Mechanical Lung Project

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Abstract

Mechanical ventilation is a useful tool in the medical field, and we can create a mathematical model of this process. Under a few assumptions we are able to create a piece-wise function that represents the volume at a given point in time during a breath: inspiration to expiration. The piece-wise function contains the volume during inspiration and a volume during expiration during different points in time of the breath in seconds. We can graph the piece-wise function to get a better idea of what is happening with some given information about a set of healthy lungs' resistance and compliance, and create a function for the flow of air in and out of the lung at a given time, and the pressure inside at any given time.

Introduction

Mechanical ventilation is a process that helps or overall replaces breathing for ill patients via a ventilator. We can create a piece-wise function that represents this process, but in order to do that we need some assumptions and starting information.

I was given the following information:

1. Compliance = 0.1 L/cm water pressure
2. Resistance = 3 cm water pressure/(L/sec)
3. Applied Pressure = 10
4. At the airway there is pressure balance: $P_r + P_{ex} + P_{app} = P_{aw}$.

The first three numbers represent the lungs of a healthy person with no underlying respiratory conditions, other groups were given numbers that have resistances and compliances that differed based on the condition. The last equation will be our differential equation which is explained in more detail later.

The Function

In order to create the function we need to do three things: find our time values base on a time constant and other data, $t_s = R \cdot C$ and $t_i = 5t_s$, and a ratio of inspiration to expiration of $1 : 5.6$, find out our $V_i(t)$, Volume during inspiration function, and $V_e(t)$, volume during expiration, and last solve for our constants using assumptions given ($V_i(0) = 0$, $V_e(t_{tot}) = 0$, and $V_i(t_i) = V_e(t_i)$).

Finding the t's

This part is straightforward using t_s we are given
$t_i = 5(3 \cdot 0.1) = 1.5$ sec
$t_e = 1.5(5.6) = 8.4$ sec
$t_{tot} = 1.5 + 8.4 = 9.9$ sec.
These numbers will come in handy later when we find the constants. The harder portion is calculating our function for volume of the lung.

Finding our Functions

At the airway, there is a pressure balance which can help us determine our V_i and V_e. This pressure balance is given via the equation:

$$P_r + P_e + P_{ex} = P_{app}$$

where,

1. P_r is the resistive pressure which is proportional to the flow giving us $P_r = R(dV/dt)$
2. P_e is the elastic pressure or $P_e = C^{-1}V$
3. P_{ex} is the residual pressure in the lung representing a constant that is equivalent for inspiration and exhalation
4. P_{app} is the applied pressure which is 10 during inspiration and zero during expiration.

With this information we can form two linear differential equations:

$$3\frac{dV_i}{dt} + 10V_i + P_{ex} = 10$$
$$3\frac{dV_e}{dt} + 10V_e + P_{ex} = 0$$

Using the linear method to solve for each differential equation, V_i and V_e can be solved for and are as follows:

$$\frac{dV_i}{dt} + \frac{10}{3}V_i = \frac{10}{3} - \frac{P_{ex}}{3} => I(x) = e^{\frac{10}{3}t} => e^{\frac{10}{3}t}\frac{dV_i}{dt} + \tfrac{10}{3}e^{\frac{10}{3}t}V_i = e^{\frac{10}{3}t}\left(\tfrac{10}{3} - \tfrac{P_{ex}}{3}\right)$$
$$\int(e^{\frac{10}{3}t}V_i)' = \int e^{\frac{10}{3}t}\left(\tfrac{10}{3} - \tfrac{P_{ex}}{3}\right)dt => e^{\frac{10}{3}t}V_i = e^{\frac{10}{3}t}\left(1 - \tfrac{P_{ex}}{10}\right) + C_i$$

$$V_i = 1 - \frac{1}{10}P_{ex} + C_i e^{-\frac{10}{3}t}$$

The idea is the same for the one below except we dont have to deal with the tracking the $\frac{10}{3}$ since the $P_{app} = 0$ making the value there 0.

$$V_e = -\frac{1}{10}P_{ex} + C_e e^{-\frac{10}{3}t}.$$

Now we have an issue: the constants need to be solved for while keeping in mind that P_{ex} need to stay consistent. Luckily we have some initial conditions to solve for them.

Finding the Constants

The constants can be found using the initial conditions or our assumptions:

$V_i(0) = 0 \qquad V_e(t_{tot}) = 0 \quad$ (Volume is 0 at the start and end of a breath)

When we apply these initial conditions we can find our C_i and C_e

$$C_i = \frac{P_{ex}}{10} - 1 \qquad C_e = \frac{P_{ex}e^{33}}{10}$$

With the new found constants in terms of our P_{ex} and $V_i(t_i) = V_e(t_i)$ (at the end of inspiration and the beginning of expiration the volumes are equivalent) solving for the rest is now possible help us find our P_{ex}. After much algebra we find

$$P_{ex} = \frac{e^5 - 1}{e^{33} - 1}.$$

Now that the equation is solved the next step is to see what all this actually means as a function and how to apply it to the real world mechanical lung.

Our Final Function

After finding our V_i, V_e, and constants (C_i, C_e, and P_{ex}) we can compute our final piece-wise function:

$$V(t) = \begin{cases} V_i(t) = 1 - \dfrac{e^5 - 1}{e^{33} - 1} + \left(\dfrac{e^5 - 1}{e^{33} - 1} - 1\right) e^{-\frac{10}{3}t} & 0 \le t \le 1.5 \\[2ex] V_e(t) = -\dfrac{e^5 - 1}{e^{33} - 1} + \left(\dfrac{e^{38} - e^{33}}{e^{33} - 1}\right) e^{-\frac{10}{3}t} & 1.5 \le t \le 9.9 \end{cases}$$

This $V(t)$ represents the total volume over time of a single breath via a ventilator. This function can be graphed as shown below. According to the graph the Volume of the lung at $t = 1.5$ is 1.0 L and makes a sharp decrease in volume for $1.5 \le t \le 9.9$. This is to be expected as the ventilator is supposed to simulate a human lung. Medical professionals see these readings as them monitor individuals on ventilators.

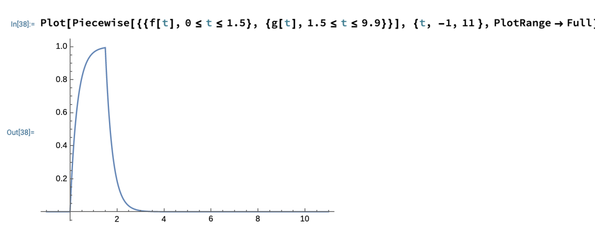

Figure 1: Graph of Volume Function

Two other things medical professionals monitor while the ventilators are running are Flow of air and Pressure which are important number to track as they should stay consistent and make sense across the volume function.

Flow and Pressure Functions

Flow can be represented as the derivative of our function.

$$F = \begin{cases} V_i'(t) = -\dfrac{10}{3}\left(\dfrac{e^5-1}{e^{33}-1} - 1\right) e^{-\frac{10}{3}t} & 0 \leq t \leq 1.5 \\[2ex] V_e'(t) = -\dfrac{10}{3}\left(\dfrac{e^{38}-e^{33}}{e^{33}-1}\right) e^{-\frac{10}{3}t} & 1.5 \leq t \leq 9.9 \end{cases}$$

Just like with our volume function, we can graph our flow function to get a better idea of what it's doing. The graph represents the flow of air in and out of the lungs. It is positive and decreasing during inspiration which tells us that as time increases he amount of that air fills our lungs decreases. During expiration it is negative and increasing which tells us the exact opposite. This makes sense since air flowing out will decrease the volume, therefore the rate will be negative.

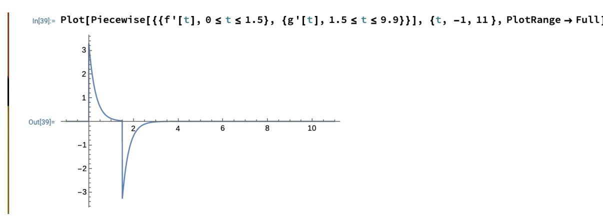

Figure 2: Graph of Flow Function

Pressure is just the function multiplied by the reciprocal of the compliance.

$$P = \begin{cases} V_i(t) = 10 - \dfrac{10e^5-10}{e^{33}-1} + \left(\dfrac{10e^5-10}{e^{33}-1} - 10\right) e^{-\frac{10}{3}t} & 0 \leq t \leq 1.5 \\[2ex] V_e(t) = -\dfrac{10e^5-10}{e^{33}-1} + \left(\dfrac{10e^{38}-10e^{33}}{e^{33}-1}\right) e^{-\frac{10}{3}t} & 1.5 \leq t \leq 9.9 \end{cases}$$

Since the pressure function is just the volume function multiplied by a scalar constant it makes sense that our graphs would look similar to each other. As time increases during inspiration the pressure in the lung increases until expiration starts

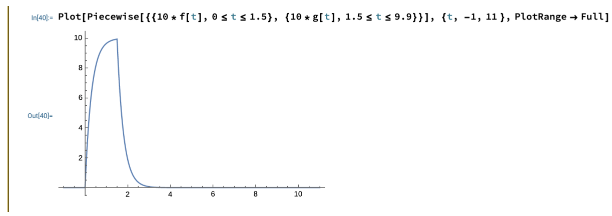

Figure 3: Graph of Pressure Function

then decreases sharply. All this information and a few mathematical tricks later and a function that represents the volume of a mechanical lung along with the flow and pressure associated with it. It is just one of the many cool things mathematics can do.

Conclusion

Mathematical models can be really useful when linking it to real world scenarios as we saw play out at the Sim center and comparing it to our graphs and numbers with our own compliance and resistance and other compliances and resistances which made the time of a single breath go up or down or the capacity of the lung go up or down. It is fascinating that functions are able to help us predict volume of lung during ventilation with a few known facts and assumptions, even if it doesn't account for assistance from the person or excess air in the lungs after an entire breath (continuous breaths versus a single breath). These factor make it so that the mathematical models are not 100% accurate to a real ventilator system, but pretty close.