AIMO Progress Prize 3 – Reference Problems and Solutions*

November 2025

Overview

This document contains a collection of 10 reference problems, the "AIMO3 Reference Bench", each accompanied by a detailed solution. These correspond to the problems provided in `reference.csv` on Kaggle. The purpose of this set is threefold:

1. To verify that your Kaggle notebook works correctly before submitting it to run on the public[1] problem set.
2. To familiarise you with the style and format of AIMO3 problems, including answer extraction conventions (note the move from 3 to 5 digits in the final answer) and the types of intermediate calculations that may be required.
3. To provide a compact benchmark for preliminary testing of models. (Because of its small size, this set is obviously not suitable for model training or fine-tuning.)

Before publication, we evaluated these problems using leading models available via API and ollama. Since these problems had never been exposed to any model before testing, the results represent uncontaminated performance. The results are shown on the next page.

About the Problems

With the exception of **Problem 10**, which adapts an existing problem, all problems in this reference set are entirely original and have not been publicly released before the launch of AIMO3.

Problems 1–4 are drawn from the AIMO2 benchmark. Each includes a brief remark summarising observed results from the AIMO2 Kaggle competition, and for Problems 2 and 3, references to the OpenAI evaluation on the public AIMO2 set (OpenAI x AIMO eval: The gap is shrinking). Problems 3 and 4 are taken from the private AIMO2 set which was not part of the OpenAI evaluation. The six-letter problem identifiers (eg "SWEETS") correspond to those used in the linked analysis. Problem 1 is deliberately straightforward—it is mainly intended as a quick check and should be solvable by any competent model.

Problems 5–9 are challenging, novel problems designed to reflect the style and difficulty of the upcoming AIMO3 dataset. Their difficulty is intentionally higher than that of AIMO2 to ensure the problems continue to challenge the strongest current open-source models.

Problem 10 is an adaptation of a problem from the International Mathematical Olympiad (IMO) Shortlist. It illustrates the process of converting a traditional olympiad-style question into the *answer-only* format used in AIMO. All problems in the AIMO public and private sets are fully original; we chose to adapt this one to preserve the hardest original material for the competition datasets.

About the Solutions

Solutions are written for clarity rather than brevity. Where a result depends on a standard theorem or lemma, a brief reference is provided when possible. The goal is to make each argument straightforward to follow, while acknowledging that the problems themselves are challenging. Only one solution path is shown for each problem, though multiple valid approaches may exist.

*To cite these problems, please refer to the citation of the associated Kaggle competition.

[1] Public is somewhat of a misnomer here as the problem set is **not** publicly available. Instead, 'public' refers to the problems being used for the public leaderboard on Kaggle.

Model Evaluation Results

We evaluated 14 leading language models on the 10-problem AIMO3 Reference Bench, including both commercial APIs and open-weight models. The results reveal a significant performance gap that AIMO3 aims to close.[2]

The strongest commercial models—`GPT-5 Pro` and `GPT-5.1 (high)`—solved all 10 problems, with `GPT-5 (high)`, `Grok-4`, and `Gemini 2.5 Pro` each solving 9 out of 10 (failing only on the hardest Problem 10). In contrast, most open-weight models struggled beyond the easier AIMO2 problems (Problems 1–4). The best-performing non-DeepSeek open-weight model, `gpt-oss-120b`, solved only these four problems and none of the harder Problems 5–10.

A notable exception is `DeepSeek-v3.1-terminus (thinking)`, which matched the performance of second-tier commercial models by solving 9 out of 10 problems. However, this is a substantially larger model (671B parameters) compared to other open-weight entries, and still falls short of the top commercial systems on the hardest problem.

Creating problems that challenge frontier models while working within AIMO's format presents a significant design challenge. The competition requires IMO-style problems with non-guessable integer answers for automated evaluation. Most olympiad problems, however, are proof-based and either have no numerical answer at all, or have answers that are easily guessable (eg, small integers) or binary (yes/no). While challenging problems meeting AIMO's criteria do exist, they are much harder to identify and create. Proof-based evaluation would open up significantly more options for difficult olympiad problems, but remains challenging to implement in a scalable and reliable way.

The near-perfect performance of frontier commercial models on this reference set demonstrates what's possible: closing this gap across the full 50-problem public set could yield scores in the high 40s. This would represent a substantial leap from the mid-30s winning scores in AIMO1 and AIMO2, marking a major milestone in open-source mathematical reasoning capabilities.

Closing this gap is the core challenge of AIMO3. Can your submission demonstrate that open-weight models can match frontier commercial performance on advanced mathematical reasoning?

[2]Model outputs are non-deterministic, so results may vary across runs. Additionally, models accessed via API may be updated over time, potentially affecting performance.

Problem 1

Problem: Alice and Bob are each holding some integer number of sweets. Alice says to Bob: "If we each added the number of sweets we're holding to our (positive integer) age, my answer would be double yours. If we took the product, then my answer would be four times yours." Bob replies: "Why don't you give me five of your sweets because then both our sum and product would be equal." What is the product of Alice and Bob's ages?

Answer: 50

Solution: Let Alice and Bob's ages be x_A and x_B, respectively. After Alice gives Bob five sweets, let Alice have y_A sweets and Bob have y_B sweets. We have

$$\begin{cases} S = x_A + y_A = x_B + y_B \\ P = x_A y_A = x_B y_B \end{cases}.$$

The unordered pairs $\{x_A, y_A\}$ and $\{x_B, y_B\}$ each form the set of roots of the quadratic $z^2 - Sz + P = 0$. Hence the sets are equal (possibly with the elements swapped). We consider the two cases.

Case 1 $x_A = x_B = x$, $y_A = y_B = y$

Before the exchange, Alice holds $y + 5$ sweets and Bob $y - 5$ so the conditions become

$$\begin{cases} (x + y + 5) = 2(x + y - 5) \implies x + y = 15 \\ x(y + 5) = 4x(y - 5) \implies x(3y - 25) = 0 \end{cases}.$$

The second condition forces either $x = 0$ or $3y = 25$, both of which violate the ages being positive integers.

Case 2 $x_A = y_B = x$, $y_A = x_B = y$

Before the exchange, Alice holds $y + 5$ sweets and Bob $x - 5$ so the conditions become:

$$\begin{cases} (x + y + 5) = 2(x + y - 5) \implies x + y = 15 \\ x(y + 5) = 4y(x - 5) \implies 3xy - 20y - 5x = 0 \end{cases}$$
$$\implies 0 = 3x(15 - x) - 20(15 - x) - 5x = -3(x - 10)^2 \implies x = 10.$$

We then have $y = 15 - x = 5$. It is easy to verify that if at the start, Alice is aged 10 and holds 10 sweets and Bob is aged 5 and holds 5 sweets then the conditions are satisfied. Therefore, the answer we report is $5 \times 10 = \boxed{50}$.

Remark: This problem was part of the AIMO2 private set ("SWEETS") and was solved by approximately 85% of submissions, including all of the top 100. We include it as an example of a problem that current models are expected to solve easily, which may assist in testing your model using the reference set on Kaggle. This problem is easier than any problem used in AIMO3, so the problems below should be used instead to give an indication of difficulty.

Problem 2

Problem: A 500×500 square is divided into k rectangles, each having integer side lengths. Given that no two of these rectangles have the same perimeter, the largest possible value of k is $\mathcal{K}$. What is the remainder when $\mathcal{K}$ is divided by 10^5?

Answer: 520

Solution: We first show that the square can be divided into 520 rectangles subject to the constraints, and then prove that this is the maximum number possible.

In the bottom 249 rows, place 249 pairs of rectangles $1 \times i$ and $1 \times (500 - i)$ for $i = 1, \ldots, 249$. Then, place a single 1×500 rectangle in the row above to complete the bottom half. For the top half, place a 1×250 rectangle vertically along the left side and then 20 more horizontal rectangles $i \times 499$ for $i = 3, 4, \ldots, 22$ (note that $3 + 4 + \cdots + 22 = \frac{1}{2} \cdot 22 \cdot 23 - 3 = 250$).

The rectangles in the bottom half have perimeters $4, 6, \ldots, 500, 504, 506, \ldots, 1000, 1002$ whilst the rectangles in the top half have perimeters $502, 1004, 1006, \ldots, 1042$ so all these perimeters are different. In total we have placed $(2 \times 249) + 1 + 1 + 20 = 520$ rectangles proving the lower bound.

To show that 520 is optimal, assume we can place $k \geq 521$ rectangles subject to the constraint.

First, note that the perimeters are all even integers and equal to $2s$ where s is the semi-perimeter of the rectangle. If the two sides of the rectangle are x and y, then $s = x + y$ and the area of the rectangle is $A = xy = x(s - x)$. Since this is a downwards-pointing quadratic in x, A is minimised when x takes either its minimum or maximum value. Noting that $1 \leq x \leq 500$ and $1 \leq y = s - x \leq 500$, we see

$$\max\{1, s - 500\} \leq x \leq \min\{500, s - 1\}.$$

If $s \leq 500$, this bound is simply $1 \leq x \leq s - 1$ and we see both the minimum and maximum values for x give $A = s - 1$ meaning that $A \geq s - 1$ whenever $s \leq 500$.

If $s \geq 501$, the bound becomes $s - 500 \leq x \leq 500$ and we see both the minimum and maximum values for x give $A = 500(s - 500)$ meaning that $A \geq 500(s - 500)$ whenever $s \geq 501$.

Define

$$f(s) = \begin{cases} s - 1 & 1 \leq s \leq 500 \\ 500(s - 500) & s \geq 501 \end{cases}.$$

Observing that $500 - 1 = 499 < 500(501 - 500)$, we see that $f(s)$ is an increasing function. Furthermore, the above shows a rectangle with semi-perimeter s has area $A \geq f(s)$.

Now suppose the 521 rectangles have semi-perimeters $2 \leq s_1 < s_2 < \cdots < s_{521}$ and areas $A_1, \ldots, A_{521}$. By considering the total area of the rectangles and using that $f(s)$ is increasing, we have

$$500^2 = A_1 + \cdots + A_{521} \geq f(s_1) + \cdots + f(s_{521}) \geq f(2) + f(3) + \cdots + f(522).$$

We can then calculate

$$500^2 \geq \sum_{s=2}^{500} f(s) + \sum_{s=501}^{522} f(s) = \sum_{s=2}^{500} (s - 1) + \sum_{s=501}^{522} 500(s - 500) = 124{,}750 + 126{,}500 > 500^2$$

which is a contradiction.

Thus, $\boxed{520}$ is indeed the optimal number of rectangles and this is the answer we report.

Remark 1: As this problem demonstrates, some answers are already smaller than the modulus for which the remainder is requested. In this case, the model should simply return the answer with no further calculations required.

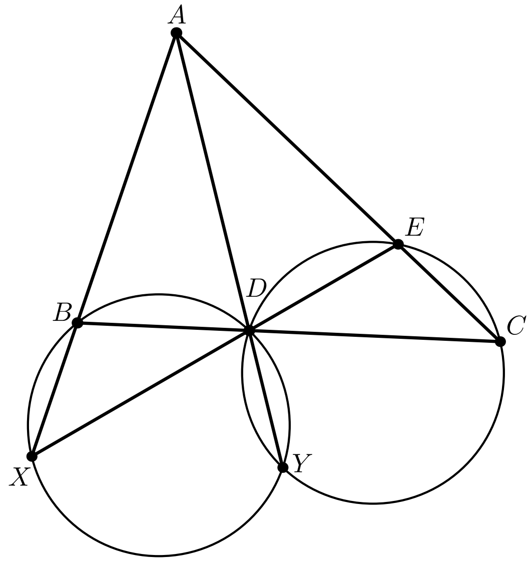
Remark 2: This problem was part of the AIMO2 public set ("RECTIL") and was solved by only a handful of the top 100 teams, and by none of the ultimate top 5. Interestingly, in the OpenAI x AIMO eval, the high compute run did not return the correct answer in its top response though lower compute runs did return the correct answer.

Problem 3

Problem: Let ABC be an acute-angled triangle with integer side lengths and $AB < AC$. Points D and E lie on segments BC and AC, respectively, such that $AD = AE = AB$. Line DE intersects AB at X. Circles BXD and CED intersect for the second time at $Y \neq D$. Suppose that Y lies on line AD. There is a unique such triangle with minimal perimeter. This triangle has side lengths $a = BC$, $b = CA$, and $c = AB$. Find the remainder when abc is divided by 10^5.

Answer: 336

Solution: As Y lies on line AD, we see A lies on line DY which is the radical axis of circles BXD and CED. We therefore have $AE \cdot AC = AB \cdot AX$, which gives $AX = AC$ since $AE = AB$. These two results together show pairs (B, E) and (X, C) are reflections across the bisector of $\angle BAC$ so D must be the intersection of this bisector with BC. We can run this argument in reverse to see that D lying on the angle bisector is also a sufficient condition for Y to lie on line AD.



If we define D' as the intersection of the internal angle bisector of $\angle BAC$ with side BC, the above shows that Y lies on AD if and only if $AD' = AB$ (and therefore $D = D'$).

From angle bisector theorem, we have $BD' = \frac{ac}{b+c}$ and $CD' = \frac{ab}{b+c}$. Define $d = AD'$. We can apply Stewart's theorem to triangle ABC with cevian AD to get

$$a \cdot \frac{ac}{b+c} \cdot \frac{ab}{b+c} + ad^2 = b^2 \cdot \frac{ac}{b+c} + c^2 \cdot \frac{ab}{b+c} = abc \implies d^2 = \frac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2}.$$

$AD' = AB$ is equivalent to $d = c$ which we can express as

$$c^2 = \frac{bc\left((b+c)^2 - a^2\right)}{(b+c)^2} \iff \left(\frac{a}{b+c}\right)^2 = \frac{b-c}{b}.$$

We are told that a, b, c are positive integers with $c < b$. We want to find the minimum such values that are the sides of a triangle and satisfy these conditions.

Write $g = \gcd(b, c)$ and $b = gb'$, $c = gc'$ where $\gcd(b', c') = 1$. The above then becomes

$$\cdots \iff \left(\frac{a}{g\left(b'+c'\right)}\right)^2 = \frac{b'-c'}{b'}.$$

Since the numerator and denominator are coprime and positive, we must have $b' - c' = y^2$ and $b' = x^2$ for coprime positive integers x and y with $x > y$ then

$$\cdots \implies \frac{a}{g\left(2x^2-y^2\right)} = \frac{y}{x} \implies a = \frac{gy\left(2x^2-y^2\right)}{x}.$$

Note that $\gcd\bigl(2x^2 - y^2, x\bigr) = \gcd\bigl(y^2, x\bigr) = 1$ so, for a to be an integer, we must have $x \mid g$. Write $g = kx$ for a positive integer k. We can then summarise the solutions as

$$(a, b, c) = \left(ky\left(2x^2-y^2\right), kx^3, kx\left(x^2-y^2\right)\right). \tag{$\blacksquare$}$$

Triangle inequality is equivalent to:

$$\begin{aligned}
b + c > a &\iff (x-y)\left(2x^2-y^2\right) > 0\\
a + c > b &\iff y\left(2x^2-xy-y^2\right) > 0\\
a + b > c &\iff y(x+y)(2x-y) > 0.
\end{aligned}$$

These will always be satisfied for coprime positive integers x and y with $x > y$. We can also check

$$b > c \iff x^2 > x\left(x^2-y^2\right) \iff xy^2 > 0$$

which will be satisfied for $x, y \geq 1$.

Lastly, note that all the steps above are reversible so the family given in ($\blacksquare$) exactly characterises triangles with the desired property.

What remains is to find the triangle with minimal perimeter. Since k simply scales the perimeter, we must have $k = 1$ for the triangle with minimal perimeter and then

$$a + b + c = y\left(2x^2-y^2\right) + x^2 + x\left(x^2-y^2\right) = (x+y)\left(2x^2-y^2\right).$$

This is an increasing function of x so to minimise the perimeter, we will choose $x = y + 1$ (which means x and y will be coprime). The above becomes

$$\cdots = (2y+1)\left(y^2+4y+2\right)$$

which is an increasing function of y so is minimised at $y = 1$.

The pair $x = 2$, $y = 1$ corresponds to $(a, b, c) = (7, 8, 6)$ which has all of the required properties (the triangle is acute since $8^2 < 7^2 + 6^2$). The answer we report is

$$7 \times 8 \times 6 = \boxed{336}.$$

Remark: This problem was also part of the AIMO2 public set (“MINPER”) and was again solved by only a handful of the top 100 teams, and by none of the ultimate top 5. This problem was solved in the OpenAI x AIMO eval by all compute levels.

Problem 4

Problem: Let $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ be a function such that for all positive integers m and n,

$$f(m) + f(n) = f(m + n + mn).$$

Across all functions f such that $f(n) \leq 1000$ for all $n \leq 1000$, how many different values can $f(2024)$ take?

Answer: 580

Solution: Let $\mathbb{Z}_{\geq 2} = \{2, 3, \ldots\}$ be the set of positive integers greater than or equal to 2. Define a function $g\colon \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 1}$ by $g(n) = f(n-1)$ for $n \geq 2$. From the given relation for f we have, for $n, m \geq 2$,

$$g(m) + g(n) = f(m-1) + f(n-1) = f(m-1+n-1+(m-1)(n-1)) = f(mn-1) = g(mn). \quad (*)$$

Repeatedly applying $(*)$, we see that if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorisation of $n \geq 2$ then

$$g(n) = g(p_1^{\alpha_1}) + \cdots + g(p_k^{\alpha_k}) = \alpha_1 \cdot g(p_1) + \cdots + \alpha_k \cdot g(p_k). \quad (\blacktriangle)$$

We have $f(2024) = g(2025)$ and the prime factorisation of 2025 is $2025 = 3^4 \cdot 5^2$ so

$$f(2024) = g(2025) = 4g(3) + 2g(5).$$

We are left to consider the possible values for the above quantity subject to the condition that $g(n) = f(n-1) \leq 1000$ for $2 \leq n \leq 1001$.

Claim: $1 \leq g(3) \leq 166$ and $1 \leq g(5) \leq 250$. For each choice of $g(3)$ and $g(5)$ in these ranges, we can construct a function g satisfying $(*)$.

Proof. Firstly, from $(\blacktriangle)$, we have

$$\begin{aligned} 1000 \geq g(729) = g\big(3^6\big) = 6g(3) &\implies g(3) \leq \left\lfloor \frac{1000}{6} \right\rfloor = 166 \\ 1000 \geq g(625) = g\big(5^4\big) = 4g(5) &\implies g(5) \leq \left\lfloor \frac{1000}{4} \right\rfloor = 250 \end{aligned}$$

which proves the bounds.

Next, let $1 \leq s \leq 166$ and $1 \leq t \leq 250$. For a positive integer $n \geq 2$, let $n = 3^\alpha \cdot 5^\beta \cdot p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ be its prime factorisation with $p_i \neq 3, 5$ and possibly with α or β being 0. Define a function g by

$$g(n) = g\big(3^\alpha \cdot 5^\beta \cdot p_1^{\gamma_1} \cdots p_k^{\gamma_k}\big) = \alpha s + \beta t + \gamma_1 + \cdots + \gamma_k.$$

This can easily be seen to satisfy $(*)$ and also has $g(3) = s$, $g(5) = t$.

Let $n \leq 1001$, it remains to check $g(n) \leq 1000$. Define $\gamma = \gamma_1 + \cdots + \gamma_k$ then

$$1001 \geq n \geq 2^\gamma \cdot 3^\alpha \cdot 5^\beta \quad \text{and} \quad g(n) = \alpha s + \beta t + \gamma \leq 166\alpha + 250\beta + \gamma. \quad (\blacksquare)$$

We now consider the possible values for β. Since $5^\beta \leq n \leq 1001$, we have $0 \leq \beta \leq 4$.

- $\beta = 4$: Since $5^4 = 625$, $(\blacksquare)$ forces $\alpha = \gamma = 0$ so $g(n) \leq 250 \times 4 = 1000$.

- $\beta = 3$: Similarly, $5^3 = 125$ so from (■), $\alpha \leq 1$ and $\gamma \leq 3$ so

$$g(n) \leq 166 \times 1 + 250 \times 3 + 3 = 919 \leq 1000.$$

- $\beta = 2$: If $\alpha = 3$, then $1001 \geq n = 2^\gamma \cdot 3^3 \cdot 5^2 = 675 \cdot 2^\gamma$ so $\gamma = 0$. From (■),

$$g(n) \leq 3 \times 166 + 2 \times 250 + 0 = 998 \leq 1000.$$

  Otherwise, $\alpha \leq 2$ and, as $\gamma \leq 5$,

$$g(n) \leq 166 \times 2 + 250 \times 2 + 5 = 837 \leq 1000.$$

- $\beta = 1$: $5^1 = 5$ and from (■), $\alpha \leq 4$, $\gamma \leq 7$ so

$$g(n) \leq 166 \times 4 + 250 \times 1 + 7 = 921 \leq 1000.$$

- $\beta = 0$: If $\alpha = 6$, then $\gamma = 0$ and $g(n) \leq 6 \times 166 = 996 \leq 1000$. Otherwise, $\alpha \leq 5$ and also $\gamma \leq 9$ leading to

$$g(n) \leq 166 \times 5 + 250 \times 0 + 9 = 839 \leq 1000.$$

In any case, we see the condition on g will be satisfied so we can reverse the process to get a valid function f. □

From the Claim, we have $f(2024) = 4s + 2t = 2(2s + t)$ with $1 \leq s \leq 166$ and $1 \leq t \leq 250$. $(2s + t)$ can take any value between 3 and $250 + 2 \times 166 = 582$ and $f(2024)$ is equal to double this so, in total, there are $582 - 3 + 1 = \boxed{580}$ possible values which is the answer we report.

Remark: This problem was part of the AIMO2 private set ("FUNVAL") and was solved by about 2% of all submissions, including roughly 20% of the top 100 (though not all of the top 5). This highlights the trade-offs made by top-ranking models, which may fail to solve certain problems that a larger fraction of lower-ranked models can, yet they still achieve higher overall performance.

Problem 5

Problem: A tournament is held with 2^{20} runners each of which has a different running speed. In each race, two runners compete against each other with the faster runner always winning the race. The competition consists of 20 rounds with each runner starting with a score of 0. In each round, the runners are paired in such a way that in each pair, both runners have the same score at the beginning of the round. The winner of each race in the i^{th} round receives 2^{20-i} points and the loser gets no points.

At the end of the tournament, we rank the competitors according to their scores. Let N denote the number of possible orderings of the competitors at the end of the tournament. Let k be the largest positive integer such that 10^k divides N. What is the remainder when k is divided by 10^5?

Answer: 21818

Solution: Since the points received in a given round are strictly greater than the points available in all subsequent rounds (from $2^k > 2^k - 1 = 2^{k-1} + 2^{k-2} + \cdots + 1$), each runner must be paired with another runner with an identical win/loss record in the preceding rounds (so they have the same score). Thus, in round i there are 2^{i-1} groups (based on the possible win/loss sequences in the previous $i-1$ rounds) each consisting of 2^{21-i} runners. Also, a runner's (ordered) win/loss record across the rounds uniquely determines their final position by uniqueness of binary expansions.

Consider a group with the same score in a particular round and let there be $2n$ runners in that group. We need to count the number of valid ways to choose n winners from these $2n$ runners. Label the runners from 1 (fastest) to $2n$ (slowest), and record the outcome of each race in a left-to-right string of parentheses, where:

- `(` indicates that runner wins the race,
- `)` indicates that runner loses.

In a valid string, each `(` must match with a later `)`, corresponding to a race between a faster and slower runner. For example, when $n = 3$, the string $(()())$ represents the matches: 1 beats 6, 2 beats 3, and 4 beats 5.

It is well-known that the number of such valid parenthesis strings is the n^{th} Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.$$

Now in round i, we have 2^{i-1} groups each consisting of 2^{21-i} runners. By the above, there are

$$\left(C_{2^{20-i}}\right)^{2^{i-1}} = \left(\frac{\left(2^{21-i}\right)!}{(2^{20-i})!\,(2^{20-i}+1)!}\right)^{2^{i-1}}$$

ways to choose the winners in this round. Since each choice of winners across the different groups will lead to a different final position (by the comment at the top about the win/loss records uniquely determining the final position), the number of possible orderings is equal to the product of the above quantity over the rounds $1 \le i \le 20$ which is

$$N = \prod_{i=1}^{20} \left(\frac{\left(2^{21-i}\right)!}{(2^{20-i})!\,(2^{20-i}+1)!} \right)^{2^{i-1}} = \prod_{i=1}^{20} \frac{1}{(2^{20-i}+1)^{2^{i-1}}} \cdot \frac{\left((2^{21-i})!\right)^{2^{i-1}}}{((2^{20-i})!)^{2^i}}$$
$$= \left[\prod_{i=1}^{20} \frac{1}{(2^{20-i}+1)^{2^{i-1}}} \right] \cdot \frac{\left((2^{20})!\right)^{2^0}}{((2^{19})!)^{2^1}} \cdot \frac{\left((2^{19})!\right)^{2^1}}{((2^{18})!)^{2^2}} \cdots \frac{\left((2^1)!\right)^{2^{19}}}{((2^0)!)^{2^{20}}}$$
$$= \left(2^{20}\right)! \cdot \prod_{i=1}^{20} \frac{1}{(2^{20-i}+1)^{2^{i-1}}}$$

We are then left to determine the highest power of 10 dividing N. Applying Legendre's Formula, we can compute the highest power of 2 and 5 dividing $\left(2^{20}\right)!$ as

$$\nu_2\left((2^{20})!\right) = \sum_{k=1}^{\infty} \left\lfloor \frac{2^{20}}{2^k} \right\rfloor = 2^{19} + 2^{18} + \cdots + 2^1 + 2^0 = 2^{20} - 1$$
$$\nu_5\left((2^{20})!\right) = \sum_{k=1}^{\infty} \left\lfloor \frac{2^{20}}{5^k} \right\rfloor = \left\lfloor \frac{2^{20}}{5} \right\rfloor + \left\lfloor \frac{2^{20}}{5^2} \right\rfloor + \cdots + \left\lfloor \frac{2^{20}}{5^8} \right\rfloor = 262{,}140\,.$$

We can also see that all the terms in the denominator in the expression for N are odd except the final one for $i = 20$. Thus,

$$\nu_2(N) = \left(2^{20} - 1\right) - \nu_2\left(2^{2^{20-1}}\right) = \left(2^{20} - 1\right) - 2^{19} = 2^{19} - 1\,.$$

Since powers of 2 cycle with period 4 modulo 5, we have that $2^{20-i} + 1$ will be divisible by 5 if and only if $20 - i \equiv 2 \bmod 4$ which is equivalent to $i = 4k + 2$ for some non-negative integer k. In this case, we can write

$$2^{20-i} + 1 = 2^{20-(4k+2)} + 1 = 2^{2(9-2k)} + 1 = 4^{9-2k} + 1\,.$$

We can apply the Lifting the Exponent Lemma (or just directly compute given the small number of possibilities for k) that

$$\nu_5\left(4^{9-2k} + 1\right) = \nu_5(4+1) + \nu_5(9-2k) = 1 + \nu_5(9-2k) = \begin{cases} 1 & k \in \{0,1,3,4\} \\ 2 & k = 2 \end{cases}\,.$$

Now introducing the powers in the denominator, we get

$$\nu_5\left(\prod_{i=1}^{20} \left(2^{20-i}+1\right)^{2^{i-1}} \right) = 2^{2-1} + 2^{6-1} + 2^{10-1} \cdot 2 + 2^{14-1} + 2^{18-1} = 140{,}322\,.$$

Thus,

$$\nu_5(N) = 262{,}140 - 140{,}322 = 121{,}818\,.$$

To get a factor of 10 in N, we must have a factor of 2 and a factor of 5. Noting that $2^{19} - 1 = 524{,}287 > 121{,}818$, we see that factors of 5 are the limiting ones. We therefore have

$$k = \nu_5(N) = 121{,}818 \equiv \boxed{21818} \pmod{10^5}$$

which is the answer we report.

Problem 6

Problem: Define a function $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by

$$f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} j^{1024} \left\lfloor \frac{1}{j} + \frac{n-i}{n} \right\rfloor .$$

Let $M = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and let $N = f\big(M^{15}\big) - f\big(M^{15} - 1\big)$. Let k be the largest non-negative integer such that 2^k divides N. What is the remainder when 2^k is divided by 5^7?

Answer: 32951

Solution: We make use of two identities. Define $\sigma_k(n) = \sum\limits_{d \mid n} d^k$.

Claim 1: (Hermite's Identity). For any $x \in \mathbb{R}$ and positive integer n, we have

$$\sum_{i=1}^{n} \left\lfloor x + \frac{n-i}{n} \right\rfloor = \lfloor nx \rfloor .$$

Proof. Let

$$d(x) = \lfloor nx \rfloor - \sum_{i=1}^{n} \left\lfloor x + \frac{n-i}{n} \right\rfloor .$$

Then,

$$d\left(x + \frac{1}{n}\right) = \lfloor nx + 1 \rfloor - \sum_{i=1}^{n} \left\lfloor x + \frac{n+1-i}{n} \right\rfloor = \lfloor nx \rfloor + 1 - \left(\sum_{i=1}^{n} \left\lfloor x + \frac{n-i}{n} \right\rfloor \right) - 1 = d(x),$$

hence d is periodic with period $\frac{1}{n}$. Since all floors are 0 for $x \in \big[0, \frac{1}{n}\big)$, this gives $d \equiv 0$ in general. $\square$

Claim 2: For any positive integer n,

$$\sum_{j=1}^{n} \sigma_k(j) = \sum_{d=1}^{n} d^k \left\lfloor \frac{n}{d} \right\rfloor . \qquad (\blacksquare)$$

Proof. We count

$$S = \sum_{j=1}^{n} \sum_{d \mid j} d^k$$

in two ways. The LHS of ($\blacksquare$) comes from the definition of σ_k.

For $d = 1, 2, \ldots, n$, we have $\left\lfloor \frac{n}{d} \right\rfloor$ multiples of d at most n, hence this is the number of times d^k is summed. This gives the RHS of ($\blacksquare$). Thus, both sides of ($\blacksquare$) are equal to S so in particular, are equal to each other. $\square$

Flipping the order of summation in f, we have

$$f(n) = \sum_{j=1}^{n} j^{1024} \sum_{i=1}^{n} \left\lfloor \frac{1}{j} + \frac{n-i}{n} \right\rfloor \overset{\textbf{Claim 1}}{=} \sum_{j=1}^{n} j^{1024} \left\lfloor \frac{n}{j} \right\rfloor \overset{\textbf{Claim 2}}{=} \sum_{j=1}^{n} \sigma_{1024}(j).$$

We therefore have

$$N = f\left(M^{15}\right) - f\left(M^{15} - 1\right) = \sigma_{1024}\big(M^{15}\big).$$

Define $P = \{2, 3, 5, 7, 11, 13\}$ so $M = \prod_{p \in P} p$. By considering the possibilities for the prime factorisation of d in the definition of σ_{1024} and raising these to the power of k we get

$$\begin{aligned} N = \sigma_{1024}\big(M^{15}\big) &= \prod_{p \in P} \left(1 + p^{1024} + p^{2 \cdot 1024} + \cdots + p^{15 \cdot 1024}\right) \\ &= \prod_{p \in P} \frac{p^{16 \cdot 1024} - 1}{p^{1024} - 1}. \end{aligned}$$

We want to count the number of factors of 2 in this expression. For $p = 2$, the expression will be odd so it suffices to consider the odd p. We apply the Lifting the Exponent Lemma in the even power case to get

$$\nu_2\left(\left(p^{1024}\right)^{16} - 1\right) - \nu_2\big(p^{1024} - 1\big) = \nu_2\big(p^{1024} + 1\big) + \nu_2(16) - 1 = 1 + 4 - 1 = 4.$$

where we have used $p^2 \equiv 1 \bmod 4$ for all odd p so $p^{1024} + 1 \equiv 2 \bmod 4$ is divisible by 2 but not 4.

Since there are 5 odd elements of P, we get

$$k = \nu_2(N) = 5 \cdot 4 = 20.$$

Lastly, we compute

$$2^k = 2^{20} \equiv \boxed{32951} \pmod{5^7}$$

which is the answer we report.

Remark: The choice of 5^7 as the modulus in this problem is intended to emphasise that models should be able to handle a variety of moduli—not just 10^5 or 99991 which are the most commonly used (the latter is used when a prime modulus is desirable). For AIMO3, all problem statements have the modulus included explicitly (unlike for AIMO1 and AIMO2 where there was an implicit step of taking your answer modulo 1000).

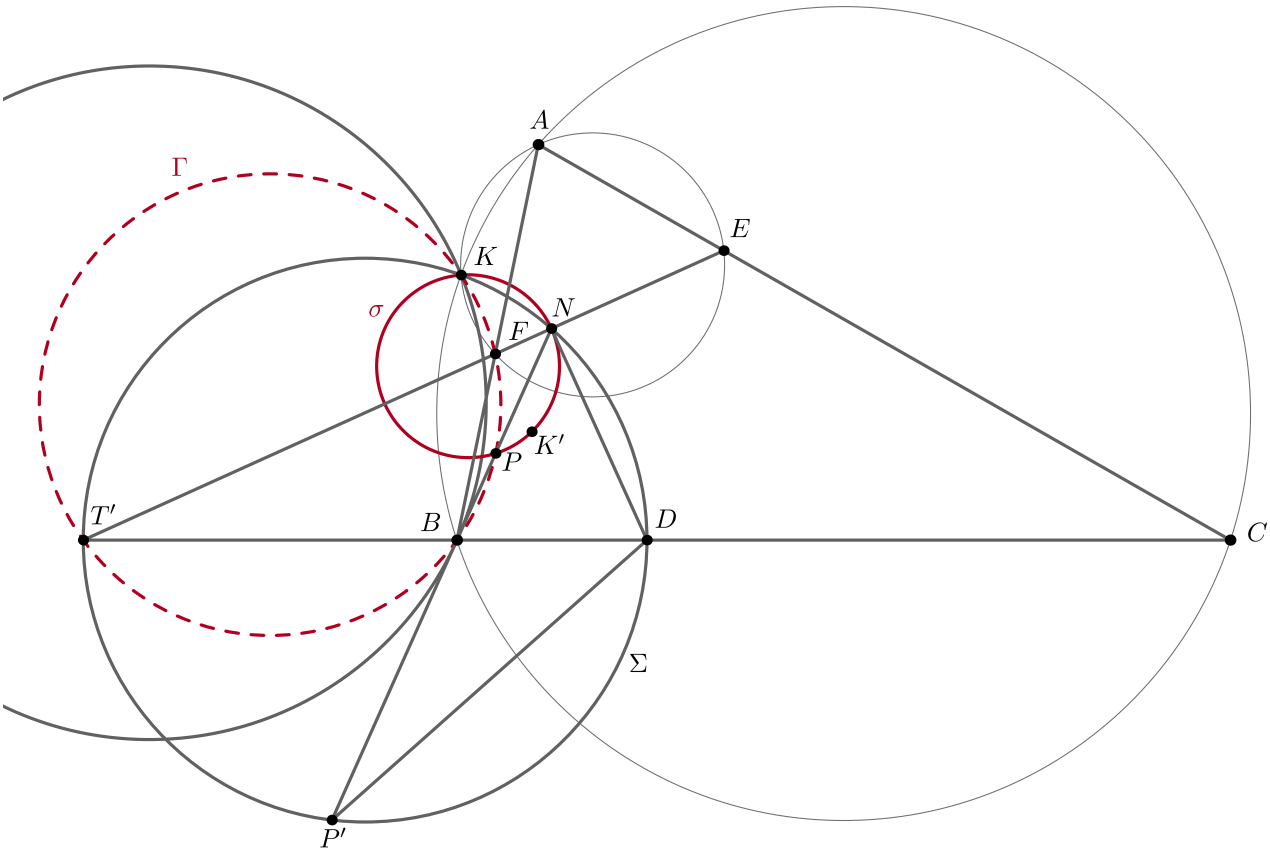
Problem 7

Problem: Let ABC be a triangle with $AB \neq AC$, circumcircle Ω, and incircle ω. Let the contact points of ω with BC, CA, and AB be D, E, and F, respectively. Let the circumcircle of AFE meet Ω at K and let the reflection of K in EF be K'. Let N denote the foot of the perpendicular from D to EF. The circle tangent to line BN and passing through B and K intersects BC again at $T \neq B$.

Let sequence $(F_n)_{n\geq 0}$ be defined by $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Call ABC *n-tastic* if $BD = F_n$, $CD = F_{n+1}$, and $KNK'B$ is cyclic. Across all n-tastic triangles, let a_n denote the maximum possible value of $\frac{CT\cdot NB}{BT\cdot NE}$. Let α denote the smallest real number such that for all sufficiently large n, $a_{2n} < \alpha$. Given that $\alpha = p + \sqrt{q}$ for rationals p and q, what is the remainder when $\lfloor p^{q^p} \rfloor$ is divided by 99991?

Answer: 57447

Solution: We assume $AB < AC$ since all n-tastic triangles have this property and that is where we are working towards. For now however, we consider all triangles with this property and will impose the n-tastic condition later.



Denote circle KNK' and KND by σ and Σ, respectively. Let EF intersect BC at T'. Next, let σ and Σ intersect line BN again at $P \neq N$ and $P' \neq N$, respectively.

From its definition, K is the centre of spiral similarity taking $FE \to BC$. Denote this spiral similarity by f so $f(F) = B$, $f(E) = C$, and $f(K) = K$. We recall some well-known results:

- K is also the centre of spiral similarity taking $BF \to CE$.
- $T'BFK$ is cyclic—denote this circle by Γ.

Claim 1: $f(N) = D$.

Proof. This is a well-known result but for completeness, we provide a proof.

Firstly, recall that lines AD, BE, and CF concur at the Gergonne point of triangle ABC. Thus, we can apply Ceva's and Menelaus's theorem to get

$$\frac{BT'}{T'C} \overset{\textbf{Menelaus}}{=} -\frac{AE}{EC} \cdot \frac{BF}{FA} \overset{\textbf{Ceva}}{=} -\frac{BD}{DC}$$

where we are using directed lengths so if a point lies outside the segment, the ratio is negative. This means that $(B, C; T', D) = -1$ where the term in brackets denotes the cross ratio of the four points.

We have $\angle T'ND = 90°$ and it is well-known (eg see here) that these two properties combined imply that ND and $T'N$ bisect $\angle BNC$ (internally and externally, respectively).

From this, we get $\angle FNB = \angle CNE$. Since $AE = AF$, $\angle AEF = \angle EFA$ which implies $\angle BFN = \angle NEC$. Combining these two results shows triangle BFN and CEN are similar hence

$$\frac{FN}{NE} = \frac{BF}{CE} = \frac{BD}{DC}$$

where we have used that BF and BD are both tangents from B to the incircle so have equal length, and similarly for C.

Since $f(FE) = BC$ and N and D split these segments in the same ratio, the claim follows. $\square$

A consequence of Claim 1 and properties of spiral similarities is that Σ passes through T'. Since $\angle T'ND = 90°$, this also shows Σ has diameter $T'D$.

Claim 2: P lies on Γ.

Proof. Firstly, note that σ is uniquely defined as the circle passing through K and N that has centre on line EF (which is equivalent to σ passing through K'). Applying the transformation f, we see $f(\sigma)$ is uniquely defined as the circle passing through $f(K) = K$ and $f(N) = D$ (Claim 1) that has centre on line $f(EF) = BC$. But from the comment above, this precisely defines Σ, thus $f(\sigma) = \Sigma$.

Next, we see that

$$\angle FNP = \angle T'NP' = \angle T'DP' = \angle BDP'.$$

Since $f(FN) = BD$ and $f(\sigma) = \Sigma$, this shows $f(P) = P'$. Thus,

$$\angle FPK = \angle f(F)f(P)f(K) = \angle BP'K = \angle NP'K = \angle NT'K = \angle FT'K$$

which shows that P lies on circle $FT'K$ ie Γ. $\square$

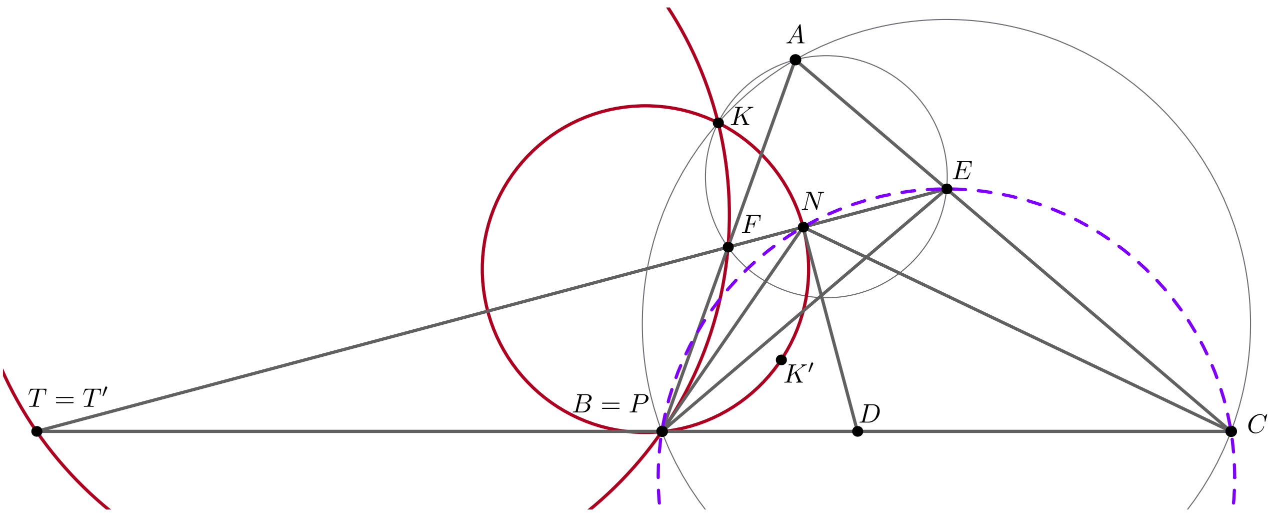
Claim 3: For an n-tastic triangle, $T = T'$ and $BNEC$ is cyclic.

Proof. For these triangles B lies on σ so we must in fact have $P = B$. It then follows (by considering P moving towards B along line BN) that Γ is tangent to BN at B. Since Γ also passes through K, this must be the circle described at the bottom of the first paragraph in the question. Since Γ intersects BC again at $T' \neq B$, the first part of the claim follows.

For the second part, applying the alternate segment theorem using the tangency of Γ to BN we get

$$\angle CBN = \angle TBP' = \angle TFB = \angle EFA = \angle AEF = 180^\circ - \angle NEC$$

which proves that $BNEC$ is cyclic. $\square$



We now consider n-tastic triangles and use T and T' interchangeably since they are the same point. Recalling from Claim 1 that triangles BFN and CEN are similar and then using Claim 3 we get

$$\angle EBN = \angle ECN = \angle NBF = \angle BTN$$

where in the last step we used that BN is tangent to Γ. We also have

$$\angle CBE = \angle CNE = \angle FNB = 180^\circ - \angle BNE = \angle ECB$$

so $EC = EB$.

Applying the sine rule to triangles EBN and ETB, we get

$$\frac{NB}{NE} = \frac{\sin \angle NEB}{\sin \angle EBN} = \frac{\sin \angle TEB}{\sin \angle BTE} = \frac{BT}{BE} = \frac{BT}{CE} = \frac{BT}{CD}.$$

We can use this to rewrite the target expression in the question as

$$\frac{CT \cdot NB}{BT \cdot NE} = \frac{CT}{BT} \cdot \frac{BT}{CD} = \frac{CT}{CD}.$$

We are told that $BD = F_n$ and $CD = F_{n+1}$ so $BC = BD + CD = F_{n+2}$. Returning to the calculations in Claim 1 (and this time using undirected lengths), we have

$$\frac{CD}{BD} = \frac{F_{n+1}}{F_n} = \frac{CT}{BT} = \frac{CT}{CT - BC} = \frac{CT}{CT - F_{n+2}} \implies CT = \frac{F_{n+1}F_{n+2}}{F_{n-1}}.$$

This allows us to write (noting that the quantity in the question is constant across all n-tastic triangles)

$$a_n = \frac{CT \cdot NB}{BT \cdot NE} = \frac{CT}{CD} = \frac{F_{n+2}}{F_{n-1}}.$$

Let $\varphi = (1+\sqrt{5})/2$ be the golden ratio. Using Binet's formula, we can write

$$a_n = \frac{\varphi^{n+2} - (-\varphi)^{-(n+2)}}{\varphi^{n-1} - (-\varphi)^{-(n-1)}}.$$

Now specialising to n even, we have

$$a_{2n} = \frac{\varphi^{2n+2} - \varphi^{-(2n+2)}}{\varphi^{2n-1} + \varphi^{-(2n-1)}} = \varphi^3 \cdot \underbrace{\frac{1 - \varphi^{-4n-4}}{1 + \varphi^{-4n+2}}}_{(*)} < \varphi^3.$$

Since $\varphi > 1$, we have $(*) \to \frac{1}{1} = 1$ as $n \to \infty$, thus the smallest real number α with the desired property is φ^3. We can compute

$$\varphi^3 = 2 + \sqrt{5} \implies p = 2\ ,\ q = 5$$

so the answer we report is

$$2^{5^2} = 33554432 \equiv \boxed{57447} \pmod{99991}.$$

Remark: Because of the problem's conditional structure—and that it concerns a family of triangles rather than one specific configuration—brute-force approaches that attempt to construct one or more diagrams and compute all lengths numerically are unlikely to succeed.

Problem 8

Problem: On a blackboard, Ken starts off by writing a positive integer n and then applies the following move until he first reaches 1. Given that the number on the board is m, he chooses a base b, where $2 \leq b \leq m$, and considers the unique base-b representation of m,

$$m = \sum_{k=0}^{\infty} a_k \cdot b^k$$

where a_k are non-negative integers and $0 \leq a_k < b$ for each k. Ken then erases m on the blackboard and replaces it with $\sum\limits_{k=0}^{\infty} a_k$.

Across all choices of $1 \leq n \leq 10^{10^5}$, the largest possible number of moves Ken could make is M. What is the remainder when M is divided by 10^5?

Answer: 32193

Solution: Consider the directed graph G with positive integers as vertices and a directed edge from n to every output of a move from n. We completely determine G.

Claim 1: The outgoing neighbours of n are exactly $\{\lceil n/2 \rceil, \lceil n/2 \rceil - 1, \ldots, 2, 1\}$.

Proof. These are achievable by bases $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots, n$, respectively since for a base $\lfloor n/2 \rfloor + 1 \leq b \leq n$, n has two digits in base-b, 1 and $n - b$, so the digit sum is $n + 1 - b$.

Next we show that there are no other neighbours. Equivalently, we need to prove that if there is an edge from n to r (that is, if for some base b with $2 \leq b \leq n$ the sum of the base-b digits of n equals r), then $r \leq \lceil n/2 \rceil$. We can verify this directly for $n \leq 10$, so henceforth assume $n > 10$.

Consider n in base-b, where $2 \leq b < n$. Since we wish to maximise the sum of the base-b digits, we can replace b^k with b copies of b^{k-1} as this will always increase the sum of the digits. Repeating this process for all powers b^k for $k \geq 2$, we reduce n to having only 2 base-b digits, the largest of which must be $\lfloor n/b \rfloor$ so we can write

$$n = \left\lfloor \frac{n}{b} \right\rfloor b + \left(n - b \left\lfloor \frac{n}{b} \right\rfloor \right).$$

Thus, in general

$$r = \text{Sum of base-}b\text{ digits of } n \leq \left\lfloor \frac{n}{b} \right\rfloor + \left(n - b \left\lfloor \frac{n}{b} \right\rfloor \right) = \left\lfloor \frac{n}{b} \right\rfloor + (n \bmod b)$$

where the final term should take the remainder in the range 0 to $b - 1$.

For $b = 2$, this gives $\lceil n/2 \rceil$.

For $3 \leq b \leq \lfloor n/2 \rfloor - 2$, we have the bound

$$r \leq \left\lfloor \frac{n}{b} \right\rfloor + (n \bmod b) \leq \frac{n}{b} + b - 1.$$

For fixed n, consider $f(b) = (n/b) + b$. Since

$$f(b+1) - f(b) = -\frac{n}{b(b+1)} + 1,$$

f will be decreasing up to a certain value (possibly no values) and then will be increasing from that point on. This means $f(b)$ will be maximised at an endpoint. By rearranging, we have

$$f(3) = (n/3) + 3 \leq \lceil n/2 \rceil + 1 \iff \begin{cases} n \geq 12 & n \text{ even} \\ n \geq 9 & n \text{ odd} \end{cases}$$

which are both true since $n > 10$.

For $b = \lfloor n/2 \rfloor - 2$, we have

$$f(\lfloor n/2 \rfloor - 2) = \frac{n}{\lfloor n/2 \rfloor - 2} + \lfloor n/2 \rfloor - 2 \leq \lceil n/2 \rceil + 1 \iff \begin{cases} n \geq 12 & n \text{ even} \\ n \geq 10 & n \text{ odd} \end{cases}$$

which are both true since $n > 10$. Thus, we have shown $f(b) \leq \lceil n/2 \rceil + 1$ for b in the given range meaning

$$r \leq f(b) - 1 \leq \lceil n/2 \rceil$$

as desired.

We are left to consider $\lfloor n/2 \rfloor - 1 \leq b \leq n$.

For $\lfloor n/2 \rfloor - 1 \leq b \leq \lfloor n/2 \rfloor$, n in base-b will have two digits, the first being a 2 and the second being 0, 1, 2, or 3. Thus the sum of the digits is $\leq 5 \leq \lceil n/2 \rceil$ (since $n > 10$).

For $\lfloor n/2 \rfloor + 1 \leq b \leq n$, n in base-b will be two digits, the first of which is 1 and the second of which is $n - b$. Thus,

$$r = 1 + (n - b) \leq n + 1 - (\lfloor n/2 \rfloor + 1) = n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$$

since $b \geq \lfloor n/2 \rfloor + 1$. This completes the proof. $\square$

Claim 2: The length of the longest path from n to 1 has length $\lceil \log_2(n) \rceil$.

Proof. Note that a move strictly decreases the value on the blackboard. Let $f(n)$ denote the length of the longest path from n to 1 in G. By considering the first move and using Claim 1, we get

$$f(n) = 1 + \max\{f(\lceil n/2 \rceil), f(\lceil n/2 \rceil - 1), \ldots, f(2), f(1)\}.$$

This Claim then follows by induction, using $\lceil \log_2(2n+1) \rceil = \lceil \log_2(2n) \rceil = 1 + \lceil \log_2(n) \rceil$ for $n \geq 1$ since $2n + 1$ is not a power of 2. $\square$

Thus, we extract the answer

$$\left\lceil \log_2 \left(10^{10^5} \right) \right\rceil = \lceil 10^5 \cdot \log_2 (10) \rceil = 332193 \equiv \boxed{32193} \pmod{10^5}$$

which is the number we report.

Remark: In order to prevent direct enumeration approaches, many of the numbers that appear in problems for AIMO3 are very large. We expect models to be able to handle calculations such as the above when calculating answers.

Problem 9

Problem: Let $\mathcal{F}$ be the set of functions $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ for which there are only finitely many $n \in \mathbb{Z}$ such that $\alpha(n) \neq 0$.

For two functions α and β in $\mathcal{F}$, define their product $\alpha \star \beta$ to be $\sum_{n\in\mathbb{Z}} \alpha(n) \cdot \beta(n)$. Also, for $n \in \mathbb{Z}$, define a shift operator $S_n\colon \mathcal{F} \to \mathcal{F}$ by $S_n(\alpha)(t) = \alpha(t+n)$ for all $t \in \mathbb{Z}$.

A function $\alpha \in \mathcal{F}$ is called *shifty* if

- $\alpha(m) = 0$ for all integers $m < 0$ and $m > 8$ and
- There exists $\beta \in \mathcal{F}$ and integers $k \neq l$ such that for all $n \in \mathbb{Z}$

$$S_n(\alpha) \star \beta = \begin{cases} 1 & n \in \{k, l\} \\ 0 & n \notin \{k, l\} \end{cases}.$$

How many shifty functions are there in $\mathcal{F}$?

Answer: 160

Solution: For $\alpha \in \mathcal{F}$, we define two functions

$$P_\alpha(x) = \sum_{k\in\mathbb{Z}} \alpha(k) \cdot x^k \qquad Q_\alpha(x) = \sum_{k\in\mathbb{Z}} \alpha(k) \cdot x^{-k}.$$

The conditions on $\mathcal{F}$ ensure that both sums only have finitely many non-zero terms and so P_α is a polynomial of degree at most 8 and Q_α is a finite Laurent polynomial (an extension of a normal polynomial to include negative powers of x).

Let $\alpha \in \mathcal{F}$ be shifty and let β be such that $S_n(\alpha) \star \beta$ is of the form defined in the question statement. We have

$$S_n(\alpha) \star \beta = \sum_{k\in\mathbb{Z}} S_n(\alpha)(k)\beta(k) = \sum_{k\in\mathbb{Z}} \alpha(k+n)\beta(k).$$

We can write

$$P_\alpha(x) = \sum_{k\in\mathbb{Z}} \alpha(k+n)x^{k+n} = x^n \sum_{k\in\mathbb{Z}} \alpha(k+n)x^k.$$

For a Laurent polynomial R and integer p, let $[x^p]\, R$ denote the x^p coefficient in R. Using the above, we can write

$$S_n(\alpha) \star \beta = [x^n]\, P_\alpha \cdot Q_\beta \implies x^k + x^l = \sum_{n\in\mathbb{Z}} S_n(\alpha) \star \beta = P_\alpha(x) Q_\beta(x)$$

where k and l are as in the condition in the problem statement.

Note that multiplying Q by a power of x (which is the same as shifting β to the right) will just increase the integers k and l (by the same amount). Thus, we can assume that Q_β is in fact a polynomial and k and l are non-negative integers.

The question now becomes how many polynomials P of degree at most 8 divide a polynomial of the form $x^k + x^l$ for non-negative integers $0 \leq k < l$. Setting $b = l - k \geq 1$, this is equivalent to dividing a polynomial of the form

$$x^a \left(x^b + 1\right)$$

for non-negative integers a and b.

Let $\Phi_n(x)$ denote the n^{th} cyclotomic polynomial which has integer coefficients and degree $\varphi(n)$ where φ is the Euler totient function. This is well-known to be irreducible and also we have

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Therefore, we can write

$$x^a\left(x^b+1\right) = x^a \cdot \frac{x^{2b}-1}{x^b-1} = x^a \cdot \prod_{d|2b\,,\, d\nmid b} \Phi_d(x).$$

As the Φ_d are all irreducible, a polynomial P will divide a polynomial of the above form if and only if it can be written as

$$P(x) = \pm x^m \prod_{d \in S} \Phi_d(x) \tag{$\blacksquare$}$$

for a non-negative integer m and a (possibly empty) set S which has the property that all its elements have the same largest power of 2 dividing them and that power is at least 2^1 (to enable us to choose a positive integer n such that $S \subset \{d : d|2n,\, d \nmid n\}$). We will count the number of choices with a $+$ sign and then double the count at the end to allow for the choices with a $-$ sign. Note that P is not identically 0 so these choices of sign will yield different polynomials.

Claim: $\varphi(n) \le 8$ if and only if $1 \le n \le 10$ or

$$n \in \{12, 14, 15, 16, 18, 20, 24, 30\}.$$

Proof. It is easy to check all the n in the given set give $\varphi(n) \le 8$. Now, for contradiction, assume there is a positive integer n not on our list with $\varphi(n) \le 8$.

If n has at least three odd prime divisors $p_1 < p_2 < p_3$ then

$$\varphi(n) \ge (p_1-1)(p_2-1)(p_3-1) \ge (3-1)(5-1)(7-1) = 48 > 8.$$

If n has two odd prime divisors $p_1 < p_2$ then if $p_2 \ge 7$ we have

$$\varphi(n) \ge (p_1-1)(p_2-1) \ge (3-1)(7-1) = 12 > 8.$$

Otherwise, $p_1 = 3$ and $p_2 = 5$ so $15 \mid n$. $n \in \{15, 30\}$ are already on our list so $n/15$ must be divisible by at least one of 3, 4 or 5 which means

$$\varphi(n) \ge 2 \cdot \varphi(n/15) \ge 2(3-1)(5-1) = 16 > 8.$$

If n has one odd prime divisor, say p with $\nu_p(n) = k$ for a positive integer k then

$$\varphi(n) \ge \varphi\left(p^k\right) = p^{k-1}(p-1) \ge 2 \cdot 3^{k-1}.$$

For $k \ge 3$, this is > 8.

For $k = 2$, if $p \ge 5$, this is $\ge 5 \cdot (5-1) > 8$ so the only possibility is $p = 3$. $n \in \{9, 18\}$ are already on our list so n must be divisible by 4 as well as 9 which means

$$\varphi(n) \ge 2 \cdot \varphi(9) = 2 \cdot 6 > 8.$$

For $k=1$, if $p \geq 11$ then we have $\varphi(n) \geq p-1 > 8$. Checking the remaining possibilities $p \in \{3,5,7\}$ and noting that for $\varphi(n)$, the largest power of 2 dividing n is at most 2^3, 2^2, and 2^1, respectively, we recover the n already on our list.

The last case is $n = 2^\alpha$ and since $\varphi(2^\alpha) = 2^{\alpha-1}$, we require $\alpha \leq 4$ for $\varphi(n) \leq 8$ and we get the remaining n from our list. □

We now split into cases based on the largest power of 2 dividing elements of S (which is the same for all elements of S as noted above). To avoid double-counting across cases, we exclude the 9 polynomials corresponding to S empty ($P(x) = 1, x, \ldots, x^8$) and add these back in at the end.

Case 1: All elements of S are divisible by 2 but not 4

From the Claim above, the only options for $s \in S$ such that $\varphi(s) \leq 8$ are in $S_1 = \{2, 6, 10, 14, 18, 30\}$. This is in turn equal to

$$\sum_{t=0}^{8} [y^t] \left[(1 + y + \cdots + y^8) \cdot \left(\prod_{s \in S_1} \left(1 + y^{\varphi(s)}\right) \right) \right].$$

Here the choice of t correspond to the degree of P, the choice of a summand in $(1 + y + \cdots + y^8)$ corresponds to m in (■), and the choice of 1 or $y^{\varphi(s)}$ in $\left(1 + y^{\varphi(s)}\right)$ corresponds to whether $\Phi_s(x)$ appears in the factorization of $P(x)$ (since $\deg \Phi_s = \varphi(s)$). For now we include the 9 choices for P where S is empty and will exclude these at the end.

Expanding the polynomial above, discarding terms of degree ≥ 9 (denoted by $\mathcal{O}(y^9)$) we get

$$\begin{aligned}
\cdots &= \left(1 + y + \cdots + y^8\right)(1+y)\left(1+y^2\right)\left(1+y^4\right)\left(1+y^6\right)^2\left(1+y^8\right) \\
&= \left(1 + y + \cdots + y^8\right)\left(1 + y + y^2 + y^3\right)\left(1+y^4\right)\left(1 + 2y^6 + y^8\right) + \mathcal{O}(y^9) \\
&= \left(1 + y + \cdots + y^8\right)\left(1 + y + y^2 + y^3\right)\left(1 + y^4 + 2y^6 + y^8\right) + \mathcal{O}(y^9) \\
&= \left(1 + y + \cdots + y^8\right)\left(1 + y + y^2 + y^3 + y^4 + y^5 + 3y^6 + 3y^7 + 3y^8\right) + \mathcal{O}(y^9).
\end{aligned}$$

We can then sum all coefficients of terms of degree ≤ 8 to get

$$9 + 8 + 7 + 6 + 5 + 4 + 3(3 + 2 + 1) = 57.$$

We must now subtract the 9 polynomials that have S empty giving $57 - 9 = 48$ possibilities for P from this case.

Case 2: All elements of S are divisible by 4 but not 8

From the Claim above, the only options for $s \in S$ such that $\varphi(s) \leq 8$ are in $S_2 = \{4, 12, 20\}$. The degrees of Φ_s for $s \in S_2$ is 2, 4, and 8, respectively. To ensure P has degree at most 8, we can choose

$$S = \{4\}, \{12\}, \{20\}, \{4, 12\}$$

which gives (counting the possibilities for m of which there are 9 minus the sum of the degrees of Φ_s since we can have $m = 0$)

$$7 + 5 + 1 + 3 = 16$$

possibilities for P.

Case 3: All elements of S are divisible by 8 but not 16

From the Claim, we must choose $S \subset \{8, 24\}$ and we have $\varphi(8) = 4$, $\varphi(24) = 8$. S must therefore be a singleton set and we get $5 + 1 = 6$ possibilities (counting the possibilities for m as above).

Case 4: All elements of S are divisible by 16 but not 32

In this case we have $S = \{16\}$ giving a single possibility for P since $\varphi(16) = 8$ so $m = 0$.

Combining the four cases and adding in the 9 possibilities for P where S is empty, we get

$$48 + 16 + 6 + 1 + 9 = 80$$

possibilities for P with a positive leading coefficient. We then need to double this to allow for a negative leading coefficient giving

$$2 \times 80 = \boxed{160}$$

choices in total which is the answer we report.

Problem 10

Problem: Let $n \geq 6$ be a positive integer. We call a positive integer n-Norwegian if it has three distinct positive divisors whose sum is equal to n. Let $f(n)$ denote the smallest n-Norwegian positive integer. Let $M = 3^{2025!}$ and for a non-negative integer c define

$$g(c) = \frac{1}{2025!}\left\lfloor \frac{2025!f(M+c)}{M} \right\rfloor.$$

We can write

$$g(0) + g(4M) + g(1848374) + g(10162574) + g(265710644) + g(44636594) = \frac{p}{q}$$

where p and q are coprime positive integers. What is the remainder when $p+q$ is divided by 99991?

Answer: 8687

Solution: In our solution to this problem, we include an extended commentary at the end that explains how we design problems to effectively evaluate a model's mathematical understanding within the constraints of an answer-only competition.

Let N be n-Norwegian for n odd and write the three divisors of N as

$$n = \frac{N}{p} + \frac{N}{q} + \frac{N}{r} \quad \text{where} \quad 1 \leq p < q < r \quad \text{and} \quad p, q, r \mid N.$$

Minimising N is equivalent to maximising $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$ subject to the resulting equation having an integer solution for N with $p, q, r \mid N$. We will provide a classification of the minimal N for all n odd.

We first consider the case when $p = 1$ and $q = 2$. We consider the possibilities for r based on whether it is odd and, in the even case, its remainder mod 4.

$$N = n \cdot \frac{2(2s+1)}{6s+5} \qquad (r = 2s+1)$$

$$N = n \cdot \frac{4s}{6s+1} \qquad (r = 4s)$$

$$N = n \cdot \frac{2s+1}{3s+2} \qquad (r = 4s+2)$$

Note in the last case, we have N odd (since n and $2s+1$ are odd) so $q = 2 \nmid N$ meaning this will not give an integer solution for N.

In the first two cases, the fraction is written in lowest terms so the denominator must divide n. Provided this is satisfied, we have $q, r \mid N$ so this will yield a valid N.

Thus, if d_1 and d_5 are the smallest divisors of n that are ≥ 6 and are 1 and 5 mod 6, respectively then we have two candidates for minimal N from choosing $d_1 = 6s+1$ or $d_5 = 6s+5$ (since we want to minimise r which is equivalent to minimising s). These give the following values for N:

$$\frac{2}{3} \cdot \frac{d_1 - 1}{d_1} \cdot n \quad \text{or} \quad \frac{2}{3} \cdot \frac{d_5 - 2}{d_5} \cdot n. \qquad (\blacktriangle)$$

Now we turn our attention to the case $p = 1$ and $q = 3$. Working through the small cases:

- $r = 4$ gives $N = \frac{12n}{19}$ which forces $19 \mid n$. We get the same answer then by taking $p = 1$, $q = 2$, and $r = 12$ so this is already covered by the first case.
- $r = 5$ gives $N = \frac{15n}{23}$ which forces $23 \mid n$. We get a smaller $N = \frac{14n}{23}$ by taking $p = 1$, $q = 2$, and $r = 7$ so this is already covered by the first case.
- $r = 6$ gives $N = \frac{2n}{3}$. For the divisibility condition $q, r \mid N$ to be satisfied, we require $9 \mid n$.
- If $r \geq 7$, then we have

$$N = n \cdot \frac{3r}{4r+3} = n\left(\frac{3}{4} - \frac{9}{16r+12}\right) \geq n\left(\frac{3}{4} - \frac{9}{16 \cdot 7 + 12}\right) = \frac{21n}{31} > \frac{2n}{3}$$

Putting the cases for $q = 2$ together with the case for $q = 3$ and $r = 6$, we have shown that if n has a divisor ≥ 6 that is ± 1 mod 6 or if $9 \mid n$ then we can find an n-Norwegian integer $\leq \frac{2n}{3}$. For this not to be the case, the only possibilities for prime factors of odd n are 3 and 5 and these must each occur with multiplicity 1 (otherwise we could use $5^2 \equiv 1 \bmod 6$). Thus, the only odd $n \geq 6$ that doesn't satisfy this condition is $n = 15$ and we can manually check that the minimum 15-Norwegian integer is 12 ($12 + 2 + 1 = 15$).

In all other cases, we claim one of the cases we've covered is optimal. This is true because

- We have already covered the cases for $p = 1$, $q \in \{2, 3\}$ that give $N \leq \frac{2n}{3}$.
- If $p = 1$, $q \geq 4$ then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 + \frac{1}{4} + \frac{1}{5} < \frac{3}{2}$ so $N > \frac{2n}{3}$.
- If $p \geq 2$ then $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} < \frac{3}{2}$ so $N > \frac{2n}{3}$.

Define d_1 and d_5 to be equal to ∞ if no such divisors of n exist. Taking the optimal cases from what we have considered above, we can write

$$f(n) = \begin{cases} 12 & \text{if} \quad n = 15 \\ \min\left\{\frac{2n}{3}, \frac{2}{3} \cdot \frac{d_1 - 1}{d_1} \cdot n, \frac{2}{3} \cdot \frac{d_5 - 2}{d_5} \cdot n\right\} & \text{if} \quad 9 \mid n \\ \min\left\{\frac{2}{3} \cdot \frac{d_1 - 1}{d_1} \cdot n, \frac{2}{3} \cdot \frac{d_5 - 2}{d_5} \cdot n\right\} & \text{otherwise} \end{cases}$$

where we have used that if $9 \nmid n$ and $n \neq 15$, then n has at least one factor that is ± 1 mod 6 by a similar argument to the one used above.

We can further simplify this. If we define p_1 and p_5 to be the smallest prime divisors of n that are ≥ 7 and are 1 and 5 mod 6, respectively (and set $p_i = \infty$ if these don't exist) then

$$f(n) = \begin{cases} 12 & \text{if} \quad n = 15 \\ 2n/3 & \text{if} \quad n = 3^\alpha, 5 \cdot 3^\alpha \text{ for } \alpha \geq 2 \\ 16n/25 & \text{if} \quad 25 \mid n \text{ and } p_1 \geq 31 \text{ and } p_5 \geq 53 \\ \min\left\{\frac{2}{3} \cdot \frac{p_1 - 1}{p_1} \cdot n, \frac{2}{3} \cdot \frac{p_5 - 2}{p_5} \cdot n\right\} & \text{otherwise} \end{cases} \tag{$\blacksquare$}$$

The bounds in the third case come from considering first when $d_1 = 25$. This will occur if and only if $p_1 > 25$ which is the same as $p_1 \geq 31$ (the first prime that is 1 mod 6 and greater than 25). In this case, setting $d_1 = 25$ in ($\blacktriangle$) gives

$$N = \frac{2}{3} \cdot \frac{d_1 - 1}{d_1} \cdot n = \frac{2}{3} \cdot \frac{25 - 1}{25} \cdot n = \frac{16}{25} \cdot n.$$

We then consider when d_5 could give a smaller value of N in ($\blacktriangle$) which requires

$$\frac{2}{3} \cdot \frac{d_5 - 2}{d_5} < \frac{16}{25} \iff d_5 < 50.$$

This will happen if and only if $p_5 < 50$ (noting that $125 > 50$ so we cannot get a suitable d_5 from powers of 5). Thus, for $N = 16n/25$ to be optimal, we require $p_5 \geq 50$ which is equivalent to $p_5 \geq 53$ as that is the smallest prime that is 5 mod 6 and greater than or equal to 50.

When setting this question, we want to test that the models can produce a correct answer across a wide range of cases. This reduces the risk of a model simply getting the correct answer via crude pattern spotting from small values of n.

From the classification in ($\blacksquare$), there are at least six cases that we would want to test:

(i) $n = 3^\alpha$ for $\alpha \geq 2$;

(ii) $n = 5 \cdot 3^\alpha$ for $\alpha \geq 3$ (there is little value in testing $n = 15$ since this falls to a direct computation);

(iii) $n = 25k$ where k has only 'large' prime divisors (so the third case holds);

(iv) Case where $25 \mid n$ but n is also divisible by a smaller prime so we move into the fourth case;

(v) Case where the minimum is the first term in the fourth case;

(vi) Case where the minimum is the second term in the fourth case (and we can create further challenge by choosing an example where $p_5 > p_1$).

For cases *(iii)* to *(vi)*, we may be tempted to use factorials to ensure no small prime divisors eg ask for $f(2025! + 125)$. However, this suffers from the issue that this is also equal to $f(125)$ so a model could stumble on the correct answer by simply omitting the factorial term. Instead, we appeal to modular arithmetic starting with $M = 3^{2025!}$ which, by the Fermat–Euler Theorem, has $M \equiv 1 \pmod{k}$ for $k \leq 2025$ with $\gcd(k, 3) = 1$ since $\varphi(k) \leq 2025$ and hence $\varphi(k) \mid 2025!$.

Thus, for a positive integer c and positive integer $k \leq 2025$ with $\gcd(k, 3) = 1$, we have

$$k \mid M + c \iff k \mid 1 + c.$$

Also, since $3 \mid M$ and $3 \nmid c$, $3 \nmid M + c$. The factors of $M + c$ that are ≤ 2025 will therefore be the same as the factors of $1 + c$. This allows us to calculate the 'small' factors of $n = M + c$ even though M is very large which in turn allows us to determine which case we fall into in ($\blacksquare$).

Now we select n to cover each of the cases described above:

(i) $n = M$ — $f(n) = 2n/3$;

(ii) $n = 5M$ — $f(n) = 2n/3$;

(iii) $n = M + 1848374$ ($1 + 1848374 = 3^2 \cdot 5^3 \cdot 31 \cdot 53$) — $f(n) = 16n/25$;

(iv) $n = M + 10162574$ ($1 + 10162574 = 3^2 \cdot 5^2 \cdot 31^2 \cdot 47$) — $f(n) = (2/3)(45/47)n$;

(v) $n = M + 265710644$ ($1 + 265710644 = 3^3 \cdot 5 \cdot 97 \cdot 103 \cdot 197$) — $f(n) = (2/3)(96/97)n$;

$$\frac{197 - 2}{197} = \frac{195}{197} > \frac{96}{97} = \frac{97 - 1}{97}.$$

(vi) $n = M + 44636594$ $(1 + 44636594 = 3 \cdot 5 \cdot 103 \cdot 167 \cdot 173)$ — $f(n) = (2/3)(165/167)n$.

$$\frac{167-2}{167} = \frac{165}{167} < \frac{102}{103} = \frac{103-1}{103}.$$

All models are allowed access to a calculator (and much more) which makes the factorisations straightforward (this would still be possible for a human given the modest prime factors but this would be somewhat arduous). We choose to keep the numbers large so even if the model has the idea of replacing M with 1, the cases are still challenging to compute directly.

In cases *(iii)* to *(vi)*, the expressions are of the form $f(M+c) = (s/t)(M+c)$ for positive integers s and t. Noting that $M \gg c$, $M \gg 2025!$ and $t \mid 2025!$ we have

$$\begin{aligned} g(c) &= \frac{1}{2025!}\left\lfloor \frac{2025! f(M+c)}{M} \right\rfloor \\ &= \frac{1}{2025!}\left\lfloor \frac{2025!}{t} \cdot s + \frac{2025! sc}{tM} \right\rfloor & \left(0 < \tfrac{2025! sc}{tM} < 1\right) \\ &= \frac{1}{2025!} \cdot \frac{2025!}{t} \cdot s \\ &= \frac{s}{t}. \end{aligned}$$

We constructed g in the problem statement precisely to have this property of extracting the fraction in front of n which is the 'interesting' part of the answer.

In cases *(i)* and *(ii)*, we have

$$\begin{aligned} g(0) &= \frac{1}{2025!}\left\lfloor \frac{2025! f(M)}{M} \right\rfloor = \frac{1}{2025!}\left\lfloor \frac{2025!(2M/3)}{M} \right\rfloor = \frac{1}{2025!}\left\lfloor \frac{2 \cdot 2025!}{3} \right\rfloor = \frac{1}{2025!} \cdot \frac{2 \cdot 2025!}{3} = \frac{2}{3} \\ g(4M) &= \frac{1}{2025!}\left\lfloor \frac{2025! f(5M)}{M} \right\rfloor = \frac{1}{2025!}\left\lfloor \frac{2025!(10M/3)}{M} \right\rfloor = \frac{1}{2025!}\left\lfloor \frac{10 \cdot 2025!}{3} \right\rfloor = \frac{1}{2025!} \cdot \frac{10 \cdot 2025!}{3} = \frac{10}{3}. \end{aligned}$$

Putting this all together,

$$\begin{aligned} &g(0) + g(4M) + g(1848374) + g(10162574) + g(265710644) + g(44636594) \\ = &\frac{2}{3} + \frac{10}{3} + \frac{16}{25} + \frac{2}{3} \cdot \frac{45}{47} + \frac{2}{3} \cdot \frac{96}{97} + \frac{2}{3} \cdot \frac{165}{167} \\ = &\frac{125561848}{19033825}. \end{aligned}$$

We then have

$$p = 125{,}561{,}848\,,\; q = 19{,}033{,}825 \implies p + q = 144{,}595{,}673 \equiv \boxed{8687} \pmod{99991}$$

which is the answer we report.

Remark 1: This problem was adapted from Problem N1 of the *International Mathematical Olympiad (IMO) Shortlist 2022* which can be found here. The original version simply asked for $f(2022)$.

All problems used in AIMO3 are entirely original; this adaptation is included solely to illustrate the process behind designing a challenging problem within the answer-only format. We reserve our original hard problems for the public and private Kaggle leaderboards, given the significant effort involved in creating them. The original problem itself is unsuitable for an answer-only setting, as it can be solved directly by enumerating possible positive integers and their divisor sums, leading straightforwardly to the correct answer of 1344.

Remark 2: Careful examination of the official solution provided in the shortlist reveals that it ultimately depends on expressing 2022 as $6(6k+1)$, where $6k+1$ is prime. The answer is then $24k$ (with $24k + 12k + 6 = 6(6k+1)$). Motivated by this, a simpler reformulation that removes any straightforward computational path is:

> **Problem:** $p = 2 \cdot 3^{30} + 1$ is prime. A positive integer is called Norwegian if it has three distinct positive divisors whose sum is equal to $6p$. Let N be the smallest Norwegian positive integer. What is the remainder when N is divided by 99991?
>
> **Answer:** 15245

However, this version tests only a single family of values for $f(n)$. A model might still conjecture a pattern by examining smaller primes, though this is complicated by the differing answers depending on whether $p \equiv \pm 1 \bmod 6$. The final version of the problem in the reference set generalises across a broader range of cases, making it extremely difficult to identify a solution pattern even from extensive numerical experimentation.