MATH 311 Homework 2.3

Will Townsend

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Problem 7

$$x\frac{dy}{dx} = y^{-3} => y^3dy = x^{-1}dx => \int y^3dy = \int x^{-1}dx$$

$$\frac{y^4}{4} = \ln x + C => y^4 = 4\ln x + C$$

Problem 8

$$\frac{dx}{dt} = 3xt^2 => \int \frac{dx}{3x} = \int t^2dt => \frac{\ln|3x|}{3} = \frac{t^3}{3} + C$$

$$=> x = e^{\frac{t^3}{3}+C} => x = Ae^{\frac{t^3}{3}}$$

Problem 9

$$\frac{dx}{dt} = \frac{t}{xe^{t+2x}} => e^{t+2x}dx = \frac{t}{x}dt => t + 2x\ dx = \ln t - \ln x\ dt =>$$

$$\int 2x + \ln x\ dx = \int \ln t - t\ dt => x^2 + x\ln x - x = t\ln t - t - \frac{t^2}{2} + C$$

Problem 10

$$\frac{dy}{dx} = \frac{x}{y^2\sqrt{x+1}} => \int y^2\,dy = \int \frac{x}{\sqrt{x+1}}\,dx => \frac{y^3}{3} = \frac{2(x+1)^{\frac{3}{2}}}{3} - 2\sqrt{x+1} + C$$

$$y^3 = 2(x+1)^{\frac{3}{2}} - 6\sqrt{x+1} + C$$

Integration (using u sub where $u = x+1 \quad du = dx$):

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int \sqrt{u} - (u)^{-\frac{1}{2}}du => \frac{2(x+1)^{\frac{3}{2}}}{3} - 2\sqrt{x+1} + C$$

Problem 13

$$\frac{dy}{dx} = 3x^2(1+y^2)^{\frac{3}{2}} => \int (1+y^2)^{-\frac{3}{2}}\,dy = \int 3x^2\,dx => \frac{y}{\sqrt{y^2+1}} = x^3 + C$$

Integration (using trig sub $y = \tan\theta$):

$$\int \frac{dy}{\sqrt{(y^2+1)^3}} = \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{y}{\sqrt{y^2+1}} + C$$

Problem 14

$$\frac{dx}{dt} - x^3 = x => \frac{dx}{dt} = x^3 + x => \int \frac{dx}{x^3+x} = \int dt =>$$

$$\ln\left(\frac{|x|}{\sqrt{x^2+1}}\right) = t + C => \frac{|x|}{\sqrt{x^2+1}} = Ae^t$$

Integration (using partial fractions):

$$\int \frac{dx}{x^3+x} = \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1}dx = \ln\left(\frac{|x|}{\sqrt{x^2+1}}\right) + C$$

Problem 15

$$(x + xy^2)\,dx + e^{x^2} y\,dy = 0 => \frac{(1+y^2)}{y}dx = -\frac{e^{x^2}}{x}dy =>$$

$$\int -\frac{x}{e^{x^2}}\,dx = \int \frac{y}{1+y^2}\,dy => \frac{e^{-x^2}}{2} + C = \ln\left(\sqrt{y^2+1}\right) =>$$

$$e^{\frac{e^{-x^2}}{2}} + C = \sqrt{y^2+1}$$

Integration (simple u subs)

Problem 16

$$y^{-1}\,dy + ye^{\cos x}\sin x\,dx = 0 => -y^{-2}\,dy = e^{\cos x}\sin x\,dx =>$$

$$\int -2\ln y\,dy = \int \cos x \ln(\sin x)\,dx => -y\ln y + y = \sin x \ln(\sin x) - \sin x + C$$

Integration (u sub)

Problem 17

$$\frac{dy}{dx} = (1+y^2)\tan x \qquad y(0) = \sqrt{3}$$

$$\int (1+y^2)^{-1}\,dy = \int \tan x\,dx => \tan^{-1} y = \ln|\cos x| + C$$

Find C:

$\tan^{-1} 0 = \ln|\cos\sqrt{2}| + C => C = -\ln|\cos\sqrt{2}|$

Final Solution:

$\tan^{-1} y = \ln|\cos x| - \ln|\cos\sqrt{2}| => y = \ln\left(\frac{|\cos x|}{|\cos\sqrt{2}|}\right)$

Problem 18

$$y' = x^3(1-y) \qquad y(0) = 3$$

$$\int \frac{1}{1-y}\,dy = \int x^3\,dx => -\ln|1-y| = \frac{x^4}{4} + C => (1-y)^{-1} = Ae^{\frac{x^4}{4}}$$

Find A:

$(1-3)^{-1} = Ae^{\frac{0^4}{4}} => -\frac{1}{2} = A$

Final Solution:

$(1-y)^{-1} = -\frac{1}{2}e^{\frac{x^4}{4}}$

Problem 19

$$\frac{1}{2}\frac{dy}{dx} = \sqrt{y+1}\cos x \qquad y(\pi) = 0$$

$$\frac{1}{2}\int \frac{1}{\sqrt{y+1}}\,dy = \int \cos x\,dx => \sqrt{y+1} = \sin x + C$$

Find C:

$\sqrt{0+1} = \sin\pi + C => C = 1$

Final Solution:

$\sqrt{y+1} = \sin x + 1$

Problem 20

$$x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)} \qquad y(1) = 1$$

$$\int y + 1\, dy = \int \frac{4x^2 - x - 2}{x^2(x+1)}\, dx => \frac{1}{2}y^2 + y = \frac{2}{x} + \ln|x| + 3\ln|x+1| + C$$

Find C:

$\frac{3}{2} = 2 + 0 + 3\ln 2 + C => C = -\ln 8 - \frac{1}{2}$

Final Solution:

$\frac{1}{2}y^2 + y = \dfrac{2}{x} + \ln|x| + 3\ln|x+1| - \ln 8 - \frac{1}{2}$

Problem 34

$$\frac{dT}{dt} = k(M - T)$$

$$\int (M-T)^{-1}\, dT = \int k\, dt => \ln\left[(M-T)^{-1}\right] = kt + C => M - T = Ae^{-kt}$$

$T(t) = M - Ae^{-kt}$

$T(0) = 100 => 100 = 70 - A => A = -30$

$T(6) = 80 => 80 = 70 + 30e^{-6k} => \dfrac{1}{3} = e^{-6k} => -\ln(3) = \dfrac{1}{6k} =>$

$k = -\dfrac{1}{6\ln(3)}$

$T(20) = 70 + 30e^{-\frac{10}{3\ln(3)}} \approx 71.44°F$

Problem 35

Find our constants:

$40 = 120 - A => A = 80$

$$90 = 120 - 80e^{-45k} => \ln\left(\frac{3}{8}\right) = -45k => k = -\frac{\ln\left(\frac{3}{8}\right)}{45}$$

Part A ($M = 100$):

$90 = 100 - 80e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}} => \frac{1}{8} = e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}} => t = 45\ln\frac{1}{8}(\ln\frac{3}{8})^{-1} \approx 95.40$ min

Part B ($M = 140$):

$90 = 140 - 80e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}} => \frac{5}{8} = e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}} => t = 45\ln\frac{5}{8}(\ln\frac{3}{8})^{-1} \approx 21.56$ min

Part C ($M = 80$):

$90 = 80 - 80e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}} => -\frac{1}{8} = e^{\frac{t\ln\left(\frac{3}{8}\right)}{45}}$ (Solution Impossible to find)