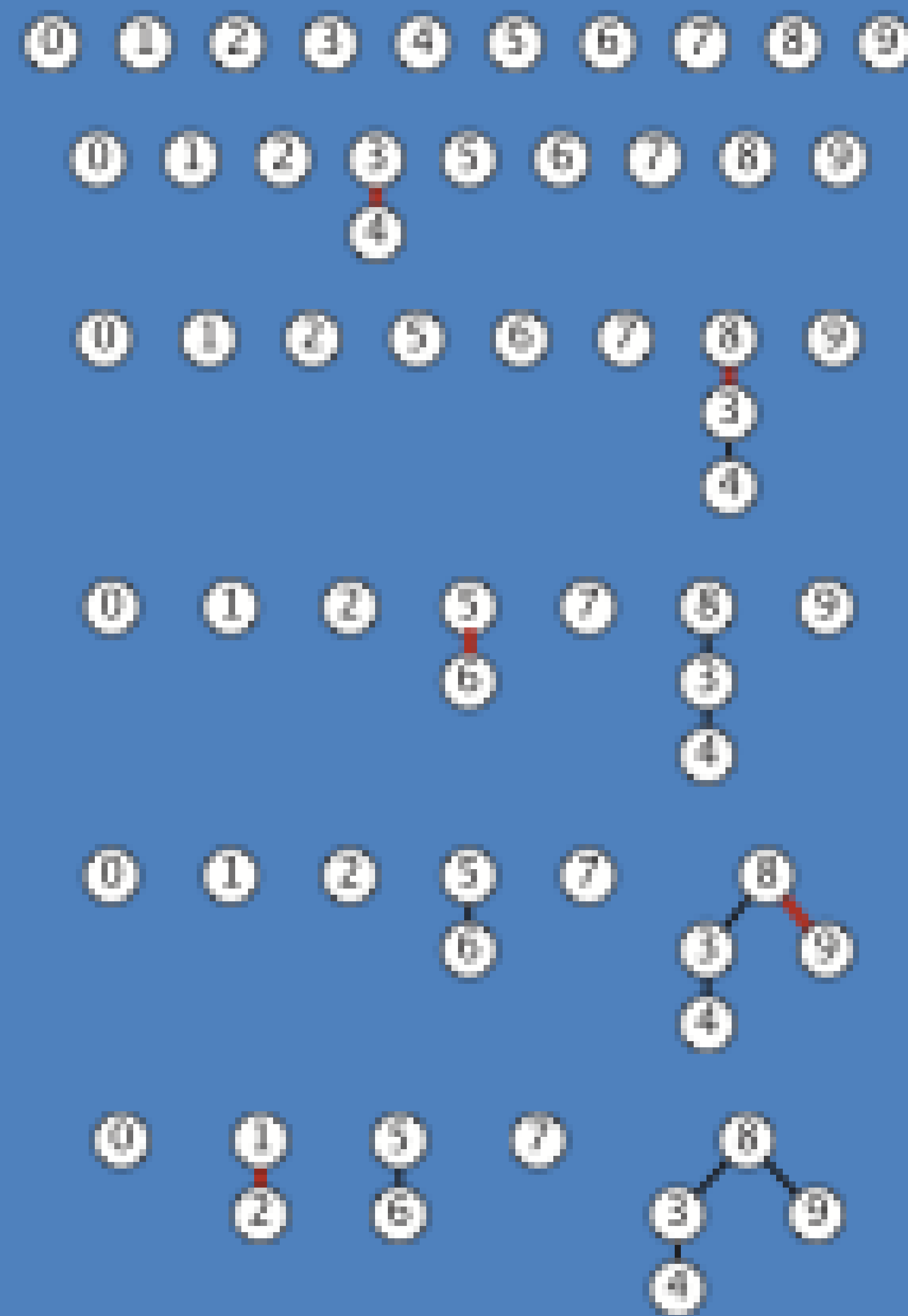
| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| id[i] | 0 | 1 | 9 | 4 | 9 | 6 | 6 | 7 | 8 | 6 |

only one value changes



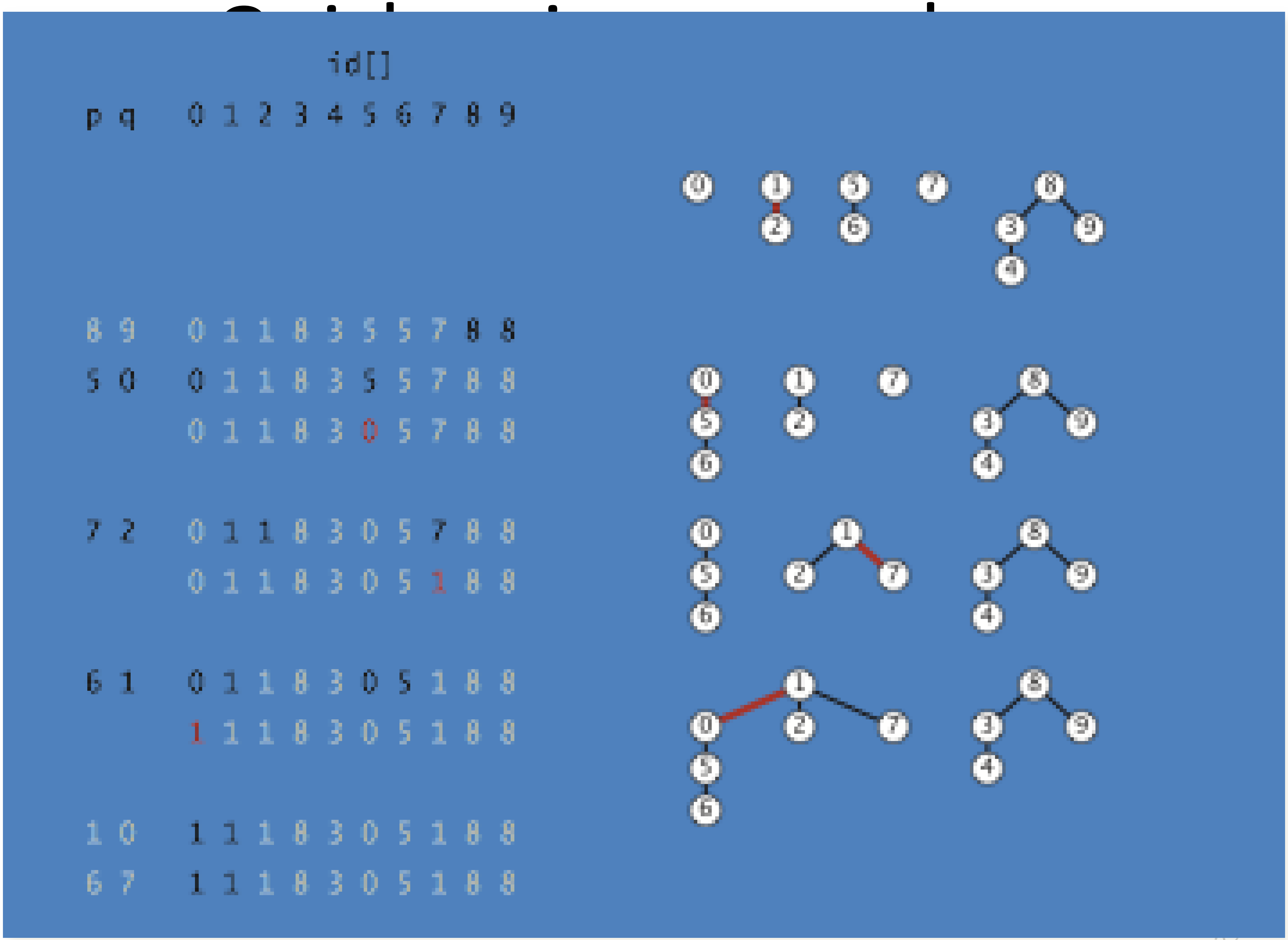
# Quick-union example

| p q | id[] | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| 3 8 | 0 | 1 | 2 | 3 | 3 | 5 | 6 | 7 | 8 | 9 |
| | 0 | 1 | 2 | 8 | 3 | 5 | 6 | 7 | 8 | 9 |
| 6 5 | 0 | 1 | 2 | 8 | 3 | 5 | 6 | 7 | 8 | 9 |
| | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 9 |
| 9 4 | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 9 |
| | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| 2 1 | 0 | 1 | 2 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |
| | 0 | 1 | 1 | 8 | 3 | 5 | 5 | 7 | 8 | 8 |





http://algs4.cs.princeton.edu/lectures/15DemoQuickUnion.mov


id[]
p q 0 1 2 3 4 5 6 7 8 9
8 9 0 1 1 8 3 5 5 7 8 8
5 0 0 1 1 8 3 5 5 7 8 8
0 1 1 8 3 0 5 7 8 8
7 2 0 1 1 8 3 0 5 7 8 8
0 1 1 8 3 0 5 1 8 8
6 1 0 1 1 8 3 0 5 1 8 8
1 1 1 8 3 0 5 1 8 8
1 0 1 1 1 8 3 0 5 1 8 8
6 7 1 1 1 8 3 0 5 1 8 8

# Quick-union: Java implementation

```
public class QuickUnionUF{
private int[] id;

public QuickUnionUF(int N)   {
      id = new int[N];
      for (int i = 0; i < N; i++) id[i] = i;
}

private int root(int i)   {
      while (i != id[i]) i = id[i];
      return i;
}

public boolean connected(int p, int q)   {
return root(p) == root(q);
}

public void union(int p, int q)   {
      int i = root(p), j = root(q);
      id[i] = j;   }
}
```

set id of each object to itself
(N array accesses)

chase parent pointers until reach root
(depth of i array accesses)

check if p and q have same root
(depth of p and q array accesses)

change root of p to point to root of q
(depth of p and q array accesses)

# Quick-union is also too slow

- Cost model. Number of array accesses (for read or write).

| algorithm | init | union | find |
|---|---|---|---|
| quick-find | N | N | 1 |
| quick-union | N | N † | N |

† includes cost of finding root

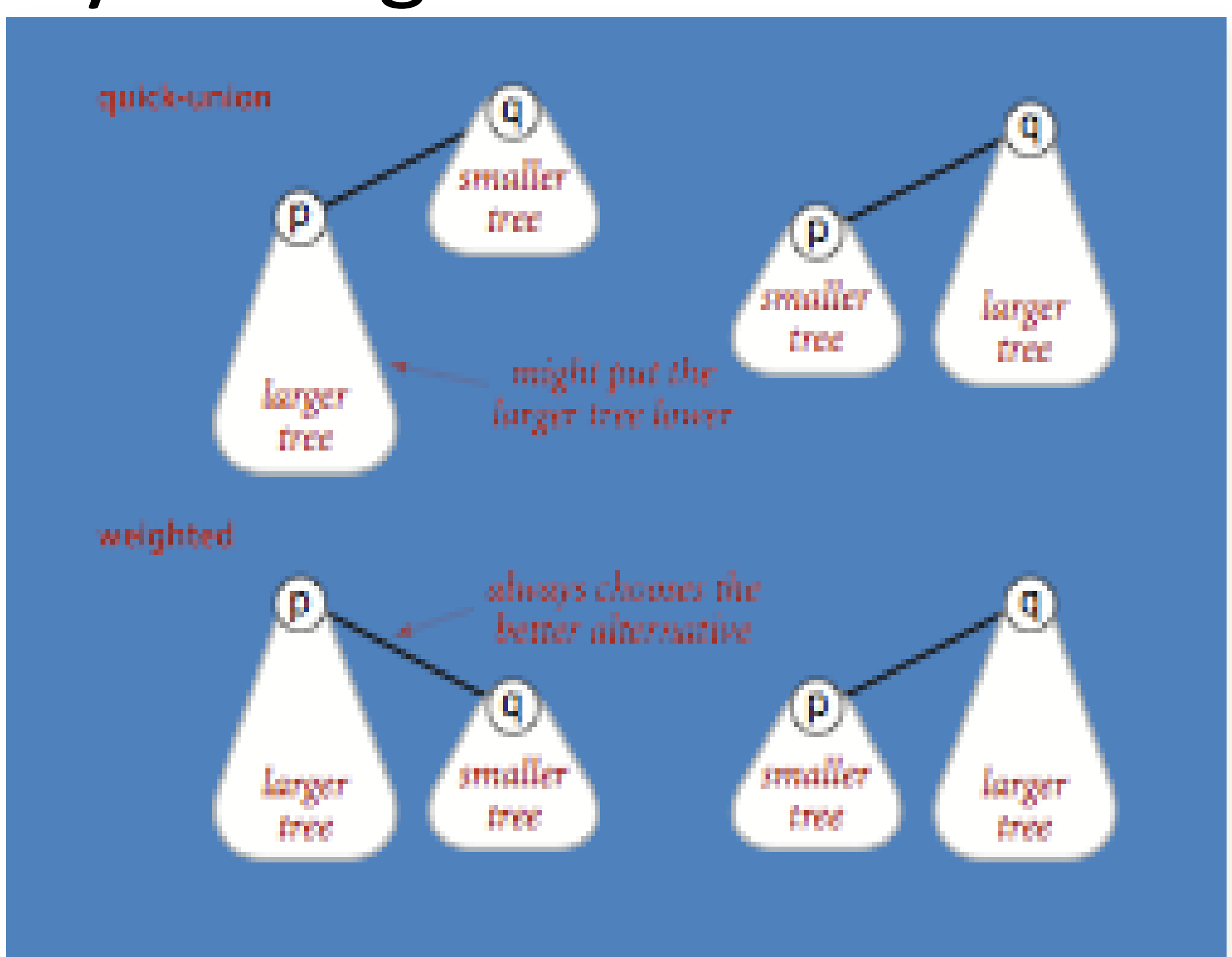
worst case

- Quick-find defect.
  - Union too expensive ($N$ array accesses).
  - Trees are flat, but too expensive to keep them flat.

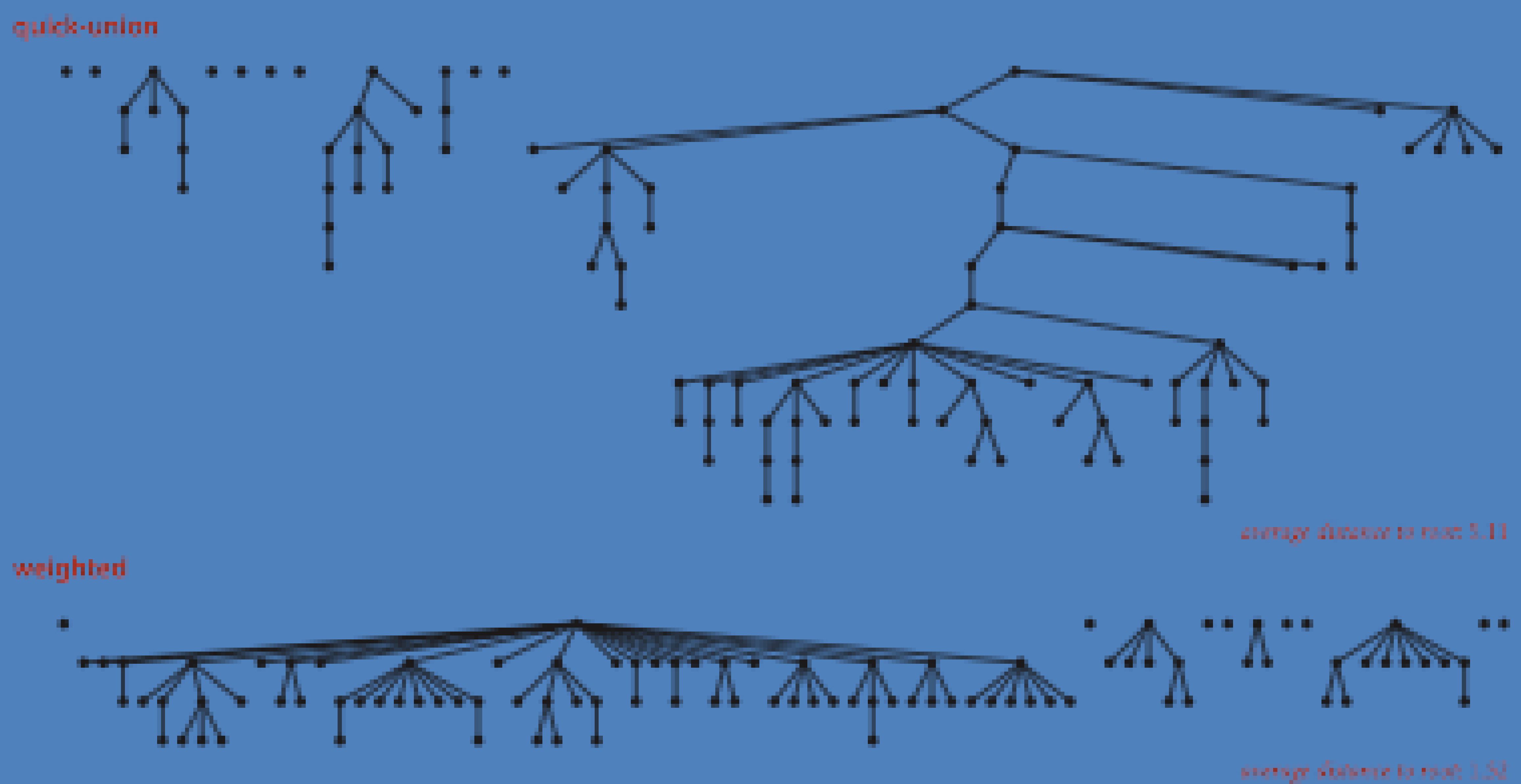
- Quick-union defect.
  - Trees can get tall.
  - Find too expensive (could be $N$ array accesses).

# Improvement 1: weighting

- Weighted quick-union.
  - Modify quick-union to avoid tall trees.
  - Keep track of size of each tree (number of objects).
  - Balance by linking small tree below large one.



# Quick-union and weighted quick-union example




Quick-union and weighted quick-union (100 sites, 88 union() operations)

# Weighted quick-union: Java implementation

- Data structure. Same as quick-union, but maintain extra array **`sz[i]`**
to count number of objects in the tree rooted at **`i`**.

- Find. Identical to quick-union.

```
return root(p) == root(q);
```

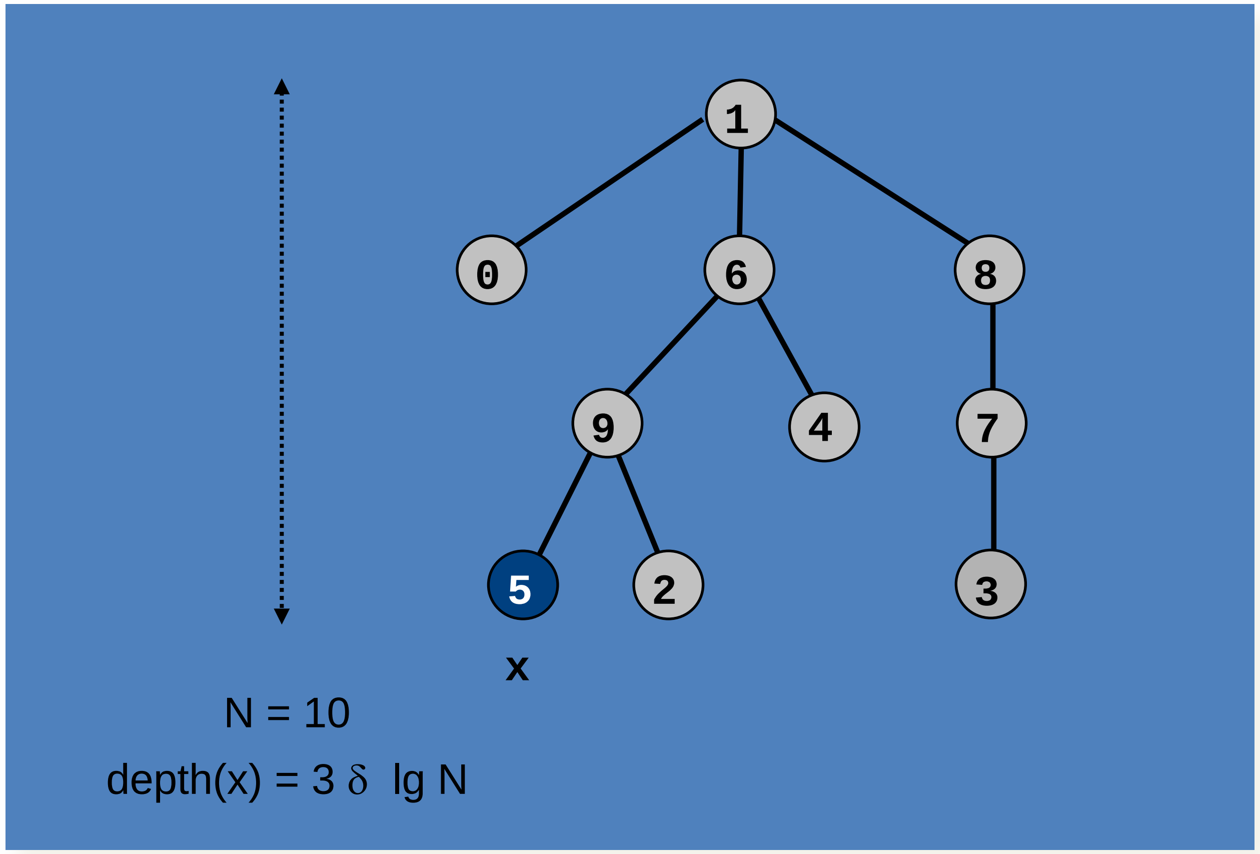
- Union. Modify quick-union to:
  - Merge smaller tree into larger tree.
  - Update the `sz[]` array.

```
int i = root(p);
int j = root(q);
if  (sz[i] < sz[j]) { id[i] = j; sz[j] += sz[i]; }
else                { id[j] = i; sz[i] += sz[j]; }
```

# Weighted quick-union analysis

- Running time.
  - Find: takes time proportional to depth of $p$ and $q$.
  - Union: takes constant time, given roots.

- Proposition. Depth of any node $x$ is at most $\lg N$.



# Weighted quick-union analysis

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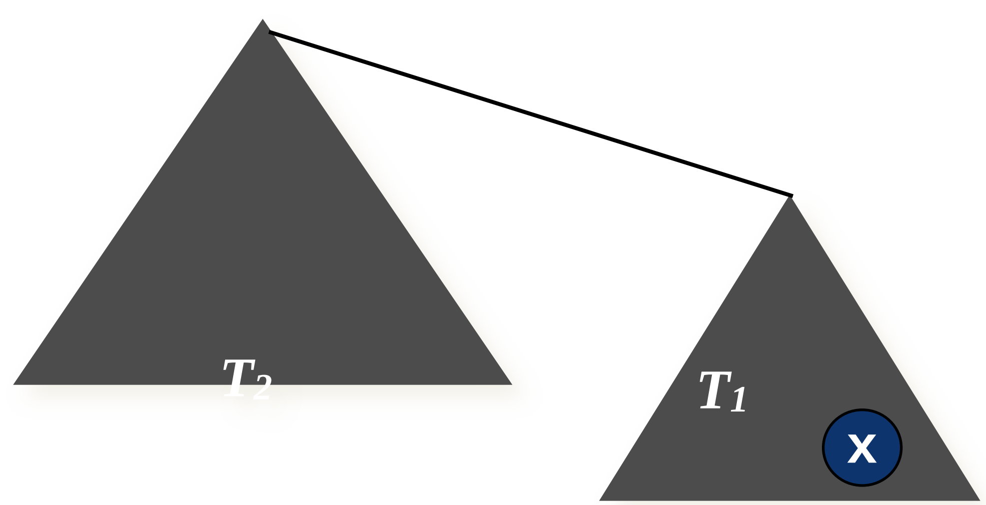
- Proposition. Depth of any node $x$ is at most $\lg N$.
- Pf. When does depth of $x$ increase?
- Increases by $1$ when tree $T_1$ containing $x$ is merged into another tree $T_2$.
  - The size of the tree containing $x$ at least doubles since $|T_2| > |T_1|$.
  - Size of tree containing $x$ can double at most $\lg N$ times. Why?



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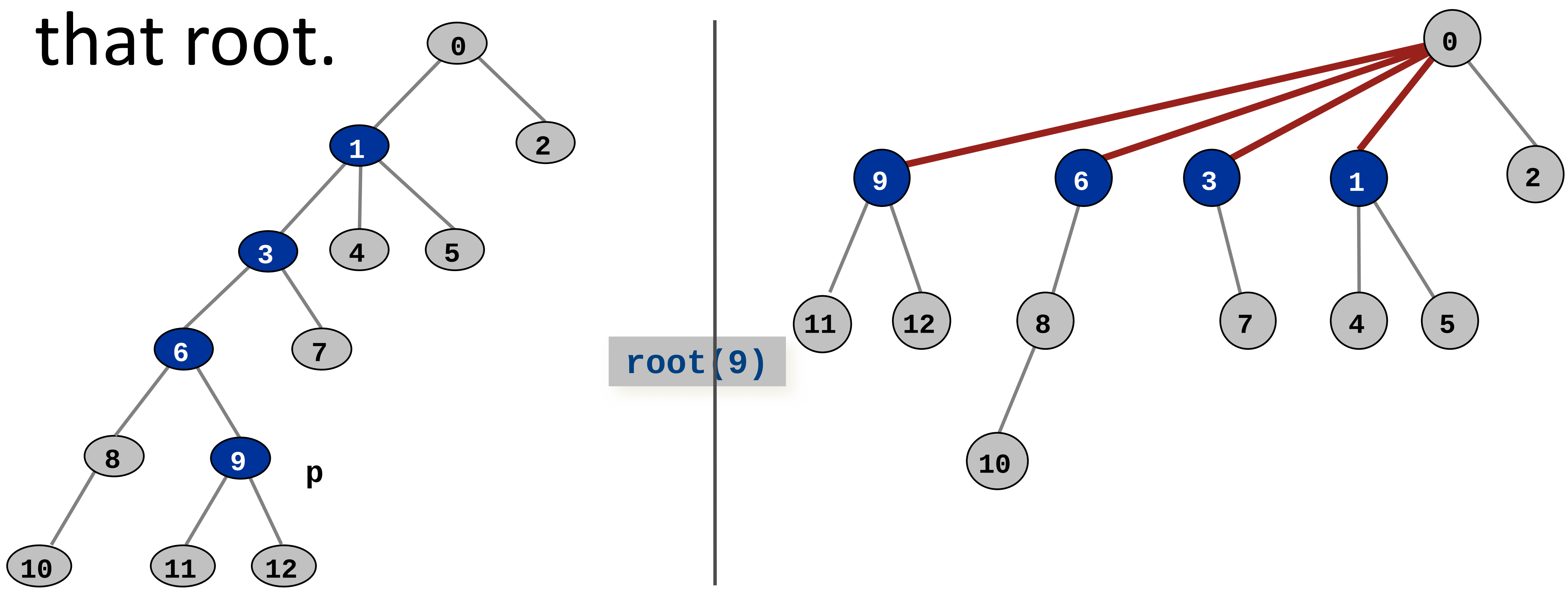
| algorithm | init | union | find |
|---|---|---|---|
| quick-find | N | N | 1 |
| quick-union | N | N † | N |
| weighted QU | N | lg N † | lg N |

† includes cost of finding root

- Q. Stop at guaranteed acceptable performance?
- A. No, easy to improve further.

# Improvement 2: path compression

- Quick union with path compression. Just after computing the root of $p$,
- set the id of each examined node to point to that root.



# Path compression: Java implementation

- Standard implementation: add second loop to `find()` to set the `id[]` of each examined node to the root.

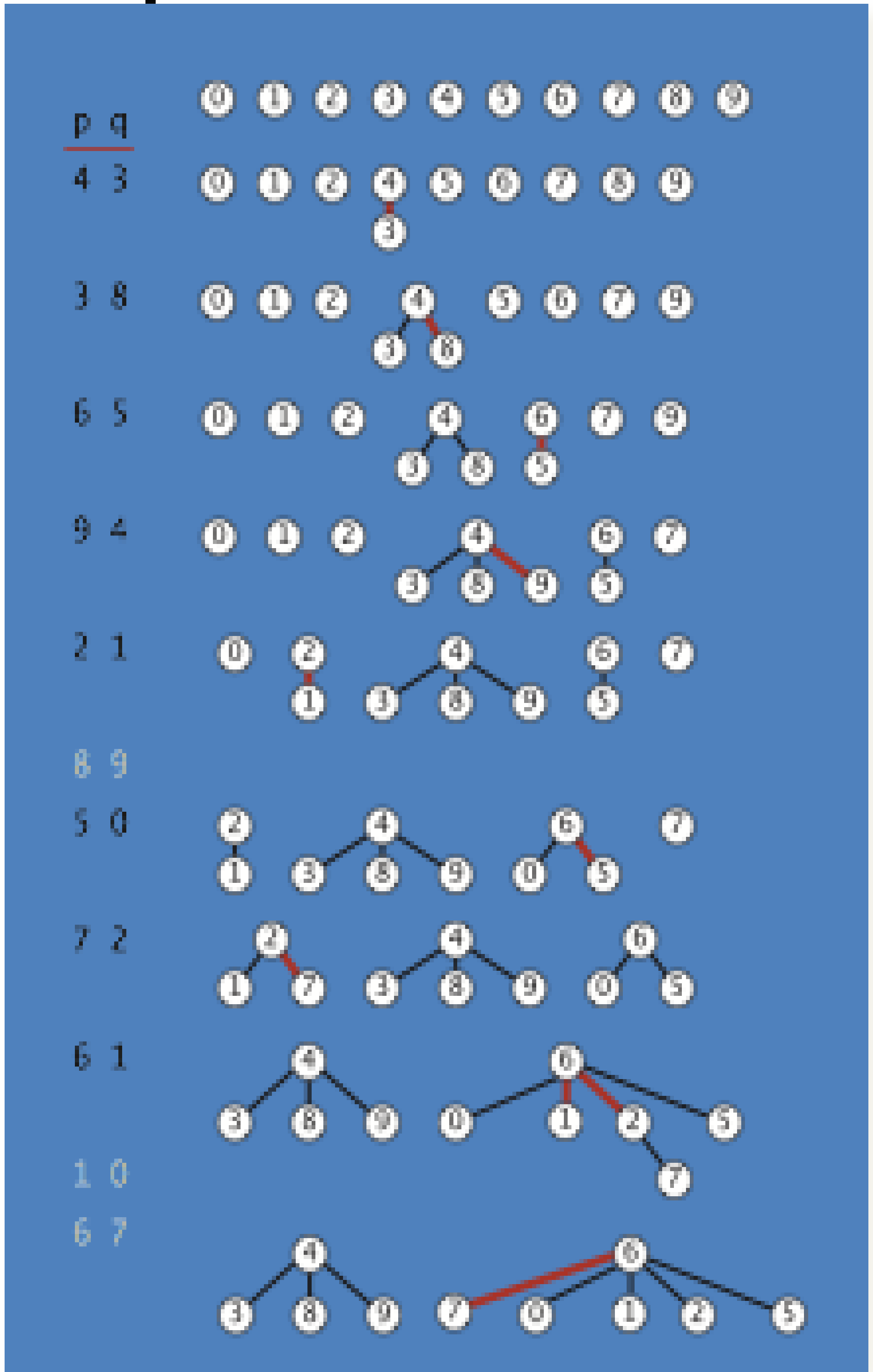
- Simpler one-pass variant: halve the path length by making every other node in path point to its grandparent.

```
public int root(int i)
{
   while (i != id[i])
   {
      id[i] = id[id[i]];
      i = id[i];
   }
   return i;
}
```

only one extra line of code !

- In practice. No reason not to! Keeps tree almost completely flat.

# Weighted quick-union with path compression example




1 linked to 6 because of path compression

7 linked to 6 because of path compression

# Weighted quick-union with path compression: amortized analysis

- Proposition. Starting from an empty data structure,
- any sequence of $M$ union–find operations on $N$ objects makes at most proportional to $N + M \lg^* N$ array accesses.
  - Proof is very difficult.
  - Can be improved to $N + M \langle(M, N)$.
  - But the algorithm is still simple!

- Linear-time algorithm for $M$ union-find ops on $N$ objec
  - Cost within constant factor of reading in the data.
  - In theory, WQUPC is not quite linear.
  - In practice, WQUPC is linear.

because $\lg^* N$ is a constant in this universe

| N | [illegible] N |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 4 | 2 |
| 16 | 3 |
| 65536 | 4 |
| $2^{65536}$ | 5 |

**lg* function**

- Amazing fact. No linear-time algorithm exists.

in "cell-probe" model of computation

# Summary

- Bottom line. WQUPC makes it possible to solve problems that could not otherwise be addressed.

| algorithm | worst-case time |
|---|---|
| quick-find | M N |
| quick-union | M N |
| weighted QU | N + M log N |
| QU + path compression | N + M log N |
| weighted QU + path compression | N + M lg* N |

**M union–find operations on a set of N objects**

- Ex. [$10^9$ unions and finds with $10^9$ objects]
  - WQUPC reduces time from 30 years to 6 seconds.
  - Supercomputer won't help much; good algorithm enables solution.

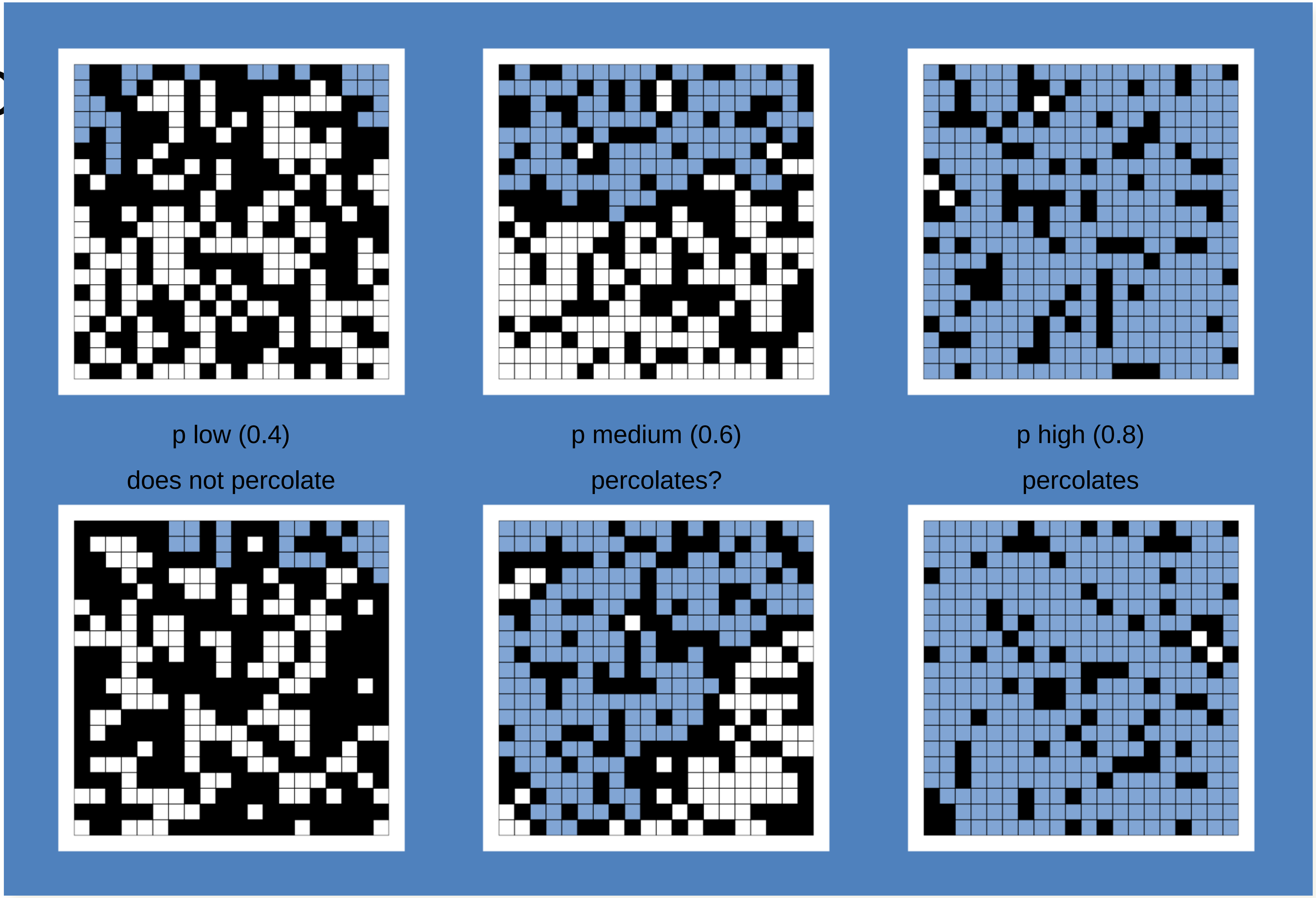
# Percolation

- A model for many physical systems:
  - $N$-by-$N$ grid of sites.
  - Each site is open with probability $p$ (or blocked with

| model | system | vacant site | occupied site | percolates |
|---|---|---|---|---|
| electricity | material | conductor | insulated | conducts |
| fluid flow | material | empty | blocked | porous |
| social interaction | population | person | empty | communicates |

# Likelihood of percolation

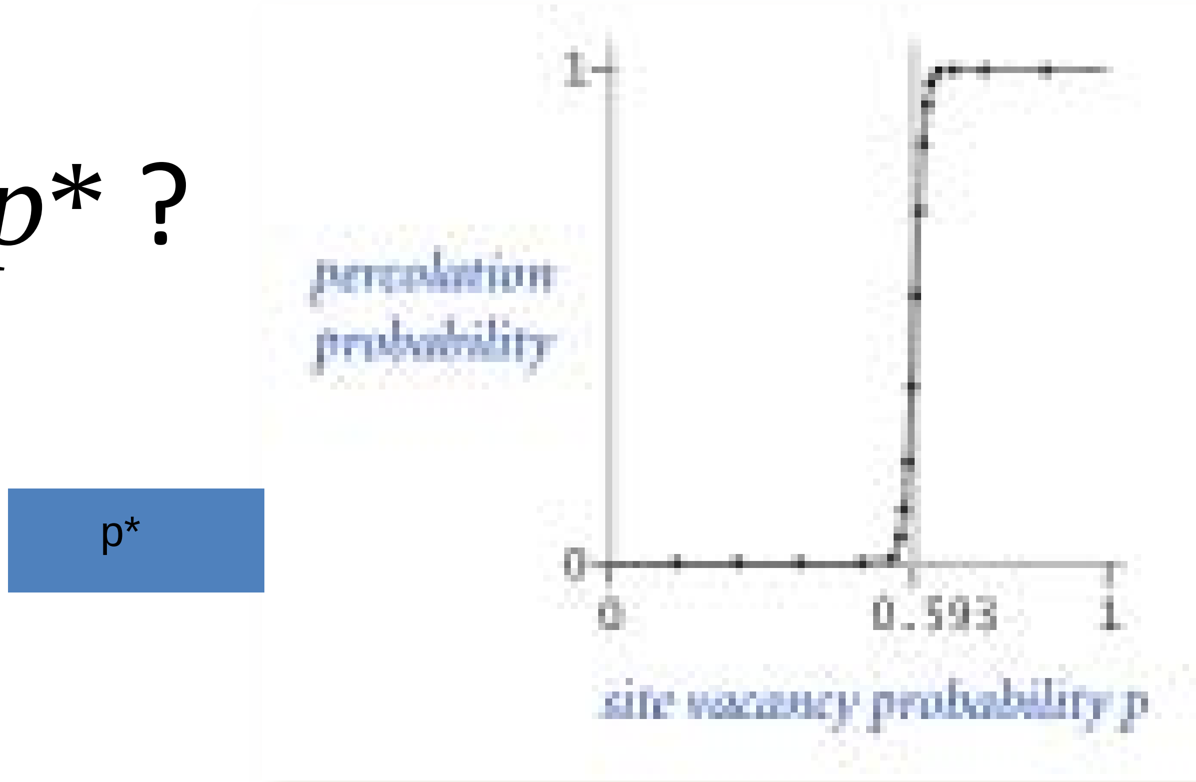
•D



# Percolation phase transition

- When $N$ is large, theory guarantees a sharp threshold $p^*$.
  - $p > p^*$: almost certainly percolates.
  - $p < p^*$: almost certainly does not percolate.

- Q. What is the value of $p^*$ ?




$N = 100$



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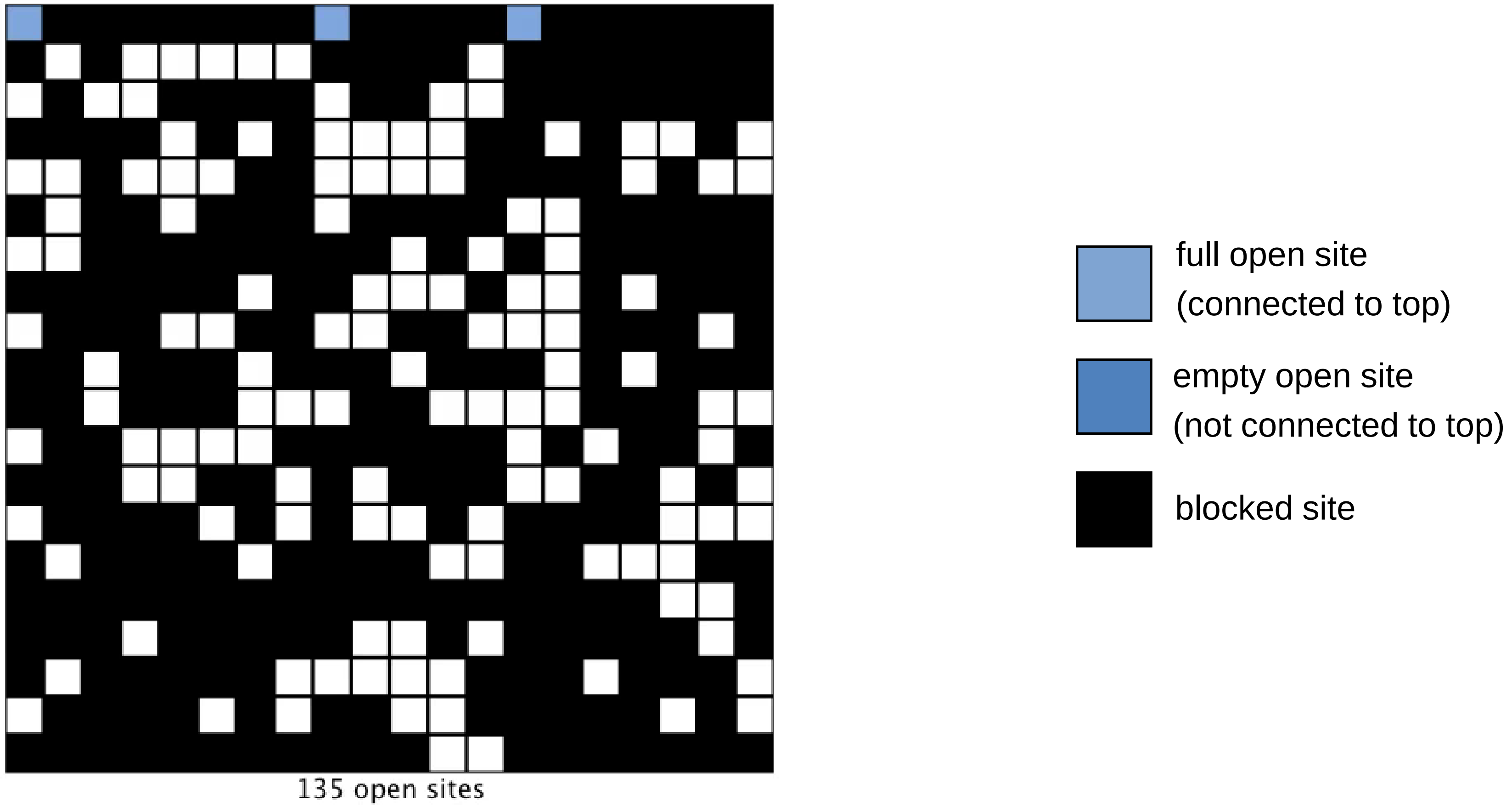
# Monte Carlo simulation

– Initialize $N$-by-$N$ whole grid to be blocked.
– Declare random sites open until top connected to bottom.
– Vacancy percentage estimates $p*$.

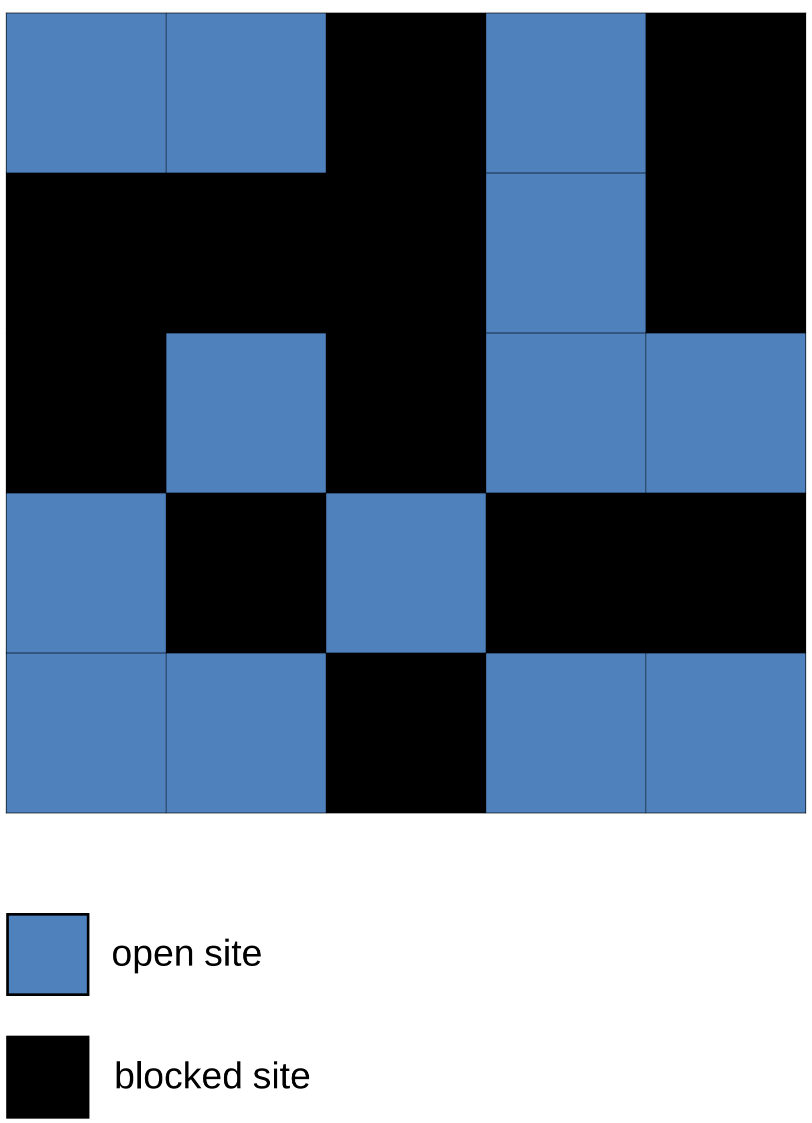



$N = 20$

# Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an $N$-by-$N$ system percolates?

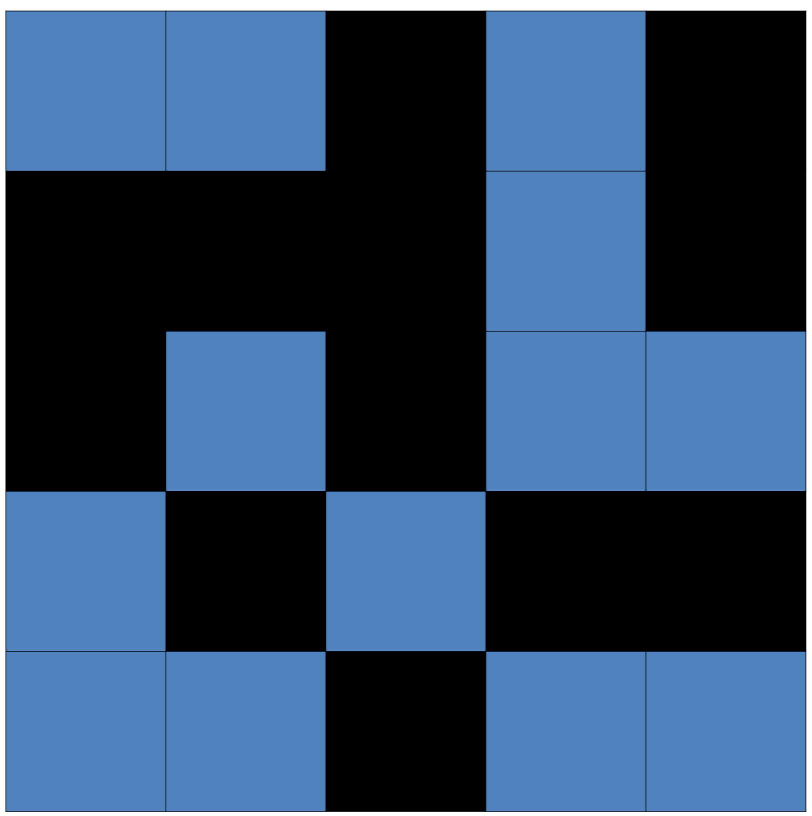
$N = 5$

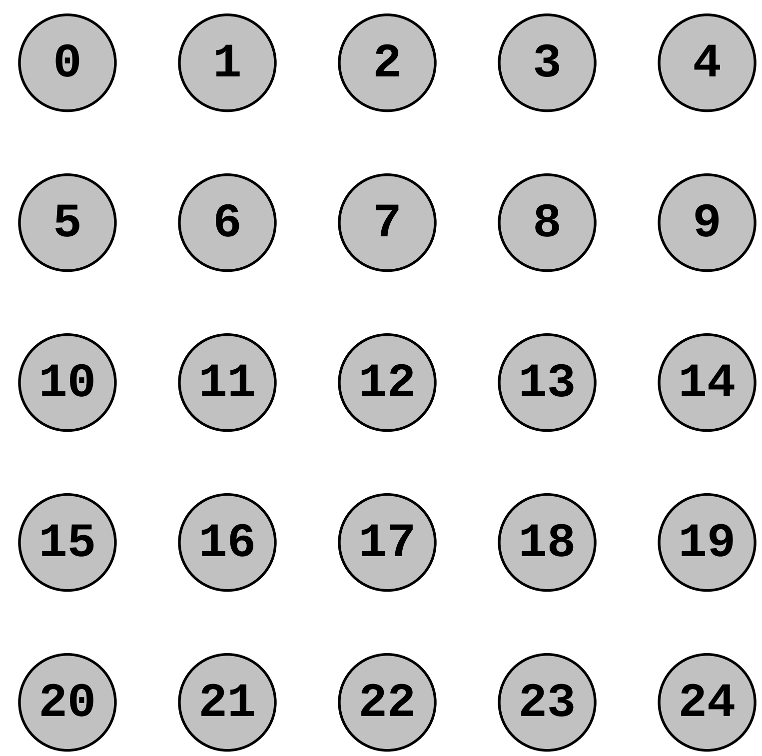


# Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an $N$-by-$N$ system percolates?
  - Create an object for each site and name them $0$ to $N^2 - 1$.

$N = 5$





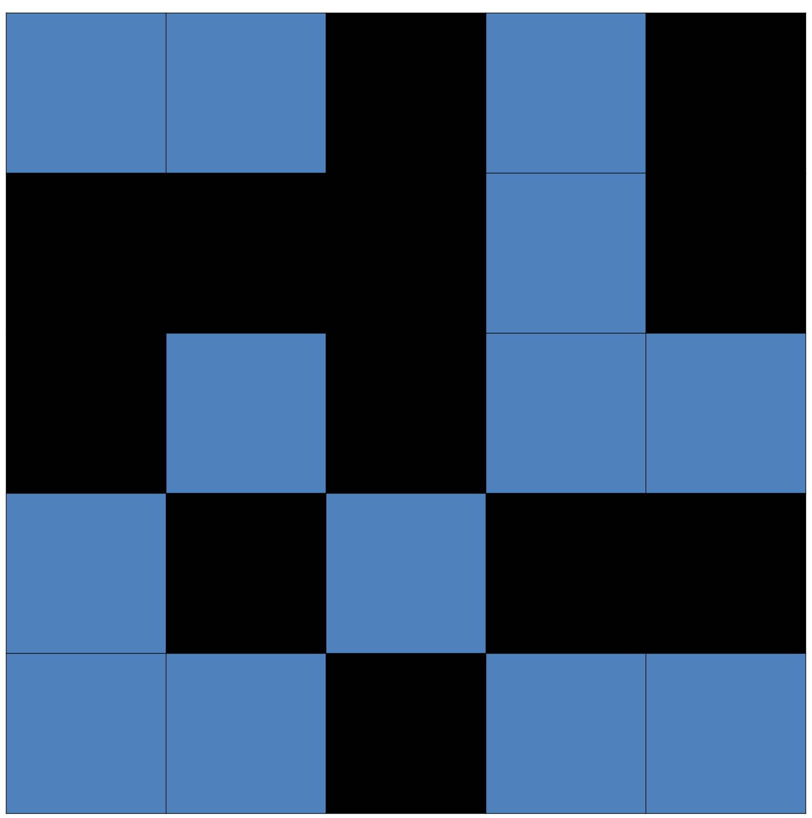

open site

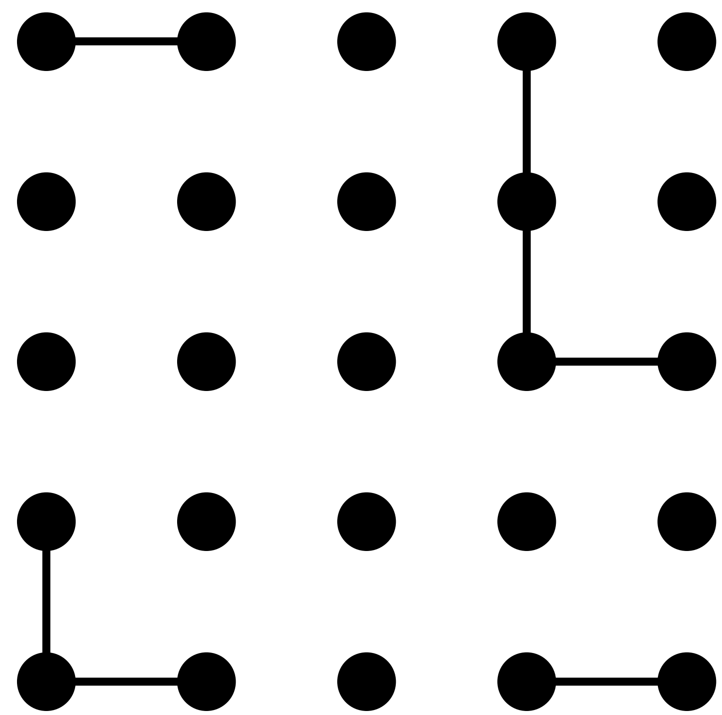
blocked site

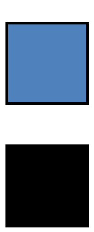
# Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an $N$-by-$N$ system percolates?
  - Create an object for each site and name them $0$ to $N^2 - 1$.
  - Sites are in same set if connected by open sites.

$N = 5$



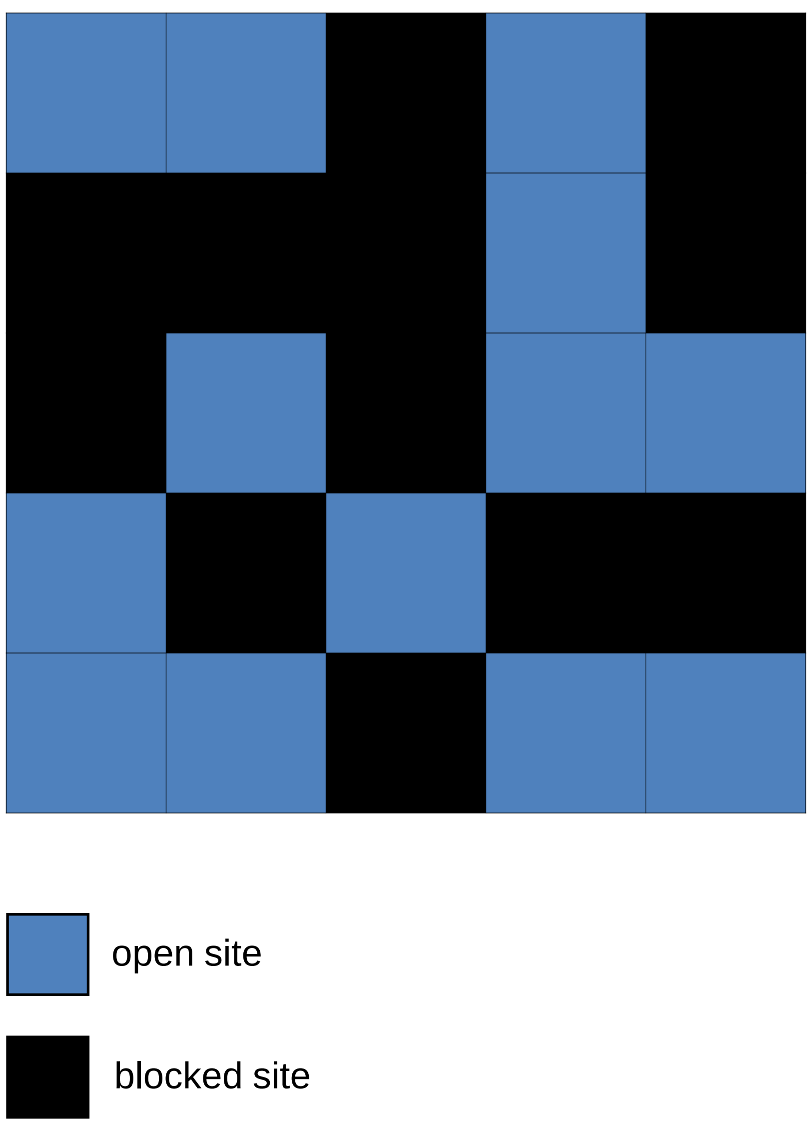


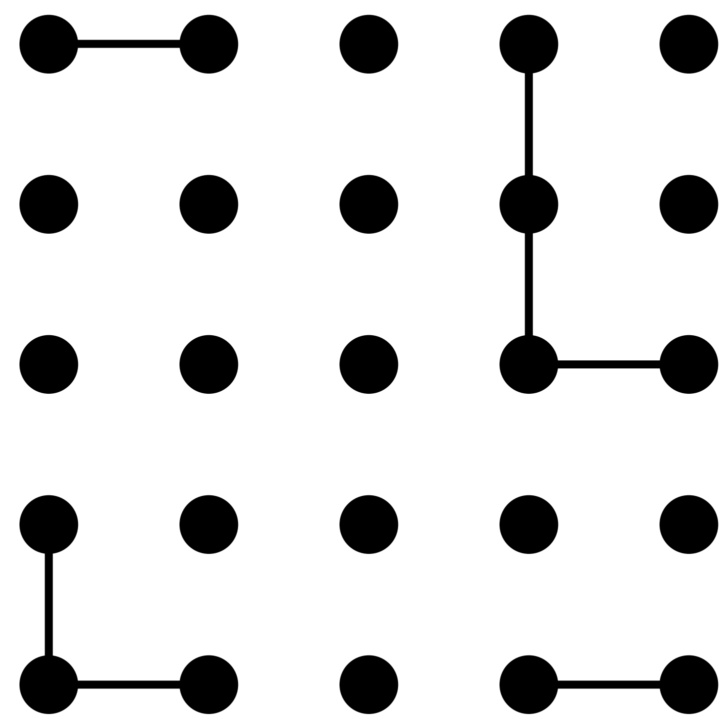


# Dynamic connectivity solution to estimate percolation threshold

- Q. How to check whether an $N$-by-$N$ system percolates?
  - Create an object for each site and name them $0$ to $N^2 - 1$.
  - Sites are in same set if connected by open sites.
  - Percolates iff any site on bottom row is connected to site on top row.

$N = 5$



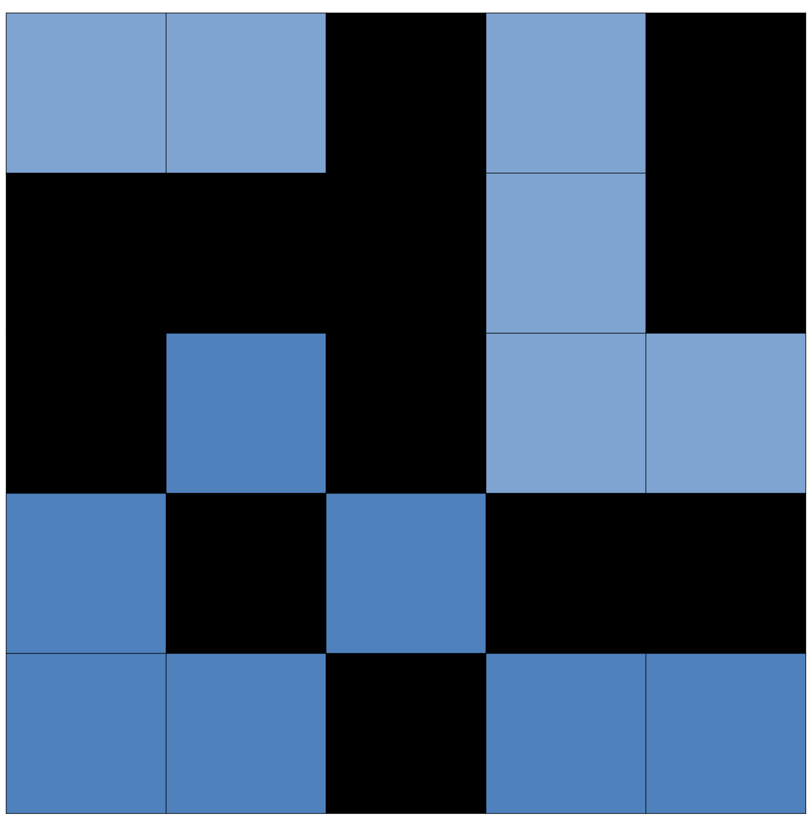
brute-force algorithm: $N^2$ calls to connected()

# Dynamic connectivity solution to estimate percolation threshold

- Clever trick. Introduce two virtual sites (and connections to top and bottom).
  - Percolates iff virtual top site is connected to virtual bottom site.
  - Open site is full iff connected to virtual top site.

needed only for visualization

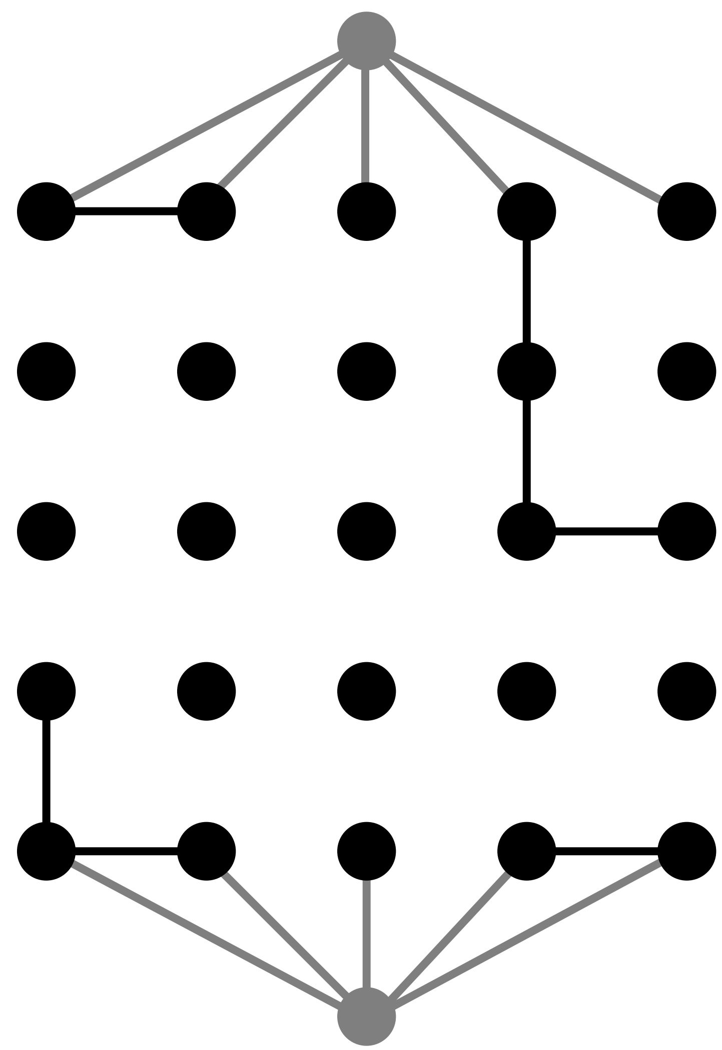
$N = 5$



empty open site
(not connected to top)

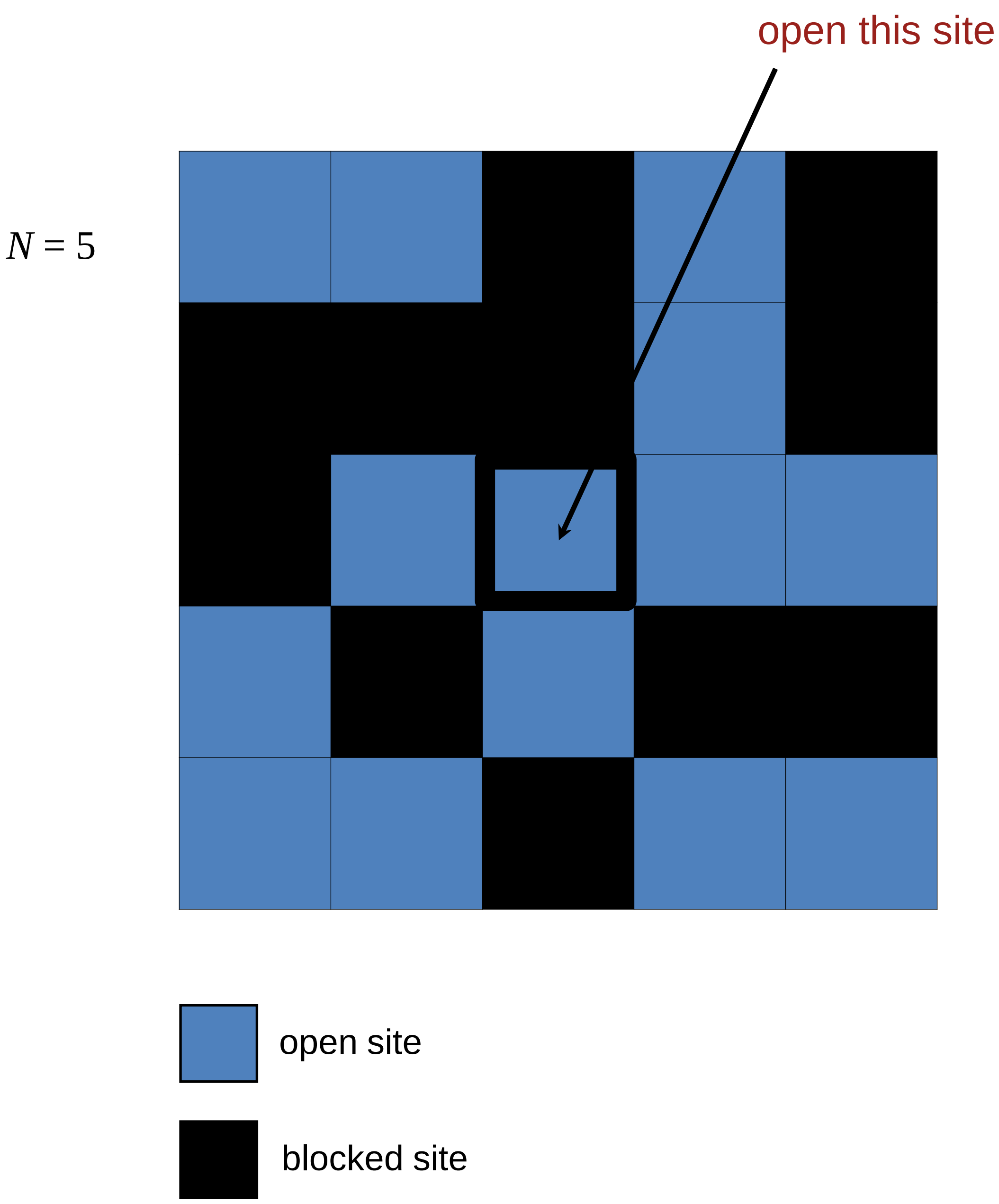
full open site
(connected to top)

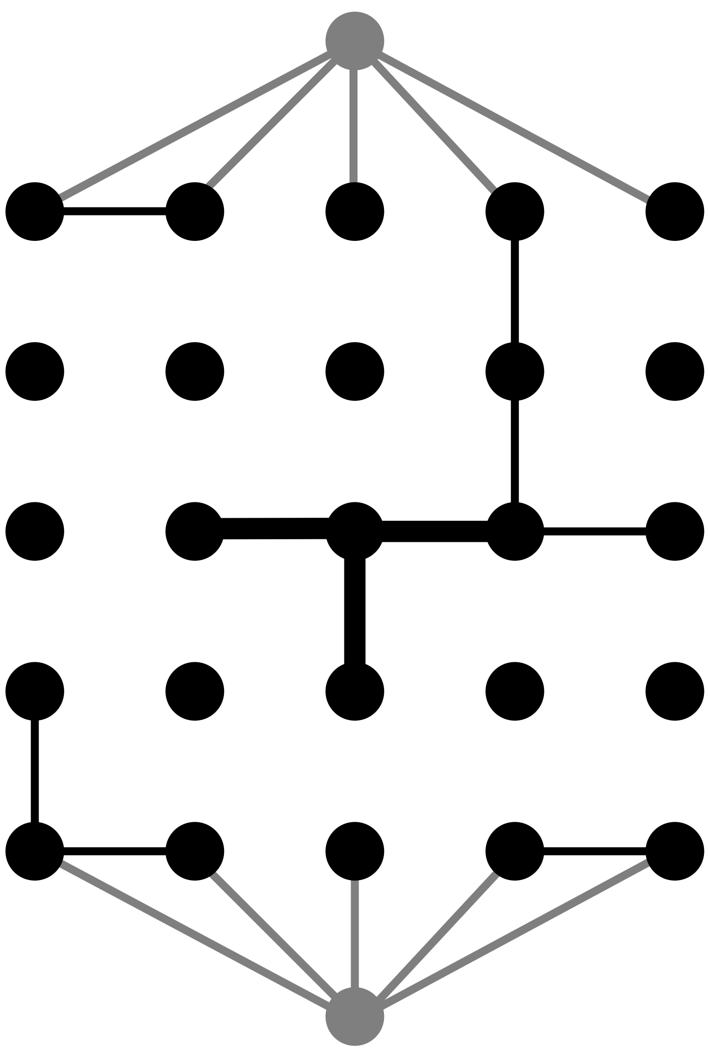
blocked site



# Dynamic connectivity solution to estimate percolation threshold

- Q. How to model as dynamic connectivity problem when opening a new site?
- A. Connect new site to all of its adjacent open sites.



# Subtext of today's lecture (and this course)

- Steps to developing a usable algorithm.
  - Model the problem.
  - Find an algorithm to solve it.
  - Fast enough? Fits in memory?
  - If not, figure out why.
  - Find a way to address the problem.
  - Iterate until satisfied.

- The scientific method.

- Mathematical analysis.